EXISTENCE AND ANALYTICITY

OF MANY BODY SCATTERING AMPLITUDES

AT LOW ENERGIES

by

Jan Dereziński

Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Mathematics

APPROVED:

G. Hagedorn, Chairman

W. Greenberg

M. Klaus

P. Fletcher

P. Zweifel

August 1985

Blacksburg, Virginia

Acknowledgements

It is a pleasure to be able to acknowledge my enormous debt to G. Hagedorn for numerous helpful and friendly discussions. I also appreciate a discussion with I. Herbst to whom I am indebted for the proof of lemma 5.1 and I am grateful to I. M. Sigal for his valuable remarks. Moreover, I owe my gratitude to M. Klaus for pointing out a mistake in an early version of my dissertation.

I thank all the members of the Center for Transport Theory and Mathematical Physics for the supportive and stimulating atmosphere that they surrounded me with during my graduate study at Va Tech. I also owe my gratitude to W. Kohler for providing me with a financial support during the summer 1984 under the grant 230-11-110-049-3529031.

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1. Introduction

There is no question that the many body Schrödinger equation is one of the most fundamental equations in modern physics. It is widely believed that most phenomena in chemistry, solid state, statistical and atomic physics can be in principle derived from it. So we should not be surprised that a lot of effort has been devoted to obtaining rigorous results on this subject. These results fall into two broad categories: those concerning bound states and the scattering theory, which involves the continuous part of the spectrum.

The many body scattering theory is quite a difficult subject but in the energy range below the lowest 3-cluster threshold it becomes much easier. Only 2-cluster scattering is possible in this energy range, which makes it possible to use effectively 2-body techniques. This paper is chiefly devoted to proving various results on scattering in this energy range by time-independent methods.

We think that the best introduction to the subject will be a brief survey of the 2-body scattering theory by the time independent method. We will follow closely [RS3].

In the 2-body case the Schrödinger Hamiltonian after separation of center of motion variables has the form

$$H = -\Delta/2\mu + V = H_0 + V$$

where typically $V(x)$ goes to zero as $|x|$ goes to infinity.

In the time independent approach the most important intermediate step is a study of the resolvent

$$R(z) = (z-H)^{-1}$$

as z approaches the continuous spectrum of H, which is located on the positive real semiaxis. This can be achieved by using the following equation:

Equation 1.1.

$$R(z)$$
$$= R_0(z)|V|^{1/2}\left(1-(V)^{1/2}R_0(z)|V|^{1/2}\right)^{-1}(V)^{1/2}R_0(z) + R_0(z)$$

where $R_0(z) = (z-H_0)^{-1}$.

The importance of this equation stems from the fact that the right hand side is expressed by the free resolvent whose properties are relatively easy to study. In particular, if we assume the potential to fall off exponentially then we can prove that $(V)^{1/2}R_0(z)|V|^{1/2}$ can be continued analytically across the positive real semiaxis onto the so-called nonphysical sheet of the complex plane. If the dimension of the physical space is odd the point zero turns out to be a branch point of the square root type, if the dimension is even this branch point is logarithmic. Moreover, using the fact that V decays in all directions we can prove that $(V)^{1/2}R_0(z)|V|^{1/2}$ is compact and goes to zero for large negative z. This enables us to use the analytic Fredholm theorem, which implies that

$\left(1-(V)^{1/2}R_0(z)|V|^{1/2}\right)^{-1}$ can be continued meromorphically across the positive real semiaxis. The above argument is the most essential part of the proof of the following proposition.

Proposition 1.2. Define ρ to be the operator of multiplication by $\exp(-b|x|)$. Suppose that $V\rho \in L^{\infty}$. Then $\rho R(z)\rho$ can be continued meromorphically across the positive real semiaxis. If the dimension is odd then $\rho R(z)\rho$ has a branch point of the square root type at zero, and if the dimension is even then this branch point is logarithmic.

A very similar line of argument will prove the next proposition.

Proposition 1.3. Define γ to be the operator of multiplication by $(1+|x|^2)^{-\delta/2}$ where $\delta > 1/2$. Suppose that $V\gamma^{-2} \in L^{\infty}$. Then $\gamma R(z)\gamma$ can be extended continuously up to the positive real semiaxis except for a closed set $\mathbb{E}$ of measure zero.

Both the above results can be improved to accomodate some singularities of the potential (in fact, assumption 3.1 will work in the case of proposition 1.2 and assumption 6.1 in the case of proposition 1.3). Moreover, by a certain more sophisticated argument due to S. Agmon (see [A] and [RS4]) one can show that the set $\mathbb{E}$ is discrete and equal to the set of bound states of H. Incidentally, this implies the absence of the singular continuous spectrum.

The above propositions have important physical consequences. Before stating them let us recall basic definitions from the 2-body scattering theory. First we define the wave operators:

$$W^{\pm} = \text{s-}\lim_{t\to\pm\infty} \exp(-iHt)\exp(iH_0 t).$$

In the time independent theory another definition is also used, which for practical purposes is equivalent to the previous one:

$$W^{\pm} = \text{w-}\lim_{\epsilon\to 0^{\pm}} \epsilon/\pi\int_{\mathbb{R}} R(\lambda+i\epsilon)R_0(\lambda-i\epsilon)d\lambda.$$

It is easy to show that for potentials that decay like $r^{-1-\epsilon}$ these wave operators exist.

Another important object is the scattering matrix defined as

$$S = W^{+*}W^{-}.$$

One can compute the kernel of the matrix $T = S - 1$ in momentum space. This kernel is equal to $\delta((k_1)^2-(k_2)^2)t(k_1,k_2)$, where

$$t(k_1,k_2) = \left(\Phi(k_1),V\Phi(k_2)\right) + \lim_{\epsilon\to 0^+}\left(\Phi(k_1),VR(\lambda+i\epsilon)V\Phi(k_2)\right)$$

and $\Phi(k)(x) = \exp(ikx)$. Now we can easily see that propositions 1.2 and 1.3 imply the following properties of the scattering amplitudes.

Proposition 1.4. Suppose that $V\rho \in L^{\infty}$. Fix two unit vectors: $\hat{e}_1$ and $\hat{e}_2$. Then $t((2\mu z)^{1/2}\hat{e}_1,(2\mu z)^{1/2}\hat{e}_2)$ can be extended to a function meromorphic in a neighborhood of the positive real semiaxis. If the dimension is odd then the point zero is its branch point of the square root type, and if the dimension is even then zero is its logarithmic branch point.

Proposition 1.5. Suppose that $V\gamma^{-2} \in L^{\infty}$. Then $t(k_1,k_2)$ exists for all $(k_1)^2/2\mu = (k_2)^2/2\mu = \lambda \in \mathbb{R}_+ - \mathbb{E}$ as a kernel of an operator on the space of square-integrable functions on a sphere.

It is easy to show that the ranges of the wave operators are contained in the subspace that belongs to the absolutely continuous part of the spectrum of H. If their ranges are equal to this subspace then we say that the wave operators are asymptotically complete. Under the same assumptions on the potential as in proposition 1.3 and by very similar methods one can prove the asymptotic completeness of the 2-body scattering. As an extra bonus one obtains a theorem on a generalized eigenfunction expansion for the energies outside of $\mathbb{E}$.

We should mention here that there exists yet another quite successful approach to proving the asymptotic completeness in the 2-body case. It is due to V. Enss and uses time dependent ideas and anologies taken from classical mechanics (see [En2], [RS3] and [P]). As a byproduct of this method one can also show the absence of the singular continuous spectrum, but one obtains much less information on the existence of the scattering amplitudes.

In our dissertation we reproduce almost all the results of the 2-body scattering theory in the setting of many body scattering theory for energies below the lowest 3-cluster threshold. The main tool in our approach is equation 3.8, which is a close analogue of equation 1.1 suited to the study of the many body resolvent below the lowest 3-cluster threshold. It is closely related to the Weinberg-Van Winter equation (see [V], [W] and [RS4]) and to equations used to prove asymptotic completeness for 3- and 4-body systems by stationary methods (see [F], [GM] and [Ha1]). Seemingly involved expressions containing the resolvents of cluster Hamiltonians that are characteristic of equation 3.8 turn out to be relatively easy to manage by use of some algebraic identities which we prove in chapter 4.

Probably the most interesting result of our paper is the proof that threshold singularities of scattering amplitudes below the lowest 3-cluster threshold are of the square root type. Chapters 3, 4 and 5 are chiefly devoted to this proof. We assume there that the 2-body potentials fall off exponentially, which is the same assumption that appears in an analoguous context in the 2-body case. In fact, in our method we express the N-body resolvent in terms of essentially 2-body objects whose analytic properties we understand better. We prove that many body scattering amplitudes can also be meromorphically continued across the real axis onto the non-physical sheet. Our proof is valid only for the energy range below the lowest 3-cluster threshold except for the thresholds where square root or logarithmic singularities may occur. By a square root singularity we mean that the function can be locally continued analytically onto the Riemann surface of the square root function; in fact, we prove that for odd dimensions the scattering amplitude can be locally expressed as $t(z) = \sum_{n=m}^{\infty} a_n (z-\omega)^{n/2}$ where m is some unknown integer. Our method can handle potentials with singularities characteristic of the form boundedness condition; we explain it in chapter 5.

It should be noted that results similar to ours have been obtained by E. Balslev in [B2]. They are resticted to the case of N=3 whereas we can handle an arbitrary finite number of particles.

Various authors have studied analytic properties of 2-cluster-2-cluster amplitudes before. This was done by E. Balslev in [B1] and [B2], G. Hagedorn in [Ha2], W. Hunziker and I. Sigal in [HS] and I. Sigal in [Sig4]. Their methods though, with the exception of [B2], went in a different direction. They assumed both the dilation analyticity and an exponential decay of the potentials and did not obtain the information on threshold singularities that we have. On the other hand, the dilation analyticity assumption allowed them to study the scattering amplitude for the whole energy range. They proved that if the potentials both decay exponentially and are dilation analytic then the 2-cluster-2-cluster scattering amplitudes can be continued meromorphically in a neighborhood of the real axis everywhere except for the thresholds (see [Ha1], [HS] and [Sig4]). In chapter 7 we join their methods with our technique based on equation 3.8 to obtain information on threshold singularities above the lowest 3-cluster threshold. We also assume the potentials to decay exponentially and to be dilation analytic. We prove that 2-cluster-2-cluster amplitudes can be meromorphically continued around two cluster thresholds that do not coincide with more-than-2-cluster thresholds onto the Riemann surface of the square root for an odd dimension and of the logarithm for an even dimension. Chapter 7 is the only place in our dissertation where we deal with energies above the lowest 3-cluster threshold. We have included it here because it is a nice application of equation 3.8.

