# APPROXIMATION OF THE LQR CONTROL PROBLEM FOR SYSTEMS GOVERNED BY PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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(ABSTRACT)



We present an abstract framework for state space formulation and a generalized theorem on well-posedness which can be applied to a class of partial functional differential equations which arise in the modeling of viscoelastic and certain thermo-viscoelastic systems. Examples to which the theory applies include both second- and fourth-order equations with a variety of boundary conditions. The theory presented here allows for singular kernels as well as flexibility in the choice of state space.

We discuss an approximation scheme using splines in the spatial variable and an averaging scheme in the delay variable. We compare a uniform mesh to a non-uniform mesh and give numerical results which indicate that the non-uniform mesh, which gives a better approximation of the kernel near the singularity, yields faster convergence. We give a proof of convergence of the simulation problem for singular kernels and of the control problem for bounded kernels. We use techniques of semigroup theory to establish the results on well-posedness and convergence.

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## TABLE OF CONTENTS

# Chapter I Introduction and Notation

## 1.1 Introduction.

In this paper we consider a general class of linear quadratic optimal control problems. Our goal is to minimize a quadratic cost functional of the form

$$J(z,u) = \int_0^\infty [\langle \mathcal{W}z, z\rangle_Z + \langle Ru, u\rangle_U]\, dt \tag{1.1.1}$$

subject to the linear system dynamics

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \tag{1.1.2}$$

$$z(0) = z_0. \tag{1.1.3}$$

Usually, the control space $U$ is $\mathbf{R}^m$, and $R$ is a diagonal matrix where $r_{ii} > 0$ is the weight or penalty on the $i$th controller. The operator $\mathcal{W} : Z \to Z$ is self-adjoint and nonnegative. Frequently $\mathcal{W}$ has the form $\mathcal{W} = \mathcal{C}^*\mathcal{C}$ where $\mathcal{C}$ is an output operator which measures various quantities related to the system. For example, if (1.1.2) describes the motion of a vibrating beam and the goal is to drive the beam to a state of rest, then $\mathcal{C}$ could be chosen to measure the deflection and velocity at several locations on the beam.

It can be shown (see [7]) that the unique control $u^* \in L_2(0,\infty;\mathbf{R}^m)$ which minimizes (1.1.1) is given by state feedback; in particular, $u^*(t)$ has the form

$$u^*(t) = -R^{-1}\mathcal{B}^*\mathcal{Q}z(t), \quad 0 \le t < \infty \tag{1.1.4}$$

where $\mathcal{Q}$ is the solution to the algebraic Riccati equation

$$\mathcal{A}^*\mathcal{Q} + \mathcal{Q}\mathcal{A} - \mathcal{Q}\mathcal{B}R^{-1}\mathcal{B}^*\mathcal{Q} + \mathcal{W} = 0. \tag{1.1.5}$$

Since in actual applications we are usually dealing with infinite dimensional systems, it is necesary to find a sub-optimal control $u_n^*(t)$ which, when applied to the infinite dimensional system, results in a stable closed-loop system whose response is "close" to optimal. We do this by approximating the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{W}$ by $A_n$, $B_n$ and $W_n$ and then solving the corresponding finite dimensional algebraic Riccati equation

$$A_n^TQ_n + Q_nA_n - Q_nB_nR^{-1}B_n^TQ_n + W_n = 0. \tag{1.1.6}$$

In [7], Gibson showed that $u_n^*(t)$ given by

$$u_n^*(t) = -R^{-1}B_n^*Q_nz_n(t) \tag{1.1.7}$$

converges in an appropriate sense to $u^*(t)$ if the $Q_n$'s converge strongly. In order to guarantee strong convergence of the $Q_n$'s, it is sufficient to show that $W_n \to \mathcal{W}$ and $B_n \to \mathcal{B}$ and that $T_n(\cdot)$ and $T_n^*(\cdot)$, the semigroups generated by $A_n$ and $A_n^*$, converge strongly to $T(\cdot)$ and $T^*(\cdot)$, the semigroups generated by $\mathcal{A}$ and $\mathcal{A}^*$, respectively.

### 1.2 Relationship to Previous Work.

The main emphasis of this paper is to study the control problem (1.1.1) – (1.1.3) for a general class of partial functional differential equations arising in the modeling of viscoelastic systems. The results that we present below can be extended to include models of thermo-viscoelasticity; for example (see [9]), coupled equations of the form

$$\sigma\frac{\partial^2}{\partial t^2}y(t,x) = \frac{\partial}{\partial x}\left[\tau\frac{\partial}{\partial x}y(t,x) + \int_{-r}^{0} g(s)\frac{\partial}{\partial x}y(t+s,x)ds\right] - \gamma\frac{\partial}{\partial x}\theta(t,x) + b(x)u(t), \tag{1.2.1}$$

$$\frac{\partial}{\partial t}\theta(t,x) = \kappa\frac{\partial^2}{\partial x^2}\theta(t,x) - \gamma_0\theta_0\frac{\partial^2}{\partial x\partial t}y(t,x). \tag{1.2.2}$$

Several researchers have studied coupled systems of hyperbolic and parabolic partial functional differential equations such as (1.2.1) - (1.2.2) (see [9, Chapter 6] for a summary of the literature). Most of the previous work is devoted to the basic questions of existence, uniqueness and continuous dependence of solutions and to the qualitative theory of these solutions and, moreover, is based on the assumption that the kernel function $g(s)$ is smooth on $[-r,0]$. In [9, p. 361] one can find a theorem on the well-posedness of an abstract form of such equations. This theorem and its proof, however, are incorrect.

Equations of viscoelasticity (e.g., (1.2.1) with $\gamma = 0$) have a special structure which we can exploit. For example, (1.2.1) with $\gamma = 0$ can be written as

$$\sigma\frac{\partial^2}{\partial t^2}y(t,x) = \frac{\partial^2}{\partial x^2}\left[\tau y(t,x) + \int_{-r}^{0} g(s)y(t+s,x)ds\right] + b(x)u(t), \tag{1.2.3}$$

or, in abstract form, as

$$\ddot{y} + A\left[\tau y + \int_{-r}^{0} g(s)y(t+s)ds\right] = f(t), \tag{1.2.4}$$

where $A$ is a positive definite, self-adjoint, closed linear operator on a Hilbert space $Y$. In [6] Fabiano and Ito consider equations of this form with singular kernels (i.e., $g \in L_1(-r,0)$) and establish well-posedness when the state space is taken to be $\mathcal{D}(A^{1/2}) \times Y \times L_2(-r,0;\mathcal{D}(A^{1/2}))$. Unfortunately, the thermo-viscoelastic equations (1.2.1) – (1.2.2) cannot be written in the form of (1.2.4), so the results in [6] cannot be applied.

In Section 2.2 we develop an abstract framework and a generalized well-posedness theorem which applies to equations of the form (1.2.4) and can also be applied to other more general equations. Our approach allows a singular kernel, and it also has the advantage that it does not require explicit knowledge of the domain of $A^{1/2}$ in order to write down the state space. This property can be useful in applications where $A^{1/2}$ is not a differential operator. We also remark that our general framework can be applied to certain finite delay systems similar in form to the infinite delay systems considered by Miller and Desch in [10]. Miller and Desch prove well-posedness for a class of equations in which the kernel is completely monotonic.

Although we present a generalized well-posedness theorem, the main goal of this paper is to develop a practical computational scheme for optimal control problems in which the system is governed by an equation of the form of (1.2.4). Approximation of such systems generally consists of two steps: first approximate the spatial variable (e.g., by means of splines) to reduce the system to a hereditary differential system on $\mathbf{R}^n$, then use the averaging scheme considered by Banks and Burns ([1]) to approximate the "history" or "memory" term (i.e., the integral term in (1.2.4)). The idea of the "AVE" scheme is essentially to approximate the kernel $g(s)$ by a step function: partition $[-r, 0]$ into $M$ subintervals and take the integral average in each subinterval. Fabiano and Ito show that the approximation scheme converges for an $L_1$ kernel using a uniform partition of $[-r, 0]$, but they give numerical results which indicate that a different partition using a finer mesh near the singularity at zero yields much faster convergence. In Chapter 3 we modify the proof given by Fabiano and Ito for singular kernels and a uniform mesh to include singular kernels and the non-uniform mesh. We also establish convergence of the adjoint system under the restriction that $g \in L_2(-r, 0)$. Convergence of the adjoint for $L_1$ kernels is not yet established and remains an open question. In Chapter 4 we present some numerical results illustrating the theory developed in Chapters 2 and 3.

### 1.3 Notation.

We will use the following notation. If $X$ is a linear space, then $\|\cdot\|_X$ and $\langle \cdot, \cdot \rangle_X$ will denote the norm and inner product on $X$. If $T$ is a linear operator from a space $X$ to a space $Y$, we denote the domain of $T$ by $\mathcal{D}(T)$ and the range of $T$ by $\mathcal{R}(T)$. We denote the resolvent set of $T$ by $\rho(T)$. For a Hilbert space $Z$, the set of all square integrable functions defined on $[a,b]$ with values in $Z$ will be denoted by $L_2(a,b;Z)$. The space of all absolutely continuous functions $f \in L_2(a,b;Z)$ with $j$th derivative $f^{(j)}$ absolutely continuous for $j = 1,2,\ldots,k-1$ and $f^{(k)} \in L_2(a,b;Z)$ is denoted by $H^k(a,b;Z)$. The symbol $H^1_L(a,b;Z)$ denotes the set of all $H^1$ functions which vanish at the left end-point of the interval; i.e., $H^1_L(a,b;Z) \equiv \{f \in H^1(a,b;Z) \mid f(a) = 0\}$. Similarly, $H^1_R(a,b;Z) \equiv \{f \in H^1(a,b;Z) \mid f(b) = 0\}$. For a function $x : [-r,\alpha) \to X$, $r,\alpha > 0$, the symbol $x_t$ for $t \in [0,\alpha)$ represents the function $x_t : [-r,0] \to X$ defined by $x_t(s) \equiv x(t+s)$. If $A$ is the infinitesimal generator of a $C_0$ semigroup $T(\cdot)$ on a Hilbert space $Z$ satisfying $\|T(t)\|_Z \leq Me^{\beta t}$, then we write $A \in G(M,\beta)$. Finally, $z_n \xrightarrow{s} z$ means that $z_n$ converges strongly to $z$.

# Chapter II Well-Posedness

In this chapter we present a well-posedness theorem for a general class of abstract ordinary differential equations. We then apply this result to a general class of partial functional differential equations (PFDE's) which arise, for example, in the Boltzmann model of linear viscoelastic systems. Finally we show how the theory developed in this chapter applies to several specific examples; namely, the distortion of a uniform bar, the torsional vibrations of a viscoelastic shaft with an attached tip-mass, and the transverse vibrations of an Euler-Bernoulli beam, also with a tip-mass.

## 2.1 State Space Formulation and Results from Semigroup Theory.

Our approach, given a PFDE, will be to reformulate the system as an abstract Cauchy problem on an appropriately chosen Hilbert space; i.e., to cast the problem in the form

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t) \tag{2.1.1}$$

$$z(0) = z_0 \tag{2.1.2}$$

where $z$ is an element of some Hilbert space $Z$ (the state space), and $u$ is in some control space $U$. The system (2.1.1) – (2.1.2) is well-posed; i.e., it has a unique solution which depends continuously on the initial data, if and only if $\mathcal{A}$ generates a $C_0$ semigroup on $Z$. A standard result from the theory of semigroups often used to establish well-posedness is the Lumer-Phillips Theorem:

THEOREM 2.1.1. *Let $A$ be a densely defined operator on a Banach space $X$. If $A$ is dissipative and there exists $\lambda_0 > 0$ such that the range of $\lambda_0 I - A$ is all of $X$, then $A$ is the infinitesimal generator of a $C_0$ semigroup of contractions on $X$.*

The proof of this theorem may be found in **[11]**. We will use the following corollary to prove the general result on well-posedness.

COROLLARY 2.1.2. *Let $A$ be a closed densely defined linear operator on a Hilbert space $H$. If there exists $\beta \in \mathbf{R}$ such that $\langle Ax, x\rangle \leq \beta\langle x, x\rangle$ for all $x \in \mathcal{D}(A)$, and $\mathcal{R}(\lambda_0 I - A)$ is dense in $H$ for some $\lambda_0 > \beta$, then $A$ is the infinitesimal generator of a $C_0$ semigroup $T(t)$ on $H$ satisfying $\|T(t)\| \leq e^{\beta t}$.*

PROOF: The operator $B$ defined by

$$\mathcal{D}(B) \equiv \mathcal{D}(A), \qquad B \equiv A - \beta I$$

is closed and densely defined. If $x \in \mathcal{D}(B)$, then $\langle Bx, x\rangle = \langle Ax, x\rangle - \beta\langle x, x\rangle \leq 0$, so $B$ is dissipative. Set $\lambda_1 \equiv \lambda_0 - \beta$. Then $\lambda_1 > 0$, and

$$\mathcal{R}(\lambda_1 I - B) = \mathcal{R}[(\lambda_0 - \beta)I - (A - \beta I)] = \mathcal{R}(\lambda_0 I - A)$$

is dense in $H$. Set $T \equiv \lambda_1 I - B = \lambda_0 I - A$. If $x \in \mathcal{D}(A)$ and $x \neq 0$, then $\langle Ax, x\rangle \leq \ \beta\,\langle x, x\rangle < \lambda_0\,\langle x, x\rangle$ which implies that $0 < \langle(\lambda_0 I - A)x, x\rangle = \langle Tx, x\rangle$.

Therefore, $T$ is injective. Now, for $x \in \mathcal{D}(T) = \mathcal{D}(B)$,

$$\begin{aligned}\|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \lambda_1^2\|x\|^2 - \lambda_1 \langle Bx, x\rangle - \lambda_1 \langle x, Bx\rangle + \|Bx\|^2 \\ &\geq \lambda_1^2\|x\|^2,\end{aligned}$$

so $T^{-1}$ is continuous. Since $T^{-1}$ is also closed, its domain must be closed. Thus, $\mathcal{R}(\lambda_1 I - B)$ is both closed and dense in $H$, so it must be all of $H$. Hence, by Theorem 2.1.1, $B$ generates a $C_0$ semigroup of contractions, $S(t)$, on $H$. If we define $T(t) \equiv e^{\beta t}S(t)$, then $T(t)$ is a $C_0$ semigroup whose infinitesimal generator is $A$, and $\|T(t)\| \leq e^{\beta t}$. ∎

In order to motivate the statement of the general theorem, let us consider the viscoelastic shaft with tip-mass mentioned at the beginning of this chapter. The equation describing the motion of the shaft (see [4] where the kernel $g(s)$ is assumed to be in $H^1$) is

$$\sigma\frac{\partial^2}{\partial t^2}y(t,x) = \frac{\partial}{\partial x}\left[\tau\frac{\partial}{\partial x}y(t,x) + \int_{-r}^{0} g(s)\frac{\partial}{\partial x}y(t+s,x)ds\right] + b(x)u_1(t), \tag{2.1.3}$$

while the boundary conditions are given by

$$y(t,0) = 0, \tag{2.1.4}$$

$$I_m\frac{\partial^2}{\partial t^2}y(t,l) = -\left[\tau\frac{\partial}{\partial x}y(t,l) + \int_{-r}^{0} g(s)\frac{\partial}{\partial x}y(t+s,l)ds\right] + u_2(t). \tag{2.1.5}$$

Here $\sigma$ is the product of the density of the shaft with its polar moment of inertia, $\tau$ is the product of the shear modulus and the polar moment of inertia, $I_m$ is the moment of inertia of the tip mass, and the delay $r > 0$ is assumed to be finite. The function $g : [-r, 0) \to \mathbf{R}$ satisfies the conditions

(1) $g < 0,\ g' \leq 0$ on $[-r, 0)$,

(2) $g \in L_1(-r, 0)$, and $\alpha \equiv \tau + \int_{-r}^{0} g(s)ds > 0$.

Set $w(t,s,x) \equiv y(t,x) - y(t+s,x)$, and let $g_\alpha(s) \equiv -\frac{1}{\alpha}g(s)$. We can then rewrite (2.1.3) and (2.1.5) and combine them into the the single equation

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} y(t,l) \\ y(t,x) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\frac{\alpha}{I_m}\delta_l D_x \\ 0 & \frac{\alpha}{\sigma}D_x^2 \end{pmatrix}\left[\begin{pmatrix} y(t,l) \\ y(t,x) \end{pmatrix} + \int_{-r}^{0} g_\alpha(s)\begin{pmatrix} w(t,s,l) \\ w(t,s,x) \end{pmatrix} ds\right] + \begin{pmatrix} \frac{1}{I_m}u_2(t) \\ \frac{1}{\sigma}b(x)u_1(t) \end{pmatrix} \tag{2.1.6}$$

where $D_x \equiv \dfrac{\partial}{\partial x}$ and $\delta_l \varphi \equiv \varphi(l)$ for $\varphi \in C(0,l)$. Observe that (2.1.6) has the form

$$\ddot{\widehat{y}} = \widehat{A}(\widehat{y} + \widehat{C}\widehat{w}) + Bu \tag{2.1.7}$$

where $\widehat{w}$ satisfies

$$\dot{\widehat{w}} = \dot{\widehat{y}} + \widehat{D}_S\widehat{w}. \tag{2.1.8}$$

We wish to recast (2.1.7) - (2.1.8) as a first order system. Let $Y$ be the Hilbert space $\mathbf{R} \times L_2(0,l)$, let $X \equiv H_L^1(0,l)$, and let $W \equiv L_2(-r, 0; X)$. Let $S \subseteq Y$ be given by

$$S \equiv \left\{ \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \in Y \;\middle|\; \psi \in H_L^1(0,l), \psi(l) = \gamma \right\}.$$

Define $j : S \to X$ by $j\begin{pmatrix} \gamma \\ \psi \end{pmatrix} \equiv \psi$, and $i : X \to W$ by $ix \equiv x$. If we set $x = j\widehat{y}$, $y = \dot{\widehat{y}}$

and $\widehat{w}(s,x) = \begin{pmatrix} w(s,l) \\ w(s,x) \end{pmatrix}$ and define $Cw \equiv j\widehat{C}\widehat{w}$, then

$$\dot{x} = jy, \tag{2.1.9}$$

$$\begin{aligned} \dot{y} = \dot{\hat{y}} &= \widehat{A}(j^{-1}x + \widehat{C}\widehat{w}) + Bu \\ &= \widehat{A}j^{-1}(x + j\widehat{C}\widehat{w}) + Bu = A(x + Cw) + Bu, \end{aligned} \tag{2.1.10}$$

and

$$\frac{\partial}{\partial t}\begin{pmatrix} w(s,l) \\ w(s,x) \end{pmatrix} = \dot{\widehat{w}} = \dot{\hat{y}} + \widehat{D}_s\widehat{w} = y + \begin{pmatrix} \frac{\partial}{\partial s}w(s,l) \\ \frac{\partial}{\partial s}w(s,x) \end{pmatrix}$$

which implies

$$\dot{w} = ijy + Dw. \tag{2.1.11}$$

Thus, if we set $Z \equiv X \times Y \times W$ and define the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq Z \to Z$ by

$$\mathcal{D}(\mathcal{A}) \equiv \left\{ \begin{pmatrix} x \\ y \\ w \end{pmatrix} \in Z \;\middle|\; \begin{matrix} y \in S, w \in \mathcal{D}(C) \cap \mathcal{D}(D), \\ (x + Cw) \in \mathcal{D}(A) \end{matrix} \right\},$$

$$\mathcal{A}\begin{pmatrix} x \\ y \\ w \end{pmatrix} \equiv \begin{pmatrix} jy \\ A(x + Cw) \\ ijy + Dw \end{pmatrix},$$

then we can write (2.1.3) – (2.1.5) in the form of (2.1.1) where $\mathcal{B} : U \to Z$ is given by

$$\mathcal{B}u \equiv \begin{pmatrix} 0 \\ Bu \\ 0 \end{pmatrix}.$$

Observe that the boundary condition (2.1.4) is incorporated into the definition of the state space $Z$.

**2.2 A General Theorem on Well-Posedness.**

We now turn our attention to the question of well-posedness of systems of the form (2.1.1) – (2.1.2). Suppose that $X$, $Y$ and $W$ are Hilbert spaces, and set $Z \equiv X \times Y \times W$. Let $S$ be a subspace of $Y$, and suppose we have the following linear operators:

$$A_0 : \mathcal{D}(A_0) \subseteq Y \to Y, \qquad A_1 : \mathcal{D}(A_1) \subseteq X \to Y,$$

$$C_1 : \mathcal{D}(C_1) \subseteq W \to Y, \qquad D : \mathcal{D}(D) \subseteq W \to W,$$

$$i : X \to W, \qquad j : S \to X,$$

where $i$ is continuous and $j$ is injective with $j^{-1} : \mathcal{R}(j) \to Y$ continuous. Define $A$ and $C$ by $A \equiv A_0A_1$, $C \equiv A_0C_1$, and define $A'$ by

$$\mathcal{D}(A') \equiv \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in X \times W \;\middle|\; x \in \mathcal{D}(A_1), w \in \mathcal{D}(C_1), A_1x + C_1w \in \mathcal{D}(A_0) \right\}$$

$$A' \begin{pmatrix} x \\ w \end{pmatrix} \equiv A_0(A_1x + C_1w).$$

Define $\mathcal{A}$ by

$$\mathcal{D}(\mathcal{A}) \equiv \left\{ \begin{pmatrix} x \\ y \\ w \end{pmatrix} \in Z \;\middle|\; \begin{matrix} y \in S, w \in \mathcal{D}(D), \\ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathcal{D}(A') \end{matrix} \right\},$$

$$\mathcal{A} \begin{pmatrix} x \\ y \\ w \end{pmatrix} \equiv \begin{pmatrix} jy \\ A'\begin{pmatrix} x \\ w \end{pmatrix} \\ ijy + Dw \end{pmatrix}.$$

Finally, for $\lambda \in \rho(D)$, define $L_\lambda : \mathcal{D}(L_\lambda) \subseteq X \to Y$ by

$$\mathcal{D}(L_\lambda) \equiv \left\{ x \in \mathcal{R}(j) \mid (x, \lambda j^{-1}x, (\lambda I - D)^{-1} i\lambda x)^T \in \mathcal{D}(\mathcal{A}) \right\},$$

$$L_\lambda x \equiv \lambda^2 j^{-1}x - A' \begin{pmatrix} x \\ (\lambda I - D)^{-1} i\lambda x \end{pmatrix}.$$

We are now ready to state the main result of this chapter.

THEOREM 2.2.1. *Suppose*

(1) $\mathcal{D}(A)$ *is dense in* $X$, $S$ *is dense in* $Y$ *and* $\mathcal{D}(C) \cap \mathcal{D}(D)$ *is dense in* $W$.

(2) $j(S)$ *is closed in* $X$.

(3) $A'$ *and* $D$ *are closed.*

(4) *There exists* $\beta \in \mathbf{R}$ *such that* $\langle z, \mathcal{A}z \rangle_Z \leq \beta \langle z, z \rangle_Z$ *for all* $z \in \mathcal{D}(\mathcal{A})$.

(5) *There exists* $\lambda_0 > \beta$, $\lambda_0 \in \rho(D)$, *such that* $\mathcal{R}(L_{\lambda_0})$ *is dense in* $Y$.

(6) $(\lambda_0 I - D)[\mathcal{D}(C) \cap \mathcal{D}(D)]$ *is dense in* $W$.

*Then* $\mathcal{A}$ *is the infinitesimal generator of a* $C_0$ *semigroup* $T(t)$ *on* $Z$ *satisfying* $\|T(t)\| \leq e^{\beta t}$.

PROOF: We must show that $\mathcal{A}$ is closed and densely defined and that $\mathcal{R}(\lambda_0 I - \mathcal{A})$ is dense in $Z$. Set $\mathcal{D} \equiv \mathcal{D}(A) \times S \times (\mathcal{D}(C) \cap \mathcal{D}(D))$. Then $\mathcal{D} \subseteq \mathcal{D}(\mathcal{A})$ and $\mathcal{D}$ is dense in $Z$, so $\mathcal{D}(\mathcal{A})$ is dense in $Z$.

Next, let $\begin{pmatrix} x_n \\ y_n \\ w_n \end{pmatrix} \in \mathcal{D}(\mathcal{A})$, $\begin{pmatrix} x_n \\ y_n \\ w_n \end{pmatrix} \to \begin{pmatrix} x \\ y \\ w \end{pmatrix}$ and $\mathcal{A}\begin{pmatrix} x_n \\ y_n \\ w_n \end{pmatrix} = \begin{pmatrix} \varphi_n \\ \psi_n \\ \theta_n \end{pmatrix} \to \begin{pmatrix} \varphi \\ \psi \\ \theta \end{pmatrix}$.
Then $y_n \in S$ and $jy_n = \varphi_n \to \varphi$. Since $j(S)$ is closed, $\varphi \in j(S)$; i.e., there exists $\widehat{y} \in S$ such that $j\widehat{y} = \varphi$. But $j^{-1}$ is bounded, so

$$\begin{aligned} \|\widehat{y} - y\| &\leq \|\widehat{y} - y_n\| + \|y_n - y\| \\ &= \|j^{-1}\varphi - j^{-1}\varphi_n\| + \|y_n - y\| \\ &\leq \|j^{-1}\| \cdot \|\varphi - \varphi_n\| + \|y_n - y\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $y = \hat{y}$; i.e., $y \in S$ and $jy = \varphi$. Now, $\begin{pmatrix} x_n \\ w_n \end{pmatrix} \in \mathcal{D}(A')$, $\begin{pmatrix} x_n \\ w_n \end{pmatrix} \to \begin{pmatrix} x \\ w \end{pmatrix}$, and $A' \begin{pmatrix} x_n \\ w_n \end{pmatrix} = \psi_n \to \psi$ as $n \to \infty$. Thus, $\begin{pmatrix} x \\ w \end{pmatrix} \in \mathcal{D}(A')$ and $A' \begin{pmatrix} x \\ w \end{pmatrix} = \psi$. Since $jy_n \to jy$ and $i$ is continuous, we have $ijy_n \to ijy$. We also have $ijy_n + Dw_n \to \theta$. Thus, $Dw_n \to \theta - ijy$. But $D$ is closed, so $w \in \mathcal{D}(D)$ and $Dw = \theta - ijy$ which implies that $ijy + Dw = \theta$. Therefore $\begin{pmatrix} x \\ y \\ w \end{pmatrix} \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A} \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ \theta \end{pmatrix}$; i.e., $\mathcal{A}$ is closed.

Finally, let $\begin{pmatrix} \varphi \\ \psi \\ \theta \end{pmatrix} \perp \mathcal{R}(\lambda_0 I - \mathcal{A})$; i.e.,

$$\langle \varphi, \lambda_0 x - jy \rangle_X + \left\langle \psi, \lambda_0 y - A' \begin{pmatrix} x \\ w \end{pmatrix} \right\rangle_Y + \langle \theta, (\lambda_0 I - D)w - ijy \rangle_W = 0$$

for all $\begin{pmatrix} x \\ y \\ w \end{pmatrix} \in \mathcal{D}(\mathcal{A})$. Let $x \in \mathcal{D}(L_{\lambda_0})$. Then $\begin{pmatrix} x \\ \lambda_0 j^{-1} x \\ (\lambda_0 I - D)^{-1} i \lambda_0 x \end{pmatrix} \in \mathcal{D}(\mathcal{A})$, so $\left\langle \psi, \lambda_0^2 j^{-1} x - A' \begin{pmatrix} x \\ (\lambda_0 I - D)^{-1} i \lambda_0 x \end{pmatrix} \right\rangle_Y = \langle \psi, L_{\lambda_0} x \rangle_Y = 0$ for all $x \in \mathcal{D}(L_{\lambda_0})$. By (5), $\psi = 0$. Thus, $\langle \varphi, \lambda_0 x - jy \rangle_X + \langle \theta, (\lambda_0 I - D)w - ijy \rangle_W = 0$ for all $\begin{pmatrix} x \\ y \\ w \end{pmatrix} \in \mathcal{D}(\mathcal{A})$. Let $x = 0$, $y = 0$. Then $\langle \theta, (\lambda_0 I - D)w \rangle_W = 0$ for all $w \in \mathcal{D}(C) \cap \mathcal{D}(D)$. Thus, $\theta = 0$ by (6). Now let $x \in \mathcal{D}(A), y = 0, w = 0$. Then $\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \in \mathcal{D}(\mathcal{A})$, so $\langle \varphi, \lambda_0 x \rangle_X = 0$ for all $x \in \mathcal{D}(A)$. By (1) this implies that $\varphi = 0$. Therefore, $\mathcal{R}(\lambda_0 I - \mathcal{A})$ is dense in $Z$. ∎

Observe that if $C_1$ can be factored as $C_1 = A_1 \widetilde{C}$ where $\widetilde{C} : \mathcal{D}(\widetilde{C}) \subseteq W \to X$, then

$$A' \begin{pmatrix} x \\ w \end{pmatrix} = A_0(A_1 x + C_1 w) = A_0(A_1 x + A_1 \widetilde{C} w) = A(x + \widetilde{C} w),$$

and $\mathcal{A}$ has the form given in the example in Section 2.1.

## 2.3 Well-Posedness of a General Class of PFDE's.

Let $Y$ be a Hilbert space and consider the equation

$$\ddot{y} + \widetilde{A}\left[\tau y + \int_{-r}^{0} g(s) y_s ds\right] = f(t) \qquad (2.3.1)$$

where $g < 0$, $g' \leq 0$ on $[-r, 0)$, $g \in L_1(-r, 0)$, and $\alpha \equiv \tau + \int_{-r}^{0} g(s)ds > 0$. Set $\widetilde{w}(s) \equiv y - y_s$ and $g_\alpha(s) \equiv -\frac{1}{\alpha} g(s)$. Substituting into (2.3.1) we get

$$\ddot{y} + \alpha\widetilde{A}\left[y + \int_{-r}^{0} g_\alpha(s)\widetilde{w}(s)ds\right] = f(t). \qquad (2.3.2)$$

Assume that $\widetilde{A}$ is a closed, densely defined, positive, self-adjoint, injective linear operator.

A standard technique (see e.g., [6]) for reformulating (2.3.2) as an abstract Cauchy problem is essentially to set $X = \mathcal{D}(\widetilde{A}^{1/2})$ where the inner product on $X$ satisfies $\langle x_1, x_2 \rangle_X = \left\langle \widetilde{A}^{1/2} x_1, \widetilde{A}^{1/2} x_2 \right\rangle_Y$ and to take the state space to be

$$Z = X \times Y \times L_2(-r, 0; X).$$

The approach we take here is similar except that it avoids the necessity of discussing the square root of $\widetilde{A}$. It is only necessary to know that the square root exists. Our approach also allows more flexibility in the choice of state space in that it is not necessary that $X$ be contained in $Y$, although $X$ will be in one-to-one correspondence with a subspace of $Y$.

