ASYMPTOTIC PHASE DIAGRAMS FOR LATTICE SPIN SYSTEMS

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(ABSTRACT)

We present a method of constructing the phase diagram at low temperatures, using the low temperature expansions. We consider spin lattice systems described by a Hamiltonian with a d-dimensional perturbation space. We prove that there is a one-one correspondence between subsets of the phase diagram and extremal elements of some family of convex sets. We also solve a linear programming problem of the phase diagram for a set of affine functionals.

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Chapter 2

$\mathbb{D}(U)$	algebra of formal series with domain U, 19
$\mathbb{D}_0$	algebra of formal series with zero first term, 19
e_G	energy of the ground state G, 13
$\mathcal{G}(H)$	set of periodic ground states of H, 12
H_Λ	Hamiltonian in a finite volume Λ, 11
$H(X\|Y)$	relative Hamiltonian, 11
$\mathbb{L}$	lattice, 10
$\mathcal{L}$	perturbation space, 13
$\mathring{p}^G$	formal pressure for a ground state G, 19
p_t^G	cut-off pressure in order t for a ground state G, 17
S	set of configurations at a lattice point, 10
$\mathcal{X}_\Lambda$	configuration space in a finite volume Λ, 10
$\mathcal{X}$	configuration space, 10
$\mathcal{X}^Y$	set of configurations equal to Y at infinity, 11
$\mathcal{X}^{per}$	set of periodic configurations, 10
Φ	interaction, 11
$\sigma(H)$	spectrum of H, 12

Chapter 3

$A_k^G(F)$	50
$B(c,r)$	ball centered at c and with radius r
$B(F_k,c)$	54
$\mathbb{F}_k$	convex structure in order k, 50
$\mathbb{F}_k(F)$	subset of $\mathbb{F}_k$ corresponding to a face F, 50
$\mathcal{G}_k(F)$	50
$\mathcal{G}_k(E)$	68
max $W_k(F)$	set of maxima of $W_k(F)$, 50

CHAPTER 1: Introduction.

The major problem of statistical physics consists in determining the number and the properties of phases of a system. Here we use the term "phase" to mean an equilibrium state [1]. We will be interested in the following situation. Suppose that a system with a known set of phases is described by a Hamiltonian H_o. Next we allow H_o to be perturbed by an element from a d-dimensional perturbation space $\mathcal{L}$, and ask the following question: how are phases of the perturbed system described by phases of the non-perturbed one? In particular, we want to determine the phase diagram. Since we know, at least in principle, how to derive the zero-temperature phase diagram, we expect some simplification of the problem at low temperatures. Therefore we restrict our attention to the low temperature region.

The framework of this paper is described in Section 2.1. We consider translation-invariant lattice spin systems. In addition, we assume that H_o, and all elements of $\mathcal{L}$, have finite ranges, and that H_o has a finite number of periodic ground states. Some remarks concerning the systems with an infinite number of ground states are comprised in Section 4.4.

As the first step in describing phase diagrams at low temperatures, one uses often the low temperature (LT) expansions. We review the most important elements of the LT expansion technique in Section 2.2. For any periodic ground state G of H_o, and small perturbations L, this technique yields the series in variables $\{e^{-\beta E_i}\}$ $(\beta \to \infty)$, denoted as $\dot{p}^G$. Here $0<E_1<E_2<\ldots$ are energies of excitations of the ground state G. For ferromagnetic, nonperturbed systems, the $\dot{p}^G(0)$'s are convergent [2]. However, nothing is known in the general case. Therefore we will use the series $\dot{p}^G$ only within the formal series framework. The formal series are discussed in Section 2.3.

Within the formal series framework we define a formal phase diagram as a family of solutions for systems of equations of the type:

$$- \langle H,e_G\rangle + \dot{p}^G(H) = - \langle H,e_{G'}\rangle + \dot{p}^{G'}(H) \qquad G,G' \in \mathcal{G}' \quad (1.1)$$

with the additional condition:

$$- \langle H,e_G\rangle + \dot{p}^G(H) > - \langle H,e_{\tilde{G}}\rangle + \dot{p}^{\tilde{G}}(H) \qquad G \in \mathcal{G}', \ \tilde{G} \notin \mathcal{G}' \quad (1.1a)$$

Here $\mathcal{G}'$ is a subset of the set of ground states $\mathcal{G}$ of H_o, and for any G, e_G is a linear functional on $\mathcal{L}$ defined at every L as energy per spin for the Hamiltonian L.

For special systems, the relation between the formal diagram and the "real life" phase diagram has been described by Slawny [3]. Consider the system for which:

a) The Hamiltonian H_o has n ground states, and satisfies Peierls condition;

b) the perturbation space $\mathcal{L}$ is (n-1)-dimensional and the functionals $\{e_G\}$ span the space dual to $\mathcal{L}$.

Pirogow and Sinai ([4],[5]) proved that at low temperatures and for small perturbations, the phase diagram for this type of system is a perturbation of the zero-temperature phase diagram. Using their contour expansions, Slawny showed that the formal phase diagram is asymptotic (as $\beta \to \infty$) to the rigorous phase diagram. One can extend this result to the cases where dim $\mathcal{L}$ is less than n-1. However, there are no corresponding results for the systems with an infinite number of ground states.

Setting aside the problem of the relation between the formal and rigorous phase diagrams, we will concentrate on the description of the former one. For practical reasons, we cannot deal with the formal series $\dot{p}^G$. Hence for any ground state G, we consider a function defined by the first m terms of the formal series, called the cut-off pressure p_m^G in order m (Section 2.2).

The phase diagram in order m is defined by cut-off pressures by formulae similar to (1.1) and (1.1a): for any subset $\mathcal{S}'$ of the set of ground states $\mathcal{S}$ we consider the system of equations:

$$- \langle \beta L, e_G \rangle + p_m^G(\beta L, \beta) = - \langle \beta L, e_{G'} \rangle + p_m^{G'}(\beta L, \beta) \qquad G, G' \in \mathcal{S}' \qquad (1.2)$$

with the following dominance condition:

$$- \langle \beta L, e_G \rangle + p_m^G(\beta L, \beta) > - \langle \beta L, e_{\tilde{G}} \rangle + p_m^{\tilde{G}}(\beta L, \beta) \quad G \in \mathcal{S}', \ \tilde{G} \notin \mathcal{S}' \qquad (1.2a)$$

If (1.2a) is satisfied, we will say that G dominates $\tilde{G}$ at the point (L, β). A subset of the phase diagram defined by (1.2) and (1.2a) is called the stratum corresponding to the family of phases $\mathcal{S}'$.

In this paper we present a method of constructing the phase diagram in any order m. This construction is relatively easy in simple cases: the main difficulty consists in generating the low temperature expansions. However, with growing dimension d of the perturbation space, and increasing difference between the number of ground states n, and d+1, the situation becomes complicated, even if the expansion coefficients are known. Our method makes possible the order-by-order description of the phase diagram. We are also able to determine in which order mhis description becomes complete, so that higher terms of the LT expansions yield no additional information.

The construction of the phase diagram is described in Chapter 3. Section 3.1 contains a definition of the phase diagram in order m, and deals with general properties of the strata arising from the analycity of cut-off pressures. In Section 3.2 we consider the following situation:

a) H_0 has d+1 ground states;

b) perturbation space is d-dimensional and $\{e_G\}$ spans the space dual to $\mathcal{L}$.

In any order m, the phase diagram for this type of systems has the following property: Let $\mathcal{S}'$ be any k-element subset of $\mathcal{S}$. Then there exists in $\mathcal{L}$ a (d-k)-dimensional surface such that any point (L,β) in this surface satisfies equation (1.2) for phases in $\mathcal{S}'$, and also the dominance condition (1.2a). This result corresponds to the rigorous result of ([4],[5]), thus supporting the formal (asymptotic) diagrams approach.

In the remaining part of Chapter 3 we deal with the situation when H_o has strictly more than d+1 ground states. The essence of our method consists of an order by order, local approximation of cut-off pressures by constants. Let us fix order m, and consider the first two steps of this construction. In zero order all constants are zero, and

$$|p_m^G(\beta L,\beta)| < \text{const } e^{-\beta\epsilon}, \quad \epsilon > 0 \tag{1.3}$$

for L in some region around zero, and β large. Using this fact we show that:

1. If G is not a ground state for $H_o + \sum_{i=1}^{d} L_i$ with $L = (L_1,\ldots,L_d) \neq 0$ then there exists a family of open balls $\{B(0,r_1(\beta)),\beta\}$ with

$r_1(\beta) = O(e^{-\beta E_1})$ such that outside this family, other phases dominate the phase G.

2. With some additional hypotheses the phase diagram outside $\{B(0,r_1(\beta)),\beta\}$ looks like the zero-order (zero-temperature) phase diagram, i.e. there is a one-one correspondence between surfaces:

$$\langle L, e_G - e_{G'}\rangle = 0, \quad G,G' \in \mathcal{S}' \subset \mathcal{S}$$

$$\langle L, e_G - e_{\tilde{G}}\rangle < 0, \quad G \in \mathcal{S}', \ \tilde{G} \notin \mathcal{S}'$$

and the surfaces defined by (1.2) and (1.2a) for $\mathcal{S}'$.

Moreover, the corresponding surfaces are close to one another in the sense that for any large β, their distance is proportional to $e^{-\beta E_1}$.

In the next step we construct the phase diagram only inside $\{B(0,r_1(\beta)),\beta\}$. Without loss of generality, let us suppose that in the first order of the LT expansion, the coefficients $n_1^G(0)$ differ for some ground states. The cut-off pressure for a ground state G is approximated by $n_1^G(0)$: if L is in $B(0,r_1(\beta))$, then

$$|p_m^G(\beta L,\beta) - n_1^G(0)| < \text{const } e^{-\beta(E_1+\epsilon)} \tag{1.4}$$

In this step we cannot use the zero temperature phase diagram. Instead we construct a phase diagram for the set of affine functionals $\{(e_G,n_1^G(0))\}$. Let $W_1 = \text{conv } \{(e_G,n_1^G(0))\}$. Then we can show that:

1. if $(e_G,n_1^G(0))$ lies inside a d-dimensional face of max W_1, then there exists a family of balls $\{B(0,r_2(\beta)),\beta\}$ (with $r_2(\beta) = O(e^{-\beta E_2})$ and depending on the face), such that other phases dominate G outside this family.

2. with some additional hypotheses the phase diagram outside $\{B(0,r_2(\beta)),\beta\}$ looks like the phase diagram for affine functionals $(e_G,n_1^G(0))$, i.e. there is one-one correspondence between surfaces:

$$- \langle\beta L,e_G\rangle + n_1^G(0) = - \langle\beta L,e_{G'}\rangle + n_1^{G'}(0), \quad G,G' > \mathcal{S}' \subset \mathcal{S}$$

$$- \langle\beta L,e_G\rangle + n_1^G(0) > - \langle\beta L,e_{\tilde{G}}\rangle + n_1^{\tilde{G}}(0), \quad G \in \mathcal{S}', \tilde{G} \notin \mathcal{S}'$$

and the surfaces defined by (1.2) and (1.2a). Again, these surfaces are close to one another in the sense that for any large β, their distance is proportional to $e^{-\beta E_2}$.

In the following step we construct the phase diagram only inside sets $\{B(0,r_2(\beta)),\beta\}$, for each face of max W_1. This construction becomes more and more complicated with increasing order. In particular, the constants which approximate cut-off pressures consist not only of LT expansion coefficients, but also of products of their derivatives of different orders. However, one can see the emerging pattern. In every order m we construct the phase diagram in sets with diameter of order m. In each of these sets we can use estimations similar to (1.3) and (1.4). The phase diagram is then described through a one-one correspondence between its strata and subsets of the phase diagrams for some set of affine functionals.

An important element of the above construction is determining a phase diagram for a set of affine functionals. We deal with this problem in Section 3.3. This is also a new problem in linear programming: for a given set Γ of affine functionals we construct a phase diagram, i.e. the family of surfaces defined by:

$$\rho_i(x) = \rho_j\,(x) \qquad \rho_i,\rho_j \in \Gamma' \quad (\Gamma' \subset \Gamma)$$

$$\rho_i(x) > \rho(x) \qquad \rho_i \in \Gamma',\ \rho \notin \Gamma'.$$

We show that there is one-one correspondence between extremal elements (faces, edges, points) of the set: max(conv Γ), and subsets of the phase diagram. Namely, to any extremal point ρ_i there corresponds an open region in the perturbation space: at any L in this region, ρ_i dominates all other functionals. Next, for any extremal edge $\{\lambda\rho_1 + (1-\lambda)\rho_2\}$ there exists a (d-1)-dimensional flat region. For any L in this region,

$$\rho(L) = \rho_1(L) \qquad \text{if } \rho = \lambda'\rho_1 + (1-\lambda')\rho_2 \ , \quad \lambda' \in (0,1)$$

and $\rho_1(L) > \tilde{\rho}(L)$ if $\tilde{\rho}$ does not belong to the edge spanned by ρ_1 and ρ_2. Similar results hold for other extremal elements of max(conv Γ).

In the remaining sections of Chapter 3 we prove a corresponding result for the phase diagram in order m. The set max(conv Γ) is substituted by a more complicated family of convex sets: the convex structure in order m. We define this family in Section 3.4.

Section 3.5 is preliminary for the main result. We use the convex structure to reduce the phase diagram construction to finding the phase diagram in a family of open sets. We also introduce affine functionals and prove several approximation results. The main result is proved in Section 3.6. We show that, with some additional hypotheses about the Hamiltonian, the phase diagram in order m is described by the convex structure in order m in the same manner as the phase diagram for affine functionals is described by max(conv Γ): For every extremal point of the convex structure, corresponding to the ground state G, there exists an open region in which G dominates all other phases. For any extremal edge spanned by points corresponding to ground states G_1,G_2, there exists a (d-1)-dimensional surface on which cut-off pressures for G_1, G_2 are equal to one another, and G_1, G_2 dominate other phases. Analogous statements hold for other subsets of the phase diagram. Section 3.6 contains also a detailed description of the phase diagram construction.

In Section 3.7 we generalize the main result of the preceding section to the cases when the full Hamiltonian has additional symmetry, other than translational invariance.

Chapter 4 contains examples. In Section 4.1 we describe the Blume-Capel model ([6],[7]), which is used extensively throughout this paper to illustrate our method. Another class of models is obtained by complicating the interaction. In this way we can create models with arbitrary properties. However, with the increasing complexity of the potential, it becomes difficult to derive the LT expansions. In Section 4.2 we present a simple case of such a model with three ground states. Section 4.3 contains a general argument about the phase diagrams for generalized ferromagnetic models. Finally, in Section 4.4 we describe models with stabilization, with the antiferromagnet on the f.c.c. lattice serving as an example. Since these models arise from systems with an infinite number of ground states, we use this example to illustrate the application of our method to the systems for which the number of ground states is not finite.

The chapters and sections in this work are numbered by arabic numerals. The formulae and propositions are numbered by double arabic numerals, the first numeral denoting the chapter number. The list of symbols is placed before the introduction.

CHAPTER 2 : Framework.

In this chapter we introduce a framework for the problem of phase diagrams. Section 1 contains a description of a lattice system and a perturbation space together with the concept of transversality. In Section 2 a pressure and a cut-off pressure are introduced, and properties of low-temperature expansions of the pressure are discussed. In Section 3 we discuss formal and asymptotic series, and their relationship to low-temperature expansions.

1. Definitions.

A. Lattice.

Let $\mathbb{L} \subset \mathbb{R}^\nu$ be a discrete, $\mathbb{Z}^\nu$-invariant subset. $\mathbb{L}$ is called a <u>lattice</u>. Let S be a finite set. Define

$$\mathcal{X} = S^{\mathbb{L}} \quad , \qquad \mathcal{X}_\Lambda = S^\Lambda \quad \text{if } \Lambda \subset \mathbb{L}.$$

$\mathcal{X}$ is called a <u>configuration space</u>. It is a topological space with the product topology.

The action of $\mathbb{Z}^\nu$ on $\mathcal{X}$ is defined as follows:

if $g \in \mathbb{Z}^\nu$ and $X \in \mathcal{X}$, then $\forall\, a \in \Lambda \quad (gX)_a = X_{g(a)}$.

A configuration X in $\mathcal{X}$ is called <u>periodic</u> if there exists a cofinite subgroup $\mathbb{G} \subset \mathbb{Z}^\nu$ (i.e. $\mathbb{Z}^\nu/\mathbb{G}$ is finite) such that

$$\forall\, g \in \mathbb{G} \quad gX = X$$

We will denote the set of periodic configurations by $\mathcal{X}^{per}$.

B. Interaction

Let Φ be a function on finite subsets of $\mathbb{L}$, assigning to each Λ a function $\Phi_\Lambda : \mathcal{X}_\Lambda \to \mathbb{R}$, which is continous in the product topology on $\mathcal{X}_\Lambda$. Φ is called an <u>interaction</u>. We assume that Φ is $\mathbb{Z}^\nu$-invariant, i.e.

$$\forall\, g \in \mathbb{Z}^\nu \ , \ \ \forall\, \Lambda \text{ finite} \ , \ \ \Phi_{g\Lambda}(gX) = \Phi_\Lambda(X)$$

For the purpose of this work Φ is assumed to have finite range. This means that for any $a \in \mathbb{L}$ there exist only a finite number of sets M containing a and such that $\Phi_M \neq 0$.

For any finite Λ we define a function

$$H_\Lambda(X) = \sum_{M\subset\Lambda} \Phi_M(X) \tag{2.1}$$

H_Λ is a <u>Hamiltonian</u> in the finite volume Λ. One would like to drop the condition "$M\subset\Lambda$" in the summation. However, the resulting expression would be meaningless. In general one is not interested in the value of $H_\Lambda(X)$ itself, but rather in its value relative to $H_\Lambda(Y)$ for some Y fixed. For any Y in $\mathcal{X}$, define

$$\mathcal{X}^Y = \{X \in \mathcal{X} : X_a \neq Y_a \text{ only for finite number of points a in } \mathbb{L}\}$$

If $X \in \mathcal{X}^Y$, we write $X \backsim Y$ and say that X is <u>equal to Y at infinity</u>. If $X \backsim Y$, then $\operatorname{supp}X = \{a \in \mathbb{L} : X_a \neq Y_a\}$. Now, for $X \backsim Y$, the expression

$$H(X|Y) = \sum \Phi_M(X) - \Phi_M(Y) \tag{2.2}$$

(with the sum over finite M) is well defined. We call it the <u>relative Hamiltonian</u>.

<u>Note</u>: We say that a Hamiltonian H has finite range if the potential Φ entering into (2.1) or (2.2) has finite range.

C. Ground states

A periodic configuration G is called a <u>ground state</u> of H if $\forall\, Y \in \mathcal{X}^G$,

$$H(Y|G) \geqslant 0$$

The set of ground states is denoted by $\mathcal{S}(H)$.

If $Y \sim G$, then Y is called an <u>excitation</u> of G. Let $Y_1, Y_2 \in \mathcal{X}^G$ be such that $\mathrm{supp}Y_1 \cap \mathrm{supp}Y_2 = \emptyset$. Denote by $Y_1 \vee Y_2$ the element of $\mathcal{X}^G$:

$(Y_1 \vee Y_2)_a = (Y_i)_a$ if $a \in \mathrm{supp}Y_i$, $(Y_1 \vee Y_2)_a = G_a$ otherwise.

If $Y = Y_1 \vee Y_2$, then Y is called <u>reducible</u>. Y_1, Y_2 are <u>components</u> of Y. If Y cannot be represented as a union of two components, then Y is <u>irreducible</u>. For any $Y \sim G$, there is a unique decomposition of Y into irreducible components.

We will be interested in values of $H(X|G)$ for G in $\mathcal{S}(H)$. Thus we define the <u>spectrum</u> of H:

$$\sigma(H) = \bigcup_{G \in \mathcal{S}} \{\, H(X|G),\ X \sim G \,\}$$

$\sigma(H)$ is countable. We will assume that elements of the spectrum are ordered increasingly, hence $\sigma(H) = \{E_1, E_2, \ldots\}$

Let Λ be finite. Define

$$\partial\Lambda = \{a \in \Lambda : \exists\, M : a \in M\ ,\ M \cap \Lambda^c \neq \emptyset \text{ and } \Phi_M \neq 0\}$$

We say that a sequence $\{\Lambda_n\}$ goes to infinity <u>in the sense of Fisher</u> ([1]) if $|\partial\Lambda_n|/|\Lambda_n| \xrightarrow{n\to\infty} 0$.

The proof of the following result can be found in [4].

Proposition 2.1

$\forall\, G \in \mathcal{X}^{per}$, the limit (in the sense of Fisher) exists:

$$e_G(H) = \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} H_\Lambda(G) \tag{2.3}$$

The function $H \to e_G(H)$ is linear. Moreover, if $G \in \mathcal{G}(H)$, then

$$e_G(H) = \inf_{X\in\mathcal{X}^{per}} e_X(H)$$

$e_G(H)$ is called the (specific) energy of the ground state G.

D. Perturbations.

Let $\mathcal{L}$ be a d-dimensional space of Hamiltonians on $\mathcal{X}$. For the consistency with section 1, an interaction for every element in $\mathcal{L}$ is assumed to have finite range. We will be interested in Hamiltonians of the form

$$H(L) = H_o + \sum_{i=1}^{d} L_i \qquad L = (L_1,\ldots,L_d)$$

where H_o is some fixed Hamiltonian with finite range, and $L \in \mathcal{L}$. $\mathcal{L}$ is called a perturbation space, and L in $\mathcal{L}$ a perturbation.

Let $G \in \mathcal{G}(H)$. Then the function $L \to e_G(L)$ (cf. (2.3)) defines an element of $\mathcal{L}^*$ (the space dual to $\mathcal{L}$).

Definition: $\mathcal{L}$ is transversal to $\mathcal{G}(H)$ if for any fixed $G_o \in \mathcal{G}(H)$, $\{e_G - e_{G_o} : G \in \mathcal{G}(H)\}$ spans $\mathcal{L}^*$.

Obviously $\mathcal{L}$ is transversal to $\mathcal{G}(H)$ only if $|\mathcal{G}(H)| \geqslant d+1$. Henceforth, we will assume that $\mathcal{L}$ is transversal to $\mathcal{G}(H)$.

2. The low-temperature expansions.

Suppose that the system on the lattice $\mathbb{L}$ is described by the Hamiltonian H_o with the set of ground states $\mathcal{G}$. Let $\mathcal{L}$ be a perturbation space, transversal to $\mathcal{G}(H)$. For any $G \in \mathcal{G}(H)$, $L \in \mathcal{L}$ and Λ finite we define the statistical sum in volume Λ with boundary conditions G by:

$$Z_\Lambda^G(\beta H_o+\beta L) = \sum_{\substack{X\sim G \\ \mathrm{supp}X\subset\Lambda}} e^{-\beta(H_o(X|G) + L(X|G))} \tag{2.4}$$

and the corresponding pressure:

$$p_\Lambda^G(\beta H_o+\beta L) = \frac{1}{|\Lambda|} \log Z_\Lambda^G(\beta H_o+\beta L) \tag{2.5}$$

Consider Hamiltonians of the form

$$H = \beta H_o + \tilde{L}$$

(i.e. with perturbations $L = \frac{\tilde{L}}{\beta}$ which are β-dependent). Then $Z_\Lambda^G(H)$ can be resummed by grouping terms with the same value of $H_o(X|G)$:

$$Z_\Lambda^G(H) = \sum_{j=1}^{N} \tilde{n}_j^G(\Lambda,\tilde{L})\ e^{-\beta E_j} \tag{2.6}$$

(N is some natural number) with

$$\tilde{n}_j^G(\Lambda,\tilde{L}) = \sum_{\substack{X\sim G \\ \mathrm{supp}X\subset\Lambda \\ H_o(X|G)=E_j}} e^{-\tilde{L}(X|G)} \tag{2.7}$$

Using the expansion of log around 1, one obtains similar expression for p_Λ^G:

$$p_\Lambda^G(H) = \frac{1}{|\Lambda|} \sum_{j=1}^{\infty} \tilde{n}_j^G(\Lambda,\tilde{L})\, e^{-\beta E_j}. \qquad (2.8)$$

$\tilde{n}_j^G$ is a polynomial in $|\Lambda|$ plus some boundary terms [3].

For the next result to hold, we need an additional restriction on H_o: $\forall$ G $\mathcal{G}(H_o)$,

$$H_o(X|G) \longrightarrow \infty \quad \text{if } |\mathrm{supp}X| \longrightarrow \infty \qquad (2.9)$$

The above condition is not easy to prove in the general case. For a certain class of models this task is simplified by the notion of an m-potential.

An interaction Φ is called an <u>m-potential</u> if there exists $X \in \mathfrak{X}$ such that $\forall\, \Lambda$:

$$\Phi_\Lambda(X) = \inf_{Y \in \mathfrak{X}} \Phi_\Lambda(Y)$$

Holsztynski and Slawny [8] showed that if an interaction is an m-potential and $\mathcal{G}(H)$ is finite, then condition (2.9) is satisfied.

For some systems, the original interaction Φ may not be an m-potential, but by grouping terms one can create a new interaction Φ' such that $\forall$ X,Y $\in \mathfrak{X}$: X $\sim$ Y:

$$H_\Phi(X|Y) = H_{\Phi'}(X|Y)$$

(The subscript denotes interaction entering into (2.2).) Φ' and Φ are called <u>equivalent</u>. It is obvious that in this case, (2.9) also holds.

Proposition 2.2.

$\forall\ j\ ,\ \frac{1}{|\Lambda|}\ n_j^G(\Lambda,\tilde{L}) \longrightarrow n_j^G(\tilde{L})$ as $|\Lambda| \to \infty$ (in the sense of Fisher).

Moreover, $n_j^G(\tilde{L})$ is the coefficient in $\tilde{n}_j^G(\Lambda,\tilde{L})$ of the linear term in $|\Lambda|$. $n_j^G(\tilde{L})$ has the form:

$$n_j^G(\tilde{L}) = \sum_{i=1}^{s_j} n_{j,i}^G \ e^{-\mu_{j,i}(\tilde{L})} \tag{2.10}$$

where s_j and $\mu_{j,i}$ depend on G, and $\mu_{j,i}$ is linear.

For the proof of this result, cf. [3] and references contained therein.

Obviously, by varying $\tilde{L}$ with fixed β, one can recover the dependence of n_j^G's on L.

It has been shown that for any ground state G the limit (in the sense of Fisher) exists:

$$p(H) + \langle H,e_G\rangle = \lim_{n\to\infty} p_\Lambda^G(H) \tag{2.11}$$

and is independent of G ([1]). p(H) is a <u>pressure</u> of the system. On the other hand, the series with coefficients $n_j^G(\tilde{L})$ is, in general, divergent. Though no proof of this fact exists, the following heuristic argument strongly suggests that it is true:

Suppose that the series with coefficients $n_j^G(\tilde{L})$ is convergent. Then

$$p(H) = \sum_{j=1}^{\infty} n_j^G(\tilde{L})e^{-\beta E_j} - \langle\tilde{L},e_G\rangle$$

But the left-hand side is G-independent, while the right-hand side is strongly G-dependent (as can be checked in numerous examples).

The set of coefficients $\{n_j^G(\tilde{L})\}$ is used to define the cut-off pressure. Let m be a natural number. For G in $\mathcal{S}(H)$ and Λ finite, define

$$Z^G_{\Lambda,m}(\tilde{L},\beta) = \sum e^{-(\beta H_o+\tilde{L})(X|G)} \tag{2.12}$$

where the summation is over those $X \in \mathcal{X}^G$, for which $H_o(X_k) \leqslant E_m$ for any irreducible component X_k of X. Then the <u>cut-off pressure</u> is

$$\tilde{p}^G_m(\tilde{L},\beta) = \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \log Z^G_{\Lambda,m}(\tilde{L},\beta) \tag{2.13}$$

The proof that this limit exists is similar to the proof of existence of pressure p(H) (2.11).

