HALF-BOUND STATES OF A ONE-DIMENSIONAL DIRAC SYSTEM: THEIR EFFECT ON THE TITCHMARSH-WEYL M(λ)-FUNCTION AND THE SCATTERING MATRIX

by

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(ABSTRACT)

We study the effect of the so-called half-bound states on the Titchmarsh-Weyl M(λ)-function and the S-matrix for a one dimensional Dirac system. For short range potentials with finite first (absolute) moments, we give an M(λ) characterization of half bound states and, as a corollary, we deduce the behavior of the spectral function near the spectral gap endpoints. Further, we establish continuity of the $\mathcal{S}$-matrix in momentum space and prove the Levinson theorem as a corollary to this analysis. We also obtain explicit asymptotics of the $\mathcal{S}$-matrix for power-law potentials.

Included is also an appendix establishing an eigenfunction expansion and the validity of the $\mathcal{S}$-matrix.

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Thanks and praises to JAH RASTAFARI - the Almighty. *Selah!*

DEDICATION

This work is dedicated with love to

Dready got a job to do

And he's got to fulfill that mission

and to the loving memory of

One bright morning

When my work is over

I will fly away home - - *Seláh*

CONTENTS

CHAPTER I

INTRODUCTION

In this paper we consider the Dirac system

$$y' = [C(\lambda) + P(x)]y, \tag{1.1}$$

with $C(\lambda) = \begin{pmatrix} 0 & \lambda + c \\ -\lambda + c & 0 \end{pmatrix}$ and $P(x) = \begin{pmatrix} p(x) & v_1(x) \\ -v_2(x) & -p(x) \end{pmatrix}$, on the real line, i. e. , for $x \in (-\infty, \infty)$, where λ is a complex spectral parameter, c is constant, p, v_1 and v_2 are real valued functions of x. A solution of (1.1) is a 2 x 1 vector

$$y(x, \lambda) = \begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix}.$$

A familiar Dirac system is the problem in relativistic quantum mechanics described by $p(x) = k/x, v_1(x) = v_2(x) = v(x)$, say, on $0 < x < \infty, k$ constant. This of course is the radial wave equation for a particle of mass c moving in a field of potential $V(x)$ ([2], [41], for example). Note that to obtain the physicists' usual notation ([2] for example), one needs to make the transformations $\lambda \rightarrow -E$ and $c \rightarrow m$. Other contexts in which (1.1) is physically relevant may be found in [1], [37], [38], and [47] for example.

With the two potentials $v_1 \neq v_2$, (1.1) has also found its place in the physics literature in the context of the inverse scattering problem for Dirac particles ([6],[34]). Specifically, the two potentials (actually, $v_1 - v_2$ and $v_1 + v_2$) result from the Gel'fand-Levitan reconstruction procedure and appear in the so-called "canonical form" of the Dirac equation ([6]), to which all other equations of Dirac type may be reduced by a unitary transformation.

System (1.1) induces a self-adjoint operator H, whose spectrum has been well studied

([12]–[16],[40]–[43]). Actually the spectrum of H is studied by regarding H as $H = H_+ \cup H_-$ where H_+ and H_- are, respectively, the self - adjoint operators induced by (1.1) on, say, $[0,\infty)$ and $(-\infty, 0]$ ([15]). Nonetheless, the spectrum of H is real and, under our hypothesis, continuously differentiable in the complement of $[-c, c]$ and discrete in $(-c, c)$, where it consists of eigenvalues of H. For the basic spectral theory of and eigenfunctions expansions associated with (1.1), an excellent reference is [25] (see also [4]).

The endpoints, $\lambda = \pm c$, of the spectral gap require special attention in studying the spectrum of H. At these points, the Jost solutions (see Chapter III) can become linearly dependent. When such behavior occurs, the point $\lambda = \pm c$ is said to be a half bound state. These (half bound) states feature significantly in the study of the inverse problem ([28]). The purpose of the present paper is to study these states. In particular, we seek to characterize them by, and study their effect on, some well known quantities of spectral interest.

The organization of this dissertation is as follows. In this chapter, we introduce the quantities we shall be working with, which are the Titchmarsh - Weyl $M(\lambda)$ – coefficient, the spectral function, the Dirac Lippmann-Schwinger solutions and the S - matrix, and point out some of their properties. Then in Chapter II, we explain some notation and give precise definition of our operator H, its resolvent set and the various components of its spectrum. We then introduce our assumptions on the potential function V and state our main results. Chapter III consists of results preliminary to the proof of our main results. There we study the $w \to 0$ limit, i. e., $\lambda \to \pm c$, of the so - called Jost solutions of (1.1). We also define the Jost functions and study their asymptotics for small and large λ. We prove our main results in Chapters IV and V. Included is also an Appendix devoted to eignfunction expansions and the S - matrix.

1. The $M(\lambda)$-coefficient.

Associated with (1.1) is the so-called Titchmarsh-Weyl $M(\lambda)$ – coefficient, which is a matrix-valued function of λ. Its importance is realized in the construction of solutions of

$$y' = [C(\lambda) + P(x)]y + If \tag{1.2}$$

which are of square integerable, as well as in the investigation of the spectra of operators associated with (1.1). Although the study of the $M(\lambda)$ – coefficient was first introduced by Titchmarsh ([41]) for a particular case of (1.1) on $0 < x < \infty$, it has only been recently that Hinton and Shaw ([10]–[14]) have developed a theory of $M(\lambda)$ functions for Hamiltonian systems — of which (1.1) is a special case. By comparison, there exists an expansive m – coefficient theory for the Schrödinger equation (see [7], [25], [4]). Hinton and Shaw's theory is applicable in the limit-circle case as well, although we describe their presentation of the $M(\lambda)$ theory only for the limit-point case.

We begin with a fundamental solution $Y(x,\lambda)$ of (1.1) determined by the initial value $Y(0,\lambda) = I$ for all λ. Partition $Y(x,\lambda)$ as $Y(x,\lambda) = \begin{pmatrix} \theta_1(x,\lambda) & \phi_1(x,\lambda) \\ \theta_2(x,\lambda) & \phi_2(x,\lambda) \end{pmatrix}$. Then the Titchmarsh-Weyl m - coefficients, $m_+(\lambda)$ and $m_-(\lambda)$, at $x = +\infty$ and $x = -\infty$, respectively, are defined to be

$$m_+(\lambda) = -\lim_{x\to+\infty} \frac{\theta_1(x,\lambda)}{\phi_1(x,\lambda)}, \Im\lambda \neq 0$$

$$\text{and} \tag{1.3}$$

$$m_-(\lambda) = -\lim_{x\to-\infty} \frac{\theta_1(x,\lambda)}{\phi_1(x,\lambda)}, \Im\lambda \neq 0.$$

The existence of these limits is established in [10]. Let us mention here that $m_+(\lambda)$ is just a systems version of Weyl's m-function for the Schrödinger equation on $0 \leq x < \infty$. Also, a matrix version of $m_+(\lambda)$ for ordinary n^{th} order differential equations has been given by Naimark [30].

The following properties about $m_\pm(\lambda)$ are well known ([10], [14]). It is also instructive to compare these properties with those of the m – coefficient for the Schrödinger equation ([4]). Let $m_+(\lambda)$ and $m_-(\lambda)$ be defined by (1.3). Let $\Psi_+(x,\lambda) = \theta(x,\lambda) + m_+(\lambda)\phi(x,\lambda)$ and $\Psi_-(x,\lambda) = \theta(x,\lambda) + m_-(\lambda)\phi(x,\lambda)$. Let $\lambda \in \{\lambda \mid \Im\lambda \neq 0\}$. Then

1) $m_\pm(\lambda)$ are analytic, $(\Im m_+(\lambda))(\Im\lambda) > 0, (\Im m_-(\lambda))(\Im\lambda) < 0, m_\pm(\bar{\lambda}) = \overline{m_\pm(\lambda)}$,

2) $m_+(\lambda) - m_-(\lambda) \neq 0$, and in fact $\Im(m_+(\lambda) - m_-(\lambda))\Im\lambda > 0$, and

3) $\Psi_+(x,\lambda) \in L^2(0,\infty)$ and $\Psi_-(x,\lambda) \in L^2(-\infty,0)$.

We also make the following observations about $m_\pm(\lambda)$:

4) It is possible that one or both of m_+ and m_- extend continuously onto parts of the real axis.

5) $\Psi_+(x,\lambda)$ and $\Psi_-(x,\lambda)$ are the unique, up to constant multiples, $L^2(0,\infty)$ and $L^2(-\infty,0)$, respectively, solutions of (1.1). Namely, $m_+(\lambda)$ picks out a basis of $L^2(0,\infty)$ solutions to (1.1), and similarly for $m_-(\lambda)$.

The Green's function for (1.2) is defined as

$$G(x,t;\lambda) = \begin{cases} \Psi_+(x,\lambda)(m_-(\lambda) - m_+(\lambda))^{-1}\Psi_-^*(t,\overline{\lambda}), & x > t, \\ \Psi_-(x,\lambda)(m_-(\lambda) - m_+(\lambda))^{-1}\Psi_+^*(t,\overline{\lambda}), & x < t, \end{cases} \tag{1.4}$$

where $\Im\lambda \neq 0$. For $f \in L^2(-\infty,\infty)$, define the operator $\mathcal{G}(\cdot,\lambda,\cdot)$ by

$$\mathcal{G}(x,\lambda,f) = \int_{-\infty}^{\infty} G(x,t;\lambda)f(t)dt. \tag{1.5}$$

The following properties of the operator $\mathcal{G}(\cdot,\lambda,\cdot)$ are established in [14] (cf [4], Ch. 9).

6) For $f \in L^2(-\infty,\infty)$, $\Im\lambda \neq 0$, equation (1.2) is uniquely solved by $y(x,\lambda) = \mathcal{G}(x,\lambda;f)$.