With this discussion in mind, let $S$ be a subspace of $Y$ such that $S \supseteq \mathcal{D}(\widetilde{A})$, and let $\sigma(\cdot, \cdot)$ be a symmetric bilinear form on $S$ such that $\sigma(y_1, y_2) = \left\langle \alpha\widetilde{A}\, y_1,\ y_2 \right\rangle_Y$ whenever $y_1 \in \mathcal{D}(\widetilde{A})$ and $y_2 \in S$. Let $X$ be a Hilbert space and $j : S \to X$ a bijective linear operator such that $j^{-1} : X \to Y$ is continuous and $\langle x_1, x_2 \rangle_X = \sigma(j^{-1}x_1, j^{-1}x_2)$. Let $W \equiv L_2(-r, 0; X)$ with $\langle w_1, w_2 \rangle_W = \int_{-r}^{0} g_\alpha(s) \langle w_1(s), w_2(s) \rangle_X \, ds$. Set

$$Z \equiv X \times Y \times W \quad \text{with} \quad \|z\|_Z^2 = \left\| \begin{pmatrix} x \\ y \\ w \end{pmatrix} \right\|_Z^2 \equiv \|x\|_X^2 + \|y\|_Y^2 + \|w\|_W^2.$$

Define $A : \mathcal{D}(A) \subseteq X \to Y$ by

$$\mathcal{D}(A) \equiv \left\{ x \in X \,\middle|\, j^{-1}x \in \mathcal{D}(\widetilde{A}) \right\}, \qquad A \equiv -\alpha \widetilde{A} j^{-1}.$$

Then $z(t) \in Z$ satisfies

$$\frac{d}{dt} z(t) = \mathcal{A} z(t) + \mathrm{col}(0, f(t), 0), \tag{2.3.3}$$

where $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) \equiv \left\{ \begin{pmatrix} x \\ y \\ w \end{pmatrix} \in Z \,\middle|\, \begin{array}{c} y \in S, \quad w \in H_R^1(-r, 0; X), \\ \left( x + \int_{-r}^{0} g_\alpha(s) w(s) ds \right) \in \mathcal{D}(A) \end{array} \right\},$$

$$\mathcal{A} \begin{pmatrix} x \\ y \\ w \end{pmatrix} \equiv \begin{pmatrix} jy \\ A\left( x + \int_{-r}^{0} g_\alpha(s) w(s) ds \right) \\ jy + \frac{\partial w}{\partial s} \end{pmatrix}.$$

LEMMA 2.3.1. *The operator $D$ given by*

$$\mathcal{D}(D) = H_R^1(-r, 0; X), \qquad D = \frac{\partial}{\partial s}$$

*is dissipative in $W$.*

The following proof is due to Fabiano and Ito (see [**6**]).

PROOF: For $w \in \mathcal{D}(D)$,

$$\begin{aligned}\langle Dw, w\rangle_W &= \int_{-r}^{0} g_\alpha(s)\left\langle \frac{\partial}{\partial s}w(s), w(s)\right\rangle_X ds\\ &= \frac{1}{2}\int_{-r}^{0} g_\alpha(s)\frac{\partial}{\partial s}\|w(s)\|_X^2\, ds.\end{aligned}$$

Let $\epsilon > 0$ and consider

$$\begin{aligned}I_\epsilon &= \frac{1}{2}\int_{-r}^{-\epsilon} g_\alpha(s)\frac{\partial}{\partial s}\|w(s)\|_X^2\, ds\\ &= \frac{1}{2}g_\alpha(-\epsilon)\|w(-\epsilon)\|_X^2 - \frac{1}{2}g_\alpha(-r)\|w(-r)\|_X^2 - \frac{1}{2}\int_{-r}^{-\epsilon} g_\alpha'(s)\|w(s)\|_X^2\, ds\\ &\leq \frac{1}{2}g_\alpha(-\epsilon)\|w(-\epsilon)\|_X^2 .\end{aligned}$$

Since $w(-\epsilon) = w(0) - \int_{-\epsilon}^{0} Dw(s)ds = -\int_{-\epsilon}^{0} Dw(s)ds$, by the Cauchy-Schwarz inequality

$$\|w(-\epsilon)\|_X^2 \leq \int_{-\epsilon}^{0}\frac{ds}{g_\alpha(s)}\int_{-\epsilon}^{0} g_\alpha(s)\|Dw(s)\|_X^2 ds.$$

Note that $g_\alpha(-\epsilon)\int_{-\epsilon}^{0}\frac{ds}{g_\alpha(s)} = \int_{-\epsilon}^{0}\frac{g_\alpha(-\epsilon)}{g_\alpha(s)}ds \leq \epsilon$. Thus, we obtain

$$I_\epsilon \leq \epsilon\int_{-\epsilon}^{0} g_\alpha(s)\|Dw(s)\|_X^2\, ds$$

for all $\epsilon > 0$. Therefore

$$\langle Dw, w\rangle_W = \lim_{\epsilon\downarrow 0} I_\epsilon \leq 0. \quad \blacksquare$$

THEOREM 2.3.2. *$\mathcal{A}$ generates a $C_0$ semigroup on $Z$.*

PROOF: Let $A$, $D$ and $j$ be as above and consider the operators:

$$\mathcal{D}(\widetilde{C}) = W, \qquad \widetilde{C}w = \int_{-r}^{0} g_\alpha(s)w(s)ds;$$

$i : X \to W$ given by $ix = x$.

Clearly $i$ is continuous. By assumption, $j^{-1} : \mathcal{R}(j) = X \to Y$ is continuous. In order to apply Theorem 2.2.1, we must show that $A$ can be factored as $A = A_0A_1$. Since $\widetilde{A} : \mathcal{D}(\widetilde{A}) \subseteq Y \to Y$ is positive and self-adjoint, it has a positive square root $\widetilde{A}^{1/2}$. Define $A_0$ and $A_1$ by

$$\mathcal{D}(A_0) \equiv \mathcal{D}(\widetilde{A}^{1/2}), \qquad A_0 \equiv -\alpha\widetilde{A}^{1/2}$$

$$\mathcal{D}(A_1) \equiv \{x \in X \mid j^{-1}x \in \mathcal{D}(\widetilde{A}^{1/2})\}, \qquad A_1 \equiv \widetilde{A}^{1/2}j^{-1}.$$

Clearly, $A = A_0A_1$.

We now verify the six conditions of Theorem 2.2.1.

(1) $\widetilde{A}$ is densely defined and $S \supseteq \mathcal{D}(\widetilde{A})$, so $S$ is dense in $Y$. Suppose $\widehat{x} \perp \mathcal{D}(A)$. Then for all $x \in \mathcal{D}(A)$,

$$0 = \langle x, \widehat{x}\rangle_X = \sigma(j^{-1}x, j^{-1}\widehat{x}) = \alpha\left\langle \widetilde{A}j^{-1}x, j^{-1}\widehat{x}\right\rangle_Y,$$

which implies that $\left\langle \widetilde{A}y, j^{-1}\widehat{x}\right\rangle_Y = 0$ for all $y \in \mathcal{D}(\widetilde{A})$. But $\widetilde{A}$ is self-adjoint and one-to-one, so $\mathcal{R}(\widetilde{A})$ is dense in $Y$ ([**13**, Theorem 13.11]). Thus $j^{-1}\widehat{x} = 0$ which implies that $\widehat{x} = 0$. Therefore, $\mathcal{D}(A)$ is dense in $X$. Finally, $\mathcal{D}(C) \cap \mathcal{D}(D) = \mathcal{D}(A\widetilde{C}) \cap \mathcal{D}(D) = H^1_R(-r, 0; \mathcal{D}(A))$ which is dense in $W$.

(2) Obvious since $\mathcal{R}(j) = X$.

(3) $D$ is densely defined and dissipative. Let $h \in W$ and set $w(s) \equiv e^s \int_s^0 e^{-\sigma}h(\sigma)d\sigma$. Then $(I - D)w = h$, so $D$ generates a $C_0$ semigroup of contractions on $W$ by the Lumer-Phillips Theorem. In particular, $D$ is closed.

To show that $A'$ is closed, let $\begin{pmatrix} x_n \\ w_n \end{pmatrix} \in \mathcal{D}(A')$, $\begin{pmatrix} x_n \\ w_n \end{pmatrix} \to \begin{pmatrix} x \\ w \end{pmatrix}$, and $A'\begin{pmatrix} x_n \\ w_n \end{pmatrix} = y_n \to y$. Set

$$\widehat{y}_n = j^{-1}\left(x_n + \int_{-r}^{0} g_\alpha(s) w_n(s) ds\right)$$

and

$$\widehat{y} = j^{-1}\left(x + \int_{-r}^{0} g_\alpha(s) w(s) ds\right).$$

Since $\|w_n - w\|_W \to 0$ as $n \to \infty$, $\int_{-r}^{0} g_\alpha(s) \|w_n(s) - w(s)\|_X^2 \, ds \to 0$ as $n \to \infty$, which implies that $\|w_n(s) - w(s)\|_X \to 0$ as $n \to \infty$ for a.e. $s \in [-r, 0)$. Thus, $\int_{-r}^{0} \|w_n(s) - w(s)\|_X \, ds \to 0$ as $n \to \infty$ by the Dominated Convergence Theorem. Now, $\|\widehat{y}_n - \widehat{y}\|_Y \le \|j^{-1}\| \cdot \left[\|x_n - x\|_X + \int_{-r}^{0} g_\alpha(s) \|w_n(s) - w(s)\|_X \, ds\right] \to 0$ as $n \to \infty$. Also, $\widehat{y}_n \in \mathcal{D}(\widetilde{A})$ and $-\alpha\widetilde{A}\, \widehat{y}_n = A'\begin{pmatrix} x_n \\ w_n \end{pmatrix} = y_n \to y$. Since $\widetilde{A}$ is closed, $\widehat{y} \in \mathcal{D}(\widetilde{A})$ and $-\alpha\widetilde{A}\widehat{y} = y$; i.e., $x + \int_{-r}^{0} g_\alpha(s) w(s) ds \in \mathcal{D}(A)$ and $A\left(x + \int_{-r}^{0} g_\alpha(s) w(s) ds\right) = y$. Thus, $A'$ is closed.

(4) Let $\begin{pmatrix} x \\ y \\ w \end{pmatrix} \in \mathcal{D}(\mathcal{A})$. Then, using the definition of $\langle \cdot, \cdot \rangle_X$,

$$\begin{aligned} \left\langle \mathcal{A}\begin{pmatrix} x \\ y \\ w \end{pmatrix}, \begin{pmatrix} x \\ y \\ w \end{pmatrix} \right\rangle_Z &= \langle jy, x \rangle_X + \left\langle A\left(x + \int_{-r}^{0} g_\alpha(s) w(s) ds\right), y \right\rangle_Y \\ &\quad + \int_{-r}^{0} g_\alpha(s) \left\langle jy + \frac{\partial}{\partial s} w(s), w(s) \right\rangle_X ds \end{aligned}$$

$$= \left\langle jy, x + \int_{-r}^{0} g_\alpha(s) w(s) ds \right\rangle_X$$

$$- \alpha \left\langle \widetilde{A} j^{-1} \left( x + \int_{-r}^{0} g_\alpha(s) w(s) ds \right), y \right\rangle_Y$$

$$+ \int_{-r}^{0} g_\alpha(s) \left\langle \frac{\partial}{\partial s} w(s), w(s) \right\rangle_X ds$$

$$= \int_{-r}^{0} g_\alpha(s) \left\langle \frac{\partial}{\partial s} w(s), w(s) \right\rangle_X ds.$$

But $\int_{-r}^{0} g_\alpha(s) \left\langle \frac{\partial}{\partial s} w(s), w(s) \right\rangle_X ds \le 0$ by Lemma 2.3.1. Thus $\langle \mathcal{A} z, z \rangle_Z \le 0$ for all $z \in \mathcal{D}(\mathcal{A})$.

(6) We will take $\lambda_0 = 1$. Since $D$ generates a $C_0$ semigroup of contractions, $\lambda_0 \in \rho(D)$ by the Hille-Yosida Theorem (see [11]). Let $h \in \mathcal{D}(A\widetilde{C})$ and set $w(s) = e^s \int_s^0 e^{-\sigma} h(\sigma) d\sigma$. Then $w \in H_R^1(-r, 0; \mathcal{D}(A)) = \mathcal{D}(A\widetilde{C}) \cap \mathcal{D}(D)$, and $(I - D)w = h$. Thus, $\mathcal{D}(A\widetilde{C}) \subseteq (I - D)[\mathcal{D}(A\widetilde{C}) \cap \mathcal{D}(D)]$, and $\mathcal{D}(A\widetilde{C})$ is dense in $W$ since $\mathcal{D}(A)$ is dense in $X$.

(5) Let $x \in \mathcal{D}(L_1)$. Set $w(s) \equiv (1 - e^s)x$. Then $w \in \mathcal{D}(D)$, and $Dw = -e^s x$, so $(I - D)w = x$, or $(I - D)^{-1} x = (1 - e^s)x$. Thus, $\begin{pmatrix} x \\ j^{-1}x \\ (1 - e^s)x \end{pmatrix} \in \mathcal{D}(\mathcal{A})$, and

$$L_1 x = j^{-1}x - A\left( x + \int_{-r}^{0} g_\alpha(s)(1 - e^s) x ds \right)$$

$$= j^{-1}x + \alpha \widetilde{A} j^{-1} \left( x + \int_{-r}^{0} g_\alpha(s)(1 - e^s) x ds \right)$$

$$= j^{-1}x + \alpha_0 \widetilde{A} j^{-1} x$$

where $\alpha_0 \equiv \alpha - \int_{-r}^{0} g(s)(1 - e^s) ds = \tau + \int_{-r}^{0} g(s) e^s ds > 0$. Let $T$ be defined by

$$\mathcal{D}(T) \equiv \{ y \in S \mid jy \in \mathcal{D}(L_1) \}, \qquad T \equiv I + \alpha_0 \widetilde{A}.$$

It is easy to see that $\mathcal{D}(\widetilde{A}) = \mathcal{D}(T)$ and $\mathcal{R}(T) = \mathcal{R}(L_1)$. By the Cauchy-Schwarz Inequality, for $y \in \mathcal{D}(T)$, $\|Ty\| \cdot \|y\| \geq |\langle Ty, y\rangle| = \left|\left\langle I + \alpha_0\widetilde{A}y, y\right\rangle\right| = \|y\|^2 + \alpha_0\left\langle \widetilde{A}y, y\right\rangle \geq \|y\|^2$ which implies $\|Ty\| \geq \|y\|$, and so $T$ is one-to-one. Also, $T^* = (I + \alpha_0\widetilde{A})^* = I + \alpha_0\widetilde{A}^* = I + \alpha_0\widetilde{A} = T$. Thus, by [**13**, Theorem 13.11], $\mathcal{R}(T)$ is dense in $Y$. ∎

## 2.4 Examples.

In this section we will use Theorem 2.3.2 to establish well-posedness of the three examples mentioned at the beginning of the chapter. We begin with the viscoelastic shaft which satisfies equations (2.1.3) – (2.1.5). Let $Y = \mathbf{R} \times L_2(0, l)$ with

$$\|y\|_Y^2 = \left\|\begin{pmatrix}\gamma\\ \psi\end{pmatrix}\right\|_Y^2 = I_m\gamma^2 + \sigma\int_0^l \psi^2.$$

Then (2.1.3) – (2.1.5) can be written as

$$\ddot{y} + \widetilde{A}\left[\tau y + \int_{-r}^{0} g(s) y_s ds\right] = f(t) \tag{2.4.1}$$

where $f(t) = \begin{pmatrix}\frac{1}{I_m}u_2(t)\\ \frac{1}{\sigma}b(x)u_1(t)\end{pmatrix}$, and

$$\mathcal{D}(\widetilde{A}) = \left\{\begin{pmatrix}\gamma\\ \psi\end{pmatrix} \in Y \;\middle|\; \psi \in H_L^1(0,l) \cap H^2(0,l), \psi(l) = \gamma\right\},$$

$$\widetilde{A}\begin{pmatrix}\gamma\\ \psi\end{pmatrix} = \begin{pmatrix}\frac{1}{I_m}\psi'(l)\\ -\frac{1}{\sigma}\psi''\end{pmatrix}.$$

Clearly, $\mathcal{D}(\widetilde{A})$ is dense in $Y$. Let $\begin{pmatrix} \gamma \\ \psi \end{pmatrix} \in \mathcal{D}(\widetilde{A})$. Then

$$\left\langle \widetilde{A}\begin{pmatrix} \gamma \\ \psi \end{pmatrix}, \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \right\rangle = \psi'(l)\psi(l) - \int_0^l \psi''\psi = \int_0^l (\psi')^2$$
$$\geq \frac{2}{l^2}\int_0^l \psi^2 \geq \frac{2}{\sigma l^2}\left\| \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \right\|^2 \geq 0.$$

Thus, $\widetilde{A}$ is positive and $\frac{2}{\sigma l^2}\left\| \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \right\|^2 \leq \left\| \widetilde{A}\begin{pmatrix} \gamma \\ \psi \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \right\|$, so $\widetilde{A}$ is one-to-one (and $\widetilde{A}^{-1}$ is continuous). Let $\begin{pmatrix} \gamma_1 \\ \psi_1 \end{pmatrix}$, $\begin{pmatrix} \gamma_2 \\ \psi_2 \end{pmatrix} \in \mathcal{D}(\widetilde{A})$. Then

$$\begin{aligned}\left\langle \widetilde{A}\begin{pmatrix} \gamma_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \gamma_2 \\ \psi_2 \end{pmatrix} \right\rangle &= \psi_1'(l)\psi_2(l) - \int_0^l \psi_1''\psi_2 \\ &= \int_0^l \psi_1'\psi_2' \\ &= \psi_1(l)\psi_2'(l) - \int_0^l \psi_1\psi_2'' = \left\langle \begin{pmatrix} \gamma_1 \\ \psi_1 \end{pmatrix}, \widetilde{A}\begin{pmatrix} \gamma_2 \\ \psi_2 \end{pmatrix} \right\rangle,\end{aligned}$$

so $\widetilde{A}$ is symmetric.

Let $\begin{pmatrix} \widehat{\gamma} \\ \widehat{\psi} \end{pmatrix} \in Y$. Define $\psi(x) \equiv \int_0^x \left[\int_t^l \sigma\widehat{\psi}(\xi)d\xi + I_m\widehat{\gamma}\right] dt$. Then $\psi(0) = 0$, $\psi'(x) = \int_x^l \sigma\widehat{\psi}(\xi)d\xi + I_m\widehat{\gamma}$ which implies $\psi'(l) = I_m\widehat{\gamma}$, and $\psi''(x) = -\sigma\widehat{\psi}(x)$. Thus, $\begin{pmatrix} \psi(l) \\ \psi \end{pmatrix} \in \mathcal{D}(\widetilde{A})$, and $\widetilde{A}\begin{pmatrix} \psi(l) \\ \psi \end{pmatrix} = \begin{pmatrix} \widehat{\gamma} \\ \widehat{\psi} \end{pmatrix}$, so $\mathcal{R}(\widetilde{A}) = Y$. Therefore, by Theorem 13.11 in [**13**], $\widetilde{A}^* = \widetilde{A}$. Finally, $\mathcal{D}(\widetilde{A}^{-1}) = \mathcal{R}(\widetilde{A}) = Y$, so $\widetilde{A}^{-1}$ is closed, and hence $\widetilde{A}$ is closed. Now let $S = \left\{ \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \in Y \,\middle|\, \psi \in H_L^1(0,l), \quad \psi(l) = \gamma \right\}$, and define $\sigma\left(\begin{pmatrix} \gamma_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \gamma_2 \\ \psi_2 \end{pmatrix}\right) \equiv \alpha\int_0^l \psi_1'\psi_2'$ for $\begin{pmatrix} \gamma_1 \\ \psi_1 \end{pmatrix}$, $\begin{pmatrix} \gamma_2 \\ \psi_2 \end{pmatrix} \in S$. Then, $S \supseteq \mathcal{D}(\widetilde{A})$ and $\sigma(y_1, y_2) = \left\langle \alpha\widetilde{A}y_1, y_2 \right\rangle_Y$ whenever $y_1 \in \mathcal{D}(\widetilde{A}), y_2 \in S$. Let $X = H_L^1(0,l)$ with $\langle x_1, x_2\rangle_X = \alpha\int_0^l x_1'x_2'$, and define $j : S \to X$ by $j\begin{pmatrix} \psi(l) \\ \psi \end{pmatrix} \equiv \psi$. Clearly $j$ is a bijective linear operator, and

$$\langle x_1, x_2 \rangle_X = \alpha \int_0^l x_1' x_2' = \sigma \left( \begin{pmatrix} x_1(l) \\ x_1 \end{pmatrix}, \begin{pmatrix} x_2(l) \\ x_2 \end{pmatrix} \right) = \sigma \left( j^{-1} x_1, j^{-1} x_2 \right).$$

For $x \in X$, $\|j^{-1}x\|_Y^2 = I_m x^2(l) + \sigma \int_0^l x^2 \leq \left( \frac{2I_m l + \sigma l^2}{2\alpha} \right) \|x\|_X^2$. Thus, $j^{-1}$ is continuous.

Now, define $Z = H_L^1(0,l) \times \mathbf{R} \times L_2(0,l) \times L_2(-r, 0; H_L^1(0,l))$ with

$$\left\| \begin{pmatrix} \varphi \\ \gamma \\ \psi \\ w \end{pmatrix} \right\|_Z^2 = \alpha \int_0^l (\varphi')^2 + I_m \gamma^2 + \sigma \int_0^l \psi^2 + \int_{-r}^0 g_\alpha(s) \int_0^l \left( \frac{\partial}{\partial x} w(s) \right)^2 dx ds,$$

and define

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} \varphi \\ \gamma \\ \psi \\ w \end{pmatrix} \in Z \;\middle|\; \begin{array}{c} \psi \in H_L^1(0,l), \quad \psi(l) = \gamma, \\ w \in H_R^1(-r, 0; H_L^1(0,l)), \\ (\varphi'(x) + \int_{-r}^0 g_\alpha(s) \frac{\partial}{\partial x} w(s,x) ds) \in H^1(0,l) \end{array} \right\},$$

$$\mathcal{A} \begin{pmatrix} \varphi \\ \gamma \\ \psi \\ w \end{pmatrix} = \begin{pmatrix} \psi \\ -\frac{\alpha}{I_m} \left[ \varphi'(l) + \int_{-r}^0 g_\alpha(s) \frac{\partial}{\partial x} w(s,l) ds \right] \\ \frac{\alpha}{\sigma} \frac{\partial}{\partial x} \left[ \varphi'(x) + \int_{-r}^0 g_\alpha(s) \frac{\partial}{\partial x} w(s,x) ds \right] \\ \psi + \frac{\partial w}{\partial s} \end{pmatrix}.$$

The operator $\mathcal{A}$ generates a $C_0$ semigroup on $Z$ by Theorem 2.3.2.

Now suppose we replace equation (2.1.5) by

$$I_m \frac{\partial^2}{\partial t^2} y(t,l) = -\left[ k y(t,l) + \tau \frac{\partial}{\partial x} y(t,l) + \int_{-r}^0 g(s) \frac{\partial}{\partial x} y(t+s, l) ds \right] + u_2(t), \quad (2.4.2)$$

where $k > 0$; i.e., we include a term representing a spring at the end of the shaft. We can formulate the system (2.1.3), (2.1.4), (2.4.2) as the abstract Cauchy problem

$$\dot{z}(t) = \tilde{\mathcal{A}} z + \mathcal{B} u$$

$$z(0) = z_0$$

where $\mathcal{D}(\widetilde{\mathcal{A}}) = \mathcal{D}(\mathcal{A})$ and

$$\widetilde{\mathcal{A}} \begin{pmatrix} \varphi \\ \gamma \\ \psi \\ w \end{pmatrix} = \begin{pmatrix} \psi \\ -\frac{1}{I_m}\left[k\varphi(l) + \alpha\varphi'(l) + \alpha\int_{-r}^{0} g_\alpha(s)\frac{\partial}{\partial x}w(s,l)ds\right] \\ \frac{\alpha}{\sigma}\frac{\partial}{\partial x}\left[\varphi'(x) + \int_{-r}^{0} g_\alpha(s)\frac{\partial}{\partial x}w(s,x)ds\right] \\ \psi + \frac{\partial w}{\partial s} \end{pmatrix}.$$

Observe that $\widetilde{\mathcal{A}} = \mathcal{A} + \mathcal{A}_1$ where $\mathcal{D}(\mathcal{A}_1) = Z$, and

$$\mathcal{A}_1 \begin{pmatrix} \varphi \\ \gamma \\ \psi \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{k}{I_m}\varphi(l) \\ 0 \\ 0 \end{pmatrix}.$$

Now, $\|\mathcal{A}_1 z\|_Z^2 = \frac{k^2}{I_m}\varphi^2(l) \le \frac{k^2 l}{I_m\alpha}\|\varphi\|_X^2 \le \frac{k^2 l}{I_m\alpha}\|z\|_Z^2$. Thus, $\|\mathcal{A}_1\| \le \sqrt{\frac{k^2 l}{I_m\alpha}}$, so $\widetilde{\mathcal{A}}$ is a bounded perturbation of $\mathcal{A}$. By Theorem 1.1 in [**11**, Chapter 3], $\widetilde{\mathcal{A}}$ generates a $C_0$ semigroup $S(t)$ on $Z$ satisfying $\|S(t)\| \le e^{\sqrt{k^2 l/I_m\alpha}\, t}$.

For our next example we will consider the longitudinal distortion of a uniform bar. This problem has been studied by Fabiano (see [**3**], [**5**]) using a formulation in which the state variable $z(t)$ corresponds to

$$\begin{pmatrix} \frac{\partial y}{\partial t} \\ \frac{\partial y}{\partial x} \\ \frac{\partial}{\partial x}(y_t) \end{pmatrix}.$$

For this problem the PFDE describing the motion of the bar is the same as in the previous example (i.e., equation (2.1.3)), while the boundary conditions are

$$y(t,0) = 0 = y(t,l). \tag{2.4.3}$$

We take $Y$ to be $L_2(0,l)$ with $\langle y_1, y_2\rangle_Y = \sigma \int_0^l y_1 y_2$, and define $\widetilde{A}$ by

$$\mathcal{D}(\widetilde{A}) \equiv H_0^1(0,l) \cap H^2(0,l), \qquad \widetilde{A} \equiv -\frac{1}{\sigma}\frac{\partial^2}{\partial x^2}.$$

Using the same arguments as before, we can show that $\widetilde{A}$ satisfies the conditions of Theorem 2.3.2. Set $S = H_0^1(0,l)$, and define $\sigma(y_1, y_2) \equiv \alpha \int_0^l y_1' y_2'$ for $y_1, y_2 \in S$. Take $X = H_0^1(0,l)$ with $\langle x_1, x_2\rangle_X = \alpha \int_0^l x_1' x_2'$, and define $j : S \to X$ by $jy = y$. As in the previous example, $\sigma(\cdot,\cdot)$ satisfies $\langle x_1, x_2\rangle_X = \sigma(j^{-1}x_1, j^{-1}x_2)$ and $\sigma(y_1, y_2) = \left\langle \alpha\widetilde{A}y_1, y_2\right\rangle_Y$ whenever $y_1 \in \mathcal{D}(\widetilde{A})$, and $j^{-1}$ is continuous. Thus, if we set

$$Z = H_0^1(0,l) \times L_2(0,l) \times L_2(-r,0; H_0^1(0,l))$$

and define $\mathcal{A}$ by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} \in Z \;\middle|\; \begin{array}{c} \psi \in H_0^1(0,l), \\ w \in H_R^1(-r,0; H_0^1(0,l)), \\ (\varphi'(x) + \int_{-r}^0 g_\alpha(s)\frac{\partial}{\partial x}w(s,x)ds) \in H^1(0,l) \end{array} \right\},$$

$$\mathcal{A}\begin{pmatrix} \varphi \\ \psi \\ w \end{pmatrix} = \begin{pmatrix} \psi \\ \frac{\alpha}{\sigma}\frac{\partial}{\partial x}\left[\varphi'(x) + \int_{-r}^0 g_\alpha(s)\frac{\partial}{\partial x}w(s,x)ds\right] \\ \psi + \frac{\partial w}{\partial s} \end{pmatrix},$$

then $\mathcal{A}$ generates a $C_0$ semigroup on $Z$.

For our final example we consider the transverse vibrations of an Euler-Bernoulli beam with a tip-mass. The beam deflection is given by the equation

$$\rho\frac{\partial^2}{\partial t^2}y(t,x) = -\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2}{\partial x^2}y(t,x) + \int_{-r}^0 g(s)\frac{\partial^2}{\partial x^2}y(t+s,x)ds\right]. \tag{2.4.4}$$

The left-hand boundary conditions are

$$y(t,0) = 0 = \frac{\partial}{\partial x}y(t,0), \tag{2.4.5}$$

and at the right-hand end we have the equations giving the rotation of the tip-mass:

$$I_c \frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial x} y(t,l)\right) = EI \frac{\partial^2}{\partial x^2} y(t,l) + \int_{-r}^{0} g(s) \frac{\partial^2}{\partial x^2} y(t+s,l) ds, \qquad (2.4.6)$$

and the transverse motion of the tip-mass:

$$M_c \frac{\partial^2}{\partial t^2} y(t,l) = \frac{\partial}{\partial x}\left[EI \frac{\partial^2}{\partial x^2} y(t,l) + \int_{-r}^{0} g(s) \frac{\partial^2}{\partial x^2} y(t+s,l) ds\right]. \qquad (2.4.7)$$

To cast (2.4.4) – (2.4.7) in the form of (2.3.1), we set

$$Y = \mathbf{R} \times \mathbf{R} \times L_2(0,l) \text{ with } \left\| \begin{pmatrix} \xi \\ \eta \\ \psi \end{pmatrix} \right\|_Y^2 = I_c \xi^2 + M_c \eta^2 + \rho \int_0^l \psi^2,$$

let $\tau = EI$, and define $\widetilde{A}$ by

$$\mathcal{D}(\widetilde{A}) = \left\{ \begin{pmatrix} \xi \\ \eta \\ \psi \end{pmatrix} \in Y \,\middle|\, \psi \in H_L^2(0,l) \cap H^4(0,l), \quad \psi(l) = \eta, \quad \frac{\partial}{\partial x}\psi(l) = \xi \right\},$$

$$\widetilde{A}\begin{pmatrix} \xi \\ \eta \\ \psi \end{pmatrix} = \begin{pmatrix} -\frac{1}{I_c}\frac{\partial^2}{\partial x^2}\psi(l) \\ -\frac{1}{M_c}\frac{\partial^3}{\partial x^3}\psi(l) \\ \frac{1}{\rho}\frac{\partial^4}{\partial x^4}\psi \end{pmatrix}.$$

Once again it is easy to see that $\widetilde{A}$ is densely defined, closed, positive, self-adjoint and one-to-one. We set

$$S = \left\{ \begin{pmatrix} \xi \\ \eta \\ \psi \end{pmatrix} \in Y \,\middle|\, \psi \in H_L^2(0,l), \quad \psi(l) = \eta, \quad \frac{\partial}{\partial x}\psi(l) = \xi \right\},$$

and define $j : S \to X \equiv H_L^2(0,l)$ by $j\begin{pmatrix} \xi \\ \eta \\ \psi \end{pmatrix} = \psi$. Here the inner product on $X$ is given by $\langle x_1, x_2 \rangle_X = \alpha \int_0^l x_1'' x_2''$. The operator $\mathcal{A}$ defined on

$$Z \equiv H_L^2(o,l) \times \mathbf{R}^2 \times L_2(0,l) \times L_2(-r,0; H_L^2(0,l))$$

by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} \varphi \\ \xi \\ \eta \\ \psi \\ w \end{pmatrix} \in Z \;\middle|\; \begin{array}{c} \psi \in H_L^2, \quad \psi(l) = \eta, \quad \frac{\partial}{\partial x}\psi(l) = \xi, \\ w \in H_R^1(-r, 0; H_L^2), \\ [\varphi'' + \int_{-r}^{0} g_\alpha(s) \frac{\partial^2}{\partial x^2} w(s,x) ds] \in H^2 \end{array} \right\},$$

$$\mathcal{A} \begin{pmatrix} \varphi \\ \xi \\ \eta \\ \psi \\ w \end{pmatrix} = \begin{pmatrix} \psi \\ \frac{\alpha}{I_c}[\varphi''(l) + \int_{-r}^{0} g_\alpha(s) \frac{\partial^2}{\partial x^2} w(s,l) ds] \\ \frac{\alpha}{M_c} \frac{\partial}{\partial x}[\varphi''(l) + \int_{-r}^{0} g_\alpha(s) \frac{\partial^2}{\partial x^2} w(s,l) ds] \\ -\frac{\alpha}{M_c} \frac{\partial^2}{\partial x^2}[\varphi''(x) + \int_{-r}^{0} g_\alpha(s) \frac{\partial^2}{\partial x^2} w(s,x) ds] \\ \psi + \frac{\partial w}{\partial s} \end{pmatrix}$$

generates a $C_0$ semigroup on $Z$.

