Robust Stabilization of Linear Time-invariant Uncertain Systems via Lyapunov Theory

by

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(ABSTRACT)

This dissertation is concerned with the problem of synthesizing a robust stabilizing feedback controller for linear time-invariant systems with constant uncertainties that are not required to satisfy matching conditions. Only the bounds on the uncertainties are required and no statistical property of the uncertainties is assumed.

The systems under consideration are described by linear state equations with uncertainties. i.e. $\dot{x}(t) = \overline{A}(\gamma)x(t) + \overline{B}(\gamma)u(t)$, where $\overline{A}(\gamma)$ is an n×n matrix and $\overline{B}(\gamma)$ is an n×m matrix. Lyapunov theory is exploited to establish the conditions for stabilizability of the closed loop system. We consider a Lyapunov function with an uncertain symmetric positive definite matrix P. The uncertain matrix P satisfies the Lyapunov equation $A^TP + PA + Q = 0$, where the matrix A is in companion form and the matrix Q is symmetric and positive definite. In the solution of the Lyapunov euqation, m rows of the matrix P are fixed in our approach of designing a robust controller. We derive necessary and sufficient conditions on these fixed m rows of the matrix P such that for given positive definite and symmetric Q the solution of the Lyapunov equation yields a positive definite matrix P and a companion matrix A that is Hurwitz. A discontinuous robust stabilizing controller is given.

Linear controller design is also investigated in this research. Under the same assumptions for the existence of a stabilizing discontinuous controller, we show that a linear robust stabilizing controller always exists.

The dissertation includes three examples to illustrate the design procedures for robust controllers. Example 2 shows that the design procedure may be applied to time-varying nonlinear systems.

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Symbols and Notations

$\mathbb{R}$	denotes the real numbers
$\mathbb{R}^n$	denotes real n-space
x^T	denotes the transpose of the vector x
A^T	denotes the transpose of the matrix A
$\|x\|$	denotes the l^2 norm of the vector x
$\|A\|$	denotes the l^2 induced matrix norm of the matrix A, that is
	$\|A\| = \sup\{\|Ax\|:\|x\| = 1\}$
A^{-1}	denotes the inverse of the square matrix A
$\dot{x}(t)$	denotes the time derivative of the vector function x(t) with respect to t
I_j	denotes the j×j dimensional identity matrix

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Chapter 1 Introduction and Related Literature

1.1 Literature

In the design of a control system, an assumed model of the plant is often used. By simplification techniques such as linearization or approximation, one usually ends up with a linear time invariant system which is the nominal model of the plant. However, the model usually contains some uncertain parameters. These uncertainties may be due to imperfect knowledge of the system or due to the nonlinear characteristics of the system. Even if the parameters are known precisely, the designer may want to design a controller for a class of systems with different parameters instead of designing a specific controller for a specific system. Thus, the design and analysis of robust controllers is an important area of research in control theory, [1]. A controller is said to be *robust* with respect to a prescribed class of perturbations if the closed loop stability is invariant under parametric perturbations from the nominal plant model. In the study of robust control the issue of uncertainty is characterized as *structured* versus *unstructured*. Consider the plant $\dot{x} = Ax + Bu$. If the uncertainty in the plant such as A can be modelled as $A + \delta A$, where only the norm of δA is known, then the uncertainty is called unstructured. If the structure of the uncertainty in the matrix A and B are known then the uncertainty is structured. Structured uncertainty specifies how the uncertainty enters into the plant.

The earliest solution for the design of a system with significant uncertainty by feedback and large-loop gains appears in 1927, [25]. Later, Nyquist frequency domain stability criterion [30], and Bode plots [21] were used to investigate the robustness of a system. Then gain and phase margins were often used as measures of robustness. That classical approach can be found in [5] - [6]. The state-variable concepts were introduced in the early 1960's [28], its major results can be found in [2] and [3].

Using state-variable concepts, the analysis of multivariable systems in the frequency domain had made significant progress. The concept of coprime matrix fractorization description of multivariable systems was introduced by papers [19], [20] in 1976 and 1980 as a design tool to cope with plant uncertainty. The Nyquist stability criterion was generalized to multivariable systems in [22], [29].

Zames, [7], introduced the concept of "small gain" which plays a key role in robust stability criteria. Kalman, [28], showed that for SISO systems the optimal LQ state-feedback control law had infinite gain margin and 60 degree phase margins. The extension of the gain and phase margin to MIMO systems was shown in [31], [10]. Then, Doyle showed that the robustness properties might disappear if LQG state-estimate feedback is considered, [26]. However, by selection of Kalman filter in the feedback loop, the desired loop return-difference properties of the optimal LQG control law could be recovered, [27]. The extension of the robustness results of this method is called LQG/LTR (Linear-Quadratic-Gaussian loop transfer recovery) approach, [9].

An optimal H^∞ sensitivity design approach was introduced in [8]. Using the Nevanlinna-Pick interpolation theory, the optimal H^∞ for the SISO system was solved in [14]. The solutions to the MIMO systems were shown in [18]. Using Nevanlinna-Pick interpolation theory, Tannenbaum, [34], synthesized stable controllers which maximized the plant gain factor in worst plant perturbations. Solutions to MIMO systems were shown in [24], [12] and [33]. Adaptive control theory is another approach to robust control. Please see [35] and references therein.

The Lyapunov function techniques have been applied to the design of robust stabilizing controllers, [36] - [76]. In 1976, Gutman and Leitmann, [52], gave a robust stabilizing controller for the systems which satisfy the matching conditions. They showed that a Lyapunov function can be build so that a robust stabilizing controller was available. When matching conditions are satisfied the Lyapunov function method provides a min-max control law which is similar to the sliding mode control as shown in [84] and [82]. In addition, a linear controller was shown to exist, [38]. See papers [36], [37], [44]-[46], [51]-[54], [57]-[59] and [81]. The necessary and sufficient conditions for matching conditions was given by Stalford, [69]. In [46], Chen presented a general design approach for various classes of robust stabilizing controllers for systems with matched uncertainty. The restriction of matching conditions motivated many researchers to design robust stabilizing controllers for systems without matching conditions. see [36], [43] - [48], [62] - [67], [70] - [76]. In 1982, Barmish and Leitmann, [36], decomposed the uncertainty into two parts -- the matched part and the mismatched part. A certain measure of mismatch was defined and a certain mismatch threshold M^* was given. A general method for the estimation of the bound of mismatched uncertainty and its maximum allowable size was given by Chen and Leitmann, [43]. Barmish, [39], provided a necessary and sufficient condition for a robust controller to stabilize a plant via Lyapunov function method. Using algebraic Riccati equations, Petersen and Hollot, [66], provided a stabilizing control law for systems with the uncertainty in the system matrix (the A matrix). A generalization of their results to the systems with uncertainty in both system matrix and input connection matrix (the B matrix) was given by Schmitendorf, [80]. Most previous mentioned controllers were based on full state feedback. For robust output feedback controllers, Chen, [44], [45], gave a direct design and an indirect design for systems with matched uncertainty.

In 1987, Stalford, [69], presented a new approach to the design of robust stabilizing controllers for linear time-invariant uncertain systems with scalar input. He transformed the system matrix to canonical form and then derived the sufficient conditions for the existence of robust stabilizing controllers. This dissertation follows his approach and provides the necessary and sufficient conditions useful in the development of this approach. We extend the approach to MIMO and present a procedure to build a Lyapunov function so that several kinds of robust stabilizing controllers, either linear or nonlinear, are available.

1.2 Dissertation Outline

This dissertation concerns the design of robust stabilizing controllers for systems with time-invariant uncertainties. We consider structured uncertainty in the plant. The key point is the design of an uncertain matrix P(γ) for the Lyapunov function. Necessary and sufficient conditions on m fixed rows of the matrix P are derived so that in the design of the matrix P for the Lyapunov equation it is possible to make the matrix P positive definite. It appears that this is the first time that this property of the Lyapunov equation has been investigated.

In Chapter 2, we define the model of interest and the assumptions to be used throughout this dissertation. The coordinate transformation matrix used in this dissertation is also described.

In Chapter 3, we present a necessary and sufficient condition on the first row of the symmetric matrix P which satisfies the Lyapunov equation $A^TP + PA + Q = 0$, where A is in companion form, Q is a given positive definite symmetric matrix. We show that the fixed first row of the matrix P satisfies the so-called n-1 stable condition if and only if P is positive definite and A is Hurwitz.

In Chapter 4, design procedures for robust stabilizing controllers ,both nonlinear and linear, are given for scalar input systems.

Chapter 5 is the main part of this dissertation. In this chapter, we state and prove necessary and sufficient conditions on m fixed rows of the matrix P in order that the solutions P and A of the Lyapunov equation satisfy P being positive definite and A being Hurwitz. This is an extention to the results in Chapter 3.

In Chapter 6, the nonlinear and linear controllers for multi-input systems are derived. The necessary and sufficient conditions given in Chapter 5 are used to design these controllers.

Chapter 7 contains three examples to illustrate the theory derived in this dissertation. Example 2 exhibits an application of the design approach in this dissertation to nonlinear systems. Example 3 is an example for the design of robust controllers for multi-input systems.

Summary and discussions are given in Chapter 8.

Chapter 2 The Problem Definitions and Assumptions

2.1 Dynamical Systems

We mainly consider the robust control of linear time-invariant uncertain systems. The plant considered can be described by the following model

$$\dot{x}(t) = \overline{\mathrm{A}}(\gamma)x(t) + \overline{\mathrm{B}}(\gamma)u(t) \tag{Σ}$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ is the state vector, $\overline{\mathrm{A}}(\gamma) \in \mathbb{R}^{n \times n}$ is an n×n matrix with uncertain elements, $\overline{\mathrm{B}}(\gamma) \in \mathbb{R}^{n \times m}$ is an n×m matrix with uncertain elements, $u(t) \in \mathbb{R}^m$ is the control, $\gamma \in \Gamma$ is the constant uncertainty and Γ is a compact subset of $\mathbb{R}^p$. In the rest of this chapter we represent $\overline{\mathrm{A}}(\gamma)$ as $\overline{\mathrm{A}}$ and $\overline{\mathrm{B}}(\gamma)$ as $\overline{\mathrm{B}}$ where it is understood that $\overline{\mathrm{A}}$and $\overline{\mathrm{B}}$ depend on $\gamma \in \Gamma$.

Consider the system (Σ). We introduce the following assumptions.

Assumption 1. The linear uncertain system (Σ) is completely controllable for each uncertainty γ.

Assumption 2. The matrix $\overline{\mathrm{B}}(\gamma)$ has rank m for each uncertainty γ which implies $m \leq n$.

Assumption 3. There exists a constant m×n matrix $F \in \mathbb{R}^{m \times n}$ with rank m. Such that

(1) the zeros of the following equation

$$\det[F[sI - \overline{\mathrm{A}}]^{-1}\overline{\mathrm{B}}] = 0 \tag{2.1}$$

have negative real part for each uncertainty γ.

(2) $(F, \overline{A}, \overline{B})$ is a minimum realization of the transfer function $[F[sI - \overline{A}]^{-1}\overline{B}]$ for all $\gamma \in \Gamma$. Let $\Phi = \det[sI - \overline{A}]$. Note that, Φ depends on $\gamma \in \Gamma$ and that the assumption under (2) implies that Equation(2.1) and Φ have no common zeros for all uncertainty γ.

Assumption 4. $\det[F\overline{B}(\gamma)] \neq 0$ for each uncertainty γ, where F is defined in Assumption 3.

Assumption 5. For non-singular matrix $\Pi(\gamma)$, $\Pi \in \mathbb{R}^{m \times m}$, there exists an $m \times m$ nonsingular matrix $\tilde{C}$ such that for an $m \times m$ matrix $\tilde{E}(\gamma)$ defined by

$$\tilde{E}(\gamma) \equiv \tilde{C}\Pi(\gamma)\text{-}I \qquad \gamma \in \Gamma \tag{2.2}$$

then $\|\tilde{E}(\gamma)\| < 1$ for all $\gamma \in \Gamma$.

Definition 1 : Let the uncertainty γ be fixed. The Kronecker invariants of the matrices $\{\overline{A}, \ \overline{B}\}$ are defined by searching the columns of the controllability matrix

$$U(\overline{A}, \ \overline{B}) = [\overline{B} \ \ \overline{AB} \ldots \overline{A}^{n-1}\overline{B}] \tag{2.3}$$

from left to right. Let $\overline{b}_i = i$'th column of $\overline{B}$. Define k_i, i = 1,2..,m, where $\overline{A}^{k_i-1}\overline{b}_i$ is linearly independent with respect to all the previous columns of the matrix U while $\overline{A}^{k_i}\overline{b}_i$ is linearly dependent with respect to all the previous columns. Then $\{k_i\}$ is defined to be the kronecker invariants for fixed $\gamma \in \Gamma$.

Assumption 6. There exists a set of Kronecker invariants k_i, , i = 1,2,...,m, of the pair $\{\overline{A}, \ \overline{B}\}$, which are constant for all uncertainties γ.

Assumption 7. The boundary $\partial\Gamma$ of the compact set Γ containing the uncertainty γ is assumed known.

2.2 Transformation

The system (Σ) satisfying Assumptions 1-7 can be transformed to canonical companion controllable form using a modified Popov's scheme described as follows :

Define the coefficients $\alpha_{ijl} \in \mathbb{R}$, [78], as follows

$$\overline{\mathrm{A}}^{k_i}\bar{b}_i = \sum_{j=1}^{i-1} \sum_{l=0}^{\min(k_i,\, k_j-1)} \alpha_{ijl} A^l \bar{b}_j + \sum_{j=i}^{m} \sum_{l=0}^{\min(k_i,\, k_j)-1} \alpha_{ijl} A^l \bar{b}_j \tag{2.4}$$

where i = 1,2,..., m, j = 1,2,..,i-1, $l = 0,1,\ldots,\min(k_i, k_j - 1)$, and where k_i, i = 1,..,m, are the Kronecker invariants of the matrices $\overline{\mathrm{A}}$, $\overline{\mathrm{B}}$. Then define the coordinate transformation matrix T which is composed of column vectors v_{ij} as follows

$$T = [v_{1,k_1-1}\ v_{1,k_1-2}\ \cdots\ v_{10}\ v_{2,k_2-1}\ v_{2,k_2-2}\ \cdots\ v_{20}\ \cdots\ v_{m,k_m-1}\ \cdots\ v_{m0}\ ]^{-1} \tag{2.5}$$

where the column vectors v_{ij} are defined by the following

$$\bar{A}\, v_{i0} = \sum_{j=1}^{m} \alpha_{ij0}\bar{b}_j, \tag{2.6}$$

$$v_{l,l-1} - \bar{A}\, v_{il} = -\sum_{\substack{j=1 \\ k_j > l}}^{m} \alpha_{ijl}\bar{b}_j, \tag{2.7}$$

$$v_{l,k_i-1} = -\sum_{\substack{j=1 \\ k_j > k_i}}^{l-1} \alpha_{\mathrm{ij}k_i}\bar{b}_j + \bar{b}_i \tag{2.8}$$

where l = $1,2,\ldots,k_l - 1$, i = 1,2,...,m. The symbol $\sum_{\substack{j=1 \\ k_j > l}}^{m}$ from (2.7) means that the summation involves the integers j∈{1,...,m} and $k_j > l$; the corresponding symbol from (2.8) has a similar meaning. T is always nonsingular,[78].

Let T(γ) be the transformation matrix such that

$$z = T(\gamma)x \tag{2.9}$$

The system $(\sum)$ becomes

$$\dot{z} = A_z(y)z + B_z u \tag{2.10}$$

where $A_z(y) = T(y)\overline{A}(y)T^{-1}(y)$, $B_z = T(y)\overline{B}(y)$ The matrix A_z of Equation (2.10) has the following block matrix form as shown in Appendix B : as :

$$A_z = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix} \tag{2.11}$$

where A_{ii} is a $k_i \times k_i$ dimensional companion matrix having the following controllable canonical form

$$A_{ii} = \begin{bmatrix} \times, \times, .., & \times \\ I_{k_i - 1} & 0 \end{bmatrix} \tag{2.12}$$

where I_j is the j×j identity matrix. If j = 0 then I_j and its associated rows are null. For non-diagonal element A_{ij}, i≠j, we have

$$A_{ij} = \begin{bmatrix} \times \times \ldots \times \times \\ 0\ 0 \ldots 0\ 0 \\ 0\ 0 \ldots 0\ 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0\ 0 \ldots 0\ 0 \end{bmatrix} \tag{2.13}$$

where except for the first row, all the elements of A_{ij} are zeros and the symbol × in equation (2.13) means that the term is not necessarily zero.

For given Kronecker invariants k_i, i = 1,2,...,m, we define r_i, i = 1,...,m, as follows

$$r_i = \begin{cases} 1 & i = 1 \\ \sum_{j=1}^{i-1} k_j + 1 & i > 1 \end{cases} \tag{2.14}$$

The $n \times m$ matrix B_z is

$$B_z = \begin{bmatrix} 1 & \times & \times & . & . & . & \times \\ 0 & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & 1 & \times & . & . & . & \times \\ . & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & . & . \end{bmatrix} \tag{2.15}$$

where the symbol $\times$ which appears in r_i' row in equation (2.13) means that the term is not necessarily zero.

Let e_i be an n-dimensional unit column vector with 1 as its r_i 'th element, then define

$$\mathrm{B} = [e_1, e_2, \dots, e_m] \tag{2.16}$$

In block matrix form the $n \times m$ matrix B is

$$B = \text{ block diag}[\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_m] = \begin{bmatrix} \tilde{b}_1 & 0 & 0 & . & . & . & 0 \\ 0 & \tilde{b}_2 & 0 & . & . & . & 0 \\ . & . & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \tilde{b}_m \end{bmatrix} \tag{2.17}$$

where $\tilde{b}_i^T = [1,0,0,\dots,0]$ has k_i elements.

The companion form used through out this dissertation for an $n \times n$ dynamics matrix is of the form in equation (2.11) and for an $n \times m$ input matrix is of the form in (2.16). We say that A_z and B_z are in compatible companion form.

Chapter 3 Linear Systems with Scalar Input

3.1 Introduction

Consider a linear time-invariant uncertain dynamical system with scalar control input

$$\dot{x}(t) = \overline{A}(\gamma)x(t) + \overline{B}(\gamma)u(t) \tag{Σ}$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ is the state vector, $\overline{A}(\gamma) \in \mathbb{R}^{n\times n}$ is an $n\times n$ matrix, $\overline{B}(\gamma) \in \mathbb{R}^n$ is an $n\times 1$ vector, $u(t) \in \mathbb{R}$ is the control, $\gamma \in \Gamma$, is the constant uncertainty. Γ is a compact subset of $\mathbb{R}^p$. The matrix elements $a_{ij}(\gamma)$ of $\overline{A}(\gamma)$, have prescribed bounds on their uncertainties. The system (Σ) is assumed to be controllable for any uncertainty γ. Matrix $\overline{A}(\gamma)$ and $\overline{B}(\gamma)$ are functions of γ, for brevity, we will write $\overline{A}$ for $\overline{A}(\gamma)$ and $\overline{B}$ for $\overline{B}(\gamma)$ when no confusion occur. Any function of $\overline{A}$ and/or $\overline{B}$ will be treated similarly. Define the matrix U as

$$U = [\overline{B}, \overline{A}\overline{B}, ..., \overline{A}^{n-1}\overline{B}] \tag{3.1}$$

then rank $U = n$.

Our objective is to design a robust stabilizing controller such that system (Σ) can be stabilized against any uncertainty $\gamma \in \Gamma$.

The first step we take is to map the system (Σ) into canonical form. For this purpose let

$$z = T(\gamma)x \tag{3.2}$$

where T(γ) is defined as

$$T(\gamma) = [q_1(\gamma), q_2(\gamma), \dots, q_n(\gamma)]^{-1} \tag{3.3}$$

where

$$\begin{aligned}
q_1(\gamma) &= \overline{B} \\
q_2(\gamma) &= \overline{A}(\gamma)q_1(\gamma) + a_1(\gamma)q_1(\gamma) \\
&\quad \cdot \\
&\quad \cdot \\
q_i(\gamma) &= \overline{A}(\gamma)q_{i-1} + a_{i-1}(\gamma)q_1(\gamma) \qquad\qquad i = 2,..,n \\
&\quad \cdot \\
q_n(\gamma) &= \overline{A}(\gamma)q_{n-1}(\gamma) + a_{n-1}(\gamma)q_1(\gamma)
\end{aligned} \tag{3.4}$$

where $a_i(\gamma)$, the coefficient of the characteristic polynomial of the matrix $\overline{\text{A}}$, is defined by

$$\det[sI - \overline{A}] = s^n + a_1(\gamma)s^{n-1} + a_2(\gamma)s^{n-2} + \cdots + a_n(\gamma) \tag{3.5}$$

Then (Σ) becomes

$$\dot{z} = \text{A}_z(\gamma)z + B_z u \tag{3.6}$$

where $\text{A}_z(\gamma) = T(\gamma)\overline{\text{A}}(\gamma)T^{-1}(\gamma)$, $B_z = T(\gamma)\overline{B}(\gamma)$, $\text{A}_z(\gamma)$ and B_z are in controllable canonical form as

$$\text{A}_z(\gamma) = \begin{bmatrix} -a_1(\gamma), \dots, -a_n(\gamma) \\ \text{I}_{n-1} \qquad 0 \end{bmatrix} \tag{3.7}$$

$$B_z^T = [1\ 0 \dots 0] \tag{3.8}$$

3.2 Existence of a Nonlinear Robust Stabilizing Controller

For the system $(\sum)$ a nonlinear robust stabilizing controller is given in [71]. Here we show the approach of its development.

