STRUCTURAL MODIFICATION UTILIZING BEAM ELEMENTS

by

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(ABSTRACT)

This study presents a concept that provides a structural dynamicist the ability to analyze the effects of making sophisticated (beam-type) structural changes to a structural system whose modal database is known. The modification technique combines the Dual Modal Space Modification Method (DMSM) and the Transfer-Matrix Method to institute general beam modifications. The DMSM method is employed to implement the beam-type modification, while the transfer-matrix method is used to formulate the modification element. The use of transfer-matrix methods provides the ability to model virtually any beam modification a designer might consider in terms of the two points being connected without the loss of any dynamic information between the points. The result is a modification scheme which is both flexible and universal.

Two numerical examples are considered. One example demonstrated the performance of the modification scheme in instituting a severe structural change. The second example demonstrated a change to a complex structure. In both cases, continuum beams were used as modification elements. The results of these two examples show that the modification scheme provides very promising results, providing an adequate modal database was used. Modal truncation was determined to be the primary source of error.

To

"Forget That Blind Ambition,
And Learn to Trust Your Intuition.
I Went Straight Ahead, Come What May,
And There's a Cowboy in the Jungle."

From "Cowboy in the Jungle"
by Jimmy Buffett

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NOMENCLATURE

Matrix and Vector Variables

$\underline{C}_p$ - coordinate transformation matrix - forces

$\underline{C}_T$ - coordinate transformation matrix - element to global

$\underline{C}_u$ - coordinate transformation matrix - displacements

$\underline{d}$ - dynamic stiffness matrix - physical space

$\underline{\Delta d}$ - dynamic stiffness modification matrix - physical space

$\underline{\bar{d}}$ - modified dynamic stiffness matrix - physical space

$\underline{D}$ - dynamic stiffness matrix - modal space

$\underline{\Delta D}$ - dynamic stiffness modification matrix - modal space

$\underline{\bar{D}}$ - modified dynamic stiffness matrix - modal space

$\utilde{f}$ - generalized force vector

$\underline{F}$ - field transfer matrix

$\underline{k}$ - stiffness matrix - physical space

$\underline{\Delta k}$ - stiffness modification matrix - physical space

$\underline{\bar{k}}$ - modified stiffness matrix - physical space

$\underline{K}$ - stiffness matrix - modal space

$\underline{\Delta K}$ - stiffness modification matrix - modal space

$\underline{\bar{K}}$ - modified stiffness matrix - modal space

$\underline{L}$ - system transfer matrix

$\underline{L}_m$ - concentrated mass point transfer matrix

$\underline{L}_s$ - spring field transfer matrix

$\underline{m}$ - mass matrix - physical space

$\underline{\Delta m}$ - mass modification matrix - physical space

$\underline{\bar{m}}$ - modified mass matrix - physical space

$\underline{M}$ - mass matrix - modal space

NOMENCLATURE (continued)

$\underline{\Delta M}$ - mass modification matrix - modal space

$\underline{\bar{M}}$ - modified mass matrix - modal space

$\underline{P}$ - modal matrix or point transfer matrix

$\underset{\sim}{p}$ - vector of internal forces

$\underset{\sim}{q}$ - vector of principle coordinates - functions of time

$\underset{\sim}{q}$ - vector of generalized displacements - transfer matrix

$\underset{\sim}{Q}$ - vector of principle coordinates - coefficients

$\underset{\sim}{u}$ - vector of generalized displacements - stiffness matrix

$\underset{\sim}{x}$ - vector of generalized displacements - physical space

$\underset{\sim}{Y}$ - dummy vector of modal displacements

$\underset{\sim}{Y}_s$ - subvector of $\underset{\sim}{Y}$

$\underset{\sim}{z}$ - state vector

<u>Scalar Variables</u>

C - constant of integration

EI - modulus of rigidity

f - generalized force

k - stiffness

Δk - stiffness change

ℓ - number of modes

ℓ_i - length

m - mass

Δm - mass change

M - moment

n - number of degrees-of-freedom

N - axial force

NOMENCLATURE (continued)

t	-	time
T	-	torque
u	-	axial displacement or a generalized displacement
v	-	transverse displacement - out-of-plane
V	-	shear
w	-	transverse displacement - in-plane

Greek Variables

$\underline{\lambda}$	-	direction cosine matrix
λ	-	natural circular frequency squared
ω	-	natural circular frequency
ϕ	-	axial twist
ρ	-	weight density
ψ	-	rotation - out-of-plane
θ	-	rotation - in-plane
ξ	-	dummy variable of integration

Superscripts

(i)	-	i^{th} modal vector
(j)	-	j^{th} modal vector
L	-	leftside
R	-	rightside
T	-	matrix transpose
-1	-	matrix inverse
^	-	element coordinate system

Subscripts

A	-	submatrix index

NOMENCLATURE (continued)

B	-	submatrix index
g	-	submatrix index
h	-	submatrix index
i	-	index
j	-	index
k	-	index
s	-	pseudo-stiffness matrix
x	-	x direction
y	-	y direction
z	-	z direction
I	-	modal space I
II	-	modal space II

CHAPTER 1 INTRODUCTION

1.1 Overview

Currently many companies are actively using experimental modal analysis to perform structural dynamic analysis. The results from the experimental modal analysis are being used to directly solve structural vibration problems, verify the results from finite-element analyses, and analyze the dynamic effects of simple structural changes. The last area, structural modification, is increasing in popularity for use as an inexpensive and quick tool to aid the designer in solving structural vibration problems. At present, the modifications available to the designer have been limited to the use of lumped masses, translational springs, and/or translational dampers. It is intended that this study will broaden the library to include the use of general beam elements.

1.2 Research Needs

Today, three methods for performing structural modification are being implemented by industry. These methods all involve performing and predicting the effect of a simple modification on an existing experimentally derived database. The process of estimating structural modifications using experimentally derived databases was initially investigated by Klosterman [1]. This work involves using the measured frequency response functions of the structural system to institute structural changes. Initially, limitations in data acquisition techniques prohibited the casual use of the technique [2]. However, with the advancement of the "state of the art", the technique has been resurrected and is in use today [2,3,4]. Another technique called eigenvalue modification or local eigenvalue modification was originally developed as a structural reanalysis technique attributed to Weissenburger [5]. Weissenburger's technique was re-examined and extended to include damping and substructuring by work carried out at Michigan Technological University [6,7]. The implementation of this technique using experimentally derived data was performed by Structural Measurement Systems Corporation [8,9,10]. The modification is introduced into the modal database as $\eta \underset{\sim}{x} \underset{\sim}{x}^T$, where η is the magnitude of change; and $\underset{\sim}{x}$ is a column vector designating the location of the change. The third method, called Dual Modal Space Modification, was developed by Luk and Mitchell [11,12,13,14] just after Structural Measurement Systems introduced their technique. Luk and Mitchell represent the modification by $\underline{P}^T \underline{\Delta} \underline{P}$, where $\underline{P}$ is the original modal matrix and $\underline{\Delta}$ is a matrix representing the change in physical space.

All three types of structural modification schemes have a common link in the type of modifications they will perform. The types of modifications are: concentrated mass changes at a single point; truss-type springs between two points; and/or translational-type dampers between two points. This leaves the designer the limited ability to quickly and inexpensively examine the effects of simple changes in axial stiffness or axial damping between two points, or a change in mass at a single point. When one considers these types of modifications, one realizes that an engineer rarely makes such simple modifications. Real world structural modifications usually involve the addition of beam-type supports, braces, or gussets [15]. The limited types of structural modification elements available are not a function of the modification schemes themselves. Instead, the limitation is a result of an incomplete database. Beams, plates, braces, and gussets are fourth order elements. They require four boundary conditions; two translations and two rotations. Since the modal database, extracted using today's methods, carries no rotational information it is impossible to add beam or plate elements properly.

The rotational degree-of-freedom problem is being studied from several avenues. First, an appeal has been made to the instrument/transducer industry [16]. There has not been an adequate response to date; i.e., a cheap versatile angular transducer. However, a paper by Licht of Brüel and Kjaer [17] shows the instrument/transducer companies are working in this direction. Next, researchers are developing measurement methods which are aimed at synthesizing angular rotations [18,19,20].

This study has resulted from a "chicken and egg" dilemma. There is no need for beam (rotational dependent) modification technology since there is no rotational information available. On the other hand, since no one "needs" the rotational information, why develop the rotational measurement technology? Since it appears that others are trying to develop adequate rotational methods, it was decided that an effort would be made to develop the technology necessary to implement sophisticated structural modifications, such as generalized beam elements.

1.3 Research Objective

The primary objective of this study will be to develop a concept that provides an experimental structural dynamicist the ability to analyze the effects of making sophisticated (beam-type) structural changes to a system whose modal database is known. The incentive for providing this ability is to augment and reduce the costly theoretical structural reanalysis and prototype testing phase of product development.

Some attempts have already been made to implement beam modifications using experimentally derived modal databases. Maleci and Young [21] performed an analysis to study the effect of rotational degrees-of-freedom on a beam modification. They use a commercial program that combines beam modification with the frequency response modification method to perform a structural change using an experimentally derived database. They analyzed two cases: a beam modification but ignoring all rotational information; and a beam modification including rotational information derived through the process in Ref. [19]. Maleci and Young reported that by ignoring rotations, the modified structure was overly flexible with high errors in the predicted natural frequencies. Unfortunately, the modification results using synthesized rotations were inconclusive due to measurement error.

O'Callahan and Chou [22,23] have developed a beam modification routine based on the Local Eigenvalue Modification Procedure. The actual modification procedure required a single modification for each pair of corresponding degrees-of-freedom between the connection points being changed. Therefore, for a three-dimensional beam modification, six modifications are made. The modification elements were similar to finite-element stiffness and mass matrices. O'Callahan and Chou's results vary. For simple (not severe) modifications, they report good performance of their technique. However, when dealing with severe changes, such as linking a point to ground, they observed that high errors result when using a truncated database. O'Callahan and Chou's analysis was performed using a theoretically generated database. However, they comment that similar results were observed using experimental derived data with rotations synthesized using a technique outlined in Ref. [20].

Evaluation of the frequency response technique is difficult because the implementation is proprietary information. Based on the results presented by Maleci and Young [21], one would have to conclude the technique is unsatisfactory for performing structural changes. On the other hand, the method presented by O'Callahan and Chou [22,23] seems a viable approach to performing structural modification. However, practically this algorithm is limited in several ways. First, a modification analysis must be performed for each pair of corresponding degrees-of-freedom between the connection points effected by the modification. For a three-dimensional beam modification, six analyses would have to be performed. Second, the type of beam elements are limited. O'Callahan and Chou use finite-element stiffness modification elements and lumped mass type of mass modification elements. How does one handle a complicated beam modification involving several changes in cross sectional areas? How does one handle distributed masses in the modification element? Although the work of O'Callahan and Chou is a significant step in structural modifications, there is still room for rethinking and improvement.

The study reported here will specifically address the questions above. The structural modification technique will attempt to satisfy the following criteria:

1. Allow unlimited potential for building and using any type of beam structure as a modification element.
2. The use of continuum beam modification elements.
3. The ability to perform severe as well as simple modifications to a structural system.
4. A modification process that is easy to implement/adopt into today's structural analysis programs.

As a final note, it is not the intention, nor should it be, that structural modification replace structural reanalysis techniques. The conceptual use of structure modification is a design aid or tool. It augments the theoretical process; it does not replace it.

1.4 Definition of Terminology

In dealing with the development of beam structural modifications, terminology can become confusing and may be misinterpreted. Therefore, to prevent misinterpretation, the following terms have been defined.

Physical Space - The space defined by the physical coordinates (degrees-of-freedom) which are used to describe the motion of the structural system. In this study, these physical coordinates are translations and rotations.

Modal Space I - The space defined by the principal or normal coordinates of the original structural system. The modal space is related to the physical space through a linear transformation. Generally, when the equations of motion of a structural system are written in modal space I, they represent a set of independent equations.

Modal Space II - If a structural modification is added to a structure system in physical space, the addition can also be performed in modal space I. Generally the addition of this change destroys the independence of the equations of motion in modal space I. Thereby, modal space II is a space defined by a new set of principal coordinates which are a linear transformation of the modal space I coordinates.

Modal Truncation - In a discretized model of a physical structure, the structure's motion is represented by n number of degrees-of-freedom. Therefore, the system has n modes of vibration. Usually only a small fraction of these n modes are available for performing a structural modification. The reduction in the number of modes is referred to as modal truncation.

Modal Insufficiency - Modal insufficiency refers to the truncation of modal information due to the discretization of a system; i.e., the reduction in the number of modes from infinity to n.

Modal Assurance - "The modal assurance criterion is a scalar constant relating the causal relationship between two modal vectors. The constant will take on values from zero, representing no consistent correspondence, to one, representing a consistent correspondence [24]." Allemang indicates that a modal assurance greater than 0.9 corresponds to a good correspondence between modes being compared [25].

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CHAPTER 2 REVIEW OF BASIC THEORY

2.1 Review of System Modification Via The Dual Modal Space Modification Technique

2.1.1 Overview

The Dual Modal Space Modification (DMSM) technique for structural modification was developed for performing structural modifications using an experimental database. The method has been developed by Luk and Mitchell [1,2,3,4], and the effort has produced an industrial computer code called Zonic Modification®*. This code comprises one of the three most popular modification routines available in industry today.

The DMSM modification procedure uses a dynamic model obtained from experiments (modal space I) and calculates the predicted dynamic characteristics which result from simple changes in physical mass, stiffness, and/or damping properties of the structure, thus allowing the capability to systematically solve vibration problems via design modifications. In the present study, the DMSM procedure provides the vehicle for instituting more complicated structural modifications: beams. If a beam structural change can be described in terms of the physical mass and stiffness matrices or a combined dynamic stiffness matrix, then the DMSM method can be used to examine the effect of that change.

There are several reasons for using the DMSM method as a modification vehicle. The primary reason for using DMSM is that the actual physical change is converted to an equivalent change in modal space I. This has a two-fold benefit. First: the modification process can

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* Trademark of Zonic Corporation, Milford, Ohio.

directly take place using experimental data, which is already in modal space I. This would eliminate the need for using a pseudo inverse to transform the experimental data from modal space to physical space. This inverse transformation process can be very cumbersome and time consuming [1,2,5]. Second: there is a reduction in problem size. Usually the modal space problem size is orders of magnitude smaller than the physical system size. This provides savings both in computational speed and memory. A secondary, but very important, reason for using the DMSM is that it is more general and thereby more flexible than other modification schemes such as the Local Eigenvalue Modification Procedure (LEMP) [6,7,8], or the Frequency Response Method [9,10]. The DMSM does not restrict the type of modification nor its severity. It should be pointed out that the LEMP is a general method for installing modifications. Indeed, O'Callahan and Chou have implemented beam modifications using this technique [11,12]. However, the beam modification was limited to a modification element which is similar to a finite element frame element. This study goes further and allows much more complicated elements to be introduced through transfer-matrix theory.

2.1.2 DMSM Background

The theory for DMSM is well established in previous work by Luk and Mitchell [1,2,3]. However, since the present study relies heavily on the DMSM procedure, it would be prudent to review this theory. For simplicity damping will not be considered in this development.

The dynamic characteristics of a structural system can be defined in terms of its mass and stiffness properties as follows:

$$\underline{m}\,\ddot{\tilde{x}} + \underline{k}\,\tilde{x} = \tilde{f} \tag{2.1}$$

This system of equations represents the structure's equations of motion in physical space. Transforming Eq. (2.1) into modal space I by using the relation

$$\tilde{x} = \underline{P}_I\, \tilde{q}_I \tag{2.2}$$

where $\underline{P}_I$ is the modal matrix which decouples Eq. (2.1) and $\tilde{q}_I$ is a vector of principal coordinates in modal space I, results in the following equation

$$\underline{M}_I\, \ddot{\tilde{q}}_I + \underline{K}_I\, \tilde{q}_I = \underline{P}_I^T\, \tilde{f} \tag{2.3}$$

Now $\underline{M}_I$ and $\underline{K}_I$ are the modal mass and stiffness matrices defined by the following relations

$$\left.\begin{aligned} \underline{M}_I &= \underline{P}_I^T\, \underline{m}\, \underline{P}_I \qquad \text{(Diagonal)} \\ \underline{K}_I &= \underline{P}_I^T\, \underline{k}\, \underline{P}_I \qquad \text{(Diagonal)} \end{aligned}\right\} \tag{2.4}$$

The natural frequencies of the system can be defined as

$$\omega_i = \sqrt{\frac{K_{ii}}{M_{ii}}}\ ; \quad i = 1,2,3,\ldots,n \tag{2.5}$$

The system of Eq. (2.3) is typically the result of an experimental analysis.

The effects of a structural modification may be examined. Simple changes to the structure such as point mass changes, Δm, or stiffness changes between points, Δk, can be represented as:

$$\underline{\bar{m}} = \underline{m} + \underline{\Delta m} \quad ; \quad \underline{\bar{k}} = \underline{k} + \underline{\Delta k} \tag{2.6}$$

When these types of changes are imposed on a system, a new set of equations of motion will result

$$\underline{\bar{m}}\, \ddot{\underset{\sim}{x}} + \underline{\bar{k}}\, \underset{\sim}{x} = \underset{\sim}{f} \tag{2.7}$$

If this new system is premultiplied by $\underline{P}_I^T$ and if Eq. (2.2) is substituted for $\ddot{\underset{\sim}{x}}$ and $\underset{\sim}{x}$, these changes can be driven into modal space I. The result of such an operation is

$$\underline{\bar{M}}_I\, \ddot{q}_I + \underline{\bar{K}}_I\, q_I = \underline{P}_I^T\, \underset{\sim}{f} \tag{2.8}$$

where

$$\left.\begin{aligned} \underline{\bar{M}}_I &= \underline{P}_I^T\, \underline{\bar{m}}\, \underline{P}_I = \underline{P}_I^T\, \underline{m}\, \underline{P}_I + \underline{P}_I^T\, \underline{\Delta m}\, \underline{P}_I = \underline{M}_I + \underline{\Delta M} \\ \underline{\bar{K}}_I &= \underline{P}_I^T\, \underline{\bar{k}}\, \underline{P}_I = \underline{P}_I^T\, \underline{k}\, \underline{P}_I + \underline{P}_I^T\, \underline{\Delta k}\, \underline{P}_I = \underline{K}_I + \underline{\Delta K} \end{aligned}\right\} \tag{2.9}$$

Note the system defined by Eq. (2.8) and (2.9) is no longer diagonal.

Certainly, by definition, $\underline{M}_I$ and $\underline{K}_I$ are diagonal. However, $\underline{\Delta M}$ and $\underline{\Delta K}$ are not diagonal except under very special circumstances. Thus, $\underline{\overline{M}}_I$ and $\underline{\overline{K}}_I$ are not diagonal. Equation (2.8) represents the dynamic motion of the modified structure in modal space I. An eigenanalysis may be performed on this new system to determine the natural frequencies and mode shapes of the modified structure. This second eigenanalysis, in effect, transforms the system of Eq. (2.8) from modal space I to modal space II. The relation between the principal coordinates of modal space I and those of modal space II can be expressed as

$$q_I = \underline{P}_{II}\, q_{II} \tag{2.10}$$

where $\underline{P}_{II}$ is the modal matrix of the modified structure in modal space II. To transform this new modal matrix back into physical space, Eqs. (2.2) and (2.10) may be combined to result in

$$\underset{\sim}{x} = \underline{P}_I\, \underline{P}_{II}\, q_{II} \tag{2.11}$$

Therefore, the mode shape representing the modal motion of the modified structure can be obtained by the product of the matrices of modal space I and space II. Notice that the modification mode shapes are described by a linear sum of the original mode shapes. This can be illustrated by rewriting Eq. (2.11) as

$$\underline{P} = [\underset{\sim}{P}_I^{(1)}, \underset{\sim}{P}_I^{(2)}, \underset{\sim}{P}_I^{(3)}, \ldots, \underset{\sim}{P}_I^{(n)}] \begin{bmatrix} P_{II1,1}, P_{II1,2}, \ldots, P_{II1,j}, \ldots, P_{II1,n} \\ P_{II2,1} \qquad\qquad\qquad\qquad \cdot \\ P_{II3,1} \qquad\qquad\qquad\qquad \cdot \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ P_{IIn,1}, \ \ldots, \ P_{IIn,j}, \ \ldots, \ P_{IIn,n} \end{bmatrix} \tag{2.12}$$

Therefore, the j^{th} mode shape of the modified structure is

$$\underset{\sim}{P}^{(j)} = P_{II1,j}\, \underset{\sim}{P}_I^{(1)} + P_{II2,j}\, \underset{\sim}{P}_I^{(2)} + \ldots + P_{IIj,j}\, \underset{\sim}{P}_I^{(j)} + \ldots + P_{IIn,j}\, \underset{\sim}{P}_I^{(n)} \tag{2.13}$$

This clearly shows that the mode shape of the modified structure is assumed by this technique to be totally dependent upon the modal information contained in the original database.

The derivation of the actual mass and stiffness matrices are not important to the present work. These derivations may be found in references [1,3]. However, it is instructive to examine the properties of these matrices. The mass modification matrix is a diagonal matrix with non zero entries at the locations where the mass modification is to be affected. Physically, this is equivalent to a change in mass at a discretized point of the system. The mass matrix does not allow changes between points or the inclusion of distributed mass between points. The stiffness change matrix is a more complex matrix similar to the finite element truss matrix [1]. The stiffness member is assumed to be uniform, pin-connected at its ends, linearly elastic, and axially loaded. The assumptions of pin-connected and axial loading are physically restrictive limitations. This stiffness formulation does not allow for bending or welded connections. It should be pointed out that these limitations are not inherent in the DMSM theory but, instead, result from the lack of rotational degrees-of-freedom in the original database. If rotational degrees-of-freedom were added, any change could take place as long as it could be described in terms of the two points between which the modification will take place.

2.2 Review of Transfer-Matrix Methods

2.2.1 Overview

The Transfer-Matrix Method is an approach developed to analyze the static and dynamic behavior of structural systems. The method is based on a line-solution development using element matrices which describe the dynamic and elastic properties of subsystems of a whole structure. These element matrices are referred to as transfer matrices. They may be thought of as representing the solution of a subsystem with transferable boundaries. Physically, the transfer matrix relates the state vector (a vector of displacements and forces) at one boundary of the subsystem to the state vector at another boundary of the subsystem. The formulation of the complete system's description takes place by successively multiplying the transfer matrices of the subsystems. This process makes the method particularly well suited for, but not limited to, the analysis of structure whose geometries are periodic: beams, truss structures, arches, membranes, plates, grillages, and shells. Well-known examples of transfer-matrix applications are aircraft structures, turbine/axial compressor assemblies, general rotating equipment, crankshafts, and drilling platforms. In this study, a complete discussion of the transfer-matrix method would be unwarranted. For a more thorough discussion of the transfer-matrix method, the reader is referred to Pestal and Leckie [13]. However, as a primer this section will present an example of the transfer-matrix method by developing the transfer matrices for a beam/concentrated mass assembly.