# Chapter III Approximation

In this chapter we consider the problem of finding approximate solutions to equation (2.3.1). We do this by approximating the operator $\mathcal{A}$ in (2.3.3). We construct a sequence of operators $\mathcal{A}^n$ such that $\mathcal{A}^n$ generates a $C_0$ semigroup $T^n(t)$. We then show that $T^n(t) \to T(t)$, the semigroup generated by $\mathcal{A}$, using the Trotter-Kato theorem. We are also interested in approximating the feedback gain, but this requires that we approximate $T^*(t)$ as well as $T(t)$. Thus, in Section 3.2 we discuss the convergence of the adjoint semigroup. We then apply the theory to the example of the viscoelastic shaft considered in the previous chapter.

## 3.1 An Approximation Scheme for the Abstract PFDE.

We will use the following version of the Trotter-Kato theorem:

THEOREM 3.1.1. *Let $A \in G(M,\beta)$ be the infinitesimal generator of a $C_0$ semigroup $T(t)$ on a Hilbert space $Z$. For $n = 1, 2, \ldots$, let $Z^n$ be a finite dimensional subspace of $Z$ such that $P^n \xrightarrow{s} I$ as $n \to \infty$ where $P^n$ is the orthogonal projection of $Z$ onto $Z^n$. Suppose*

H1) *$A^n \in G(M,\beta)$ is the infinitesimal generator of a $C_0$ semigroup $T^n(t)$ on $Z^n$ for $n = 1, 2, \ldots$.*

H2) *For all $z \in Z$, $(\lambda I - A^n)^{-1}P^n z \to (\lambda I - A)^{-1} z$ as $n \to \infty$.*

*Then for all $z \in Z$, $T^n(t)P^n z \to T(t)z$ as $n \to \infty$, and the convergence is uniform on bounded t-intervals.*

This theorem follows from Theorem 4.2 in [**11**, Chapter 3]. Recall that the operator $\mathcal{A}$ we seek to approximate is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ y \\ w \end{pmatrix} \in Z \;\middle|\; \begin{matrix} y \in S, \;\; w \in H^1_R(-r, 0; X), \\ \left(x + \int_{-r}^0 g_\alpha(s) w(s) ds\right) \in \mathcal{D}(A) \end{matrix} \right\},$$

$$\mathcal{A}\begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} jy \\ A\left(x + \int_{-r}^0 g_\alpha(s) w(s) ds\right) \\ jy + \dfrac{\partial w}{\partial s} \end{pmatrix}.$$

Recall also that $g_\alpha \in L_1(-r, 0)$, $g_\alpha > 0$, and $g'_\alpha \geq 0$ on $[-r, 0)$.

If we define the operator $A_0 : \mathcal{D}(A_0) \subseteq X \times Y \to X \times Y$ by

$$\mathcal{D}(A_0) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X \times Y \;\middle|\; x \in \mathcal{D}(A), \; y \in S \right\},$$

$$A_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} jy \\ Ax \end{pmatrix},$$

then we can write $\mathcal{A}$ in the form

$$\mathcal{A}\begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} A_0 \left[ \begin{pmatrix} x \\ y \end{pmatrix} + \displaystyle\int_{-r}^0 g_\alpha(s) \begin{pmatrix} w(s) \\ 0 \end{pmatrix} ds \right] \\ jy + \dfrac{\partial w}{\partial s} \end{pmatrix},$$

which suggests a two-stage approximation of $\mathcal{A}$: We first approximate $A_0$ by discretizing the spatial variable, typically by means of spline functions. We then approximate $\dfrac{\partial w}{\partial s}$ by discretizing the delay variable. In this paper we will use an averaging scheme for the second stage.

Let us now proceed with the first stage of the approximation. Define the bilinear form $\sigma_0(\cdot,\cdot)$ on $X \times S$ by

$$\sigma_0\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \equiv \sigma\left(y_1, j^{-1}x_2\right) - \sigma\left(j^{-1}x_1, y_2\right)$$

where $\sigma$ is the bilinear form on $S$ discussed in Section 2.3. Observe that for $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in \mathcal{D}(A_0)$, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in X \times S$,

$$\left\langle A_0 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right\rangle_{X\times Y} = \sigma_0\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right).$$

Now for each positive integer $N$, let $X^N$ and $Y^N$ be finite dimensional subspaces of $X$ and $Y$ with $Y^N \subseteq S$, and define $W^N \equiv L_2(-r,0;X^N)$. We define $A_0^N : X^N \times Y^N \to X^N \times Y^N$ by restricting $\sigma_0$ to $X^N \times Y^N$; i.e.,

$$\langle A_0^N u^N, v^N\rangle_{X\times Y} \equiv \sigma_0\left(u^N, v^N\right) \quad \text{for } u^N, v^N \in X^N \times Y^N.$$

Now set $Z^N \equiv X^N \times Y^N \times W^N$ and define $\mathcal{A}^N : \mathcal{D}(\mathcal{A}^N) \subseteq Z^N \to Z^N$ by

$$\mathcal{D}(\mathcal{A}^N) \equiv X^N \times Y^N \times H_R^1(-r,0;X^N),$$

$$\mathcal{A}^N \begin{pmatrix} x^N \\ y^N \\ w^N \end{pmatrix} \equiv \begin{pmatrix} A_0^N \left[\begin{pmatrix} x^N \\ y^N \end{pmatrix} + \int_{-r}^{0} g_\alpha(s) \begin{pmatrix} w^N(s) \\ 0 \end{pmatrix} ds\right] \\ jy^N + \dfrac{\partial}{\partial s} w^N \end{pmatrix}.$$

For each positive integer $M$ partition the interval $[-r,0]$ into subintervals $[t_j^M, t_{j-1}^M]$, $j = 1,2,\ldots,M$, where

$$-r = t_M^M < t_{M-1}^M < \cdots < t_1^M < t_0^M = 0. \tag{3.1.1}$$

We will say more later about how the $t_j^M$ are chosen. Set $\alpha_j^M \equiv t_{j-1}^M - t_j^M$ for $j = 1, 2, \ldots, M$, let $\chi_j^M$ denote the characteristic function of $[t_j^M, t_{j-1}^M)$ for $j = 2, \ldots, M$, and let $\chi_1^M$ denote the characteristic function of $[t_1^M, 0]$. Let $B_i^M(t)$, $i = 0, 1, \ldots, M$ be the usual linear spline functions satisfying $B_i^M(t_j^M) = \delta_{ij}$. Define the finite dimensional subspaces $W^{N,M}$ and $\widetilde{W}^{N,M}$ of $W$ by

$$\begin{aligned} W^{N,M} &\equiv \left\{ w \in W \;\middle|\; w = \sum_{i=1}^{M} a_i^M \chi_i^M,\ a_i^M \in X^N \right\}, \\ \widetilde{W}^{N,M} &\equiv \left\{ w \in W \;\middle|\; w = \sum_{i=1}^{M} b_i^M B_i^M,\ b_i^M \in X^N \right\}. \end{aligned} \tag{3.1.2}$$

Define the operator $\widetilde{D}^{N,M} : \widetilde{W}^{N,M} \to W^{N,M}$ by

$$\widetilde{D}^{N,M} w^{N,M} \equiv \sum_{i=1}^{M} \frac{1}{\alpha_i^M} \left( b_{i-1}^M - b_i^M \right) \chi_i^M \tag{3.1.3}$$

where $w^{N,M} = \sum_{i=1}^{M} b_i^M B_i^M$ and $b_0^M = 0$. Define the isomorphism $\imath^{N,M} : \widetilde{W}^{N,M} \to W^{N,M}$ by

$$\imath^{N,M} w^{N,M} \equiv \sum_{i=1}^{M} b_i^M \chi_i^M.$$

Now define $D^{N,M} : W^{N,M} \to W^{N,M}$ by $D^{N,M} \equiv \widetilde{D}^{N,M} \left(\imath^{N,M}\right)^{-1}$. To complete the approximation, set $Z^{N,M} \equiv X^N \times Y^N \times W^{N,M}$, and for $z^{N,M} = \left(x^N, y^N, w^{N,M}\right)^T \in Z^{N,M}$, define

$$\mathcal{A}^{N,M} z^{N,M} \equiv \begin{pmatrix} A_0^N \left[ \begin{pmatrix} x^N \\ y^N \end{pmatrix} + \int_{-r}^{0} g_\alpha(s) \begin{pmatrix} w^{N,M}(s) \\ 0 \end{pmatrix} ds \right] \\ jy^N + D^{N,M} w^{N,M} \end{pmatrix}.$$

If $w^{N,M} = \sum_{i=1}^{M} w_i^M \chi_i^M$, then

$$\mathcal{A}^{N,M} z^{N,M} = \begin{pmatrix} A_0^N \begin{pmatrix} x^N + \sum_{i=1}^{M} (g_\alpha)_i^M w_i^M \\ y^N \end{pmatrix} \\ jy^N + \sum_{i=1}^{M} \dfrac{w_{i-1}^M - w_i^M}{\alpha_i^M} \chi_i^M \end{pmatrix} \tag{3.1.4}$$

where $(g_\alpha)_i^M \equiv \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s)ds$.

In order to prove convergence of our approximation scheme, we must impose some conditions upon the spaces $X^N$ and $Y^N$ and upon the partitions of $[-r, 0]$. Thus, we make the following assumptions:

A1) Let $P_X^N$ and $P_Y^N$ be the orthogonal projections of $X$ and $Y$ onto $X^N$ and $Y^N$, respectively. Then $P_X^N \xrightarrow{s} I$, and $P_Y^N \xrightarrow{s} I$ where $I$ is the appropriate identity operator.

A2) For each positive integer $M$ let $\Pi^M = \{t_j^M \mid j = 0, 1, \ldots, M\}$ be a partition of $[-r, 0]$ satisfying (3.1.1), and set $\Lambda^M \equiv \{1, 2, \ldots, M\}$. Then there exist positive constants $\epsilon_1$, $\epsilon_2$ and $C$ independent of $M$ such that $\Lambda^M = \Lambda_1^M \cup \Lambda_2^M$ where

$$\Lambda_1^M = \left\{ j \in \Lambda^M \;\middle|\; \alpha_j^M \leq rM^{-(1+\epsilon_1)/2} \right\}.$$

If $j \in \Lambda_2^M$, then $(g_\alpha)_j^M \leq \dfrac{C}{M}$, and $\Lambda_2^M$ contains at most $M^{1-\epsilon_2}$ elements of $\Lambda^M$. Furthermore, $\alpha_{j-1}^M (g_\alpha)_j^M \leq (g_\alpha)_{j-1}^M \alpha_j^M$ for $j = 2, 3, \ldots, M$, and if $j \in \Lambda_1^M$, then $1, 2, \ldots, j-1 \in \Lambda_1^M$.

REMARK 3.1.2. *Suppose* $t_j^M = \dfrac{-jr}{M}$ *for* $j = 0, 1, \ldots, M$. *Then* $\alpha_j^M = \dfrac{r}{M}$ *for all* $j \in \Lambda^M$, *so A2) is satisfied with* $\epsilon_1 = 1$, $\epsilon_2, C > 0$ *arbitrary since* $\Lambda_2^M = \emptyset$ *and* $(g_\alpha)_j^M \le (g_\alpha)_{j-1}^M$.

LEMMA 3.1.3. *Let* $C = \int_{-r}^0 g_\alpha(s)ds$, *and suppose that the* $t_j^M$ *are chosen so that* $(g_\alpha)_j^M = \dfrac{C}{M}$ *for* $j = 1, 2, \ldots, M$. *Then for* $\epsilon_1 = \frac{1}{2}$, $\epsilon_2 = \frac{1}{4}$ *the partition* $\Pi^M$ *satisfies A2) for all positive integers* $M$.

PROOF: Since $g_\alpha > 0$, $g'_\alpha \ge 0$ it is clear that $\alpha_M^M \ge \alpha_{M-1}^M \ge \cdots \ge \alpha_1^M$. Thus, if $\Lambda_2^M \ne \emptyset$, then $\Lambda_1^M = \{1, 2, \ldots, n-1\}$ and $\Lambda_2^M = \{n, n+1, \ldots, M\}$ for some $n \ge 1$. If for some $M$, $\Pi^M$ does not satisfy A2), then $\Lambda_2^M$ contains more than $M^{3/4}$ elements. Thus,

$$
\begin{aligned}
0 \ge t_{n-1}^M &= \left(t_{n-1}^M - t_n^M\right) + \left(t_n^M - t_{n+1}^M\right) + \cdots + \left(t_{M-1}^M - t_M^M\right) + t_M^M \\
&= \alpha_n^M + \alpha_{n+1}^M + \cdots + \alpha_M^M - r \\
&> rM^{-3/4} + rM^{-3/4} + \cdots + rM^{-3/4} - r \\
&> M^{3/4}\left(rM^{-3/4}\right) - r = 0. \quad \blacksquare
\end{aligned}
$$

For the remainder of this section we will assume that we have a partition which satisfies A2).

LEMMA 3.1.4. $\mathcal{A}^{N,M} \in G(1,0)$ *for all* $N$, $M$.

PROOF: It is sufficient to show that $\mathcal{A}^{N,M}$ is dissipative in $Z^{N,M}$. Let $z^{N,M} =$

$\begin{pmatrix} x^N \\ y^N \\ w^{N,M} \end{pmatrix}$ with $w^{N,M} = \sum_{i=1}^{M} w_i^M \chi_i^M$. Then

$$\langle \mathcal{A}^{N,M} z^{N,M}, z^{N,M} \rangle_Z = \sigma_0 \left( \begin{bmatrix} x^N + \sum_{i=1}^{M} (g_\alpha)_i^M w_i^M \\ y^N \end{bmatrix}, \begin{bmatrix} x^N \\ y^N \end{bmatrix} \right)$$

$$+ \int_{-r}^{0} g_\alpha(s) \left\langle jy^N + \sum_{i=1}^{M} \frac{w_{i-1}^M - w_i^M}{\alpha_i^M} \chi_i^M, \sum_{i=1}^{M} w_i^M \chi_i^M \right\rangle_X ds$$

$$= -\sigma \left( j^{-1} \left( \sum_{i=1}^{M} (g_\alpha)_i^M w_i^M \right), y^N \right) + \int_{-r}^{0} g_\alpha(s) \left\langle jy^N, \sum_{i=1}^{M} w_i^M \chi_i^M \right\rangle_X ds$$

$$+ \int_{-r}^{0} g_\alpha(s) \left\langle \sum_{i=1}^{M} \frac{w_{i-1}^M - w_i^M}{\alpha_i^M} \chi_i^M, \sum_{i=1}^{M} w_i^M \chi_i^M \right\rangle_X ds$$

$$= \sum_{i=1}^{M} \frac{1}{\alpha_i^M} (g_\alpha)_i^M \langle w_{i-1}^M - w_i^M, w_i^M \rangle_X$$

$$\leq \sum_{i=1}^{M} \frac{(g_\alpha)_i^M}{\alpha_i^M} \left[ \|w_{i-1}^M\|_X \cdot \|w_i^M\|_X - \|w_i^M\|_X^2 \right]$$

$$\leq \frac{1}{2} \sum_{i=1}^{M} \frac{(g_\alpha)_i^M}{\alpha_i^M} \left[ \|w_{i-1}^M\|_X^2 - \|w_i^M\|_X^2 \right]$$

$$= \frac{1}{2} \left[ \sum_{i=1}^{M-1} \|w_i^M\|_X^2 \left( \frac{(g_\alpha)_{i+1}^M}{\alpha_{i+1}^M} - \frac{(g_\alpha)_i^M}{\alpha_i^M} \right) - \|w_M^M\|_X^2 \frac{(g_\alpha)_M^M}{\alpha_M^M} \right] \leq 0$$

where we used the Cauchy-Schwarz Inequality and the inequality $2ab \leq a^2 + b^2$, and from A2) the fact that $(g_\alpha)_{i+1}^M / \alpha_{i+1}^M \leq (g_\alpha)_i^M / \alpha_i^M$ for $i = 1, 2, \ldots, M-1$. ∎

LEMMA 3.1.5. *$P_W^{N,M} h \to h$ as $N, M \to \infty$ for all $h \in W$.*

PROOF: Recall from the proof of Theorem 2.3.1 that the operator $D$ given by $\mathcal{D}(D) = H_R^1(-r, 0; X)$, $Dw = \frac{\partial w}{\partial s}$, generates a $C_0$ semigroup of contractions on $W$. Thus, $\mathcal{D}(D^2)$ is dense in $W$ by Theorem 2.7 in [**11**, Chapter 1]. If $w \in \mathcal{D}(D^2)$ and

$s \le \hat{s}$, then

$$\begin{aligned} g_\alpha(s)\left\|Dw(\hat{s})\right\|_X^2 &= g_\alpha(s)\left\|\int_{\hat{s}}^0 D^2w(\xi)d\xi\right\|_X^2 \\ &= g_\alpha(s)\left\|\int_{\hat{s}}^0 \frac{1}{\sqrt{g_\alpha(\xi)}}\sqrt{g_\alpha(\xi)}D^2w(\xi)d\xi\right\|_X^2 \\ &\le \int_{\hat{s}}^0 \frac{g_\alpha(s)}{g_\alpha(\xi)}d\xi \int_{\hat{s}}^0 g_\alpha(\xi)\left\|D^2w(\xi)\right\|_X^2 d\xi \le r\left\|D^2w\right\|_W^2. \end{aligned}$$

For $w \in \mathcal{D}(D^2)$, set $w^{N,M}(t) \equiv \sum_{i=1}^{M} w^N(t_{i-1}^M)\chi_i^M(t)$ where $w^N = P_X^N w$. Note that for $w \in \mathcal{D}(D)$,

$$w^N(t) = P_X^N w(t) = P_X^N \int_0^t Dw(\xi)d\xi = \int_0^t P_X^N Dw(\xi)d\xi$$

which implies that $w^N \in \mathcal{D}(D)$ and $Dw^N = P_X^N Dw$. Thus, if $w \in \mathcal{D}(D^2)$, then $w^N \in \mathcal{D}(D^2)$ and $\left\|D^2w^N\right\|_W \le \|D^2w\|_W$. Now, for $w \in \mathcal{D}(D^2)$,

$$\left\|w - w^{N,M}\right\|_W \le \left\|w - w^N\right\|_W + \left\|w^N - w^{N,M}\right\|_W.$$

Estimating the first term on the right-hand side,

$$\left\|w - w^N\right\|_W^2 = \int_{-r}^0 g_\alpha(s)\left\|w - P_X^N w\right\|_X^2 ds \to 0 \text{ as } N \to \infty$$

by the Dominated Convergence Theorem. For the second term,

$$\begin{aligned} \left\|w^N - w^{N,M}\right\|_W^2 &= \int_{-r}^0 g_\alpha(s)\left\|w^N(s) - w^{N,M}(s)\right\|_X^2 ds \\ &= \sum_{i=1}^{M}\int_{t_i^M}^{t_{i-1}^M} g_\alpha(s)\left\|w^N(s) - w^N(t_{i-1}^M)\right\|_X^2 ds = S_1 + S_2 \end{aligned}$$

where $S_j = \sum_{i\in\Lambda_j^M} \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) \left\|w^N(s) - w^N(t_{i-1}^M)\right\|_X^2 ds$ for $j = 1, 2$. By the Mean Value Theorem, for $s \in (t_i^M, t_{i-1}^M)$ there exists $\xi(s) \in (s, t_{i-1}^M)$ such that $w^N(s) - w^N(t_{i-1}^M) = Dw^N(\xi)(s - t_{i-1}^M)$. Thus,

$$
\begin{aligned}
S_1 &= \sum_{i\in\Lambda_1^M} \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) \left\|Dw^N(\xi)\right\|_X^2 (s - t_{i-1}^M)^2 ds \\
&\le r \left\|D^2 w\right\|_W^2 \sum_{i\in\Lambda_1^M} \int_{t_i^M}^{t_{i-1}^M} (s - t_{i-1}^M)^2 ds \\
&= r \left\|D^2 w\right\|_W^2 \sum_{i\in\Lambda_1^M} \frac{1}{3} (\alpha_i^M)^3 \\
&\le r \left\|D^2 w\right\|_W^2 \cdot M \cdot \frac{1}{3} \left(\frac{r}{M^{(1+\epsilon_1)/2}}\right)^3 \to 0 \text{ as } M \to \infty.
\end{aligned}
$$

By an argument similar to the proof that $g_\alpha(s) \left\|Dw(\hat{s})\right\|_X^2 \le r \left\|D^2 w\right\|_W^2$, there exists a constant $\gamma$ independent of $M$ such that

$$
\begin{aligned}
S_2 &\le 2 \sum_{i\in\Lambda_2^M} \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) \left[\left\|w^N(s)\right\|_X^2 + \left\|w^N(t_{i-1}^M)\right\|_X^2\right] ds \\
&\le 2\gamma \left\|Dw\right\|_W^2 \sum_{i\in\Lambda_2^M} \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) ds \\
&\le 2\gamma \left\|Dw\right\|_W^2 \sum_{i\in\Lambda_2^M} \frac{C}{M} \le \frac{2\gamma C \left\|Dw\right\|_W^2}{M^{\epsilon_2}} \to 0 \text{ as } M \to \infty.
\end{aligned}
$$

Hence, for $w \in \mathcal{D}(D^2)$,

$$
\begin{aligned}
\left\|w - P_W^{N,M} w\right\|_W &\le \left\|w - w^{N,M}\right\|_W + \left\|P_W^{N,M}(w - w^{N,M})\right\|_W \\
&\le 2 \left\|w - w^{N,M}\right\|_W \to 0 \text{ as } N, M \to \infty.
\end{aligned}
$$

Since $\mathcal{D}(D^2)$ is dense in $W$ and $\left\|P_W^{N,M}\right\| \leq 1$,

$$\left\|h - P_W^{N,M}h\right\|_W \to 0 \text{ as } N, M \to \infty \text{ for all } h \in W. \quad \blacksquare$$

Let $P_W^{N,M}$ be the orthogonal projection of $W$ onto $W^{N,M}$. If $P_Z^{N,M}$ denotes the orthogonal projection of $Z$ onto $Z^{N,M}$, then for $z = (x, y, w)^T \in Z$, $P_Z^{N,M} z = \left(P_X^N x, P_Y^N y, P_W^{N,M} w\right)^T$. Thus, $P_Z^{N,M} \xrightarrow{s} I$ as $N, M \to \infty$ by assumption A1) and the previous lemma.

If we define $A^N : X^N \to Y^N$ by $\langle A^N x^N, y^N\rangle_Y \equiv -\sigma\left(j^{-1}x^N, y^N\right)$, then it is easy to see that $A_0^N \begin{pmatrix} x^N \\ y^N \end{pmatrix} = \begin{pmatrix} jy^N \\ A^N x^N \end{pmatrix}$ for $\begin{pmatrix} x^N \\ y^N \end{pmatrix} \in X^N \times Y^N$. Define the bilinear form $a(\cdot,\cdot)$ on $\mathcal{D}(D) \times W$ by $a(w, h) \equiv \int_{-r}^0 g_\alpha(s) \langle Dw, h\rangle_X \, ds$, and the bilinear form $a^{N,M}(\cdot,\cdot)$ on $\widetilde{W}^{N,M} \times W^{N,M}$ by $a^{N,M}\left(w^{N,M}, h^{N,M}\right) \equiv \left\langle \widetilde{D}^{N,M} w^{N,M}, h^{N,M}\right\rangle_W$ where $\widetilde{D}^{N,M}$ is given by (3.1.3). Using the definition it is clear that $a^{N,M}\left(w^{N,M}, h^{N,M}\right) = \int_{-r}^0 g_\alpha(s) \langle Dw^{N,M}, h^{N,M}\rangle_X \, ds$.

For $z = \begin{pmatrix} x \\ y \\ w \end{pmatrix} \in \mathcal{D}(\mathcal{A})$ and $\text{Re}\lambda > 0$, consider the equation $(\lambda I - \mathcal{A})z = \begin{pmatrix} \varphi \\ \psi \\ h \end{pmatrix}$, or equivalently,

$$\lambda x - jy = \varphi, \tag{3.1.5}$$

$$\lambda y - A\left(x + \int_{-r}^0 g_\alpha(s) w(s) ds\right) = \psi, \tag{3.1.6}$$

$$\lambda w - jy - \frac{\partial w}{\partial s} = h. \tag{3.1.7}$$

From (3.1.7), $w(s) = \int_s^0 e^{\lambda(s-\xi)}(jy + h(\xi))d\xi$, and from (3.1.5), $jy = \lambda x - \varphi$, or

$y = \lambda j^{-1}x - j^{-1}\varphi$. Substituting into (3.1.6) we get

$$\lambda^2 j^{-1}x - A\left[x + \int_{-r}^{0} g_\alpha(s)\lambda \int_{s}^{0} e^{\lambda(s-\xi)}x\,d\xi ds\right]$$
$$= \psi + \lambda j^{-1}\varphi - A\int_{-r}^{0} g_\alpha(s)\int_{s}^{0} e^{\lambda(s-\xi)}(\varphi - h(\xi))d\xi ds.$$

Using the fact that $\int_s^0 \lambda e^{\lambda(s-\xi)}d\xi = 1 - e^{\lambda s}$ we obtain

$$\Delta(\lambda)x = \psi + \lambda j^{-1}\varphi - A\int_{-r}^{0} g_\alpha(s)(\lambda I - D)^{-1}(\varphi - h(s))ds \tag{3.1.8}$$

where

$$\Delta(\lambda) = \lambda^2 j^{-1} - \frac{1}{\alpha}A\left(\tau + \int_{-r}^{0} e^{\lambda s}g(s)ds\right). \tag{3.1.9}$$

Now, for $z^{N,M} = \left(x^N, y^N, w^{N,M}\right)^T \in Z^{N,M}$ and $\mathrm{Re}\lambda > 0$, consider the equation $\left(\lambda I - \mathcal{A}^{N,M}\right)z^{N,M} = \left(\varphi^N, \psi^N, h^{N,M}\right)^T$. We have the equations

$$\lambda x^N - jy^N = \varphi^N, \tag{3.1.10}$$

$$\lambda y^N - A^N\left(x^N + \sum_{i=1}^{M}(g_\alpha)_i^M w_i^M\right) = \psi^N, \tag{3.1.11}$$

$$\lambda w_i^M - jy^N - \frac{1}{\alpha_i^M}(w_{i-1}^M - w_i^M) = h_i^M \text{ for } i = 1, 2, \ldots, M \tag{3.1.12}$$

where $w^{N,M} = \sum_{i=1}^{M} w_i^M \chi_i^M$ and $h^{N,M} = \sum_{i=1}^{M} h_i^M \chi_i^M$. From (3.1.12),

$$\left(\lambda + \frac{1}{\alpha_i^M}\right)w_i^M = \frac{1}{\alpha_i^M}w_{i-1}^M + jy^N + h_i^M,$$

or

$$w_i^M = \left(1 + \alpha_i^M\lambda\right)^{-1}\left[w_{i-1}^M + \alpha_i^M\left(jy^N + h_i^M\right)\right] \text{ for } i = 1, 2, \ldots, M,$$

where $w_0^M = 0$. By induction, $w_i^M = \sum_{k=1}^{i}\left[\prod_{l=k}^{i}\left(1+\alpha_l^M\lambda\right)^{-1}\right]\alpha_k^M\left(jy^N + h_k^M\right)$. From (3.1.10), $jy^N = \lambda x^N - \varphi^N$ which implies $y^N = \lambda j^{-1}x^N - j^{-1}\varphi^N$. Substituting into (3.1.11) we obtain

$$\lambda^2 j^{-1}x^N - A^N\left(x^N + \lambda\sum_{i=1}^{M}(g_\alpha)_i^M\sum_{k=1}^{i}\left[\prod_{l=k}^{i}\left(1+\alpha_l^M\lambda\right)^{-1}\right]\alpha_k^M x^N\right)$$
$$= \psi^N + \lambda j^{-1}\varphi^N - \sum_{i=1}^{M}(g_\alpha)_i^M A^N\sum_{k=1}^{i}\left[\prod_{l=k}^{i}\left(1+\alpha_l^M\lambda\right)^{-1}\right]\alpha_k^M\left(\varphi^N - h_k^M\right). \quad (3.1.13)$$

By induction, $\lambda\sum_{k=1}^{i}\left[\prod_{l=k}^{i}\left(1+\alpha_l^M\lambda\right)^{-1}\right]\alpha_k^M = 1 - \prod_{k=1}^{i}\left(1+\alpha_k^M\lambda\right)^{-1}$. Define

$$\Delta^{N,M}(\lambda) \equiv \lambda^2 j^{-1} - A^N\left[1 + \sum_{i=1}^{M}(g_\alpha)_i^M\left(1 - \prod_{k=1}^{i}\left(1+\alpha_k^M\lambda\right)^{-1}\right)\right].$$