The following result comes from [3].

<u>Proposition 2.3.</u>

$\forall\, m \in \mathbb{N}$, $\forall\, G \in \mathcal{S}$, $\exists\, c_m(G) > 0$ and $\beta_m(G)$: $\forall\, \beta>\beta_m(G)$,

$\forall\, L \in \mathcal{L}$: $\|L\| < c_m(G)$

$$\tilde{p}^G_m(\beta L,\beta) = \sum_{i=1}^{\infty} n^G_{i,m}(\beta L)\ e^{-\beta E_i} \tag{2.14}$$

with the right-hand side absolutely convergent. Moreover, if $i \leqslant m$, then $n^G_{i,m}(\beta L) = n^G_i(\beta L)$, where n^G_i are given by prop.2.2.

<u>Remark</u>: $c_m(G)$ is choosen in the following way. For any G and t there exists only a finite number (modulo translations) of irreducible excitations of G, say $X_1,\ldots,X_K$, for which $H_o(X_k|G) \leqslant E_m$. Let $c_m(G)$ be

such that $\forall X_i$: $H_o(X_i|G) \geq c_m(G) > 0$. Then for any L for which $\|L\| < c_m(G)$,

$$H_o(X_i|G) + L(X_i|G) \geq c_m(G) > 0. \tag{2.15}$$

This condition arises from the requirement similar to (2.9) and is neccesary for the existence of the expansion (2.14).

<u>Notation</u>: Let

$$p_m^G(\beta L,\beta) = \sum_{E_j \leq E_m} n_j^G(\beta L) e^{-\beta E_j} \quad , \quad \|L\| \leq c_m(G) \tag{2.16}$$

For our purpose we need p_m^G, and not the full expression (2.14). Henceforth, p_m^G will also be called the cut-off pressure. Since we do not use $\tilde{p}_m^G$, there will be no misunderstanding on this point.

One additional advantage of using p_m^G is that this function is defined for all $\beta>0$.

<u>Remark</u>: Let G be a ground state, and gG be its translate ($g \in Z^\nu$). Since Φ is Z^ν-invariant, gG is also a ground state. Moreover, since

$$H(gX|gG) = H(X|G) \quad , \quad \text{any } X \sim G,$$

for any j , $n_j^{gG}(\tilde{L}) = n_j^G(\tilde{L})$.

Thus in the framework we use, translates of G are indistinguishable from one another. Henceforth, we will understand the phrase: "ground state G" to mean "set of translates of ground state G".

3. Formal series and asymptotic series.

Let $\{E_i, i=1,2,\ldots\}$ be a discrete, additive subset of $\mathbb{R}_+$, with $0=E_0<E_1<E_2\ldots$, and $E_n \to \infty$ as $n \to \infty$. Let also $\{f_i, i=1,2,\ldots\}$ be a family of functions:

$$f_i : B(0,r) \to \mathbb{R} \quad , \quad B(0,r) \subset \mathbb{R}^k$$

with each f_i analytic in $B(0,r)$. The expression

$$\dot{f}(x) = \sum_{i=0}^{\infty} f_i(x)\, e^{-\beta E_i} \qquad \beta>0,\ x \in B(0,r) \tag{2.17}$$

is called the <u>formal</u> <u>Dirichlet</u> <u>(functional)</u> <u>series</u>. Functions f_i are <u>coefficients</u> of the series. The set of formal Dirichlet series is denoted by $\mathbb{D}(B(0,r))$. $\mathbb{D}$ is an algebra with addition defined componentwise, and multiplication $*$ defined by

$$(\dot{f}(x) * \dot{g}(x))_K = \sum_{E_i+E_j=E_K} f_i(x) g_j(x)$$

The unit element $\dot{E} \in \mathbb{D}$ has components $(\dot{E})_j = \delta_{oi}$.
By $\mathbb{D}^n(B(0,r))$ we will denote the direct sum of n copies of $\mathbb{D}(B(0,r))$: if $\dot{f} \in \mathbb{D}^n$, then its components are $\mathbb{R}^n$-valued functions on $B(0,r)$.
Let $\mathbb{D}_o^n = \{\dot{f} \in \mathbb{D}^n : f_o \equiv 0\}$

<u>Remark</u>: If a set of exponents $\{E_i\}$ is equal to the spectrum $\sigma(H_o)$ of H_o (cf. Section 1C), then coefficients $n_i^G(\tilde{L})$, defined for each phase by Proposition 1.2, determine an element of $\mathbb{D}_o(B(0,c_m(G))$, denoted by $\dot{\rho}^G(\tilde{L})$. Explicitly,

$$\dot{\rho}^G(\tilde{L}) = \sum_{i=1}^{\infty} n_i^G(\tilde{L})\, e^{-\beta E_i} \tag{2.18}$$

In $\mathbb{D}_o^n$ one can perform an operation of substituting a formal series for an argument of $\dot{f}(x)$. Suppose that f_i has an expansion around zero:

$$f_i(x) = \sum_{\alpha} f_{i,\alpha}\, x^{\alpha}$$

where α is a multiplicity function on $\{1,\dots,n\}$, and $x^{\alpha} = \Pi\, x_i^{\alpha(i)}$

If $\dot{x} = (\dot{x}_1, \dot{x}_2, \dots, \dot{x}_K) \in \mathbb{D}_o^K$, then we denote by $f_i(\dot{x})$ an element of $\mathbb{D}^n(B(0,r))$ of the form:

$$f_i(\dot{x}) = \sum_{\alpha} f_{i,\alpha}\, \dot{x}^{\alpha}$$

where $\dot{x}^{\alpha} = \dot{x}_1^{\alpha(1)} * \dot{x}_2^{\alpha(2)} * \dots * \dot{x}_K^{\alpha(k)}$. Then

$$\dot{f}(\dot{x}) = \sum_{i=1}^{\infty} f_i(\dot{x}).$$

<u>Definition (asymptotic series)</u>

Let $f : B(0,r)\times[M,\infty) \longrightarrow \mathbb{R}^n$ $(M>0)$ be analytic in the first variable in $B(0,r)$. We say that $\dot{f} \in \mathbb{D}^n(B(0,r))$ is a <u>formal series asymptotic to f</u> (denoted as $\dot{f} \sim f$) if $\forall\, m \in \mathbb{N}$, $\forall\, x \in B(0,r)$,

$$\|f(x,\beta) - \sum_{i=1}^{m} f_i(x)\, e^{-\beta E_i}\|\, e^{\beta E_m} \xrightarrow{\beta\to\infty} 0. \tag{2.19}$$

Let $g(x,\beta)$ be such that $\forall\, x \in B(0,r)$

$$\|f(x,\beta) - g(x,\beta)\|\, e^{\beta E_m} \xrightarrow{\beta\to\infty} 0$$

Then g is called an <u>m-th approximation of f</u>. Note that each m-th approximation of f has the form:

$$g(x,\beta) = \sum_{i=1}^{m} f_i(x)\, e^{-\beta E_i} + h(x,\beta)$$

where

$$\|h(x,\beta)\|\, e^{\beta E_m} \xrightarrow{\beta\to\infty} 0.$$

Let us consider the following problem. Suppose that $f : B(0,r)\times[M,\infty) \longrightarrow \mathbb{R}^n$ is analytic in the first variable, and that $\dot{f} \sim f$ with $\dot{f} \in \mathbb{D}_0^n(B(0,r))$. Consider equations:

$$x = f(x,\beta) \tag{2.20}$$

$$\dot{x} = \dot{f}(\dot{x}) \tag{2.20a}$$

The question is: if x_0 is a solution of (2.20), and $\dot{x}_0$ a solution of (2.20a), is $\dot{x}_0$ asymptotic to x_0? Furthermore, if g is an m-th approximation of f, and y is a solution of the equation

$$x = g(x,\beta), \tag{2.20b}$$

is y an m-th approximation to x?

Let us first address the problem of existence of solutions. For β large enough, the existence of $x_0(\beta)$ and $y(\beta)$ is assured by the implicit function theorem. The following proposition (cf. [3]) asserts the existence of $\dot{x}_0$ as well.

<u>Proposition 2.4.</u>

There exists in $\mathbb{D}_0^n$ a unique solution $\dot{x}_0$ of (2.20a). It is given by the formula:

$$\dot{x}_0 = \lim_{n\to\infty} \dot{f}^{(n)}(0) \tag{2.21}$$

where $\dot{f}^{(n)}$ stands for n-fold composition of $\dot{f}$ with itself. If j is such that $E_j \leqslant nE_1$, then

$$(\dot{x}_o)_j = (\dot{f}^{(n)}(0))_j$$

Zero in (2.17) can be substituted by any other element of $\mathbb{D}_o^n$.

Next we prove the following technical lemma which we will later use in the proof of Proposition 3.12.

<u>Lemma 2.5.</u>

Let $\dot{x} \in \mathbb{D}_o^K$, $\dot{x} = \sum_{i=1}^{\infty} x_i \; e^{-\beta E_i}$, and $\tilde{x}_m(\beta) = \sum_{i=1}^{m} e^{-\beta E_i} \; x_i$ $(\tilde{x}_m(\beta) \in \mathbb{R}^K)$.

Then $\exists \; c_m > 0$ and β_o such that $\forall \; \beta > \beta_o$

$$\| \sum_{i=1}^{m} (\dot{f}(\dot{x}))_i e^{-\beta E_i} - \sum_{i=1}^{m} f_i(\tilde{x}_m) \; e^{-\beta E_i} \| < c_m \; e^{E_{m+1}}$$

<u>Proof</u>:

First note that $\tilde{x}_m(\beta)$ can be treated both as an $\mathbb{R}^K$-valued function and as an element of $\mathbb{D}^K$. Hence we can write:

$$\sum_{i=1}^{m} f_i(\tilde{x}_m) \; e^{-\beta E_i} \equiv \dot{y} = \sum_{s=1}^{\infty} y_s e^{-\beta E_s}$$

with $y_s = \sum_{i=1}^{s} (f_i(\tilde{x}))_{i(s)}$.

Here $i(s)$ is such that $E_{i(s)} + E_i = E_s$.

Note that for any multiplicity function α,

$$(\dot{x}^{\alpha})_s = (\tilde{x}^{\alpha})_s \quad \text{if } s < m+1.$$

Hence

$$(f_i(\dot{x}))_s = (f_i(\tilde{x}))_s \quad \text{if } \; s < m+1$$

It follows that

$$y_s = (\dot{f}(\dot{x}))_s \quad \text{if } s < m+1.$$

Furthermore, since $\dot{y}$ is absolutely convergent, $\exists \; c_m > 0$:

$$\| \sum_{s=m+1}^{\infty} y_s e^{\beta E_s} \| \leqslant c_m e^{\beta E_{m+1}}.$$

■

Proposition 2.6.

1) $\dot{x}_o \backsim x_o(\beta)$

2) $y(\beta)$ is a m-th approximation to $x_o(\beta)$.

Proof

1) Let $\tilde{x}(\beta) = \sum_{i=1}^{m} (\dot{x}_o)_i e^{-\beta E_i}$. Then

$$\|\tilde{x}(\beta) - x_o(\beta)\| = \| \sum_{i=1}^{m} (\dot{f}(\dot{x}_o))_i e^{-\beta E_i} - f(x_o,\beta)\| \leqslant$$

$$\leqslant \|g(\tilde{x},\beta) - f(x_o,\beta)\| + \|g(\tilde{x},\beta) - \sum_{i=1}^{m} (\dot{f}(\dot{x}_o))_i e^{-\beta E_i}\| \leqslant$$

$$\leqslant \|g(\tilde{x},\beta) - g(x_o,\beta)\| + \|g(x_o,\beta) - f(x_o\beta)\| +$$

$$+ \|g(\tilde{x},\beta) - \sum_{i=1}^{m} (\dot{f}(\dot{x}))_i e^{-\beta E_i}\|$$

The second and third terms go to zero faster than $e^{-\beta E_m}$ (by definition of m-th approximation and Lemma 2.5.). The first term is evaluated as follows:

$$\|g(\tilde{x},\beta) - g(x_o,\beta)\| \leqslant \|\tilde{x}(\beta) - x_o(\beta)\| \sup_{x\in B(0,r)} \|D_x g(x,\beta)\| \leqslant$$

$$\leqslant \|\tilde{x}(\beta) - x_o(\beta)\| \sum_{i=1}^{m} \sup_{x\in B(0,r)} \|D_x f_i(x)\| e^{-\beta E_i} \leqslant c \|\tilde{x}(\beta) - x_o(\beta)\| e^{-\beta E_1}$$

If β is such that $ce^{-\beta E_1} < 1$, then

$$\|\tilde{x}(\beta) - x_o(\beta)\| e^{\beta E_m} < \frac{1}{1-ce^{-\beta E_1}} \{\|g(x_o,\beta) - f(x_o,\beta)\| +$$

$$+ \|g(\tilde{x},\beta) - \sum_{i=1}^{m} (\dot{f}(\dot{x}))_i e^{-\beta E_i}\|\} e^{\beta E_m} \xrightarrow{\beta\to\infty} 0$$

Again, the first term vanishes since g is a m-th approximation of f, and the second term vanishes by Lemma 2.5.

2) $\|y(\beta) - x_0(\beta)\| = \|g(y,\beta) - f(x_0,\beta)\| \leq$

$$\leq \|g(y,\beta) - g(x_0,\beta)\| + \|g(x_0,\beta) - f(x_0,\beta)\| \leq$$

$$\leq c\,\|y(\beta) - x_0(\beta)\|\, e^{-\beta E_1} + \|g(x_0,\beta) - f(x_0,\beta)\|$$

Here c is as in part i) of the proof. Hence for β such that $ce^{-\beta E_1} < 1$, we have

$$\|y(\beta) - x_0(\beta)\|\, e^{\beta E_m} < \frac{1}{1-ce^{-\beta E_1}} \|g(x_0,\beta) - f(x_0,\beta)\|\, e^{\beta E_m} \xrightarrow{\beta\to\infty} 0.$$

<u>Remark</u>: Since $y(\beta)$ is a m-th approximation to $x_0(\beta)$,

$$y(\beta) = \tilde{x}_m(\beta) + z(\beta)$$

with $\|z(\beta)\|\, e^{\beta E_m} \xrightarrow{\beta\to\infty} 0$

■

CHAPTER 3 : Asymptotic Phase Diagrams

In this chapter we present method of constructing asymptotic phase diagrams. Section 1 contains definitions. We also study the general properties of phase diagrams which arise from analycity of cut-off pressures. Next we investigate two special cases. The "Pirogov-Sinai" case is worked out in Section 2: the dimension of the perturbation space is smaller by one than the number of ground states. Section 3 contain the phase digram construction for a family of affine functionals.

In Section 4 we introduce the convex structure of order m. This structure is used in Section 5 to reduce the phase diagram construction to finding the phase diagram separately in some family of open sets. In Section 6 we show that the phase digram is described by the convex structure introduced in Section 4. Finally, Section 7 contains remarks about the modifications of the method in the presence of symmetries.

1. Definitions.

Suppose that a system on a lattice $\mathbb{L}$ is given by a Hamiltonian H_o satisfying condition (2.9). $\mathcal{G}$ is the set of ground states of H_o, and $\mathcal{L}$ is a d-dimensional perturbation space. The norm on $\mathcal{L}$ is given by $\|.\|$:

$$\|x\| = \sum_{i=1}^{d} |x_i| .$$

Let $m \in \mathbb{N}$. For any $G \in \mathcal{G}$ and $\beta > 0$ we define:

$$\tau_m^G(\beta L,\beta) = - \langle \beta L, e_G \rangle + p_m^G(\beta L,\beta) \tag{3.1}$$

with e_G being an element of $\mathcal{L}^*$ introduced by (2.3), and p_m^G being defined by (2.16) for L satisfying condition (2.15). Let

$$\epsilon_m = \min_{\mathcal{G}} c_m(G) \tag{3.2}$$

and define $O = B(0,\epsilon_m)$, with $\bar{O}$ denoting the closure of O.
The set O is the common domain for all functions τ_m^G.
Let $\beta_o > 0$ be fixed. We will specify its value later. We define:

$$\Omega_{m,\beta} = \bigcup_{G,G'\in\mathcal{G}} \{L\in\bar{O} : \tau_m^G(\beta L,\beta) = \tau_m^{G'}(\beta L,\beta) \geq \tau_m^{G''}(\beta L,\beta) \text{ all } G''\in\mathcal{G}\} \tag{3.3}$$

The family $\underline{\Omega_m} = \{\underline{\Omega_{m,\beta}}\ ,\ \beta>\underline{\beta_o}\}$ is called the <u>asymptotic phase diagram in order m.</u>
Let $\mathcal{G}' \subset \mathcal{G}$. The <u>stratum</u> of Ω_m <u>corresponding to $\mathcal{G}'$</u> is the family of sets:

$$\Omega_m(\mathcal{G}') = \{\Omega_{m,\beta}(\mathcal{G}'),\ \beta>\beta_o\}$$

with

$$\Omega_{m,\beta}(\mathcal{G}') = \{L\in\bar{O} : \tau_m^G(\beta L,\beta) = \tau_m^{G'}(\beta L,\beta) > \tau_m^{G''}(\beta L,\beta)$$
$$\text{for all } G,G'\in\mathcal{G}',\ G''\notin\mathcal{G}'\} \tag{3.4}$$

We will also use the term "stratum" to mean a member $\Omega_{m,\beta}(\mathcal{G}')$ of the family, with $\beta>\beta_o$ fixed.
The stratum coresponding to $\mathcal{G}'$ is of <u>order</u> k if k is the largest natural number such that

$$\operatorname{diam}(\Omega_{m,\beta}(\mathcal{G}')) < c(\mathcal{G}')\, e^{-\beta E_k}$$

for some positive constant $c(\mathcal{G}')$, and all $\beta>\beta_o$.

Let $G \in \mathcal{G}$. The domain of G is the family of sets

$$\Omega_m(G) = \{\Omega_{m,\beta}(G)\ ,\ \beta > \beta_0\}$$

with

$$\Omega_{m,\beta}(G) = \{L \in \bar{O}\ :\ \pi_m^G(L,\beta) \geqslant \pi_m^{G'}(L,\beta)\ ,\ \text{all } G' \in \mathcal{G}\} \tag{3.5}$$

Note that in contradistinction to strata, domains are closed sets. We define the order of the domain in the same way as for strata.

There are no restrictions on L other than (2.15). We pay the price for this in the form of some restrictions on β_0, which generally arise as conditions for the existence of strata. Factors which determine β_0 are described at the end of Section 6. Here we note that as long as β_0 remains finite, its value is of no interest in the asymptotic diagrams approach.

As mentioned in the introduction, we expect that the asymptotic phase diagram will yield qualitative features of the "true" phase diagram. Here we understand the "true" phase diagram to mean the phase diagram constructed by rigorous methods (for example, by Pirogov-Sinai theory ([4],[5]), or by reflection positivity [9],[10]). By studying Ω_m we can obtain two types of information about this true phase diagram:

1. One can determine, which strata are not empty, whether or not this property is stable (i.e. if $\Omega_m(\mathcal{G}') \neq \emptyset$, then $\forall\ s \geqslant m$, $\Omega_s(\mathcal{G}') \neq \emptyset$), and what is the order of the stratum.

2. The localization of the nonempty stratum in the perturbation space can be described by a local approximation (in some order) by some piecewise linear set.

The easiest way to approach these problems is to study the members $\Omega_{m,\beta}$ of the asymptotic phase diagram for some β fixed. Let us perform the change of variables:

$$\bar{O} \longrightarrow \beta\bar{O} : L \longrightarrow \beta L \equiv x \qquad (\beta > \beta_0 \text{ fixed})$$

As one can see from definition (3.1), the pair of variables (x,β) is natural for the problem (cf. also (2.16)). Henceforth, the strata of $\Omega_{m,\beta}$ for β fixed will be expressed in terms of the variable x.

The definition (3.4) of a stratum involves a system of equations and a set of inequalities. The stratum is "cut out" from the solution set of the system by these inequalities. The first question arises, for which subfamilies $\mathcal{G}'$, the system of equations has the solution for all β large enough. Before we answer this question, let us prove a technical lemma.

<u>Lemma 3.1.</u>

$\forall\ \beta > 0\ ,\ \forall\ m \in \mathbb{N}\ ,\ \forall\ G \in \mathcal{G}\ \ \exists\ A_m(G) > 0\ :\ \forall\ x \in \beta\bar{O}$

$$\| D_x p_m^G(x,\beta) \| < A_m(G) e^{-\epsilon_m \beta} \tag{3.6}$$

Hence $\forall\ x, x' \in \beta\bar{O}$

$$| p_m^G(x,\beta) - p_m^G(x',\beta) | \leq \| x - x' \|\ A_m(G)\ e^{-\epsilon_m \beta} \tag{3.7}$$

<u>Proof</u>:

Using the form of $p_m^G(x,\beta)$ (cf. (2.10) and (2.16)) we obtain an inequality:

$$\|D_x p_m^G(y,\beta)\| \leqslant \sum_{j=1}^{m} e^{-\beta E_j} \sum_{l=1}^{s_j} |n_{j,l}^G| \, \|\mu_{j,l}\| \, e^{-\mu_{j,l}(y)}$$

Note that as long as $y \in \beta\bar{O}$,

$$\beta E_j + \mu_{j,l}(y) \geqslant \beta\epsilon_m$$

(cf. definition of $\bar{O}$). Hence

$$\|D_x p_m^G(x,\beta)\| \leqslant \sum_{j=1}^{m} e^{-\beta\epsilon_m} \sum_{l=1}^{s_j} |n_{j,l}^G| \, \|\mu_{j,l}\| < A_m(G) e^{-\beta\epsilon_m}$$

where

$$A_m(G) = \sum_{j=1}^{m} \sum_{l=1}^{s_j} |n_{j,l}^G| \, \|\mu_{j,l}\|$$

By the mean value theorem

$$|p_m^G(x,\beta) - p_m^G(x',\beta)| \leqslant \|x - x'\| \, \|D_x p_m^G(y,\beta)\|$$

where y is between x and x'. ∎

<u>Definition</u>: A family of points $x_1, x_2, \ldots, x_s$ in a linear vector space is said to be <u>linearly</u> <u>independent</u> if for any choice of i, $(1 \leqslant i \leqslant s)$, the set of vectors $\{x_k - x_i,\ k \neq i\}$ is linearly independent.

Proposition 3.2.

Suppose that $\mathcal{G}_o \equiv \{G_0,G_1,\ldots,G_s\} \subset \mathcal{G}$ is such that $\{e_{G_i},\ i=1,\ldots,s\}$ is linearly independent in $\mathcal{L}^*$. Let $N = \bigcup_{i=1}^{s} \mathrm{Ker}(e_{G_i} - e_{G_0})$, and $\mathcal{L} = N \oplus M$. Consider a system of equations:

$$\pi_m^{G_i}(x,\beta) - \pi_m^{G_o}(x,\beta) = 0 \tag{3.8}$$

Then there exists $\beta_m(\mathcal{G}_o)$ such that:

1. $\forall\ \beta>\beta_m(\mathcal{G}_o)$ the solution $y : \beta O \cap N \times (\beta_m(\mathcal{G}_o),\infty) \to M$ exists and is analytic in first coordinate,

2. $\forall\ z \in \beta O \cap N$
$$y(z,\beta) = \sum_{j=1}^{\infty} y_j(z) e^{-\beta E_j} \tag{3.9}$$

3. $\exists\ a>0 : \forall\ \beta>\beta_m(\mathcal{G}_o)\ ,\ \forall\ z \in \beta O \cap N :$
$$\|y(z,\beta)\| < a \tag{3.10}$$

Proof:

For β large enough and $x \in \beta O$, $n_j^{G_i}(x)$ can be written in the following form (cf.(2.10)):

$$n_j^{G_i}(x) = \sum_{l=1}^{r_j} n_{j,l}^{G_i}\ e^{-\beta\mu_{j,l}(x)}$$

where $\mu_{j,l} \in \{\mu_{j,l}^{i},\ i=0,\ldots,s\}$, and $n_{j,l}^{G_i} = 0$ if $\mu_{j,l}$ does not enter into (2.10) for the phase G. Let $r = \sum_{j=1}^{m} r_j$. Define a function

$$F : M \times \mathbb{R}^r \to M : F_i(y,u) = -\langle y, e_{G_i} - e_{G_o}\rangle +$$

$$+ \sum_{j=1}^{m} \sum_{l=1}^{r_j} (n_{j,l}^{G_i} - n_{j,l}^{G_o})\ e^{-\beta\mu_{j,l}(y)}\ u_{jl}$$

where $u = (u_{11}, u_{12}, \ldots, u_{1r_1}, \ldots, u_{mr_m})$

F satisfies conditions of the implicit function theorem: $F(0,0) = 0$, and $\forall\, h \in \mathcal{L}$: $D_y F(0,0)(h) = (\langle h, e_{G_o} - e_{G_i}\rangle)_{i=1}^{s}$, so $D_y F(0,0)$ is invertible since $\{e_{G_i}, i=0,..,s\}$ is linearly independent. Hence there exists an open ball $B(0,q) \subset \mathbb{R}^{r}$ in which the solution: $y : B(0,q) \longrightarrow M$ of the equation: $F(y,u) = 0$ exists. Moreover, since F is analytic, y is analytic in u. Let I be a multiplicity function : $I : \{(j,l) : 1 \leq j \leq m,\ 1 \leq l \leq r_j\} \to \mathbb{N}$, and γ a set of all multiplicity functions. We can write $y(u)$ in the following way: $\exists\ q_1 < q$:

$\forall\ u \in B(0,q_1)$

$$y(u) = \sum_{I \in \gamma} y_I u^I \tag{3.11}$$

where $u^I = \prod_{i=1}^{m} \prod_{l=1}^{r_j} u_{jl}^{I(j,l)}$.

Obviously, there exists $a>0$ such that for any u in $B(0,q_1)$ we have:

$$\|y(u)\| \leq a.$$

Consider now the subset of $B(0,q_1)$:

$$\{u \in \mathbb{R}^{r} : u_{jl} = e^{-\beta(E_j + \mu_{j,l}(L))},\ L \in O,\ \beta \in \mathbb{R}_+\} \cap B(0,q_1)$$

Let $\beta_m(\mathcal{S}_o)$ be such that $\epsilon_m\, \beta_m(\mathcal{S}_o) > -\ln q_1$, with ϵ_m as in (3.2). For $\beta > \beta(\mathcal{S}_o)$ define

$$y(z,\beta) = y((e^{-\beta E_j + \mu_{j,l}(z)})_{jl}) \qquad z \in \beta O \cap N,$$

Then $y(z,\beta)$ exists and is the solution of (3.8). Obviously y is bounded on $\beta O \cap N$ for all $\beta > \beta_m(\mathcal{S}_o)$):

$$\|y(z,\beta)\| \leq a$$

The substitution: $u_{jl} = e^{-\beta E_j + \mu_{j,l}(z)}$ in (3.9) yields

$$y(z,\beta) = \sum_{j=1}^{\infty} y_j(z)\, e^{-\beta E_j}$$

where

$$y_j(z) = \sum_{I} y_I \exp[-\sum_{(n,l)} I(n,l)\, \mu_{n,l}(z)]$$

and the first sum is taken over all multiplicity functions I such that

$$\sum_{(n,l)} I(n,l) E_n = E_j$$ ∎

Note: Conditions imposed on $\mathcal{S}_o$ may be relaxed in the presence of symmetries (cf. Section 7).

In general, if $\mathcal{S}_o$ does not satisfy the conditions of the above proposition, the solution of (3.8), and hence $\Omega_m(\mathcal{S}_o)$, does not exist. In order to avoid this problem, we will be forced to adopt an additional assumption (cf. Assumption 3.20).