This study employs the use of transfer matrices in the creation of modification elements. These modification elements will represent the actual changes applied to the unmodified system. At first, the rationale for using transfer matrices as modification elements may seem questionable. The transfer matrix is based on state vectors of the boundaries. These state vectors contain both displacement and force information. On the other hand, the database to be modified is in a stiffness formulation which relates displacements to forces. It will be shown in Chapter 3 that the stiffness formulation is just a matrix transform of the transfer matrix formulation. Therefore, if the transfer matrix of a particular subsystem is known, the stiffness matrix can be found.

There are several reasons for employing transfer matrices for the modification matrix development. The primary reason for using transfer matrices is that the characteristics of a subsystem may be completely described in terms of its boundaries. This is a very important feature when dealing with structural modifications. Modifications usually take place between two locations of the structure. Therefore, being able to describe the modification in terms of two points is of prime interest. Another benefit in using transfer matrices is flexibility. Complex modification elements can be modeled using a variety of transfer-matrix elements which may be successively multiplied to produce a "super" transfer-matrix representation of the modification element. Notice that intermediate conditions and the number of degrees-of-freedom present no difficulty since they have no effect on the order of the transfer-matrices required. The order of the transfer matrix is dependent only on the order of the differential equations governing the behavior of the elements. Any number of elements may be combined without increasing the problem size. This eliminates the need for condensation procedures which would have to be used if a finite element approach were used. This does not mean the displacement information inside the modification is lost. The information can be synthesized from knowledge of the displacement information at the boundaries of the modification element. A further advantage of the transfer-matrix approach is the ability to use exact continuum elements as modification elements. Unlike the finite-element method, the transfer-matrix method provides the ability to formulate exact elements [14].

2.2.2 Transfer-Matrix Review

Basic Transfer-Matrix Theory is well established and may be found in several references [13,15,16]. This section is intended to familiarize the reader in the development of transfer matrices. As an example the transfer-matrix theory will be applied to the plane flexural vibrations of a straight beam. For simplicity, the beam will be considered massless with concentrated masses distributed across the beam.

Consider the system shown in Fig. 2.1. This system can be broken up into two types of elements: an element representing the massless beam and an element representing the concentrated mass. The first step is to isolate the elements of the system. Figure 2.2(a) illustrates the beam element and the state variable positive sign convention. Figure 2.2(b) shows the concentrated mass and the state variable positive sign convention. The fundamental equation of motion for a uniform simple beam is

$$(EI)_i \frac{d^4 w(x)}{dx^4} = f(x) \tag{2.14}$$

where f(x) is some applied force. The shearing force, bending moment, and rotation can be expressed as

$$\left.\begin{aligned} V(x) &= -(EI)_i \frac{d^3 w(x)}{dx^3} \\ M(x) &= -(EI)_i \frac{d^2 w(x)}{dx^2} \\ \theta(x) &= -\frac{dw(x)}{dx} \end{aligned}\right\} \tag{2.15}$$

Combining Eq. (2.15) with successive integrations of Eq. (2.14) yields:

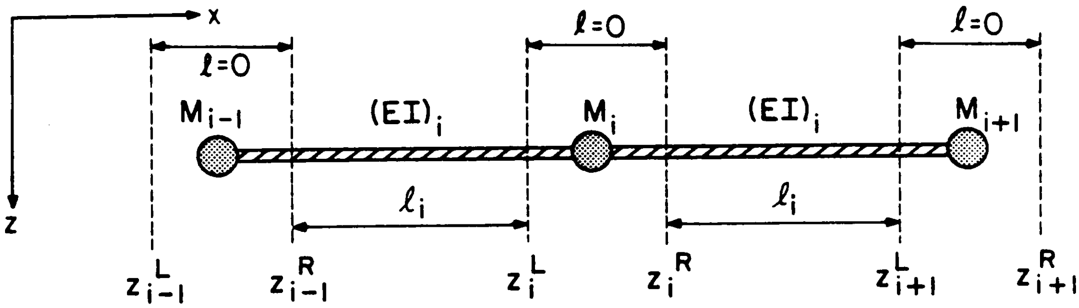
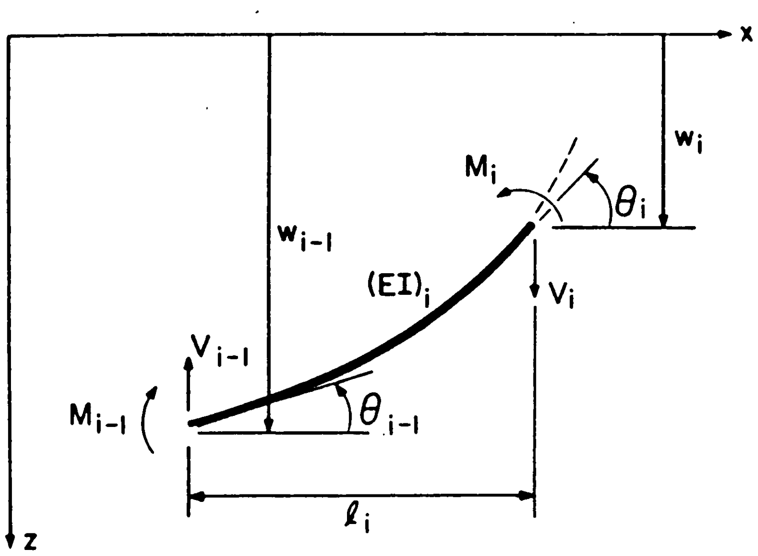
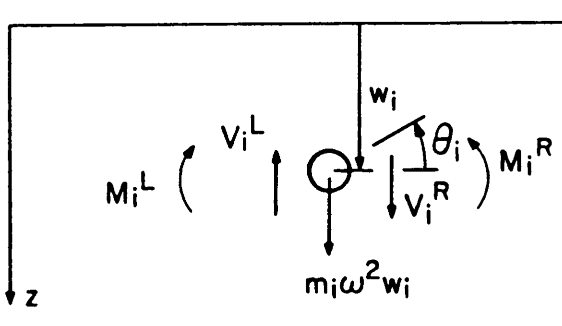


Figure 2.1 Beam with Concentrated Masses



a) MASSLESS BEAM



b) CONCENTRATED MASS

Figure 2.2 Positive State Variable Sign Conventions for Massless Beam Element and Concentrated Mass

$$\left.\begin{aligned}
-V(x) &= C_1 + \int_o^x f(\xi)\, d\xi \\
-M(x) &= C_2 + C_1\, x + \iint_o^x f(\xi)\, d\xi \\
-(EI)_i\,\theta(x) &= C_3 + C_2\, x + C_1 \frac{x^2}{2} + \iiint_o^x f(\xi)\, d\xi \\
(EI)_i\, w(x) &= C_4 + C_3\, x + C_2 \frac{x^2}{2} + C_1 \frac{x^3}{6} + \iiiint_o^x f(\xi)\, d\xi
\end{aligned}\right\} \quad (2.16)$$

where C_1, C_2, C_3, and C_4 are constants of integration. By assuming there are no applied forces, the equation above can be written as

$$\left.\begin{aligned}
w(x) &= \frac{C_4}{(EI)_i} + \frac{C_3\, x}{(EI)_i} + \frac{C_2\, x^2}{2(EI)_i} + \frac{C_1\, x^3}{6(EI)_i} \\
\theta(x) &= \qquad\quad - \frac{C_3}{(EI)_i} - \frac{C_2\, x}{(EI)_i} - \frac{C_1\, x^2}{2(EI)_i} \\
M(x) &= \qquad\qquad\qquad\quad - C_2 \quad\; - C_1\, x \\
V(x) &= \qquad\qquad\qquad\qquad\qquad\;\; - C_1
\end{aligned}\right\} \quad (2.17)$$

The boundary conditions shown in Fig. 2.2(a) can be represented by the following relations:

$$\left.\begin{aligned}
w(0) &= w_{i-1} & \qquad w(\ell_i) &= w_i \\
\theta(0) &= \theta_{i-1} & \qquad \theta(\ell_i) &= \theta_i \\
M(0) &= M_{i-1} & \qquad M(\ell_i) &= M_i \\
V(0) &= V_{i-1} & \qquad V(\ell_i) &= V_i
\end{aligned}\right\} \quad (2.18)$$

Substituting the boundary conditions above into the system of Eq. (2.17) yields the following system

$$\left.\begin{aligned}
w_i &= w_{i-1} - \theta_{i-1}\ell_i - \frac{M_{i-1}\,\ell_i^2}{2(EI)_i} - \frac{V_{i-1}\,\ell_i^3}{6(EI)_i} \\
\theta_i &= \theta_{i-1} + \frac{M_{i-1}\,\ell_i}{(EI)_i} + \frac{V_{i-1}\,\ell^2}{2(EI)_i} \\
M_i &= M_{i-1} + V_{i-1}\,\ell_i \\
V_i &= V_{i-1}
\end{aligned}\right\} \tag{2.19}$$

In matrix form, the equations can be written as

$$\begin{Bmatrix} w \\ \theta \\ M \\ V \end{Bmatrix}_i = \begin{bmatrix} 1 & -\ell_i & -\frac{\ell_i^2}{2(EI)_i} & -\frac{\ell^3{}_i}{6(EI)_i} \\ 0 & 1 & \frac{\ell_i}{(EI)_i} & \frac{\ell_i^2}{2(EI)_i} \\ 0 & 0 & 1 & \ell_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} w \\ \theta \\ M \\ V \end{Bmatrix}_{i-1} \tag{2.20}$$

or simply

$$\underset{\sim}{z}_i = \underline{F}_i\,\underset{\sim}{z}_{i-1} \tag{2.21}$$

where $\underset{\sim}{z}_i$ is called a state vector containing the boundary information at the right end of the beam, $\underline{F}_i$ is called a field transfer matrix describing the massless beam, and $\underset{\sim}{z}_{i-1}$ is the state vector describing the state variable information at the input (left-hand side) of the beam.

The transfer matrix for the concentrated mass can be found by noting that the deflection, slope, and moment are continuous across a pure point mass. Hence,

$$w_i^R = w_i^L \qquad \theta_i^R = \theta_i^L \qquad M_i^R = M_i^L \tag{2.22}$$

However, the shear is not continuous across the pure mass due to the inertia of the mass. Applying Newton's second law and assuming harmonic motion yields

$$V_i^R = V_i^L - m_i\, \omega^2\, w_i \tag{2.23}$$

Combining Eqs. (2.22) and (2.23) into matrix form produces

$$\begin{Bmatrix} w \\ \theta \\ M \\ V \end{Bmatrix}_i^R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -m_i\omega^2 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} w \\ \theta \\ M \\ V \end{Bmatrix}_i^L \tag{2.24}$$

or

$$\underset{\sim}{z}_i^R = \underline{P}_i\, \underset{\sim}{z}_i^L \tag{2.25}$$

Since the element has no length associated with it, $\underline{P}_i$ is called a point transfer matrix.

Reconsider the system of Fig. 2.1, this system's transfer matrix can be formulated by noting that

$$\underset{\sim}{z}_{i+1}^{R} = \underline{L}\ \underset{\sim}{z}_{i-1}^{L} \tag{2.26}$$

Successive substitution of point and field transfer matrices yields

$$\underset{\sim}{z}_{i+1}^{R} = \underline{P}_{i+1}\ \underline{F}_{i}\ \underline{P}_{i}\ \underline{F}_{i}\ \underline{P}_{i-1}\ \underset{\sim}{z}_{i-1}^{L} \tag{2.27}$$

Therefore, the system transfer matrix is

$$\underline{L} = \underline{P}_{i+1}\ \underline{F}_{i}\ \underline{P}_{i}\ \underline{F}_{i}\ \underline{P}_{i-1} \tag{2.28}$$

Two observations are worthwhile. First: the transfer-matrix description of the system of Fig. 2.1 can be completely defined in terms of its end conditions. Second: the degrees-of-freedom which were associated with the ends of each element are eliminated through the transfer-matrix multiplication. The order of this problem is four. If a stiffness model was used using two frame elements, each having four degrees-of-freedom, the order of the problem size would have increased to six.

Equation (2.28) shows that the system transfer matrix is composed of two types of transfer matrices; a massless beam field transfer matrix and a concentrated point mass transfer matrix. This demonstrates the flexibility of the transfer matrix theory. Catalogs of transfer matrix elements exist [13,17,18] which can be used to formulate an almost infinite number of system transfer matrices. Figure 2.3 illustrates

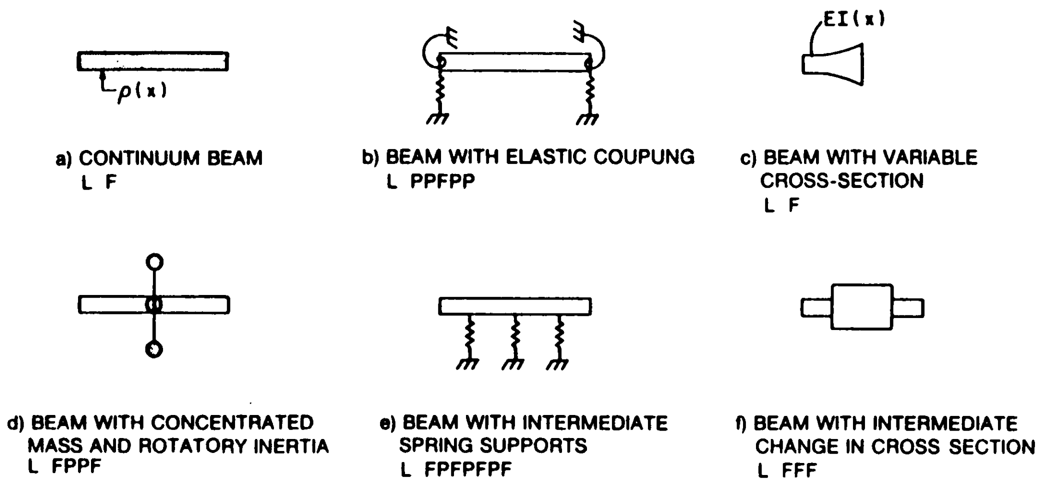


Figure 2.3 Examples of Possible Transfer-Matrix Modification Element

several different types of systems which may be formulated into a modification element. In each case, the order of the transfer matrix is four and the system is completely defined in terms of its boundary state vectors.

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CHAPTER 3 BEAM STRUCTURAL MODIFICATIONS

3.1 Overview

Now that a background in DMSM and Transfer-Matrix Methods has been presented the proposed beam modification method can be developed. The basis for this beam modification algorithm relies on the combination of the DMSM technique and transfer-matrix methods to institute structural changes using existing mathematical models. However, before proceeding with this development, consider the actual purpose of providing structural modifications: namely, to predict the effect of making modifications or design changes to a structure whose mathematical model already exists. To meet these purposes, a modification scheme must possess several important characteristics. These are:

1. The ability to provide a variety of modification elements which represent realistic structural changes such as beam-type supports, braces, and gussets.
2. An algorithm that can handle any structural change; not only small, but also large system changes.
3. A fast algorithm so that changes can be implemented and examined quickly. This would enable the designer to "play what-if games" when attempting to bring about specified dynamic modifications.

In short, the combination of these characteristics requires a theory which would have to be very flexible and also easy to implement.

The theory presented in this chapter is an attempt at satisfying these characteristics. The DMSM technique is employed because it can easily be adapted to handle any modification which can be described in terms of mass and/or stiffness matrices; no restriction is placed on the severity of the structural change. A second reason for using this technique is that it is already in use by industry [1]. Therefore, beam modifications can be incorporated by industry with a minimum of effort. The requirement that the algorithm handle any user-conceived structural change caused the selection of the transfer-matrix method for formulating the structural change description. Transfer-matrix methods have the unique advantage that virtually any modification element may be described in terms of its end points. The method automatically "condenses" the full dynamic information to the ends of the structural modification. This means the designer can build a modification element from a list of members which can be combined in any order to produce a particular modification. For example, the designer might build a continuous distributed mass beam with a concentrated mass at its center and rotational/translational stiffness at its ends. Such a beam would be assembled through transfer matrix operations to produce an element fully defined by the displacements and forces at the points where the element connects to the structure.

A primary weakness of this approach is speed. The actual implementation of this approach involves the solution of a nonstandard eigenvalue problem. This nonstandard eigenvalue problem arises because the mass and stiffness matrices are irrevocably entwined. Indeed, for the case of continuum beams the eigenvalue problem becomes a transcendental eigenvalue problem. The solution to the nonstandard eigenvalue problem is usually performed by a determinant search method, which is usually slower than standard eigenanalysis routines. A second weakness of this modification approach is the requirement of rotational degrees-of-freedom for making beam modifications. At present, rotational degrees-of-freedom are nonexistent in the original mathematical model, as extracted by the usual experimental modal analysis. However, as pointed out in Chapter 1, it may not be long before rotational information may be gained from an experimental test.

3.2 Beam Modification Theory

3.2.1 Modification of the DMSM Method

The DMSM method presented in Chapter 2 provides the vehicle for implementing beam modifications. However, before the DMSM method can be used, it must be reformulated to suit the adoption of transfer-matrix elements. The homogeneous form of the physical system transformed and decoupled into modal space I is a modification of Eq. (2.8); i.e.,

$$\underline{M}_I \ddot{\underset{\sim}{q}}_I + \underline{K}_I \underset{\sim}{q}_I = \underset{\sim}{0} \tag{3.1}$$

For simple harmonic response, $\underset{\sim}{q}_I(t) = \underset{\sim}{Q}_I e^{i\omega t}$, the equation becomes

$$(\underline{K}_I - \omega^2 \underline{M}_I) \underset{\sim}{Q}_I = \underset{\sim}{0} \tag{3.2}$$

Now Eq. (3.2) can be recast in terms of the physical system parameters by using Eq. (2.4), as

$$\underline{P}_I^T \; (\underline{k} - \omega^2 \; \underline{m}) \; \underline{P}_I \; \underset{\sim}{Q}_I = \underset{\sim}{0} \tag{3.3}$$

or

$$\underline{P}_I^T \; \underline{d}(\omega) \; \underline{P}_I = \underset{\sim}{0} \tag{3.4}$$

where $\underline{d}(\omega)$, the dynamic stiffness matrix, is

$$\underline{d}(\omega) = (\underline{k} - \omega^2 \; \underline{m}) \tag{3.5}$$

In the DMSM method, changes are mathematically described by changing either or both the mass and/or stiffness matrix, Eq. (2.6). Instead of making a singular change in the mass and/or stiffness matrix, any change to the system could be described in terms of a change to the dynamic stiffness matrix, $\underline{d}(\omega)$. It may seem strange to do this, but this changed formulation has resulted from the form of the sophisticated transfer-matrix elements. For example, a continuum beam transfer matrix has a formulation including trigonometric and hyperbolic functions which have the mass and stiffness properties entwined in their arguments. Therefore, when the transfer matrix is converted to a "stiffness" matrix, that matrix would be an inseparable composite of both the stiffness, $\underline{k}$, and mass, $\underline{m}$, matrices. Thus, the structural modification problem is cast into a change in the dynamic stiffness matrix, $\underline{d}(\omega)$.

Realizing structural changes now take place using the dynamic stiffness matrix, a system change, in terms of physical space, can be written as

$[K - m\omega^2]\underset{\sim}{x} = \underset{\sim}{0} \rightarrow d(\omega)\underset{\sim}{x} = \underset{\sim}{0} \rightarrow [d(\omega) + \underset{\sim}{\Delta d}(\omega)]\underset{\sim}{x} = \underset{\sim}{0}$

$$[\underline{d}(\omega) + \underline{\Delta d}(\omega)]\ \underset{\sim}{x} = \underset{\sim}{0} \tag{3.6}$$

where $\underline{\Delta d}(\omega)$ is the matrix containing the characteristics of the structural change. Formulating Eq. (3.6) in terms of modal space I yields

$$\underline{P}_I^T\ [\underline{d}(\omega) + \underline{\Delta d}(\omega)]\ \underline{P}_I\ \underset{\sim}{Q}_I = \underset{\sim}{0} \tag{3.7}$$

Expanding this equation results in

$$[\underline{P}_I^T\ \underline{d}(\omega)\ \underline{P}_I + \underline{P}_I^T\ \underline{\Delta d}(\omega)\ \underline{P}_I]\ \underset{\sim}{Q}_I = \underset{\sim}{0} \tag{3.8}$$

or

$$[\underline{D}_I(\omega) + \underline{\Delta D}_I(\omega)]\ \underset{\sim}{Q}_I = \underset{\sim}{0} \tag{3.9}$$

Note the system described by Eq. (3.9) is no longer modal; i.e., it is no longer diagonal. A closer look shows that $\underline{D}_I(\omega)$ is certainly diagonal. However, $\underline{\Delta D}_I(\omega)$ is not diagonal except under extraordinary circumstances. Thus, the modification augments the diagonal dynamic stiffness matrix so that modal space I is no longer modal in nature. This means that the system defined by Eq. (3.9) must go through an eigenvalue extraction process to define the new natural frequencies of the modified structure. This extraction process drives the system into modal space II.

Combining the sum in Eq. (3.9) yields

$$\bar{\underline{D}}_I(\omega)\ \underset{\sim}{Q}_I = \underset{\sim}{0} \tag{3.10}$$

Because of the possibility of a combined dynamic stiffness modification, extraction processes that depend upon a constant mass and stiffness matrix can no longer be used. Simply stated the system defined by Eq. (3.10) cannot be formulated into the standard eigenvalue problem: $\lambda\ \underline{A}\ \underset{\sim}{x} = \underline{B}\ \underset{\sim}{x}$. The eigenextraction problem of Eq. (3.10) is known as a nonstandard or transcendental eigenvalue problem. This eigenvalue problem has been studied by several investigators [2-8]. The result of this work is a directed determinant zero search called the Wittrick-Williams Method [8]. This method is similar to determinant zero search methods using a Sturm sequence described in Meirovitch [9] and Bathe and Wilson [10].

