THE MOMENTS AND DISTRIBUTION FOR AN ESTIMATE OF THE SHANNON INFORMATION MEASURE AND ITS APPLICATION TO ECOLOGY

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CHAPTER I

INTRODUCTION

1. Background

Williams [55] gives an account of a thirteenth century naturalist who was interested in the numbers of individuals, and relative abundance of species. Even though man has been interested in the concept of diversity for many centuries, the quantitative approach is of comparatively recent growth.

Odum [31] states it is generally assumed, but without much scientific evidence, that the "advantage" of a diversity of species -- that is, the survival value to the community -- lies in increased stability. The more species present, the greater the possibilities for adaptation to changing conditions.

One only has to look at the journals to satisfy one's self that diversity is of prime importance to the ecologist at the present time. The journals will also indicate that there are a rather large number of indices being used to measure diversity.

Diversity to the ecologists usually means some measure that incorporates both the number of species and their relative abundance.

2. Indices of Diversity

In 1943, Fisher, Corbet and Williams [5] found a measurement of diversity which arises out of the discovery that the logarithmic series showed a very close fit to the frequency distribution of species with different numbers of individuals in a random sample from an * insect population. Briefly, they observed that the number of species with two representatives, the number with three representatives, and etc. could be written in terms of the number with one representative. Thus the sequence $n_1, \frac{n_1 x}{2}, \frac{n_1 x^2}{3}, \ldots$, where x is less than unity, was found to be representative of their samples. The sum of the sequence therefore should be S, the total number of species present

$$S = \frac{-n_1}{x} \log (1-x).$$

It then follows that the sample size N is

$$N = n_1 + n_1 x + n_1 x^2 + \ldots$$

$$N = \frac{n_1}{1-x}.$$

Fisher called $\alpha = \frac{n_1}{x}$ the 'Index of Diversity' of the population. This index of diversity is dependent on the existence of a logarithmic series distribution.

In 1944, Yule [57] in studying the frequency distribution of the use of different nouns in samples from the writings of different authors, suggested as a 'Characteristic' the value

$$10{,}000(M_2 - M_1)/(M_1)^2$$

where M_1 and M_2 are the first and second moments of the distribution and 10 000 is an arbitrary factor. Williams [55] points out that when the logarithmic series is applicable, Yule's Characteristic is proportional to the reciprocal of the Fisher, Corbet and Williams index of diversity.

In 1949, Simpson [48] suggest using $\lambda = \sum_{i=1}^{s} (\frac{n_i}{N})^2$ as an index of diversity. This index can be simply interpreted as the probability that two individuals chosen at random and independently from the population will be found to belong to the same group.

In 1953, Good [6] considered the following expression, which he regarded as measures of heterogeneity of the population,

$$C_{m,n} = \sum_{i=1}^{s} p_i^m (-\log p_i)^n \quad m,n = 0,1,2,\ldots \quad .$$

If one chooses m = 2, n = 0, and uses $\frac{n_i}{N}$ as an estimate of p_i we see that we have Simpson's index. Also, if one chooses m = 1 and n = 1 we have,

$$C_{1,1} = -\sum_{i=1}^{s} p_i \log p_i$$

which Good calls Shannon's entropy [44], and this is the basis for the entire field of Information Theory.

2.1 Information Theory

Information theory is based on the concept that information is measurable. This idea is not new. In physics, the notion of a measurable relation between information and degree of orderliness (entropy) dates back to Boltzmann's work in 1872.

In 1948, the communication engineer C. E. Shannon published an article on the mathematical theory of communication. This article appeared in a specialized journal, "The Bell Systems Technical Journal," and it pertained to no other field than telecommunication. It certainly did not look like an article destined to reach wide popularity among psychologists, linguists, mathematicians, biologists, and economists. Yet this is what happened.

It seems reasonable to equate the amount of information acquired, as a result of an event, to the amount of uncertainty which its occurrence has abolished. Information is a measurable abstract quantity; its value does not depend on what the information is about, just as length or weight have values which do not depend on the nature of the thing which is long or heavy.

The ecologist wishes to be able to compare events with quite different probability sets. For instance, he wishes to be able to say which uncertainty is greater, that associated with a situation with 3 equiprobable alternatives, or that where there are four possibilities .8, .10, .05, and .05.

3. The Problem

There are many more indices of diversity than those listed above. The one that is receiving the greatest attention is Shannon's and it is Shannon's index of diversity that we choose to examine. We do not in any way attempt to show that it is the best measure of diversity. The problem is to find the moments and the distribution of

$$\overline{H} = -\sum_{i=1}^{s} \frac{n_i}{N} \log \frac{n_i}{N} .$$

4. Equipment Used

The majority of the computations were performed on the IBM 360 model 65 utilizing the remote terminal. The language used on the remote terminal is Conversational Programming System which is a subset of PL/1. The remainder of the computations were performed on the IBM 1130, the IBM 7094, and the IBM 1620.

Natural logarithms are used throughout this work.

CHAPTER II

MULTIVARIATE MOMENTS

1. Introduction

In most ecological work one is faced with a large number of species and a large number of individuals. Lloyd and Ghelardi [22] have programmed a computer to work with a model for two hundred species and a million individuals. We have worked with more than one hundred species and several million individuals. Some ecologists use a vacuum type sampler and they often pick up two hundred species and up to two thousand individuals. It should be clear that any serious statistical study of ecological problems leads one straight away into multivariate moments.

In obtaining multivariate moments one encounters tedious algebra in which it is easy to make errors, cumbersome expressions, and notational problems.

A brief discussion of some known methods is given and two methods will be presented which may not be new but at the same time they seem not to have appeared in print.

2. Summary of Some Approaches to Multivariate Moments

2.1 Generating Functions

The multivariate moment of a set of discrete random variables of order r_1, r_2, ... r_k about an arbitrary origin is defined as

$$\mu'_{r_1 r_2 \ldots r_k} = \sum_x p(x_1, x_1, x_2, \ldots x_k)\, x_1^{r_1}\, x_2^{r_2} \ldots x_k^{r_k}$$

the sum being a multiple one taken over all possible values of each x. If in the probability generating function t is replaced by exp(t) we have the crude moment generating function. Thus

$$M'(t_1 t_2 \ldots t_k) = \sum_x p(x_1, x_2, \ldots, x_k) \exp(t_1 x_1 + t_2 x_2 + \ldots + t_k x_k)$$

$$= \sum_{r_1=0} \cdots \sum_{r_k=0} \mu'_{r_1 r_2 \ldots r_k} \frac{t_1^{r_1}}{r_1!} \cdots \frac{t_k^{r_k}}{r_k!} .$$

For the multinomial, the probability generating function is,

$$(p_1 t_1 + p_2 t_2 + \ldots + p_k t_k)^n$$

so that the crude moment generating function is,

$$(p_1 e^{t_1} + p_2 e^{t_2} + \ldots + p_k e^{t_k})^n =$$

$$(1 + \sum_1^k p_i t_i + \frac{1}{2!} \sum_1^k p_i t_i^2 + \ldots)^n .$$

By expanding the generating function

$$(\Sigma p_i e^{t_i})^n = 1 + n\,(\Sigma p_i t_i + \frac{1}{2!} \Sigma p_i t_i^2 + \ldots) +$$

$$\frac{n(n-1)}{2!} (\Sigma p_i t_i + \frac{1}{2!} \Sigma p_i t_i^2 + \ldots)^2 +$$

$$\frac{n(n-1)(n-2)}{3!} (\Sigma p_i t_i + \frac{1}{2!} \Sigma p_i t_i^2 + \ldots)^3 + \ldots$$

the crude moments can be gotten by looking for the coefficients of $\frac{t_i^\alpha}{\alpha!}$

such as

$$\mu'_{\ldots 1 \ldots} = np_i \qquad\qquad i = 1, 2, \ldots k$$

$$\mu'_{\ldots 2 \ldots} = np_i + n(n-1)p_i^2 \qquad i = 1,2, \ldots k$$

$$\mu'_{\ldots 11 \ldots} = n(n-1)p_i p_j \qquad i \neq j,$$

and so on.

Moments about the means are obtained from moments about the origins. Thus

$$M(t_1 t_2 \ldots t_k) = \sum_x p(x_1, \ldots, x_k) \exp\left[\sum_1^k t_i (x_i - \mu'_{\ldots 1 \ldots})\right]$$

$$= \exp(-\sum_1^k t_i \mu'_{\ldots 1 \ldots}) \sum_r \mu'_{r_1 \ldots r_k} \frac{t_1^{r_1}}{r_1!} \cdots \frac{t_k^{r_k}}{r_k!}$$

$$= \sum_r \mu_{r_1 \ldots r_k} \frac{t_1^{r_1}}{r_1!} \cdots \frac{t_k^{r_k}}{r_k!} .$$

2.2 Wishart's Method

Let $(p_0 + \sum_1^k p_i t_i)^n$ be the probability generating function of the multinomial distribution where $p_0 = 1 - \Sigma p_i$. The moment generating function is

$$(p_0 + p_1 e^{t_1} + p_2 e^{t_2} + \ldots + p_k e^{t_k})^n = W = E \exp(\sum_1^k t_i x_i).$$

Put $p_i/p_0 = a_i$, $i = 1, 2, \ldots, k$, so that

$$\frac{p_1 + p_2 + \ldots + p_k}{p_0} = a_1 + a_2 + \ldots + a_k$$

$$= (1-p_0)/p_0.$$

Also, $1/p_0 = 1 + \sum_1^k a_i$ and let $1 + \sum_1^k a_i = a_0$.

Hence,

$$W = (1+a_1e^{t_1} + a_2e^{t_2} + \dots + a_ke^{t_k})^n/(1 + \sum_1^k a_1)^n \qquad (2.2.1)$$

and differentiating (2.2.1) with respect to a_i and then with respect to t_i;

$$(1/W)\partial W/\partial a_i = ne^{t_i}/(1+\Sigma a_ie^{t_i}) - n/(1+\Sigma a_i))$$

$$(1/W)\partial W/\partial t_i = na_ie^{t_i}/(1+\Sigma a_ie^{t_i}) .$$

But

$$\left.\frac{\partial W}{\partial t_i}\right|_{t_i = 0} = \mu'_{\dots 1 \dots} = \frac{na_i}{1+\Sigma a_i} ,$$

hence

$$\frac{1}{W}\frac{\partial W}{\partial a_i} + \frac{n}{1+\Sigma a_i} = \frac{1}{a_iW}\frac{\partial W}{\partial t_i} \quad \text{and}$$

$$\frac{\partial W}{\partial t_i} = a_i\frac{\partial W}{\partial a_i} + W\mu'_{\dots 1 \dots} . \qquad (2.2.2)$$

Differentiating (2.2.2) r_i times with respect to t_i for all $i = 1,2,\dots,k$ and set all t_i equal to zero, thus

$$\mu'_{r_1r_2\dots r_i + 1\dots r_k} = \mu'_{\dots 1 \dots}\, \mu'_{r_1r_2\dots r_k} + a_i\frac{\partial(\mu'_{r_1r_2\dots r_k})}{\partial a_i} .$$

Example 2.2.1

Let $r_1 = r_i = 0$ and $r_2 = 1$,

$$\mu'_{11} = \mu'_{10}\ \mu'_{01} + a_i \frac{\partial(\mu_{01})}{\partial a_i}$$

$$= np_i\ np_j + a_i \frac{\partial(np_j)}{\partial a_i}$$

$$= n^2 p_i p_j + a_i \frac{\partial \dfrac{na_j}{a_0}}{\partial a_i}$$

$$= n^2 p_i p_j + na_i \frac{\partial \dfrac{a_j}{1+a_i+a_j+\ldots+a_k}}{\partial a_i}$$

$$= n^2 p_i p_j + na_i \frac{-a_j}{a_0^2}$$

$$= n^2 p_i p_j + n \frac{a_i}{a_0} \frac{-a_j}{a_0}$$

$$= n^2 p_i p_j - np_i p_j$$

$$= n(n-1) p_i p_j;$$

Example 2.2.2

$$\mu'_{21} = \mu'_{10}\mu'_{11} + a_i \frac{\partial \mu'_{11}}{\partial a_i}$$

$$= np_i n(n-1)p_i p_j + a_i \frac{\partial n(n-1)p_i p_j}{\partial a_i}$$

$$= n^2(n-1)p_i^2 p_j + n(n-1)a_i \frac{\partial \frac{a_i}{a_0}\frac{a_j}{a_0}}{\partial a_i}$$

$$= n(n-1)(n-2)p_i^2 p_j + n(n-1)p_i p_j \ .$$

The central moments can be derived by this method. Let

$$V = \exp\ (-n\sum_1^k p_i t_i)(p_0+\Sigma p_i e^{t_i})^n$$

or in the notation set out above

$$V = \exp(-n\Sigma a_i t_i/a_0)\left(\frac{1+\Sigma a_i e^{t_i}}{a_0}\right)^n . \tag{2.2.3}$$

Upon taking logarithms of (2.2.3) we get

$$\log V = -n\Sigma a_i t_i/a_0 + n \log(1+\Sigma\ a_i\ e^{t_i}) - n \log a_0. \tag{2.2.4}$$

Differentiating (2.2.4) with respect to t_i gives

$$\frac{1}{V}\frac{\partial V}{\partial t_i} = -\frac{na_i}{a_0} + \frac{na_i e^{t_i}}{1+\Sigma a_i e^{t_i}} \ . \tag{2.2.5}$$

Differentiating (2.2.4) with respect to a_i gives

$$\frac{1}{V}\frac{\partial V}{\partial a_i} = -\frac{nt_i}{a_0} + \frac{n\Sigma a_i t_i}{a_0^2} + \frac{ne^{t_i}}{1+\Sigma a_i e^{t_i}} - \frac{n}{a_0},$$

multiplying this last equation by a_i and substituting (2.2.5) gives

$$\frac{a_i}{V}\frac{\partial V}{\partial a_i} = \frac{1}{V}\frac{\partial V}{\partial t_i} + \frac{na_i}{a_0^2}\left(\sum_1^k a_i t_i - t_i a_0\right). \tag{2.2.6}$$

Solving (2.2.6) for $\frac{\partial V}{\partial t_i}$,

$$\frac{\partial V}{\partial t_i} = a_i \frac{\partial V}{\partial a_i} + \frac{Vna_i}{a_0^2}\left(t_i a_0 - \sum_1^k a_i t_i\right). \tag{2.2.7}$$

Now differentiate (2.2.7) r_i times with respect to t_i for all $i = 1,2, \ldots, k$ and then set all t_i's equal to zero. The second term on the right side of (2.2.7) can best be differentiated by using Leibniz's formula, $d^n uv = u^n + \binom{n}{1} u^{n-1} v$. As an illustration we will differentiate the second term r_1 times. Thus

$$\frac{na_i}{a_0^2}\left(\frac{\partial^{r_1} V}{\partial t_1^{r_1}}(t_i a_0 - \Sigma a_i t_i) + \binom{r_1}{1}\frac{\partial^{r_1 - 1} V}{\partial t_1^{r_1 - 1}}(-a_1)\right)$$

and now differentiating this last expression r_2 times

$$\frac{na_i}{a_0^2}\left(\frac{\partial^{r_1+r_2}V}{\partial t_1^{r_1} t_2^{r_2}}(t_i a_0 - \Sigma a_i t_i) + \binom{r_2}{1}\frac{\partial^{r_1+r_2-1}V}{\partial t_1^{r_1}\partial t_2^{r_2-1}}(-a_2) - \right.$$

$$\left. a_1\binom{r_1}{1}\frac{\partial^{r_1+r_2-1}V}{\partial t_1^{r_1-1}\partial t_2^{r_2}}\right)$$. Generalize and set the t_i's equal to zero

we have

$$\mu_{r_1 r_2 \ldots r_i+1 \ldots r_k} = a_i\frac{\partial\mu_{r_1 r_2 \ldots r_k}}{\partial a_i} + \frac{na_i}{a_0^2}(-a_1 r_1 \mu_{r_1-1\, r_2 \ldots r_k} -$$

$$a_2 r_2 \mu_{r_1 r_2-1\, r_3 \ldots r_k} -$$

$$a_{i-1} r_{i-1}\mu_{r_1 \ldots r_{i-1}-1\, r_i \ldots r_k} +$$

$$(a_0 - a_i) r_i \mu_{r_1 \ldots r_i-1\, r_{i+1} \ldots r_k} -$$

$$a_{i+1} r_{i+1}\mu_{r_1 \ldots r_{i+1}-1, \ldots r_k}).$$

<u>Example 2.2.3</u>

Let $r_1 = 1$ and $r_i = r_2 = 1$ then

$$\mu_{12} = \frac{na_2}{a_0^2}(-a_1(1)\mu_{01} - a_2(1)\mu_{10}) + a_2\frac{\partial\mu_{11}}{\partial a_2}$$

$$= np_2(-p_1\mu_{01} - p_2\mu_{10}) + a_2\frac{\partial\mu_{11}}{\partial a_2}$$

$$= np_1(-p_1(0) - p_2(0)) + a_2 \frac{\partial(-np_1p_2)}{\partial a_2}$$

$$= na_2 \frac{\partial \frac{a_1}{a_0} \frac{a_2}{a_0}}{\partial a_2}(-1)$$

$$= -na_1a_2 \frac{a_0-2a_2}{a_0^3}$$

$$= -np_1p_2(1-2p_2).$$

<u>Example 2.2.4</u>

Let $r_1 = 2$, $r_2 = r_i = 0$, $r_3 = 1$ then

$$\mu_{211} = \frac{na_2}{a_0^2} (-a_1(2)\mu_{101} - a_2(0)\mu_{2,-1,1} - a_3(1)\mu_{200}) +$$

$$a_2 \frac{\partial \mu_{201}}{\partial a_2}$$

$$= np_2(-2p_1\mu_{101} - p_3\mu_{200}) + a_2 \frac{\partial \mu_{201}}{\partial a_2}$$

$$= -n(n-2)p_1p_2p_3(1-3p_1).$$

For reference on the Wishart Method see [9], [10], [39], and [56].

2.3 Small Sample Method

Suppose t is a statistic defined over a discrete sample space and the

$$E\{f(t)\} = \frac{A_1}{n^\lambda} + \frac{A_2}{n^{\lambda+1}} + \ldots + \frac{A_m}{n^{\lambda+m-1}} \text{, where } \lambda \text{ is a positive integer.}$$

We have "m" coefficients to determine. We have

$$E\{f(t)\}_{n=1} = A_1 + A_2 + \ldots + A_m,$$

$$E\{f(t)\}_{n=2} = \frac{A_1}{2^\lambda} + \frac{A_2}{2^{\lambda+1}} + \ldots + \frac{A_m}{2^{\lambda+m-1}},$$

$$\vdots$$

$$E\{f(t)\}_{n=m} = \frac{A_1}{m^\lambda} + \frac{A_2}{m^{\lambda+1}} + \ldots + \frac{A_m}{m^{\lambda+m-1}}.$$

Hence,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{2^\lambda} & \frac{1}{2^{\lambda+1}} & \cdots & \frac{1}{2^{\lambda+m-1}} \\ \frac{1}{3^\lambda} & \frac{1}{3^{\lambda+1}} & \cdots & \frac{1}{3^{\lambda+m-1}} \\ \vdots & & & \\ \frac{1}{m^\lambda} & \frac{1}{m^{\lambda+1}} & & \frac{1}{m^{\lambda+m-1}} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} E\{f(t)\}_{n=1} \\ E\{f(t)\}_{n=2} \\ \vdots \\ E\{f(t)\}_{n=m} \end{bmatrix}.$$

Solving for $[A_1 A_2 \ . \ . \ . \ A_m]'$,

$$\begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{bmatrix} = \begin{bmatrix} 1 & 1 & & 1 \\ \frac{1}{2^\lambda} & \frac{1}{2^{\lambda+1}} & \cdots & \frac{1}{2^{\lambda+m-1}} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \frac{1}{m^\lambda} & \frac{1}{m^{\lambda+1}} & \cdots & \frac{1}{m^{\lambda+m-1}} \end{bmatrix}^{-1} \begin{bmatrix} E\{f(t)\}_{n=1} \\ E\{f(t)\}_{n=2} \\ \\ \\ \\ E\{f(t)\}_{n=m} \end{bmatrix} .$$

Example 2.3.1

Let $\varepsilon_i = \dfrac{n_i - np_i}{n}$ and consider the $E(\varepsilon_1^{\alpha_1} \varepsilon_2^{\alpha_2} \ldots \varepsilon_k^{\alpha_k})$.

Here λ = integer part of $\left(\dfrac{1+\sum_1^k \alpha_i}{2}\right)$ and $m = \sum_1^k \alpha_i - \lambda$.

$E(\varepsilon_i^4) = \dfrac{A_1}{n^2} + \dfrac{A_2}{n^3}$, since $[\frac{1+4}{2}] = 2$ and

$$\begin{bmatrix} A_1 \\ \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \\ \frac{1}{4} & \frac{1}{8} \end{bmatrix}^{-1} \begin{bmatrix} (1-p_i)^4 p_i + (-p_i)^4 (1-p_i) \\ \\ (1-p_i)^4 p_i^{\ 2} + (-p_i)^4 (1-p_i)^2 + (\frac{1}{2} - p_i)^4 \ 2p_i q_i \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ \\ A_2 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ \\ 2 & -8 \end{bmatrix} \begin{bmatrix} p_i q_i \ (1-3p_i q_i) \\ \\ p_i^2 q_i^2 (q_i^2 + p_i^2) + \frac{p_i q_i}{8} (1-2p_i)^4 \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ \\ A_2 \end{bmatrix} = \begin{bmatrix} 3p_i^2 q_i^2 \\ \\ p_i q_i - 6p_i^2 q_i^2 \end{bmatrix} .$$

Therefore,

$$E(\varepsilon_i^4) = \frac{3p_i^2q_i^2}{n^2} + \frac{p_iq_i}{n^3}\left(\frac{1-6p_iq_i}{\quad}\right) .$$

Example 2.3.2

In the expectation of $\varepsilon_i^2\varepsilon_j^2$ we have $\lambda = [\frac{5}{2}] = 2$ and $m = 4-2 = 2$, therefore,

$$E(\varepsilon_i^2\varepsilon_j^2) = \frac{A_1}{n^2} + \frac{A_2}{n^3} .$$

$$E(\varepsilon_i^2\varepsilon_j^2)_{n=1} = (1-p_i)^2(-p_j)^2p_i + (-p_i)^2(1-p_j)^2p_j + (-p_i)^2(-p_j)^2(1-p_i-p_j)$$

$$= p_ip_j(p_i+p_j-3p_ip_j) \quad \text{and}$$

$$E(\varepsilon_i^2\varepsilon_j^2)_{n=2} = (1-p_i)^2(-p_j)^2p_i^2 + (-p_i)^2(1-p_j)^2p_j^2 + (-p_i)^2(-p_j)^2(1-p_i-p_j)^2 +$$

$$(\tfrac{1}{2}-p_i)^2(\tfrac{1}{2}-p_j)^2 2p_ip_j + (\tfrac{1}{2}-p_i)^2(-p_j)^2 2p_i(1-p_i-p_j) +$$

$$(-p_i)^2(\tfrac{1}{2}-p_j)^2 2p_j(1-p_i-p_j)$$

$$= \frac{p_ip_j}{8} . \quad \text{Hence}$$

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 2 & -8 \end{bmatrix} \begin{bmatrix} p_ip_j(p_i+p_j-3p_ip_j) \\ p_ip_j(\frac{1}{8}) \end{bmatrix} \qquad \text{and}$$

$$E(\varepsilon_i^2\varepsilon_j^2) = \frac{p_i p_j}{n^2}(1-p_i-p_j+3p_ip_j) - \frac{p_ip_j}{n^3}(1-2p_i-2p_j+6p_ip_j).$$

In the book "Combinatorial Chance" by David and Barton [4], page 146, there is an error in $E(\varepsilon_i^2\varepsilon_j^2)$.

2.4 Q-Statistic Method

Myers [29] introduced the concept of orthogonal polynomials with respect to a statistical distribution function. This method is as follows.

Let $f(x,\theta)$ be a given parent distribution for which all moments exists. Let t be a statistic which is a function of the crude sample moments. Then there exists a set of orthogonal polynomials $\{q_r(x)\}$ such that

$$\int_{-\infty}^{\infty} q_r(x)q_s(x)f(x,\theta)dx = \phi_r, \quad r = s$$

$$= 0\ , \quad r \neq s,$$

where $\phi_r > 0, \phi_0 = 1$, and the coefficient of x^r in $q_r(x)$ is unity. The r^{th} polynomial of such a set can be found from the determinantal equation.

$$q_r(X) = (-1)^r \begin{vmatrix} 1 & X & X^2 & \cdots & X^r \\ 1 & \mu_1 & \mu_2 & \cdots & \mu_r \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{r+1} \\ \mu_2 & \mu_3 & \mu_4 & \cdots & \mu_{r+2} \\ & & \vdots & & \\ \mu_{r-1} & \mu_r & \mu_{r+1} & \cdots & \mu_{2r-1} \end{vmatrix} \div \begin{vmatrix} 1 & \mu_1 & \mu_2 & \cdots & \mu_{r-1} \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_r \\ \mu_2 & \mu_3 & \mu_4 & \cdots & \mu_{r+1} \\ & & \vdots & & \\ \mu_{r-1} & \mu_r & \mu_{r+1} & \cdots & \mu_{2r-2} \end{vmatrix}$$

where $X = x-\mu_1$, and μ_j is the j^{th} central moment of $f(x,\theta)$.

The polynomial $Q_r(x)$ is defined as

$$Q_r(x) = \frac{\sum_1^n q_r(x_i)}{n}.$$

Thus, $Q_r(x)$ is a function of the crude sample moments. The set of equations $\{Q_r(x)\}$, $r = 1, 2, \ldots s$, can be solved for the crude moments.

We can now express the statistic t in terms of the Q's. Consider the joint moment generating function of the Q's,

$$E(\exp(\alpha_r Q_r+\alpha_s Q_s+\ldots))=(E\ \exp(\frac{\alpha_r q_r+\alpha_s q_s+\ldots}{n}))^n \tag{2.4.1}$$

where r, s, t,..., are distinct integers and expand both sides and equate coefficients of the appropriate products of powers of the α's and obtain the desired expected Q-products. The notation used is,

$$EQ_r^\alpha = (r^\alpha)$$

and

$$Eq_r^\alpha = [r^\alpha].$$

We can now write

$$E(Q_r^\alpha Q_s^\beta\ldots) = (r^\alpha s^\beta\ldots) = \sum_{\lambda=a}^{b} (r^\alpha s^\beta\ldots)_\lambda/n^\lambda,$$

where a = integer part of $(\frac{1+\alpha+\beta+\ldots}{2})$ and $b = -1+\alpha+\beta+\ldots$. Tables of the expected Q-products as functions of the expected q-products are given by Myers [29]. These tables have been extended by means of the IBM 1620 by Shenton, Bowman, and Reinfelds [46]. For example, from these tables we find

$$E(Q_1^2Q_2^2) = \frac{E(q_1^2)E(q_2^2)}{n^2} + \frac{E(q_1^2q_2^2) - E(q_1^2)E(q_2^2)}{n^3} .$$

Expanding (2.4.1) and use the notation described above gives,

$$E[1 + \sum_{i=r} \alpha_i Q_i + \frac{(\sum\limits_{i=r} \alpha_i Q_i)^2}{2!} + \frac{(\sum\limits_{i=r} \alpha_i Q_i)^3}{3!} + \ldots] =$$

$$1 + n \{ \sum_{i=r} \frac{\alpha_i^2[i^2]}{n^2 2!} + \frac{\sum\limits_{i=r} \alpha_i^3[i^3]}{n^3 3!} + \ldots\} + \frac{n(n-1)}{2} \{\frac{\sum\limits_{i=r} \alpha_i^2[i^2]}{n^2 2!} + \frac{\sum\limits_{i=r} \alpha_i^3[i^3]}{n^3 3!} + \ldots\}^2 +$$

$$\frac{n(n-1)(n-2)}{6} \{\frac{\sum\limits_{i=r} \alpha_i^2[i^2]}{n^2 2!} + \frac{\sum\limits_{i=r} \alpha_i^3[i^3]}{n^3 3!} + \ldots\}^3 + \ldots +$$

$$\frac{n(n-1)\ldots(n-k+1)}{k!} \{\frac{\sum\limits_{i=r} \alpha_i^2[i^2]}{n^2 2!} + \frac{\sum\limits_{i=r} \alpha_i^3[i^3]}{n^3 3!} + \ldots\}^k + \ldots \quad . \qquad (2.4.2)$$

Equating coefficients of α's in (2.4.2) we see that

$$EQ_i = 0$$

$$E\frac{Q_i^2}{2!} = \frac{n[i^2]}{n^2 2!} \quad \longrightarrow \quad (i^2) = \frac{[i^2]}{n}$$

$$E\frac{Q_i^3}{3!} = \frac{n[i^3]}{n^3 3!} \quad \longrightarrow \quad (i^3) = \frac{[i^3]}{n^2}$$

$$E\frac{Q_i^4}{4!} = \frac{n[i^4]}{n^4 4!} + \frac{n(n-1)}{2} \frac{[i^2]^2}{(n^2 2!)^2} \quad \longrightarrow \quad (i^4) = \frac{3[i^2]^2}{n^2} + \frac{[i^4]-3[i^2]^2}{n^3}$$

and etc. where $i = r,s,\ldots$.

Kendall's [18] symbolic operator can be applied to (r^j) where $j = 2,3,\ldots$, for example if we start with

$$(r^4) = \frac{3[r^2]^2}{n^2} + \frac{[r^4]-3[r^2]^2}{n^3} \text{ and apply } s\frac{\partial}{\partial r} \text{ we can arrive at,}$$

$$(r^3s) = \frac{3[r^2][rs]}{n^2} + \frac{[r^3s]-3[r^2][rs]}{n^3}\text{, and applying it again gives,} \tag{2.4.3}$$

$$(r^2s^2) = \frac{2[rs]^2+[r^2][s^2]}{n^2} + \frac{[r^2s^2]-2[rs]^2-[r^2][s^2]}{n^3} \text{ and etc.} \tag{2.4.4}$$

It is this repeated application of Kendall's symbolic operator that has been automated. If one feeds an initial expression into the Computer it will produce the desired products (see [18]).

Due to the orthogonality of the q's equation (2.4.3) is zero.

A modification of this method will give the multinomial moments. We will not insist on the $\{q_r(x)\}$ being orthogonal. Let $e_j = q_r(x_j) = x_j - p_j$, where $pr(x_j=1) = p_j$ and $pr(x_j=0) = 1-p_j$.

Using this definition of q gives $E(e_i) = 0$, $E(e_ie_j) = -p_ip_j$,

$E(e_j^2) = p_jq_j$, $E(e_i^2e_j^2) = p_ip_j(p_i+p_j-3p_ip_j)$ and etc.

Substituting these values into (2.4.4) yields

$$E(\frac{n_i}{n} - p_i)^2(\frac{n_j}{n} - p_j)^2 = (r^2s^2) = \frac{p_ip_j}{n^2}(1-p_i-p_j+3p_ip_j)\frac{p_ip_j}{n^3}(1-2p_i-2p_j+6p_ip_j).$$

2.5 Conclusions

Since the table of expectation of Q-products has been automated and the expectation of $(e_i e_j e_k \ldots)$, where $e_i = x_i - p_i$ and $pr(x_i=1) = p_i$ and $pr(x_i=0) = 1-p_i$, is straightforward algebra the Q-Statistic Method has a decided advantage over other known methods of finding multinomial moments of high order. In addition, this method holds the promise of being completely automated. However, the Small Sample Method holds some promise of being automated. In fact we have a program that will evaluate $E(f(t))_{n=x}$ where x is an integer and the distribution is multinomial with equal probabilities.

CHAPTER III

The Mean and the Variance of $\bar{H}$

1. $\bar{H}$ and the Multinomial

Let us suppose a mixed species population has $n_1, n_2, \ldots, n_k$ individuals of k different types, where

$$N = \sum_{i=1}^{k} n_i \ .$$

The number of ways one can partition N objects into k categories with n_1 elements alike of one kind, n_2 elements alike of a second kind, and so on for s kinds, is

$$\frac{N!}{n_1!n_2!\ldots n_s!} = \frac{N!}{\prod_{i=1}^{s} n_i!} \ .$$

Let, $H = \log \dfrac{N!}{\prod_{i=1}^{s} n_i!}$

$$= \log N! - \sum_{i=1}^{s} \log n_i!.$$

Assuming a reasonably large sample, then

$\log N! \doteq N(\log N-1)$ and $\log n_i! \doteq n_i\,(\log n_i-1)$.

Now, $H \doteq N \log N - N - (\sum_{i=1}^{s} n_i \log n_i - \Sigma n_i)$, thus

$$H \doteq -\left[\Sigma n_i \log \frac{n_i}{N}\right] .$$

If one takes H to be the diversity of the population, then the diversity per individual is

$$\overline{H} = -\left[\Sigma \frac{n_i}{N} \log \frac{n_i}{N}\right] .$$

2. Expected Value of $\overline{H}$

The frequency generating function of the multinomial distribution is

$$(p_1t_1 + p_2t_2 \ldots\ldots + p_st_s)^N = \Sigma \frac{N!}{n_1!n_2!\ldots n_s!} (p_1t_1)^{n_1} (p_2t_2)^{n_2} \ldots (p_st_s)^{n_s}, \tag{3.2.1}$$

where $\sum_1^s p_i = 1$ and $n_1 + n_2 + \ldots + n_s = N$. We now perform a series of mathematical operations on (3.2.1) in order to produce the desired expectation.