7) $y(x,\lambda) = \mathcal{G}(x,\lambda;f) \in L^2(-\infty,\infty)$; in particular $\|y\| \leq \frac{1}{\Im\lambda}\|f\|$.

The Green's function $G(x,t;\lambda)$ may be written in a different way as

$$G(x,t;\lambda) = \begin{cases} Y(x,\lambda)M_1Y^*(t,\overline{\lambda}), & x > t, \\ Y(x,\lambda)M_2Y^*(t,\overline{\lambda}), & x < t, \end{cases}$$

where

$$M_1 = \begin{pmatrix} (m_- - m_+)^{-1} & (m_- - m_+)^{-1}m_- \\ m_+(m_- - m_+)^{-1} & m_+(m_- - m_+)^{-1}m_- \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} (m_- - m_+)^{-1} & (m_- - m_+)^{-1}m_+ \\ m_-(m_- - m_+)^{-1} & m_-(m_- - m_+)^{-1}m_+ \end{pmatrix},$$

and we have suppressed the λ - dependence.

The Titchmarsh-Weyl $M(\lambda)-$ coefficient for (1.1) is then defined as

$$M(\lambda) = \frac{1}{2}(M_1(\lambda) + M_2(\lambda)). \tag{1.6}$$

The following observations are in order.

8) (6) above establishes (1.5) as the resolvent operator for problem (1.2) for $\Im\lambda \neq 0$.

9) By the uniqueness and square integrability of the solution given by (1.5) above, we see that $M(\lambda)$ picks out a basis of $L^2(-\infty,\infty)$ solutions to (1.2).

Next, we state the connection between the $M(\lambda)-$ coefficient and the spectrum of the operator H. We mention here that the relationship $M(\lambda)$ bears to the spectrum of H is the same as that borne by $m_+(\lambda)$ to the operator induced by (1.1) on $0 \leq x < \infty$ (compare [12] and [13]).

Let $\rho(H)$, $P(H)$ $C(H)$ and $PC(H)$ denote respectively the *resolvent set*, *point spectrum*, *continuous spectrum*, and *point−continuous spectrum of* H. The following classification holds:

10) The point $\lambda_0 \in \rho(H)$ iff $M(\lambda)$ is analytic at λ_0. Then the resolvent operator at such points is given by (1.5).

11) The point $\lambda_0 \in P(H)$ iff $M(\lambda)$ has a simple pole at λ_0.

12) The point $\lambda_0 \in C(H)$ iff $M(\lambda)$ is not analytic at λ_0 and $\lim_{\nu\to 0}\nu M(\lambda_0+i\nu)=0$.

13) The point $\lambda_0 \in PC(H)$ iff $\lim_{\nu\to 0}\nu M(\lambda_0+i\nu)=S\neq 0$ and $M(\lambda)-i(\lambda-\lambda_0)^{-1}S$ is not analytic at λ_0.

Moreover, the spectrum of H is the support of a (matrix valued) measure $d\tau(\lambda)$, where $\tau(\lambda)$ is a real matrix valued step function, nondecreasing and right continuous, with jump discontinuities at the eigenvalues of H. $\tau(\lambda)$ is called the spectral function for H and is related to the $M(\lambda)$ function by the Titchmarsh-Kodaira formula ([13])

$$\tau(\lambda_1)-\tau(\lambda_2)=\frac{1}{\pi}\lim_{\epsilon\to 0}\int_{\lambda_1}^{\lambda_2}\Im M(\lambda+i\epsilon)d\lambda, \tag{1.7}$$

at points of continuity λ_1, λ_2 of $\tau(\lambda), \lambda_1 < \lambda_2$.

2. The Scattering Matrix.

The properties of quantum-mechanical systems are most conveniently described by the scattering operator, or S - matrix, S, which is defined in the following manner ([33], [36]). Define the wave operators

$$\Omega_{\pm} = \Omega_{\pm}(H, H_0) \equiv s - \lim_{t \to \mp\infty} \exp(itH)\exp(-itH_0),$$

where the $s - \lim$ denotes the strong limit. If $\Omega_{\pm}$ exist, the S - matrix is defined by $S = (\Omega_+)^* \Omega_-$ (here $*$ denotes the adjoint). The goal is then to find an expression for S in a representation where H_0 is diagonal. To this end, one solves a boundary value problem for functions that are not square integrable — the Lippmann - Schwinger solutions — and expresses the S - matrix in terms of these solutions ([33],[36]).

We have included an appendix, where we derive the H_0-spectral representation of S starting from the definition $S = (\Omega_+)^* \Omega_-$.

Below we briefly describe the S - matrix in terms of the Dirac Lippmann-Schwinger solutions. Our exposition here follows closely that of ([32]) for the Schrödinger equation. We begin by replacing (1.1) by the following matrix equation:

$$\begin{aligned}\Psi(x,w) \;=\;& \Psi_0(x,w) + \int_{-\infty}^{x} dt W(w) e^{iw(x-t)} V(t)\Psi(t,w) + \\ & \int_{x}^{\infty} dt W^T(w) e^{-iw(x-t)} V(t)\Psi(t,w),\end{aligned} \tag{1.8}$$

where $\Psi = [\Psi^{(1)}, \Psi^{(2)}]$, $\Psi_0(x,w) = \begin{pmatrix} e^{iwx} & e^{-iwx} \\ \frac{iw}{\lambda+c} e^{iwx} & \frac{-iw}{\lambda+c} e^{-iwx} \end{pmatrix}$,

$w = +\sqrt{\lambda^2 - c^2}, |\lambda| > c, W(w) = \frac{1}{2} \begin{pmatrix} \frac{\lambda+c}{iw} & -1 \\ 1 & \frac{-iw}{\lambda+c} \end{pmatrix}$, and $V(x) = -\begin{pmatrix} v_2(x) & p(x) \\ p(x) & v_1(x) \end{pmatrix}$.

We shall call (1.8) the Dirac Lippmann-Schwinger equation. The existence of its unique solution is established in the Appendix (see also [19] and [36]).

With the definition (II §1) of the operator $Q = [Q_1, Q_2]^T$ anticipated, we define the following quantities:

$$T_l = 1 + \tfrac{1}{2}\int_{-\infty}^{\infty} dt (Q_1(\Psi^{(1)}(t)) + \tfrac{\lambda+c}{iw} Q_2(\Psi^{(1)}(t))) e^{-iwt},$$

$$R_l = -\tfrac{1}{2}\int_{-\infty}^{\infty} dt (Q_1(\Psi^{(1)}(t) - \tfrac{\lambda+c}{iw} Q_2(\Psi^{(1)}(t))) e^{iwt},$$

and (1.9)

$$T_r = 1 - \tfrac{1}{2}\int_{-\infty}^{\infty} dt (Q_1(\Psi^{(2)}(t) - \tfrac{\lambda+c}{iw} Q_2(\Psi^{(2)}(t))) e^{iwt},$$

$$R_r = \tfrac{1}{2}\int_{-\infty}^{\infty} dt (Q_1(\Psi^{(2)}(t) + \tfrac{\lambda+c}{iw} Q_2(\Psi^{(2)}(t))) e^{-iwt}.$$

The quantities T_l, T_r, R_l, R_r are seen to be the transmission and reflection coefficients for incidence from the left and right, respectively, from the following asymptotic behavior of the components of Ψ :

$$\Psi^{(1)} = \begin{cases} \begin{pmatrix} 1 \\ \frac{iw}{\lambda+c} \end{pmatrix} T_l e^{iwx} + o(1) & as\ x \to +\infty, \\ \\ \begin{pmatrix} 1 \\ \frac{iw}{\lambda+c} \end{pmatrix} e^{iwx} + \begin{pmatrix} 1 \\ -\frac{iw}{\lambda+c} \end{pmatrix} R_l e^{-iwx} + o(1) & as\ x \to -\infty. \end{cases} \tag{1.10}$$

$$\Psi^{(2)} = \begin{cases} \begin{pmatrix} 1 \\ -\frac{iw}{\lambda+c} \end{pmatrix} T_r e^{-iwx} + o(1) & as\ x \to -\infty, \\ \\ \begin{pmatrix} 1 \\ -\frac{iw}{\lambda+c} \end{pmatrix} e^{-iwx} + \begin{pmatrix} 1 \\ \frac{iw}{\lambda+c} \end{pmatrix} R_r e^{iwx} + o(1) & as\ x \to +\infty. \end{cases} \tag{1.11}$$

Let

$$A = \begin{pmatrix} T_l - 1 & R_r \\ R_l & T_r - 1 \end{pmatrix}.$$

Then we see that we can write

$$A = \frac{\lambda + c}{2iw} \int_{-\infty}^{\infty} dt \Psi_0^*(t, w) V(t) \Psi(t, w), \tag{1.12}$$

where * denotes complex conjugate transpose. Using (1.10)–(1.11) to evaluate the Wronskian determinant, $W[\Psi^{(1)}, \Psi^{(2)}]$, we find that

$$W[\Psi^{(1)}, \Psi^{(2)}] = -\frac{2iw}{\lambda + c} T_r = -\frac{2iw}{\lambda + c} T_l,$$

whence we conclude $T_l = T_r = T$. The scattering matrix S is defined by

$$S = I + A = \begin{pmatrix} T & R_r \\ R_l & T \end{pmatrix}, \text{ where } I \text{ is the identity matrix.} \tag{1.13}$$

Now since $W[\Psi^{(1)}, \Psi^{(2)}] \neq 0$, then we may express the solution $\Psi(x, -w)$ of (1.8) as a matrix-multiple of $\Psi(x, w)$. Let us write $\Psi(x, -w) = \Psi(x, w)M(w)$, and let $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then Wronskian evaluations reveal that $M(w) = BS(-w)$. Noting that for $w \in \Re$, $\Psi(x, -w) = \overline{\Psi(x, w)}$, we arrive at, by (1.12) and (1.13)

$$\overline{\Psi(x, w)} = \Psi(x, w)B\overline{S(w)}. \tag{1.14}$$

Equation (1.14) and the relations (1.10)–(1.11) result in the fact that S is unitary, i. e. , $SS^* = S^*S = I$. In terms of the elements of S, we may, for future reference, express the unitary condition as (cf. [32])

$$|R_l|^2 = |R_r|^2 = 1 - |T|^2, \tag{1.15}$$

$$\frac{R_l \overline{R_r}}{|R_l|^2} = \frac{\overline{R_l} R_r}{|R_r|^2} = -\frac{T^2}{|T|^2}. \tag{1.16}$$