Equation(3.6) can be represented as

$$\dot{z} = \hat{A}_z(\gamma)z + B_z[u + B_z^T(A - \hat{A})z] = \hat{A}_z(\gamma)z + B_z(u + \sigma z) \tag{3.9}$$

where $\hat{A}_z$ has the same form as A_z. That is

$$\hat{A}_z(\gamma) = \begin{bmatrix} -\hat{a}_1(\gamma), \ldots, -\hat{a}_n(\gamma) \\ \mathrm{I}_{n-1} \quad\quad 0 \end{bmatrix} \tag{3.10}$$

and $\sigma = B_z^T(A_z(\gamma) - \hat{A}_z(\gamma)) = [\hat{a}_1(\gamma) - a_1(\gamma), \ldots, \hat{a}_n(\gamma) - a_n(\gamma)]$. We define $\hat{a}(\gamma) = (\hat{a}_1(\gamma), \ldots, \hat{a}_n(\gamma))$.

Consider the Lyapunov equation

$$P(\gamma)\hat{A}_z(\gamma) + \hat{A}_z^T(\gamma)P(\gamma) + Q(\gamma) = 0 \tag{3.11}$$

where $Q(\gamma)\in\mathbb{R}^{n\times n}$,$P(\gamma)\in\mathbb{R}^{n\times n}$ are nxn symmetric positive definite matrix for all $\gamma\in\Gamma$ and where the vector $\hat{a}(\gamma)$ are arbitrary quantities.

Consider the system (3.9), the time derivative of the Lyapunov function V

$$V = z^T P(\gamma)z \tag{3.12}$$

is given by

$$\dot{V} = -z^T Q(\gamma)z + 2[Fx]^T(u + \sigma(\gamma)z) \tag{3.13}$$

where $F = B_z^T P(\gamma)T(\gamma)$. A basic assumption of this method is that

Assumption 1 For the matrix $T(\gamma)$ defined in (3.3), there exists $\hat{a}(\gamma)$,$\gamma\in\Gamma$, such that the solution $P(\gamma)$ of (3.11) is a symmetric positive definite matrix and satisfies the constraint $F = B_z^T P(\gamma)T(\gamma)$ $\gamma\in\Gamma$, where F is a constant matrix.

We consider the discontinuous control law

$$u(x) = \begin{cases} -\rho(x) & Fx > 0 \\ \rho(x) & Fx < 0 \\ 0 & Fx = 0 \end{cases} \tag{3.14}$$

where

$$\rho(x) = \max_{\gamma \in \Gamma} \|\sigma(\gamma)T(\gamma)x\| \tag{3.15}$$

Then for Fx $\neq 0$

$$[Fx]^T(u + \sigma(\gamma)z) \leq 0 \qquad \forall \gamma \in \Gamma \tag{3.16}$$

Hence $\dot{V}$ is always smaller than zero. Since P,Q > 0, by the Lyapunov theory, the control (3.14) stabilize system ($\sum$) for all uncertainty γ.

In the procedure described above, we need to design a positive definite and symmetric matrix P which satisfies Assumption 1. Let P_1 be the first row of P. Then $P_1 = FT^{-1}$. For given Q and P_1, Lyanupov equation (3.11) provides n(n + 1)/2 equations and unknowns. Hence the matrix P and $\hat{A}_z$ are function of Q and P_1. Given the matrix Q, and the constraint $P_1 = FT^{-1}$, we seek the necessary and sufficient conditions on P_1 such that the solutions of (3.11) yield P > 0 and $\hat{A}_z$ Hurwitz so that we can select F to meet the requirements. Such conditions address the existence of a stabilizing F for the controller (3.14).

3.3 A Necessary and Sufficient Condition

Definition 3.1 The vector **a** = $(a_1, a_2, \ldots, a_n)$ is said to be n-stable provided the following polynomial is Hurwitz

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n \tag{3.17}$$

Definition 3.2 The vector $P_1 = (p_{11}, \ldots, p_{1n})$ with $P_{11} > 0$ is said to be n-1 stable provided the following polynomial is Hurwitz.

$$p_{11}s^{n-1} + p_{12}s^{n-2} + \dots + p_{1n} \tag{3.18}$$

Theorem 3.1 (necessity)

Let the matrix A be Hurwitz and in the following companion form

$$\mathrm{A} = \begin{bmatrix} -a_1, \dots, -a_n \\ \mathrm{I}_{n-1} \quad 0 \end{bmatrix} \tag{3.19}$$

Let Q be symmetric positive definite and P be symmetric. Then the first row P_1 of the solution P of the Lyapunov equation

$$P\mathrm{A} + \mathrm{A}^T P + Q = 0 \tag{3.20}$$

is n-1 stable.

Proof :

Consider the scalar control system

$$\dot{x}(t) = \mathrm{A}x(t) + Bu(t) \tag{Λ}$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ is the state vector, $B \in \mathbb{R}^n$ is an $n \times 1$ vector, $u(t) \in \mathbb{R}$ is the control. $B^T = [1, 0, \dots, 0]$. Let Q be symmetric and positive definite matrix and P be a solution of the equation (3.20). Then P is known to be positive definite. The matrices Q and P $\in \mathbb{R}^{n \times n}$. Define the Lyapunov function

$$V = x^T P x \tag{3.21}$$

In view of the system (Λ), the time derivative of the Lyapunov function (3.21) yields

$$\dot{V} = -x^T Q x + 2[P_1 x]^T u \tag{3.22}$$

where $P_1 = B^T P$. Consider the linear control

$$u(x) = \sum_{i=1}^{n-1} (a_i - \frac{p_{1\,i+1}}{p_{11}}) x_i + a_n x_n \tag{3.23}$$

The equation $P_1\, x = 0$ defines an n-1 dimensional linear manifold M, which is a subspace, in n-dimensional space. We show that given any $x_0 \in$M, there exists a solution of (Λ) under the control (3.23) with $x(0) = x_0$ and x(t)$\in$M for all t $\geq$ 0. That is, applying control (3.23) to system (Λ) yields

$$p_{11}\dot{x}_1 = -\sum_{i=1}^{n-1} p_{1,\,i+1} x_i \tag{3.24}$$

$$\dot{x}_{i+1} = x_i \qquad\qquad i = 1,2,...,n-1. \tag{3.25}$$

Substituting (3.25) to (3.24) yields

$$\sum_{i=1}^{n} p_{1i}\,\dot{x}_i \;=\; 0 \qquad\qquad \forall t \geq 0 \tag{3.26.a}$$

or, equivalently

$$P_1 x(t) = c \qquad\qquad \forall t \geq 0 \tag{3.26.b}$$

The constant c is zero since x(0) = x_0 and $P_1 x_0$ = 0. Therefore, x(t)$\in$M for all t $\geq$0. Then, equation (3.22) becomes

$$\dot{V} = -x^T Q x \tag{3.27}$$

According to the Lyapunov theory of stability for nonlinear control systems, the system (Λ) is stable under the control defined by (3.23) for any initial condition x(0)$\in$M. The characteristic equation of the linear Equation (3.24) and (3.25) is given by (3.26.b). That is, by taking Laplace transformation of (3.24) and (3.25) we have

$$\sum_{i=1}^{n} p_{1i} s^{n-i} x_1(s) = 0 \tag{3.28}$$

Since the system is stable for any x(0)∈M, equation (3.18) must be Hurwitz. That $p_{11} > 0$ is a direct result of P being positive definite. Hence P_1 is n-1 stable.

3.4 Existence of a Solution When P_1 is n-1 Stable

In this section we establish that the Lyapunov equation admits symmetric solutions P for the case when the first row of P is prescribed to be n-1 stable.

Theorem 3.2

Let $\mathbf{a} = [a_1, a_2, \ldots, a_n]$. If P_1 is n-1 stable then there exists a symmetric solution matrix P and a solution vector $\mathbf{a}$ of the Lyapunov equation (3.20) where A is defined by equation (3.19).

Proof :

Consider the Lyapunov equation (3.20). Let the first row P_1 of P be n-1 stable. We consider P to be symmetric. The Lyapunov equation (3.20) consists of n(n + 1)/2 equations. The unknown variables P_{ij}, i = 2,3,...,n, j≥i can be solved uniquely in terms of the elements of P_1, Q and $\mathbf{a}$ resulting in exactly n(n-1)/2 unknowns and equations, assuming that $\mathbf{a}$ is known. See Appendix A. These can be substituted into the remaining n equations to yield the following linear matrix equation for the unknown vector $\mathbf{a}$

$$\mathrm{H}\ \mathbf{a}^T = \mathrm{y} \tag{3.29}$$

where y is an n-dimensional vector which depends on the elements of P_1 and Q and where H which depends on the elements of P_1 has the form

$$H = \begin{bmatrix} p_{11} & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -p_{13} & p_{12} & -p_{11} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ p_{15} & -p_{14} & p_{13} & -p_{12} & p_{11} & 0 & 0 & \ldots & 0 & 0 & 0 \\ -p_{17} & p_{16} & -p_{15} & p_{14} & -p_{13} & p_{12} & -p_{11} & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . & . & \ldots & . & . & . \\ . & . & . & . & . & . & . & \ldots & . & . & . \\ . & . & . & . & . & . & . & \ldots & . & . & . \\ 0 & . & . & . & . & . & . & \cdots & -p_{1n} & p_{1n-1} & -p_{1n-2} \\ 0 & 0 & . & . & . & . & . & \ldots & 0 & 0 & p_{1n} \end{bmatrix} \tag{3.30}$$

The i'th row of H has the general form

$$\ldots(-1)^j P_{1i+j} \ldots P_{1i} \ldots (-1)^j P_{1i-j} \tag{3.31}$$

where $1 \le j \le i-1$ and $j \le n-i$.

There exists a solution of equation (3.29) provided H has an inverse. The determinant of H is given by the product of p_{11} and the determinant of the following submatrix H_1 :

$$H_1 = \begin{bmatrix} p_{12} & -p_{11} & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -p_{14} & p_{13} & -p_{12} & p_{11} & 0 & 0 & \ldots & 0 & 0 & 0 \\ p_{16} & -p_{15} & p_{14} & -p_{13} & p_{12} & -p_{11} & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . & \ldots & . & . & . \\ . & . & . & . & . & . & \ldots & . & . & . \\ . & . & . & . & . & . & \ldots & . & . & . \\ . & . & . & . & . & . & \cdots & -p_{1n} & p_{1n-1} & -p_{1n-2} \\ 0 & . & . & . & . & . & \ldots & 0 & 0 & p_{1n} \end{bmatrix} \tag{3.32}$$

That the determinant of H_1 in nonzero follows from the premise that P_1 is n-1 stable, or, equivalently, that the roots of the characteristic Equation (3.18) have negative real part. The Routh-Hurwitz criterion states that the necessary and sufficient conditions for all roots s_j of the characteristic polynomial (3.18) to have negative real parts is that all determinants $\Delta_1^*, \Delta_2^*, \ldots, \Delta_{n-1}^*$ of the matrix H^*, which is formed from H_1 by changing all negative signs to positive signs, be positive [86].

$$\Delta_1^* = p_{12}$$

$$\Delta_2^* = \begin{vmatrix} p_{12} & p_{11} \\ p_{14} & p_{13} \end{vmatrix}$$

$$\Delta_3^* = \begin{vmatrix} p_{12} & p_{11} & 0 \\ p_{14} & p_{13} & p_{12} \\ p_{16} & p_{15} & p_{14} \end{vmatrix} \tag{3.33}$$

$$\vdots$$

$$\Delta_{n-1}^* = \det H^*$$

The negative signs in H_1 have no effect on the values of its Routh-Hurwitz determinants. They have the same values as given by (3.33). Therefore, the determinant of H_1 is equal to the determinant of H^*. As a result, the matrix H has an inverse and Equation (3.29) has a well-defined solution whenever P_1 is n-1 stable. Since the vector **a** is well-defined the elements p_{ij}, $i = 2,3,\ldots,n$, $j \geq i$ are also well-defined and consequently, a solution vector **a** and a symmetric solution matrix P exist to the Lyapunov equation (3.20) whenever P_1 is n-1 stable.

3.5 Sufficiency of P_1 being n-1 Stable

In order that the Lyapunov equation (3.20) admit an n-stable vector **a** it is necessary that the first row P_1 be n-1 stable. This result is established in Theorem 3.1. In the last section is shown that if P_1 is n-1 stable then there exist unique solutions P (symmetric) and vector **a** of the Lyapunov equation. In the next section we show that the solution vector **a** is n-stable provided P_1 is n-1 stable.

Theorem 3.3

Let P_1 be n-1 stable. Let P and **a** be the solutions of the Lyapunov equation (3.20) when the first row of P is P_1 Then P is positive definite and **a** is n-stable.

Proof :

Since **a** is n-stable if and only if P is positive definite we only need to show that P is positive definite.

Let P_1 be n-1 stable. Let P and **a** be the solution of the Lyapunov equation (3.20). Consider the scalar control system

$$\dot{x} = Ax + Bu \tag{3.34}$$

where A is defined as in (3.19) and $B^T = [1, 0, \ldots, 0]$. Define the Lyapunov function

$$V = x^T P x \tag{3.35}$$

Its derivative is given by

$$\dot{V} = -x^T Q x + 2[P_1 x]^T u \tag{3.36}$$

Consider the linear control law

$$u = \ -cP_1 x \tag{3.37}$$

where c is some positive scalar. In view of this control law we find that

$$\dot{V} \le -x^T Q x \tag{3.38}$$

We wish to show that for large enough c the differential system defined by (3.34) and (3.37) is stable. Substitution of (3.37) into (3.34) yields

$$\dot{x} = A_c x \tag{3.39}$$

where

$$\mathrm{A}_c = \begin{bmatrix} -a_{c1}, \ldots, -a_{cn} \\ \mathrm{I}_{n-1} \quad 0 \end{bmatrix} \tag{3.40}$$

Let

$$a_c = [a_{c1}, a_{c2}, \ldots, a_{cn}] \tag{3.41}$$

$$a_{ci} = a_i + cp_{1i} \qquad i = 1,2,\ldots,n \tag{3.42}$$

The stability of (3.39) is determined by the roots of the characteristic equation

$$a_{c1}s^{n-1} + a_{12}s^{n-2} + \ldots + a_{cn} = 0 \tag{3.43}$$

Consider the Hurwitz matrix

$$H_c = \begin{bmatrix} a_{c1} & 1 & 0 & 0 & \cdots & 0 \\ a_{c3} & a_{c2} & a_{c1} & 1 & \cdots & 0 \\ a_{c5} & a_{c4} & a_{c3} & a_{c2} & \cdots & 0 \\ a_{c7} & a_{c6} & a_{c5} & a_{c4} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{c,2n-1} & a_{c,2n-2} & a_{c,2n-3} & a_{c,2n-4} & \cdots & a_{cn} \end{bmatrix} \tag{3.44}$$

where all elements corresponding to subscripts r such that r > n in the above are to be replaced by zero. Construct the Routh-Hurwitz determinants.

$$\Delta_{c1} = a_{c1} \tag{3.45a}$$

$$\Delta_{c2} = \det \begin{vmatrix} a_{c1} & 1 \\ a_{c3} & a_{c2} \end{vmatrix} \tag{3.45b}$$

$$\Delta_{c3} = \det \begin{vmatrix} a_{c1} & 1 & 0 \\ a_{c3} & a_{c2} & a_{c1} \\ a_{c5} & a_{c4} & a_{c3} \end{vmatrix} \tag{3.45c}$$

.

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.

$$\Delta_{cn} = \det |H_c| \tag{3.45d}$$

The Routh-Hurwitz criterion for stability states that the necessary and sufficient conditions for the roots s_j of the characteristic polynomial to have negative real parts is that all the determinants $\Delta_{c1}, \Delta_{c2}, \dots, \Delta_{cn}$ be positive, [86]. Consider the determinants defined by the limit as $c \rightarrow \infty$.

$$\Delta_i = \underset{c \rightarrow \infty}{limit} \frac{\Delta_{ci}}{c} \tag{3.46}$$

If $\Delta_i > 0$, i = 1,...,n then for sufficiently large c it follows that $\Delta_{ci} > 0$, i = 1,2,...,n. Observe that

$$\Delta_1 = p_{11} \tag{3.47a}$$

$$\Delta_2 = \begin{vmatrix} p_{11} & 0 \\ p_{31} & p_{21} \end{vmatrix} \tag{3.47b}$$

.

.

$$\Delta_n = \det |H| \tag{3.47c}$$

where

$$H = \begin{bmatrix} p_{11} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ p_{13} & p_{12} & p_{11} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ p_{15} & p_{14} & p_{13} & p_{12} & p_{11} & 0 & 0 & \dots & 0 & 0 & 0 \\ p_{17} & p_{16} & p_{15} & p_{14} & p_{13} & p_{12} & p_{11} & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . & . & \dots & . & . & . \\ . & . & . & . & . & . & . & \dots & . & . & . \\ . & . & . & . & . & . & . & \dots & . & . & . \\ 0 & . & . & . & . & . & . & \cdots & p_{1n} & p_{1n-1} & p_{1n-2} \\ 0 & 0 & . & . & . & . & . & \dots & 0 & 0 & p_{1n} \end{bmatrix} \tag{3.48}$$

Define the determinates

$$\Delta_{0i} = \underset{c \to \infty}{limit} \Delta_{ci} \qquad i = 1,2, \dots , n \tag{3.49}$$

Note that

$$\frac{\Delta_{ci}}{c} = \Delta_i + \frac{1}{c} \Delta_{0i} \qquad i = 1,2, \dots , n \tag{3.50}$$

and

$$\underset{c \to \infty}{limit} \frac{\Delta_{ci}}{c} = \Delta_i \qquad i = 1,2, \dots , n \tag{3.51}$$

where Δ_i is defined by equation (3.47). The above limits exist since the determinant is a continuous function of its linear arguments.

Let H^* be the submatrix of H defined by removing the first row and the first column. Since the Routh-Hurwitz determinants (3.47) of H are p_{11} times the Routh-Hurwitz determinants of H^* it follows that stability of the differential system (3.39) for large c is determined by the signs of the Routh-Hurwitz determinants of H^*. But the sign of each of these is positive since P_1 is n-1 stable. That is, the roots of the characteristic equation (3.18) has H^* as its Routh-Hurwitz matrix. This equation has stable eigenvalues since P_1 in n-1 stable. Therefore, the sign of each Routh-Hurwitz determinants of H^* is positive. Consequently, the differential system is stable for sufficiently large c.

We investigate the positive definiteness of the matrix P. Assume P is not positive definite. In this case there exists $x_0 \neq 0$ such that

$$V(x_0) = x_0^T P x_0 \leq 0 \tag{3.52}$$

Let c be sufficiently large so that (3.39) is stable. From inequality (3.38) we observe that $\dot{V}$ is negative for all $x \neq 0$. Consequently, $V(x(t)) < 0$ for $t > 0$. There exist $\varepsilon > 0$ and $t_1 > 0$ such that $V(x(t)) < -\varepsilon$ for all $t \geq t_1$ Since V is a continuous function there is some ball of radius $r > 0$ about the origin

$$B(r,0) = \{x: \|x\| \leq r\} \tag{3.53}$$

such that

$$V(x) < -\varepsilon \qquad \forall x \in B(r,0) \tag{3.54}$$

This implies that x(t) cannot enter the ball B(r,0) for all $t > t_1$, for any positive c. This is a contradiction since the differential system (3.39) is stable. Thus, we have established that P_1 being n-1 stable is a sufficient condition for **a** to be n-stable.

Chapter 4 Robust Control for Scalar Control System

4.1 Introduction

The robust stability of uncertain linear systems using linear feedback control is investigated in [38], [37]. They show under certain condition of continuity, compactness and measurability and a matching condition on the input matrix B of $\dot{x} = Ax + Bu$ that a linear control suffices to stabilize systems which were previously stabilized via nonlinear control. We consider the problem of stabilizing uncertain linear systems that do not necessarily satisfy matching conditions. In our treatment, however, we consider systems possessing constant uncertainties rather than time-varying uncertainties as treated by [37], [38]. Under Assumptions I-III stated latter in this chapter, [73], demonstrates the existence of a discontinuous stabilizing controller for uncertain linear systems with constant uncertainties in the absence of matching conditions. In this chapter we show that a linear control suffices to stabilize the same class of systems that satisfy Assumptions I-III.

Consider a plant described by a linear uncertain system with scalar control

$$\dot{x}(t) = \overline{A}(\gamma)x(t) + \overline{B}(\gamma)u(t) \qquad (\Sigma)$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ is the state vector, $\overline{A}(\gamma) \in \mathbb{R}^{n \times n}$ is an n×n matrix, $\overline{B}(\gamma) \in \mathbb{R}^n$ is an n×1 vector, $u(t) \in \mathbb{R}$ is the control, $\gamma \in \Gamma$, is the constant uncertainty and Γ is a compact subset of $\mathbb{R}^p$. The matrix elements $a_{ij}(\gamma)$ of $\overline{A}(\gamma)$ have prescribed bounds on their uncertainties. The system (Σ) is assumed to be controllable for any uncertainty γ. Matrix $\overline{A}(\gamma)$ and $\overline{B}(\gamma)$ are functions of γ, for brevity, we will write $\overline{A}$ for $\overline{A}(\gamma)$ and $\overline{B}$ for $\overline{B}(\gamma)$ when no confusion occurs.

Assumption I : The linear system (Σ) is completely controllable for each uncertainty γ.

Under this assumption the system (Σ) can be transformed to controllable companion form using the following formula.

$$z = T(\gamma)x \tag{4.1}$$

where $T(\gamma)$ is defined as in Equation (3.4). The coefficient of the characteristic polynomial of the matrix $\overline{A}$, $a_i(\gamma)$, is given by

$$\det[sI - \overline{A}] = s^n + a_1(\gamma)s^{n-1} + a_2(\gamma)s^{n-2} + \cdots + a_n(\gamma) \tag{4.2}$$

In such coordinates systems (Σ) becomes

$$\dot{z} = A_z(\gamma)z + B_z u \tag{4.3}$$

where $A_z(\gamma) = T(\gamma)\overline{A}(\gamma)T^{-1}(\gamma)$, $B_z = T(\gamma)\overline{B}(\gamma)$, $A_z(\gamma)$ and B_z are in controllable canonical form as

$$A_z(\gamma) = \begin{bmatrix} -\bar{a}_1(\gamma), \ldots, -\bar{a}_n(\gamma) \\ \mathrm{I}_{n-1} \quad\quad 0 \end{bmatrix} \tag{4.4}$$

$$B_z^T = [1\,0\ldots0] \tag{4.5}$$

Let $\bar{a}(\gamma)$ be the coefficient vector of the open-loop characteristic equation $\overline{A}(\gamma)$ i.e.

$$a(\gamma) = [a_1(\gamma), \ldots, a_n(\gamma)] \tag{4.6}$$

Assumption II : There exists a row vector $F \in \mathbb{R}^n$ such that P_1 defined by the following equation

$$P_1(\gamma) = FT^{-1}(\gamma) \tag{4.7}$$

is n-1 stable for all uncertainties γ.

