STATIONARY SOLUTIONS OF ABSTRACT KINETIC EQUATIONS

by

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(ABSTRACT)

The abstract kinetic equation $T\psi' = -A\psi$ is studied with partial range boundary conditions in two geometries, in the half space $x \geq 0$ and on a finite interval $[0, \tau]$. T and A are abstract self-adjoint operators in a complex Hilbert space. In the case of the half space problem it is assumed that T is a (possibly) unbounded injection and A is a positive compact perturbation of the identity satisfying a regularity condition, while in the case of slab geometry T is a bounded injection and A is a bounded Fredholm operator with a finite dimensional negative part. Existence and uniqueness theory is developed for both models. Results are illustrated on relevant physical examples.

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CHAPTER I

INTRODUCTION

Over the past three decades, a variety of functional analytic techniques have been developed for the study of properties of many transport equations which occur in physical models, including neutron transport in nuclear reactors, polarized light transfer through planetary atmosphers, electron scattering in metals, and kinetics in rarefied gases. The most popular of these techniques (from early in the 1960's and until the present in many engineering and physics circles) has been the eigenfunction expansion method first introduced by Case in 1959 [7,8], based in part on an earlier work of van Kampen [26].

In 1973, Hangelbroek [18,19] introduced a Hilbert space technique based on a detailed analysis of certain noncommuting projections, which he applied to a one-speed neutron transport equation. Although in these articles explicit use was made of the Wiener-Hopf factorization of a dispersion function (symbol of the equation), it was later shown, in joint work with Lekkerkerker [23], that the required factorization was a corollary of some operator constructions carried out in the analysis. Because this Hilbert space technique leads to the study of abstract versions of transport equations, and because it avoids the difficulties of Wiener-Hopf factorizations (or Riemann matrix problems), it has been intensively studied in the last few years.

The common feature of stationary linear transport processes in plane parallel geometry is that each of them is described by a linear integro-differential equation which can be written

$$T\psi'(x) = -A\psi(x), \tag{1}$$

where T and A are certain self-adjoint operators on a Hilbert space, with A representing the collision process for the particular physical model. The specific form of the operators T and A depends on the nature of the problem, but several properties of these operators are shared by almost all models, such as self-adjointness of T and A, injectivity of T, and in many important cases positivity of A and compactness of I−A. Therefore it is natural to develop a unified theory for all of these physical models. This approach has led to the foundation of so called abstract kinetic theory, where Eq.(1), referred to as an abstract kinetic equation, is the central object of study.

In this dissertation we present an existence and uniqueness theory for a class of problems in abstract kinetic theory relevant to rarefied gas dynamics and neutron transport in multiplying media. First we describe briefly the type of physical problems we have in mind and which we wish to study in abstract generalization.

In rarefied gas dynamics, the BGK approximation of the Boltzmann equation leads to the kinetic equation

$$\frac{\partial\psi}{\partial t} + v\cdot\nabla\psi = \nu(v)\left[\sum_{i=0}^{4}(\psi_i,\psi)\psi_i - \psi\right],$$

where $\psi(t,x,v)$ represents the molecular density of particles having velocity v at a point x at time t, ψ_i, i=0,1,2,3,4 are the collision invariants of mass, the three momenta and energy, and $\nu(v)$ is the collision frequency. We assume that the collision invariants are normalized such that $(\psi_i,\psi_j)=\delta_{ij}$. Under the assumption of constant collision frequency the stationary version of this equation in plane parallel geometry is futher reduced to

$$v_x \frac{\partial\psi}{\partial x} = -\psi + \sum_{i=0}^{4} (\psi_i,\psi)\psi_i. \tag{2}$$

The above equation and its various modifications have been studied in many contexts (cf. [1,5,6,11,13,27,42]). An excellent introduction to the Boltzmann equation and related model equations can be found in Cercignani's monograph [10], to which we refer the reader for additional details and physical background.

Equation (2) takes the form of Eq.(1) if we define

$$(T\psi)(v) = v_x\psi(v)$$

and

$$(A\psi)(v) = \psi(v) - \sum_{i=0}^{4} (\psi_i,\psi)\psi_i(v).$$

In this example the operator T is unbounded, injective and self-adjoint on the Hilbert space $H=L^2(\mathbb{R}^3,(\pi^3)^{-1/2}e^{-v^2}dv)$, while A is a positive compact perturbation of the identity. These properties of T and A will be the assumptions under which we will study Eq.(1) in the half space $x\geq 0$. Equation (2) will serve then as an example of our study.

In the one-speed approximation, stationary neutron transport in a plane parallel homogeneous medium is described by the following equation

$$\mu\frac{\partial\psi}{\partial x} = -\psi(x,\mu) + \frac{c}{2}\int_{-1}^{1} f(\mu,\nu)\psi(x,\nu)d\nu. \tag{3}$$

Here the unknown function ψ represents the angular density of neutrons, $\mu\in[-1,1]$ is the cosine of the angle describing the direction of propagation and x is the position coordinate. The scattering kernel has the form

$$f(\mu,\nu) = (2\pi)^{-1}\int_0^{2\pi} p(\mu\nu+(\mu^2-1)^{1/2}(\nu^2-1)^{1/2}\cos\alpha)d\alpha.$$

The function p is determined by properties of the host medium. The value $cf(\mu,\nu)$ gives the probability that the collision of a particle having velocity "in the direction ν" with the host medium results in a secondary particle with velocity direction μ. Therefore we assume that p is nonnegative and satisfies the normalization condition

$$\frac{1}{2}\int_{-1}^{1} p(t)dt=1.$$

Then c is the mean number of secondaries per collision. If $0<c<1$ the host medium is absorbing, $c=1$ corresponds to a conservative medium and $c>1$ to a multiplying medium. For additional details we refer the reader to the classic text by Case and Zweifel [9]. Equation (3) has been analyzed extensively in the case of absorbing and conservative media (cf. [3,18–23,33,36]) but very few studies have dealt with the case of a multiplying medium (cf. [2,37,38]).

If we define

$$(T\psi)(\mu) = \mu\psi(\mu)$$

and

$$(A\psi)(\mu) = \psi(\mu) - \frac{c}{2}\int_{-1}^{1} f(\mu,\nu)\psi(\nu)d\nu,$$

then Eq.(3) can be rewritten in the form (1). In this example T and A are bounded self-adjoint on the Hilbert space $H=L^2[-1,1]$. If $0<c<1$, A is strictly positive; if $c=1$, A is positive but has a nontrival kernel; and if $c>1$, A is no longer positive. Indeed in this latter case the spectrum of A contains strictly negative eigenvalues. We will study Eq.(1) on a finite spatial domain $[0,\tau]$ for bounded self-adjoint Fredholm operators A which have "a finite negative part". Since relevant physical applications come from neutron transport theory, we assume for the sake of simplicity that T is bounded.

The abstract kinetic equation as well as concrete kinetic models have been studied mostly for positive collision operators A. An important exception is the recent work of Greenberg and van der Mee [16], where the half space problem for Eq.(1) with A having a finite negative part has been solved. We also note an earlier paper by Ball and Greenberg [2], where the use of Π_κ-space (Pontryagin space) theory has led to spectral analysis of the one-speed isotropic neutron transport equation in a multiplying medium. Our treatment of Eq.(1) on finite intervals $[0,\tau]$ with A having a finite negative part is new.

The theory for positive A so far developed either assumes that the operator T is bounded or seeks solutions in an "enlarged" space $H_T \supset D(T)$ (weak solutions). For T bounded, the existence and uniqueness theory for H-valued functions (strong solutions) has been developed by Hangelbroek for A a concrete rank one perturbation of the identity [18,19] and for I-A a trace class operator from neutron transport [20]. In both these studies the collision operator A was assumed invertible. The critical case (conservative medium) for the one-speed neutron transport equation, in which A has a nontrivial kernel, has been solved by Lekkerkerker [33]. Equation (1) for I-A an abstract compact operator satisfying a "regularity condition" and T bounded has been studied extensively by van der Mee [34], who obtained a complete existence and uniqueness theory for the half space and slab problems with partial range boundary conditions. In this dissertation we extend the existence and uniqueness results of van der Mee [34] to the case of T unbounded.

Existence and uniqueness theory for weak solutions was developed initially by Beals [3]. In his innovative work both half space and slab problems have been analyzed. The approach he initiated has since been applied to more general abstract kinetic equations. We note the work of Greenberg, van der Mee and Zweifel [17], where the case of positive self-adjoint Fredholm collision operator A with nontrivial kernel has been studied.

Now let us review the aim of this dissertation. We consider an abstract kinetic equation

$$T\psi'(x) = -A\psi(x) \tag{1}$$

in two geometries, in the half space $x\geq0$ and on a finite interval $[0,\tau]$.

In the first case we assume that T is a self-adjoint, possibly unbounded, injective operator on a complex Hilbert space H and A is a positive compact perturbation of the identity. We study Eq.(1) together with the partial range boundary condition at $x=0$

$$Q_+\psi(0) = \varphi_+, \tag{4}$$

where Q_+ is the orthogonal projection onto the maximal T-positive T-invariant subspace of H, and with a condition at infinity, namely one of

$$\lim_{x\to\infty} \|\psi(x)\| = 0, \tag{5a}$$

$$\lim_{x\to\infty} \|\psi(x)\| < \infty, \tag{5b}$$

$$\|\psi(x)\| = O(x) \quad \text{as } x\to\infty. \tag{5c}$$

We prove theorems describing the measure of noncompleteness (which we define as the codimension in Ran Q_+ of the subspace of boundary values $\varphi_+\in$Ran Q_+ for which the problem is solvable) and the measure of nonuniqueness (the dimension of the solution space of the corresponding homogeneous problem) for the boundary value problems (1)-(4)-(5).

In the second case, T is bounded, self-adjoint and injective but A is no longer required to be positive. We assume that A is a bounded self-adjoint Fredholm operator with a finite negative part. Now Eq.(1) is studied together with the partial range boundary conditions

$$Q_+\psi(0) = \varphi_+, \tag{4}$$

$$Q_-\psi(\tau) = \varphi_-, \tag{6}$$

where $Q_-=I-Q_+$ is the orthogonal projection onto the maximal T-negative T-invariant subspace of H and φ_+, φ_- are given vectors in the ranges of Q_+, Q_- respectively. We prove that for any incoming fluxes φ_+, φ_- the boundary value problem (1)-(4)-(6) is uniquely solvable for values of τ less then a certain critical value.

The presentation of the dissertation is organized as follows. In Chapter II we study the half space problem for strictly positive collision operators A. There we introduce the albedo operator and study its properties. The existence and uniqueness theorem is shown to be a consequence of the bounded invertibility of the albedo operator. In Chapter III we generalize the results of the previous chapter to positive collision operators A. This generalization is possible due to a decomposition of the Hilbert space which reduces the relevant transport operator. In the first part of Chapter III we show that such decomposition exists. Then in the second part we construct solutions of the half space problems (1)-(4)-(5) and prove the corresponding solvability theorems. Chapter IV is devoted to the study of the abstract slab problem (1)-(4)-(6) for collision operators having a finite negative part. First we introduce another useful decomposition of the Hilbert space, which is in fact a generalization of the one used in Chapter III. With the help of this decomposition we prove the existence and uniqueness theorems for weak and strong solutions of the slab problem. In Chapter V we demonstrate applications of our results to several physical problems. The first three examples, which come from rarefied gas dynamics, illustrate the results of Chapter III. The other two examples, both from neutron transport, illustrate the results of Chapter IV.

CHAPTER II

DISSIPATIVE MODELS: STRICTLY POSITIVE COLLISION OPERATORS

We study the abstract kinetic equation

$$(T\psi)'(x) = -A\psi(x), \quad 0<x<\infty, \tag{1}$$

along with the partial range boundary condition at $x=0$

$$Q_+\psi(0) = \varphi_+ \tag{2}$$

and the growth condition at infinity

$$\lim_{x\to\infty} \|\psi(x)\| < \infty. \tag{3}$$

We assume that A is a strictly positive compact perturbation of the identity and satisfies a regularity condition.

1. DEFINITIONS AND PRELIMINARIES

Let H be a complex Hilbert space with an inner product $(\cdot,\cdot)$ and T an injective self-adjoint operator on H. We define $Q_\pm$ as the complementary orthogonal projections onto the maximal T-positive/negative T invariant subspaces of H. Throughout this chapter A will be a strictly positive operator on H such that $B=I-A$ is compact. Then $\mathrm{Ker}A = \{0\}$ and $A^{-1}=I+C$, where $C=BA^{-1}$ is obviously compact. Since A and A^{-1} are bounded, the inner product $(\cdot,\cdot)_A$ on

H defined by

$$(f,g)_A = (Af,g). \tag{4}$$

is equivalent to the original inner product on H. We will denote by H_A the Hilbert space H endowed with the inner product $(\cdot,\cdot)_A$.

<u>LEMMA</u> <u>II.1</u>. The operator $S=A^{-1}T$ is injective and self-adjoint with respect to the H_A-inner product (4).

<u>Proof</u>: Clearly $D(S)=D(T)$. Let $f,g\in D(S)$. Since T and A are self-adjoint $(Sf,g)_A=(Tf,g)=(f,Tg)=(f,Sg)_A$ which shows that S is symmetric, i.e. $S\subset S^*$. Now let $f\in D(S^*)$. Then the functional $D(S)\ni g\to(f,Sg)_A=(f,Tg)$ is continuous on $D(S)=D(T)$. Since T is self-adjoint this implies that $f\in D(T)=D(S)$. Hence $D(S^*)\subset D(S)$ and $S=S^*$.■

Since S is self-adjoint and injective we can define $P_\pm$ as the complementary $(\cdot,\cdot)_A$- orthogonal projections of H onto the maximal S-positive/negative S-invariant subspaces of H. Moreover $P_\pm$, as well as $Q_\pm$, are invariant on D(T) and are bounded complementary projections on the complete inner product space D(T) with the graph inner product defined by

$$(f,g)_{GT} = (f,g) + (Tf,Tg). \tag{5}$$

The self-adjointness of S with respect to the inner product (4) allows the machinery of the Spectral Theorem to be introduced. If $F(\cdot)$ is the resolution of the identity associated with S, we can define the operator valued functions

$$\exp\{\mp zT^{-1}A\}P_\pm h = \pm \int_0^{\pm\infty} e^{\mp z/t}dF(t)h, \quad h\in H, \tag{6}$$

for $\mathrm{Re}z>0$. Then the restrictions of $\exp\{\mp zT^{-1}A\}P_\pm$ to Ran $P_\pm$ are bounded analytic semigroups on Ran $P_\pm$ whose infinitesimal generators are the restrictions of $\pm T^{-1}A$ to Ran $P_\pm$. From the injectivity of A and the dominated convergence theorem, we have $\lim_{x\to\infty} \|\exp\{\mp xT^{-1}A\}P_\pm h\| = 0$ for $h\in H$. Moreover, the strong (and even uniform) derivative of the expression (6) exists for $x\in(0,\infty)$, belongs to D(T) and satisfies the differential equation (1).

2. THE BOUNDARY VALUE PROBLEM

In this section we shall prove the existence and uniqueness theorem for the right (left) half space problem

$$(T\psi)'(x) = -A\psi(x), \quad 0<\pm x<\infty, \tag{1}$$

$$Q_\pm\psi(0) = \varphi_\pm, \tag{2}$$

$$\lim_{x\to\pm\infty} \|\psi(x)\| < \infty. \tag{3}$$

We define a solution of the boundary value problem (1)–(3) for any $\varphi_\pm\in Q_\pm[D(T)]$ to be a continuous function $\psi:[0,\infty)\to D(T)$ ($\psi:(-\infty,0]\to D(T)$) such that $T\psi$ is strongly differentiable on $(0,\infty)$ ($(-\infty,0)$) and Eqs. (1)–(3) are satisfied.

LEMMA II.2. $\psi(x)$ is a solution of the boundary value problem (1)–(3) if and only if

$$\psi(x) = e^{-xT^{-1}A}h_\pm, \quad 0<\pm x<\infty, \tag{7}$$

for some $h_\pm\in\text{Ran } P_\pm\cap D(T)$ with $Q_\pm h=\varphi_\pm$. Such solutions are strongly differentiable on $(0,\infty)$ ($(-\infty,0)$) and vanish at infinity with respect to the original norm on H as well as the graph norm on D(T).