Suppose that $\mathcal{S}_o$ satisfies conditions of Proposition 3.2. The stratum $\Omega_m(\mathcal{S}_o)$ is "cut out" of the solution set (3.9) by surfaces which arise as intersections of this set with solution sets for other families of phases. Our next result describes the way in which these solution sets behave in the vicinity of their common intersection.

Let $\mathcal{S}' \subset \mathcal{S}$ be such that $\{e_G,\ G \in \mathcal{S}'\}$ is linearly independent. We define the boundary $\partial\Omega_{m,\beta}(\mathcal{S}')$ of $\Omega_{m,\beta}(\mathcal{S}')$ as the set

$$\partial\Omega_{m,\beta}(\mathcal{S}') = O \cap [\overline{\Omega_{m,\beta}(\mathcal{S}')} \setminus \Omega_{m,\beta}(\mathcal{S}')]$$

Furthermore, let $\{\mathcal{S}_i\}$ be a family of subsets of $\mathcal{S}'$ with $|\mathcal{S}_i| \geq 2$.

We write $\mathcal{E}$ as a direct sum:

$$\mathcal{E} = Y_i \oplus X_i \oplus N$$

with

$$N = \bigcap_{G,G' \in \mathcal{S}'} \ker(e_G - e_{G'}),$$

$$N_i \equiv X_i \oplus N = \bigcap_{G,G' \in \mathcal{S}_i} \ker(e_G - e_{G'})$$

Assume that $\dim Y_i = r_i$.

For any $\mathcal{S}_i$, the hypotheses of Proposition 3.2. are satisfied. Let $y_i(z,\beta)$ be the solution of the system of equations (3.8) with $\mathcal{S}_o \equiv \mathcal{S}_i$.

<u>Proposition</u> <u>3.3.</u>

$\exists\ \beta_m(\mathcal{S}',\mathcal{S}_i)$, $\exists\ a(\mathcal{S}',\mathcal{S}_i)>0$: $\forall\ \beta>\beta_m(\mathcal{S}',\mathcal{S}_i)$, $\forall\ z_o \in \beta\bar{O} \cap N$,
$\forall\ z \in \beta\bar{O} \cap X_i$

$$\|y_i(z+z_o,\beta) - y_i(z_o,\beta)\| \leq \frac{ae^{-\beta\epsilon_m}}{1-ae^{-\beta\epsilon_m}} \| z \| \qquad\qquad a \equiv a(\mathcal{S}',\mathcal{S}_i)$$

(for the clarification of notation, see Fig.1)

<u>Proof</u>:

Choose $G_o \in \mathcal{S}_i$. Let

$$S \ :\ Y_i \to \mathbb{R}^{r_i} \ :\ (Sy)_j = \langle y, e_{G_o} - e_{G_j} \rangle$$

where $G_j \in \mathcal{S}_i$. S is invertible (by the definition of Y_i).

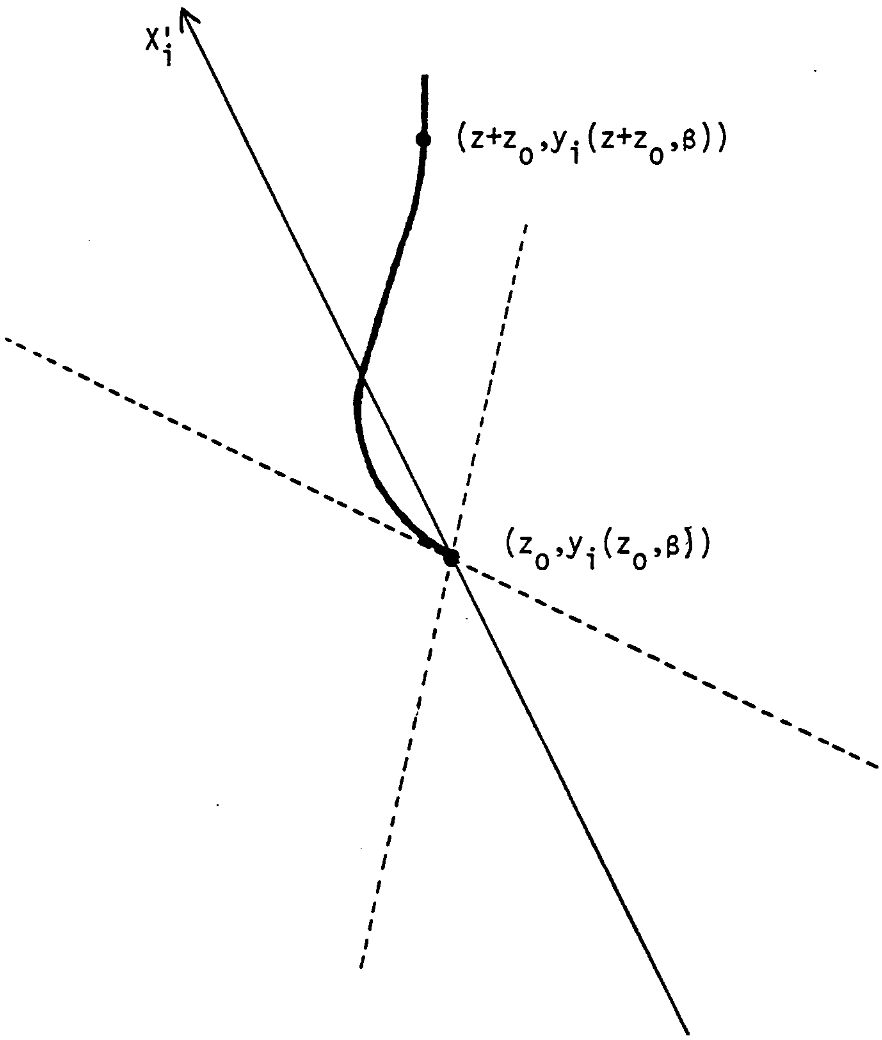


Fig.1: The restrictions on a stratum near its boundary. The figure shows the cut through L with the plane perpendicular to N at z_0. Here $X_i' = X_i + (z_0, y_i(z_0,\beta))$. The dashed lines represent regions described by Proposition 3.3.

Denote by $\Delta(z,\beta)$ the vector in $\mathbb{R}^{r_i}$ with coordinates:

$$\Delta_j(z,\beta) = p_m^{G_j}(y_i(z+z_o,\beta),z+z_o,\beta) - p_m^{G_o}(y_i(z+z_o,\beta),z+z_o,\beta).$$

Then $\forall\, y \in y_i(\beta\mathcal{O} \cap X_i,\beta)$, y satisfies the equation

$$y(z,\beta) = S^{-1}\,\Delta(z,\beta)$$

Let $y_o \equiv y_i(z_o,\beta)$, and $y \equiv y_i(z+z_o,\beta)$, $z \in \beta\bar{\mathcal{O}} \cap X_i$. Then

$$\|y_o - y\| \leqslant \|S^{-1}\|\,\|\Delta(z,\beta) - \Delta(0,\beta)\| \leqslant$$

$$\leqslant 2\|S^{-1}\|\,\|(y,z+z_o) - (y_o,z_o)\| \max_{G\in\mathcal{S}_i} A_m(G)\, e^{-\beta\epsilon_m}$$

In the last inequality (3.7) has been used.

If $\beta_m(\mathcal{S}',\mathcal{S}_i)$ is such that

$$a \equiv a(\mathcal{S}',\mathcal{S}_i) \equiv 2 \max_{G\in\mathcal{S}_i} A_m(G)\|S^{-1}\|\, e^{-\beta_m(\mathcal{S}',\mathcal{S}_i)\epsilon_m} < 1$$

then $\forall\, \beta>\beta_m(\mathcal{S}',\mathcal{S}_i)$:

$$\|y - y_o\| \leqslant \frac{ae^{-\beta\epsilon_m}}{1-ae^{-\beta\epsilon_m}}\,\|z\| \qquad \blacksquare$$

2. The Pirogov-Sinai case: $|\mathcal{S}| = \dim \mathcal{L} + 1$.

As the first case, let us consider the situation when $\mathcal{L}$ is transversal to $\mathcal{S}$ and $\dim \mathcal{L} = |\mathcal{S}|-1$. This case has been solved rigorously in ([4],[5]). It therefore is a test for the validity of the asymptotic diagrams approach. We will also use the results of this section later on in the construction of the phase diagram in the general case.

Let us suppose that the elements of $\mathcal{G}$ are ordered in some way: $\mathcal{G} = \{G_0, G_1, \ldots, G_d\}$. We consider the function:

$$\tau_m(\beta) : \beta\bar{O} \to \mathbb{R}^{d+1} \quad : \quad \tau_m(\beta)(x)_i = \tau_m^{G_i}(x,\beta) \tag{3.12}$$

The norm on $\mathbb{R}^{d+1}$ is given by $\|x\| = \sum_{i=0}^{d} |x_i|$

Proposition 3.4.

$\exists\ \beta_o,\ \exists\ a>0,\ \exists\ A>0 : \forall\ \beta>\beta_o\ ,\ \forall\ x,x' \in \beta\bar{O}$:

$$a\|x - x'\| \leqslant \|\tau_m(\beta)(x) - \tau_m(\beta)(x')\| \leqslant A\|x - x'\| \tag{3.13}$$

Proof:

The upper bound in (3.13) is obtained by applying (3.7) to each component of $\tau_m(\beta)(x) - \tau_m(\beta)(x')$. Let

$$A = (d+1) \max_{0\leqslant i\leqslant d} (A_m(G_i) + \|e_{G_i}\|).$$

To show the second inequality, let us first note that

$$\|x - x'\| = \sup_{\|e\|=1} |\langle x-x',e\rangle| \qquad \text{with } e \in \mathcal{L}^*$$

Since $\{e_{G_i} - e_{G_o}\}$ generates $\mathcal{L}^*$, $e = \sum_{i=1}^{d} c_i(g_{G_i} - e_{G_o})$. It is easy to see that $\gamma \equiv \sup_{\|e\|=1} \sum_{i=1}^{d} |c_i| < \infty$

Hence

$$\|x - x'\| \leqslant \gamma \max_{1\leqslant i\leqslant d} |\langle x - x', e_{G_i} - e_{G_o}\rangle| \leqslant 2\gamma \max_{0\leqslant i\leqslant d} |\langle x - x', e_{G_i}\rangle|$$

But

$$|\langle x - x', e_{G_i}\rangle| \leqslant |\tau_m^{G_i}(\beta)(x) - \tau_m^{G_i}(\beta)(x')| + |p_m^{G_i}(x,\beta) - p_m^{G_i}(x',\beta)| \leqslant$$

$$\leqslant \|\tau_m(\beta)(x) - \tau_m(\beta)(x')\| + A_m(G_i)\|x - x'\|\, e^{-\epsilon_m\beta}$$

Let β_o be such that $\alpha \equiv 2\gamma \max\limits_{0\leqslant i\leqslant d} A_m(G_i)e^{-\beta_o\epsilon_m} < 1$. Set $a = \dfrac{2\gamma}{1-\alpha}$ ∎

Let $\Delta = \{y \in \mathbb{R}^{d+1} : y_i = y_j, \text{ all } i,j\}$. Δ induces an equivalence relation: $y \sim y'$ if $y - y' \in \Delta$. Denote the set of equivalence classes as $\mathbb{R}^{d+1}/\Delta$. We define the function:

$\tilde{\tau}_m(\beta) : \mathcal{L} \to \mathbb{R}^{d+1}/\Delta : \tilde{\tau}_m(\beta)(x) = [\tau_m(\beta)(x)]$

In $\mathbb{R}^{d+1}/\Delta$ we introduce the norm $\|.\|_1$: $\|x\|_1 = \sum\limits_{i=1}^{d} |x_i - x_o|$.

<u>Proposition</u> <u>3.5.</u>

$\exists\ \beta_1 : \forall\ \beta > \beta_1$, $\tilde{\tau}_m(\beta)$ is a local diffeomorphism on βO.

<u>Proof</u>:

Let $\beta > \beta_o$, with β_o as in Proposition 3.4. By the same proposition, $\tilde{\tau}_m(\beta)$ is a global homeomorphism on βO. Moreover, $\forall\ x_o \in \beta O$, $\forall\ h \in \mathcal{L}$

$$\|D_x\tilde{\tau}_m(\beta)(x_o)h\| = \sum_{i=1}^{d} |\langle h, -e_{G_i} + e_{G_o} + D_x p_m^{G_i}(x_o,\beta) - D_x p_m^{G_o}(x_o,\beta)\rangle|$$

Since the mapping $h \to (\langle h, -e_{G_i} + e_{G_o}\rangle)_{i=1}^{d}$ is invertible, then $\exists\ \alpha > 0$:

$$\sum_{i=1}^{d} |\langle h, e_{G_i} - e_{G_o}\rangle| > \alpha\ \|h\|$$

Let β_1 be such that $2 \max\limits_{0\leqslant i\leqslant d} A_m(G_i)e^{-\beta_1\epsilon_m} < \alpha$, where $A_m(G_i)$ is given by (3.6), and ϵ_m by (3.2). Then

$$\|D_x\tilde{\tau}_m(\beta)(x_o)h\| \geqslant (\alpha - 2 \max_{0\leqslant i\leqslant d} A_m(G_i)e^{-\beta\epsilon_m})\|h\| >$$

$$> (\alpha - 2 \max_{0\leqslant i\leqslant d} A_m(G_i)e^{-\beta_1\epsilon})\|h\|$$

for all x_o in βO. ∎

Let us define the subsets of $\mathbb{R}^{d+1}/\Delta$:

$$R = \{x : x_i = x_j \geq x_k, \text{ all pairs of } i,j,\ k \neq i,j\}/\Delta \tag{3.14}$$

and for $I \subset \{1,2,\dots,d+1\}$, $|I| \geq 2$

$$R^I = \{x : x_i = x_j > x_k,\ i,j \in I,\ k \notin I\}/\Delta$$

Obviously $\forall\ \mathcal{S}^I = \{G_i,\ i \in I\}$ the following holds:

$$\Omega_{m,\beta}(\mathcal{S}^I) = \tilde{\tau}_m^{-1}(\beta)(R^I \cap \tilde{\tau}_m(\beta O)) \tag{3.15}$$

<u>Proposition 3.6.</u>

Let $\beta > \beta_1$. Then $\forall\ \mathcal{S}^I \subset \mathcal{S} : |\mathcal{S}^I| \geq 2$, $\Omega_{m,\beta}(\mathcal{S}^I)$ exists and is a differential manifold modelled on R^I. $\Omega_{m,\beta}$ is a stratified manifold modelled on R.

<u>Proof</u>:

$\Omega_{m,\beta}(\mathcal{S}^I)$ is a diffeomorphic image of R^I by Proposition 3.5. Obviously the closure of $\Omega_{m,\beta}(\mathcal{S}^I)$ contains $\Omega_{m,\beta}(\mathcal{S}^I)$ and strata of lower dimensions (note that $\partial\Omega_{m,\beta}(\mathcal{S}^I) = \bigcup_{I' \supset I} \Omega_{m,\beta}(\mathcal{S}^{I'})$). ∎

The typical phase diagram for d = 2 is presented in Fig.2. The system has three ground states: G_0, G_1, G_2. By Proposition 3.6, the phase diagram $\Omega_{m,\beta}$ has one triple point $\Omega_{m,\beta}(\mathcal{S})$, and three lines where two phases coexist: $\Omega_{m,\beta}(G_0,G_1)$, $\Omega_{m,\beta}(G_0,G_2)$, and $\Omega_{m,\beta}(G_1,G_2)$.

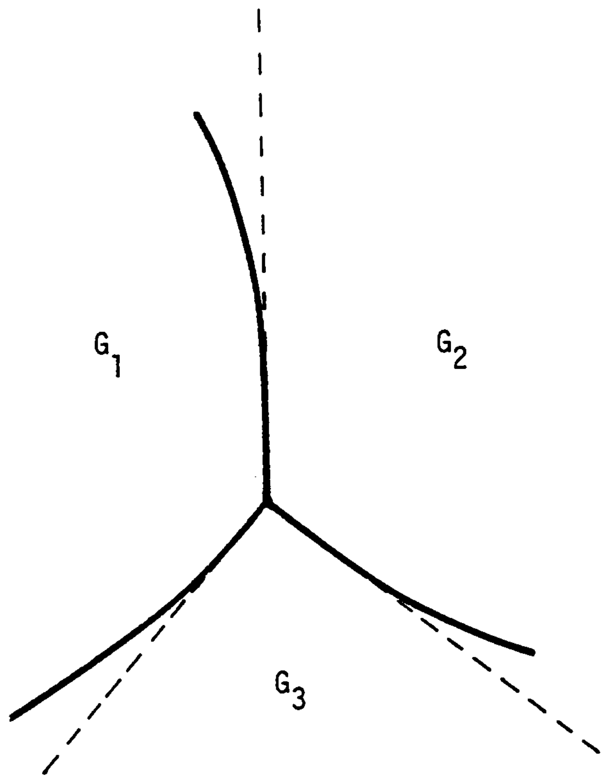


Fig.2: The generic phase diagram for the system with two dimensional perturbation space and three ground states. The dashed lines show zero order (zero temperature) phase diagram.

3. Phase diagrams for sets of affine functionals.

As the next case in the investigation of phase diagram for various systems, let us consider the phase digram for the set of affine functionals. This is also a new type of a problem in linear programming.

Suppose that $\Gamma = \{\rho_i,\ i=1,2,\dots,N\}$ is a set of affine functionals:

$$\rho_i : \mathbb{R}^d \to \mathbb{R} \ :\ \rho_i(x) = \langle x,h_i\rangle + a_i \quad , a \in \mathbb{R} \quad (d+1 \leq N).$$

Assume that $\{h_i : \rho_i \in \Gamma\}$ spans $\mathbb{R}^d$. We will also write

$$\rho_i = (h_i, a_i) \in \mathbb{R}^{d+1}.$$

If $\Gamma' \subset \Gamma$, we define

$$\Pi(\Gamma') = \{x : \rho_i(x) = \rho_j(x) > \rho_k(x) \text{ if } \rho_i,\rho_j \in \Gamma',\ \rho_k \notin \Gamma'\} \tag{3.11}$$

$$\Pi(\rho_i) = \{x : \rho_i(x) \geq \rho_j(x) \text{ all } j\neq i\} \quad \text{(closed)}$$

The set $\Pi = \bigcup \Pi(\Gamma')$, where the union is over all $\Gamma' \subset \Gamma$ such that $|\Gamma'| \geq 2$, will be called the phase diagram for Γ, and its subsets $\Pi(\Gamma')$ the strata.

Let

$$W = \operatorname{conv} \Gamma \subset \mathbb{R}^{d+1} \tag{3.17}$$

We define maxW as the set of maxima of W. We will say that $E \subset \max W$ is a face (extremal edge) of maxW if E is a face (extremal edge) of W. $\epsilon(\max W)$ denotes a set of extremal points of maxW.

<u>Theorem 3.7</u>

1. $\rho \in \epsilon(\max W) \Rightarrow \mathrm{Int}\Pi(\rho) \neq \emptyset$

2. $\rho = \sum_{i=1}^{s} \lambda_i \rho_i$ $(2 \leq s \leq d+1)$ with $\rho_i \in \epsilon(\max W)$, $\lambda_i \in (0,1)$,

   $\sum_{i=1}^{s} \lambda_i = 1 \Rightarrow \Pi(\rho) = \Pi(\{\rho_1,\ldots,\rho_s\})$

3. $\rho \notin \max W \Rightarrow \Pi(\rho) = \emptyset$

4. There exists a one-one correspondence between extremal elements of $\max W$ and strata of Π. Namely, let $E \subset \max W$ be a face (extremal edge) of dimension $d-r$. Then $\Pi(E) \neq \emptyset$, $\dim\Pi(E) = r$, and

   $\Pi(E) = \{x: \rho(x) = \rho'(x) > \tilde{\rho}(x)\ \forall \rho,\rho' \in E,\ \tilde{\rho} \in \bigcup_{F \supset E} \epsilon(F)\}$

<u>Proof</u>:

1. Let $\Gamma' = \Gamma \setminus \{\rho\}$, and $W' = \mathrm{conv}\,\Gamma'$. Suppose that $\rho \in \epsilon(\max W)$. Since W' is convex and closed, then there exists a hyperplane $P \subset \mathbb{R}^{d+1}$, strictly separating ρ from W'. Hence there exist $x_o \in \mathbb{R}^d$, y_o, $\alpha \in \mathbb{R}$ such that

$$P = \{(\eta,t) \in \mathbb{R}^{d+1} : \langle x_o,\eta\rangle + y_o t = \alpha\}$$

If ρ is such that $h \in \epsilon(\mathrm{conv}\{h' : \rho' \in \Gamma'\})$, then P may be choosen so that $y_o = 0$. (since $\mathrm{conv}\{h' : \rho' \in \Gamma'\}$ can be separated from h). Then $\exists\ \gamma > 0$ such that $\langle x_o,h\rangle = \alpha + \gamma$, and $\langle x_o,\rho'\rangle < \alpha$ for all $\rho' \in \Gamma'$. Choose $\lambda > 0$ such that $\lambda\gamma + a = \max_{\rho' \in \Gamma'} a'$.

Then

$$\rho(\lambda x_o) = \langle \lambda x_o,h\rangle + a = \lambda\alpha + \lambda\gamma + a >$$
$$> \rho'(\lambda x_o) + \lambda\gamma + a - a' \geq \rho'(\lambda x_o)$$

for all $\rho' \in \Gamma'$. By changing x_o we can assume that $\lambda = 1$.

If ρ is such that $h \notin \epsilon(\mathrm{conv}\{h' : \rho' \in \Gamma'\})$, then $y_o \neq 0$ and by adjusting x_o and α we can assume that $y_o = 1$. Note that $\rho(x_o) > \alpha$ while

$\rho'(x_o) < \alpha$ for all $\rho' \in \Gamma'$ (if $\rho(x_o) < \alpha$, then since P separates ρ from W', $\rho'(x_o) > \alpha > \rho(x_o)$ for $\rho' = (h,a')$, and hence $\rho \notin \max W$).

Let now $\epsilon > 0$ be such that $\rho(x_o) > \alpha + \epsilon$. Then $\forall x$:
$\| x - x_o\| \leqslant \frac{1}{2} \epsilon \, (\max_{\rho\in\Gamma} \|h\|)^{-1}$

$$\rho(x) = \rho(x_o) + \langle x-x_o,h\rangle \geqslant \rho(x_o) - \| x - x_o\| \, \|h\| > \alpha + \frac{\epsilon}{2}$$

$$\rho'(x) = \rho'(x_o) + \langle x-x_o,h'\rangle \leqslant \rho'(x_o) + \| x - x_o\| \, \|h\| < \alpha + \frac{\epsilon}{2}$$

Hence $\mathrm{Int}\Pi(\rho) \neq \emptyset$.

2. Let $\rho = \sum_{i=1}^{s} \lambda_i \rho_i$. Define

$$N = \{x : \rho_i(x) = \rho_j(x) \ , \ i=1,\dots,s\}$$

Note that:

a) $\forall x \in N$, $\rho(x) = \rho_i(x)$ for all i (this is evident since $\sum_{i=1}^{s} \lambda_i = 1$)

b) $\forall x \notin N \ \exists i : \rho(x) < \rho_i(x)$.

Suppose that b) does not hold. Then $\exists x_o \notin N : \forall i$,

$$\rho(x_o) \geqslant \rho_i(x_o)$$

Obviously then $\exists \epsilon > 0$, $\exists j : \rho(x_o) \geqslant \rho_j(x_o) + \epsilon$. (if $\rho(x_o) = \rho(x_o)$ for all i, then $x_o \in N$). But in this case

$$\rho(x_o) = \sum_{i=1}^{s} \lambda_i \rho_i(x_o) \leqslant \sum_{i=1}^{s} \lambda_i \rho(x_o) - \epsilon = \rho(x_o) - \epsilon$$

(contradiction)

Obviously $\Pi(\rho) = \Pi(\{\rho_1,\dots,\rho_s\})$

3. $\rho \notin \max W \Rightarrow \exists\ (h,t) \in \max W : (h,t) > \rho$. Hence $\forall\ x \in \mathbb{R}^{d+1}$,

$$\langle x,h\rangle + t > \rho(x)$$

Let $(h,t) = \sum_{i=1}^{s} \lambda_i \rho_i$. Suppose that $\exists\ x_o : \rho(x_o) > \rho'(x_o)$ for all other ρ', then $\rho(x_o) > \rho_i(x_o)$, $i=1,\dots s$, and $\rho(x_o) > \langle x_o,h\rangle + t$ (contradiction). Thus $\Pi(\rho) = \emptyset$.

4. First let us consider the case when E is an extremal point ρ_o, i.e. $r=d$. Then $\Pi(E) = \Pi(\rho_o) \neq \emptyset$ and has dimension d (part 1.). Let

$$\Gamma_o = \bigcup_{\rho_o \in F} F, \qquad \Pi_o(\rho_o) = \{x : \rho_o(x) > \rho'(x)\ ,\ \rho' \in \Gamma_o\}$$

<u>Claim</u>: $\forall\ \rho \notin \Gamma_o$, $\forall\ x \in \Pi_o : \rho_o(x) > \rho(x)$.

<u>Proof</u>: Let $\rho = (h,a)$. Consider $\rho_\lambda = (\lambda h+(1-\lambda)h_o, a_\lambda)$, $\lambda \in (0,1)$, where a_λ is such that $(h,a_\lambda) \in \max W$. Then $\exists$ F and $\lambda' \in (0,1)$ such that $\rho_o, \rho_{\lambda'} \in F$. Obviously $a_{\lambda'} > \lambda' a+(1-\lambda')a_o$, since $\rho \notin F$. But then $\forall\ x \in \Pi_o(\rho_o)$

$$\rho_o(x) > \rho_{\lambda'}(x) = \lambda'\langle x,h\rangle + (1-\lambda')\langle x,h_o\rangle + a_{\lambda'} >$$
$$> \lambda'\rho(x) + (1-\lambda')\rho_o(x)$$

i.e. $\rho_o(x) > \rho(x)$.

Finally note that if $\rho \notin \bigcup_{\rho_o \in F} \epsilon(F)$ and $\rho = \sum_{i=1}^{s} \lambda_i \rho_i$, then the condition :

$\rho_o(x) > \rho_i(x)$ for all $i=1,\dots,s$ induces the condition: $\rho_o(x) > \rho(x)$. Thus $\Pi_o(\rho_o) = \{x : \rho(x) < \rho_o(x),\ \rho \in \bigcup_{\rho_o \in F} \epsilon(F)\}$.

Now consider the general case: E has dimension d-r, r < d.

Let $\rho_0, \rho_1, \ldots, \rho_{d-r} \in \epsilon(E)$, and

$$N = \{x : \rho_i(x) = \rho_o(x),\ i=1,2,\ldots,d-r\}.$$

Since $\{\rho_i - \rho_o, i=1,\ldots,d-r\}$ is linearly independent, $E \subset [\rho_i - \rho_o]$. Thus

$\forall\, x \in N$, $\forall\, \rho' \in E : \rho' - \rho_o = \sum_{i=1}^{d-r} c_i(\rho_i - \rho_o)$, one has

$$\rho'(x) - \rho_o(x) = \sum_{i=1}^{d-r} c_i(\rho_i(x) - \rho_o(x)) = 0$$

i.e. $\Pi(E) \subset N$

Note that $N = x_o + \bigcap_{i=1}^{d-r} \mathrm{Ker}(h_i - h_o)$, where x_o is any solution of the system of equations:

$$\langle x, h_i - h_o\rangle = a_o - a_i.$$

We can choose x_o uniquely by demanding that it is orthogonal to

$\bigcap_{i=1}^{d-r} \mathrm{Ker}(h_i - h_o)$. Consider the map:

$$pr_E : \mathbb{R}^{d+1} \to \mathbb{R}^{d-r+1} \quad : \quad pr_E(\rho) \equiv \tilde{\rho} = (\tilde{h}, a + h(x_o))$$

where $\tilde{h}$ is the restriction of h to $\bigcap_{i=1}^{d-r} \mathrm{Ker}(h_i - h_o)$. If E is translated to zero, then pr_E can be interpreted as a projection along E onto its orthogonal complement. Obviously for any ρ in E, $\tilde{\rho} = \tilde{\rho}_o$.