4.2 Stabilizing Discontinuous Controller

Consider the solution P(γ) of the Lyapunov equation for each uncertainty γ

$$P(\gamma)\hat{A}(\gamma) + \hat{A}^T(\gamma)P(\gamma) + Q(\gamma) = 0 \tag{4.8}$$

where $Q(\gamma)\in\mathbb{R}^{n\times n}, P(\gamma)\in\mathbb{R}^{n\times n}$ are nxn symmetric positive definite matrix. The minimum eigenvalue of Q(γ) is λ_1. $\hat{A}$ has the companion form

$$\hat{A}(\gamma) = \begin{bmatrix} -\hat{a}_1(\gamma), \ldots, -\hat{a}_n(\gamma) \\ I_{n-1} \quad\quad 0 \end{bmatrix} \tag{4.9}$$

Since P_1 is n-1 stable, from the results in the last chapter, the resulting equations in the Lyapunov equation (4.8) can be solved uniquely for $\hat{a}(\gamma)$ and $p_{ij}(\gamma)$, i = 1,2,...,n, j $\geq$ i. The necessary and sufficient condition that $\hat{a}(\gamma)$ be stable is that P_1 as defined by (4.7) be n-1 stable. Consequently, under Assumption II, Equation (4.8) yields a set of stable coefficient vectors $\hat{a}(\gamma)$ defined on the uncertainties γ. Since $\hat{A}(\gamma)$ is Hurwitz P(γ) is symmetric, positive definite.

Let $\sigma(\gamma)$ represent the difference between the stable coefficient vector $\hat{a}(\gamma)$ and the open-loop one a(γ).

$$\sigma(\gamma) = \hat{a}(\gamma) - a(\gamma) \tag{4.10}$$

We make the following assumption.

Assumption III : The product of $\sigma(\gamma)$ and T(γ) has a finite bound on its norm Let M be defined by

$$M = \sup_{\gamma} \|\sigma(\gamma)T(\gamma)\| < \infty \tag{4.11}$$

In view of (4.8) and (4.9) we can rewrite (4.3) as

$$\dot{z} = \hat{A}(\gamma) + B[\sigma(\gamma)T(\gamma)x + u] \tag{4.12}$$

Consider the Lyapunov's function

$$V = z^T P(\gamma) z \tag{4.13}$$

and its time derivative

$$\dot{V} = -z^T Q(\gamma) z + 2[P_1(\gamma)T(\gamma)x]^T[\sigma(\gamma)T(\gamma)x + u] \tag{4.14}$$

where we have made use of (4.1) and $P_1(\gamma) = B^T P(\gamma)$.

Under Assumption II, there is a set of stable coefficient vectors $\hat{a}(\gamma)$ such that

$$P_1(\gamma)T(\gamma) = F \tag{4.15}$$

where F is a constant row vector, $P_1(\gamma) = B^T P(\gamma)$ and $P(\gamma)$ is the symmetric, positive definite solution of the Lyapunov equation (4.8).

Assumptions I-III permit the design of a discontinuous controller, [73], that stabilizes (Σ). We consider the control u be

$$u = \begin{cases} -\dfrac{Fx}{|Fx|}p(x) & \text{if } |Fx| > \varepsilon \\ 0 & \text{if } |Fx| \le \varepsilon \end{cases} \tag{4.16}$$

where ε is a small constant to reduce the chattering of the control u. For $|Fx| > \varepsilon$ Equation (4.14) becomes

$$\dot{V} \le -\lambda_1 \|x\|^2 + 2M|Fx|\,\|x\| - 2|Fx|p(x) \tag{4.17}$$

Define the function p(x) as below

$$p(x) > M\|x\| - \lambda_1 \frac{\|x\|^2}{2|Fx|} \tag{4.18}$$

then $\dot{V}$ is smaller than zero for all uncertainties γ. If $|Fx| \le \varepsilon$ under the control (4.16) Equation (4.14) becomes

$$\dot{V} \le -\lambda_1 \|x\|^2 + 2M|Fx|\,\|x\| \tag{4.19}$$

$\dot{V}$ is less that zero if $\|x\| > \frac{2M\varepsilon}{\lambda_1}$. Hence we have

$$\|x\| \leq \frac{2M\varepsilon}{\lambda_1} \tag{4.20}$$

That is, under the control defined by (4.16) and (4.18) we can keep the norm of the state x as small as we want. If we take $\varepsilon = 0$, then $\dot{V}$ is always less than zero. Since P(γ) is positive definite, the controller (4.16) and (4.18) stabilizes system ($\sum$).

The existence of a constant vector F under Assumption II is shown to hold for several systems in practice, [70], [71], [72]. Assumption II is always met for linear uncertain system ($\sum$) that satisfy the so-called matching conditions, [58], [59].

It may appear that discontinuous controllers of the type described by (4.16) can stabilize a larger class of linear uncertain system than can linear controllers. In the next section we show under the Assumption I-III, however, that if there is a stabilizing controller of the type (4.16) then there is also a stabilizing linear controller.

4.3 Stabilizing Linear Controller

We prove the following theorem

Theorem 4.1

If Assumptions I-III are met then there is a linear controller defined by

$$u = -cFx \tag{4.21}$$

that stabilizes the system ($\sum$) where c satisfies the inequality

$$c > \frac{\sup_{\gamma}\|\sigma(\gamma)T(\gamma)\|^2}{2\ \lambda_1} \tag{4.22}$$

Proof :

Consider the controller defined by (4.21) and (4.22). It suffices to show that $\dot{V}$ satisfies

$$\dot{V} \leq -\alpha\|x\|^2 \tag{4.23}$$

for some positive scalar α. Substitution of (4.15) (4.21) into (4.14) yields

$$\dot{V} = - z^T Q(\gamma) z + 2[Fx]^T [\sigma(\gamma) T(\gamma) x - cFx] \tag{4.24}$$

We need to show that $\dot{V}$ defined by (4.24) satisfies (4.23) for some $\alpha>0$ whenever c satisfies (4.22). Inequality (4.23) is met if the following inequality holds for all x for some $\alpha>0$.

$$2c\|Fx\|^2 - 2\|Fx\|M\|x\| + \lambda_1\|x\|^2 \geq \alpha\|x\|^2 \tag{4.25}$$

where M is defined by (4.11). For $\|x\| \neq 0$ define

$$y = \frac{\|\tilde{F}x\|}{\|x\|} \tag{4.26}$$

Inequality (4.25) holds for all x provided

$$2cy^2 - 2My + \lambda_1 \geq \alpha \tag{4.27}$$

for some $\alpha > 0$.

Inequality (4.27) is possible only if

$$4M^2 - 8c(\lambda_1 - \alpha) < 0 \tag{4.28}$$

That is to say

$$c > \frac{M^2}{2(\lambda_1 - \alpha)} \tag{4.29}$$

Consequently, when c satisfies Equation (4.29) there is some $\alpha > 0$ such that $\dot{V}$ satisfies Equation (4.22). Therefore, under Assumptions 1-3, the linear control defined by Equation (4.21) and Equation (4.22) stabilizes system ($\sum$). Thus, we have showed that there exists a linear control which can stabilize the system for any uncertainty γ.

Chapter 5 Linear Systems with Multi-input

5.1 Introduction

For a multi-input linear time-invariant uncertain dynamical system

$$\dot{x}(t) = \overline{A}(\gamma)x(t) + \overline{B}(\gamma)u(t) \tag{Σ}$$

as described in Chapter 2. We investigate the extension of previous results to the multi-input case. For any uncertainty $\gamma \in \Gamma$ the pair $(\overline{A}(\gamma), \overline{B}(\gamma))$ of the system (Σ) can be transformed to canonical companion controllable form $(A_z(\gamma),\ B_z(\gamma))$ as described in Chapter 2. We follow a procedure similar to the Lyapunov function approach of the scalar input case described in Chapter 3. In the designing of robust stabilizing controllers, we need to specify a constant F matrix such that $F \in \mathbb{R}^{m \times n} = B_z^T(\gamma)P(\gamma)T(\gamma)$ for all $\gamma \in \Omega$. Then m rows of the matrix P are known. Our problem is the following : What properties must these m rows of P satisfy in order that the Lyapunov equation described below provides a positive definite matrix P.

$$PA + A^T P + Q = 0 \tag{5.1}$$

where $Q \in \mathbb{R}^{n \times n}$ is an n×n symmetric positive definite matrix, $P \in \mathbb{R}^{n \times n}$ is an n×n symmetric matrix, $A \in \mathbb{R}^{n \times n}$ is an n×n matrix in the canonical companion controllable form described in Chapter 2.

In designing nonlinear robust controller to stabilize the plant (Σ) as in Chapter 4, it suffices to find a Hurwitz matrix A satisfying Lyapunov equation (5.1) when some rows of the matrix P are prescribed. Hence, we investigate the properties of the matrix P which ensure that A be Hurwitz.

5.2 *Main Results*

The following matrices introduced here are assumed to hold throughout this chapter. Let matrices A, P, and Q $\in \mathbb{R}^{n\times n}$ be nxn dimensional. We assume that Q is symmetric positive definite and that P is a symmetric matrix. The matrices A, P and Q satisfy the Lyapunov equation (5.1). The matrix A is in the canonical companion controllable form as shown in Chapter 2. Let $B \in \mathbb{R}^{n\times m}$ be an n×m matrix which is in the companion form shown in Chapter 2. The set $\{k_i\}$, i = 1,2,...,m, are the Kronecker invariants of A and B, [78]. Define $C \in \mathbb{R}^{m\times n}$ be the m×n matrix defined as follows. Let $\tilde{e}_i$ be a unit column vector with 1 as its $r_i + k_i - 1$ 'th element. Then the matrix C is defined as

$$C = [\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_m]^T \tag{5.2}$$

In block form, C is

$$C = \text{ block diag}[\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_m] = \begin{bmatrix} \tilde{c}_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \tilde{c}_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \tilde{c}_m \end{bmatrix} \tag{5.3}$$

where $\tilde{c}_i = [0,0,\dots,1]$ has k_i elements. We note that {C,A} is observable. There is a connection among the matrices A, B and C as shown in the following Lemma.

Lemma 1

For the compatible companion matrices A, B and C, we have

$$\det[C[sI - A]^{-1}B] = \frac{1}{\det[sI - A]} \tag{5.4}$$

Proof :

Consider a plant defined by

$$\dot{x} = Ax + Bu \tag{5.5}$$

then the characteristic polynomial of the system is det[sI - A]. For the m states of x, i.e. $y = Cx$, we have

$$(-1)^n \det[C[sI - A]^{-1}B] \times \det[sI - A] = \det \begin{vmatrix} A - sI & B \\ C & 0 \end{vmatrix} \tag{5.6}$$

Since b_i, the i'th column of the matrix B, has 1 on its r_i 'th element and all other elements are zero. We conclude that the any element of the r_i'th row of the matrix [A-sI] has a cofactor which determinant is zero. That is to say, if we replace the r_i'th row of the matrix [A-sI] by zeros then the determinant of the right hand side of Equation (5.6) will remain the same. After we replace the r_i'th row of the matrix [A-sI] by zeros then we can see that any element contains s in the replaced matrix has a cofactor which determinant is zero. That is to say, the determinant of the matrix on the right hand side of (5.6) is a constant. This constant can be calculated by letting s = 0 in (5.6)

$$\det \begin{vmatrix} A - sI & B \\ C & 0 \end{vmatrix} = \det \begin{vmatrix} \tilde{A} & B \\ C & 0 \end{vmatrix} \tag{5.7}$$

where $\tilde{A}$ has the same form as A except that all the r_i rows are zeroes. Define the matrix Y

$$Y = \begin{bmatrix} \tilde{A} & B \\ C & 0 \end{bmatrix} \tag{5.8}$$

The matrix Y has the property that every column and every row has one element 1 and others are zeros. The non-zero elements in the matrix B are the (n + j, r_j), j = 1,...,m, elements of Y. The non-zero elements in the matrix C are the (n + j, $r_j + k_j - 1$), j = 1,...,m, elements of Y. Its determinant can be calculated by first consider the matrix B and then the matrix C, let det[Y] = $(-1)^L$ then

$$
\begin{aligned}
L &= \sum_{j=1}^{m} n + 1 + r_j - j + 1 + \sum_{j=1}^{m} n - m + 1 + r_j + k_j - j \\
&= 2(nm + m + m(m+1)/2 + \sum_{j=1}^{m} r_j) + n + m(1-m)
\end{aligned}
\tag{5.9}
$$

Define L_1 as

$$\det[C[sI - A]^{-1}B] \times \det[sI - A] = (-1)^{L_1} \tag{5.10}$$

then

$$L_1 = 2(nm + m + m(m+1)/2 + \sum_{j=1}^{m} r_j) + m(1-m) \tag{5.11}$$

Since L_1 is even, then $(-1)^{L_1}$ is 1. Hence (5.4) is established and the lemma is proved.

Next, we introduce the matrix W defined below

$$W = \text{block diag}[w_1, w_2, \ldots, w_m] \tag{5.12}$$

where $w_i^T = adj[sI - A_{ii}]\tilde{b}_i$. We note that $adj[sI - A_{ii}]\tilde{b}_i = [s^{k_i-1}, s^{k_i-2}, \ldots, s, 1]^T$, where s is a complex number. In matrix form W is as follows :

$$
W = \begin{bmatrix}
s^{k_1-1} & 0 & \cdots & . \\
s^{k_1-2} & 0 & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
1 & . & \cdots & . \\
0 & s^{k_2-1} & \cdots & . \\
0 & s^{k_2-2} & \cdots & . \\
. & . & \cdots & . \\
. & . & \cdots & . \\
. & 1 & \cdots & . \\
. & 0 & \cdots & s^{k_m-1} \\
. & 0 & \cdots & s^{k_m-2} \\
. & . & \cdots & . \\
0 & . & \cdots & . \\
0 & 0 & \cdots & 1
\end{bmatrix}
\tag{5.13}
$$

where k_i, i = 1,...,m, are the Kronecker invariant of the matrices A and B, and where $\tilde{b}$'s are described in Chapter 2.

Define the matrix G as

$$G = B^T P W \tag{5.14}$$

we see that g_{ij}, the elements of the matrix G, are related to the matrix P by the following :

$$g_{ij} = p_{r_i, r_j} s^{k_j - 1} + p_{r_i, r_j + 1} s^{k_j - 2} + \cdots + p_{r_i, r_{j+1} - 1} \tag{5.15}$$

where i = 1,2,...,m , j = 1,2,...,m.

In designing the robust controllers for the scalar input system (Ω) via Lyapunov theory, the selection of P(γ) in the Lyapunov function V has the constraint $P_1 T = F$. For the multi-input system ($\sum$), the constraint on P becomes $B_z^T(\gamma)P(\gamma) = FT^{-1}(\gamma)$, where $B_z^T(\gamma)$ is in Companion form. The constraint on P states that certain rows of P defined by the matrix $B^T P$ are fixed as given quantities. We state this as an equation $B^T P = F^*$, where F^* is a given matrix. Specifically, these are the r_i, i = 1,2,...,m, rows of P. In this dissertation we seek conditions on these rows of P defined by $B^T P = F^*$ such that the Lyapunov equation (5.1) admits a stable solution for the A matrix which is in companion form compatible with B. The necessary and sufficient conditions on the rows of the matrix P are given in the Main Theorem below :

Main Theorem

Let the controllable pair of matrices A and B be in companion form. (Eqs.(2.11) and (2.15)) Let the matrices P and Q be symmetric, and Q be positive definite. The matrix G is defined by Equation (5.14) . The matrices A, P, and Q are assumed to satisfy the Lyapunov equation (5.1). Then the following two properties hold among matrices A, B, P, Q, and G :

(I) If the matrix A is Hurwitz, then the determinant of G is a Hurwitz polynomial.

(II) If the m rows of the matrix P defined by $B^T P = F^*$ are fixed as given quantities and

(1) the determinant of G is a Hurwitz polynomial

(2) $B^T P B$ is positive definite

then the solution P of the Lyapunov equation (5.1) is positive definite, and the solution A of Lyapunov equation (5.1) is Hurwitz.

The proof of (I) is given in Theorem 5.1 and that of (II) is given in Theorem 5.3 below.

We remark that in assertion (II) we seek solutions which satisfy the Lyapunov equation (5.1) where Q is given and m rows of P are fixed by the constraint $B^TP = F^*$, where F^* is a prescribed matrix. Recall that the companion form of A has m rows which have undefined elements × associated with them. The unknown quantities of the Lyapunov equation (5.1) are then the n-m rows of the matrix P which remain after the m rows B^TP are removed and the m rows of the matrix A which have the undefined elements as in Eqs. (2.14) and (2.15).

Theorem 5.1 (property I)

Let the controllable pair of matrices {A,B} be in companion form and let k_i, i = 1,...,m, be their Kronecker invariants. The matrices P and Q are symmetric, Q is positive definite. The matrix G is defined by Equation (5.14) . We assume matrices A, P, and Q satisfy the Lyapunov equation (5.1). If A is Hurwitz, then the determinant of G is a Hurwitz polynomial. i.e. det $|G| = 0$ has all roots s_i with negative real part.

Proof:

Assume that P and A are solutions to the Lyapunov equation (5.1) in which P is constrained by $B^TP = F^*$. Suppose A is Hurwitz. Since Q > 0, it follows that P > 0. We define the plant

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (\Pi)$$

where $t\in\mathbb{R}, x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the control. The equation $B^TPx = 0$ defines an n-m dimensional linear manifold

$$M = \{x\in\mathbb{R}^n : B^TPx = 0\} \qquad (5.16)$$

We show that there is a stabilizing control u(t) which keeps a solution x(t) in M when it starts in M. It is this stabilizing property of M that leads to the result that the determinant of G is a Hurwitz polynomial.

The time derivative of the Lyapunov function V defined as

$$V = x^T P x \tag{5.17}$$

is given by

$$\dot{V} = -x^T Q x + 2[B^T P x]^T u \tag{5.18}$$

Consider solutions x(t) which satisfy the differential form

$$B^T P \dot{x} = 0 \tag{5.19}$$

Since A, B are in cononical companion form, we have

$$\dot{x}_j = x_{j-1} \qquad 2 \le j \le n,\ j \ne r_1, r_2, \dots, r_m \tag{5.20}$$

Substituting (5.20) into (5.19), gives

$$p_{r_j r_1}\dot{x}_{r_1} + p_{r_j r_2}\dot{x}_{r_2} + \cdots + p_{r_j r_m}\dot{x}_{r_m} = q_j \tag{5.21}$$

where j = 1,2,...,m and where q_j is defined by

$$q_j = -\sum_{\substack{l=2 \\ i \ne r_p}}^{n} p_{r_j, i} x_{l-1}, \qquad p = 1,2,\dots,m. \tag{5.22}$$

Define $\dot{x}_r^T = [\dot{x}_{r_1}, \dots, \dot{x}_{r_m}]$. In matrix form, Equation (5.21) can be represented as following

$$B^T P B \dot{x}_r = q \tag{5.23}$$

where $q^T = [q_1, q_2, \dots, q_m]$. Recall that P is positive definite. Hence the matrix $B^T P B$ is positive definite and therefore non-singular. Define $\tilde{u}^T = [\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m]$ as

$$\tilde{u} = [B^T P B]^{-1} q \tag{5.24}$$

Consider the discontinuous control $u(x)^T = [u_1(x), \dots, u_m(x)]$ where $u_j(x)$ is defined by

$$u_j(x) = \begin{cases} 0 & if \;\; B^T Px \neq 0 \\ \tilde{u}_j - \sum_{l=1}^{n} a_{r_j,l} x_l & if \;\; B^T Px = 0 \end{cases} \tag{5.25}$$

where the a's are the coefficients of the matrix A.

In view of control (5.25), Equation (5.18) reduces to

$$\dot{V} = -x^T Q x \tag{5.26}$$

Since A is Hurwitz, the system ($\prod$) is also stable under the control (5.25) provided it can be shown that a solution which begins in the manifold defined by the equation

$$B^T P x = 0 \tag{5.27}$$

stays in this manifold. In this case stability of the system ($\prod$) is ensured by Theorem (1.1) in [74]. That is, according to Lyapunov theory of stability for nonlinear control systems,[74], the system ($\prod$) is stable under the control u(x) defined by Equation (5.25) provided the system is stable on the manifold defined by the equation $B^T P x = 0$. The Equation (5.27) defines an n-m dimensional linear manifold M in n-dimensional space. Let x_0 belong to M. Consider the initial condition $x(0) = x_0$. Since Equation (5.27) holds the system ($\prod$) under the control (5.25) becomes

$$\dot{x}_r = \tilde{u} = [B^T P B]^{-1} q \tag{5.28}$$

Hence

$$[B^T P B]\dot{x}_r = q \tag{5.29}$$

Using Equation (5.20) , we have

$$q_j = -\sum_{\substack{i=2 \\ i \neq r_p}}^{n} p_{r_j,i} x_{i-1} = -\sum_{\substack{i=2 \\ i \neq r_p}}^{n} p_{r_j,i} \dot{x}_i , \quad p = 1,2,\ldots,m. \tag{5.30}$$

Equation (5.29) becomes

$$B^T P\dot{x} = 0 \tag{5.31}$$

Hence

$$B^T P x = c \tag{5.32}$$

where c is a constant vector. Since x_0 satisfies $B^T P x_0 = 0$ then $c = 0$. Then the solution to system ($\prod$) begins in M and stays in M and therefore from Equation (5.26) the system is stable in M. From system ($\prod$) and Equation (5.25), the equation of motion in M satisfies

$$\sum_{i=1}^{n} p_{r_j,i} x_i(t) = 0 \qquad j = 1, 2, \dots, m \tag{5.33}$$

$$\dot{x}_j = x_{j-1} \qquad 2 \le j \le n,\ j \ne r_1, r_2, \dots, r_m \tag{5.34}$$

We examine the motion of the system ($\prod$) on the manifold (5.27). Taking Laplace transform of Equation (5.34), we have

$$s x_j(s) = x_{j-1}(s) \qquad 2 \le j \le n,\ j \ne r_1, r_2, \dots, r_m. \tag{5.35}$$

The Laplace transform of Equation (5.33) together with Equation (5.35) form the m equations

$$\sum_{i=1}^{m} \sum_{l=1}^{k_i} s^{k_i - l} p_{j, r_i + l - 1} \times y_l = 0 \tag{5.36}$$

where $j = r_1, r_2, \dots, r_m$.

$$y_l = \begin{cases} x_{r_i}(s) & \text{if } k_i = 1 \\ x_{r_{i+1}-1}(s) & \text{if } k_i \ne 1 \end{cases} \tag{5.37}$$

In matrix form Equation (5.36) can be presented as

$$Gy = 0 \tag{5.38}$$

where the elements of the matrix G are defined by Equation (5.15). Since the system is stable, we conclude that the determinant of G must be a Hurwitz polynomial and the theorem is proved.