Proof: We will consider the right half space (1)–(3) only. The proof for the left half space problem goes along the same lines. Suppose that

$\psi:[0,\infty)\mapsto D(T)$ is a solution of the boundary value problem. Using the facts that $\psi:[0,\infty)\mapsto H$ is continuous, A is bounded and $(T\psi)'(x)=-A\psi(x)$, we have that the function $(T\psi)'$ is bounded and continuous on $(0,\infty)$. Since, for $0<\epsilon,x<\infty$,

$$T\psi(x) - T\psi(\epsilon) = \int_{\epsilon}^{x}(T\psi)'(x)dx = -A\int_{\epsilon}^{x}\psi(x)dx \tag{8}$$

it follows that $T\psi$ is continuous on $(0,\infty)$ and $\|T\psi(x)\|=O(x)$ as $x\to\infty$. For all $\mathrm{Re}\lambda<0$ and $m>0$, we have $\int_{\epsilon}^{m}e^{x/\lambda}\psi(x)dx\in D(T)$ and

$$T\int_{\epsilon}^{m}e^{x/\lambda}\psi(x)dx = \int_{\epsilon}^{m}e^{x/\lambda}T\psi(x)dx,$$

whence

$$0 = \lambda\int_{\epsilon}^{m}e^{x/\lambda}[(T\psi)'(x)+A\psi(x)]dx =$$

$$= [\lambda e^{x/\lambda}T\psi(x)]_{x=\epsilon}^{x=m} - (T-\lambda A)\int_{\epsilon}^{m}e^{x/\lambda}\psi(x)dx. \tag{9}$$

Notice that from (8) it follows that $T\psi(x)$ has a strong limit as $x\to 0$. Using the fact that ψ is continuous on $[0,\infty)$ and T is a closed operator, the limit is, in fact, $T\psi(0)$. Thus one can conclude that (9) is valid for $\epsilon=0$. Noting also that $\lambda e^{x/\lambda}T\psi(x)$ vanishes as $x\to\infty$, we have

$$\int_{0}^{\infty}e^{x/\lambda}\psi(x)dx = \lambda(\lambda-A^{-1}T)^{-1}A^{-1}T\psi(0)$$

for all nonreal λ in the left half plane. The left hand side of this expression has an analytic continuation to the left half plane and so does the right hand side. But this forces $\psi(0)\in \mathrm{Ran}\, P_+\cap D(T)$ and thus Eqs. (4)-(5) specify an initial value problem on Ran P_+ whose solution must have the form (7). The identity $Q_+h=\varphi_+$ is immediate, as is the proof of the converse argument.■

Lemma II.2 provides a solution of the boundary value problems (1)-(3) if one is able to find vectors $h_\pm$ such that the conditions $Q_\pm h=\varphi_\pm$ are satisfied. Since $h_\pm\in \mathrm{Ran}\, P_\pm\cap D(T)$ one can combine these two conditions into the following single equation

$$(Q_+P_+ + Q_-P_-)h = \varphi$$

where $h=h_++h_-$ and $\varphi=\varphi_++\varphi_-$. Let $V=Q_+P_++Q_-P_-$. Clearly V maps Ran $P_\pm$ into Ran $Q_\pm$. Therefore if we show that V is invertible, $E=V^{-1}$ is bounded the choice of the vectors $h_\pm$ will be possible. One can simply take $h_\pm=E\varphi_\pm$. To secure that $h_\pm$ belong to D(T) whenever $\varphi_\pm \in D(T)$ we will also have to prove that V maps D(T) onto D(T). It is clear that the above choice will be unique and consequently the boundary value problem (1)-(3) will be uniquely solvable.

The operator V was first introduced by Hangelbroek [18,19] in the context of isotropic one-speed neutron transport. In that study a Wiener-Hopf factorization was assumed to exist for the dispersion function associated with the integral equation, and the invertibility of V was a consequence of the known factorization. It was not until 1977, in joint work with Lekkerkerker [23], that the invertibility of V was shown to be equivalent to the Wiener-Hopf factorization.

The operator $E=V^{-1}$ is sometimes refered to as the albedo operator. One can give simple physical interpretation for the action of the albedo operator. Namely, in transport theory language, if $\varphi_\pm \in$Ran $Q_\pm$ is an incoming flux at the boundary $x=0$ for the right (left) half space problem then $E\varphi_\pm$ will be the corresponding total (i.e. incoming and reflected) flux at $x=0$.

The properties of the operator V are collected in the next lemma. The proof of this lemma requires a regularity condition for the collision operator A. Since the proof is rather lengthy and technical, we will postpone it as well as the formulation of the regularity condition to Section 3. Here we limit ourselves to the statement of the lemma.

LEMMA II.3. The operator V is invertible and $E=V^{-1}$ is bounded on H and V maps $D(T)$ onto $D(T)$.

Now Lemmas II.2 and II.3 give us the principal result of this section.

THEOREM II.4. The right (left) half space problem

$$(T\psi)'(x) = -A\psi(x), \quad 0<\pm x<\infty,$$

$$Q_{\pm}\psi(0) = \varphi_{\pm},$$

$$\lim_{x\to\pm\infty} \|\psi(x)\| < \infty,$$

is uniquely solvable for each $\varphi_{\pm}\in Q_{\pm}[D(T)]$, and the solution is

$$\psi(x) = e^{-xT^{-1}A}E\varphi_{\pm}, \quad 0\le\pm x<\infty.$$

It is possible to seek solutions of the boundary value problem for all $\varphi_{\pm}\in Q_{\pm}[H]$ rather then just $\varphi_{\pm}\in Q_{\pm}[D(T)]$. However, in this case it seems necessary to reformulate the problem slightly. The differential equation (1) is replaced by

$$T\psi'(x) = -A\psi(x), \quad 0<\pm x<\infty, \tag{10}$$

and a solution is defined to be a continuous function $\psi:[0,\infty)\to H$ ($\psi:(-\infty,0]\to H$) which is continuously differentiable on $(0,\infty)$ ($(-\infty,0)$) such that $\psi'(x)\in D(T)$ for $x\in(0,\infty)$ ($x\in(-\infty,0)$), and which satisfies (10)-(2)-(3). Then one can prove the following analog of Theorem II.4.

THEOREM II.5. The right (left) half space problem

$$T\psi'(x) = -A\psi(x), \quad 0<\pm x<\infty,$$

$$Q_{\pm}\psi(0) = \varphi_{\pm},$$

$$\lim_{x\to\pm\infty} \|\psi(x)\| < \infty,$$

is uniquely solvable for each $\varphi_\pm \in Q_\pm[H]$, and the solution is

$$\psi(x) = e^{-xT^{-1}A}E\varphi_\pm, \quad 0\le\pm x<\infty.$$

3. PROPERTIES OF THE OPERATOR V

We prove here Lemma II.3 stated in the previous section. We shall establish the injectivity of V on H and the compactness of I−V on H and on D(T) equipped with the graph norm (5). Once these are proved, the Fredholm alternative gives the boundedness of V^{-1} and shows that $V[D(T)]=D(T)$.

We present four technical lemmas which will be used in the subsequent analysis. The first is a result on Bochner integrable functions. The proof of this result as well as the exposition of Bochner integral theory can be found in the monograph by Hille and Phillips [24].

LEMMA II.6. Let T be a closed linear operator on H. If f is a Bochner integrable H−valued function on Ω such that Tf is also Bochner integrable on Ω, then

$$\int_\Omega f(\lambda)d\lambda \in D(T)$$

and

$$T\left(\int_\Omega f(\lambda)d\lambda\right) = \int_\Omega Tf(\lambda)d\lambda.$$

The next lemma is a consequence of the norm closedness of the ideal of compact operators in the algebra $\mathcal{L}(H)$.

LEMMA II.7. The integral of a (norm) continuous compact operator valued function with integrable norm is a compact operator.

The following two lemmas involving fractional powers of positive operators are standard. For compeletenees we will present proofs, which have been taken from Krasnoselskii et al. [28]. The first lemma is a consequence of the Hölder inequality, while the second one was originally proved by Krein and Sobolevskii [30].

<u>LEMMA</u> <u>II.8</u>. Let A be a positive self-adjoint operator. Then for any $\tau\in(0,1)$ and any $h\in D(A)$ we have $\|A^{\tau}h\|\leq\|Ah\|^{\tau}\|h\|^{1-\tau}$.

<u>Proof</u>: Let E be the resolution of the identity for the operator A. Then if $\tau\in(0,1)$ and $h\in D(A)$, it follows from the Hölder inequality that

$$\|A^{\tau}h\|^2 = \int_0^{\infty}\lambda^{\tau}d(E(\lambda)h,h) \leq$$

$$\left(\int_0^{\infty}\lambda^2 d(E(\lambda)h,h)\right)^{\tau}\left(\int_0^{\infty}d(E(\lambda)h,h)\right)^{1-\tau} = \|Ah\|^{2\tau}\|h\|^{2(1-\tau)}.\blacksquare$$

<u>LEMMA</u> <u>II.9</u>. Let A be a strictly positive self-adjoint operator and B a closed operator satisfying $\|Bh\|\leq k\|Ah\|^{\tau}\|h\|^{1-\tau}$ for any D(A) and some $\tau\in(0,1)$. Then for all $\tau<\delta<1$, $D(A^{\delta})\subset D(B)$ and for any $h\in D(A^{\delta})$, $\|Bh\|\leq k_0\|A^{\delta}h\|$.

<u>Proof</u>: First of all note that since $\sigma(A)\subset[a,\infty)$ for same $a>0$ and $D(A)\subset D(B)$, the vector valued function

$$t\longrightarrow t^{-\tau}B(tI+A)^{-1}h = BA^{-1}t^{-\tau}A(tI+A)^{-1}h$$

is well defined and continuous on $(0,\infty)$ for any $h\in H$.

Now using the hypothesis of the lemma and the following estimates

$$\|(tI+A)^{-1}\|\leq(a+t)^{-1},$$

$$\|A(tI+A)\|\leq 1 \quad \text{for any } t\geq 0\ ,$$

we show that for $\tau<\delta<1$

$$\int_0^\infty t^{-\delta}\|B(tI+A)^{-1}h\|dt\leq$$

$$k\int_0^\infty t^{-\delta}\|A(tI+A)^{-1}h\|^{\tau}\|(tI+A)^{-1}h\|^{1-\tau}dt\leq$$

$$k\int_0^\infty t^{-\delta}(a+t)^{\tau-1}dt\|h\| =k_1\|h\|.$$

This means that $\int_0^\infty t^{-\delta}(tI+A)^{-1}hdt \in D(B)$ and

$$B\int_0^\infty t^{-\delta}(tI+A)^{-1}hdt = \int_0^\infty t^{-\delta}B(tI+A)^{-1}hdt$$

(cf. Lemma II.6). Therefore, since for any $\alpha\in(0,1)$

$$A^{-\alpha} = \pi^{-1}\sin\alpha\pi\int_0^\infty t^{-\alpha}(tI+A)^{-1}dt$$

we have the estimate $\|BA^{-\delta}h\|\leq k_0\|h\|$ for any $h\in H$, where $k_0=k_1\pi^{-1}\sin\alpha\pi$. Now this implies that $D(A^{\delta})\subset D(B)$ and $\|Bh\|\leq k_0\|A^{\delta}h\|$ for any $h\in D(A^{\delta})$. ∎

In addition to the compactness of I−A, we shall assume throughout the regularity condition:

$$\exists\alpha\in(0,1) \text{ and } \omega>\max\{\tfrac{1+\alpha}{2},\tfrac{2-\alpha}{2}\}\text{: } \mathrm{Ran}(I-A)\subset\mathrm{Ran}(|T|^{\alpha})\cap D(|T|^{1+\omega}). \tag{10}$$

<u>LEMMA</u> <u>II.10</u>. The operators $P_\pm-Q_\pm$ are compact on H and the restrictions of $P_\pm-Q_\pm$ to D(T) are compact on D(T) endowed with the graph inner product (5). Moreover $(P_\pm-Q_\pm)[H]\subset D(T)$.

Proof: Since $P_+-Q_+=-(P_--Q_-)$ we will give the proof for P_+-Q_+ only. We will prove first that P_+-Q_+ is compact on H and $(P_+-Q_+)[H]\subset D(T)$. Let $\Delta_1=\Delta(\epsilon,M)$ denote the oriented curve composed of the straight lines from $-i\epsilon$ to $-i$, from $-i$ to $M-i$, from $M+i$ to i, and from $+i$ to $+i+\epsilon$. Let $\Delta_2=\Delta(M)$ denote the oriented curve composed of the straight lines from $M-i$ to $+\infty-i$ and from $+\infty+i$ to $M+i$. Denote $\Delta=\Delta_1\cup\Delta_2$ with the orientation inherited from Δ_1 and Δ_2. We recall that the projections P_+ and Q_+ are bounded on H and on D(T) endowed with the graph inner product (5). We have the integral representations

$$P_+ = \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\Delta}(\lambda-S)^{-1}d\lambda,$$

$$Q_+ = \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\Delta}(\lambda-T)^{-1}d\lambda,$$

where the limits are in the strong topology. Let

$$P_+^{(1)} = \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\Delta_1}(\lambda-S)^{-1}d\lambda, \quad P_+^{(2)} = \frac{1}{2\pi i}\int_{\Delta_2}(\lambda-S)^{-1}d\lambda,$$

and

$$Q_+^{(1)} = \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\Delta_1}(\lambda-T)^{-1}d\lambda, \quad Q_+^{(2)} = \frac{1}{2\pi i}\int_{\Delta_2}(\lambda-T)^{-1}d\lambda.$$

Then one has

$$P_+-Q_+ = (P_+^{(1)}-Q_+^{(1)}) + (P_+^{(2)}-Q_+^{(2)}). \tag{11}$$

We will show that $P_+^{(1)}-Q_+^{(1)}$ and $P_+^{(2)}-Q_+^{(2)}$ are compact on H, and $(P_+^{(1)}-Q_+^{(1)})[H]\subset D(T)$ as well as $(P_+^{(2)}-Q_+^{(2)})[H]\subset D(T)$.

Consider first

$$P_+^{(1)}-Q_+^{(1)} = \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\Delta_1}[(\lambda-S)^{-1}-(\lambda-T)^{-1}]d\lambda. \tag{12}$$

We shall see that this limit can be taken in the norm topology. We exploit the regularity condition (10) and obtain from the Closed Graph Theorem the existence of a bounded operator D such that $B=|T|^{\alpha}D$. Then for nonreal λ

$$(\lambda-S)^{-1}-(\lambda-T)^{-1} = (\lambda-T)^{-1}(S-T)(\lambda-S)^{-1} =$$
$$= (\lambda-T)^{-1}BS(\lambda-S)^{-1} = (\lambda-T)^{-1}|T|^{\alpha}D\ S(\lambda-S)^{-1},$$

which shows that $(\lambda-S)^{-1}-(\lambda-T)^{-1}$ is a compact operator on H. Next, since S is self-adjoint on H with respect to the inner product (4), we may use the Spectral Theorem to show the norm estimate

$$\|S(i\mu-S)^{-1}\|_{\mathcal{L}(H_A)} \le \sup_{t\in\mathbb{R}} \left|\frac{t}{i\mu-t}\right| \le 1.$$

But the inner products on H and H_A are equivalent, and thus also are the $\mathcal{L}(H)$ and $\mathcal{L}(H_A)$ norms, so there is a constant c_0 such that $\|S(i\mu-S)^{-1}\|_{\mathcal{L}(H)} \le c_0$.

Likewise, from the Spectral Theorem,

$$\|\,|T|^{\alpha}(i\mu-T)^{-1}\|_{\mathcal{L}(H)} \le \sup_{t\in\mathbb{R}} \left|\frac{t^{\alpha}}{i\mu-t}\right| \le c_{\alpha}|\mu|^{\alpha-1}.$$

Thus

$$\|(\int_{\Delta(\epsilon,M)} - \int_{\Delta(\gamma,M)})[(\lambda-S)^{-1}-(\lambda-T)^{-1}]d\lambda\|_{\mathcal{L}(H)} \le$$

$$\le 2\|D\|_{\mathcal{L}(H)}c_0c_{\alpha}\int_{\epsilon}^{\gamma}\mu^{\alpha-1}d\mu$$

which shows that the limit (12) exists in the operator norm topology, and consequently proves the compactness of $P_+^{(1)}-Q_+^{(1)}$.