First multiply (3.2.1) by e^{-x} :

$$\Sigma P(\underline{n})t_1^{n_1} t_2^{n_2} \ldots t_s^{n_s} e^{-x} = e^{-x}(p_1t_1 + \ldots + p_st_s)^N . \tag{3.2.2}$$

Differentiate (3.2.2) with respect to t_1 and then multiply by t_1 ;

$$\Sigma P(\underline{n})t_1^{n_1} t_2^{n_2} \ldots t_s^{n_s} n_1 e^{-x} = Ne^{-x} t_1 p_1 (p_1t_1 + \ldots p_st_s)^{N-1} . \tag{3.2.3}$$

Returning to (3.2.1) we differentiate with respect to t_1, multiply by t_1, and replace t_1 by $e^{-x}t_1$;

$$\Sigma P(\underline{n})n_1(e^{-x}t_1)^{n_1} t_2^{n_2} \ldots t_s^{n_s} = Ne^{-x} t_1 p_1 (p_1 t_1 e^{-x} + \ldots + p_st_s)^{N-1}. \tag{3.2.4}$$

Subtract (3.2.4) from 3.2.3) and divide by x;

$$\Sigma P(\underline{n}) n_1 t_1^{n_1} \ldots t_s^{n_s} \frac{e^{-x} - e^{-n_1 x}}{x} = N t_1 p_1 \frac{e^{-x}}{x} [(p_1 t_1 + \ldots + p_s t_s)^{N-1} -$$

$$(p_1 e^{-x} t_1 + p_2 t_2 + \ldots + p_s t_s)^{N-1}] \;. \tag{3,2.5}$$

Set each t_i equal to one and integrate with respect to x ;

$$\Sigma P(\underline{n}) n_1 \int_0^\infty \frac{e^{-x} - e^{-n_1 x}}{x} dx = N p_1 \int_0^\infty \frac{e^{-x}}{x} [1 - (p_1 e^{-x} + q_1)^{N-1}] \;. \tag{3.2.6}$$

The integral on the left side of (3.2.6) can be evaluated;

$$\Sigma P(\underline{n}) n_1 \log n_1 = \int_0^\infty N p_1 \frac{e^{-x}}{x} [1 - (p_1 e^{-x} + q_1)^{N-1}] dx \;. \tag{3.2.7}$$

On generalizing (3.2.7) we have

$$\Sigma P(\underline{n}) \sum_{i=1}^{s} n_i \log n_i = N \Sigma p_i \int_0^\infty \frac{e^{-x}}{x} [1 - (p_i e^{-x} + q_i)^{N-1}] dx$$

therefore;

$$E(\sum_{i=1}^{s} n_i \log n_i) = N \Sigma p_i \int_0^\infty \frac{e^{-x}}{x} [1 - (p_i e^{-x} + q_i)^{N-1}] dx \;. \tag{3.2.8}$$

Subtract N log N from each side of (3.2.8) then divide through by a negative N, and replace one by $(p_i + q_i)$;

$$E\left(-\sum_{i=1}^{s}\frac{n_i}{N}\log\frac{n_i}{N}\right) = \log N - {}_0\int^{\infty}\frac{e^{-x}}{x}\,\Sigma p_i[(p_i+q_i)^{N-1}-(p_ie^{-x}+q_i)^{N-1}]dx. \tag{3.2.9}$$

If we use the binomial expansion on the two terms in the integrand and collect terms to form Frullanian type integrals we get, see Hardy [13],

$$E\left(-\sum_{i=1}^{s}\frac{n_i}{N}\log\frac{n_i}{N}\right) = \log N - {}_0\int^{\infty}\Big[\binom{N-1}{0}\Sigma p_i^N\frac{e^{-x}-e^{-Nx}}{x}+\binom{N-1}{1}\Sigma p_i^{N-1}q_i\frac{e^{-x}-e^{-(N-1)x}}{x}+\ldots \tag{3.2.10}$$

$$+\binom{N-1}{N-2}\Sigma p_i^2q_i^{N-2}\frac{e^{-x}-e^{-2x}}{x}+\binom{N-1}{N-1}\Sigma p_iq_i^{N-1}\frac{e^{-x}-e^{-x}}{x}\Big]dx.$$

Integrating term by term in (3.2.10);

$$E(\overline{H}) = \log N-\Big[\binom{N-1}{0}\Sigma p_i^N\log N+\binom{N-1}{1}\Sigma p_i^{N-1}q_i\log(N-1)+\ldots+$$

$$\binom{N-1}{N-2}\Sigma p_i^2q_i^{N-2}\log 2\Big],$$

$$E(\overline{H}) = \log N - \sum_{\lambda=1}^{N-1}\binom{N-1}{N-\lambda}\log(N-\lambda+1)\sum_{i=1}^{s}p_i^{N-\lambda+1}q_i^{\lambda-1},\quad N\geq 2. \tag{3.2.11}$$

The $E(\overline{H})$ for several trinomials is graphed in figure 3.1.

3. The Variance of $\overline{H}$

We again perform a series of mathematical operations comparable to those performed in the previous section. First we integrate (3.2.5);

$$\Sigma P(\underline{n})n_1(\log n_1)t_1^{n_1}\ldots t_s^{n_s} = Nt_1p_1\,{}_0\int^{\infty}\frac{e^{-x}}{x}\Big[(p_1t_1+\ldots+p_st_s)^{N-1} -$$

$$(p_1t_1e^{-x}+p_2t_2+\ldots+p_st_s)^{N-1}\Big]dx\,. \tag{3.3.1}$$

E($\bar{H}$) For Several Trinomials

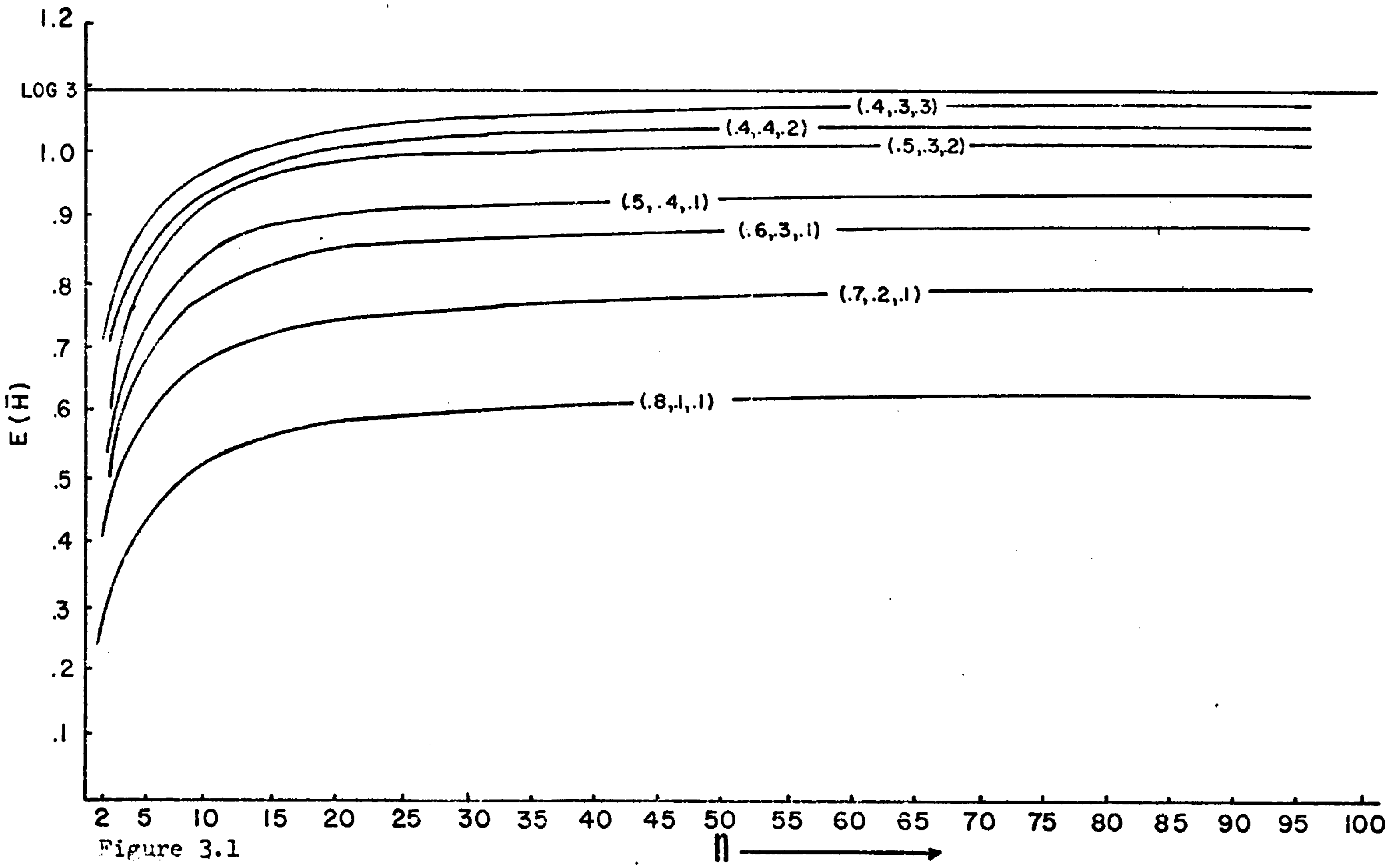

Figure 3.1

Differentiate (3.3.1) with respect to t_1 and multiply by t_1;

$$\Sigma P(\underline{n})n_1^2(\log n_1)t_1^{n_1}\cdots t_s^{n_s} = Np_1t_1\int_0^\infty \frac{e^{-x}}{x}\Big[\Big(\sum_{i=1}^{s} p_it_i\Big)^{N-1}-\Big(p_1t_1e^{-x}+$$

$$\sum_{i=2}^{s} p_it_i\Big)^{N-1}\Big]dx + N(N-1)t_1^2p_1^2\int_0^\infty \frac{e^{-x}}{x}\Big[\Big(\sum_{i=1}^{s} p_it_i\Big)^{N-2}-$$

$$\Big(p_1t_1e^{-x}+\sum_{i=2}^{s} p_it_i\Big)^{N-2}e^{-x}\Big]dx. \qquad (3.3.2)$$

Multiply (3.3.2) by e^{-y} and set each t_i equal to one;

$$\Sigma P(\underline{n})n_1^2(\log n_1)e^{-y} = Np_1\int_0^\infty \frac{e^{-x-y}}{x}[1-(p_1e^{-x}+q_1)^{N-1}]dx +$$

$$N(N-1)p_1^2\int_0^\infty \frac{e^{-x-y}}{x}[1-(p_1e^{-x}+q_1)^{N-2}e^{-x}]dx. \qquad (3.3.3)$$

Replace t_1 in (3.3.2) by $e^{-y}t_1$ and set each t_i equal to one;

$$\Sigma P(\underline{n})n_1^2(\log n_1)e^{-n_1y} = Np_1\int_0^\infty \frac{e^{-x-y}}{x}[(e^{-y}p_1+q_1)^{N-1}-(p_1e^{-x-y}+q_1)^{N-1}]dx +$$

$$N(N-1)p_1^2\int_0^\infty \frac{e^{-x-2y}}{x}[(p_1e^{-y}+q_1)^{N-2}-(p_1e^{-x-y}+q_1)^{N-2}e^{-x}]dx. \qquad (3.3.4)$$

Subtract (3.3.4) from (3.3.3) and divide by y;

$$\Sigma P(\underline{n})n_1^2(\log n_1)\frac{e^{-y}-e^{-n_1y}}{y} = Np_1\int_0^\infty \frac{e^{-x-y}}{xy}[1-(p_1e^{-x}+q_1)^{N-1}-(p_1e^{-y}+q_1)^{N-1}+$$

$$(p_1e^{-x-y}+q_1)^{N-1}]dx +$$

$$N(N-1)p_1^2\int_0^\infty \frac{e^{-x-y}}{xy}[1-(p_1e^{-x}+q_1)^{N-2}e^{-x}-(p_1e^{-y}+q_1)^{N-2}e^{-y}+(p_1e^{-x-y}+q_1)^{N-2}e^{-x-y}]dx. \qquad (3.3.5)$$

Integrate (3.3.5) with respect to y;

$$\Sigma P(\underline{n})n_1^2(\log n_1)^2 = {}_0\int^\infty {}_0\int^\infty Np_1\frac{e^{-x-y}}{xy}[1-(p_1e^{-x}+q_1)^{N-1}-(p_1e^{-y}+q_1)^{N-1}+$$

$$p_1e^{-x-y}+q_1)^{N-1}+N(N-1)p_1-(N-1)p_1e^{-x}(p_1e^{-x}+q_1)^{N-2}-(N-1)p_1e^{-y}(p_1e^{-y}+q_1)^{N-2}+$$

$$(N-1)p_1e^{-x-y}(p_1e^{-x-y}+q_1)^{N-2}]dxdy. \tag{3.3.6}$$

Generalize (3.3.6);

$$E[\Sigma n_i^2(\log n_i)^2] = N\sum_{i=1}^{s} p_i\, {}_0\int^\infty {}_0\int^\infty \frac{e^{-x-y}}{xy}[1-(p_ie^{-x}+q_i)^{N-1}-(p_ie^{-y}+q_i)^{N-1}+$$

$$(p_ie^{-x-y}+q_i)^{N-1}+(N-1)p_i-(N-1)p_ie^{-x}(p_ie^{-x}+q_i)^{N-2}-(N-1)p_ie^{-y}(p_ie^{-y}+q_i)^{N-2}+$$

$$(N-1)p_ie^{-x-y}(p_ie^{-x-y}+q_i)^{N-2}]dxdy. \tag{3.3.7}$$

Differentiate (3.3.1) with respect to t_2 and multiply by $e^{-y}t_2$;

$$\Sigma P(\underline{n})n_1n_2(\log n_1)t_1^{n_1}\ldots t_s^{n_s}\, e^{-y} = Nt_1t_2(N-1)p_1p_2\,{}_0\int^\infty \frac{e^{-x-y}}{x}[(\sum_{i=1}^{s} p_it_i)^{N-2}-$$

$$(p_1t_1e^{-x}+\sum_{i=2}^{s} p_it_i)^{N-2}]dx. \tag{3.3.8}$$

Differentiate (3.3.1) with respect to t_2, multiply by t_2 and then replace t_2 by $e^{-y}t_2$;

$$\Sigma P(\underline{n})n_1 n_2(\log n_1)t_1^{n_1}(e^{-y}t_2)^{n_2}t_3^{n_3}\cdots t_s^{n_s} = N(N-1)p_1 p_2 t_1 t_2 \int_0^{\infty}\frac{e^{-x-y}}{x}[(p_1 t_1 +$$

$$p_2 e^{-y}t_2 + \sum_{i=3}^{s} p_i t_i)^{N-2} - (p_1 t_1 e^{-x} + p_2 t_2 e^{-y} + \sum_{i=3}^{s} p_i t_i)^{N-2}]dx. \qquad (3.3.9)$$

Subtract (3.3.9) from (3.3.8), set each t_i equal to one and divide by y;

$$\Sigma P(\underline{n})n_1 n_2(\log n_1)\frac{e^{-y}-e^{-n_2 y}}{y} = N(N-1)p_1 p_2 \int_0^{\infty}\frac{e^{-x-y}}{xy}[1-(p_1 e^{-x}+q_1)^{N-2}-$$

$$(p_2 e^{-y}+q_2)^{N-2}+(p_1 e^{-x}+p_2 e^{-y}+\sum_{i=3}^{s} p_i)^{N-2}]dx. \qquad (3.3.10)$$

Integrate (3.3.10) with respect to y;

$$P(\underline{n})n_1 n_2 \log n_1 \log n_2 = N(N-1)p_1 p_2 \int_0^{\infty}\int_0^{\infty}\frac{e^{-x-y}}{xy}[1-(p_1 e^{-x}+q_1)^{N-2}-$$

$$(p_2 e^{-y}+q_2)^{N-2}+(p_1 e^{-x}+p_2 e^{-y}+\sum_{i=3}^{s} p_i)^{N-2}]dxdy. \qquad (3.3.11)$$

Generalize (3.3.11);

$$E(\sum_{i\neq j}\sum n_i \log n_i n_j \log n_j) = N(N-1)\sum_{i\neq j}\sum p_i p_j \int_0^{\infty}\int_0^{\infty}\frac{e^{-x-y}}{xy}[1-$$

$$(p_i e^{-x}+q_i)^{N-2}-(p_j e^{-y}+q_j)^{N-2}+(p_i e^{-x}+p_j e^{-y}+q_i+q_j-1)^{N-2}]dxdy. \qquad (3.3.12)$$

We will now make use of the following identity,

$$E\left(\sum_{i=1}^{s} n_i \log n_i\right)^2 = E\left[\sum_{i=1}^{s} n_i^2(\log n_i)^2\right] + E\left[\sum_{i\neq j}\sum(n_i \log n_i)(n_j \log n_j)\right]. \qquad (3.3.13)$$

Substitute (3.3.7) and (3.3.12) into (3.3.13);

$$E\left(\sum_{i=1}^{s} n_i \log n_i\right)^2 = N\sum_{i=1}^{s} p_i \int_0^\infty\int_0^\infty \frac{e^{-x-y}}{xy}\,[1-(p_ie^{-x}+q_i)^{N-1}-(p_ie^{-y}+q_i)^{N-1}+$$

$$(p_ie^{-x-y}+q_i)^{N-1}]dxdy + N(N-1)\Sigma p_i^2 \int_0^\infty\int_0^\infty \frac{e^{-x-y}}{xy}\,[1-$$

$$e^{-x}(p_ie^{-x}+q_i)^{N-2}-e^{-y}(p_ie^{-y}+q_i)^{N-2}+e^{-x-y}(p_ie^{-x-y}+q_i)^{N-2}]dxdy +$$

$$N(N-1)\sum_{i\neq j}\sum p_ip_j \int_0^\infty\int_0^\infty \frac{e^{-x-y}}{xy}\,[1-(p_ie^{-x}+q_i)^{N-2}-(p_je^{-y}+q_j)^{N-2} +$$

$$(p_ie^{-x}+p_je^{-y}+q_i+q_j-1)^{N-2}dxdy. \qquad (3.3.14)$$

By definition,

$$\mathrm{var}(\overline{H}) = \frac{1}{N^2}\,[E(\Sigma n_i \log n_i)^2-(E\Sigma n_i \log n_i)^2]. \qquad (3.3.15)$$

Substitute (3.3.14) and (3.2.8) into (3.3.15);

$$\mathrm{var}(\overline{H}) = \int_0^\infty\int_0^\infty \frac{e^{-x-y}}{xy}\sum_{i=1}^{s} p_i \sum_{j=1}^{s} p_j[(p_ie^{-x-y}+q_i)^{N-1} -$$

$$(p_ie^{-x}+q_i)^{N-1}(p_je^{-y}+q_j)^{N-1}]dxdy-\frac{(N-1)}{N}\int_0^\infty\int_0^\infty \frac{e^{-x-y}}{xy}\sum_{i\neq j}\sum p_ip_j[(p_ie^{-x-y}+q_i)^{N-2}-$$

$$(p_ie^{-x}+p_je^{-y}+q_i+q_j-1)^{N-2}]dxdy. \qquad (3.3.16)$$

For convenience we will adopt the following notation;

$$u \equiv e^{-x}\ ,\ v \equiv e^{-y},\ \text{and}\ \sum_1^s p_j(p_j+q_j)^{N-1} = 1.$$

First Term of Var($\overline{H}$)

Using the notation just mentioned the first term without the factor $\frac{e^{-x-y}}{xy}$ becomes,

$$\sum_1^s p_i(p_i uv+q_i)^{N-1} \sum_1^s p_j(p_j+q_j)^{N-1} - \sum_1^s p_i(p_i u+q_i)^{N-1} \sum_1^s p_j(p_j v+q_j)^{N-1}. \tag{3.3.17}$$

Apply the binomial expansion formula to (3.3.17);

$$\sum_1^s p_i[\sum_{b=0}^{N-1} \binom{N-1}{b}(p_i uv)^{N-1-b} q_i^b] \sum_1^s p_j[\sum_{b=0}^{N-1} \binom{N-1}{b} p_j^{N-1-b} q_j^b] -$$

$$\sum_1^s p_i[\sum_{b=0}^{N-1} \binom{N-1}{b}(p_i u)^{N-1-b} q_i^b] \sum_1^s p_j[\sum_{b=0}^{N-1} \binom{N-1}{b}(p_j v)^{N-1-b} q_j^b]. \tag{3.3.18}$$

Expression (3.3.18) can be written as;

$$\sum_{a=0}^{N-1} \binom{N-1}{a} \sum_{i=1}^s p_i^{N-a} q_i^a (uv)^{N-1-a} [\sum_{b=0}^{N-1} \binom{N-1}{b} \sum_{j=1}^s p_j^{N-b} q_j^b] -$$

$$\sum_{a=0}^{N-1} \binom{N-1}{a} \sum_{i=1}^s p_i^{N-a} q_i^a (v)^{N-1-a} [\sum_{b=0}^{N-1} \binom{N-1}{b} \sum_{j=1}^s p_j^{N-b} q_j^b (u)^{N-1-b}]. \tag{3.3.19}$$

We are going to expand each term of (3.3.19) and collect terms in such a way as to form Frullanian type integrals. We will use the additional notation:

$$C_r \equiv \binom{N-1}{r}.$$

Thus the first term of (3.3.19) is

$$\begin{array}{lll}
C_0^2 \Sigma p_i^N \Sigma p_i^N (uv)^{N-1} + & \cdots & + C_0 C_{N-1} \Sigma p_i^N \Sigma p_i q_i^{N-1} (uv)^{N-1} + \\
C_1 C_0 \Sigma p_i^{N-1} q_i \Sigma p_i^N (uv)^{N-2} + & \cdots & + C_1 C_{N-1} \Sigma p_i^{N-1} q_i \Sigma p_i q_i^{N-1} (uv)^{N-2} + \\
C_2 C_0 \Sigma p_i^{N-2} q_i^2 \Sigma p_i^N (uv)^{N-3} + & \cdots & + C_2 C_{N-1} \Sigma p_i^{N-2} q_i^2 \Sigma p_i q_i^{N-1} (uv)^{N-3} + \\
\vdots & & \vdots \\
C_{n-1} C_0 \Sigma p_i q_i^{N-1} \Sigma p_i^N + & \cdots & + C_{N-1}^2 \Sigma p_i q_i^{N-1} \Sigma p_i q_i^{N-1} .
\end{array} \qquad (3.3.20)$$

The second term of (3.3.19) is

$$\begin{array}{lll}
C_0^2 \Sigma p_i^N \Sigma p_i^N (uv)^{N-1} + & \cdots & + C_0 C_{N-1} \Sigma p_i^N \Sigma p_i q_i^{N-1} v^{N-1} + \\
C_1 C_0 \Sigma p_i^{N-1} q_i \Sigma p_i^N u^{N-1} v^{N-2} + & \cdots & + C_1 C_{N-1} \Sigma p_i^{N-1} q_i \Sigma p_i q_i^{N-1} v^{N-2} + \\
C_2 C_0 \Sigma p_i^{N-2} q_i^2 \Sigma p_i^N u^{N-1} v^{N-2} + & \cdots & + C_2 C_{N-1} \Sigma p_i^{N-2} q_i^2 \Sigma p_i q_i^{N-1} v^{N-3} + \\
\vdots & & \vdots \\
C_{N-1} C_0 \Sigma p_i q_i^{N-1} \Sigma p_i^N u^{N-1} + & \cdots & + C_{N-1}^2 \Sigma p_i q_i^{N-1} \Sigma p_i q_i^{N-1} .
\end{array} \qquad (3.3.21)$$

It is best to think of (3.3.20) and (3.3.21) as being in the form of $[a_{ij}]-[b_{ij}]$. By collecting terms in the following manner; $a_{12}+a_{21}-b_{12}-b_{21}$, $a_{13}+a_{31}-b_{13}-b_{31}$, $a_{23}+a_{32}-b_{23}-b_{32}$, and etc. and noticing that $a_{ii}-b_{ii}=0$ we can collect all the terms into a series of Frullanian integrals. Therefore, the first term of the var$(\overline{H})$ is

$$
\begin{aligned}
{}_0\int^{\infty}{}_0\int^{\infty} \frac{uv}{xy} \{ & \Sigma p_i^N [C_1 \Sigma p_i^{N-1} q_i (uv)^{N-2}(uv+1-u-v)+\ldots+ \\
& C_{N-1}\Sigma p_i q_i^{N-1}(u^{N-1}v^{N-1}+1-u^{N-1}-v^{N-1})]+ \\
& C_1 \Sigma p_i^{N-1} q_i [C_1 \Sigma p_i^{N-2} q_i^2 (uv)^{N-3}(uv+1-u-v)+\ldots+ \\
& C_{N-1}\Sigma p_i q_i^{N-1}(u^{N-2}v^{N-2}+1-u^{N-2}-v^{N-2})]+ \\
& \vdots \\
& +C_{N-1}C_{N-2}\Sigma p_i q_i^{N-1} \Sigma p_i^2 q_i^{N-2}(uv+1-u-v)\}dxdy. \qquad (3.3.22)
\end{aligned}
$$

Integrating (3.3.22);

$$
\begin{aligned}
& \Sigma p_i^N [C_1 \Sigma p_i^{N-1} q_i \log^2 \frac{N}{N-1} + C_2 \Sigma p_i^{N-2} q_i^2 \log^2 \frac{N}{N-2} +\ldots+ \\
& C_{N-1}\Sigma p_i q_i^{N-1} \log^2 N]+C_1 \Sigma p_i^{N-1} q_i [C_2 \Sigma p_i^{N-2} q_i^2 \log^2 \frac{N-1}{N-2} + \\
& C_3 \Sigma p_i^{N-3} q_i^3 \log^2 \frac{N-1}{N-3} +\ldots+C_{N-1}\Sigma p_i q_i^{N-1} \log^2 (N-1)]+\ldots+ \\
& C_{N-2}C_{N-1}\Sigma p_i^2 q_i^{N-2} \Sigma p_i q_i^{N-1} \log^2 2. \qquad (3.3.23)
\end{aligned}
$$

Simplify (3.3.23):

$$\sum_{a=0}^{N-2} \binom{N-1}{a} \sum_{i=1}^{s} p_i^{N-a} q_i^{a} \Big[\sum_{b=a+1}^{N-1} \binom{N-1}{b} \sum_{i=1}^{s} p_i^{N-b} q_i^{b} \log^2 \frac{N-a}{N-b} \Big] \qquad (3.3.24)$$

<u>Second Term of Var($\overline{H}$)</u>

The integrand of the second term of (3.3.16), without the factor

$\frac{e^{-x-y}}{xy} \sum\sum_{i \neq j} p_i p_j$, is

$$[(p_i uv+p_j)+(1-p_i-p_j)]^{N-2}-[(p_i u+p_j v)+(1-p_i-p_j)]^{N-2}. \qquad (3.3.25)$$

Expanding and collecting terms in (3.3.25) we get

$$\binom{N-2}{0}(1-p_i-p_j)^0[(p_i uv+p_j)^{N-2}-(p_i u+p_j v)^{N-2}]+$$

$$\binom{N-2}{1}(1-p_i-p_j)^1[(p_i uv+p_j)^{N-3}-(p_i u+p_j v)^{N-3}]+$$

$$\binom{N-2}{2}(1-p_i-p_j)^2[(p_i uv+p_j)^{N-4}-(p_i u+p_j v)^{N-4}]+\ldots+$$

$$\binom{N-2}{N-3}(1-p_i-p_j)^{N-3}[(p_i uv+p_j)-(p_i u+p_j v)] \;. \qquad (3.3.26)$$

Upon examining the first term of (3.3.26) we see that

$$(p_i uv+p_j)^{N-2}-(p_i u+p_j v)^{N-2} =$$

$$[p_i^{N-2}(uv)^{N-2}+\binom{N-2}{1}p_i^{N-3}p_j(uv)^{N-3}+\ldots+\binom{N-2}{N-3}p_i p_j^{N-3}uv+p_j^{N-2}]-$$

$$[p_i^{N-2}u^{N-2}+\binom{N-2}{1}p_i^{N-3}p_j u^{N-3}v+\ldots+\binom{N-2}{N-3}p_i p_j^{N-3}uv^{N-3}+p_j^{N-2}v^{N-2}] \,. \qquad (3.3.27)$$

We can form the Frullanian Integrals by collecting the first and last terms inside each of the brackets in (3.3.27) and now $C_r=\binom{N-2}{r}$;

$$[p_i^{N-2}(uv)^{N-2}+p_j^{N-2}-p_i^{N-2}u^{N-2}-p_j^{N-2}v^{N-2}], \qquad (3.3.28)$$

and moving in from each end for the next grouping,

$$[C_1 p_i^{N-3}p_j(uv)^{N-3}+C_{N-3}p_i p_j^{N-3}uv-C_1 p_i^{N-3}p_j u^{N-3}v-C_{N-3}p_i p_j^{N-3}uv^{N-3}] \,.$$

Let us now integrate (3.3.28);

$${}_0\int^\infty {}_0\int^\infty \frac{uv}{xy} \sum_{i\neq j}\sum p_i p_j[(p_i uv)^{N-2}+p_j^{N-2}-(p_i u)^{N-2}-(p_j v)^{N-2}]dxdy =$$

$${}_0\int^\infty {}_0\int^\infty \sum_{i\neq j}\sum p_i^{N-1}p_j uv(1-u^{N-2})(1-v^{N-2})dxdy =$$

$$\sum_{i\neq j}\sum p_i^{N-1}p_j \log^2 (N-1).$$

One can do likewise for each of the above binomials in (3.3.26) and the second term of the $\mathrm{Var}(\overline{H})$ is

$$\frac{N-1}{N}\{[\sum_{i\neq j}\sum p_i^{N-1}p_j\ \log^2(N-1)+C_1\sum\sum p_i^{N-2}p_j^2\ \log^2(\frac{N-2}{2})+\ldots+$$

$$C_{[\frac{N-2}{2}]}\sum\sum p_i^{N-[\frac{N-3}{2}]-1}p_j^{[\frac{N-3}{2}]+1}\log^2\frac{N-1-[\frac{N-3}{2}]}{[\frac{N-3}{2}]+1}]+$$

$$C_1[\sum_{i\neq j}\sum p_i^{N-2}p_j(1-p_i-p_j)\log^2\ (N-2)+C_1\sum\sum p_i^{N-3}p_j{}^2(1-p_i-p_j)\log^2\frac{N-3}{2}+\ldots+$$

$$C_{[\frac{N-2}{2}]}\sum\sum p_i^{N-[\frac{N-3}{2}]-2}p_j^{[\frac{N-3}{2}]+1}(1-p_i-p_j)\log^2\frac{N-2-[\frac{N-3}{2}]}{[\frac{N-3}{2}]+1}]+\ldots+$$

$$C_{N-3}[\sum_{i\neq j}\sum p_i^2p_j(1-p_i-p_j)^{N-3}\ \log^2 2]\}$$ (notice the integral part of symbol).

(3.3.29)

We can rewrite (3.3.29) as

$$\frac{N-1}{N}\sum_{b=0}^{N-3}\binom{N-2}{b}[\sum_{a=0}^{[\frac{N-b-2}{2}]}\binom{N-2-b}{a}\sum_{i\neq j}\sum p_i^{N-1-a-b}p_j^{a+1}(1-p_i-p_j)^b\ \log^2\frac{N-a-b-1}{a+1}].$$

(3.3.30)

Combining (3.3.24) and (3.3.30) we have

$Var(\overline{H})$ =

$$\sum_{a=0}^{N-2}\binom{N-1}{a}\sum_{i=1}^{s}p_i^{N-a}q_i^a\ [\sum_{b=a+1}^{N-1}\binom{N-1}{b}\sum_{i=1}^{s}\ p_i^{N-b}q_i^b\ \log^2\frac{N-a}{N-b}]\ -$$

$$\frac{N-1}{N}\sum_{b=0}^{N-3}\binom{N-2}{b}[\sum_{a=0}^{[\frac{N-b-2}{2}]}\binom{N-2-b}{a}\sum_{i\neq j}\sum p_i^{N-1-a-b}p_j^{a+1}(1-p_i-p_j)^b\ \log^2\frac{N-a-b-1}{a+1}]\ .$$

(3.3.31)

The formulas for $E(\overline{H})$ and $Var(\overline{H})$ are tabulated in Appendix A.

CHAPTER IV

Asymptotic Expressions for the Mean and Variance of $\overline{H}$

1. Introduction

Even though we have expressions for computing the exact values for the mean and variance of $\overline{H}$ we find it desirable to have some other form for finding these values. The computing time for finding the variance can be significant. For instance, it takes one hour and forty-seven minutes on the IBM 360 model 65 to find the variance of a sample with forty three individuals and thirteen species. Also, due to the factorials involved in the formula for the variance, maximum word length on most computers would be exceeded in many instances.