3. Eigenfunction Expansions.

Our "formal scattering theory" exposition would not be complete without establishing an eigenfunction expansion. We proceed much as done in [23] and [36]. Consider first the unperturbed Hamiltonian H_0. Let $w \in \Re$ and define $\lambda = +\sqrt{w^2+c^2}$. Then the (generalized) eigenfunctions

$$u_0(x,w) = \begin{pmatrix} 1 \\ \frac{iw}{\lambda+c} \end{pmatrix} e^{iwx},$$

$$v_0(x,w) = \begin{pmatrix} 1 \\ \frac{\lambda+c}{iw} \end{pmatrix} e^{iwx}, \tag{1.17}$$

satisfy

$$H_0 u_0(x,w) = \lambda u_0(x,w),$$

$$H_0 v_0(x,w) = -\lambda v_0(x,w),$$

i. e. u_0 is a (generalized) eigenfunction for positive λ, whereas v_0 is a (generalized) eigenfunction for negative λ. The eigenfunctions (1.17) also satisfy the (formal) orthogonality relations

$$\begin{aligned} < u_0(x,w'), u_0(x,w) > &= 2\pi \tfrac{\lambda\delta(w-w')}{\lambda+c} =< v_0(x,w'), v_0(x,w) >, \\ < u_0(x,w'), v_0(x,w) > &=< v_0(x,w'), u_0(x,w) > = 0. \end{aligned} \tag{1.18}$$

Now, for any $f \in (L^2(\Re))^2$ define $\hat{f} = (\hat{f}_+, \hat{f}_-)^T$ by

$$\begin{aligned} \hat{f}_+(w) &= \tfrac{1}{\sqrt{2\pi}}\, l.i.m. \int_{\Re} dx (u_0(x,w))^* f(x) r(w), \\ \hat{f}_-(w) &= \tfrac{1}{\sqrt{2\pi}}\, l.i.m. \int_{\Re} dx (v_0(x,w))^* f(x) r(w), \end{aligned} \tag{1.19}$$

where $r(w) = \sqrt{\frac{\lambda+c}{2\lambda}}$.

Then for any Borel set $B \subset (-\infty,-c] \cup [c,\infty)$, the spectral projection $P_0(B)$ for B is given by

$$(P_0(B)f)(x) = \frac{1}{2\pi}\, l.i.m. \left[\int_{B_+} dw\, u_0(x,w)\hat{f}_+(w)r(w) + \int_{B_-} dw\, v_0(x,w)\hat{f}_-(w)r(w)\right],$$

where $B_\pm = \{w \in \Re \mid \pm\lambda(w) \in B\}$. In particular, we have the eigenfunction expansion

$$f(x) = \frac{1}{\sqrt{2\pi}}\, l.\,i.\,m. \int_{\Re} dw\, [u_0(x,w)\hat{f}_+(w) + v_0(x,w)\hat{f}_-(w)]r(w). \tag{1.20}$$

The mapping $f \mapsto \hat{f}$ is "easily" shown to define a unitary transformation of $(L^2(\Re))^2$ onto itself, i. e. ,

$$\int_{\Re} dx\, [|f_1(x)|^2 + |f_2(x)|^2] = \int_{\Re} dw\, [|\hat{f}_+(w)|^2 + |\hat{f}_-(w)|^2] \tag{1.21}$$

and $Ran\,\hat{} = (L^2(\Re))^2$. Moreover, one finds that if $f \in \mathcal{D}(H_0)$, then

$$(H_0 f)\hat{}\,(w) = \lambda A \hat{f}(w), \quad \text{where } A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.22}$$

Similar results for the operator $H = H_0 + V$, V subject to Assumption (2.1) are true (see Appendix). In this case the (generalized) eigenfunctions are given by

$$\begin{aligned} u(x,w) &= u_0(x,w) + \textstyle\int_{-\infty}^{x} dt W_+(w) e^{iw(x-t)} V(t) u(t) + \\ &\quad \textstyle\int_x^{\infty} W_+^T(w) e^{-iw(x-t)} V(t) u(t,w), \end{aligned} \tag{1.23}$$

$$\begin{aligned} v(x,w) &= v_0(x,w) + \textstyle\int_{-\infty}^{x} dt W_-(w) e^{iw(x-t)} V(t) v(t) + \\ &\quad \textstyle\int_x^{\infty} W_-^T(w) e^{-iw(x-t)} V(t) v(t,w), \end{aligned} \tag{1.24}$$

where $u_0(x,w)$ and $v_0(x,w)$ are the same as above and

$$W_+(w) = \frac{1}{2}\begin{pmatrix} \frac{\lambda+\epsilon}{iw} & -1 \\ 1 & -\frac{iw}{\lambda+\epsilon} \end{pmatrix}, W_-(w) = \frac{1}{2}\begin{pmatrix} \frac{iw}{\lambda+\epsilon} & 1 \\ -1 & -\frac{\lambda+\epsilon}{iw} \end{pmatrix}.$$

The eigenfunctions u and v satisfy:

$$H u(x,w) = \lambda u(x,w),$$

$$H v(x,w) = -\lambda v(x,w).$$

The eigenfunction $u(x,w)$ given by (1.23) is just the solution $\Psi^{(1)}(x,w)$ of (1.8) if $w \geq 0$, while it is the solution $\Psi^{(2)}(x,w)$ of (1.8) if $w \leq 0$. If we write $\Psi^{(1)}(x,w) = \Psi^{(1)}(x,\lambda)$, $\lambda = +\sqrt{w^2+c^2}$, then we see that $v(x,w) = \Psi^{(1)}(x,-\lambda)$ for $w \geq 0$, and likewise $v(x,w) = \Psi^{(2)}(x,-\lambda)$, for $w \leq 0$. In particular, for $w \in \Re$, $u(x,w)$ defines, by way of (1.12), the S - matrix for $\lambda > c$, whereas $v(x,w)$ defines the S - matrix for $\lambda < -c$; that is to say, with $\lambda = +\sqrt{w^2+c^2}$, $w \in \Re$, we have

$$A(\lambda) = \frac{\lambda + c}{2iw}\int_{-\infty}^{\infty} dt\,[u_0(t,w), u_0(t,-w)]V(t)[u(t,w), u(t,-w)],$$

and similarly for $A(-\lambda)$. Using the relations (1.10) and (1.11), adapted to $u(x,w)$ and $v(x,w)$, (1.15),(1.16) and the relation

$$W[z(x,w), \overline{y(x,w)}]\,\Big|_{x=c}^{x=d} = (\lambda_z - \overline{\lambda_y})\int_c^d dx\,[(y(x,w)^* z(x,w)], \tag{1.25}$$

where $Hy = \lambda_y y$, $Hz = \lambda_z z$, it is only a matter of computation to verify the following orthogonality relations (which have to be interpreted in the sense of distributions);

$$\begin{aligned} &< u(x,w'), u(x,w) >= \tfrac{4\pi\lambda}{\lambda+c}\delta(w-w') =< v(x,w'), v(x,w) >, \\ &< u(x,w'), v(x,w) >=< v(x,w'), u(x,w) >= 0. \end{aligned} \tag{1.26}$$

Next for any $f \in (L^2(\Re))^2$, define $f^{\#} = (f_+^{\#}, f_-^{\#})^T$ by:

$$\begin{aligned} f_+^{\#}(w) &= \tfrac{1}{\sqrt{2\pi}}\, l.i.m.\ \textstyle\int_{\Re} dx (u(x,w))^* f(x) r(w), \\ f_-^{\#}(w) &= \tfrac{1}{\sqrt{2\pi}}\, l.i.m.\ \textstyle\int_{\Re} dx (v(x,w))^* f(x) r(w). \end{aligned} \tag{1.27}$$

With $B, B_{\pm}$ as before, we have (see Appendix)

$$(P(B)f)(x) = \frac{1}{\sqrt{2\pi}}\, l.i.m.\,[\int_{B_+} dw\, u(x,w) f_+^{\#}(w) r(w) + \int_{B_-} dw\, v(x,w) f_-^{\#}(w) r(w)]$$

where $r(w) = \sqrt{\frac{\lambda+c}{2\lambda}}$. Hence we obtain the eigenfunction expansion (cf. [44], Thm 17.C.2)

$$(P(H_{ac})f)(x) = l.i.m.\ \int_{\Re} dw\, r(w)[u(x,w)f_+^{\#}(w) + v(x,w)f_-^{\#}(w)], \tag{1.28}$$

where H_{ac} denotes the absolutely continuous subspace of H.

Chapter II

NOTATION AND STATEMENT OF RESULTS

1. Notation and Definitions.

The equation we consider is the Dirac system

$$y' = [C(\lambda) + P(x)]y$$

for $x \in (-\infty, \infty)$, where $C(\lambda)$ and $P(x)$ are the matrices

$$C(\lambda) = \begin{pmatrix} 0 & \lambda + c \\ -\lambda + c & 0 \end{pmatrix}, P(x) = \begin{pmatrix} p(x) & v_1(x) \\ -v_2(x) & -p(x) \end{pmatrix},$$

λ is a complex spectral parameter and c is a constant. We shall also write $V(x) = -J\,P(x)$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \text{ Also let } R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $\lambda \in S = \{\lambda \mid \Im\lambda \geq 0\}$, we set $w = (\lambda^2 - c^2)^{\frac{1}{2}}$, taking w to be the principal branch of $\sqrt{\ }$ on $S\backslash(-\infty, -c)$, and define it so as to be continuous on $(-\infty, -c)$.