We should mention also that there are some interesting though less complete results on analytic properties of other kinds of many body scattering amplitudes - see [B1], [B2], [Sig2], [Sig3], [Sig4] and [HS].

Asymptotic completeness has been regarded as one of the more difficult problems in the theory of the many body Schrödinger equation and its solution has been found only very recently (see [SigSof]). But if we restrict our attention to the energies below the lowest 3-cluster threshold it becomes much easier. A proof of the asymptotic completeness in this energy range was found already by J. M. Combes in [C] and B. Simon in [Sim1] and [Sim2] for some restricted classes of potentials. A proof of this fact for potentials that decay like $r^{-1-\epsilon}$ was given by V. Enss (see [En1] and [P]). In chapter 6 we give an alternative proof of the asymptotic completeness below the lowest 3-cluster threshold for potentials that decay like $r^{-1-\epsilon}$. Enss used the time-dependent method whereas our approach is time-independent. As a byproduct of our approach we obtain a theorem on a generalized eigenfunction expansion for all energies in the studied range except for a closed set $\mathbb{E}$ of measure zero. This result to our knowledge cannot be obtained by the Enss method. On the other hand Enss proves the absence of singular continuous spectrum below the lowest 3-cluster threshold - a result that we could not obtain. (We were not able to reproduce in our setting Agmon's argument that relates the exceptional set $\mathbb{E}$ to the set of eigenvalues, see [A] and [RS4]). The time-independent technique in the many body scattering that we apply has been used and developed by L. D. Faddeev ([F]), J. Ginibre and M. Moulin ([GM]), J. S. Howland ([Ho]), I. Sigal ([Sig1]) and G. Hagedorn ([Ha1]).

Another result that we obtain is the existence and the continuity of scattering amplitudes below the lowest 3-cluster threshold outside of a closed set $\mathbb{E}$ of measure zero for potentials that decay like $r^{-1-\epsilon}$. We should mention here that a very similar result can be obtained by using the so-called commutator methods due to E. Mourre ([M]) and P. Perry, I. Sigal and B. Simon ([PSS]). But to be able to apply these techniques one has to impose on the potentials an additional assumption, for instance $V = V_1 + V_2 + V_3$ with x^2V_1, $x^2\nabla V_2$, $x^2\nabla V_3 \in L^\infty + L^u$ where $u=d/2$ for $d>4$, $u>2$ for $d=4$ and $u=2$ for $d=1, 2, 3$, which narrows the class of potentials significantly. On the other hand, unlike our method, the commutator technique gives information about boundary values of the resolvent above the lowest 3-cluster threshold and relates the exceptional set $\mathbb{E}$ to the set of bound states and thresholds.

2. Notation.

We study a many body Schrödinger operator acting on $L^2(\mathbb{R}^{dN})$ defined by

$$\tilde{H} = -\sum_{i=1}^{N} \tilde{\Delta}_i/2m_i + \sum_{\substack{i,j=1 \\ i<j}}^{N} V_{ij}(x_i - x_j)$$

$$= -\sum_{i=1}^{N} \tilde{\Delta}_i/2m_i + V$$

where x_i is a d-dimensional vector pointing at the position of the i-th particle with mass m_i and $\tilde{\Delta}_i$ is the Laplacian in x_i.

Throughout the paper we will assume the potentials to be form bounded with respect to the free Hamiltonian with an arbitrarily small bound, which implies that the Hamiltonian is self-adjoint (see [RS2]).

Now we have to introduce some concepts that belong to the standard folklore of many body Schrödinger operators. They are discussed in more detail in numerous papers on the scattering theory, notably in [Ha1], [Sig1] and [RS3].

First we remove the center-of-mass motion to obtain the Hamiltonian

$$H = -\sum_{i=1}^{N-1} \Delta_i/2\mu_i + \sum_{\substack{i,j=1 \\ i<j}}^{N} V_{ij}(x_i - x_j) = H_0 + V$$

where Δ_i is the Laplacian corresponding to the i-th coordinate in some system of Jacobi coordinates and μ_i is the corresponding reduced mass for $i=1,...N-1$. This Hamiltonian acts on $L^2(X)$ where X denotes the space isomorphic to $\mathbb{R}^{d(N-1)}$ that describes the relative motion of N particles.

We let $R(z) = (z-H)^{-1}$ and $R_0(z) = (z-H_0)^{-1}$.

A cluster decomposition is a partition of the set $\{1,2,...N\}$ into nonempty disjoint subsets called clusters. A cluster decomposition will be denoted by capital letters such as D, B. A subscript on a cluster decomposition denotes the number of clusters in a given partition. $D_i \subset D_j$ means that D_i refines D_j, i.e., the clusters of D_i are obtained by further partitioning the clusters of D_j. Note that $D_i \subset D_j$ implies $i \geq j$ and that for any D_i we have $D_i \subset D_i$. A cluster decomposition with N-1 clusters may be also called a pair and denoted by a Greek letter such as σ or by (ij).

For each cluster decomposition we define

$$H_D = -\sum_{i=1}^{N-1} \Delta_i/2\mu_i + \sum_{(i,j)\subset D} V_{ij} = -\sum_{i=1}^{N-1} \Delta_i/2\mu_i + V_D$$

$$R_D(z) = (z-H_D)^{-1}.$$

The Hamiltonian obtained from H_{D_i} by separating the cluster center of motion variables will be denoted by H^{D_i}. This separation results in a decomposition of the space $L^2(X)$ into $L^2(X^{D_i}) \otimes L^2(X_{D_i})$ where X^{D_i} is isomorphic to $\mathbb{R}^{d(N-i)}$ and X_{D_i} is isomorphic to $\mathbb{R}^{d(i-1)}$. Here the first variables, denoted by x^{D_i}, stand for intracluster degrees of freedom and the latter, denoted by x_{D_i}, for intercluster degrees of freedom. If we represent the original Hilbert space as the above tensor product we can write our cluster Hamiltonian as:

$$H_{D_i} = H^{D_i} \otimes 1 + 1 \otimes T_{D_i}$$

where T_{D_i} is the kinetic energy of the c.m. motion of the clusters.

Eigenvalues of H^{D_i} for $1<i<N$ are called i-cluster thresholds. The point zero is the only N-cluster threshold. We denote the lowest i-cluster threshold for $i>2$ by ξ. The set of 2-cluster thresholds will be denoted by Ω. ω_1 will mean the lowest 2-cluster threshold, which is at the same time the bottom of the continuous spectrum. Elements of $L^2(X^{D_i})$ that are the eigenvectors of H^{D_i} we denote by ϕ_α and call channels. We sometimes denote the threshold corresponding to the channel ϕ_α by ω_α, the corresponding reduced mass of intercluster motion by μ_α and the corresponding cluster decomposition by $D(\alpha)$. $\nu_\alpha(\lambda)$ will denote $(2\mu_\alpha(\lambda-\omega_\alpha))^{1/2}$. The generalized eigenvector of $H_{D(\alpha)}$ corresponding to the channel ϕ_α with the intercluster momentum k we denote by $\Phi_\alpha(k)$, explicitly

$$\Phi_\alpha(k)(x)=\phi_\alpha(x^{D(\alpha)})\exp(ikx_{D(\alpha)}).$$

We define also

$$T_\alpha = \omega_\alpha + T_{D(\alpha)} = \omega_\alpha - \Delta_{D(\alpha)}/2\mu_\alpha.$$

The scattering amplitude for the $\alpha-\beta$ scattering at energy λ is given by the formula

$$t_{\alpha\beta}(k_1,k_2)= \left(\Phi_\alpha(k_1),(V-V_{D(\alpha)})\Phi_\beta(k_2)\right)$$

$$+ \lim_{\epsilon\to 0^+}\left(\Phi_\alpha(k_1),(V-V_{D(\alpha)})R(\lambda+i\epsilon)(V-V_{D(\beta)})\Phi_\beta(k_2)\right)$$

where $T_\alpha\Phi_\alpha(k_1) = \lambda\Phi_\alpha(k_1)$, $T_\beta\Phi_\beta(k_2) = \lambda\Phi_\beta(k_2)$ and $(.)$ denotes the scalar product. (See [N]; for a rigorous derivation of this formula see [HS]).

We denote by $P^\epsilon_{D_2}$ the orthogonal projection onto the part of spectrum of H^{D_2} with energies below $\xi-\epsilon$. We need to use the family of projections $P^\epsilon_{D_2}$ instead of the projection $P^0_{D_2}$ because of a possible occurence of the Efimov effect. If this effect occurs in a more-than-3-particle system then there may exist infinitely many 2-cluster thresholds below ξ (see [Ef], [Y] and [OS]).

$|x|$ means some fixed Euclidean norm of the vector x. The symbols ρ^b, $\rho^{D,b}$ and $\rho_D{}^b$ will denote the operators of multiplication by $\exp(-b(|x|^2+1)^{1/2})$, $\exp(-b(|x^D|^2+1)^{1/2})$ and $\exp(-b(|x_D|^2+1)^{1/2})$ resp.

3. Main Results.

In chapters 3, 4 and 5 we will require the potentials to decay exponentially, which is expressed in the following assumption.

Assumpion 3.1. We assume that for any i and j, $|V_{ij}|^{1/2}(\rho^{ij,c})^{-1}(1-\Delta^{ij})^{-1}(\rho^{ij,c})^{-1}|V_{ij}|^{1/2}$ is bounded on $L^2(X^{ij})$ for some $c>0$. In other words, we assume the operator of multiplication by $V_{ij}(x^{ij})\exp(2c(|x^{ij}|^2+1)^{1/2})$ to be form compact with respect to the Laplacian.

The main result of our paper is contained in the following two theorems.

Theorem 3.2. For any $b>0$ $\rho^bR(z)\rho^b$ can be continued meromorphically across the real line below ξ outside of Ω. If $\omega \in \Omega$, $\omega<\xi$ and the dimension is odd then $\rho^bR(z)\rho^b$ can be continued meromorphically onto a neighborhood of ω on the Riemann surface of $(z-\omega)^{1/2}$. If the dimension is even then the same is true with $\log(z-\omega)$ replacing $(z-\omega)^{1/2}$.

Theorem 3.3. Fix two unit vectors $\hat{e}_1$ and $\hat{e}_2$. Fix two channels ϕ_α and ϕ_β. Then the scattering amplitude $t_{\alpha\beta}[(2\mu_\alpha(z-\omega_\alpha))^{1/2}\hat{e}_1,(2\mu_\beta(z-\omega_\beta))^{1/2}\hat{e}_2]$ can be continued meromorphically in z across the real line below ξ outside of Ω. If $\omega \in \Omega$, $\omega<\xi$ and the dimension is odd then the scattering amplitude can be continued meromorphically onto a neighborhood of ω on the Riemann surface of $(z-\omega)^{1/2}$. If the dimension is even then the same is true with $\log(z-\omega)$ replacing $(z-\omega)^{1/2}$. This means that at each 2-cluster threshold that lies below the lowest at least 3-cluster threshold the scattering amplitude has at worst a square root branch point singularity for odd dimensions and a logarithmic singularity for even dimensions.