With the natural frequencies defined, their associated eigenvectors, $\underset{\sim}{P}_{II}^{(j)}$, can be found through a variety of methods illustrated in references [9,10,11]. One method described in Craig [11], which assumes there are no repeated natural frequencies, is based on the solution of the following equation

$$\bar{\underline{D}}_I(\omega_j)\ \underset{\sim}{Y}^{(j)} = \underset{\sim}{0} \tag{3.11}$$

By assuming one element of $\underset{\sim}{Y}^{(j)}$ is unity:

$$\underset{\sim}{Y}^{(j)} = \frac{1}{\underset{\sim}{Y}_s^{(j)}} \tag{3.12}$$

The modified dynamic stiffness matrix can be partitioned as

$$\bar{\underline{D}}_I(\omega_j)\ \underset{\sim}{Y}^{(j)} = \left[\begin{array}{c|c} \underline{D}_{AA}(\omega_j) & \underline{D}_{AB}(\omega_j) \\ \hline \underline{D}_{BA}(\omega_j) & \underline{D}_{BB}(\omega_j) \end{array}\right] \left\{\begin{array}{c} 1 \\ \hline \underset{\sim}{Y}_s^{(j)} \end{array}\right\} \tag{3.13}$$

Solving the equation for $\underset{\sim}{Y}_s^{(j)}$ yields

$$\underset{\sim}{Y}_s^{(j)} = -\ \underline{D}_{BB}^{-1}(\omega_j)\ \underline{D}_{BA}(\omega_j) \tag{3.14}$$

The j^{th} eigenvector can be assembled using Eq. (3.12) and the process is repeated until all eigenvectors are found. A new modal matrix (modal space II) can be formulated from the $\underset{\sim}{Y}^{(j)}$ as $\underline{P}_{II}$. Note that $\underline{P}_{II}$ is not the modal matrix for the physical system; i.e., the physical mode shapes. Instead $\underline{P}_{II}$ is the modal matrix of modal space II. With this in mind $\underset{\sim}{Q}_I$ can be written as the expansion

$$\underset{\sim}{Q}_I = \underline{P}_{II}\ \underset{\sim}{Q}_{II} \tag{3.15}$$

Using Eq. (2.10), the physical system mode shapes can be written as

$$\underline{P} = \underline{P}_I\ \underline{P}_{II} \tag{3.16}$$

This equation shows, as pointed out in Chapter 2, that the modified physical system's mode shapes are linear summations of the original system modes, with $\underline{P}_{II}$ acting as a weighting function (Eq. (2.13)). Therefore, the modified mode shapes are totally dependent on the original modal database. The dependence of the modification results can also be seen in the derivation of the change matrix, $\underline{\Delta D}(\omega)$. It can be shown, Appendix A, that each element of $\underline{\Delta D}(\omega)$ can be expressed by

$$\Delta D(i,j,\omega) = \underset{\sim}{P}_I^T(g,i)\ [\underline{\Delta d}_{gg}(\omega)\ \underset{\sim}{P}_I(g,j) + \underline{\Delta d}_{gh}(\omega)\ \underset{\sim}{P}_I(h,j)]$$
$$+ \underset{\sim}{P}_I^T(g,i)\ [\underline{\Delta d}_{hg}(\omega)\ \underset{\sim}{P}_I(g,j) + \underline{\Delta d}_{hh}(\omega)\ \underset{\sim}{P}_I(h,j)] \tag{3.17}$$

$$i,j = 1,2,3,\ldots,\ell$$

where $\Delta D(i,j,\omega)$ - i,j element of $\underline{\Delta D}(\omega)$

$\underset{\sim}{P}_I(g,i)$ - modal information from the degrees-of-freedom at point g for the i^{th} mode

$\underset{\sim}{P}_I(h,i)$ - modal information from the degrees-of-freedom at point h for the i^{th} mode

$\underline{\Delta d}$ - submatrices of the change matrix in physical space.

ℓ - number of modes in the system.

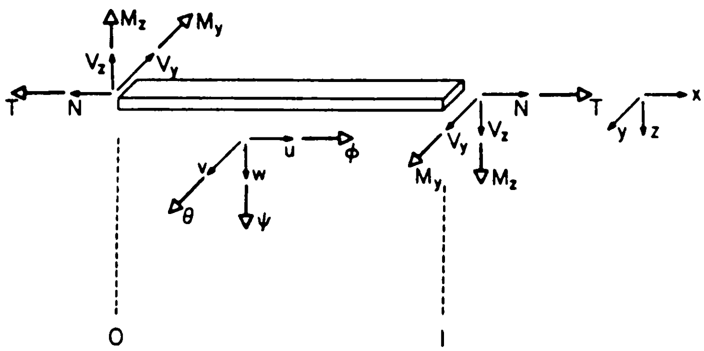
This equation shows that the modification matrix in modal space is only a function of the change dynamic stiffness matrix between the two connection points and the modal space I displacement information at these points.

This dependency becomes very important when dealing with results from an experimental analysis. Most experimental analyses produce truncated mathematical models--more degrees-of-freedom than modes. The primary effect of using truncated systems is the occurrence of modal truncation. Equations (3.16) and (3.17) clearly show that if a particular mode of the modified structure is dependent upon a missing mode in the original modal database, the modification has little hope of obtaining a good estimate of that mode. There is insufficient information to form that particular mode. A second observation is that modal truncation can affect any mode of the modification. Although no definitive study on modal truncation applied to an experimental database has been performed, it is generally believed that modal truncation affects only the modes at the extremes of the analysis range [12]. This is not true. Modal truncation could affect the modes in the middle of the analysis range while not affecting the modes at the extremes.

3.2.2 Development of the Modification Elements

Now that a process for instituting transfer matrix based modifications has been defined, the actual derivation of the transfer matrix modification element, $\underline{\Delta d}(\omega)$, can be addressed. In order to draw upon the extensive and versatile library of transfer-matrix elements [13-16], a means is necessary where one can convert from a transfer matrix to a stiffness matrix formulation. This conversion process is not unique [15,17-22]. However, a slightly different approach is taken here. Differences between transfer matrix/stiffness matrix coordinate systems and conversions between element/global coordinate systems will be considered.

Recall from Chapter 2 that the transfer matrix representation of the beam shown in Fig. 3.1 can be written as:



$$\underset{\sim}{z}_0 = \begin{Bmatrix} u \\ v \\ w \\ \phi \\ \theta \\ \psi \\ M_z \\ M_y \\ T \\ V_z \\ V_y \\ N \end{Bmatrix}_0 \qquad \underset{\sim}{z}_1 = \begin{Bmatrix} u \\ v \\ w \\ \phi \\ \theta \\ \psi \\ M_z \\ M_y \\ T \\ V_z \\ V_y \\ N \end{Bmatrix}_1$$

Figure 3.1 General Modification Beam Element

$$\hat{\tilde{z}}_1 = \hat{\underline{L}}\ \hat{\tilde{z}}_0 \tag{3.18}$$

where the hats, '^', denote an element coordinate system. This equation in partitioned form, separating the displacements from the forces, yields

$$\left\{\begin{array}{c}\hat{\tilde{q}} \\ \hline \hat{\tilde{f}}\end{array}\right\}_1 = \left[\begin{array}{c|c}\hat{\underline{L}}_1 & \hat{\underline{L}}_2 \\ \hline \hat{\underline{L}}_3 & \hat{\underline{L}}_4\end{array}\right]\left\{\begin{array}{c}\hat{\tilde{q}} \\ \hline \hat{\tilde{f}}\end{array}\right\}_0 \tag{3.19}$$

Expanding this matrix equation results in the following two equations

$$\left.\begin{aligned}\hat{\tilde{q}}_1 &= \hat{\underline{L}}_1\ \hat{\tilde{q}}_0 + \hat{\underline{L}}_2\ \hat{\tilde{f}}_0 \\ \hat{\tilde{f}}_1 &= \hat{\underline{L}}_3\ \hat{\tilde{q}}_0 + \hat{\underline{L}}_4\ \hat{\tilde{f}}_0\end{aligned}\right\} \tag{3.20}$$

Solving the first equation for $\hat{\tilde{f}}_0$ and substituting this result into the second equation produces

$$\left.\begin{aligned}\hat{\tilde{f}}_0 &= -\ \hat{\underline{L}}_2^{-1}\ \hat{\underline{L}}_1\ \hat{\tilde{q}}_0 + \hat{\underline{L}}_2^{-1}\ \hat{\tilde{q}}_1 \\ \hat{\tilde{f}}_1 &= (\hat{\underline{L}}_3 - \hat{\underline{L}}_4\ \hat{\underline{L}}_2^{-1}\ \hat{\underline{L}}_1)\ \hat{\tilde{q}}_0 + \hat{\underline{L}}_4\ \hat{\underline{L}}_2^{-1}\ \hat{\tilde{q}}_1\end{aligned}\right\} \tag{3.21}$$

In partitioned matrix form Eq. (3.21) becomes

$$\begin{Bmatrix} \hat{\tilde{f}}_0 \\ \hline \hat{\tilde{f}}_1 \end{Bmatrix} = \left[\begin{array}{c|c} -\hat{\underline{L}}_2\,\hat{\underline{L}}_1 & \hat{\underline{L}}_2^{-1} \\ \hline (\hat{\underline{L}}_3 - \hat{\underline{L}}_4\,\hat{\underline{L}}_2^{-1}\,\hat{\underline{L}}_1) & \hat{\underline{L}}_4\,\hat{\underline{L}}_2^{-1} \end{array}\right] \begin{Bmatrix} \hat{\tilde{q}}_0 \\ \hline \hat{\tilde{q}}_1 \end{Bmatrix} \tag{3.22}$$

or

$$\hat{\tilde{f}} = \hat{\underline{d}}_s\,\hat{\tilde{q}} \tag{3.23}$$

where $\hat{\underline{d}}_s$ is called the local pseudo-dynamic stiffness matrix. Equation (3.23) appears to be in a stiffness form. However, its development was carried out using a transfer-matrix sign convention, which, in general, is not the same as the sign convention of the stiffness matrix. Figure 3.2 shows the comparative sign conventions used in the two approaches. A second problem arises from the ordering of the force and displacement vectors. Figure 3.3 illustrates the necessary changes which need to be applied to transform the pseudo-dynamic stiffness matrix into a form consistent with the conventional stiffness matrix. A coordinate transformation is needed to perform this mapping. If the transformation matrices $\hat{\underline{C}}_p$ and $\hat{\underline{C}}_u$ give such a mapping, then

$$\hat{\tilde{f}} = \hat{\underline{C}}_p\,\hat{\tilde{p}} \quad ; \quad \hat{\tilde{q}} = \hat{\underline{C}}_u\,\hat{\tilde{u}} \tag{3.24}$$

The transformation matrices $\hat{\underline{C}}_p$ and $\hat{\underline{C}}_u$ for three-dimensional beams are shown in Appendix B. Substituting Eq. (3.24) into Eq. (3.23) and noting that, because $\hat{\underline{C}}_p$ is a set of orthogonal vectors, $\hat{\underline{C}}_p^{-1}$ equals $\hat{\underline{C}}_p^{T}$ and

$$\hat{\tilde{p}} = \hat{\underline{C}}_p^{T}\,\hat{\underline{d}}_s\,\hat{\underline{C}}_u\,\hat{\tilde{u}} \tag{3.25}$$

or

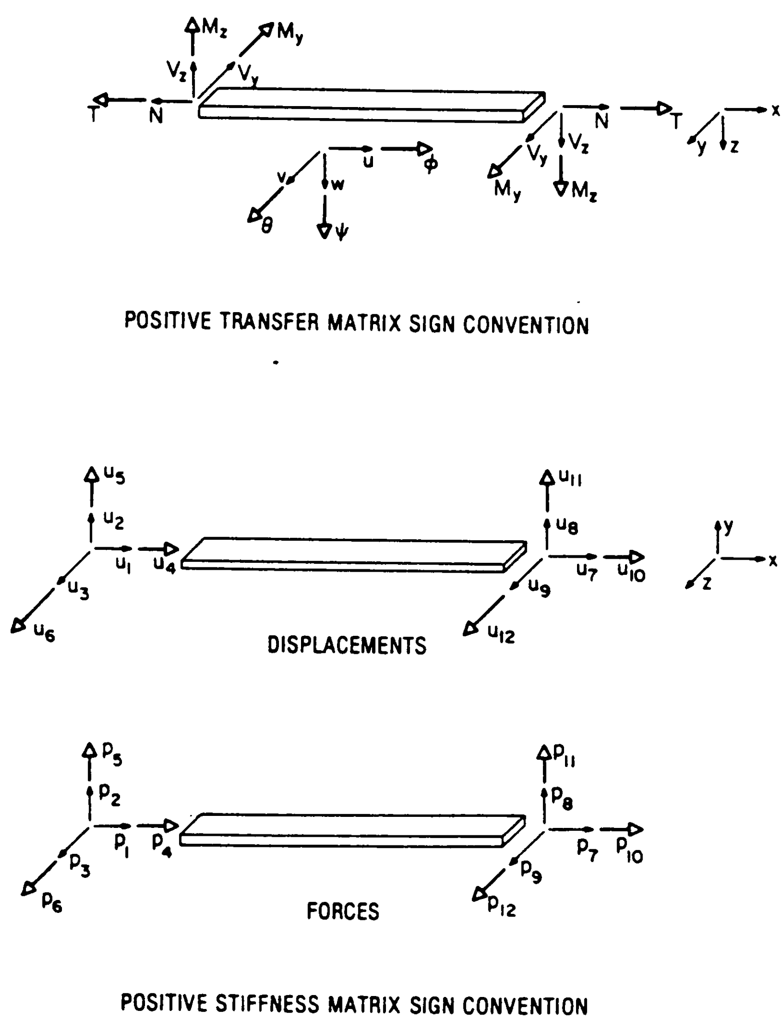


Figure 3.2 Comparative Study of the Transfer-Matrix Stiffness Matrix Sign Convention

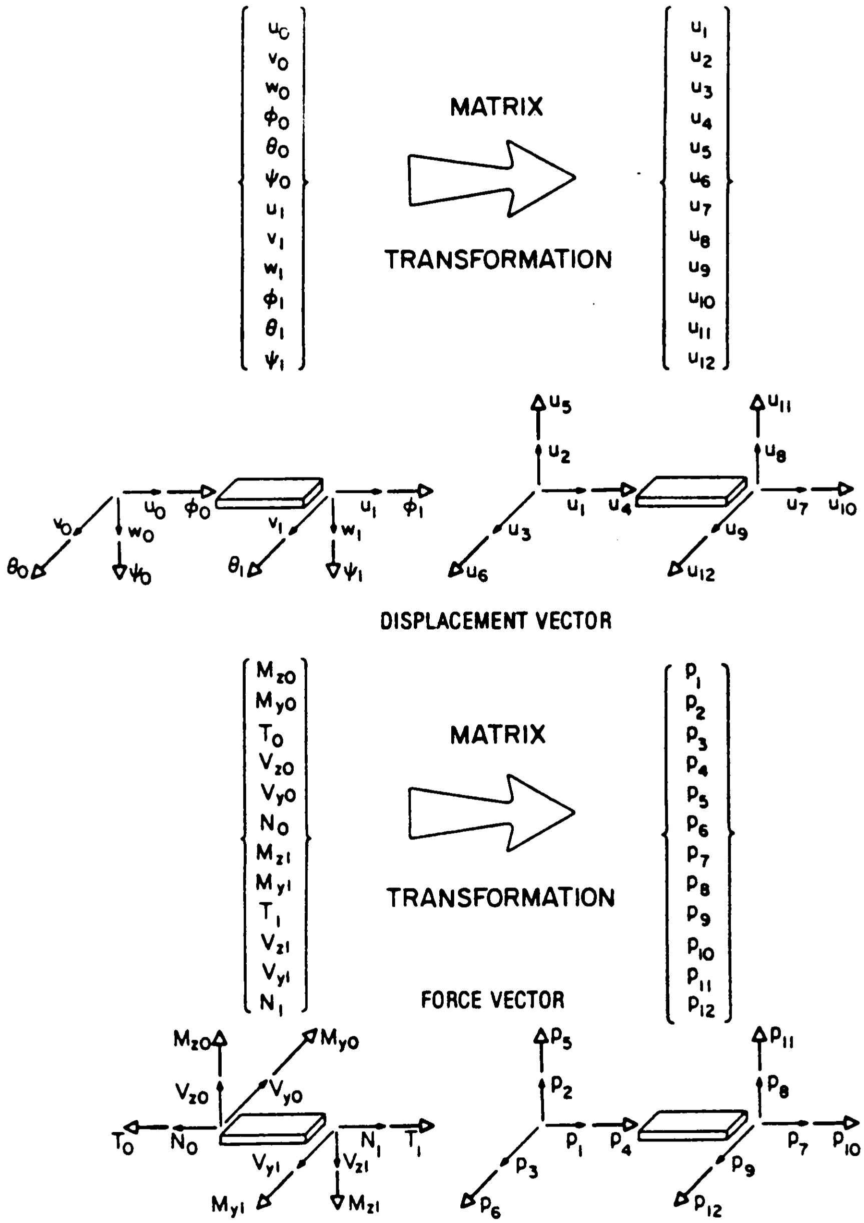


Figure 3.3 Pseudo-Stiffness Matrix to Conventional Stiffness Matrix Coordinate Transformation

$$\hat{\underset{\sim}{p}} = \hat{\underline{d}}\ \hat{\underset{\sim}{u}} \tag{3.26}$$

where $\hat{\underline{d}}$ is a conventional local dynamic stiffness matrix.

The next step is to convert the system of Eq. (3.26) from the element to the global coordinate system. The element-to-global coordinate system transformation used here is no different than that used in finite-element theory. Several references [11,23-25] present this transformation in terms of the direction cosine matrix, $\underline{\lambda}$. The element coordinate system can be expressed in terms of the global coordinate system as

$$\hat{\underset{\sim}{p}} = \underline{C}_T\ \underset{\sim}{p} \quad ; \quad \hat{\underset{\sim}{u}} = \underline{C}_T\ \underset{\sim}{u} \tag{3.27}$$

where $\underline{C}_T$ is the coordinate transformation matrix defined in Appendix B. Applying Eq. (3.27) to Eq. (3.26) and realizing that $\underline{C}_T^{-1} = \underline{C}_T^T$ results in

$$\underset{\sim}{p} = \underline{C}_T^T\ \hat{\underline{d}}\ C_T\ \underset{\sim}{u} \tag{3.28}$$

or

$$\underset{\sim}{p} = \underline{d}\ \underset{\sim}{u} \tag{3.29a}$$

where

$$\underline{d} = \underline{C}_T^T\ \hat{\underline{C}}_p^T\ \hat{\underline{d}}_s\ \hat{\underline{C}}_u\ C_T \tag{3.29b}$$

Now $\underline{d}$ is a dynamic stiffness matrix in the global coordinate system which was derived from a transfer-matrix description of a beam element.

The final step in this process is to produce a modification matrix from $\underline{d}$. Systematically, this can be done by first mapping the structural change matrix, $\underline{d}$, into a matrix the size of the physical system. Next, this matrix could be driven into modal space I. The end result is the modification matrix $\underline{\Delta D}(\omega)$ of Eq. (3.9). Realistically, Eq. (3.17) can be used to drive the structural change matrix $\underline{d}(\omega)$ into the modification matrix $\underline{\Delta D}(\omega)$. This would save time in eliminating two matrix multiplications which are of the order of the number of degrees-of-freedom of the physical system.

It may be instructive to pause a moment and examine the structure and properties of the dynamic stiffness matrix $\underline{d}(\omega)$. An important characteristic of this modification matrix is that it is a complete description of the beam modification between its connection points. At first this may seem a trivial point, but it is a very powerful point. It has to be remembered that the coupling matrix is formulated from a transfer matrix or matrices, which can be an exact solution of the beam between its ends. The structural change matrix is not an approximation --it contains information for an infinite number of degrees-of-freedom.

Consider the derivation of $\underline{d}(\omega)$. This structural change matrix is formulated from a transfer matrix through a linear process which is not practically concerned with the content of the transfer-matrix. The transfer-matrix can be as complex as necessary to describe the modification taking place. Continuum beams, beams with rotatory inertia, shear deformation, or point masses, beams with bolted connections, or any combination of beams may be assembled in terms of a single transfer matrix. In turn, this transfer matrix is formulated into a dynamic stiffness matrix which is a complete description of the change. This is the very strength and uniqueness of this approach. The only limitation imposed on this process is that the upper right quarter of the element transfer matrix be invertible (see Eq. (3.22)). Physically, this is not a problem. The only instance where the inversion is invalid is when the element transfer matrix is a point transfer matrix. An example of a point transfer matrix would be a point mass change. Point mass changes occur at a single location not between locations. Equation (3.22) is developed for changes between locations not changes at single locations. Therefore, it is not surprising the theory excludes single-point type changes.

Finally, consider the structure of the structural change matrix. This matrix is derived from a transfer matrix which may contain both elastic and inertial properties. Therefore, $\underline{d}$ has both stiffness and mass properties entwined in its structure. If $\underline{d}$ is derived from a transfer-matrix description of a massless beam, it would indeed be equivalent to a standard finite-element representation of a beam (frame) element [26]. However, if $\underline{d}$ is derived from a transfer matrix description of a continuum beam, it would be an inseparable composite of stiffness and mass terms. In fact, even the frequency term, ω, can not be separated from the structural change matrix because of the hyperbolic and trigonometric terms. The complexity of the modification matrix combined with the lack of knowledge of how significant the change will be eliminates the use of methods such as the local eigenvalue modification procedure [12,26] or perturbation methods [9]. The local eigenvalue modification procedure relies on separate mass and stiffness matrices, while perturbation methods require the effects of the modification to be a small change to the system. There is no guarantee that industrial use will be constrained to small eigen changes; in fact, it is likely that major changes will be attempted.

3.2.3 Synthesis of the Modification's Mode Shape

In performing a beam modification using the process outlined above, no information about the modification's motion is produced. All the modification's displacement information is expressed in terms of its end displacements. However, information concerning the modal displacement of the modification can be obtained through a postprocessing procedure.