We will denote by $E(x, \lambda)$ the fundamental matrix for the free problem, i. e. ,

$$E(x, \lambda) = \begin{pmatrix} \cos wx & \frac{\lambda+c}{w} \sin wx \\ -\frac{w}{\lambda+c} \sin wx & \cos wx \end{pmatrix}.$$

If $y(x, \lambda) = [y_1(x, \lambda), y_2(x, \lambda)]^{\mathrm{T}}$ is a solution of (1.1), we define an operator Q by

$$Q(y(x, \lambda)) = [Q_1(y(x, \lambda)), Q_2(y(x, \lambda))]^{\mathrm{T}} \equiv P(x)y(x, \lambda).$$

When no confusion is possible, we shall write $Q(x)$ or Q for $Q(y(x, \lambda))$ and $Q^0(x)$ or Q^0 for $Q(y(x, c))$. For any quantity q considered, the symbols q^*, q^{T} and $\bar{q}$ will denote, respectively, the complex conjugate transpose,the transpose and the complex conjugate of q. We shall denote by $y(x, \lambda)$ and $z(x, \lambda)$ the Jost solutions of (1.1) which, for $\Im\lambda > 0$, decay exponentially as $x \to +\infty$ and as $x \to -\infty$, respectively. Also, we write $W[y, z]$ for the Wronskian determinant of y and z.

If M is a matrix, we use the notation $\Im M$ to mean the quantity $\frac{1}{2i}(M - M^*)$. Also, if $M = (m_{ij})$, then by the modulus $|\cdot|$ of M we shall mean the sum of the moduli of m_{ij}. Often, we will also write K for unspecified (not necessarily equal) positive constants.

Let us now proceed to cast our problem in operator-theoretic language. Let $(L^2(\Re))^2$ denote the Hilbert space of all (2 x 1) square integrable vector functions $f = [f_1, f_2]^T$, that is to say f with $\|f\|^2 < \infty$, where the inner product $< \cdot, \cdot >$ and norm are given by

$$< f, g >= \int_{\Re}[f(x)]^* g(x)dx \text{ and } \|f\|^2 =< f, f > .$$

We shall write $L^2(\Re)$ or simply L^2 for $(L^2(\Re))^2$. The Dirac operator induced by (1.1) is defined on $L^2(\Re))$ by

$$H(y) = Jy' + (Rc + V)y : D(H) \to (L^2(\Re))^2$$

and

$$D(H) = \{y \in L^2(\Re) \mid \ y \textit{ is locally absolutely continuous on } \Re, H(y) \in L^2(\Re))\}.$$

Then (1.1) is equivalent to $Hy = \lambda y$. We also denote the unperturbed Hamiltonian by H_0, i. e., $H_0 = H - V$.

The following definitions are standard: If for some complex number $\lambda, (H - \lambda)^{-1}$: $L^2(\Re) \to L^2(\Re)$ exists and is bounded, we call $R_\lambda(H) = (H - \lambda I)^{-1}$ the *resolvent operator* corresponding to λ. The set $\rho(H)$ of all such points is called the *resolvent set* of H. The *spectrum* $\sigma(H)$ of H is the set of all complex numbers not contained in $\rho(H)$. The set of isolated points of $\sigma(H)$ is the *point spectrum* of H, denoted by $P(H)$. The set of all limit points of the spectrum of H is called the *essential spectrum* of H, denoted by $E(H)$. The subset $PC(H) \subset E(H) \backslash ((E(H) \cap P(H))$ of embedded eigenvalues, those λ for which (1.1) has a nontrivial solution $y \in D(H)$, is called the *point continuous spectrum.* The set $C(H) = E(H) \backslash PC(H)$ is the *continuous spectrum* of H.

Central to our study is the notion of a *half-bound state,* which we shall often abbreviate HBS. These states, under our hypotheses, can occur only at $\lambda = \pm c$, and are defined to be those λ_0 such that $\lambda_0 \notin P(H)$ but the Jost solutions (Chapter III) are linearly dependent.

2. Assumptions and Main Results.

In this section we introduce our assumptions on the potential V and present explicit statements of our main results. Our first theorem, and the corollary, deals with the Titchmarsh-Weyl $M(\lambda)$ – coefficient.

Assumption (2.1): *We assume that the potential V is (componentwise) real valued, Lebesgue measurable and satisfies the integrability condition*

$$\int_{-\infty}^{\infty} |V(x)| (1+ |x|) dx < \infty.$$

Under this assumption, (1.1) is in the limit point case at both $x = -\infty$ and $x = +\infty$, and hence H is self-adjoint on $D(H)$ ([17],[25],[44]). Moreover, the spectrum of H consists of a finite point spectrum $P(H) \subset (-c, c)$ and a continuously differentiable continuous spectrum $C(H) = \Re\backslash(-c,c)$([11],[14], [15],[25]). The assumption also allows us to prove the existence and unitarity of the scattering matrix, hence the existence of the wave operators associated with H (see Appendix).

Theorem (2.2): *Suppose V satisfies Assumption (2.1). Let $M(\lambda)$ denote the Titchmarsh-Weyl matrix m - coefficient for (1.1). Then the point $\lambda = \lambda_0$ is a half-bound state if and only if there exists a nonzero matrix $S(\lambda_0)$ such that*

$$\lim_{\nu\to 0} \nu^{\frac{1}{2}} M(\lambda_0 + i\nu) = S(\lambda_0),\ where\ \lambda_0 = \pm c.$$

The nonzero, complex − valued matrices $S(c) = (s_{ij})$ and $S(-c) = (s_{ij}^-)$ are given by

$$s_{11} = \alpha_0^{\phi}\beta_0^{\phi}(2ci\gamma_1^2)^{-\frac{1}{2}}.$$

$$s_{12} = s_{21} = -\tfrac{1}{2}(\alpha_0^{\phi}\beta_0^{\theta} + \alpha_0^{\theta}\beta_0^{\phi})(2ci\gamma_1^2)^{-\frac{1}{2}},$$

$$s_{22} = \alpha_0^{\theta}\beta_0^{\theta}(2ci\gamma_1^2)^{-\frac{1}{2}},$$

$$s_{11}^- = \alpha_{-1}^{\phi}\beta_{-1}^{\phi}(-2ci\gamma_{-1}^2)^{-\frac{1}{2}},$$

$$s_{12}^- = s_{21}^- = -\tfrac{1}{2}(\alpha_{-1}^{\phi}\beta_{-1}^{\theta} + \alpha_{-1}^{\theta}\beta_{-1}^{\phi})(-2ci\gamma_{-1}^2)^{-\frac{1}{2}},$$

$$s_{22}^- = \alpha_{-1}^{\theta}\beta_{-1}^{\theta}(-2ci\gamma_{-1}^2)^{-\frac{1}{2}},$$

where $\gamma_1 = \frac{1}{2ci}\left(\alpha + \frac{1}{\alpha}\right)$ and $\gamma_{-1} = -\frac{1}{2ci}\left(\beta + \frac{1}{\beta}\right)$, with $\alpha = \lim_{x\to-\infty} y_1(x,c)$ and $\beta = \lim_{x\to\infty} y(x,-c)$, and the other constants are defined in the proof of Lemma (3.9).

Corollary (2.3): *Let $\tau(\lambda)$denote the spectral function for H.Let $\lambda, S(\lambda_0), \gamma_1$ and γ_{-1}be as in Theorem (2.2). Then*

$$\lim_{|\lambda|\downarrow\lambda_0} \frac{d\tau(\lambda)}{d\lambda} = 0\ \ if\ \ \lambda_0\ is\ not\ a\ \mathrm{HBS}$$

and

$$\lim_{|\lambda|\downarrow\lambda_0} (\lambda-\lambda_0)^{\frac{1}{2}}\frac{d\tau(\lambda)}{d\lambda} = -\frac{1}{\pi}(2\lambda_0)^{-\frac{1}{2}}\tilde{S}(\lambda_0)\ \ if\ \ \lambda_0\ is\ a\ \mathrm{HBS},.$$

where $|\lambda| \downarrow \lambda_0$ *indicates* $\lambda \to \lambda_0$ *with* $|\lambda| > |\lambda_0|$ $(\lambda \in \Re)$, *and* $\tilde{S}(\lambda_0) =$
$-\sqrt{2i\lambda_0}\left(\frac{\Im(\gamma(\lambda_0))}{\gamma(\lambda_0)}\right) S(\lambda_0)$, $\gamma(c) = \gamma_1, \gamma(-c) = \gamma_{-1}$.

Remarks:

1) Theorem (2.2) is a further refinement to the four part classification of the spectrum of H given by (10)—(13) of Chapter I. In particular, if λ_0 is a HBS, then we retain the characterization ((12) of [1]) of λ_0 as an element of $C(H)$ and obtain further asymptotic behavior about $M(\lambda)$ at λ_0.

2) Hinton, Klaus and Shaw ([17]) recently obtained an analogous asymptotic condition for equation (1.1) on $0 \leq x < \infty$. Their result is that our conclusion holds if $M(\lambda)$ is replaced by $m_+(\lambda)$, with an appropriate scalar S, an analogy which is not completely unexpected considering the similarity of the four part resolvent-spectrum classification of H and H_+ ([12]).

Our next result addresses the behavior of the scattering matrix (or S-matrix), particularly as $\lambda \to c$ and as $\lambda \to -c$, and the corollary relates this behavior to the bound states of H.