In the next theorem, we show that a linear control exists under certain conditions. This theorem is useful in proving the converse of Theorem 5.1. We consider a system (Λ) defined as

$$\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t) \tag{Λ}$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control, $\mathrm{A} \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix, $\mathrm{B} \in \mathbb{R}^{n \times m}$ is an $n \times m$ input matrix. A, B are both in compatible companion form as defined in (2.13) and (2.17). In this dissertation, we define a surface to be stable as below.

Definition 1 : For a constant matrix $F \in \mathbb{R}^{m \times n}$, the linear manifold defined by the equation F x = 0 is said to be a stable n-m dimensional surface if $\det[\, F[sI - \mathrm{A}]^{-1}\mathrm{B}\,]$ has n-m zeros with negative real part.

Let $\bar{y} = Fx$, then

$$\bar{y}(s) = Fx(s) = F[sI - \mathrm{A}]^{-1}\mathrm{B}u(s) \tag{5.39}$$

If F x = 0 defines a stable surface, then the roots of the equation $\det[\, F[sI - \mathrm{A}]^{-1}\mathrm{B}\,] = 0$ must have negative real parts by definition. Let

$$\det[sI - \mathrm{A}] \equiv \phi(s) \tag{5.40}$$

Then, we have

$$\psi(s) \equiv \det\begin{bmatrix} sI - \mathrm{A} & \mathrm{B} \\ F & 0 \end{bmatrix} = \det[sI - \mathrm{A}]\det[-F[sI - \mathrm{A}]^{-1}\mathrm{B}] = (-1)^m \phi(s)\det[F[sI - \mathrm{A}]^{-1}\mathrm{B}] \tag{5.41}$$

is a n-m order Hurwitz polynomial. Then

$$\det[F[sI - \mathrm{A}]^{-1}\mathrm{B}] = (-1)^m \frac{\psi(s)}{\phi(s)} \tag{5.42}$$

Theorem 5.2

Consider the system (Λ). Suppose that there is a constant matrix $F \in \mathbb{R}^{m \times n}$, such that the manifold defined by the equation F x = 0 is a stable n-m dimensional surface and that the matrix $-[FB]$ is Hurwitz. Then there exists a constant c such that the linear control u = -cFx stabilizes the system (Λ).

Proof:

Let f_{ij} be the elements of the matrix F. Then under the linear control

$$u = -\, cFx \tag{5.43}$$

the closed loop system becomes

$$\dot{x} = (\mathrm{A} - c\mathrm{B}F)x = \tilde{\mathrm{A}}x \tag{5.44}$$

Then $\tilde{\mathrm{A}} = (\mathrm{A}\text{-}c\mathrm{BF})$ has the same form as A, with elements

$$\tilde{a}_{r_j i} = a_{r_j i} - cf_{ji} \qquad\qquad j = 1,\dots,m,\ i = 1,\dots,n \tag{5.45}$$

Taking the Laplace transform of Equation (5.44), we have m equations :

$$s^{k_j} y_j(s) = \sum_{i=1}^{m} \sum_{l=1}^{k_i} (\tilde{a}_{r_j,\, r_i + l - 1} s^{k_i - l}) y_i(s) \tag{5.46}$$

where $j = 1, 2, \dots, m.$, and

$$y_i(s) = x_{r_i + k_i - 1}(s) \tag{5.47}$$

or in matrix form :

$$y(s) = Cx(s) \tag{5.48}$$

The relation between y(s) and x(s) is

$$x(s) = Wy(s) \tag{5.49}$$

We divide Equation (5.46) by a positive constant c and select the magnitude of c as large as necessary, so that $|\frac{a_{r_{ji}}}{c}| << |f_{ij}|$. Let $\varepsilon = 1/c$, then Equation (5.46) becomes

$$\varepsilon s^{k_j} y_j(s) + \sum_{l=1}^{m} \sum_{l=1}^{k_i} (f_{j,r_i+l-1} s^{k_i - l}) y_i(s) = 0 \tag{5.50}$$

where j = 1,...,m.

Define the matrix H as

$$H \equiv FW \tag{5.51}$$

where W is defined as Equation (5.13). The elements h_{ij}, i,j = 1,2,...,m, of the matrix H, satisfy

$$h_{ij} = f_{i,r_j} s^{k_j - 1} + f_{i,r_j+1} s^{k_j - 2} + \cdots + f_{i,r_{j+1}-1} \tag{5.52}$$

Define the coefficients β_i, i = 0,...,n-m, as

$$\det |H| = \sum_{i=0}^{n-m} \beta_i s^i = \beta_{n-m} s^{n-m} + \beta_{n-m-1} s^{n-m-1} + \cdots + \beta_0 \tag{5.53}$$

From (5.49) and (5.51) we have

$$Fx(s) = FWy(s) = Hy(s) \tag{5.54}$$

where y(s) is defined in Equation (5.47). Since $B^T B = I_m$ the Laplace transformation of system (Λ) yields

$$u(s) = B^T [sI - \mathrm{A}] x(s) = B^T [sI - \mathrm{A}] Wy(s) \tag{5.55}$$

and we have, from Appendix B,

$$\det[B^T[sI - A]W] = \det[sI - A] = \phi \tag{5.56}$$

From system (Λ) and (5.54), we have

$$Fx(s) = F[sI - A]^{-1}Bu(s) = Hy(s) \tag{5.57}$$

Hence substituting (5.55) into (5.57) yields

$$H = [\, F[sI - A]^{-1}B\,]\,[B^T[sI - A]W] \tag{5.58}$$

The determinant of H satisfies

$$\begin{aligned}\det[H] &= \det[F[sI - A]^{-1}B] \times \det[B^T[sI - A]W] \\ &= (-1)^m \frac{\psi}{\phi}\phi \\ &= (-1)^m \psi\end{aligned} \tag{5.59}$$

Consequently the assumption of the manifold defined by Fx = 0 being stable implies that the determinant of H is Hurwitz. Hence the polynomial (5.53) is Hurwitz. Without loss of generality, we may assume $\beta_i > 0$ and $c > 0$. (If $\beta_i < 0$, we select $c < 0$). Equation (5.50) can be written in matrix form as

$$\tilde{H}y(s) = 0 \tag{5.60}$$

where matrix $\tilde{H}$ has elements as follows

$$\begin{aligned}\tilde{h}_{ij} &= \delta_{ij}\varepsilon s^{k_j} + h_{ij} \\ &= \delta_{ij}\varepsilon s^{k_j} + f_{i,r_j}s^{k_j-1} + \cdots + f_{i,r_{j+1}-1}\end{aligned} \tag{5.61}$$

where $\delta_{ij} = 1$ *if* $i = j$, otherwise $\delta_{ij} = 0$.

The stability of Equation (5.60) depends on whether all roots of the following equation

$$\det|\tilde{H}| = 0 \tag{5.62}$$

have negative real part.

For very large c , we have $\varepsilon << 1$. If we drop the higher order ε terms in each coefficient of Equation (5.62), then

$$\det |\tilde{H}| \cong s^{n-m} \det \begin{bmatrix} \varepsilon s + f_{1r_1} & f_{1r_2} & \cdots & f_{1r_m} \\ f_{2r_1} & \varepsilon s + f_{2r_2} & \cdots & f_{2r_m} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ f_{mr_1} & f_{mr_2} & \cdots & \varepsilon s + f_{mr_m} \end{bmatrix}$$

$$+ \beta_{n-m-1}s^{n-m-1} + \beta_{n-m-2}s^{n-m-2} + \cdots + \beta_0 \tag{5.63}$$

The derivation of the above equation is given in the Section 5.4. Note that

$$FB = \begin{bmatrix} f_{1r_1} & f_{1r_2} & \cdots & f_{1r_m} \\ f_{2r_1} & f_{2r_2} & \cdots & f_{2r_m} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ f_{mr_1} & f_{mr_2} & \cdots & f_{mr_m} \end{bmatrix} \tag{5.64}$$

By assumption, -FB is Hurwitz. Hence the determinant of the matrix $[sI_m + FB]$ is Hurwitz. It is easy to see that the determinant of the matrix $[\varepsilon sI_m + FB]$ is also Hurwitz for positive ε. Define coefficients d_j, j = 0,...,m-1, as

$$\det \begin{vmatrix} \varepsilon s + f_{1r_1} & f_{1r_2} & \cdots\cdot & f_{1r_m} \\ f_{2r_1} & \varepsilon s + f_{2r_2} & \cdots\cdot & f_{2r_m} \\ \cdot & \cdot & \cdots\cdot & \cdot \\ \cdot & \cdot & \cdots\cdot & \cdot \\ \cdot & \cdot & \cdots\cdot & \cdot \\ f_{mr_1} & f_{mr_2} & \cdots\cdot & \varepsilon s + f_{mr_m} \end{vmatrix} \cong \varepsilon^m s^m + \varepsilon^{m-1} d_{m-1} s^{m-1} + \cdots + \varepsilon d_1 s + d_0 \quad (5.65)$$

We note that for the coefficient of the s^j term the minimum order of ε is j. Here, we have neglected higher order ε terms in each coefficient of Equation (5.65).

Note : $\beta_{n-m} = d_0$.

From Equation (5.62), (5.63) and (5.65) we conclude that the following polynomial p(s) = det $|\tilde{H}|$, i.e.

$$p(s) = \varepsilon^m s^n + \varepsilon^{m-1} d_{m-1} s^{n-1} + \cdots + d_0 s^{n-m} + \beta_{n-m-1} s^{n-m-1} + \cdots + \beta_1 s + \beta_0 \quad (5.66)$$

must be Hurwitz in order for the system to be stable. We investigate the roots of p(s) as $\varepsilon << 1$. First consider the roots s_i where the magnitude of s_i is finite and of order ε^0. Let ε be as small as necessary, then p(s) reduces to

$$p(s) \cong \det |H| = \sum_{i=0}^{n-m} \beta_i s^i = \beta_{n-m} s^{n-m} + \beta_{n-m-1} s^{n-m-1} + \cdots + \beta_0 \quad (5.67)$$

Hence p(s) has n-m stable roots close to the roots of the determinant of the matrix H. In Equation (5.66) , let $S = \varepsilon s$, then the equation becomes

$$p(S) = S^n + d_{m-1} S^{n-1} + \cdots + d_0 S^{n-m} + \varepsilon \beta_{n-m-1} S^{n-m-1} + \cdots + \varepsilon^{n-m-1} \beta_1 S + \varepsilon^{n-m} \beta_0 \quad (5.68)$$

We consider the roots S which are finite and of order ε^0. Let ε be small as necessary, we have

$$p(S) \cong S^n + d_{m-1} S^{n-1} + d_{m-2} S^{n-2} + \cdots + d_0 S^{n-m} \quad (5.69)$$

Then p(S) has m stable roots which are the roots of the polynomial Equation (5.65) . (note : the n-m roots of Equation (5.69) are zero. Since we assume S to be much larger that ε, these zero roots are not the ones of interest.) From the above analysis, we conclude that m roots of p(s) are near those of the determinant of the Hurwitz matrix H, and n-m roots of p(s) are close to the roots of the Hurwitz polynomial Equation (5.65) divided by ε. Hence p(s) is a Hurwitz polynomial.

Theorem 5.3 (property II)

Let the controllable pair of matrices A and B be in companion form. (Eqs.(2.11) and (2.15)) Let the matrices P and Q be symmetric, and Q be positive definite. The matrix G is defined by Equation (5.14) . The matrices A, P, and Q are assumed to satisfy the Lyapunov equation (5.1). If the rows of the matrix P defined by $B^TP = F^*$ are fixed as given quantities and

(1) the determinant of G is a Hurwitz polynomial

(2) B^TPB is positive definite

then the solution P of Lyapunov equation (5.1) is positive definite, and the solution A of Lyapunov equation (5.1) is Hurwitz.

Proof :

Consider the plant

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (\Upsilon)$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ is the state vector.

Let $F = B^TP$. Then by Assumption (1) the determinant of the matrix $G = B^TPW = FW$ is a Hurwitz polynomial. From Equation (5.47) we have F x(s) = F W y(s) = G y(s), and GC = FWC = F. Then

$$\det[F[sI - A]^{-1}B] = \det[GC[sI - A]^{-1}B] = \det[G]\det[C[sI - A]^{-1}B] = \frac{\det[G]}{\det[sI - A]} \qquad (5.70)$$

Since det[G] is Hurwitz, det[F adj[sI-A] B] must be Hurwitz. That the matrix $B^TPB = FB$ is positive definite assures that -FB is Hurwitz. Hence by Theorem 5.2, the linear control (5.43) stabilize the system (Ψ) for some sufficiently large c. Define the Lyapunov function

$$V = x^T P x \tag{5.71}$$

The time derivative of V is

$$\dot{V} = -x^T Q x + 2[B^T P x]^T u \tag{5.72}$$

Consider the linear control

$$u = -\, cB^T P x \tag{5.73}$$

where c is a sufficiently large number to stabilize (Ψ)

$$\dot{V} = -x^T Q x - 2c[B^T P x]^T [B^T P x] = -x^T Q x - 2c\|B^T P x\|^2 \le 0 \tag{5.74}$$

Assume P is not positive definite. Then there exists $x_0 \neq 0$ such that

$$V(x_0) = x_0^T P x_0 \le 0 \tag{5.75}$$

From Equation (5.74) we see that for $x_0 \neq 0$, $\dot{V}(x_0) < 0$. Consequently, $V(x(t)) < 0$ for $t > 0$. There exits $\varepsilon > 0$ and $t_1 > 0$ such that $V(x(t)) < -\varepsilon$ for all $t > t_1$. Since V is a continuous function there is some ball of radius $r > 0$ about the origin

$$B(r,0) = \{x : \|x\| \le r\} \tag{5.75}$$

such that

$$V(x) < -\, \varepsilon \qquad \qquad for\ \ \forall x \in B(r,0) \tag{5.76}$$

This implies that x(t) cannot enter the ball B(r,0) for all $t > t_1$, for any positive c. This contradicts that the control (5.73) stabilizes (Ψ). Therefore, P must be positive definite and A must be Hurwitz.

5.3 The Derivation of Equation (5.63)

Let

$$\det|\tilde{H}| = \sum_{i=0}^{n} c_i s^i \tag{5.77}$$

From the definition of $\tilde{H}$ defined by Equation (5.61) , the coefficients c_i, i = 1,...,n, are polynomials of ε. Since ε can be as small as we want, we only need to consider the minimum order terms of ε in each coefficient c_i. We have, for ε = 0, Equation (5.77) becomes

$$\det|\tilde{H}| = \det|H| = \sum_{i=0}^{n-m} \tilde{b}_i s^i = \tilde{b}_{n-m} s^{n-m} + \tilde{b}_{n-m-1} s^{n-m-1} + \cdots + \tilde{b}_0 \tag{5.78}$$

Hence as ε approaches zero, we have $c_i \cong \tilde{b}_i$, i = 0,1,...,n-m. Now we only need to consider the coefficients c_i , i > n-m. Let i = n-m + j, then the minimum order of ε in the coefficient c_i must be no less than j. This is due to the observation that the difference between the matrix H and $\tilde{H}$ is that the diagonal element in $\tilde{H}$ has an additional term εs^{k_j}. Due to the contribution of these terms, with respect to s, the order of the matrix $\tilde{H}$ is one and only one order higher than the matrix H on each diagonal element. Since $c_i s^i$ needs at least j'th terms of εs^{k_j} in order to have power of s as i, the minimum order of ε for c_i is j. i.e.

$$c_i = d_j \varepsilon^j + O(\varepsilon^{j+1}) \tag{5.79}$$

where d_j is some constant which is not a function of ε. We now consider the terms which contribute to

$$c_i s^l \cong d_j \varepsilon^j s^l \tag{5.80}$$

Let M = n! = n*(n-1)*...*2*1. Then the integer set {1,2,...,n} has M different permutations. Let K_p, p = 1,2,...,M. denote the p'th permutation, and $K(p,l)$ denote the l'th element of the p'th permutation. we have, by the definition of determinant

$$\det |\tilde{H}| = \sum_{p=1}^{p=M} \prod_{l=1}^{l=n} (-1)^{(p+K(p,l))} \tilde{h}_{l,K(p,l)} \tag{5.81}$$

Those terms contributed to d_j in the product have j diagonal elements of $\tilde{H}$. The coefficients of the term $\tilde{h}_{l,K(p,l)}$ which has order less that $s^{k_{K(p,l)}-1}$ have no contribution to d_j. Define the element of the matrix Y

$$\tilde{y}_{ij} = \delta_{ij} \varepsilon s^{k_j} + f_{l,r_j} s^{k_j - 1} \tag{5.82}$$

then d_j is the coefficient of the s^l term of the determinant of the matrix Y. So, we conclude that

$$s^{n-m} \det \begin{bmatrix} \varepsilon s + f_{1r_1} & f_{1r_2} & \cdots & f_{1r_m} \\ f_{2r_1} & \varepsilon s + f_{2r_2} & \cdots & f_{2r_m} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ f_{mr_1} & f_{mr_2} & \cdots & \varepsilon s + f_{mr_m} \end{bmatrix}$$

$$= \varepsilon^m s^n + \varepsilon^{m-1} d_{m-1} s^{n-1} + \cdots + \varepsilon d_1 s^{n-m+1} + d_0 s^{n-m} \tag{5.83}$$

where $d_0 = \tilde{b}_{n-m}$ Hence we prove Equation (5.63) .

Chapter 6 Linear and Nonlinear Control

6.1 Introduction

In this chapter, we show the existence of the robust nonlinear and linear controller for a system satisfying Assumption 1-7 described in Chapter 2. The dynamical system considered is

$$\dot{x}(t) = \overline{\mathrm{A}}(\gamma)x(t) + \overline{\mathrm{B}}(\gamma)u(t) \tag{Σ}$$

as described in Chapter 2.

We assume that the system (Σ) satisfying Assumptions 1-7 described in Chapter 2, hence it can be transformed to canonical companion controllable form using the transformation described in Chapter 2. That is, there exists a matrix $\mathrm{T}(\gamma)$ with the coordinate transformation

$$z = T(\gamma)x \tag{6.1}$$

that the system (Σ) becomes

$$\dot{z} = \mathrm{A}_z(\gamma)z + \mathrm{B}_z u \tag{6.2}$$

where $\mathrm{A}_z(\gamma) = T(\gamma)\overline{\mathrm{A}}(\gamma)T^{-1}(\gamma)$, $\mathrm{B}_z = T(\gamma)\overline{\mathrm{B}}(\gamma)$. We define the $m \times m$ matrix B_s as follows

$$\mathrm{B}_s = B^T B_z \tag{6.3}$$

6.2 Stabilizing Discontinuous Controller

Theorem 6.1

For a dynamical system described by system (Σ), there exits a discontinuous robust stabilizing controller which stabilizes the system for each uncertainty.

Proof :

We transform the system to Equation (6.2). Let $\tilde{A}(\gamma)$ be a Hurwitz matrix which has the same form as $A(\gamma)$ in Equation (2.11) . Then system (6.2) can be written as

$$\dot{z} = \tilde{A}(\gamma)z + B_z(\gamma)[\sigma(\gamma)z + u] \tag{6.4}$$

where σ is defined as follows

$$\sigma(\gamma) = B_s^{-1}(\gamma)B^T[A(\gamma) - \tilde{A}(\gamma)] \tag{6.5}$$

We have used the relations that

$$B_z(\gamma) = BB_s(\gamma) \tag{6.6}$$

and

$$B\,B^T[A(\gamma) - \tilde{A}(\gamma)] = [A(\gamma) - \tilde{A}(\gamma)] \tag{6.7}$$

Let $P(\gamma) \in \mathbb{R}^{n\times n}$ and $Q(\gamma) \in \mathbb{R}^{n\times n}$ be uncertain positive definite symmetric matrices, and let $\tilde{A}(\gamma)$, $P(\gamma)$ and $Q(\gamma)$ satisfy the Lyapunov equation :

$$P(\gamma)\tilde{A}(\gamma) + \tilde{A}^T(\gamma)P(\gamma) + Q(\gamma) = 0 \tag{6.8}$$

Consider the Lyapunov function

$$V = z^T P(\gamma) z \tag{6.9}$$

The time derivative of V is

$$\dot{V} = -z^T Q(\gamma) z + 2[B_z^T(\gamma) P(\gamma) z]^T [\sigma(\gamma) z + u] \tag{6.10}$$

In order to utilize Equation (6.10) in designing a controller, we need $\tilde{A}(\gamma)$ to be Hurwitz and $P(\gamma)$ to be symmetric and positive definite. Hence, we need to show that it is possible to assign the values of the m rows $B_z^T P(\gamma)$ such that the two properties of P defined in Theorem 6.1 are satisfied.