Let $h\in H$. Since $[(\lambda-S)^{-1}-(\lambda-T)^{-1}]h$ and $T[(\lambda-S)^{-1}-(\lambda-T)^{-1}]h = T(\lambda-T)^{-1}BS(\lambda-S)^{-1}h$ are bounded and continuous functions on Δ_1, $\frac{1}{2\pi i}\int_{\Delta_1}[(\lambda-S)^{-1}-(\lambda-T)^{-1}]hd\lambda\in D(T)$ and

$$T(\frac{1}{2\pi i}\int_{\Delta_1}[(\lambda-S)^{-1}-(\lambda-T)^{-1}]hd\lambda) =$$

$$= \frac{1}{2\pi i}\int_{\Delta_1}T[(\lambda-S)^{-1}-(\lambda-T)^{-1}]hd\lambda.$$

Now, note that

$$\|(\int_{\Delta(\cdot\ M)} - \int_{\Delta(\gamma,M)})T[(\lambda-S)^{-1}-(\lambda-T)^{-1}]d\lambda\|_{\mathcal{L}(H)} \le$$

$$\leq 2c_0\|B\|_{\mathcal{L}(H)}|\epsilon-\gamma|$$

implies the existence of the limit

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\Delta(\epsilon,M)}T[(\lambda-S)^{-1}-(\lambda-T)^{-1}]d\lambda$$

in the operator norm topology. Therefore, by the closedness of T,

$$(P_+^{(1)}-Q_+^{(1)})h = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\Delta_1}[(\lambda-S)^{-1}-(\lambda-T)^{-1}]hd\lambda\in D(T),$$

which proves the inclusion $(P_+^{(1)}-Q_+^{(1)})[H]\subset D(T)$.

Next let us consider

$$P_+^{(2)}-Q_+^{(2)} = \frac{1}{2\pi i}\int_{\Delta_2}[(\lambda-S)^{-1}-(\lambda-T)^{-1}]d\lambda. \tag{13}$$

Since, for nonreal λ, $(\lambda-S)^{-1}-(\lambda-T)^{-1} = (\lambda-T)^{-1}BS(\lambda-S)^{-1}$ is compact, it is enough to show the integrability of this operator. We can rewrite the operator in the following form:

$$(\lambda-S)^{-1}-(\lambda-T)^{-1} = (\lambda-T)^{-1}BS(\lambda-T)^{-1}[(\lambda-T)(\lambda-S)^{-1}]$$
$$= (\lambda-T)^{-1}CT(\lambda-T)^{-1}[(\lambda-T)(\lambda-S)^{-1}].$$

Evidently $\mathrm{Ran}\,(\lambda-S)^{-1}=D(\lambda-T)$ and by the Closed Graph Theorem $(\lambda-T)(\lambda-S)^{-1}$ is a bounded operator on H. In fact we will show that the norm of this operator is uniformly bounded for $\lambda\in\Delta_2$. We have the identity $(\lambda-T)(\lambda-S)^{-1} = I+BS(\lambda-S)^{-1} = I+CT(\lambda-S)^{-1}$. By virtue of the estimate $\|(\lambda-S)^{-1}\|_{\mathcal{L}(H)}\leq c_0$ for $\lambda\in\Delta_2$, it is enough to show that CT is bounded on $D(T)=D(S)$. But by the regularity condition (10), $\mathrm{Ran}\,C = \mathrm{Ran}\,B \subset D(|T|^{1+\omega})\subset D(T)$ and then by the Closed Graph Theorem, the operator TC is bounded on H, thus $CT\subset (TC)^*$ is bounded on D(T). Finally, for any $\lambda\in\Delta_2$ we have $\|(\lambda-T)(\lambda-S)^{-1}\|_{\mathcal{L}(H)}\leq 1+\|(TC)^*\|_{\mathcal{L}(H)}c_0$, which provides a λ-uniform bound, as claimed.

Therefore it is sufficient to show the integrability of $F(\lambda)$ =

$(\lambda-T)^{-1}CT(\lambda-T)^{-1}$. Let Q_0 be a spectral projection belonging to the spectral decomposition of the self-adjoint operator T such that the resolvent set of $Q_1=I-Q_0$ contains a real neighborhood of zero. We can decompose $F(\lambda)$ as follows:

$$\begin{aligned}F(\lambda) = \\ &(\lambda-T)^{-1}|T|^{-\omega}Q_1|T|^{\omega}C|T|^{1+\nu}|T|^{-1-\nu}T(\lambda-T)^{-1}Q_1+\\ &+(\lambda-T)^{-1}|T|^{-\omega}Q_1|T|^{\omega}CT(\lambda-T)^{-1}Q_0+\\ &+(\lambda-T)^{-1}Q_0C|T|^{1+\nu}|T|^{-1-\nu}T(\lambda-T)^{-1}Q_1+\\ &+(\lambda-T)^{-1}Q_0CT(\lambda-T)^{-1}Q_0,\end{aligned}$$

where $\nu=\frac{\alpha}{2}$ and $2\omega>\max\{1+\alpha,2-\alpha\}$, and we may choose $\omega<1+\frac{\alpha}{2}$. Note that $\nu+\omega>1$. For $\lambda\in[M\pm i,\ \infty\pm i)$ we have the following estimates:

$$\|(\lambda-T)^{-1}|T|^{-\omega}Q_1\|_{\mathcal{L}(H)} \le \text{const.}(\mathrm{Re}\,\lambda)^{-\omega}, \tag{14}$$

$$\||T|^{-1-\nu}T(\lambda-T)^{-1}Q_1\|_{\mathcal{L}(H)} \le \text{const.}(\mathrm{Re}\,\lambda)^{-\nu}, \tag{15}$$

$$\|(\lambda-T)^{-1}Q_0\|_{\mathcal{L}(H)} \le \text{const.}(\mathrm{Re}\,\lambda)^{-1}, \tag{16}$$

$$\|T(\lambda-T)^{-1}Q_0\|_{\mathcal{L}(H)} \le \text{const.}(\mathrm{Re}\,\lambda)^{-1}. \tag{17}$$

Moreover, since $\mathrm{Ran}\ C = \mathrm{Ran}\ B \subset D(|T|^{1+\omega}) \subset D(|T|^{1+\alpha}) \subset D(|T|^{1+\nu}) \subset D(|T|^{\omega})$, both $|T|^{\omega}C$ and $(C|T|^{1+\nu})^{*}=|T|^{1+\nu}C$ are bounded, thus also $C|T|^{1+\nu}$ (on $D(|T|^{1+\nu})$). So we must consider $|T|^{\omega}C|T|^{1+\nu}$.

Choose $\sigma\in(0,1)$. Clearly it is enough to consider $|T|$ on $\mathrm{Ran}\ Q_1$. As $C|T|^{1+\omega}$ is bounded on $D(|T|^{1+\omega})$, we have $\|Ch\|\le k\||T|^{-1-\omega}h\|$ for all $h\in D(|T|^{-1-\omega})=\mathrm{Ran}(|T|^{1+\omega})$. Then, by Lemma II.8, for $h\in D(|T|^{-1-\omega})$ we have $\||C|^{\sigma}h\|\le k^{\sigma}\||T|^{-1-\omega}h\|^{\sigma}\|h\|^{1-\sigma}$. Hence, by Lemma II.9, $\||C|^{\sigma}h\|\le k_0\||T|^{-\delta(1+\omega)}h\|$ for all $h\in D(|T|^{-\delta(1+\omega)})$ and any $\sigma<\delta<1$. Therefore $|T|^{\delta(1+\omega)}|C|^{\sigma}$ and $|C|^{\sigma}|T|^{\delta(1+\omega)}$ are bounded.

For $\delta=\frac{\omega}{1+\omega}$ and $\delta=\frac{1+\nu}{1+\omega}$, respectively, and $\sigma=\frac{1}{2}$ we recover $|T|^{\omega}|C|^{1/2}$ and $|C|^{1/2}|T|^{1+\nu}$ as bounded operators. Then, using the polar decomposition $C=U|C|$, we can represent $|T|^{\omega}C|T|^{1+\nu}$ as a composition of bounded operators; one has

$$|T|^{\omega}C|T|^{1+\nu} = |T|^{\omega}|C|^{1/2}U|C|^{1/2}|T|^{1+\nu}.$$

Now we can estimate the norm of $F(\lambda)$:

$$\|F(\lambda)\|_{\mathcal{L}(H)}\leq c((\mathrm{Re}\lambda)^{-\nu-\omega}+(\mathrm{Re}\lambda)^{-1-\omega}+(\mathrm{Re}\lambda)^{-1-\nu}+(\mathrm{Re}\lambda)^{-2})\leq$$
$$\leq c(\mathrm{Re}\lambda)^{-s},$$

where c is a constant and $s=\min\{\nu+\omega,1+\omega,1+\nu,2\}$. This estimate, along with the uniform boundedness of $(\lambda-T)(\lambda-S)^{-1}$ for $\lambda\in\Delta_2$, shows the integrability of $(\lambda-S)^{-1}-(\lambda-T)^{-1}$ on Δ_2 and completes the proof of the compactness of $P_{+}^{(2)}-Q_{+}^{(2)}$.

Let $h\in H$. Note that $[(\lambda-S)^{-1}-(\lambda-T)^{-1}]h\in D(T)$ for any $\lambda\in\Delta_2$. In order to prove that $(P_{+}^{(2)}-Q_{+}^{(2)})h\in D(T)$ it is enough to show that $T[(\lambda-S)^{-1}-(\lambda-T)^{-1}]h$ is Bochner integrable on Δ_2 (cf. Lemma II.6). Since

$$\|T[(\lambda-S)^{-1}-(\lambda-T)^{-1}]h\| = \|T(\lambda-T)^{-1}BS(\lambda-S)^{-1}h\| =$$
$$= \|T(\lambda-T)^{-1}CT(\lambda-T)^{-1}[(\lambda-T)(\lambda-S)^{-1}h]\| \leq$$
$$\leq (1+\|(TC)^{*}\|_{\mathcal{L}(H)}c_0)\| T(\lambda-T)^{-1}CT(\lambda-T)^{-1}\|_{\mathcal{L}(H)}\|h\| =$$
$$= (1+\|(TC)^{*}\|_{\mathcal{L}(H)}c_0)\|TF(\lambda)\|_{\mathcal{L}(H)}\|h\|,$$

it is sufficient to prove the integrability of $\|TF(\lambda)\|_{\mathcal{L}(H)}$ on Δ_2. This can be done in the same way as in the case of $\|F(\lambda)\|_{\mathcal{L}(H)}$, the only change being that one must use $|T|^{-1-\gamma}|T|^{1+\gamma}$ with $\max\{\frac{1+\alpha}{2},\frac{2-\alpha}{2}\}<\gamma<\omega$ instead of $|T|^{-\omega}|T|^{\omega}$ in the decomposition of $TF(\lambda)$. Namely, we decompose $TF(\lambda)$ as follows: $TF(\lambda)$ =

$$T(\lambda-T)^{-1}|T|^{-1-\gamma}Q_1|T|^{1+\gamma}C|T|^{1+\nu}|T|^{-1-\nu}T(\lambda-T)^{-1}Q_1$$
$$+T(\lambda-T)^{-1}|T|^{-1-\gamma}Q_1|T|^{1+\gamma}CT(\lambda-T)^{-1}Q_0$$
$$+T(\lambda-T)^{-1}Q_0C|T|^{1+\nu}|T|^{-1-\nu}T(\lambda-T)^{-1}Q_1$$
$$+T(\lambda-T)^{-1}Q_0CT(\lambda-T)^{-1}Q_0$$

and use the estimates (15)-(17) and

$$\|T(\lambda-T)^{-1}|T|^{-1-\gamma}Q_1\|_{\mathcal{L}(H)} \leq \text{const.}(\mathrm{Re}\,\lambda)^{-\gamma}$$

along with the boundedness of $|T|^{1+\gamma}C$, $C|T|^{1+\nu}$ and $|T|^{1+\gamma}C|T|^{1+\nu}$ which are proved in the same way as before. This ends the proof of the inclusion $(P_+^{(2)}-Q_+^{(2)})[H] \subset D(T)$.

Now (11) implies that P_+-Q_+ is compact on H and $(P_+-Q_+)[H]\subset D(T)$.

It remains to prove that the restriction of P_+-Q_+ to D(T) is compact with respect to the graph norm (5). If $P_+^\dagger=AP_+A^{-1}$, then $P_+^\dagger-Q_+=P_+-Q_++P_+C-BP_+-BP_+C$ is compact in H and $(P_+^\dagger-Q_+)Th=T(P_+-Q_+)h$ for $h\in D(T)$. Let $\{h_n\}$ be a $\|\cdot\|_{GT}$-bounded sequence in D(T). Then the squences $\{h_n\}$ and $\{Th_n\}$ are bounded in H-norm. Since P_+-Q_+ and $P_+^\dagger-Q_+$ are compact on H, we can choose a subsequence $\{h_{n_m}\}$ $\subset \{h_n\}$ such that both sequences $\{(P_+-Q_+)h_{n_m}\}$ and $\{(P_+^\dagger-Q_+)Th_{n_m}\}$ are Cauchy in H. Now since

$$\|(P_+-Q_+)h_{n_m}\|_{GT}^2 = \|(P_+-Q_+)h_{n_m}\|^2 + \|T(P_+-Q_+)h_{n_m}\|^2 =$$

$$= \|(P_+-Q_+)h_{n_m}\|^2 + \|(P_+^\dagger-Q_+)Th_{n_m}\|^2 ,$$

it is clear that $\{(P_+-Q_+)h_{n_m}\}$ is a Cauchy sequence in the graph norm . This proves that P_+-Q_+ is is compact on D(T) endowed with the graph norm and completes the proof of the lemma.∎

From the identity $I-V=(Q_--Q_+)(P_+-Q_+)$ and Lemma II.10, we have the following corollary.

COROLLARY. I−V is compact (in both topologies under consideration).

The injectivity of V remains to be demonstrated. We have

LEMMA II.11. The operator V has zero null space.

Proof: Suppose $Vh=0$ for some $h\in H$. Then $Q_+P_+h=-Q_-P_-h=0$ yields that (cf. the previous lemma)

$$P_+h = (P_+-Q_+)P_+h\in D(T),$$

$$P_-h = (P_--Q_-)P_-h\in D(T),$$

whence $h=P_+h+P_-h\in D(T)$. Thus $\mathrm{Ker}V \subset D(T)$. Following [23], we note $P_+h\in \mathrm{Ran}\ Q_-\cap \mathrm{Ran}\ P_+$, so $(TP_+h,P_+h)\leq 0$ and $(TP_+h,P_+h)=(A^{-1}TP_+h,P_+h)_A\geq 0$. Thus we conclude that $P_+h=0$. In a similar way one may derive $P_-h=0$.∎

Finally we prove

LEMMA II.3. The operator V is invertible and $E=V^{-1}$ is bounded on H and V maps D(T) onto D(T).

Proof: Since V is invertible and I−V is compact on H, from the Fredholm alternative follows that the equation $Vg=f$ has a unique solution for any $f\in H$, i.e. $\mathrm{Ran}V=H$. Then by the Closed Graph Theorem $E=V^{-1}$ is bounded on H. The proof that V maps D(T) onto D(T) uses the same argument i.e. the Fredholm alternative in the space D(T) endowed with the graph norm (5).∎

CHAPTER III

CONSERVATIVE MODELS: COLLISION OPERATORS WITH NONTRIVIAL KERNEL

In the previous chapter we have assumed that A is a strictly positive operator. Requiring that $\mathrm{Ker}A=\{0\}$ excludes from consideration many physically important problems, in particular linearized gas kinetic equations, where conservation laws result in the collision operator A having a nontrivial kernel, consisting of collision invariants (cf. [10]). In this chapter we will generalize the results of Chapter II to the case where A is positive but has a nontrivial kernel. As before we will assume that A is a compact perturbation of the identity and satisfies a regularity condition which is a slightly strengthened version of condition II.(10). The condition we henceforth impose reads as follows:

$$\exists\alpha\in(0,1) \text{ and } \omega>\max\{\tfrac{1+\alpha}{2},\tfrac{2-\alpha}{2}\}\text{: } \mathrm{Ran}(I-A)\subset\mathrm{Ran}(|T|^{\alpha})\cap D(|T|^{3+\omega}). \tag{1}$$

1. DECOMPOSITIONS AND REDUCTION

Let L be a linear operator on H. We define the zero root linear manifold $Z_0(L)$ of L by

$$Z_0(L) = \{ f\in D(L) \mid f\in D(L^n) \text{ and } L^n f=0 \text{ for some } n\in\mathbb{N} \}.$$

In addition to the regularity condition above we will assume that the zero root manifold $Z_0(T^{-1}A)$ is contained in $D(T)$. The following lemma characterizes $Z_0(T^{-1}A)$ and $Z_0(AT^{-1})$, and yields useful decompositions of H. For isotropic neutron transport, these results are due to Lekkerkerker [33] and for more general cases with T bounded to van der Mee [34] and Greenberg et al. [17].