Most of our paper will be devoted to proving these results. Throughout this chapter though we will not give full proofs. We will assume in this chapter that $V_{ij}(\rho^{ij,c})^{-1} \in L^{\infty}$. The case of singular potentials will be studied in chapter 5. Moreover, we defer some technical lemmas to chapter 4.

Lemma 3.4. We can find a>0 such that

$(\rho^{D_2,a})^{-1}P^{\epsilon}_{D_2}(\rho^{D_2,a})^{-1}$ is bounded.

Proof. By the HVZ theorem, the part of the spectrum of H^{D_2} associated with the range of $P^{\epsilon}_{D_2}$ is pure point and lies at least a distance ϵ below the bottom of the continuous spectrum. It is well known (see [RS4] theorems XIII.39 and 40) that the eigenvectors with such energies belong to the domain of the multiplication by $\exp(a|x^{D_2}|)$ for some $a>0$ (a may depend on ϵ). QED.

From now on we will usually omit a in $\rho^{D_2,a}$ and we will assume that it has a value determined for a given ϵ by the above lemma. We will also usually omit z in $R(z)$ and $R_D(z)$ and ϵ in $P^{\epsilon}_{D_2}$.

Lemma 3.5. Let ϕ_α be a 2-cluster channel, ω_α the corresponding threshold and $\sigma\not\subset D(\alpha)$. Then we can find $b>0$ such that for any unit vector $\hat{e}$, the function

$$z \longrightarrow V_\sigma(\rho^b)^{-1}\Phi_\alpha[(2\mu_\alpha(\omega_\alpha-z))^{1/2}\hat{e}]$$

defined for real z greater than ω_α can be continued analytically onto a neighborhood of the real line outside of ω_α. It can also be continued onto a neighborhood of ω_α on the Riemann surface of $(z-\omega_\alpha)^{1/2}$.

Proof. We easily see that ϕ_α and V_σ with $\sigma\not\subset D_2(\alpha)$ have enough fall-off to make up for the growth of $\exp[i(\omega_\alpha-z)^{1/2}\hat{e}x_{D_2(\alpha)}]$ and $(\rho^b)^{-1}$. QED.

Lemma 3.6. a) If $i\geqslant 3$ then $(V_\sigma)^{1/2}R_{E_i}|V_{\sigma'}|^{1/2}$,

$(V_\sigma)^{1/2}R_{E_i}P_{E_2}|V_{\sigma'}|^{1/2}$ and $(V_\sigma)^{1/2}R_{E_i}P_{E_2}(\rho^{E_2})^{-1}|V_{\sigma'}|^{1/2}$

are analytic on $\mathbb{C} - [\xi,\infty)$.

b) $(V_\sigma)^{1/2} R_{E_2} (1-P_{E_2}) |V_{\sigma'}|^{1/2}$ is analytic on $\mathbb{C} - [\xi-\epsilon,\infty)$.

Proof. a) By the HVZ theorem the spectrum of an at least 3-cluster Hamiltonian belongs to $[\xi,\infty)$. b) is similarly obvious. QED.

Lemma 3.7. The following expressions can be continued analytically onto a neighborhood of the real line outside of the eigenvalues of H^{F_2}. If ω is an eigenvalue of H^{F_2} then they can be continued onto a neighborhood of ω on the Riemann surface of $(z-\omega)^{1/2}$ for an odd d and of $\log(z-\omega)$ for an even d.

a) $(V_\sigma)^{1/2} R_{F_2} P_{F_2} (\rho^{F_2})^{-1} |V_{\sigma'}|^{1/2}$ where $\sigma,\sigma' \not\subset F_2$,

b) $\rho^{E_2} (V_\sigma)^{1/2} R_{F_2} P_{F_2} (\rho^{F_2})^{-1} |V_{\sigma'}|^{1/2}$ where $\sigma' \not\subset F_2$, $E_2 \neq F_2$

and c) $(V_\sigma)^{1/2} P_{E_2} R_{F_2} P_{F_2} (\rho^{F_2})^{-1} |V_{\sigma'}|^{1/2}$ where $E_2 \neq F_2$, and $\sigma' \not\subset F_2$.

Proof. The lemma follows easily from the proof lemma 1 of the Appendix to §XI.6 of [RS3] which says the following: $\rho^b (z+\Delta)^{-1} \rho^b$ can be analytically continued across the positive real axis onto the nonphysical sheet as long as $\mathrm{Im}(z^{1/2}) > -b$. See also [KS]. QED.

Now we want to introduce our basic equation for the resolvent. If $k>l$ and $D_k \subset D_l$ we define

$$L_{D_l, D_k}$$

$$= \sum_{D_k \subset D_{k-1} \subset \dots \subset D_l} R_{D_k}(V_{D_{k-1}} - V_{D_k})R_{D_{k-1}} \dots R_{D_{l+1}}(V_{D_l} - V_{D_{l+1}})R_{D_l} .$$

Now we can write

$$R = \sum_{m=1}^{\infty} \sum_{k=2}^{N-1} \sum_{D^1_{N-1} \subset D^1_2 \not\supset D^2_{N-1} \subset \dots \subset D^{m-1}_2 \not\supset D^m_{N-1} \subset D^m_k}$$

$$R_0 V_{D^1_{N-1}} L_{D^1_{N-1}, D^1_2} V_{D^2_{N-1}} L_{D^2_{N-1}, D^2_2} \dots L_{D^{m-1}_{N-1}, D^{m-1}_2} V_{D^m_{N-1}} L_{D^m_{N-1}, D^m_k} .$$

The above formula was studied in section III of [Ha1] where the reader will find a more detailed derivation and discussion of it. It is closely related to the Weinberg-Van Winter equation (see lemma 4.2 and §XII5 of [RS4]). The formula is valid for z sufficiently large and negative, for which z the series converges.

A sequence $D_k \subset D_{N-2} \subset \dots \subset D_1$ will be called a string.

Now we transform our expression for R in the following way: for $i>1$ each $V_{D^i_{N-1}}$ that appears between $R_{D^{i-1}_2}$ and $R_{D^i_{N-1}}$ we replace with

$$\Big[(1-P_{D_2^{i-1}})\Big|V_{D_{N-1}^i}\Big|^{1/2}\cdot(V_{D_{N-1}^i})^{1/2}$$

$$+\ P_{D_2^{i-1}}(\rho^{D_2^{i-1}})^{-1}\Big|V_{D_{N-1}^i}\Big|^{1/2}\times\rho^{D_2^{i-1}}(V_{D_{N-1}^i})^{1/2}\Big]$$

where both "$\cdot$" and "x" denote multiplication. We expand all the square brackets. Each summand of our series we factor by "cutting" it in the following places:

- at each "x",
- at each "$\cdot$" that "belongs" to $V_{D_{N-1}^i}$ unless there is "x" at $V_{D_{N-1}^{i+1}}$.

Our aim is to convert the series into some matrix formula. We introduce a square matrix M(z), a row vector A(z), a column vector B(z) and a scalar C(z) with entries from $B(L^2(X))$ and with indices of the form (D_2,E_{N-1}) or (E_{N-1}).

$$M_{(D_2,E_{N-1})(F_2,G_{N-1})}=\sum_{D_2\not\supset E_{N-1}\subset E_2\not\supset F_{N-1}\subset F_2\not\supset G_{N-1}}$$

$$\rho^{D_2}(V_{E_{N-1}})^{1/2}L_{E_{N-1},E_2}(1-P_{E_2})$$

$$\times\ V_{F_{N-1}}L_{F_{N-1},F_2}P_{F_2}(\rho^{F_2})^{-1}\Big|V_{G_{N-1}}\Big|^{1/2}$$

$$+ \sum_{D_2 \ddagger E_{N-1} C F_2 \ddagger G_{N-1}} \rho^{D_2} (V_{E_{N-1}})^{1/2} L_{E_{N-1},F_2} P_{F_2} (\rho^{F_2})^{-1} \left| V_{G_{N-1}} \right|^{1/2}$$

$$M_{(E_{N-1})(F_2,G_{N-1})} = \sum_{E_{N-1} C E_2 \ddagger F_{N-1} C F_2 \ddagger G_{N-1}}$$

$$(V_{E_{N-1}})^{1/2} L_{E_{N-1},E_2} (1-P_{E_2}) V_{F_{N-1}} L_{F_{N-1},F_2} P_{F_2} (\rho^{F_2})^{-1} \left| V_{G_{N-1}} \right|^{1/2}$$

$$M_{(D_2,E_{N-1})(G_{N-1})}$$

$$= \sum_{D_2 \ddagger E_{N-1} C E_2 \ddagger G_{N-1}} \rho^{D_2} (V_{E_{N-1}})^{1/2} L_{E_{N-1},E_2} (1-P_{E_2}) \left| V_{G_{N-1}} \right|^{1/2}$$

$$M_{(E_{N-1})(G_{N-1})}$$

$$= \sum_{E_{N-1} C E_2 \ddagger G_{N-1}} (V_{E_{N-1}})^{1/2} L_{E_{N-1},E_2} (1-P_{E_2}) \left| V_{G_{N-1}} \right|^{1/2}$$

$$A_{(F_2,G_{N-1})} = \sum_{E_{N-1} C E_2 \ddagger F_{N-1} C F_2 \ddagger G_{N-1}}$$

$$R_0 V_{E_{N-1}} L_{E_{N-1},E_2} (1-P_{E_2}) V_{F_{N-1}} L_{F_{N-1},F_2} P_{F_2} (\rho^{F_2})^{-1} \left| V_{G_{N-1}} \right|^{1/2}$$

$$+ \sum_{F_{N-1} C F_2 \ddagger G_{N-1}} R_0 V_{F_{N-1}} L_{F_{N-1},F_2} P_{F_2} (\rho^{F_2})^{-1} \left| V_{G_{N-1}} \right|^{1/2}$$

$$A_{(G_{N-1})} = \sum_{E_{N-1} \subset E_2 \not\supset G_{N-1}} R_0 V_{E_{N-1}} L_{E_{N-1},E_2} (1-P_{E_2}) \left| V_{G_{N-1}} \right|^{1/2}$$

$$B_{(D_2,E_{N-1})} = \sum_{i=2}^{N-1} \sum_{D_2 \not\supset E_{N-1} \subset E_i} \rho^{D_2} (V_{E_{N-1}})^{1/2} L_{E_{N-1},E_i}$$

$$B_{(E_{N-1})} = \sum_{i=2}^{N-1} \sum_{E_{N-1} \subset E_i} (V_{E_{N-1}})^{1/2} L_{E_{N-1},E_i}$$

$$C = R_0 + \sum_{i=2}^{N-1} \sum_{D_{N-1} \subset D_i} R_0 V_{D_{N-1}} L_{D_{N-1},D_i}$$

Using geometric series to resum the expression for R we obtain:

Equation 3.8.

$$R(z) = A(z) \frac{1}{1-M(z)} B(z) + C(z).$$

The above formula strongly resembles the one used by G. Hagedorn in [Ha1] to prove asymptotic completeness for the case of 3 and 4 bodies.