From transfer-matrix theory, the state vector at any point 'x' can be expressed as

$$\left\{\begin{array}{c} \underline{q} \\ \hline \underset{\sim}{f} \end{array}\right\}_x^{(j)} = \left[\begin{array}{c|c} \underline{L}_1(x,\omega_j) & \underline{L}_2(x,\omega_j) \\ \hline \underline{L}_3(x,\omega_j) & \underline{L}_4(x,\omega_j) \end{array}\right] \left\{\begin{array}{c} \underline{q} \\ \hline \underset{\sim}{f} \end{array}\right\}_0^{(j)} \tag{3.30}$$

In other words, the state vector, which contains the displacement information, at any point 'x' is a function of the transfer matrix up to that point 'x' and the state vector at the beginning of the beam. In the case of the modification element, the transfer matrix $\underline{L}(x)$ is defined through the modification process. However, the initial state vector, $\underset{\sim}{z}_0$, is not fully defined. The transfer-matrix description of the element may also be written as

$$\begin{Bmatrix} \underset{\sim}{q} \\ \underset{\sim}{f} \end{Bmatrix}_1^{(j)} = \left[\begin{array}{c|c} \underset{\sim}{L}_1(\ell,\omega_j) & \underset{\sim}{L}_2(\ell,\omega_j) \\ \hline \underset{\sim}{L}_3(\ell,\omega_j) & \underset{\sim}{L}_4(\ell,\omega_j) \end{array}\right] \begin{Bmatrix} \underset{\sim}{q} \\ \underset{\sim}{f} \end{Bmatrix}_0^{(j)} \tag{3.31}$$

Both $\underset{\sim}{q}_0$, $\underset{\sim}{q}_1$ and $\underset{\sim}{L}$ are known from the modification process. However, $\underset{\sim}{f}_0$ and $\underset{\sim}{f}_1$ are unknown. Solving for $\underset{\sim}{f}_0$ yields

$$\underset{\sim}{f}_0^{(j)} = \underset{\sim}{L}_2^{-1}(\ell,\omega_j)\ [\underset{\sim}{q}_1^{(j)} - \underset{\sim}{L}_1(\ell,\omega_j)\ \underset{\sim}{q}_0^{(j)}] \tag{3.32}$$

By knowing $\underset{\sim}{q}_0$ and $\underset{\sim}{f}_0$, Eq. (3.30) can be used to find displacements at any point along the modification. The displacements, which are actually modal displacements, can be added to the modal vector corresponding to the mode ω_j. The process is repeated for each mode until all mode shapes have been recovered.

3.3 Modification Demonstration

In the previous section, an algorithm for the application of beam modification elements was developed. The verification of this algorithm is very cumbersome and complex to perform without the aid of a digital computer. This verification is left to the following chapter. However, if the modification theory were applied to a simpler system, say, a spring-mass system, a better physical understanding of the process might be achieved. The conversion of the beam modification procedure to a simple spring-mass system is straightforward and is considered below.

Consider changing the system shown in Fig. 3.4(a) to the system shown in Fig. 3.4(b) by adding the modification shown in Fig. 3.4(c). This modification demonstrates a spring and mass change between two

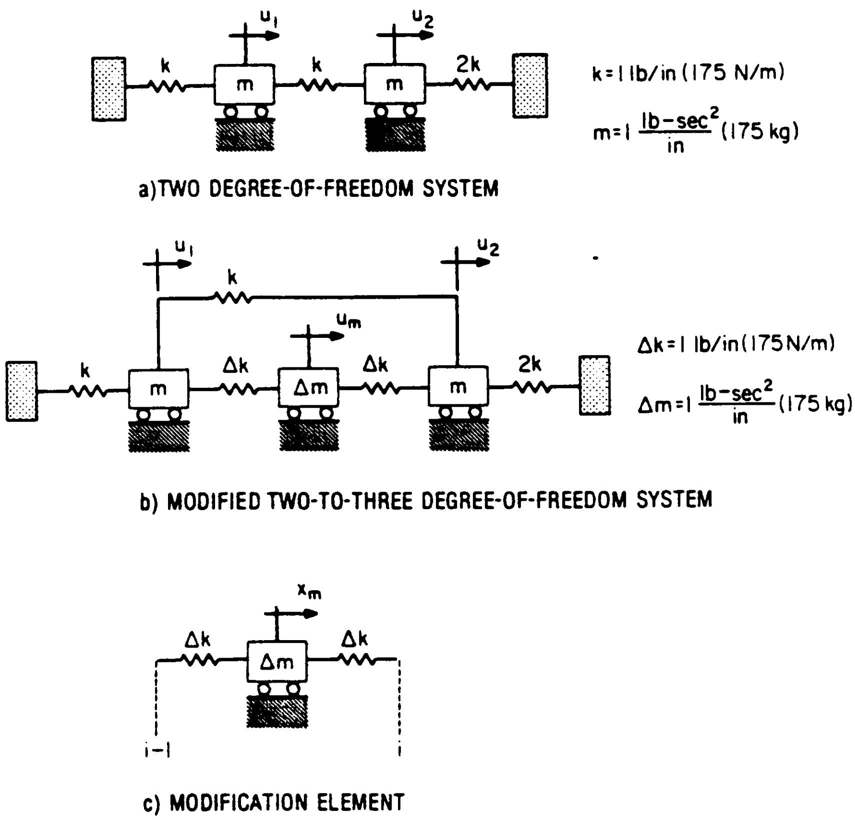


Figure 3.4 Spring-Mass System Modification

points in the unmodified system of Fig. 3.4(a). Note this mass change is actually an addition of a degree-of-freedom. However, this third degree-of-freedom is hidden in the modification element until it is recovered through a postprocess procedure.

The system of equations which describe the free vibration of the unmodified system of Fig. 3.4(a) are

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.33}$$

By assuming harmonic motion, $u(t) = Ue^{i\omega t}$, and by substituting the values given in Fig. 3.4(a), the system can be written as

$$\begin{bmatrix} 2 - \omega^2 & -1 \\ -1 & 3 - \omega^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.34}$$

Solving this system for its free response results in the following natural frequencies and modal matrix

$$\begin{aligned} \omega_1 &= 1.176 \text{ rad/sec} \\ \omega_2 &= 1.902 \text{ rad/sec} \end{aligned} \qquad \underline{P}_I = \begin{bmatrix} 1.000 & -0.6180 \\ 0.6180 & 1.000 \end{bmatrix} \tag{3.35}$$

The system can now be driven into modal space I using Eq. (3.4).

$$\begin{bmatrix} 1.910 - 1.382\,\omega^2 & 0 \\ 0 & 5.000 - 1.382\,\omega^2 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}_I = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.36}$$

The system is now in suitable form to apply the modification.

The transfer-matrix equation defining the motion of the element of Fig. 3.4(c) is

$$\hat{\underline{z}}_i = \hat{\underline{L}}_s \, \hat{\underline{L}}_m \, \hat{\underline{L}}_s \, \hat{\underset{\sim}{z}}_{i-1} \qquad (3.37)$$

where $\underline{L}_s$ and $\underline{L}_m$ are transfer matrices for a spring and mass element, respectively. These matrices are shown below:

$$\hat{\underline{L}}_s = \begin{bmatrix} 1 & 1/\Delta k \\ 0 & 1 \end{bmatrix} \; ; \quad \hat{\underline{L}}_m = \begin{bmatrix} 1 & 0 \\ -\Delta m\omega^2 & 1 \end{bmatrix} \qquad (3.38)$$

Carrying out the matrix multiplications and expanding the state vectors of Eq. (3.37) yields the following transfer-matrix representation of the modification element.

$$\begin{Bmatrix} \hat{x}_i \\ \hat{f}_i \end{Bmatrix} = \begin{bmatrix} 1 - \dfrac{\Delta m\omega^2}{\Delta k} & \dfrac{2\Delta k - \Delta m\omega^2}{\Delta k^2} \\ -\Delta m\omega^2 & \dfrac{1 - \Delta m\omega^2}{\Delta k} \end{bmatrix} \begin{Bmatrix} \hat{x}_{i-1} \\ \hat{f}_{i-1} \end{Bmatrix} \qquad (3.39)$$

or

$$\hat{\underset{\sim}{z}}_i = \hat{\underline{L}} \, \hat{\underset{\sim}{z}}_{i-1} \qquad (3.40)$$

Now that the transfer-matrix representation of the modification element has been defined, the pseudo-dynamic stiffness matrix, $\hat{\underline{d}}_s$, can be developed. By using Eq. (3.22), the transfer matrix in Eq. (3.40) is transformed to

$$\begin{Bmatrix} \hat{f}_{i-1} \\ \hat{f}_i \end{Bmatrix} = \begin{bmatrix} -\dfrac{\Delta k(\Delta k - \Delta m\omega^2)}{(2\Delta k - \Delta m\omega^2)} & \dfrac{\Delta k^2}{(2\Delta k - \Delta m\omega^2)} \\ -\dfrac{\Delta k^2}{(2\Delta k - \Delta m\omega^2)} & \dfrac{\Delta k(\Delta k - \Delta m\omega^2)}{(2\Delta k - \Delta m\omega^2)} \end{bmatrix} \begin{Bmatrix} \hat{x}_{i-1} \\ \hat{x}_i \end{Bmatrix} \qquad (3.41)$$

Careful examination of Fig. 3.4 reveals that the sign conventions are inconsistent. The sign convention of Fig. 3.4(a) is that of a conventional stiffness formulations, while the sign convention of Fig. 3.4(c) is that of the transfer-matrix method. The coordinate transformation between these two sign conventions, Eq. (3.24), is

$$\begin{Bmatrix} \hat{x}_{i-1} \\ \hat{x}_i \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{u}_{i-1} \\ \hat{u}_i \end{Bmatrix} ; \begin{Bmatrix} \hat{f}_{i-1} \\ \hat{f}_i \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{p}_{i-1} \\ \hat{p}_i \end{Bmatrix} \qquad (3.42)$$

Applying Eq. (3.25) produces the following element dynamic stiffness matrix

$$\begin{Bmatrix} \hat{p}_{i-1} \\ \hat{p}_i \end{Bmatrix} = \begin{bmatrix} \dfrac{\Delta k(\Delta k - \Delta m\omega^2)}{(2\Delta k - \Delta m\omega^2)} & -\dfrac{\Delta k^2}{(2\Delta k - \Delta m\omega^2)} \\ -\dfrac{\Delta k^2}{(2\Delta k - \Delta m\omega^2)} & \dfrac{\Delta k(\Delta k - \Delta m\omega^2)}{(2\Delta k - \Delta m\omega^2)} \end{bmatrix} \begin{Bmatrix} \hat{u}_{i-1} \\ u_i \end{Bmatrix} \qquad (3.43)$$

The final step to deriving the modification matrix is to transform from the element to the global coordinate system. In this case, the two coordinate systems are equivalent and no transformation is necessary. Therefore, the dynamic stiffness matrix of Eq. (3.43) is the physical space modification matrix $\underline{\Delta d}(\omega)$.

Notice that the modification matrix of Eq. (3.43) exhibits all the characteristics examined in the previous section. Although this matrix is simple compared to a continuum beam modification matrix, it is a good tool to show the characteristics of a modification matrix. The modification matrix contains information pertaining to a third degree-of-freedom and both mass and stiffness properties. These mass and stiffness properties are inseparable in terms of separate mass and stiffness matrices. Also notice the frequency term, ω, is located in the denominator of all the elements. The effect of this term is the addition of a pole in the characteristic equation. The development of this pole will be shown in the following paragraphs.

Substituting the modification values of Fig. 3.4(b) into Eq. (3.43) results in the following modification matrix

$$\underline{\Delta d}(\omega) = \frac{1}{2 - \omega^2} \begin{bmatrix} 1 - \omega^2 & -1 \\ -1 & 1 - \omega^2 \end{bmatrix} \tag{3.44}$$

Driving this matrix into modal space I yields

$$\underline{\Delta D}_I(\omega) = \frac{1}{2 - \omega^2} \begin{bmatrix} 0.1459 - 1.382\ \omega^2 & -\ 0.6180 \\ -\ 0.6180 & 2.618 - 1.382\ \omega^2 \end{bmatrix} \tag{3.45}$$

Adding this matrix to the system matrix yields a dynamic description of the modified system

$$\bar{\underline{D}}(\omega) = \frac{1}{2 - \lambda}\begin{bmatrix} 3.966 - 6.056\lambda + 1.382\lambda^2 & -0.6180 \\ -0.6180 & 12.62 - 9.146\lambda + 1.382\lambda^2 \end{bmatrix} \tag{3.46}$$

where $\lambda = \omega^2$. The characteristic equation for this system is

$$0 = \frac{26.00 - 59.00\lambda + 41.00\lambda^2 - 11.00\lambda^3 + \lambda^4}{(2 - \lambda)} \tag{3.47}$$

This characteristic equation contains both poles and zeros of the system. The zeros correspond to the natural frequencies of the modified system. The poles correspond to the natural frequencies of the modification element with grounded end conditions [13,28].

The solution to this characteristic equation and the corresponding unit normalized modal matrix is

$$\begin{aligned} \omega_1 &= 0.8864 \text{ rad/sec} \\ \omega_2 &= 1.881 \text{ rad/sec} \\ \omega_3 &= 2.162 \text{ rad/sec} \end{aligned} \qquad \underline{P}_{II} = \begin{bmatrix} 1.0000 & -0.3949 & 0.1055 \\ 0.09833 & 1.000 & 1.0000 \end{bmatrix} \tag{3.48}$$

The physical space modal matrix can be produced by using Eq. (3.16)

$$\underline{P} = \begin{bmatrix} 1.0000 & 1.0000 & -0.4812 \\ 0.7627 & 0.3154 & 1.0000 \end{bmatrix} \tag{3.49}$$

By using the process outlined in Subsection 3.2.3, the motion of the third degree-of-freedom can be synthesized. First, the initial state vector $\hat{\tilde{z}}_{i-1}$ of Eq. (3.40) must be found. Equation (3.49) defines the deflection end conditions of the modification element. By using Eq. (3.32), an equation for the unknown force f_{i-1} can be defined as

$$f_{i-1} = -\frac{\Delta k(\Delta k - \Delta m\omega^2)}{2\Delta k - \Delta m\omega^2} x_{i-1} + \frac{\Delta k^2}{2\Delta k - \Delta m\omega^2} x_i \tag{3.50}$$

Therefore,

$$\left.\begin{aligned} f_{i-1}(\omega_1) &= 0.4516 \\ f_{i-1}(\omega_2) &= -1.855 \\ f_{i-1}(\omega_3) &= 0.2873 \end{aligned}\right\} \tag{3.51}$$

By using Fig. 3.4(c), an equation for the displacement of the third mass is

$$\tilde{z}_m = \underline{L}_s \tilde{z}_{i-1} \tag{3.52}$$

or

$$\begin{Bmatrix} x_m \\ f_m \end{Bmatrix} = \begin{bmatrix} 1 & 1/\Delta k \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_{i-1} \\ f_{i-1} \end{Bmatrix} \tag{3.53}$$

Therefore, the modal displacement of the third mass is

$$x_m = x_{i-1} + \frac{F_{i-1}}{\Delta k} \tag{3.54}$$

and it follows that

$$\left.\begin{aligned} x_m^{(1)} &= 1.452 \\ x_m^{(2)} &= -0.8546 \\ x_m^{(3)} &= -0.1939 \end{aligned}\right\} \tag{3.55}$$

Substituting into the modal matrix and unit normalizing results in the complete modified system modal matrix of:

$$\underline{P} = \begin{bmatrix} 0.6889 & 1.0000 & -0.4812 \\ 1.0000 & -0.8546 & -0.1939 \\ 0.5254 & 0.3154 & 1.0000 \end{bmatrix} \tag{3.56}$$

The natural frequencies of Eq. (3.48) and the modal matrix above solves the free vibration problem of the modified system shown in Fig. 3.4(b). For comparison the system of Fig. 3.4(b) was solved using a standard eigenanalysis procedure [29]. The results of this comparison are shown in Fig. 3.5. For this simple modification, the procedure developed in Section 3.2 gives exact results.

NATURAL FREQUENCIES (RAD/SEC)

MODE NUMBER	EXACT [29]	MODIFICATION
1	0.8864	0.8864
2	1.881	1.881
3	2.162	2.162

MODEL MATRICES:

$$P_{exact} = \begin{bmatrix} 0.6889 & 1.000 & -0.4812 \\ 1.000 & -0.8546 & -0.1939 \\ 0.5254 & 0.3154 & 1.000 \end{bmatrix}$$

$$P_{mod} = \begin{bmatrix} 0.6889 & 1.000 & -0.4812 \\ 1.000 & -0.8546 & -0.1939 \\ 0.5254 & 0.3154 & 1.000 \end{bmatrix}$$

Figure 3.5 Comparison of Exact Result to the Results of the Modification Process of Figure 3.4

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CHAPTER 4 NUMERICAL EXAMPLES

4.1 Overview

The previous chapter presented an algorithm for installing beam modifications to a structure whose modal space database is known. It was pointed out that the verification of this algorithm using beam elements is difficult without the aid of a digital computer. This chapter presents two examples for which beam modifications are considered. The first example is a modification of a cantilevered beam to a fixed-fixed beam by the addition of a continuum beam element fixed to ground. This example demonstrates a radical change in the original structure's dynamic behavior. The second example is a modification of a Portal Arch to a Vierendeel Truss by the addition of a continuum beam element across the legs of the arch. This example is designed to demonstrate a change to a complex structure.

These two examples are intended to demonstrate the characteristics which a modification scheme must possess: i.e., modifications which represent realistic changes and an algorithm that can handle any degree of change. Both examples use continuum beam modification elements representing real changes. The change from a cantilevered to a fixed-fixed beam represents a large system change while the change from a Portal Arch to a Vierendeel Truss represents a moderate change to a complex structure.

In both cases, the example's modal database was developed through finite-element modeling. The modifications were applied through a computer program using the procedure outlined in Chapter 3. A general flow chart for this program is shown in Fig. 4.1. For the first example, the resulting natural frequencies and mode shapes were compared to a continuous solution. The results of the second example were compared to results obtained through a finite-element model of the Vierendeel Truss.

4.2 Cantilevered to Fixed-Fixed Beam Modification

The first example analyzed was the modification of a cantilevered beam to a fixed-fixed beam. Figure 4.2(a) shows the cantilevered beam's configuration. This beam was modified to the fixed-fixed beam shown in Fig. 4.2(b). The modification was accomplished using a continuum beam element fixed to ground as shown in Fig. 4.2(c).

This particular example was chosen for several reasons. First, the modification represents a major change in the structure's dynamic behavior. The change in the first natural frequency is approximately 340 percent. Second, the results are easily verified. The dynamic database for the cantilevered beam as well as the modification results can be verified using continuous solutions.

The modal database for the cantilevered beam was developed using finite-element techniques. A three-dimensional finite-element model of the cantilevered beam is shown in Fig. 4.3. The model consists of ten frame elements for a total of sixty-six degrees-of-freedom. The consistent mass matrix and the stiffness matrix for the frame element were obtained from Craig [1]. These matrices represent frame elements which neglect rotatory inertia and shear deformation. The couplings between in-plane bending, out-of-plane bending, axial deflections, and