<u>LEMMA</u> <u>III.1</u>. (i) The Jordan chains of $T^{-1}A$ at the zero eigenvalue have at most length two, i.e. if $f\in Z_0(T^{-1}A)$, then there exists $g\in Z_0(T^{-1}A)$ such that $T^{-1}Af=g$ and $T^{-1}Ag=0$. Hence $Z_0(T^{-1}A) = \mathrm{Ker}((T^{-1}A)^2)$.

(ii) $T[Z_0(T^{-1}A)] = Z_0(AT^{-1})$. (2)

(iii) The following decompositions hold true:

$$Z_0(T^{-1}A) \oplus Z_0(AT^{-1})^{\perp} = H, \tag{3a}$$

$$Z_0(AT^{-1}) \oplus Z_0(T^{-1}A)^{\perp} = H. \tag{3b}$$

(iv) $A[Z_0(AT^{-1})^{\perp}] = Z_0(T^{-1}A)^{\perp} = \overline{T[Z_0(AT^{-1})^{\perp}\cap D(T)]}$. (4)

<u>Proof</u>: (i) Let $f\in Z_0(T^{-1}A)$ and $g=T^{-1}Af$. Clearly $g\in Z_0(T^{-1}A)$. Let $h=T^{-1}Ag$. We will show that if $T^{-1}Ah=0$ then $h=0$. Assume that $T^{-1}Ah=0$. Then $(Ag,g)=(Th,g)=(h,Tg)=(h,Af)=(Ah,f)=0$. Since A is positive this implies that $Th=Ag=0$ and consequently that $h=0$.

(ii) First we will show the inclusion $T[Z_0(T^{-1}A)]\subset Z_0(AT^{-1})$. Let $f\in\mathrm{Ker}A$ then $Tf\in D(AT^{-1})$ and $AT^{-1}(Tf)=0$ so $Tf\in T[\mathrm{Ker}A]\subset \mathrm{Ker}(AT^{-1})\subset Z_0(AT^{-1})$. Moreover if $f\in D(T^{-1}A)$ and $T^{-1}Af\in\mathrm{Ker}A$ then $Tf\in D(AT^{-1})$ and $Af=T(T^{-1}Af)\in\mathrm{Ker}(AT^{-1})$. Therefore $(AT^{-1})^2(Tf)=AT^{-1}Af=0$ so $Tf\in Z_0(AT^{-1})$. This shows that indeed $T[Z_0(T^{-1}A)]\subset Z_0(AT^{-1})$. Now we will prove that $Z_0(AT^{-1}) \subset T[Z_0(T^{-1}A)]$. Let $f \in \mathrm{Ker}(AT^{-1})$. Then $T^{-1}f \in \mathrm{Ker}A$ implies that $f \in T[\mathrm{Ker}A] \subset T[Z_0(T^{-1}A)]$. Next assume that $f \in \mathrm{Ker}(AT^{-1})^2$ and $AT^{-1}f \neq 0$. Then $AT^{-1}f = Tg$ for some $g \in \mathrm{Ker}A$. Since $Tg \in \mathrm{Ran}A$, there exists $h \in \mathrm{Ran}A$ such that $Tg = Ah$. Then $h \in \mathrm{Ker}((T^{-1}A)^2) \subset D(T)$ and $T^{-1}f = h$ which shows that $f \in T[\mathrm{Ker}(T^{-1}A)^2] = T[Z_0(T^{-1}A)]$. Similary using the induction one shows that $\mathrm{Ker}((AT^{-1})^n) \subset T[\mathrm{Ker}(T^{-1}A)^n] \subset T[Z_0(T^{-1}A)]$ for any $n \in \mathbb{N}$. Then this implies that $Z_0(AT^{-1}) \subset T[Z_0(T^{-1}A)]$ and ends the proof of (ii).

(iii) First we will show that $Z_0(T^{-1}A)$ is nondegenerate, i.e.,

$$\{ f \in Z_0(T^{-1}A) \mid (Tf,g)=0 \text{ for all } g \in Z_0(T^{-1}A) \} = \{0\}.$$

Let $f \in Z_0(T^{-1}A)$ and $(Tf,g)=0$ for all $g \in Z_0(T^{-1}A)$. Then $Tf \in Z_0(T^{-1}A)^{\perp} \subset (\mathrm{Ker}A)^{\perp} = \mathrm{Ran}A$. Thus $f = T^{-1}Ah$ for some $h \in H$. But then h must belong to $Z_0(T^{-1}A)$ and $(Ah,h)=(Tf,h)=0$. Therefore $Tf=Ah=0$ and conseqently $f=0$.

Now let $f \in Z_0(T^{-1}A) \cap Z_0(AT^{-1})^{\perp}$. Then (2) implies that $(Tf,g)=0$ for all $g \in Z_0(T^{-1}A)$. Since $Z_0(T^{-1}A)$ is nondegenerate, $f=0$. This shows that $Z_0(T^{-1}A) \cap Z_0(AT^{-1})^{\perp} = \{0\}$. Likewise if $f \in Z_0(AT^{-1}) \cap Z_0(T^{-1}A)^{\perp}$ then $f=Th$ for some $h \in Z_0(T^{-1}A)$ and $(Th,g)=0$ for all $g \in Z_0(T^{-1}A)$. Hence $h=0$ and then $f=0$ showing that $Z_0(AT^{-1}) \cap Z_0(T^{-1}A)^{\perp} = \{0\}$. Now we use a simple dimension argument. We have

$$\operatorname{codim} Z_0(AT^{-1})^{\perp} = \dim Z_0(AT^{-1}) = \dim Z_0(T^{-1}A)$$

and similary

$$\operatorname{codim} Z_0(T^{-1}A)^{\perp} = \dim Z_0(T^{-1}A) = \dim Z_0(AT^{-1})$$

which imply the decompositions (3a) and (3b) respectively.

(iv) Note that T maps $Z_0(AT^{-1})^{\perp} \cap D(T)$ into $Z_0(T^{-1}A)^{\perp}$ and $Z_0(T^{-1}A)$ onto $Z_0(AT^{-1})$. Now using the fact that T has dense range one proves that

$$Z_0(T^{-1}A)^{\perp} = \overline{T[\, Z_0(AT^{-1})^{\perp} \cap D(T)\,]}.$$

Clearly A maps $Z_0(T^{-1}A)$ into $Z_0(AT^{-1})$ and $Z_0(AT^{-1})^\perp$ into $Z_0(T^{-1}A)^\perp$. Since $\mathrm{Ker}A \subset Z_0(T^{-1}A)$, the codimension of $A[Z_0(T^{-1}A)]$ in the subspace $Z_0(AT^{-1})$ is equal to dim KerA. But in the first place codim RanA=dim KerA and therefore, since the restriction of A to the subspace $Z_0(AT^{-1})^\perp$ has closed range in $Z_0(T^{-1}A)^\perp$, one must have

$$A[Z_0(AT^{-1})^\perp] = Z_0(T^{-1}A)^\perp,$$

which ends the proof.■

The decompositions (3) will enable us to reduce a boundary value problem with given A (having nontrivial kernel) to one with a strictly positive collision operator. This reduction, in fact, follows immediately from the following proposition.

<u>PROPOSITION</u> <u>III.2</u>. Let β be an invertible operator on $Z_0(T^{-1}A)$ satisfying

$$(T\beta h,h) \geq 0, \; h \in Z_0(T^{-1}A). \tag{5}$$

Let P be the projection of H onto $Z_0(AT^{-1})^\perp$ along $Z_0(T^{-1}A)$. If A is a positive, compact perturbation of the identity with nontrivial kernel and satisfies the regularity condition (1), then A_β defined by

$$A_\beta = T\beta^{-1}(I-P) + AP \tag{6}$$

is a strictly positive operator satisfying

$$A_\beta^{-1}T = \beta \oplus (T^{-1}A \restriction Z_0(AT^{-1})^\perp)^{-1}. \tag{7}$$

A_β is also a compact perturbation of the identity satisfying the condition

$$\mathrm{Ran}(I-A_\beta) \subset \mathrm{Ran}(|T|^\alpha) \cap D(|T|^{1+\omega}), \tag{8}$$

with α and ω as in (1).

Proof: The identity (7) follows immediately from the definition of A_β. Moreover, for $g \in H$ we have

$$(A_\beta g,g) = (APg,Pg) + (T\beta^{-1}(I-P)g,(I-P)g) \geq 0.$$

We have used here (5) for $h=\beta^{-1}T(I-P)g$. Since $\sigma(A) \subset \{0\} \cup [\epsilon,\infty)$ for some $\epsilon>0$ and $Z_0(T^{-1}A)$ has finite dimension, we must have strict positivity for A_β from the obvious triviality of its kernel.

Next, since $A_\beta - A = (A_\beta - A)(I-P)$ has finite rank, A_β must be a compact perturbation of the identity. Furthermore,

$$I-A_\beta = (I-A)-(A_\beta-A)(I-P) = (I-A)-T(\beta^{-1}-T^{-1}A)(I-P) \tag{9}$$

and therefore

$$\mathrm{Ran}(I-A_\beta) \subset \mathrm{Ran}(|T|^\alpha). \tag{10}$$

Also, using that $\mathrm{Ran}(I-A) \subset D(|T|^{3+\omega}) \subset D(|T|^{2+\omega})$ we find that for $h \in Z_0(T^{-1}A)$ and $g=T^{-1}Ah$

$$h = (I-A)h+Tg = (I-A)h+T(I-A)g \in D(|T|^{2+\omega}).$$

But this and (9) imply that

$$\mathrm{Ran}(I-A_\beta) \subset D(|T|^{1+\omega}). \tag{11}$$

Now from (10) and (11) we obtain (8).∎

As in Chapter II one can construct the Hilbert space H_{A_β}. Due to boundedness and strict positivity of A_β the H_{A_β}-norm is equivalent to the original norm on H. The equivalence of these norms also shows that the topology of H_{A_β} does not depend on the choice of the operator β in Proposition III.2. We may define $P_\pm$ as the H_{A_β}-orthogonal complementary projections of H onto the maximal $A_\beta^{-1}T$-positive/negative $A_\beta^{-1}T$-invariant subspaces. Since the decomposition (3a) reduces completely the operators $T^{-1}A$ and $A_\beta^{-1}T$, it clear that PP_+, PP_- and $I-P$ form a family of complementary projections commuting with $T^{-1}A$. One can also show that they are β-independent. Moreover using the results of Section II.1 one shows that the operator $T^{-1}A_\beta$ is densly defined in H. Thus $D(T^{-1}A_\beta \restriction Z_0(AT^{-1})^\perp)=D(T^{-1}A \restriction Z_0(AT^{-1})^\perp)$ is dense in $Z_0(AT^{-1})^\perp$. Since $Z_0(T^{-1}A)$ is finite dimensional, $D(T^{-1}A)=D(T^{-1}A \restriction Z_0(AT^{-1})^\perp)\oplus Z_0(T^{-1}A)$ and $T^{-1}A=(AT^{-1})^*$, the operator $T^{-1}A$ is closed and densly defined in H.

The next proposition gives a decomposition of $Z_0(T^{-1}A)$ into T-positive/negative subspaces and a characterization of β compatible with the intended boundary value problems. A proof of this proposition can be found in [34] and [17] for bounded T; the unbounded case introduces some technicalities connected with D(T).

<u>PROPOSITION</u> <u>III.3</u>. The subspaces

$$M_\pm = \{\mathrm{Ran}\ PP_\mp \oplus \mathrm{Ran}\ Q_\pm\} \cap Z_0(T^{-1}A) \tag{12}$$

satisfy the condition

$$\pm(Tf,f) > 0, \quad 0 \neq f \in M_\pm, \tag{13}$$

while

$$M_+ \oplus M_- = Z_0(T^{-1}A) \tag{14}$$

and

$$\{M_+ \cap \mathrm{Ker}A\} \oplus \{M_- \cap \mathrm{Ker}A\} \oplus T^{-1}A[Z_0(T^{-1}A)] = \mathrm{Ker}A. \tag{15}$$

Moreover, it is possible to choose the operator β such that

$$\mathrm{Ran}\ P_+ \subset \mathrm{Ran}\ PP_+ \oplus \mathrm{Ker}A. \tag{16}$$

<u>Proof</u>: Let us first distinguish between

$$M_\pm = \{PP_\mp[D(T)] \oplus Q_\pm[D(T)]\} \cap Z_0(T^{-1}A) \tag{17}$$

and

$$N_{\pm} = \{PP_{\mp}[H] \oplus Q_{\pm}[H]\} \cap Z_0(T^{-1}A) \tag{18}$$

Since $M_{\pm} \subset N_{\pm}$ are finite dimensional and we have the orthogonal complements

$$(PP_{\pm}[D(T)])^{\perp} = (PP_{\pm}[H])^{\perp},$$

$$(Q_{\pm}[D(T)])^{\perp} = (Q_{\pm}[H])^{\perp},$$

it is clear that $M_{\pm}=N_{\pm}$. Therefore, we shall interpret $M_{\pm}$ as in (17), but use the shorthand notation (12).

Every $f_0 \in M_+$ can be written as $f_0=g_-+h_+$, where $g_- \in PP_-[D(T)]$ and $h_+ \in Q_+[D(T)]$. Thus,

$$0 \le (Th_+,h_+) = (Tf_0,f_0)+(Tg_-,g_-)-(Tf_0,g_-)-(Tg_-,f_0) =$$

$$= (Tf_0,f_0)+(A_{\beta}^{-1}Tg_-,g_-)_{A_{\beta}} \le (Tf_0,f_0),$$

where we have deleted two vanishing terms by using (4). Also, $(Tf_0,f_0)=0$ would imply $(Th_+,h_+)=(A_{\beta}^{-1}Tg_-,g_-)_{A_{\beta}}=0$, thus $h_+=g_-=0$ and $f_0=0$, which settles (13) for M_+. The proof for M_- is analogous.

Next, observe that

$$(TPP_{\mp}[D(T)])^{\perp} = PP_{\pm}[H] \oplus Z_0(T^{-1}A),$$

$$(TQ_{\pm}[D(T)])^{\perp} = Q_{\mp}[H],$$

$$(T[Z_0(T^{-1}A)])^{\perp} = Z_0(AT^{-1})^{\perp}.$$

One easily computes that

$$(T[M_{\pm}])^{\perp} = \{[\mathrm{Ran}\ PP_{\pm} \oplus Z_0(T^{-1}A)] \cap \mathrm{Ran}\ Q_{\mp}\}+Z_0(AT^{-1})^{\perp}.$$

Note that

$$(I-P)[\{\mathrm{Ran}\ PP_{\pm} \oplus Z_0(T^{-1}A)\} \cap \mathrm{Ran}\ Q_{\mp}] =$$

$$\{\mathrm{Ran}\ PP_{\pm} \oplus \mathrm{Ran}\ Q_{\mp}\} \cap Z_0(T^{-1}A).$$

Then we have

$$(T[M_{\pm}])^{\perp} = (I-P)[\{\mathrm{Ran}\ PP_{\pm} \oplus Z_0(T^{-1}A)\} \cap \mathrm{Ran}\ Q_{\mp}] +$$

$$+ P[\{\mathrm{Ran}\ PP_{\pm} \oplus Z_0(T^{-1}A)\} \cap \mathrm{Ran}\ Q_{\mp}] + Z_0(AT^{-1})^{\perp} =$$

$$= \{\mathrm{Ran}\ PP_{\pm} \oplus \mathrm{Ran}\ Q_{\mp}\} \cap Z_0(T^{-1}A) \oplus Z_0(AT^{-1})^{\perp} = M_{\mp} \oplus Z_0(AT^{-1})^{\perp}.$$

Hence, (14) holds true.