Proof of Theorem 3.2. Fix $\epsilon > 0$. First we prove that for any $b>0$ $M(z)$, $\rho^b A(z)$, $B(z)\rho^b$ and $\rho^b C(z)\rho^b$ can be continued analytically across the real line below $\xi-\epsilon$ outside of Ω and if $\omega \in \Omega$ and $\omega<\xi-\epsilon$ then they can be continued analytically onto a neighborhood of ω on the Riemann surface of $(z-\omega)^{1/2}$ or $(z-\omega)^{1/2}$ (depending on the dimension). A direct application of lemma 3.6 proves this fact for the following terms: $M_{(D_2,E_{N-1})(G_{N-1})}$, $M_{(E_{N-1})(G_{N-1})}$ and $\rho^b A_{(G_{N-1})}$. All of the remaining terms contain $R_{D_2} P_{D_2}$ and have a complicated structure involving one or two strings. To prove their analyticity we have to apply also Lemma 3.7. But before this we have to do some algebraic manipulations on such expressions. These manipulations involve a repeated use of the resolvent identity and some combinatorics - proofs are given in the next chapter. In a standard way - by studying the Hilbert-Schmidt norm - we show that $M(z)$ is compact and goes to zero for large negative z. By the analytic Fredholm theorem $\frac{1}{1-M(z)}$ is meromorphic on the domain of analyticity of $M(z)$. Thus by equation 3.8 and by the fact that we can take ϵ arbitrarily small, we see that $\rho R(z)\rho$ has the desired analytic properties. QED.

Proof of Theorem 3.3. We apply theorem 3.2, lemma 3.5 and the definition of $t_{\alpha\beta}$. QED.

4. One- and Two-String Expressions.

We begin with an essentially combinatoric lemma.

Lemma 4.1. a) Fix D_k and D_j. Then

$$\sum_{D_k \subset D_{k-1} \subset D_j} (V_{D_{k-1}} - V_{D_k}) = V_{D_j} - V_{D_k}.$$

b) Fix D_a, b and c. Then for some numbers $A(a,b,c)$ we have

$$\sum_{D_a \subset D_b \subset D_c} (V_{D_b} - V_{D_a}) = A(a,b,c)(V - V_{D_a}).$$

Proof. a) If $\sigma \subset D_k$ or $\sigma \not\subset D_j$ then V_σ does not belong to $V_{D_{k-1}} - V_{D_k}$. Assume it is not the case. Then σ belongs to exactly one D_{k-1} such that $D_k \subset D_{k-1} \subset D_j$.

b) If $\sigma \subset D_a$ then $V_{D_b} - V_{D_a}$ does not contain V_σ. Let $\sigma \not\subset D_a$. The number of D_b's such that $D_a \subset D_b$ and $\sigma \subset D_b$ is equal to the number of partitions of an a-1-element set into b nonempty subsets. The number of D_c-s such that $D_b \subset D_c$ is equal to the number of partitions of a b-element set into c nonempty subsets. $A(a,b,c)$ equals the product of both these numbers. QED.

The terms in $\rho^b A$, M, $B\rho^b$ and $\rho^b C \rho^b$ that we study fall into two categories: in the first one there are only products involving one long string, in the second one there are sums of products involving two strings. First we study a typical expression involving one string.

Lemma 4.2. Fix D_k and D_{k-m}. Then

$$L_{D_k,D_{k-m}} = \sum_{D_k \subset D_j \subset D_{k-m}} C(k,j,k-m) R_{D_j}$$

where C() are some numerical coefficients.

Proof. We prove our lemma by induction on m. For m=0 the lemma is obvious. Assume it to be true for some m.

$$L_{D_k,D_{k-m-1}} = \sum_{D_k \subset D_{k-1} \subset D_{k-m-1}} R_{D_k}(V_{D_{k-1}} - V_{D_k}) L_{D_{k-1},D_{k-m-1}}$$

$$= \sum_{D_k \subset D_{k-1} \subset D_j \subset D_{k-m-1}} C(k-1,j,k-m-1) R_{D_k}(V_{D_{k-1}} - V_{D_k}) R_{D_j}$$

$$= \sum_{D_k \subsetneq D_j \subset D_{k-m-1}} C(k-1,j,k-m-1) R_{D_k}(V_{D_j} - V_{D_k}) R_{D_j}$$

$$= \sum_{D_k \subsetneq D_j \subset D_{k-m-1}} C(k-1,j,k-m-1)(R_{D_j} - R_{D_k})$$

We used in the following order: the induction step, lemma 4.1 a) and the resolvent identity. QED.

Consequence 4.3. Let $\epsilon>0$. Then for any $b>0$ the first term in $M_{(D_2,E_{N-1})(F_2,G_{N-1})}$, the first term in

$\rho^b A_{(F_2,G_{N-1})}$, $B\rho^b$ and $\rho^b C\rho^b$ can be continued analytically across the real line below $\xi-\epsilon$ outside of Ω. Moreover, if $\omega \in \Omega$ and $\omega<\xi-\epsilon$ then they can be continued analytically onto a neighborhood of ω on the Riemann surface of $(z-\omega)^{1/2}$ for an odd d and of $\log(z-\omega)$ for an even d.

Now we look closer at two-string expressions. Cluster decompostions that belong to the left hand side string will be denoted by B_i and those that belong to the right hand side string - by D_i. We break up our study into a series of lemmas.

Lemma 4.4. Fix B_m and D_2. Then

$$\sum_{B_m \subset B_3 \subset B_2} L_{B_m,B_3}(V_{B_2}-V_{B_3})R_{D_2} = \sum_{\sigma \not\subset D_2} Q_\sigma V_\sigma R_{D_2} + Y + ZR_{D_2}$$

where Q_σ and Y are sums of R_{D_j}'s with j greater than 2 and Z is some number.

Proof. Consider the expression on the left hand side in the lemma. By lemma 4.2 it is equal to

$$\sum_{B_m \subset B_j \subset B_3 \subset B_2} C(m,j,3)R_{B_j}(V_{B_2}-V_{B_3})R_{D_2}.$$

By lemma 4.1 b) it can be rewritten as

$$\sum_{B_m \subset B_j} C(m,j,3)A(j,3,2)R_{B_j}(V-V_{B_j})R_{D_2}.$$

Now: $R_{B_j}(V-V_{B_j})R_{D_2} = R_{B_j}(V-V_{D_2})R_{D_2} + R_{D_2} - R_{B_j}$. QED.

Lemma 4.5. Fix B_{N-1} and D_2. Then

$$\sum_{B_{N-1} \subset B_3 \subset B_2 \neq D_2} V^{1/2}_{B_{N-1}} L_{B_{N-1},B_3}(V_{B_2}-V_{B_3})R_{D_2}$$

$$= \sum_{\sigma \not\subset D_2} C_\sigma V_\sigma^{1/2} R_{D_2} + C$$

where C_σ's and C are sums of products of V's, $V^{1/2}$'s and R_{D_l}'s with l greater than 2.

Proof. Assume first that $B_{N-1} \subset D_2$. The expression can be rewritten as

$$\sum_{m=2}^{N-2} \sum_{\substack{B_{N-1} \subset B_{m+1} \subset B_m \subset B_3 \subset B_2 \\ B_{m+1} \subset D_2;\ B_m \not\subset D_2}}$$

$$L_{B_{N-1},B_{m+1}}(V_{B_m}-V_{B_{m+1}})L_{B_m,B_3}(V_{B_2}-V_{B_3})R_{D_2}.$$

Now we apply lemma 4.4 and notice that $V_{B_m}-V_{B_{m+1}}$ consists of V_σ's with $\sigma \not\subset D_2$.

Now let $B_{N-1} \not\subset D_2$. Then we can apply lemma 4.4 immediately. QED.

Lemma 4.6. Fix B_{N-1} and D_2. Then

$$\sum_{B_{N-1}\subset B_2\not\supset D_{N-1}\subset D_2} V^{1/2}_{B_{N-1}} L_{B_{N-1},B_2}(1-P_{B_2})V_{D_{N-1}} L_{D_{N-1},D_2}$$

$$=\sum_{\sigma\not\supset D_2} Q_\sigma V^{1/2}_\sigma R_{D_2} + Y + \sum_{B_2\neq D_2} Z_{B_2} P_{B_2} R_{D_2}$$

where Q_σ's, Y and Z_{B_2}'s are sums of products of V's, $V^{1/2}$'s, $(1-P_{B_2})R_{B_2}$'s and R_{D_j}'s with j greater than 2.

Proof. First we apply lemma 4.2 to L_{D_{N-1},D_2}. We get a sum that includes only terms R_{D_j} with j greater than 2 and R_{D_2}. The former we will include in Y, what is left has the form

$$\sum_{B_{N-1}\subset B_3\subset B_2\not\supset D_{N-1}\subset D_2} V^{1/2}_{B_{N-1}} C(N-1,2,2)L_{B_{N-1},B_3}(V_{B_2}-V_{B_3})$$

$$\times R_{B_2}(1-P_{B_2})V_{D_{N-1}} R_{D_2}$$

$$= \sum_{B_{N-1}\subset B_3\subset B_2\neq D_2} V^{1/2}_{B_{N-1}} C(N-1,2,2)L_{B_{N-1},B_3}(V_{B_2}-V_{B_3})$$

$$\times \left[R_{D_2} - P_{B_2}R_{D_2} - (1-P_{B_2})R_{B_2} + (1-P_{B_2})R_{B_2}\sum_{B_2\supset\sigma\not\supset D_2} V_\sigma R_{D_2}\right].$$

After expanding the square bracket we can include the third term in Y and the fourth one in Q_σ. The second one will constitute Z. Up to a constant we are left with

$$\sum_{B_{N-1}CB_3CB_2\neq D_2} V^{1/2}_{B_{N-1}} L_{B_{N-1},B_3}(V_{B_2}-V_{B_3})R_{D_2}.$$

The above expression is taken care of by lemma 4.5. QED.