2. Review

Basharin [2] gives the mean and variance of $\overline{H}$ to order N^{-1}. Briefly, his work is given here. Let $H = -\sum_1^s p_i \log p_i$ and $\overline{H} = -\sum_1^s \frac{n_i}{N} \log \frac{n_i}{N}$, where $\Sigma n_i = N$. Expand $\overline{H}$ in a Taylor series about the point $(p_1, p_2, \ldots, p_s)$, confining ourselves to derivatives of fourth order. The value of the derivative of H at the point $(p_1, \ldots p_s)$ is the following:

$$\frac{\partial H}{\partial p_i} = -1 - \log p_i, \quad \frac{\partial^k H}{\partial p_i^k} = (-1)^{(k-1)}(k-2)!\,p_i^{(-k+1)}, \quad i = 1,2,\ldots s, k \geq 2.$$

Mixed derivatives of all orders of H vanish at $(p_1, \ldots, p_s)$. Basharin gives the expansion with Lagrange's remainder term in the following form:

$$\overline{H} = H - \sum_{1}^{s} (\frac{n_i}{N} - p_i)(1+\log p_i) - \frac{1}{2}\sum_{1}^{s} (\frac{n_i}{N} - p_i)^2/p_i +$$

$$\frac{1}{6}\sum_{1}^{s} (\frac{n_i}{N} - p_i)^3/p_i^2 - \frac{1}{12}\sum_{1}^{s} (\frac{n_i}{N} - p_i)^4/[p_i+\theta(\frac{n_i}{N} - p_i)]^3, \qquad (4.2.1)$$

where $0<\theta<1$.

It is necessary to compute the central moments of $\frac{n_i}{N}$ in order to determine the moments of $\overline{H}$. Basharin gives the following values for the first few moments:

$$E(\frac{n_i}{N}) = p_i, \; E(\frac{n_i}{N} - p_i)^2 = \frac{p_i q_i}{N},$$

$$E(\frac{n_i}{N} - p_i)(\frac{n_j}{N} - p_j) = -\frac{p_i p_j}{N}, \quad i \neq j,$$

$$E(\frac{n_i}{N} - p_i)^3 = \frac{2p_i^3 - 3p_i^2 + p_i}{N^2},$$

$$E(\frac{n_i}{N} - p_i)^2(\frac{n_j}{N} - p_j) = -\frac{p_i p_j(1-2p_i)}{N^2}, \quad i \neq j,$$

$$E(\frac{n_i}{N} - p_i)(\frac{n_j}{N} - p_j)(\frac{n_k}{N} - p_k) = 0(\frac{1}{N^2}), \quad i \neq j \neq k,$$

$$E(\frac{n_i}{N} - p_i)^4 = 0(\frac{1}{N^2}), \qquad (4.2.2)$$

$$E(\frac{n_i}{N} - p_i)^2(\frac{n_j}{N} - p_j)^2 = 0(\frac{1}{N^2}), \quad i \neq j,$$

$$E(\frac{n_i}{N} - p_i)^3(\frac{n_j}{N} - p_j) = 0(\frac{1}{N^2}), \quad i \neq j,$$

$$E(\frac{n_i}{N} - p_i)^2(\frac{n_j}{N} - p_j)(\frac{n_k}{N} - p_k) = 0(\frac{1}{N^2}), \quad i \neq j \neq k,$$

$$E(\frac{n_i}{N} - p_i)^6 = 0(\frac{1}{N^3}) .$$

The remaining moments have order of magnitude not greater than N^{-2}.

With the results of (4.2.2) and (4.2.1) he writes:

$$E(\overline{H}) = H - \frac{s-1}{2N} + \frac{2 - 3s + \sum_1^s \frac{1}{p_i}}{6N^2} - \frac{1}{12}\sum_1^s E[(\frac{n_i}{N} - p_i)^4/(p_i(1-\theta)+\theta p_i)^3].$$

The order of magnitude of the remainder is less than N^{-2} therefore:

$$E(\overline{H}) = H - \frac{s-1}{2N} + 0(\frac{1}{N^2}) \ . \qquad (4.2.3)$$

For the variance of $\overline{H}$, he makes use of the following relation:

$$\mathrm{var}(\overline{H}) = E(\overline{H} - H + \frac{s-1}{2N} + 0(\frac{1}{N^2}))^2.$$

Using only third order derivatives, he obtains

$$\mathrm{var}(\overline{H}) = \frac{1}{N}(\sum_1^s p_i \ln^2 p_i - H^2) + 0(\frac{1}{N^2}) \ . \qquad (4.2.4)$$

3. Expected Value of $\overline{H}$

Let $\varepsilon_i = \frac{n_i}{N} - p_i$ where $i = 1,2,\ldots,k$. Then

$$\overline{H} = - \sum_1^k((Np_i + N\varepsilon_i)/N)\log((Np_i + N\varepsilon_i)/N). \qquad (4.3.1)$$

On expanding (4.3.1) we get that

$$\overline{H} = - \sum_1^k(p_i + \varepsilon_i)(\log p_i + \varepsilon_i/p - \varepsilon_i^2/2p_i^2 + \varepsilon_i^3/3p_i^3 - \ldots) \ . \qquad (4.3.2)$$

Simplifying (4.3.2) and adopting the following notation:

$$H = -\Sigma p_i \log p_i = \phi_0, \phi_1 = -\Sigma \varepsilon_i \log p_i, \text{ and}$$

$$\phi_a = \frac{(-1)^{a-1}}{(a-1)a} \sum_{i=1}^{k} \frac{\varepsilon_i^a}{p_i^{a-1}}, \quad a = 2,3,\ldots \text{ thus :}$$

$$\overline{H} = H - \sum_1^k \varepsilon_i \log p_i + \sum_{a=2}^{\infty} \phi_a . \tag{4.3.3}$$

In order to determine the expectation of (4.3.3) it is necessary to compute the central moments of $\frac{n_i}{N}$. In order to compute these moments we employ the Q-product method described earlier:

$$E\phi_o = H,\ E\phi_1 = 0,\ E\phi_2 = -\frac{k-1}{2N},$$

$$E\phi_3 = \frac{2 - 3k + \Sigma p_i^{-1}}{6N^2},$$

$$E\phi_4 = -\frac{1}{4N^2}(1-2k + \Sigma p_i^{-1}) - \frac{1}{12N^3}(-6 + 12k - 7\Sigma p_i^{-1} + \Sigma p_i^{-2}),$$

$$E\phi_5 = \frac{1}{2N^3}(\Sigma p_i^{-2} - 4\Sigma p_i^{-1} + 5k - 2) +$$

$$\frac{1}{20N^4}(24 - 60k + 50\Sigma p_i^{-1} - 15\Sigma p_i^{-2} + \Sigma p_i^{-3}),$$

$$E\phi_6 = -\frac{1}{2N^3}(\Sigma p_i^{-2} - 3\Sigma p_i^{-1} + 3k - 1) - \frac{1}{6N^4}\Sigma \frac{q_i}{p_i^3}(5p_i^2 - 16p_i q_i + 5q_i^2) -$$

$$\frac{1}{30N^5}\Sigma \frac{q_i}{p_i^4}(p_i^4 - 26p_i^3 q_i + 66p_i^2 q_i^2 - 26p_i q_i^3 + q_i^4) .$$

Upon taking the expectation of (4.3.3) and using the above moments we have:

$$E(\overline{H}) = H - \frac{k-1}{2N} - \frac{\Sigma p_i^{-1}-1}{12N^2} - \frac{\Sigma p_i^{-2}-\Sigma p_i^{-1}}{12N^3} + O(\frac{1}{N^4}) \quad . \tag{4.3.4}$$

By computing $E\phi_7$ and $E\phi_8$ we could extend (4.3.4) to the N^{-4} term.

4. Variance of $\overline{H}$

Let $r_i = \log p_i$ in addition to the notation above and

$$\begin{aligned} \mathrm{var}(\overline{H}) &= E(\overline{H}-\phi_0-E(\overline{H}-\phi_0))^2 \\ &= E(\overline{H}-\phi_0)^2-(E(\overline{H}-\phi_0))^2 \\ &= E(\phi_1+\phi_2+\ldots)^2-(E(\overline{H}-\phi_0))^2. \end{aligned} \tag{4.4.1}$$

In order to evaluate (4.4.1) to order N^{-2} we need;

$$\begin{aligned} E(\phi_1^2) &= (\Sigma p_i r_i^2-H^2)/N \quad , \\ E(\phi_2^2) &= (k^2-1)/4N^2+(\Sigma p_i^{-1}-2k-k^2+2)/4N^3, \\ E(2\phi_1\phi_2) &= (kH+\Sigma r_i)/N^2, \quad \text{and} \\ E(2\phi_1\phi_3) &= (kH+\Sigma r_i)/N^2+2(kH+\Sigma(r_i-\frac{Hp_i^{-1}}{6} - \frac{r_ip_i^{-1}}{6}))/N^3 \quad . \end{aligned} \tag{4.4.2}$$

Substituting (4.4.2) into (4.4.1) yields,

$$\mathrm{var}(\overline{H}) = \frac{\Sigma p_i \log^2 p_i-H^2}{N} + \frac{(k-1)}{2N^2} + O(\frac{1}{N^3}) \quad . \tag{4.4.3}$$

Figures 4.5 and 4.6 give errors for a special case of (4.4.3).

5. Expectation of $\overline{H}$: Equiprobable case

From (4.3.3) we have, where $p_1 = p_2 = \ldots = p_k = \frac{1}{k}$ that

$$\overline{H} = H - \Sigma\varepsilon_i \log\frac{1}{k} + \sum_2^{\infty} \phi_a .$$

Here $\phi_a = \frac{(-1)^{a-1}}{a(a-1)} k^{a-1} \sum_1^k \varepsilon_i^a$, $a = 2,3,\ldots$ and $\varepsilon_i = (\frac{n_i}{N} - \frac{1}{k})$ therefore,

$$E(\overline{H}) = \log k - \frac{1}{2} k \sum_1^k E(\varepsilon_i^2) + \frac{1}{6}k^2 E(\Sigma\varepsilon_i^3) - \ldots \quad . \tag{4.5.1}$$

In evaluating these expectations we use the Small Sample Method described earlier. The expectations in (4.5.1) give the binomial central moments. The following work was performed on the IBM 7094 using a program developed by Hearn [14]. The moments are;

$$-\frac{2}{k} E\phi_2 = [1][\frac{(k-1)^2}{k^2} + \frac{k-1}{k^2}] \quad \text{and}$$

$$E\phi_2 = \frac{k-1}{2N} \quad ,$$

$$\frac{6}{k^2} E\phi_3 = [1][\frac{(k-1)^3}{k^3} - \frac{k-1}{k^3}] \quad \text{and}$$

$$E\phi_3 = \frac{(k-1)(k-2)}{6N^2} \quad ,$$

$$-\frac{12}{k^3}E\phi_4 = \begin{bmatrix} -1 & 8 \\ \\ 2 & -8 \end{bmatrix}\begin{bmatrix} (k^3-4k+6k-3)/k^3 \\ \\ (k-1)/8k \end{bmatrix}$$ and

$$E\phi_4 = -\frac{(k-1)^2}{4N^2} - \frac{(k-1)(k^2-6k+6)}{12N^3},$$

$$\frac{42}{k^6}E\phi_7 = \begin{bmatrix} \frac{1}{2} & -64 & \frac{729}{2} \\ -\frac{5}{2} & 256 & -\frac{2187}{2} \\ 3 & -192 & 729 \end{bmatrix}\begin{bmatrix} (k-1)(k^5-6k^4+15k^3-20k^2+15k-6)/k^6 \\ (k-1)(k^5+50k^4-300k^3+680k^3-720k+288)/64k^6 \\ (k-1)(k^5+106k^4-405k^3+540k^2-405k+162)/729k^6 \end{bmatrix}$$

$$E\phi_7 = \frac{5(k-1)^3(k-2)}{2N^4} + \frac{(k-1)^2(k-2)(4k^2-33k+33)}{3N^5} +$$

$$\frac{(k-1)(k-2)(k^4-60k^3+420k^2-720k+360)}{42N^6}$$, and so on.

It should be mentioned that the above expectations can be gotten from a recursive scheme for Q products [46]. The formula is

$$A_{s+1} = \sum_{r=0}^{s}\binom{s}{r}a_{r+1}A_{s-r}\{N^{-r} - \frac{s-r}{(r+1)N^{r+1}}\}.$$

Letting $K = k-1$, $e_i = x_i - \frac{1}{k}$, $E(\varepsilon_i^s) = A_s$, and

$$a_r = E(e_i^r) = E(x_i - \frac{1}{k})^r = \sum_{x=0}^{1}(\frac{1}{k})^x(\frac{k-1}{k})^{1-x}(x-\frac{1}{k})^r, \quad r = 1,2,\ldots .$$

also,

$$a_1 = 0,\ A_0 = 1 \text{ and } A_1 = 0 \ .$$

We will compute a few by this method;

$$A_2 = \sum_{r=0}^{1} \binom{1}{r} a_r{+1}\ A_{1-r}\{N^{-r}-(1-r)/[(r+1)N^{r+1}]\}$$

$$= \binom{1}{0} a_1 A_1 \{1-\frac{1}{N}\} + \binom{1}{1} a_2\ A_0 \{\frac{1}{N} - 0\}$$

$$= \frac{K}{Nk^2} \ ,$$

$$A_3 = \sum_{r=0}^{2} \binom{2}{r} a_{r+1}\ A_{2-r} \{N^{-r} - \frac{(2-r)}{(r+1)N^{r+1}}\}$$

$$= \frac{K(K-1)}{N^2k^3} \ ,$$

$$A_7 = \frac{105K^3(K-1)}{N^4k^7} + \frac{14K^2(K-1)(4K^2-25K+4)}{N^5k^7} +$$

$$\frac{K(K-1)(K^4-56K^3+246K^2-56K+1)}{N^6k^7} \quad \text{and so on.}$$

The table of binomial moments given in Appendix B were computed with this formula.

Hence,

$$
\begin{aligned}
E(\overline{H}) = {} & \log k - K/(2N) - K(K+2)/(12N^2) - K(K+1)^2/(12N^3) - K(19K^3 + \\
& 46K^2 + 34K + 6)/(120N^4) - K(K + 1)^2(27K^2 + 24K + 2)/(60N^5) - \\
& K(863K^5 + 2910K^4 + 3600K^3 + 1930K^2 + 387K + 12)/(504N^6) - \\
& K(1375K^6 + 5310K^5 + 7899K^4 + 5576K^3 + 1823K^2 + 214K + \\
& 3)/(168N^7) - K(33{,}953K^7 + 147{,}874K^6 + 257{,}034K^5 + \\
& 224{,}818K^4 + 101{,}774K^3 + 21{,}594K^2 + 1{,}538K + 10)/(720N^8) - \\
& K(57{,}218K^8 + 277{,}858K^7 + 552{,}623K^6 + 576{,}316K^5 + 333{,}236K^4 + \\
& 103{,}060K^3 + 14{,}744K^2 + 652K + 2)/(180N^9) - \ldots \quad .
\end{aligned}
\tag{4.5.2}
$$

Thus (4.5.2) represents the $E(\overline{H})$ in the equiprobable case. Each term in the series is positive and from Table 4.1 the series appears to be divergent. The divergent property of the series is further emphasized in figure 4.1. Figures 4.2, 4.3, and 4.4 demonstrate the accuracy of the series.

Table 4.1

Values of Individual Terms in $E(\overline{H})$

N	1	4	25	50
k	2	2	10	25
N^{-1}	.500	.125	.180	.240
N^{-2}	.250	.016	.013	.021
N^{-3}	.333	.005	.005	.010
N^{-4}	.875	.003	.003	.009
N^{-5}	3.533	.003	.004	.013
N^{-6}	19.250	.004	.005	.024
N^{-7}	132.143	.008	.009	.056
N^{-8}	1095.271	.016	.021	.159
N^{-9}	10643.177	.041	.054	.525

Table 4.2

Percentage Error in $E(\overline{H})$ Using Partial Sums

k \ N	5	10	20	30
	2.79703	0.47284	0.10094	0.04309
2	1.06395	0.08229	0.00730	0.00201
	0.60181	0.03022	0.00106	0.00019
	7.05498	4.18221	1.19097	0.45403
8	9.24259	1.01959	0.49358	0.15548
	32.43236	1.22938	0.24562	0.08471
	3.98978	4.70058	1.70514	0.71138
10	20.47691	0.03802	0.69547	0.28251
	64.96182	4.20067	0.23654	0.15255
	25.71505	1.53199	3.21632	2.05725
20	115.88210	14.97036	0.18739	0.64988
	459.37580	46.40340	3.42900	0.24369
	67.00285	7.33459	2.70077	2.70645
30	264.29330	42.91369	4.48666	0.22355
	1409.85080	146.20783	14.92004	3.05905

Remarks on this table: The entries in the columns are percentage errors in using terms up to and including (1) n^{-1} term (2) n^{-2} term (3) n^{-3} term.

$E(\bar{H})$ Using Partial Sums

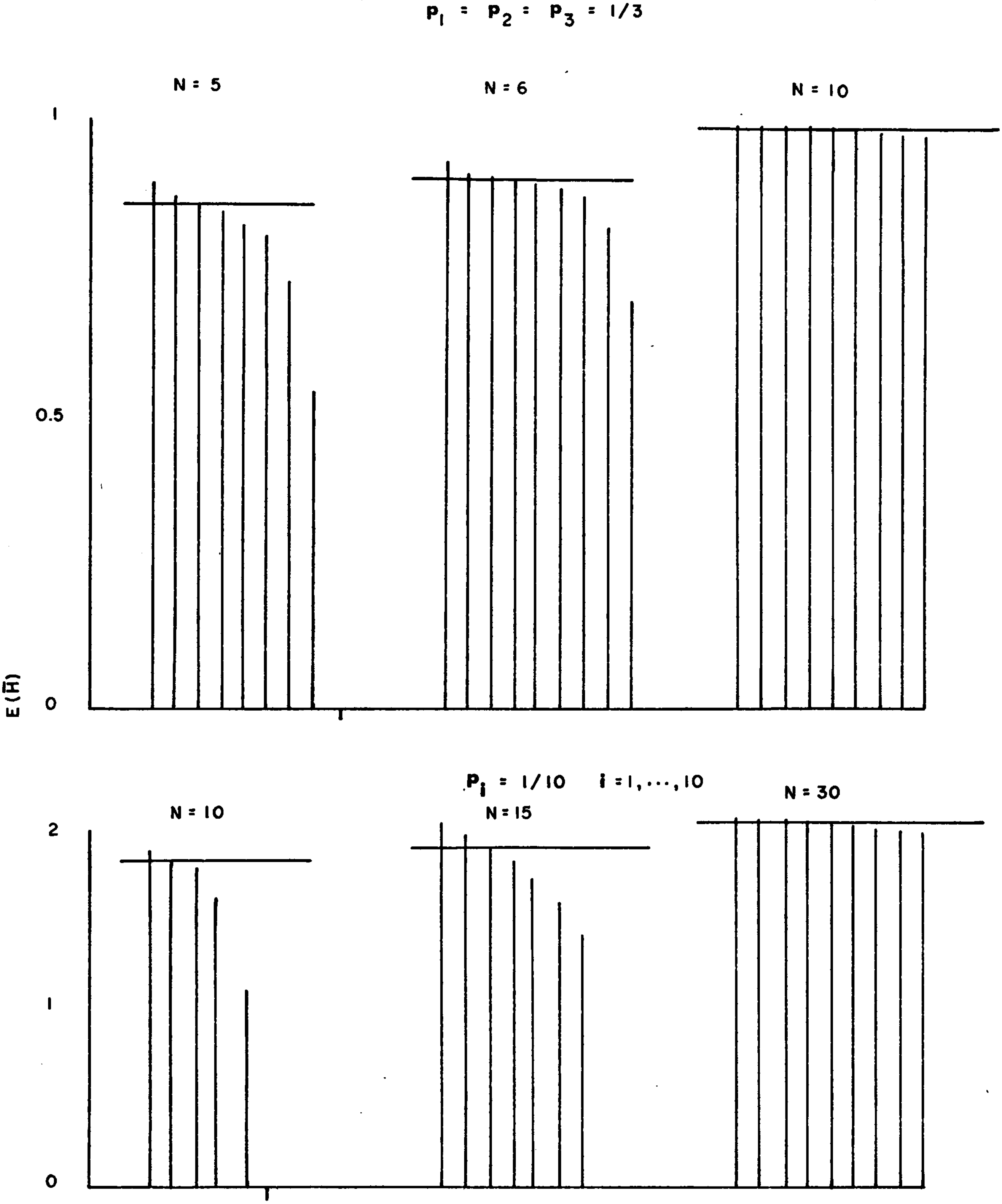

Figure 4.1 THE FIRST ABSCISSA IS LOG $k - K/(2n)$, THE SECOND ABSCISSA IS LOG $k - K/(2n) - K(K+2)/(12n^2)$, AND ETC. THE HORIZONTAL LINE IS THE TRUE VALUE OF $E(\bar{H})$.

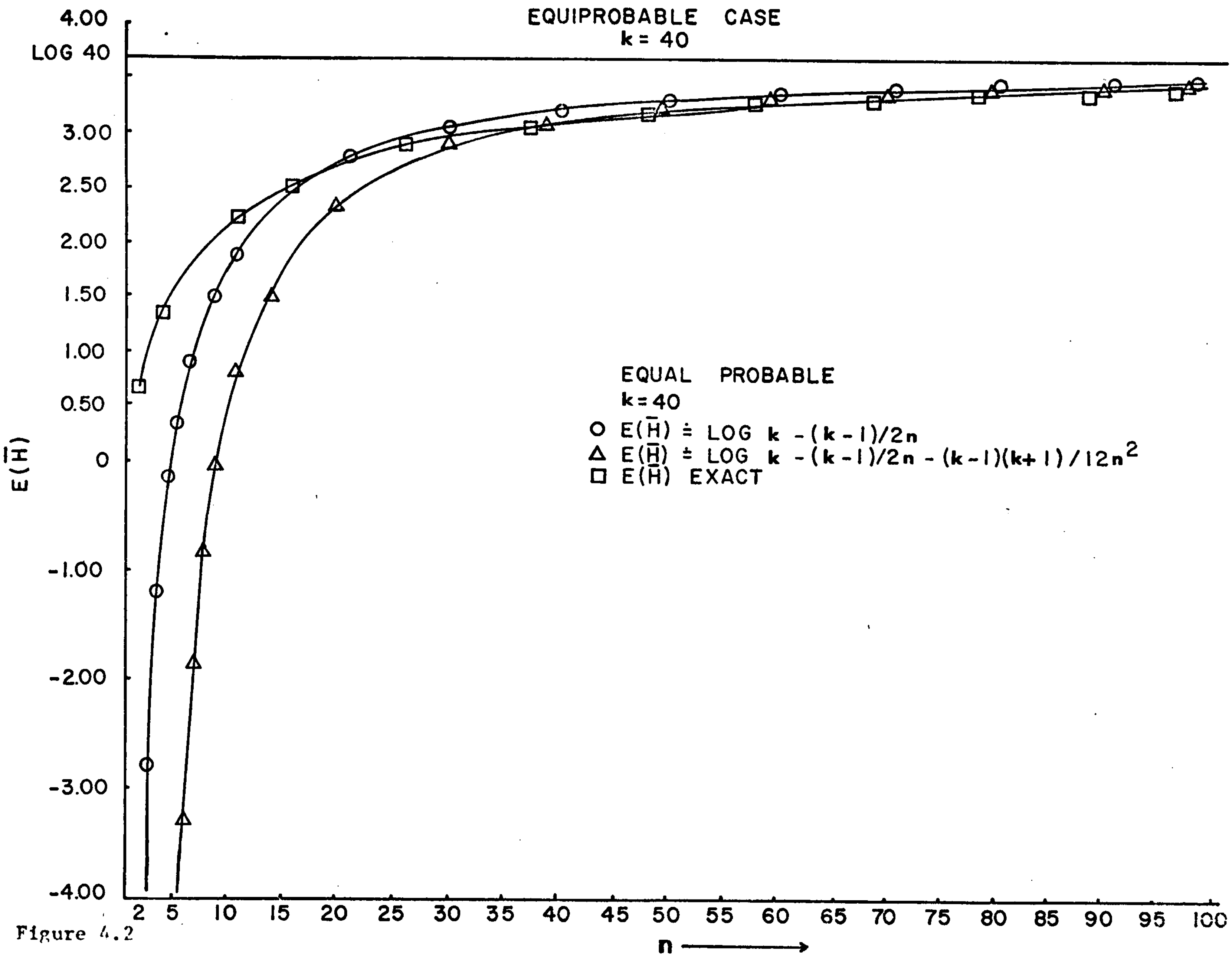
EQUIPROBABLE CASE
k = 40
4.00
LOG 40
3.00
2.50
2.00
1.50
1.00
0.50
0
-1.00
-2.00
-3.00
-4.00
E(H̄)
EQUAL PROBABLE
k = 40
○ E(H̄) ≐ LOG k - (k - 1)/2n
△ E(H̄) ≐ LOG k - (k - 1)/2n - (k - 1)(k + 1)/12n²
□ E(H̄) EXACT
2 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100
n →

Figure 4.2

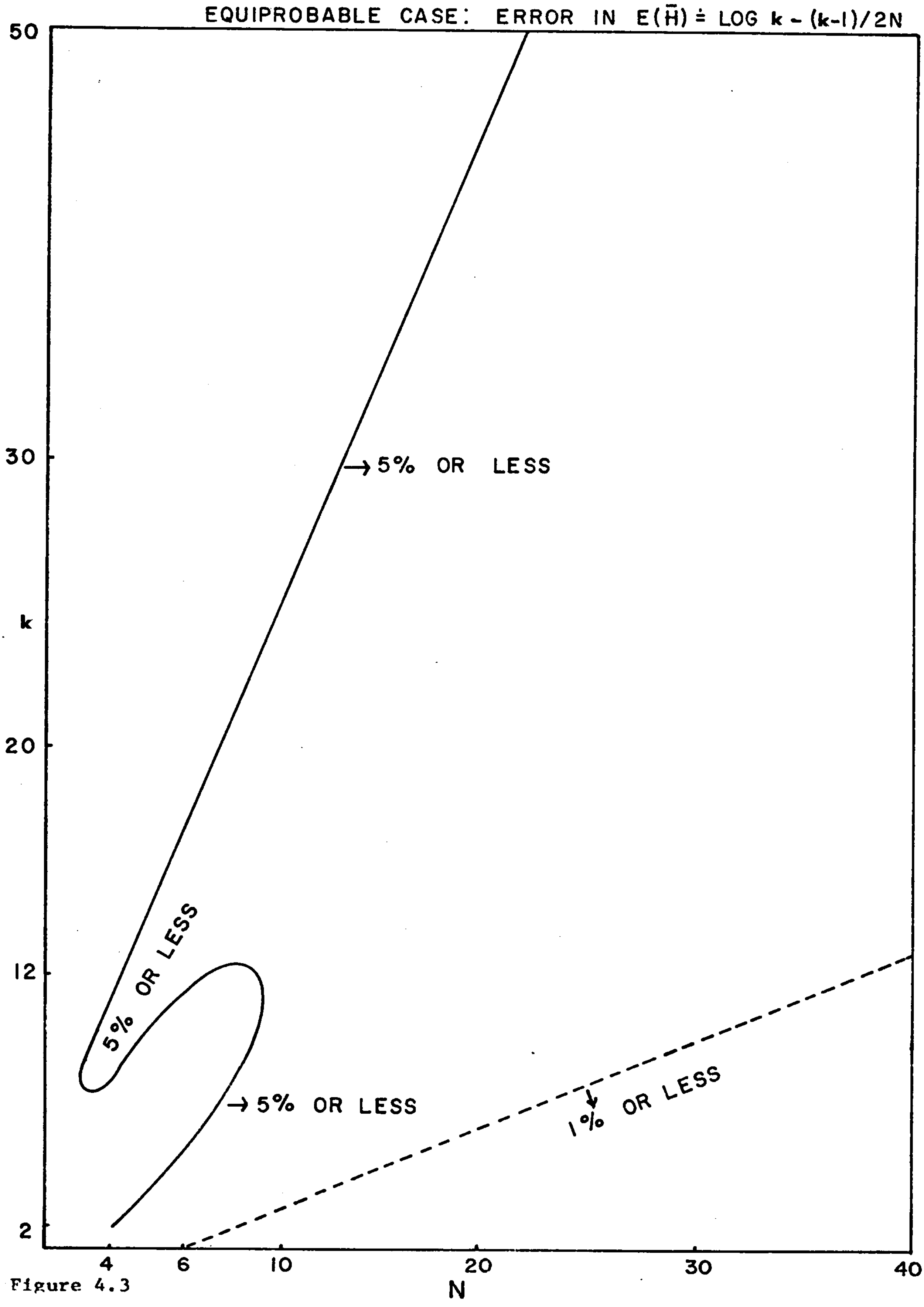
EQUIPROBABLE CASE: ERROR IN E($\bar{H}$) ≐ LOG k - (k-1)/2N
50
30
k
20
12
2
4
6
10
20
30
40
N
→ 5% OR LESS
5% OR LESS
→ 5% OR LESS
1% OR LESS

Figure 4.3

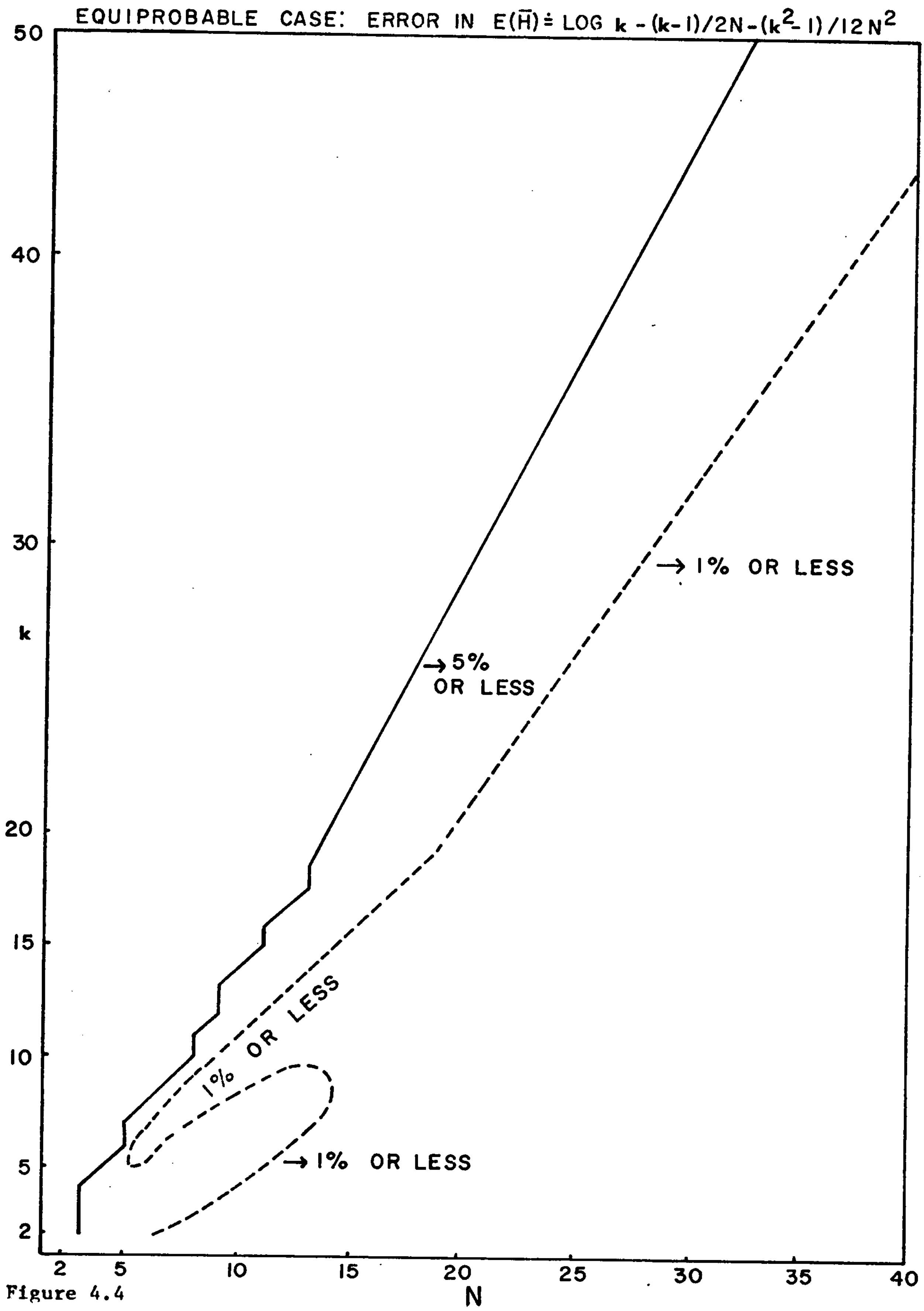
EQUIPROBABLE CASE: ERROR IN $E(\bar{H}) \doteq \text{LOG } k - (k-1)/2N - (k^2-1)/12N^2$
50
40
30
k
20
15
10
5
2
→ 1% OR LESS
→ 5% OR LESS
1% OR LESS
→ 1% OR LESS
2
5
10
15
20
25
30
35
40
N