Let $\tilde{W} = pr_E W$.

<u>Claim</u>: $\tilde{\rho}_o \in \epsilon(\max\tilde{W})$, and $\forall$ face $\tilde{F} \subset \max\tilde{W}$ such that $\tilde{\rho}_o \in \tilde{F}$,

$\exists$ a face $F \supset E$ such that $\tilde{F} \supset pr_E F$

<u>Proof</u>: First note that if $\tilde{E}'$ is a plane of dimension k in $\tilde{W}$, then $\tilde{E}' = pr_E E'$ with E' being a plane in W of dimension not less than k.

Now suppose that $\tilde{\rho}_o = \lambda\tilde{\rho}_1 + (1-\lambda)\tilde{\rho}_2$. Let $E' = \mathrm{conv}(E,\{\rho_1,\rho_2\})$. Then $\mathrm{pr}_E E' = \mathrm{conv}(\{\tilde{\rho}_1,\tilde{\rho}_2\})$, hence E' is a plane in W. If $\rho_1, \rho_2 \notin E$, then E is not extremal. Thus $\rho_1, \rho_2 \in E$, i.e. $\tilde{\rho}_1 = \tilde{\rho}_2 = \tilde{\rho}_o$ and $\tilde{\rho}_o \in \epsilon(\max\tilde{W})$.

Second part of the claim is obvious.

Now we apply part 1. of the theorem to $\tilde{W}$ and conclude that $\tilde{\Pi}(\tilde{\rho}_o) \neq \emptyset$. Using the case when E is an extremal point, we can find the bounds on $\tilde{\Pi}(\tilde{\rho}_o)$. But since $\Pi(E) \subset N$, we have: $\mathrm{pr}_E\Pi(E) \equiv \tilde{\Pi}(\tilde{\rho}_o) = \Pi(E)$. ∎

The phase diagram Π for Γ is now constructed as follows: to any d-dimensional face F of $\max W$ there corresponds (by part 4 of Theorem 3.7) a point $v(F)$ which is the unique element of $\Pi(F)$. Furthermore, for any face E of F (of dimension d-2) there exists a 1-dim line on which elements belonging to E coexist. This line either goes to infinity (if some elements of $\epsilon(E)$ are such that their linear parts belong to $\mathrm{conv}\{h, \rho\in\Gamma\}$), or it terminates at another point of coexistence $v(F')$, for some face F' sharing E with F. The process then continues for faces of Π with lower dimensions. After Π has been found, the domain $\Pi(\rho)$ for each ρ is determined by parts 1, 2 and 3 (existence), and part 4 (localization) of theorem 3.7.

<u>Remark</u> <u>3.8</u>:

If Γ is a set of linear functionals $\{e_G, G \in \mathcal{G}(H_o)\}$, then the application of Theorem 3.7 results in a zero-temperature phase diagram.

4. The convex structure in order m.

We will now consider the general situation: dim $\mathcal{L} \leq |\mathcal{S}|-1$, and cut-off pressures are not neccesarily constant. We will show that the properties of the phase diagram $\Omega_{m,\beta}$ can be described with the help of some family of convex sets (a convex structure) in $\mathcal{L}^{*}\times\mathbb{R}$. This family is a generalization of the set maxW defined in Section 3 (cf. (3.17)).

The definition of the convex structure is the subject of this section.

Example: Blume-Capel model ([6],[7])

The following example will be used throughout this section and the following ones.

Let $\mathbb{L} = \mathbb{Z}^2$, $S = \{-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$

$$H_o = \sum_{\langle a,b\rangle} (s_a - s_b)^2 \qquad s_a(X) = X_a$$

where $\langle a,b\rangle$ denotes a pair of nearest neighbors.

The set of ground states of H_o is:

$$\mathcal{S} = \{(-5),(-3),(-1),(1),(3),(5)\}$$

with (k) denoting the configuration: $\forall\, a \in \mathbb{Z}^2$, $(k)_a = \frac{k}{2}$.

We will describe this model extensively in Section 4.1, here we only cite some properties of formal pressure coefficients for first few orders.

1. $E_1 = 4$, $n_1^{(5)}(0) = n_1^{(-5)}(0) = 1$

$$n_1^{(k)}(0) = 2 \quad \text{if } k \neq -5,\ 5$$

2. $E_2 = 6$, $n_2^{(5)}(0) = n_1^{(-5)}(0) = 2$

$$n_1^{(k)}(0) = 4 \quad \text{if } k \neq -5, 5$$

3. The order 7 is the lowest order in which for any (k) there exists an excitation s such that for some $a \in \mathbb{Z}^2$, $|s_a - k| > 1$. In this order

$$n_7^{(-1)}(0) = n_7^{(1)}(0) \quad , \quad n_7^{(3)}(0) = n_7^{(-3)}(0)$$

$$n_7^{(1)}(0) - n_7^{(3)}(0) \equiv a(3) > 0$$

4. $\forall\, i < 7$, $n_i^{(k)}(0) = n_i^{(1)}(0) \qquad k = -3, 3, -1$

The perturbation space is generated by interactions:

$$(\Phi_1)_a = s_a \quad , \quad (\Phi_2)_a = s_a^3 \ , \ (\Phi_i)_A = 0 \quad \text{if } A \neq \{a\}$$

In the base generated in $\mathcal{L}^*$ by Φ_1, Φ_2

$$e_{(k)} = \frac{1}{8}\,(4k \ , \ k^3)$$

<u>The convex structure in order m</u>

For any $G \in \mathcal{S}$, let $A_1^G = \dot{p}^G(0)_1 \equiv n_1^G(0)$. Define

$$W_1 \subset \mathcal{L}^* \times \mathbb{R} : W_1 = \mathrm{conv}\{(e_G, A_1^G),\ G \in \mathcal{S}\}$$

In W_1 we consider the set of maxima of W_1: $\max W_1$. The set $E \subset \max W_1$ is an <u>s-dimensional</u> <u>face</u> <u>(edge)</u> of $\max W_1$ if it is an s-dimensional face (edge) of W_1.

Let $\mathbb{F}_1$ denote the set of all faces of $\max W_1$. If $F \in \mathbb{F}_1$, we define:

$$\mathcal{S}_1(F) = \{G : (e_G, n_1^G(0)) \in F\}$$

There exists a unique vector $v_1(F)$ in $\mathcal{L}$ such that $\forall\, G, G_o \in \mathcal{S}_1(F)$:

$$-\langle v_1(F), e_G - e_{G_o}\rangle + A_1^G - A_1^{G_o} = 0$$

If $G_o, G_1, \ldots, G_d$ are any phases corresponding to elements of $\epsilon(F)$, then

$v_1(F)$ is defined as the solution of the system of equations:

$$- \langle v_1(F), e_{G_i} - e_{G_0} \rangle + A_1^{G_i} - A_1^{G_o} = 0 \qquad (3.18)$$

Example:

In our example, $\max W_1$ has five faces (cf. Fig.3a). They are listed below, every face P together with elements of $\mathcal{S}_1(P)$, and $v_1(P)$.

F : (-1),(1),(-3),(3) , $v_1(F) = 0$

G : (5),(3),(-1) , $v_1(G) = (\frac{1}{6}, -\frac{2}{21});$

H : (5),(-1),(-3) , $v_1(H) = (\frac{13}{21}, -\frac{4}{21});$

G′ : (-5),(-3),(1) , $v_1(G') = (-\frac{1}{6}, \frac{2}{21});$

H′ : (-5),(1),(3) , $v_1(H') = (-\frac{13}{21}, \frac{4}{21}).$

For any $F \in \mathbb{F}_1$, we define the following quantities:

$$\tilde{v}_1(F,\beta) = e^{-\beta E_1} v_1(F) \qquad (3.19)$$

$$\dot{v}_1 \in \mathbb{D}^d : (\dot{v}_1)_1 = v_1, \ (\dot{v}_1)_k = 0 \text{ if } k \geqslant 2$$

Let $A_2^G(F) = \dot{p}^G(\dot{v}_1(F))_2$, and

$$W_2(F) = \text{conv } \{(e_G, A_2^G(F)), \ G \in \mathcal{S}_1(F)\}$$

We denote the set of faces of $\max W_2(F)$ by $\mathbb{F}_2(F)$. The set

$$\mathbb{F}_2 = \bigcup_{F \in \mathbb{F}_1} \mathbb{F}_2(F)$$

is called the <u>convex structure in order 2</u>.

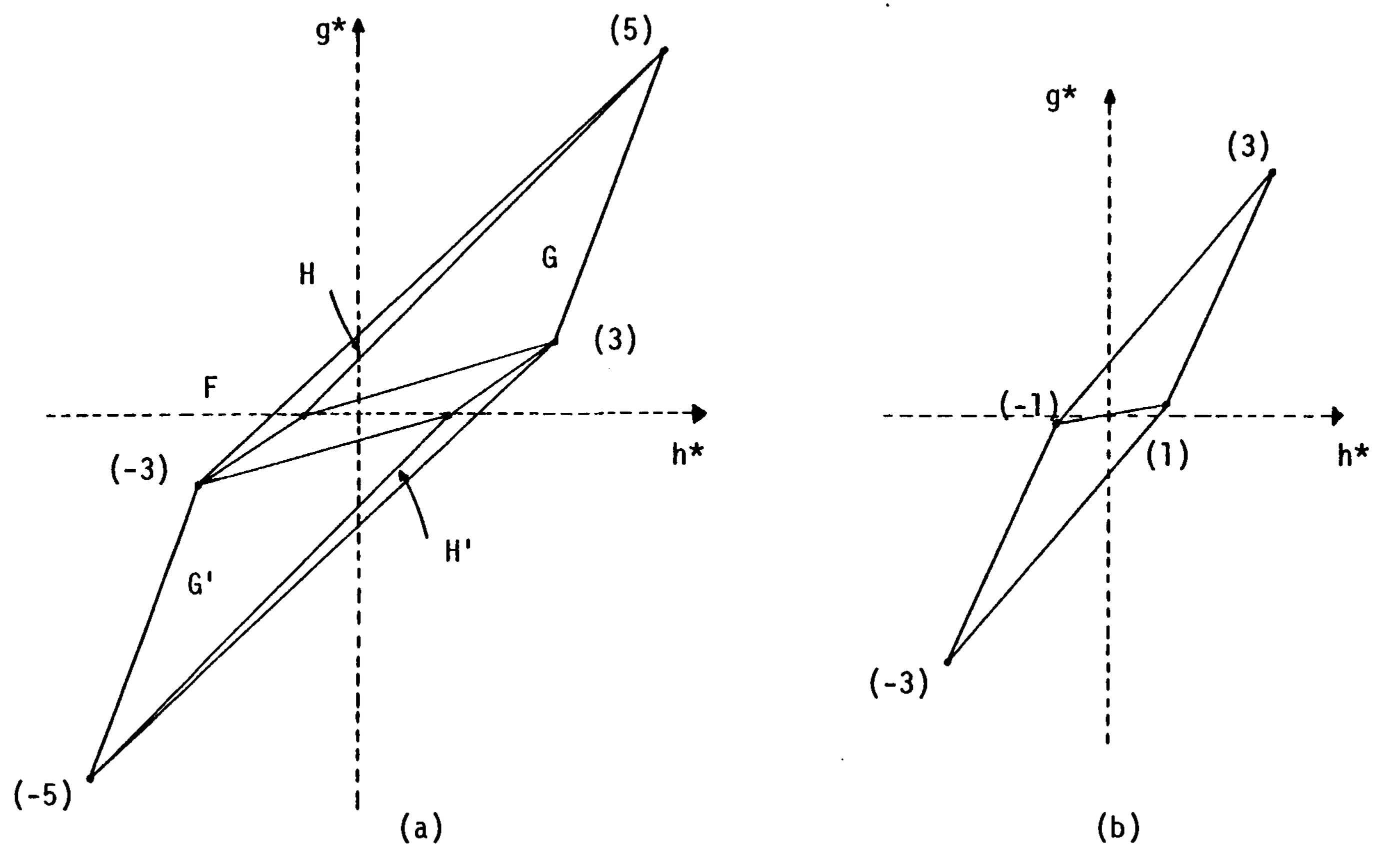


Fig.3: The projection on L^* of (a) max W_1; (b) max $W_{i(3)}(F)$

Example:

In order 2 ($E_2 = 6$) one has

$$\dot{p}^{(k)}(\dot{v}_1(P))_2 = \dot{p}^{(k)}(0)_2$$

for any face P. Hence

$$A_2^{(k)}(P) = n_2^{(k)}(0) = \begin{cases} 4 \ , & k \neq -5,\ 5 \\ 2 \ , & k = -5,\ 5 \end{cases}$$

Thus $\max W_2(P)$ has only one element which is isomorphic (as a convex set) to P. We will denote it also as P. Moreover, it is easy to see that $v_2(P) = 2v_1(P)$. Obviously $\mathbb{F}_2 \sim \mathbb{F}_1$ (as collections of convex sets, cf. Remark 3.11 below).

The convex structure in order m is defined by induction.
If $F' \in \mathbb{F}_{m-1}$ is given, then for any $F \in \mathbb{F}_{m-1}(F')$ one defines $v_{m-1}(F)$ by means of formula (3.18), and

$$\tilde{v}_{m-1}(F,\beta) = \tilde{v}_{m-2}(F',\beta) + e^{-\beta E_{m-1}} v_{m-1}(F)$$

$$\dot{v}_{m-1} \in \mathbb{D}^d : (\dot{v}_{m-1}(F))_k = (\dot{v}_{m-2}(F'))_k \text{ if } k \neq m-1 \ ,$$

$$(\dot{v}_{m-1}(F))_{m-1} = v_{m-1}(F)$$

$$\mathcal{G}_{m-1}(F) = \{G \in \mathcal{G}_{m-2}(F') : (e_G, A^G_{m-1}(F')) \in F \}$$

Let $A^G_m(F) = \dot{p}^G(\dot{v}_{m-1}(F))_m$. We define $W_m(F)$, $\max W_m(F)$ and $\mathbb{F}_m(F)$ as in order 2. The <u>convex structure in order m</u> is the set

$$\mathbb{F}_m = \bigcup_{F \in \mathbb{F}_{m-1}} \mathbb{F}_m(F)$$

For the completion, $F_o = W_o = \text{conv}\ \{e_G,\ G \in \mathcal{G}\}$

Remark 3.9.

Let $\tilde{A}_m^G(F) = A_m^G(F) + c$, and $\tilde{W}_m(F) = \text{conv}\ \{(e_G, \tilde{A}_m^G(F)),\ G \in \mathcal{S}_{m-1}(F)\}$. Then $\tilde{W}_m(F)$ is a translate of $W_m(F)$ and hence there is a trivial correspondence between extremal edges and faces of both sets. We will often use $c = -A_m^{G_o}(F)$ for some $G_o \in \mathcal{S}_{m-1}(F)$. Note that due to the fact mentioned above, we can reduce the amount of information needed about the formal series coefficients for deducing the form of convex structure: it is not neccesary to know the absolute values of coefficients but rather their values relative to coefficients of some fixed phase.

Remark 3.10.

For any F in $\mathbb{F}_m$, there exists a unique set of faces $\{F_0, F_1, \ldots, F_m\}$ such that $F_i \in \mathbb{F}_i(F_{i-1})$, and $F_m = F$.

Remark 3.11.

Suppose that $|\mathcal{S}(F)| = d+1$ for some $F \in \mathbb{F}_m$. Then $\mathbb{F}_{m+1}(F) = \{F_{m+1}\}, \ldots, \mathbb{F}_{m+s}(F_{m+s-1}) = \{F_{m+s}\}$ for all s, and $F, F_{m+1}, \ldots, F_{m+s}$ are isomorphic (as convex sets in $\mathcal{L}^* \times \mathbb{R}$, i.e. that there is a one-one correspondence between extremal elements of both sets).

Example:

Let P be any element of $\mathbb{F}_2$. In order 3 ($E_3 = 8$)

$$\dot{p}^{(k)}(\dot{v}_2(P))_3 = \dot{p}^{(k)}(\dot{v}_1(P))_3 = n_3^{(k)}(0) + \langle v_1(P), dn_1^{(k)}(0)\rangle$$

Since the exact form of these expressions is cumbersome and of little importance, we will not reproduce it here. We note that $W_3(P)$ is again isomorphic to $W_2(P)$: if P is not F, then this holds because $\mathcal{S}_3(P)$ has three elements, and for F one has: $v_2(F) = 0$, so $W_3(F)$ is a translate of $W_2(F)$. Since in any order s, higher than 2, $\mathbb{F}_s(P)$ is isomorphic to $\mathbb{F}_2(P)$ (P not equal to F), we will not investigate $\mathbb{F}_s(P)$.

Let us study $W_i(F)$ for $i > 3$. As we have already observed,

$$\dot{p}^{(K)}(\dot{v}_2)_3 = n_3^{(K)}(0) = \text{constant if } k = -3,-1,1,3$$

Hence

$$v_3(F) = 0$$

By the inductive argument, $v_i(F) = 0$ if $i < 7$. In order 7

$$n_7^{(1)}(0) - n_7^{(3)}(0) \equiv a(3) > 0$$

Max $W_7(F)$ has two faces (Fig.3b). These are listed below together with corresponding vectors in $\mathcal{L}$:

$F_1 : \mathcal{S}_7(F_1) = \{(3),(1),(-1)\}$, $v_7(F_1) = (\frac{a(3)}{12} , - \frac{a(3)}{3})$

$F_2 : \mathcal{S}_7(F_2) = \{(-3),(-1),(1)\}$, $v_7(F_2) = (- \frac{a(3)}{12} , \frac{a(3)}{3})$

One does not have to investigate convex structures in higher orders ($s\geq 7$) since $\mathbb{F}_s$ is isomorphic to $\mathbb{F}_7$ (in the sense of Remark 3.11)

5. Applications of convex structures.

The general strategy of constructing the phase diagram $\Omega_{m,\beta}$ is based on the following observation: Suppose that $\mathcal{S}'$ is a subset of $\mathcal{S}$, and let U be an open set contained in $\bigcup_{G\in\mathcal{S}'}\Omega_{m,\beta}(G)$. Define $\tilde{\Omega}_{m,\beta}$ as a phase diagram for phases in $\mathcal{S}'$ (i.e. with other phases neglected). Then

$$\Omega_{m,\beta} \cap U = \tilde{\Omega}_{m,\beta} \cap U$$

We will find the covering of βO with the family of sets such that in each of these sets, $\Omega_{m,\beta}$ is described by $\tilde{\Omega}_{m,\beta}$ for some subset $\mathcal{G}'$ of $\mathcal{G}$.

<u>Definition</u>:

Let $F \in \mathbb{F}_m$, and $\{F_0, F_1, \ldots, F\}$ be as in Remark 3.10. For any G in $\mathcal{G}$, we define an affine functional:

$$\rho^G_{m,\beta}(F) : \mathcal{L} \longrightarrow \mathbb{R}$$

$$\rho^G_{m,\beta}(F,x) = - \langle x, e_G \rangle + \sum_{j=1}^{m} A^G_j(F_{j-1}) e^{-\beta E_j} \tag{3.20}$$

<u>Remark 3.12</u>

$\forall\, G \in \mathcal{G}_m(F),\ \forall\, j < m$,

$$A^G_j(F_{j-1}) - \langle v_j(F_j,\beta), e_G \rangle = A^{G_o}_j(F_{j-1}) - \langle v_j(F_j,\beta), e_{G_o} \rangle$$

with $G_o \in \mathcal{G}_m(F)$ being any fixed element.

Let $\{F_o, \ldots, F_m\}$ be a sequence of faces as in the remark 3.10. With every element of this sequence we associate a set

$$U_k(F_k) = B(\tilde{v}_k(F_k,\beta), c_k(F_k)\, e^{-\beta E_{k+1}})$$

The constant $c_k(F_k)$ will be specified later in this section. Slightly abusing this notation, we set $U_o(F_o) \equiv \beta 0$.

We will define $U_k(F_k)$ in four steps, using an induction in the order k. Suppose that $U_{k-1}(F_{k-1})$ is defined. Then we show that:

1. in $U_{k-1}(F_{k-1})$, the function τ^G_m is approximated by $\rho^G_{k,\beta}(F_{k-1})$, so that their difference is of order k;
2. in $U_{k-1}(F_{k-1})$, the phase diagram is given by phases in $\mathcal{G}_{k-1}(F_{k-1})$ (i.e. with neglecting other phases);
3. if $\rho^G_{k,\beta}(F_{k-1}) \in \text{Int}\, F_k$ for some $F_k \in \mathbb{F}_k(F_{k-1})$, then $\Omega_{m,\beta}(G)$ is of

order at least $k+1$;

4. In step 4 we define $U_k(F_k)$, $F_k \in \mathbb{F}_k(F_{k-1})$.

<u>Step 1</u>: Affine functionals (3.20) approximate functions (3.1).

<u>Proposition 3.13.</u>

Let $F \in \mathbb{F}_m$, and $\{F_0, F_1, \ldots, F\}$ be the family of faces corresponding to F (cf. remark (3.10)). For $0 \leqslant k \leqslant m$ and $c>0$ we define

$$B(F_k,c) = \{x : \|x - \tilde{v}_k(F_k,\beta)\| \leqslant ce^{-\beta E_k}\}$$

Then $\exists\, \alpha_m > 0 : \forall\, G \in \mathcal{S}_m(F) \;\; \exists\, d_k(G) > 0 : \forall\, \beta > 0 \,,\, \forall\, x \in B(F_k,c)$:

$$|\rho^G_{m,\beta}(F,x) - \pi^G_m(x,\beta)| \leqslant d_k(G)e^{-\beta(E_k+\alpha_m)} \tag{3.21}$$

<u>Proof</u>:

$$|\rho^G_{m,\beta}(F,x) - \pi^G_m(x,\beta)| = |\sum_{j=1}^{m} \dot{p}^G(\dot{v}_m(F))_j e^{-\beta E_j} - p^G_m(x,\beta)| \leqslant$$

$$\leqslant |\sum_{j=1}^{m} \dot{p}^G(\dot{v}_m(F))_j e^{-\beta E_j} - p^G_m(\tilde{v}_m(F,\beta),\beta)| +$$

$$+ |p^G_m(\tilde{v}_m(F,\beta),\beta) - p^G_m(x,\beta)| \leqslant$$

$$\leqslant c_m(G)e^{-\beta E_{m+1}} + A_m(G)\|x - \tilde{v}_m(F,\beta)\|\, e^{-\beta\epsilon_m}$$

The estimation of the first term is given by Lemma 2.5., and of the second one by (3.7). By the hypothesis

$$\|x - \tilde{v}_m(F,\beta)\| \leqslant ce^{-\beta E_k} + \|\tilde{v}_k(F_k,\beta) - \tilde{v}_m(F,\beta)\| \leqslant$$

$$\leqslant ce^{-\beta E_k} + e^{-\beta E_{k+1}} \sum_{j=l+1}^{m} \|v_j(F_j,\beta)\|$$

Let $\alpha_m = \min\limits_{k \leqslant m} (\epsilon_m, \dfrac{E_{k+1}-E_k}{2})$, with ϵ_m defining the set O (cf. (3.2)).

Set $d_k(G) = c_m(G) + A_m(G)[c + \sum_{j=l+1}^{m} \|v_j(F_j,\beta)\|]$. ∎

<u>Note</u>: In the zero order the estimation is not as good, since we demand that (3.21) holds on $\beta\bar{O}$. Hence

$\|x\| \leqslant \beta c$

Then

$|\rho^{G}_{m,\beta}(F,x) - \pi^{G}_{m}(x,\beta)| \leqslant \beta d_o(G) e^{-\beta\alpha_o}$

The next proposition is an immediate consequence of Proposition 3.13.

<u>Proposition 3.14.</u>

Let $\mathcal{S}_o \equiv \{G_0, G_1,\ldots,G_s\} \subset \mathcal{S}_m(F)$ be such that $\{e_{G_i}, i=1,\ldots,s\}$ is linearly independent. Consider the solutions: $y(z,\beta)$ ($\beta > \beta_m(\mathcal{S}_o)$) of the system of equations (3.8) (cf. Proposition 3.2.), and $y_o(z,\beta)$ of the system

$$\rho^{G_i}_{m,\beta}(F,y,z) = \rho^{G_o}_{m,\beta}(F,y,z) \qquad i=1,\ldots,s \tag{3.22}$$

Here $x \in \bigcap_{i=1}^{s} \mathrm{Ker}(e_{G_i} - e_{G_o})$

Let $B(F_k,c)$ be as in Proposition 3.13. Then $\forall\, k \leqslant m$, $\forall\, c > 0$, $\exists\, a_k(c)$ such that if $\beta > \beta_m(\mathcal{S}_o)$ and $(y_o,z) \in B(F_k,c)$, then

$$\|y(z,\beta) - y_0(z,\beta)\| < a_K(c)\, e^{-\beta E_{K+1}} \tag{3.23}$$

<u>Note</u>: Proposition 3.14 states that inside $B(F_K,c)$, the solutions of (3.8) and (3.22) are close to one another: their distance is of order $K+1$.

<u>Proof</u>:

The proof is by induction in K. Suppose that the estimate (3.23) holds in the ball $B(F_{K-1},r)$ for some radius $r>0$. Let $c' > c$ be such that if $(y_0,z) \in B(F_K,c')$, then $(y(z,\beta),z) \in B(F_K,c)$ (if $a_{K-1}(r)$ is given by the estimation (3.23) for order $K-1$, we take $c' = c + a_{K-1}(r)$).

Note that both (3.8) and (3.22) can be written in the form:

$$\langle y, e_{G_i} - e_{G_0}\rangle = p_m^{G_i}(y,z,\beta) - p_m^{G_0}(y,z,\beta) \qquad i=1,\dots,s$$

$$\langle y, e_{G_i} - e_{G_0}\rangle = \sum_{j=1}^{m} [A_j^{G_i}(F_{j-1}) - A_j^{G_0}(F_{j-1})]\, e^{-\beta E_j} \qquad i=1,\dots,s$$

Hence

$$|\langle y-y_0, e_{G_i} - e_{G_0}\rangle| \leq |\pi_m^{G_i}(y,z,\beta) - \rho_{m,\beta}^{G_0}(F,y,z)| +$$

$$+ |\pi_m^{G_i}(y,z,\beta) - \rho_{m,\beta}^{G_0}(F,y,z)| \leq 2d_K e^{-\beta(E_K+\alpha_m)} \tag{3.24}$$

Next note that

$$y(z,\beta) = \sum_{j=1}^{\infty} y_j(z) e^{-\beta E_j}$$

(cf. (3.9)). Obviously

$$y_0(z,\beta) = \sum_{j=1}^{m} y_{0,j} e^{-\beta E_j}$$

(is z-independent).

By (3.24), $\langle y_{0,j} - y_j(z)\, ,\, e_{G_i} - e_{G_0}\rangle = 0$ if $j \leq m$. Therefore for all z:

$(y_o,z) \in B_K(c')$, $y_{o,j} = y_j(z)$ if $j \leqslant m$. Finally,

$$\|y(z,\beta) - y_o(\beta)\| \leqslant \sum_{j=K+1}^{\infty} \|y_j(z)\| \, e^{-\beta E_j} \leqslant a_K(c) e^{-\beta E_{K+1}}$$

where $a_K(c) = \sup \sum_{j=K+1}^{\infty} \|y_j(\beta K)\| \, e^{-\beta(E_j - E_{K+1})}$ and sup is taken over all L in $\bar{O}$ and $\beta > \beta_m$. $a_K(c)$ is finite because of estimation (3.10). ∎

<u>Step 2</u>: The phase diagram in $U_K(F_K)$ is given by the set of phases $\mathcal{G}_K(F_K)$ (i.e. we may disregard other phases).