Consequence 4.7. Let $\epsilon>0$. Then for any $b>0$ the second term in $M_{(D_2,E_{N-1})(F_2,G_{N-1})}$, $M_{(E_{N-1})(F_2,G_{N-1})}$, the second term in $\rho^b A_{(F_2,G_{N-1})}$ can be continued analytically across the real line below $\xi-\epsilon$ outside of Ω. Moreover, if $\omega \in \Omega$ and $\omega<\xi-\epsilon$ then they can be continued analytically onto a neighborhood of ω on the Riemann surface of $(z-\omega)^{1/2}$ for an odd d and of $\log(z-\omega)$ for an even d.

5. Singular Potentials.

This chapter shows how to modify the proofs of theorems 3.2 and 3.3 if the potentials are singular and satisfy only assumption 3.1. First we introduce the so called Sobolev spaces $\mathbb{H}_m(\mathbb{R}^k)=(1-\Delta_k)^{-m/2}L^2(\mathbb{R}^k)$ where Δ_k is the k-dimensional Laplacian. We will often write $\mathbb{H}_m$ instead of $\mathbb{H}_m(\mathbb{R}^k)$ if it does not lead to confusion.

Lemma 5.1. Suppose the potentials are form bounded with respect to the Laplacian with a zero bound. Let ϕ be an eigenvector of H^D with the energy below the continuous spectrum. Then for some a>0 we have $(\rho^{D,a}\phi)^{-1} \in \mathbb{H}_1(X^D)$.

Proof. It is well known that for some a>0 we have $(\rho^{D,a})^{-1}\phi \in L^2$ (see [RS4]). We will drop all the reference to D in our computations. We denote $(\rho^a)^{-1}$ by exp(F) and exp(F)ϕ by ϕ_F. We easily compute the following formula (see [FH]):

$$\Delta\phi_F = \left[\nabla(\nabla F) + (\nabla F)\nabla\right]\phi_F - (\nabla F)^2\phi_F + \exp(F)\Delta\phi.$$

We apply it to our Hamiltonian.

$$2\left(\phi_F,H\phi_F\right) = \left(H\phi_F,\phi_F\right) + \left(\phi_F,H\phi_F\right)$$

$$= \left(e^F H\phi,\phi_F\right) - \left(\left[\nabla(\nabla F) + (\nabla F)\nabla\right]\phi_F,\phi_F\right) + \left((\nabla F)^2\phi_F,\phi_F\right)$$

$$+ \left(\phi_F,e^F H\phi\right) - \left(\phi_F,\left[\nabla(\nabla F) + (\nabla F)\nabla\right]\phi_F\right) + \left(\phi_F,(\nabla F)^2\phi_F\right)$$

$$= 2E\left(\phi_F,\phi_F\right) + 2\left(\phi_F,(\nabla F)^2\phi_F\right) < \infty$$

Since $(\nabla F)^2 = (a\nabla|x|)^2 = a^2$ is bounded we can see that ϕ belongs to $\mathbb{H}_1$. QED.

Lemma 5.2. For any $b>0$ $\rho^b(z+\Delta)^{-1}\rho^b$ can be extended to an analytic function on the part of the Riemann surface of the square root defined by $\mathrm{Im}(z^{1/2})>-b$, with values in bounded operators from $\mathbb{H}_{-1}$ to $\mathbb{H}_1$.

Proof. First we see that

$$(-\Delta+1)\rho^b(z+\Delta)^{-1}\rho^b$$
$$= \left[(-\Delta\rho^b) - 2(\nabla\rho^b)\nabla + \rho^b(-\Delta+1)\right](z+\Delta)^{-1}\rho^b$$
$$= \left[(-\Delta\rho^b) + (z+1)\rho^b\right](z+\Delta)^{-1}\rho^b + (-2\nabla\rho^b)\nabla(z+\Delta)^{-1}\rho^b - (\rho^b)^2.$$

By mimmicking the proof of lemma 1 of the appendix to §XI.6 of [RS3] we show that the above expression extends to an analytic family of bounded operators from L^2 to L^2 on the desired complex domain. This means that $\rho^b(z+\Delta)^{-1}\rho^b$ is analytic on the same domain as a function with values in bounded operators from L^2 to $\mathbb{H}_2$ and, by an analoguous argument, as a function with values in bounded operators from $\mathbb{H}_{-2}$ to L^2. Now we apply interpolation. QED.

Equipped with these two lemmas, we can easily modify the proofs from chapter 3 to include the singular potentials. For instance, to prove lemma 3.7 a) we write

$$(V_\sigma)^{1/2}R_{F_2}P_{F_2}(\rho^{F_2,a})^{-1}\left|V_{\sigma'}\right|^{1/2}$$

$$= \left[(V_\sigma)^{1/2} P_{F_2} (\rho_{F_2}{}^{b})^{-1}\right] \left[\rho_{F_2}{}^{b} P_{F_2} R_{F_2} \rho_{F_2}{}^{b}\right]$$

$$\times \left[(\rho_{F_2}{}^{b})^{-1} P_{F_2} (\rho^{F_2,a})^{-1} \left|V_{\sigma'}\right|^{1/2}\right]$$

$$= QY(z)Z$$

Now for some $a,b>0$ the term Z maps L^2 into $\mathbb{H}_{-1}$, $Y(z)$ maps $\mathbb{H}_{-1}$ into $\mathbb{H}_1$ and has the desired analytic properties and Q maps $\mathbb{H}_1$ into L^2, which proves lemma 3.7 a).

6. Asymptotic Completeness and Existence of Scattering Amplitudes.

In this chapter instead of being interested in an analytic continuation of the resolvent onto the nonphysical sheet we are studying here just its continuous limit up to the real axis. That allows us to weaken assumptions on potentials, which instead of decaying exponentially have to decay only as $r^{-1-\epsilon}$. The main link of this chapter with the preceding ones is theorem 6.4, which makes an extensive use of methods developed there. It deals with continuing the resolvent up to the real axis. Related properties of the resolvent of the Schrödinger equation were sometimes called a "limiting absorbtion principle" or a "limiting similarity principle" and were studied in the context of the scattering theory or of the absolute continuity of the spectrum (see for instance in [A], [Ha1], [Ho], [Sig1] and [Sig5]). By using standard methods of the stationary scattering theory and theorem 6.4 we are able to obtain various kinds of information about the scattering below the lowest 3-cluster threshold, such as the existence of a generalized eigenfunction expansion (theorem 6.6), asymptotic completeness (theorem 6.7) and the existence and the continuity of the scattering amplitude outside an exceptional set (theorem 6.8).

We need some additional definitions. If ϕ_α is a channel then we define the imbedding J_α of $L^2(X_{D(\alpha)})$ in $L^2(X)$ by the following formula:

$$J_\alpha(f) = f\otimes\phi_\alpha.$$

Then we define the channel wave operators:

$$W_\alpha^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} \exp(-iHt)\exp(iH_{D(\alpha)}t)J_\alpha.$$

$E_\alpha(B)$ will denote the spectral projection of T_α corresponding to a measurable set $B \subset \mathbb{R}$. $E_{ac}(B)$ will denote the spectral projection onto the absolutely continuous part of the spectrum of H belonging to B.

Throughout this chapter δ will be a fixed number greater than 1/2. Let γ, γ_D and γ^D be multiplication operators by $\left[1+|x|^2\right]^{-(1/2)\delta}$, $\left[1+|x_D|^2\right]^{-(1/2)\delta}$ and $\left[1+|x^D|^2\right]^{-(1/2)\delta}$ resp.

We need some formalism enabling us to restrict Fourier transforms to d-1-spheres. Let $\mathcal{S}(\mathbb{R}^d)$ denote the space of Schwartz functions on $\mathbb{R}^d$. For $\nu>0$ we define $\pi(\nu):\mathcal{S}(\mathbb{R}^d)\to L^2(S^{d-1},d\hat{e})$ by $(\pi(\nu)f)(\hat{e})=\nu^{(d-1)/2}\hat{f}(\nu\hat{e})$, where $\hat{e}$ belongs to the unit sphere S^{d-1}, $\hat{f}$ is the Fourier transform of f, and $d\hat{e}$ is the invariant surface measure on $S^{d-1}\subset\mathbb{R}^d$. If $\mathbb{R}^d$ in the above definition is equal to X_D for some 2-cluster decomposition D then such a $\pi(\nu)$ will be denoted by $\pi_D(\nu)$.

$\mathbb{H}_{m,n}$ denotes the space $(1+|x|^2)^{-n/2}(1-\Delta)^{-m/2}L^2$.

Assumption 6.1. $|V_\sigma|^{1/2}(\gamma^\sigma)^{-1}$ is bounded from H_1 to L^2. (In other words $|V_\sigma|^{1/2}$ is bounded from $H_{1,\delta}$ to L^2).

Proposition 6.2. Suppose $\delta>1/2$. Then $\pi(\nu)$ extends to a bounded mapping of $H_{0,\delta}(\mathbb{R}^d)$ into $L^2(S^{d-1},d\hat{e})$. Moreover $\nu\to\pi(\nu)$ is norm Hölder continuous.

Proof. See [GM]. QED.

We note also that if $f \in L^2$ then $\pi(\nu)f$ makes sense for almost all ν as an $L^2(S^{d-1},d\hat{e})$-valued measurable function.

Lemma 6.3. a) $(\lambda+i\epsilon+\Delta_d)^{-1}$ viewed as an operator from $\mathbb{H}_{1,\delta}$ into $\mathbb{H}_{-1,-\delta}$ can be written as a weak Riemann integral

$$\int d\nu(\lambda+i\epsilon-\nu^2)^{-1}\pi(\nu)^*\pi(\nu).$$

Besides if λ is not equal 0 and $\epsilon\to 0^+$ then $(\lambda+i\epsilon+\Delta_d)^{-1}$ has a weak limit as an operator from $\mathbb{H}_{1,\delta}$ into $\mathbb{H}_{1,\delta}$.

Proof. By the spectral theorem and properties of the Fourier transformation this representation is obvious as a map from $\mathcal{D}$ into $\mathcal{D}'$. Once we know this the first result follows by the density argument and proposition 6.2. Now let $f,g \in H_{1,\delta}$. Let χ be the characteristic function of some neighborhood of λ.

$$(f,(\lambda+i\epsilon-\Delta)^{-1}g)$$
$$= ((1-\Delta)^{-1/2}f,(1-\Delta)(\lambda+i\epsilon-\Delta)^{-1}\chi(\Delta)(1-\Delta)^{-1/2}g)$$

$+ ((1-\Delta)^{-1/2}f, (1-\Delta)(\lambda+i\epsilon-\Delta)^{-1}(1-\chi(\Delta))(1-\Delta)^{-1/2}g)$

Note that $(1-\Delta)^{-1/2}f$ and $(1-\Delta)^{-1/2}g$ belong to $\mathbb{H}_{0,\delta}$. The second term causes no problem. To show the existence of the limit of the first term we study its integral representation. In the neighborhood of $\nu=\lambda^{1/2}$ we use the Hölder continuity of $\pi(\nu)$, which takes care of the singularity. (See also [GM]). QED.