Figure 4.4

6. The Variance of $\overline{H}$: Equiprobable Case

Returning to (4.4.1) and setting the p_i's $= \frac{1}{k}$, we can use the Small Sample Method to find the expectations of the cross products. Thus we have

$$-\frac{6}{k^3} E2\phi_2\phi_3 = \begin{bmatrix} -1 & 16 \\ 2 & -16 \end{bmatrix} \begin{bmatrix} (k-1)^2(k-2)/k^3 \\ (k-1)(k^2-4)/(8k^3) \end{bmatrix}$$

$$E2\phi_2\phi_3 = -\frac{(k-1)(k-2)(k+5)}{6N^3} + \frac{(k-1)(k-2)}{N^4} ,$$

$$\frac{4}{k^2} E\phi_2^2 = \begin{bmatrix} -1 & 8 \\ 2 & -8 \end{bmatrix} \begin{bmatrix} (k^2-2k+1)/k^2 \\ (k-1)/(4k) \end{bmatrix}$$

$$E\phi_2^2 = \frac{k^2-1}{4N^2} - \frac{k-1}{2N^3} ,$$

$$\frac{36}{k^4} E\phi_3^2 = \begin{bmatrix} \frac{1}{2} & -32 & \frac{243}{2} \\ -\frac{5}{2} & 128 & -\frac{729}{2} \\ 3 & -96 & 243 \end{bmatrix} \begin{bmatrix} (k^4-6k^3+13k^2-12k+4)/k^4 \\ (k^4+3k^3-32k^2+60k-32)/(16k^4) \\ (k^4+6k^3-43k^2+72k-36)/81k^4) \end{bmatrix}$$

$$E\phi_3^2 = \frac{(k-1)(k-2)}{6N^3} + (k-1)(k-2)(k^2+15k-52)/36N^4) -$$

$$(k-1)(k-2)(3k-8)/6N^5) ,$$

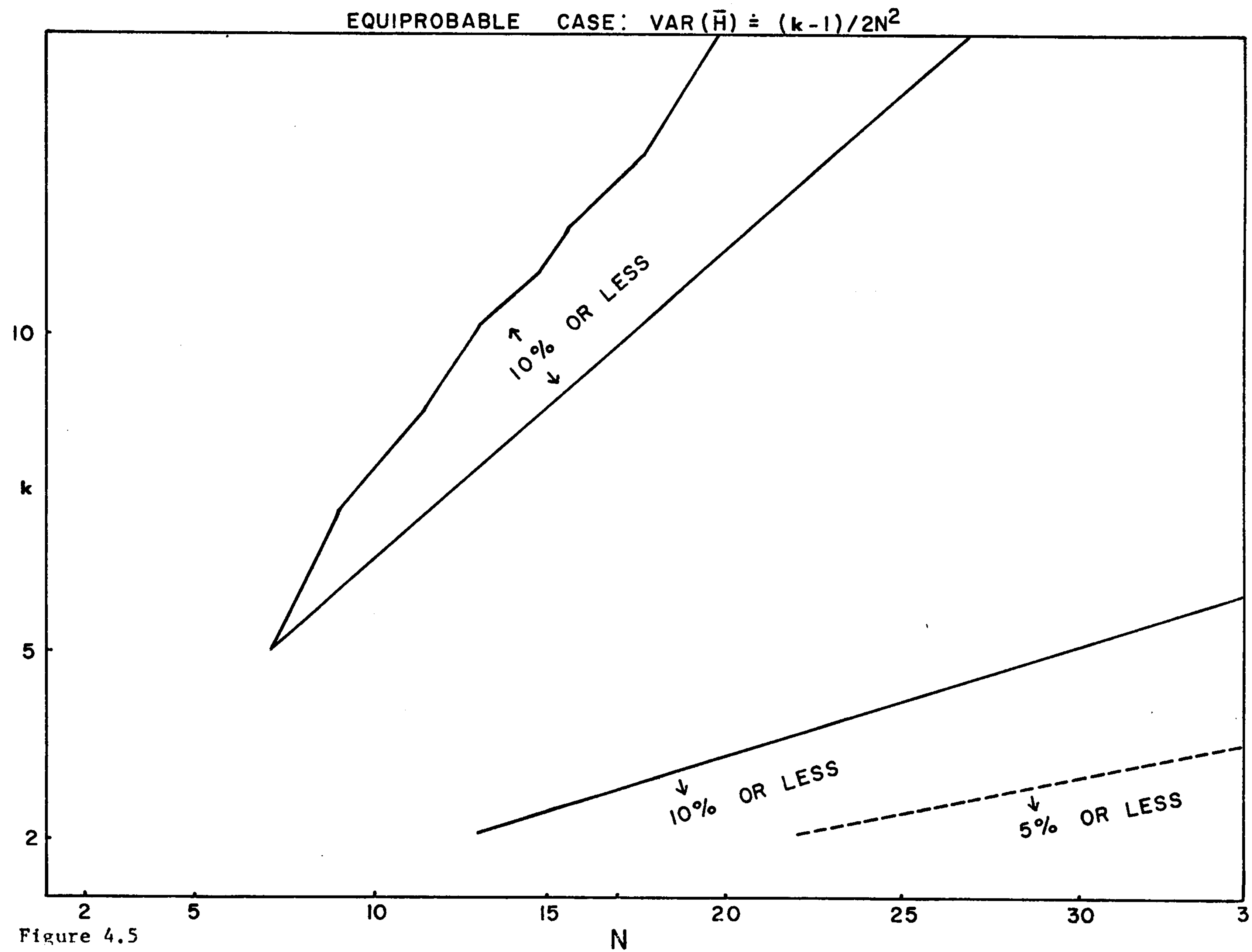
EQUIPROBABLE CASE: VAR($\bar{H}$) ≐ (k-1)/2N^2
10% OR LESS
10% OR LESS
5% OR LESS
k
10
5
2
N
2
5
10
15
20
25
30
35

Figure 4.5

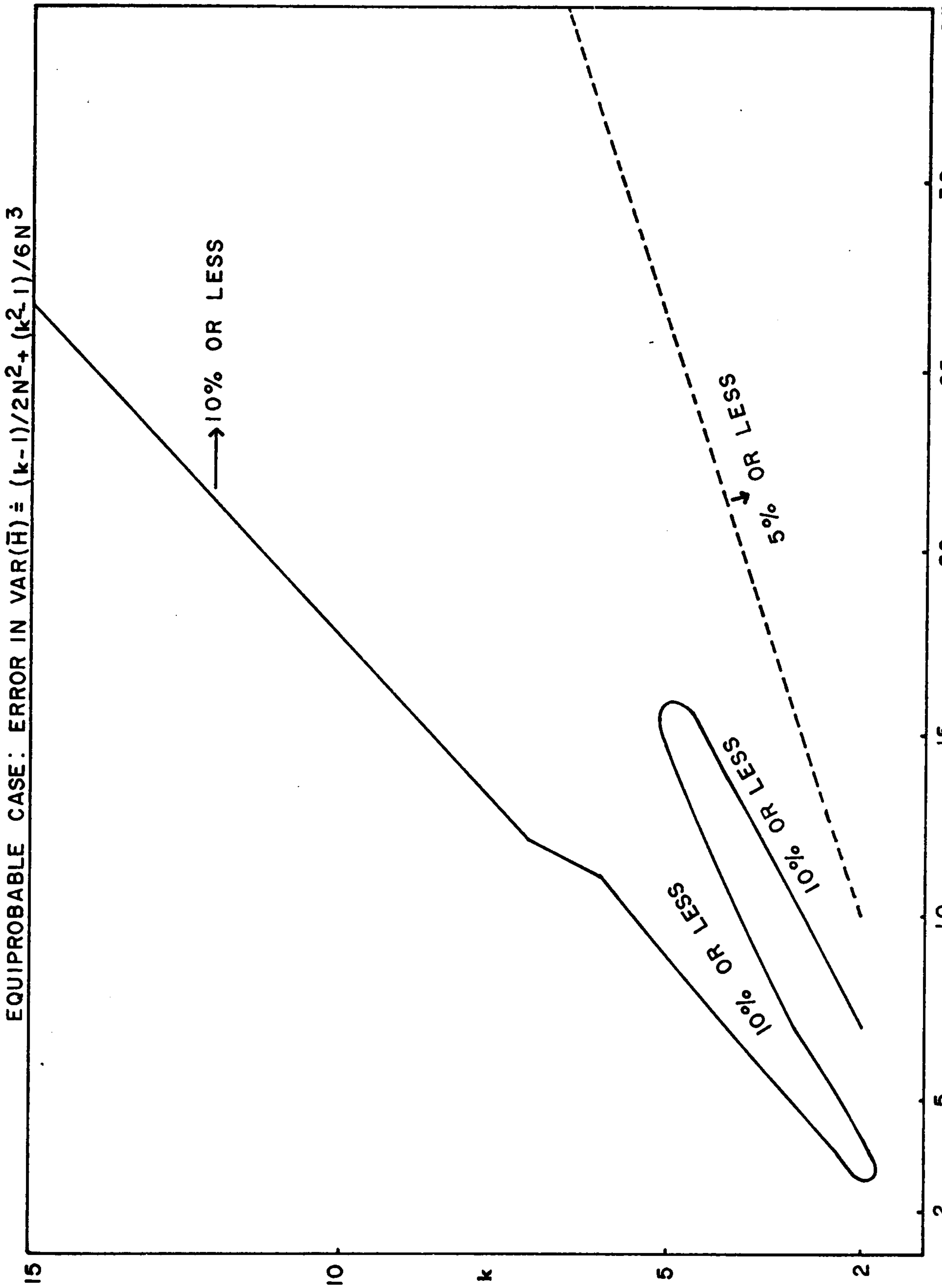
EQUIPROBABLE CASE: ERROR IN VAR($\bar{H}$) ≐ $(k-1)/2N^2 + (k^2-1)/6N^3$
→10% OR LESS
10% OR LESS
10% OR LESS
5% OR LESS
k
15
10
5
2
N
2
5
10
15
20
25
30
35

Figure 4.6

$$\frac{24}{k^4}E\phi_2\phi_4 = \begin{bmatrix} \frac{1}{2} & -4 & \frac{243}{2} \\ -\frac{5}{2} & 128 & -\frac{729}{2} \\ 3 & -96 & 243 \end{bmatrix} \begin{bmatrix} (k^4-5k^3+10k^2-9k+3)/k^4 \\ (k^4+5k^3-30k^2+48k-24)/2k^4 \\ (3k^4+31k^3-124k^2+171k-81)/9k^4 \end{bmatrix}$$

$$E\phi_2\phi_4 = (k-1)^2(k+3)/(8N^3)+(k-1)(k^3+7k^2-72k+78)/(24N^4) - (k-1)(7k^2-36k+36)/(12N^5) \ .$$

Therefore,

$$\mathrm{var}(\overline{H}) = \frac{k-1}{2N^2} + \frac{k^2-1}{6N^3} + 0\left(\frac{1}{N^4}\right) .$$

Some significant work on the moments of $\overline{H}$ was done by Rogers, M.S. and Green B. F., "The Moments of Sample Information when the Alternatives are Equally Likely" "Symposium on Information Theory in Psychology" 1955, University of Illinois Press (edited by H. Quastler). It should be pointed out that the second term in the series for the var($\overline{H}$) is not correct in the paper by Rogers and Green. Also in the 1955 symposium mentioned above is a paper by Miller, G.A., "Note on the Bias of Information Estimates." A paper by Carlton, A. G. was written some time after 1955 and sent to me by the author, "On the Bias of Information Estimates," I do not know where this paper was published.

7. Classification of Moments

$\overline{H} = \phi_0 + \phi_1 + \phi_2 + \ldots$ and let $\nu_s = E(\overline{H}-\phi_0)^s$,

then the following table gives the terms needed in order to evaluate the moments of $\overline{H}$ to the desired order.

Table 4.3 Power of n^{-1}: General Case

	N^{-1}	N^{-2}	N^{-3}	N^{-4}
ν_1	ϕ_2	ϕ_3,ϕ_4	ϕ_4,ϕ_5,ϕ_6	$\phi_5,\phi_6,\phi_7,\phi_8$
ν_2	ϕ_1^2	$\phi_1\phi_2$	$\phi_1\phi_3,\phi_1\phi_4$	$\phi_1\phi_4,\phi_1\phi_5,\phi_1\phi_6$
		$\phi_1\phi_3$	$\phi_1\phi_5,\phi_2\phi_3$	$\phi_1\phi_7,\phi_2\phi_3,\phi_2\phi_4$
		ϕ_2^2	$\phi_2\phi_4,\phi_2^2,\phi_3^2$	$\phi_2\phi_6,\phi_3\phi_4,\phi_3\phi_5$
				$\phi_3^2,\phi_4^2,\phi_2\phi_5$
ν_3		$\phi_1^2\phi_2$	$\phi_1^2\phi_2,\phi_1^2\phi_3$	$\phi_1^2\phi_3,\phi_1^2\phi_4,\phi_1^2\phi_5$
		ϕ_1^3	$\phi_1^2\phi_4,\phi_1\phi_2^2$	$\phi_1^2\phi_6,\phi_1\phi_2^2,\phi_1\phi_3^2$
			$\phi_1\phi_2\phi_3,\phi_2^3$	$\phi_2^2\phi_3,\phi_2^2\phi_4,\phi_2\phi_3^2$
				$\phi_1\phi_2\phi_3,\phi_1\phi_2\phi_4$
				$\phi_1\phi_2\phi_5,\phi_1\phi_3\phi_4,\phi_2^3$
ν_4		ϕ_1^4	$\phi_1^4,\phi_1^2\phi_2^2$	$\phi_1^2\phi_2^2,\phi_1^2\phi_3^2,\phi_1^2\phi_2\phi_3$
			$\phi_1^3\phi_2,\phi_1^3\phi_3$	$\phi_1^2\phi_2\phi_4,\phi_1^3\phi_2,\phi_1^3\phi_3$
				$\phi_1^3\phi_4,\phi_1^3\phi_5,\phi_1\phi_2^2\phi_3$
				$\phi_1\phi_2^3,\phi_2^4$

Table 4.4 Power of n^{-1}: Equiprobable Case

	N^{-1}	N^{-2}	N^{-3}	N^{-4}	N^{-5}
ν_1	ϕ_2	ϕ_3,ϕ_4	ϕ_4,ϕ_5,ϕ_6	$\phi_5,\phi_6,\phi_7,\phi_8$	$\phi_6,\phi_7,\phi_8,\phi_9$
					ϕ_{10}
ν_2		ϕ_2^2	$\phi_2^2,\phi_2\phi_3$	$\phi_2\phi_3,\phi_2\phi_4,\phi_3^2$	$\phi_2\phi_4,\phi_2\phi_5$
			$\phi_2\phi_4,\phi_3^2$	$\phi_3\phi_4,\phi_3\phi_5,\phi_4^2$	$\phi_2\phi_6,\phi_2\phi_4$
				$\phi_2\phi_5,\phi_2\phi_6$	$\phi_2\phi_8,\phi_3\phi_4$
					$\phi_3\phi_5,\phi_3\phi_5$
					$\phi_3\phi_7,\phi_4\phi_5$
					$\phi_4\phi_5,\phi_3^2,\phi_4^2$
					$\phi_5^2,\phi_2\phi_7$
					$\phi_3\phi_6,\phi_4\phi_6$
ν_3			ϕ_2^3	$\phi_2^2\phi_3,\phi_2^2\phi_4$	$\phi_2^2\phi_3,\phi_2^2\phi_4$
				$\phi_2\phi_3^2,\phi_2^3$	$\phi_2^2\phi_5,\phi_2^2\phi_6$
					$\phi_3^2\phi_4,\phi_4\phi_2\phi_3$
					$\phi_2\phi_4^2,\phi_2^3$
					$\phi_3^3,\phi_2\phi_3^2$
					$\phi_2\phi_3\phi_5$
ν_4				ϕ_2^4	$\phi_2^2\phi_3^2,\phi_2^3\phi_3$
					$\phi_2^3\phi_4,\phi_2^4$

CHAPTER V

SOME REMARKS ON THE DISTRIBUTION OF $\overline{H}$

1. Introduction

In this chapter we will look at the exact higher moments of $\overline{H}$ of several multinomials in order to assess the asymptotic distribution of $\overline{H}$. The same technique is used that Uppuluri and Bowman [49] used in their study of the Likelihood Ratio Test Criterion. In fact some of the same multinomial distributions will be used in order to show the relationship between $\overline{H}$ and $-2 \log \Lambda$.

Basharin [2] states that $\overline{H}$ is asymptotically normal. We will verify this in the general case and show that $\overline{H}$ is asymptotically chi-square in the equiprobable case.

2. Higher Moments of $\overline{H}$

To explain what was done in obtaining μ_3 and μ_4 the following example is used.

Example:

Given the multinomial distribution

$$\frac{4!}{n_1!n_2!n_3!}\left(\frac{1}{3}\right)^{n_1}\left(\frac{1}{3}\right)^{n_2}\left(\frac{1}{3}\right)^{n_3},\quad n_1+n_2+n_3 = 4,$$

we take all possible three part partitions of four and compute the $\overline{H}$ for each partition and find its associated probability. There will be $\binom{n+k-1}{k-1}$ partitions in general and for the example there is $\binom{6}{2}$. These are given in the following table.

Table 5.1

Sample Configuration

partitions			$\overline{H}$	probability
4	0	0	0.0	1/81
0	4	0	0.0	1/81
0	0	4	0.0	1/81
3	1	0	0.5623351446	4/81
3	0	1	0.5623351446	4/81
1	3	0	0.5623351446	4/81
0	3	1	0.5623351446	4/81
1	0	3	0.5623351446	4/81
0	1	3	0.5623351446	4/81
2	2	0	0.6931471805	6/81
2	0	2	0.6931471805	6/81
0	2	2	0.6931471805	6/81
2	1	1	1.0397207708	12/81
1	2	1	1.0397207708	12/81
1	1	2	1.0397207708	12/81

There are only four distinct $\overline{H}$s, thus;

$\overline{H}$	probability
0.0	1/27
0.5623351446	8/27
0.6931471805	2/9
1.0397207708	4/9

We now compute the crude moments of $\overline{H}$;

$$\mu_1' = \sum_{i=1}^{4} \overline{H}_i \, p_i, \quad \mu_2' = \Sigma \, \overline{H}_i^2 \, p_i, \qquad (5.2.1)$$

$$\mu_3' = \Sigma \, \overline{H}_i^3 \, p_i \text{ and } \mu_4' = \Sigma \, \overline{H}_i^4 \, p_i$$

From (5.2.1) we have for the central moments;

$$\mu_1 = \mu_1' = \qquad 0.7827486478$$

$$\mu_2 = \mu_2' - \mu_1'^2 = \qquad 0.0682199610 \qquad (5.2.2)$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = \qquad -0.0135532990$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = \qquad 0.0165552278.$$

From (5.2.2) we get,

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = -0.7606376512,$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3.5572323043.$$

We have computed the moments of a sample of size four from a trinomial. We need to compute the moments when the sample size becomes large. At this point we made use of a program written by Dr. Niels Eskelson for the IBM 1130 which will perform every operation set forth in the example above. The following tables were computed using this program and figures 5.1 and 5.2 give some indication of the distribution of $\overline{H}$ for fixed sample sizes.

Table 5.1 Moments of $\overline{H}$ Computed From the Exact Distribution of the Binomial; $p_1 = .2$ and $p_2 = .8$ with Varying Sample Sizes

N	μ_1	μ_2	μ_3	μ_4	$\mu_3/\mu_2^{3/2}$	μ_4/μ_2^2
2	0.22180709	0.10454657	0.02608781	0.01743975	0.77174363	1.59558823
8	0.42718724	0.04866675	-0.00952345	0.00638796	-0.88704586	2.69710328
10	0.44318199	0.03809972	-0.00723027	0.00472170	-0.97223676	3.25278090
16	0.46652745	0.02207970	-0.00311675	0.00190517	-0.94997618	3.90794406
18	0.47064046	0.01926595	-0.00243640	0.00145606	-0.91109397	3.92284082
20	0.47386308	0.01707952	-0.00194330	0.00113617	-0.87061780	3.89487302
30	0.48314799	0.01091092	-0.00080622	0.00043139	-0.70740191	3.62367710
40	0.48759314	0.00803700	-0.00043923	0.00022271	-0.60961179	3.44797209
60	0.49193829	0.00527331	-0.00019085	0.00009152	-0.49841030	3.29117886
70	0.49316461	0.00450061	-0.00013950	0.00006581	-0.46204122	3.24920113
100	0.49535680	0.00312682	-0.00006779	0.00003103	-0.38771521	3.17472853
150	0.49704909	0.00207271	-0.00002995	0.00001339	-0.31743026	3.11718328
200	0.49789123	0.00155019	-0.00001680	0.00000742	-0.27538765	3.08969923
500	0.49940058	0.00061703	-0.0000269	0.00000116	-0.17574166	3.06106564

Table 5.2 Moments of $\overline{H}$ Computed From the Exact Distribution of the Trinomial; $p_1 = .33333333$, $p_2 = .33333333$, $p_3 = .33333334$ with Varying Sample Sizes

N	μ_1	μ_2	μ_3	μ_4	$\mu_3/\mu_2^{3/2}$	μ_4/μ_2^2
3	0.66847884	0.09144710	-0.01552810	0.02979495	-0.56151790	3.56289302
5	0.85329967	0.04976730	-0.01128238	0.00935463	-1.01621287	3.77692767
7	0.93278037	0.02724501	-0.00668232	0.00372403	-1.48592575	5.01694275
9	0.97463284	0.01608789	-0.00370630	0.00172569	-1.81631941	6.66753306
11	0.99979412	0.01028894	-0.00208996	0.00085725	-2.00254876	8.09781273
13	1.01642468	0.00705653	-0.00123437	0.00044973	-2.08237568	9.03172273
15	1.02820716	0.00512239	-0.00077018	0.00024907	-2.10080725	9.49248350
17	1.03699606	0.00388795	-0.00050707	0.00014559	-2.09164167	9.63175273
19	1.04381138	0.00305518	-0.00035026	0.00008965	-2.07416620	9.60484384
21	1.04925643	0.00246691	-0.00025204	0.00005789	-2.05706820	9.51269749
23	1.05371015	0.00203543	-0.00018761	0.00003898	-2.04310215	9.40989451
25	1.05742262	0.00170911	-0.00014362	0.00002722	-2.03274180	9.31995562
27	1.06056575	0.00145605	-0.00011251	0.00001960	-2.02512787	9.24783851
29	1.06326179	0.00125566	-0.00008987	0.00001449	-2.01992957	9.19575359
31	1.06560013	0.00109417	-0.00007296	0.00001096	-2.01608929	9.16063188
33	1.06764774	0.00096206	-0.00006807	0.00000844	-2.01309578	9.12430529

Table 5.3 Moments of $\overline{H}$ Computed From the Exact Distribution of the Multinomial; $p_1 = p_2 = p_3 = p_4 = 1/4$, with Varying Sample Sizes

N	μ_1	μ_2	μ_3	μ_4	$\mu_3/\mu_2^{3/2}$	μ_4/μ_2^2
4	0.91771969	0.07288890	-0.01141471	0.01907567	-0.58026315	3.59051840
6	1.08077338	0.04630908	-0.00760445	0.00776007	-0.76267587	3.61854191
8	1.16537916	0.02944575	-0.00516263	0.00343148	-1.02232697	3.95764734
10	1.21528242	0.01938279	-0.00339344	0.00171825	-1.25863614	4.57355703
12	1.24742825	0.01330519	-0.00221362	0.00093869	-1.44300625	5.30249143
14	1.26956093	0.00952126	-0.00145431	0.00054360	-1.57074941	5.99649057
16	1.28561730	0.00707890	-0.00098210	0.00032880	-1.64895729	6.56148442
18	1.29776147	0.00544298	-0.00067258	0.00020629	-1.68984174	6.96318796
21	1.30725965	0.00430732	-0.00048421	0.00013379	-1.70580862	7.21121164
22	1.31489247	0.00349235	-0.00035333	0.00008954	-1.70717305	7.34212986
24	1.32116338	0.00288968	-0.00026423	0.00006164	-1.70101084	7.38249441
26	1.32640996	0.00243204	-0.00020292	0.00004365	-1.69193867	7.38025236
28	1.33086654	0.00207636	-0.00015220	0.00003168	-1.68262747	7.34983667
30	1.33470067	0.00179430	-0.00012425	0.00002354	-1.67424607	7.31278082
32	1.33803533	0.00156671	-0.00010335	0.00001785	-1.66670988	7.27576002

Table 5.4 Moments of $\overline{H}$ Computed From the Exact Distribution of the Trinomial; $p_1 = .11111111$, $p_2 = .22222222$, $p_3 = .66666667$ with Varying Sample Sizes

N	μ_1	μ_2	μ_3	μ_4	$\mu_3/\mu_2^{3/2}$	μ_4/μ_2^2
3	0.48569849	0.12339086	-0.01059003	0.03142064	-0.24432776	2.06371205
5	0.61819839	0.09760162	-0.01550083	0.02584429	-0.50835796	2.71300541
7	0.68211688	0.07519923	-0.01305933	0.01699191	-0.63328745	3.00479927
11	0.74431135	0.04836873	-0.00779718	0.00771198	-0.73297742	3.29637656
21	0.79693715	0.02352700	-0.00257762	0.00193178	-0.71428365	3.48999962
23	0.80181063	0.02121937	-0.00215616	0.00157270	-0.69756168	3.49146450
25	0.80585963	0.01931003	-0.00182461	0.00130023	-0.67997939	3.48703453
31	0.81469900	0.01517655	-0.00117281	0.00079516	-0.62729314	3.45233069
35	0.81881315	0.01327366	-0.00090968	0.00060271	-0.59484966	3.42083134
40	0.82273516	0.01147479	-0.00068670	0.00044509	-0.55866852	3.38032449
50	0.82811962	0.00903220	-0.00042949	0.00026995	-0.50034431	3.30901184
75	0.83512159	0.00590567	-0.00018555	0.00011172	-0.40885015	3.20327776*
150	0.84196433	0.00290272	-0.00004563	0.00002626	-0.29182567	3.11745326**

* Approximately one hour on IBM 1130; 2926 different partitions and 507 different $\overline{H}$'s; sum of the probabilities was .99999989

** Approximately eight and one half hours on IBM 1130; 11476 different partitions and 1951 different $\overline{H}$'s; sum of the probabilities was 0.99999972

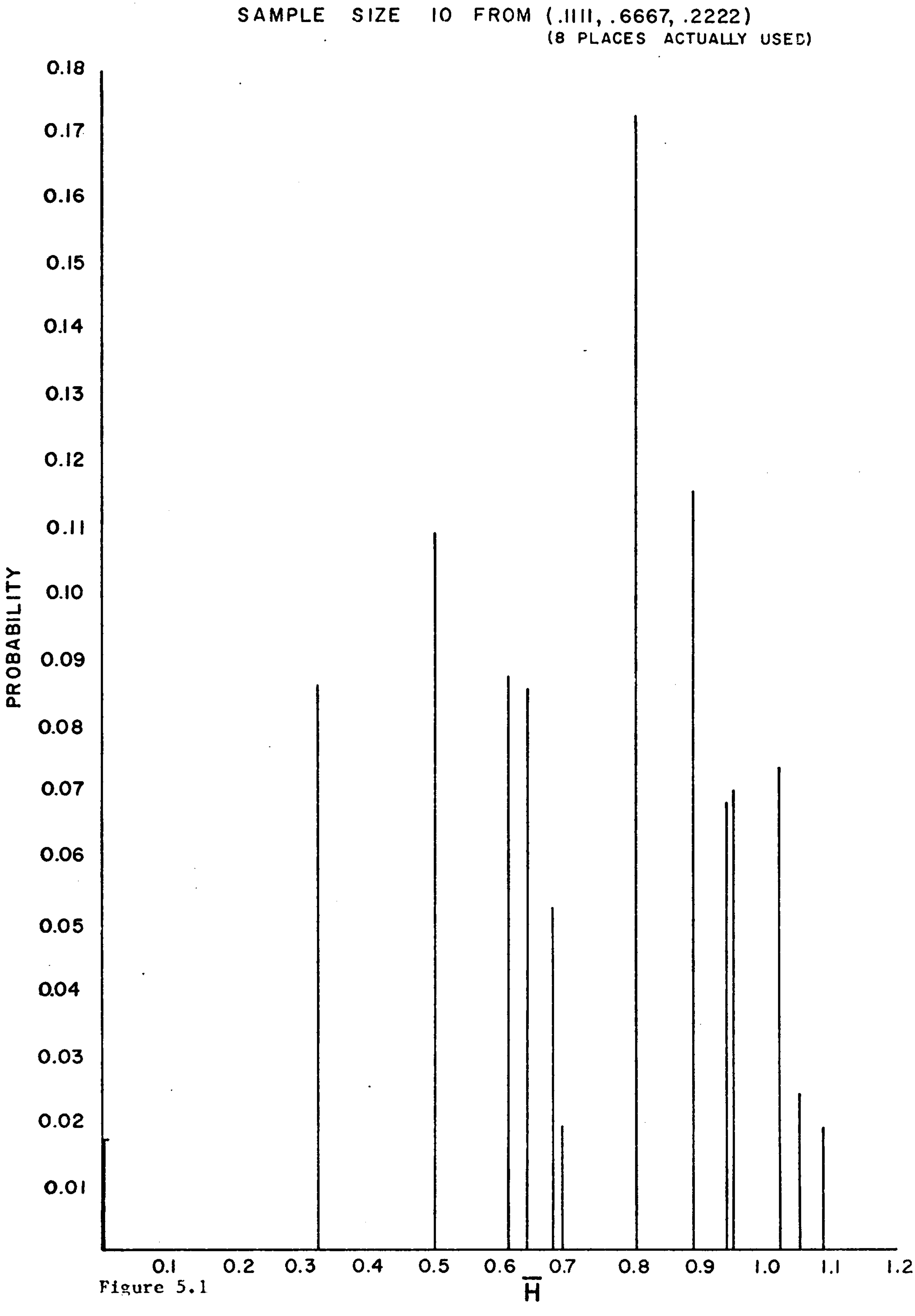
SAMPLE SIZE 10 FROM (.1111, .6667, .2222)
(8 PLACES ACTUALLY USED)
PROBABILITY
0.18
0.17
0.16
0.15
0.14
0.13
0.12
0.11
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
$\overline{H}$

Figure 5.1

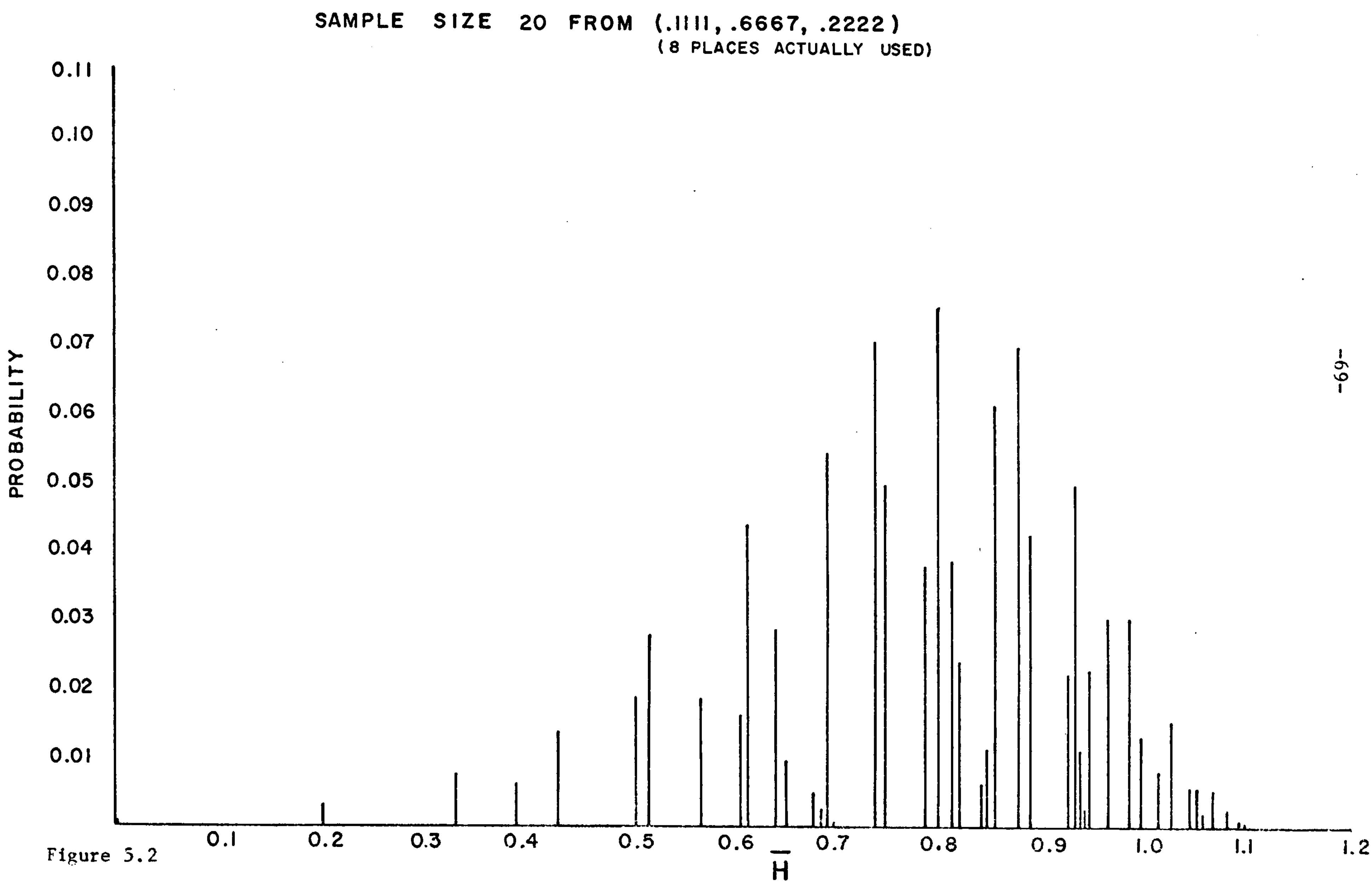
SAMPLE SIZE 20 FROM (.1111, .6667, .2222)
(8 PLACES ACTUALLY USED)
PROBABILITY
0.11
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
$\overline{H}$