Then, since $\sum_{i=1}^{M}(g_\alpha)_i^M = -\frac{1}{\alpha}\int_{-r}^{0}g(s)ds = \frac{\tau}{\alpha} - 1$,

$$\Delta^{N,M}(\lambda) = \lambda^2 j^{-1} - \frac{1}{\alpha}A^N\left[\tau + \int_{-r}^{0}g(s)e^M(\lambda,s)ds\right] \quad (3.1.14)$$

where

$$e^M(\lambda,s) \equiv \sum_{i=1}^{M}\left(\prod_{k=1}^{i}\left(1+\alpha_k^M\lambda\right)^{-1}\right)\chi_i^M(s).$$

Let $\left(\lambda I - D^{N,M}\right)^{-1}\left(\varphi^N - h^{N,M}\right) = \sum_{i=1}^{M}\xi_i^M\chi_i^M$. Then

$$\varphi^N - h^{N,M} = \left(\lambda I - D^{N,M}\right)\sum_{i=1}^{M}\xi_i^M\chi_i^M = \sum_{i=1}^{M}\left[\lambda\xi_i^M - \frac{1}{\alpha_i^M}\left(\xi_{i-1}^M - \xi_i^M\right)\right]\chi_i^M$$

which implies

$$\varphi^N - h_i^M = \lambda\xi_i^M - \frac{1}{\alpha_i^M}\left(\xi_{i-1}^M - \xi_i^M\right) \quad \text{for } i = 1, 2, \ldots, M.$$

Thus, $\xi_i^M = \left(1 + \lambda\alpha_i^M\right)^{-1}\left[\xi_{i-1}^M + \alpha_i^M\left(\varphi^N - h_i^M\right)\right]$, or, by induction,

$$\xi_i^M = \sum_{k=1}^{i}\left[\prod_{l=k}^{i}\left(1 + \lambda\alpha_l^M\right)^{-1}\right]\alpha_k^M\left(\varphi^N - h_k^M\right).$$

Now, $\int_{-r}^{0} g_\alpha(s)\sum_{i=1}^{M}\xi_i^M\chi_i^M = \sum_{i=1}^{M}(g_\alpha)_i^M\xi_i^M$, so

$$\int_{-r}^{0} g_\alpha(s)\left(\lambda I - D^{N,M}\right)^{-1}\left(\varphi^N - h^{N,M}\right) ds$$

$$= \sum_{i=1}^{M}(g_\alpha)_i^M\sum_{k=1}^{i}\left[\prod_{l=k}^{i}\left(1 + \lambda\alpha_l^M\right)^{-1}\right]\alpha_k^M\left(\varphi^N - h_k^M\right). \quad (3.1.15)$$

Therefore, from (3.1.13), (3.1.14) and (3.1.15) we obtain

$$\Delta^{N,M}(\lambda)x^N = \psi^N + \lambda j^{-1}\varphi^N - A^N\int_{-r}^{0} g_\alpha(s)\left(\lambda I - D^{N,M}\right)^{-1}\left(\varphi^N - h^{N,M}\right) ds. \quad (3.1.16)$$

LEMMA 3.1.6. *For $\lambda > 0$, $\left(\lambda I - D^{N,M}\right)^{-1} P_W^{N,M} h \to (\lambda I - D)^{-1}h$ for all $h \in W$.*

PROOF: Recall from the proof of Lemma 3.1.4 that $D^{N,M}$ is dissipative in $W^{N,M}$. Since $\dim(W^{N,M}) < \infty$, $\left\|\left(\lambda I - D^{N,M}\right)^{-1}\right\| \le \frac{1}{\lambda}$. Let $w = (\lambda I - D)^{-1}h$ and $w^{N,M} = \left(\lambda I - D^{N,M}\right)^{-1} P_W^{N,M} h$, or equivalently $(\lambda I - D)w = h$ and $(\lambda I - D^{N,M})w^{N,M} = P_W^{N,M} h$. Thus,

$$\lambda\langle w, \gamma\rangle_W - a(w, \gamma) = \langle h, \gamma\rangle_W \text{ for all } \gamma \in W, \quad (3.1.17)$$

and for all $\gamma^{N,M} \in W^{N,M}$,

$$\lambda\left\langle i^{N,M}\widetilde{w}^{N,M}, \gamma^{N,M}\right\rangle_W - a^{N,M}\left(\widetilde{w}^{N,M}, \gamma^{N,M}\right) = \left\langle P_W^{N,M}h, \gamma^{N,M}\right\rangle_W$$

where $\widetilde{w}^{N,M} \equiv (i^{N,M})^{-1} w^{N,M}$. Choosing $\gamma = \gamma^{N,M}$ in (3.1.17) we obtain

$$\lambda \left\langle w, \gamma^{N,M} \right\rangle_W - a\left(w, \gamma^{N,M}\right) = \lambda \left\langle i^{N,M} \widetilde{w}^{N,M}, \gamma^{N,M} \right\rangle_W - a^{N,M}\left(\widetilde{w}^{N,M}, \gamma^{N,M}\right)$$

which implies that for every $\widehat{w}^{N,M} \in \widetilde{W}^{N,M}$

$$\lambda \left\langle i^{N,M}\left(\widehat{w}^{N,M} - \widetilde{w}^{N,M}\right), \gamma^{N,M} \right\rangle_W - a^{N,M}\left(\widehat{w}^{N,M} - \widetilde{w}^{N,M}, \gamma^{N,M}\right)$$
$$= a\left(w, \gamma^{N,M}\right) - a^{N,M}\left(\widehat{w}^{N,M}, \gamma^{N,M}\right) - \lambda \left\langle w - i^{N,M}\widehat{w}^{N,M}, \gamma^{N,M} \right\rangle_W$$

for all $\gamma^{N,M} \in W^{N,M}$. Using the definitions of $a^{N,M}$ and $D^{N,M}$ we see that

$$\left\langle \left(\lambda I - D^{N,M}\right) i^{N,M}\left(\widehat{w}^{N,M} - \widetilde{w}^{N,M}\right), \gamma^{N,M} \right\rangle_W$$
$$= a\left(w, \gamma^{N,M}\right) - a^{N,M}\left(\widehat{w}^{N,M}, \gamma^{N,M}\right) - \lambda \left\langle w - i^{N,M}\widehat{w}^{N,M}, \gamma^{N,M} \right\rangle_W.$$

Let $\widehat{h}^{N,M} \equiv \left(\lambda I - D^{N,M}\right) i^{N,M}\left(\widehat{w}^{N,M} - \widetilde{w}^{N,M}\right)$. Then

$$i^{N,M}\left(\widehat{w}^{N,M} - \widetilde{w}^{N,M}\right) = \left(\lambda I - D^{N,M}\right)^{-1} \widehat{h}^{N,M}$$

which implies that $\left\| i^{N,M}\left(\widehat{w}^{N,M} - \widetilde{w}^{N,M}\right) \right\|_W \leq \frac{1}{\lambda} \left\| \widehat{h}^{N,M} \right\|_W$. Define $\Gamma_1^{N,M} \subseteq W^{N,M}$ by

$$\Gamma_1^{N,M} \equiv \left\{ \gamma^{N,M} \in W^{N,M} \;\middle|\; \left\|\gamma^{N,M}\right\|_W \leq 1 \right\}.$$

Then,

$$\begin{aligned} \left\| \widehat{h}^{N,M} \right\|_W &= \sup_{\gamma^{N,M} \in \Gamma_1^{N,M}} \left| \left\langle \widehat{h}^{N,M}, \gamma^{N,M} \right\rangle_W \right| \\ &\leq \sup_{\gamma^{N,M} \in \Gamma_1^{N,M}} \Big[ \left| a\left(w, \gamma^{N,M}\right) - a^{N,M}\left(\widehat{w}^{N,M}, \gamma^{N,M}\right) \right| \\ &\qquad + \lambda \left| \left\langle w - i^{N,M}\widehat{w}^{N,M}, \gamma^{N,M} \right\rangle_W \right| \Big] \end{aligned}$$

which implies

$$\left\|i^{N,M}\left(\widehat{w}^{N,M}-\widetilde{w}^{N,M}\right)\right\|_W$$
$$\leq \frac{1}{\lambda}\left[\sup_{\gamma^{N,M}\in\Gamma_1^{N,M}}\left|a\left(w,\gamma^{N,M}\right)-a^{N,M}\left(\widehat{w}^{N,M},\gamma^{N,M}\right)\right|\right.$$
$$\left.+\lambda\left\|P_W^{N,M}\left(w-i^{N,M}\widehat{w}^{N,M}\right)\right\|_W\right]. \tag{3.1.18}$$

Take $\widehat{w}^{N,M}=\int_0^s P_W^{N,M}(Dw)d\xi$. Then

$$a\left(w,\gamma^{N,M}\right)-a^{N,M}\left(\widehat{w}^{N,M},\gamma^{N,M}\right)=\int_{-r}^0 g_\alpha(s)\left\langle Dw-P_W^{N,M}(Dw),\gamma^{N,M}\right\rangle_X ds=0$$

for all $\gamma^{N,M}\in W^{N,M}$. For this $\widehat{w}^{N,M}$,

$$\left\|w-i^{N,M}\widehat{w}^{N,M}\right\|_W\leq\left\|w-\widehat{w}^{N,M}\right\|_W+\left\|\widehat{w}^{N,M}-i^{N,M}\widehat{w}^{N,M}\right\|_W$$
$$=\left\|\int_0^s\left(Dw-P_W^{N,M}(Dw)\right)d\xi\right\|_W+\left\|\widehat{w}^{N,M}-i^{N,M}\widehat{w}^{N,M}\right\|_W.$$

Now,

$$\left\|\int_0^s\left[Dw-P_W^{N,M}(Dw)\right]d\xi\right\|_W^2=\int_{-r}^0 g_\alpha(s)\left\|\int_0^s\left[Dw-P_W^{N,M}(Dw)\right]d\xi\right\|_X^2 ds$$
$$\leq\int_{-r}^0 g_\alpha(s)\left(\int_s^0\frac{1}{\sqrt{g_\alpha(\xi)}}\sqrt{g_\alpha(\xi)}\left\|Dw-P_W^{N,M}(Dw)\right\|_X d\xi\right)^2 ds$$
$$\leq\int_{-r}^0 g_\alpha(s)\int_s^0\frac{d\xi}{g_\alpha(\xi)}\int_s^0 g_\alpha(\xi)\left\|Dw-P_W^{N,M}(Dw)\right\|_X^2 d\xi ds$$
$$\leq\int_{-r}^0\int_s^0\frac{g_\alpha(s)}{g_\alpha(\xi)}d\xi ds\left\|Dw-P_W^{N,M}(Dw)\right\|_W^2$$
$$\leq\frac{r^2}{2}\left\|Dw-P_W^{N,M}(Dw)\right\|_W^2\to 0 \text{ as } N,M\to\infty$$

by Lemma 3.1.5. Set $P_W^{N,M}(Dw) = \sum_{i=1}^{M} \xi_i^M \chi_i^M$ and $\widehat{w}^{N,M}(s) = \sum_{i=1}^{M} b_i^M B_i^M(s)$. Then, for $j = 1,2,\ldots,M$, $b_j^M = \widehat{w}^{N,M}(t_j^M) = \int_0^{t_j^M} \sum_{i=1}^{M} \xi_i^M \chi_i^M(\xi) d\xi = -\sum_{i=1}^{j} \xi_i^M \alpha_i^M$. Now, $i^{N,M}\widehat{w}^{N,M} = \sum_{i=1}^{M} b_i^M \chi_i^M$. Since $\widehat{w}^{N,M} = \sum_{i=1}^{M} b_i^M B_i^M$,

$$\widehat{w}^{N,M}(s) = b_j^M + \frac{b_{j-1}^M - b_j^M}{\alpha_j^M}\left(s - t_j^M\right) \text{ for } s \in [t_j^M, t_{j-1}^M).$$

Hence, $\widehat{w}^{N,M} - i^{N,M}\widehat{w}^{N,M} = \frac{b_{j-1}^M - b_j^M}{\alpha_j^M}\left(s - t_j^M\right)$ on $[t_j^M, t_{j-1}^M)$ for $j = 1,2,\ldots,M$. Thus,

$$\left\|\widehat{w}^{N,M} - i^{N,M}\widehat{w}^{N,M}\right\|_W^2 = \sum_{i=1}^{M} \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) \left\| \frac{b_{i-1}^M - b_i^M}{\alpha_i^M}\left(s - t_i^M\right)\right\|_X^2 ds.$$

But, $\frac{1}{\alpha_i^M}\left(b_{i-1}^M - b_i^M\right) = \frac{1}{\alpha_i^M}\left(\sum_{j=1}^{i} \xi_j^M \alpha_j^M - \sum_{j=1}^{i-1} \xi_j^M \alpha_j^M\right) = \xi_i^M$, so

$$\left\|\widehat{w}^{N,M} - i^{N,M}\widehat{w}^{N,M}\right\|_W^2 = \sum_{i=1}^{M} \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) \left\|\xi_i^M\right\|_X^2 \left(s - t_i^M\right)^2 ds = S_1 + S_2.$$

We estimate the term $S_1$ by

$$\begin{aligned} S_1 &\le \sum_{i \in \Lambda_1^M} \left(\alpha_i^M\right)^2 \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) \left\|\xi_i^M\right\|_X^2 \\ &\le \frac{r^2}{M^{1+\epsilon_1}} \sum_{i=1}^{M} \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) \left\|\xi_i^M\right\|_X^2 ds \\ &\le \frac{r^2}{M^{1+\epsilon_1}} \left\|Dw\right\|_W^2 \to 0 \text{ as } M \to \infty. \end{aligned}$$

Define the norm $\|\cdot\|_1$ on $W$ by $\|w\|_1^2 \equiv \int_{-r}^{0} \|w(s)\|_X^2 ds$. Then $\left\|P_W^{N,M}(Dw)\right\|_1^2 = \int_{-r}^{0} \left\|\sum_{i=1}^{M} \xi_i^M \chi_i^M\right\|_X^2 ds = \sum_{i=1}^{M} \int_{t_i^M}^{t_{i-1}^M} \left\|\xi_i^M\right\|_X^2 ds = \sum_{i=1}^{M} \alpha_i^M \left\|\xi_i^M\right\|_X^2$. For $w \in W$, $\|w\|_1^2 =$

$\int_{-r}^{0} \|w(s)\|_X^2 \, ds \le \frac{1}{g_\alpha(-r)} \int_{-r}^{0} g_\alpha(s) \|w(s)\|_X^2 \, ds = \frac{1}{g_\alpha(-r)} \|w\|_W^2$. Thus

$$\alpha_j^M \|\xi_j^M\|_X^2 \le \left\| P_W^{N,M}(Dw) \right\|_1^2 \le \frac{1}{g_\alpha(-r)} \left\| P_W^{N,M}(Dw) \right\|_W^2 \le \frac{1}{g_\alpha(-r)} \|Dw\|_W^2 .$$

Thus, for $s \in [t_j^M, t_{j-1}^M)$, $\|\xi_j^M\|_X^2 (s - t_j^M)^2 \le (\alpha_j^M)^2 \|\xi_j^M\|_X^2 \le \frac{r}{g_\alpha(-r)} \|Dw\|_W^2$.

Hence,

$$S_2 \le \sum_{i \in \Lambda_2^M} \frac{r}{g_\alpha(-r)} \|Dw\|_W^2 \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) ds \le \frac{r \|Dw\|_W^2 C}{g_\alpha(-r) M^{\epsilon_2}} \to 0 \text{ as } M \to \infty,$$

so $\|w - i^{N,M} \widehat{w}^{N,M}\|_W \to 0$ as $N, M \to \infty$. Thus, from (3.1.18) it follows that $\|w^{N,M} - i^{N,M} \widehat{w}^{N,M}\|_W \to 0$ as $N, M \to \infty$. Therefore,

$$\begin{aligned} \|w - w^{N,M}\|_W &\le \|w - i^{N,M} \widehat{w}^{N,M}\|_W \\ &\quad + \|i^{N,M} \widehat{w}^{N,M} - w^{N,M}\|_W \to 0 \text{ as } N, M \to \infty. \quad \blacksquare \end{aligned}$$

LEMMA 3.1.7. $\int_{-r}^{0} g_\alpha(s) \left| e^{\lambda s} - e^M(\lambda, s) \right| ds \to 0$ as $M \to \infty$.

PROOF: $\int_{-r}^{0} g_\alpha(s) \left| e^{\lambda s} - e^M(\lambda, s) \right| ds = \sum_{i=1}^{M} \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) \left| e^{\lambda s} - \prod_{j=1}^{i} \left(1 + \alpha_j^M \lambda\right)^{-1} \right| ds$

$= S_1 + S_2$. Let $\epsilon > 0$. Choose $M_0$ so that if $M \ge M_0$, then $\frac{2C}{M^{\epsilon_2}} < \epsilon$,

$e^{\lambda(r/M^{(1+\epsilon_1)/2})} - 1 < \frac{\epsilon}{2}$ and $\frac{(r\lambda)^2 e^{r\lambda}}{2M^{\epsilon_1}} < \frac{\epsilon}{2}$. For $M \ge M_0$,

$$\begin{aligned} S_2 &\le \sum_{i \in \Lambda_2^M} \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) \left[ e^{\lambda s} + \prod_{j=1}^{i} (1 + \alpha_j^M \lambda)^{-1} \right] ds \\ &\le 2 \sum_{i \in \Lambda_2^M} \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) ds \le \frac{2C}{M^{\epsilon_2}} < \epsilon. \end{aligned}$$

Let $\Lambda_1^M = \{1, 2, \ldots, n\}$ and suppose $s \in [t_n^M, 0]$. Then for some $i \in \Lambda_1^M$, $t_i^M \le s \le t_{i-1}^M$ which implies that $e^{\lambda t_i^M} \le e^{\lambda s} \le e^{\lambda t_{i-1}^M}$, so $0 \le e^{\lambda s} - e^{\lambda t_i^M} \le e^{\lambda t_i^M}\left(e^{\lambda \alpha_i^M} - 1\right) < \frac{\epsilon}{2}$. But, $t_i^M = -\sum_{j=1}^i \alpha_j^M$ which implies

$$\left|e^{\lambda t_i^M} - \prod_{j=1}^i (1+\alpha_j^M\lambda)^{-1}\right| = \left|\prod_{j=1}^i e^{-\lambda\alpha_j^M} - \prod_{j=1}^i (1+\alpha_j^M\lambda)^{-1}\right|$$
$$\le \left|\prod_{j=1}^i (1+\alpha_j^M\lambda) - \prod_{j=1}^i e^{\lambda\alpha_j^M}\right|.$$

Now, $e^{\lambda\alpha_j^M} = (1+\lambda\alpha_j^M) + \frac{1}{2}e^{\lambda\xi_j}\left(\lambda\alpha_j^M\right)^2$ for some $\xi_j$ between 0 and $\alpha_j^M$. Set $a_j = (1+\lambda\alpha_j^M)$ and $b_j = \frac{1}{2}e^{\lambda\xi_j}\left(\lambda\alpha_j^M\right)^2$. Then

$$\left|\prod_{j=1}^i (1+\alpha_j^M\lambda) - \prod_{j=1}^i e^{\lambda\alpha_j^M}\right| = \left|\prod_{j=1}^i a_j - \prod_{j=1}^i (a_j+b_j)\right|.$$

By induction, $\prod_{j=1}^i a_j - \prod_{j=1}^i (a_j+b_j) = -\sum_{j=1}^i b_j \prod_{k=j+1}^i (a_k+b_k)\prod_{l=1}^{j-1} a_l$. Thus,

$$\left|\prod_{j=1}^i (1+\alpha_j^M\lambda) - \prod_{j=1}^i e^{\lambda\alpha_j^M}\right| = \sum_{j=1}^i \frac{1}{2}e^{\lambda\xi_j}\left(\lambda\alpha_j^M\right)^2 \prod_{k=j+1}^i e^{\lambda\alpha_k^M}\prod_{l=1}^{j-1}(1+\alpha_l^M\lambda)$$
$$\le \frac{1}{2}e^{\lambda r}\sum_{j=1}^i \left(\lambda\alpha_j^M\right)^2 \le \frac{1}{2}e^{\lambda r}\cdot M\cdot\left(\frac{\lambda r}{M^{(1+\epsilon_1)/2}}\right)^2 = \frac{e^{\lambda r}(\lambda r)^2}{2M^{\epsilon_1}} < \frac{\epsilon}{2},$$

since $e^{\lambda\xi_j}\prod_{k=j+1}^i e^{\lambda\alpha_k^M}\prod_{l=1}^{j-1}(1+\alpha_l^M\lambda) \le e^{\lambda\alpha_j^M}\prod_{k=j+1}^i e^{\lambda\alpha_k^M}\prod_{l=1}^{j-1} e^{\lambda\alpha_l^M} = e^{\lambda t_i^M} \le e^{\lambda r}$. Hence,

$$S_1 \le \sum_{i\in\Lambda_1^M}\int_{t_i^M}^{t_{i-1}^M} g_\alpha(s)\left[\left|e^{\lambda s} - e^{\lambda t_i^M}\right| + \left|e^{\lambda t_i^M} - \prod_{j=1}^i (1+\alpha_j^M\lambda)^{-1}\right|\right] ds$$
$$\le \epsilon\sum_{i=1}^M \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s)ds = \epsilon\int_{-r}^0 g_\alpha(s)ds. \quad \blacksquare$$

We are now ready to prove the main result of this section.

**THEOREM 3.1.8.** *For all $z \in Z$, $e^{\mathcal{A}^{N,M}t}P_Z^{N,M}z \to T(t)z$ as $N, M \to \infty$, uniformly on bounded $t$-intervals.*

**PROOF:** We have only to establish H2) of Theorem 3.1.1. Define the bilinear forms $\mu(\cdot,\cdot)$ and $\mu^M(\cdot,\cdot)$ on $X$ by

$$\mu(x_1, x_2) \equiv \left\langle \Delta(\lambda)x_1, j^{-1}x_2 \right\rangle_Y,$$

$$\mu^M(x_1, x_2) \equiv \left\langle \Delta^{N,M}(\lambda)x_1, j^{-1}x_2 \right\rangle_Y.$$

Then

$$\mu(x_1, x_2) = \lambda^2 \left\langle j^{-1}x_1, j^{-1}x_2 \right\rangle_Y + \left(\frac{\tau}{\alpha} - \int_{-r}^{0} e^{\lambda s} g_\alpha(s)ds\right) \sigma(j^{-1}x_1, j^{-1}x_2),$$

and

$$\mu^M(x_1, x_2) = \lambda^2 \left\langle j^{-1}x_1, j^{-1}x_2 \right\rangle_Y + \left(\frac{\tau}{\alpha} - \int_{-r}^{0} e^M(\lambda, s) g_\alpha(s)ds\right) \sigma(j^{-1}x_1, j^{-1}x_2)$$

for $x_1, x_2 \in X$. Thus,

$$\begin{aligned} \left|\mu^M(x_1, x_2) - \mu(x_1, x_2)\right| &\le \left(\int_{-r}^{0} g_\alpha(s) \left|e^{\lambda s} - e^M(\lambda, s)\right| ds\right) \left|\sigma(j^{-1}x_1, j^{-1}x_2)\right| \\ &\le \left(\int_{-r}^{0} g_\alpha(s) \left|e^{\lambda s} - e^M(\lambda, s)\right| ds\right) \|x_1\|_X \cdot \|x_2\|_X , \end{aligned} \tag{3.1.19}$$

and

$$\mu^M(x, x) \ge \lambda^2 \left\|j^{-1}x\right\|_Y^2 + \omega \|x\|_X^2 \qquad \text{for some } \omega > 0. \tag{3.1.20}$$

Now, for $z = \begin{pmatrix} \varphi \\ \psi \\ h \end{pmatrix} \in Z$, let $(\lambda I - \mathcal{A})^{-1} z = \begin{pmatrix} x \\ y \\ w \end{pmatrix}$, and $(\lambda I - \mathcal{A}^{N,M})^{-1} P_Z^{N,M} z = \begin{pmatrix} x^N \\ y^N \\ w^{N,M} \end{pmatrix}$. Then, by (3.1.8), for all $u \in X$,

$$\mu(x,u) = \langle \psi + \lambda j^{-1}\varphi, j^{-1}u \rangle_Y \\ + \sigma\left( j^{-1} \int_{-r}^{0} g_\alpha(s)\,(\lambda I - D)^{-1}\,(\varphi - h(s))\,ds, j^{-1}u \right),$$

and by (3.1.16), for all $u^N \in X^N$,

$$\mu^M(x^N,u^N) = \langle \psi^N + \lambda j^{-1}\varphi^N, j^{-1}u^N \rangle_Y \\ + \sigma\left( j^{-1} \int_{-r}^{0} g_\alpha(s)\,\left(\lambda I - D^{N,M}\right)^{-1}\,\left(\varphi^N - h^{N,M}(s)\right)ds, j^{-1}u^N \right),$$

where $P_Z^{N,M} z = \begin{pmatrix} \varphi^N \\ \psi^N \\ h^{N,M} \end{pmatrix}$. Let $\widehat{x}^N = P_X^N x$. Then, taking $u = u^N$ in the first equation, we get

$$\begin{aligned} &\mu^M\left(\widehat{x}^N - x^N, u^N\right) \\ &\quad= \mu^M\left(\widehat{x}^N, u^N\right) - \mu^M\left(x, u^N\right) + \mu^M\left(x, u^N\right) - \mu\left(x, u^N\right) \\ &\qquad+ \langle \psi - \psi^N, j^{-1}u^N \rangle_Y + \lambda \left\langle j^{-1}\left(\varphi - \varphi^N\right), j^{-1}u^N \right\rangle_Y \\ &\qquad+ \sigma\left( j^{-1} \int_{-r}^{0} g_\alpha(s)\left[(\lambda I - D)^{-1}(\varphi - h(s))\right.\right. \\ &\qquad\qquad\left.\left. - (\lambda I - D^{N,M})^{-1}(\varphi^N - h^{N,M}(s))\right] ds, j^{-1}u^N \right). \end{aligned} \tag{3.1.21}$$

Now, by (3.1.20), $\omega \left\|\widehat{x}^N - x^N\right\|_X^2 \leq \mu^M(\widehat{x}^N - x^N, \widehat{x}^N - x^N)$. Taking $u^N = \widehat{x}^N - x^N$

in (3.1.21) we get

$$\begin{aligned}
&\mu^M(\widehat{x}^N - x^N, \widehat{x}^N - x^N) \\
&\quad \le \left|\mu^M(\widehat{x}^N - x, \widehat{x}^N - x^N)\right| + \left|\mu^M(x, \widehat{x}^N - x^N) - \mu(x, \widehat{x}^N - x^N)\right| \\
&\qquad + \left\|\psi - \psi^N\right\|_Y \cdot \left\|j^{-1}\right\| \cdot \left\|\widehat{x}^N - x^N\right\|_X + \lambda \left\|j^{-1}\right\|^2 \cdot \left\|\varphi - \varphi^N\right\|_X \cdot \left\|\widehat{x}^N - x^N\right\|_X \\
&\qquad + \int_{-r}^0 g_\alpha(s) \Big\|(\lambda I - D)^{-1}(\varphi - h) \\
&\qquad\qquad\qquad - (\lambda I - D^{N,M})^{-1} P_W^{N,M}(\varphi - h)\Big\|_X \, ds \cdot \left\|\widehat{x}^N - x^N\right\|_X .
\end{aligned}$$

Estimating the first term in the right-hand side we get,

$$\left|\mu^M(\widehat{x}^N - x, \widehat{x}^N - x^N)\right| \le \left(\lambda^2 \left\|j^{-1}\right\|^2 + \frac{r}{\alpha}\right) \left\|\widehat{x}^N - x\right\|_X \cdot \left\|\widehat{x}^N - x^N\right\|_X .$$

By (3.1.19),

$$\begin{aligned}
&\left|\mu^M(x, \widehat{x}^N - x^N) - \mu(x, \widehat{x}^N - x^N)\right| \\
&\qquad\qquad \le \left(\int_{-r}^0 g_\alpha(s) \left|e^{\lambda s} - e^M(\lambda, s)\right| ds\right) \cdot \|x\|_X \cdot \left\|\widehat{x}^N - x^N\right\|_X .
\end{aligned}$$

By Hõlder's Inequality

$$\begin{aligned}
&\int_{-r}^0 g_\alpha(s) \left\|(\lambda I - D)^{-1}(\varphi - h) - (\lambda I - D^{N,M})^{-1} P_W^{N,M}(\varphi - h)\right\|_X ds \\
&\quad \le \left(\int_{-r}^0 g_\alpha(s) ds\right)^{1/2} \left\|(\lambda I - D)^{-1}(\varphi - h) - (\lambda I - D^{N,M})^{-1} P_W^{N,M}(\varphi - h)\right\|_W .
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\left\|\hat{x}^N - x^N\right\|_X \\
&\quad \leq \frac{1}{\omega}\Bigg[\left(\lambda^2 \left\|j^{-1}\right\|^2 + \frac{\tau}{\alpha}\right)\left\|\hat{x}^N - x\right\|_X + \left(\int_{-r}^{0} g_\alpha(s)\left|e^{\lambda s} - e^M(\lambda, s)\right|\right)\|x\|_X \\
&\quad + \left\|j^{-1}\right\| \cdot \left\|\psi - \psi^N\right\|_Y + \lambda\left\|j^{-1}\right\|^2 \cdot \left\|\varphi - \varphi^N\right\|_X + \left(\int_{-r}^{0} g_\alpha(s)ds\right)^{1/2} \cdot \\
&\quad \cdot \left\|(\lambda I - D)^{-1}(\varphi - h) - (\lambda I - D^{N,M})^{-1}P_W^{N,M}(\varphi - h)\right\|_W\Bigg] \to 0
\end{aligned}
$$

as $N, M \to \infty$, so $\left\|x - x^N\right\|_X \leq \left\|x - \hat{x}^N\right\|_X + \left\|\hat{x}^N - x^N\right\|_X \to 0$ as $N, M \to \infty$. Now, $y = \lambda j^{-1}x - j^{-1}\varphi$, and $y^N = \lambda j^{-1}x^N - j^{-1}\varphi^N$ which implies

$$
\left\|y - y^N\right\|_Y \leq \left\|j^{-1}\right\|\left(\lambda\left\|x - x^N\right\|_X + \left\|\varphi - \varphi^N\right\|_X\right) \to 0 \text{ as } N, M \to \infty.
$$

Finally, $(\lambda I - D)w = h + jy = h + \lambda x - \varphi$, and $(\lambda I - D^{N,M})w^{N,M} = h^{N,M} + jy^N = P_W^{N,M}(h - \varphi) + \lambda x^N$. Thus,

$$
\begin{aligned}
\left\|w - w^{N,M}\right\|_W \leq &\left\|(\lambda I - D)^{-1}(h - \varphi) - (\lambda I - D^{N,M})^{-1}P_W^{N,M}(h - \varphi)\right\|_W \\
&+ \lambda\left\|(\lambda I - D)^{-1}x - (\lambda I - D)^{-1}x^N\right\|_W \\
&+ \lambda\left\|(\lambda I - D)^{-1}x^N - (\lambda I - D^{N,M})^{-1}P_W^{N,M}x^N\right\|_W \to 0
\end{aligned}
$$

as $N, M \to \infty$. ∎

### 3.2 Convergence of the Adjoint Semigroup.

In this section we will restrict the kernel function $g(s)$ to be in $L_2(-r, 0)$ rather than in $L_1(-r, 0)$. With this restriction the norm $\|\cdot\|_e$ on $W$ defined by $\|w\|_e^2 \equiv$

$\int_{-r}^{0} \|w(s)\|_X^2\, ds$ is equivalent to the original norm $\|\cdot\|_W$ on $W$. We define the norm $\|\cdot\|_{Z_e}$ on $Z$ by $\|z\|_{Z_e}^2 \equiv \|x\|_X^2 + \|y\|_Y^2 + \|w\|_e^2$ where $z = (x, y, w)^T \in Z$. Since this norm is equivalent to the norm $\|\cdot\|_Z$ on $Z$, we will use it to prove convergence of the adjoint semigroup.