Now we state the main technical theorem of this chapter.

Theorem 6.4. Suppose that assumption 6.1 holds. Then there exists a closed set $\mathbb{E}\subset[\omega_1,\xi]$ of measure zero such that

$R(z)$

$$= \sum_{\substack{\sigma\not\subset D_2 \\ \sigma'\not\subset D_2'}} P_{D_2} R_{D_2}(z)|V_\sigma|^{1/2} F_{D_2,\sigma,D_2',\sigma'}(z)(V_{\sigma'})^{1/2} R_{D_2'}(z) P_{D_2'}$$

$$+ \sum_{\sigma\not\subset D_2} P_{D_2} R_{D_2}(z)|V_\sigma|^{1/2} F_{D_2,\sigma,\mathrm{reg}}(z)$$

$$+ \sum_{\sigma'\not\subset D_2'} F_{\mathrm{reg},D_2',\sigma'}(z)(V_{\sigma'})^{1/2} R_{D_2'}(z) P_{D_2'} + F_{\mathrm{reg},\mathrm{reg}}(z)$$

where $F_{D_2,\sigma,D_2',\sigma'}(z)$, $F_{D_2,\sigma,\mathrm{reg}}(z)$, $F_{\mathrm{reg},D_2',\sigma'}(z)$ and $F_{\mathrm{reg},\mathrm{reg}}(z)$ are analytic functions with values in bounded operators from $L^2(X)$ to $L^2(X)$ that can be extended continuously up to $(\omega_1,\xi)-\mathbb{E}$.

Proof. The proof is basically identical to that of theorem 3.2. The most essential differences consist in replacing $\rho^{D,b}$ by γ^D; moreover, we should replace "a function that can be continued analytically onto..." by "a function continuous up to $(\omega_1,\xi)-\Omega$". Instead of the analytic Fredholm theorem we have to use its modification from [RS3] chapter XI.6. By virtue of this theorem "a function that can be continued meromorphically onto..." should be replaced by "a function continuous up to $(\omega_1,\xi)-\mathbb{E}$". Besides, by using lemma 4.2 one can rewrite $A(z)$ so that all its singular terms begin with $P_{D_2}R_{D_2}(z)|V_\sigma|^{1/2}$. Similar algebraic transformation we have to do on $C(z)$ and $B(z)$. QED.

The above theorem is an important step in proving asymptotic completeness of wave operators; as it is well known their existence can be shown much easier and in greater generality.

Theorem 6.5. Let $\epsilon>0$. Suppose that $V_\sigma = U_\sigma W_\sigma$ where $U_\sigma(1-\Delta)^{-1/2}$ and $W_\sigma(1-\Delta)^{-1/2}$ are bounded and $W_\sigma \in (1+|x^\sigma|)^{-\epsilon}L^2$ for $d=1$ and $W_\sigma \in L^{d-\epsilon}$ for $d>1$. Then all the channel operators $W_\alpha^{\pm}$ exist.

Proof. Mimick the proof of the analoguous result from

[RS3]. (See proofs of theorems XI.6, XI.16, XI.26, and XI.34 of [RS3]). QED.

Now we state the main results of this chapter. Their proofs will be given at the end of this chapter. The next two theorems follow from theorem 6.4 by the standard techniques of the stationary scattering theory that can be found in [Ha1], [Ho] or [Sig1].

Theorem 6.6. Let V satisfy the assumption 6.1 and let ϕ_α be a 2-cluster channel. Denote also for convenience

$$\sum_{\sigma' \not\subset D_2'} F_{D_2,\sigma,D_2',\sigma'}(z)(V_{\sigma'})^{1/2} R_{D_2'}(z) P_{D_2'}$$

$+ \quad F_{D_2,\sigma,reg}(z)$ by $Z_{D_2,\sigma}(z)$.

Then for $\lambda \in (\omega_1,\xi)-\mathbb{E}$ the limit

$$\text{w-}\lim_{\epsilon\to 0^+} \sum_{\sigma\not\subset D(\alpha)} \pi_{D(\alpha)}(\nu_\alpha(\lambda))J_\alpha(V_\sigma)^{1/2} Z_{D(\alpha),\sigma}(\lambda+i\epsilon)\gamma$$

exists and for almost all λ is equal to $\pi_{D(\alpha)}(\nu_\alpha(\lambda))W_\alpha^{\pm *}\gamma$.

Theorem 6.7. Suppose assumption 6.1 holds. Let ξ be the lowest at least 3-cluster threshold. Then

$$E_{ac}(\omega_1,\xi) = \sum_\alpha W_\alpha^{\pm} E_\alpha(\omega_1,\xi) W_\alpha^{\pm *}.$$

The following theorem may be regarded as an analogue of theorem 3.3 in the case when the potentials fall off like $r^{-1-\epsilon}$.

Theorem 6.8. Let ϕ_α and $\phi_{\alpha'}$ be two-cluster channels. Define the T-matrix for the $\alpha-\alpha'$ scattering by the following formula (see [HS]):

$$T_{\alpha\alpha'}(\lambda) = \Big(\phi_\alpha, \pi_{D(\alpha)}(\nu_\alpha(\lambda))(V-V_{D(\alpha)})\pi_{D(\alpha')}(\nu_{\alpha'}(\lambda))^*\phi_{\alpha'}\Big)$$

$$+ \text{w-}\lim_{\epsilon\to 0^+}\Big(\phi_\alpha, \pi_{D(\alpha)}(\nu_\alpha(\lambda))(V-V_{D(\alpha)})$$

$$\times R(\lambda+i\epsilon)(V-V_{D(\alpha')})\pi_{D(\alpha')}(\nu_{\alpha'}(\lambda))^*\phi_{\alpha'}\Big).$$

Suppose also that assumption 6.1 holds. Then for $\lambda \in (-\infty,\xi)-\mathbb{E}$, $\lambda \to T_{\alpha\alpha'}(\lambda)$ is a continuous function with values in bounded operators from $L^2(S^{d-1},d\hat{e})$ into itself.

It is convenient to introduce the so-called stationary wave operators

$$W^\epsilon_\alpha(A) = \epsilon/\pi\int_A R(\lambda+i\epsilon)R_{D(\alpha)}(\lambda-i\epsilon)J_\alpha d\lambda \text{ and } W^\epsilon_\alpha = W^\epsilon_\alpha(\mathbb{R}).$$

Propositon 6.9. a) $W^\epsilon_\alpha(A)$ is well defined.

b) If $W_\alpha^{\pm}$ exists then $W_\alpha^{\pm} = \text{w-}\lim_{\epsilon\to 0^\pm} W^\epsilon_\alpha$.

c) If $\text{w-}\lim_{\epsilon\to 0^\pm} W^\epsilon_\alpha$ exists then $W_\alpha^{\pm}E_\alpha(A) = \text{w-}\lim_{\epsilon\to 0^\pm} W^\epsilon_\alpha(A)$.

Proof. See [Ha1], [Ho] or [Sig1]. QED.

Lemma 6.10. If ϕ_α is a 2-cluster channel then the following operators map $L^2(X)$ continuously into $\mathbb{H}_{1,\delta}(X_{D(\alpha)})$ and their adjoints map $\mathbb{H}_{-1,-\delta}(X_{D(\alpha)})$ into $L^2(X)$:

$J_\alpha(\gamma^{D(\alpha)})^{-1}|V_\sigma|^{1/2}$ with $\sigma\not\subset D(\alpha)$, $\gamma_{D(\alpha)}J_\alpha$ and $J_\alpha\gamma^E|V_\sigma|^{1/2}$ with $D(\alpha)\neq E$.

Proof. It follows from the decay properties of the eigenfunctions. QED.

Lemma 6.11. If ϕ_α is a channel and $\lambda\neq\omega_\alpha$ then

$$\epsilon(\pi)^{-1}(\lambda+i\epsilon-T_\alpha)^{-1}(\lambda-i\epsilon-T_\alpha)^{-1}$$

considered as a map from $\mathbb{H}_{1,\delta}(X_{D(\alpha)})$ to $\mathbb{H}_{-1,-\delta}(X_{D(\alpha)})$ converges weakly to

$$\pi_{D(\alpha)}(\nu_\alpha(\lambda))^*\pi_{D(\alpha)}(\nu_\alpha(\lambda))\frac{\mu_\alpha}{\nu_\alpha(\lambda)}$$

as $\epsilon\to0^+$. The convergence is uniform on compacts not containing ω_α.

Proof. We drop α from the expressions below. We can write our expression as

$$(\pi)^{-1}\int d\nu\ \epsilon\left[(\lambda-\nu^2/2\mu-\omega)^2+\epsilon^2\right]^{-1}\pi_D(\nu)^*\pi_D(\nu).$$

We change the variables in this expression and obtain

$$(\pi)^{-1}\int\mu d(\nu^2/2\mu)\epsilon\left[(\lambda-\nu^2/2\mu-\omega)^2+\epsilon^2\right]^{-1}\pi_D(\nu)^*\pi_D(\nu)\frac{\mu}{\nu}.$$

Now if we notice that $\epsilon(\pi)^{-1}\left[(\lambda-\nu^2/2\mu-\omega)^2+\epsilon^2\right]^{-1}$ behaves like an approximate delta function we easily complete the proof of the lemma. QED.

Lemma 6.12. If assumption 6.1 holds and $\lambda \in (\omega, \xi) - \mathbb{E}$ then $R(\lambda + i\epsilon)$ has a limit as $\epsilon \to 0^+$ as a map from $\mathbb{H}_{1,\delta}(X)$ to $\mathbb{H}_{-1,\delta}(X)$.

Proof. Using theorem 6.4 we write

$$R(z) = \sum_{\substack{\sigma \not\subset D_2(\alpha) \\ \sigma' \not\subset D_2(\alpha')}} J_\alpha^*(z - T_\alpha)^{-1} J_\alpha |V_\sigma|^{1/2}$$

$$\times F_{D_2(\alpha), \sigma, D_2(\alpha'), \sigma'}(z)(V_{\sigma'})^{1/2} J_{\alpha'}^*(z - T_{\alpha'})^{-1} J_{\alpha'}$$

+ other terms.

Now we note that $J_{\alpha'}$ maps $H_{1,\delta}(X)$ into $H_{1,\delta}(X_{D(\alpha')})$, $(z - T_{\alpha'})^{-1}$ has a limit between $\mathbb{H}_{1,\delta}(X_{D(\alpha')})$ and $\mathbb{H}_{-1,-\delta}(X_{D(\alpha')})$, $(V_{\sigma'})^{1/2} J_{\alpha'}$ maps it into L^2, etc. The other terms are handled in a similar fashion. QED.