<u>Lemma 3.15</u>

Let $F \in \mathbb{F}_K$. Then $\exists\, c > 0$, $\exists\, \gamma(c) > 0$, $\exists\, \beta_K(F) : \forall\, \beta > \beta_K(F)$,

$\forall\, \tilde{G} \notin \mathcal{G}_K(F)$, $\forall\, x \in B(\tilde{v}_K(F_K,\beta), ce^{-\beta E_K})$:

$$\rho^{G}_{K,\beta}(F,x) - \rho^{\tilde{G}}_{K,\beta}(F,x) > \gamma(c) e^{-\beta E_K} \tag{3.25}$$

for some $G \in \mathcal{G}_K(F_K)$. Hence $U_K(F_K) \subset \cup\, \Omega_{m,\beta}(G)$
with the union over elements of $\mathcal{G}_K(F_K)$.

<u>Proof:</u>

Let $\mathcal{G}^j$ be the set of phases $\tilde{G} \notin \mathcal{G}_K(F)$ such that j is the highest order for which $\tilde{G} \in \mathcal{G}_j(F_j)$. Denote

$$\Delta_t(G) \equiv A^G_t(F_{t-1}) - \langle v_t(F_t,\beta), e_G\rangle \qquad \text{if } t \leqslant K$$

Then $\forall\, \tilde{G} \in \mathcal{G}^j$, $\forall\, G \in \mathcal{G}_K(F)$

$\rho^G_{j,\beta}(F_{j-1}, x+\tilde{v}_j) - \rho^{G}_{j,\beta}(F_{j-1}, x+\tilde{v}_j) =$

$$= - \langle x, e_G - e_{\tilde{G}}\rangle + [\Delta_j(G) - \Delta_j(\tilde{G})] e^{-\beta E_j}$$

(cf. Remark 3.12.)

Note that by the definition of $\tilde{v}_j(F_j,\beta)$,

$$\Delta_j(G) - \Delta_j(\tilde{G}) > 0$$

Let $\beta_j(\tilde{G})$ be such that

$$\delta(\tilde{G}) \equiv \Delta_j(G) - \Delta_j(\tilde{G}) - \sum_{i=j+1}^{k} |\Delta_i(G) - \Delta_i(\tilde{G})|\ e^{-\beta_j(\tilde{G})(E_i - E_j)} > 0$$

Define

$$c_j = \frac{1}{2} \max \|e_G - e_{\tilde{G}}\|^{-1} \delta(\tilde{G}) \qquad \beta_j = \max \beta_j(\tilde{G})$$

with both maxima taken over $\mathcal{G}^j$. Then $\forall\ x \in B(\tilde{v}_k, c_j e^{-\beta E_j})$,

$\forall\ \tilde{G} \in \mathcal{G}^j$,

$$\rho^{G}_{k,\beta}(F,x) - \rho^{\tilde{G}}_{k,\beta}(F,x) = -\langle x-\tilde{v}_k, e_G - e_{\tilde{G}}\rangle + \sum_{i=j}^{k} [\Delta_i(G) - \Delta_i(\tilde{G})]\ e^{-\beta E_i} \geqslant$$

$$\geqslant e^{-\beta E_j} [\delta(\tilde{G}) - \|e_G - e_{\tilde{G}}\|\ c_j] \geqslant \frac{1}{2}\ c_j e^{-\beta E_j}$$

Now let $\beta_k(F) > \max_{j \leqslant k} \beta_j$ be such that $\forall\ \beta > \beta_k(F)$

$$c_j e^{-\beta E_j} > c_k e^{-\beta E_k}.$$

Set $c = c_k$ and $\gamma(c) = \frac{1}{2}\ c_k$.

Combining Proposition 3.13 and Lemma 3.15 we see that the inequality (3.25) also holds for the cut-off pressures with the slight redefinition of $\gamma(c)$. Obviously for any $c > 0$

$$U_k(F_k) \subset B(\tilde{v}_k(F_k), ce^{-\beta E_k})$$

if β is large enough.

■

Step 3: If ρ_K^G is an interior point of F_K, then $\Omega_{m,\beta}(G)$ has order at least $K+1$.

Lemma.3.16.

Let $\mathcal{S}_o$ be as in Proposition 3.14. Suppose U is a subset of the solution set for the system of equations (3.22), homeomorphic to a ball (U is connected and has no holes). Furthermore, let W be an open bounded subset of the solution set for the system of equations (3.8). Assume also that $\exists\ F \in \mathbf{F}_K$, $\exists c > 0$:

$$U \subset B(F_K, c)$$

If there exists $d > 0$ such that

$$\partial W \equiv \bar{W} \setminus W \subset \{z : \mathrm{dist}(z,U) \leqslant de^{-\beta E_K}\}$$

then $\forall\ \beta > \beta_m(\mathcal{S}_o)$ (cf. Proposition 3.2)

$$W \subset \{z : \mathrm{dist}(z,U) \leqslant (a_K(c)+d)e^{-\beta E_{K+1}}\}$$

where $a_K(c)$ is given by Proposition 3.14.

Proof:

Let $S = \{z : \mathrm{dist}(z,U) \leqslant de^{-\beta e_K}\}$, and $N = \bigcap_{G,G' \in \mathcal{S}_o} \ker(e_G - e_{G'})$

The solution set of the system (3.22) is $N + w$, where w is any solution of (3.22). Since $\partial W \subset S$, then $pr_N\ \partial W \subset pr_N\ S$. By hypothesis, W is bounded, and hence $pr_N\ W \subset pr_N\ S$. But then $\forall\ (y,z) \in W$,

$$\mathrm{dist}((y,z),U) \leqslant \mathrm{dist}(y,w+N) + \mathrm{dist}(z,pr_N\ S) \leqslant$$

$$\leqslant (a_K(c)+d)e^{-\beta E_{K+1}}$$

■

Proposition 3.17.

Let $F \in \mathbb{F}_m(F')$ and G be such that $\rho^G_{m,\beta}(F') \in \mathrm{Int}F$. Then $\forall\, s \geq m$ $\exists\, \beta_s(G)$: $\forall\, \beta > \beta_s(G)$, $\Omega_{s,\beta}(G)$ is of order not lower than m, i.e. $\exists\, r_s > 0$:

$$\Omega_{s,\beta}(G) \subset B(\tilde{v}_m(F,\beta), r_s e^{-\beta E_{m+1}}) \qquad (3.26)$$

Proof:

We will prove the proposition for order m only. The proof for $s \geq m$ is similar.

Since $\rho^G_{m,\beta} \in \mathrm{Int}F$, then there exist d+1 elements of $\mathcal{G}$: $G_0, G_1, \ldots, G_d$, such that $\{e_{G_i}\}$ is linearly independent, and

$$\rho^G_{m,\beta} = \sum_{i=0}^{d} \lambda_i \rho^{G_i}_{m,\beta} \qquad \text{with } \sum_{i=0}^{d} \lambda_i = 1 \;,\; \lambda_i \in (0,1)$$

Let $\beta_m(G) = \max\limits_{0 \leq i \leq d} \beta_m(\{G,G_i\})$, with $\beta_m(\{G,G_i\})$ as in Proposition 3.2.

Define $\tilde{\Omega}_{m,\beta}$ to be the phase diagram in order m for the set of phases $\{G, G_0, \ldots, G_d\}$. Obviously $\Omega_{m,\beta}(G) \subset \tilde{\Omega}_{m,\beta}(G)$. We will apply induction in k, i.e. we want to show that $\forall\, k \leq m \;\; \exists\, r'_k > 0$:

$$\tilde{\Omega}_{m,\beta}(G) \subset Z_k(r'_k) \equiv B(\tilde{v}(F_k,\beta), r'_k e^{-\beta E_{k+1}}) \qquad (3.27)$$

Suppose that (3.27) holds for k-1. In order to show that it holds also for order k, we will use inductively Lemma 3.16.

First let us consider the solution y_i of the system of equations:

$$\pi^G_m(x,\beta) = \pi^{G_t}_m(x,\beta) \;,\; t \neq i \qquad x \in Z_{k-1}(r'_{k-1})$$

By Proposition 3.14., $\exists\, a_i > 0$:

$$\|y_i - \tilde{v}_k(F,\beta)\| \leq a_i e^{-\beta e_{k+1}}$$

Let $a^1 = \max_i a_i$

Next, let

$$M_{i,j} = \{(z,y_{i,j}),\ z \in \bigcap_{t \neq i,j} \ker(e_G - e_{G_t})\}$$

be the solution set of the system of equations:

$$\pi_m^G(x,\beta) = \pi_m^{G_t}(x,\beta) \quad , \quad t \neq i,j$$

Denote by W the subset of $M_{i,j}$ bounded by y_i and y_j.

If $(z,y_{i,j}) \in W$, by Lemma 3.16 one has:

$$\|(z,y_{i,j}) - \tilde{v}_j(F,\beta)\| \leq (a^1 + a_{i,j})e^{-\beta E_{k+1}}$$

where $a_{i,j}$ is obtained by applying Proposition 3.14. to the set $\{G,G_t,\ t\neq i,j\}$.

Let $a^2 = a^1 + \max_{\{i,j\}} a_{i,j}$

We continue in the same manner for elements of $\partial\tilde{\Omega}_{m,\beta}(G)$ of higher dimensions. Finally, there exists a^d such that $\forall\, x \in \partial\tilde{\Omega}_{m,\beta}(G)$

$$\|x - \tilde{v}_k(F,\beta)\| \leq a^d\, e^{-\beta E_{k+1}}$$

Hence by applying Lemma 3.16. once again, we obtain (3.27).

Set $r_m = r'_m$ ∎

The next corollary deals with the case when in some order, $\rho^G_{m-1,\beta}(F') \in \mathrm{Int}\, F$ $(F \in \mathbb{F}_{m-1}(F'))$, but $\rho^G_{m,\beta}(F)$ is neither an extremal point of $\max W_m(F)$, nor an interior point of any face from $\mathbb{F}_{m+1}(F)$. The conclusion of the corollary is the only statement that we can make about the domain of G.

Corollary 3.18.

Let $\{F_o,\ldots,F_m\}$ be a sequence of faces as in Remark 3.10. Suppose that G is such that:

1. $\rho^G_{m-1,\beta}(F_{m-2}) \in \mathrm{Int}\, F_{m-1}$,

2. $\rho^G_{m,\beta}(F_{m-1}) = \sum_{i=1}^{r} \lambda_i \rho^{G_i}_{m,\beta}(F_{m-1})$, $r < d+1$

   where $\rho^{G_i}_{m,\beta}(F_{m-1}) \in \epsilon_m(F_m)$, $\sum_{i=1}^{r} \lambda_i = 1$, and $\lambda_i \in (0,1)$

Then $\forall\, s \geq m \;\; \exists\, d_m > 0$, $\exists\, \beta_m(G) : \forall\, \beta > \beta_m(G)$

$$\Omega_{m,\beta}(G) \subset \{z : \mathrm{dist}(z, \Pi(G_1,\ldots,G_r)) \leq d_m e^{-\beta E_{m+1}}\} \tag{3.28}$$

where

$$\Pi(G_1,\ldots,G_r) = \{x : \rho^{G_i}_{m,\beta}(F_{m-1},x) = \rho^{G}_{m,\beta}(F_{m-1},x) > \rho^{G'}_{m,\beta}(F_{m-1},x), \text{ all } G' \neq G, G_i\}$$

Proof:

We will prove this Corollary for $\Omega_{m,\beta}(G)$ only. The proof for higher orders is similar.

Let $\mathbb{F}' = \{F \in \mathbb{F}_m(F_{m-1}) : \rho^G_{m,\beta}(F_{m-1}) \in F\}$

Consider the set of phases:

$$\mathcal{S}_o = \{G\} \cup \{G' : \rho^{G'}_{m,\beta}(F_{m-1}) \in \bigcup_{F \in \mathbb{F}'} \epsilon_m(F)\}$$

Define

$$\beta_m(G) = \max_{G' \in \mathcal{S}_o} \beta_m(\{G,G'\})$$

Let $\tilde{\Omega}_{m,\beta}$ be a phase diagram for the phases in this set. Now we use the inductive argument as in the proof of the Proposition 3.17. Here $U = \Pi(G, G_1, \ldots, G_m)$ ∎

<u>Step 4</u>: The definition of sets $U_k(F_k)$.

Suppose that $F_k \in \mathbf{F}_k$. Define $r_k(F_k)$ to be the smallest number such that

$U'_k(F_k) \equiv B(\tilde{v}_k(F_k,\beta), r_k(F_k)e^{-\beta E_{k+1}})$ contains sets described below.

1. $\Omega_{m,\beta}(G)$ if $\rho^{G}_{k,\beta}(F_{k-1}) \in \mathrm{Int}\, F_k$.

2. If F has more than d+1 extremal points, let us consider any d-element collection of pairs $\{(G_{1,i}, G_{2,i}), i=1,\ldots,d\}$, where any phase corresponds to an extremal point of F, and some elements in different pairs may be the same. For any pair $(G_{1,i}, G_{2,i})$, let N_i be the solution set for the equation

$$\rho^{G_{1,i}}_{k,\beta}(F_{k-1},x) = \rho^{G_{2,i}}_{k,\beta}(F_{k-1},x)$$

Define

$$S_i = \{x : \mathrm{dist}(x,N_i) \leqslant a_i e^{-\beta E_{k+1}}\}$$

where a_i is given by Proposition 3.14. applied to the set $\{G_{1,i}, G_{2,i}\}$. It is easy to see that for any such d-element collection of pairs, $\bigcap_{i=1}^{d} S_i$ is of order k+1. We require $r_k(F_k)$ to be such that for any collection of pairs as described above, $\bigcap_{i=1}^{d} S_i \subset U'_k(F_k)$.

If F_k contains exactly d+1 functionals $\rho^{G}_{k+1}(F_{k-1})$, then we set $U_k = \emptyset$.

<u>Example</u>:

The construction of the set $U'_o(F_o)$ for our example is shown in Fig.4. We choose the collection of pairs $\{(-5),(3)\},\{(-3),(3)\}$. The thin

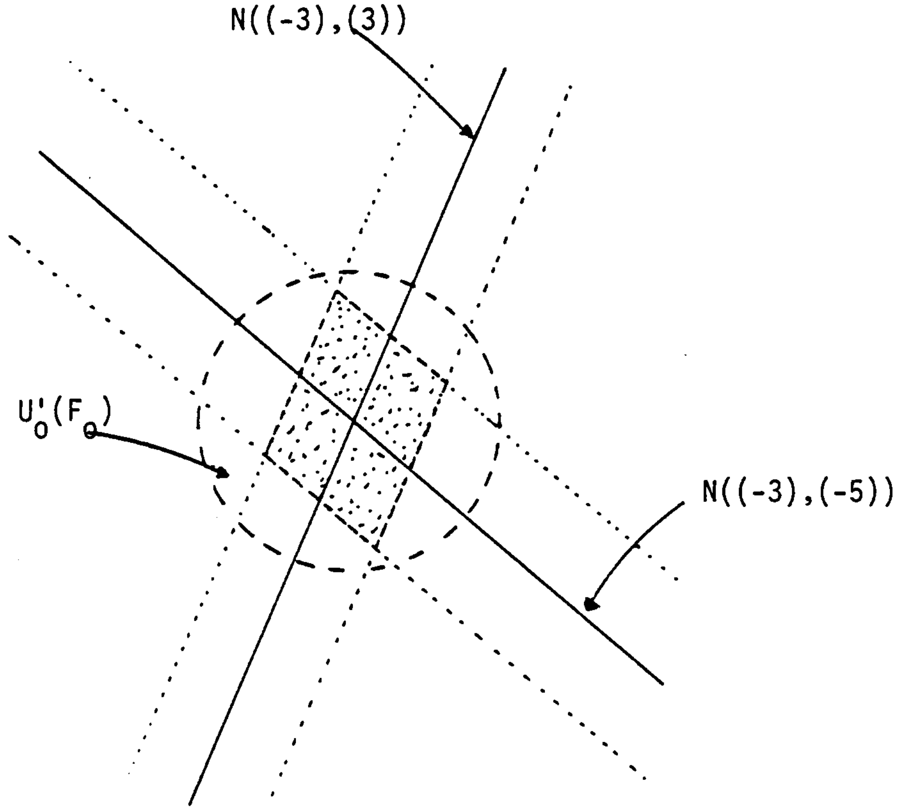


Fig.4: The construction of the set $U_0'(F_0)$ (dashed circle). The thin solid lines are N((-3),(-5)) and N((-3),(3)). The dotted lines represent sets S((-3),(-5)) and S((-3),(3)). The dotted region is their intersection.

solid lines represent sets $N((-5),(-3))$ and $N((3),(-3))$. The dotted lines show regions restricted by Proposition 3.13. The dotted region is $S((-5),(-3)) \cap S((3),(-3))$

Now let $\gamma > 0$ be any fixed number. We define $c_k(F_k) = r_k(F_k) + \gamma$. In the applications γ will be choosen in such a way that its value comprises a small fraction of any $r_k(F_k)$, $k \leqslant m$.

Let $\beta_U(\gamma)$ be such that if $\beta>\beta_U(\gamma)$, then the following conditions hold:

a) $U_k(F_k) \subset U'_{k-1}(F_{k-1})$

b) $U_k(F_k) \subset B(F_k,c)$ (with c defined by Lemma 3.15) (3.29)

c) $\forall\, F,F' \in \mathbb{F}_k(F_{k-1})$, $U_k(F) \cap U_k(F') = \emptyset$

If $\beta>\beta_U(\gamma)$, we can define a covering of βO by a family of sets:

$\{U_k(F_k),\ F_k \in \mathbb{F}_k,\ k=0,\dots,m\}$.

The phase diagram $\Omega_{m,\beta}$ in each of the sets $U_k(F_k)$ is the same as the phase diagram $\tilde{\Omega}_{m,\beta}$ for the set of phases $\mathcal{S}_k(F_k)$. We observe some sort of scaling: the procedure of determining $\tilde{\Omega}_{m,\beta}$ (or $\Omega_{m,\beta}$ in the set $U_k(F_k)$) is exactly the same as an original problem of finding $\Omega_{m,\beta}$ in the set βO. The only differences are that we deal with a smaller set of phases $\mathcal{S}_k(F_k)$, and all strata which do not intersect the boundary $U_k(F_k)$, are of order at least k+1 (in the original problem such strata are of order at least 1). The last property follows trivially from the fact that $U_k(F_k)$ is of order k+1.

In the connection with the scaling property mentioned above, let us note the following result. We will use this result in the proof of

Theorem 3.22 (next section) to determine orders of strata.

Let $\Pi(F_K)$ be a phase diagram for the set of affine functionals

$$\{\rho^G_{K+1,\beta}(F_K),\ G \in \mathcal{S}_K(F_K)\}.$$

We introduce the notation: if $\mathcal{S}' \in \mathcal{S}$, then

$$\Pi(F_K,\mathcal{S}') \equiv \Pi(\{\rho^G_{K+1,\beta}(F_K),\ G \in \mathcal{S}'\})$$

Note that for all phases in $\mathcal{S}_K(F_K)$

$$\rho^G_{K+1,\beta}(F_K,x+\tilde{v}_K(F_K,\beta)) = -\langle x,e_G\rangle + A^G_{K+1}(F_K)e^{-\beta E_{K+1}}$$

This fact allows us to draw conclusions about orders of strata of $\Pi(F_K)$.

Proposition 3.19.

Let E be a n-dimensional face of max $W_{K+1}(F_K)$ such that

$$\Pi(F_{K-1},\{G : \rho^G_{K,\beta}(F_{K-1}) \in E\}) = \{\tilde{v}_K(F_K,\beta)\}$$

Then

$$\{\Pi(F_K,\{G : \rho^G_{K+1,\beta}(F_K) \in E\}),\ \beta>\beta_K(F_K)\}$$

is of order k+1. $\beta_K(F_K)$ is given by Lemma 3.15.

Proof:

We will prove the proposition for $E = \{\rho^G_{K+1,\beta}(F_K)\}$, since in general case we can use the projection method introduced in the proof of Theorem 3.7, part 4.

Let $\rho^G_{K+1,\beta}(F_K) \in \epsilon(\max W_{K+1}(F_K))$. Since $\Pi(F_K,\{G\}) = \{\tilde{v}_K(F_K,\beta)\}$, then $\rho^G_{K,\beta}(F_{K-1}) \in \operatorname{Int} F_K$, and hence $\exists\ G_0, G_1,\ldots,G_d \in \mathcal{S}_K(F_K)$: $\rho^G_{K,\beta}(F_{K-1})$ is the convex combination of $\rho^G_{K,\beta}(F_{K-1})$'s. Let $\tilde{v}'_{K+1}$ be such that

$$\rho^{G_i}_{K+1,\beta}(F_K,\tilde{v}'_{K+1}) = \rho^{G_0}_{K+1,\beta}(F_K,\tilde{v}'_{K+1})$$

For any x in the boundary $\partial\Pi(F_K,\{G\})$ of $\Pi(F_K\{G\})$, there exists G_i such that

$$\rho^{G}_{k+1,\beta}(F_k,x) = \rho^{G_i}_{k+1,\beta}(F_k,x).$$

Then

$$\langle x - \tilde{v}'_{k+1}, e_G - e_{G_i}\rangle =$$

$$= [A^{G}_{k+1}(F_k) - \langle v'_{k+1}, e_G\rangle - A^{G_i}_{k+1}(F_k) + \langle v'_{k+1}, e_{G_i}\rangle]\, e^{-\beta E_{k+1}}$$

Hence

$$\|x - \tilde{v}'_{k+1}\| \leqslant$$

$$\leqslant \|e_G - e_{G_i}\|^{-1} \|A^{G}_{k+1}(F_k) - \langle v'_{k+1}, e_G\rangle - A^{G_i}_{k+1}(F_k) + \langle v'_{k+1}, e_{G_i}\rangle\|\, e^{-\beta E_{k+1}}$$

and $\{\Pi(F_{k+1},\{G\}),\ \beta>\beta_{k+1}(F_k)\}$ is of order at least k+1.

By Lemma 3.15, $\Pi(F_k,\{G\})$ does not intersect $\Pi(F_k,\{\tilde{G}\})$ if $\tilde{G} \notin \mathcal{S}_k(F_k)$.

Let x_o be such that $e^{-\beta E_{k+1}} x_o + \tilde{v}'_{k+1} \in \Pi(F_k,\{G\})$. Then $\forall\ \tilde{G} \in \mathcal{S}_k(F_k)$

$$A^{G}_{k+1}(F_k) - \langle x_o, e_G\rangle - A^{\tilde{G}}_{k+1}(F_k) + \langle x_o, e_{\tilde{G}}\rangle \equiv \gamma(\tilde{G}) > 0$$

For any $x \in \partial\Pi(F_k,\{G\})$ $\exists$ $\tilde{G}$ such that

$$\gamma(\tilde{G}) e^{-\beta E_{k+1}} = \rho^{G}_{k+1,\beta}(F_k,x_o) - \rho^{\tilde{G}}_{k+1,\beta}(F_k,x_o) =$$

$$= \rho^{G}_{k+1,\beta}(F_k,x_o) - \rho^{G}_{k+1,\beta}(F_k,x) - \rho^{\tilde{G}}_{k+1,\beta}(F_k,x_o) + \rho^{\tilde{G}}_{k+1,\beta}(F_k,x) \leqslant$$

$$\leqslant \|e_G - e_{\tilde{G}}\|\ \|x - x_o\|$$

Thus $\{\Pi(F_{k+1},\{G\}),\ \beta>\beta_k(F_k)\}$ is of order not higher than k+1. ∎

Example:

Let us apply Theorem 3.7 to the set $\{(e_{(k)}, (k) \in \mathcal{S}\}$. We obtain zero-temperature (and zero-order) phase diagram (Fig.6a, p.78). Next let us consider the set $\{(e_{(k)}, A_1^{(k)}e^{-4\beta}), (k) \in \mathcal{S}\}$. The resulting phase diagram is represented in Fig.7. Finally, Fig.9a contain the blow-up of the central part (around zero) of the phase diagram for the set $\{\rho_7^{(k)}(F), (k) \in \mathcal{S}\}$.

In our example all lines

$$\Pi(F_0, \{(k),(j)\}) \;, \; j,k \neq -1,1$$

are of order 0. The lines

$$\Pi(F_1, \{(1),(k)\}) \text{ and } \Pi(F_1, \{(-1),(k)\}) \quad , \quad k \neq -1,1$$

are of order 1. The line

$$\Pi(F_7, \{(1),(-1)\}$$

is of order 7.

6. Phase diagrams in order m.

In this section we present the construction of the phase diagram in order m. We show that if the set maxW of Section 3 is replaced by the convex structure in order m, the result holds for $\Omega_{m,\beta}$, similar to Theorem 3.7.

Let us first describe the restrictions of the method. The first problem arises in connection with the existence of strata. This problem has been already mentioned in Section 1. Suppose that E is a n-dimensional face of max $W_{k+1}(F_k)$. By $\mathcal{S}_{k+1}(E)$ we denote the set of phases G such that $\rho^G_{K+1,\beta}(F_K) \in E$. Then $\Pi(F_K, \mathcal{S}_{K+1}(E))$ exists. However, if $\mathcal{S}_E$ does not satisfy conditions imposed by Proposition 3.2. (or, in the presence of symmetries, by Proposition 3.24), then the solution for the system of equations (3.8) (and hence $\Omega_{m,\beta}(\mathcal{S}_{K+1}(E))$) does not generally exist. Therefore we have to make an additional assumption about the convex structure:

<u>Assumption</u> <u>3.20</u>:

Let $k \leqslant m$. If E is a face (extremal edge) of $\max W_1(F)$ ($F \in \mathbb{F}_{1-1}$) with $\dim E < d$, E contains only $\dim E + 1$ functionals $\rho^G_{K,\beta}(F_{K-1})$).

The example of the system, in which the above assumption does not hold, is given in Section 4.1A.

The next problem is as follows. Let us consider $U_m(F)$, $F \in \mathbb{F}_m$. Inside $U_m(F)$, the form of τ^G_m is determined by terms of order higher than m. These terms have no correlation to formal pressure coefficients. Hence the phase diagrams in higher orders will generally have different properties than $\Omega_{m,\beta}$. Thus information contained inside $U_m(F)$ is superfluous. We dispose of it by the means of the following definition.

<u>Definition</u>: Let $\mathcal{S}' \subset \mathcal{S}$. We say that the function $x(\beta)$ is K-equivalent with constant b to $\Omega_{m,\beta}(\mathcal{S}')$ if $\forall\, \mathcal{S}'' \subset \mathcal{S}' : \Omega_{m,\beta}(\mathcal{S}'') \neq \emptyset$,

$$\Omega_{m,\beta}(\mathcal{S}'') \subset B(x(\beta), be^{-\beta E_{K+1}}).$$

<u>Note</u>: In general $\Omega_{m,\beta}(\mathcal{S}')$ does not exist.

We have already shown (Section 5) that the phase diagram $\Omega_{m,\beta}$ can be constructed separately in sets $U'_k(F_k) \setminus \cup U_k(F)$, with the union over faces in $\mathbb{F}_{k+1}(F_k)$. Note that if the system satisfies Assumption 3.20, then $\Omega_{m,\beta}$ has a natural structure of a stratified manifold: each stratum is a manifold by Proposition 3.2, and evidently the closure of any $\Omega_{m,\beta}(\mathcal{S}_o)$ contains this stratum and all $\Omega_{m,\beta}(\mathcal{S}')$ with $\mathcal{S}' \supset \mathcal{S}_o$ (which are of lower dimension). $\Pi(F_k)$ also is a stratified manifold. We will show that there is a relation between $\Omega_{m,\beta}$ and $\Pi(F_k)$.