THEOREM 3.2.1. *The adjoint of $\mathcal{A}$ with respect to the norm $\|\cdot\|_{Z_e}$ is given by*

$$\mathcal{D}(\mathcal{A}^*) \equiv \left\{ \begin{pmatrix} x \\ y \\ w \end{pmatrix} \in Z \;\middle|\; \begin{matrix} y \in S,\ x + \int_{-r}^{0} w(s)ds \in \mathcal{D}(A), \\ w \in H_L^1(-r, 0; X) \end{matrix} \right\},$$

$$\mathcal{A}^* \begin{pmatrix} x \\ y \\ w \end{pmatrix} \equiv \begin{pmatrix} -jy \\ -A\left(x + \int_{-r}^{0} w(s)ds\right) \\ -g_\alpha(s)jy - \frac{\partial w}{\partial s} \end{pmatrix}.$$

PROOF: Let $(x, y, w)^T \in \mathcal{D}(\mathcal{A}^*)$. Then there exists $(\varphi, \psi, h)^T \in Z$ such that

$$\left\langle \mathcal{A}\begin{pmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{w} \end{pmatrix}, \begin{pmatrix} x \\ y \\ w \end{pmatrix} \right\rangle_{Z_e} = \left\langle \begin{pmatrix} \widehat{x} \\ \widehat{y} \\ \widehat{w} \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \\ h \end{pmatrix} \right\rangle_{Z_e}$$

for all $\widehat{z} = (\widehat{x}, \widehat{y}, \widehat{w})^T \in \mathcal{D}(\mathcal{A})$, and in this case, $\mathcal{A}^* \begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ h \end{pmatrix}$. Thus,

$$\langle j\widehat{y}, x\rangle_X + \left\langle A\left(\widehat{x} + \int_{-r}^{0} g_\alpha(s)\widehat{w}(s)ds\right), y\right\rangle_Y + \int_{-r}^{0} \left\langle j\widehat{y} + \frac{\partial}{\partial s}\widehat{w}(s), w(s)\right\rangle_X ds$$
$$= \langle \widehat{x}, \varphi\rangle_X + \langle \widehat{y}, \psi\rangle_Y + \int_{-r}^{0} \langle \widehat{w}(s), h(s)\rangle_X\, ds. \quad (3.2.1)$$

Suppose $\widehat{y} = 0$ and $\widehat{w} = 0$. Then $\langle A\widehat{x}, y\rangle_Y = \langle \widehat{x}, \varphi\rangle_X$ for all $\widehat{x} \in \mathcal{D}(A)$. But, $\langle \widehat{x}, \varphi\rangle_X = \sigma(j^{-1}\widehat{x}, j^{-1}\varphi) = \langle -A\widehat{x}, j^{-1}\varphi\rangle_Y$, so $\langle A\widehat{x}, y\rangle_Y = \langle A\widehat{x}, -j^{-1}\varphi\rangle_Y$ for all $\widehat{x} \in \mathcal{D}(A)$. Since $\mathcal{R}(A)$ is dense in $Y$ (see proof of Theorem 2.3.1), $y = -j^{-1}\varphi$ which implies $y \in S$ and $\varphi = -jy$. Hence,

$$\left\langle A\left(\widehat{x}+\int_{-r}^{0} g_\alpha(s)\widehat{w}(s)ds\right), y\right\rangle_Y = -\sigma\left(j^{-1}\left(\widehat{x}+\int_{-r}^{0} g_\alpha(s)\widehat{w}(s)ds\right), y\right)$$
$$= \left\langle \widehat{x}+\int_{-r}^{0} g_\alpha(s)\widehat{w}(s), \varphi\right\rangle_X.$$

Substituting into (3.2.1) we get

$$\langle j\widehat{y}, x\rangle_X - \int_{-r}^{0} g_\alpha(s)\langle \widehat{w}(s), jy\rangle_X\,ds + \int_{-r}^{0}\left\langle j\widehat{y}+\frac{\partial}{\partial s}\widehat{w}(s), w(s)\right\rangle_X ds$$
$$= \langle \widehat{y}, \psi\rangle_Y + \int_{-r}^{0}\langle \widehat{w}(s), h(s)\rangle_X\,ds \quad (3.2.2)$$

for all $\widehat{z} \in \mathcal{D}(\mathcal{A})$. Now suppose $\widehat{w} = 0$. Then $\langle j\widehat{y}, x\rangle_X + \int_{-r}^{0}\langle j\widehat{y}, w(s)\rangle_X\,ds = \langle \widehat{y}, \psi\rangle_Y$ for all $\widehat{y} \in S$ which implies $\sigma\left(\widehat{y}, j^{-1}\left(x+\int_{-r}^{0} w(s)ds\right)\right) = \langle \widehat{y}, \psi\rangle_Y$. In particular, for $\widehat{y} \in \mathcal{D}(\widetilde{A})$, $\langle \widehat{y}, \psi\rangle_Y = \left\langle \alpha\widetilde{A}\widehat{y}, j^{-1}\left(x+\int_{-r}^{0} w(s)ds\right)\right\rangle_Y$. Thus, $j^{-1}\left(x+\int_{-r}^{0} w(s)ds\right) \in \mathcal{D}(\widetilde{A}^*)$ and $\psi = \alpha\widetilde{A}^* j^{-1}\left(x+\int_{-r}^{0} w(s)ds\right)$; i.e.,

$$\left(x+\int_{-r}^{0} w(s)ds\right) \in \mathcal{D}(A) \quad \text{and} \quad \psi = -A\left(x+\int_{-r}^{0} w(s)ds\right).$$

Substituting into (3.2.2) we obtain

$$\int_{-r}^{0}\left[\langle \widehat{w}(s), g_\alpha(s)jy+h(s)\rangle_X - \left\langle \frac{\partial}{\partial s}\widehat{w}(s), w(s)\right\rangle_X\right] ds = 0$$

for all $\widehat{w} \in H_R^1(-r,0;X)$. By the Fundamental Lemma of the Calculus of Variations, $w \in H_L^1(-r,0;X)$ and $-\frac{\partial}{\partial s}w(s) = g_\alpha(s)jy + h(s)$ which implies $h(s) = -g_\alpha(s)jy - \frac{\partial w}{\partial s}$. ∎

Next, let $(x^N, y^N, w^{N,M})^T \in Z^{N,M}$. Then, since $\dim(Z^{N,M}) < \infty$,

$$(\mathcal{A}^{N,M})^* z^{N,M} = (\mathcal{A}^{N,M})^*\begin{pmatrix} x^N \\ 0 \\ 0\end{pmatrix} + (\mathcal{A}^{N,M})^*\begin{pmatrix} 0 \\ y^N \\ 0\end{pmatrix} + (\mathcal{A}^{N,M})^*\begin{pmatrix} 0 \\ 0 \\ w^{N,M}\end{pmatrix}.$$

Let $(\mathcal{A}^{N,M})^* \begin{pmatrix} x^N \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi^N \\ \psi^N \\ h^{N,M} \end{pmatrix}$. Then, for all $\widehat{z}^{N,M} = (\widehat{x}^N, \widehat{y}^N, \widehat{w}^{N,M})^T \in Z^{N,M}$,

$$\left\langle \mathcal{A}^{N,M}\widehat{z}^{N,M}, \begin{pmatrix} x^N \\ 0 \\ 0 \end{pmatrix} \right\rangle_{Z_e} = \left\langle \widehat{z}^{N,M}, \begin{pmatrix} \varphi^N \\ \psi^N \\ h^{N,M} \end{pmatrix} \right\rangle_{Z_e};$$

i.e., $\langle j\widehat{y}^N, x^N\rangle_X = \langle \widehat{x}^N, \varphi^N\rangle_X + \langle \widehat{y}^N, \psi^N\rangle_Y + \langle \widehat{w}^{N,M}, h^{N,M}\rangle_e$. Taking $\widehat{x}^N = 0$, $\widehat{y}^N = 0$, we see that $h^{N,M} = 0$, and then taking $\widehat{y}^N = 0$ we see that $\varphi^N = 0$. Thus, $\langle j\widehat{y}^N, x^N\rangle_X = \langle \widehat{y}^N, \psi^N\rangle_Y$ for all $\widehat{y}^N \in Y^N$ which implies $\langle \widehat{y}^N, \psi^N\rangle_Y = \sigma\left(\widehat{y}^N, j^{-1}x^N\right) = -\langle \widehat{y}^N, A^N x^N\rangle_Y$. Hence $\psi^N = -A^N x^N$.

Next, let $(\mathcal{A}^{N,M})^* \begin{pmatrix} 0 \\ y^N \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi^N \\ \psi^N \\ h^{N,M} \end{pmatrix}$. As above

$$\left\langle A^N \left( \widehat{x}^N + \sum_{i=1}^{M} (g_\alpha)_i^M \widehat{w}_i^M \right), y^N \right\rangle_Y = \langle \widehat{x}^N, \varphi^N\rangle_X + \langle \widehat{y}^N, \psi^N\rangle_Y + \langle \widehat{w}^{N,M}, h^{N,M}\rangle_e$$

for all $\widehat{z}^{N,M} \in Z^{N,M}$ where $\widehat{w}^{N,M} = \sum_{i=1}^{M} \widehat{w}_i^M \chi_i^M$. Taking $\widehat{x}^N = 0$, $\widehat{w}^{N,M} = 0$, we get $\psi^N = 0$. Taking $\widehat{w}^{N,M} = 0$, we get $\langle A^N \widehat{x}^N, y^N\rangle_Y = \langle \widehat{x}^N, \varphi^N\rangle_X$ for all $\widehat{x}^N \in X^N$ which implies $\langle \widehat{x}^N, \varphi^N\rangle_X = -\langle \widehat{x}^N, jy^N\rangle_X$, and so $\varphi^N = -jy^N$. Thus, $\left\langle A^N \sum_{i=1}^{M} (g_\alpha)_i^M \widehat{w}_i^M, y^N \right\rangle_Y = \sum_{i=1}^{M} \alpha_i^M \langle \widehat{w}_i^M, h_i^M\rangle_X$ where $h^{N,M} = \sum_{i=1}^{M} h_i^M \chi_i^M$. Hence, for each $i = 1, 2, \ldots, M$,

$$\alpha_i^M \langle \widehat{w}_i^M, h_i^M\rangle_X = \langle A^N (g_\alpha)_i^M \widehat{w}_i^M, y^N\rangle_Y = -(g_\alpha)_i^M \langle \widehat{w}_i^M, jy^N\rangle_X$$

for all $\widehat{w}_i^M \in X^N$ which implies $h_i^M = -\dfrac{(g_\alpha)_i^M}{\alpha_i^M} jy^N$ which implies $h^{N,M} = -g_\alpha^M jy^N$ where $g_\alpha^M \equiv \sum_{i=1}^{M} \dfrac{(g_\alpha)_i^M}{\alpha_i^M} \chi_i^M$.

Finally, let $\left(\mathcal{A}^{N,M}\right)^* \begin{pmatrix} 0 \\ 0 \\ w^{N,M} \end{pmatrix} = \begin{pmatrix} \varphi^N \\ \psi^N \\ h^{N,M} \end{pmatrix}$, where $w^{N,M} = \sum_{i=1}^{M} w_i^M \chi_i^M$. Then,

$$\left\langle j\widehat{y}^N + \sum_{i=1}^{M} \frac{1}{\alpha_i^M}\left(\widehat{w}_{i-1}^M - \widehat{w}_i^M\right)\chi_i^M, w^{N,M}\right\rangle_e$$
$$= \left\langle \widehat{x}^N, \varphi^N\right\rangle_X + \left\langle \widehat{y}^N, \psi^N\right\rangle_Y + \left\langle \widehat{w}^{N,M}, h^{N,M}\right\rangle_e .$$

Taking $\widehat{y}^N = 0$, $\widehat{w}^{N,M} = 0$, we get $\varphi^N = 0$. Now let $\widehat{w}^{N,M} = 0$. Then, $\left\langle j\widehat{y}^N, w^{N,M}\right\rangle_e$ $= \left\langle \widehat{y}^N, \psi^N\right\rangle_Y$ for all $\widehat{y}^N \in Y^N$; i.e.,

$$\left\langle \widehat{y}^N, \psi^N\right\rangle_Y = \left\langle j\widehat{y}^N, \sum_{i=1}^{M} \alpha_i^M w_i^M\right\rangle_X = -\left\langle \widehat{y}^N, A^N \sum_{i=1}^{M} \alpha_i^M w_i^M\right\rangle_Y$$

which implies $\psi^N = -A^N \sum_{i=1}^{M} \alpha_i^M w_i^M = -A^N \int_{-r}^{0} w^{N,M}(s)ds$. Finally,

$$\sum_{i=1}^{M} \left\langle \widehat{w}_{i-1}^M - \widehat{w}_i^M, w_i^M\right\rangle_X = \sum_{i=1}^{M} \alpha_i^M \left\langle \widehat{w}_i^M, h_i^M\right\rangle_X$$

which implies $\alpha_i^M \left\langle \widehat{w}_i^M, h_i^M\right\rangle_X = \left\langle \widehat{w}_i^M, w_{i+1}^M - w_i^M\right\rangle_X$ so $h_i^M = -\frac{1}{\alpha_i^M}\left(w_i^M - w_{i+1}^M\right)$ for $i = 1, 2, \ldots, M$ where $w_{M+1}^M = 0$. Thus, we have proved

THEOREM 3.2.2. *For* $z^{N,M} = \left(x^N, y^N, w^{N,M}\right)^T \in Z^{N,M}$

$$\left(\mathcal{A}^{N,M}\right)^* z^{N,M} = \begin{pmatrix} -jy^N \\ -A^N\left(x^N + \sum_{i=1}^{M} \alpha_i^M w_i^M\right) \\ -g_\alpha^M jy^N - \sum_{i=1}^{M} \frac{1}{\alpha_i^M}\left(w_i^M - w_{i+1}^M\right)\chi_i^M \end{pmatrix}.$$

Let $W^{N,M}$ be as in (3.1.2) and define $\widetilde{W}_*^{N,M} \subseteq W$ by

$$\widetilde{W}_*^{N,M} \equiv \left\{ w \in W \;\middle|\; w = \sum_{i=0}^{M-1} b_i^M B_i^M,\ b_i^M \in X^N \right\}.$$

Define $\widetilde{D}_*^{N,M} : \widetilde{W}_*^{N,M} \to W^{N,M}$ by

$$\widetilde{D}_*^{N,M} w^{N,M} \equiv \sum_{i=1}^{M} \frac{1}{\alpha_i^M} \left(b_{i-1}^M - b_i^M\right) \chi_i^M$$

where $w^{N,M} = \sum_{i=0}^{M-1} b_i^M B_i^M$ and $b_M^M = 0$. Define the isomorphism $i_*^{N,M} : \widetilde{W}_*^{N,M} \to W^{N,M}$ by

$$i_*^{N,M} w^{N,M} \equiv \sum_{i=1}^{M} b_{i-1}^M \chi_i^M.$$

Now define $D_*^{N,M} : W^{N,M} \to W^{N,M}$ by $D_*^{N,M} \equiv \widetilde{D}_*^{N,M} \left(i_*^{N,M}\right)^{-1}$. We can now write $\left(\mathcal{A}^{N,M}\right)^*$ in the form

$$\left(\mathcal{A}^{N,M}\right)^* \begin{pmatrix} x^N \\ y^N \\ w^{N,M} \end{pmatrix} = \begin{pmatrix} -A_0^N \left[ \begin{pmatrix} x^N \\ y^N \end{pmatrix} + \int_{-r}^{0} \begin{pmatrix} w^{N,M}(s) \\ 0 \end{pmatrix} ds \right] \\ -g_\alpha^M j y^N - D_*^{N,M} w^{N,M} \end{pmatrix}.$$

We wish to prove convergence of the adjoint semigroup using the method of the previous section. Thus, we assume A1) and A2). Let $W_e$ and $Z_e$ be $W$ and $Z$ equipped with the norms $\|\cdot\|_e$ and $\|\cdot\|_{Z_e}$, respectively, and let $P_{W_e}^{N,M}$ and $P_{Z_e}^{N,M}$ be the orthogonal projections of $W_e$ onto $W^{N,M}$ and $Z_e$ onto $Z^{N,M}$. Since $\|\cdot\|_W$ and $\|\cdot\|_e$ are equivalent norms and $P_W^{N,M} \xrightarrow{s} I$ with respect to the norm $\|\cdot\|_W$, we have

LEMMA 3.2.3. *$P_{Z_e}^{N,M} \xrightarrow{s} I$ with respect to the norm $\|\cdot\|_{Z_e}$.*

Let $D_*$ be defined by $\mathcal{D}(D_*) \equiv H_L^1(-r,0;X)$, $D_* \equiv \dfrac{\partial}{\partial s}$. Define the bilinear form $a_*(\cdot,\cdot)$ on $\mathcal{D}(D_*) \times W$ by $a_*(w,h) \equiv -\int_{-r}^{0} \langle D_* w, h \rangle_X \, ds$, and the bilinear form $a_*^{N,M}(\cdot,\cdot)$ on $\widetilde{W}_*^{N,M} \times W^{N,M}$ by $a_*^{N,M}(w^{N,M}, h^{N,M}) \equiv \left\langle -\widetilde{D}_*^{N,M} w^{N,M}, h^{N,M} \right\rangle_e = -\int_{-r}^{0} \left\langle D_* w^{N,M}, h^{N,M} \right\rangle_X ds$.

For $z = \begin{pmatrix} x \\ y \\ w \end{pmatrix} \in \mathcal{D}(\mathcal{A}^*)$ and $\mathrm{Re}\lambda > 0$, consider the equation $(\lambda I - \mathcal{A}^*)\, z = \begin{pmatrix} \varphi \\ \psi \\ h \end{pmatrix}$,

or

$$\lambda x + jy = \varphi, \tag{3.2.3}$$

$$\lambda y + A\left(x + \int_{-r}^{0} w(s)ds\right) = \psi, \tag{3.2.4}$$

$$\lambda w + g_\alpha jy + \frac{\partial w}{\partial s} = h. \tag{3.2.5}$$

From (3.2.5) $w(s) = -\int_{-r}^{s} e^{\lambda(\xi - s)}\left(g_\alpha(\xi)jy - h(\xi)\right)d\xi$, and from (3.2.3), $jy = -\lambda x + \varphi$, or $y = -\lambda j^{-1}x + j^{-1}\varphi$. Substituting into (3.2.4) we get

$$\Delta(\lambda)x = -\psi + \lambda j^{-1}\varphi - A\int_{-r}^{0}(\lambda I + D_*)^{-1}\left[g_\alpha(s)\varphi - h(s)\right]ds \tag{3.2.6}$$

where $\Delta(\lambda)$ is given by (3.1.9).

Now, for $z^{N,M} = \left(x^N, y^N, w^{N,M}\right)^T \in Z^{N,M}$ and $\mathrm{Re}\lambda > 0$, consider the equation $\left(\lambda I - \left(\mathcal{A}^{N,M}\right)^*\right) z^{N,M} = \left(\varphi^N, \psi^N, h^{N,M}\right)^T$. We have

$$\lambda x^N + jy^N = \varphi^N, \tag{3.2.7}$$

$$\lambda y^N + A^N\left(x^N + \sum_{i=1}^{M}\alpha_i^M w_i^M\right) = \psi^N, \tag{3.2.8}$$

$$\lambda w_i^M + \frac{(g_\alpha)_i^M}{\alpha_i^M} jy^N + \frac{1}{\alpha_i^M}\left(w_i^M - w_{i+1}^M\right) = h_i^M \quad \text{for} \quad i = 1, 2, \ldots, M \tag{3.2.9}$$

where $w^{N,M} = \sum_{i=1}^{M} w_i^M \chi_i^M$ and $h^{N,M} = \sum_{i=1}^{M} h_i^M \chi_i^M$. From (3.2.9),

$$\left(\lambda + \frac{1}{\alpha_i^M}\right) w_i^M = \frac{1}{\alpha_i^M} w_{i+1}^M - \frac{(g_\alpha)_i^M}{\alpha_i^M} jy^N + h_i^M$$

or

$$w_i^M = \left(1 + \lambda\alpha_i^M\right)^{-1} \left[w_{i+1}^M - (g_\alpha)_i^M j y^N + \alpha_i^M h_i^M\right], \; i = 1, 2, \ldots, M$$

where $w_{M+1}^M = 0$. In order to find a formula for $w_i^M$ in terms of the $h_i^M$, we first observe that

$$w_{M-1}^M = \left(1 + \lambda\alpha_{M-i}^M\right)^{-1} \left[w_{M-(i-1)}^M - (g_\alpha)_{M-i}^M \, j y^N + \alpha_{M-i}^M h_{M-i}^M\right],$$

for $i = 0, 1, \ldots, M-1$. Then, by induction,

$$w_{M-i}^M = \sum_{k=0}^{i} \left[\prod_{l=k}^{i} \left(1 + \lambda\alpha_{M-l}^M\right)^{-1}\right] \cdot \left[\alpha_{M-k}^M h_{M-k}^M - (g_\alpha)_{M-k}^M \, j y^N\right].$$

Thus, $w_i^M = \sum_{k=i}^{M} \left[\prod_{l=i}^{k} \left(1 + \lambda\alpha_l^M\right)^{-1}\right] \cdot \left[\alpha_k^M h_k^M - (g_\alpha)_k^M j y^N\right]$. From (3.2.7), $j y^N = \varphi^N - \lambda x^N$ which implies $y^N = j^{-1}\varphi^N - \lambda j^{-1} x^N$. Substituting into (3.2.8) we obtain

$$\begin{aligned} &\lambda^2 j^{-1} x^N - A^N \left(x^N + \sum_{i=1}^{M} \alpha_i^M \sum_{k=i}^{M} \left[\prod_{l=i}^{k} \left(1 + \lambda\alpha_l^M\right)^{-1}\right] \lambda (g_\alpha)_k^M x^N\right) \\ &= -\psi^N + \lambda j^{-1} \varphi^N - A^N \sum_{i=1}^{M} \alpha_i^M \sum_{k=i}^{M} \left[\prod_{l=i}^{k} \left(1 + \lambda\alpha_l^M\right)^{-1}\right] \cdot \left[(g_\alpha)_k^M \varphi^N - \alpha_k^M h_k^M\right]. \end{aligned}$$

By reversing the order of summation on the left-hand side, we get

$$\Delta^{N,M}(\lambda) x^N = -\psi^N + \lambda j^{-1} \varphi^N - A^N \int_{-r}^{0} \left(\lambda I + D_*^{N,M}\right)^{-1} \left(g_\alpha^M \varphi^N - h^{N,M}\right) ds \quad (3.2.10)$$

where $\Delta^{N,M}(\lambda)$ is given by (3.1.14).

LEMMA 3.2.4. *For $\lambda > 0$, $\left(\lambda I + D_*^{N,M}\right)^{-1} P_{W_*}^{N,M} h \to \left(\lambda I + D_*\right)^{-1} h$ for all $h \in W$.*

PROOF: We first show that $-D_*^{N,M}$ is dissipative. Let $w^{N,M} = \sum_{i=1}^{M} w_i^M \chi_i^M$. Then

$$\begin{aligned}
\langle -D_*^{N,M} w^{N,M}, w^{N,M} \rangle_e &= -\int_{-r}^{0} \left\langle \sum_{i=1}^{M} \frac{1}{\alpha_i^M} \left(w_i^M - w_{i+1}^M\right) \chi_i^M, \sum_{i=1}^{M} w_i^M \chi_i^M \right\rangle_X ds \\
&= \sum_{i=1}^{M} \langle w_{i+1}^M - w_i^M, w_i^M \rangle_X \\
&\leq \sum_{i=1}^{M} \left[ \|w_{i+1}^M\|_X \cdot \|w_i^M\|_X - \|w_i^M\|_X^2 \right] \\
&\leq \frac{1}{2} \sum_{i=1}^{M} \left[ \|w_{i+1}^M\|_X^2 - \|w_i^M\|_X^2 \right] = -\frac{1}{2} \|w_1^M\|_X^2 \leq 0.
\end{aligned}$$

Thus, $\left\| \left(\lambda I + D_*^{N,M}\right)^{-1} \right\|_e \leq \frac{1}{\lambda}$. Let

$$w = (\lambda I + D_*)^{-1} h \quad \text{and} \quad w^{N,M} = \left(\lambda I + D_*^{N,M}\right)^{-1} P_{W_*}^{N,M} h,$$

or equivalently

$$(\lambda I + D_*) w = h \quad \text{and} \quad \left(\lambda I + D_*^{N,M}\right) w^{N,M} = P_{W_*}^{N,M} h.$$

Set $\widetilde{w}^{N,M} \equiv \left(i_*^{N,M}\right)^{-1} w^{N,M}$ and $\Gamma_e^{N,M} \equiv \left\{ \gamma^{N,M} \in W^{N,M} \;\middle|\; \|\gamma^{N,M}\|_e \leq 1 \right\}$. Following the proof of Lemma 3.1.6 we can show that

$$\begin{aligned}
\left\| i_*^{N,M} \left(\widehat{w}^{N,M} - \widetilde{w}^{N,M}\right) \right\|_e \leq \frac{1}{\lambda} &\sup_{\gamma^{N,M} \in \Gamma_e^{N,M}} \left| a_* \left(w, \gamma^{N,M}\right) - a_*^{N,M} \left(\widehat{w}^{N,M}, \gamma^{N,M}\right) \right| \\
&+ \left\| P_{W_*}^{N,M} \left(w - i_*^{N,M} \widehat{w}^{N,M}\right) \right\|_e \qquad (3.2.11)
\end{aligned}$$

for all $\widehat{w}^{N,M} \in \widetilde{W}_*^{N,M}$. Take $\widehat{w}^{N,M} = \int_{-r}^{s} P_{W_*}^{N,M}(D_* w) d\xi$. Then

$$a_* \left(w, \gamma^{N,M}\right) - a_*^{N,M} \left(\widehat{w}^{N,M}, \gamma^{N,M}\right) = \int_{-r}^{0} \left\langle P_{W_*}^{N,M}(D_* w) - D_* w, \gamma^{N,M} \right\rangle_X ds = 0$$

for all $\gamma^{N,M} \in W^{N,M}$. Since $\|\cdot\|_W$ and $\|\cdot\|_e$ are equivalent norms there exist constants $\gamma_1, \gamma_2 > 0$ such that $\gamma_1 \|w\|_e \leq \|w\|_W \leq \gamma_2 \|w\|_e$ for all $w \in W$. Thus,

$$\begin{aligned}\gamma_1 \left\|w - i_*^{N,M}\widehat{w}^{N,M}\right\|_e &\leq \left\|w - i_*^{N,M}\widehat{w}^{N,M}\right\|_W \\ &\leq \left\|w - \widehat{w}^{N,M}\right\|_W + \left\|\widehat{w}^{N,M} - i_*^{N,M}\widehat{w}^{N,M}\right\|_W.\end{aligned}$$

Again following the proof of Lemma 3.1.6 we see that

$$\begin{aligned}\left\|w - \widehat{w}^{N,M}\right\|_W &\leq \frac{r^2}{2}\left\|D_* w - P_{W_*}^{N,M}(D_* w)\right\|_W^2 \\ &\leq \frac{(r\gamma_2)^2}{2}\left\|D_* w - P_{W_*}^{N,M}(D_* w)\right\|_e^2 \to 0 \text{ as } N, M \to \infty.\end{aligned}$$

Set $P_{W_*}^{N,M}(D_* w) = \sum\limits_{i=1}^{M} \xi_i^M \chi_i^M$ and $\widehat{w}^{N,M}(s) = \sum\limits_{i=0}^{M-1} b_i^M B_i^M(s)$. Then

$$b_j^M = \widehat{w}^{N,M}(t_j^M) = \int_{-r}^{t_j^M} \sum_{i=1}^{M} \xi_i^M \chi_i^M(\xi) d\xi = \sum_{i=j+1}^{M} \xi_i^M \alpha_i^M \quad \text{for } j = 0, 1, \ldots, M-1.$$

Now, $i_*^{N,M}\widehat{w}^{N,M} = \sum\limits_{i=1}^{M} b_{i-1}^M \chi_i^M$. Since $\widehat{w}^{N,M} = \sum\limits_{i=0}^{M-1} b_i^M B_i^M$,

$$\widehat{w}^{N,M}(s) = b_{j-1}^M - \frac{b_{j-1}^M - b_j^M}{\alpha_j^M}\left(t_{j-1}^M - s\right) \quad \text{for } s \in [t_j^M, t_{j-1}^M).$$

Hence, $\widehat{w}^{N,M} - i_*^{N,M}\widehat{w}^{N,M} = -\dfrac{b_{j-1}^M - b_j^M}{\alpha_j^M}\left(t_{j-1}^M - s\right)$ on $[t_j^M, t_{j-1}^M)$ for $j = 1, 2, \ldots, M$.