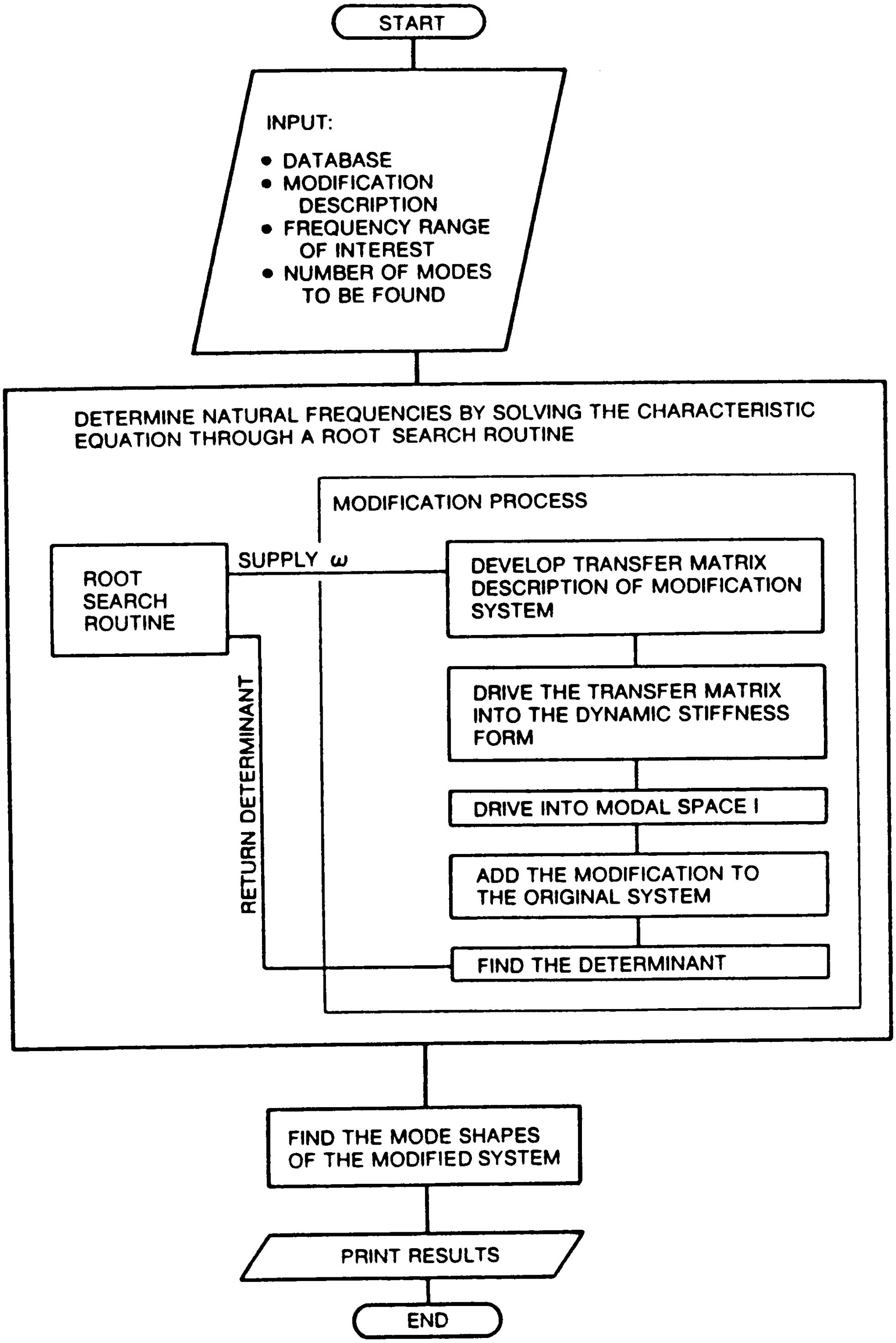


Figure 4.1 Beam Modification Flow Chart

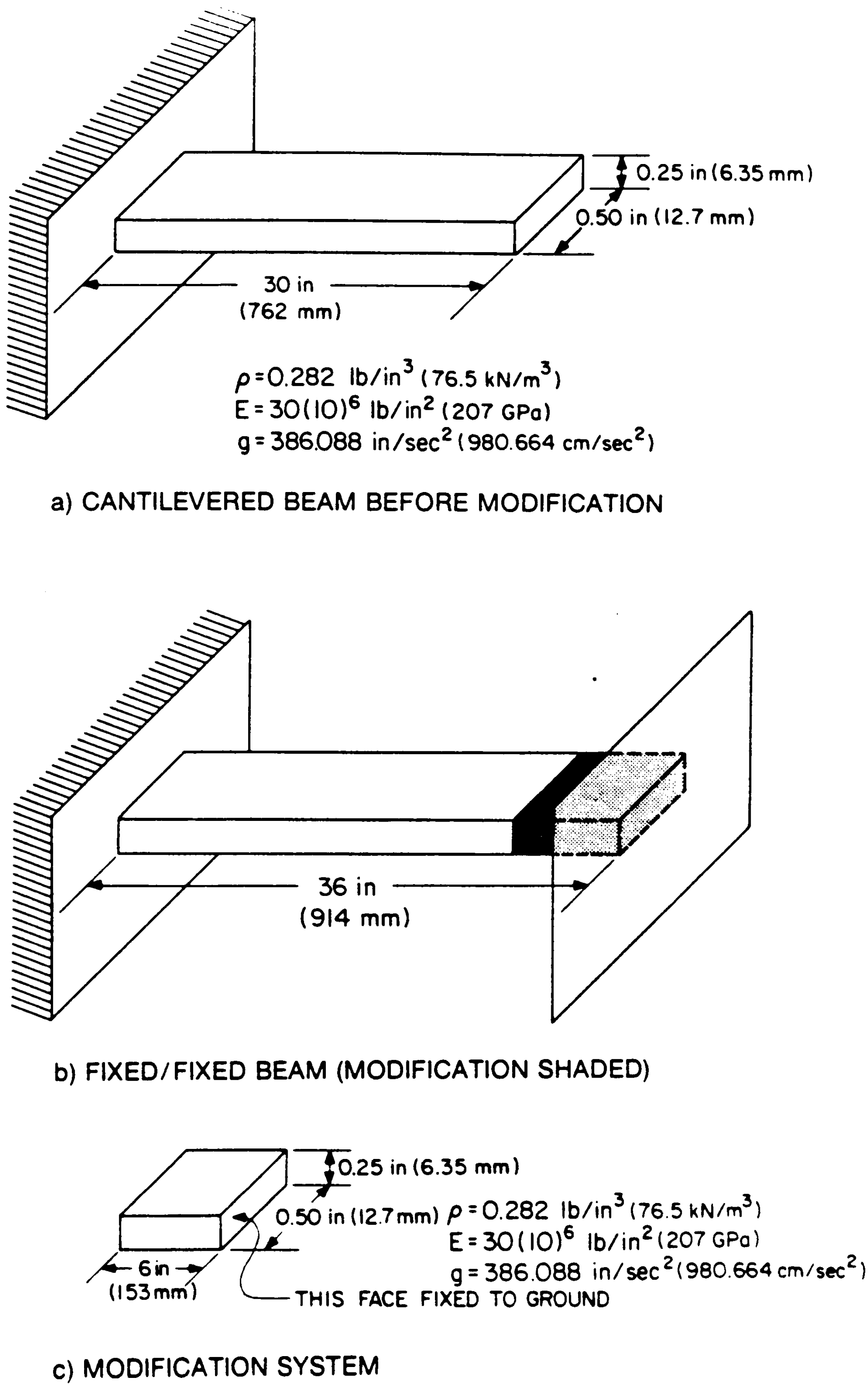
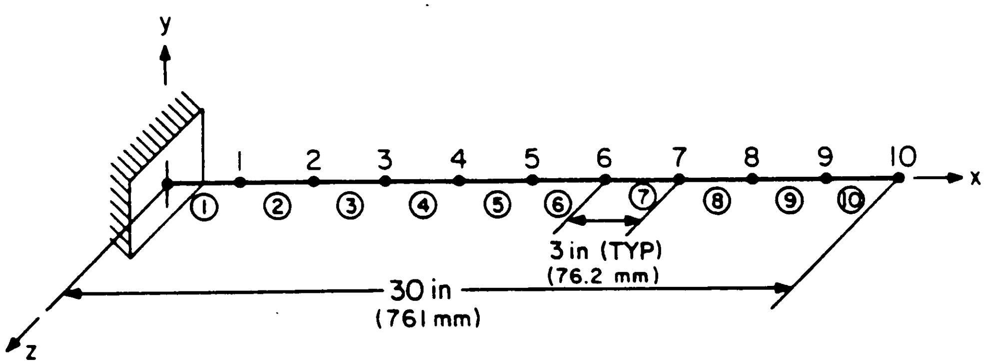
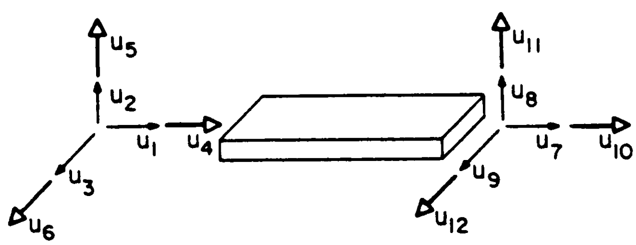


Figure 4.2 Modification of a Cantilevered Beam to a Fixed/Fixed Beam



a) FINITE-ELEMENT MODEL—CANTILEVERED BEAM



b) TYPICAL THREE-DIMENSIONAL FRAME ELEMENT

Figure 4.3 Finite-Element Model of the Cantilevered Beam of Figure 4.2

torsion are also ignored. The frame elements were assembled into constrained global mass and stiffness matrices using a structural analysis program called MAPMODES [2]. The first ten natural frequencies and mode shapes were found using an eigensolver called EIGENR [3]. The modal space I mass and stiffness matrices were produced using Eq. (2.4). The results of this analysis is shown in Table 4.1 along with the natural frequencies derived from a continuous solution obtained from Bishop and Johnson [4]. The modal assurance between the finite-element-produced mode shapes and the mode shapes derived using a continuous solution [4] are presented in the last column of Table 4.1.

The results of the finite-element eigenanalysis corresponds well with the results of the continuous solution. The relative differences in the first ten natural frequencies is less than 0.8 percent. The modal assurance between the mode shapes of the finite-element and the mode shapes produced through a continuous solution is one except for mode ten for which the modal assurance is 0.999998. These results show that the mode shapes between models are very similar if not the same. These combined results show that the discrete mathematical model is a reasonable representation of the dynamic characteristics for the first ten modes of the cantilevered beam.

Having established an adequate database, the modification process can take place. Using the modification routine outlined in Fig. 4.1, the cantilevered beam's dynamic database was modified and the first ten natural frequencies and mode shapes of the resulting fixed-fixed beam were found. The natural frequencies derived through this modification process along with the exact natural frequencies computed using Bishop

Table 4.1 Results of the Cantilevered Beam Eigenanalysis

Mode Number	Natural Frequency* (FEM)	Natural Frequency (Exact, Ref.[4])	Modal Assurance (FEM vs. Exact.)
1	56.6779 rad/sec	57.1396 rad/sec	1.00000
2	114.289	114.279	1.00000
3	358.078	358.088	1.00000
4	716.192	716.175	1.00000
5	1002.93	1002.76	1.00000
6	1966.71	1964.81	1.00000
7	2005.82	2005.31	1.00000
8	3256.18	3247.97	1.00000
9	3933.35	3929.61	1.00000
10	4878.11	4851.89	0.999998

*The author recognizes that some of the FEM results do not support the inclusion principle (separation theorem). However, the FEM model was considered sufficient to demonstrate the modification concepts presented here.

and Johnson [4] are presented in Table 4.2. The modification's mode shapes (dashed) are plotted over the exact mode shapes (solid) in Fig. 4.4 and Fig. 4.5. Figure 4.4 shows the in-plane bending modes, while Fig. 4.5 shows the out-of-plane bending modes. The modal assurances of Table 4.2 were computed between the modification's mode shapes and the exact mode shapes [4].

At first glance, the results of the modification process appear to be overly stiff and inconsistent. There appears to be no order in the relative error between the modification's natural frequencies and the exact natural frequencies. The maximum error occurs in the ninth mode (26.8%) while minimal errors occur at the extremes; mode one (0.249%) and mode ten (0.788%). Similar observations can be made for the modal assurances. At first, it is difficult to explain such inconsistencies - one might have expected the error to increase with mode. However, since the two planes of bending are uncoupled, the results can be viewed separately in terms of two planes of bending, as shown in Table 4.3. By viewing the results in this manner, more insight into the cause of the errors becomes evident. Table 4.3 shows that the relative error is generally greater for the out-of-plane bending modes than for the in-plane bending modes. The results obtained for in-plane bending are good with relative error less than eight percent and modal assurances greater than 0.94. However, the results for out-of-plane bending are poor with relative errors as high as 26.8 percent and modal assurance as low as 0.77.

It is interesting to note that the six in-plane bending and the four out-of-plane bending modes of the modification were produced using

Table 4.2 Cantilevered to Fixed/Fixed Beam Modification Results - Ten Mode Database

Mode Number	Natural Frequency (Modification)	Natural Frequency (Exact, Ref. [4])	Relative Error	Modal Assurance (Exact vs. Modification)
1	253.123 rad/sec	252.496 rad/sec	0.249%	0.999941
2	506.511	504.991	0.301	0.999931
3	706.880	696.013	1.56	0.997562
4	1438.12	1365.46	5.40	0.977736
5	1441.62	1392.03	3.56	0.988663
6	2423.29	2255.52	7.44	0.946682
7	3035.80	2728.92	11.3	0.895919
8	3518.90	3369.38	4.44	0.965455
9	5712.89	4511.05	26.8	0.769651
10	4743.04	4705.98	0.788	0.997512

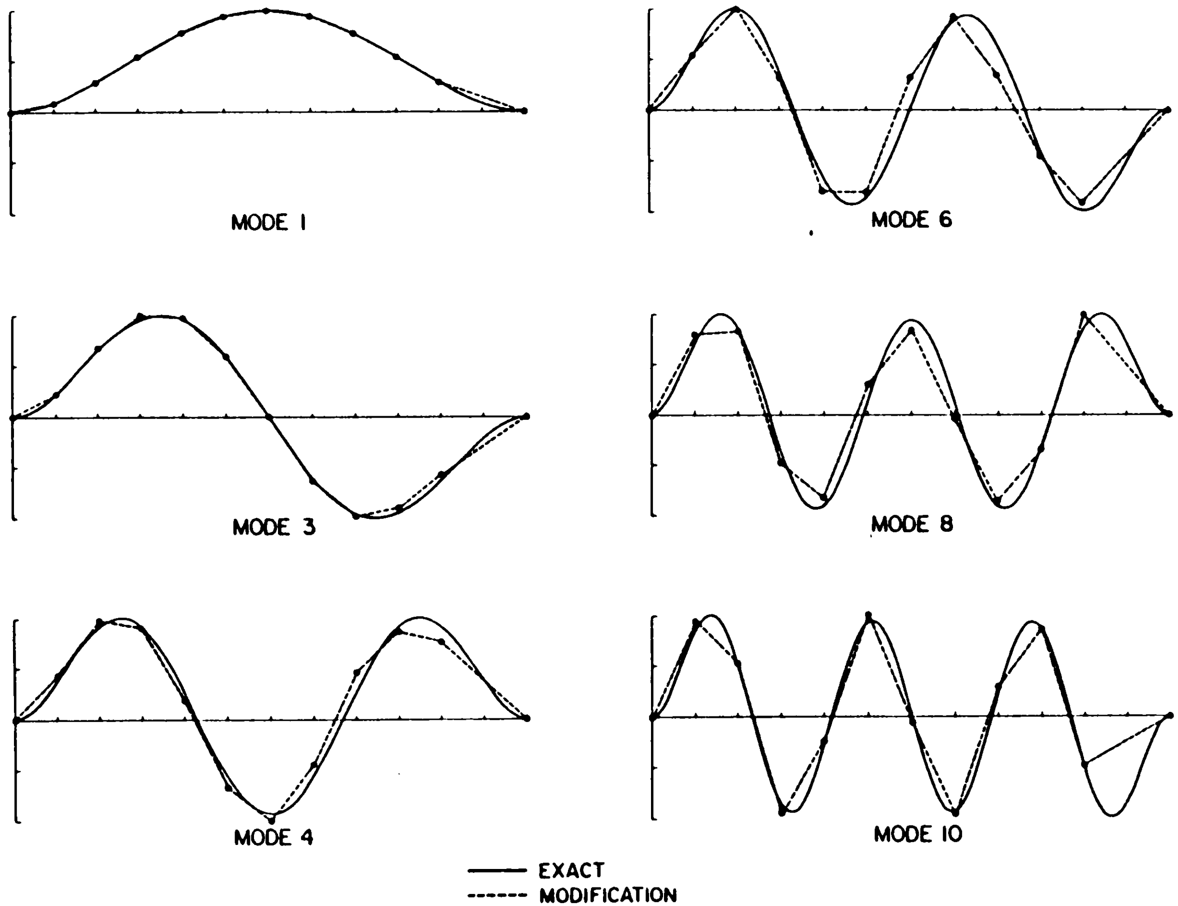


Figure 4.4 Comparison of the In-Plane Bending Mode Results - Modification vs. Continuous Solutions

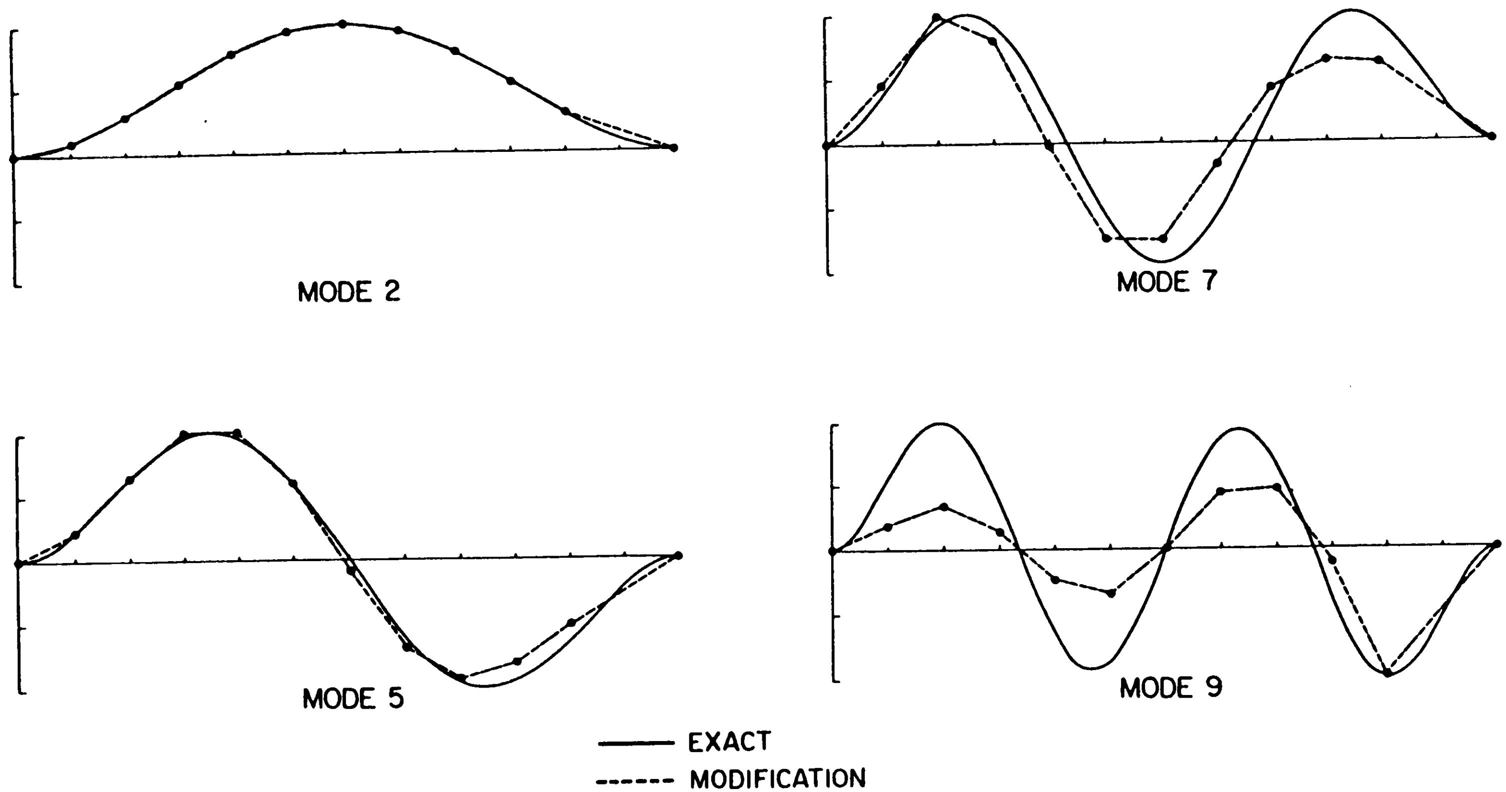


Figure 4.5 Comparison of the Out-of-Plane Bending Mode Results - Modification vs. Continuous Solution

Table 4.3 Cantilever to Fixed/Fixed Beam Modification Results - Ten Mode Database

a) In-Plane Bending Modes

Mode Number	Natural Frequency (Modification)	Natural Frequency (Exact,Ref.[4])	Relative Error	Modal Assurance (Exact vs. Modification)
1	253.123 rad/sec	252.496 rad/sec	0.249%	0.999941
3	706.880	696.013	1.56	0.997562
4	1438.12	1364.46	5.40	0.977736
6	2423.29	2255.52	7.44	0.946682
8	3518.90	3369.38	4.44	0.965455
10	4743.04	4705.98	0.788	0.997512

b) Out-of-Plane Bending Modes

Mode Number	Natural Frequency (Modification)	Natural Frequency (Exact,Ref.[4])	Relative Error	Modal Assurance (Exact vs. Modification)
2	506.511 rad/sec	504.991 rad/sec	0.301%	0.999931
5	1441.62	1392.03	3.56	0.988663
7	3035.80	2728.92	11.3	0.895919
9	5721.89	4511.05	26.8	0.769651

a database which contained six in-plane bending and four out-of-plane bending modes. Perhaps the four out-of-plane bending modes of the cantilevered beam's database were an insufficient number of modes from which to draw upon to successfully generate the out-of-plane bending modes. If true, this would be a case of modal truncation. Two tests were performed to assess the effect of modal truncation:

1) the cantilivered beam's database was extended to fourteen modes and

2) the cantilevered beam's database was truncated to six modes.

The first test was developed to demonstrate that by extending the database, the relative errors of the first ten modification modes would improve. The second test was developed to demonstrate that by truncating the database, the relative errors of the modification results would rise.

In the first test, the expansion of the cantilevered beam's database represented an addition of two in-plane and two out-of-plane bending modes. Table 4.4 shows the results of this test. This table shows that when the cantilevered beam's database was increased, both the in-plane and out-of-plane bending mode's relative errors were reduced. The relative errors of the in-plane bending modes were moderately reduced (7.44% down to 4.83% for mode six), while significant reductions (26.8% down to 7.44% for mode nine) were achieved in the out-of-plane bending modes. It appears that the contribution of the additional cantilevered out-of-plane bending modes is more significant than the additional cantilevered in-plane bending modes. It is interesting to notice here that by using a cantilever modal database which includes six

Table 4.4 Cantilever to Fixed/Fixed Beam Modification Results - Fourteen Mode Database

a) In-Plane Bending Modes

Mode Number	Natural Frequency (Modification)	Natural Frequency (Exact,Ref.[4])	Relative Error	Modal Assurance (Exact vs. Modification)
1	253.062 rad/sec	252.496 rad/sec	0.224%	0.999961
3	702.350	696.013	0.911	0.999075
4	1412.02	1364.46	3.49	0.990782
6	2364.42	2255.52	4.83	0.977980
8	3475.35	3369.38	3.15	0.984483
10	4740.10	4705.98	0.725	0.998918
11	6376.69	6265.36	1.78	0.975826
13	8507.16	8047.50	5.71	0.854851

b) Out-of-Plane Bending Modes

Mode Number	Natural Frequency (Modification)	Natural Frequency (Exact,Ref.[4])	Relative Error	Modal Assurance (Exact vs. Modification)
2	506.485 rad/sec	504.991 rad/sec	0.296%	0.999941
5	1413.80	1392.03	1.56	0.997562
7	2876.22	2728.92	5.40	0.977735
9	4846.53	4511.05	7.44	0.946681
12	7037.77	6738.75	4.44	0.965453
14	9486.03	9411.96	0.787	0.997512

out-of-plane bending modes, the relative errors and modal assurances of the out-of-plane modification results match the relative errors and modal assurances of the in-plane modification results obtained using a modal database which includes six in-plane bending modes. This is not surprising. Since the planes of bending are uncoupled, the out-of-plane bending modes are the same as the in-plane bending modes, only frequency shifted. Therefore, out-of-plane modification results should be similar to the in-plane modification results when an equal number of out-of-plane or in-plane bending modes exist in the cantilevered beam's modal database.

In the second test, two in-plane and two out-of-plane bending modes were deleted from the cantilevered beam's modal database. Table 4.5 shows that by truncating the cantilevered beam's database, the relative errors rise. The relative errors for the in-plane bending modes rose significantly (from 7.44% to 26.8% for mode six). The first out-of-plane bending mode (mode two) more than doubles in relative error (0.301% to 0.883%). However, the second out-of-plane bending mode (mode five) was not recovered. Again, it is interesting to notice that the relative errors and modal assurance of the in-plane bending results obtained using a cantilevered database containing four in-plane bending modes match the relative errors and modal assurances of the out-of-plane bending results of Table 4.3. This occurs for the same reason as pointed out in the preceding paragraph.

The results of this analysis clearly show that the accuracy of the modified structure's natural frequencies and mode shapes are very dependent on which modes are included in the original (unmodified)

Table 4.5 Cantilever to Fixed/Fixed Beam Modification Results - Six Mode Database

a) In-Plane Bending Modes

Mode Number	Natural Frequency (Modification)	Natural Frequency (Exact, Ref.[4])	Relative Error	Modal Assurance (Exact vs. Modification)
1	253.136 rad/sec	252.496 rad/sec	0.254%	0.999931
3	720.794	696.013	3.56	0.988557
4	1517.91	1364.46	11.3	0.895919
6	2860.94	2255.52	26.8	0.787994

b) Out-of-Plane Bending Modes

Mode Number	Natural Frequency (Modification)	Natural Frequency (Exact,Ref.[4])	Relative Error	Modal Assurance (Exact vs. Modification)
2	509.451 rad/sec	504.991 rad/sec	0.883%	0.998691

structure's modal database. However, quantification of the effect of modal truncation is not possible through the use of this model. In other words, there does not seem to be any sort of trend in these results whereby a rule can be stated giving the number of modes necessary to produce a successful modification. In fact, it may be possible that the total number of modes contained in a database is not as important as which modes are present. This is consistent with Eq. (2.13) which states that a modal vector of the modified structure is a weighted linear sum of the modal vectors present in the database. If a particular modification modal vector is highly dependent on one or more modal vectors of the database, which are not present, then the modification modal vector will not be successfully created.