By Assumption 4, there is a matrix F such that $\det[F\overline{B}] \neq 0$ for each uncertainty γ. It follows that the matrix $FT^{-1}B$ is not singular. The existence of the inverse of $[FT^{-1}B]$ can be shown below. Since

$$T(\gamma)\overline{B} = B_z(\gamma) = BB_s(\gamma) \tag{6.11}$$

We have

$$T^{-1}(\gamma)B = \overline{B}B_s^{-1}(\gamma) \tag{6.12}$$

Hence

$$\det[FT^{-1}(\gamma)B] = \det[F\overline{B}B_s^{-1}(\gamma)] = \det[F\overline{B}]\det[B_s^{-1}] = \det[F\overline{B}] \tag{6.13}$$

Since $\det[F\overline{B}] \neq 0$. Then the matrix $[FT^{-1}B]$ must be non-singular. Hence we can select a symmetric and positive definite m×m matrix $D(\gamma)$ define a non-singular m×m matrix $\Pi(\gamma)$ such that

$$\Pi(\gamma) \equiv B_s^T(\gamma) D(\gamma) [FT^{-1}(\gamma)B]^{-1} \tag{6.14}$$

We define

$$\tilde{P}(\gamma) \equiv B_s^{-T}(\gamma)[\Pi(\gamma) FT^{-1}(\gamma)] \tag{6.15}$$

Then we have

$$\begin{aligned} \tilde{P}(\gamma)B &= B_s^{-T}(\gamma)\Pi(\gamma)FT^{-1}(\gamma)B \\ &= B_s^{-T}B_s^T D(\gamma)[FT^{-1}B]^{-1}FT^{-1}B \\ &= D(\gamma) \end{aligned} \tag{6.16}$$

Hence $\tilde{P}(\gamma)B$ is symmetric and positive definite. Using Popov's theorem ,[78], we have

$$[sI - \bar{A}]T^{-1}W = \bar{B}R(s) \tag{6.17}$$

Hence

$$T^{-1}W = [sI - \bar{A}]^{-1}\bar{B}R(s) \tag{6.18}$$

We have

$$FT^{-1}W = F[sI - \bar{A}]^{-1}\bar{B}R(s) \tag{6.19}$$

then

$$\det[FT^{-1}W] = \det[F[sI - \bar{A}]^{-1}\bar{B}]\det[R(s)] = \det[F[sI - \bar{A}]^{-1}\bar{B}]\det[sI - A] \tag{6.20}$$

Since we assume that det $[F[sI - \bar{A}]^{-1}\bar{B}]$ has non-minimum phase zeros then det [F T^{-1} W] is Hurwitz. Then

$$\det[\tilde{P}W] = \det[B_s^{-T}[\Pi(\gamma)FT^{-1}]W] = \det[\Pi]\det[FT^{-1}W] \tag{6.21}$$

Hence $\det[\tilde{P}W]$ is a Hurwitz polynomial. Then the matrix $\tilde{P} = B_s^{-T}[\Pi(\gamma)FT^{-1}]$ has the desired two properties required by the matrix $[B^TP(\gamma)]$. So, we may determine m rows of P by the following equation.

$$[B^TP(\gamma)] = B_s^{-T}[\Pi(\gamma)FT^{-1}(\gamma)] \tag{6.22}$$

For the given matrix $[B^TP(\gamma)]$ and $Q(\gamma)$, we note that the matrix $\tilde{A}(\gamma)$ which is the solution of the Lyapunov equation (6.8) is not unique. However, its existence is shown in Appendix A for scalar input. Then from (6.22)

$$[B_z^TP(\gamma)] = B_s^TB^TP(\gamma) = [\Pi(\gamma)FT^{-1}(\gamma)] \tag{6.23}$$

If $E(\gamma) = \Pi(\gamma) - I$ with $\|E(\gamma)\| > 1$ then by Assumption 5, we can find a non-singular constant matrix $\tilde{C}$, such that $\tilde{E}(\gamma) = \Pi(\gamma)\tilde{C} - I$, satisfies $\|\tilde{E}(\gamma)\| < 1$. We make the definitions

$$\tilde{\Pi}(\gamma) \equiv \Pi(\gamma)\tilde{C} \tag{6.24}$$

$$\tilde{F} \equiv \tilde{C}^{-1}F \tag{6.25}$$

from which we have the following

$$\tilde{E}(\gamma) \equiv \tilde{\Pi}(\gamma) - I \tag{6.26}$$

hence

$$\begin{aligned} B_z^T P(\gamma) &= \Pi(\gamma)FT^{-1}(\gamma) \\ &= \Pi(\gamma)\tilde{C}\tilde{C}^{-1}FT^{-1}(\gamma) \\ &= \tilde{\Pi}(\gamma)\tilde{F}T^{-1}(\gamma) \end{aligned} \tag{6.27}$$

Let M be defined by

$$M = \sup_{\gamma}\|\sigma(\gamma)T(\gamma)\| < \infty \tag{6.28}$$

Since the norm of $\tilde{E}(\gamma)$ is bounded by unity and Γ is compact, there is a constant $\rho > 0$,such that

$$\rho = \sup_{\gamma}\|\tilde{E}(\gamma)\| < 1 \tag{6.29}$$

For the solutions $\tilde{A}(\gamma)$ and $P(\gamma)$ of Lyapunov equation (6.8) we have using (6.26)

$$\begin{aligned} \dot{V} &= -z^T Q(\gamma)z + 2[\tilde{\Pi}\tilde{F}x]^T[\sigma z + u] \\ &= -z^T Q(\gamma)z + 2[\tilde{F}x]^T\tilde{\Pi}^T(\gamma)[\sigma z + u] \\ &= -z^T Q(\gamma)z + 2[\tilde{F}x]^T\tilde{\Pi}^T(\gamma)\sigma T(\gamma)x + 2[\tilde{F}x]^T u + 2[\tilde{F}x]^T\tilde{E}^T(\gamma)u \end{aligned} \tag{6.30}$$

Since $T^T(\gamma)Q(\gamma)T(\gamma)$ is positive definite,there exists a non-negative constant λ such that

$$z^T Q(\gamma)z = x^T T^T(\gamma)Q(\gamma)T(\gamma)x > \lambda\|x\|^2 \tag{6.31}$$

Let the discontinuous control law be

$$u(x) = \begin{cases} -\dfrac{\tilde{F}x}{\|\tilde{F}x\|}p(x) & \text{if } \|\tilde{F}x\| > \varepsilon \\ 0 & \text{if } \|\tilde{F}x\| \leq \varepsilon \end{cases} \tag{6.32}$$

then by the Equation (6.30) , for the case $\|\tilde{F}x\| > \varepsilon$, we have

$$\dot{V} \leq -\lambda\|x\|^2 + 2M(1+\rho)\|\tilde{F}x\|\,\|x\| - 2\|\tilde{F}x\|(p(x) - \rho\,|p(x)|) \tag{6.33}$$

Define pp(x) as follows

$$pp(x) = \frac{M(1+\rho)}{(1-\rho)}\|x\| - \frac{\lambda\|x\|^2}{2(1-\rho)\|\tilde{F}x\|} \tag{6.34}$$

If p(x) is defined by the following equation

$$\begin{aligned} p(x) &> pp(x) & \text{if } pp(x) \geq 0 \\ &> \frac{(1-\rho)}{(1+\rho)}pp(x) & \text{if } pp(x) < 0 \end{aligned} \tag{6.35}$$

then $\dot{V}$ is less than zero. When $\|\tilde{F}x\| \leq \varepsilon$ then the control (6.32) will yield

$$\dot{V} \leq -\lambda\|x\|^2 + 2M(1+\rho)\|\tilde{F}x\|\|x\| \tag{6.36}$$

Then $\dot{V}$ will be less than 0 if $\|x\| > \dfrac{2M(1+\rho)\varepsilon}{\lambda}$. Hence for $Vdor$ to be no less than zero, we have

$$\|x\| \leq \frac{2M(1+\rho)\varepsilon}{\lambda} \tag{6.37}$$

So, by selection of ε, we can keep the norm of the state x as small as we want. If we take $\varepsilon = 0$, then $\dot{V}$ is always less than zero. Since the matrices P and Q > 0, the contol law (6.32) stabilizes the system. Hence this Theorem is proved.

6.3 Stabilizing Linear Controller

Theorem 6.2

For a dynamical system described as in system ($\sum$), there exits a linear robust stabilizing controller which stabilizes the system for each uncertainty.

Proof :

Let our control be linear, i.e.

$$u = -c\tilde{F}x \tag{6.38}$$

Then (6.30) yields

$$\dot{V} \leq -z^T Q(y) z + 2\|\tilde{F}x\|(1+\rho) M \|x\| - 2c\|\tilde{F}x\|^2 + 2c\rho\|\tilde{F}x\|^2 \tag{6.39}$$

We want to design the control u so that

$$\dot{V} \leq -\alpha\|x\|^2 \tag{6.40}$$

for some positive scalar α. Inequality (6.31) implies

$$2(\rho-1)c\|\tilde{F}x\|^2 + 2(1+\rho)M\|\tilde{F}x\|\,\|x\| - \lambda\|x\|^2 \leq \alpha\|x\|^2 \tag{6.41}$$

where M,λ and ρ are defined by (6.28),(6.31) and (6.29), respectively. For $\|x\| \neq 0$ define

$$y = \frac{\|\tilde{F}x\|}{\|x\|} \tag{6.42}$$

Inequality (6.31) holds for all x provided

$$2(1-\rho)cy^2 - 2(1+\rho)My + \lambda \geq \alpha \tag{6.43}$$

Inequality (6.43) is possible only if

$$4(1+\rho)^2M^2 - 8(1-\rho)c(\lambda-\alpha) < 0 \tag{6.44}$$

That is to say

$$c > \frac{(1+\rho)^2 M^2}{2(1-\rho)(\lambda-\alpha)} \tag{6.45}$$

Consequently, when c satisfies Equation (6.45) there is some $\alpha > 0$ such that $\dot{V}$ satisfies Equation (6.40). Therefore, under Assumptions 1-7, the linear control defined by Equation (6.38) and Equation (6.45) stabilizes system ($\sum$). Thus, we have showed that there exists a linear control which stabilizes the system for any uncertainty γ.

Chapter 7 Examples

7.1 Introduction

In this chapter, we present some examples to illustrate the applications of the theory developed in this research. Example I is a scalar input system with three state variables. It serves as an illustration of the design process of a controller for a scalar input system. Example II is an example to show how the theory may apply to a nonlinear time-varying system. We show how to design a robust controller for a multi-input system in Example III.

7.1 Example I : Scalar Input System

This example utilizes the wing rock model described in Stalford[72]. Here we assume all the uncertainties are time-invariant. The model is

$$\dot{x} = \overline{A}x + \overline{B}u \tag{7.1}$$

where $\overline{A}$ is

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & -1/\tau \end{bmatrix} \tag{7.2}$$

and where $\overline{B}$ is

$$\overline{B} = \begin{bmatrix} 0 \\ 0 \\ -1/\tau \end{bmatrix} \tag{7.3}$$

The uncertainties have the following bounds
$a_{21} \in [-16, -4]$, $a_{22} \in [-24, 3.5]$, $a_{23} \in -10.$, $\tau = 0.1$. The first and second state variables denote roll angle and roll rate, respectively. The third state variable represents an actuator with the time constant τ. The control variable represents the input control voltage to the actuator. The characteristic polynomial of this system is

$$\det[sI - \overline{A}] = s^3 + a_1 s^2 + a_2 s + a_3 \tag{7.4}$$

where we have $a_1 = 1/\tau - a_{22}$, $a_2 = -a_{21} - a_{22}/\tau$, and $a_3 = -a_{21}/\tau$.
Taking a coordinate transformation z = Tx. Here we define T^{-1} as described in Chapter 3, and we have

$$T^{-1} = \begin{bmatrix} 0 & 0 & -\frac{a_{23}}{\tau} \\ 0 & -\frac{a_{23}}{\tau} & 0 \\ \frac{-1}{\tau} & \frac{a_{22}}{\tau} & \frac{a_{21}}{\tau} \end{bmatrix} \tag{7.5}$$

Then the transformation matrix T is

$$T = \tau \begin{bmatrix} \frac{a_{21}}{a_{23}} & -\frac{a_{22}}{a_{23}} & -1 \\ 0 & \frac{-1}{a_{23}} & 0 \\ \frac{-1}{a_{23}} & 0 & 0 \end{bmatrix} \tag{7.6}$$

and $A_z = TAT^{-1}$ is

$$A_z = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{7.7}$$

We find that for the matrix F defined below

$$F = [0,1,-1] \tag{7.8}$$

det[F adj[sI-A] B] is Hurwitz for any uncertainty. Note FB = $1/\tau > 0$. Let the first row of the matrix P be

$$P_1 = [p_{11}, p_{12}, p_{13}] = FT^{-1} = [1/\tau,\ \ -a_{23}/\tau - a_{22}/\tau,\ \ -a_{21}/\tau] \tag{7.9}$$

Consider the Lyapunov equation

$$\hat{A}^T P + P\hat{A} + Q = 0 \tag{7.10}$$

where P,Q are symmetric positive definite and where $\hat{A}$ is

$$\hat{A} = \begin{bmatrix} -\hat{a}_1 & -\hat{a}_2 & -\hat{a}_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{7.11}$$

then let Q be diagonal. $q_{11} = 2p_{12}$, $q_{22} = 2p_{11}$ and $q_{33} = 2p_{13}$. Then

$$\begin{aligned} \hat{a}_1 &= -2(a_{22} + a_{23}) \\ \hat{a}_2 &= -2a_{21} - \frac{2}{a_{22} + a_{23}} \\ \hat{a}_3 &= 1 \end{aligned} \tag{7.12}$$

The minimum eigenvalue of the matrix Q, λ_1, is the minimum of q_{11}, q_{22}, and q_{33} . We have For the Lyapunov function V

$$V = z^T P z \tag{7.13}$$

The time derivative of V is

$$\dot{V} = -z^T Q z + 2[Fx]^T[\sigma T x + u] \tag{7.14}$$

Let $M = \max \|\sigma T\| = 75.2$. We consider the control u as

$$u = \begin{cases} -\dfrac{Fx}{|Fx|}p(x) & \text{if } |Fx| > \varepsilon \\ 0 & \text{if } |Fx| \leq \varepsilon \end{cases} \tag{7.15}$$

where ε is a small constant to reduce the chattering of the control u. For $|Fx| > \varepsilon$ Equation (7.14) becomes

$$\dot{V} \leq -\lambda_1 \|x\|^2 + 2M|Fx|\,\|x\| - 2|Fx|p(x) \tag{7.16}$$

Define the function p(x) as below

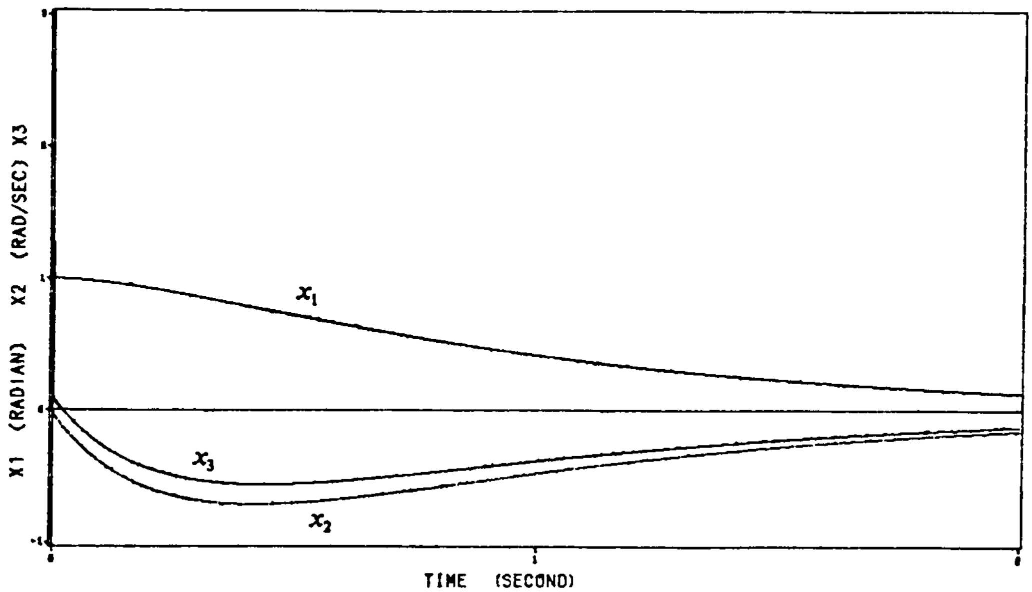
$$p(x) = M\|x\| - \lambda_1 \frac{\|x\|^2}{2|Fx|} \tag{7.17}$$

then $\dot{V}$ is smaller than zero, and the controller defined by (7.15) and (7.17) can stabilize the system. Since $\lambda_1 = 20$ and $M = 75.2$, we take

$$p(x) = 80\|x\| - 10\frac{\|x\|^2}{|Fx|} \tag{7.18}$$

Some of the numerical results are shown in the following figures. For Figure 1, $a_{21} = -4$. $a_{22} = 3.5$ initial condition $x^T = [1,\ 0,\ 0]$. For Figure 2, $a_{21} = -4$. $a_{22} = 3.5$ initial condition $x^T = [0,\ 1,\ 0]$.

(a)



(b)

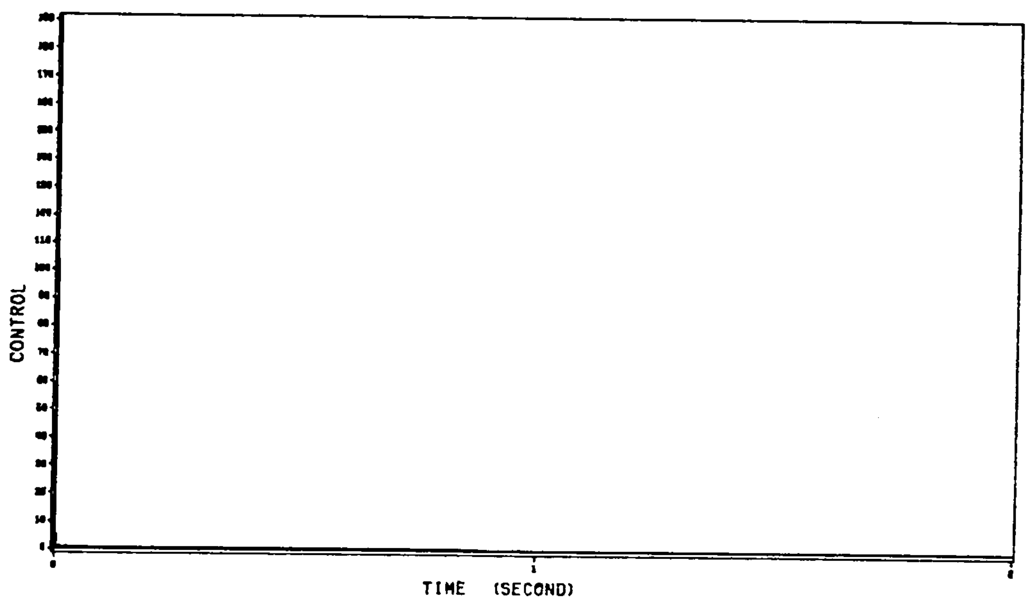


Figure 1 : Simulation of Example 1 with $a_{21} = -4$ $a_{22} = 3.5$ $x_0 = [1, 0, 0]^T$
(a) States Versus Time
(b) Control Input Versus Time

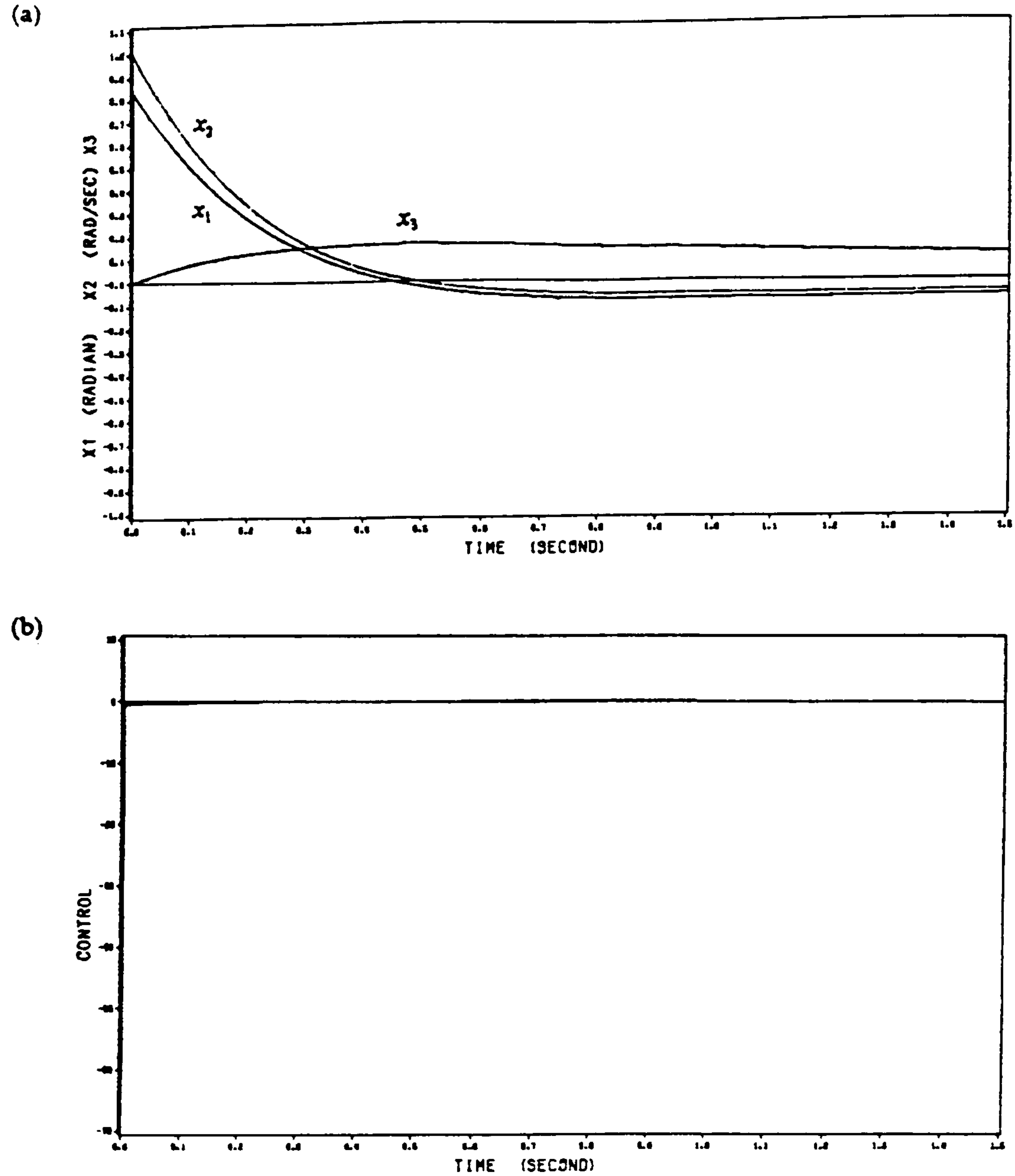


Figure 2 : Simulation of Example 1 with $a_{21} = -4$ $a_{22} = 3.5$ $x_0 = [0,\ 1,\ 0]^T$
(a) States Versus Time
(b) Control Input Versus Time