Thus,

$$\left\|\widehat{w}^{N,M} - i_*^{N,M}\widehat{w}^{N,M}\right\|_W^2 = \sum_{i=1}^{M} \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) \left\|\frac{b_{i-1}^M - b_i^M}{\alpha_i^M}\left(t_{i-1}^M - s\right)\right\|_X^2 ds.$$

But, $\dfrac{b_{i-1}^M - b_i^M}{\alpha_i^M} = \dfrac{1}{\alpha_i^M}\left(\sum\limits_{j=i}^{M} \xi_j^M \alpha_j^M - \sum\limits_{j=i+1}^{M} \xi_j^M \alpha_j^M\right) = \xi_i^M$, so

$$\left\|\widehat{w}^{N,M} - i_*^{N,M}\widehat{w}^{N,M}\right\|_W^2 = \sum_{i=1}^{M} \int_{t_i^M}^{t_{i-1}^M} g_\alpha(s) \left\|\xi_i^M\right\|_X^2 \left(t_{i-1}^M - s\right)^2 ds = S_1 + S_2.$$

Once again, as in the proof of Lemma 3.1.6, $S_1 \to 0$ and $S_2 \to 0$ as $M \to \infty$, so $\left\| w - i_*^{N,M} \widehat{w}^{N,M} \right\|_W \to 0$ as $N, M \to \infty$ which implies $\left\| w - i_*^{N,M} \widehat{w}^{N,M} \right\|_e \to 0$ as $N, M \to \infty$. Therefore, from (3.2.11), $\left\| w^{N,M} - i_*^{N,M} \widehat{w}^{N,M} \right\|_e \to 0$ as $N, M \to \infty$, and so $\left\| w - w^{N,M} \right\|_e \to 0$ as $N, M \to \infty$. ∎

**THEOREM 3.2.5.** *For all $z \in Z$, $e^{(\mathcal{A}^{N,M})^* t} P_{Z_*}^{N,M} z \to T^*(t)z$ as $N, M \to \infty$, uniformly on bounded $t$-intervals.*

**PROOF:** Let $\mu(\cdot,\cdot)$ and $\mu^M(\cdot,\cdot)$ be as in the proof of Theorem 3.1.8. For $z = \begin{pmatrix} \varphi \\ \psi \\ h \end{pmatrix} \in Z$, let $(\lambda I - \mathcal{A}^*)^{-1} z = \begin{pmatrix} x \\ y \\ w \end{pmatrix}$, and $(\lambda I - (\mathcal{A}^{N,M})^*)^{-1} P_{Z_*}^{N,M} z = \begin{pmatrix} x^N \\ y^N \\ w^{N,M} \end{pmatrix}$. Then, by (3.2.6), for all $u \in X$,

$$\mu(x,u) = \langle -\psi + \lambda j^{-1}\varphi, j^{-1}u \rangle_Y + \sigma\left( j^{-1} \int_{-r}^{0} (\lambda I + D_*)^{-1} \left[ g_\alpha(s)\varphi - h(s) \right] ds, j^{-1}u \right),$$

and by (3.2.10), for all $u^N \in X^N$,

$$\mu^M(x^N, u^N) = \langle -\psi^N + \lambda j^{-1}\varphi^N, j^{-1}u^N \rangle_Y + \sigma\left( j^{-1} \int_{-r}^{0} (\lambda I + D_*^{N,M})^{-1} \left[ g_\alpha^M(s)\varphi^N - h^{N,M}(s) \right] ds, j^{-1}u^N \right),$$

where $P_{Z_*}^{N,M} z = \begin{pmatrix} \varphi^N \\ \psi^N \\ h^{N,M} \end{pmatrix}$. The rest of the proof is analogous to the proof of Theorem 3.1.8 once we observe that $P_{W_*}^{N,M}(g_\alpha(s)\varphi) = g_\alpha^M(s)\varphi^N$. To see that this is true, let $u^N \in X^N$. Then for $i = 1, 2, \ldots, M$,

$$\langle g_\alpha(s)\varphi, u^N \chi_i^M(s)\rangle_e = \left(\int_{t_i^M}^{t_{i-1}^M} g_\alpha(s)ds\right)\langle \varphi, u^N\rangle_X = (g_\alpha)_i^M \langle \varphi^N, u^N\rangle_X$$

$$= \int_{-r}^{0} \langle g_\alpha^M(s)\varphi^N, u^N \chi_i^M\rangle_X \, ds.$$

Thus, $\langle g_\alpha(s)\varphi, h^{N,M}\rangle_e = \langle g_\alpha^M(s)\varphi^N, h^{N,M}\rangle_e$ for all $h^{N,M} \in W^{N,M}$. ∎

### 3.3 Approximation of a Viscoelastic Shaft.

Recall from Section 2.4 that the system described by (2.1.3) – (2.1.5) can be written in the form

$$\ddot{y} + \widetilde{A}\left[\tau y + \int_{-r}^{0} g(s) y_s ds\right] = f(t)$$

on the Hilbert space $Y \equiv \mathbf{R} \times L_2(0, l)$ where

$$\mathcal{D}(\widetilde{A}) = \left\{ \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \in Y \;\middle|\; \psi \in H_L^1(0, l) \cap H^2(0, l),\ \psi(l) = \gamma \right\},$$

$$\widetilde{A}\begin{pmatrix} \gamma \\ \psi \end{pmatrix} = \begin{pmatrix} \frac{1}{I_m}\psi'(l) \\ -\frac{1}{\sigma}\psi'' \end{pmatrix},$$

and

$$f(t) = \begin{pmatrix} \frac{1}{I_m} u_2(t) \\ \frac{1}{\sigma} b(x) u_1(t) \end{pmatrix}.$$

We will take the kernel function $g$ to be $g(s) \equiv -e^{5s}/(5\sqrt{-s})$. Clearly $g \in L_1(-r, 0)$, $g < 0$ and $g' < 0$ on $[-r, 0)$. In order to satisfy A2) we require that $\tau$ be chosen large enough that $\alpha \equiv \tau + \int_{-r}^{0} g(s)ds > 0$. We will use the uniform mesh described in Remark 3.1.2 and the non-uniform mesh described in Lemma 3.1.3, so A2) will be satisfied.

The space $X$ is $H^1_L(0,l)$ with norm $\|\varphi\|_X^2 = \alpha \int_0^l (\varphi')^2$. The space $Y$ is as above with norm $\left\| \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \right\|_Y^2 = I_m \gamma^2 + \sigma \int_0^l \psi^2$. For the first stage of the approximation we use a linear spline scheme as follows. For each positive integer $N$ let $x_i^N = \dfrac{il}{N}$ for $i = 0, 1, \ldots, N$, and let $h_i^N(\cdot)$ be the unique continuous, piecewise linear function on $[0, l]$ satisfying $h_i^N(x_j^N) = \delta_{ij}$. For simplicity of notation we denote $h_i^N(\cdot)$ by $h_i$ and $\dfrac{d}{dx}(h_i^N(\cdot))$ by $h_i'$. Define the spaces $X^N$ and $Y^N$ by

$$X^N \equiv \left\{ \varphi \in X \;\middle|\; \varphi = \sum_{i=1}^{N} \alpha_i h_i \right\},$$

$$Y^N \equiv \left\{ \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \in Y \;\middle|\; \psi = \sum_{i=1}^{N} \beta_i h_i,\ \gamma = \psi(l) = \beta_N \right\}.$$

Observe that $Y^N \subseteq S$.

We now show that $P_X^N \xrightarrow{s} I$ and $P_Y^N \xrightarrow{s} I$.

**Definition 3.3.1.** *For $f \in C(0,l)$ let $f_1^N$ denote the piecewise linear interpolate of $f$; i.e., $f_1^N$ is linear on each interval $[x_{i-1}^N, x_i^N]$, and $f_1^N(x_i^N) = f(x_i^N)$, $i = 0, 1, \ldots, N$.*

**Lemma 3.3.2.** *There exist positive constants $\gamma_1, \gamma_2$ such that if $f \in C^2(0,l)$, then*

(1) $\|f_1^N - f\|_{L_2(0,l)} \le \dfrac{\gamma_1}{N^2} \|f''\|_{L_2(0,l)}$,

(2) $\left\| \dfrac{d}{dx}(f_1^N - f) \right\|_{L_2(0,l)} \le \dfrac{\gamma_2}{N} \|f''\|_{L_2(0,l)}$.

**Proof:** See [15, Theorem 2.5].

**Theorem 3.3.3.** (1) *If $\varphi \in X$, then $\|P_X^N \varphi - \varphi\|_X \to 0$ as $N \to \infty$.*

(2) *If $\begin{pmatrix} \gamma \\ \psi \end{pmatrix} \in Y$, then $\left\| P_Y^N \begin{pmatrix} \gamma \\ \psi \end{pmatrix} - \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \right\|_Y \to 0$ as $N \to \infty$.*

PROOF: (1) Let $\widehat{\varphi} \in C^2(0,l) \cap H_L^1(0,l)$. Then $\|\widehat{\varphi} - P_X^N \widehat{\varphi}\|_X \leq \|\widehat{\varphi} - \widehat{\varphi}_1^N\|_X = \alpha^{1/2} \left\|\frac{d}{dx}\left(\widehat{\varphi} - \widehat{\varphi}_1^N\right)\right\|_{L_2} \leq \frac{\alpha^{1/2}\gamma_2}{N} \|\widehat{\varphi}''\|_{L_2} \to 0$ as $N \to \infty$. Let $\epsilon > 0$. Since $C^2 \cap H_L^1$ is dense in $H_L^1$, there exists $\widehat{\varphi} \in C^2 \cap H_L^1$ such that $\|\varphi - \widehat{\varphi}\|_X < \frac{\epsilon}{3}$. Thus,

$$\begin{aligned}\|\varphi - P_X^N \varphi\|_X &\leq \|\varphi - \widehat{\varphi}\|_X + \|\widehat{\varphi} - P_X^N \widehat{\varphi}\|_X + \|P_X^N \widehat{\varphi} - P_X^N \varphi\|_X \\ &< \frac{2\epsilon}{3} + \|\widehat{\varphi} - P_X^N \widehat{\varphi}\|_X < \epsilon \quad \text{for } N \text{ sufficiently large.}\end{aligned}$$

(2) Let $\widehat{\psi} \in C^2(0,l)$. Define $\widehat{\psi}_\gamma^N$ to be the piecewise linear function satisfying $\widehat{\psi}_\gamma^N(x_j^N) = \widehat{\psi}(x_j^N)$ for $j = 0, 1, \ldots, N-1$, and $\widehat{\psi}_\gamma^N(x_N^N) = \gamma$. Thus, $\widehat{\psi}_\gamma^N = \widehat{\psi}_1^N$ on $[0, x_{N-1}^N]$.

Now,

$$\begin{aligned}\left\|\begin{pmatrix}\gamma \\ \widehat{\psi}\end{pmatrix} - P_Y^N \begin{pmatrix}\gamma \\ \widehat{\psi}\end{pmatrix}\right\|_Y &\leq \left\|\begin{pmatrix}\gamma \\ \widehat{\psi}\end{pmatrix} - \begin{pmatrix}\gamma \\ \widehat{\psi}_\gamma^N\end{pmatrix}\right\|_Y = \sigma^{1/2} \left\|\widehat{\psi} - \widehat{\psi}_\gamma^N\right\|_{L_2} \\ &\leq \sigma^{1/2}\left(\left\|\widehat{\psi} - \widehat{\psi}_1^N\right\|_{L_2} + \left\|\widehat{\psi}_1^N - \widehat{\psi}_\gamma^N\right\|_{L_2}\right) = I_1 + I_2.\end{aligned}$$

By Lemma 3.3.2, $I_1 \leq \frac{\sigma^{1/2}\gamma_1}{N^2}\left\|\widehat{\psi}''\right\|_{L_2} \to 0$ as $N \to \infty$. Let $k \equiv \left|\gamma - \widehat{\psi}(l)\right|$. Then

$$I_2^2 = \sigma \int_0^l \left|\widehat{\psi}_1^N - \widehat{\psi}_\gamma^N\right| dx = \sigma \int_{x_{N-1}^N}^l \left|\widehat{\psi}_1^N - \widehat{\psi}_\gamma^N\right| dx \leq \frac{\sigma k l}{N} \to 0 \quad \text{as } N \to \infty.$$

By a density argument similar to that used in part (1),

$$\left\|P_Y^N \begin{pmatrix}\gamma \\ \psi\end{pmatrix} - \begin{pmatrix}\gamma \\ \psi\end{pmatrix}\right\|_Y \to 0 \text{ as } N \to \infty \quad \text{for all } \begin{pmatrix}\gamma \\ \psi\end{pmatrix} \in Y. \quad \blacksquare$$

Thus, A1) and A2) both hold so the approximation converges. In the next chapter we will discuss implementation of this algorithm.

REMARK 3.3.4. *The above spline estimates are standard. A similar argument can be used to prove convergence if we use cubic splines instead of linear splines. The above arguments can also be applied to the other examples given in Section 2.4.*

If we wish to include the spring term in the right-hand boundary condition (i.e., use (2.4.2)), then in order to approximate the resulting system, we simply add the operator $\mathcal{A}_1$ to $\mathcal{A}^{N,M}$. The semigroups generated by $\mathcal{A}^{N,M} + \mathcal{A}_1$ converge to the semigroup generated by $\widetilde{\mathcal{A}}$ as the following theorem shows.

THEOREM 3.3.5. *Suppose $A, Z, A^n, Z^n$ and $P^n$ satisfy Theorem 3.1.1 and $B$ is a bounded linear operator on $Z$. If $S(t)$ is the $C_0$ semigroup generated by $A + B$ and $S^n(t)$ is the $C_0$ semigroup generated by $A^n + B$ on $Z^n$, then for all $z \in Z$, $S^n(t)P^n z \to S(t)z$ as $n \to \infty$, uniformly on bounded t-intervals.*

PROOF: $A, A^n \in G(M,\beta)$. By Theorem 13.4.1 in [8], $S(t) = \sum_{k=0}^{\infty} S_k(t)$ where $S_0(t) = T(t)$ and $S_k(t)x = \int_0^t T(t-s)BS_{k-1}(s)x\,ds$ for $k = 1, 2, \ldots$, and $S^n(t) = \sum_{k=0}^{\infty} S_k^n(t)$ where the $S_k^n$ are defined similarly. Furthermore,

$$\left\| S^n(t) - \sum_{k=0}^{K} S_k^n(t) \right\| \leq M(M\|B\|)^{K+1} t^{K+1} e^{\beta_1 t}/(K+1)!$$

where $\beta_1 = \beta + M \cdot \|B\|$. Let $\epsilon > 0$. Since $C^K/K! \to 0$ as $K \to \infty$, there exists a constant $K_0$ independent of $n$ such that for fixed $z \in Z$,

$$\left\| S^n(t)P^n z - \sum_{k=0}^{K_0} S_k^n(t)P^n z \right\| < \frac{\epsilon}{4} \qquad \text{and} \qquad \left\| S(t)z - \sum_{k=0}^{K_0} S_k(t)z \right\| < \frac{\epsilon}{4}.$$

Thus, $\|S^n(t)P^nz - S(t)z\| < \frac{\epsilon}{2} + \sum_{k=0}^{K_0} \|S_k^n(t)P^nz - S_k(t)z\|$. Define

$$f(\beta) \equiv \begin{cases} t + \frac{\|B\|\cdot M(e^{\beta t}-1)}{\beta}, & \text{if } \beta \neq 0, \\ t(1+\|B\|\cdot M), & \text{if } \beta = 0. \end{cases}$$

Choose $\delta > 0$ so that $\delta \sum_{k=0}^{K_0} [f(\beta)]^k < \frac{\epsilon}{2}$. Since $T^n(t)P^n\widehat{z} \to T(t)\widehat{z}$ uniformly on bounded $t$-intervals for all $\widehat{z} \in Z$, there exists $N_0$ such that if $n > N_0$, then

$$\|T^n(s)P^nz - T(s)z\| < \delta$$

and

$$\|T^n(t-s)P^nBS_k(s)z - T(t-s)BS_k(s)z\| < \delta$$

for $k = 0, 1, \ldots, K_0 - 1$ and for all $s \in [0, t]$. Suppose $\|S_k^n(s)P^nz - S_k(s)z\| < C$ where $C > \delta$ for all $s \in [0, t]$. Then

$$\begin{aligned}
&\left\|S_{k+1}^n(s)P^nz - S_{k+1}(s)z\right\| \\
&\quad \leq \int_0^s [\|T^n(s-\sigma)BS_k^n(\sigma)P^nz - T^n(s-\sigma)P^nBS_k(\sigma)z\| \\
&\qquad\qquad + \|T^n(s-\sigma)P^nBS_k(\sigma)z - T(s-\sigma)BS_k(\sigma)z\|]\, d\sigma \\
&\quad \leq \int_0^s \left[C \cdot \|B\| \cdot Me^{\beta(s-\sigma)} + \delta\right] d\sigma \\
&\quad \leq \int_0^t \left[C \cdot \|B\| \cdot Me^{\beta(t-\sigma)} + \delta\right] d\sigma \\
&\quad \leq C \int_0^t \left[\|B\| \cdot Me^{\beta(t-\sigma)} + 1\right] d\sigma = Cf(\beta).
\end{aligned}$$

Thus, $\|S_k^n(t)P^nz - S_k(t)z\| \leq \delta\,[f(\beta)]^k$ for $k = 1, 2, \ldots, K_0$. Therefore,

$$\|S^n(t)P^nz - S(t)z\| < \frac{\epsilon}{2} + \delta \sum_{k=0}^{K_0} [f(\beta)]^k < \epsilon. \qquad \blacksquare$$

# Chapter IV Numerical Results

In this chapter we discuss the implementation of the Finite Element/Averaging approximation scheme for the viscoelastic shaft with tip-mass. We describe how to construct the matrices for the simulation problem and the control problem, and we compare the results obtained using the uniform mesh to the results from the non-uniform mesh. The computer codes described in this chapter were implemented on a Vax 8800 and an IBM 3090.

## 4.1 Simulation.

We consider the system given by the equation

$$\sigma\frac{\partial^2}{\partial t^2}y(t,x) = \frac{\partial}{\partial x}\left[\tau\frac{\partial}{\partial x}y(t,x) + \int_{-r}^{0} g(s)\frac{\partial}{\partial x}y(t+s,x)ds\right] + b(x)u_1(t), \tag{4.1.1}$$

with boundary conditions

$$y(t,0) = 0, \tag{4.1.2}$$

$$I_m\frac{\partial^2}{\partial t^2}y(t,l) = -\left[\tau\frac{\partial}{\partial x}y(t,l) + \int_{-r}^{0} g(s)y(t+s,l)ds\right] + u_2(t), \tag{4.1.3}$$

and initial conditions

$$y(0,x) = s(x), \qquad 0 \le x \le l, \tag{4.1.4}$$

$$\frac{\partial}{\partial t}y(0,x) = v(x), \qquad 0 \le x \le l, \tag{4.1.5}$$

$$y_0(s,x) = h(s,x), \qquad 0 \le x \le l, \quad -r \le s \le 0. \tag{4.1.6}$$

Recall that the spaces $X$ and $Y$ are given by

$$X = H_L^1(0,l) \quad \text{with norm } \|\varphi\|_X^2 = \alpha \int_0^l [\varphi'(x)]^2 \, dx,$$

$$Y = \mathbf{R} \times L_2(0,l) \quad \text{with norm } \left\| \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \right\|_Y^2 = I_m \gamma^2 + \sigma \int_0^l \psi^2(x) dx,$$

and that $S = \left\{ \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \in Y \;\middle|\; \psi \in H_L^1(0,l),\ \psi(l) = \gamma \right\}$. The first step in the approximation is to construct finite dimensional subspaces $X^N \subseteq X$ and $Y^N \subseteq S$, then to compute the matrix representation of $A_0^N$ with respect to a basis for $X^N \times Y^N$. Since we are using linear splines, we choose as basis elements

$$e_i^N = \begin{pmatrix} h_i \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad e_{N+i}^N = \begin{pmatrix} 0 \\ h_i(l) \\ h_i \end{pmatrix}$$

for $i = 1, 2, \ldots, N$ and set $X^N \times Y^N \equiv \text{span}\{e_i^N \mid i = 1, 2, \ldots, 2N\}$. Observe that this definition agrees with the choice of $X^N$ and $Y^N$ given in Section 3.3. Suppose $u^N \in X^N \times Y^N$ is given by $u^N = \sum_{i=1}^{2N} \alpha_i^N e_i^N$, and let $v^N = A_0^N u^N$ where $v^N = \sum_{i=1}^{2N} \beta_i^N e_i^N$. Then, for $i = 1, 2, \ldots, 2N$,

$$\langle v^N, e_i^N \rangle_{X \times Y} = \langle A_0^N u^N, e_i^N \rangle_{X \times Y} = \sigma_0 \left( u^N, e_i^N \right),$$

or, equivalently,

$$\sum_{j=1}^{2N} \beta_j^N \langle e_j^N, e_i^N \rangle_{X \times Y} = \sum_{j=1}^{2N} \alpha_j^N \sigma_0 \left( e_j^N, e_i^N \right). \tag{4.1.7}$$

Define the $N \times N$ matrices $D$ and $H$ by

$$D_{ij} \equiv \langle e_j^N, e_i^N \rangle_{X \times Y} = \alpha \int_0^l h_i'(x) h_j'(x) dx,$$

$$H_{ij} \equiv \langle e_{N+j}^N, e_{N+i}^N \rangle_{X \times Y} = I_m h_j(l) h_i(l) + \sigma \int_0^l h_j(x) h_i(x) dx$$

for $i = 1, 2, \ldots, N$. Then we can rewrite (4.1.7) as the $2N \times 2N$ system $M\vec{\beta} = F\vec{\alpha}$ where

$$M = \begin{bmatrix} D & 0 \\ 0 & H \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix}.$$

Thus, $[A_0^N]$, the matrix representation of $A_0^N$, is given by

$$[A_0^N] = \begin{bmatrix} 0 & I \\ -H^{-1}D & 0 \end{bmatrix}. \tag{4.1.8}$$

If we take $g(s) \equiv 0$ in (4.1.1) and (4.1.3) and set $\sigma = \tau = I_m = l = 1$, then we arrive at the system

$$\frac{\partial^2}{\partial t^2} y(t,x) = \frac{\partial^2}{\partial x^2} y(t,x) + b(x)u_1(t), \tag{4.1.9}$$

$$y(t,0) = 0, \tag{4.1.10}$$

$$\frac{\partial^2}{\partial t^2} y(t,1) = -\frac{\partial}{\partial x} y(t,1) + u_2(t). \tag{4.1.11}$$

Using the standard separation of variables technique, we find that the eigenvalues of the system (4.1.9) – (4.1.11) are $i\lambda_j$ where $\lambda_j$ are the solutions of the equation $\lambda = \cot \lambda$. In Table 4.1.1 we compare the first eight eigenvalues of the system to the computed eigenvalues of $[A_0^N]$ for several values of $N$. Observe that approximating the higher frequencies requires large values of $N$. Using a large value of $N$ (e.g., $N = 64$) to approximate the system with damping included (i.e., $g(s) \not\equiv 0$) would result in a prohibitively large system of equations. Thus, since we are interested in controlling the modes corresponding to the lower frequencies, we will use small values of $N$ (e.g.; $N = 8$ or 16) for computations estimating the eigenvalues of the damped system. Note that the error in the fourth frequency is 5.8% for $N = 8$ and only 1.5% for $N = 16$.

For the second stage of the approximation, we choose a positive integer $M$ and construct the matrix representation of the operator $\mathcal{A}^{N,M}$ given by (3.1.4). We will denote this matrix by $A^{N,M}$. Using (4.1.8) it is clear that $A^{N,M}$ has the form

$$A^{N,M} = \begin{bmatrix} 0 & I & 0 & 0 & \cdots & \cdots & 0 \\ -H^{-1}D & 0 & -H^{-1}D_1 & -H^{-1}D_2 & \cdots & \cdots & -H^{-1}D_M \\ 0 & I & -\frac{1}{\alpha_1^M}I & 0 & \cdots & \cdots & 0 \\ 0 & I & \frac{1}{\alpha_2^M}I & -\frac{1}{\alpha_2^M}I & & & \vdots \\ 0 & I & 0 & \frac{1}{\alpha_3^M}I & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & -\frac{1}{\alpha_{M-1}^M}I & 0 \\ 0 & I & 0 & \cdots & 0 & \frac{1}{\alpha_M^M}I & -\frac{1}{\alpha_M^M}I \end{bmatrix}$$

where $D_i = (g_\alpha)_i^M D$ for $i = 1, 2, \ldots, M$. Notice that $A^{N,M}$ is a square matrix of order $(M+2) \cdot N$.

If we replace (4.1.3) by (2.4.2), then to approximate the resulting system we must approximate the operator $\mathcal{A}_1$ defined in Section 2.4 and add its matrix representation to $A^{N,M}$. Using the same method used above to find $[A_0^N]$, it is easy to see that $A_1^{N,M}$ has the form

$$A_1^{N,M} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -H^{-1}K & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right]$$

where $K_{ij} = kh_i(l)h_j(l)$.

We can compare the computed eigenvalues of $[A_0^N]$ to the eigenvalues of the undamped system with the spring added just as we did for the system (4.1.9) – (4.1.11). Note that the right-hand boundary condition is given by

$$\frac{\partial^2}{\partial t^2}y(t,1) = -\left(y(t,1) + \frac{\partial}{\partial x}y(t,1)\right) + u_2(t). \tag{4.1.12}$$

Applying separation of variables to (4.1.9), (4.1.10) and (4.1.12), we see that the imaginary parts of the eigenvalues are given by the solutions to $\lambda/(\lambda^2 - 1) = \tan\lambda$. Table 4.1.2 gives the approximations to the eigenvalues of the system with $k = 1$. Again we observe that $N = 16$ yields a reasonably accurate approximation of the first four modes. Observe also that in the higher frequencies, the spring term has virtually no impact.

As mentioned in Remark 3.3.4, we can use cubic splines instead of linear splines. The procedure is exactly the same as above except for the way in which we incorporate the boundary condition (4.1.2). For linear splines we simply discard $h_0$ since it is the only one of the "hat" functions for which $h_i(0) \neq 0$. Let $B_i$ denote the $i$th cubic spline basis function, $i = -1, 0, \ldots, N+1$; i.e.,

$$B_i(x) \equiv \frac{1}{h^3}\begin{cases} (x - x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x \in [x_{i-1}, x_i] \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x \in [x_i, x_{i+1}] \\ (x_{i+2} - x)^3, & x \in [x_{i+1}, x_{i+2}] \end{cases}$$

where $h = l/N$ (see [12]). Observe that $B_{-1}(0) = B_1(0) = 1$ and $B_0(0) = 4$. Thus, if we set $b_1(\cdot) \equiv B_0(\cdot) - 4B_{-1}(\cdot)$, $b_2(\cdot) \equiv 4B_1(\cdot) - B_0(\cdot)$ and $b_i(\cdot) \equiv B_{i-1}(\cdot)$

TABLE 4.1.1 Eigenfrequencies of $A_0$ with $k = 0$ using linear splines

| $j$ | $\lambda_j$ | $\lambda_j^4$ | $\lambda_j^8$ | $\lambda_j^{16}$ | $\lambda_j^{32}$ | $\lambda_j^{64}$ | $\lambda_j^{128}$ |
|---|---|---|---|---|---|---|---|
| 1 | .86033 | .86078 | .86045 | .86036 | .86034 | .86034 | .86033 |
| 2 | 3.4256 | 3.5142 | 3.4479 | 3.4312 | 3.4270 | 3.4260 | 3.4257 |
| 3 | 6.4373 | 7.0856 | 6.6033 | 6.4788 | 6.4477 | 6.4399 | 6.4379 |
| 4 | 9.5293 | 11.327 | 10.083 | 9.6675 | 9.5638 | 9.5379 | 9.5315 |
| 5 | 12.645 | — | 13.937 | 12.972 | 12.727 | 12.666 | 12.650 |
| 6 | 15.771 | — | 18.177 | 16.409 | 15.930 | 15.811 | 15.781 |
| 7 | 18.902 | — | 22.552 | 20.004 | 19.177 | 18.971 | 18.919 |
| 8 | 22.036 | — | 26.213 | 23.779 | 22.472 | 22.145 | 22.064 |

TABLE 4.1.2 Eigenfrequencies of $A_0$ with $k = 1$ using linear splines

| $j$ | $\lambda_j$ | $\lambda_j^4$ | $\lambda_j^8$ | $\lambda_j^{16}$ | $\lambda_j^{32}$ | $\lambda_j^{64}$ | $\lambda_j^{128}$ |
|---|---|---|---|---|---|---|---|
| 1 | 1.2078 | 1.2091 | 1.2081 | 1.2079 | 1.2078 | 1.2078 | 1.2078 |
| 2 | 3.4482 | 3.5363 | 3.4705 | 3.4538 | 3.4496 | 3.4486 | 3.4483 |
| 3 | 6.4410 | 7.0887 | 6.6069 | 6.4824 | 6.4513 | 6.4435 | 6.4416 |
| 4 | 9.5305 | 11.328 | 10.084 | 9.6687 | 9.5650 | 9.5391 | 9.5326 |
| 5 | 12.646 | —— | 13.937 | 12.972 | 12.727 | 12.666 | 12.651 |
| 6 | 15.772 | —— | 18.177 | 16.409 | 15.930 | 15.811 | 15.781 |
| 7 | 18.903 | —— | 22.552 | 20.004 | 19.177 | 18.971 | 18.920 |
| 8 | 22.037 | —— | 26.213 | 23.779 | 22.472 | 22.145 | 22.064 |

for $i = 2, 3, \ldots, N+2$ then $b_i(0) = 0$ for all $i$, so we can choose as our basis for $X^N \times Y^N$ the set $\{e_i^N \mid i = 1, 2, \ldots, 2N+4\}$ where

$$e_i^N = \begin{pmatrix} b_i(\cdot) \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad e_{N+2+i}^N = \begin{pmatrix} 0 \\ b_i(l) \\ b_i(\cdot) \end{pmatrix}$$

for $i = 1, 2, \ldots, N+2$. The $N+2 \times N+2$ matrices $D$ and $H$ are defined exactly as above. Observe that the matrix $A^{N,M}$ has order $(M+2) \cdot (N+2)$ when using cubic splines. In order to compare the results obtained using cubic splines to the results from linear splines we will use two fewer cubic spline elements at each step so that the corresponding matrices will be the same size for both schemes. Table 4.1.3 gives the eigenvalues of system (4.1.9) – (4.1.11) (i.e., $k = 0$) on the left and the eigenvalues for the system with $k = 1$ on the right. Comparing Table 4.1.3 to Tables 4.1.1 and 4.1.2, we see that cubic splines give a significant improvement over linear splines. For the case with $k = 0$, using six cubic spline elements gives a better approximation to the first five modes than using sixteen linear spline elements, and fourteen cubic spline elements does better than 128 linear spline elements in the first eight modes. A similar improvement can be observed in the case with $k = 1$.