Theorem (2.4): *Suppose that Assumption(2.1) holds. Let $S(\cdot)$ be the scattering matrix in the position representation. Then $S(\lambda)$ is continuous for $\lambda \in \Re\backslash(-c, c)$. In particular, the behavior of $S(\lambda)$ at $\pm c$ is the followomg. Let y and y^- denote y(x,c) and y(x,-c), repectively. Let $\gamma = W[y(x,c), z(x,c)]$, $\gamma_- = W[y(x,-c), z(x,-c)]$ and let α and β be as in Theorem (2.2). Then* if $\lambda = c$ *is not a* HBS,

$$T(\lambda) = \frac{iw}{c\gamma} + o(w) \text{ as } \lambda \to c,$$

$$\lim_{\lambda \to c} R_l(\lambda) = 1; \ \lim_{z \to c} R_r(\lambda) = -1,$$

and if $\lambda = c$ *is a* HBS,

$$\lim_{\lambda \to c} S(\lambda) = \frac{1}{\alpha^2 + 1} \begin{pmatrix} 2\alpha & \alpha^2 - 1 \\ 1 - \alpha^2 & 2\alpha \end{pmatrix};$$

if $\lambda = -c$ *is not a* HBS,

$$T(\lambda) = \frac{iw}{c\gamma_-} + o(w) \text{ as } \lambda \to -c,$$

$$\lim_{\lambda \to -c} R_l(\lambda) = 1, \ lim_{z \to -c} R_r(\lambda) = -1,$$

and if $\lambda = -c$ *isa* HBS,

$$\lim_{\lambda \to -c} S(\lambda) = \frac{1}{\beta^2 + 1} \begin{pmatrix} 2\beta & \beta^2 - 1 \\ 1 - \beta^2 & 2\beta \end{pmatrix}.$$

Corollary (2.5) (Levinson's Theorem): *Let* $T(\lambda)$ *denote the transmission coefficient for the system* (1.1) *and write* $T(\lambda) = |T(\lambda)| \, exp\Phi(\lambda)$, $\lambda \in \Re$. *Let* N *be the number of eigen-values of the operator* H. *Then* N *is finite and the following formula holds* :

$$\Phi(-c) - \Phi(c) = \begin{cases} N\pi & \text{if both of } \lambda = \pm c \text{ are HBS's}, \\ (N + \frac{1}{2})\pi & \text{if exactly one of } \pm c \text{ is an HBS}, \\ (N + 1)\pi & \text{if neither one of } \lambda = \pm c \text{ is a HBS}. \end{cases}$$

Remarks:

1) Theorem (2.4) is the analog of results recently established by Klaus ([22]) and Newton ([31]) for the Schrödinger operator.

2) Corollary (2.5) is not a new result. It is the analog of the theorem of Levinson ([24]) which, for the Schrödinger operator, relates, for each partial wave, the scattering phase shifts at zero energy to the number of bound states. The first such result for the Dirac operator was obtained by Barthélémy ([2]) for the physical problem. More recently, versions of the theorem were given for charged Dirac particles moving in a background monopole field ([9],[26],[27],[29],[45]) under various hypotheses, all of which are more restrictive than ours. Ma and Ni ([27]) seems to be the only correct paper which takes half - bound states into account, and also points out an error in [2]. Hinton, Klaus and Shaw ([17]) also recently proved a version of the theorem for (1.1) on $0 \leq x < \infty$, under hypotheses similar to ours.

Chapter III

PRELIMINARY RESULTS

1. The Jost solutions and existence of their limits.

As a special case of section 3 and 4 of [2], one obtains the Jost solutions, $y(x,\lambda), z(x,\lambda)$, for (1.1), which are defined by their asymptotic behavior

$$\lim_{x\to\infty} y(x,\lambda) = \begin{pmatrix} 1 \\ \frac{iw}{\lambda+c} \end{pmatrix} e^{iwx}, \quad \lim_{x\to-\infty} z(x,\lambda) = \begin{pmatrix} 1 \\ \frac{-iw}{\lambda+c} \end{pmatrix} e^{-iwx}.$$

These solutions are analytic for $\Im\lambda > 0$ and continuous for $\Im\lambda \geq 0$ (see below for $\lambda = \pm c$), and are constructed by iterating the Volterra equations

$$y(x,\lambda) = \begin{pmatrix} 1 \\ \frac{iw}{\lambda+c} \end{pmatrix} e^{iwx} - \int_x^\infty E(x-t,\lambda)P(t)y(t,\lambda)\,dt \tag{3.1}$$

and

$$z(x,\lambda) = \begin{pmatrix} 1 \\ \frac{-iw}{\lambda+c} \end{pmatrix} e^{-iwx} + \int_{-\infty}^x E(x-t,\lambda)P(t)z(t,\lambda)\,dt, \tag{3.2}$$

where $E(x,\lambda)$ is defined in II §1.

We observe that although Barthélémy ([2]) makes the assumption that

$$\int_{-\infty}^\infty |V(x)|\,(1+|x|^2)dx < \infty,$$

it is not used in establishing the existence and analyticity of the Jost solutions. Assumption (2.1) suffices to establish their analyticity properties.

Having the Jost solutions at our disposal for $\Im\lambda \geq 0, \lambda \neq \pm c$, the next natural step is to ask, what happens at $\lambda = \pm c$? We study this question in the next series of lemmas.

We begin by considering solutions of the equation

$$H\,y = I\,c\,y, \tag{3.3}$$

i. e. the solutions of (2.1) at $\lambda = c$. We have

Lemma (3.1) (cf. [31] Lemma (2.1)): *Under Assumption (2.1), (3.3) has unique solutions $y(x), z(x)$, which satisfy the boundary conditions*

$$\begin{aligned} \lim_{x\to\infty} y(x) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \lim_{x\to-\infty} z(x) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.4}$$

Let $\gamma \equiv W[y,z]$, the Wronskian determinant of $y(x)$ and $z(x)$. The solutions $y(x)$ and $z(x)$ have the following behavior

$$y(x) = \begin{pmatrix} -2\gamma c x + o(x) \\ -\gamma + o(1) \end{pmatrix}, x \to -\infty,$$

$$z(x) = \begin{pmatrix} 2\gamma c x + o(x) \\ \gamma + o(1) \end{pmatrix}, x \to \infty.$$

<ins>Proof</ins>: Under assumption (2.1), the Volterra integral equations

$$y(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_x^\infty E(x-t,c)P(t)y(t)\,dt, \tag{3.5}$$

$$z(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_\infty^x E(x-t,c)P(t)z(t)\,dt, \tag{3.6}$$

can be solved by iteration. That gives uniqueness and continuity of y and z.

Let us consider the solution $z(x)$. The arguments for $y(x)$ follow, *mutatis mutandis,* in the same vein. Equation (3.6) yields

$$|z(x)| \leq 1 + 2\int_{-\infty}^{x} dt(1 + c\,|x - t|)\,|P(t)|\,|z(t)|,$$

so if $x \leq 0$, then

$$|z(x)| \leq 1 + 2\int_{-\infty}^{\infty} dt(1 + 2c\,|t|)\,|P(t)|\,|z(t)|\,.$$

So Gronwall's Lemma and Assumption (2.1) imply that for $x \leq 0$,

$$|z(x)| \leq K, \qquad\qquad K \in \Re^{+}. \tag{3.7}$$

Using (3.7), we find that for $x \geq 0$,

$$\left|\int_{-\infty}^{0} E(x - t, c)P(t)z(t)\,dt\right| \leq 2\int_{-\infty}^{0} dt(1 + 2c\,|x - t|)\,|P(t)|\,|z(t)|$$

$$\leq 2\int_{-\infty}^{-x} dt(1 + 2c\,|t|)\,|P(t)|\,|z(t)| + 2\int_{-x}^{0} dt(1 + 2cx)\,|P(t)|\,|z(t)|$$

$$\leq K + K(1 + 2cx) \leq K(1 + 2cx),$$

where we have used K for undetermined positive constants. Hence

$$|z(x)| \leq 1 + K(1 + 2cx) + 2\int_{0}^{x} dt(1 + 2cx)\,|P(t)|\,|z(t)|$$

$$\leq K(1 + 2cx) + 2(1 + 2cx)\int_{0}^{x} dt\,|P(t)|\,|z(t)|,$$

so that

$$\frac{|z(x)|}{1 + 2cx} \leq K + 2\int_{0}^{x} dt(1 + 2ct)\,|P(t)|\,\frac{|z(t)|}{1 + 2ct}.$$

Appealing once again to Gronwall's Lemma, we find that

$$\frac{|z(x)|}{1 + 2cx} \leq K \qquad\qquad \text{for } x \geq 0. \tag{3.8}$$

Therefore, we have, combining (3.8) and (3.7)

$$|z(x)| \leq \begin{cases} K, & x \leq 0, \\ K(1+2cx), & x \geq 0. \end{cases} \tag{3.9}$$

Now, we see that we may write

$$\gamma = W[y,z] = \int_{-\infty}^{\infty} dt\, Q_2(z(t)) = \int_{-\infty}^{\infty} dt\, Q_2(y(t)),$$

and therefore

$$z_2(x) = \int_{-\infty}^{x} dt\, Q_2(z(t)) = \gamma - \int_{x}^{\infty} dt\, Q_2(z(t)).$$

Taking account of (3.9) thus implies that

$$z_2(z) = \gamma + o(1), \text{as } x \to \infty.$$

From (3.3), we have that $z_1'(x) = 2cz_2(x) + Q_1(z(x))$. And so, from the behavior of $z_2(x)$ and (3.9), we conclude that $z_1(x) = 2c\gamma x + o(x)$, as $x \to \infty$. Also, since $Q_2(z) \in L^1(-\infty,\infty)$, we see that

$$z_2(x) = o(1), \qquad as\ x \to -\infty.$$

Using (3.9), we also find that, for $x < 0$,

$$\left| x \int_{-\infty}^{x} dt\, Q_2(z(t)) \right| \leq \int_{-\infty}^{x} dt\, (|t||Q_2(z(t))|) < \infty,$$

so that

$$x \int_{-\infty}^{x} dt\, Q_2(z(t)) \to 0,\ as\ x \to -\infty.$$

This fact and the representation

$$z_1(x) = 1 + 2cx \int_{-\infty}^{x} dt\, Q_2(z(t)) + \int_{-\infty}^{x} dt (Q_1(z(t)) - 2ctQ_2(z(t)))$$

therefore yield

$$z_1(x) = 1 + o(1), x \to -\infty.$$

This completes the proof of the lemma. ■

A parallel situation obtains at $\lambda = -c$. There, we consider solutions of

$$\mathrm{Hy}=\text{-cIy}. \tag{3.3$'$}$$

We denote the solutions by $y^-(x)$ and $z^-(x)$. These satisfy the Volterra equations

$$y^-(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_x^{\infty} E(x-t,-c)P(t)y^-(t)\,dt,$$

$$z^-(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^{x} E(x-t,-c)P(t)z^-(t)\,dt.$$

We put $W[y^-, z^-] = \gamma_- = \int_{-\infty}^{\infty} dtQ_1(y^-(t))\,dt = \int_{-\infty}^{\infty} dtQ_1(z^-(t))$.