Figure 5.2

Table 5.5 Moments of the Statistic $-\Sigma \frac{n_i}{N} \log \frac{n_i}{N}$

for $p_1 = 1/9$, $p_2 = 2/9$ and $p_3 = 6/9$ (to sixteen decimal digits)*

N	μ_1'	μ_2	μ_3	μ_4	β_1	β_2
10	0.73321406	0.05343612	-0.00888467	0.00926332	0.51734202	3.24411983
11	0.74431137	0.04836873	-0.00779718	0.00771199	0.53725604	3.29637751
12	0.75359150	0.04406321	-0.00686127	0.00648366	0.55027217	3.33938251
13	0.76145049	0.04037799	-0.00605882	0.00550211	0.55762475	3.37473575
14	0.76817921	0.03720036	-0.00537105	0.00471022	0.56037194	3.40366989
15	0.77399589	0.03444119	-0.00478060	0.00406528	0.55941102	3.42716222
16	0.77906722	0.03202940	-0.00427232	0.00353520	0.55549600	3.44600448
17	0.78352256	0.02990803	-0.00383327	0.00309570	0.54925595	3.46084901
18	0.78746368	0.02803113	-0.00345259	0.00272829	0.54121267	3.47224080
19	0.79097166	0.02636142	-0.00312123	0.00241878	0.53179684	3.48064073
20	0.79411178	0.02486840	-0.00283166	0.00215615	0.52136253	3.48644285
25	0.80585965	0.01931002	-0.00182460	0.00130023	0.46236937	3.48701170
30	0.81348698	0.01573994	-0.00125557	0.00085707	0.40426820	3.45948916

(continued)

* From unpublished work by Dr. K. O. Bowman, Oak Ridge National Laboratory; This work was done on the IBM 360 - model 65

continued Table 5.5

35	0.81881315	0.01327364	-0.00090968	0.00060270	0.35383851	3.42075142
40	0.82273521	0.01147477	-0.00068668	0.00044506	0.31209044	3.38009979
45	0.82574169	0.01010688	-0.00053579	0.00034139	0.27806169	3.34212292
50	0.82811967	0.00903216	-0.00042947	0.00026992	0.25031383	3.30860547
55	0.83004806	0.00816544	-0.00035193	0.00021868	0.22749925	3.27983733
60	0.83164382	0.00745149	-0.00029372	0.00018076	0.20851657	3.25542530
65	0.83298658	0.00685301	-0.00024892	0.00015192	0.19251571	3.23474096
70	0.83413235	0.00634396	-0.00021369	0.00012948	0.17885571	3.21713907
75	0.83512170	0.00590559	-0.00018549	0.00011167	0.16705620	3.20204733
80	0.83598474	0.00552408	-0.00016256	0.00009731	0.15675575	3.18899154
85	0.83674428	0.00518899	-0.00014364	0.00008556	0.14767934	3.17759194
90	0.83741794	0.00489231	-0.00012786	0.00007581	0.13961736	3.16754920
95	0.83801953	0.00462779	-0.00011455	0.00006765	0.13240369	3.15862848
100	0.83856006	0.00439044	-0.00010323	0.00006073	0.12590868	3.15064516
110	0.83949174	0.00398206	-0.00008509	0.00004974	0.11467713	3.13693570
120	0.84026629	0.00364325	-0.00007136	0.00004194	0.10529853	3.12556536
130	0.84092040	0.00335760	-0.00006070	0.00003513	0.09734514	3.11596660

(continued)

continued Table 5.5

140	0.84148015	0.00311352	-0.00005227	0.00003013	0.09051298	3.10774685
150	0.84196460	0.00290253	-0.00004548	0.00002612	0.08457948	3.10062438
160	0.84238799	0.00271833	-0.00003993	0.00002287	0.07937767	3.09439083
170	0.84276119	0.00255612	-0.00003534	0.00002018	0.07477972	3.08888821
180	0.84309261	0.00241218	-0.00003150	0.00001794	0.07068604	3.08399429
190	0.84338891	0.00228360	-0.00002825	0.00001606	0.06701783	3.07961289

Table 5.6 **Moments of the Normal and Chi - Square Distributions**

	μ_1	μ_2	μ_3	μ_4	$\mu_3/\mu_2^{3/2}$	μ_4/μ_2^2
Normal	m	σ^2	0	$3\sigma^4$	0	3
Chi - Square ν degrees of freedom	ν	2ν	8ν	$48\nu + 12\nu^2$	$\frac{2\sqrt{2\nu}}{\nu}$	$\frac{12 + 3\nu}{\nu}$

Table 5.7

Moments of $\overline{H}$ Computed by the Formulas (3.2.11) and (3.3.31) with $p_1 = 1/9$, $p_2 = 2/9$, and $p_3 = 6/9$

N	μ_1	μ_2
2	.34229490	.1200949463
3	.48569894	.1233908630
4	.56610929	.1111645253
5	.61819839	.0976016252
6	.65487638	.0854770560
7	.68211688	.0751992379
8	.70311212	.0666113817
9	.71974654	.0594436457
10	.73321405	.0534361244
11	.74431136	.0483687333
12	.75359149	.0440633064
13	.76145048	.0403779902
14	.76817920	.0372003564
15	.77399588	.0344411900
16	.77906721	.0320294930
17	.78352255	.0299080309
18	.78746367	.0280311351
19	.79097165	.0263614194
20	.79411177	.0248683990
21	.79693716	.0235269898
22	.79949142	.0223164164
23	.80181065	.0212193640
24	.80392496	.0202213138
25	.80585965	.0193100218
26	.80763611	.0184751045
27	.80927255	.0177077106
28	.81078455	.0170002567
29	.81218549	.0163462151
30	.81348697	.0157399414
31	.81469903	.0151765343
32	.81583045	.0146517203
33	.81688891	.0141617591
34	.81788117	.0137033650
35	.81881318	.0132736414
36	.81969022	.0128700269
37	.82051698	.0124902487
38	.82129764	.0121322841
39	.82203593	.0117943284
40	.82273520	.0114747661
45	.82574168	.0101068798
50	.82811967	.0090321637

From table (5.1) observe that μ_4/μ_2^2 is approaching three as N is increasing. Comparing table (5.1) with the normal distribution in table (5.6) we reach the conclusion that $\overline{H}$, with $p_1 = .2$ and $p_2 = .8$ is asymptotically normal. Comparing table (5.2) with the chi-square distribution in table (5.6) we can state that $\overline{H}$, with $p_1 = .33333333$, $p_2 = .33333333$, and p_3 .33333334 is asymptotically chi-square with two degrees of freedom. From table (5.3) we get that $\overline{H}$, with $p_1 = p_2 = p_3 = p_4 = 1/4$ is aymptotically chi-square with three degrees of freedom. Table (5.4) and table (5.5) tell us that $\overline{H}$, with $p_1 = 1/9$, $p_2 = 2/9$, abd $p_3 = 6/9$ is aymptotically normal.

For further work into the distribution of $\overline{H}$, one is referred to "Comments on the Distribution of Indices of Diversity" by Shenton, L. R.; Bowman, K. O., Hutcheson, K., and Odum, E. P. International Symposium on Ecology, August 1969, New Haven, Connecticutt.

3. $\overline{H}$ and $-2 \log \Lambda$

$-2 \log \Lambda$ is referred to as the likelihood ratio test criterion, where

$$\Lambda = (Np_1/n_1)^{n_1} (Np_2/n_2)^{n_2} \ldots (Np_s/n_s)^{n_s} .$$

Schaffer 43 gives approximate expressions for the first two moments of

$$L = -2 \log \Lambda = 2\sum_{1}^{s} \log (n_i/Np_i)^{n_i}, \qquad (5.3.1)$$

as

$$\begin{aligned} \mu_1 (L) &\doteq (s-1)+(R_1-1)/6N+(R_2-R_1)/6N^2 \text{ and} \\ \mu_2 (L) &\doteq 2(s-1)+2(R_1-1)/3N+4(R_2-R_1)/3N^2, \end{aligned} \qquad (5.3.2)$$

where

$$R_m = \sum_{1}^{s} (1/p_i)^m.$$

By multiplying and dividing (5.3.1) by N we get

$$L = -2N(\overline{H} + \Sigma(n_i/N) \log p_i), \tag{5.3.3}$$

and in the notation of Chapter IV this becomes

$$L = -2N(\phi_2+\phi_3+\phi_4+ \cdot \cdot \cdot). \tag{5.3.4}$$

If we assume equiprobability in (5.3.3) we have

$$L = 2N(\log s-\overline{H}). \tag{5.3.5}$$

Thus,

$$\begin{aligned} \mu_1(L) &= 2N(\log s-\mu_1(\overline{H})) \text{ and} \\ \mu_2(L) &= 4N^2\mu_2(\overline{H}). \end{aligned} \tag{5.3.6}$$

For a comparison we give the following table, part of which came from Uppuluri and Bowman [49].

Table 5.8

Moments of $-2 \log \Lambda$

with $p_1 = p_2 = p_3 = 1/3$

	Uppuluri and Bowman exact moments		Schaffer's Approximation		Moments from (5.3.6)	
N	μ_1	μ_2	μ_1	μ_2	μ_1	μ_2
10	2.1996	5.0976	2.1633	4.7733	2.1996	5.0976
11	2.1740	4.9799	2.1460	4.6832	2.1740	4.9799
12	2.1535	4.8692	2.1319	4.6111	2.1535	4.8692
13	2.1369	4.7702	2.1203	4.5523	2.1369	4.7702
14	2.1233	4.6840	2.1105	4.5034	2.1233	4.6840
15	2.1122	4.6102	2.1022	4.4622	2.1122	4.6102
20	2.0774	4.3793	2.0742	4.3267	2.0774	4.3793
40	2.0355	4.1519	2.0352	4.1483	2.0355	4.1519
80	2.0172	4.0708	2.0171	4.0704	2.0172	4.0708

This chapter can be summarized in the following chart.

Table 5.9 Asymptotic Moments of $\overline{H}$ and $-2 \log \Lambda$

Statistic	Mean	Variance	Asymptotic Distribution
$\overline{H}=-\sum_{1}^{s} \frac{n_i}{N} \log \frac{n_i}{N}$	$-\Sigma p_i \log p_i - \frac{s-1}{2N} - \frac{\Sigma p_i^{-1}-1}{12N^2} - \frac{\Sigma p_i^{-2}\ p_i^{-1}}{12N^3} + 0(\frac{1}{N^4})$	$\frac{1}{N}(\Sigma p_i \log^2 p_i-(-\Sigma p_i \log p_i)^2) + \frac{s-1}{2N^2} + 0\ (\frac{1}{N^3})$	For fixed s, $\overline{H}$ converges to the normal as $N\to\infty$ (Basharin, 1959)
$\overline{H} = \log s$ if $p_1=p_2= \cdots =p_s$	$\log s - \frac{s-1}{2N} - \frac{s^2-1}{12N^2} - \frac{s^2(s-1)}{12N^3} + 0(\frac{1}{N^4})$	$\frac{s-1}{2N^2} + \frac{s^2-1}{6N^3} + 0(\frac{1}{N^4})$	For fixed s, $\overline{H}$ converges to the chi-square as $N\to\infty$
$L = 2\sum_{1}^{s} \log (\frac{n_i}{Np_i})^{n_i}$ or $L =-2N(\phi_2+\phi_3+\phi_4+\ldots)$	$(s-1) + \frac{1}{6N} (\Sigma p_i^{-1}-1) + \frac{1}{6N^2}(\Sigma p_i^{-2}-\Sigma p_i^{-1}) + 0(\frac{1}{N^3})$	$2(s-1) + \frac{2}{3N}(\Sigma p_i^{-1}-1) + \frac{4}{3N^2}(\Sigma p_i^{-2} - \Sigma p_i^{-1}) + 0(\frac{1}{N^3})$	For fixed s, L converges to the chi-square as $N\to\infty$ (Schaffer, 1957)

CHAPTER VI

SUMMARY AND DISCUSSION

1. Introduction

This work is primarily directed at the problem of obtaining the moments and distribution of the statistic

$$\bar{H} = -\sum_{i=1}^{s} \frac{n_i}{N} \log \frac{n_i}{N}$$

which is used as measure of statistical diversity in an ecological system. N represents the number of individuals with n_1 belonging to one species, n_2 belonging to a second species, and so on for s species and $\sum_{i=1}^{s} n_i = N$.

$\bar{H}$ measures departure from a uniform state of nature which would be found in a situation in which all species were represented equally. It achieves its maximum value when $n_1 = n_2 = \ldots = n_s$ and its minimum value when there is only one species (category) present.

It can be interpreted as an index of diversity achieved by some aspect of natural phenomena at a given place and time, in a non-repeatable situation. This is sometimes referred to as the <u>local</u> interpretation of diversity.

Another interpretation concerns its <u>global evaluation</u>, and here the interest is in its sampling fluctuations, assuming something about randomness in species distribution.

Exact values for the first four moments are given by sample configuration methods. Monte-Carlo simulation is used to give approximate moments, and asymptotic moments are developed. Expressions are given for the exact mean and variance of $\overline{H}$. The variance of $\overline{H}$ is a new result and the form for the mean is probably new.

Previous work is this area has been primarily the asymptotic development with no more than two terms in the series approximations.

2. Expected Value of $\overline{H}$ and Variance of $\overline{H}$

Beginning with the frequency generating function for the multinomial distribution,

$$(p_1t_1+p_2t_2+\ldots+p_st_s)^N = \Sigma \frac{N!}{n_1!n_2!\ldots n_s!} (p_1t_1)^{n_1}(p_2t_2)^{n_2}\ldots(p_st_s)^{n_s}$$

where $\Sigma p_i=1$ and $\Sigma n_i=N$, we perform some lengthy algebra in order to arrive at the $E(\overline{H})$. The final result is

$$E(\overline{H}) = \log N - \Sigma p_i \int_0^\infty \frac{e^{-x}}{x} [1 - (p_ie^{-x}+q_i)^{N-1}]\, dx\,.$$

This can be evaluated by replacing one by $(p_i+q_i)^{N-1}$, then use the binomial expansion and collect terms to form Frullanian type integrals.

$$E(\overline{H}) = \log N - \int_0^\infty \frac{e^{-x}}{x} \Sigma p_i[(p_i+q_i)^{N-1} - (p_ie^{-x}+q_i)^{N-1}]\, dx$$

$$= \log N - \int_0^\infty [\binom{N-1}{0} \Sigma p_i^N \frac{e^{-x}-e^{-N}}{x} + \binom{N-1}{1}\Sigma p_i^{N-1}q_i\frac{e^{-x}-e^{-(N-1)x}}{x} + \ldots$$

$$+ \binom{N-1}{N-1} \Sigma p_iq_i^{N-1} \frac{e^{-x}-e^{-x}}{x}]\, dx.$$

Integrating,

$$E(\overline{H}) = \log N - [\binom{N-1}{0}\Sigma p_i^N \log N + \binom{N-1}{1}\Sigma p_i^{N-1} q_i \log (N-1) + \ldots +$$

$$\binom{N-1}{N-2} \Sigma p_i^2 q_i^{N-2} \log 2].$$

With still more of the same type algebra we get

$\mathrm{var}(\overline{H}) =$

$$\sum_{a=0}^{N-2} \binom{N-1}{a} \sum_{i=1}^{s} p_i^{N-a} q_i^a [\sum_{b=a}^{N-1} \binom{N-1}{b} \sum_{i=1}^{s} p_i^{N-b} q_i^b \log^2 \frac{N-b}{N-a}] -$$

$$\frac{N-1}{N} \sum_{b=0}^{N-3} \binom{N-2}{b} [\sum_{a=0}^{B} \binom{N-b-2}{a} \Sigma \sum_{i \neq j} p_i^{N-a-b-1} p_j^{a+1} (1-p_i-p_j)^b \log^2 \frac{N-a-b-1}{a+1}]$$

where B = integer part of (N-b-2)/2.

These expressions are readily computerized, and can be used as long as the parameters do not exceed the capacity of the machine. On the IBM 360/65 one can set N and s up to about 200.

3. Asymptotic Expressions for the Mean and Variance of $\overline{H}$

Let $\varepsilon_i = n_i/N - p_i$ where $i = 1,2,\ldots s$. Then,

$$\overline{H} = - \sum_{i=1}^{s} ((Np_i + N\varepsilon_i)/N) \log ((Np_i + N\varepsilon_i)/N).$$

Expanding this expression,

$$\overline{H} = - \Sigma p_i \log p_i - \Sigma \varepsilon_i \log p_i - \frac{1}{2} \Sigma \varepsilon_i^2/p_i + \ldots,$$

where $\sum_{i=1}^{s} \varepsilon_i = 0$. When $p_1 = p_2 = \dots = p_s = 1/s$ this reduces to

$$\bar{H} = \log s - \frac{1}{2} s \quad \Sigma\varepsilon_i^2 + \frac{1}{6} s^2 \, \Sigma\varepsilon_i{}^3 - \dots .$$

Taking expectations,

$$E(\bar{H}) = - \Sigma p_i \log p_i - (s-1)/2N - (\Sigma p_i{}^{-1}-1)/12N^2 -$$

$$(\Sigma p_i{}^{-2} - \Sigma p_i{}^{-1})/12N^3 + 0\ (1/N^4) .$$

For the variance let $H = - \sum_{i=1}^{s} p_i \log p_i$ and

$$\text{var}(\bar{H}) = E(\bar{H} - H - E(\bar{H}-H))^2$$

$$= E(\bar{H}-H)^2 - (E(H-H))^2$$

$$= (\Sigma p_i \log^2 p_i - H^2)/N + (s-1)/2N^2 + 0\ (1/N^3) .$$

In the equiprobable case

$$E(\bar{H}) = \log s - \sum_{i=1}^{9} a_i/N^i + 0\ (1/N^{10})$$

where the a_i's are given on page 49 and

$$\text{var}(\bar{H}) = (s-1)/2N^2 + (s^2-1)/6N^3 + 0\ (1/N^4).$$

The series for $E(\bar{H})$ can be proved to be divergent. In fact it can be proved that the terms (after the first) are one signed and ultimately increase without limit, no matter how large the sample. The tendency to instability is more emphatic the larger the number of categories, and it seems likely that divergence is also affected by the occurrence of one or more categories with small associated probabilities. The behavior of the terms can be envisaged as the ordinates of a parabolic arc (concave upward) there being a decreasing segment when k is near to n, almost all decreasing when $n \gg k$ (but then ultimately switches to an all-increasing segment), and all increasing when $k>n$.

Similar remarks doubtless apply to μ_2, μ_3 and μ_4 although we have not studied these in detail because of the complexity of the terms involved, and this complexity increases with the order of the moment.

In using the series to approximate $E(\bar{H})$ a gemeral rule is to use terms through the smallest a_i. The error involved is approximately five percent or less if $N \geq s$. There may be cases where the best estimate is log s. This situation would occur if $\log s - a_1/N$ is negative.

4. Moments and Distribution of $\bar{H}$ by Sample Configuration

By constructing all sample configurations it is possible to compute the moments and skewness parameters $\sqrt{\beta_1}$ and β_2. This method provides exact moments machine-wise but it is not feasible for large N and s.

On page 64 results are given for s=2 and N as great as 500. The β_2 in this case approaches 3. Results are given for trinomials (s=3) with N as great as 190 and again β_2 approaches 3. The case of equal probabilities was studied on page 66, for s=4. In this case β_2 approaches 7. The β_2 for the Chi-Square is $(12 + 3\nu)/\nu$, where ν is the degrees of freedom, and here ν is 3.

If the probabilities are equal the distribution of $\overline{H}$ approaches a Chi-Square and if the probabilities are not equal the distribution approaches a Normal.

The computing time is significant due to the many partitions. For example,

s	N	Partitions
3	150	11,476
3	75	2,926
5	100	4,598,126
7	50	32,468,436
9	25	13,884,156
11	15	3,268,760

so that computing time even on large machines becomes an important factor.

5. Moments and Distribution of $\overline{H}$ by Monte-Carlo Simulation

The Monte-Carlo simulation is quite suitable for evaluating the moments of $\overline{H}$. We merely draw samples from categorized data and compute $\overline{H}$ and print out the moments or any other desirable information such as percentage points. Of course, the computing time necessary may become critical. For s=15 and N=200 (using 50,000 cycles) requires approximately one hour on the IBM 360/65. An analysis is made of the accuracy of this method and a table of moments, $\sqrt{\beta_1}$ and β_2 is given in Appendix D.

5.1 Pielou's Sequential Approach to the Assessment of $\bar{H}$

An ecologist using $\bar{H}$ as a measure of diversity is assuming that his collection, on which the $\bar{H}$ was computed, may be regarded as a random sample from some much larger parent population and represents the average conditions in it. If one could regard one's collection as being respresentative of a large parent population, it is unsatisfactory to accept a single sample value of n_i/N as a reliable estimate of p_i. It is unusual for a collection not to contain at least some species with only one or two individuals [55]. An attempt to increase the values of the n_i by enlarging the collection is no help. When the sample is enlarged one usually obtains not only more members of species already present but also, in ones and twos, individuals of species not previously represented. Precise estimates of all the p_i are not easy to come by (Good [6]). How then can the population diversity be estimated? Pielou [34] says a possible method is as follows.

Draw cumulative samples and evaluate the successive values $\bar{H}_1, \bar{H}_2, \ldots, \bar{H}_r$ of the diversity. If one continues this the result is ultimately the true diversity of the population. Pielou has suggested a stopping rule such as sampling until

$$|\bar{H}_r - \bar{H}_{r-1}| \leq \theta \bar{H}_{r-1}, \quad 0<\theta<1$$

in certain sample units. The object of this interpretation of Pielou's stopping rule is to gain some little insight into what the process involves. There are several tables in Appendix D using MacArthur's Model [23]. It has been found that the final sample size N* has a mean which changes from 19 to 28 as s goes from 5 to 100 and the standard deviation of N^* is about 6 and the distribution of N^* is near normal. The mean of $\bar{H}^*$ increases from 1.2 to 3 and has a small standard deviation and is not as near the Normal as that of N^*.

There is thus at least a suggestion here that if MacArthur's Model holds then the 'stopping' rule would not involve a large sample and it seems conceivable that one might quit with a value of $\bar{H}^*$ much lower than the true value.

6. An Exploration Into the Higher Moments of $\bar{H}$

The exact results in the general case for the mean and variance of $\bar{H}$ is given in Chapter III. These results required evaluating a single and a double integral respectively. The evaluation of these integrals proved to be tedious. In Appendix E the integrals for the first four moments have been given with the hope that they too may be evaluated. Perhaps Monte-Carlo methods would be sucessful. The author feels that methods unlike those of Chapter III will have to be brought to bear in order to accomplish this task.

7. Multinomial Type Moments

In obtaining the asymptotic results given in Chapter IV the multinomial moments had to be computed to a higher order than had been published here-to-fore. The methods used to produce these moments are given in Chapter II. The so called Q-Statistic Method [47] has been computerized. The first fourteen Central Binomial Moments and the Central Multinomial Moments through order six are given in Appendix B.

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BIBLIOGRAPHY

[1] Barton, D. E., and David, F. N. (1956), "Some Notes on Ordered Random Intervals," Royal Statistical Society Journal, Vol. 18, p. 79-94.

[2] Basharin, G.P. (1959), "On a Statistical Estimate for the Entropy of a Sequence of Independent Random Variables," in N Artin (ed.), Theory of Probability and Its Applications, Vol. IV, (Translation of Teoriya Veroyatnostei i ee Pvimeneniya) Society for Industrial and Applied Mathematics, Philadelphia, p. 333-336.

[3] Brillouin, L. (1960), Science and Information Theory, (2nd ed.), Academic Press, New York.

[4] David, F.N., and Barton, D.E. (1962), Combinatorial Chance, Hafner Publishing Co., New York.

[5] Fisher, R. A., Corbet, A. Steven, and Williams, C.B. (1943), "The Relation between the Number of Species and the Number of Individuals in a Random Sample of an Animal Population," Journal of Animal Ecology, Vol. 12, p. 42-58.

[6] Good, I.J. (1953), "The Population Frequencies of Species and the Estimation of Population Parameters," Biometrika, Vol. 40, p. 237-264.

[7] Good, I. J. and Toulmin, G.H.(1956), "The Number of New Species, and the Increase in Population Coverage, when a Sample is Decreased," Biometrika, Vol. 43, p. 45-63.

[8] Good, I.J., "Distribution Word Frequencies," Nature, Vol. 179, p. 595.

[9] Gotaas, P., (1936), "Formulae of Recurrence for the Semivariants at a Class of Frequency Functions of More Variables," Skand. AktuarTidskr, Vol. 19, p. 200-211.

[10] Guldberg, S. (1935),"Recurrence Formulae for the Semi-invariants of Some Discontinuous Frequency Functions of n Variables," Skand. Aktuartidskr, Vol. 18, p. 270-278.

[11] Hadley, Herschel N. (1958), Comparison of Exact and Asymptotic Distributions of the Information-Statistic for Samples from Multinomial Populations, M.S. Thesis, George Washington University.

[12] Hairston, Nelson G. (1959). "Species Abundance and Community Organization," Ecology, Vol. 40, p. 404-416.

[13] Hardy, G. H., (1901-02). "On the Frullanian Integral," Quarterly Journal of Mathematics, Vol. 33, p. 113-144.

[14] Hearn, A. C. (1967), Reduce Users' Manual. Stanford Artificial Intelligence Project, Memo No. 50, Institute of Theoretical Physics, ITP-247.

[15] Herdan, G. (1958),"The Mathematical Relation Between Greenberg's Index of Linguistic Diversity and Yule's Characteristic," Biometrika, Vol. 45, p. 268-270.

[16] Hirschman, I.I., Jr. (1957), "A Note on Entropy," American Journal of Mathematics, Vol. 79, p. 152-156.

[17] Kempthorne, Oscar, (1966), "Some Aspects of Experimental Inference," Amer. Stat. Assoc. Jr., Vol. 61, p. 11-34.

[18] Kendall, M. G., (1940),"The Derivation of Multivariate Sampling Formulae from Univariate Formulae by Symbolic Operation," Ann. Eugen., Vol. 10, p. 392.

[19] Khinchin, A. J., (1957), Mathematical Foundations of Information Theory, Dover Publications, New York.

[20] King, James C., (1961), "Inbreeding Heterosis and Information Theory," The American Naturalist, XCV, p. 345-364.

[21] Kullback, S. (1967), "The Two Concepts of Information," Amer. Stat. Assoc. Jr. Vol. 62, p. 685-686.

[22] Lloyd, Monte, and Ghelardi, R. J., (1964), "A Table for Calculating the 'Equitability' Component of Species Diversity," J. Anim. Ecol., Vol 33, p. 217-225.

[23] MacArthur, Robert H. (1957),"On the Relative Abundance of Bird Species," Proc. Nat. Acad. of Sc., Vol. 43, p. 293-295.

[24] MacArthur, Robert H., and MacArthur, John W. (1961), "On Bird Species Diversity," Ecology, Vol. 42, p. 594-598.

[25] MacArthur, Robert H. (1964)"Envirgonmental Factors Affecting Bird Species Diversity," The Am. Naturalist, Vol XCVIII, p. 387-414.

[26] Margolef, D. R. (1957), "La Teoria de la Information in Ecologia," Mem. R. Acad. Cien. Artes, Vol. 32, p. 373-449, English translation by Hall W. (1957). Information Theory in Ecology. Gen. System. Vol. 3, p. 36-71.

[27] McIntosh, Robert P. (1967), "An Index of Diversity and the Relation of Certain Concepts to Diversity," Ecology, Vol. 48, p. 392-404.

[28] McMillan, Brockway, (1953), "The Basic Theorems of Information Theory," Annais of Math. Stat., Vol. 24, p. 196-219.

[29] Myers , R. H., (1963), Orthogonal Statistics and Some Sampling Properties of Moment Estimators for the Negative Binomial Distributions, Doctoral dissertation, Virginia Polytechnic Institute.

[30] Monk, Carl D. (1967), "Tree Species Diversity in the Eastern Deciduous Forest with Particular Reference to North Central Florida," Amer. Natur. Vol. 101,p173-187.

[31] Odum, Eugene P. (1966), Ecology, Holt, Rinehart and Winston, New York, Toronto and London.

[32] Patten, Bernard C. (1962), "Species Diversity in Net Phytoplankton of Raritan Bay," J. Marine Res. Vol. 20, p. 57-75.

[33] Pielou, E.C. (1966a), "The Measurement of Diversity in Different Types of Biological Collections," J. Theoret. Biol. Vol. 13, p. 131-144.

[34] Pielou, E. C. (1966b), The Use of Information Theory in the Study of the Diversity of Biological Populations, Fifth Berkeley Symposium, p. 163-177.

[35] Pielou, E. C. (1966c),"Species-Diversity and Pattern-Diversity in the Study of Ecological Succession," J. Theoret. Biol., Vol. 10, p. 370-383.

[36] Pielou, E. C. (1966d). "Shannon's Formula as a Measure of Specific Diversity: Its Use and Misuse," The Am. Naturalist, Vol. 100, p. 463-465.

[37] Preston, F. W. (1948), "The Commonness, and Rarity, of Species," Ecology, Vol. 29, p. 254-283.

[38] Quastler, H. (1953), Information Theory in Biology, University of Illinois Press, Urbana.

[39] Qvale, Paul (1932), "Remarks on Semi-invariants and Incomplete Moments," Skand. Aktuartidskr, Vol. 15, p. 196-210.

[40] Rao, C. Radhakrishna, (1957), "Maximum Likelihood Estimation for the Multinomial Distribution," Sankhya, Vol. 18, p. 139-148.

[41] Rao, C. Radhakrishna, (1958), "Maximum Likelihood Estimation for the Multinomial Distribution Distribution with Infinite Number of Cells," Sankhya, Vol. 20, p. 211-218.

[42] Reza, Fazlollah M. (1961), An Introduction to Information Theory, McGraw-Hill, New York.

[43] Schaffer, K. A. (1957), "The Likelihood Test for Fit," Mittbl. Math. Statistik, Vol. 9, p. 27-54.

[44] Shannon, C.E. and Weaver, W. (1963), The Mathematical Theory of Communication, University of Illinois Press, Urbana

[45] Sheehan, D. M. (1967), The Computational Approach to Sampling Moments: A Study of Certain Estimators for the Negative Binomial Distributions, Doctoral Dissertation, Virginia Polytechnic Institute.

[46] Shenton, L. R., Bowman, K. O. and Reinfelds, Juris (1967), "Sampling Moments of Moment and Maximum Likelihood Estimators for Discrete Distributions," 36th Session of the International Statistical Institute, Sydney, Australia.

[47] Shenton, L. R. and Myers, R. (1963), Orthogonal Statistics, Proceedings of the International Symposium at Montreal, Pergamon Press, p. 445-458.

[48] Simpson, E. H. (1949), "Measurement of Diversity," Nature, Vol. 163, p. 688.

[49] Uppuluri, V. R. and Bowman, Kimiko O. (1966a), Likelihood Ratio Test Criterion for Small Samples from Multinomial Distributions, ORNL-3991, p. 1-47.

[50] Uppuluri, V. R., Rao, C. R., and Bowman, K. O. (1966b), Data on a Set of Trinomial Distributions, ORNL-4004.

[51] Wiener, N. (1948), Cybernetics, Wiley and Sons, Inc., New York,.

[52] Wilks, S. S. (1938-39), "The Large-Sample Distribution of the Likelihood Ratio for Testing Composite Hypotheses," Annals of Math. Stat., 9-10, 60-62.

[53] Williams, C. B. (1944), "Some Applications of the Logarithmic Series and the Index of Diversity to Ecological Problems," Jr. Ecol. Vol. 32, p. 1-44.

[54] Williams, C. B. (1946), Yule's 'Characteristic' and the 'Index of Diversity,'" Nature, Vol. 157, p. 482.

[55] Williams, C. B. (1964), Patterns in the Balance of Nature, Academic Press, London and New York.

[56] Wishart, John (1949),"Cumulants of Multivariate Multinomial Distributions," Biometrika, Vol. 36, p. 47-58.

[57] Yule, Udny G. (1944), The Statistical Study of Literary Vocabulary, Cambridge University Press.