We next compute the eigenvalues of the AVE matrix (i.e., $A^{N,M}$ when $k = 0$ and $A^{N,M} + A_1^{N,M}$ when $k = 1$) to see the effect of damping and to compare the rate of convergence for the two mesh schemes discussed above. We set $g(s) \equiv -e^{5s}/(5\sqrt{-s})$. Note that $g \in L_1(-r, 0)$ and that $g$ has a singularity at zero. We use cubic splines with $N = 6$ and set $k = 0$. Figure 4.1.4 shows the open loop eigenvalues

TABLE 4.1.3 Eigenfrequencies of $A_0$ using Cubic Splines

| | $k = 0$ | | | $k = 1$ | | |
|---|---|---|---|---|---|---|
| $j$ | $\lambda_j$ | $\lambda_j^6$ | $\lambda_j^{14}$ | $\lambda_j$ | $\lambda_j^6$ | $\lambda_j^{14}$ |
| 1 | .86033 | .86033 | .86033 | 1.2078 | 1.2078 | 1.2078 |
| 2 | 3.4256 | 3.4256 | 3.4256 | 3.4482 | 3.4482 | 3.4482 |
| 3 | 6.4373 | 6.4375 | 6.4373 | 6.4410 | 6.4412 | 6.4410 |
| 4 | 9.5293 | 9.5349 | 9.5294 | 9.5305 | 9.5360 | 9.5305 |
| 5 | 12.645 | 12.695 | 12.645 | 12.646 | 12.696 | 12.646 |
| 6 | 15.771 | 15.954 | 15.772 | 15.772 | 15.954 | 15.772 |
| 7 | 18.902 | 22.181 | 18.906 | 18.903 | 22.181 | 18.906 |
| 8 | 22.036 | 23.770 | 22.050 | 22.037 | 23.770 | 22.050 |

of AVE using the uniform mesh with $M = 4, 8, 12, 16, 24$ and 32. Observe that the plots for $M = 24$ and $M = 32$ are "close" together. Figure 4.1.5 shows the eigenvalues obtained using the non-uniform Mesh with $M = 4, 8$ and 12. The plots for $M = 16$ and $M = 24$ are nearly indistinguishable from the plot for $M = 12$, so the scheme appears to have "converged." However, the plot obtained using the uniform mesh is considerably to the left of the plot from the non-uniform mesh. Since both schemes are approximating the same system, we expect to get the same eigenvalues, so one of the schemes must not have converged. In Figure 4.1.6 we show the eigenvalues for the uniform mesh with $M = 48, 64, 80$ and 96. Observe that the eigenvalues are moving toward those in Figure 4.1.5, but that even for $M = 96$ they are still too far to the left. It is interesting to note that the smallest interval in the case with $M = 12$ using the non-uniform mesh has length .001924. Achieving the same size interval using the uniform mesh would require setting $M = 520$. Thus, it appears that the non-uniform mesh is far superior to the uniform mesh.

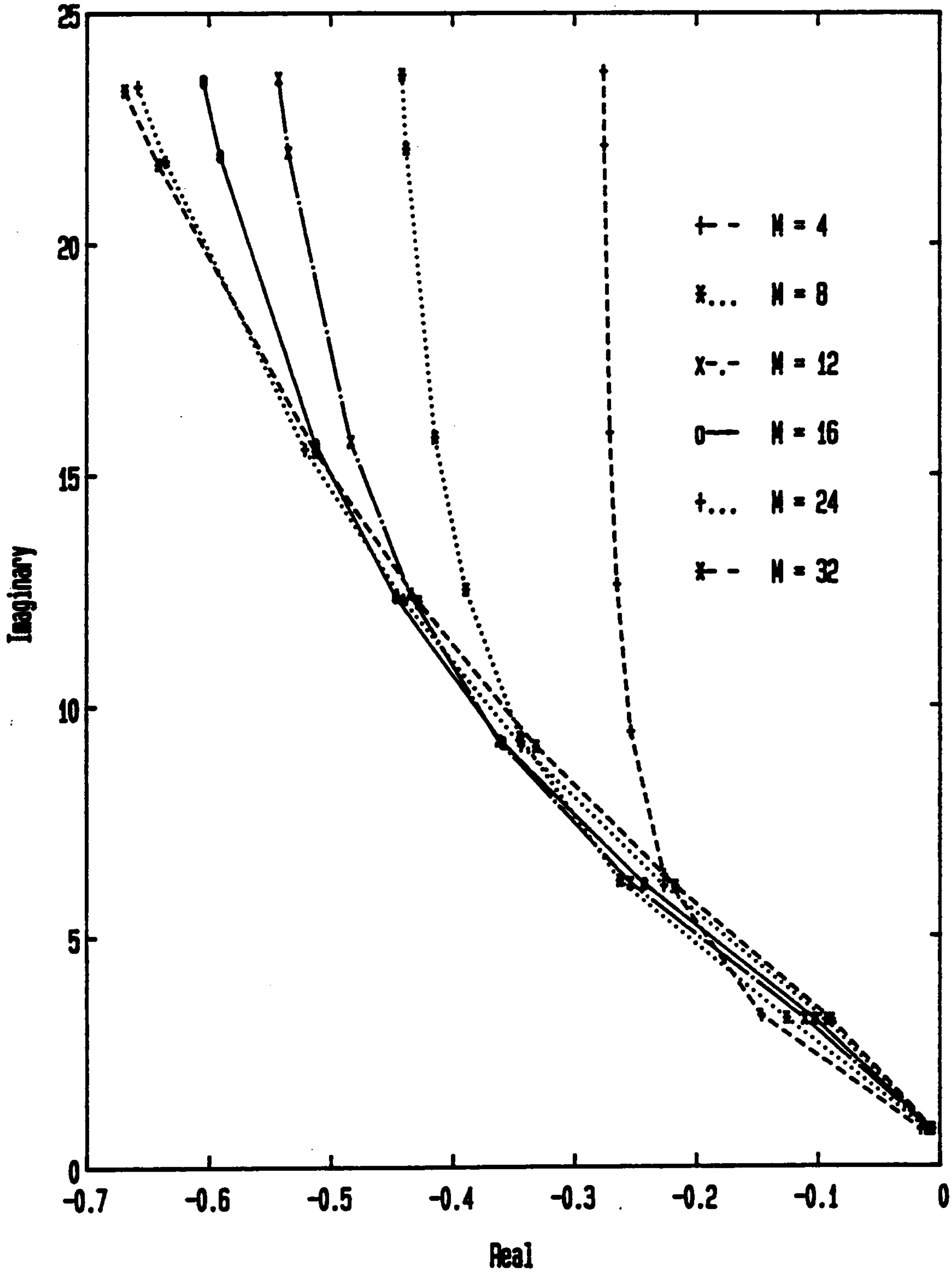


FIGURE 4.1.4 Uniform mesh, 6 cubic splines

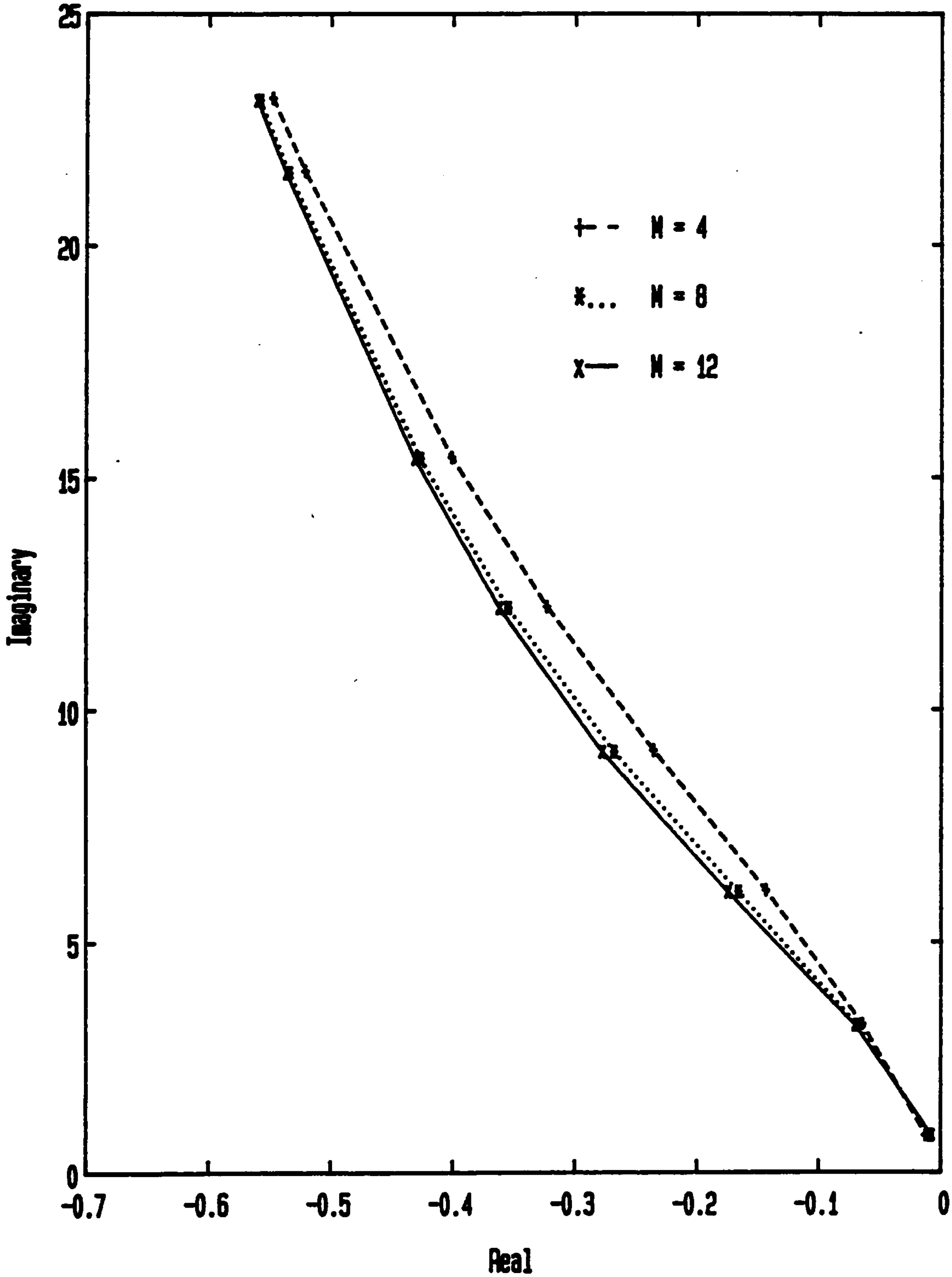


FIGURE 4.1.5 Non-uniform mesh, 6 cubic splines

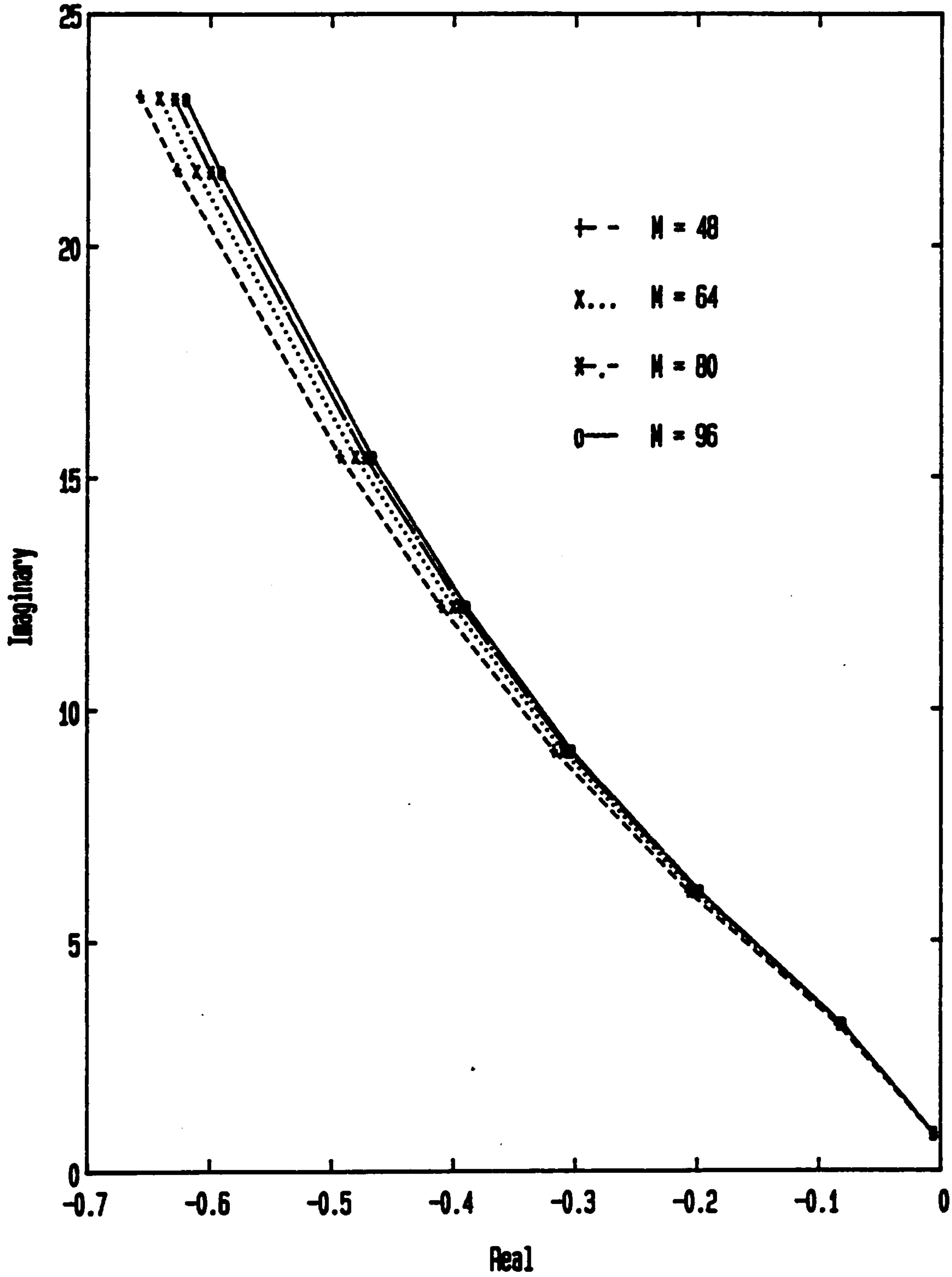


FIGURE 4.1.6 Uniform mesh, 6 cubic splines

## 4.2 Control.

For the control problem we must approximate the operator $\mathcal{A}^*$ as well as $\mathcal{A}$. Since the theory does not guarantee convergence of the adjoint for an $L_1$ kernel, we replace $g$ by the function $g_p$ defined by

$$g_p(s) \equiv \begin{cases} g(s), & \text{for } -r \le s \le -p, \\ g(-p) + g'(-p)\cdot(s+p), & \text{for } -p \le s \le 0 \end{cases}$$

where $0 < p < r$. We will take $p = 2^{-10}$.

The operator $\mathcal{B} : \mathbf{R}^2 \to Z$ is given by

$$\mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1/I_m \\ 1/\sigma\, b(x) & 0 \\ 0 & 0 \end{pmatrix}.$$

For this paper, we will take $b(x) = x^2$. Let $\mathcal{B}\begin{pmatrix} 0 \\ u_2 \end{pmatrix} = \sum \alpha_j f_j$ where $\{f_i\}$ is a basis for $Z^{N,M}$. Then $\sum \alpha_j \langle f_j, f_i \rangle_Z = \left\langle \mathcal{B}\begin{pmatrix} 0 \\ u_2 \end{pmatrix}, f_i \right\rangle_Z$ for all $i$, which implies $\sum \alpha_j \langle e_j, e_i \rangle_{X\times Y} = \left\langle \begin{pmatrix} 0 \\ 1/I_m u_2 \\ 0 \end{pmatrix}, e_i \right\rangle_{X\times Y}$ where $\{e_i\}$ is a basis for $X^N \times Y^N$. Thus, $\vec{\alpha} = \begin{pmatrix} 0 \\ \frac{1}{I_m} u_2 H^{-1} B_2 \end{pmatrix}$ where $(B_2)_i = I_m h_i(l)$.

Now, let $\mathcal{B}\begin{pmatrix} u_1 \\ 0 \end{pmatrix} = \sum \beta_j f_j$. Then, $\sum \beta_j \langle e_j, e_i \rangle_{X\times Y} = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1/\sigma\, b(x) u_1 \end{pmatrix}, e_i \right\rangle_{X\times Y}$, which implies $\vec{\beta} = \begin{pmatrix} 0 \\ \frac{1}{\sigma} u_1 H^{-1} B_1 \end{pmatrix}$ where $(B_1)_i = \sigma \int_0^l b(x) h_i(x)\,dx$. Thus,

$$B^{N,M} = \begin{pmatrix} 0 & 0 \\ \frac{1}{\sigma} H^{-1} B_1 & \frac{1}{I_m} H^{-1} B_2 \\ 0 & 0 \end{pmatrix}.$$

We will take $\mathcal{W} = \mathcal{C}^*\mathcal{C}$ where

$$\mathcal{C}\begin{pmatrix}\varphi\\ \gamma\\ \psi\\ w\end{pmatrix} = \begin{pmatrix}p_{-1}\gamma\\ p_0\varphi(l)\\ PM_\epsilon\psi\\ PM_\epsilon\varphi\end{pmatrix}$$

where $P = \mathrm{diag}(p_1, \ldots, p_n)$, $p_i > 0$ for $i = -1, 0, \ldots, n$, and the operator $M_\epsilon : L_2(0,l) \to \mathbf{R}^n$ is defined by

$$M_\epsilon f = \begin{pmatrix}\frac{1}{2\epsilon}\int_{x_1-\epsilon}^{x_1+\epsilon} f(x)dx\\ \vdots\\ \frac{1}{2\epsilon}\int_{x_n-\epsilon}^{x_n+\epsilon} f(x)dx\end{pmatrix}$$

where $0 < x_1 < \cdots < x_n < l$. Thus, $M_\epsilon\psi$ measures the average displacement at each of the $x_i$, and $M_\epsilon\varphi$ measures the average velocity. In order to construct the approximation to $\mathcal{C}$, we merely have to integrate the "approximate delta functions" $\delta_i^\epsilon$ defined by

$$\delta_i^\epsilon = \frac{1}{2\epsilon}\chi_{[x_i-\epsilon, x_i+\epsilon]}, \qquad i = 1, 2, \ldots, n$$

against each of the basis functions. For the runs described below, we used $n = 4$ with $x_1 = .25$, $x_2 = .32$, $x_3 = .5$ and $x_4 = .677$. We set $\epsilon = .01$ and took all the weights to be 1.

We use Potter's method (see [14]) to solve the finite dimensional algebraic Riccati equation (1.1.6). To simplify notation, let $A, B$ and $C$ denote the approximations of $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively, and set $W = C^TC$. The first step in Potter's method is to form the matrix

$$P = \begin{bmatrix}A^T & W\\ BR^{-1}B^T & -A\end{bmatrix}.$$

Next, find the eigenvalues and eigenvectors of P and form the matrix $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ where the columns of $Z$ are the eigenvectors of $P$ corresponding to the eigenvalues with positive real part. When eigenvalues occur in complex conjugate pairs, so do the eigenvectors. In this case, the real and the imaginary part of the eigenvector each forms a column of $Z$. Finally, the solution to the Riccati equation is given by $Q = XY^{-1}$. Once we have found $Q$, we can compute the "gain" matrix $K = -R^{-1}B^TQ$ and the "closed-loop" matrix $A + BK$.

Figure 4.2.1 shows the closed-loop eigenvalues for 6 cubic splines and the non-uniform mesh with $M = 4, 8, 12$ and 16. As expected based on our experience with the open-loop eigenvalues, there is a negligible difference between the case with $M = 12$ and the case with $M = 16$. Since Potter's method involves computing the eigenvectors of an order $2 \cdot (N + 2) \cdot (M + 2)$ matrix, we want to keep $M$ small while increasing $N$. Figure 4.2.1 indicates that we can expect reasonably accurate results for $M$ as small as 8. Figure 4.2.2 shows the closed-loop eigenvalues for larger values of $N$.

Although it may appear that the scheme is converging, the ultimate goal is to find approximations to the gain operator $\mathcal{K}$, so the real test for convergence is whether the gains converge. Given the gain matrix $K$, we wish to find a representation of the gain operator $\mathcal{K}$. Since $u^*(t) = \begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} = \mathcal{K}z(t)$, there exist operators $K_i : Z \to \mathbf{R}$ such that $u_i^*(t) = \mathcal{K}_i z(t)$ for $i = 1, 2$. By the Riesz-Representation Theorem, there exist

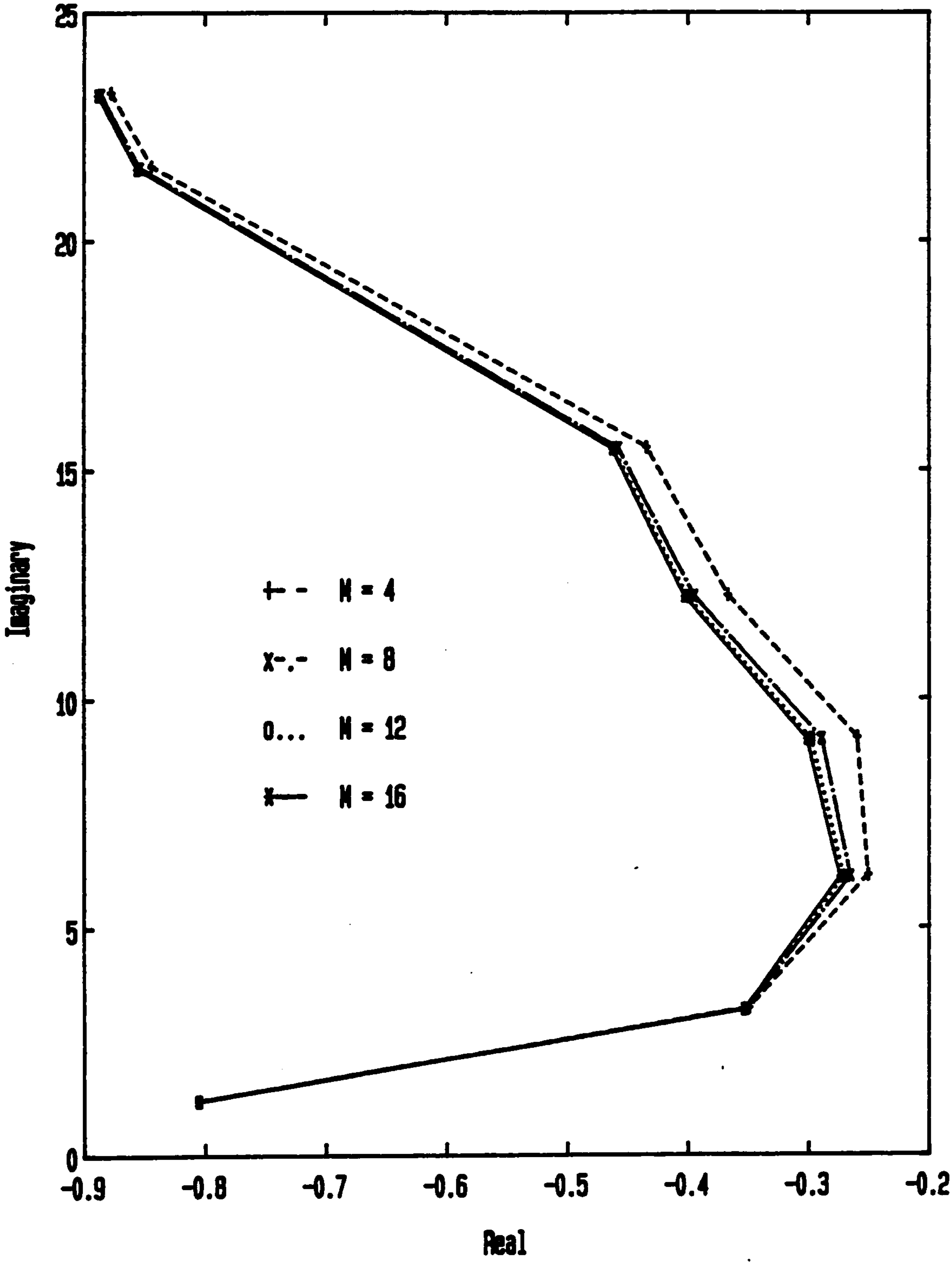


FIGURE 4.2.1 Non-uniform mesh, 6 cubic splines

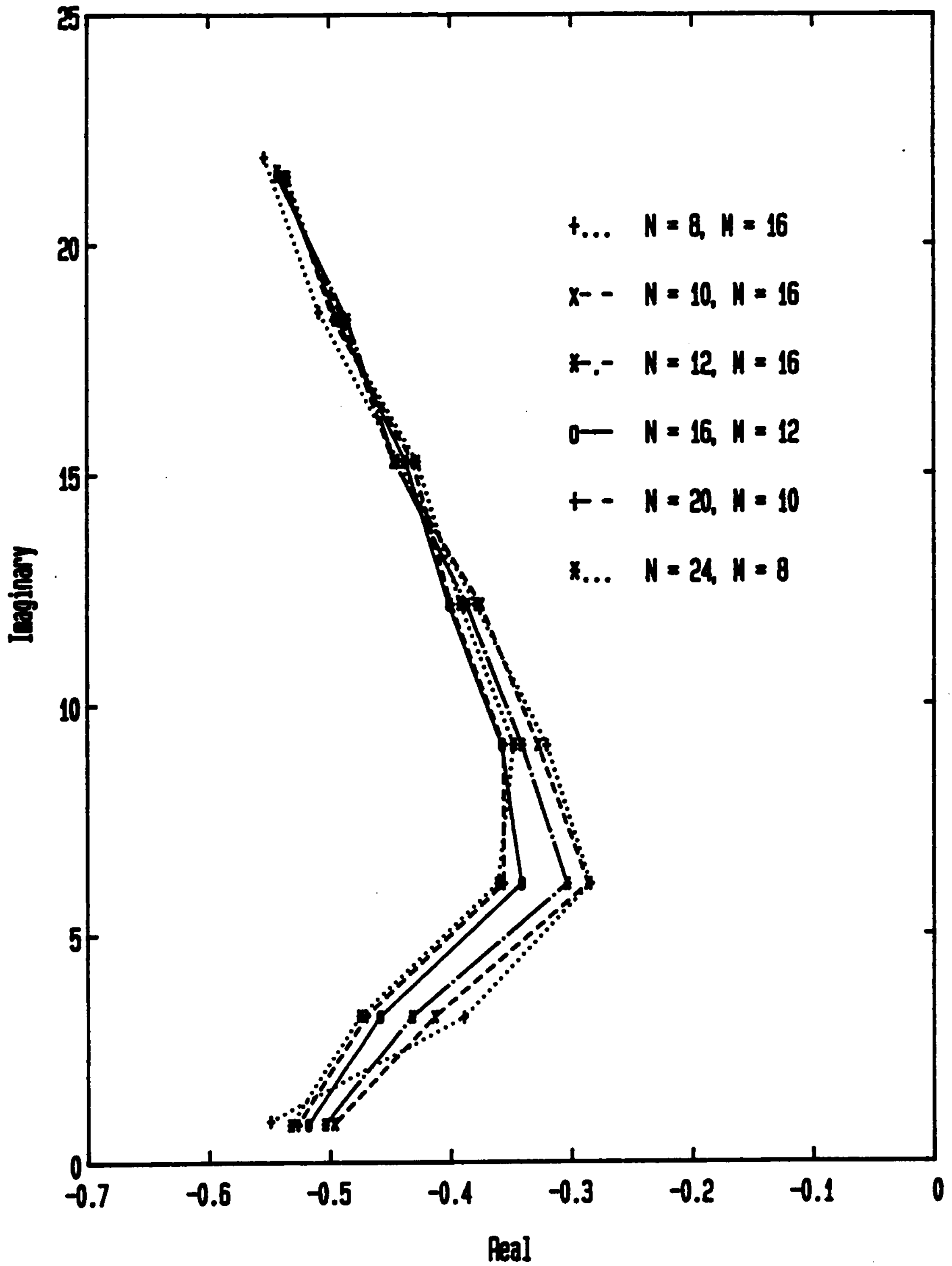


FIGURE 4.2.2 Non-uniform mesh, N cubic splines

$\widehat{z}_i \in Z$ such that $\mathcal{K}_i z(t) = \langle \widehat{z}_i, z(t) \rangle_Z$. Let $\widehat{z}_i = \begin{pmatrix} K_{i,1}(\cdot) \\ K_{i,0} \\ K_{i,2}(\cdot) \\ K_{i,3}(\cdot,\cdot) \end{pmatrix}$. Our goal is to find approximations to the "ODE gains" $K_{i,0}$ and the "functional gains" $K_{i,1}, K_{i,2}$ and $K_{i,3}$. Let $\left\{ f_j^{N,M} \right\}$ be a basis for $Z^{N,M}$ and let $K^{N,M} = \sum \beta_j^{N,M} f_j^{N,M}$ be given. Suppose we wish to find a matrix representation, which we will denote by $\left[K^{N,M}\right]$, of the operator $T : Z^{N,M} \to \mathbf{R}$ defined by $Tz = \left\langle K^{N,M}, z \right\rangle_Z$. Let $z = \sum \alpha_i^{N,M} f_i^{N,M}$. Then $Tz = \left\langle \sum \beta_j^{N,M} f_j^{N,M}, \sum \alpha_i^{N,M} f_i^{N,M} \right\rangle_Z = \left(\vec{\beta}^{N,M}\right)^T G \vec{\alpha}^{N,M}$ where $G_{ij} = \left\langle f_j^{N,M}, f_i^{N,M} \right\rangle_Z$. Thus, $\left[K^{N,M}\right] = \left(\vec{\beta}^{N,M}\right)^T G$ which implies $\left[K^{N,M}\right]^T = G^T \vec{\beta}^{N,M} = G\vec{\beta}^{N,M}$. Now, suppose that we know the matrix representation $\left[K^{N,M}\right]$ with respect to the basis $\left\{ f_j^{N,M} \right\}$. Then the coefficients $\beta_j^{N,M}$ are given by $\vec{\beta}^{N,M} = G^{-1}\left[K^{N,M}\right]^T$. Using the basis for $W^{N,M}$ that we used in Section 4.1 to compute the projection operator, it is easy to see that the matrix $G$ is given by

$$G = \begin{bmatrix} D & 0 & 0 & \dots & 0 \\ 0 & H & 0 & & 0 \\ 0 & 0 & (g_\alpha)_1^M D & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \dots & 0 & (g_\alpha)_M^M D \end{bmatrix}.$$

In Table 4.2.3 we show the "ODE" gains $K_{1,0}$ and $K_{2,0}$, and in Figures 4.2.4 – 4.2.10, we show plots of several of the gain functionals. All runs used the non-uniform mesh. From Table 4.2.3 it appears that $M$ does not have to be very large for the non-uniform mesh. However, $N$, the number of subdivisions for the spatial variable, must be quite large before we observe convergence. We observe the same phenomena in the plots of the functional gains. Thus, although a small number of elements gives a very good approximation of the eigenvalues, a good approximation of the optimal control requires a large number of elements. Observe also from the plots of $K_{1,1}(x)$ that a small number of elements yields a result radically different from the "converged" gain functional.