Let us consider a covering of $\beta\bar{O}$ by the family of sets defined in a following way. Let $F \in \mathbb{F}_k(F_{k-1})$, and $\mathcal{S}' \subset \mathcal{S}_k(F)$. We say that $\mathcal{S}'$ is <u>normal</u> if:

1. $|\mathcal{S}'| = d+1$
2. all elements of $\mathcal{S}'$ correspond to extremal points of F_k;
3. $O'(F,\mathcal{S}') \equiv \mathrm{Int}\,[\, U_{k-1}(F') \cap \bigcup_{G \in \mathcal{S}'} \Pi(F,G)\,]$ is connected.

For any normal subfamily of $\mathcal{S}_k(F_k)$ we define

$$O(F,\mathcal{S}') = \cap\, \{x \in O' : \mathrm{dist}\,(x,\Pi(F,G')) \leq a_k(G')\, e^{-\beta E_{k+1}}\} \setminus U'_k(F_k)$$

The intersection here is over all $G' \notin \mathcal{S}'$ which have common boundary with some $G \in \mathcal{S}'$, and $a_1(G')$ is given by Proposition 3.14 for the family $\mathcal{S}_o \equiv \{G,G'\}$.

Obviously the family of sets:

$\{O(F,\mathcal{S}'),\ F \in \mathbb{F}_k,\ \mathcal{S}'$ normal subset of $\mathcal{S}_k(F),\ k=0,\ldots,m\}$

covers $\beta\bar{O}$.

<u>Example</u>:

In our example the covering is as follows. Only $U_o(F_o)$ and $U_7(F)$ are nonempty. The normal subfamilies of $\mathcal{S}$ are: $\{(5),(3),(-3)\}$, $\{(3),(-5),(-3)\}$ $\{(-5),(-3),(5)\}$, $\{(-5),(3),(5)\}$.

The set $O(F_o,\{(5),(3),(-3)\})$ is dotted on Fig.5a.

Note that $\mathcal{S}_1(H_1)$, $\mathcal{S}(G_1)$, $\mathcal{S}(H')$ and $\mathcal{S}(G')$ are normal.

The normal subfamilies of $\mathcal{S}_1(F)$ are: $\{(-1),(3),(1)\}$, $\{(3),(1),(-3)\}$, $\{(1),(-3),(-1)\}$, $\{(-3),(-1),(1)\}$

The set $O(F;\{(-1),(3),(1)\})$ is dotted on Fig.5b.

Finally both $\mathcal{S}_7(F_1)$ and $\mathcal{S}_7(F_2)$ are normal.

<u>Lemma 3.21</u>

Let $Z = U_k'(F_k) \setminus \cup U_{k+1}(F)$, with the union over faces in $\mathbf{F}_{k+1}(F_k)$. Then $\exists\ \beta(F_k)$: $\forall\ \beta > \beta(F_k)$ the following holds:

1. There is a one-one correspondence between strata of $\Pi(F_k) \cap Z$ and strata of $\Omega_{m,\beta} \cap Z$: $\Pi(F_k,\mathcal{S}_o) \to \Omega_{m,\beta}(\mathcal{S}_o)$. This correspondence preserves the closure, i.e. the elements of the closure of $\Pi(F_k,\mathcal{S}_o)$ correspond to the elements of the closure of $\Omega_{m,\beta}(\mathcal{S}_o)$.

2. $\exists\ a(F_k) > 0$: $\mathrm{dist}(\Pi(\mathcal{S}') \cap Z\ ,\ \Omega_{m,\beta}(\mathcal{S}') \cap Z) \leqslant a(F_k)e^{-\beta E_{k+1}}$.
3. If $\Pi(F_k,\mathcal{S}_o) \subset U_k'(F_k)$, then $\Omega_{m,\beta}(\mathcal{S}_o)$ is of order $k+1$.

<u>Proof</u>:

The family of sets $\{O(F_k,\mathcal{S}'),\ \mathcal{S}'$ normal in $\mathcal{S}_k(F_k)\}$ covers Z. By Proposition 3.6, $\Omega_{m,\beta}$ and $\Pi(F_k)$ are diffeomorphic in every $O(F_k,\mathcal{S}')$. Consider a stratum $\Pi(F_k,\mathcal{S}_o)$ with dimension $r < d$. There is a face of $\max W_{k+1}(F_k)$ of dimension $d-r$, corresponding to $\Pi(F_k,\mathcal{S}_o)$. By Assumption 3.19, $|\mathcal{S}_o| \leqslant d$, and it contains only phases corresponding to extremal points of $\max W_{k+1}(F_k)$. Hence there exists a normal subset $\mathcal{S}'$ such that $\mathcal{S}_o \subset \mathcal{S}'$. By Proposition 3.6, $\Omega_{m,\beta}(\mathcal{S}_o) \cap O(F_k,\mathcal{S}')$ exists. The

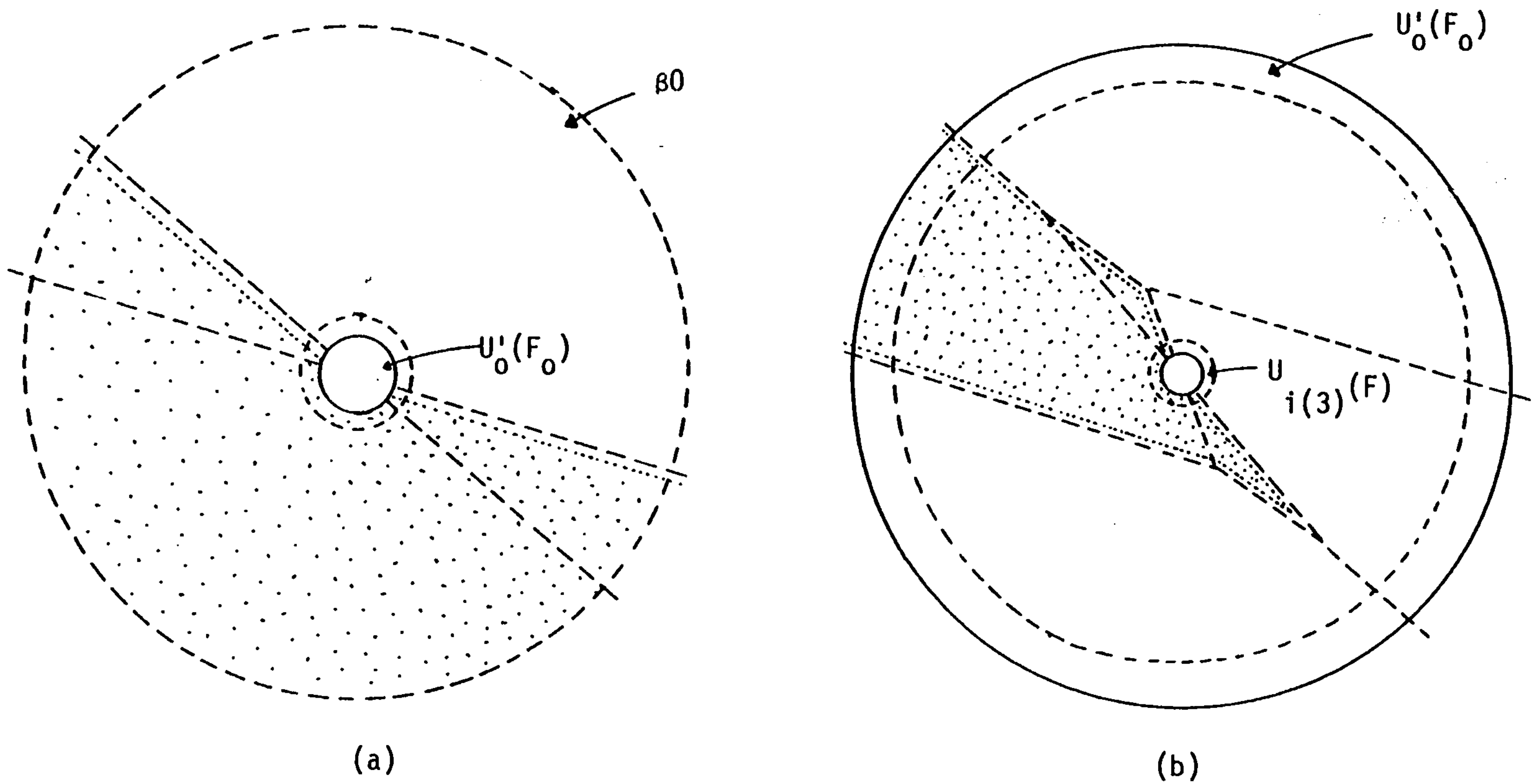


Fig.5: The elements of covering of βO: (a) $O(F_o,\{(5),(3),(-3)\})$;
(b) $O(F,\{(-1),(3),(1)\})$.
The dashed lines separate domains for different phases. The dotted lines show restrictions imposed by Proposition 3.14.

elements of the closure of $\Omega_{m,\beta}$, which lie inside $O(F_k,\mathcal{S}')$, obviously correspond to the elements of the closure of $\Pi(F_k,\mathcal{S}_o)$ which are inside $O(F_k,\mathcal{S}')$ (since $\Omega_{m,\beta}$ and $\Pi(F_k)$ are diffeomorphic inside $O(F_k,\mathcal{S}')$). Note that $\Omega_{m,\beta}(\mathcal{S}_o) \subset \bigcup_{\mathcal{S}'\supset\mathcal{S}_o} \Omega_{m,\beta}(\mathcal{S}')$, and the same holds for $\Pi(F_k,\mathcal{S}_o)$. Hence for every element of the closure of $\Omega_{m,\beta}(\mathcal{S}_o)$ we can find a set $O(F_k,\mathcal{S}')$ which contains this element. Thus by the above argument, there exists the corresponding element of the closure of $\Pi(F_k,\mathcal{S}_o)$.

If F_k contais only d+1 functionals $\rho_k^G(F_{k-1})$, the there is an obvious correspondence between $\Pi(F_k,\mathcal{S}_k(F_k))$ and $\Omega_{m,\beta}(\mathcal{S}_k(F_k))$ (both sets contain one point).

The bound on the distance of corresponding strata can be easily established by applying Proposition 3.14 to any $\mathcal{S}_o$ for which $\Omega_{m,\beta}(\mathcal{S}_o)$ is nonempty. Here $a(F_k)$ is a maximum over a_k's for all such $\mathcal{S}_o$ (a_k is given by Proposition 3.14).

The combination of Lemma 3.16 and Proposition 3.18 provides for the proof of the statement about orders of the strata. ∎

The following theorem is the main result of this paper.

Theorem 3.22.

Suppose that the system satisfies Assumption 3.20 for all orders up to the order m. Then $\exists\ \beta_m : \forall\ \beta>\beta_m$ the following holds:

1. $\forall\ F \in \mathbb{F}_m \quad \exists!\ v(F)$:

   a) if F contains only d+1 fonctionals $\rho_{m,\beta}^G(F_{m-1})$, then $v(F)$ is the unique element of $\Omega_{m,\beta}(\mathcal{S}_m(F))$,

b) If F is other than in a), then $\exists\, b > 0$ such that $v(F)$ is m-equivalent with constant b to $\Omega_{m,\beta}(\mathcal{S}_m(F))$.

2. Let $E \subset \max W_m(F')$ $(F' \in \mathbb{F}_{m-1})$ be such that $\dim E = d-r$, $r \geq 1$. Then $\Omega_{m,\beta}(\mathcal{S}_m(E)) \neq \emptyset$, and has dimension r. In particular, $\Omega_{m,\beta}(G) \neq \emptyset$ if and only if $\rho^G_{m,\beta}(F') \in \epsilon(\max W_m(F'))$.

3. The order of stratum $\Omega_{m,\beta}(\mathcal{S}_m(E))$ is given by

   $k = \min\{t : E \subset \max W_t(F),\ F \in \mathbb{F}_{t-1}\}$

4. $\forall\, s \geq m \quad \exists\, \beta_s : \forall\, \beta > \beta_s$, 1a), 2 and 3 hold for corresponding strata of $\Omega_{s,\beta}$.

Proof:

First note that due to the Assumption 3.19, if E is a r-dimensional face of $\max W_{k+1}(F_k)$ $(r<d)$, then for any $s \geq k+1$, E is a r-dimensional face of $\max W_{s+1}(F_s)$. The same holds for a d-dimensional face F if F contain only d+1 functionals ρ^G_k. Let k be the lowest order in which E is an extremal element of $\max W_k(F_{k-1})$. By Theorem 3.7, there exists a stratum $\Pi(F_k, \mathcal{S}_k(E))$ of $\Pi(F_k)$ which corresponds to E. By Lemma 3.21, there is a stratum $\Omega_{m,\beta}(\mathcal{S}_k(E))$ which corresponds to $\Pi(F_k, \mathcal{S}_k(E))$. Moreover, $\Omega(F_k, \mathcal{S}_k(E))$ has order $k+1$ (by Proposition 3.18) and hence $\Omega_{m,\beta}(\mathcal{S}_k(E))$ also has order $k+1$ (by Lemma 3.21).

If E contais more than d+1 affine functionals, then in general there is no correspondence. However, in higher orders E is replaced by $\max W_{k+1}(E)$.

In the last order we are left with sets $U_m(F)$, $F \in \mathbb{F}_m$, in which Lemma 3.21 cannot be applied. But since every $U_m(F)$ is of order $m+1$, then $\Omega_{m,\beta} \cap U_m(F)$ is m-equivalent with constant $c_m(F)$ to the line of coexistence $\tilde{v}_m(F)$.

With the change of order m we have to redefine sets $U_k(F_k)$ and sets $O(F_k,\mathcal{S}')$ since there will be a change in estimation (3.23). By lowering temperature we can compensate for these changes, so the above considerations hold for a construction of $\Omega_{s,\beta}$ with $s \geqslant m$. The only exception is the statement 1b., since now we can investigate phase diagram in any $U_m(F_m)$ in the same manner as we did in sets $U_k(F_k)$ in lower orders.

The value of β_m is determined as the maximum of the following:

1. β_1' given as the condition that the sets $U_k(F_k)$ do not intersect (cf.(3.29));
2. $\beta(F_k)$ defined by Lemma 3.21;
3. $\beta_m(\{G,G'\})$ (as in Proposition 3.2.) for all pairs of phases, corresponding to extremal points which share a 1-dimensional extremal edge. The condition: $\beta>\beta_m(\{G,G'\})$ assures the existence of stratum $\Omega_{m,\beta}(G,G')$ and hence validates an approximation (3.23) which enters into the definition of $O(F,\mathcal{S}')$.

Obviously β_m is finite. ∎

<u>Remark</u> <u>3.23</u>

Let us assume that $\mathbf{F}_m$ has the following property: $\forall\ F \in \mathbf{F}_m(F_{m-1})$, F contains only d + 1 functionals $\rho^G_{m,\beta}(F_{m-1})$. By Remark 3.11, for any $s \geqslant m$, $\mathbf{F}_s$ is isomorphic (as a collection of convex sets) to $\mathbf{F}_m$. Also, since every $U_m(F_m) = \emptyset$, the phase diagram Ω_s is completely determined by extremal properties of $\mathbf{F}_m$. We will say that order m is <u>conclusive</u>.

Note that all features of phases diagrams in orders lower than m are obtained from $\Omega_{m,\beta}$ in a following way: one finds sets $U_k(F_k)$, $F_k \in \mathbb{F}_k$. All strata of $\Omega_{m,\beta}$ which lie inside $U_k(F_k)$, are then identified with the point $\tilde{v}_k(F_k)$. The remaining strata of $\Omega_{m,\beta}$ correspond to strata of $\Omega_{k,\beta}$.

<u>The description of the phase diagram in order m.</u>

Let $\beta_s' > \beta_s$ be such that if $\beta > \beta_s'$, Proposition 3.3 holds for any nonempty stratum $\Omega_{s,\beta}(\mathcal{G}')$, and any subset $\mathcal{G}''$ of $\mathcal{G}'$ with $|\mathcal{G}''| \geq 2$. Obviously, β_s' is finite. The phase diagram in order m is described as follows. For every $F \in \mathbb{F}_m$, there exists a point $v(F)$ which plays the role of the unique point of coexistence of phases from $\mathcal{G}_m(F)$. Next, for any (d-1)-dimensional face E_1 of F, there is a 1-dimensional surface of coexistence of phases in $\mathcal{G}_{E_1}$ which either terminates at the boundary of

$\beta\bar{O}$ (if some phases in $\mathcal{G}_E$ correspond to elements of $\epsilon(\mathrm{conv}\{e_G, G \in \mathcal{G}\})$), or terminates at another point of coexistence $v(F')$ (with $F' \supset E_1$). Furthermore, for any (d-2)-dimensional edge E_2, there exists a 2-dimensional surface of coexistence of phases from $\mathcal{G}_{E_2}$. This surface is bounded by the set of lines which are the surfaces of coexistence of phases from $\mathcal{G}_{E'}$, for any $E' \supset E_2$. In addition, if some element of $\mathcal{G}_{E_2}$ corresponds to the element of $\epsilon(\mathrm{conv}\{e_G, G \in \mathcal{G}\})$, then one of the boundaries is the boundary of $\beta\bar{O}$. One can make similar statements about strata of higher dimensions.

The most convenient way of representing the phase diagram is to show it separately in the blow-ups of sets $U_k(F)$, $F \in \mathbb{F}_k$. In each of these sets we can apply Lemma 3.21 to obtain the topology and localization of strata. Finally, Proposition 3.3. gives more detailed description of strata near their boundaries.

Example:

The phase diagram for our example is shown in Fig.6b,8 and 9b. Since $\Omega_{m,\beta}$ is constructed order by order, we represent it in the succesive blow-ups of sets $U_k(F_k)$.

As it has been discussed before (Section 4), W_o has four extremal points: (-5),(5),(-3),(3), and four edges: {(-5),(-3)} , {(-5),(3)} , {(5),(-3)} , {(5),(3)} (Fig.3a).

Hence on $\beta O \setminus U_o(F_o)$, $\Omega_{m,\beta}$ has four lines of two-phase coexistence: $\Omega_{m,\beta}((-5),(-3))$, $\Omega_{m,\beta}((5),(-3))$, $\Omega_{m,\beta}((5),(3))$, $\Omega_{m,\beta}((5),(3))$ (Fig.6b). The shaded areas show the regions, to which strata listed above are restricted by Proposition 3.13. Numbers appearing close to strata denote their orders (in Fig.6b all orders are zero).

Next, $\Omega_{m,\beta} \cap U_o(F_o)$ is shown in Fig.8. It is easy to see the correspondence between strata of $\Omega_{m,\beta}$, strata of $\Pi(F_o)$, and extremal features of $\max W_1$ (cf. Fig.3b). Finally, the blow-up of $U_7(F)$ is shown in Fig.9b. Since $\mathbb{F}_i \sim \mathbb{F}_7$ if $i \geq 7$, the phase diagram of Fig.6b,8 and 9b is representative for all orders higher than 7.

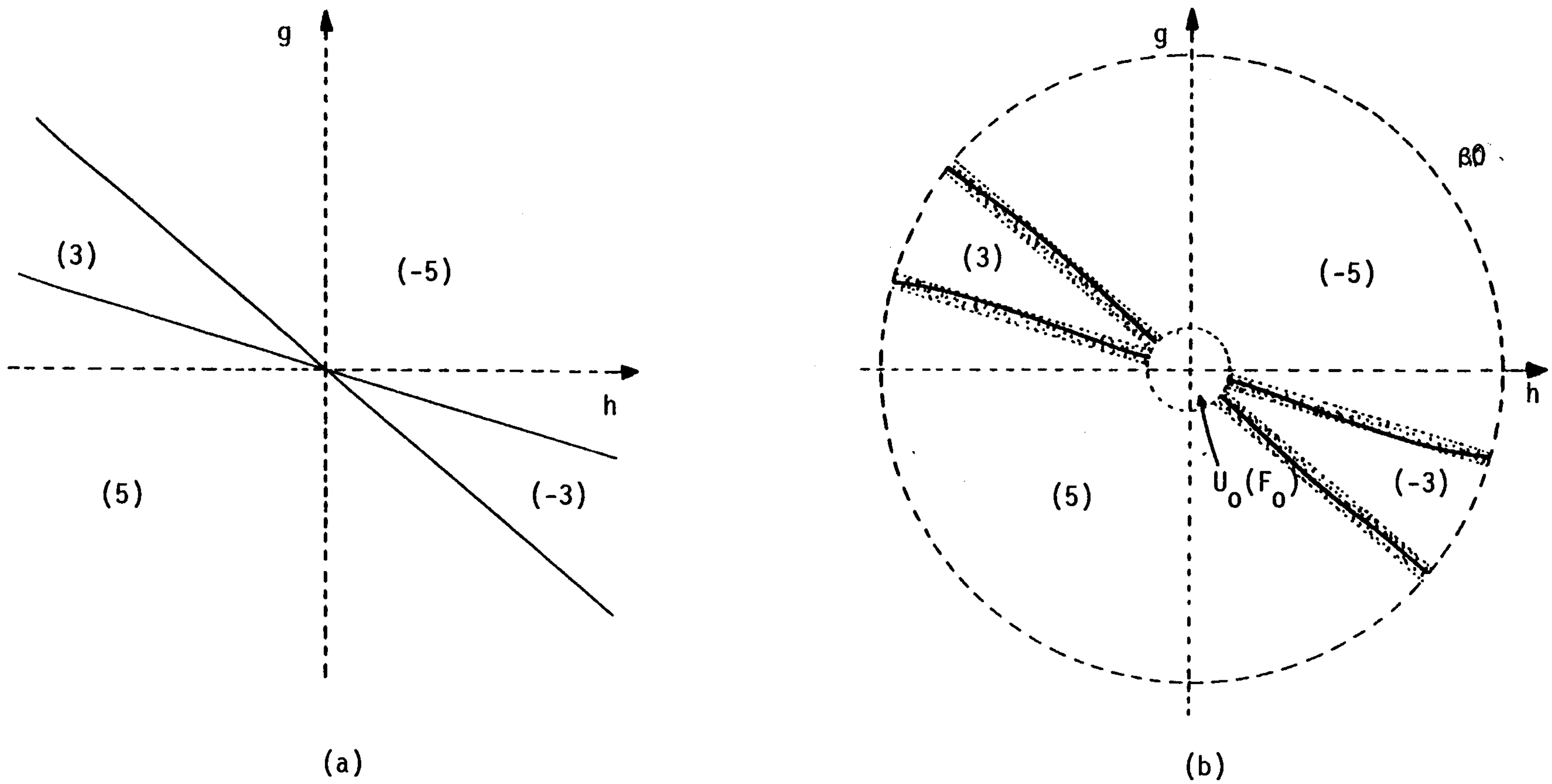


Fig.6: (a) The phase diagram Π for the set of functionals $\{e_{(k)}\}$;
(b) The phase diagram $\Omega_{m,\beta}$ in $\beta 0 \quad U_o'(F_o)$. The dotted regions represent the restrictions imposed by Proposition 3.14.

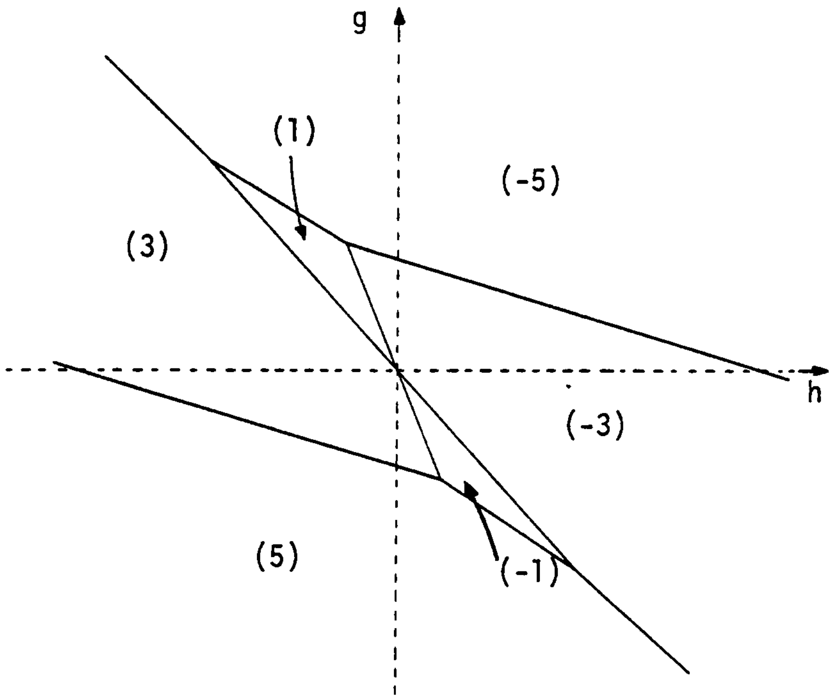


Fig.7: The phase diagram $\Pi(F_o)$ for the set of affine functionals $\{(e_{(k)}, A_1^{(k)})\}$

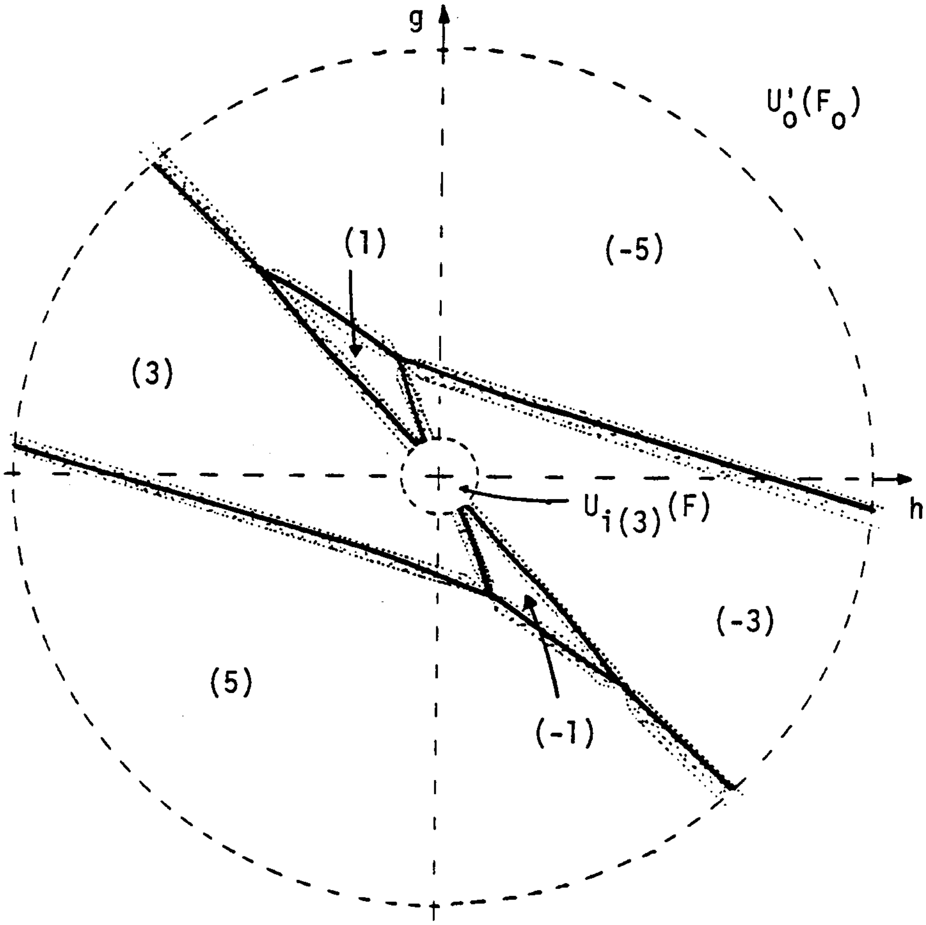


Fig.8. The phase diagram $\Omega_{m,\beta}$ in the set $U_o'(F_o)$ $U_{i(3)}(F)$

The dotted regions represent the restrictions imposed by Proposition 3.14.