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APPENDICES

Appendix A

Table A.1

μ_1 and μ_2 of $\overline{H}$: General Case

Sample size	species	Prob.	μ_1	μ_2
4	2	.5,.5	.541098	.045493
4	2	.4,.6	.520269	.053222
5	2	.4,.6	.555789	.036129
5	2	.1,.9	.218896	.071030
10	2	.1,.9	.266578	.045637
10	2	.4,.6	.619632	.010002
5	3	.3,.3,.4	.843791	.052452
10	3	.3,.3,.4	.978559	.014854
10	3	.1111,.2222,.6667	.733169	.053442
15	3	.1111,.2222,.6667	.773951	.034445
20	3	.1111,.2222,.6667	.794067	.024872
25	3	.1111,.2222,.6667	.805815	.019313
50	3	.1111,.2222,.6667	.828075	.009033
4	4	.01,.02,.07,.9	.208707	.088644
4	4	.1,.2,.3,.4	.842752	.084953
5	4	.1,.2,.3,.4	.927729	.071353
7	4	.05,.45,.35,.15	.928180	.055588
10	4	.05,.05,.1,.8	.546537	.089247
15	4	.1,.2,.3,.4	1.169085	.019987
15	4	.05,.05,.1,.8	.594568	.065642
25	4	.05,.05,.1,.8	.639383	.041214
10	5	.1,.1,.2,.3,.3	1.273364	.036911
20	5	.01,.02,.03,.44,.5	.843477	.022672
20	5	.1,.2,.2,.4,.1	1.360470	.019774
10	10	.01,.02,.03,.04,.2 .25,.3,.05,.025,.075	1.409747	.056647
20	10	.01,.02,.03,.04,.2 .25,.3,.05,.025,.075	1.591188	.035088

Table A.2

μ_1 and μ_2 of $\overline{H}$: Equiprobable Case

k \ N		2	3	4	5	6	7
2	μ_1	.34657	.47739	.54110	.57701	.59945	.61463
	μ_2	.12011	.07597	.04549	.02840	.01877	.01311
3		.46210	.66848	.78275	.85330	.90008	.93278
		.10677	.09145	.06822	.04922	.03653	.02725
4		.51986	.77002	.91772	1.01401	1.08077	1.12914
		.09008	.08757	.07289	.05837	.04637	.03681
5		.55452	.83286	1.00357	1.11871	1.20100	1.26220
		.07687	.08015	.07110	.06031	.05039	.04192
10		.62383	.96286	1.18681	1.34877	1.47231	1.56995
		.04324	.05130	.05143	.04898	.04563	.04204
15		.64694	1.00747	1.25145	1.43207	1.57300	1.68686
		.02989	.03692	.03849	.03908	.03682	.03518
20		.56849	1.03022	1.28446	1.47504	1.62546	1.74834
		.02282	.02374	.03055	.03080	.03034	.02953
25		.66542	1.04362	1.30449	1.50125	1.65763	1.78625
		.01845	.02351	.02528	.02578	.02568	.02526
30		.67004	1.05272	1.31793	1.51891	1.67937	1.81195
		.01548	.01988	.02154	.02213	.02221	.02201
35		.67334	1.05924	1.32758	1.53161	1.69505	1.83053
		.01334	.01722	.01875	.01937	.01955	.01948
40		.67582	1.06413	1.33485	1.54119	1.70689	1.84458
		.01171	.01518	.01660	.01722	.01745	.01745
45		.67774	1.06794	1.34050	1.54866	1.71614	1.85558
		.01044	.01358	.01490	.01550	.01575	.01580
50		.67928	1.07100	1.34504	1.55467	1.72358	1.86443
		.00942	.01228	.01350	.01409	.01435	.01443
55		.68054	1.07350	1.34877	1.55959	1.72968	1.87170
		.00858	.01120	.0.235	.01291	.01318	.01328
60		.68159	1.07558	1.35187	1.56370	1.73348	1.87777
		.00787	.01030	.01138	.01191	.01218	.01230
75		.68390	1.08017	1.35872	1.57277	1.74605	1.89121
		.00632	.00830	.00920	.00967	.00992	.01006
100		.68390	1.08477	1.06558	1.58188	1.75738	1.90474
		.00476	.00627	.00698	.00736	.00758	.00771
200		.68968	1.09168	1.37591	1.59561	1.77450	1.92522

Continued Table A.2

k \ N	8	9	10	11	12	13
2	.62553 .00962	.63372 .00734	.64012 .00579	.64526 .00469	.64947 .00388	.65300 .00327
3	.95663 .02072	.97463 .01609	.98863 .01274	.99979 .01029	1.00888 .00845	1.01642 .00706
4	1.16538 .02945	1.19329 .02377	1.21528 .01938	1.23296 .01597	1.24743 .01331	1.25944 .01120
5	1.30909 .03490	1.34590 .02916	1.37536 .02448	1.39934 .02068	1.41914 .01757	1.43571 .01503
10	1.64910 .03852	1.71451 .03518	1.76941 .03208	1.81606 .02924	1.85611 .02664	1.89081 .02429
15	1.78114 .03338	1.86068 .03154	1.92877 .02973	1.98775 .02798	2.03932 .02630	2.08480 .02471
20	1.85122 .02853	1.93896 .02744	2.01486 .02631	2.08128 .02518	2.13996 .02407	2.19220 .02298
25	1.89466 .02367	1.98772 .02399	2.06875 .02325	2.14011 .02248	2.20354 .02171	2.26037 .02094
30	1.92420 .02166	2.02099 .02121	2.10564 .02070	2.18050 .02016	2.24734 .01961	2.30747 .01904
35	1.94561 .01926	2.04515 .01896	2.13248 .01860	2.20996 .01821	2.27935 .01780	2.34196 .01738
40	1.96183 .01733	2.06348 .01712	2.15288 .01687	2.23239 .01657	2.30376 .01626	2.36831 .01593
45	1.97433 .01574	2.07787 .01560	2.16891 .01541	2.25003 .01519	2.32298 .01495	2.38908 .01469
50	1.98477 .01441	2.08946 .01432	2.18184 .01418	2.26428 .01401	2.33852 .01382	2.40589 .01362
55	1.99318 .01329	2.09900 .01323	2.19250 .01312	2.27602 .01299	2.35134 .01284	2.41977 .01268
60	2.00022 .01232	2.10699 .01229	2.20142 .01221	2.28587 .01211	2.36209 .01199	2.43142 .01186
75	2.01580 .01012	2.12469 .01012	2.22121 .01010	2.30772 .01005	2.38599 .00991	2.52278 .00983
100	2.03151 .00778	2.14256 .00782	2.24122 .00783	2.32985 .00782	2.41021 .00780	2.48363 .00777
200	2.05533	2.16969	2.27165	2.36355	2.44717	2.52383

Continued Table A.2

k \ N	14	15	16	17	18	19
2	.65600 .00297	.65858 .00241	.66082 .00210	.66279 .00185	.66453 .00165	.66608 .00147
3	1.02278 .00597	1.02821 .00512	1.03290 .00444	1.03700 .00389	1.04061 .00343	1.04381 .00306
4	1.26956 .00957	1.27819 .00817	1.28562 .00708	1.29208 .00618	1.29776 .00544	1.30278 .00483
5	1.44973 .01294	1.46173 .01122	1.47208 .00978	1.48110 .00858	1.48901 .00758	1.49600 .00673
10	1.92110 .02217	1.94771 .02025	1.97125 .01851	1.99217 .01695	2.01086 .01554	2.02764 .01427
15	2.12517 .02321	2.16123 .02180	2.19362 .02047	2.22283 .01923	2.24930 .01807	2.27338 .01699
20	2.23902 .02193	2.28124 .02092	2.31950 .01995	2.35432 .01903	2.38615 .01814	2.41533 .01730
25	2.31162 .02019	2.35810 .01945	2.40047 .01873	2.43925 .01803	2.47490 .01735	2.50777 .01670
30	2.36192 .01848	2.41152 .01793	2.45690 .01738	2.49862 .01684	2.53710 .01631	2.57273 .01580
35	2.39884 .01695	2.45079 .01652	2.49848 .01609	2.54244 .01567	2.58312 .01525	2.62088 .01484
40	2.42708 .01560	2.48089 .01526	2.53039 .01492	2.57612 .01458	2.61854 .01425	2.65800 .01391
45	2.44938 .01443	2.50468 .01415	2.55564 .01388	2.60281 .01360	2.64663 .01333	2.68747 .01305
50	2.46743 .01340	2.52396 .01318	2.57613 .01295	2.62448 .01272	2.66947 .01250	2.71145 .01227
55	2.48235 .01250	2.53990 .01232	2.59308 .01213	2.64243 .01194	2.68839 .01175	2.73134 .01155
60	2.49488 .01171	2.55331 .01156	2.60734 .01140	2.65753 .01124	2.70433 .01108	2.74810 .01091
75	2.52278 .00983	2.58317 .00973	2.63915 .00963	2.69126 .00953	2.73994 .00943	2.78559 .00932
100	2.55114 .00773	2.61356 .00768	2.67156 .00763	2.72566 .00758	2.77633 .00752	2.82393 .00746
200	2.59455	2.66017	2.72134	2.77861	2.83241	2.88313

Continued Table A.2

k \ N	20	21	22	23	24	25
2	.66747 .00131	.66873 .00120	.66987 .00109	.67090 .00099	.67185 .00091	.67272 .00084
3	1.04668 .00274	1.04926 .00247	1.05159 .00224	1.05371 .00204	1.05565 .00186	1.05742 .00171
4	1.30726 .00431	1.31127 .00387	1.31489 .00349	1.31817 .00317	1.32116 .00289	1.32390 .00265
5	1.50223 .00601	1.50780 .00539	1.51281 .00487	1.51735 .00441	1.52148 .00401	1.52525 .00367
10	2.04275 .01312	2.05643 .01208	2.06885 .01114	2.08016 .01029	2.09050 .00951	2.09997 .00881
15	2.29535 .01598	2.31547 .01504	2.33394 .01416	2.35094 .01334	2.36664 .01257	2.38116 .01185
20	2.44218 .01650	2.46696 .01574	2.48989 .01501	2.51115 .01432	2.53092 .01367	2.54934 .01305
25	2.53817 .01607	2.56638 .01547	2.59262 .01488	2.61708 .01432	2.63993 .01379	2.66132 .01327
30	2.60582 .01530	2.63663 .01482	2.66539 .01435	2.69230 .01389	2.71753 .01345	2.74123 .01303
35	2.65605 .01444	2.68889 .01405	2.71963 .01367	2.74848 .01329	2.77559 .01293	2.80113 .01257
40	2.69483 .01359	2.72929 .01326	2.76162 .01295	2.79202 .01264	2.82066 .01233	2.84769
45	2.72566 .01278	2.76145 .01251	2.79508 .01225	2.82676 .01199	2.85665	2.88491
50	2.75076 .01204	2.78766 .01181	2.82238 .01159	2.85512 .01137	2.88606	2.91536
55	2.77160 .01136	2.80943 .01117	2.84507 .01098	2.87872	2.91055	2.94072
60	2.78917 .01075	2.82780 .01058	2.86422 .01042	2.89865	2.93124	2.96217
75	2.82850 .00921	2.86895 .00910	2.90718	2.94339	2.97774	3.01040
100	2.86879	2.91117	2.95730	2.98939	3.02561	3.06012
200	2.93109	2.97655	3.01974	3.06087	3.10012	3.13764

Continued Table A.2

k \ N	50	60	70	80	90	100
2	.68304 .00020	.68474 .00014	.68595 .00010	.68686 .00008	.68756 .00006	.68812 .00005
3	1.07833 .00041	1.08175 .00028	1.08417 .00021	1.08601 .00016	1.08742	1.08854
4	1.35576 .00062	1.36093 .00043	1.36460	1.36734	1.36947	1.37117
5	1.56856 .00084	1.57550	1.58043	1.58411	1.58696	1.58923
10	2.20812	2.22470	2.23628			
15	2.55594	2.58355				
20	2.78281					
25	2.94395					
30	3.06466					
35						
40						
45						
50						
55						
60						
75						
100						
200						

Appendix B

Table B.1

Expectation of Q-Products

$(a^2)_1 = +[a^2]$

$(ab)_1 = [a\ b]$

$(a^3)_2 = [a^3]$

$(a^2b)_2 = [a^2\ b]$

$(abc)_2 = [a\ b\ c]$

$(a^4)_2 = 3[a^2]^2$

$(a^3b)_2 = 3[a^2]\ [a\ b]$

$(a^2b^2)_2 = 2[a\ b]^2+[a^2]\ [b^2]$

$(a^2bc)_2 = 2[a\ b]\ [a\ c]+[a^2]\ [b\ c]$

$(abcd)_2 = [b\ d]\ [a\ c]+[a\ b]\ [c\ d]+[a\ d]\ [b\ c]$

Continued Table B.1

$$(a^4)_3 = [a^4]-3[a^2]^2$$

$$(a^3b)_3 = [a^3\ b]-3[a^2]\ [a\ b]$$

$$(a^2b^2)_3 = [a^2\ b^2]-2[a\ b]^2-[a^2]\ [b^2]$$

$$(a^2bc)_3 = [a^2\ b\ c]-2[a\ b]\ [a\ c]-[a^2]\ [b\ c]$$

$$(abcd)_3 = +[a\ b\ c\ d]-[b\ d]\ [a\ c]-[a\ b]\ [c\ d]-[a\ d]\ [b\ c]$$

Continued Table B.1

$(a^5)_3 =$

$10[a^2]\ [a^3]$

$(a^4b)_3 =$

$4[a\ b]\ [a^3]+6[a^2]\ [a^2\ b]$

$(a^3b^2)_3 =$

$[b^2]\ [a^3]+6[a\ b]\ [a^2\ b]+3[a^2]\ [a\ b^2]$

$(a^3bc)_3 =$

$[b\ c]\ [a^3]+3[a\ c]\ [a^2\ b]+3[a\ b]\ [a^2\ c]+3[a^2]\ [a\ b\ c]$

$(a^2b^2c)_3 =$

$[b^2]\ [a^2\ c]+2[b\ c]\ [a^2\ b]+4[a\ b]\ [a\ b\ c]+2[a\ c]\ [a\ b^2]+[a^2]\ [b^2\ c]$

$(a^2bcd)_3 =$

$[b\ d]\ [a^2\ c]+[c\ d]\ [a^2\ b]+[b\ c]\ [a^2\ d]+2[a\ d]\ [a\ b\ c]+2[a\ b]\ [a\ c\ d]$

$+2[a\ c]\ [a\ b\ d]+[a^2]\ [b\ c\ d]$

$(abcde)_3 =$

$[b\ d]\ [a\ c\ e]+[c\ d]\ [a\ b\ e]+[b\ c]\ [a\ d\ e]+[d\ e]\ [a\ b\ c]+[a\ d]\ [b\ c\ e]$

$+[b\ e]\ [a\ c\ d]+[a\ b]\ [c\ d\ e]+[c\ e]\ [a\ b\ d]+[a\ c]\ [b\ d\ e]+[a\ e]\ [b\ c\ d]$

Continued Table B.1

$(a^5)_4 =$

$[a^5]-10[a^2]\,[a^3]$

$(a^4 b)_4 =$

$[a^4\ b]-4[a\ b]\,[a^3]-6[a^2]\,[a^2\ b]$

$(a^3 b^2)_4 =$

$[a^3\ b^2]-[b^2]\,[a^3]-6[a\ b]\,[a^2\ b]-3[a^2]\,[a\ b^2]$

$(a^3 bc)_4 =$

$[a^3\ b\ c]-[b\ c]\,[a^3]-3[a\ c]\,[a^2\ b]-3[a\ b]\,[a^2\ c]-3[a^2]\,[a\ b\ c]$

$(a^2 b^2 c)_4 =$

$[a^2\ b^2\ c]-[b^2]\,[a^2\ c]-2[b\ c]\,[a^2\ b]-4[a\ b]\,[a\ b\ c]-2[a\ c]\,[a\ b^2]$

$-[a^2]\,[b^2\ c]$

$(a^2 bcd)_4 =$

$[a^2\ b\ c\ d]-[b\ d]\,[a^2\ c]-[c\ d]\,[a^2\ b]-[b\ c]\,[a^2\ d]-2[a\ d]\,[a\ b\ c]$

$-2[a\ b]\,[a\ c\ d]-2[a\ c]\,[a\ b\ d]-[a^2]\,[b\ c\ d]$

$(abcde)_4 =$

$+[a\ b\ c\ d\ e]-[b\ d]\,[a\ c\ e]-[c\ d]\,[a\ b\ e]-[b\ c]\,[a\ d\ e]-[d\ e]\,[a\ b\ c]$

$-[a\ d]\,[b\ c\ e]-[b\ e]\,[a\ c\ d]-[a\ b]\,[c\ d\ e]-[c\ e]\,[a\ b\ d]-[a\ c]\,[b\ d\ e]$

$-[a\ e]\,[b\ c\ d]$

Continued Table B.1

$(a^6)_3 =$

$15[a^2]^3$

$(a^5b)_3 =$

$15[a^2]^2 [a\ b]$

$(a^4b^2)_3=$

$12[a^2] [a\ b]^2+3[a^2]^2 [b^2]$

$(a^4bc)_3 =$

$12[a^2] [a\ b] [a\ c]+3[a^2]^2 [b\ c]$

$(a^3b^3)_3 =$

$6[a\ b]^3+9[a^2] [a\ b] [b^2]$

$(a^3b^2c)_3 =$

$6[a\ b]^2 [a\ c]+3[a^2] [a\ c] [b^2]+6[a^2] [a\ b] [b\ c]$

$(a^3bcd)_3 =$

$6[a\ b] [a\ d] [a\ c]+3[a^2] [a\ c] [b\ d]+3[a^2] [a\ d] [b\ c]+3[a^2] [a\ b] [c\ d]$

$(a^2b^2c^2)_3 =$

$2[c^2] [a\ b]^2+8[a\ c] [a\ b] [b\ c]+2[a^2] [b\ c]^2+2[a\ c]^2 [b^2]+[a^2] [c^2] [b^2]$

Continued Table B.1

$(a^2b^2cd)_3 =$

$2[c\ d]\ [a\ b]^2+4[a\ d]\ [a\ b]\ [b\ c]+4[a\ c]\ [a\ b]\ [b\ d]+2[a^2]\ [b\ c]\ [b\ d]$
$+2[a\ c]\ [a\ d]\ [b^2]+[a^2]\ [c\ d]\ [b^2]$

$(a^2bcde)_3 =$

$2[c\ d]\ [a\ b]\ [a\ e]+2[a\ d]\ [a\ e]\ [b\ c]+2[a\ d]\ [a\ b]\ [c\ e]+2[a\ c]\ [a\ e]\ [b\ d]$
$+2[a\ c]\ [a\ b]\ [d\ e]+[a^2]\ [c\ e]\ [b\ d]+[a^2]\ [b\ c]\ [d\ e]+2[a\ c]\ [a\ d]\ [b\ e]$
$+[a^2]\ [c\ d]\ [b\ e]$

$(abcdef)_3 =$

$[c\ d]\ [b\ f]\ [a\ e]+[c\ d]\ [a\ b]\ [e\ f]+[d\ f]\ [a\ e]\ [b\ c]+[a\ d]\ [e\ f]\ [b\ c]$
$+[d\ f]\ [a\ b]\ [c\ e]+[a\ d]\ [b\ f]\ [c\ e]+[c\ f]\ [a\ e]\ [b\ d]+[a\ c]\ [e\ f]\ [b\ d]$
$+[c\ f]\ [a\ b]\ [d\ e]+[a\ c]\ [b\ f]\ [d\ e]+[a\ f]\ [c\ e]\ [b\ d]+[a\ f]\ [b\ c]\ [d\ e]$
$+[c\ f]\ [a\ d]\ [b\ e]+[a\ c]\ [d\ f]\ [b\ e]+[a\ f]\ [c\ d]\ [b\ e]$

Continued Table B.1

$(a^6)_4 =$

$10[a^3]^2+15[a^4]\ [a^2]-45[a^2]^3$

$(a^5b)_4 =$

$10[a^3]\ [a^2\ b]+10[a^3\ b]\ [a^2]+5[a^4]\ [a\ b]-45[a^2]^2\ [a\ b]$

$(a^4b^2)_4 =$

$6[a^2\ b]^2+4[a^3]\ [a\ b^2]+6[a^2\ b^2]\ [a^2]+8[a^3\ b]\ [a\ b]+[a^4]\ [b^2]-36[a^2]\ [a\ b]^2$

$-9[a^2]^2\ [b^2]$

$(a^4bc)_4 =$

$6[a^2\ b]\ [a^2\ c]+4[a^3]\ [a\ b\ c]+6[a^2\ b\ c]\ [a^2]+4[a^3\ c]\ [a\ b]+4[a^3\ b]\ [a\ c]$

$+[a^4]\ [b\ c]-36[a^2]\ [a\ b]\ [a\ c]-9[a^2]^2\ [b\ c]$

$(a^3b^3)_4 =$

$9[a^2\ b]\ [a\ b^2]+[a^3]\ [b^3]+3[a\ b^3]\ [a^2]+9[a^2\ b^2]\ [a\ b]+3[a^3\ b]\ [b^2]$

$-18[a\ b]^3-27[a^2]\ [a\ b]\ [b^2]$

$(a^3b^2c)_4 =$

$3[a^2\ c]\ [a\ b^2]+6[a^2\ b]\ [a\ b\ c]+[a^3]\ [b^2\ c]+3[a\ b^2\ c]\ [a^2]+6[a^2\ b\ c]\ [a\ b]$

$+3[a^2\ b^2]\ [a\ c]+[a^3\ c]\ [b^2]+2[a^3\ b]\ [b\ c]-18[a\ b]^2\ [a\ c]-9[a^2]\ [a\ c]\ [b^2]$

$-18[a^2]\ [a\ b]\ [b\ c]$

Continued Table B.1

$(a^3bcd)_4 =$

$3[a^2\ c]\ [a\ b\ d]+3[a^2\ d]\ [a\ b\ c]+3[a^2\ b]\ [a\ c\ d]+[a^3]\ [b\ c\ d]$

$+3[a\ b\ c\ d]\ [a^2]+3[a^2\ c\ d]\ [a\ b]+3[a^2\ b\ c]\ [a\ d]+3[a^2\ b\ d]\ [a\ c]$

$+[a^3\ c]\ [b\ d]+[a^3\ d]\ [b\ c]+[a^3\ b]\ [c\ d]-18[a\ b]\ [a\ d]\ [a\ c]-9[a^2]\ [a\ c]\ [b\ d]$

$-9[a^2]\ [a\ d]\ [b\ c]-9[a^2]\ [a\ b]\ [c\ d]$

$(a^2b^2c^2)_4 =$

$4[a\ b\ c]^2+2[a^2\ b]\ [b\ c^2]+2[a\ c^2]\ [a\ b^2]+2[a^2\ c]\ [b^2\ c]+[b^2\ c^2]\ [a^2]$

$+4[a\ b^2\ c]\ [a\ c]+[a^2\ b^2]\ [c^2]+4[a\ b\ c^2]\ [a\ b]+4[a^2\ b\ c]\ [b\ c]$

$+[a^2\ c^2]\ [b^2]-6[c^2]\ [a\ b]^2-24[a\ c]\ [a\ b]\ [b\ c]-6[a^2]\ [b\ c]^2-6[a\ c]^2\ [b^2]$

$-3[a^2]\ [c^2]\ [b^2]$

$(a^2b^2cd)_4 =$

$4[a\ b\ c]\ [a\ b\ d]+2[a^2\ b]\ [b\ c\ d]+2[a\ c\ d]\ [a\ b^2]+[a^2\ d]\ [b^2\ c]$

$+[a^2\ c]\ [b^2\ d]+[b^2\ c\ d]\ [a^2]+2[a\ b^2\ d]\ [a\ c]+2[a\ b^2\ c]\ [a\ d]$

$+[a^2\ b^2]\ [c\ d]+4[a\ b\ c\ d]\ [a\ b]+2[a^2\ b\ d]\ [b\ c]+2[a^2\ b\ c]\ [b\ d]$

$+[a^2\ c\ d]\ [b^2]-6[c\ d]\ [a\ b]^2-12[a\ d]\ [a\ b]\ [b\ c]-12[a\ c]\ [a\ b]\ [b\ d]$

$-6[a^2]\ [b\ c]\ [b\ d]-6[a\ c]\ [a\ d]\ [b^2]-3[a^2]\ [c\ d]\ [b^2]$

Continued Table B.1

$(a^2bcde)_4 =$

2[a c e] [a b d]+2[a b c] [a d e]+[a^2 e] [b c d]+[a^2 b] [c d e]

+2[a c d] [a b e]+[a^2 d] [b c e]+[a^2 c] [b d e]+[b c d e] [a^2]

+2[a b d e] [a c]+2[a b c e] [a d]+[a^2 b e] [c d]+2[a c d e] [a b]

+2[a b c d] [a e]+[a^2 d e] [b c]+[a^2 b d] [c e]+[a^2 c e] [b d]

+[a^2 b c] [d e]+[a^2 c d] [b e]-6[c d] [a b] [a e]-6[a d] [a e] [b c]

-6[a d] [a b] [c e]-6[a c] [a e] [b d]-6[a c] [a b] [d e]-3[a^2] [c e] [b d]

-3[a^2] [b c] [d e]-6[a c] [a d] [b e]-3[a^2] [c d] [b e]

$(abcdef)_4 =$

[c e f] [a b d]+[a c e] [b d f]+[b c f] [a d e]+[a b c] [d e f]

+[a e f] [b c d]+[a b f] [c d e]+[c d f] [a b e]+[a c d] [b e f]

+[a d f] [b c e]+[a c f] [b d e]+[b c d e] [a f]+[b d e f] [a c]

+[a b d e] [c f]+[b c e f] [a d]+[a b c e] [d f]+[a b e f] [c d]

+[c d e f] [a b]+[a c d e] [b f]+[b c d f] [a e]+[a b c d] [e f]

+[a d e f] [b c]+[a b d f] [c e]+[a c e f] [b d]+[a b c f] [d e]

+[a c d f] [b e]-3[c d] [b f] [a e]-3[c d] [a b] [e f]-3[d f] [a e] [b c]

-3[a d] [e f] [b c]-3[d f] [a b] [c e]-3[a d] [b f] [c e]-3[c f] [a e] [b d]

-3[a c] [e f] [b d]-3[c f] [a b] [d e]-3[a c] [b f] [d e]-3[a f] [c e] [b d]

-3[a f] [b c] [d e]-3[c f] [a d] [b e]-3[a c] [d f] [b e]-3[a f] [c d] [b e]

Continued Table B.1

$(a^6)_5 =$

$[a^6]-15[a^4]\ [a^2]-10[a^3]^2+30[a^2]^3$

$(a^5 b)_5 =$

$[a^5\ b]-10[a^3\ b]\ [a^2]-5[a^4]\ [a\ b]-10[a^3]\ [a^2\ b]+30[a^2]^2\ [a\ b]$

$(a^4 b^2)_5 =$

$[a^4\ b^2]-6[a^2\ b^2]\ [a^2]-8[a^3\ b]\ [a\ b]-[a^4]\ [b^2]-6[a^2\ b]^2-4[a^3]\ [a\ b^2]$
$+24[a^2]\ [a\ b]^2+6[a^2]^2\ [b^2]$

$(a^4 bc)_5 =$

$[a^4\ b\ c]-6[a^2\ b\ c]\ [a^2]-4[a^3\ c]\ [a\ b]-4[a^3\ b]\ [a\ c]-[a^4]\ [b\ c]$
$-6[a^2\ b]\ [a^2\ c]-4[a^3]\ [a\ b\ c]+24[a^2]\ [a\ b]\ [a\ c]+6[a^2]^2\ [b\ c]$

$(a^3 b^3)_5 =$

$[a^3\ b^3]-3[a\ b^3]\ [a^2]-9[a^2\ b^2]\ [a\ b]-3[a^3\ b]\ [b^2]-9[a^2\ b]\ [a\ b^2]$
$-[a^3]\ [b^3]+12[a\ b]^3+18[a^2]\ [a\ b]\ [b^2]$

$(a^3 b^2 c)_5 =$

$[a^3\ b^2\ c]-3[a\ b^2\ c]\ [a^2]-3[a^2\ b^2]\ [a\ c]-6[a^2\ b\ c]\ [a\ b]-2[a^3\ b]\ [b\ c]$
$-[a^3\ c]\ [b^2]-6[a^2\ b]\ [a\ b\ c]-3[a^2\ c]\ [a\ b^2]-[a^3]\ [b^2\ c]+12[a\ c]\ [a\ b]^2$
$+12[a^2]\ [a\ b]\ [b\ c]+6[a^2]\ [a\ c]\ [b^2]$

Continued Table B.1

$(a^3bcd)_5 =$

$[a^3\ b\ c\ d]-3[a\ b\ c\ d]\ [a^2]-3[a^2\ b\ d]\ [a\ c]-3[a^2\ c\ d]\ [a\ b]-3[a^2\ b\ c]\ [a\ d]$
$-[a^3\ d]\ [b\ c]-[a^3\ b]\ [c\ d]-[a^3\ c]\ [b\ d]-3[a^2\ d]\ [a\ b\ c]-3[a^2\ b]\ [a\ c\ d]$
$-3[a^2\ c]\ [a\ b\ d]-[a^3]\ [b\ c\ d]+12[a\ c]\ [a\ b]\ [a\ d]+6[a^2]\ [a\ d]\ [b\ c]$
$+6[a^2]\ [a\ b]\ [c\ d]+6[a^2]\ [a\ c]\ [b\ d]$

$(a^2b^2c^2)_5 =$

$[a^2\ b^2\ c^2]-[b^2\ c^2]\ [a^2]-4[a\ b^2\ c]\ [a\ c]-[a^2\ b^2]\ [c^2]-4[a\ b\ c^2]\ [a\ b]$
$-4[a^2\ b\ c]\ [b\ c]-[a^2\ c^2]\ [b^2]-4[a\ b\ c]^2-2[a^2\ b]\ [b\ c^2]-2[a\ c^2]\ [a\ b^2]$
$-2[a^2\ c]\ [b^2\ c]+4[c^2]\ [a\ b]^2+16[a\ c]\ [a\ b]\ [b\ c]+4[a^2]\ [b\ c]^2$
$+4[a\ c]^2\ [b^2]+2[a^2]\ [c^2]\ [b^2]$

$(a^2b^2cd)_5 =$

$[a^2\ b^2\ c\ d]-[b^2\ c\ d]\ [a^2]-2[a\ b^2\ d]\ [a\ c]-2[a\ b^2\ c]\ [a\ d]-[a^2\ b^2]\ [c\ d]$
$-4[a\ b\ c\ d]\ [a\ b]-2[a^2\ b\ d]\ [b\ c]-2[a^2\ b\ c]\ [b\ d]-[a^2\ c\ d]\ [b^2]$
$-4[a\ b\ c]\ [a\ b\ d]-2[a^2\ b]\ [b\ c\ d]-2[a\ c\ d]\ [a\ b^2]-[a^2\ d]\ [b^2\ c]$
$-[a^2\ c]\ [b^2\ d]+4[c\ d]\ [a\ b]^2+8[a\ d]\ [a\ b]\ [b\ c]+8[a\ c]\ [a\ b]\ [b\ d]$
$+4[a^2]\ [b\ c]\ [b\ d]+4[a\ c]\ [a\ d]\ [b^2]+2[a^2]\ [c\ d]\ [b^2]$

Continued Table B.1

$(a^2bcde)_5 =$

$[a^2\ b\ c\ d\ e]-[b\ c\ d\ e]\ [a^2]-2[a\ b\ d\ e]\ [a\ c]-2[a\ b\ c\ e]\ [a\ d]$

$-[a^2\ b\ e]\ [c\ d]-2[a\ c\ d\ e]\ [a\ b]-2[a\ b\ c\ d]\ [a\ e]-[a^2\ d\ e]\ [b\ c]$

$-[a^2\ b\ d]\ [c\ e]-[a^2\ c\ e]\ [b\ d]-[a^2\ b\ c]\ [d\ e]-[a^2\ c\ d]\ [b\ e]$

$-2[a\ c\ e]\ [a\ b\ d]-2[a\ b\ c]\ [a\ d\ e]-[a^2\ e]\ [b\ c\ d]-[a^2\ b]\ [c\ d\ e]$

$-2[a\ c\ d]\ [a\ b\ e]-[a^2\ d]\ [b\ c\ e]-[a^2\ c]\ [b\ d\ e]+4[c\ d]\ [a\ b]\ [a\ e]$

$+4[a\ d]\ [a\ e]\ [b\ c]+4[a\ d]\ [a\ b]\ [c\ e]+4[a\ c]\ [a\ e]\ [b\ d]+4[a\ c]\ [a\ b]\ [d\ e]$

$+2[a^2]\ [c\ e]\ [b\ d]+2[a^2]\ [b\ c]\ [d\ e]+4[a\ c]\ [a\ d]\ [b\ e]+2[a^2]\ [c\ d]\ [b\ e]$

$(abcdef)_5 =$

$[a\ b\ c\ d\ e\ f]-[b\ c\ d\ e]\ [a\ f]-[b\ d\ e\ f]\ [a\ c]-[a\ b\ d\ e]\ [c\ f]$

$-[b\ c\ e\ f]\ [a\ d]-[a\ b\ c\ e]\ [d\ f]-[a\ b\ e\ f]\ [c\ d]-[c\ d\ e\ f]\ [a\ b]$

$-[a\ c\ d\ e]\ [b\ f]-[b\ c\ d\ f]\ [a\ e]-[a\ b\ c\ d]\ [e\ f]-[a\ d\ e\ f]\ [b\ c]$

$-[a\ b\ d\ f]\ [c\ e]-[a\ c\ e\ f]\ [b\ d]-[a\ b\ c\ f]\ [d\ e]-[a\ c\ d\ f]\ [b\ e]$

$-[c\ e\ f]\ [a\ b\ d]-[a\ c\ e]\ [b\ d\ f]-[b\ c\ f]\ [a\ d\ e]-[a\ b\ c]\ [d\ e\ f]$

$-[a\ e\ f]\ [b\ c\ d]-[a\ b\ f]\ [c\ d\ e]-[c\ d\ f]\ [a\ b\ e]-[a\ c\ d]\ [b\ e\ f]$

$-[a\ d\ f]\ [b\ c\ e]-[a\ c\ f]\ [b\ d\ e]+2[c\ d]\ [b\ f]\ [a\ e]+2[c\ d]\ [a\ b]\ [e\ f]$

$+2[d\ f]\ [a\ e]\ [b\ c]+2[a\ d]\ [e\ f]\ [b\ c]+2[d\ f]\ [a\ b]\ [c\ e]+2[a\ d]\ [b\ f]\ [c\ e]$

$+2[c\ f]\ [a\ e]\ [b\ d]+2[a\ c]\ [e\ f]\ [b\ d]+2[c\ f]\ [a\ b]\ [d\ e]+2[a\ c]\ [b\ f]\ [d\ e]$

$+2[a\ f]\ [c\ e]\ [b\ d]+2[a\ f]\ [b\ c]\ [d\ e]+2[c\ f]\ [a\ d]\ [b\ e]+2[a\ c]\ [d\ f]\ [b\ e]$

$+2[a\ f]\ [c\ d]\ [b\ e]$

Table B.2

Moments of $e_j = x_j - p_j$ When Sample Size is One

$\mathrm{Prob}(x_j=1) = p_j$ and $\mathrm{Prob}(x_j=0) = 1-p_j$

$$E\, e_j = 0$$

$$E\, e_j^2 = p_j q_j$$

$$E\, e_j e_k = -p_j p_k$$

$$E\, e_j^3 = p_j q_j (1-2p_j)$$

$$E\, e_j^2 e_k = -p_j p_k (1-2p_j)$$

$$E\, e_j e_k e_l = 2p_j p_k p_l$$

$$E\, e_j^4 = p_j q_j (1-3p_j q_j)$$

$$E\, e_j^3 e_k = -p_j p_k (1-3p_j+3p_j^2)$$

$$E\, e_j^2 e_k^2 = p_j p_k (p_j+p_k-3p_j p_k)$$

$$E\, e_j^2 e_k e_l = p_j p_k p_l (1-3p_j)$$

$$E\, e_j e_k e_l e_m = -3p_j p_k p_l p_m$$

$$E\, e_j^5 = p_j q_j (1-2p_j)(1-2p_j q_j)$$

$$E\, e_j^4 e_k = -p_j p_k (1-2p_j)(1-2p_j q_j)$$

$$E\, e_j^3 e_k^2 = p_j p_k (p_k-3p_j p_k+4p_j^2 p_k-p_j^2)$$

$$E\, e_j^3 e_k e_l = p_j p_k p_l (1-3p_j q_j+p_j^2)$$

$$E\, e_j^2 e_k^2 e_l = -p_j p_k p_l (p_j+p_k-4p_j p_k)$$

$$E\, e_j^2 e_k e_l e_m = -p_j p_k p_l p_m (1-4p_j)$$

Continued Table B.2

$E\ e_j e_k e_l e_m e_n = 4p_j p_k p_l p_m p_n$

$E\ e_j^6 = p_j q_j(1-5p_j q_j+5p_j^2 q_j^2)$

$E\ e_j^5 e_k = -p_j p_k(1-5p_j q_j+5p_j^2 q_j^2)$

$E\ e_j^4 e_k^2 = p_j p_k(p_j^3 q_k+p_k q_j^4-p_j^4 p_k)$

$E\ e_j^4 e_k e_l = p_j p_k p_l(1-4p_j+6p_j^2-5p_j^3)$

$E\ e_j^3 e_k^3 = -p_j p_k(p_j^2+p_k^2-3p_j^2 p_k-3p_j p_k^2+5p_j^2 p_k^2)$

$E\ e_j^3 e_k^2 e_l = -p_j p_k p_l(p_k-3p_j p_k q_j-p_j^2+2p_j^2 p_k)$

$E\ e_j^3 e_k e_l e_m = -p_j p_k p_l p_m(1-3p_j q_j+2p_j^2)$

$E\ e_j^2 e_k e_l e_m e_n = p_j p_k p_l p_m p_n(1-5p_j)$

$E\ e_j^2 e_k^2 e_l^2 = p_j p_k p_l(p_j p_k+p_j p_l+p_k p_l-5p_j p_k p_l)$

$E\ e_j e_k e_l e_m e_n e_o = -5p_j p_k p_l p_m p_n p_o$

Table B.3

Multinomial Central Moments

μ_r^s is the Coefficient of N^s in the Central Moment of μ_r.