TABLE 4.2.3 ODE Gains

| $N$ | $M$ | $K_{1,0}$ | $K_{2,0}$ |
|---|---|---|---|
| 6 | 8 | −.7046 | −2.520 |
| 6 | 12 | −.7054 | −2.521 |
| 6 | 16 | −.7057 | −2.522 |
| 8 | 16 | −.4982 | −1.714 |
| 12 | 16 | −.4790 | −1.582 |
| 16 | 12 | −.5093 | −1.630 |
| 20 | 10 | −.5271 | −1.658 |
| 24 | 8 | −.5382 | −1.676 |
| 27 | 8 | −.5447 | −1.687 |
| 30 | 7 | −.5496 | −1.695 |
| 34 | 6 | −.5546 | −1.703 |
| 40 | 5 | −.5601 | −1.712 |
| 45 | 4 | −.5628 | −1.716 |
| 47 | 4 | −.5641 | −1.719 |

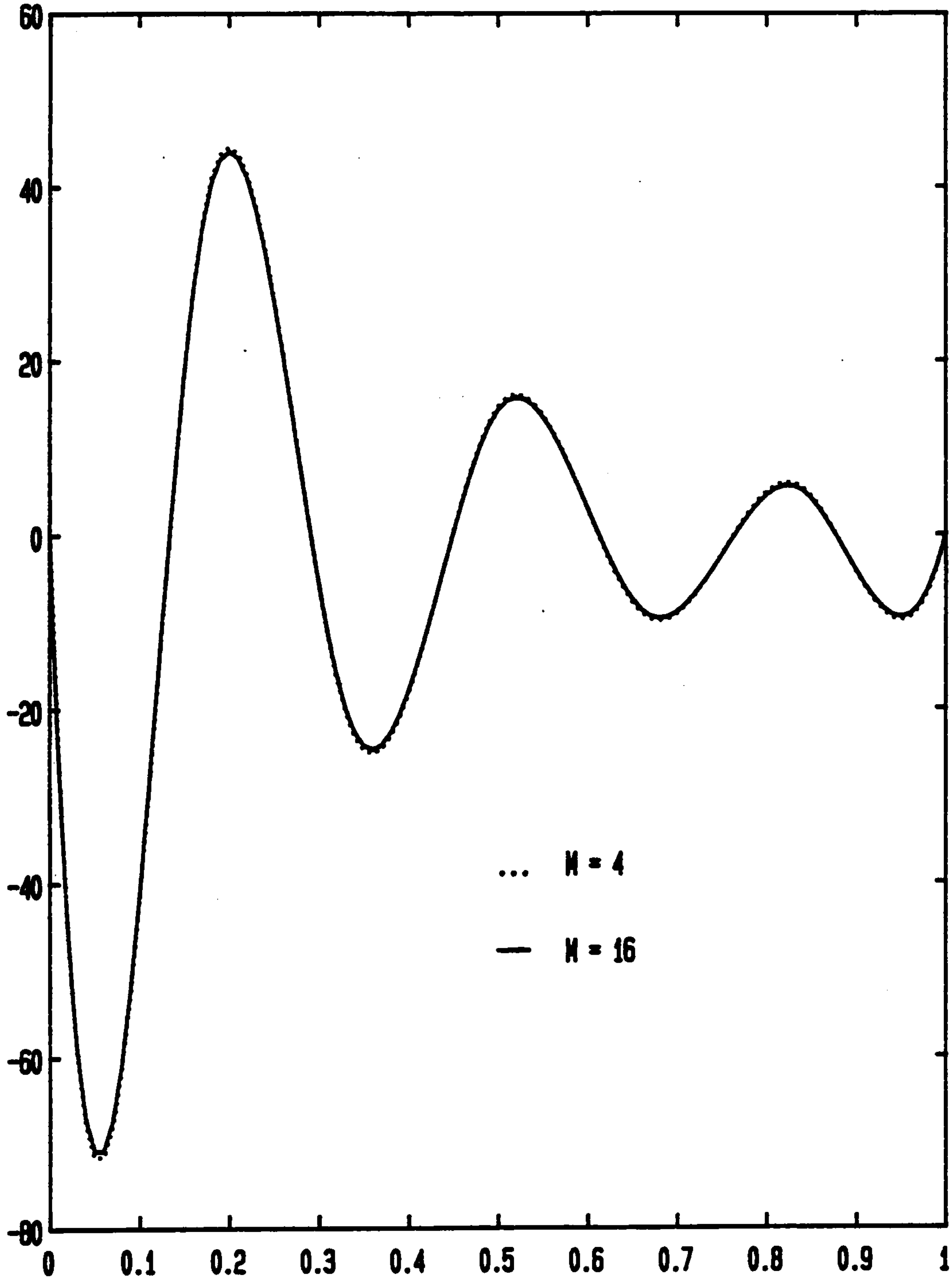


FIGURE 4.2.4  K1,1(x), Non-uniform mesh, 6 cubic splines

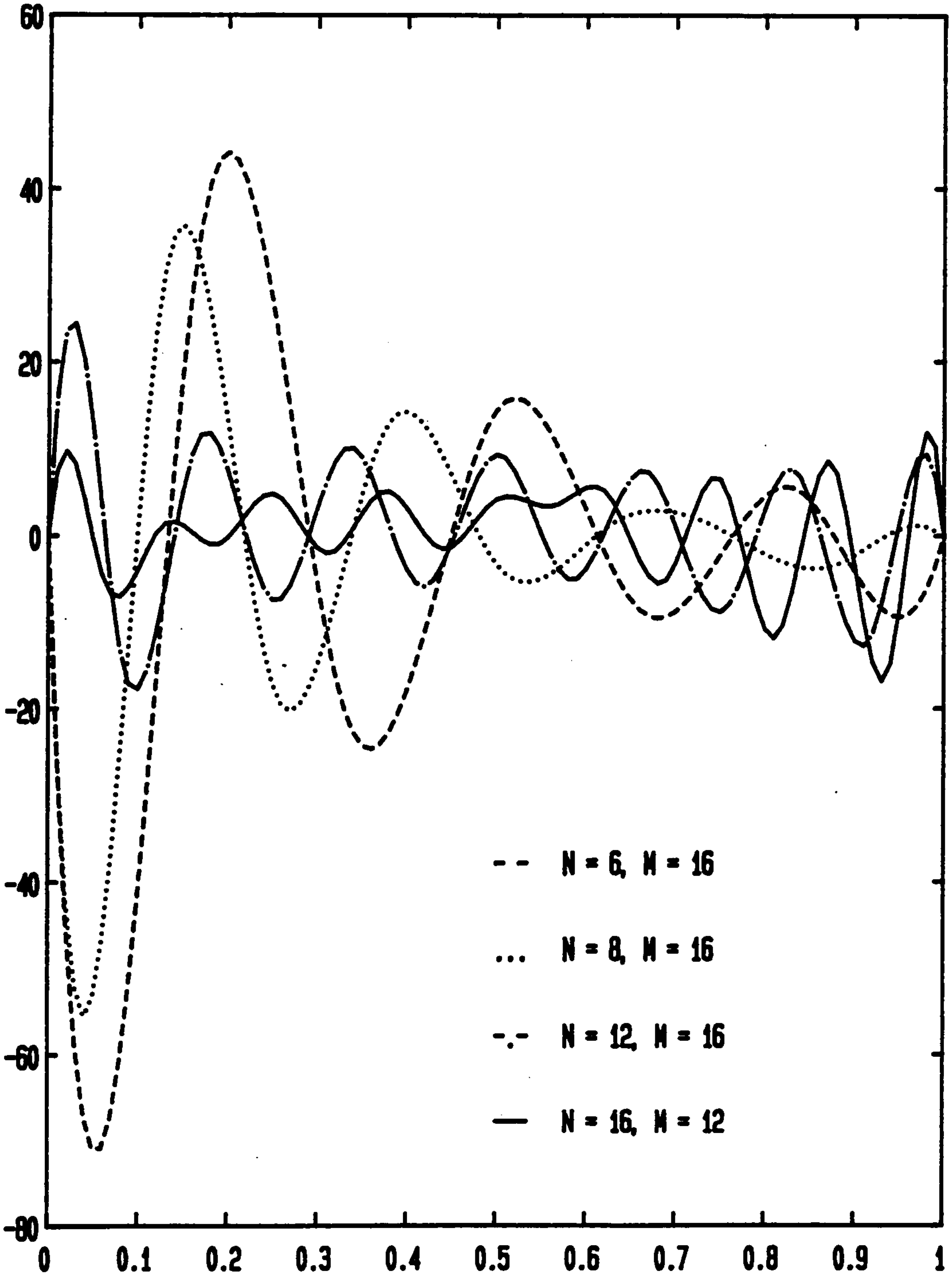


FIGURE 4.2.5 K1,1(x), Non-uniform mesh, N cubic splines

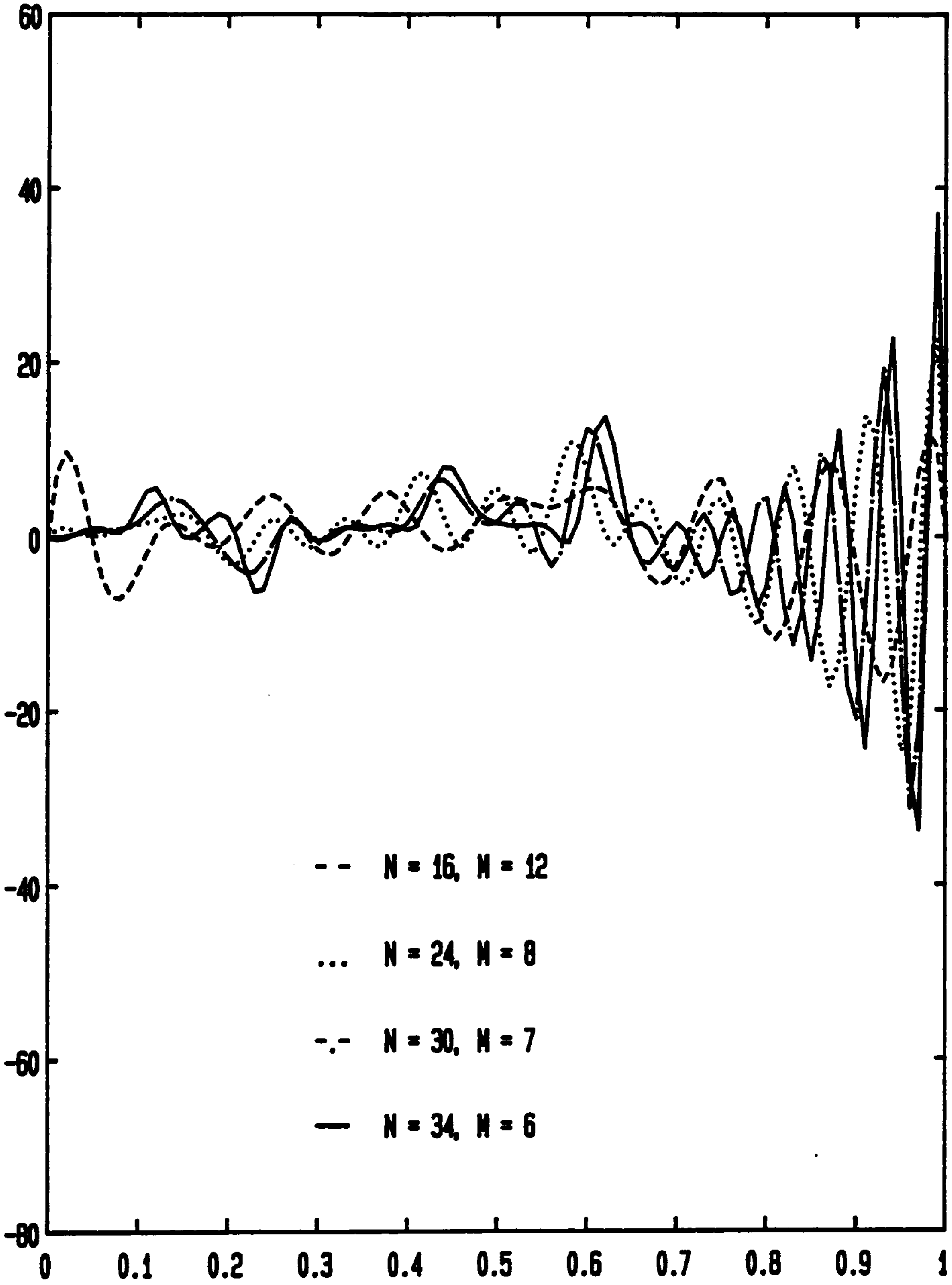


FIGURE 4.2.6 K1,1(x), Non-uniform mesh, N cubic splines

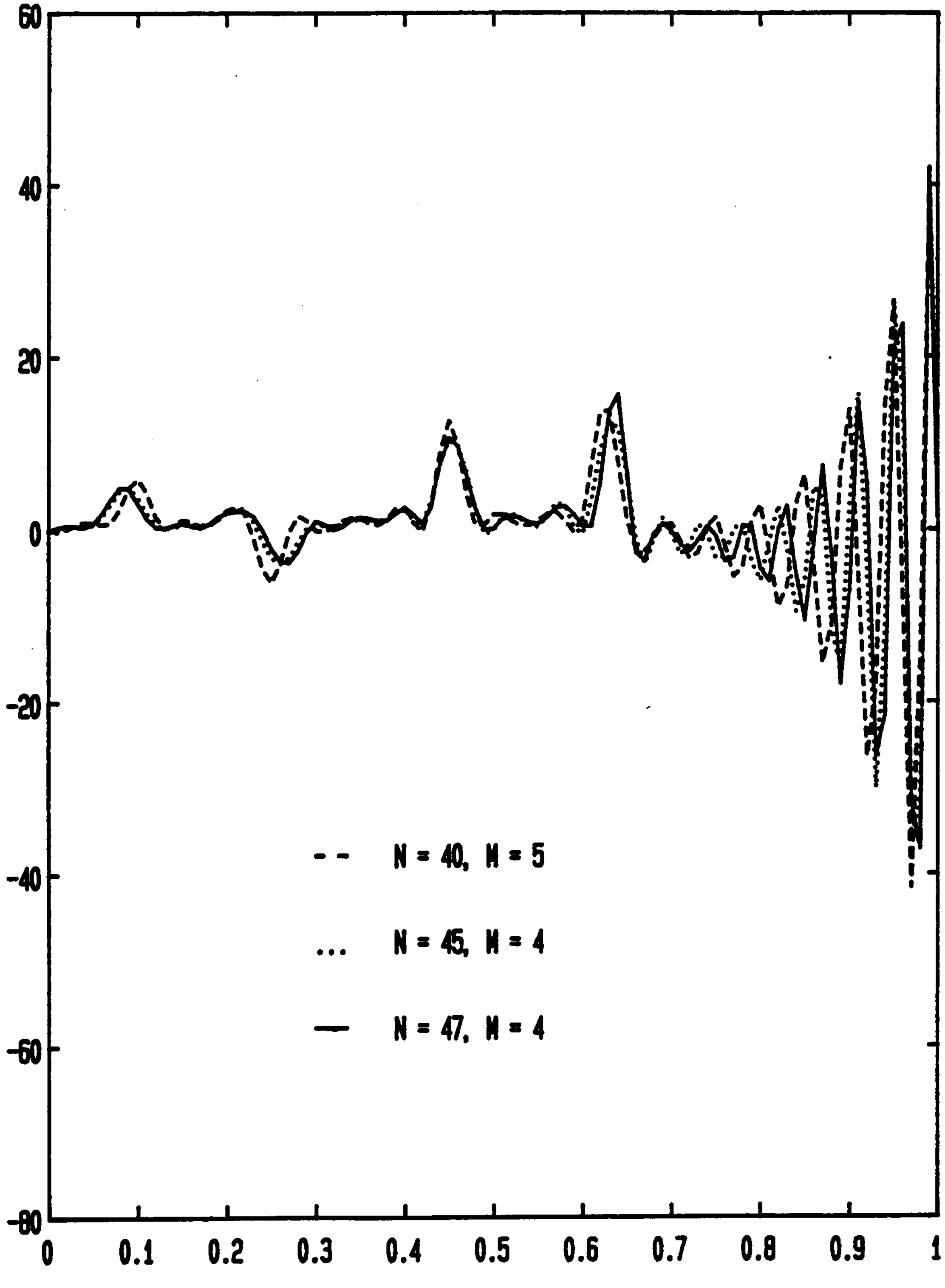


FIGURE 4.2.7 K1,1(x), Non-uniform mesh, N cubic splines

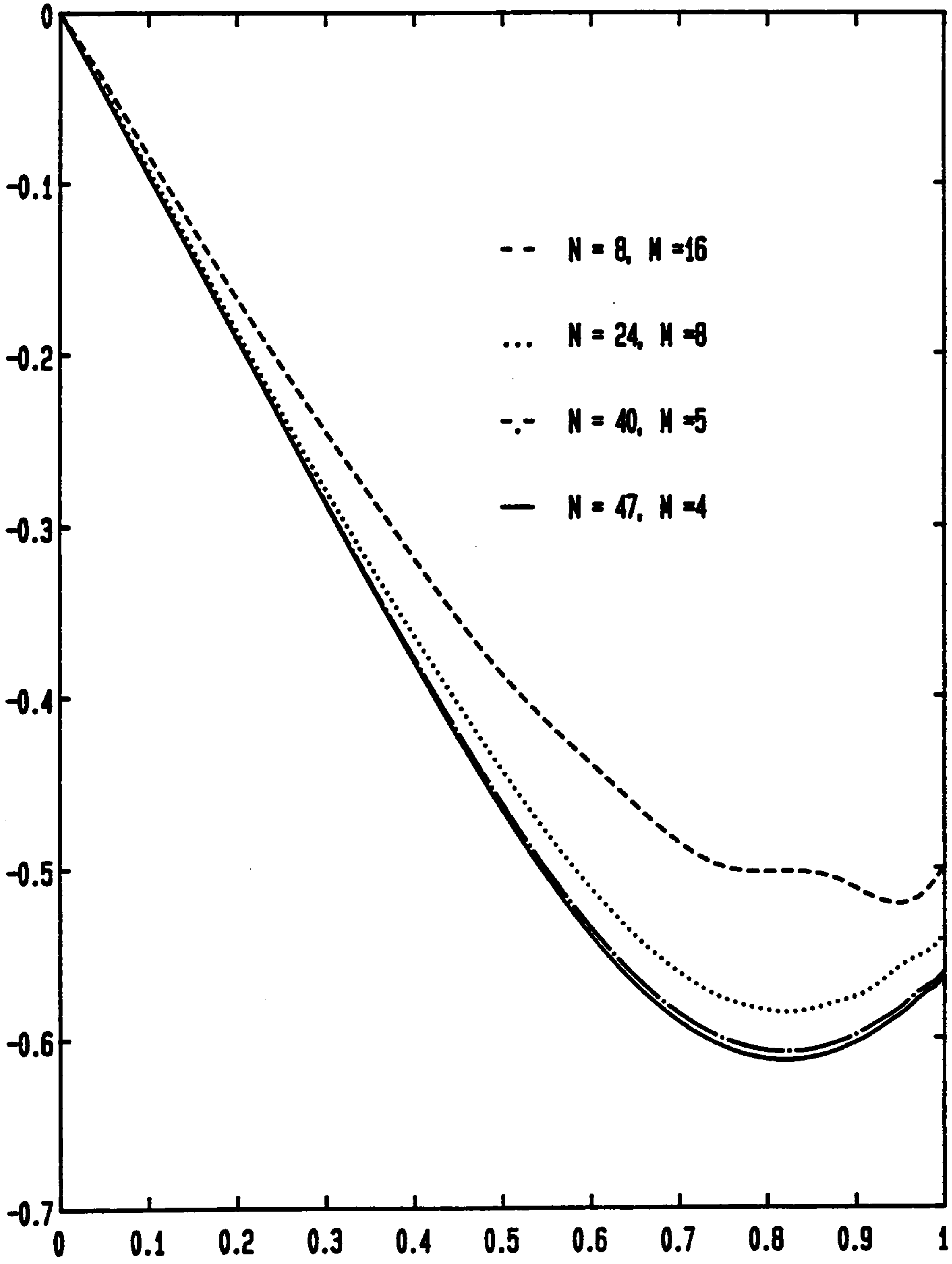


FIGURE 4.2.8 K1,2(x), Non-uniform mesh, N cubic splines

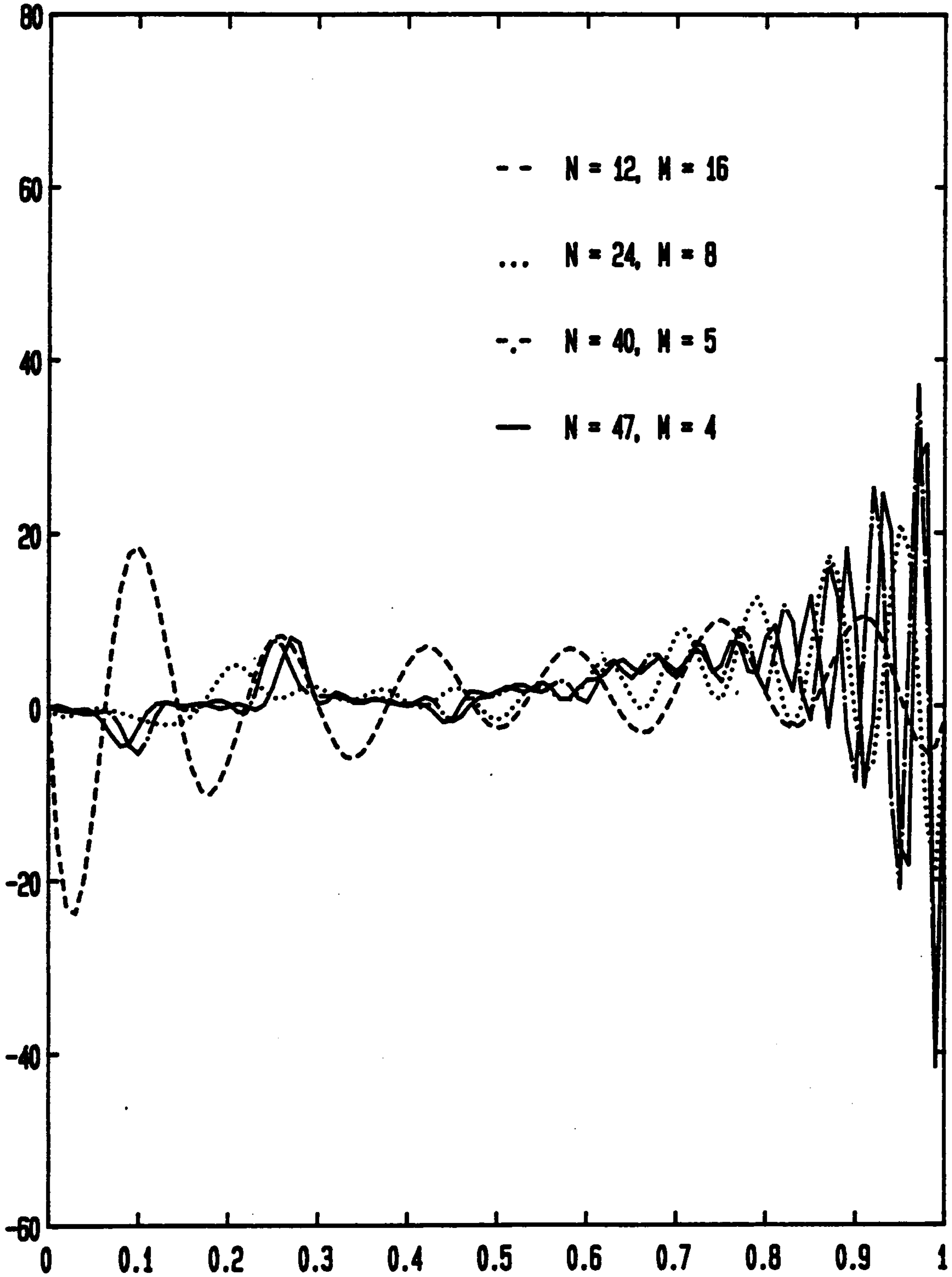


FIGURE 4.2.9 K2,1(x), Non-uniform mesh, N cubic splines

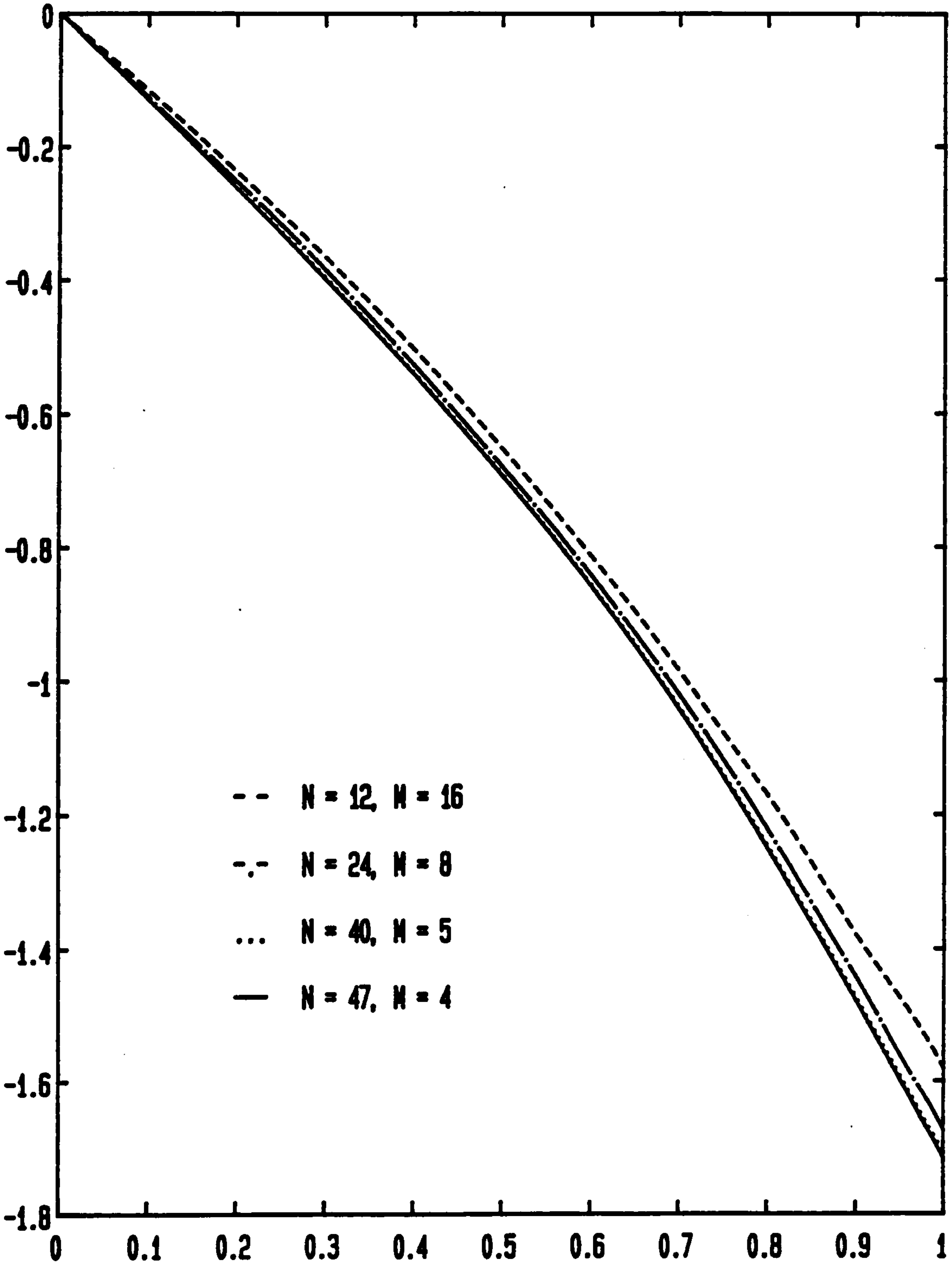


FIGURE 4.2.10 K2,2(x), Non-uniform mesh, N cubic splines

### 4.3 Conclusions.

In this paper we have developed an abstract framework for state space formulation and proved a generalized result on well-posedness which we applied to an abstract viscoelastic model with a singular kernel in the memory term. We also developed an approximation scheme and showed that the "non-uniform" mesh proposed in [6] yields a convergent scheme. We also showed that the adjoint system converges for bounded kernels. We applied our scheme to a viscoelastic shaft with tip-mass and developed computer codes to estimate the eigenvalues and solve the closed-loop system.

There is much room for further study. In particular we plan to

(1) plot the gain functionals $K_{i,3}$ to see if they appear to converge at the same rate as $K_{i,1}$ and $K_{i,2}$;

(2) investigate the convergence of the adjoint for singular kernels;

(3) find an estimate on the rate of convergence for the non-uniform mesh;

(4) study some of the other examples (e.g., beams, thermo-viscoelastic systems) mentioned in the paper.

REFERENCES


[1] Banks, H. T., and J. A. Burns. "Hereditary Control Problems: Numerical Methods Based on Averaging Approximations." *SIAM J. Control Optimization*, 16, No. 2, (1978), 169–208.

[2] Banks, H. T., and F. Kappel. "Spline Approximations for Functional Differential Equations." *J. Differential Equations,* 34, (1979), 496–522.

[3] Burns, J. A., and R. H. Fabiano. "Feedback Control of a Hyperbolic Partial-Differential Equation with Viscoelastic Damping," LCDS/CCS Report No. 88-5, Brown University, Providence, R.I., February, 1988.

[4] Burns, J. A., R. E. Miller, and E. M. Cliff. "Control of a Viscoelastic Shaft with Attached Tip Mass." $26^{th}$ *IEEE Conference on Decision and Control*, 1987, 997–1001.

[5] Fabiano, R. H. "Approximation of Integro-Partial Differential Equations of Hyperbolic Type." Ph. D. Dissertation. Virginia Polytechnic Institute and State University, 1986.

[6] Fabiano, R. H., and K. Ito. "Semigroup Theory and Numerical Approximation for Equations in Linear Viscoelasticity," LCDS/CCS Report No. 88-12, Brown University, Providence, July, 1988.

[7] Gibson, J. S. "Linear-Quadratic Optimal Control of Hereditary Differential Systems: Infinite Dimensional Riccati Equations and Numerical Approximations." *SIAM J. Control Optimization*, 21, No. 1, (1983), 95–139.

[8] Hille, E., and R. S. Phillips. *Functional Analysis and Semi-Groups.* Providence: AMS Colloquium Publications, Vol. 31, 1957.

[9] Marsden, J. E., and T. J. R. Hughes. *Mathematical Foundations of Elasticity.* Englewood Cliffs, New Jersey: Prentice-Hall, 1983.

[10] Miller, R. K., and W. Desch. "Exponential Stabilization of Volterra Integral Equations with Singular Kernels." to appear.

[11] Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* New York: Springer, 1983.

[12] Prenter, P. M. *Splines and Variational Methods.* New York: Wiley, 1975.

[13] Rudin, W. *Functional Analysis.* New York: McGraw-Hill, 1973.

[14] Russell, D. L. *Mathematics of Finite Dimensional Control Systems, Theory and Design.* New York: Marcel Dekker, 1979.

[15] Schultz, M. H. *Spline Analysis.* Englewood Cliffs: Prentice-Hall, 1973.

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