7.2 Example II : Control of a Pendulum

We consider the simple pendulum illustrated in Figure 3. This problem has been considered in [51],[37], and [67]. Here we consider the case in which the mass and the length are known only within given bounds. The governing equation of this system is

$$\ddot{\theta} = \frac{-g}{l} \sin \theta(t) + \frac{\hat{u}}{ml^2} \tag{7.19}$$

where $\theta(t)$ is the angle of the pendulum arm, g is the gravitational constant, $\hat{u}$ is the control. The length of the pendulum arm is l. The mass of the swinging object is m. Assume

$$m^- \leq m \leq m^+ \tag{7.20}$$

$$l^- \leq l \leq l^+ \tag{7.21}$$

Letting $x_1(t) = \dot{\theta}(t)$, $x_2(t) = \theta(t)$, equation (7.19) leads to the following state equation

$$\begin{aligned} \dot{x}_1 &= \frac{-g}{l} \sin x_2 + \frac{\hat{u}}{ml^2} \\ \dot{x}_2 &= x_1 \end{aligned} \tag{7.22}$$

Equation (7.22) can be represented as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [\sigma + \frac{\hat{u}}{ml^2}] \tag{7.23}$$

where

$$\sigma = a_1 x_1 + a_2 x_2 - \frac{g}{l} \sin x_2 \tag{7.24}$$

and where the coefficient a_1, a_2 are defined to positive so that the matrix A defined as

$$A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \tag{7.25}$$

is Hurwitz. Let $x = [x_1, x_2]^T$, and $B = [1,\ 0]^T$. Then consider the Lyapunov equation

$$A^T P + PA + Q = 0 \tag{7.26}$$

where P,Q are symmetric positive definite. For the Lyapunov function

$$V = x^T P x \tag{7.27}$$

the time derivative of V is

$$\dot{V} = -x^T Q x + 2[B^T P x]^T [\sigma + \frac{\hat{u}}{ml^2}] \tag{7.28}$$

We have

$$B^T P = [p_{11},\ p_{12}] = [f_1,\ f_2] \equiv F \tag{7.29}$$

To satisfy the n-1 stable conditions of $[p_{11}, p_{12}]$, we select $p_{11} = 1$, $p_{12} = 1$. We let the matrix F $= [f_1, f_2] = [1,1]$. Let Q be diagonal. Then the Lyapunov equation (7.26) has a solution for the matrix A. We have

$$a_1 = p_{12} - \frac{q_{11}}{2} \tag{7.30}$$

$$a_2 = \frac{q_{22}}{2p_{12}} \tag{7.31}$$

We select $q_{11} = q_{22} = p_{12} = 1$ then $a_1 = 0.5$, and $a_2 = 0.5$. Then let our controller be

$$\hat{u} = \begin{cases} -\dfrac{Fx}{|Fx|} p(x) & if\ |Fx| > \varepsilon \\ 0 & if\ |Fx| \leq \varepsilon \end{cases} \tag{7.32}$$

where ε is a very small constant to reduce the chattering of the control $\hat{u}$. For $|Fx| > \varepsilon$ Equation (7.28) becomes

$$\dot{V} = -\|x\|^2 + 2[Fx]\sigma - 2|Fx|\frac{\hat{u}}{ml^2} \tag{7.33}$$

Define the function p(x) as below

$$p(x) = M^+l^{+2}(0.5x_1 - 0.5x_2 + \frac{g}{l^-}|\sin(x_2)| - \frac{\|x\|^2}{2|Fx|}) \tag{7.34}$$

then $\dot{V}$ is less than zero, and the controller defined by (7.32) and (7.33) stabilize the system. In the simulations of this example, we let $M^+ = 20$, $M^- = 10$, $l^+ = 50$, and $l^- = 10$. We call the controller defined by Equation (7.34) Control (A). Since negative p(x) makes $\dot{V}$ less negative, we consider another controller called Control (B) which is defined by Equation (7.34) except that p(x) $= 0$ if the right hand side of Equation (7.34) is less that zero.

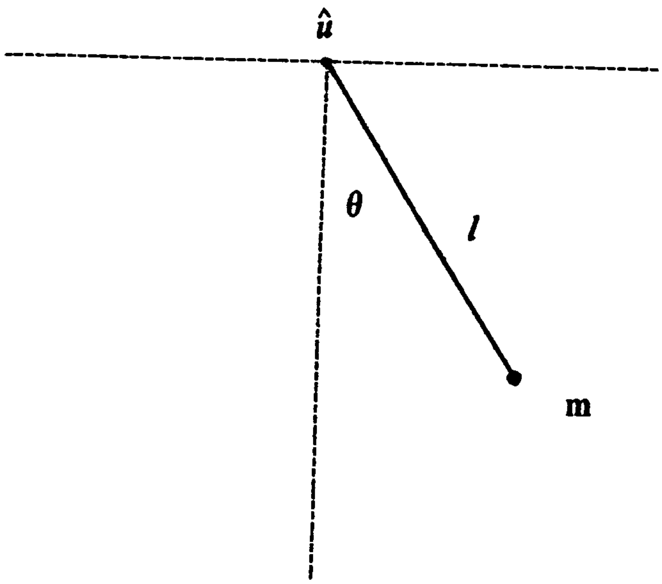


Figure 3 : The Pendulum considered in Example 2

(a)

(b)

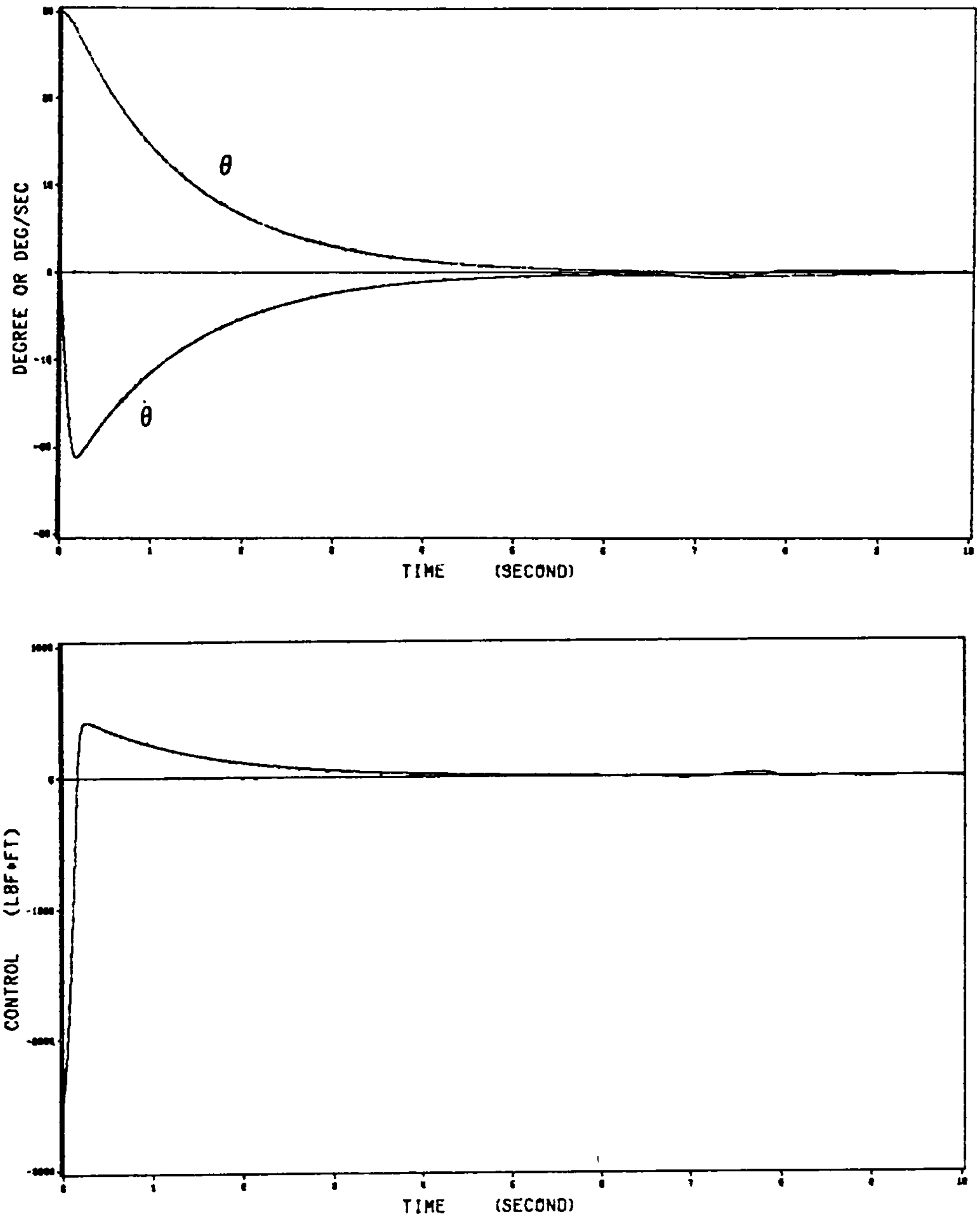


Figure 4 : Simulation of Pendulum with mass = 10. length = 50.
Initial rest at $\theta = 30°$. Under control (A).
(a) States Versus Time
(b) Control Input Versus Time

(a)

(b)

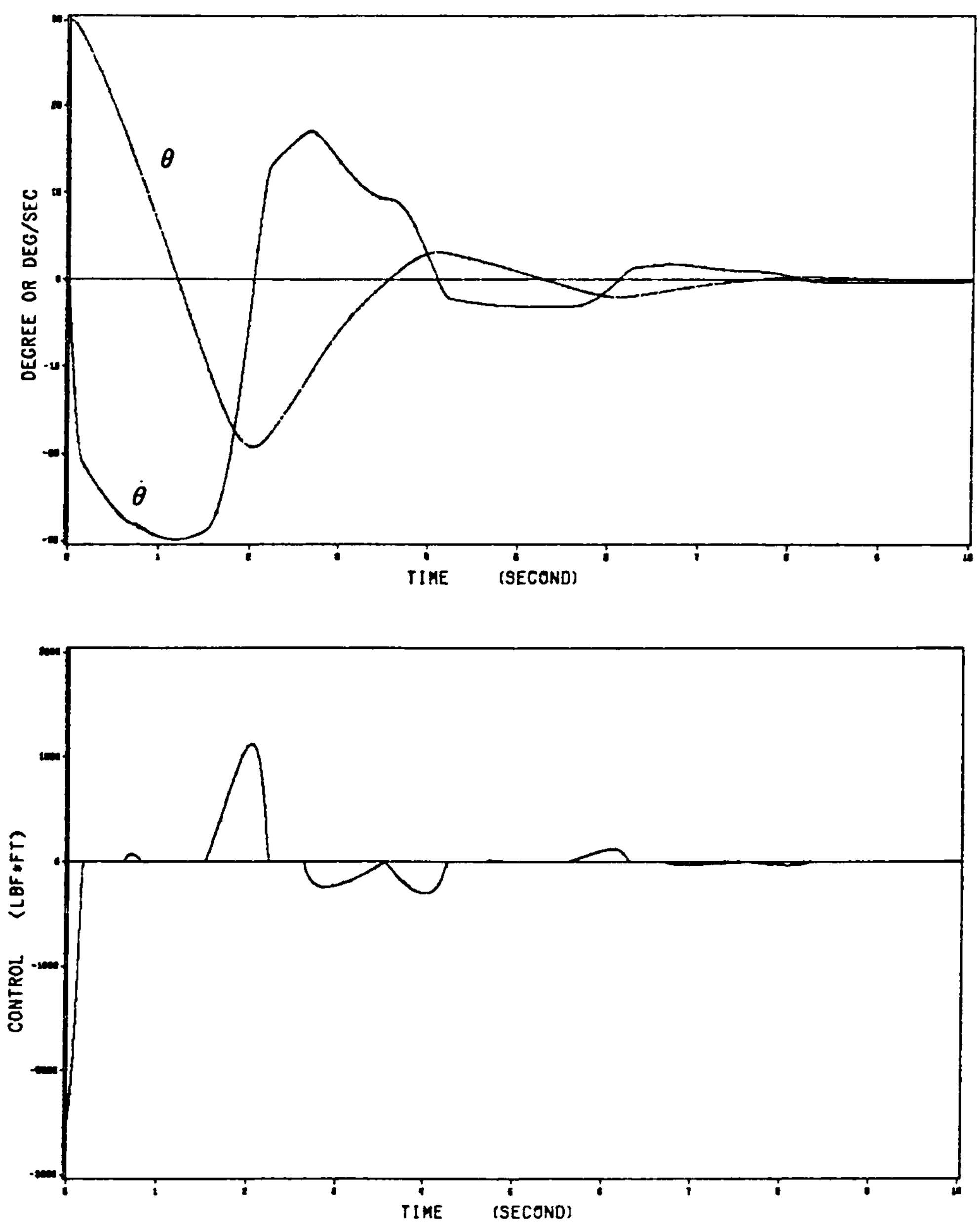


Figure 5 : Simulation of Pendulum with mass = 10. length = 50.
Initial rest at $\theta = 30°$. Under control (B).
(a) States Versus Time
(b) Control Input Versus Time

(a)

(b)

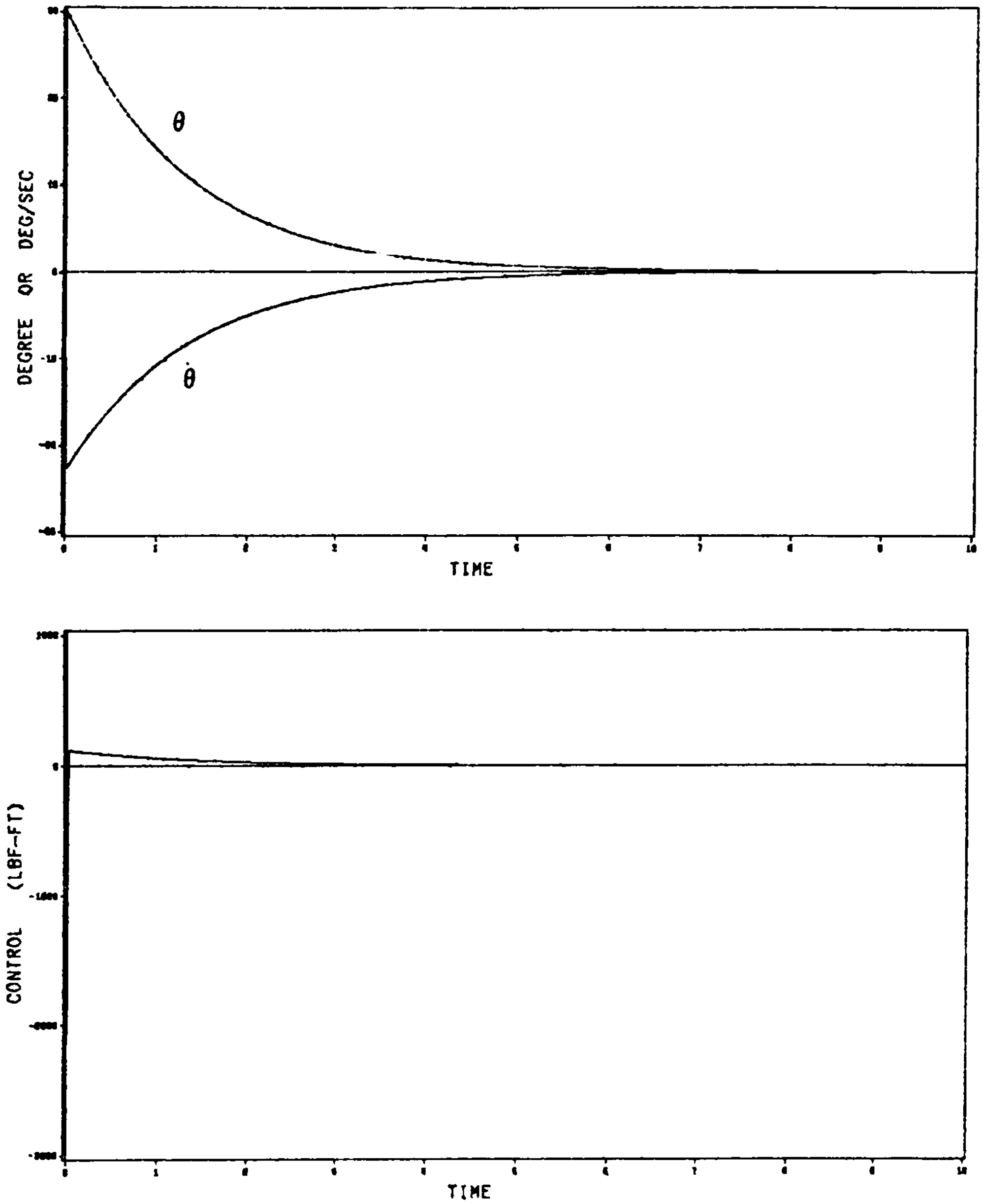
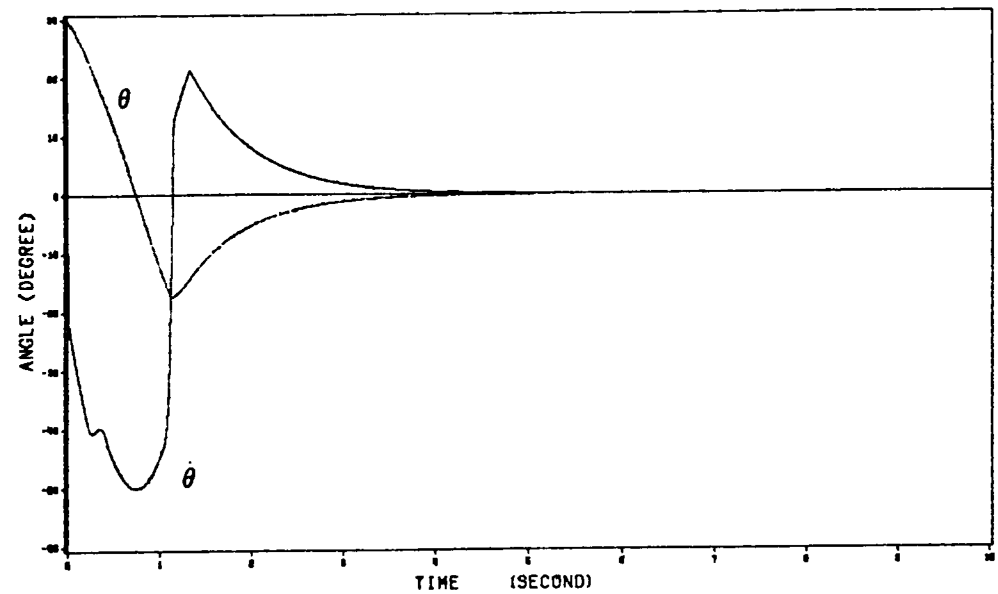


Figure 6 : Simulation of Pendulum with mass = 20. length = 10.
Initial rest at $\theta = 30°$. Under control (A).
(a) States Versus Time
(b) Control Input Versus Time

(a)



(b

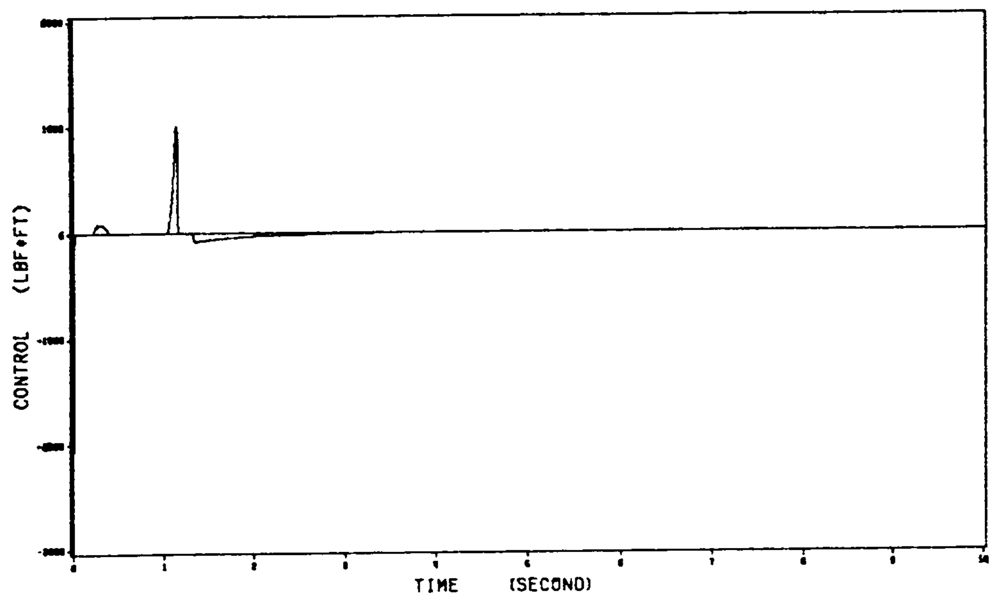


Figure 7 : Simulation of Pendulum with mass = 20. length = 10.
Initial rest at $\theta = 30$ degree. Under control (B).
(a) States Versus Time
(b) Control Input Versus Time

7.3 Example III : Two Input System

Consider the following plant

$$\dot{x} = \bar{A}x + \bar{B}u \tag{7.35}$$

where

$$\bar{A} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix} \tag{7.36}$$

$$\bar{B} = \begin{bmatrix} 0 & 0 \\ c_1 & 0 \\ 0 & c_2 \end{bmatrix} \tag{7.37}$$

where the uncertainties have the following bounds

$a_{11} \in [-1,1]$, $a_{12} \in [1,2]$, $a_{21} \in [10,15]$, $a_{31} \in [-20,-10]$, $a_{23} \in [-6,-5]$, $a_{33} \in [10,20]$,

and $c_1 \in [1,2]$, $c_2 \in [2,10]$. All the uncertainties are independent. For these uncertainties, the open loop system is always unstable. Let b_1 and b_2 be the first and second column of the matrix $\bar{B}$. Then we have

$$\bar{A}^2 b_1 = \alpha_{110} b_1 + \alpha_{120} b_2 + \alpha_{111} \bar{A} b_1 \tag{7.38}$$

$$\bar{A} b_2 = \alpha_{210} b_1 + \alpha_{220} b_2 + \alpha_{211} \bar{A} b_1 \tag{7.39}$$

where $\alpha_{110} = a_{12}a_{21}$, $\alpha_{120} = \dfrac{a_{12}a_{31}c_1}{c_2}$, $\alpha_{111} = a_{11}$, $\alpha_{210} = \dfrac{a_{23}c_2}{c_1}$, $\alpha_{220} = a_{33}$, $\alpha_{211} = 0$.