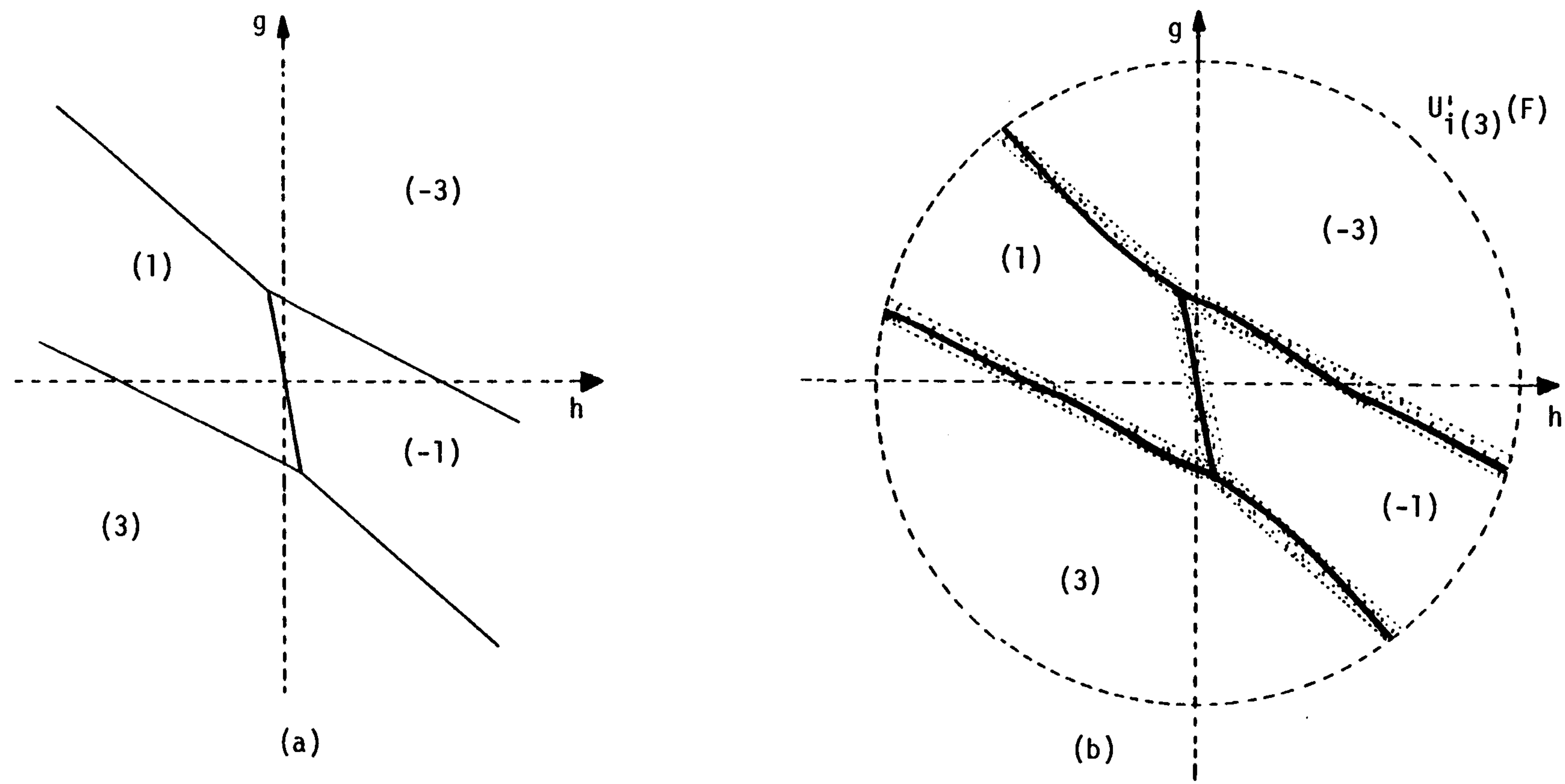


Fig.9: (a) The phase diagram for the set of affine functionals $\{(e_{(k)}, A^{(k)}_{i(3)}(F))\}$

(b) The phase diagram $\Omega_{m,\beta}$ inside the set $U'_{i(3)}(F)$

7. Phase diagrams in the presence of symmetries.

Assumption 3.20 can be relaxed in the presence of symmetries of an original Hamiltonian H_o.

Let R be the transformation group acting on the lattice, and Q be the group of transformation acting on $\mathcal{X}$ pointwise: if Q_o is a group of transformations of S, then $Q = Q_o^{\mathbf{L}}$. The transformations of $\mathcal{X}$ form a group given by the semidirect product R*Q with the action defined by

$$(r_1,q_1)*(r_2,q_2) = (r_1+ r_2,q_1+ r_1(q_2))$$

where $r(q) \in Q$ is given by $(r(q))_a = q_{r(a)}$

The subgroup $\Theta \subset R*Q$ is the <u>symmetry group</u> of the Hamiltonian H_o if $\forall\ \theta \in \Theta$, $\forall\ Y\sim X$

$$H_o(\theta X|\theta Y) = H_o(X|Y)$$

Θ induces the group of transformations T acting on $\mathcal{L}$ in the following way: if $\theta \in \Theta$ and $L \in \mathcal{L}$, then $\forall\ X\sim Y$

$$T_\theta L(X|Y) = L(\theta X|\theta Y)$$

T in turn induces the group of transformations T^* of $\mathcal{L}^* \times \mathbb{R}$:

if $\theta \in \Theta$, then

$$T_\theta^*(h,a) \equiv (T_\theta^* h,a)$$

where $T_\theta^* h$ is defined as follows: $\forall\ x \in \mathcal{L}$

$$\langle x,T_\theta^* h\rangle = \langle T_\theta x,h\rangle$$

It is easy to see that $\forall\ G \in \mathcal{X}^{per}$ $\langle T_\theta L,e_G\rangle = \langle L,e_{\theta G}\rangle$.

Obviously $\forall\ G \in \mathcal{G}$, $\forall\ \theta \in \Theta$, $\forall\ x \in \beta O$

$$n_m^{\theta G}(x) = n_m^G(T_\theta x).$$

Therefore for any m

$$\pi_m^{\theta G}(x,\beta) = \pi_m^{G}(T_\theta x,\beta)$$

It follows that $\Omega_{m,\beta}$ is invariant with respect to T.

Suppose now that $F \in \mathbb{F}_m(F_{m-1})$, and let $\Theta(F)$ be a symmetry group of $\mathcal{S}_m(F)$: $\forall\ \theta \in \Theta(F)$, $T_\theta{}^*F = F$

Define $\mathcal{L}(F) = \{L : \forall\ \theta \in \Theta(F)\ T_\theta L = L\}$, $\mathcal{L} = \mathcal{L}(F) \oplus \mathcal{L}'(F)$.

<u>Claim</u>: If $\mathcal{S}_m(F)$ has p+1 orbits with respect to $\Theta(F)$,

then $p - \dim \mathcal{L}(F) \geq 0$

<u>Proof</u>: Let n_i+1 be the number of elements in the i-th orbit. Then

$$\sum_{i=0}^{p}(n_i+1) = |\mathcal{S}_m(F)| \geq d+1$$

(since a face has at least d+1 extremal points). Also

$$d - \dim \mathcal{L}(F) \leq \max_{0\leq i\leq p} n_i$$

Then

$$p - \dim \mathcal{L}(F) \geq p - \dim \mathcal{L}(F) + \sum_{i=0}^{p} n_i - (d - \dim \mathcal{L}(F)) =$$

$$= p + \sum_{i=0}^{p} n_i - d = \sum_{i=0}^{p}(n_i+1) - (d+1) \geq 0$$

Suppose now that $\mathcal{S}_m(F)$ has $p = \dim \mathcal{L}(F) + 1$ orbits with respect to $\Theta(F)$. Let G_i, $i=0,\dots,p$ be representatives of the orbits.

Proposition 3.24.

$\exists\ \beta_m(F) : \forall\ \beta > \beta_m(F)$, $\Omega_{m,\beta}(G_m(F))$ exists and is contained in $\mathcal{L}(F)$.

Proof:

First note that $\forall\ \theta \in \Theta$, $\forall\ G \in \mathcal{S}_m(F)$, $\mathcal{L}(F) \subset \ker(e_G - e_{\theta G})$. Suppose that $L \in \mathcal{L}'(F)$ and $\forall\ \theta \in \Theta(F)$, $\forall\ G \in \mathcal{S}_m(F)$, $e_G(L) = e_{\theta G}(L)$, i.e.

$$\langle \theta L - L, e_G \rangle = 0$$

Note that $\{e_{G_i},\ i=0,\dots,p\}$ is linearly independent, and it spans $\mathcal{L}'(F)^*$. Hence $\theta L = L$, i.e. $L = 0$. Thus $\mathcal{L}(F) = \bigcap_{G,\theta} \ker(e_G - e_{\theta G})$.

Consider now the system of equations:

$$\pi_m^G(x,\beta) - \pi_m^{G'}(x,\beta) = 0 \qquad G,G' \in \mathcal{S}_m(F)$$

This system can be separated into two sets of equations:

$$\langle x, e_G - e_{G_i} \rangle, \quad i=0,\dots,p \quad G \text{ in } i\text{-th orbit} \tag{3.30}$$

and

$$\pi_m^{G_i}(x,\beta) - \pi_m^{G_o}(x,\beta) = 0 \qquad i=1,\dots,p$$

The solution set of the first system of equations is $\mathcal{L}(F)$, hence one can choose $d - \dim \mathcal{L}(F)$ linearly independent equations of type (3.30). The second set is linearly independent. Hence by the dimension argument one can apply the implicit function theorem. ∎

Note that $\mathbb{F}_{m+1}(F) = \{F_{m+1}\}$, and $F_{m+1} \sim F$, so the proposition holds by induction for all $s \geq m$.

Next suppose that the following holds:

<u>Assumption 3.25.</u>

Let E be a r-dimensional face (extremal edge) of F, $\Theta(E) \subset \Theta(F)$ be the symmetry group for E, and $\mathcal{L}(E)$ a subspace invariant with respect to $\Theta(E)$. Then $\mathcal{G}_m(E)$ has $s = r + 1 - \dim \mathcal{L}(E)$ orbits.

If the above assumption is satisfied, then by the argument similar as in Proposition 3.24, $\Omega_{m,\beta}(\mathcal{G}_E)$ exists and is contained in $\mathcal{L}(E)$.

Now assume that every edge $E \subset \max W_K(F_{K-1})$, $F_{K-1} \in \mathbb{F}_{K-1}$, $\mathcal{G}_E$ satisfies the above condition. Then Theorem 3.22. holds with the small modification of the covering:

$$O(F,\mathcal{G}') = O'(F,\mathcal{G}') \setminus \Delta'$$

where

$$\Delta' = \{x \in O'(F,\mathcal{G}') : \text{dist}\,(x,\Pi(F_K,G') < a_K(G,G')e^{-\beta E_{K+1}}$$
$$\text{unless } G' = \theta G \ , \ \theta \in \Theta(F)\}$$

here $a_K(G,G')$ is given by Proposition 3.14.

Since the stratum $\Omega_{m,\beta}(\{G,\theta G\})$ is contained in $\ker(e_G - e_{\theta G})$, this definition of $O(F,\mathcal{G}')$ provides for the set in which Proposition 3.6 can be applied. The example of the system with symmetries is given in Section 4.1B.

CHAPTER 4: Examples

This chapter contains examples. In Section 1 we provide the full argument about the LT expansions for the Blume-Capel model. Using different perturbation spaces with this model, we also present examples for systems, where Assumption 3.20 does not hold (Section 4.1A), and for the system with symmetries (Section 4.1B). The next section contains a simple case of another class of models, in which a complication of interaction produces arbitrary properties of the phase diagram. In Section 3 we present a general discussion of phase diagrams for the generalized ferromagnetic models. Finally, in Section 4 we describe models with stabilization, and discuss the application of our method to the systems with infinite number of ground states.

1. The Blume-Capel model

This model has been first discussed in [6],[7]. Since we have used this model throughout the previous chapter to illustrate our method, we present here all information about the LT expansions which is necessary to determine the phase diagram.

Let $\mathbb{L} = \mathbb{Z}^2$, $S = \{-q, -q+1, \ldots, q-1, q\}$. Define interaction Φ:

$$\Phi_{\{a,b\}} = (s_a - s_b)^2 \text{ if } a,b \text{ are nearest neighbors, } \Phi_\Lambda = 0 \text{ otherwise.}$$

The Hamiltonian has the form:

$$H_o = \sum_{\langle a,b\rangle} (s_a - s_b)^2$$

where $\langle a,b\rangle$ denotes a pair of nearest neighbors.

Φ is obviously $\mathbb{Z}^2$-invariant. It is also invariant with respect to the operation of the spin flip:

$$f: \mathcal{X} \to \mathcal{X} : \ (fs)_a = -s_a \ \forall a \in \mathbb{L}$$

The group $\{e,f\}$, where e is identity, in the only pointwise acting group of symmetries of H_o.

The set of periodic ground states is:

$$\mathcal{G} = \{G_k \equiv (k),\ k = -q,\ldots,q : \forall a \in \mathbb{Z}^2\ (G_k)_a = k\}$$

By (m) we will denote the ground state corresponding to the minimal $|k|$ ($m = 0$ if q is integer, $m = \frac{1}{2}$ if q is half-integer).

Let $(k) \in \mathcal{G}$. Define

$$\mathcal{X}_r^{(k)} = \{X \in \mathcal{X}^{(k)} : \max_{\mathrm{supp}X} |X_a - k| = r\}$$

<u>Claim</u>: $q - |k| \geqslant r \Rightarrow \mathcal{X}_r^{(m)} \sim \mathcal{X}_r^{(m)}$ (set isomorphism)

<u>Proof</u>: Let $(k) \in \mathcal{G}$, without loss of generality we may assume that $k \geqslant 0$ (the case $k \leqslant 0$ is obtained by symmetry). Consider the map

$$\varphi : \mathcal{X}_r^{(m)} \to \mathcal{X}_r^{(k)} : \varphi(X)_a = X_a - m + k.$$

Obviously φ is an isomorphism.

<u>Note</u>: If $q - k < r$, then the above claim fails. Suppose that $r \leqslant q - m$. Let $X \in \mathcal{X}_r^{(m)}$. Define

$$\bar{X} \in \mathcal{X}^{(m)} : |\bar{X}_a - X_a| = 2|X_a - m|$$

Let a be such that $|X_a - m| = r$. Then

$$|\varphi(\bar{X})_a - k + (\varphi(X)_a - k)| = 2r$$

and since $|\varphi(X)_a - k| \leqslant q - k < r$, then

$$|\varphi(\bar{X})_a - k| > r \geqslant q - k$$

Hence $\varphi(\bar{X}) \notin \mathcal{X}$, and φ is not into.

<u>Low temperature expansion for H_o (no perturbation)</u>

First note that H_o satisfies condition (2.9). Let $(k) \in \mathcal{S}$. For any $X \in \mathcal{X}^{(k)}$, let $\tilde{X}$ be such that

$$\forall\ a \in \text{supp } X \quad \tilde{X}_a = k + 1 \quad , \quad \tilde{X}_a = k \quad \text{otherwise}$$

(if $k = q$, then take $\tilde{X}_a = q - 1$). Obviously

$$H_o(\tilde{X}|(k)) \leqslant H_o(X|(k))$$

But H_o restricted to the set $\{X : |X_a - k| \leqslant 1\}$ is isomorphic to the Hamiltonian for the Ising model, and $\{\tilde{X} : X \in \mathcal{X}^{(k)}\}$ is isomorphic to the set of excitations from the ground state of Ising model. Hence

$$H_o(\tilde{X}|(k)) \longrightarrow \infty \qquad \text{if } |\text{supp}X| \to \infty$$

Consider now $(k) \in \mathcal{S}$ and let $X \in \mathcal{X}^{(k)}$. If $X \in \mathcal{X}_r^{(k)}$ with $r \leqslant q - |k|$, then by the Claim, X contributes to $n_i^{(k)}$ in the same way as corresponding excitation contributes to $n_i^{(m)}$. Let $E_{i(k)}$ be the lowest energy for which $\exists\ X \in \mathcal{X}_r^{(k)}$:

$$H_o(X|(k)) = E_{i(k)}$$

and such that $r > q - |k|$. If $i < i(k)$, then $\forall\ w \leqslant k$

$$n_i^{(w)}(0) = n_i^{(k)}(0) = n_i^{(m)}(0)$$

<u>Proposition</u> <u>4.1.</u>

$$a_k \equiv n^{(m)}_{i(k)}(0) - n^{(k)}_{i(k)}(0) > 0$$

<u>Proof</u>:

Let $t = q - |k| + 1$. Consider $s \in \mathcal{X}^{(k)}_r$ with $r > t$. Define

$\bar{s} \in \mathcal{X}^{(k)}_t$: supp $\bar{s}$ = supp s, if $|s_a - k| \leqslant t$ then $\bar{s}_a = s_a$,

if $|s_a - k| > t$, then $\bar{s}_a - k = t$ for $k < 0$, $\bar{s}_a - k = -t$ for $k \geqslant 0$

Let $U = \{a \in \mathbb{L} : \bar{s}_a \neq s_a\}$

Then

$$H_o(s|\bar{s}) = \sum_{(a,b)\cap U \neq \emptyset} [(s_a - s_b)^2 - (\bar{s}_a - \bar{s}_b)^2] =$$

$$= \sum_{a\in U, b\notin U} [(s_a - s_b)^2 - (\bar{s}_a - \bar{s}_b)^2] + \sum_{a,b\in U} (s_a - s_b)^2$$

In the second sum $\bar{s}_a - \bar{s}_b = 0$ by the definition of U. Note that if $a \in U$, $b \notin U$, then

$$|s_a - k| > t = |\bar{s}_a - k| \geqslant |s_b - k| = |\bar{s}_b - k|$$

Hence

$$\sum_{a\in U, b\notin U} [(s_a - s_b)^2 - (\bar{s}_a - \bar{s}_b)^2] =$$

$$= \sum_{a\in U, b\notin U} [(|s_a - k| - |s_b - k|)^2 - (|\bar{s}_a - k| - |s_b - k|)^2] \geqslant$$

$$\geqslant \sum_{a\in U, b\notin U} (|s_a - k| - t)^2$$

Obviously for any excitation there is at least one bond for which $a \in U$, $b \notin U$, therefore always

$$H_o(s|\bar{s}) = H_o(s|(k)) - H_o(\bar{s}|(k)) > 0$$

Hence one needs to investigate excitations from $\mathcal{X}^{(k)}_t$ only.

Let $X \in \mathcal{X}_t^{(m)}$ be such that $H_o(X|(m)) = E_{i(k)}$. Note that X is irreducible, hence it contributes positively to $n_i^{(m)}(0)$ (cf. algebraic method of calculating $n_i^{(k)}(0)$'s in [11]). Moreover, $\mathcal{X}_t^{(k)}$ is always isomorphic to a subset of $\mathcal{X}_t^{(m)}$. Obviously there are irreducible excitations in $\mathcal{X}_t^{(m)}$ such that

$$H_o(X|(m)) = E_{i(k)}$$

but there are no corresponding excitations in $\mathcal{X}_t^{(k)}$. ∎

The above considerations allow us to make a general statement about the form of convex structures for any type of perturbations of H_o.

Let $\mathcal{L}$ be a perturbation space transversal to $\mathcal{S}$ and such that

$$\dim \mathcal{L} \leqslant |\mathcal{S}| - 1.$$

<u>Proposition</u> <u>4.2.</u>

$\forall\, k \quad \forall\, i < i(k) \quad \exists\, \mathbb{F} \in \mathbb{F}_i$:

1. $\forall\, w : |w| \leqslant |k| \quad , (W) \in \mathcal{S}_i(F)$

2. unless order $i+1$ is conclusive, $v_i(F,\beta) \equiv 0$

<u>Proof</u>: (by induction in i)

Suppose that the thesis holds for $i < i(k)-1$.

If $i < i(s) - 1$ for some s: $|s| \leqslant |k|$, then

$$\dot{p}^{\,(w)}(0)_{i+1} \equiv n_{i+1}^{(w)}(0) = n_{i+1}^{(m)}(0)$$

and $F_{i+1} \backsim F_i$

Now suppose that $i = i(s) - 1$. Then

$$\dot{p}^{\,(w)}(0)_{i(s)} \equiv n_{i(s)}^{(w)}(0) = n_{i(s)}^{(m)}(0) - a_s\delta_{w,s}$$

Hence $\max W_{i(s)}(F_i)$ has one face which contains $\rho_{i(s)}^{(w)}(F_i)$ if $|w| < |s|$. This face is determined by the vector v which satisfies the system of equations:

$$\langle v, e_{(w)} - e_{(m)} \rangle = 0 \qquad (w) \in \mathcal{S}_{i(s)}(F_{i(s)})$$

Since by hypothesis there are at least dim $\mathcal{L}$ + 1 linearly independent points $\rho^{G}_{i(s)}$ in $\mathcal{S}_{i(s)}(F_{i(s)})$, then $v \equiv 0$.

Obviously $\rho^{(s)}_{i(s)}, \rho^{(-s)}_{i(s)} \notin F$. ∎

A) $\mathcal{L}$ is generated by interactions:

$(\Phi_1)_a = s_a$, $(\Phi_2)_a = s_a^3$, $(\Phi_i)_A = 0$ if $A \neq \{a\}$.

For $q = \frac{5}{2}$ this example has been used in Chapter 3 to illustrate the method. Let us consider the case when q is integer. This case serves as a good example of the system in which assumption 3.20. does not hold.

W_o is as follows:

if $|w| > \frac{q-1}{2}$, then $e_{(w)} \in \epsilon(W_o)$

if $|w| < \frac{k-1}{2}$, then $e_{(w)} \in \text{Int conv } \{e_{(w)}, |w| \leqslant K\}$

if $k = 2n+1$, then

$$e_{(n)} = \frac{2}{3K+1} e_{(-K)} + \frac{3K-1}{3K+1} e_{(n+1)} ,$$

$$e_{(-n)} = \frac{2}{3K+1} e_{(K)} + \frac{3K-1}{3K+1} e_{(-n-1)}$$

The above facts combined with the form of $n_i^{(K)}(0)$ imply that for any $k = 2n+1$ one has

$$\rho^{(n)}_{i(K)} = \frac{2}{3K+1} \rho^{(-K+1)}_{i(K)} + \frac{3K-1}{3K+1} \rho^{(n+1)}_{i(K)}$$

and similarly for $(-n)$.

B) $\mathcal{L}$ is generated by interaction:

$(\Phi_1)_a = s_a$, $(\Phi_2)_a = s_a^2$, $\quad (\Phi_i)_\Lambda = 0$ if $\Lambda \neq \{a\}$

Let $q = 2$.

In the base induced in $\mathcal{L}$ by Φ_1, Φ_2, one has $e_{(k)} = (k,k^2)$.

It is easy to see that $\forall\, k$, $e_{(k)} \in \epsilon(W_o)$. max W_1 has two faces:

H containing points corresponding to : (2),(-2),(1),(-1) ;

$$v_1(H) = (\ 0,\ -\frac{1}{3}\ )$$

F : (1),(-1),(0) , $v_1(F) = 0$

Note that $\{e,f\}$ is a symmetry group for $\mathcal{S}_1(H)$. It has two orbits: $\{(-2),(2)\}$ and $\{(-1),(1)\}$. Hence by Proposition 3.24. one can apply Theorem 3.22. Obviously order 1 is conclusive as to the existence of the strata. In addition, if order i(1) is examined, one can see that $v_{i(1)}(F) = (\ 0\ ,\ -\ 1\ )$.

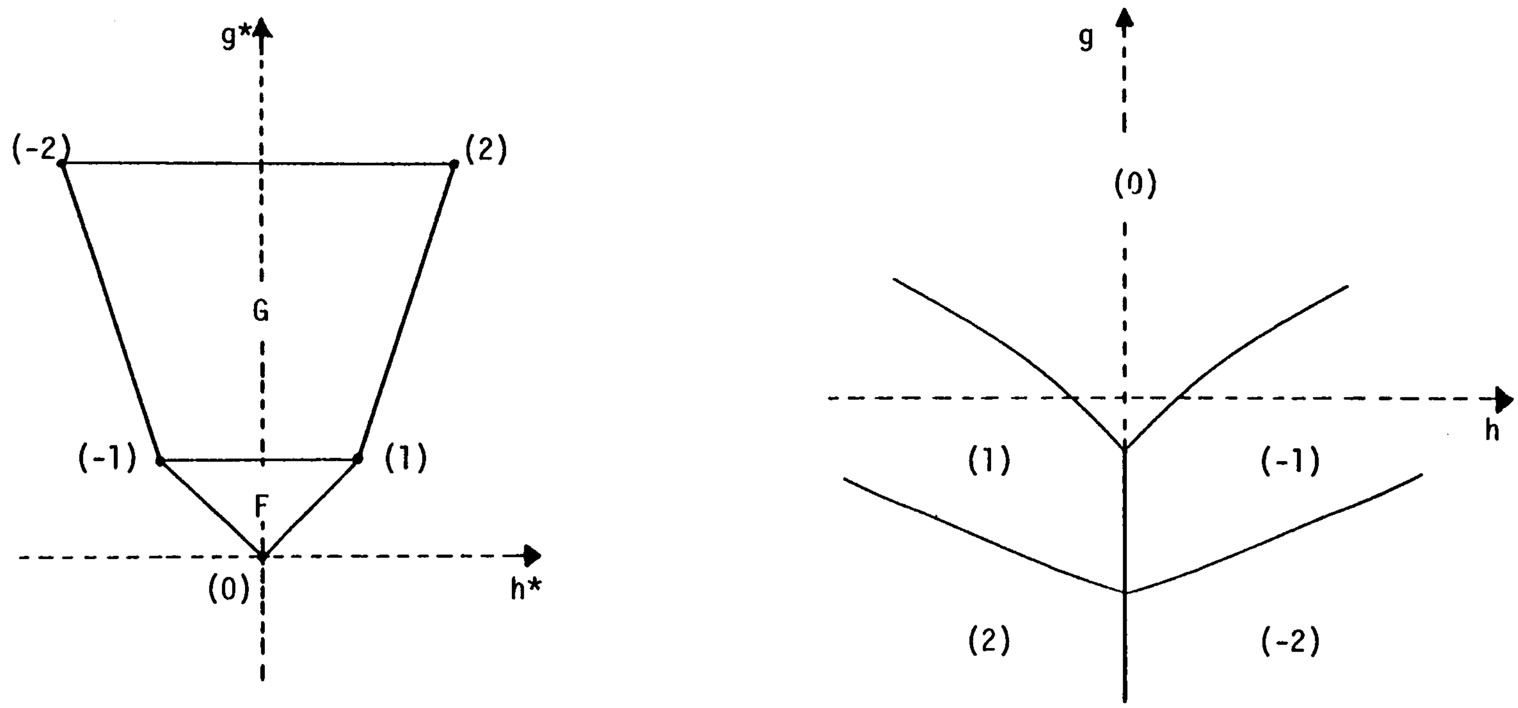
The convex structure $\mathbb{F}_1$ is shown in Fig.10a, and the phase diagram $\Omega_{s,\beta}$ for $s > i(1)$ is presented in Fig.10b. The indeces appearing at strata of $\Omega_{s,\beta}$ show their orders.