In much the same way as in Lemma (3.1), we are able to prove

Lemma (3.2): *Under assumption (1.2),* (3.3)$'$ has unique continuous solutions $y^-(x)$ *and* $z^-(x)$ *which satisfy the boundary conditions*

$$\lim_{x\to\infty} y^-(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\lim_{x\to-\infty} z^-(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solutions y^- *and* z^- *have the following behaviors*

$$y^-(x) = \begin{pmatrix} -\gamma_- + o(1) \\ -2c\gamma_- x + o(x) \end{pmatrix}, \qquad \text{as } x \to -\infty,$$

and

$$z^{-}(x) = \begin{pmatrix} \gamma_{-} + o(1) \\ 2c\gamma_{-}x + o(x) \end{pmatrix}, \qquad \text{as } x \to \infty.$$

On account of (3.1) and (3.2), we see that the only possible half-bound states can occur at $\lambda = \pm c$. And, by Lemma (3.1), $\lambda = c$ is a half-bound if and only if $\gamma = 0$, and similarly for $\lambda = -c$.

Let us now turn to the continuity of the Jost solutions at $\lambda = \pm c$. In particular, Lemma (3.5) tells us how $y(x,\lambda)$ approaches $y(x) = y(x,c)$ as $\lambda \to c$, with similar behavior for $z(x,\lambda)$ and corresponding statement as $\lambda \to -c$. First we have

Lemma (3.3): *Let $\alpha(x) = 1 - [(\sin x)/x]e^{ix}$ and $y(x)$ be the solution in Lemma (3.1). Let*

$$h(x,\lambda) = \int_x^\infty dt\alpha(w(t-x))E(x-t,c)P(t)y(t).$$

Then

$$|h(x,\lambda)| \le \begin{cases} K\nu(w), & x \ge 0, \\ K\left[\nu(w) + \frac{2w|x|}{1+2w|x|}\right](1+|x|), & x < 0, \end{cases}$$

where $\nu(w)$ is bounded, independent of x and $\nu(w) = o(1)$ as $\lambda \to c$.

Proof: One observes that $|\alpha(x)| \le K[|x|/(1+|x|)]$. Then, using the bounds established in Lemma (3.1) it is easy to show that for $x \ge 0$,

$$|h(x,\lambda)| \le K\int_0^\infty dy\,|P(t)|\left(\frac{wt}{1+wt}\right)(1+t) \equiv \nu(w),$$

which is easily shown to approach zero, as $w \to 0$, by dominated convergence.

For $x < 0$, the inequalities

$$\left|\int_x^0 s(t)\,dt\right| \le K\frac{2w|x|^2}{1-2w|x|}, \left|\int_0^{|x|} s(t)\,dt\right| \le K\frac{2w|x|^2}{1-2w|x|}, \text{ and } \left|\int_{|x|}^\infty s(t)\,dt\right| \le \nu(w),$$

where $s(t) = \alpha(w(t-x))E(x-t,\lambda)Q_y(t)$, are easily proven. This completes the proof of the lemma. ∎

Similar arguments (see [31]) also establish the following result.

Lemma (3.4): *Let* $h(x,\lambda)$ *be the function defined in Lemma (3.3) and let*

$$g(x,\lambda) = h(x,\lambda) - \int_x^\infty E(x-t,\lambda)e^{iw(t-x)}P(t)g(x,t)\,dt.$$

Then this Volterra equation has a unique solution $g(x,\lambda)$ *which satisfies the same inequalities given for* $h(x,\lambda)$ *in Lemma (3.3).*

The next result tells us how $y(x,z)(z(x,\lambda))$ approaches $y(x)(z(x))$ as $\lambda \to c$.

Lemma (3.5): *Suppose Assumption (2.1) holds and let* $y(x,\lambda), z(x,\lambda)$ *and* $y(x), z(x)$ *be the solutions established above.*

Then

$$e^{-iwx}y(x,\lambda) = y(x) + g_1(x,\lambda)$$

and

$$e^{iwx}z(x,\lambda) = z(x) + g_2(x,\lambda),$$

where

$$|g_1(x,\lambda)| \le \begin{cases} K\nu(x) & x \ge 0, \\ K\left[\nu(w) - \frac{2|w|x}{1-2|w|x}\right](1-x), & x < 0, \end{cases}$$

and

$$|g_2(x,\lambda)| \le \begin{cases} K\nu(x) & x \le 0, \\ K\left[\nu(w) + \frac{2|w|x}{1-2|w|x}\right](1+x), & x > 0, \end{cases}$$

where $\nu(w)$ *is as in Lemma(3.3).*

Proof: *For $y(x,\lambda)$,* we multiply (3.1) by e^{-iwx} and subtract (3.5) from the result. This shows that $g(x,\lambda) = y(x,\lambda)e^{-iwx} - y(x)$ satisfies the Volterra equation of Lemma (3.4). The argument for $z(x,\lambda)$ follows similarly. ■

2. The Jost functions and their asymptotics.

For $\lambda \in S \equiv \{\lambda | \Im\lambda \geq 0, \lambda \neq \pm c\}$, the variation of parameters formula gives for (2.1)

$$y(x,\lambda) = E(x,\lambda)y(0,\lambda) + \int_0^x E(x-t,\lambda)P(t)y(t)\,dt. \tag{3.10}$$

Let $\epsilon > 0$ and set $S_\epsilon = \{\lambda \mid \lambda \in S, |\lambda - c| \geq \epsilon, |\lambda + c| \geq \epsilon\}$. Writing $z(x,\lambda) = e^{iwx}y(x,\lambda)$ results in the equation

$$z(x,\lambda) = E(x,\lambda)z(0,\lambda) + \int_0^x e^{iw(x-t)}E(x-t,\lambda)P(t)z(t)\,dt.$$

Then an application of the Gronwall inequality leads to

$$|z(x,\lambda)| \leq K, \text{ where } K = K(\epsilon, y(0,\lambda)),$$

and hence

$$|y(x,\lambda)| \leq Ke^{\tau x}, \tau = \Im w, \lambda \in S_\epsilon, 0 \leq x < \infty. \tag{3.11}$$

Lemma (3.6): *Let $\Im w > 0$ and define*

$$A_y(\lambda) \equiv \left(\frac{w}{i(\lambda + c)}, 1\right) y(0,\lambda) + \int_0^\infty e^{iwt} \left(\frac{w}{i(\lambda + c)}, 1\right) Q(y(t))\,dt \tag{3.12}$$

and

$$B_y(\lambda) \equiv \left(\frac{w}{i(\lambda+c)}, -1\right) y(0,\lambda) - \int_{-\infty}^{0} e^{-iwt}\left(\frac{w}{i(\lambda+c)}, -1\right) Q(y(t))\,dt. \tag{3.13}$$

Then

$$\lim_{x\to\infty} e^{iwx} y(x,\lambda) = A_y(\lambda)\begin{pmatrix} \frac{i(\lambda+c)}{2w} \\ \frac{1}{2} \end{pmatrix}$$

and

$$\lim_{x\to-\infty} e^{-iwx} y(x,\lambda) = B_y(\lambda)\begin{pmatrix} \frac{i(\lambda+c)}{2w} \\ -\frac{1}{2} \end{pmatrix}.$$

In particular, $A_y(\lambda)$ and $B_y(\lambda)$ are analytic on $\Im\lambda > 0$, are bounded on S_ϵ and each has a continuous extension to $(-\infty,-c)\cup(c,\infty)$.

Proof: The integral representation of $A_y(\lambda)$ and $B_y(\lambda)$ yield the analyticity (cf. [17]). The bound (3.11) (and straightforward computation) yields the $(x\to\infty)$ limit and shows that $A_y(\lambda)$ has a continuous extension to $(-\infty,-c)\cup(c,\infty)$. Boundedness of $A_y(\lambda)$ is clear from the relationship between $y(x,\lambda)$ and $A_y(\lambda)$. Corresponding statements for $B_y(\lambda)$ are established similarly, with the inequality $|y(x,\lambda)| \le Ke^{-\tau x}, \tau = \Im w, \lambda \in S_\epsilon$,
$-\infty < x \le 0$ replacing (3.1).∎

The integral representation of $A_y(\lambda)$ shows that it is real valued on $(-c,c)$. We therefore obtain an analytic continuation by defining $A_y(\bar\lambda)$ on $\Im\lambda < 0$ by $A_y(\lambda) = \overline{A_y(\bar\lambda)}$. Similarly we obtain an analytic continuation, into $\Im\lambda < 0$, for $B_y(\lambda)$.

Lemma (3.7): *Let $\lambda \in (-\infty,-c)\cup(c,\infty)$. Then*

$$y(x,\lambda) = Re\left\{e^{-iwx}A_y(\lambda)\begin{pmatrix} \frac{i(\lambda+c)}{w} \\ 1 \end{pmatrix}\right\} + o(1), \text{ as } x\to+\infty$$

and

$$y(x,\lambda) = Re\left\{e^{iwx}B_y(\lambda)\begin{pmatrix}\frac{i(\lambda+c)}{w}\\ 1\end{pmatrix}\right\} + o(1), \text{ as } x \to -\infty.$$

Proof: Using (3.11), we can write

$$y(x,\lambda) = E(x,\lambda)y(0,\lambda) + \int_0^\infty E(x-t,\lambda)Q(y(t))\,dt + o(1)\ (x \to \infty)$$

and the first equality is established.