Since $(T[\mathrm{Ker}A])^{\perp} \cap Z_0(T^{-1}A) = T^{-1}A[Z_0(T^{-1}A)]$, one obtains easily

$$\{T[M_{\pm} \cap \mathrm{Ker}A]\}^{\perp} \cap \mathrm{Ker}A = \{(T[M_{\pm}])^{\perp} + (T[\mathrm{Ker}A])^{\perp}\} \cap \mathrm{Ker}A =$$

$$= \{M_{\mp} + T^{-1}A[Z_0(T^{-1}A)] \oplus Z_0(AT^{-1})^{\perp}\} \cap \mathrm{Ker}A =$$

$$= \{M_{\mp} \cap \mathrm{Ker}A\} \oplus T^{-1}A[Z_0(T^{-1}A)],$$

where we have used that $h \in T^{-1}A[Z_0(T^{-1}A)] \subset \mathrm{Ker}A$ satisfies $(Th,h)=0$, in combination with (13). This then implies (15).

Let us construct β in such a way that

$$\mathrm{Ran}\ P_{+} \subset \mathrm{Ran}\ PP_{+} \oplus \mathrm{Ker}A. \tag{16}$$

Choose a basis $x_1,\dots,x_k$ of the subspace $T^{-1}A[Z_0(T^{-1}A)] \subset \mathrm{Ker}A$. Since $(Tx,z)=(Ay,z)=(y,Az)=0$ for all $z \in \mathrm{Ker}A$ and $x=T^{-1}Ay \in T^{-1}A[Z_0(T^{-1}A)]$, we have $(Tx_i,x_j)=0$. Next choose a basis $z_1,\dots,z_m$ of any complement of $T^{-1}A[Z_0(T^{-1}A)]$ in KerA in such a way that $(Tz_i,z_j)=0$ for $i\neq j$ and $(Tz_i,x_j)=0$ for all $i\leq m$ and $j\leq k$. Since $Z_0(T^{-1}A)$ is nondegenerate, we must have $(Tz_i,z_i) \equiv \tau(z_i) \neq 0$ for $i \leq m$. Using an induction argument we then select $y_1,\dots,y_k$ satisfying $T^{-1}Ay_i=x_i$ as well as $(Ty_i,z_j)=0$ for $j\leq m$ and $(Ty_i,x_j)=(Ty_i,y_j)=0$ for $j\leq i-1$, and $(Ty_i,y_i)<0$ for $i=1,\dots,k$. As a result we obtain a special basis $\{x_1,\dots,x_k,y_1,\dots,y_k,z_1,\dots z_m\}$ of the zero root linear manifold $Z_0(T^{-1}A)$. We now define the invertible matrix β on $Z_0(T^{-1}A)$ by $\beta x_i = x_i$, $\beta y_i = -y_i$, and $\beta z_j = \frac{\tau(z_j)}{|\tau(z_j)|} z_j$. The inclusion (16) is immediate.∎

2. EXISTENCE AND UNIQUENESS THEORY FOR HALF SPACE PROBLEMS

In this section we will analyze the boundary value problems

$$(T\psi)'(x) = -A\psi(x), \quad 0 < \pm x < \infty, \tag{19}$$

$$Q_{\pm}\psi(0) = \varphi_{\pm}, \tag{20}$$

along with a condition at infinity, namely, one of

$$\lim_{x\to\pm\infty} \|\psi(x)\| = 0, \tag{21}$$

$$\|\psi(x)\| = O(1) \ (x\to\pm\infty), \tag{22}$$

$$\|\psi(x)\| = O(x) \ (x\to\pm\infty). \tag{23}$$

As there is a complete symmetry between left and right half space problems, we will consider the right half space problem only. By a solution of the various boundary value problems for $\varphi_{+}\in Q_{+}[D(T)]$ we shall mean a continuous function ψ: $[0,\infty)\to D(T)\subset H$ such that $T\psi$ is strongly differentiable on $(0,\infty)$ and Eqs. (19), (20) and (21) ((22), (23), resp.) are satisfied.

First we outline the procedure which will be used to construct solutions to these boundary value problems. Let us reduce the half space problem (19)-(20) to two subproblems. Writing $\psi_1 = P\psi$ and $\psi_0 = (I-P)\psi$, Eq.(19) may be decomposed as follows:

$$(T\psi_1)'(x) = -A\psi_1(x), \quad 0<x<\infty, \tag{24}$$

and

$$\psi_0'(x) = -T^{-1}A\psi_0(x), \quad 0<x<\infty.$$

The second equation is an evolution equation on the finite dimensional space $Z_0(T^{-1}A)$, and therefore admits an elementary solution of the form

$$\psi_0(x) = e^{-xT^{-1}A}\psi_0(0) = (I-xT^{-1}A)\psi_0(0),$$

where Lemma III.1 has been used. Next consider Eq.(24). Let us add to this equation the dummy equation

$$(T\phi_0)'(x) = -A_{\beta}\phi_0(x), \quad 0<x<\infty, \tag{25}$$

on $Z_0(T^{-1}A)$, where A_{β} is given by (6) for some β. The solution of Eq.(25) is easy to compute but does not concern us, as it will be projected out shortly. However, defining $\phi = \phi_0 + \psi_1$ we can combine Eqs. (24) and (25) to obtain

$$(T\phi)'(x) = -A_\beta \phi(x), \quad 0<x<\infty. \tag{26}$$

Now since A_β is strictly positive (and a compact perturbation of the identity satisfying (8)), we can apply the results of Chapter II to Eq.(26) and obtain its solution in the form

$$\phi(x) = e^{-xT^{-1}A_\beta}Eg_+, \quad 0\le x<\infty,$$

where $g_+ \in Q_+[D(T)]$ and $E=(Q_+P_+ + Q_-P_-)^{-1}$. Then projecting $\phi(x)$ onto $Z_0(AT^{-1})^\perp$ along $Z_0(T^{-1}A)$ and adding $\psi_0(x)$, we obtain a solution of Eq.(19) in the form

$$\psi(x) = e^{-xT^{-1}A}PEg_+ + \psi_0(x), \quad 0\le x<\infty. \tag{27}$$

Now we have to fit the boundary condition (20). To do this we must find $g_+ \in Q_+[D(T)]$ and $\psi_0(0) \in Z_0(T^{-1}A)$ such that

$$Q_+(PEg_+ + \psi_0(0)) = \varphi_+. \tag{28}$$

Note that if $\psi_0(0)=0$ ($\psi_0(0)\in \mathrm{Ker}A$, $\psi_0(0)\in Z_0(T^{-1}A)$, resp.) then the right half space condition at infinity (21) ((22), (23), resp.) is satisfied.

Let us define the measure of noncompleteness for any of the boundary value problems to be the codimension in Ran Q_+ of the subspace of boundary values $\varphi_+ \in$ Ran Q_+ for which the problem is solvable, and the measure of nonuniqueness to be the dimension of the solution space of the corresponding homogeneous problem. The principal results of this chapter are the following existence and uniqueness theorems:

<u>THEOREM</u> <u>III.4</u>. The boundary value problem

$$(T\psi)'(x) = -A\psi(x), \quad 0<x<\infty, \tag{19}$$

$$Q_+\psi(0) = \varphi_+, \tag{20}$$

$$\lim_{x\to\infty} \|\psi(x)\| = 0, \tag{21}$$

has at most one solution for every $\varphi_+ \in Q_+[D(T)]$, and the measure of noncompleteness for solutions of this problem coincides with the maximal number of lineary independent vectors $g_1,\dots,g_n \in \mathrm{Ker}A$ satisfying

$$(Tg_i,g_j) = 0, \quad 1\le i,j\le n,\ i\ne j, \tag{29}$$

$$(Tg_i,g_i) \ge 0, \quad 1\le i\le n. \tag{30}$$

The solution, if it exists, is given by

$$\psi(x) = e^{-xT^{-1}A}PEg_+ \tag{31}$$

where g_+ is the unique solution of

$$Q_+PEg_+ = \varphi_+. \tag{32}$$

THEOREM III.5. The boundary value problem

$$(T\psi)'(x) = -A\psi(x), \quad 0<x<\infty, \tag{19}$$

$$Q_+\psi(0) = \varphi_+, \tag{20}$$

$$\|\psi(x)\| = 0(1) \ (x\to\infty), \tag{22}$$

has at least one solution for every $\varphi_+\in Q_+[D(T)]$, and the measure of nonuniqueness for solutions of this problem coincides with the maximal number of linearly independent vectors $h_1,\dots,h_k \in \mathrm{Ker}A$ satisfying

$$(Th_i,h_j) = 0, \quad 1\le i,j\le k,\ i\ne j, \tag{33}$$

$$(Th_i,h_i) < 0, \quad 1\le i\le k. \tag{34}$$

The solutions have the form

$$\psi(x) = e^{-xT^{-1}A}PEh_+ + h_0 \tag{35}$$

where $h_0\in\{\mathrm{Ran}\ PP_+\oplus \mathrm{Ran}\ Q_-\}\cap \mathrm{Ker}A$ and h_+ is the unique solution of

$$Q_+PEh_+ + Q_+h_0 = \varphi_+. \tag{36}$$

THEOREM III.6. The boundary value problem

$$(T\psi)'(x) = -A\psi(x), \quad 0<x<\infty, \tag{19}$$

$$Q_+\psi(0) = \varphi_+, \tag{20}$$

$$\|\psi(x)\| = 0(x) \quad (x\to\infty), \tag{23}$$

has at least one solution, and the measure of nonuniqueness for solutions of this problem coincides with the maximal number of linearly independent vectors $e_1,\dots,e_m \in Z_0(T^{-1}A)$ satisfying

$$(Te_i,e_j) = 0, \quad 1\le i,j\le m, \ i\neq j, \tag{37}$$

$$(Te_i,e_i) < 0, \quad 1\le i\le m. \tag{38}$$

The solutions have the form

$$\psi(x) = e^{-xT^{-1}A}PEf_+ + (I-xT^{-1}A)f_0, \tag{39}$$

where $f_0 \in \{\mathrm{Ran}\ PP_+ \oplus \mathrm{Ran}\ Q_-\} \cap Z_0(T^{-1}A)$ and f_+ is the unique solution of

$$Q_+PEf_+ + Q_+f_0 = \varphi_+. \tag{40}$$

<u>Proof of Theorems III.4, III.5 and III.6</u>: Consider first the boundary value problem (19), (20) and (22). A solution of this boundary value problem exists if one is able to find $g_+ \in \mathrm{Ran}\ Q_+$ and $\psi_0(0) \in \mathrm{Ker}A$ such that the equation $Q_+(PEg_+ + \psi_0(0)) = \varphi_+$ is fulfilled. Since $E\varphi_+ \in \mathrm{Ran}\ P_+ \subset \mathrm{Ran}\ PP_+ \oplus \mathrm{Ker}A$, one can simply choose $g_+ = \varphi_+$ and $\psi_0(0) = (I-P)E\varphi_+$. This establishes the existence of solutions to the boundary value problem. To study uniqueness, let us suppose that ψ is a solution of (19), (20) and (22) corresponding to $\varphi_+ = 0$. Then ψ has the form (27) with $PEg_+ + \psi_0(0) \in \mathrm{Ran}\ Q_-$ for some $g_+ \in \mathrm{Ran}\ Q_+$ and $\psi_0(0) \in \mathrm{Ker}A$. Therefore $\psi_0(0) \in \{\mathrm{Ran}\ PP_+ \oplus \mathrm{Ran}\ Q_-\} \cap \mathrm{Ker}A = M_- \cap \mathrm{Ker}A$. Conversely, if $\psi_0(0) \in M_- \cap \mathrm{Ker}A$, then we can find $g_+ \in \mathrm{Ran}\ Q_+$ and $h_- \in \mathrm{Ran}\ Q_-$ such that $\psi_0(0) = PEg_+ - h_-$, i.e., $Q_+(PEg_+ + \psi_0(0)) = 0$. Hence the measure of nonuniqueness of solutions of the boundary value problem (19), (20) and (22) is equal to $\dim (M_- \cap \mathrm{Ker}A)$. Since by Proposition III.3 M_- is a subspace of $Z_0(T^{-1}A)$ maximal with regard to vectors h satisfying $(Th,h)<0$ for $h\neq 0$, $M_- \cap \mathrm{Ker}A$ is also maximal in this respect as a subspace of KerA. One can show that the dimension

of such a maximal subspace does not depend on the specific choice of this subspace. Therefore, the dimension of $M_-\cap\mathrm{Ker}A$ equals the dimension of the subspace spanned by h_i satisfying (33) and (34), which completes the proof of Theorem III.5.

Consider now the boundary value problem (19), (20) and (23). By virtue of the above considerations the existence of solutions is clear. To study uniqueness let us assume that ψ is a solution of (19), (20) and (23) with $\varphi_+=0$. Then ψ has the form (39) and $\psi_0(0)\in\{\mathrm{Ran}\ PP_+\oplus\mathrm{Ran}\ Q_-\}\cap Z_0(T^{-1}A) = M_-$. Conversely, if $\psi_0(0)\in M_-$, then there exists a vector $f_+\in\mathrm{Ran}\ Q_+$ such that $Q_+(PEf_++\psi_0(0))=0$ which leads to a solution of Eqs. (19), (20) and (23) with $\varphi_+=0$. Therefore, the measure of nonuniqueness equals dim M_-, which coincides with the dimension of the subspace spanned by vectors e_i satisfying (37) and (38). This ends the proof of Theorem III.6.

It remains to demonstrate the proof of Theorem III.4. The uniqueness of solutions of the boundary value problem (19), (20) and (21) follows from the uniqueness of solutions of the problem

$$(T\varphi)'(x) = -A_\beta\varphi(x), \quad 0<x<\infty, \tag{41}$$

$$Q_+\varphi(0) = \varphi_+, \tag{42}$$

$$\lim_{x\to\infty} \|\varphi(x)\| = 0, \tag{43}$$

which was proved in Chapter II (cf. Theorem II.4). For, in fact, every solution of (19), (20) and (21) has its initial value $\psi(0)$ in Ran PP_+ and therefore must be a solution of Eqs. (41)-(43). To analyze existence for Eqs. (19) (20) and (21), we consider the possibility of finding $g_+\in\mathrm{Ran}\ Q_+$ such that

$$\varphi_+ = Q_+PEg_+ = PEg_+-Q_-PEg_+,$$

or, equivalently, $\varphi_+\in\{\mathrm{Ran}\ PP_+\oplus\mathrm{Ran}\ Q_-\}\cap\mathrm{Ran}\ Q_+$. Therefore the measure of noncompleteness is equal to the codimension of $\{\mathrm{Ran}\ PP_+\oplus\mathrm{Ran}\ Q_-\}\cap\mathrm{Ran}\ Q_+$ in

Ran Q_+. Let us compute:

$$\delta_+ = \dim \frac{\text{Ran } Q_+}{\{\text{Ran } PP_+ \oplus \text{Ran } Q_-\} \cap \text{Ran } Q_+} = \dim \frac{H}{\text{Ran } PP_+ \oplus \text{Ran } Q_-}.$$

Since $\text{Ran}V = H$, $V = Q_+P_+ + Q_-P_- = P_+ + Q_-(P_- - P_+)$, and $\text{Ran } P_+ \cap \text{Ran } Q_- = \{0\}$, we conclude that $H \subset \text{Ran } P_+ \oplus \text{Ran } Q_-$ and consequently that $\text{Ran } PP_+ \oplus \text{Ran } Q_- + \text{Ker}A = H$ (cf. Proposition III.3). Then

$$\delta_+ = \dim \frac{\text{Ran } PP_+ \oplus \text{Ran } Q_- + \text{Ker}A}{\text{Ran } PP_+ \oplus \text{Ran } Q_-} =$$

$$\dim \frac{\text{Ker}A}{\{\text{Ran } PP_+ \oplus \text{Ran } Q_-\} \cap \text{Ker}A} = \dim \frac{\text{Ker}A}{M_- \cap \text{Ker}A} =$$

$$\dim\, [\{M_+ \cap \text{Ker}A\} \oplus T^{-1}A[Z_0(T^{-1}A)]],$$

where we have used (15). Note that the subspace $\{M_+ \cap \text{Ker}A\} \oplus T^{-1}A[Z_0(T^{-1}A)]$ has the property $(Tf,f) \geq 0$ for all f contained therein, and any subspace of KerA containing $\{M_+ \cap \text{Ker}A\} \cap T^{-1}A[Z_0(T^{-1}A)]$ and having the same property coincides with the latter. Using standard Pontryagin space theory (cf.[4], Chapter IX), we conclude that δ_+ equals the dimension of a subspace of KerA spanned by vectors g_i satisfying (29) and (30). This completes the proof of Theorem III.4.∎

If the differential equation (19) is replaced by

$$T\psi'(x) = -A\psi(x), \quad 0<x<\infty, \tag{44}$$

one can seek solutions of the boundary value problems with $\varphi_+ \in Q_+[H]$. Here by a solution we mean a continuous function $\psi:[0,\infty) \mapsto H$ which is continuously differentiable on $(0,\infty)$, $\psi'(x) \in D(T)$ for $x \in (0,\infty)$ and such that Eq.(44), the boundary condition $Q_+\psi(0) = \varphi_+$ and an appropriate condition at infinity are satisfied. Then one can prove the analogs of Theorems III.4, III.5 and III.6, where one has to substitute Eq.(44) for Eq.(19), and φ_+ may belong to $Q_+[H]$ rather then just to $Q_+[D(T)]$. In this setting the spaces $M_\pm$ appearing in the proofs of the analogs of these theorems must be read as in (18). The equivalence of (17) and (18) causes the measures of nonuniqueness and noncompleteness to coincide with those for Eqs. (19)–(20) with appropriate condition at infinity. We note, however, that unlike the earlier boundary value problem, Eq.(44) may not lead to an equivalent vector valued convolution equation [35].