If q is arbitrary, the conclusive order is i(1). The construction of $\mathbb{F}_{i(1)}$ has several characteristic steps occuring at orders i(k). For any k there exists $F' \in \mathbb{F}_{i(k)-1}$ such that $\mathbb{F}_{i(k)}(F')$ has two faces:

H : (-k),(k),(-k+1),(k-1) , $v_{i(k)}(H) = (\ 0\ ,\ -\ \frac{1}{2k-1}\ )$

F : all other phases , $v_{i(k)}(F) = 0$

The group $\{e,f\}$ is a symmetry group for $\mathcal{S}_{i(k)}(H)$. Hence one can apply Theorem 3.22. to this system and obtain phase diagram in the manner similar to that presented for case q = 2.



(a) (b)

Fig.10: The phase diagram construction for the Blume-Capel model with the perturbation space generated by s_a and s_a^2 . (a) the projection of max W_1 on L^*;
(b) the phase diagram $\Omega_{s,\beta}$ for $s>i(3)$.

2. The model M.

The large class of models is obtained by complicating the potential. In this way we can construct examples with arbitrary properties. The problem is that with the increasing complexity of an interaction, the difficulties with generating the low-temperature expansions also increase. The following example, suggested to me by J.Miekisz, is one of the simplest in this category. We will denote it as "model M". It belongs to the class of so called "stacked" systems.

Let $\mathbb{L} = Z^2$, $S = \{-1,1\} \equiv \{-,+\}$. Introduce the base in Z^2: $\{e_1,e_2\}$. The system consists of lines, stacked one onto another and interacting by ferromagnetic potential: in the direction of e_2 (vertical):

$$\Phi_{\{a,a+e_2\}} = -3\, s_a s_{a+e_2}$$

The interaction inside each line (in the direction of e_1 - horizontal) is given by:

$$\Phi_{\{a-e_1,a,a+e_1\}} = -\frac{1}{2}\, s_{a-e_1} s_a s_{a+e_1} + \frac{1}{2}\, s_a - 3\, s_{a-e_1} s_{a+e_1}$$

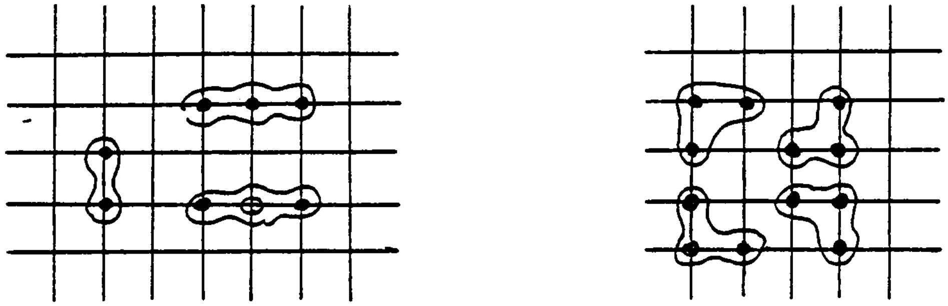
(see Fig.11a)

It is easy to see that Φ is an m-potential. First note that the configuration of any ground state has to be ferromagnetic in the direction of e_2. To find the configuration in the direction of e_1, let us consider X_Λ with $\Lambda = \{-e_1,0,e_1\}$.

a) $\Phi_\Lambda(+++) = \Phi_\Lambda(---) = \Phi_\Lambda(+-+) = \Phi_\Lambda(-+-) = -3$

b) $\Phi_\Lambda(+--) = \Phi_\Lambda(--+) = 2$

c) $\Phi_\Lambda(++-) = \Phi_\Lambda(-++) = 4$



H_0 perturbation (A)

(a)

+	−	+	−	+
+	−	+	−	+
+	−	+	−	+
+	−	+	−	+
+	−	+	−	+

A

+	+	+	+	+
+	+	+	+	+
+	+	+	+	+

E

Fig.11. (a) The bounds for the model M: for the Hamiltonian H_0 (left), and the perturbation (A) (right)

(b) The ground states: A (left) and E (right)

The ground state F is obtained from E by replacing all + by -.

The ground states are (Fig.11b):

E: $E_a = 1$ for all a;

F: $F_a = -1$ for all a;

A: $A_a = 1$ if $a = 2me_1 + ne_2$,

$A_a = -1$ if $a = (2m+1)e_1 + ne_2$, $m,n \in Z$

τA : a translate of A by e_1. We identify τA with A.

<u>The low-temperature expansion</u>

Let $X \sim G$ ($G \in \mathcal{S}$) be connected. Suppose that the projection of suppX on $\{ne_1,\ n \in \mathbb{Z}\}$ has k elements, and the projection of suppX on $\{me_2,\ m \in \mathbb{Z}\}$ has h elements. Then

$$H_o(X|G) \geqslant 10h + 12k \, .$$

This fact may be verified by counting the number of broken bonds (Λ is a bond if $\Phi_\Lambda \neq 0$. Λ is broken if $\Phi_\Lambda(X)$ is not minimal). X has at least 2h broken horizontal bonds, with minimal energy change equal 5 per bond (configuration b) above) and at least 2k broken vertical bonds, each producing the energy change 6.

H_o satisfies condition (2.9), since it has m-potential and finite number of ground states.

As we will show, order 8 ($E_8 = 44$) suffices for the determination of the phase diagram (in the sense of Remark 3.23). Hence if X is irreducible and $H_o(X|G) \leqslant E_8$, then $k \leqslant 2$ and $h \leqslant 2$, or $h = 3$ and $k = 1$. The possible configurations with length k and height h satisfying these conditions are listed in the table. From this table one can see that:

$n_s^A(0) = \frac{1}{2}$ if $s < 8$

$n_s^E(0) = 1$ if $s = 2,5,6$, $n_s^E(0) = 0$ if $s = 1,3,4,7$;

$n_s^F(0) = 0$ if $s = 2,5,6$, $n_s^F(0) = 1$ if $s = 1,3,4,7$;

$n_8^A(0) = -\frac{3}{4}$, $n_8^E(0) = 0$, $n_8^F(0) = -\frac{5}{2}$

A) Let $\mathcal{L}$ be one-dimensional perturbation space generated by interaction:

$$\Psi_{\{a,a+e_1,a+e_2\}} = \frac{1}{4} \times s_a s_{a+e_1} s_{a+e_2} \text{ , } \Psi_A = 0 \text{ otherwise.}$$

Then $e_E = 1$, $e_F = -1$, $e_A = 0$

In order 1, $\max W_1$ has one face P, since $e_A = \frac{1}{2} e_E + \frac{1}{2} e_F$,

and $n_1^A(0) = \frac{1}{2} n_1^E(0) + \frac{1}{2} n_1^F(0)$.

The corresponding point in $\mathcal{L}$ is:

$$v_1(P) = -\frac{1}{2} \text{ , } \tilde{v}_1(P) = -\frac{1}{2} e^{-22\beta}$$

Hence

$\dot{\rho}^G(\dot{v}_s)_s = \dot{\rho}^G(0)_s$ if $s < 8$

Note that for $s < 8$,

$$n_s^A(0) = \frac{1}{2} n_s^E(0) + \frac{1}{2} n_s^F(0)$$

Therefore max W_s has one face isomorphic to P.

Next,

$$A_8^G(0) = n_8^G(0) + \langle v_1(P), dn_1^G(0)\rangle.$$

By straightforward calculation,

$$n_1^A(x) = \frac{1}{2} e^{-2x} \text{ , } n_1^F(x) = e^{-2x} \text{ , } n_1^E(x) \equiv 0$$

Hence $A_8^A(P) = -\frac{1}{4}$, $A_1^F(P) = -\frac{3}{2}$, $A_1^E(P) = 0$

max $W_8(P)$ has two faces:

P_1: $\mathcal{S}_8(P_1) = \{A,F\}$, $v_8(P_1) = \frac{5}{4}$

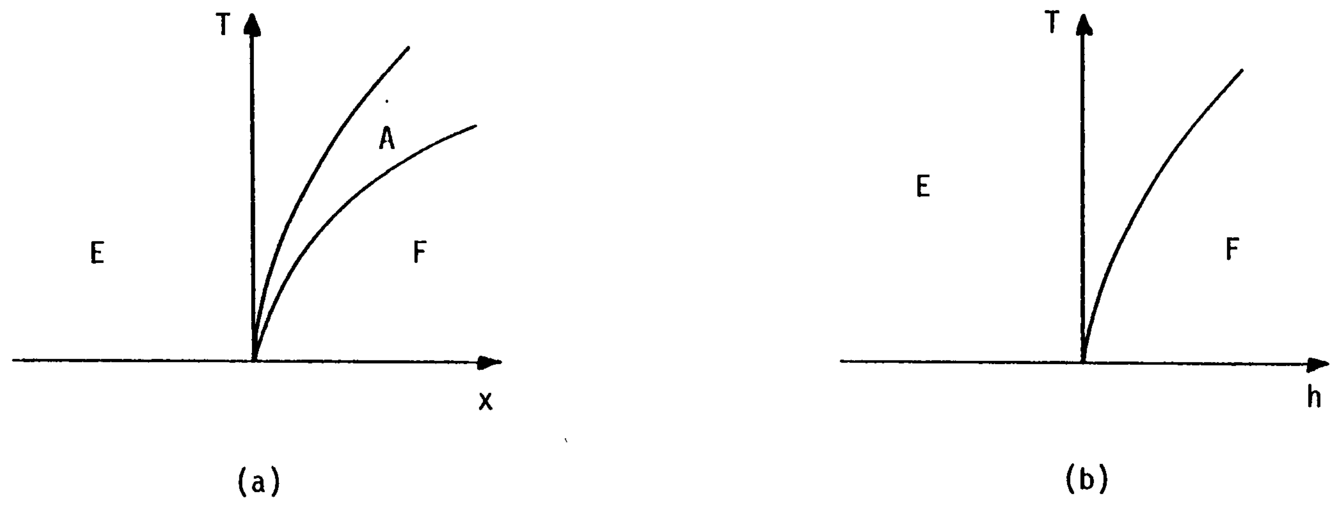


Fig.12: The phase diagram for model M: (a) in the case (A) - perturbation is generated by triangle interaction; (b) in the case (B)- perturbation is generated by one spin interaction with an external field.

P_2: $\mathcal{S}_8(P_2) = \{A,E\}$, $v_8(P_2) = \frac{1}{4}$

The phase diagram is shown in Fig.12a.

B) Let $\mathcal{L}$ be generated by an interaction:

$$\Psi_{\{a\}} = hs_a \ , \ \Psi_\Lambda = 0 \text{ otherwise}$$

Then $e_E = 1$, $e_F = -1$, $e_A = 0$

Up to order 7, all calculations for the case A hold here. However, now

$$n_1^A(x) = \frac{1}{2} e^{2x} \ , \quad n_1^F(x) = e^{-2x} \ , \ n_1^E(x) \equiv 0$$

and $A_8^A(P) = -\frac{5}{4}$, $A_8^F(P) = -\frac{3}{2}$, $A_8^E(P) = 0$

Therefore max $W_8(P)$ has only one face P_1, and $\rho_8^A(P) \notin P_1$. The phase diagram for this case is shown in Fig.12b.

TABLE

The low-energy excitations (up to order 8) for model M. Circles in column "suppX" denote flipped spins, and dashes in between denote spins which are not flipped. Numbers in the left column are energies of an excitation with given support, and numbers in the right column are multiplicities.

suppX	E		F		A			
o	26	1	22	1	26 (-)	$\frac{1}{2}$	22 (+)	$\frac{1}{2}$
oo	48	1	48	1	48	1		
o-o	38	1	34	1	38 (--)	$\frac{1}{2}$	34 (++)	$\frac{1}{2}$
o o	40	1	32	1	40 (-)	$\frac{1}{2}$	32 (+)	$\frac{1}{2}$
o o o	54	1	42	1	54 (-)	$\frac{1}{2}$	42 (+)	$\frac{1}{2}$
oo oo	72	1	72	1	72	1		
o-o o-o	52	1	44	1	52 (-)	$\frac{1}{2}$	44 (+)	$\frac{1}{2}$
two indep.	52	$-\frac{7}{2}$	44	$-\frac{7}{2}$	52 (-,-) 44 (+,+)	$-\frac{5}{4}$ $-\frac{5}{4}$	48 (-,+)	-1

3. Generalized Ferromagnetic Systems.

Let $\mathbb{L}$ be any lattice, $S = \{-q,-q+1,\dots,q\}$, Let $F \in \mathcal{X}$ be such that $\forall\, a \in \mathbb{L}$, $F_a = q$. The Hamiltonian is:

$$H_o = -\sum_B J(B)s_B \qquad J(B) \geq 0$$

where $B : \mathbb{L} \to \mathbb{N}$ is a multiplicity function (non-zero for finite number of points only), and

$$s_B = \prod_{a\in\mathbb{L}} s_a^{B(a)}$$

Such a system is called a generalized ferromagnetic system.

Let $Q = \{e,f\}$ be a group of transformations of S of the form:

$$es = s \;,\; fs = -s$$

Define a subgroup of $Q^{\mathbb{L}}$:

$$\Theta = \{\theta \in Q^{\mathbb{L}} : \text{if } J(B) \neq 0, \text{ then } \prod_{a\in\mathbb{L}} \theta_a^{B(a)} = 1\}$$

It is easy to see that Θ is a subgroup of the symmetry group of H_o. With this definition one can show that ([2])

$$\mathcal{G}(H_o) = \{gF,\ g \in \Theta\}$$

Hence in the absence of perturbation, the low-temperature expansion for all ground states is the same.

Assume now that Θ is finite, and let $\mathcal{L}$ be any perturbation space satisfying the transversality condition. Then the following hold:

1. $\forall\, s$, $\mathbf{F}_s$ contains only one element F_s. F_s is a translate of F_o, and for the corresponding element of $\mathcal{L}$, $\nu_s(F_s) = 0$.

2. If $e_G \in \text{Int } F_o$, then $\forall\, s\ \Omega_{s,\beta}(G) \subset U_s(F_s)$.

Because of 1., the zero-th order is conclusive.

The phase diagram $\Omega_{m,\beta}$ can be now described in terms of zero-th order (zero-temperature) phase diagram Ω_0. First note that Assumption 3.20. (or 3.25.) imposes restriction on $\mathcal{L}$:

1. $\forall \in \mathcal{S}$, $e_G \in \varepsilon(F_0)$ or $e_G \in \text{Int } F_0$.
2. any face (extremal edge) of F_0 contains the proper number of functionals e_G (prescribed by 3.25.).

The description of $\Omega_{m,\beta}$ is now as follows. For any stratum $\Omega_0(\mathcal{S}')$ (with exception of the point $\Omega_0(\mathcal{S})$ of coexistence of all phases) there exists a stratum $\Omega_{m,\beta}(\mathcal{S}')$. $\Omega_{m,\beta}(\mathcal{S}')$ is "close" to $\Omega_0(\mathcal{S}')$ in terms of Proposition 3.13. The domains of all phases for which $e_G \in \text{Int } F_0$ are m-equivalent (with the constant equal to the radius of $U_m(F_0)$) to the point $\{0\}$.

4. The antiferromagnet on the f.c.c. lattice.

The important class of models consists of so called models with stabilizations. We will use an example from this class to demonstrate how our method works if there is an infinite number of ground states.

Suppose that H_0 has an infinite number of ground states. We introduce an additional Hamiltonian, which forces spins in some fixed distance one from another to be aligned ferromagnetically:

$$H_1(m_1,\ldots,m_\nu) = -\sum_{i=1}^{\nu} \varepsilon_i \sum_{a\in\mathbb{L}} s_a s_{a+m_i e_i}$$

where $\{e_i\}$ is the canonical base in $\mathbb{R}^\nu$, and ε_i are small positive. Then the Hamiltonian $H_0 + H_1(m_1,\ldots,m_\nu)$ has finite number of ground states, namely these ground states of H_0 which are invariant with respect to

$m_1 Z \oplus m_2 Z \oplus \ldots \oplus m_\nu Z$. We can apply our method to this system as long as other hypotheses are satisfied.

As an example let us consider the antiferromagnet on the f.c.c. lattice in $\mathbb{R}^3$. $\mathbb{L}$ contains four sublattices: $\mathbb{Z}^3$, $\frac{1}{2}(e_1+e_2) + \mathbb{Z}^3$, $\frac{1}{2}(e_1+e_3) + \mathbb{Z}^3$, $\frac{1}{2}(e_2+e_3) + \mathbb{Z}^3$. The configuration set S is $\{-1,1\}$. The Hamiltonian H_o is given by

$$H_o = - \sum_{\langle a,b \rangle} s_a s_b$$

where $\langle a,b \rangle$ denotes a pair of nearest neighbors. We add one-dimensional perturbation:

$$L(J) = J \sum s_a s_b$$

with the sum over pairs of next nearest neighbors (cf.Fig 13a).

The reader will find an extensive description of this model in [3]. Here we cite results only, without proofs.

The ground states of H_o are as follows. There is a class of completely symmetric ground states: we choose any two sublattices and assign +1 to every point. To every point of the remaining two sublattices we assign -1. This class has six members. Every other ground state is obtained from the completely symmetric ones in the following way. We choose one of the base vectors, say e_1. Starting from any of the completely symmetric ground states, we flip spins in arbitrary finite number of lattice planes perpendicular to the e_1-axis. Then we repeat this flipping in a periodic fasion. It is evident that all ground states differing by the choice of the axis are related by a symmetry of the full Hamiltonian H_o+L. We identify these states with G.

Henceforth we assume that the axis of changes is the e_1-axis (x-axis). Thus every ground state can be viewed as a sequence of antiferromagnetically ordered planes, with no a priori relation between spin orientations in different planes (other than induced by periodicity).

Let $G \in \mathcal{G}(H_0)$. Consider a pair of planes perpendicular to the x-axis: $\{P, gP\}$, where $g \in \mathbb{Z}^3$ is the translation by vector e_1. We say that this pair is an β-pair if $\forall\, a \in P$, $G_a = G_{a+e_1}$.
If $G_a = -\,G_{a+e_1}$, then the pair is an α-pair (cf. Fig.13b). Let L be the period of G in the direction of x. We define

$$p_\alpha(G) = \frac{1}{L}\,\mathrm{Card}\{\alpha\text{-pairs with the first plane intersecting } \{0, e_1, 2e_1, \ldots, (L-1)e_1\}\}$$

p_α is a concentration of α-pairs in the ground state G. We define $p_\beta(G)$ in the similar fasion. By $\alpha\alpha$ we will denote three planes P, gP, g^2P such that $\{P, gP\}$ and $\{gP, g^2P\}$ are α-pairs. Then $p_{\alpha\alpha}(G)$ is a concentration of $\alpha\alpha$ triples. We will also use this notation to describe longer sequences of planes. One has obvious relations for concentrations:

$$p_\alpha + p_\beta = 1 \quad , \quad p_{\alpha\alpha} + p_{\alpha\beta} = p_{\alpha\alpha} + p_{\beta\alpha} = p_\alpha \quad ,$$
$$p_{\beta\beta} + p_{\alpha\beta} = p_{\beta\beta} + p_{\beta\alpha} = p_\beta$$

and so on.

Let us first study the system without stabilization. We note that the Hamiltonian H_0 satisfies condition (2.9). The argument is long and will not be reproduced here. The first four terms of low-temperature expansions for any ground state can be expressed in terms of concentrations p in the following way:

$E_1 = 8 \quad n_1(0) = 1$

$E_2 = 12 \quad n_2(0) = 4$

$E_3 = 16 \quad n_3(0) = \frac{29}{2} + p_\beta(G)$

$E_4 = 20 \quad n_4^G(0) = 60 + 12p_\beta(G) + 2p_{\beta\beta}(G)$

The expression for the last coefficient differs from the expression obtained by Mackenzie and Young [12]. Our result is in agreement with calculations by Styer [13].

Next we add perturbation L(J). It is easy to see that

$$e_G = 2 + p_\beta(G) - p_\alpha(G) = 1 + 2p_\beta(G).$$

Hence for $J < 0$, the only ground states are these, for which $p_\beta = 1$, i.e. the only plane pairs are β-pairs. These configurations are exactly the completely symmetric ground states described before. Their class will be denoted by (β).

For $J > 0$, the concentrations for the ground states satisfy the condition: $p_\beta = 0$. These ground states are described as follows: starting from any ground state (β), we flip spins in every other plane. This class has twelve elements, and will be denoted as (α).

<u>The convex structure in low orders</u>:

<u>order 1</u>: $\rho^G_{1,\beta} = (-e_G, 1)$. There is one face F parallel to max W_o. $v_1(F) = 0$.

<u>order 2</u>: $\rho^G_{2,\beta} = (-e_G, 4)$. Again max W_2 has one face F parallel to max W_o. $v_2(F) = 0$.

<u>order 3</u>: $\rho^G_{3,\beta} = (-1, \frac{29}{2}) + p_\beta(G)(-2,1)$. Hence all functionals lie on

the same line. There is one face F'. $v_3(F')$ is defined by the equation:

$$\rho^{(\alpha)}_{3,\beta}(x) = \rho^{(\beta)}_{3,\beta}(x)$$

i.e. $v_3(F') = \frac{1}{2}$, $\tilde{v}_3(F') = \frac{1}{2}\, e^{-16\beta}$.

<u>order</u> 4: $\rho^{G}_{4,\beta} = (-\, e_G \ ,\ 60+12p_\beta+2p_{\beta\beta})$.

Let $G \neq (\alpha),(\beta)$. Then $e_G = p_\alpha(G)e_{(\alpha)} + p_\beta(G)e_{(\beta)}$, and

$$p_\alpha(G)\rho^{(\alpha)}_{4,\beta} + p_\beta(G)\rho^{(\beta)}_{4,\beta} = (-\, e_G \ ,\ 60+12p_\beta+2p_\beta) > \rho^{G}_{4,\beta}.$$

The last inequality follows from the fact that

$$p_\beta(G) = p_{\beta\beta}(G) + \frac{1}{2}\,(p_{\beta\alpha} + p_{\alpha\beta}) > p_{\beta\beta}$$

if $G \neq (\alpha),(\beta)$. Hence max W_4 has only one face $\tilde{F}$, and no phase other than (α) and (β) belong to $S_4(\tilde{F})$.

Suppose now that we add to H_o the stabilization $H_1(m)$. Then the forms of convex structures described above do not change. We conclude that for β large enough, and J small enough, there are only two phases: (α) for larger J, and (β) for smaller J. The phase diagram in any order higher than 3 consists only of the line separating these two phases (Fig.13c).

Let us return now to the original system without stabilization. We have already derived the convex structures in this case. When one examines estimations in Section 3.4, one can see that in orders 1,2 and 3 the bound (3.21) is G-independent. Hence there exists a ball $B(0,re^{-\beta E_4})$ such that for any $G \neq (\alpha),(\beta)$:

$$\Omega_{m,\beta}(G) \subset B(0,re^{-\beta E_4})$$

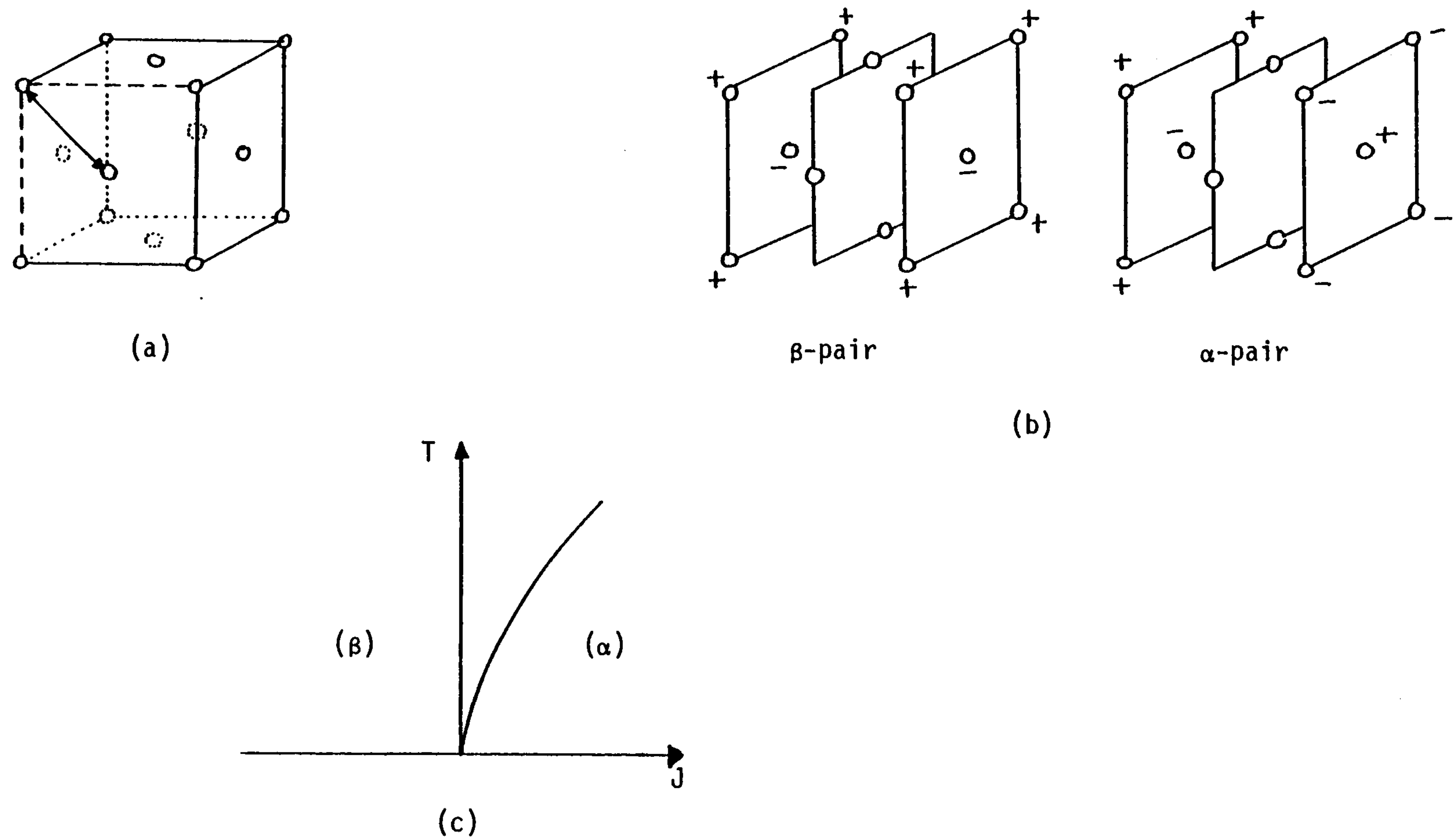


Fig.13: (a) The bounds for the antiferromagnet on the f.c.c. lattice. Arrows show the nearest neighbors bound, the dashed line- the next nearest neighbors bound.
(b) α-pair and β-pair of planes; (c) The phase diagram for any stabilization.

Inside this ball

$$|\rho^G_{4,\beta}(x) - \pi^G_t(x,\beta)| < ce^{-\beta E_4}$$

with c independent of G. From this fact, and the form of the convex structure in order 4, we conclude that $\forall\ G \neq (\alpha),(\beta)\ \ \exists\ \beta_t(G)$: $\forall\ \beta > \beta_t(G)$, $\Omega_{t,\beta}(G) = \emptyset$. This is the most general statement which we can make about the domains. Note that $\sup_G \beta_t(G) = \infty$. This is because if

$\tilde{v}_4(\tilde{F})$ corresponds to the face of max W_4, then $\forall\ \epsilon\ \exists\ G$:

$$\rho^G_{4,\beta}(\tilde{v}_4(\tilde{F},\beta)) - \rho^G_{4,\beta}(\tilde{v}_4(\tilde{F},\beta)) = \frac{1}{2}\,(p_{\beta\alpha} + p_{\alpha\beta}) < \epsilon.$$

Hence if the phase diagram is taken in the sense of the definition (3.2), we cannot describe it by using our method.

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