Evidence of this hypothesis can be seen in a comparison of the modification results obtained using databases which contained ten and fourteen modes. It is important here to remember that the mode shapes of the in-plane and out-of-plane bending modes are the same. Also, the out-of-plane natural frequencies are double the in-plane natural frequencies. This is true for both the cantilevered beam and the fixed-fixed beam. The original example used a database containing information about ten modes: six in-plane bending modes and four out-of-plane bending modes. The resulting modification produced six in-plane bending modes with low relative errors and four out-of-plane bending modes, three of these having high relative errors. In the second analysis, the database was extended to contain fourteen modes; eight in-plane bending modes and six out-of-plane bending modes. The modification process resulted in eight in-plane modes with low relative errors and six out-of-plane bending modes with low relative errors. The fourteen mode database represents an expansion of two in-plane and two out-of-plane bending modes. However, the relative errors associated with the in-plane bending mode results improved modestly, while the relative errors associated with the out-of-plane bending mode results improved significantly (see Tables 4.3 and 4.4). It appears that the second through the sixth out-of-plane bending modes of the modification are very dependent on the fifth and sixth out-of-plane bending modes of the database.

An analogous demonstration can be made by looking at a comparison of the modification results obtained using the six and ten mode databases. By truncating the ten mode database down to six modes, two in-plane and two out-of-plane bending modes were deleted. Deletion of the two in-plane bending modes resulted in the in-plane modification results deteriorating significantly. For this case, it appears that the fifth and sixth in-plane modes of the data are necessary to produce the first four in-plane bending modes of the modification. The deletion of the two out-of-plane bending modes from the database eliminates the possibility of determining the second out-of-plane bending mode of the modification. However, the first out-of-plane bending mode was obtained quite accurately with only the two remaining out-of-plane bending modes. These observations show that the accuracy of modification process is more a function of which modes are present in the database instead of the total number of modes in the database.

This example specifically addresses the application of a severe modification to a structural system. It is interesting to note that if fourteen out of a possible sixty modes are used in the cantilevered beam's database, fourteen of the fixed/fixed beam's modes were found. These results represent a significant improvement over results presented by O'Callahan and Chou [5,6]. O'Callahan and Chou observed that their modification scheme performed poorly when they combined a severe modification with a truncated database. They state that:

> "If the modification is very severe such as changing to fixed end boundary condition or adding a large stiffness, then the results of the modification will have significant error unless a near complete modal database is used in the analysis" [5].

This study shows errors less than eight percent in performing a severe modification. This study also shows that the errors of this example do not seem to be related to the severity of change, but instead the errors are based upon using a database which contains the "right" modes. That is, prediction is hampered by modal truncation.

4.3 Portal Arch to Vierendeel Truss Beam Modification

The second example analyzed was the modification of a Portal Arch to a Vierendeel Truss. The modification was carried out by adding a continuum beam between the legs of the arch attached at half-way points along their length. Figure 4.6 illustrates this modification process. This example was chosen to demonstrate the ability to modify a complicated structure.

The dynamic database for the Portal Arch was derived using a three-dimensional finite-element model. This model is shown in Fig. 4.7. It consists of twelve frame elements for a total of seventy-eight degrees-of-freedom. Twelve of these degrees-of-freedom are constrained to

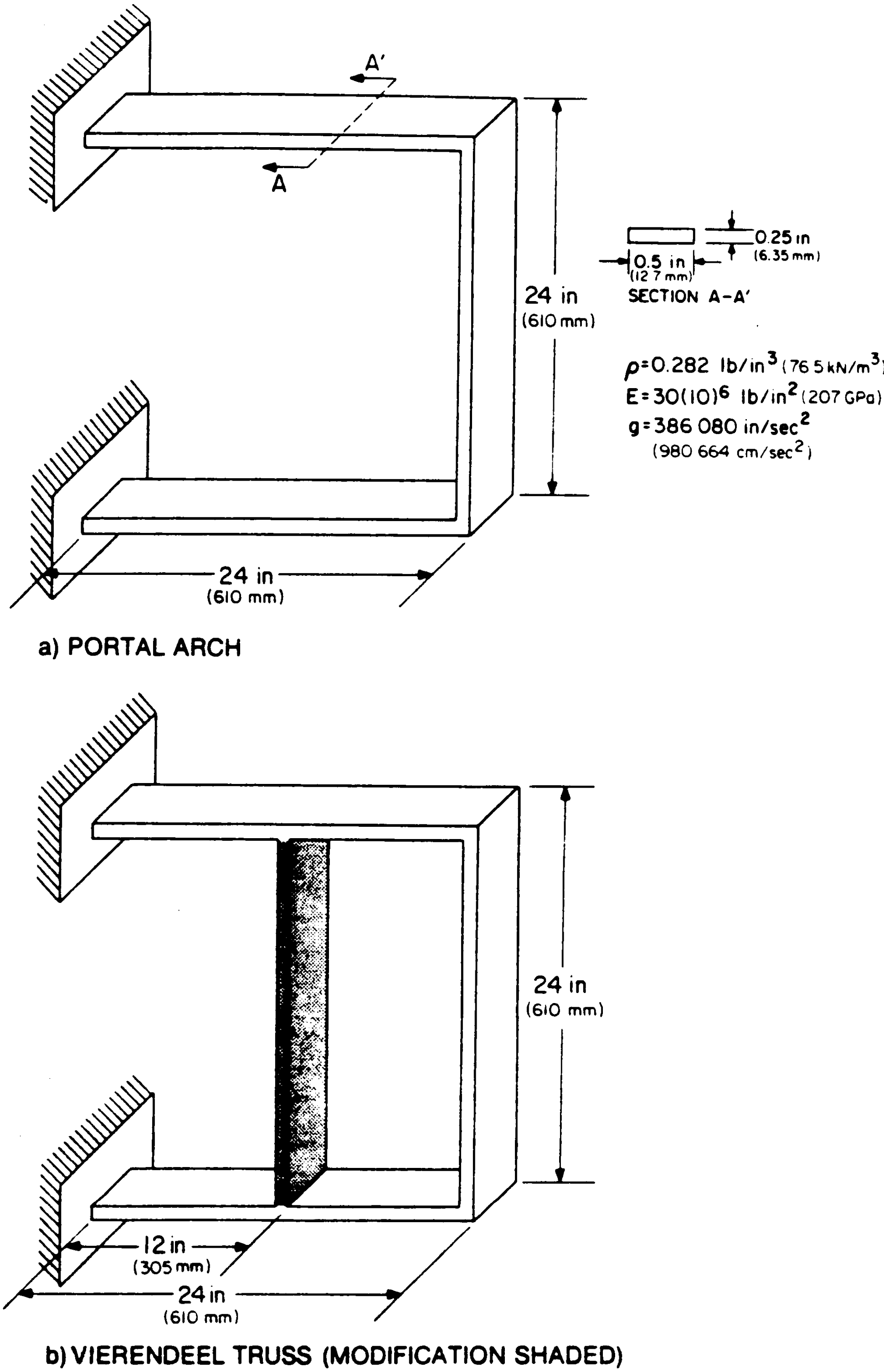


Figure 4.6 Modification of a Portal Arch to a Vierendeel Truss

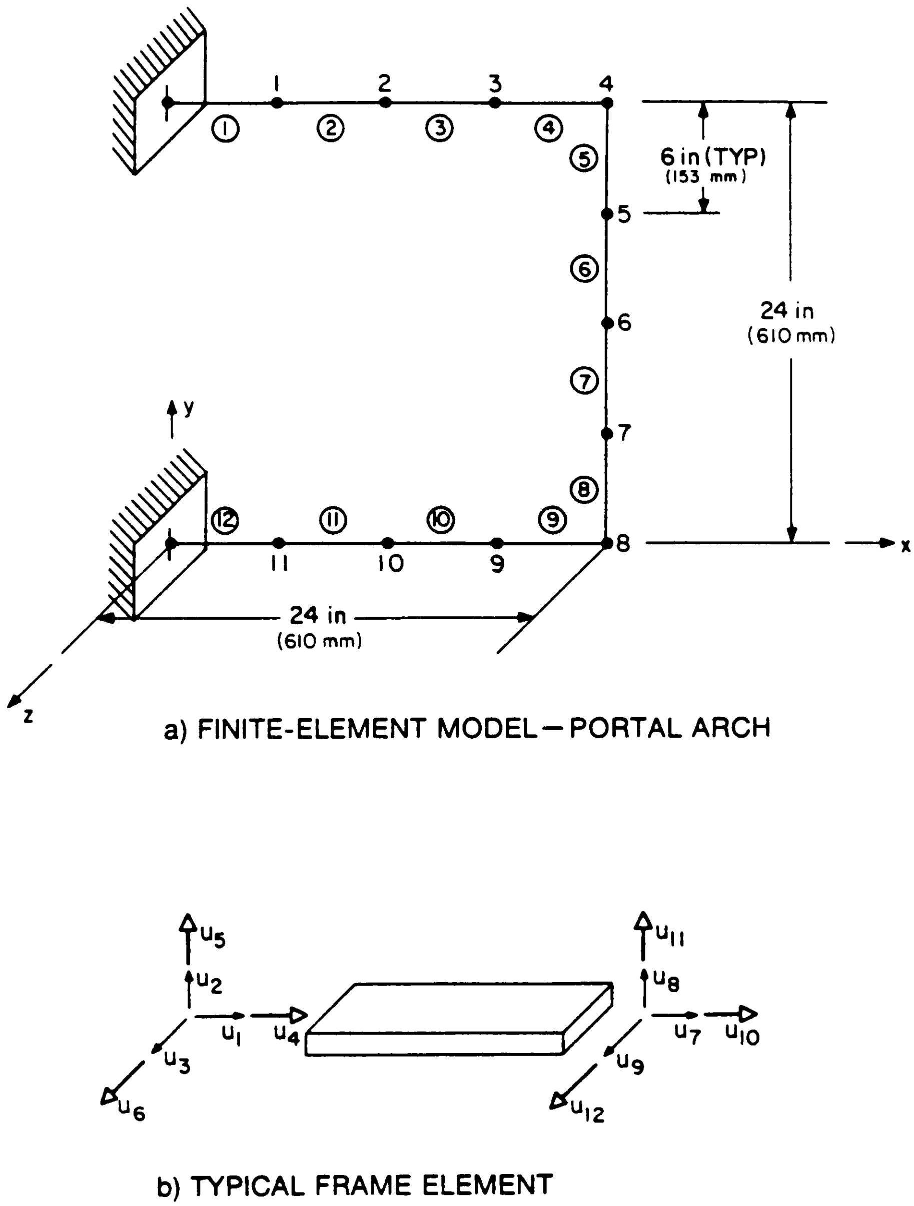


Figure 4.7 Finite-Element Model of the Portal Arch

ground. The dynamic solution of this model was carried out in the same manner as the cantilevered beam finite-element analysis of the last section. The first ten natural frequencies from this eigenanalysis are shown in Table 4.6. For comparative purposes, the Portal Arch was modeled using transfer-matrix methods. The modeling process for the transfer-matrix solution followed the method presented by Pestel and Leckie [7]. The continuum transfer matrix needed for this process was generated using methods outlined in Pilkey [8]. The natural frequencies and mode shapes for this model were obtained using a computer program called LSD [9]. The natural frequencies of the transfer-matrix solution and the modal assurances between the transfer-matrix solution and the finite-element solution are shown in Table 4.6. Since continuous transfer-matrix elements were used, the transfer-matrix solution represents an exact solution of this problem. It may be noted that the transfer-matrix solution matches the results of Bishop and Johnson [4], and Jara-Almonte [10] for the in-plane bending portion of the Portal Arch's dynamic behavior. The results presented in Table 4.6 show that the finite-element model is an adequate representation of the dynamic characteristics of the Portal Arch. The relative errors of the natural frequencies are all below one-half of one percent, while the modal assurances are above 0.97.

Applying the modification algorithm illustrated in Fig. 4.1, the modification of Fig. 4.6 was performed. A continuum beam was installed between grid points 2 and 10 of the finite-element model of Fig. 4.7. Ten modes of the finite-element model were used as a database for the modification process. The first ten natural frequencies and mode shapes

Table 4.6 Results of the Portal Arch Eigenanalysis

Mode Number	Natural Frequency** (FEM)	Natural Frequency (TM)	Relative Error*	Modal Assurance (TM vs. FEM)
1	81.1329 rad/sec	81.3702 rad/sec	- 0.293%	1.00000
2	101.440	101.410	0.029	0.998754
3	195.543	195.467	0.039	0.998609
4	321.242	321.103	0.043	1.00000
5	524.295	523.811	0.092	0.999999
6	568.647	567.892	0.133	1.00000
7	683.237	682.882	0.052	0.996906
8	993.606	992.603	0.101	0.978615
9	1152.77	1146.94	0.505	0.998031
10	1156.68	1155.24	0.125	0.999950

* The transfer matrix (TM) solution is considered an exact solution.

**The author recognizes that some of the FEM results do not support the inclusion principle (separation theorem). However, the FEM model was considered sufficient to demonstrate the modification concepts presented here.

were found. The natural frequencies of this modification process are presented in Table 4.7 and the mode shapes are shown in Figs. 4.8 through 4.12.

To check the results of this modification, a finite-element of the Vierendeel Truss was created. Figure 4.13 shows this model. There are sixteen elements which represent ninety-six degrees-of-freedom. Twelve of these degrees-of-freedom are constrained to ground. The first ten natural frequencies and mode shapes are presented alongside the modification results (Table 4.7 and Figs. 4.8-4.12). The results of this finite-element model were checked using a transfer-matrix model similar to the model used to check the Portal Arch. Comparisons of natural frequencies between the finite-element truss and the transfer-matrix truss are shown in Table 4.8. The relative error in the natural frequencies between the two models is small which indicates that the finite-element model is an adequate model to be used as a guide to check the results of the modification analysis.

The results of this modification process show the same trends as the results of the cantilever-fixed/fixed beam modification. The results are stiffer and there does not seem to be a pattern for the errors. Small errors occur for all modes except modes six and seven. The relative errors associated with these modes are an order of magnitude higher than the surrounding modes. It is interesting to note that while the relative errors for modes six and seven were high (21.8%, 18.5%), the modal assurances were 0.95, 0.91, respectively. The modification produced an inadequate estimate for the natural frequency. However, it produced a good approximation of the respective mode

Table 4.7 Arch to Truss Modification Results*

Mode Number	Natural Frequency (MOD)	Natural Frequency (FEM)	Relative Error**	Modal Assurance (FEM vs. MOD)
1	98.0609 rad/sec	97.9451 rad/sec	0.118%	0.999999
2	114.292	106.970	6.85	0.999528
3	211.969	209.523	1.17	0.999853
4	379.220	377.444	0.471	0.999639
5	403.042	397.515	1.39	0.970253
6	576.478	476.178	21.1	0.948587
7	581.724	490.770	18.5	0.910574
8	796.620	769.806	3.48	0.939496
9	874.385	860.394	1.63	0.996535
10	1107.34	1104.27	0.278	0.978037

* Arch database had 10 modes from Table 4.6.

**FEM results are used as the error reference.

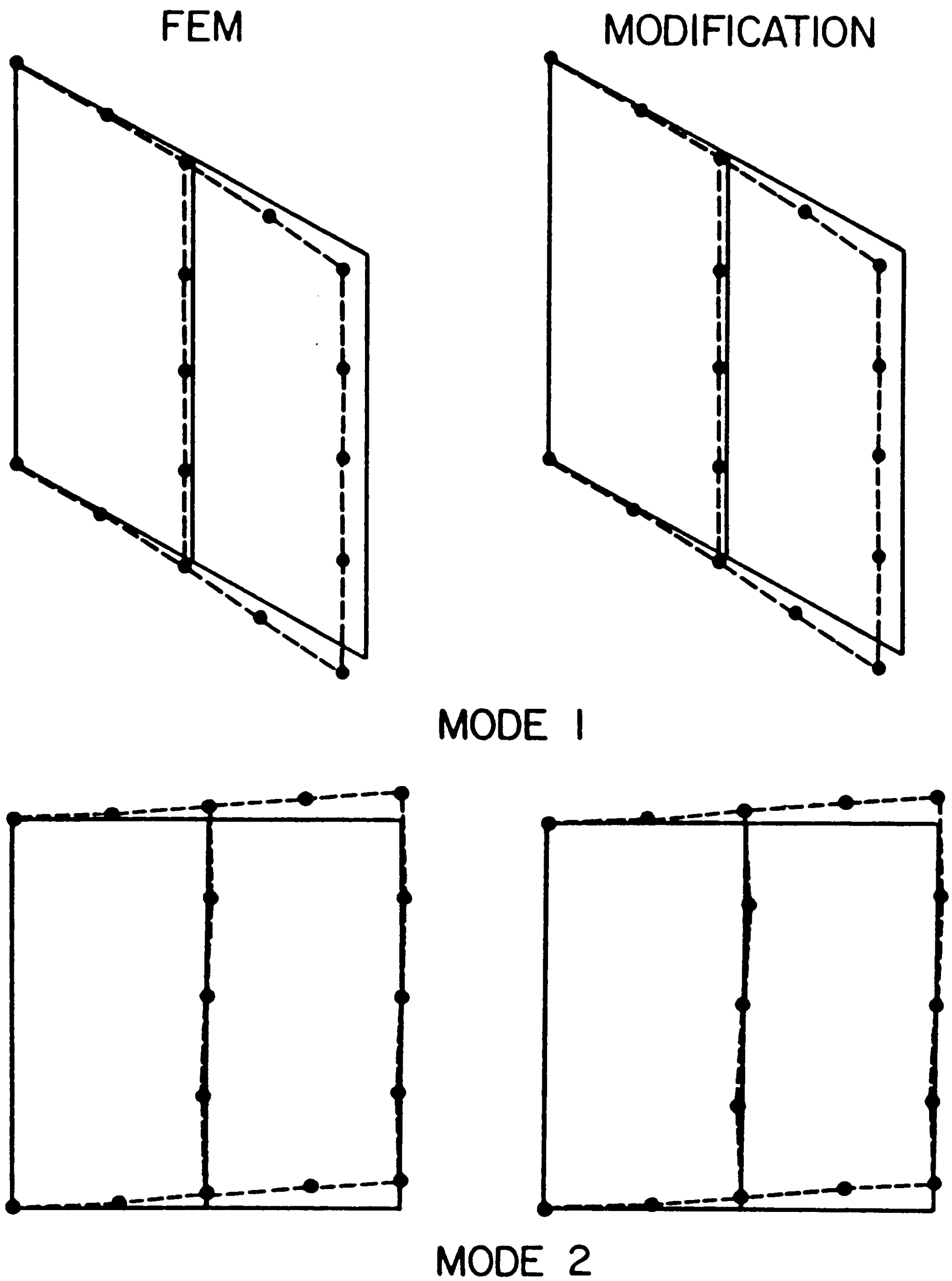


Figure 4.8 Comparison of the Modified Arch Mode Shape to the FEM Truss Mode Shape

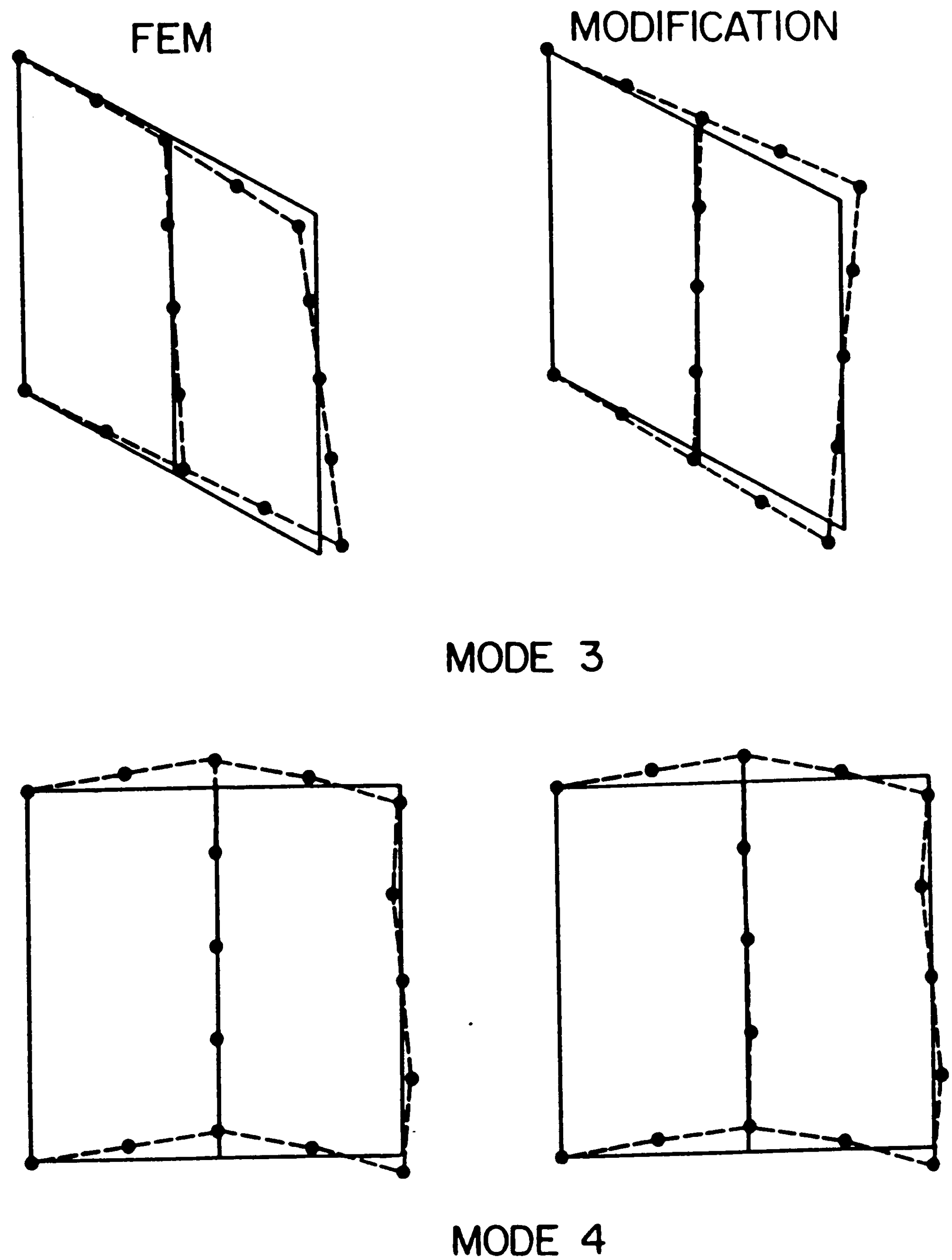


Figure 4.9 Comparison of the Modified Arch Mode Shape to the FEM Truss Mode Shape