Lemma 6.13. Assume all the hypotheses of theorem 6.6 and write D, μ, ω and $\nu(\lambda)$ instead of $D(\alpha)$, μ_α, ω_α and $\nu_\alpha(\lambda)$. Then if B is a compact subset of $(\omega_1, \xi) - \mathbb{E}$ the following is true:

$$\gamma_D E_\alpha(B) W_\alpha^{\pm *} \gamma = \sum_{\sigma \not\subset D} \int_{\nu^2/2\mu + \omega \in B}$$

$$d\nu \gamma_D \pi_D^*(\nu) \pi_D(\nu) J_\alpha^*(V_\sigma)^{1/2} Z_{D,\sigma}(\nu^2/2\mu + \omega + i0)\gamma.$$

Proof.

$$\gamma_D E_\alpha(B) W_\alpha^{\pm *}\gamma = \underset{\epsilon\to 0^+}{\text{w-lim}}\ \gamma_D W_\alpha^{\epsilon *}(B)\gamma = \underset{\epsilon\to 0^+}{\text{w-lim}}\int_B \sum_{\sigma\not\subset D(\beta)} d\lambda$$

$$\times\ \epsilon(\pi)^{-1}\gamma_D J_\alpha^{*}(\lambda+i\epsilon-T_\alpha)^{-1}(\lambda-i\epsilon-T_\beta)^{-1}P_\beta(V_\sigma)^{1/2}Z_{D(\beta),\sigma}(\lambda+i\epsilon)\gamma$$

First we notice that by theorem 6.4 the expression $Z_{D(\beta),\sigma}(\lambda+i\epsilon)\gamma$ has continuous boundary values as ϵ goes to 0^+. Next we look at 3 possible kinds of terms that appear in the above sum.

1) D is not equal to $D(\beta)$. We will prove that this term goes to zero. Take $f \in L^2(X_D)$ and $g \in L^2(X)$. Denote $J_\alpha\gamma_D f$ by f_1 and $P_\beta(V_\sigma)^{1/2}Z_{D(\beta),\sigma}(\lambda+i\epsilon)\gamma g$ by $g_1(\lambda)$. Note that f_1 and $g_1(\lambda)$ belong to $\mathbb{H}_{1,\delta}(X_D)\otimes\phi_\alpha$ and $\mathbb{H}_{1,\delta}(X_{D(\beta)})\otimes\phi_\beta$ resp. We can estimate

$$\epsilon(\pi)^{-1}\Big(f,\gamma_D J_\alpha^{*}(\lambda+i\epsilon-T_\alpha)^{-1}(\lambda-i\epsilon-T_\beta)^{-1}$$

$$\times\ P_\beta(V_\sigma)^{1/2}Z_{D(\beta),\sigma}(\lambda+i\epsilon)\gamma g\Big)$$

$$\leqslant\ (\pi)^{-1}\|f_{1,\epsilon}\|\|g_{1,\epsilon}(\lambda)\|$$

where $f_{1,\epsilon}=(\epsilon)^{1/2}(\lambda+i\epsilon-T_\alpha)^{-1}f_1$ and $g_{1,\epsilon}(\lambda)=(\epsilon)^{1/2}(\lambda+i\epsilon-T_\beta)^{-1}g_1(\lambda)$. Now using the continuity of $g_1(\lambda)$ in λ and a standard argument involving Stone's formula we prove that the integral of the right hand side is uniformly bounded as ϵ goes to 0^+. Next we will prove that the integrand goes to zero pointwise. We estimate

$$\epsilon\Big(f_1(\lambda+i\epsilon-T_\alpha)^{-1}(\lambda+i\epsilon-T_\beta)^{-1}g_1(\lambda)\Big)$$

$\leqslant \|(1-E_\alpha(K_r))f_{1,\epsilon}\|\|(1-E_\beta(K_r))g_{1,\epsilon}(\lambda)\|$

$+ \|E_\alpha(K_r)f_{1,\epsilon}\|\|(1-E_\beta(K_r))g_{1,\epsilon}(\lambda)\|$

$+ \|(1-E_\alpha(K_r))f_{1,\epsilon}\|\|E_\beta(K_r)g_{1,\epsilon}(\lambda)\|$

$+ \|E_\alpha(K_r)E_\beta(K_r)f_{1,\epsilon}\|\|E_\alpha(K_r)E_\beta(K_r)g_{1,\epsilon}(\lambda)\|$

where K_r is the set of those $x \in \mathbb{R}$ which satisfy $|x-\lambda|<r$. Now note that $(1-E_\alpha(K_r))f_{1,\epsilon}$ goes to zero, and $E_\alpha(K_r)f_{1,\epsilon}$ goes to $(\pi\mu)^{1/2}(\nu(\lambda))^{-1/2}\pi_D(\nu_\alpha(\lambda))f_1$. Moreover by some abuse of notation we can write that $E_\alpha(K_r)E_\beta(K_r)f_{1,\epsilon}$ goes to $E_\beta(K_r)\pi_D(\nu_\alpha(\lambda))f_1$ as ϵ goes to 0^+. Since $T_\alpha-T_\beta$ has a zero kernel this term goes to zero as r goes to zero. Similar statements are true about the terms involving g_1. This proves that the whole studied expression goes to zero as ϵ goes to 0^+.

2) $D= D(\beta)$ and α is different than β. This term is zero because $P_\alpha P_\beta = 0$.

3) $\alpha = \beta$. Lemma 6.11 shows that in this case we have

$$\operatorname*{w-lim}_{\epsilon\to 0^+}\int_B \sum_{\sigma\notin D(\beta)} d\lambda\, \epsilon(\pi)^{-1}\gamma_D J_\alpha^*(\lambda+i\epsilon-T_\alpha)^{-1}$$

$$\times\ (\lambda-i\epsilon-T_\beta)^{-1}P_\beta(V_\sigma)^{1/2}Z_{D(b),\sigma}(\lambda+i\epsilon)\gamma$$

$$= \int_{\nu^2/2\mu+\omega\ \in\ B} \sum_{\sigma\notin D} d(\nu^2/2\mu+\omega)$$

$$\times\ \gamma_D J_\alpha^*\pi_D^*(\nu)\pi_D(\nu)P_\alpha(V_\sigma)^{1/2}Z_{D,\sigma}(\nu^2/2\mu+\omega+i0)\gamma\mu(\nu)^{-1}$$

$$= \sum_{\sigma\notin D}\int_{\nu^2/2\mu+\omega\ \in\ B} d\nu$$

$\times\ \gamma_D \pi_D^*(\nu)\pi_D(\nu)J_\alpha^*(V_\sigma)^{1/2}Z_{D,\sigma}(\nu^2/2\mu+\omega+i0)\gamma$. QED.

Proof of theorem 6.6. Let $f \in \mathcal{D}(X_D)$ and $g \in L^2$. Then $(\gamma_D)^{-1}f \in L^2(X_D)$; so by lemma 6.13 we see that

$$\int_{\nu^2/2\mu+\omega\ \in\ B} d\nu\Big(\pi_D(\nu)f, \pi_D(\nu)E_\alpha(B)W_\alpha^{\pm *}\gamma g\Big)$$

$$= \int d\nu\Big(\pi_D(\nu)f, \pi_D(\nu)E_\alpha(B)W_\alpha^{\pm *}\gamma g\Big) = \Big(f, E_\alpha(B)W_\alpha^{\pm *}\gamma g\Big)$$

$$= \sum_{\sigma\not\subset D}\ \int_{\nu^2/2\mu+\omega\in B} d\nu\Big(\pi_D(\nu)f, \pi_D(\nu)J_\alpha^*(V_\sigma)^{1/2}Z_{D,\sigma}(\nu^2/2\mu+\omega+i0)\gamma g\Big)$$

We can easily check that the space of functions on B of the form $\nu \longrightarrow \pi_D(\nu)f$ where $f \in \mathcal{D}(X_D)$ is dense in $L^2(B, L^2(S^{d-1}, d\omega))$. Consequently

$$\pi_D(\nu)E_\alpha(B)W_\alpha^{\pm *}\gamma g$$

$$= \sum_{\sigma\not\subset D} d\nu\pi_D(\nu)J_\alpha^*(V_\sigma)^{1/2}Z_{D,\sigma}(\nu^2/2\mu+\omega+i0)\gamma g$$

for almost all ν. Since our Hilbert space is separable also

$$\pi_D(\nu)E_\alpha(B)W_\alpha^{\pm *}\gamma$$

$$= \sum_{\sigma\not\subset D} d\nu\pi_D(\nu)J_\alpha^*(V_\sigma)^{1/2}Z_{D,\sigma}(\nu^2/2\mu+\omega+i0)\gamma$$

holds for almost all ν. QED.

Proposition 6.14. Suppose all the assumptions from the lemma 6.13 hold. Then

$$E(B) = \sum_\alpha W_\alpha^\pm E_\alpha(B) W_\alpha^{\pm *}.$$

Proof. It suffices to prove that

$$\gamma E(B)\gamma = \gamma \sum_\alpha W_\alpha^\pm E_\alpha(B) W_\alpha^{\pm *}\gamma.$$

Apply first the Stone formula and then theorem 6.4.

$$\gamma E(B)\gamma = \operatorname*{s-lim}_{\epsilon\to 0^+} \epsilon/\pi \int_B d\lambda \gamma R(\lambda+i\epsilon)R(\lambda-i\epsilon)\gamma$$

$$= \operatorname*{s-lim}_{\epsilon\to 0^+} \epsilon/\pi \int_B \sum_{\sigma\not\subset D(\alpha)} \sum_{\sigma'\not\subset D(\alpha')} d\lambda \gamma Z^*_{D(\alpha),\sigma}(\lambda+i\epsilon)(V_\sigma)^{1/2}P_\alpha$$

$$\times\ (\lambda+i\epsilon-T_\alpha)^{-1}(\lambda-i\epsilon-T_{\alpha'})^{-1}P_{\alpha'}(V_{\sigma'})^{1/2}Z_{D(\alpha'),\sigma'}(\lambda+i\epsilon)\gamma$$

Now we repeat the argument from lemma 6.13 that shows that all the terms with $D(\alpha)\neq D(\alpha')$ or $\alpha\neq\alpha'$ vanish. Therefore we get

$$\operatorname*{s-lim}_{\epsilon\to 0^+} \int_B \sum_{\sigma,\sigma'\not\subset D(\alpha)} d\lambda\, \epsilon \gamma Z^*_{D(\alpha),\sigma}(\lambda+i\epsilon)(V_\sigma)^{1/2}P_\alpha(\lambda+i\epsilon-T_\alpha)^{-1}$$

$$\times\ (\lambda-i\epsilon-T_\alpha)^{-1}P_\alpha(V_{\sigma'})^{1/2}Z_{D(\alpha),\sigma'}(\lambda+i\epsilon)\gamma$$

Now we use lemma 6.11 and theorem 6.6 to obtain

$$\int_{\nu^2/2\mu+\omega\ \in\ B} \sum_{\sigma,\sigma'\not\subset D(\alpha)} d\nu$$

$$\times \ \gamma Z^{*}_{D(\alpha),\sigma}(\nu^2/2\mu+\omega+i\epsilon)(V_\sigma)^{1/2}P_\alpha\pi^{*}_{D(\alpha)}(\nu)\pi_{D(\alpha)}(\nu)$$

$$\times \ P_\alpha(V_{\sigma'})^{1/2}Z_{D(\alpha),\sigma'}(\nu^2/2\mu+\omega+i\epsilon)\gamma$$

$$= \int_{\nu^2/2\mu+\omega \ \in \ B} \sum_\alpha d\nu\gamma W_\alpha^{\pm}\pi^{*}_{D(\alpha)}(\nu)\pi_{D(\alpha)}(\nu)P_\alpha W_\alpha^{\pm *}\gamma$$

$$= \sum_\alpha \gamma W_\alpha^{\pm}E_\alpha(B)W_\alpha^{\pm *}\gamma .$$

QED.