We have the Kronecker invariants $k_1 = 2$, $k_2 = 1$. Define the following matrices (see Appendix B for details)

$$v_{11} = b_1 \tag{7.40}$$

$$v_{20} = b_2 - \alpha_{211} b_1 \tag{7.41}$$

$$v_{10} = \bar{A} v_{11} - \alpha_{111} b_1 \tag{7.42}$$

Then the coordinate transformation matrix T is

$$T = [v_{11},\ v_{10},\ v_{20}]^{-1} = \begin{bmatrix} \frac{a_{11}}{a_{12}c_1} & \frac{1}{c_1} & 0 \\ \frac{1}{a_{12}c_1} & 0 & 0 \\ 0 & 0 & \frac{1}{c_2} \end{bmatrix} \tag{7.43}$$

The transformed matrix $B_z = T\bar{B}$ is

$$B_z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.44}$$

We have $B_z = I_2$. Applying Popov's Theorem, the transformed matrix $A_z = T\bar{A}T^{-1}$ has the property

$$B^T[sI - A_z]W = R(s) = \begin{bmatrix} s^2 - \alpha_{111}s - \alpha_{110} & -\alpha_{210} \\ -\alpha_{120} & s - \alpha_{220} \end{bmatrix} \tag{7.45}$$

Let A_z be

$$A_z = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13} \\ 1 & 0 & 0 \\ \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} \end{bmatrix} \tag{7.46}$$

The left hand side of (7.45) is

$$B^T[sI - A_z]W = \begin{bmatrix} s^2 - \hat{a}_{11}s - \hat{a}_{12} & -\hat{a}_{13} \\ -\hat{a}_{31}s - \hat{a}_{32} & s - \hat{a}_{33} \end{bmatrix} \tag{7.47}$$

compare (7.47) and (7.45) we have

$$\begin{aligned}
\hat{a}_{11} &= \alpha_{111} \\
\hat{a}_{12} &= \alpha_{110} \\
\hat{a}_{13} &= \alpha_{210} \\
\hat{a}_{31} &= 0 \\
\hat{a}_{32} &= \alpha_{120} \\
\hat{a}_{33} &= \alpha_{220}
\end{aligned} \tag{7.48}$$

Next we want to find the matrix F, such that det[F adj(sI-$\overline{A}$)$\overline{B}$] is Hurwitz.

$$\det[\mathrm{F}\ adj(sI-\overline{A})\overline{B}] = \det\begin{vmatrix} a_{11}-s & a_{12} & 0 & 0 & 0 \\ a_{21} & -s & a_{23} & c_1 & 0 \\ a_{31} & 0 & a_{33}-s & 0 & c_2 \\ f_{11} & f_{12} & f_{13} & 0 & 0 \\ f_{21} & f_{22} & f_{33} & 0 & 0 \end{vmatrix} \tag{7.49}$$

$$= c_1 \times c_2 \times \det\begin{vmatrix} a_{11}-s & a_{12} & 0 \\ f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{33} \end{vmatrix} \tag{7.50}$$

Then for the matrix F below

$$F = \begin{bmatrix} 4 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \tag{7.51}$$

the equation (7.50) is Hurwitz. Then let the positive definite symmetric matrix D be

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \tag{7.52}$$

where d_1 and d_2 are positive. Then define the matrix Π as

$$\Pi = B_s^T D[FT^{-1}B]^{-1} = \begin{bmatrix} \frac{d_1}{c_1} & -\frac{d_1}{c_1} \\ 0 & \frac{d_2}{c_2} \end{bmatrix} \tag{7.53}$$

We select $d_1 = c_1$ and $d_2 = c_2$. Then

$$\Pi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \tag{7.54}$$

Define the matrix C as

$$C = \begin{bmatrix} 1.5 & 1.5 \\ 0 & 1.5 \end{bmatrix} \tag{7.55}$$

Then

$$\tilde{\Pi} = \Pi C = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \tag{7.56}$$

$$\tilde{E} \equiv \tilde{\Pi} - I = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \tag{7.57}$$

We have $\|\tilde{E}\| = 0.5$. Then the matrix P is

$$B^T P = \Pi F T^{-1} = \begin{bmatrix} c_1 & (5a_{12} - a_{11})c_1 & 0 \\ 0 & -a_{12}c_1 & c_2 \end{bmatrix} \tag{7.58}$$

Thus, we have assigned two rows of the matrix P. Then consider the solution of the Lyapunov equation

$$P\tilde{A} + \tilde{A}^T P + Q = 0 \tag{7.59}$$

Let Q be diagonal, and

$$q_{11} = 2c_1 a_{12} \tag{7.60}$$

$$q_{22} = 2c_1^2 a_{12}^2(4a_{12} - a_{11})/c_2 \tag{7.61}$$

$$q_{33} = 2c_2 a_{12} \tag{7.62}$$

Then one of the solution of the matrix $\tilde{A}$ is

$$\tilde{A} = \begin{bmatrix} (-6a_{12} + a_{11}) & \dfrac{(-2c_1 a_{12}^2)}{c_2} & a_{12} \\ 1 & 0 & 0 \\ 0 & -\dfrac{c_1}{c_2} a_{12}(6a_{12} - a_{11}) & -a_{12} \end{bmatrix} \tag{7.63}$$

Now

$$\sigma T = B^T[A_z - \tilde{A}]T = \begin{bmatrix} -\dfrac{(6a_{11} - a_{21})}{c_1} + \dfrac{2a_{12}}{c_2} & \dfrac{-6a_{12}}{c_1} & \dfrac{a_{23}}{c_1} - \dfrac{a_{12}}{c_2} \\ \dfrac{(a_{31} + 6a_{12} - a_{11})}{c_2} & 0 & \dfrac{(a_{33} + a_{12})}{c_2} \end{bmatrix} \tag{7.64}$$

Then we have

$$M = \sup_{\gamma} \|\sigma T\| = 27.1 \tag{7.65}$$

The sliding surface $\tilde{F}$ is

$$\tilde{F} = C^{-1}F = \begin{bmatrix} \dfrac{10}{3} & \dfrac{1}{1.5} & 0 \\ \dfrac{-1}{1.5} & 0 & \dfrac{1}{1.5} \end{bmatrix} \tag{7.66}$$

For the Lyapunov function $V = z^T P z$, its time derivative becomes

$$\begin{aligned} \dot{V} &= -z^T Q(\gamma) z + 2[\tilde{\Pi}\tilde{F}x]^T[\sigma z + u] \\ &= -z^T Q(\gamma) z + 2[\tilde{F}x]^T \tilde{\Pi}^T(\gamma)[\sigma z + u] \\ &= -z^T Q(\gamma) z + 2[\tilde{F}x]^T \tilde{\Pi}^T(\gamma)\sigma T(\gamma) x + 2[\tilde{F}x]^T u + 2[\tilde{F}x]^T \tilde{E}^T(\gamma) u \end{aligned} \tag{7.67}$$

We have $\|\tilde{\Pi}\| = 1.5$ and $\|\tilde{E}\| = 0.5$. Let the control u be

$$u(x) = \begin{cases} -\dfrac{\tilde{F}x}{\|\tilde{F}x\|}p(x) & if\ \|\tilde{F}x\| > \varepsilon \\ 0 & if\ \|\tilde{F}x\| \le \varepsilon \end{cases} \tag{7.68}$$

then by the equation (7.67), for the case $\|Fx\| > \varepsilon$, we have

$$\begin{aligned} \dot{V} &\le -z^T Q(y)z + 2\|\tilde{F}x\| 1.5 \times M \times \|x\| - 2\|\tilde{F}x\|(1 - \|\tilde{E}\|)p(x) \\ &= -z^T Q(y)z + 2\|\tilde{F}x\|(1.5M\|x\| - 0.5p(x)) \end{aligned} \tag{7.69}$$

The minimum eigenvalue of the matrix Q, λ_1, is $\min[q_{11}, q_{22}, q_{33}] = 0.6$. We have

$$\|z^T Qz\| = \|x^T(T^T QT)x\| \ge \lambda_1 \|x\|^2 \tag{7.70}$$

Here we use the fact that the eigenvalues of the matrix Q are not affected by the coordinate transformation matrix T. Then (7.69) becomes

$$\dot{V} \le -\lambda_1 \|x\|^2 + 2\|\tilde{F}x\|(1.5M\|x\| - 0.5p(x)) \tag{7.71}$$

Define pp(x) as below

$$pp(x) = 3M\|x\| - \frac{\lambda_1 \|x\|^2}{\|\tilde{F}x\|} \tag{7.72}$$

Then p(x) is defined as

$$\begin{aligned} p(x) &> pp(x) && \text{if } pp(x) \ge 0 \\ &> pp(x)/3. && \text{if } pp(x) < 0 \end{aligned} \tag{7.73}$$

then $\dot{V}$ is less than zero. Since P is positive definite, the control defined by (7.68) (7.73) stabilizes the system. In this example, we select pp(x) as

$$pp(x) = 82\|x\| - \frac{0.6\|x\|^2}{\|\tilde{F}x\|} \tag{7.74}$$

For Figure 8, $a_{11} = -1$, $a_{12} = 1$, $a_{21} = 10$, $a_{31} = -20$, $a_{23} = -6$, $a_{33} = 10$, $c_1 = 1$, $c_2 = 10$, initial states $x_0^T = [1, 0, 0]$.

For Figure 9, $a_{11} = 1$, $a_{12} = 1$, $a_{21} = 15$, $a_{31} = -20$, $a_{23} = -6$, $a_{33} = 20$, $c_1 = 2$, $c_2 = 8$, initial states $x_0^T = [1, 0, 0]$.

(a)

(b)

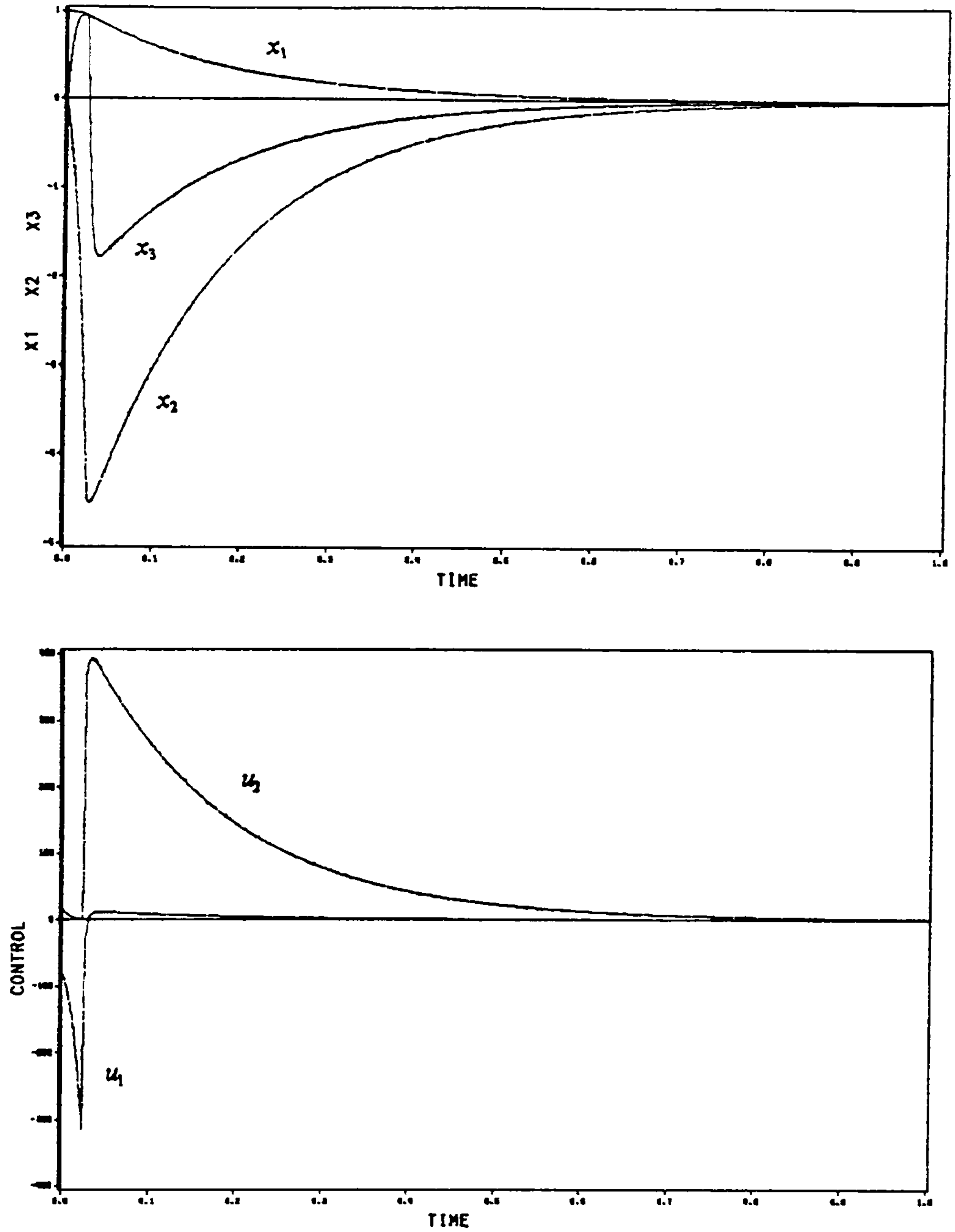
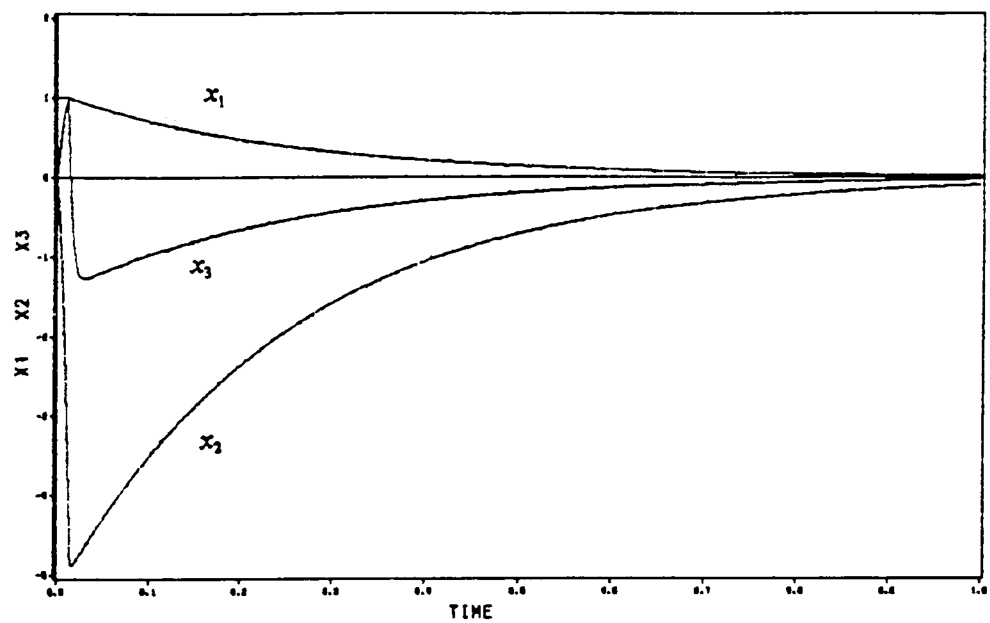


Figure 8 : Simulation of Example III with $a_{11} = -1$,
$a_{23} = -6$ $a_{33} = 10$ $c_1 = 1$ $c_2 = 10$
(a) States Versus Time
(b) Control Inputs Versus Time

(a)



(b)

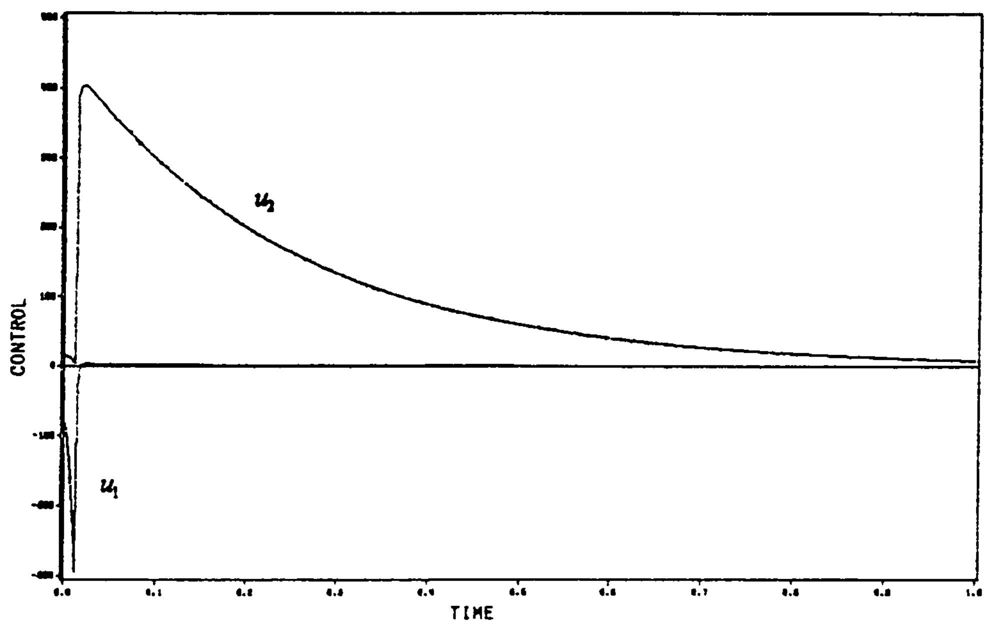


Figure 9 : Simulation of Example III with $a_{11} = 1$,
$a_{23} = -6$ $a_{33} = 20$ $c_1 = 2$ $c_2 = 8$
(a) States Versus Time
(b) Control Inputs Versus Time

Chapter 8 Summary and Discussion

8.1 Summary

In this dissertation, we present a theory on designing robust stabilizing controllers for linear systems with time-invariant uncertainties. The uncertainties are structured and only their bounds are used, no statistical information about the uncertainties is required. The uncertainties do not have to satisfy the matching conditions. One condition for the existence of the robust stabilizing controller presented is that there exists a constant matrix F such that y = Fx has minimum phase property. Then, the theorem of the necessary and sufficient conditions on the m rows of the matrix P which satisfies the Lyapunov equation $A^T P + PA + Q = 0$ assures that from the m rows of the matrix P and given matrix Q > 0, one can construct a positive definite matrix P and the matrix A. This approach is used to construct the uncertain matrix P. Then from the time derivative of the Lyapunov function $V = z^T P(\gamma) z$ we can design robust stabilizing controllers which are either linear or nonlinear.

8.1 Discussion and Further Research

In the design procedure, it is required that a constant matrix F exists such that $\det[FT^{-1}(\gamma)W]$ is Hurwitz. This is equivalent to the condition that $\det[\ [F\,\mathrm{adj}[sI - \overline{A}(\gamma)]\overline{B}(\gamma)]\ ]$ be Hurwitz. It is of interest to investigate the application of Kharitonov like theorems [40] to the existence of such matrix F. In the examples given, we find that the magnitude of the control is always very large in a very short period of time. Then the control decreases very quickly. In Example 2, we show the application of the design approach given in this dissertation to a nonlinear system. The extension of the design procedure to nonlinear time-varying systems is of main concern in further research. Hence the transformations of nonlinear systems to companion form such as [87] are under investigation.

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Appendix A. Unique Solution of the Lyapunov Equation

Consider Lyapunov equation (3.20) where A is in the companion form (2.11), Q is symmetric positive definite and P is symmetric. We assume that the first row P_1 of P is a given n-1 stable vector as definition 2 in chapter 3. The quantities P_1 and Q are given. The vector **a** and the variables P_{ij}, i = 2,...,n, j≥i are the unknowns.