CHAPTER IV

NONDISSIPATIVE MODELS: COLLISION OPERATORS WITH NEGATIVE PART

In this chapter we study the abstract kinetic equation

$$T\psi'(x) = -A\psi(x), \quad 0<x<\tau, \tag{1}$$

relevant to transport in a multiplying medium confined within the slab $[0,\tau]$. The multiplying character of the medium manifests itself mathematically in allowing A to have a finite dimensional negative part. The equation is studied along with the partial range boundary conditions

$$Q_+\psi(0) = \varphi_+, \tag{2}$$

$$Q_-\psi(\tau) = \varphi_-, \tag{3}$$

where $Q_\pm$ are the complemetary orthogonal projections onto the maximal T-positive/negative subspaces of H and φ_+, φ_- are given vectors in the ranges of Q_+ and Q_- respectively.

1. PRELIMINARIES AND DECOMPOSITIONS

Let T be a bounded self-adjoint injective operator on a complex Hilbert space H. Let A be a bounded self-adjoint Fredholm operator on H such that the spectrum of A in $(-\infty,0]$ consists of a finite number of eigenvalues of finite multiplicity. Let us introduce the (indefinite) inner product on H by

$$[f,g]_A = (Af,g). \tag{4}$$

We will denote by H_A the space H equipped with the $[\cdot,\cdot]_A$-inner product. Then H_A is an indefinite inner product space with the fundamental decomposition

$$H_A = H_A^- \oplus H_A^0 \oplus H_A^+ \tag{5}$$

where $H_A^{\pm}$ is strictly positive/strictly negative subspace of H_A while H_A^0 is neutral. One can take for example $H_A^- = \bigoplus_{i=1}^{k} \mathrm{Ker}(\lambda_i - A)$ where $\lambda_1, \ldots, \lambda_k$ are the strictly negative eigenvalues of A, $H_A^0 = \mathrm{Ker}A$ and H_A^+ to be the the H-orthogonal complement of $H_A^- \oplus H_A^0$. If $\mathrm{Ker}A = \{0\}$, then H_A is a Π_κ-space where κ is the sum of the multiplicities of the negative eigenvalues of A (cf. [25,32]). One can easily show that the operator $T^{-1}A$ is self-adjoint with respect to the indefinite inner product (4). Then $T^{-1}A$ has at most κ nonreal eigenvalues (multiplicities taken into account) occuring in complex conjugate pairs with pairwise coinciding Jordan structures, while the length of a Jordan chain for a real eigenvalue of $T^{-1}A$ does not exceed $2\kappa+1$. Moreover there is a resolution of the identity for the real spectrum of $T^{-1}A$, possibly with finitely many critical points at certain eigenvalues (cf. [25,32]). If A has a nontrivial kernel, then H_A is no longer Π_κ-space. It is not even a Krein space. This complicates the spectral analysis of the operator $T^{-1}A$. However under certain additional assumptions we will be able to deduce spectral properties of $T^{-1}A$ vital for the subsequent analysis.

Let L be a linear operator on H and λ be a complex number. We define the λ root linear manifold $Z_\lambda(L)$ by

$$Z_\lambda(L) = \{\, h \in H \mid h \in D(L^n) \text{ and } (\lambda - L)^n h = 0 \text{ for some } n \in \mathbb{N} \,\}.$$

DEFINITION. The operator A will be called T-regular if

(a) $Z_0(T^{-1}A)$ is finite dimensional and nondegenerage with respect to the indefinite inner product $[f,g]_T=(Tf,g)$, i.e.,

$$\{ f \in Z_0(T^{-1}A) \mid (Tf,g)=0 \text{ for all } g \in Z_0(T^{-1}A) \} = \{0\}.$$

(b) for any real nonzero λ $Z_\lambda(T^{-1}A)$ is nondegenerate with respect to the indefinite inner product $[\cdot,\cdot]_A$.

Using the condition (a) and a simple dimension argument one finds that

$$Z_0(T^{-1}A) \oplus (T[Z_0(T^{-1}A)])^\perp = H \tag{6}$$

Since the neutral subspace $H_A^0=\mathrm{Ker}A$ is contained in $Z_0(T^{-1}A)$, the subspace $(T[Z_0(T^{-1}A)])^\perp$ is a Π_κ-space invariant under $T^{-1}A$. Now one can formulate a spectral theorem for the restriction of $T^{-1}A$ to $(T[Z_0(T^{-1}A)])^\perp$. Condition (b) guarantees that the critical points in the spectral resolution of $T^{-1}A \restriction (T[Z_0(T^{-1}A)])^\perp$ are regular (cf. [25,32]).

The assumption on A we have just made allows us to construct a decomposition of the Hilbert space H into $T^{-1}A$ invariant subspaces in such a way that A can be deformed to a strictly positive operator. Such a decomposition and the corresponding modification of the collision operator is provided by the following proposition.

PROPOSITION IV.1. Let A be T-regular. Then there exists a $T^{-1}A$ invariant finite dimensional subspace $Z(T^{-1}A)$ of H with the following properties:

(i) $Z(T^{-1}A) \oplus (T[Z(T^{-1}A)])^\perp = H,$ (7)

(ii) The subspace $(T[Z(T^{-1}A)])^\perp$ is $T^{-1}A$-invariant and strictly positive with respect to $[\cdot,\cdot]_A$,

(iii) The constituent subspaces in the decomposition (7) are

$[\cdot,\cdot]_A$-orthogonal.

Moreover, if P denotes the projection of H onto $(Z[T(T^{-1}A)])^{\perp}$ along $Z(T^{-1}A)$ and β is an inveritible $[\cdot,\cdot]_T$-positive operator on $Z(T^{-1}A)$, then the bounded operator

$$A_\beta = AP + T\beta^{-1}(I-P)$$

is strictly positive with respect to H-inner product, Fredholm and

$$A_\beta^{-1}T = \beta \oplus (T^{-1}A \restriction (T[Z(T^{-1}A)])^{\perp})^{-1}.$$

Finally, if I-A is compact and satisfies the regularity condition

$$\mathrm{Ran}(I-A) \subset \mathrm{Ran}\,|T|^{\gamma} \text{ for some } \gamma \in (0,1), \tag{8}$$

then $I-A_\beta$ is also compact and satisfies the same regularity condition.

The proof of this proposition has been given by Greenberg and van der Mee in [16]. Here we present a new version of this proof and, in fact, a slightly different Hilbert space decomposition.

<u>Proof</u>: Since $Z_0(T^{-1}A)$ is finite dimensional and nondegenerate with respect to $[\cdot,\cdot]_T$-inner product, we have the following decomposition of the space H

$$Z_0(T^{-1}A) \oplus (T[Z_0(T^{-1}A)])^{\perp} = H.$$

The space $(T[Z_0(T^{-1}A)])^{\perp}$ is a Π_κ-space with respect to $[\cdot,\cdot]_A$-inner product and the restriction of $T^{-1}A$ to this subspace is self-adjoint in $[\cdot,\cdot]_A$-inner product. Then there is a decomposition of the space $(T[Z_0(T^{-1}A)])^{\perp}$ into the $[\cdot,\cdot]_A$-orthogonal direct sum of the spectral subspaces of $T^{-1}A \restriction (T[Z_0(T^{-1}A)])^{\perp}$

$$(T[Z_0(T^{-1}A)])^{\perp} = H_{im} \oplus H_{re}\ ,$$

such that $\sigma(T^{-1}A \restriction H_{re})$ is purely real and the spectrum of $T^{-1}A \restriction H_{im}$ consists of finite number of nonreal eigenvalues $\lambda_i, \bar{\lambda}_i$, $i=1,\ldots,m \le \kappa$ of finite mulitiplicity (cf. [39]). In fact we have

$$H_{im} = \overset{m}{\underset{i=1}{\oplus}} (Z_{\lambda_i}(T^{-1}A) \oplus Z_{\bar{\lambda}_i}(T^{-1}A))$$

and dim $H_{im}=2\bar{\kappa}\leq 2\kappa$ where $\bar{\kappa}$ is the sum of the algebraic multiplicities of the eigenvalues of $T^{-1}A$ in the open upper (lower) half plane. H_{im} is neutral with respect to $[\cdot,\cdot]_A$-inner product. The space H_{re} is a $\Pi_{\kappa'}$-space and $\kappa'=\kappa-\bar{\kappa}$ (cf. [4,25]). If ξ_i, $i=1,\dots,n$ are all critical points of the spectral resolution of $T^{-1}A \restriction H_{re}$ then H_{re} can be decomposed into the $[\cdot,\cdot]_A$-orthogonal direct sum of the $T^{-1}A$-invariant subspaces

$$H_{re} = H_0 \oplus (\overset{n}{\underset{i=1}{\oplus}} Z_{\xi_i}(T^{-1}A))$$

where $Z_{\xi_i}(T^{-1}A)$ are $\Pi_{\kappa(\xi_i)}$-spaces, while H_0 is a Hilbert space and

$\sum_{i=1}^{n} \kappa(\xi_i)=\kappa'$ (cf. [25,29]). Moreover, since ξ_i are regular critical points , the subspaces $Z_{\xi_i}(T^{-1}A)$ are nondegenerate and finite dimensional. Now we put

$$Z'(T^{-1}A) = H_{im} \oplus (\overset{n}{\underset{i=1}{\oplus}} Z_{\xi_i}(T^{-1}A)).$$

Clearly $Z'(T^{-1}A)$ is finite dimensional, $T^{-1}A$-invariant,

$$Z'(T^{-1}A) \oplus H_0 = (T[Z_0(T^{-1}A)])^{\perp}$$

and the subspaces $Z'(T^{-1}A)$ and H_0 are $[\cdot,\cdot]_A$-orthogonal.

Now we will show that $H_0=(T[Z'(T^{-1}A)])^{\perp}$, by proving that $(T[Z_0(T^{-1}A)])^{\perp}$ can be decomposed into $[\cdot,\cdot]_A$-orthogonal direct sum of the subspaces $Z'(T^{-1}A)$ and $(T[Z'(T^{-1}A)])^{\perp}$. This decomposition will follow immediately from a simple dimension argument if we show that

$$Z'(T^{-1}A)\cap(T[Z'(T^{-1}A)])^{\perp} = \{0\}.$$

Since $Z'(T^{-1}A)\cap(T[Z'(T^{-1}A)])^{\perp}$ can be written as an intersection of the subspaces $Z_\eta(T^{-1}A)\cap[TZ_\eta(T^{-1}A)]^\perp$, where η is a complex eigenvalue or a critical point of $T^{-1}A$, it is enuogh to show that at least one of them contains only the zero vector. In fact this holds true for any of these subspaces. Let $f\in Z_\eta(T^{-1}A)\cap[TZ_\eta(T^{-1}A)]^\perp$. Then for any $g\in Z_\eta(T^{-1}A)$

$$[f,g]_A=(Af,g)=(f,T(T^{-1}A)g)=0.$$

Since $Z_\eta(T^{-1}A)$ is nondegenerate with respect to $[\cdot,\cdot]_A$-inner product, this implies that $f=0$. Therefore

$$(T[Z_0(T^{-1}A)])^\perp = Z'(T^{-1}A) \oplus (T[Z'(T^{-1}A)])^\perp.$$

It remains to show that the constituent subspaces in this decomposition are $[\cdot,\cdot]_A$-orthogonal. If $f\in Z'(T^{-1}A)$ and $g\in(T[Z'(T^{-1}A)])^\perp$ then we have

$$[f,g]_A=(Af,g)=(T(T^{-1}A)f,g)=0.$$

Now we define

$$Z(T^{-1}A) = Z_0(T^{-1}A) \oplus Z'(T^{-1}A).$$

Since

$$(T[Z(T^{-1}A)])^\perp = (T[Z_0(T^{-1}A)] \oplus T[Z'(T^{-1}A)])^\perp =$$
$$= (T[Z_0(T^{-1}A)])^\perp\cap(T[Z'(T^{-1}A)])^\perp = (T[Z'(T^{-1}A)])^\perp,$$

we have the follwing decomposition of the space H

$$Z(T^{-1}A) \oplus (T[Z(T^{-1}A)])^\perp = H.$$

Clearly the constituent subspaces in (7) have all the properties stated in the first part of the propostion.

The proof of the remaining part is exactly the same as the proof of Proposition III.2.■

Since A_β is strictly positive on H and A_β^{-1} is bounded, the inner product $(\cdot,\cdot)_{A_\beta}$ defined by

$$(f,g)_{A_\beta} = (A_\beta f,g)$$

is equivalent to the original inner product on H. We will denote by H_{A_β} the space H endowed with the $(\cdot,\cdot)_{A_\beta}$-inner product. It is clear that the topology on H_{A_β} does not depend on β. Now we can easily show that the operator $S_\beta = A_\beta^{-1}T$ is bounded, injective and self-adjoint on H_{A_β}. Then we define $P_\pm$ as the $(\cdot,\cdot)_{A_\beta}$-orthogonal complementary projections of H_{A_β} onto the maximal $(\cdot,\cdot)_{A_\beta}$-positive/negative $A_\beta^{-1}T$-invariant subspaces of H_{A_β}. Note that $P_\pm$ depend on β. However, since the decomposition (7) reduces the operator $A_\beta^{-1}T$ and $A_\beta^{-1}T \restriction (T[Z(T^{-1}A)])^\perp$ does not depend on β, the operators PP_+, PP_- and $I-P$ form a set of β-independent complementary projections on H.

Employing the self-adjointness of $T^{-1}A_\beta$ with respect to the $(\cdot,\cdot)_{A_\beta}$-inner product and the self-adjointness of the restriction of $T^{-1}A$ to the subspace $(T[Z(T^{-1}A)])^\perp$ one defines the semigroups $\exp\{\mp xT^{-1}A_\beta\}P_\pm$ and $\exp\{\mp xT^{-1}A\}PP_\pm$. Moreover one can show that they can be extended to contraction analytic semigroups on appropriate subspaces.