The above proposition immediately implies theorem 6.7 when we notice that the set $\mathbb{E}$ is closed and of measure zero.

Proof of theorem 6.8. The first term of $T_{\alpha\alpha'}(\lambda)$ causes no problems. The second one is taken care of by lemma 6.12. QED.

7. Threshold singularities of the 2-cluster-2-cluster amplitudes above the lowest 3-cluster threshold.

Unlike in previous chapters, in this one we have to use the dilation analyticity technique. So first let us introduce a family of unitary operators on $L^2(\mathbb{R}^d)$ defined by

$$(\Gamma(\theta)f)(x) = \exp(d\theta/2)f(\exp(\theta)x).$$

Assumption 7.1. The potentials V_{ij} can be factored into $W_{ij}^{(1)}W_{ij}^{(2)}$ such that for some $c>0$ and $\gamma>0$ $\Gamma(\theta)W_{ij}^{(i)}\Gamma(-\theta)(\rho^{ij,c})^{-1}(1-\Delta^{ij})^{-1/2}$ extend to analytic families in $|\mathrm{Im}\theta|<\gamma$ with values in bounded operators.

The above assumption is a kind of a unification of the dilation analyticity condition (see [BC] and [RS4]) and of the condition of an exponential decay. It appeared before in a similar situation in [Ha2], [HS] and [Sig4]. For simplicity we will conduct parts of our reasoning under a stronger assumption.

Assumption 7.2. For some $c>0$ and $\gamma>0$ $\Gamma(\theta)V_{ij}\Gamma(-\theta)(\rho^{ij,2c})^{-1}$ extends to an analytic family in $|\mathrm{Im}\theta|<\gamma$ with values in L^∞.

It is easy to extend our proof to cover singular potentals satisfying assumption 7.1 by following the ideas of chapter 5.

In this chapter P_D will mean the projection onto the point spectrum of H^D.

Now we state the main theorem of this chapter.

Theorem 7.3. Suppose assumption 7.1 holds. Fix two channels ϕ_α and ϕ_β that correspond to 2-cluster decompositions $D(\alpha)$ and $D(\beta)$ respectively. Assume that the threshold energies ω_α and ω_β that correspond to these channels do not coincide with thresholds of $H_{D(\alpha)}$ and $H_{D(\beta)}$ respectively. Then the following is true.

a) The scattering amplitude $t_{\alpha\beta}[\nu_\alpha(z)\hat{e}_1,\nu_\beta(z)\hat{e}_2]$ exists and can be extended to a meromorphic function of z in a neighborhood of the real axis outside the thresholds of H.

b) If ω is a 2-cluster threshold which does not coincide with a more-than 2-cluster threshold and d is odd then $t_{\alpha\beta}[\nu_\alpha(z)\hat{e}_1,\nu_\beta(z)\hat{e}_2]$ can be extended to a meromorphic function of z in a neighborhood of ω on the Riemann surface of $(z-\omega)^{1/2}$. If d is even then the same is true except that $\log(z-\omega)$ replaces $(z-\omega)^{1/2}$.

Remark 7.4.a) The result a) has been known before (see [HS] and [Sig4]).

b) If we are interested just in the existence of the 2-cluster-2-cluster amplitude we do not need to assume either the dilation analyticity or the exponential decay of the potentials. Instead we can apply the results of [PSS] and get the existence for a much wider class, of potentials.

Now we present the main facts used in the proof of theorem 7.3. Unless stated otherwise we will suppose assumption 7.1 to be true.

Theorem 7.5. (Balslev and Combes, see [BC] or [RS4]) $\Gamma(\theta)H_D\Gamma(-\theta)$ defined for real θ can be extended to an analytic family for $|\mathrm{Im}\theta|<\gamma$. Let us denote this family by $H_D(\theta)$. The continuous spectrum of $H_D(\theta)$ is equal to

$$\bigcup_{\omega_i} \omega_i + \exp(2\theta)\mathbb{R}_+$$

where ω_i runs over the set of all the thresholds of H_D.

By $R(\theta,z)$, $\phi_\alpha(\theta)$, etc we will denote the unique analytic continuation of $\Gamma(\theta)R(z)\Gamma(-\theta)$, $\Gamma(\theta)\phi_\alpha$, etc.

Lemma 7.6. Suppose the channel ϕ_α corresponds to the threshold ω_α and ω_α is not a threshold of $H^{D(\alpha)}$. Then for some $a>0$ and $\gamma'>0$ the vector $(\rho^{D(\alpha),a})^{-1}\phi_\alpha(\theta)$ belongs to L^2 uniformly for $|\mathrm{Im}\theta|<\gamma'$.

Proof. See theorem XII.41 of [RS4]. QED.

The next lemma follows easily by the standard dilation analyticity techniques (see [BC] and [RS4]).

Lemma 7.7. Suppose $0<\mathrm{Im}\theta<\gamma$. Let ω be a 2-cluster threshold that does not coincide with a more-than-2-cluster threshold. Then

a) ω is isolated in the set of the thresholds of H,

b) the functions $R_{D_2}(\theta,z)(1-P_{D_2}(\theta))$ and $R_{D_k}(\theta,z)$ for $k>2$ are analytic in z in a neighborhood of ω.

Lemma 7.8. Under the same assumptions as above if $b>0$ and d is odd the operator valued function

$\rho^{D_2,b}R_{D_2}(\theta,z)P_{D_2}(\theta)\rho^{D_2,b}$ defined for all z in a neighborhood of ω except for the cut $\omega + \exp(2\theta)\mathbb{R}_+$ can be extended analytically onto a neighborhood of ω on the Riemann surface of the function $(z-\omega)^{1/2}$. If d is even the same is true except that $\log(z-\omega)$ replaces $(z-\omega)^{1/2}$.

Proof. Note that

$$R_{D_2}(\theta,z)P_{D_2}(\theta) = \sum_{D(\alpha)=D_2} P_\alpha(\theta)\otimes\left(z-\omega_\alpha-\exp(2\theta)T_{D_2}\right)^{-1}$$

where P_α is the projection corresponding to a channel α and ω_α - its respective threshold. In the above sum all the terms with $\omega_\alpha \neq \omega$ are analytic around ω. The terms with $\omega_\alpha = \omega$ are treated as in lemma 3.7. QED.

Proof of theorem 2.3. We are only going to outline the argument since its basic ingredients are already given in detail in chapter 3 and [HS].

We use the formula for $t_{\alpha\beta}$ and equation 3.8 and get:

$$t_{\alpha\beta}(k_1,k_2) = \left(\Phi_\alpha(k_1),(V-V_{D(\alpha)})\Phi_\beta(k_2)\right)$$

$$+ \lim_{\epsilon\to 0^+}\left(\rho^{-1}(V-V_{D(\alpha)})\Phi_\alpha(k_1),\rho\left[A(\lambda+i\epsilon)(1-M(\lambda+i\epsilon))^{-1}B(\lambda+i\epsilon)\right.\right.$$

$$\left.\left.+ C(\lambda+i\epsilon)\right]\rho\rho^{-1}(V-V_{D(\beta)})\Phi_\beta(k_2)\right).$$

The first term on the right hand side is easy to handle; we will concentrate on the second one. We fix $\epsilon>0$. The function

$$\theta\longrightarrow\left(\rho^{-1}(\theta)(V(\theta)-V_{D(\alpha)}(\theta))\Phi_\alpha(\theta,k_1),\right.$$

$$\rho(\theta)\left[A(\theta,\lambda+i\epsilon)(1-M(\theta,\lambda+i\epsilon))^{-1}B(\theta,\lambda+i\epsilon) + C(\theta,\lambda+i\epsilon)\right]\rho(\theta)$$
$$\left.\times\ \rho^{-1}(\theta)(V(\theta)-V_{D(\beta)}(\theta))\Phi_\beta(\theta,k_2)\right)$$

does not depend on θ for θ real and is analytic for $|\mathrm{Im}\theta| < \gamma$; consequently it does not depend on θ at all. Now fix θ with $|\mathrm{Im}\theta|<\gamma$. Suppose for simplicity that assumption 7.2 holds. By lemmas 7.6, 7.7 and 7.8 and methods from chapter 3 and 4, $\rho^{-1}(\theta)(V(\theta)-V_{D(\alpha)}(\theta))\Phi_\alpha(\theta,k_1)$, $\rho(\theta)(A(\theta,\lambda+i\epsilon)$, $M(\theta,\lambda+i\epsilon)$, $B(\theta,\lambda+i\epsilon)\rho(\theta)$, $\rho(\theta)C(\theta,\lambda+i\epsilon)\rho(\theta)$ and

$\rho^{-1}(\theta)(V(\theta)-V_{D(\beta)}(\theta))\Phi_{\beta}(\theta,k_2)$ can be extended to analytic functions on a neighborhood of ω on the Riemann surface of either $(z-\omega)^{-1}$ or $\log(z-\omega)$. Now we apply the analytic Fredholm theorem to the term $(1-M(\theta,\lambda+i\epsilon))^{-1}$, which completes our proof. QED.

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EXISTENCE AND ANALYTICITY
OF MANY BODY SCATTERING AMPLITUDES
AT LOW ENERGIES

by

Jan Dereziński

Committee Chairman: George Hagedorn

Mathematics

(Abstract)

We study elastic and inelastic (2 cluster) - (2 cluster) scattering amplitudes for N-body quantum systems. For potentials falling off like $r^{-1-\epsilon}$ we prove that below the lowest 3-cluster threshold these amplitudes exist, are continuous and that asymptotic completeness holds. Moreover, if potentials fall off exponentially we prove that these amplitudes can be meromorphically continued in the energy, with square root branch points at the 2 cluster thresholds.