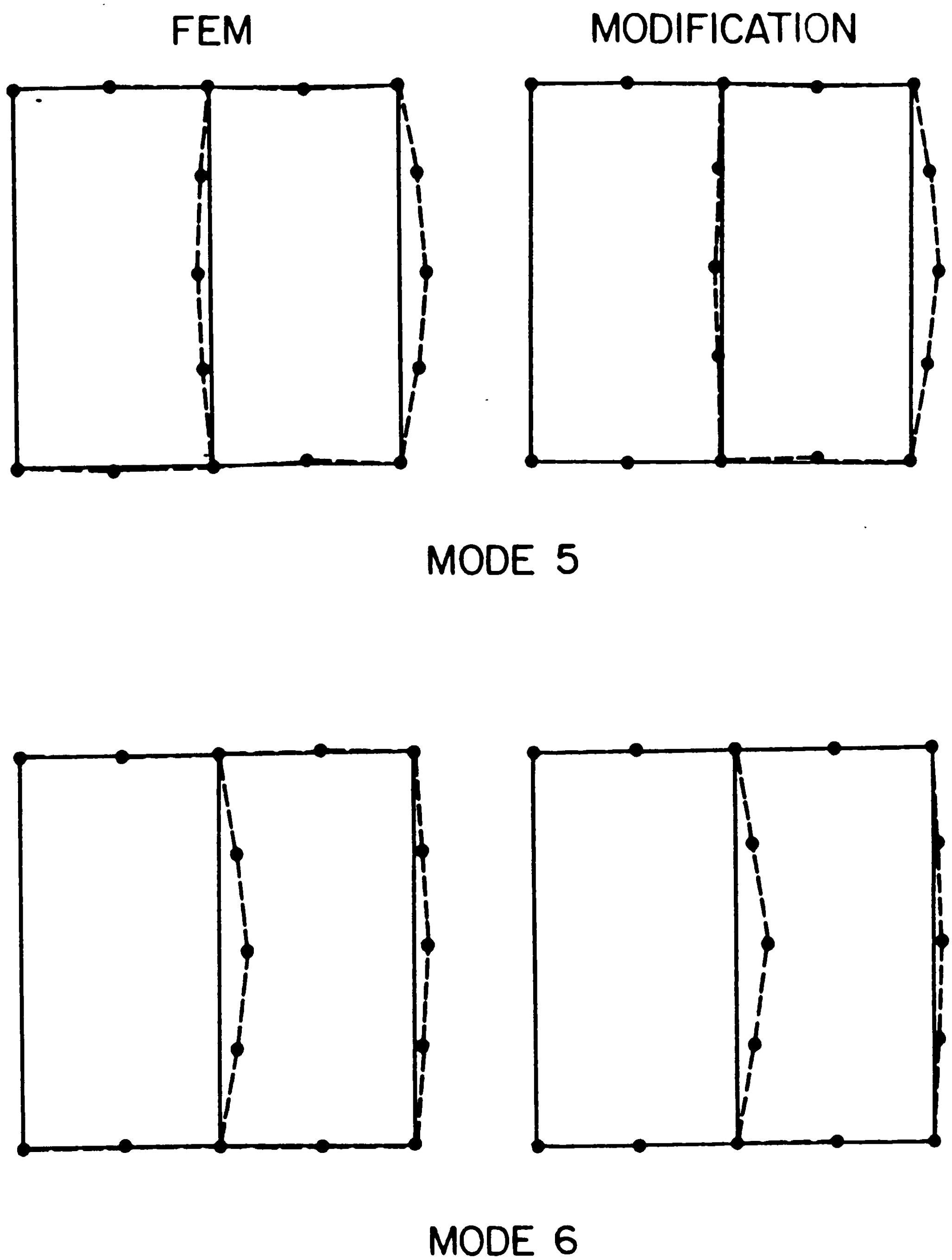


Figure 4.10 Comparison of the Modified Arch Mode Shape to the FEM Truss Mode Shape

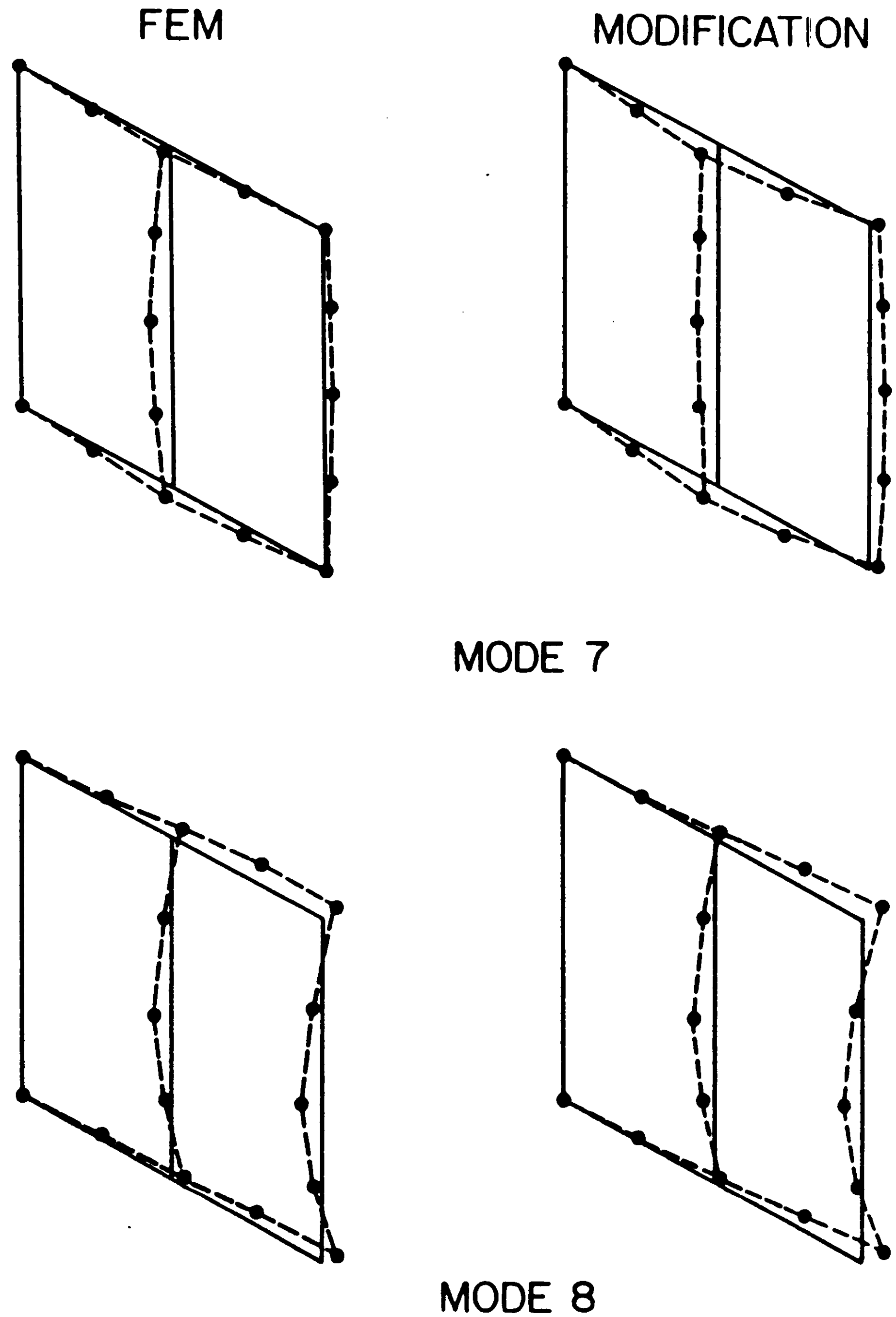


Figure 4.11 Comparison of the Modified Arch Mode Shape to the FEM Truss Mode Shape

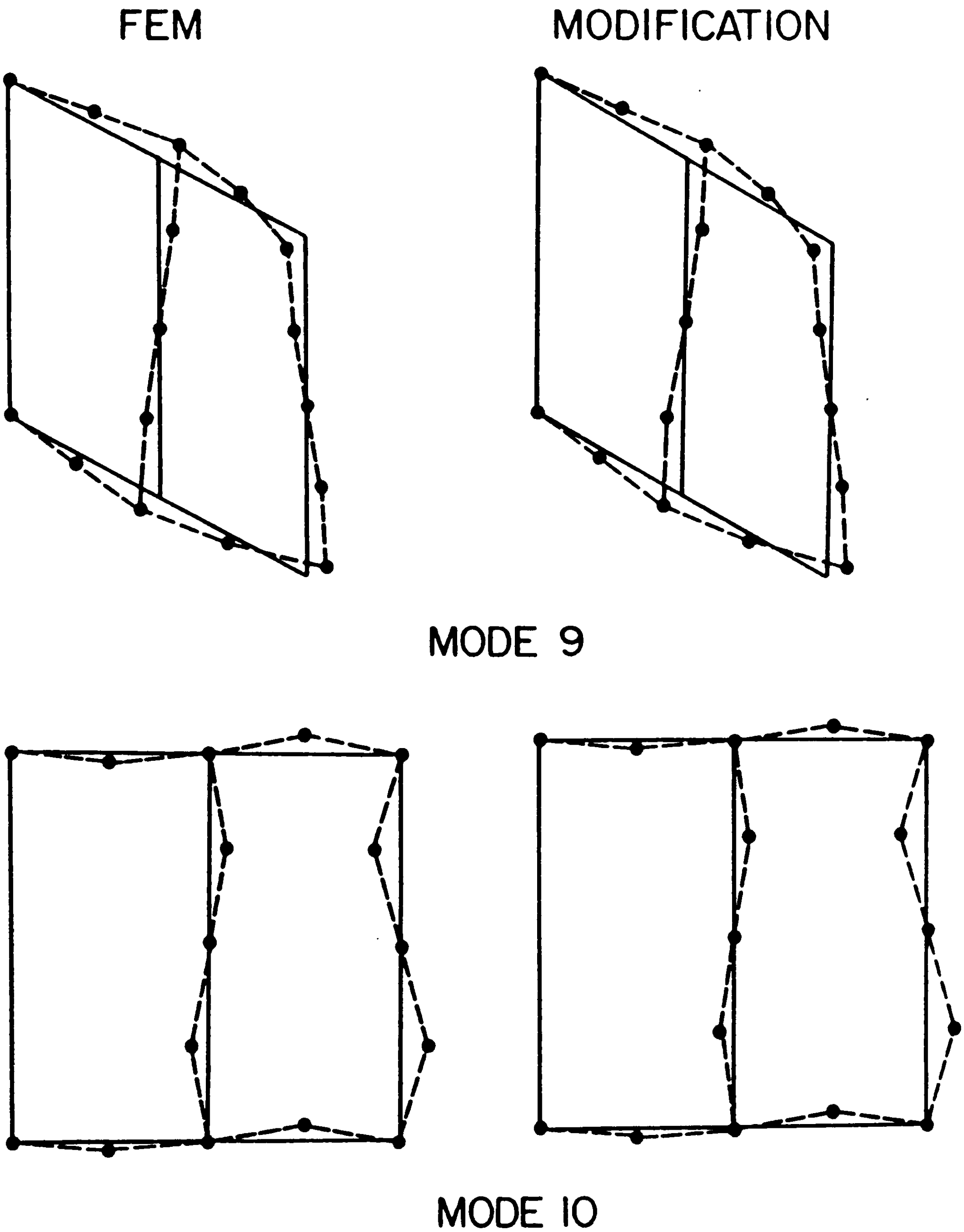
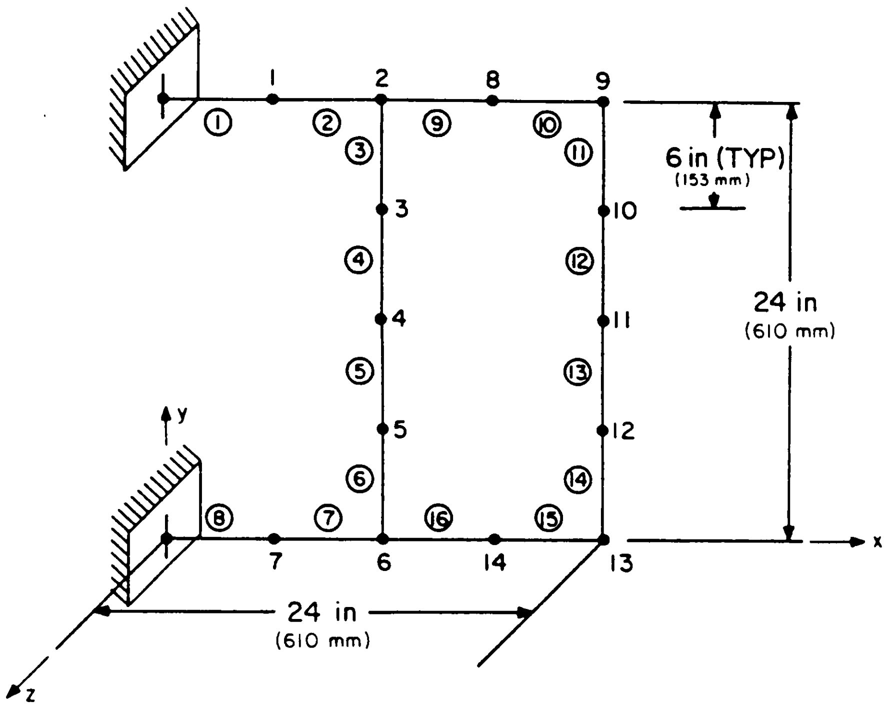
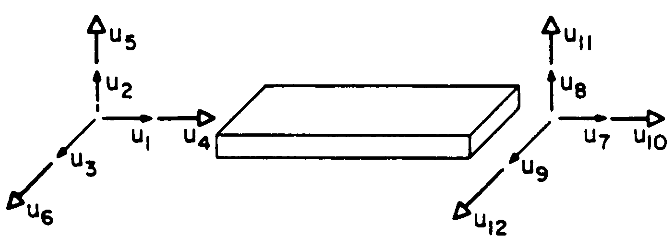


Figure 4.12 Comparison of the Modified Arch Mode Shape to the FEM Truss Mode Shape



a) FINITE-ELEMENT MODEL—VIERENDEEL TRUSS



b) TYPICAL FRAME ELEMENT

Figure 4.13 Finite-Element Model of the Vierendeel Truss

Table 4.8 Comparison Study Between the Eigenanalysis of the FEM Truss Model and the Dynamic Analysis of the Continuum Transfer-Matrix Truss

Mode Number	Natural Frequency** (FEM)	Natural Frequency (TM)	Relative Error*
1	97.9451 rad/sec	98.0292 rad/sec	$-85.8\ (10)^{-3}\%$
2	106.970	107.197	-0.212
3	209.523	209.522	$0.5\ (10)^{-3}$
4	377.444	377.459	$-4.0\ (10)^{-3}$
5	397.515	397.255	0.066
6	476.178	475.733	0.0935
7	490.770	490.692	0.016
8	769.806	769.414	0.051
9	860.394	859.772	0.0723
10	1104.27	1099.29	0.453

* The transfer matrix (TM) solution is considered an exact solution.

**The author recognizes that some of the FEM results do not support the inclusion principle (separation theorem). However, the FEM model was considered sufficient to demonstrate the modification concepts presented here.

shape. At first, this modal assurance result does not seem surprising when comparing the mode shape of Figs. 4.10 and 4.11. There does not seem to be much difference in the mode shapes between the finite-element model and the modified arch result. However, when the sixth mode shape was magnified, as shown in Fig. 4.14, deviations became apparent. The cross-members of each model (FEM, modification) are in-phase while the sides of each model move out-of-phase. It appears that the sixth mode is in greater error than the seventh mode. This is surprising considering the modal assurance for mode six is higher than the modal assurance for mode seven. The modal assurance seems to be giving an erroneous result. Careful analysis of the shape of mode six shows the reason for the breakdown of the modal assurance criterion. The sides of the modification are in error while the cross-member seems correct. Also, the cross-member modal displacements are an order of magnitude greater than the modal displacement of the sides. Since the modal assurance depends on a "sum of squares" process, the differences in magnitudes cause the modal assurance to be more influenced by the large and dominating shape of the cross-member. Therefore, since the cross-member shapes of the finite-element truss and the modification truss are similar, the modal assurance calculation produces an unexpectedly high estimation of confidence. This may mean a new modal accuracy measure should be sought.

In the cantilever and fixed/fixed beam example, it was determined that modal truncation was the cause of gross errors occurring through the modification process. To determine whether modal truncation was the cause of high relative error associated with modes six and seven, the

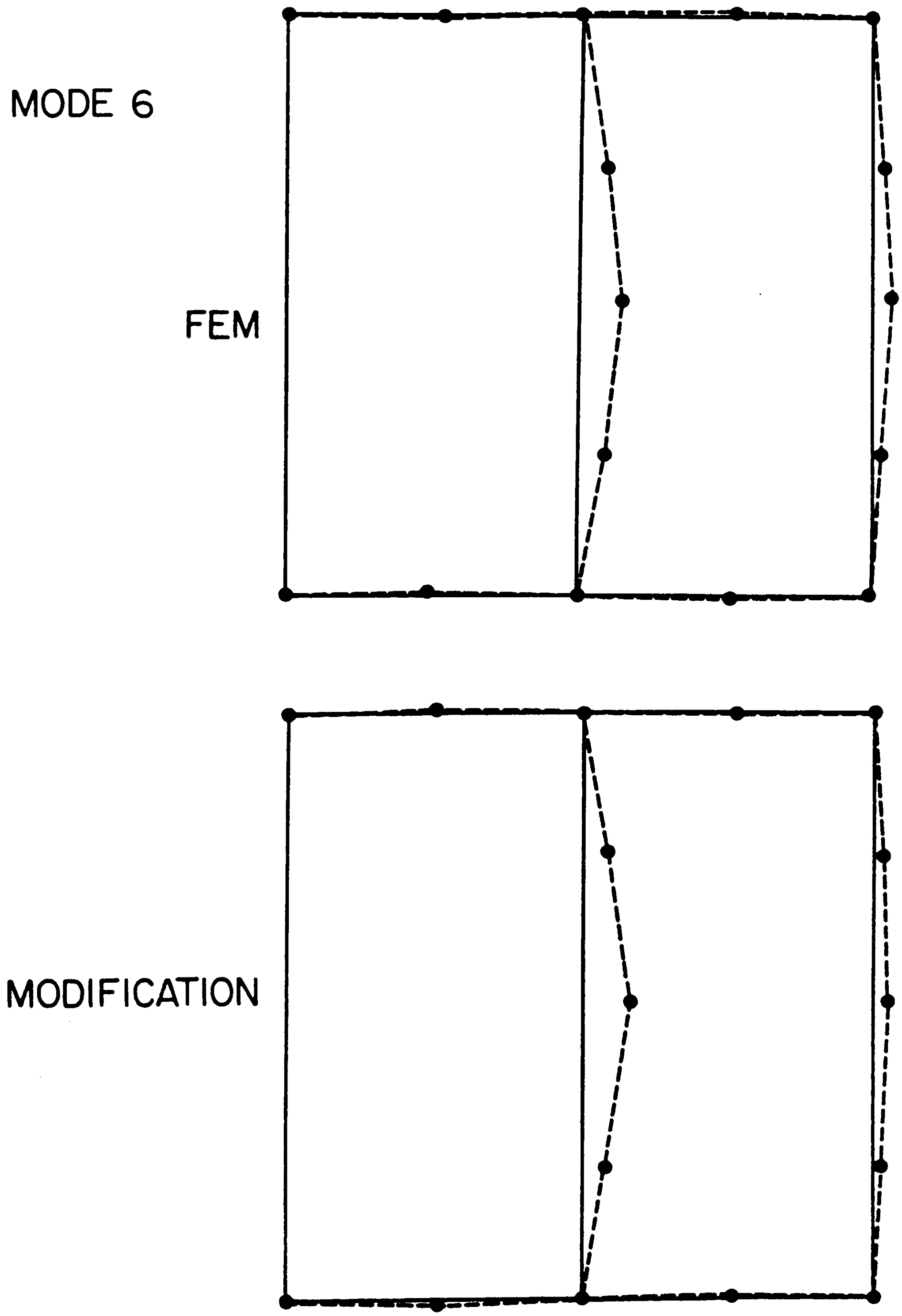


Figure 4.14 Expanded View of the Sixth Mode of the Modified Arch and FEM Truss

modal database of the Portal Arch was successively expanded from ten up to nineteen modes--the limit of the modification program's capability. Particular attention was paid to the behavior of the sixth and seventh modes of the modification. The results of this analysis is shown in Table 4.9. The natural frequency for the sixth mode improved consistently from 21.1%, using a ten mode database, to 6.21%, using a ninteen mode database. Rapid improvement occurs with the incorporation of the eleventh mode into the database (21.1% to 7.02%). Further improvement occurred with the addition of the fourteenth and seventeenth modes into the database. The behavior of the seventh mode is not as encouraging; the improvement in natural frequency is negligible. The modal assurance associated with these modes improve slightly: mode six rises from 0.95 to 0.99; mode seven rises from 0.910 to 0.911.

Mode six of the modified arch shows the effect of modal truncation. The information contained in the ten mode database is insufficient to create the sixth mode of the modified arch. However, by expanding the database to include one more mode of the Portal Arch, the modification process produced an acceptable result. On the other hand, the seventh mode did not show signs of improving. The obvious question is: How many more, or better--which, modes must be included in the original database to adequately determine this mode? To attempt to answer this question, one might consider the possibility that this mode will never be generated unless an infinite number of modes are included in the original database. This assertion leads to a concept slightly different than that of modal truncation, termed here modal insufficiency. Remember, modal truncation refers to errors which result

Table 4.9 Study of the Effect of Increasing the Modal Database for the Arch to Truss Modification

Number of Modes in the Arch's Database	Natural Frequency of the Modified Arch	
	Mode 6	Mode 7
10	576.478 rad/sec	581.724 rad/sec
11	509.596	581.724
12	509.596	581.724
13	509.596	581.724
14	505.775	581.724
15	505.775	581.590
16	505.775	581.590
17	505.759	581.590
18	505.759	581.590
19	505.759	581.590

TM (Exact) Values for Modes Six and Seven:

Mode 6 - 475.733 rad/sec

Mode 7 - 490.692 rad/sec

from using a database, which is a subset of the complete database. This complete database is a function of the number of degrees-of-freedom used to model the system. Modal insufficiency refers to the completeness of the complete database. In actuality, modal insufficiency is a restatement of the completeness principle [11,12,13]. This concept of modal insufficiency may be an important error when dealing with truncated systems. If modal insufficiency is a significant problem in a modification process, one would be unable to generate a database for which the dual space modification scheme could be used. Perhaps this is the situation for mode seven.