2. SOLUTION OF THE BOUNDARY VALUE PROBLEM

In this section we will prove existence and uniqueness theorems for the boundary value problem

$$T\psi'(x) = -A\psi(x), \ 0<x<\tau, \tag{1}$$

$$Q_+\psi(0) = \varphi_+, \tag{2}$$

$$Q_-\psi(\tau) = \varphi_-, \tag{3}$$

where $Q_{\pm}$ are the complementary orthogonal projections of H onto the maximal T-positive/negative T-invariant subspaces of H and the vectors $\varphi_{\pm} \in \mathrm{Ran}\, Q_{\pm}$ are given. In the first part of this section we will study solutions in some extension space H_S of H (weak solutions). This approach was introduced by Beals [3] and then developed further in [14,17]. In the second part we will strengthen the results in the case when A is a compact perturbation of the identity satisfying the regularity condition (8).

Let H_{S_β} be the completion of H_{A_β} with respect to the inner product

$$(f,g)_{S_\beta} = (|S_\beta| f,g)_{A_\beta}. \tag{9}$$

Since $Z(T^{-1}A)$ is finite dimensional, the topology of H_{S_β} does not depend on β and all inner products (9) are equivalent. In fact, one can show that H_{S_β} is topologically isomorphic to H_T, the completion of H with respect to the inner product $(f,g)_T = (|T| f,g)$ (cf. the proof of Lemma IV.3). Therefore we will suppress the subscript β in H_{S_β} and write H_S. Since the extension of $T^{-1}A_\beta$ to the space H_S is self-adjoint one can define the projections $P_{\pm}$ and the contraction semigroups $\exp\{\mp xT^{-1}A_\beta\}P_{\pm}$ with the help of the spectral theorem. Moreover one can extend $Q_{\pm}$ and P to bounded projections acting in H_S. Now P is the projectin of H_S onto the completion of $(T[Z(T^{-1}A])^{\perp}$ along $Z(T^{-1}A)$. The problem of extending $T^{-1}A$ to H_S is more delicate. Since the decomposition (7) reduces completely the operators $T^{-1}A$ and $T^{-1}A_\beta$ and these operators coincide on the subspace $(T[Z(T^{-1}A)])^{\perp}$ one can extend the restriction of $T^{-1}A$ to $(T[Z(T^{-1}A)])^{\perp}$ with respect to the inner product (9) to the self-adjoint operator on the completion of $(T[Z(T^{-1}A)])^{\perp}$. Since $Z(T^{-1}A)$ is finite dimensional, the restriction of $T^{-1}A$ to $Z(T^{-1}A)$ can be defined in the natural way as the bounded operator with respect to the norm induced by the inner product (9). We will denote by K the resulting extension of $T^{-1}A$ to H_S. Now one can define the complementary projections of $P[H_S]$ onto the maximal K-positive/negative K-invariant subspaces of $P[H_S]$. Since P and $P_\pm$ commute one can show that they coincide with $PP_\pm$. Further, using the spectral theorem for the restriction of K to $P[H_S]$ one defines the semigroups $\exp\{\mp xK\}PP_\pm$. Clearly $\exp\{\mp xK\}PP_\pm = \exp\{\mp xT^{-1}A_\beta\}PP_\pm$.

After this introduction we can precise what we shall mean by a (weak) solution of the boundary value problem (1)-(3). A continuous function $\psi:[0,\tau]\to H_S$ is a weak solution of the boundary value problem (1)-(3) if

(i) for all $x\in(0,\tau)$ $\psi(x)$ belongs to the domain of K,

(ii) ψ is continuously differentiable on $(0,\tau)$ and $\psi'=-K\psi$,

(iii) $Q_+\psi(0)=\varphi_+$ and $Q_-\psi(\tau)=\varphi_-$.

Solving Eq.(1) on the subspaces Ran PP_+, Ran PP_- and $\mathrm{Ran}(I-P)=Z(T^{-1}A)$ with the help of standard semigroup theory, we obtain

$$\psi(x) = e^{-xK}PP_+\psi(0) + e^{(\tau-x)K}PP_-\psi(\tau) + e^{-xT^{-1}A}(I-P)\psi(0).$$

Putting $h=PP_+\psi(0)+PP_-\psi(\tau)+(I-P)\psi(0)$ one can write

$$\psi(x) = [e^{-xK}PP_+ + e^{(\tau-x)K}PP_- + e^{-xT^{-1}A}(I-P)]h. \tag{10}$$

Hence $\psi(x)$ will solve the boundary value problem (1)-(3), if the vector h is chosen in such a way that the boundary conditions (2) and (3) are satisfied. Fitting Eqs. (2) and (3) one sees that h must be a solution of

$$V_\tau h = \varphi$$

where $\varphi=\varphi_++\varphi_-$ and

$$V_\tau = Q_+[PP_++e^{\tau K}PP_-+(I-P)] + \\ + Q_-[e^{-\tau K}PP_++PP_-+e^{-\tau T^{-1}A}(I-P)]. \tag{11}$$

Therefore the unique solvability of the boundary value problem (1)-(3) is equivalent to the bounded invertibility of the operator V_τ on H_S. We will proof that this is in fact the case. We have

<u>THEOREM</u> <u>IV.2</u>. Let A be T-regular. Then there exists $\tau_c>0$ such that for all $0<\tau<\tau_c$ and every $\varphi_+\in Q_+[H_S]$ and $\varphi_-\in Q_-[H_S]$ the boundary value problem

$$T\psi'(x) = -A\psi(x),\ 0<x<\tau, \tag{1}$$

$$Q_+\psi(0) = \varphi_+, \tag{2}$$

$$Q_-\psi(\tau) = \varphi_-, \tag{3}$$

has a unique solution. The solution is given by

$$\psi(x) = [e^{-xK}PP_+ + e^{(\tau-x)K}PP_- + e^{-xT^{-1}A}(I-P)]V_\tau^{-1}\varphi \tag{12}$$

where $\varphi=\varphi_+ + \varphi_-$ and V_τ is the invertible operator on H_S defined by (11).

The proof of Theorem IV.2 employs certain properties of the operator

$$V_{\tau,\beta} = Q_+[P_+ + e^{\tau T^{-1}A}\beta P_-] + Q_-[e^{-\tau T^{-1}A}\beta P_+ + P_-]$$

which are described in the following lemma.

<u>LEMMA</u> <u>IV.3</u>. For any $\tau>0$ the operator $V_{\tau,\beta}$ is invertible on H_S and

$$\| V_{\tau,\beta}^{-1} \|_{\mathcal{L}(H_S)} \le \frac{\| V_\beta^{-1} \|_{\mathcal{L}(H_S)}}{1-\| V_\beta^{-1} - I \|_{\mathcal{L}(H_S)}} \tag{13}$$

where $V_\beta = Q_+P_+ + Q_-P_-$ has a bounded inverse on H_S satisfying

$$\| V_\beta^{-1} - I \|_{\mathcal{L}(H_S)} < 1. \tag{14}$$

<u>Proof</u>: The following proof is adopted from an argument of Beals [3]. We will show first that the Hilbert spaces H_S and H_T are topologically isomorphic. As

the first step in this prove we will demonstrate the equivalence of the norms $\|\cdot\|_S$ and $\|\cdot\|_T$ on H.

Let $h \in P_+[H_{A_\beta}]$ and $u(x) = \exp\{-xT^{-1}A_\beta\}P_+h$ for $x \geq 0$. Clearly u is continuous on $[0,\infty)$, differentiable on $(0,\infty)$ and $Tu'(x) = -A_\beta u(x)$. Therefore

$$\left|\frac{d}{dx}(|T|u(x),u(x))\right| = |2\mathrm{Re}(|T|u'(x),u(x))| =$$
$$= |2\mathrm{Re}((Q_+-Q_-)Tu'(x),u(x))| = |2\mathrm{Re}((Q_+-Q_-)A_\beta u(x),u(x))| \leq$$
$$\leq 2C\|u(x)\|^2_{A_\beta}.$$

Here we have used the fact that the norms $\|\cdot\|$, $\|\cdot\|_{A_\beta}$ and $\|A_\beta(\cdot)\|$ are equivalent . C denotes a constant. Then, since $\|u(x)\|_T \to 0$ as $x \to \infty$,

$$\|h\|^2_T = \|u(0)\|^2_T = -\int_0^\infty \frac{d}{dx}\|u(x)\|^2_T dx \leq$$

$$\leq 2C\int_0^\infty \|u(x)\|^2_{A_\beta} dx = 2C\int_0^\infty (\exp\{-2xT^{-1}A_\beta\}P_+h,h)_{A_\beta} dx.$$

But from the spectral theorem

$$2\int_0^\infty (\exp\{-2xT^{-1}A_\beta\}P_+h,h)_{A_\beta} dx = (A_\beta^{-1}Th,h)_{A_\beta} = \|h\|^2_S ,$$

hence $\|h\|^2_T \leq C\|h\|^2_S$.

Likewise if $h \in P_-[H_{A_\beta}]$ we prove that $\|h\|^2_T \leq C\|h\|^2_S$ in the very analogous way. Now if $h \in H_{A_\beta}$, let $h_\pm = P_\pm h$. Then

$$\|h\|^2_T \leq 2(\|h_+\|^2_T + \|h_-\|^2_T) \leq 2C(\|h_+\|^2_S + \|h_-\|^2_S) = 2C\|h\|^2_S.$$

Now, let us suppose that $h \in H_{A_\beta}$. Then

$$\|h\|^2_S = (|S_\beta|h,h)_{A_\beta} = (S_\beta h,(P_+-P_-)h)_{A_\beta} = (Th,(P_+-P_-)h) =$$
$$= ((Q_+-Q_-)h,(P_+-P_-)h)_T \leq \|(Q_+-Q_-)h\|_T\|(P_+-P_-)h\|_T \leq$$
$$\leq \|h\|_T(\|P_+h\|_T + \|P_-h\|_T) \leq 2(C)^{1/2}\|h\|_T\|h\|_S ,$$

so $\|h\|_S \leq 2(C)^{1/2}\|h\|_T$. This ends the prove of the equivalence of H_S and H_T norms.

In the next step we will show that $V_\beta = Q_+P_+ + Q_-P_-$ establishes the topological isomorphism of H_S and H_T. Let $W_\beta = I - V_\beta = Q_+P_- + Q_-P_+$. Then for any $h \in H_{A_\beta}$ we have

$$\|V_\beta h\|_T^2 - \|W_\beta h\|_T^2 = (TP_+h, P_+h) - (TP_-, P_-h) - (TP_-h, P_+h) +$$
$$(TP_+h, P_-h) = (T(P_+ - P_-)h, h) = (S_\beta(P_+ - P_-)h, h)_{A_\beta} = \|h\|_S^2. \qquad (15)$$

Hence $\|h\|_S \leq \|V_\beta h\|_T$. Now , using a density argument we extend this inequality to all H_S. This shows that V_β is injective. To prove that V_β^{-1} exists as a bounded operator it is enough to show that $V_\beta^*: H_T \to H_S$ is also injective. Recall that $P_\pm$ are self-adjoint in H_S and consider $Q_\pm$ as mapping H_S to H_T. Then

$$(Q_+^* g, h)_S = (g, Q_+h)_T = (Q_+g, h)_T = (S_\beta Q_+ g, h)_{A_\beta} = ((P_+ - P_-)Q_+ g, h)_S$$

implies that $Q_+^* = (P_+ - P_-)Q_+$. Likewise, one proves that $Q_-^* = (P_- - P_+)Q_-$. Then we compute the following identities

$$V_\beta^* = (Q_+P_+ + Q_-P_-)^* = P_+Q_+^* + P_-Q_-^* = P_+Q_+ + P_-Q_-,$$
$$W_\beta^* = I - V_\beta^* = -(P_+Q_- + P_-Q_+),$$

and similary as in (15)

$$\|V_\beta^* g\|_S^2 - \|W_\beta^* g\|_S^2 = \|g\|_T^2. \qquad (16)$$

Therefore V_β^* is injective and consequently $\mathrm{Ran}V_\beta = H_T$. Hence $V_\beta: H_S \to H_T$ as well as $V_\beta^*: H_T \to H_S$ are topological isomorphisms.

Now we can proceed to the proof of the inequality (14). Note that

$$\|V_\beta^{-1} - I\|_{\mathcal{L}(H_S)} = \|V_\beta^{-1} W_\beta\|_{\mathcal{L}(H_S)} = \|W_\beta^*(V_\beta^*)^{-1}\|_{\mathcal{L}(H_S)}.$$

Moreover from (16) follows that

$$\|W_\beta^*(V_\beta^*)^{-1}h\|_S^2 = \|h\|_S^2 - \|(V_\beta^*)^{-1}h\|_T^2. \qquad (17)$$

Since V_β^* is a topological isomorphism, $\|(V_\beta^*)^{-1}h\|_T \geq C\|h\|_S$ for some constant $C>0$. Finally it is clear from (17) that $C\leq 1$ and

$$\|V_\beta^{-1}-I\|_{\mathcal{L}(H_S)} = \|W_\beta^*(V_\beta^*)^{-1}\|_{\mathcal{L}(H_S)} \leq (1-C^2)^{1/2} < 1.$$

Now we will prove the inequality (13). Using simple algebra and the fact that $\exp\{\mp\tau T^{-1}A_\beta\}P_\pm = P_\pm\exp\{-\tau\,|T^{-1}A_\beta|\}$ we show

$$V_{\tau,\beta} = V_\beta(I+(V_\beta^{-1}-I)e^{-\tau|T^{-1}A_\beta|}).$$

Then since

$$\|(V_\beta^{-1}-I)e^{-\tau|T^{-1}A_\beta|}\|_{\mathcal{L}(H_S)} < \|V_\beta^{-1}-I\|_{\mathcal{L}(H_S)} < 1,$$

the inequality (13) follows by mere computation.

This completes the proof.■

The analogs of the operators V_β and $V_{\tau,\beta}$ play important role in the analysis of the abstract half space and slab problems relevant to subcritcal media, i.e. for A strictly positive and bounded. The proofs of the statements contained in this lemma except the inequality (13) can be found in many sources. In addition to Beals [3] we cite a recent paper by Hangelbroek [22], where different proofs are given.

<u>Proof of Theorem IV.2</u>: We will show that for sufficiently small τ

$$\|V_\tau-V_{\tau,\beta}\|_{\mathcal{L}(H_S)} < \frac{1}{\|V_{\tau,\beta}^{-1}\|_{\mathcal{L}(H_S)}}, \tag{18}$$

which guarantees the bounded invertibility of V_τ. Simple algebra shows that

$$V_\tau-V_{\tau,\beta} = Q_+[I-e^{\tau\beta^{-1}}](I-P)P_- + Q_-[I-e^{-\tau\beta^{-1}}](I-P)P_+ +$$

$$+ Q_-[e^{-\tau(T^{-1}A\restriction Z(T^{-1}A))_{-1}}](I-P).$$

Therefore

$$\| V_\tau - V_{\tau,\beta} \|_{\mathcal{L}(H_S)} \le$$

$$\le \text{const.}(e^{\tau} \| \beta^{-1} \|_{-1}) + \text{const.}(e^{\tau} \| T^{-1}A \restriction Z(T^{-1}A) \|_{-1}) \equiv f(\tau).$$

Clearly $f(\tau) \to 0$ as $\tau \to 0$. Let

$$\tau_c = \sup_{\tau > 0} \{ \tau \mid f(\tau) < (1 - \| V_\beta^{-1} - I \|_{\mathcal{L}(H_S)}) \| V_\beta^{-1} \|_{\mathcal{L}(H_S)}^{-1} \}.$$

Now using the inequality (13) from Lemma IV.3 one shows that the inequality (18) holds true for any $\tau \in (0, \tau_c)$. ∎

In the case where the collision operator A is a compact perturbation of the identity satisfying the regularity condition

$$\text{Ran}(I-A) \subset \text{Ran} \mid T \mid^{\gamma} \text{ for some } \gamma \in (0,1) \tag{8}$$

one can improve the statement of Theorem IV.2. First of all one does not have to extend the original Hilbert space H to obtain a solution $\psi(x)$. All operators involved act on H and the solution $\psi(x)$ of the boundary value problem (1)–(3) has the same structure as (12) with K replaced by $T^{-1}A$. But the crucial observation, which helps to strengthen the uniqueness and existence theorem, is contained in the following lemma.

LEMMA IV.4. For all z with $\text{Re} z > 0$ the operator $V_z - I$ is compact and the operator valued function $z \to V_z - I$ is analytic in the open right half plane.