The second equality is proven similarly using the corresponding bound for $y(x,\lambda)$ as $x \to -\infty$. ∎

We observe that Lemma (3.2) shows that $A_y(\lambda)$ (respectively, $B_y(\lambda)$) determines the asymptotic phase of $y(x,\lambda)$ as $x \to +\infty$ (respectively, $x \to -\infty$). In particular, let $\Phi_A(\lambda) = \arg A_y(\lambda)$ and $\Phi_B(\lambda) = \arg B_y(\lambda)$, then we have

$$y_2(x,\lambda) = |A_y(\lambda)|\cos(wx - \Phi_A(\lambda)) + o(1) \text{ as } x \to +\infty,$$

and

$$y_2(x,\lambda) = |B_y(\lambda)|\cos(wx - \Phi_A(\lambda)) + o(1) \text{as } x \to -\infty,$$

with similar asymptotic equalities holding for $y_1(x,\lambda)$ also.

Featured in the definition of the M(λ)-coefficient were two solutions $\theta(x,\lambda)$ and $\phi(x,\lambda)$.

Recall that these are defined by the initial conditions

$$\theta(0,\lambda) = \begin{pmatrix}1\\ 0\end{pmatrix},\ \phi(0,\lambda) = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

These solutions, which are defined and continuous for all $x \in (-\infty,\infty)$ and analytic for all complex numbers

λ will play a key role in our development.

A proof of part (i) of the next lemma can be found in [17] ([17], Lemma (2.1)). The proof presented here is very similar.

Lemma (3.8): *Let $\lambda \in S$ and let $v = (v_1 + v_2)/2$. Then as $\lambda \to \infty$, we have*

$$(i)\ A_\theta(\lambda) \to -i\exp(-i\int_0^\infty v(t)\,dt)$$

$$(ii)\ A_\phi(\lambda) \to \exp(-i\int_0^\infty v(t)\,dt)$$

$$(iii)\ B_\theta(\lambda) \to -i\exp(-i\int_{-\infty}^0 v(t)\,dt)$$

$$(iv)\ B_\phi(\lambda) \to -\exp(-i\int_{-\infty}^0 v(t)\,dt)$$

Proof: We only prove (iii) here. The other conclusions are established similarly. Consider, first $\lambda \in \Re$. First we establish that as $|\lambda| \to \infty$, we have

$$\theta(x,\lambda) = \begin{pmatrix} \cos(wx + \int_0^x v(t)\,dt) + o(1) \\ -\sin(wx + \int_0^x v(t)\,dt) + o(1) \end{pmatrix}. \tag{3.14}$$

Set

$$z \equiv \frac{1}{2}\begin{pmatrix} \frac{1}{\lambda+c} & \frac{1}{iw} \\ \frac{1}{\lambda+c} & -\frac{1}{iw} \end{pmatrix}\theta. \tag{3.15}$$

Then

$$z' = \left\{ \begin{pmatrix} iw & 0 \\ 0 & -iw \end{pmatrix} + P_0 + \frac{w}{\lambda+c}\,\frac{v_1}{2i}M_1 - \frac{\lambda+c}{w}\,\frac{v_2}{2i}M_2 \right\} z,$$

where

$$P_0 = \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix},\ M_1 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Now we have that for $|\lambda|$ sufficiently large, say $|\lambda| > \lambda_0$,

$$\left|\frac{w}{\lambda+c}-1\right| \le \frac{4c}{|\lambda|} \text{ and } \left|\frac{\lambda+c}{w}-1\right| \le \frac{4c}{|\lambda|}.$$

Therefore, we see that for $|\lambda| > \lambda_0$ the z-equation reads

$$z' = \left\{ \begin{pmatrix} iw & 0 \\ 0 & -iw \end{pmatrix} + P_0 + \frac{v_1}{2i}M_1 - \frac{v_2}{2i}M_2 + O\left(\frac{1}{\lambda}\right) v_1 M_1 + O\left(\frac{1}{\lambda}\right) v_2 M_2 \right\} z.$$

Next define $n(x) = iwx + i\int_0^x v(t)\,dt$ and set

$$\Psi(x) = \begin{pmatrix} e^{-n(x)} & 0 \\ 0 & e^{n(x)} \end{pmatrix} z(x). \tag{3.16}$$

Straightforward computation shows that

$$\begin{aligned} \Psi_1' &= \tfrac{i}{2}(v_2 - v_1)e^{-2n}\Psi_2 + pe^{-2n}\Psi_2 + O\left(\tfrac{1}{\lambda}\right) v_2(\Psi_1 + e^{-2n}\Psi_2) + \\ &\quad O\left(\tfrac{1}{\lambda}\right) v_1(e^{-2n}\Psi_2 - \Psi_1), \\ \Psi_2' &= -\tfrac{i}{2}(v_2 - v_1)e^{-2n}\Psi_1 + pe^{-2n}\Psi_1 - O\left(\tfrac{1}{\lambda}\right) v_2(\Psi_2 + e^{-2n}\Psi_1) + \\ &\quad O\left(\tfrac{1}{\lambda}\right) v_1(\Psi_2 - \Psi_1 e^{2n}). \end{aligned} \tag{3.17}$$

Noting that $z(0) = \frac{1}{2(\lambda+c)}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and letting

$$\tilde{\Psi}_i = \Psi_i - \frac{1}{2(\lambda+c)}, \qquad i = 1,2$$

we then obtain from (3.17) the following :

$$\tilde{\Psi}_1 = \frac{i}{2}\int_0^x (v_2 - v_1)e^{-2n}\tilde{\Psi}_2 - \frac{i}{2}\int_0^x (v_2 - v_1)\frac{e^{-2n}}{2(\lambda+c)} + \int_0^x pe^{-2n}\tilde{\Psi}_2 -$$

$$\int_0^x p\frac{e^-2n}{2(\lambda+c)} + O(\frac{1}{\lambda})\int_0^x v_2(\tilde{\Psi}_1 + e^{-2n}\tilde{\Psi}_2) - O\left(\frac{1}{\lambda}\right)\int_0^x v_2\left(\frac{1+e^{-2n}}{2(\lambda+c)}\right)$$

$$+O\left(\frac{1}{\lambda}\right)\int_0^x v_1(e^{-2n}\tilde{\Psi}_2-\tilde{\Psi}_1)-O\left(\frac{1}{\lambda}\right)\int_0^x v_1\left(\frac{e^{-2n}-1}{2(\lambda+c)}\right),$$

$$\tilde{\Psi}_2(x)=-\frac{i}{2}\int_0^x(v_2-v_1)e^{2n}\tilde{\Psi}_1+\frac{i}{2}\int_0^x(v_2-v_1)\frac{e^{2n}}{2(\lambda+c)}+\int_0^x pe^{2n}\tilde{\Psi}_1-\int_0^x p\frac{e^{2n}}{2(\lambda+c)}$$

$$-O\left(\frac{1}{\lambda}\right)\int_0^x v_2(\tilde{\Psi}_2+e^{2n}\tilde{\Psi}_1)+O\left(\frac{1}{\lambda}\right)\int_0^x v_2\left(\frac{1+e^{2n}}{2(\lambda+c)}\right)+$$

$$O\left(\frac{1}{\lambda}\right)\int_0^x v_1(\tilde{\Psi}_2-\tilde{\Psi}_1e^{2n})-O\left(\frac{1}{\lambda}\right)\int_0^x v_1\left(\frac{1-e^{2n}}{2(\lambda+c)}\right).$$

These expressions result in

$$\begin{aligned}|\tilde{\Psi}(x)| &\le K\int_0^x|P(t)||\tilde{\Psi}(t)|\,dt+\tfrac{1}{2|\lambda+c|}\left|\int_0^x(v_2-v_1)(e^{2n}+e^{-2n})\right|\\ &+\tfrac{1}{2|\lambda+c|}\left|\int_0^x p(e^{2n}+e^{-2n})\right|+O\left(\tfrac{1}{\lambda}\right)\tfrac{1}{2|\lambda+c|}\left|\int_0^x(v_2+v_1)(e^{2n}+e^{-2n})\right| \qquad (3.2)\\ &+O(\tfrac{1}{\lambda})\tfrac{1}{2|\lambda+c|}\left|\int_0^x(v_1+v_2)\right|.\end{aligned}$$

The 4^{th} and 5^{th} terms on the right hand side of (3.18) are clearly $o(\frac{1}{\lambda})$ as $|\lambda|\to\infty$. We want to show that the same is true for the 2^{nd} and 3^{rd} terms on the right hand side of (3.18).

Consider the term $\int_0^\infty p(t)e^{2n(t)}\,dt$. Let $\epsilon>0$ and choose $R=R(\epsilon)$ so that $\int_R^\infty |p(t)|\,dt<\epsilon$. Then

$$\sup_{x\ge 0}\left|\int_0^x p(t)e^{i\int_0^t v(s)ds}\,dt\right| \le \int_R^\infty |p(t)|\,dt+\left|\int_0^R p(t)e^{i\int_0^t v(s)ds}\,dt\right|$$

and hence, by the Riemann-Lebesgue Lemma, we get

$$\lim_{n\to\infty}\left(\sup_{x\ge 0}\left|\int_0^x p(t)e^{2n(t)}\,dt\right|\right)=0.$$

The argument for the other terms is similar.