4.4 References

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CHAPTER 5 CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

This study developed a concept that provides an experimental structural dynamicist the ability to analyze the effects of making sophisticated structural changes to a structural system whose modal database is known. The incentive for providing this ability is to augment and thereby reduce the cost of theoretical structural reanalysis and prototype development phase of product development. The uniqueness of this approach is the combination of the lumped/continuum-based Transfer-Matrix Method and the Dual Modal Space Modification Method to produce an extremely flexible structural modification algorithm. The development of this new algorithm provides the following contributions to the structural modification community:

1) The ability to use continuum beams as modification members.
2) An unlimited potential for modeling and using almost any type of beam structure as a modification element.
3) The ability to perform severe as well as simple modifications to a structural system.

Through the development of this study, several observations and conclusions can be drawn about the modification technique and on structural modification in general.

1) Chapter 2 presented a review of Transfer-Matrix Theory and the Dual Modal Space Modification Technique. It was shown that the modified modal vector is a weighted linear sum of the original modal vectors. This observation gives one an insight into the

effect of modal truncation. Clearly, if a particular mode of the modified system is dependent (i.e., has a high weighting) on one or more modes of the original modal database that has not been included, then the modified modal vector will be in error.

2) In Chapter 3, it was shown that the Dual Modal Space Modification Method could be combined with Transfer-Matrix Theory to provide an algorithm for instituting structural modifications. In pursuit of this algorithm, a complete method for transforming the transfer-matrix description of a modification element to a stiffness matrix description was shown. This method incorporates differences in transfer matrix vs. stiffness coordinate systems as well as element vs. global coordinate systems. Also a method for obtaining the mode shape of the modification element using the information of the modified modal vector was presented. The algorithm was demonstrated using a modification of a two degree-of-freedom system to a three degree-of-freedom system. This demonstration showed the modification process gave exact results.

3) Two numerical examples were analyzed in Chapter 4. The results showed that for the two problems analyzed, the modification algorithm provided very promising results providing an adequate modal database is used. These results are a significant improvement over existing methods of beam structural modifications. The errors incurred by this method were attributed to the use of a truncated modal database.

4) The first numerical example of Chapter 4 demonstrated a severe modification of a system. The example involved the modification of a cantilevered beam to a fixed/fixed beam. It was shown that fourteen fixed/fixed modes could be obtained using fourteen of sixty modes of the cantilevered beam with less than 8% relative error in the natural frequency. These results are a significant improvement over the observations of O'Callahan and Chou who state that by using a local eigenvalue modification procedure significant errors occur when a severe modification is combined with a truncated modal database.

5) The second example of Chapter 4, the modification of a Portal Arch to a Vierendeel Truss, demonstrated a modification of a complex structure. It was shown that nine out of ten truss modes could be accurately obtained using a database containing eleven arch modes (see Table 4.9). The seventh truss natural frequency could not be predicted through this modification procedure even when the the modal database was increased to almost twice its original size. It was postulated that this mode's natural frequency would never improve unless an infinite number of modes are used in the database. This concept was termed modal insufficiency.

6) The second example also presented a case where the modal assurance may give erroneously high assurance for different mode shapes. It was observed that when the motion of a mode shape is dominant by a few points and when the motion of these points are close to motion of the same points in the compari-

son's mode shape, the modal assurance will be high. This could occur despite the fact all the other displacements between mode shape have no correlation!

7) The results of both numerical examples show that modal truncation is not simply a function of the number of modes contained in the database. Modal truncation is, more importantly, a function of which modes are present. It was seen that the severity of this effect is difficult to quantify since modal truncation can effect any mode of the modified structure.

5.2 Recommendations

During the course of this study, many questions were raised which could not be answered within the scope of the study. Also, some observations deserve further investigation. These questions and observations lead directly to recommendations for further study.

1) An obvious observation is the need for rotational information in the modal database. While efforts are underway to determine these rotations through a post-process of the modal database, the instrument/transducer manufacturers seem to be slow in response to this need. It's obvious for accurate beam modifications to become possible rotational degrees-of-freedom, information must be obtained.

2) The examples used in the demonstration of this concept used databases which were theoretically generated. This leads to the following question: How will this modification technique perform using databases which include errors? A sensitivity

analysis should be carried out to determine the effect of errors in the modal database on this modification method. This analysis should include the effects of errors in the modal mass matrix, modal stiffness matrix, and the modal matrix. The results of this analysis could then be used to set specifications on the derivation of modal parameters from experimental results. The results of a sensitivity analysis on the accuracy of the angular modal information could be used to aid instrument/transducer manufacturers in developing angular transducer specifications.

3) Along the same lines as the above recommendation is an investigation into a method for determining the amount of error present in a modal database. This could then be correlated to the ability to generate modifications using poor databases.

4) In the demonstration of this technique, the results of the two examples were compared to the results of another modeling technique. In practice, the generation of a theoretical result defeats the purpose of using modal modification. However, these theoretical models were needed to find the errors of the modification process. A strong effort should be made to look into deriving some means of verifying whether a modification is correct. Remember that, theoretically, the modified modal vector is a weighted linear combination of the original modal matrix. Possibly, through a technique similar to the modal assurance criterion, the modification results can

be compared to the original modal database and some rule can be contrived about the confidence of the result.

5) Modal truncation was shown to contribute significantly to the errors of the modifications result. However, little information could be obtained from the modal analysis literature on the effects of modal truncation. These results indicate a detailed study into the effects of modal truncation is warranted. This study should encompass not only effect but also a means of detecting when modal truncation has occurred. A good place to start this study is by examining the finite-element substructuring and reanalysis fields. The modal analysis community should be informed that modal truncation can affect any mode of a modification result.

6) As well as an investigation into the effects of modal truncation, the premise of modal insufficiency should be analyzed. This analysis should include methods to determine the severity of modal insufficiency and possible rules to avoid modal insufficiency. Possibly, this recommendation could be combined with the study of modal truncation.

7) In the current study, the modal assurance criterion was found to give erroneous results in certain situations. A study should be performed to find a new form of modal assurance that is more sensitive to minor mode shape deviations.

8) Continued study using the examples of the present study is recommended for two reasons. First: the database of the Portal Arch example needs to be expanded, to say one hundred

modes, to help define the existence of modal insufficiency. This would involve refining the finite-element models to generate an adequate modal database. Also, the modification routine needs to be reconfigured to handle a larger problem. Second: the effects of truncating the low-freqency modes needs to be examined. This would simulate using an experiment database which was frequency bounded.

9) In a laboratory test situation, speed is essential. If an analysis package using beam modification is to be effective, it must be fast. The modification process developed in this study requires the solution of a nonstandard (transcendental) eigenvalue problem. The solution of this nonstandard eigenvalue problem is carried out by a sophisticated determinant search process. If more effort could be put into a solution process for nonstandard eigenvalue problems, the modification process could be made more efficient enabling the designer to quickly make recommendations based on experimental results.

10) A study related to the recommendation above should be performed to determine whether the modification process can be implemented on the small modal analyzers available today. Possibly these small analyzers could incorporate some degree of parallel processing to aid in the nonstandard eigenvalue problem.

11) A study should be performed which would extend this beam modification technique to include damping. Both the Dual

Modal Space Modification Technique and the Transfer-Matrix Method provide for handling damping. Therefore, these technologies could be combined to create a complex-variable beam modification routine.

12) At present, the modification technique presented here only provides the ability to perform a single beam-type change. This should be extended to include both multiple and successive modifications.

13) The present study implemented beam modifications using a continuum beam modification element. A library should be created, containing a variety of beam modification elements, which allows a variety of changes to be implemented by the designer.

14) An extension to building a library of different types of beam modification elements would be to develop guidelines for specifying the attachment characteristics of welded, bolted, or riveted attachments. This could be incorporated into a modification procedure which used beam elements, drawn from a library that include flexible-attachment points.

15) A natural extension of beam modifications is to develop plate or gusset modifications. Transfer-matrix methods could play an important role in developing the plate/gusset modification element.

16) Another extension of the beam modification technique is to include the technique in an optimization process. This optimization process could be used to obtain some frequency

shift or to determine the dynamically optimum modification element.

APPENDIX A EXPANSION OF THE CHANGE MATRIX $\underline{\Delta D}(\omega)$

In the modification procedure, it is required to drive the physical space change matrix, $\underline{\Delta d}(\omega)$, into modal space I using the following equation

$$\underline{\Delta D}_I(\omega) = \underline{P}_I^T \, \underline{\Delta d}(\omega) \, \underline{P}_I \tag{A.1}$$

Performing the actual transformation into modal space using this equation is inefficient. The size of $\underline{\Delta d}(\omega)$ must be a size equal to the number of degrees-of-freedom of the system being analyzed. However, $\underline{\Delta d}(\omega)$ is largely composed of zeros. This section presents an alternative procedure for calculating the components of $\underline{\Delta D}_I(\omega)$.

Consider a modification between two grid points g and h of a structure. The modification change matrix can be written as

$$\underline{\Delta d}(\omega) \, \underset{\sim}{u} = \underset{\sim}{p} \tag{A.2}$$

or in terms of the two connection points g and h

$$\left[\begin{array}{c|c} \underline{\Delta d}_{gg}(\omega) & \underline{\Delta d}_{gh}(\omega) \\ \hline \underline{\Delta d}_{hg}(\omega) & \underline{\Delta d}_{hh}(\omega) \end{array}\right] \left\{\begin{array}{c} \underset{\sim}{u}_g \\ \underset{\sim}{u}_h \end{array}\right\} = \left\{\begin{array}{c} \underset{\sim}{p}_g \\ \underset{\sim}{p}_h \end{array}\right\} \tag{A.3}$$

When the system is mapped to a global size, only entries corresponding to points g and h will be nonzero.

$$
\underline{\Delta d}(\omega) = \begin{bmatrix}
\underline{0} & \cdots & \cdots & \underline{0} & \cdots & \cdots & \cdots & \underline{0} & \cdots & \cdots & \underline{0} \\
\vdots & & & \vdots & & & & \vdots & & & \vdots \\
\vdots & & & \underline{0} & & & & \underline{0} & & & \vdots \\
\underline{0} & \cdots & \underline{0} & \underline{\Delta d}_{gg}(\omega) & \underline{0} & \cdots & \underline{0} & \underline{\Delta d}_{gh}(\omega) & \underline{0} & \cdots & \underline{0} \\
\vdots & & & \underline{0} & & & & \underline{0} & & & \vdots \\
\vdots & & & \vdots & & & & \vdots & & & \vdots \\
\vdots & & & \underline{0} & & & & \underline{0} & & & \vdots \\
\underline{0} & \cdots & \underline{0} & \underline{\Delta d}_{hg}(\omega) & \underline{0} & \cdots & \underline{0} & \underline{\Delta d}_{hh}(\omega) & \underline{0} & \cdots & \underline{0} \\
\vdots & & & \underline{0} & & & & \underline{0} & & & \vdots \\
\vdots & & & \vdots & & & & \vdots & & & \vdots \\
\underline{0} & \cdots & \cdots & \underline{0} & \cdots & \cdots & \cdots & \underline{0} & \cdots\; \underline{0} & \cdots & \underline{0}
\end{bmatrix}
\tag{A.4}
$$

This global size matrix can be mapped into modal space I using Eq. (A.1). By expanding this matrix multiplication, the following equation can be written:

$$
\Delta D_{Iij}(\omega) = \sum_{r=1}^{n} P_{Irj} \sum_{k=1}^{n} \Delta d_{rk}(\omega)\, P_{Iki} \tag{A.5}
$$

where $\quad i$ & j: $1,2,3,\ldots,n$

n = number of degrees-of-freedom in the model.

Rewriting this equation in terms of vectors containing the modal displacement information for each point in the structure yields

$$\Delta D_{Iij}(\omega) = \sum_{r=1}^{m} \underline{P}^T_{Irj} \left[\sum_{k=1}^{m} \underline{\Delta d}_{rk}(\omega) \, \underline{P}_{Iki} \right] \tag{A.6}$$

where i & j: $1,2,3,\ldots,n$

m = number of points in the structure's model

For a single modification between points g and h, all the submatrices of $\underline{\Delta d}(\omega)$ are zero except $\underline{\Delta d}_{gg}(\omega)$, $\underline{\Delta d}_{gh}(\omega)$, $\underline{\Delta d}_{hg}(\omega)$, and $\underline{\Delta d}_{hh}(\omega)$. Therefore, Eq. (A.6) reduces to

$$\begin{aligned} \underline{\Delta D}_{Iij}(\omega) = {} & \underline{P}^T_{Igi} \left[\underline{\Delta d}_{Igg}(\omega) \, \underline{P}_{Igi} + \underline{\Delta d}_{gh}(\omega) \, \underline{P}_{Ihj} \right] \\ & + \underline{P}^T_{Ihi} \left[\underline{\Delta d}_{hg}(\omega) \, \underline{P}_{Igj} + \underline{\Delta d}_{hh}(\omega) \, \underline{P}_{Ihj} \right] \end{aligned} \tag{A.7}$$

where i & j: $1,2,3,\ldots,n$.

Now, the change matrix in modal space I can be formulated using a matrix equation which is now on the order of the number of degrees-of-freedom at a point of the structure. Notice that for truncated systems n is replaced by ℓ - the number of modes in the modal matrix, $\underline{P}$.

APPENDIX B COORDINATE TRANSFORMATIONS

B.1 Pseudo-Dynamic Stiffness Matrix to Stiffness Matrix Transformation

The application of the modification algorithm requires a transformation form the transfer-matrix sign convention to the stiffness sign convention. Equation 3.24 defines this transformation as

$$\hat{\underset{\sim}{f}} = \hat{\underline{C}}_p \, \hat{\underset{\sim}{p}} \quad ; \quad \hat{\underset{\sim}{q}} = \hat{\underline{C}}_u \, \hat{\underset{\sim}{u}} \tag{3.24}$$

The transformation matrices $\hat{\underline{C}}_p$ and $\hat{\underline{C}}_u$ will be defined below for a three-dimensional beam element.

The mapping process needed to develop the coordinate transformation matrices $\hat{\underline{C}}_p$ and $\hat{\underline{C}}_u$ is shown in Fig. B.1. Equation (3.24) can be rewritten in terms of the ends of the beam as

$$\begin{Bmatrix} \hat{\underset{\sim}{f}}_0 \\ \hat{\underset{\sim}{f}}_u \end{Bmatrix} = \left[\begin{array}{c|c} \hat{\underline{C}}_{p0} & \underline{0} \\ \hline \underline{0} & \hat{\underline{C}}_{p1} \end{array}\right] \begin{Bmatrix} \hat{\underset{\sim}{p}}_0 \\ \hat{\underset{\sim}{p}}_1 \end{Bmatrix} \; ; \; \begin{Bmatrix} \hat{\underset{\sim}{q}}_0 \\ \hat{\underset{\sim}{q}}_1 \end{Bmatrix} = \left[\begin{array}{c|c} \hat{\underline{C}}_{u0} & \underline{0} \\ \hline \underline{0} & \hat{\underline{C}}_{u1} \end{array}\right] \begin{Bmatrix} \hat{\underset{\sim}{u}}_0 \\ \hat{\underset{\sim}{u}}_1 \end{Bmatrix} \tag{B.1}$$

or simply

$$\left.\begin{array}{ll} \hat{\underset{\sim}{f}}_0 = \hat{\underline{C}}_{p0} \, \hat{\underset{\sim}{p}}_0 & \qquad \hat{\underset{\sim}{q}}_0 = \hat{\underline{C}}_{u0} \, \hat{\underset{\sim}{u}}_0 \\ \hat{\underset{\sim}{f}} = \hat{\underline{C}}_{p1} \, \hat{\underset{\sim}{p}}_u & \qquad \hat{\underset{\sim}{q}}_1 = \hat{\underline{C}}_{u1} \, \hat{\underset{\sim}{u}}_1 \end{array}\right\} \tag{B.2}$$

By using Fig. B.1, the relationships between the transfer-matrix

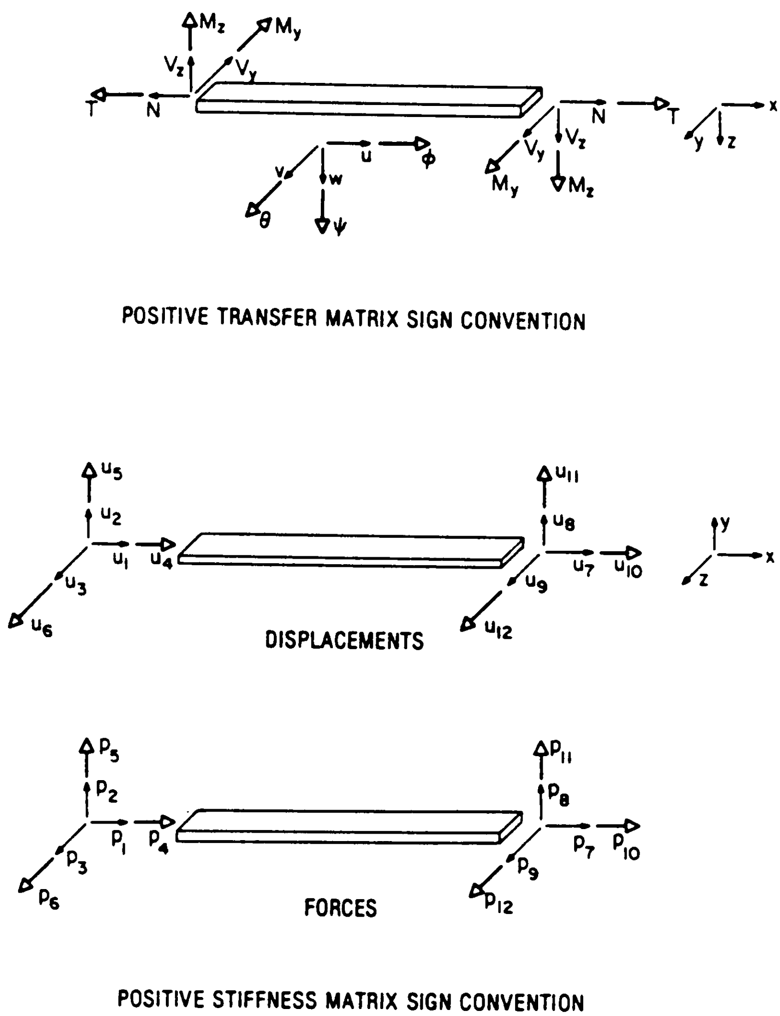

Figure B.1 Comparative Study of the Transfer Matrix/Stiffness Matrix Sign Conventions variables and the stiffness matrix variables can be expressed as

$$\left.\begin{array}{ll}
M_{z0} = p_5 & u_0 = u_1 \\
M_{y0} = -p_6 & v_0 = u_3 \\
T_0 = -p_4 & w_o = -u_2 \\
V_{z0} = p_2 & \phi_0 = u_4 \\
V_{y0} = -p_3 & \theta_0 = u_6 \\
N_0 = -p_1 & \psi_0 = -u_5 \\
M_{z1} = -p_{11} & u_1 = u_7 \\
M_{y1} = p_{12} & v_1 = u_9 \\
T_1 = p_{10} & w_1 = -u_8 \\
V_{z1} = -p_8 & \phi_1 = u_{10} \\
V_{y1} = p_9 & \theta_1 = u_{12} \\
N_1 = p_7 & \psi_1 = -u_{11}
\end{array}\right\} \quad (B.3)$$

Substituting the values of Eq. (B.3) into Eq. (B.2) yields

$$\begin{Bmatrix} M_z \\ M_y \\ T \\ V_z \\ V_y \\ N \end{Bmatrix}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} \quad (B.4)$$

$$\begin{Bmatrix} M_z \\ M_y \\ T \\ V_z \\ V_y \\ N \end{Bmatrix}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} p_7 \\ p_8 \\ p_9 \\ p_{10} \\ p_{11} \\ p_{12} \end{Bmatrix} \qquad (B.5)$$

$$\begin{Bmatrix} u \\ v \\ w \\ \phi \\ \theta \\ \psi \end{Bmatrix}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \qquad (B.6)$$

$$\begin{Bmatrix} u \\ v \\ w \\ \phi \\ \theta \\ \psi \end{Bmatrix}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{Bmatrix} \qquad (B.7)$$

Combining Eqs. (B.4) through (B.7) with Eq. (B.1) and (3.24) yields the coordinate transformations $\hat{\underline{C}}_p$ and $\hat{\underline{C}}_u$.

B.2 Element Coordinate System to Global Coordinate System Transformation

The transformation from the element coordinate system to the global coordinate system can be carried out using the standard direction cosine matrix. Following Craig's lead [1], $\underline{\lambda}$ is the direction cosine matrix between the element and global coordinate system; i.e.,

$$\begin{matrix} x \\ y \\ z \end{matrix} = \underline{\lambda} \begin{matrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{matrix} \tag{B.8}$$

Therefore, at one end of the beam element

$$\left.\begin{matrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix} = \underline{\lambda} \begin{Bmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \end{Bmatrix} & \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \underline{\lambda} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{Bmatrix} \\ \begin{Bmatrix} p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \underline{\lambda} \begin{Bmatrix} \hat{p}_4 \\ \hat{p}_5 \\ \hat{p}_6 \end{Bmatrix} & \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \underline{\lambda} \begin{Bmatrix} \hat{u}_4 \\ \hat{u}_5 \\ \hat{u}_6 \end{Bmatrix} \end{matrix}\right\} \tag{B.9}$$

Applying this process to the displacements and forces at the other end of the beam and combining results yields

$$\underline{C}_T = \left[\begin{array}{c|c|c|c} \underline{\lambda} & \underline{0} & \underline{0} & \underline{0} \\ \hline \underline{0} & \underline{\lambda} & \underline{0} & \underline{0} \\ \hline \underline{0} & \underline{0} & \underline{\lambda} & \underline{0} \\ \hline \underline{0} & \underline{0} & \underline{0} & \underline{\lambda} \end{array}\right] \qquad \text{(B.10)}$$

B.3 Reference

1. Craig, R. R., Jr., Structural Dynamics: An Introduction to Computer Methods, John Wiley and Sons, New York, New York, 1981.

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