Proof: Let $\text{Re} z > 0$. Simple computation yields

$$V_z - I = Q_-[I - e^{-zT^{-1}A}](I-P) + Q_+(PP_- - Q_-)e^{zT^{-1}A}PP_- +$$

$$+ Q_-(PP_+ - Q_+)e^{-zT^{-1}A}PP_+ - Q_+(PP_- - Q_-) - Q_-(PP_+ - Q_+). \tag{19}$$

Since PP_+-Q_+ and PP_--Q_- are compact (cf. Lemma II.10) and the semigroups $\exp\{\mp zT^{-1}A\}PP_\pm$, $\exp\{-zT^{-1}A\}(I-P)$ are bounded and analytic, one can conclude that $z\to V_z-I$ is an analytic function with values in the ideal of compact operators on H.∎

Now using the analytic version of the Fredholm alternative (cf. Reed and Simon [40]) one can infer the following result: either V_z is not invertible for any z with Rez>0 or V_z^{-1} exists for all $z\in\{z \mid \mathrm{Re}z>0\}\setminus\Delta$ where Δ is a discrete subset of the open right half plane and $z\in\Delta$ if and only if the equation $V_zh=0$ has a nontrivial solution. Since by Theorem IV.2 V_τ as a bounded operator on H has trivial kernel for any $\tau\in(0,\tau_c)$ and $V_\tau-I$ is compact, it follows that for such τ the bounded inverse of V_τ exists. But this excludes the first possibility stated in the above corollary and yields the following theorem.

<u>THEOREM</u> <u>IV.5</u>. Let A be T-regular compact perturbation of the identity satisfying the regularity condition (8). Then there exists a discrete subset Δ of the open right half plane such that:

(i) for any $\tau\in(0,\infty)\setminus\Delta$ and every $\varphi_+\in Q_+[H]$ and $\varphi_-\in Q_-[H]$ the boundary value problem (1)-(3) has a unique solution given by the equation (12), where $\varphi=\varphi_++\varphi_-$ and V_τ is the invertible operator on H defined by (11),

(ii) for any real $\tau\in\Delta$ the boundary value problem (1)-(3) with $\varphi_\pm=0$ has at least one nontrivial solution, which is given by the formula (10) where h is a nontrivial solution of $V_\tau h=0$.

If I-A is a finite rank perturbation of the identity satisfying the regularity condition we can characterize the set Δ.

THEOREM IV.6. Let A be a T-regular finite rank perturbation of the identity satisfying the regularity condition (8). Then

$\Delta = \{ z \in \mathbb{C} \mid \det[V_z] = 0 \}$.

Proof: If we show that $V_z - I$ is a trace class operator, the description of the set Δ will follow easily from the Fredholm theory of trace class operators (cf.[41], Theorem 3.3.9). It is clear from (19) that one has to prove only that $PP_{\pm} - Q_{\pm}$ are trace class operators. Moreover, since $Z(T^{-1}A)$ is finite dimensional, it is enough to prove that $P_{\pm} - Q_{\pm}$ are trace class operators. To show this we mimic the argument given in the first part of the proof of Lemma II.10. Since a closedness argument is employed in this proof and the ideal of trace class operators is closed in the $\|\cdot\|_1 = \mathrm{tr}\,|\cdot|$ norm, we have to use this norm in estimating relevant operator valued integrals. However, the estimates carry over with $\|D\|_1$ replacing $\|D\|$ where $D = |T|^{-\alpha}(I-A)$ is obviously a finite rank operator.■

CHAPTER V

APPLICATIONS

This section contains several physical models leading to equations of the form

$$T\psi'(x) = -A\psi(x). \tag{1}$$

All of them involve a time independent one dimensional transport problem in a semi-infinite medium with spatial variable $x\in[0,\infty)$ or in a slab $[0,\tau]$. The equations are studied with the boundary condition

$$Q_+\psi(0) = \varphi_+ \tag{2}$$

at $x=0$, and the condition

$$Q_-\psi(\tau) = \varphi_- \tag{3}$$

at $x=\tau$ in the case of transport in the slab $[0,\tau]$, or with a condition at infinity for transport in the semi-infinite medium $x\geq 0$. For all models we shall specify the Hilbert space H, operators T and A, and whenever it is relevant the kernel of A and the zero root linear manifold $Z_0(T^{-1}A)$. We shall state the results of the existence and uniqueness theory using the abstract theorems presented in Chapters III and IV.

The first three examples come from kinetic theory of rarefied gases and illustrate the results of Chapter III.

1. The scalar BGK equation ([6,10,11,29])

$$v\frac{\partial f}{\partial x}(x,v) = -f(x,v) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f(x,u)e^{-u^2}du, \quad v\in(-\infty,+\infty).$$

The equation is posed in the Hilbert space $H=L^2(\mathbb{R},\rho)$ where $d\rho=\pi^{-1/2}e^{-v^2}dv$. We define T and A by

$$(Tf)(v) = vf(v)$$

and

$$(Af)(v) = f(v) - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} f(x,v)e^{-v^2}dv.$$

Then T is unbounded and self-adjoint. Since $Af=f-(1,f)1$, $I-A$ is compact (in fact rank one) and A is positive, the last property being a consequence of the Bessel inequality. One checks easily that the regularity condition III.(1) is met. Further, $\mathrm{Ker}A=\mathrm{span}\{1\}$ and $Z_0(T^{-1}A)=\mathrm{span}\{1,v\}$. With respect to the sesquilinear form

$$[h,k]_T = (Th,k) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} vh(v)\overline{k(v)}e^{-v^2}dv,$$

the set $\{v+1,v-1\}$ is a basis of mutually $[\cdot,\cdot]_T$-orthogonal vectors of $Z_0(T^{-1}A)$.

Now we can apply the main theorems of Chapter III. Since $[1,1]_T=0$, Theorem III.5 guarantees the existence of a unique bounded at infinity solution of Eqs. (1)–(2), which agrees with the result of Beals [3]. On the other hand, by Theorem III.4, a solution of Eqs. (1)–(2) vanishing at infinity may not exist, the measure of noncompleteness being 1. Consider now solutions to Eqs. (1)–(2) which may grow like x at infinity. By Theorem III.6, since $[v+1, v+1]_T=1$ and $[v-1,v-1]_T=-1$, these solutions are not unique, the measure of nonuniqueness being 1. Thus, the Kramers or slip-flow problem [11], with $\varphi_+=0$ in (2), has a unique nonzero solution, up to a constant factor.

2. The one dimensional BGK model equation ([1,42])

$$v\frac{\partial f}{\partial x}(x,v) + f(x,v) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty}[1+2vu+2(v^2-\tfrac{1}{2})(u^2-\tfrac{1}{2})]f(x,u)e^{-u^2}du.$$

The equation is posed in the Hilbert space $H=L^2(\mathbb{R},\rho)$ where ρ is the same measure as in the first example. We define T and A by

$$(Tf)(v) = vf(v)$$

and

$$(Af)(v) = f(v) - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}[1+2vu+2(v^2-\tfrac{1}{2})(u^2-\tfrac{1}{2})]f(u)e^{-u^2}du.$$

Then T is unbounded and self-adjoint. Since $Af=f-(\alpha_1,f)\alpha_1-(\alpha_2,f)\alpha_2-(\alpha_3,f)\alpha_3$ and $\alpha_1=1$, $\alpha_2=\sqrt{2}v$, $\alpha_3=\sqrt{2}(v^2-\frac{1}{2})$ forms an orthonormal sequence in H, it follows from the Bessel inequality that A is positive. Moreover it is clear that I−A is compact. One checks easily that the regularity condition III.(1) is fulfilled, and that KerA = span $\{1,v,v^2\}$, $Z_0(T^{-1}A)=\text{span}\,\{1,v,v^2,v^3\}$. Now we introduce the sesquilinear form

$$[h,k]_T = (Th,k) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}vh(v)\overline{k(v)}e^{-v^2}dv.$$

To apply Theorems III.4 and III.5 one has to represent this sesquilinear form as a diagonal matrix with respect to an appropriate mutually $[\cdot,\cdot]_T$-orthogonal basis of KerA. The diagonalization of a symmetric bilinear form is a simple algebraic procedure and results in a matrix with 1, −1 and 0 on the diagonal. Then, by Theorem III.4, a solution of Eqs. (1)−(2) vanishing at infinity may not exist; more precisely, the measure of noncompleteness is 2. This reflects the fact that the BGK linearized equation under study represents perturbation of the longitudinal momentum density. Thus, conservation of mass and energy restricts the set of initial states correspondingly by two dimensions in order to have solutions vanishing at infinity, or, equivalently, to have physical densities approaching the equilibrium density at infinity. Solutions of the boundary value problem bounded at infinity always exist by Theorem III.5, and indeed have measure of nonuniqueness 1. Of course, such solutions represent physical distributions which may not converge to the Maxwellian about which the equation was initially linearized. Finally, diagonalization of the symmetric bilinear form on $Z_0(T^{-1}A)$ leads to a matrix with 1, -1, $\frac{3}{2}$ and $-\frac{3}{2}$ on the diagonal. Thus, solutions to the boundary value problem of order x at infinity have measure of nonuniqueness 2.

3. The BGK equation for heat transfer ([5,12,31])

$$v\frac{\partial}{\partial x}\begin{bmatrix} f_1(x,v) \\ f_2(x,v) \end{bmatrix} = -\begin{bmatrix} f_1(x,v) \\ f_2(x,v) \end{bmatrix} + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} D(v,u)\begin{bmatrix} f_1(x,u) \\ f_2(x,u) \end{bmatrix} e^{-u^2}du$$

with

$$D(v,u) = \begin{bmatrix} 1 + \frac{2}{3}(v^2-\frac{1}{2})(u^2-\frac{1}{2}) & \frac{2}{3}(v^2-\frac{1}{2}) \\ \frac{2}{3}(u^2-\frac{1}{2}) & \frac{2}{3} \end{bmatrix}.$$

This equation is posed in the Hilbert space $H=L^2(\mathbb{R},\rho)\oplus L^2(\mathbb{R},\rho)$ where $\frac{d\rho}{dv} = \frac{1}{\sqrt{\pi}}e^{-v^2}$. Let f be the column vector with entries f_1 and f_2. We define T and A by

$$(Tf)(v) = vf(v)$$

and

$$(Af)(v) = f(v) - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} D(v,u)f(u)e^{-u^2}du.$$

Then T is unbounded and self-adjoint. Since $Af=f-(\alpha_1,f)\alpha_1-(\alpha_2,f)\alpha_2$ where $\alpha_1=[1,0]^\perp$, $\alpha_2=(\frac{2}{3})^{1/2}[v^2-\frac{1}{2},1]^\perp$ is an orthonormal set in H, it follows from the Bessel inequality that A is positive. Moreover it is clear that I$-$A is compact.

One easily checks that the regularity condition III.(1) is satisfied. Straightforward computation gives

$$\mathrm{KerA} = \mathrm{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}v^2\\1\end{bmatrix}\right\}$$

and

$$Z_0(T^{-1}A) = \mathrm{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}v\\1\end{bmatrix}, \begin{bmatrix}v^2\\1\end{bmatrix}, \begin{bmatrix}v^3\\v\end{bmatrix}\right\}.$$

Now we introduce the sesquilinear form

$$[h,k]_T = (Th,k) =$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} vh_1(v)\overline{k_1(v)}e^{-v^2}dv + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} vh_2(v)\overline{k_2(v)}e^{-v^2}dv.$$

Since KerA is degenerate with respect to this sesquilinear form, Theorem III.5 implies that the boundary value problem (1), (2) has a unique bounded at infinity solution. On the other hand, by Theorem III.4, solutions vanishing at infinity may not exist, the measure of noncompleteness being 2, which is again a result of the conservation laws (of mass and energy). To apply Theorem III.6 one has to represent the sesquilinear form on $Z_0(T^{-1}A)$ as a diagonal matrix. One obtains a matrix with 1, -1, $\frac{5}{2}$ and $-\frac{5}{2}$ on the diagonal. Thus, solutions to the boundary value problem of order x at infinity have measure of nonuniqueness 2. The corresponding Kramers or slip-flow problem has a two dimensional manifold of solutions.

The next two examples come from neutron transport theory and illustrate the results of Chapter IV.

4. The isotropic neutron transport equation ([2,9])

The isotropic one-speed neutron transport equation is

$$\mu\frac{\partial\psi}{\partial x}(x,\mu) = -\psi(x,\mu) + \frac{1}{2}c\int_{-1}^{1}\psi(x,\nu)d\nu \tag{4}$$

where $\mu \in [-1,1]$ is the cosine of the angle describing the direction of propagation and $x \in (0,\tau)$ is the position coordinate. The number c represents the mean number of secondaries per collision. We will assume that $c>1$ which is the case for a multiplying medium. The equation is studied in the Hilbert space $H=L^2[-1,1]$. We define the operators A, T and $Q_\pm$ on H in the following manner:

$$(Af)(\mu) = f(\mu) - \tfrac{1}{2}c\int_{-1}^{1} f(\nu)d\nu \ ,$$
$$(Tf)(\mu) = \mu f(\mu),$$
$$(Q_\pm f)(\mu) = \begin{cases} f(\mu) & \pm\mu>0 \\ 0 & \pm\mu<0 . \end{cases}$$

One easily verifies that T and A are bounded self-adjoint, injective, I−A is compact (in fact rank one) and the regularity condition IV.(8) is satisfied with any $\gamma \in (0,\frac{1}{2})$. It also has been shown in [16] that A is T-regular. Therefore we can use Theorem IV.5 to conclude that for sufficiently small diameter the slab problem for the isotropic neutron transport equation with $c>1$ has a unique solution for any incoming fluxes at the boundaries $x=0$ and $x=\tau$. Since ψ represents the neutron density, it is clear that only nonnegative solutions are physically acceptable. Using the integral equation corresponding to Eq.(4) one can easily show that the solution is in fact nonnegative whenever incoming fluxes are nonnegative.

5. <u>The symmetric multigroup transport equation</u> ([16,34])

The symmetric multigroup approximation with isotropic scattering leads to the following neutron transport equation:

$$\mu\frac{\partial\psi_i}{\partial x}(x,\mu) = -\sigma_i\psi_i(x,\mu) + \frac{1}{2}\sum_{j=1}^{N} C_{ij}\int_{-1}^{1}\psi_j(x,\nu)d\nu \quad i=1,..,N \tag{5}$$

where $\mu \in [-1,1]$ and $x \in (0,\tau)$. Here $\psi_i(x,\mu)$ is the angular density of neutrons with speed $\frac{1}{\sigma_i}$ (in units of the largest speed). We will assume that

$C=(C_{ij})_{i,j=1}^{N}$ is a real symmetric matrix. Let Σ be the diagonal matrix with diagonal entries $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N = 1$.

Equation (5) is studied in the Hilbert space $H = \bigoplus_{i=1}^{N} L^2([-1,1],\sigma_i d\mu)$. We define operators A,T and $Q_\pm$ on H by

$$(Af)(\mu) = f(\mu) - \tfrac{1}{2}\Sigma^{-1}C\int_{-1}^{1} f(\nu)d\nu,$$

$$(Tf)(\mu) = \Sigma^{-1}\mu f(\mu),$$

$$(Q_\pm f)(\mu) = \begin{cases} f(\mu) & \pm\mu>0 \\ 0 & \pm\mu<0 . \end{cases}$$

where $f=(f_i)_{i=1}^{N} \in H$. Let $e_i \in H$ be the constant vector valued function whose all entries are zero but the i-th entry which is equal to 1. Then

$$Af = f - \tfrac{1}{2}\Sigma^{-1}C\sum_{i=1}^{N}(e_i,f)e_i.$$

Now it is clear that I−A is compact (in fact of finite rank) and A is self-adjoint in H. We will assume that $\det(\Sigma - C) \neq 0$ so that A is invertible. One verifies easily that the regularity condition IV.(8) is satisfied with any $\gamma \in (0,\frac{1}{2})$. The operator T is evidently bounded and self-adjoint in H. Moreover it has been shown in [16] that A is T-regular. Thus, by Theorem IV.5, for sufficiently small diameter the slab problem for Eq.(5) has a unique solution for any incoming fluxes.

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