Equation (3.20) can be written as

$$p_{1j}a_j + p_{ij}a_l - p_{l,j+1} - p_{j,l+1} = q_{ij} \qquad (A.1.a)$$

where $i \leq i \leq n$, $1 \leq j \leq n$ and where

$$p_{r,s} = 0 \qquad if\ \ r > n\ \ or\ s > n \qquad (A.1.b)$$

Equation (A.1) represents n(n + 1)/2 equations and unknowns. A property of Equation (A.1) is that the unknowns p_{ij} appear in one or two equations only. Our solution procedure is the following :

(i) Eliminate equations containing unknowns P_{ij} which appear in only one equation. For example, p_{ll}, l = 2,3,...,n, appear in only one equation each, so we can eliminate the equations

$$p_{1l}a_{l-1} + p_1 l + 1a_l - p_{l-1,l+1} - p_{l,l} = q_{l-1,l} \qquad (A.2)$$

where $2 \leq l \leq n$. After this elimination we have $n(n+1)/2-(n-1)$ remaining equations and unknowns in equation (A.1). Once these unknowns are obtained then they can be substituted into equation (A.2) to yield unique solutions for $p_{l,l}$

(ii) In the remaining $n(n+1)/2-(n-1)$ equation, the unknowns $p_{l-1,l+1}$, $3 \leq l \leq n-1$, appear once only. Since they appear only once in the remaining equations we eliminate them from equation (A.1). That is, the following equations are eliminated :

$$p_{1,l-2}a_{l+1} + p_1 l + 1a_{l-2} - p_{l-2,l+2} - p_{l-1,l+1} = q_{l-2,l+1} \qquad (A.3)$$

where $3 \leq l \leq n-1$.

(iii) After the eliminations established by steps (i) and (ii) are completed we can recursively eliminate the equations containing $p_{l-i,l+i}$, $2 \leq l \leq n$, $1 < i < n-1$. The number of equations and unknowns remaining after this step is given by

$$n\frac{(n+1)}{2} - (n-1) - (n-3) - (n-5) - \cdots \qquad (A.4)$$

(iv) Consider the equations in equation (A.1) defined by $i = j = 1$, $1 \leq l \leq n$. These are given by

$$p_{1l}a_l - p_{l,l+1} = \frac{q_{ll}}{2} \qquad (A.5)$$

or, equivalently, in matrix form

$$\begin{bmatrix} p_{11} & 0 & 0 & \cdots & \cdot & 0 \\ 0 & p_{12} & 0 & \cdots & \cdot & \cdot \\ 0 & 0 & p_{13} & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & p_{1,n-1} & \cdot \\ 0 & 0 & \cdots & \cdot & \cdot & p_{1n} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ a_{n-1} \\ a_n \end{bmatrix} - \begin{bmatrix} 0 \\ p_{23} \\ p_{34} \\ \cdot \\ \cdot \\ p_{n-1,n} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{q_{11}}{2} + p_{12} \\ \frac{q_{22}}{2} \\ \frac{q_{33}}{2} \\ \cdot \\ \cdot \\ \frac{q_{n-1,n-1}}{2} \\ \frac{q_{nn}}{2} \end{bmatrix} \qquad (A.6)$$

The unknowns $p_{l,l+1}$, l≤2 in equation (A.5) and (A.6) each appear in another equation of (A.1) which has not been eliminated in the steps (i)-(iii). Consider the equations defined by i = l+1, j=l-1 :

$$p_{1,l+1}a_{l-1} + p_{1,l-1}a_{l+1} - p_{l,l+1} - p_{l-1,l+2} = q_{l-1,l+1} \qquad (A.7)$$

where we have used that P is symmetric. Solving for $p_{l,l+1}$ yields :

$$p_{l,l+1} = p_{1,l+1}a_{l-1} + p_{1,l-1}a_{l+1} - p_{l-1,l+2} - q_{l-1,l+1} \qquad (A.8)$$

where 2≤l≤n, a_0 = 1 and $p_{r,s} = q_{r,s} = 0$ if either r or s≤0 or > n. Substitution of (A.8) into (A.5) gives

$$-p_{1,l+1}a_{l-1} + p_{1l}a_l - p_{1,l-1}a_{l+1} + p_{l-1,l+2} = \frac{q_{ll}}{2} - q_{l-1,l+1} \qquad (A.9)$$

In this step we have eliminated n-2 equations

(v) Equation (A.9) contains the unknown $p_{l-1,l+2}$ which we eliminate in this step. The remaining equations of (A.1) containing $p_{l-1,l+2}$ can be written as

$$p_{l-1,l+2} = p_{1,l+2}a_{l-2} + p_{1,l-2}a_{l+2} - p_{l-2,l+3} - q_{l-2,l+2} \qquad (A.10)$$

where 4≤l≤n; a_0 = 1, a_s = 0 if s < 0; and $p_{r,s} = q_{r,s} = 0$ if r,s > n or < 1.

Substitution of (A.10) into (A.9) yields

$$p_{1,l+2}a_{l-2} - p_{1,l+1}a_{l-1} + p_{1l}a_l - p_{1,l-1}a_{l+1} + p_{1,l-2}a_{l+2} - p_{l-2,l+3}$$

$$= \frac{q_{ll}}{2} - q_{l-1,l+1} + q_{l-2,l+2} \qquad (A.11)$$

In this step we have eliminated n-4 equations. (vi) Continuing the recursive process as described by the steps (iv) and (v) eliminates the unknowns $p_{l-i,l+i}$ in the final set of equations. Finally we reduce the original set of equations (A.1) to n equations which do not contain any unknowns $p_{i,j}$, i=2,3,...,n and j≥n.

We obtain the following set of equations :

$$p_{11}a_1 = \frac{q_{11}}{2} + p_{12} \tag{A.12.a}$$

$$p_{1l}a_l + \sum_{i=1}^{m}(-1)^l[p_{1,l+i}a_{l-i} + p_{1,l-i}a_{l+i}] = \frac{q_{ll}}{2} + \sum_{i=1}^{m}(-1)^l q_{l-i,l+i} \tag{A.12.b}$$

$$p_{1n}a_n = \frac{q_{nn}}{2} \tag{A.12.c}$$

where $2 \leq l \leq n-1$, $m = \min(l-1, l-n)$; $a_0 = 1$, $a_s = 0$ if $s < 0$ or $s > n$; and $p_{r,s} = q_{r,s} = 0$ if $r,s > n$ or < 1. In matrix form Eq. (A.12) is represented by Eqs. (3.29) and (3.30).

Equation (3.29) contains only unknown vector **a** which has a unique solution $p_{i,j}$ which are eliminated in the steps (i) - (vi) can be computed uniquely using the equations that eliminated them. For example, from (A.1) we have

$$p_{i+1,j} = p_{1i}a_j + p_{ij,l} - p_{i,j+1} - q_i \tag{A.13.a}$$

For $i = 1$, $2 \leq j \leq n$ we can obtain $p_{2,j}$. Then for $i = 2,3 \leq j \leq n$ we can compute $p_{3,j}$. That is, the unknowns $p_{i,j}$, $i > 2$, $i \leq k \leq n$ are computes recursively from (A.13).

Equation (A.12) resulted from the elimination of equations in steps (iv)-(vi). The number eliminated is the sum

$$(n-2) + (n-4) + (n-6) + \ldots$$

The number eliminated in steps (i)-(iii) are

$$(n-1) + (n-3) + (n-5) + \cdots$$

so that the total number eliminated is the sum

$$\sum_{i=1}^{n-1} i = n\frac{(n-1)}{2}$$

The equations remaining after the elimination process is of course n. The above solution process proves that Lyapunov Equation (3.20) has a unique solution under the assumptions placed on A, Q and P provided the matrix H has an inverse, which is established in the proof of Theorem 3.1.

Appendix B. Transformation

B.1 The Properties of Popov Transformation

This section is quoted from Popov's paper [78]. Popov's paper has three misprints, they are corrected here.

Consider a plant

$$\dot{x}(t) = \overline{A}x(t) + \overline{B}u(t) \tag{Σ}$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control, $\overline{A} \in \mathbb{R}^{n \times n}$ is an n×n matrix, $\overline{B} \in \mathbb{R}^{n \times m}$ is an n×m input matrix.

Assumptions

(1) $\quad \operatorname{rank}[\overline{B}, \overline{A}\,\overline{B}, \ldots, \overline{A}^{n-1}\overline{B}] = n$

(2) $\quad \operatorname{rank}\overline{B} = m$

Assumption (2) implies that $m \le n$.

Consider the controllability matrix

$$U = [\overline{B}, \overline{A}\,\overline{B}, ..., \overline{A}^{n-1}\overline{B}] \qquad (B.14)$$

more explicitly :

$$U = [b_1, b_2, ..., b_m, \overline{A}\,b_1, \overline{A}\,b_2, ..., \overline{A}\,b_m, ..., \overline{A}^{n-1}b_1, \overline{A}^{n-1}b_2, ..., \overline{A}^{n-1}b_m] \qquad (B.15)$$

where b_i is the i'th element of matrix $\overline{B}$.

Definition 1. The ith **Kronecker invariant** , k_i is defined as the smallest positive integer such that the vector $\overline{A}^{k_i}b_i$ is a linear combination of the column vectors situated before it.

Proposition 1. The Kronecker invariants from Definition 1 satisfy the relation

$$k_1 + k_2 + \cdots + k_m \;=\; n \qquad (B.16)$$

Corollary 1. There exists exactly one set of ordered numbers $\alpha_{ijk} \in \mathbb{R}$, defined for i = 1,2,..., m, j = 1,2,..,i-1, k = 0,1,...,min(k_i , $k_j - 1$) and for i = 1,2,...,m, j = i,i + 1,...,m, k = 0,1,...,min(k_i , k_j)-1 such that, for every i = 1,2,...,m, one has

$$\overline{A}^{k_i}b_i = \sum_{j=1}^{i-1}\ \sum_{k=0}^{\min(k_i,\,k_j-1)} \alpha_{ijk}\overline{A}^k b_j + \sum_{j=i}^{m}\ \sum_{k=0}^{\min(k_i,\,k_j)-1} \alpha_{ijk}\overline{A}^k b_j \qquad (B.17)$$

Proposition 2. The numbers k_i and α_{ijk} (Definition 1, Corollary 1) remain unchanged if $(\overline{A}, \overline{B})$ is replaced by $(\tilde{A}, \tilde{B}) = (T\overline{A}\,T^{-1}, T\overline{B})$,(det $T \neq 0$).

Theorem 1 Let Assumption (1) and (2) be satisfied for the pair of matrices $(\overline{A}, \overline{B})$. Let k_i, i = 1,2,...,m, be some positive integers satisfying Equation (B.16) . Let $\alpha_{ijk} \in \mathbb{R}$ be some numbers, defined for i = 1,2,...,m, j = 1,2,...,i-1, k = 0,1,...,min(k_i , $k_j - 1$) and for i = 1,2,...,m, j = i,i + 1,...,m, k = 0,1,...,min(k_i , k_j)-1. Then the following properties of the above numbers k_i and α_{ijk} are equivalent :

(i) There exists a set of vectors $v_{ij} \in \mathbb{R}^n$, i = 1,2,...,m, j = 0,1,...,n-1, such that the matrix

$$M = (v_{10}\; v_{11}\; \cdots\; v_{1,k_1-1}\; v_{20}\; \cdots\; v_{2,k_2-1}\; \cdots\; v_{m0}\; \cdots\; v_{m,k_m-1}) \qquad (B.18)$$

is nonsingular and the following equations are satisfied :

$$\bar{A} v_{i0} = \sum_{j=1}^{m} \alpha_{ij0} b_j, \tag{B.19}$$

$$v_{i,k-1} - \bar{A} v_{ik} = - \sum_{\substack{j=1 \\ k_j > k}}^{m} \alpha_{ijk} b_j, \tag{B.20}$$

$$v_{i,k_i-1} = - \sum_{\substack{j=1 \\ k_j > k_i}}^{i-1} \alpha_{ijk_i} b_j + b_i \tag{B.21}$$

where k $=$ $1,2,\ldots,k_i-1$, $i=1,2,\ldots,m$. The symbol $\sum\limits_{\substack{j=1 \\ k_j > k}}^{m}$ from (B.20) means that the summation involves only those integers $j \in \{1,\ldots,m\}$ for which $k_i > k$; the corresponding symbol from (B.21) has a similar meaning.

(ii) There exists a polynomial matrix

$$S(\sigma) = (v_1(\sigma)\, v_2(\sigma)\, \ldots\, v_m(\sigma)) \tag{B.22}$$

whose columns $v_i(\sigma)$ have the form

$$v_i(\sigma) = v_{i0} + v_{i1}\sigma + \cdots + v_{i,k_i-1}\sigma^{k_i-1} \tag{B.23}$$

where the coefficients $v_{ik} \in \mathbb{R}^n$ have the property

$$\det(v_{10}\, v_{11}\, \ldots\, v_{1,k_1-1}\, v_{20}\, \ldots\, v_{2,k_2-1}\, \ldots\, v_{m0}\, \ldots\, v_{m,k_m-1}) \neq 0 \tag{B.24}$$

such that

$$(\sigma I - \bar{A}) S(\sigma) = \bar{B} R(\sigma) \tag{B.25}$$

where I is the identity matrix in $\mathbb{R}^{n \times n}$ and $R(\sigma)$ is a polynomial matrix whose entries $(R(\sigma))_{ji}$ have the expressions

$$(R(\sigma))_{ii} = -\sum_{k=0}^{k_i-1} \alpha_{iik}\sigma^k + \sigma^{k_i}, \qquad i = 1,2,\ldots,m \tag{B.26}$$

$$(R(\sigma))_{ji} = -\sum_{k=0}^{\min(k_i,\, k_j-1)} \alpha_{ijk}\sigma^k \qquad i = 2,3,\ldots m, j < i \tag{B.27}$$

$$(R(\sigma))_{ji} = -\sum_{k=0}^{\min(k_i,\, k_j)-1} \alpha_{ijk}\sigma^k \qquad i = 1,2,\ldots m-1, j > i \tag{B.28}$$

B.2 Derivation of the Transformation used in this Dissertation

Consider the plant (Σ). Define a coordinate transformation matrix T such that

$$z = Tx \tag{B.29}$$

the inverse of T is defined as follows :

$$T^{-1} = (v_{1,k_1-1}\ v_{1,k_1-2}\ \cdots\ v_{10}\ v_{2,k_2-1}\ v_{2,k_2-2}\ \cdots\ v_{20}\ \cdots\ v_{m,k_m-1}\ \cdots\ v_{m0}) \tag{B.30}$$

Then (Σ) can be transformed to

$$\dot{z} = A_z z + B_z u \tag{A}$$

where $A_z = T\overline{A}T^{-1}$, $B_z = T\overline{B}$. Define the matrix T to be composed of row vectors $t_{i,j}$ as

$$T^T = (t^T_{1,k_1-1}\ t^T_{1,k_1-2}\ \cdots\ t^T_{10}\ t^T_{2,k_2-1}\ t^T_{2,k_2-2}\ \cdots\ t^T_{20}\ \cdots\ t^T_{m,k_m-1}\ \cdots\ t^T_{m0}) \tag{B.31}$$

That is

$$t_{i,j}v_{k,l} = \begin{cases} 1 & if\ i=k,\ j=l \\ 0 & otheres \end{cases} \tag{B.32}$$

Define the function δ

$$\delta(y) = \begin{cases} 1 & if\ y>0 \\ 0 & if\ y \le 0 \end{cases} \tag{B.33}$$

We investigate the properties of $t_{i,j}v_{k,k_k}$, k = 1,...,m, and where j $\neq k_i - 1$ for i = 1,2,...,m. Then applying (B.21) we have :

$$t_{i,j}v_{1,k_1-1} = t_{i,j}b_1 = 0 \tag{B.34a}$$

$$t_{i,j}v_{2,k_2-1} = t_{i,j}(-\alpha_{21k_2}\delta(k_2-k_1)b_1 + b_2) = 0 \tag{B.34b}$$

.

.

$$t_{i,j}v_{m,k_m-1} = t_{i,j}(b_m - \sum_{j=1}^{m-1}\alpha_{mjk_m}k_j - k_m b_j) \tag{B.34c}$$

where j$\neq k_k - 1$ for k = 1,2,...,m. From (B.34) we have

$$t_{i,j}b_k = 0 \qquad k = 1,2,...,m \tag{B.35}$$

where j$\neq k_k - 1$ for k = 1,2,...,m. Consider the matrix T^{-1}. For simplicity we discuss the columns $v_{p,k}$, k = 0,1,..$k_p - 1$ in T^{-1} . From equation (B.20) we have

$$\bar{A}v_{ik} = v_{i,k-1} + \sum_{\substack{j=1 \\ k_j > k}}^{m}\alpha_{ijk}b_j \tag{B.36}$$

Applying (B.36) recursively for $k = k_p - 1$ to $k = 1$, we have

$$\overline{A} v_{p,k} = v_{p,k-1} + \sum_{j=1}^{m} \alpha_{pjk} \delta(k_j - k) b_j \qquad (B.37)$$

where k = $1,2,\ldots,k_p - 1$. then one of the element of A_z is

$$t_{i,j} \overline{A} v_{p,k} = t_{i,j} v_{p,k-1} = \begin{cases} 1 & \textit{if } i = p \text{ and } j = k-1 \\ 0 & \textit{others} \end{cases} \qquad (B.38)$$

where k = $1,2,\ldots,k_p - 1$. and we have

$$t_{i,j} \overline{A} v_{p,0} = t_{i,j} \sum_{l=1}^{m} \alpha_{pl0} b_l = 0 \qquad (B.39)$$

We note that if $t_{i,j}$ is the k'th row of the matrix T then $v_{i,j+1}$ is the k-1'th column of the matrix T^{-1} we have

$$t_{i,j} \overline{A} v_{i,j+1} = t_{i,j} v_{i,j} = 1 \qquad (B.40)$$

Hence the (i,j) element of the matrix A_z is

$$\hat{a}_{ij} = \begin{cases} 1 & \textit{if } i = j+1 \\ 0 & \textit{if } i \neq j+1 \end{cases} \qquad (B.41)$$

where $i = 1,\ldots,m$. and $i \neq r_1, r_2, \ldots, r_m$. Then we conclude that A_z is of the following block matrix form

$$A_z = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix} \tag{B.42}$$

where A_{ll} is a $k_l \times k_l$ dimensional companion matrix having the form of the following controllable canonical form

$$A_{ll} = \begin{bmatrix} \times, \times, .., \times \\ I_{k_l -1} \quad 0 \end{bmatrix} \tag{B.43}$$

For non diagonal element A_{ij}, $i \neq j$, we have

$$A_{ij} = \begin{bmatrix} \times \times \ldots \times \times \\ 0\,0 \ldots 0\,0 \\ 0\,0 \ldots 0\,0 \\ \cdot \cdots \cdot \\ \cdot \cdots \cdot \\ \cdot \cdots \cdot \\ 0\,0 \ldots 0\,0 \end{bmatrix} \tag{B.44}$$

where except for the first row, all the elements of A_{ij} are zeros and $\times$ in equation (B.44) means that the term is not necessarily zero, although it is possible to be zero. We have $B_z(1,1) = t_{1,k_1-1}b_1 = 1$ and

$$t_{1,k_1-1}v_{2,k_2-1} = t_{1,k_1-1}(b_2 - \delta(k_1 - k_2)\alpha_{21k_2}b_1) = 0 \tag{B.45}$$

Hence

$$t_{1,k_1-1}b_2 = \delta(k_1 - k_2)\alpha_{21k_2} \tag{B.46}$$

Applying this procedure, we can calculate the matrix $R\overline{B} = B_z$ as

$$B_z = \begin{bmatrix} 1 & \times & \times & . & . & . & \times \\ 0 & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & 1 & \times & . & . & . & \times \\ . & 0 & . & . & . & . & . \\ . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & . & . \end{bmatrix} \tag{B.47}$$

Let e_i be a unit column vector with 1 as its r_i'th element, then define

$$\mathrm{B} = [e_1, e_2, \dots, e_m] \tag{B.48}$$

In block matrix form B is

$$B = \text{ block diag}[\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_m] = \begin{bmatrix} \tilde{b}_1 & 0 & 0 & . & . & . & 0 \\ 0 & \tilde{b}_2 & 0 & . & . & . & 0 \\ . & . & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \tilde{b}_m \end{bmatrix} \tag{B.49}$$

where $\tilde{b}_i^T = [1,0,0,\dots,0]$ has k_i elements.

satisfied. Define the $m \times m$ matrix B_s as follows

$$\mathrm{B}_s = B^T B_z \tag{B.50}$$

i.e. the i'th row of the matrix B_s is the r_i'th row of the matrix B_z

For the elements $\bar{a}_{ij}$ of the matrix A_z, where i = 1,...,m. and i = $r_1, r_2, \dots, r_m$, Popov's Theorem 1 will be very useful.

Define the matrix W as

$$W = \text{block diag}[w_1, w_2, \dots, w_m] \tag{B.51}$$

where $w_i^T = [\sigma^{k_i-1}, \sigma^{k_i-2}, \dots, \sigma, 1]^T$, where σ is a complex number. In matrix form W is as follows

$$
W = \begin{bmatrix}
\sigma^{k_1-1} & 0 & \cdots & \cdot \\
\sigma^{k_1-2} & 0 & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
1 & \cdot & \cdots & \cdot \\
0 & \sigma^{k_2-1} & \cdots & \cdot \\
0 & \sigma^{k_2-2} & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & 1 & \cdots & \cdot \\
\cdot & 0 & \cdots & \sigma^{k_m-1} \\
\cdot & 0 & \cdots & \sigma^{k_m-2} \\
\cdot & \cdot & \cdots & \cdot \\
0 & \cdot & \cdots & \cdot \\
0 & 0 & \cdots & 1
\end{bmatrix} \tag{B.52}
$$

Then the matrix $S(\sigma)$ in equation (B.22) and (B.23) is

$$S(\sigma) = T^{-1}\, W \tag{B.53}$$

then multip.ying the matrix T on both sides of equation (B.25) we have

$$T(\sigma I - \bar{A})S(\sigma) = T(\sigma I - \bar{A})T^{-1} W = (\sigma I - A_z)W \tag{B.54}$$

and right hand side

$$T\bar{B}R(\sigma) = B_z R(\sigma) \tag{B.55}$$

Hence we have

$$B^T(\sigma I - A_z)W = B_s R(\sigma) \tag{B.56}$$

The left hand side contains only the unspecified m rows of matrix A_z . Hence each coefficient can be calculated by perform the calculations at the right hand side.

Note :

$$\det[B^T(\sigma I - A_z)W] = \det[R(\sigma)] = \det[\sigma I - \bar{A}] \tag{B.57}$$

where we have $\det[B_s] = 1$.

Vita

The vita has been removed from
the scanned document