This table is calculated by substituting the entries of Table B.2 into Table B.1.

$\mu_{10} = 0$

$\mu_{20}^1 = p_j q_j$

$\mu_{11}^1 = -p_j p_k$

$\mu_{30}^1 = p_j q_j (1-2p_j)$

$\mu_{21}^1 = -p_j p_k (1-2p_j)$

$\mu_{111}^1 = 2p_j p_k p_1$

$\mu_{40}^2 = 3p_j^2 q_j^2$

$\mu_{40}^1 = p_j q_j (1-6p_j q_j)$

$\mu_{31}^2 = -3p_j^2 p_k q_j$

$\mu_{31}^1 = -p_j p_k (1-6p_j+6p_j^2\ )$

Continued Table B.3

$$\mu^2_{22} = p_j p_k(1-p_j-p_k+3p_j p_k)$$

$$\mu^1_{22} = -p_j p_k(1-2p_j-2p_k+6p_j p_k)$$

$$\mu^2_{211} = -p_j p_k p_l(1-3p_j)$$

$$\mu^1_{211} = 2p_j p_k p_l(1-3p_j)$$

$$\mu^2_{1111} = 3p_j p_k p_l p_m$$

$$\mu^1_{1111} = 6p_j p_k p_l p_m$$

$$\mu^2_{50} = 10p_j^2 q_j^2(1-2p_j)$$

$$\mu^1_{50} = p_j q_j(1-2p_j)\ (1-12p_j q_j)$$

$$\mu^2_{41} = -10p_j^2 p_k q_j(1-2p_j)$$

$$\mu^1_{41} = -p_j p_k(1-2p_j)\ (1-12p_j q_j)$$

$$\mu^2_{32} = p_j p_k(1-6p_j-p_k+15p_j p_k+5p_j^2-20p_j^2 p_k)$$

$$\mu^1_{32} = -p_j p_k(1-6p_j-2p_k+18p_j p_k+6p_j^2-24p_j^2 p_k)$$

Continued Table B.3

$$\mu^2_{311} = -p_j p_k p_l q_j (1-8p_j)$$

$$\mu^1_{311} = 2p_j p_k p_l (1-9p_j+12p_j^2)$$

$$\mu^2_{221} = -p_j p_k p_l (2-5p_j-5p_k+20p_j p_k)$$

$$\mu^1_{221} = 2p_j p_k p_l (1-3p_j-3p_k+12p_j p_k)$$

$$\mu^2_{2111} = 5p_j p_k p_l p_m (1-4p_j)$$

$$\mu^1_{2111} = -4p_j p_k p_l p_m (1-5p_j)$$

$$\mu^2_{11111} = -20p_j p_k p_l p_m p_n$$

$$\mu^1_{11111} = 24p_j p_k p_l p_m p_n$$

$$\mu^3_{60} = 15p_j^3 q_j^3$$

$$\mu^2_{60} = 5p_j^2 q_j^2 (5-26p_j+26p_j^2)$$

$$\mu^1_{60} = p_j q_j (1-30p_j+230p_j^2-240p_j^3+40p_j^4)$$

$$\mu^3_{51} = -15p_j^3 p_k q_j^2$$

$$\mu^2_{51} = -5p_j^2 p_k q_j (5-26p_j+26p_j^2)$$

Continued Table B.3

$$\mu^1_{51} = -p_j p_k(1-30p_j q_j+50p_j^2 q_j^2+70p_j^2 q_j-70p_j^3 q_j)$$

$$\mu^3_{42} = 3p_j^2 p_k q_j(1-p_j-p_k+5p_j p_k)$$

$$\mu^2_{42} = p_j p_k(1-17p_j-p_k+41p_j p_k+42p_j^2-156p_j^2 p_k-26p_j^3+130p_j^3 p_k)$$

$$\mu^1_{42} = -p_j p_k(1-14p_j-2p_k+42p_j p_k-144p_j^2 p_k+120p_j^3 p_k+36p_j^2-24p_j^3)$$

$$\mu^3_{411} = -3p_j^2 p_k p_1 q_j(1-5p_j)$$

$$\mu^2_{411} = -p_j p_k p_1(1-41p_j+150p_j^2-112p_j^3)$$

$$\mu^1_{411} = 2p_j p_k p_1(1-21p_j+77p_j^2-65p_j^3)$$

$$\mu^3_{33} = -3p_j^2 p_k^2(3-3p_j-3p_k+5p_j p_k)$$

$$\mu^2_{33} = p_j p_k(1-6p_j-6p_k+5p_j^2+5p_k^2+63p_j p_k-78p_j^2 p_k-78p_j p_k^2+130p_j^2 p_k^2)$$

$$\mu^1_{33} = -p_j p_k(1-6p_j-6p_k+6p_j^2+6p_k^2+54p_j p_k-72p_j^2 p_k-72p_j p_k^2+120p_j^2 p_k^2)$$

$$\mu^3_{321} = 3p_j^2 p_k p_1(1-p_j+30p_k-3p_j p_k)$$

$$\mu^2_{321} = p_j p_k p_1(2-21p_j-5p_k+78p_j p_k-130p_j^2 p_k+26p_j^2)$$

$$\mu^1_{321} = 2p_j p_k p_1(1-9p_j-3p_k+36p_j p_k-36p_j^2 p_k+12p_j^2)$$

Continued Table B.3

$$\mu^3_{3111} = 3p_j^2 p_k p_l p_m (3-5p_j)$$

$$\mu^2_{3111} = p_j p_k p_l p_m (5-78p_j+130p_j^2)$$

$$\mu^1_{3111} = -6p_j p_k p_l p_m (1-12p_j+22p_j^2)$$

$$\mu^3_{222} = p_j p_k p_l (1-p_j-p_k-p_l+3p_j p_k+3p_j p_l+3p_k p_l-15p_j p_k p_l)$$

$$\mu^2_{222} = -p_j p_k p_l (3-7p_j-7p_k-7p_l+26p_j p_k+26p_j p_l+26p_k p_l-130p_j p_k p_l)$$

$$\mu^1_{222} = 2p_j p_k p_l (1-3p_j-3p_k-3p_l+12p_j p_k+12p_j p_l+12p_k p_l-52p_j p_k p_l)$$

$$\mu^3_{2211} = -p_j p_k p_l p_m (1-3p_j-3p_k+15p_j p_k)$$

$$\mu^2_{2211} = p_j p_k p_l p_m (7-26p_j-26p_k+130p_j p_k)$$

$$\mu^1_{2211} = -p_j p_k p_l p_m (6-26p_j-26p_k+114p_j p_k)$$

$$\mu^3_{21111} = 3p_j p_k p_l p_m p_n (1-13p_j)$$

$$\mu^2_{21111} = -p_j p_k p_l p_m p_n (26-123p_j)$$

$$\mu^1_{21111} = 24p_j p_k p_l p_m p_n (1-5p_j)$$

$$\mu^3_{111111} = -15p_j p_k p_l p_m p_n p_o$$

Continued Table B.3

$$\mu^{2}_{111111} = 130 p_j p_k p_l p_m p_n p_o$$

$$\mu^{1}_{111111} = -120\ p_j p_k p_l p_m p_n p_o$$

Table B.4

BINOMIAL CENTRAL MOMENTS

μ_r^s is the coefficient of n^s in the central moment of μ_r.

$$\mu_2^1 = pq$$

$$\mu_3^2 = pq(q-p)$$

$$\mu_4^2 = 3p^2q^2$$

$$\mu_4^1 = pq(q^2-4qp+p^2)$$

$$\mu_5^2 = 10p^2q^2(q-p)$$

$$\mu_5^1 = pq(q-p)(q^2-10qp+p^2)$$

$$\mu_6^3 = 15p^3q^3$$

$$\mu_6^2 = 5p^2q^2(5q^2-16qp+5p^2)$$

$$\mu_6^1 = pq(q^4-26q^3p+66q^2p^2-26qp^3+p^4)$$

$$\mu_7^3 = 105q^3p^3(q-p)$$

$$\mu_7^2 = 14p^2q^2(q-p)(4q^2-25qp+4p^2)$$

$$\mu_7^1 = pq(q-p)(q^4-56q^3p+246q^2p^2-56qp^3+p^4)$$

$$\mu_8^4 = 105p^4q^4$$

$$\mu_8^3 = 70p^3q^3(7q^2-20qp+7p^2)$$

$$\mu_8^2 = 7p^2q^2(17q^4-240q^3p+530q^2p^2-240qp^3+17p^4)$$

Continued Table B.4

$\mu_8^1 = pq(q^6-120q^5p+1191q^4p^2-2416q^3p^3+1191q^2p^4-120qp^5+p^6)$

$\mu_9^4 = 1260p^4q^4(q-p)$

$\mu_9^3 = 14p^3q^3(q-p)(137q^2-670qp+137p^2)$

$\mu_9^2 = 6p^2q^2(q-p)(41q^4-994q^3p+328q^2p^2-994qp^3+41p^4)$

$\mu_9^1 = qp(q-p)(q^6-246q^5p-4047q^4p^2-11572q^3p^3+4047q^2p^4-246qp^5+p^6)$

$\mu_{10}^5 = 945p^5q^5$

$\mu_{10}^4 = 3150p^4q^4(3q^2-8qp+3p^2)$

$\mu_{10}^3 = 105p^3q^3(65q^4-684q^3p+1394q^2p^2-684qp^3+65p^4)$

$\mu_{10}^2 = 3p^2q^2(167q^6-7144q^5p+50201q^4p^2-92768q^3p^3+50201q^2p^4-7144qp^5+167p^6)$

$\mu_{10}^1 = pq(q^8-502q^7p+14608q^6p^2-88234q^5p^3+156190q^4p^4-88234q^3p^5+14608q^2p^6-502qp^7+p^8)$

$\mu_{11}^5 = 17325p^5q^5(q-p)$

$\mu_{11}^4 = 1540p^4q^4(q-p)(37q^2-155qp+37p^2)$

$\mu_{11}^3 = 55p^3q^3(q-p)(417q^4-6804q^3p+19002q^2p^2-6804qp^3+417p^4)$

$\mu_{11}^2 = p^2q^2(q-p)(1012q^6-69762q^5p+725604q^4p^2-1725724q^3p^3+725604q^2p^4-69762qp^5+1012p^6)$

$\mu_{11}^1 = pq(q-p)(q^8-1012q^7p+46828q^6p^2-408364q^5p^3+901990q^4p^4-408364q^3p^5+46828q^2p^6-1012qp^7+p^8)$

Continued Table B.4

$$\mu_{12}^{6} = 10395p^{6}q^{6}$$

$$\mu_{12}^{5} = 17325q^{5}p^{5}(11q^{2}-28qp+11p^{2})$$

$$\mu_{12}^{4} = 385q^{4}p^{4}(787q^{4}-6928q^{3}p+13398q^{2}p^{2}-6928qp^{3}+787p^{4})$$

$$\mu_{12}^{3} = p^{3}q^{3}(74316q^{6}-2007852q^{5}p+11533368q^{4}p^{2}-20093304q^{3}p^{3}+11533368q^{2}p^{4}-2007852qp^{5}+74316p^{6})$$

$$\mu_{12}^{2} = p^{2}q^{2}(2035q^{8}-226952q^{7}p+3908344q^{6}p^{2}-18794776q^{5}p^{3}+31134026q^{4}p^{4}-18794776q^{3}p^{5}+3908344q^{2}p^{6}-226952qp^{7}+2035p^{8})$$

$$\mu_{12}^{1} = pq(q^{10}-2036q^{9}p+152637q^{8}p^{2}-2203488q^{7}p^{3}+9738114q^{6}p^{4}-15724248q^{5}p^{5}+9738114q^{4}p^{6}-2203488q^{3}p^{7}+152637q^{2}p^{8}-2036qp^{9}+p^{10})$$

$$\mu_{13}^{6} = 270270p^{6}q^{6}(q-p)$$

$$\mu_{13}^{5} = 5005p^{5}q^{5}(q-p)(327q^{2}-1230q+327p^{2})$$

$$\mu_{13}^{4} = 715p^{4}q^{4}(q-p)(2080q^{4}-26628q^{3}p+66592q^{2}p^{2}-26629qp^{3}+2080p^{4})$$

$$\mu_{13}^{3} = p^{3}q^{3}(q-p)(235092q^{6}-9314448q^{5}p+73959600q^{4}p^{2}-15674736q^{3}p^{3}+73959600q^{2}p^{4}-9314448qp^{5}+235092p^{6})$$

$$\mu_{13}^{2} = p^{2}q^{2}(q-p)(4082q^{8}-709280q^{7}p+17540744q^{6}p^{2}-113162192q^{5}p^{3}+220929644q^{4}p^{4}-113162192q^{3}p^{5}+17540744q^{2}p^{6}-709280qp^{7}+4082p^{8})$$

$$\mu_{13}^{1} = pq(q-p)(q^{10}-4082q^{9}p+474189q^{8}p^{2}-9713496q^{7}p^{3}+56604978q^{6}p^{4}-105907308q^{5}p^{5}+56604978q^{4}p^{6}-9713496q^{3}p^{7}+474189q^{2}p^{8}-4082qp^{9}+p^{10})$$

$$\mu_{14}^{7} = 135135p^{7}q^{7}$$

$$\mu_{14}^{6} = 315315p^{6}q^{6}(13q^{2}-32qp+13p^{2})$$

Continued Table B.4

$$\mu^5_{14} = 35035p^5q^5(346q^4-2698q^3p+5028q^2p^2-2698qp^3+346p^4)$$

$$\mu^4_{14} = p^4q^4(6914908q^6-142117976q^5p+71405744q^4p^2-119434515q^3p^3+ 714057344q^2p^4-14211797qp^5+6914908p^6)$$

$$\mu^3_{14} = p^3q^3(731731q^8-43499456q^7p+553124572q^6p^2-2294780488q^5p^3+ 3634673042q^4p^4-2294780488q^3p^5+553124572q^2p^6-43499456qp^7+731731p^8)$$

$$\mu^2_{14} = q^2p^2(8177q^{10}-2211456q^9p+82117997q^8p^2-846410240q^7p^3+3176769986q^6p^4- 4881700928q^5p^5+3176769986q^4p^6-846410240q^3p^7+82117997q^2p^8- 2211456qp^9+8177p^{10})$$

$$\mu^1_{14} = pq(q^{12}-8178q^{11}p+1479726q^{10}p^2-45533450q^9p^3+423281535q^8p^4- 1505621508q^7p^5+2275172004q^6p^6-1505621508q^5p^7+423281535q^4p^8- 45533450q^3p^9+1479726q^2p^{10}-8178qp^{11}+p^{12})$$

Appendix C

The three programs given here are all written in Conversational Programming System. They are used on the remote terminals on a time sharing technique, the computer being the IBM 360 Model 65.

Program C.1

This program gives the diversity as measured by $\bar{H}$ and the diversity as measured by (s-1)/log N. The evenness or the normed $\bar{H}$ is given as H-hat. Also, the Lloyd and Ghelardi [22] measure of equitability is given as S'/S. The program is activated by giving the number of species, 'sp' and then the number of entries, 'qq'. The entries are then entered. Each entry is of the form; 'n1,n2' where n1 is the number of species that has n2 individuals.

Program C.2

This program gives the exact mean of a sample. The program is activated by giving the number of species 'sp', the sample size 'n', and the probabilities associated with each species.

Program C.3

This program gives the exact variance of a sample. The program is activated by giving the number of species 'S', the sample size 'N', and the probabilities associated with each species.

Program C.1

```
            DECLARE z(300);
start:      BW=0;
            GET LIST(sp);
            n=sp;
            q=0;
            p=1;
            GET LIST(qq);
lw:         DO j=1 TO qq;
              GET LIST(n1,n2);
              BW=BW+n1;
              q=q+n1;
lx:           DO i=p TO q;
                z(i)=n2;
                END lx;
              p=q+1;
              END lw;
            sig=0;
            y=0;
            kon=1;
            IF BW¬=n THEN GET LIST(WRONGn);
loop:       DO i=1 TO n;
              x=z(i);
              y=y+x;
              sig=sig+x*log(x)*kon;
              END loop;
            hbar=log(y)*kon-sig/y;
            hhat=hbar/(log(n)*kon);
            sample=y;
            specie=n;
h1:         IMAGE;
hbar=--.----- hhat=--.----- sample=-----. specie=---.;
            PUT IMAGE(hbar,hhat,sample,specie)(h1);
            r=(n-1)/log(y);
Li:         DO k=1 TO 200;
              IF M(k)<=hbar&hbar<=M(k+1) THEN GO TO RA;
              END Li;
RA:         w=(hbar-M(k))/(M(k+1)-M(k));
            k=k+w;
            x1=k/n;
H1:         IMAGE;
s_1/logN=----.----- S'=---.----- S=----.- S'/S=--.-----;
            PUT IMAGE(r,k,n,x1)(H1);
            GO TO start;
```

Continued Program C.1

```
      kk:      DO s=1 TO 200;
                  DECLARE R(0:300);
                  DECLARE M(200);
      l1:         DO i=1 TO s;
                     P=0;
      l2:            DO j=1 TO i;
                        P=P+1/(s-j+1);
                        END l2;
                     R(i)=1/s*P;
                     END l1;
                  R(0)=0;
      l3:         DO k=1 TO s;
                     R(k)=R(k-1)-R(k)*log(R(k));
                     END l3;
                  M(s)=R(s);
                  END kk;
               GO TO start;
         6121 END OF SEGMENT LISTING
```

Program C.2

```
start:  GET LIST(sp);
        DECLARE Vs(100);
        GET LIST(n);
        DECLARE P(2);
k1:     GET LIST(P);
        DECLARE V(0:500);
        K=0;
lm:     DO i=1 TO sp;
          K=K+P(i)**n;
          END lm;
        V(0)=1;
        X=log(n)*K;
LOOP:   DO s=1 TO n-1;
          K=0;
          L=s;
L2:       DO i=1 TO sp;
            K=K+P(i)**(n-L)*(1-P(i))**L;
            END L2;
          V(s)=(n-s)*V(s-1)/s;
          Vs(s)=K*V(s);
          END LOOP;
        DECLARE f(0:500);
        f(0)=log(n);
MOOP:   DO t=1 TO n-1;
          f(t)=Vs(t)*log(n-t);
          X=X+f(t);
          END MOOP;
        Y=log(n)-X;
        mean=Y;
j1:     IMAGE;
P  -.--- -.--- -.--- -.--- -.--- -.---;
j2:     IMAGE;
mean=---.------;
        PUT IMAGE(P)(j1);
        PUT IMAGE(mean)(j2);
        GO TO k1;
   6121 END OF SEGMENT LISTING
```

Program C.3

```
        DECLARE V(0:500),W(0:500),X(0:500);
        DECLARE P(3),Q(3);
        X=0;
        GET LIST(N,S);
L:      DO ik=1 TO S;
          GET LIST(P(ik));
          Q(ik)=1-P(ik);
          END L;
        NN=N-1;
        N2=N-1;
        kk=1;
        GO TO nbx;
nbx1:   V=X;
        NN=N-2;
        N2=N-2;
        kk=2;
        GO TO nbx;
nbx2:   W=X;
        var1=0;
        var2=0;
L1:     DO a=0 TO N-2;
L2:       DO i=1 TO S;
L3:         DO b=a+1 TO N-1;
L4:           DO j=1 TO S;
                B=b;
                x1=V(a)*P(i)**(N-a)*Q(i)**a;
                x2=x1*V(b)*P(j)**(N-b)*Q(j)**B;
                x3=x2*log((N-a)/(N-b))**2;
                var1=var1+x3;
                END L4;
              END L3;
            END L2;
          END L1;
        b=-1;
L8:     b=b+1;
        NN=N-2-b;
        N2=trunc((N-2-b)/2);
        kk=3;
        GO TO nbx;
L5:     DO a1=0 TO trunc((N-2-b)/2);
```

Continued Program C.3

```
     L6:      DO ii=1 TO S;
     L7:        DO jj=1 TO S;
                  IF ii=jj THEN GO TO gg;
                  IF b=0 THEN GO TO g;
                  y1=W(b)*X(a1)*P(ii)**(N-a1-1-b);
                  y2=y1*P(jj)**(a1+1);
                  y22=y2*(1-P(ii)-P(jj))**b;
                  y3=y22*log((N-a1-b-1)/(a1+1))**2;
                  var2=var2+y3;
                  GO TO gg;
     g:           z1=W(b)*X(a1)*P(ii)**(N-a1-1-b);
                  z2=z1*P(jj)**(a1+1);
                  z3=z2*log((N-a1-b-1)/(a1+1))**2;
                  var2=var2+z3;
     gg:          ;
                  END L7;
                END L6;
              END L5;
              IF b<N-3 THEN GO TO L8;
              var2=(N-1)/N*var2;
              varH=var1-var2;
     h:       IMAGE;
var1=------.---------- var2=------.----------;
     h1:      IMAGE;
varH=------.---------- N=----- S=---;
              PUT IMAGE(var1,var2)(h);
              PUT IMAGE(varH,N,S)(h1);
              GO TO L;
     nbx:     X(0)=1;
     LOOP:    DO l=1 TO N2;
                X(l)=(NN-l+1)*X(l-1)/l;
                END LOOP;
              IF kk=1 THEN GO TO nbx1;
              IF kk=2 THEN GO TO nbx2;
              IF kk=3 THEN GO TO L5;
              STOP ;
         6121 END OF SEGMENT LISTING
```

Appendix D

Higher Moments of $\bar{H}$ by Monte-Carlo Simulation

Let us draw a random sample of n from categorized data and compute the moments. We will use MacArthur's Model [23] to categorize the sample. Lloyd and Ghelardi [22] state that this model apportions individuals among the species in about as equitable a fashion as ever occurs in nature. MacArthur's formula is

$$p_{i} = 1/s \sum_{i=1}^{r} 1/(s-i+1) \ ,$$

where p_i is the theoretical proportion of individuals in the r most abundant species (r = 1, 2, ... s), each theoretical proportion itself being arrived at by summing over i items (i = 1, 2, ... , r). For example if,

$s = 2;\ p_1 = 0.25$ and $p_2 = 0.75$

$s = 3;\ p_1 = 0.1111,\ p_2 = 0.2778$ and $p_3 = 0.6111.$

The following table is a sample of the Monte-Carlo simulation from the IBM 360/65. Eight decimal places in double precision were used.

Moments of $\overline{H}$ for MacArthur's Model

by

Monte-Carlo Method

5 Categories; Sample of 20; 15,000 Cycles

Table D.1

1.2336	0.0261	-0.0023	0.0023	-0.5541	3.3339
1.2345	0.0259	-0.0023	0.0023	-0.5413	3.3896
1.2312	0.0262	-0.0022	0.0022	-0.5207	3.2207
1.2315	0.0259	-0.0021	0.0021	-0.4962	3.1740
1.2305	0.0263	-0.0021	0.0022	-0.4982	3.1736
1.2349	0.0257	-0.0021	0.0021	-0.5117	3.2139
1.2346	0.0254	-0.0022	0.0021	-0.5512	3.2655
1.2319	0.0261	-0.0023	0.0023	-0.5425	3.3930
1.2329	0.0255	-0.0021	0.0021	-0.5198	3.2455
1.2319	0.0259	-0.0023	0.0022	-0.5503	3.3635
1.2337	0.0258	-0.0022	0.0022	-0.5270	3.2842
1.2321	0.0262	-0.0024	0.0023	-0.5639	3.3527
1.2336	0.0257	-0.0021	0.0021	-0.5154	3.2461
1.2348	0.0265	-0.0023	0.0022	-0.5335	3.2092
1.2311	0.0262	-0.0023	0.0023	-0.5413	3.3050

The true mean of $\overline{H}$ for s=5 and n=20 in MacArthur's model is 1.232434 and the true variance is 0.025952 and the standard deviation is 0.161. Based on a 100 lines as in the table above we found that $\overline{\mu}_1$ = 1.232261 and the sample variance s^2 = 0.00001326 and s = 0.00364076. The interval $\overline{\mu}_1 \pm s$ contained the entire sample.

The true β_2 for the above case is 3.299. Based of the same 100 lines we found that β_2 = 3.295437 and the sample variance s^2 = 0.00604926 and s = 0.07777697. The interval $\overline{\beta}_2 \pm s$ contained exactly 67 per cent of the sample and the interval $\overline{\beta}_2 \pm 2s$ contained 96 per cent of the sample.

Thus, the Monte-Carlo simulation appears to be an adequate method of finding the moments of $\overline{H}$.

Table D.2

Monte-Carlo Simulation of the Moments
of
MacArthur's Model
(15,000 cycles)

k \ n		2	3	4	5	10	15	20	25	50
2	μ_1	0.2626	0.3606	0.4113	0.4409	0.5027	0.5262	0.5373	0.5419	0.5510
	s.d.	0.3362	0.3154	0.2853	0.2582	0.1754	0.1360	0.1143	0.0995	0.0695
	$\sqrt{\beta_1}$	0.4997	-0.2685	-0.6781	-0.9318	-1.3325	-1.2443	-1.1718	-0.9893	-0.7447
	β_2	1.2497	1.0721	1.6129	2.2598	4.4923	4.8676	4.8722	4.3085	3.8006
3		0.3759	0.5282	0.6145	0.6700	0.7871	0.8247	0.8459	0.8591	0.8799
		0.3453	0.3382	0.3135	0.2921	0.2083	0.1687	0.1413	0.1240	0.0843
		-0.1687	-0.3984	-0.5421	-0.6268	-0.8202	-0.8729	-0.7687	-0.8019	-0.5704
		1.0288	2.4165	2.9225	3.1215	3.5705	3.7160	3.4921	3.7791	3.3969
4		0.4379	0.6341	0.7466	0.8177	0.9784	1.0365	1.0637	1.0824	1.1163
		0.3343	0.3391	0.3201	0.2967	0.2219	0.1788	0.1533	0.1346	0.0914
		-0.5462	-0.5112	-0.5179	-0.5410	-0.6498	-0.5846	-0.5826	-0.5929	-0.4815
		1.2984	2.8167	3.1915	3.3034	3.5051	3.2792	3.2019	3.2886	3.2389
5		0.4809	0.7027	0.8393	0.9216	1.1187	1.1961	1.2322	1.2535	1.3015
		0.3195	0.3307	0.3125	0.3017	0.2300	0.1859	0.1624	0.1419	0.0978
		-0.8406	-0.6187	-0.5445	-0.5739	-0.5773	-0.5499	-0.5145	-0.5151	-0.4566
		1.7066	3.0560	3.3339	3.3870	3.2918	3.4185	3.2145	3.2485	3.1636
6		0.5103	0.7581	0.9007	1.0049	1.2345	1.3218	1.3688	1.3937	1.4500
		0.3054	0.3205	0.3092	0.2959	0.2307	0.1915	0.1637	0.1478	0.1016
		-1.0723	-0.7288	-0.5875	-0.5584	-0.5465	-0.4654	-0.4756	-0.4723	-0.4302
		2.1499	3.2453	3.3901	3.4502	3.4435	3.1751	3.2182	3.2327	3.2096
7		0.5314	0.7991	0.9592	1.0673	1.3238	1.4262	1.4825	1.5124	1.5803
		0.2932	0.3108	0.3036	0.2904	0.2334	0.1954	0.1689	0.1504	0.1039
		-1.2610	-0.8340	-0.6763	-0.6123	-0.4842	-0.4406	-0.4472	-0.4854	-0.4078
		2.5901	3.4146	3.5641	3.5731	3.3726	3.1936	3.1929	3.3274	3.4113

Continued Table D.2

n / k	2	3	4	5	10	15	20	25	50
8	0.5500	0.8217	1.0038	1.1154	1.4001	1.5170	1.5749	1.6120	1.6913
	0.2807	0.3034	0.2961	0.2867	0.2338	0.1954	0.1707	0.1529	0.1074
	-1.4489	-0.8969	-0.7136	-0.6232	-0.5149	-0.4482	-0.4346	-0.4141	-0.3729
	3.0992	3.5345	3.6012	3.6203	3.4828	3.3170	3.2046	3.1620	3.1672
9	0.5655	0.8488	1.0343	1.1606	1.4671	1.5947	1.6603	1.7027	1.7868
	0.2687	0.2920	0.2865	0.2816	0.2332	0.1938	0.1729	0.1541	0.1073
	-1.6399	-0.9708	-0.7525	-0.6544	-0.5003	-0.4337	-0.4247	-0.3645	-0.3635
	3.6565	3.6919	3.6808	3.6513	3.4783	3.4468	3.1873	3.1341	3.1459
10	0.5758	0.8702	1.0624	1.1923	1.5223	1.6575	1.7290	1.7774	1.8743
	0.2600	0.2819	0.2835	0.2773	0.2321	0.1962	0.1735	0.1556	0.1092
	-1.7633	-1.0323	-0.8454	-0.6824	-0.4557	-0.4171	-0.3932	-0.4082	-0.3281
	4.1094	3.8196	3.8832	3.6291	3.3384	3.2719	3.2628	3.2917	3.0936
15	0.6081	0.9398	1.1651	1.3076	1.7161	1.8916	1.9972	2.0595	2.2001
	0.2274	0.2471	0.2529	0.2509	0.2200	0.1966	0.1751	0.1585	0.1124
	-2.2995	-1.3892	-1.0238	-0.8266	-0.4495	-0.3960	-0.3902	-0.3216	-0.3336
	6.2877	4.6595	4.0867	3.9132	3.2815	3.1808	3.2358	3.1608	3.2458
20	0.6309	0.9736	1.2056	1.3777	1.8312	2.0424	2.1626	2.2443	2.4239
	0.1982	0.2251	0.2294	0.2265	0.2098	0.1889	0.1728	0.1572	0.1159
	-2.8682	-1.6698	-1.1955	-0.9323	-0.5127	-0.4086	-0.3754	-0.2842	-0.2940
	9.2263	5.5527	4.4660	3.9604	3.4340	3.2274	3.2582	3.1450	3.2098
25	0.6407	0.9990	1.2389	1.4173	1.9079	2.1390	2.2825	2.3767	2.5898
	0.1832	0.2023	0.2093	0.2133	0.1994	0.1837	0.1663	0.1541	0.1165
	-3.2108	-1.8785	-1.3621	-1.1237	-0.5640	-0.4208	-0.3400	-0.3124	-0.2778
	11.3093	6.1845	5.0663	4.5570	3.5103	3.3330	3.1985	3.1785	3.0822
50	0.6666	1.0462	1.3072	1.5059	2.0834	2.3820	2.5753	2.7088	3.0448
	0.1330	0.1511	0.1582	0.1632	0.1602	0.1548	0.1457	0.1381	0.1118
	-4.8138	-2.7755	-1.8993	-1.5811	-0.7731	-0.5169	-0.4176	-0.3639	-0.2359
	24.1722	10.3470	6.4335	5.7412	3.9254	3.3715	3.3197	3.2730	3.1308

Pielou's 'Sequential' approach to the Assessment of $\overline{H}$

To estimate the diversity ($\overline{H}$) of a population, Pielou [33] has suggested drawing cumulative samples and evaluating the successive values $\overline{H}_1$, $\overline{H}_2$, ..., $\overline{H}_r$ of the diversity, the stopping rule being based on some such rule as sampling until

$$|\overline{H}_r - \overline{H}_{r-1}| \quad \leq \theta \overline{H}_{r-1} \quad (0<\theta<1)$$

in certain sample units. Actually Pielou's method is rather more complicated than this, but we have simplified her approach with the object of gaining some little insight into what the process involves. As an illustration we have drawn sample of five from MacArthur's model with k=5, 10, 15, 25, 50, 100 and set up the usual moment parameters, (for the case =0.05) for the ultimate sample size n* and the assessed value $\overline{H}$* of $\overline{H}$. It is of considerable interest to note that the final sample size has a mean which changes from 19 to 28 as k goes from 5 to 100; moreover the s.d. of this sample size is around 6 and its distribution is nearly normal; by contrast, and as we should expect, the mean value of $\overline{H}$* increases from 1.2 to nearly 3 and the distribution of $\overline{H}$* deviates more from normality. However, the samll s.d. of $\overline{H}$* is noteworthy.

There is thus at least a suggestion here that if MacArthur's Model holds then the 'stopping'rule would not involve a large sample and it seems conceivable that one might quit with a value of $\bar{\bar{H}}^*$ much lower than the true value.

Moments of Index Diversity (H) and Sample Size (n) for MacArthur's Model under Pielou's 'sequential' approach.

Let $\bar{H}_m$ be the diversity for a sample of m, and $\bar{H}_{m+m'}$ the diversity for the cumulated sample m+m . Draw samples of m' up to the stage when

$$|\bar{H}_{m+\lambda m'} - \bar{H}_{m+\lambda m'-m'}| \leq 0.05\ \bar{H}_{m+\lambda m'-m'}$$

occurs for the first time. Let the sample size and diversity at this stage be n* and $\bar{H}$*.

Monte-Carlo Assessments of the Moments

m = m' = 5

(5,000 cycles)

Categories	Statistic	Mean	S.D.(σ)	Skewness($\sqrt{\beta_1}$)	Kurtosis(β_2)
5	$\bar{H}$*	1.2144	0.1755	-0.7568	3.9675
	n*	18.9960	7.1142	0.8147	3.7213
10	$\bar{H}$*	1.7066	0.2077	-0.7899	3.9908
	n*	19.8390	5.8471	0.5172	3.3122
15	$\bar{H}$*	1.9787	0.2205	-0.8156	4.0399
	n*	20.8060	5.6983	0.5005	3.2954
25	$\bar{H}$*	2.2978	0.2265	-0.7446	4.1393
	n*	22.3780	5.4238	0.2703	2.8004
50	$\bar{H}$*	2.6904	0.2229	-0.7094	4.0165
	n*	25.4920	5.4477	0.1115	2.7474
100	$\bar{H}$*	3.0011	0.2102	-0.7783	4.2296
	n*	28.3900	5.2353	0.0220	2.8712

(10,000 Cycles)

Categories	Statistic	Mean	S.D.(σ)	Skewness($\sqrt{\beta_1}$)	Kurtosis(β_2)
5	$\bar{H}^*$	1.2152	0.1805	-0.8681	4.2501
	n^*	18.8920	7.0109	0.7603	3.4244
10	$\bar{H}^*$	1.7079	0.2068	-0.7777	4.0440
	n^*	19.9070	5.8665	0.5375	3.2640
15	$\bar{H}^*$	1.9776	0.2156	-0.8316	4.2616
	n^*	20.6665	5.5918	0.4987	3.1894
25	$\bar{H}^*$	2.3005	0.2238	-0.7328	4.0536
	n^*	22.4015	5.4873	0.3810	2.9863
50	$\bar{H}^*$	2.6886	0.2271	-0.8112	4.4897
	n^*	25.3400	5.3856	0.1356	2.8185
100	$\bar{H}^*$	2.9998	0.2093	-0.7193	3.9724
	n^*	28.3575	5.1744	0.0572	2.9160

(50,000 Cycles)

Categories	Statistic	Mean	S.D.(σ)	Skewness($\sqrt{\beta_1}$)	Kurtosis(β_2)
5	$\bar{H}^*$	1.214	0.178	-0.809	4.138
	n^*	19.002	7.049	0.743	3.369
10	$\bar{H}^*$	1.706	0.204	-0.760	4.071
	n^*	19.887	5.838	0.578	3.483
15	$\bar{H}^*$	1.978	0.215	-0.815	4.167
	n^*	20.775	5.602	0.475	3.234
25	$\bar{H}^*$	2.301	0.224	-0.770	4.271
	n^*	22.489	5.499	0.353	2.999
50	$\bar{H}^*$	2.687	0.225	-0.768	4.210
	n^*	25.427	5.484	0.137	2.818
100	$\bar{H}^*$	2.999	0.212	-0.817	4.329
	n^*	28.323	5.228	0.030	2.911

Appendix E

An Exploration Into the Moments of $\overline{H}$

The purpose here is to give expressions for the first four moments of $\overline{H}$. Even though we can not integrate the expressions we do not conclude that they are impossible, quite the contrary.

The approach in this section is slightly different than that used in Chapter III. We feel the advantage here being that these latter expressions are more readily adaptable for machine computations.

We set out some basic results and notation to be used subsequently.

probability : $p_1\ p_2\ p_3 \ldots p_s;\ \sum_{i=1}^{s} p_i = 1$

observation: $n_1\ n_2\ n_3 \ldots n_s;\ \sum_{i=1}^{s} n_i = n$

$$E\ (n_i) = np_i$$

$$E\ (u^{n_i}) = (p_i u + q_i)^n;\ p_i + q_i = 1$$

$$E\ n_i u^{n_i - 1} = np_i (p_i u + q_i)^{n-1}$$

$$E\ n_i^2\ u^{n_i - 1} = n(n-1)p_i^2\ u(p_i u + q_i)^{n-2} + np_i(p_i u + q_i)$$

$$E\ n_i^3\ u^{n_i - 1} = np_i(p_i u + q_i)^{n-1} + 3n(n-1)p_i^2\ u(p_i u + q_i)^{n-2} +$$

$$n(n-1)(n-2)p_i^3\ u^2(p_i u + q_i)^{n-3}$$

$$E\, n_i^4\, u^{n_i-1} = np_i(p_iu + q_i)^{n-1} + 7n(n-1)p_i^2\, u(p_iu + q_i)^{n-2} +$$

$$6n(n-1)(n-2)p_i^3u^2(p_iu + q_i)^{n-3} +$$

$$n(n-1)(n-2)(n-3)p_i^4u^3(p_iu + q_i)^{n-4}$$

$$E\, u_1^{n_i}u_2^{n_j} = (p_iu_1 + p_ju_2 + 1-p_i-p_j)^n$$

$$E\, n_iu_1^{n_i-1}\, u_2^{n_j} = np_i(p_iu_1 + p_j - u_2 + 1 - p_i - p_j)^{n-1}$$

$$E\, n_in_j - u_1^{n_i-1}\, u_2^{n_j-1} = n(n-1)p_ip_j(p_iu_1 + p_ju_2 + 1-p_i-p_j)^{n-2}$$

$$E\, n_i^2\, n_j^2\, u_1^{n_i-1}\, u_2^{n_j-1} = n(n-1)p_ip_j(p_iu_1 + p_ju_2 + 1-p_i-p_j)^{n-2} +$$

$$n(n-1)(n-2)p_ip_j^2\, u_2\, (p_iu_1 + p_ju_2 + 1-p_i-p_j)^{n-3} +$$

$$n(n-1)(n-2)p_i^2p_ju_1(p_iu_1 + p_ju_2 + 1-p_i-p_j)^{n-3} +$$

$$n(n-1)(n-2)(n-3)p_i^2p_j^2\, u_1u_2(p_iu_1 + p_ju_2 + 1-p_i-p_j)^{n-4}$$

$$P_i(u) = p_iu + q_i$$

$$P_{i,j}(u_1,u_2) = p_iu_1 + p_ju_2 + 1-p_i-p_j$$

$\log b = {}_o\!\int^{\infty} \dfrac{e^{-x}-e^{-bx}}{x}\, dx$, set $u=e^{-x}$ in this integral and

$$\log b = \int_0^1 \frac{1 - u^{b-1}}{\log (1/u)} du$$

$$\underline{du} = \frac{du_a \, du_b \, du_c \, du_d}{\log(1/u_a)\log(1/u_b)\log(1/u_c)\log(1/u_d)}$$

__Bias__

$$E\, n_i \log n_i = E \int_0^1 \frac{n_i - n_i u^{n_i - 1}}{\log(1/u)} \, du$$

$$= \int_0^1 \frac{np_i - np_i(p_i u + q_i)^{n-1}}{\log (1/u)} \, du$$

$$E \sum_{i=1}^{s} n_i \log n_i = n \int_0^1 \frac{1 - \Sigma p_i(p_i u + q_i)^{n-1}}{\log (1/u)} \, du$$

$$E (\overline{H}) = \log n - \int_0^1 \frac{1 - \Sigma p_i \, P_i^{n-1}(u)}{\log (1/u)} \, du.$$

__variance of $\overline{H}$__

$$n(\log n - \overline{H}) = \sum_{i=1}^{s} n_i \log n_i$$

$$n^2(\log n - \overline{H})^2 = \Sigma n_i^2 \log^2 n_i + \sum_{i \neq j}^{s} \sum^{s} n_i \, n_j \log n_i \log n_j$$

$$n_i(\log n_i)n_j \log n_j = \int_0^1 \int_0^1 n_i \, n_j (1-u_1^{n_i - 1})(1-u_2^{n_j - 1}) \, \underline{du}$$

$$E\, n_i \, n_j \log n_i \log n_j = \int_0^1 \int_0^1 n(n-1) p_i p_j [1 - P_i^{n-2}(u_1) - P_j^{n-2}(u_2) + P_{i,j}^{n-2}(u_1, u_2)] \, \underline{du} \qquad \text{(E.1)}$$

$$n_i^2(\log n_i)^2 = {}_o\int^1 {}_o\int^1 n_i^2 \, (1-u_1^{n_i-1})(1-u_2^{n_j-1}) \, \underline{du}$$

$$E\, n_i^2(\log n_i)^2 = {}_o\int^1 {}_o\int^1 n(n-1)p_i^2[1-u_1P_i^{n-2}(u_1)-u_2P_i^{n-2}(u_2) +$$

$$u_1u_2P_i^{n-2}(u_1u_2)] \, \underline{du} \; +$$

$$ {}_o\int^1 {}_o\int^1 np_i[1-P_i^{n-1}(u_1)-P_i^{n-1}(u_2) + P_i^{n-1}(u_1u_2)] \, \underline{du} \, . \qquad (E.2)$$

From (E.1) and (E.2) above we have

$$E(\log n - \overline{H})^2 = \sum_{i=1}^{s} [\frac{A_i}{n} + (1- \frac{1}{n})B_i] + \sum_{i\neq j}^{s} \sum^{s} (1-\frac{1}{n})C_{i,j}$$

where B_i is the first term of (E.2) and A_i is the second term of (E.2) and $C_{i,j}$ is equation (E.1).

Therefore,

$$\mathrm{var}(\overline{H}) = \sum_{i=1}^{s} [\frac{A_i}{n} + (1 - \frac{1}{n})B_i] + \sum_{i\neq j}^{s} \sum^{s} (1 - \frac{1}{n})C_{ij} - [E(\overline{H})]^2.$$

<u>Third Moment of $\overline{H}$</u>

Let $\alpha_1 + \alpha_2 + \ldots + \alpha_s = n(\log n-\overline{H}) = n_1 \log n_1 + n_2 \log n_2 +$

$$\ldots + n_s \log n_s$$

then $(\alpha_1 + \alpha_2 + \ldots + \alpha_s)^3 = \sum_{i=1}^{s} \alpha_i^3 + 3 \sum_{i \neq j}^{s} \sum^{s} \alpha_i^2 \alpha_j + 6 \sum_{i \neq j \neq k}^{s} \sum^{s} \sum^{s} \alpha_i \alpha_j \alpha_k$ and

$$E[\log n - \overline{H}]^3 = (A_3 + 3A_{2,1} + 6A_{1,1,1})/n^3 .$$

$$A_3 = \frac{1}{n^2} \sum_{i=1}^{s} p_i \int_0^1 \int_0^1 \int_0^1 [1 - P_i^{n-1}(u_1) - P_i^{n-1}(u_2) - P_i^{n-1}(u_3) +$$

$$P_i^{n-1}(u_2u_3) + P_i^{n-1}(u_1u_3) + P_i^{n-1}(u_1u_2) - P_i^{n-1}(u_1u_2u_3)]\, du +$$

$$\frac{3(n-1)}{n^2} \sum_{i=1}^{s} p_i^2 \int_0^1 \int_0^1 \int_0^1 [1 - u_1P_i^{n-2}(u_1) - u_2P_i^{n-2}(u_2) - u_3P_i^{n-2}(u_3) +$$

$$u_1u_2P_i^{n-2}(u_1u_2) + u_1u_3P_i^{n-2}(u_2u_3) - u_1u_2u_3P_i^{n-2}(u_1u_2u_3)]du +$$

$$\frac{(n-1)(n-2)}{n^2} \sum_{i=1}^{s} p_i^3 \int_0^1 \int_0^1 \int_0^1 [1 - u_1^2P_i^{n-3}(u_1) u_2P_i^{n-3}(u_2) - u_3P_i^{n-3}(u_3) +$$

$$(u_1u_2)^2P_i^{n-3}(u_1u_2) + (u_1u_3)^2P_i^{n-3}(u_1u_3) + (u_2u_3)^2P_i^{n-3}(u_2u_3) -$$

$$(u_1u_2u_3)^2P_i^{n-3}(u_1u_2u_3)]\, \underline{du} .$$

$$A_{2,1} = \frac{n-1}{n^2} \sum_{i \neq j} \sum p_i p_j \int_0^1 \int_0^1 \int_0^1 [1 - P_i^{n-2}(u_1) - P_i^{n-2}(u_2) - P_j^{n-2}(u_3) + P_i^{n-2}(u_1u_2) +$$

$$P_{i,j}^{\,n-2}(u_1,u_3) + P_{i,j}^{\,n-2}(u_2,u_3) - P_{i,j}^{\,n-2}(u_1u_2,u_3)]\, \underline{du} +$$

$$\frac{(n-1(n-2)}{n^2} \sum_{i \neq j} \sum p_i^2 p_j \int_0^1 \int_0^1 \int_0^1 [1 - u_1P_i^{n-3}(u_1) - u_2P_i^{n-3}(u_2) - u_3P_j^{n-3}(u_3) +$$

$$u_1P_i^{n-3}(u_1,u_2) + u_1P_{i,j}^{\,n-3}(u_1,u_3) + u_2P_{i,j}^{\,n-3}(u_2,u_3) - u_1u_2P_{i,j}^{\,n-3}(u_1u_2,u_3)]\underline{du}.$$

$$A_{1,1,1} = \frac{(n-1)(n-2)}{n^2} \sum_{i \neq j \neq k}\sum\sum p_i p_j p_k \, {}_o\!\int^1 {}_o\!\int^1 {}_o\!\int^1 [1-P_i^{n-3}(u_1)-P_j^{n-3}(u_2) -$$

$$P_k^{n-3}(u_3) + P_k^{n-3}(u_1,u_2) + P_{i,k}{}^{n-3}(u_1,u_3) + P_{j,k}{}^{n-3}(u_2,u_3) -$$

$$P_{i,j,k}{}^{n-3}(u_1,u_2,u_3)] \, \underline{du} \, .$$

<u>Fourth Moment of $\overline{H}$</u>

We have as in third moment,

$$(\alpha_1 + \alpha_2 +\ldots+\alpha_s)^4 = \sum_{i=1}^{s} \alpha_i^4 + 4\sum_{i \neq j}\sum \alpha_i^3\alpha_j + 12\sum_{i \neq j \neq k}\sum\sum \alpha_i^2\alpha_j\alpha_k +$$

$$6 \sum_{i \neq j}\sum \alpha_i^2\alpha_j^2 \; 24 \sum_{i \neq j \neq k \neq l}\sum\sum \alpha_i\alpha_j\alpha_k\alpha_l$$

$$E[(\log n - \overline{H})]^4 = (B_4 + 4\, B_{3,1} + 12\, B_{2,1,1} + 24\, B_{1,1,1,1} + 6\, B_{2,2})/n^4$$

$$B_4 = \frac{1}{n^3} \sum_{i=1}^{s} p_i \, {}_o\!\int^1 {}_o\!\int^1 {}_o\!\int^1 {}_o\!\int^1 [1- \sum_{a=1}^{4} P_i^{n-1}(u_a) + \sum_{a<b}^{3}\sum^{4} P_i{}^{n-1}(u_a u_b) -$$

$$\sum_{a<b<c}\sum\sum P_i^{n-1} (u_a u_b u_c) + P_i^{n-1}(u_1 u_2 u_3 u_4)] \, \underline{du} \, +$$

$$\frac{7(n-1)}{n^3} \sum_{i=1}^{s} p_i^2 \, {}_o\!\int^1 {}_o\!\int^1 {}_o\!\int^1 {}_o\!\int^1 [1- \sum_{a=1}^{4} u_a \, P_i^{n-2}(u_a) + \sum_{a<b}^{3}\sum^{4} u_a u_b P_i^{n-2}(u_a u_b) -$$

$$\sum_{a<b<c}^{2}\sum^{3}\sum^{4} u_a u_b u_c P_i^{n-2}(u_a u_b u_c) + u_1 u_2 u_3 u_4 P_i^{n-2}(u_1 u_2 u_3 u_4)] \, \underline{du} \, +$$

$$\frac{6(n-1)(n-2)}{n^3} \sum_{i=1}^{s} p_i^3 \int_0^1\int_0^1\int_0^1\int_0^1 [1-\sum_{a=1}^{4} u_a^2 P_i^{n-3}(u_a) + \sum_{a<b}\sum u_a^2u_b^2P_i^{n-3}(u_au_b) -$$

$$\sum_{a<b<c}\sum\sum u_a^2u_b^2u_c^2 P_i^{n-3}(u_au_bu_c) + u_1^2u_2^2u_3^2u_4^2 P_i^{n-3}(u_1u_2u_3u_4)]\ \underline{du} +$$

$$\frac{(n-1)(n-2)(n-3)}{n^3} \sum_{i=1}^{s} P_i^4 \int_0^1\int_0^1\int_0^1\int_0^1[1-\sum_{a=1}^{4} u_a^3 P_i^{n-4}(u_a) + \sum_{a<b}\sum u_a^3u_b^3P_i^{n-4}(u_au_b) -$$

$$\sum_{a<b<c}\sum\sum u_a^3u_b^3u_c^3P_i^{n-4}(u_au_bu_c) + u_1^3u_2^3u_3^3u_4^3P_i^{n-4}(u_1u_2u_3u_4]\ \underline{du}\ .$$

$$B_{3,1} = \frac{n-1}{n^3}\sum_{i\neq j}^{s}\sum^{s} p_ip_j \int_0^1\int_0^1\int_0^1\int_0^1 [1-\sum_{a=1}^{3} P_i^{n-2}(u_a) - P_j^{n-2}(u_4) + \sum_{a<b}\sum P_i^{n-2}(u_au_b) +$$

$$\sum_{a=1}^{3} P_{i,j}^{\ n-2}(u_a,u_4) - P_i^{n-2}(u_1u_2u_3) - \sum_{a}^{2}\sum_{b}^{3} P_{i,j}^{\ n-2}(u_au_b,u_4) +$$

$$P_{i,j}^{\ n-2}(u_1u_2u_3,u_4)]\ \underline{du} +$$

$$\frac{3(n-1)(n-2)}{n^3}\sum_{i\neq j}^{s}\sum^{s} p_i^2p_j \int_0^1\int_0^1\int_0^1\int_0^1[1-\sum_{a=1}^{3} u_aP_i^{n-3}(u_a) - u_4P_j^{n-3}(u_4) +$$

$$\sum_{a<b}^{2}\sum^{3} u_aP_i^{n-3}(u_au_b) + \sum_{a=1}^{3} u_aP_{i,j}^{\ n-3}(u_a,a_4) - u_1P_i^{n-3}(u_1u_2u_3) -$$

$$\sum_{a<b}^{2}\sum^{3} u_au_bP_{i,j}^{\ n-3}(u_au_b,u_4) + u_1u_2u_3P_{i,j}^{\ n-3}(u_1u_2u_3,u_4)]\ \underline{du}\ +$$

$$\frac{(n-1)(n-2)(n-3)}{n^3} \sum_{i\neq j}^{s} \sum^{s} p_i^3 p_j \; {}_0\!\int^1 {}_0\!\int^1 {}_0\!\int^1 {}_0\!\int^1 [1- \sum_{a=1}^{3} u_a^2 \, P_i^{n-4}(u_a) - u_4^2 P_j^{n-4}(u_4) +$$

$$\sum_{a<b}^{2} \sum^{3} u_a^2 \, P_i^{n-4}(u_a u_b) + \sum_{a=1}^{3} u_a^2 \, P_{i,j}^{n-4}(u_a, u_4) - u_1^2 P_i^{n-4}(u_1 u_2 u_3) -$$

$$\sum_{a<b}^{2} \sum^{3} u_a^2 u_b^2 P_i^{n-4}(u_a u_b, u_4) + u_1^2 u_2^2 u_3^2 \, P_{i,j}^{n-4}(u_1 u_2 u_3, u_4)] \; \underline{du} \; .$$

$$B_{2,1,1} = \frac{(n-1)(n-2)}{n^3} \sum_{i\neq j\neq k} \sum \sum p_i p_j p_k \; {}_0\!\int^1 {}_0\!\int^1 {}_0\!\int^1 {}_0\!\int^1 [1-P_i^{n-3}(u_1) - P_i^{n-1}(u_2) -$$

$$P_j^{n-3}(u_3) - P_k^{n-3}(u_4) + P_i^{n-3}(u_1 u_2) + P_{i,j}^{n-3}(u_1, u_3) + P_{i,j}^{n-3}(u_2, u_3) +$$

$$P_{i,k}^{n-3}(u_1, u_4) + P_{i,k}^{n-3}(u_2, u_4) + P_{j,k}^{n-3}(u_3, u_4) - P_{i,j}^{n-3}(u_1 u_2, u_3) -$$

$$P_{i,\;k}^{n-3}(u_1 u_2, u_4) - P_{i,j,k}^{n-3}(u_1, u_3, u_4) - P_{i,j,k}^{n-3}(u_2, u_3, u_4) +$$

$$P_{i,j,k}^{n-3}(u_1 u_2, u_3, u_4)] \; \underline{du} \; +$$

$$\frac{(n-1)(n-2)(n-3)}{n^3} \sum_{i\neq j\neq k} \sum \sum p_i^2 p_j p_k \; {}_0\!\int^1 {}_0\!\int^1 {}_0\!\int^1 {}_0\!\int^1 [1-u_1 P_i^{n-4}(u_1) - u_2 P_i^{n-4}(u_2) -$$

$$u_3 P_j^{n-4}(u_3) - u_4 P_k^{n-4}(u_4) + u_1 u_2 P_i^{n-4}(u_1 u_2) + u_1 P_{i,j}^{n-4}(u_1, u_3) +$$

$$u_2 P_{i,j}^{n-4}(u_2, u_3) + u_1 P_{i,k}^{n-4}(u_1, u_4) + u_2 P_{i,k}^{n-4}(u_2, P_{i,k}^{n-4}(u_2, u_4) +$$

$$u_3 P_{j,k}^{n-4}(u_3,u_4) - u_1u_2P_{i,j}^{n-4}(u_1u_2,u_3) - u_1u_2P_{i,k}^{n-4}(u_1u_2,u_4) -$$

$$u_1P_{i,j,k}^{n-4}(u_1,u_3,u_4) - u_2P_{i,j,k}^{n-4}(u_2,u_3,u_4) +$$

$$u_1u_2P_{i,j,k}^{n-4}(u_1u_2,u_3,u_4)] \; \underline{du} \; .$$

$$B_{2,2} = \frac{n-1}{n^3}\sum_{i\neq j}^{s}\sum^{s} p_i p_j \int_0^1\int_0^1\int_0^1\int_0^1 [1-P_i^{n-2}(u_1)-P_i^{n-2}(u_2)-P_j^{n-2}(u_3)-P_j^{n-2}(u_4) +$$

$$P_i^{n-2}(u_1u_2) + P_{i,j}^{n-2}(u_1,u_3)+P_{i,j}^{n-2}(u_2,u_3)+P_{i,j}^{n-2}(u_1,u_4) +$$

$$P_{i,j}^{n-2}(u_2,u_4)+P_j^{n-2}(u_3u_4)-P_{i,j}^{n-2}(u_1u_2,u_3)-P_{i,j}^{n-2}(u_1u_2,u_4) -$$

$$P_{i,j}^{n-2}(u_1,u_3u_4)-P_{i,j}^{n-2}(u_2,u_3u_4)-P_{i,j}^{n-2}(u_1u_2,u_3u_4)] \; \underline{du} \; +$$

$$\frac{(n-1)(n-2)}{n^3}\sum_{i\neq j}\sum p_i p_j^2 \int_0^1\int_0^1\int_0^1\int_0^1 [1-P_i^{n-3}(u_1)-u_2P_i^{n-3}(u_2)-P_j^{n-3}(u_3) -$$

$$u_4P_j^{n-3}(u_4) + P_i^{n-3}(u_1u_2)+u_3P_{i,j}^{n-3}(u_1,u_3)+u_3P_{i,j}^{n-3}(u_2,u_3)+$$

$$u_4P_{i,j}^{n-3}(u_1,u_4)+u_4P_{i,j}^{n-3}(u_2,u_4) + P_j^{n-3}(u_3u_4)-u_3P_{i,j}^{n-3}(u_1u_2,u_3) -$$

$$u_4P_{i,j}^{n-3}(u_1u_2,u_4) - (u_3u_4P_{i,j}^{n-3}(u_1,u_3u_4) - u_3u_4P_{i,j}^{n-3}(u_2,u_3u_4) -$$

$$u_3u_4P_{i,j}^{\;n-3}(u_1u_2,u_3u_4)]\ \underline{du}\ +$$

$$(n-1)(n-2)\sum_{i\neq j}^{s}\sum^{s} p_i^2\ p_j\ {}_o\!\int^1{}_o\!\int^1{}_o\!\int^1{}_o\!\int^1[1-u_1P_i^{n-3}(u_1)-P_i^{n-3}(u_2)-$$

$$u_3P_j^{\;n-3}(u_3)-P_j^{n-3}(u_4)+u_1u_2P_i^{n-3}(u_1u_2)+u_1P_{i,j}^{\;n-3}(u_1,u_3)+$$

$$u_2P_{i,j}^{\;n-3}(u_2,u_3)+u_1P_{i,j}^{\;n-3}(u_1,u_4)+u_2P_{i,j}^{\;n-3}(u_2,u_4)+$$

$$u_3u_4P_j^{n-3}(u_3u_4)-u_1u_2P_{i,j}^{\;n-3}(u_1u_2,u_3)-u_1u_2P_{i,j}^{\;n-3}(u_1u_2,u_4)-$$

$$u_1P_{i,j}^{\;n-3}(u_1,u_3u_4)-u_2P_{i,j}^{\;n-3}(u_2,u_3u_4)-u_1u_2P_{i,j}^{\;n-3}(u_1u_2,u_3u_4)]\ \underline{du}\ +$$

$$(n-1)(n-2)(n-3)\sum_{i\neq j}^{s}\sum^{s} p_i^2\ p_j^2\ {}_o\!\int^1{}_o\!\int^1{}_o\!\int^1{}_o\!\int^1[1-u_1P_i^{n-4}(u_1)-$$

$$u_2P_i^{n-4}(u_2)-u_3P_j^{n-4}(u_3)-u_4P_j^{n-4}(u_4)+u_1u_2P_i^{n-4}(u_1u_2)+$$

$$u_1u_3P_{i,j}^{\;n-4}(u_1,u_3)+u_2u_3P_{i,j}^{\;n-4}(u_2,u_3)+u_1u_4P_{i,j}^{\;n-4}(u_1,u_4)+$$

$$u_2u_4P_{i,j}^{\;n-4}(u_2,u_4)+u_3u_4P_j^{n-4}(u_3u_4)-u_1u_2u_3P_{i,j}^{\;n-4}(u_1u_2,u_3)-$$

$$u_1u_2u_3P_{i,j}^{\;n-4}(u_1u_2,u_4)-u_1u_3u_4P_{i,j}^{\;n-4}(u_1,u_3u_4)-$$

$$u_2u_3u_4P_{i,j}^{\;n-4}(u_2,u_3u_4)-u_1u_2u_3u_4P_{i,j}^{\;n-4}(u_1u_2,u_3u_4)]\ \underline{du}\ .$$

$$B_{1,1,1,1} = \frac{(n-1)(n-2)(n-3)}{n^3} \sum^{s}\sum^{s}\sum^{s}\sum^{s}_{i\neq j\neq k\neq l} p_i p_j p_k p_l \int_o^1\int_o^1\int_o^1\int_o^1 [1-P_i^{n-4}(u_1) -$$

$$P_j^{n-4}(u_2) - P_k^{n-4}(u_3) - P_l^{n-4}(u_4) + P_{i,j}^{n-4}(u_1,u_2) +$$

$$P_{i,k}^{n-4}(u_1,u_3) + P_{j,k}^{n-4}(u_2,u_3) + P_{j,k}^{n-4}(u_2,u_4) +$$

$$P_{i,l}^{n-4}(u_1,u_4) + P_{k,l}^{n-4}(u_3 u_4) - P_{i,j,k}^{n-4}(u_1,u_2,u_3) -$$

$$P_{i,j,l}^{n-4}(u_1,u_2,u_4) - P_{i,k,l}^{n-4}(u_1,u_3,u_4) -$$

$$P_{j,k,l}^{n-4}(u_2,u_3,u_4) + P_{i,j,k,l}^{n-4}(u_1,u_2,u_3,u_4)] \, \underline{du} \; .$$

THE MOMENTS AND DISTRIBUTION FOR AN ESTIMATE

OF THE

SHANNON INFORMATION MEASURE

AND ITS

APPLICATION TO ECOLOGY

by

Kermit Hutcheson

This dissertation deals primarily with the moments and distribution of $\overline{H} = -\Sigma \frac{n_i}{N} \log \frac{n_i}{N}$. Some techniques of obtaining multivariate moments, in particular multinomial moments, are given.

The approach used in obtaining the moments of $\overline{H}$ was through the probability generating function of the multinomial distribution. A series of rather simple mathematical operations will produce the $E(\overline{H})$ as an integral and $Var(\overline{H})$ as a double integral. These integrals are evaluated exactly thus giving the exact mean and variance of $\overline{H}$.

The mean and variance is also given in series form. The series for the mean of $\overline{H}$ appears to be divergent. Several charts are given which indicate the percent error incurred when the series are used.

The combinatorial approach was used in finding the asymptotic distribution of $\overline{H}$. The IBM 1130 and the IBM 360 model 65 were used to do this work. The results is that $\overline{H}$ is aymptotically normal in the general case and $\overline{H}$ is asymptotically chi-square in the equiprobable case.

Tables are given for the mean and variance of $\overline{H}$ in the general case and in the equiprobable case.

Two methods are given for finding multivariate moments. The Q-Product Method due to Shenton, Bowman, and Reinfelds [36^{th} Session of the International Statistical Institute,, 1967] and the Small Sample Method. There is every indication that these methods can be completely automated. A table of the first fourteen binomial moments is given and a table through order six of the multinomial moments is given.