Thus

$$|\tilde{\Psi}(x)| \leq o(\frac{1}{\lambda}) + K\int_0^x |P(t)|\,|\tilde{\Psi}(t)|\,dt$$

and Gronwall's Lemma once again yields $|\tilde{\Psi}(x)| \leq o(\frac{1}{\lambda})$. Therefore, we obtain

$$\begin{aligned} \Psi_1(x) &= \tfrac{1}{2(\lambda+c)} + o(\tfrac{1}{\lambda}), \\ \Psi_2(x) &= \tfrac{1}{2(\lambda+c)} + o(\tfrac{1}{\lambda}). \end{aligned} \tag{3.19}$$

Taking into account (3.15) and (3.16), the asymptotics (3.19) establish (3.14).

From (3.13), we have

$$B_0(\lambda) = \frac{w}{i(\lambda+c)} + \int_{-\infty}^0 e^{iwt}\left(\frac{iw}{\lambda+c}Q_1(\theta(t)) + Q_2(\theta(t))\right)dt.$$

For $|\lambda| > \lambda_0$, (3.14) now implies that

$$B_0(\lambda) = -i + \int_{-\infty}^0 e^{-iwt}[i(p\cos\mu - v_1\sin\mu) - (v_2\cos\mu - p\sin\mu)]\,dt + o(1) \tag{3.20}$$

where $\mu = \mu(t) \equiv w(t) + \int_0^t v(s)ds$.

However

$$\begin{aligned} \textstyle\int_{-\infty}^0 e^{iwt}p(t)(i\cos\mu(t) + \sin\mu(t))\,dt &= \textstyle i\int_{-\infty}^0 e^{iwt}p(t)e^{-i\mu(t)}\,dt \\ &= \textstyle i\int_{-\infty}^0 e^{-iwt}p(t)e^{-iwt}e^{-i\int_0^t v(s)ds}\,dt \to 0, \end{aligned}$$

as $|\lambda| \to \infty$, by the Riemann- Lebesgue Lemma. Another application of the Riemann-Lebesgue Lemma produces, as $|\lambda| \to \infty$

$$\int_{-\infty}^0 e^{-iwt}(v_2\cos\mu(t) + iv_1\sin\mu(t))\,dt = -i\left[1 - \exp\left(-i\int_{-\infty}^0 v(t)\,dt\right)\right],$$

and the proof of (iii) is complete (by(3.20)) for $\lambda \in \Re$. Since $B_0(\lambda)$ is bounded for $\lambda \in S_\epsilon$ and has a limit as $|\lambda| \to \infty$ on e, an application of the Phragmén-Lindelöff theorem ([8], p.237) completes the proof of (iii). ■

The next Lemma links the asymptotic behavior of the functions $A_\theta(\lambda), A_\phi(\lambda), B_\theta(\lambda)$, $B_\phi(\lambda)$ as $\lambda \to \pm c$, with behavior of the solutions $\theta(x,\lambda)$ and $\phi(x,\lambda)$ at $\lambda = \pm c$. The proof given here is modelled after the methods of reference [17].

Lemma (3.9): *Suppose Assumption (2.1) holds. Then there are constants* $\alpha_0^\theta, \alpha_0^\phi, \beta_0^\theta, \beta_0^\phi, \alpha_{-0}^\theta, \alpha_{-0}^\phi, \beta_{-0}^\theta, \beta_{-0}^\phi \in \Re$ *and* $\alpha_1^\theta, \alpha_1^\phi, \beta_1^\theta, \beta_1^\phi, \alpha_{-1}^\theta, \alpha_{-1}^\phi, \beta_{-1}^\theta, \beta_{-1}^\phi \in i\Re$ *such that the following asymptotic relations hold: as* $\lambda \to c$

$$A_\theta(\lambda) = \begin{cases} \alpha_0^\theta + o(1) & \text{if } \theta_1(x,c) \text{ is unbounded as } x \to \infty, \\ \alpha_1^\theta w + o(w) & \text{if } \theta_1(x,c) \text{ is bounded as } x \to \infty, \end{cases}$$

$$A_\phi(\lambda) = \begin{cases} \alpha_0^\phi + o(1) & \text{if } \theta_1(x,c) \text{ is unbounded as } x \to \infty, \\ \alpha_1^\phi w + o(w) & \text{if } \theta_1(x,c) \text{ is bounded as } x \to \infty, \end{cases}$$

$$B_\theta(\lambda) = \begin{cases} \beta_0^\theta + o(1) & \text{if } \theta_1(x,c) \text{ is unbounded as } x \to -\infty, \\ \beta_1^\theta w + o(w) & \text{if } \theta_1(x,c) \text{ is bounded as } x \to -\infty, \end{cases}$$

$$B_\phi(\lambda) = \begin{cases} \beta_0^\phi + o(1) & \text{if } \theta_1(x,c) \text{ is unbounded as } x \to -\infty, \\ \beta_1^\phi w + o(w) & \text{if } \theta_1(x,c) \text{ is bounded as } x \to -\infty, \end{cases}$$

and as $\lambda \to -c$

$$\frac{i(\lambda+c)}{w} A_\theta(\lambda) = \begin{cases} \alpha_{-1}^\theta + o(1) & \text{if } \theta_2(x,-c) \text{ is unbounded as } x \to \infty, \\ \alpha_{-0}^\theta w + o(w) & \text{if } \theta_2(x,-c) \text{ is bounded as } x \to \infty, \end{cases}$$

$$\frac{i(\lambda+c)}{w} A_\phi(\lambda) = \begin{cases} \alpha_{-1}^\phi + o(1) & \text{if } \theta_2(x,-c) \text{ is unbounded as } x \to \infty, \\ \alpha_{-0}^\phi w + o(w) & \text{if } \theta_2(x,-c) \text{ is bounded as } x \to \infty, \end{cases}$$

$$\frac{i(\lambda+c)}{w} B_\theta(\lambda) = \begin{cases} \beta_{-1}^\theta + o(1) & \text{if } \theta_2(x,-c) \text{ is unbounded as } x \to -\infty, \\ \beta_{-0}^\theta w + o(w) & \text{if } \theta_2(x,-c) \text{ is bounded as } x \to -\infty, \end{cases}$$

$$\frac{i(\lambda+c)}{w}B_\phi(\lambda) = \begin{cases} \beta^{\phi}_{-1} + o(1) & \textit{if } \theta_2(x,-c) \textit{ is unbounded as } x \to -\infty, \\ \beta^{\phi}_{-0} w + o(w) & \textit{if } \theta_2(x,-c) \textit{ is bounded as } x \to -\infty. \end{cases}$$

Moreover, the following pairs are nonvanishing

$$(\alpha_0^\theta, \alpha_0^\phi), (\alpha_0^\theta, \alpha_1^\theta), (\beta_0^\theta, \beta_0^\phi), (\beta_0^\theta, \beta_1^\theta), (\alpha_0^\phi, \alpha_1^\phi), (\beta_0^\phi, \beta_1^\phi),$$

and

$$(\alpha^\theta_{-1}, \alpha^\phi_{-1}), (\alpha^\theta_{-0}, \alpha^\theta_{-1}), (\beta^\theta_{-1}, \beta^\phi_{-1}), (\beta^\theta_{-1}, \beta^\theta_{-0}), (\alpha^\phi_{-1}, \alpha^\phi_{-0}), (\beta^\phi_{-1}, \beta^\phi_{-0}).$$

Proof: We only include a proof for A_ϕ as $\lambda \to c$ here, noting that the other cases are similar. First, the following statements are easy to prove ([17]);

$$\begin{aligned}
&\theta_1(x,c) \textit{ is bounded as } x \to \infty \iff \alpha_0^\theta \equiv \textstyle\int_0^\infty Q_2(\theta(t,c))\,dt = 0, \\
&\phi_1(x,c) \textit{ is bounded as } x \to \infty \iff \alpha_0^\phi \equiv 1 + \textstyle\int_0^\infty Q_2(\phi(t,c))\,dt = 0, \\
&\theta_1(x,c) \textit{ is bounded as } x \to -\infty \iff \beta_0^\theta \equiv \textstyle\int_{-\infty}^0 Q_2(\theta(t,c))\,dt = 0, \\
&\phi_1(x,c) \textit{ is bounded as } x \to -\infty \iff \beta_0^\phi \equiv -1 + \textstyle\int_{-\infty}^0 Q_2(\phi(t,c))\,dt = 0 \qquad (3.21) \\
&\theta_2(x,-c) \textit{ is bounded as } x \to \infty \iff \alpha^\theta_{-1} \equiv -2ci(1 + \textstyle\int_0^\infty Q_1(\theta(t,-c))\,dt) = 0, \\
&\phi_2(x,-c) \textit{ is bounded as } x \to \infty \iff \alpha^\phi_{-1} \equiv -2ci \textstyle\int_0^\infty Q_1(\phi(t,-c))\,dt = 0, \\
&\theta_2(x,-c) \textit{ is bounded as } x \to -\infty \iff \beta^\theta_{-1} \equiv -2ci\left(1 - \textstyle\int_{-\infty}^0 Q_1(\theta(t,-c))\,dt\right) = 0, \\
&\phi_2(x,-c) \textit{ is bounded as } x \to -\infty \iff \beta^\phi_{-1} \equiv 2ci \textstyle\int_{-\infty}^0 Q_1(\phi(t,-c))\,dt = 0.
\end{aligned}$$

We note that if $\theta_1(x,c) = O(1)$, then $\theta_2(x,c) = o(1)$, $x \to \pm\infty$ and otherwise $\theta_1(x,c) = O(x)$ and $\theta_2(x,c) = O(1)$, $x \to \pm\infty$ with the same comments holding for $\phi(x,c)$. Also, if $\theta_2(x,-c) = O(1)$, then $\theta_1(x,-c) = o(1)$, $x \to \pm\infty$ and otherwise $\theta_2(x,-c) = O(x)$ and $\theta_1(x,-c) = O(1)$, $x \to \pm\infty$ with the same holding also for $\phi(x,-c)$. Thus relations (3.21) define constants which characterize boundedness of solutions $\theta(x,\lambda)$ and $\phi(x,\lambda)$ at $\lambda = \pm c$ as $x \to \pm\infty$. Also, since the solutions $\theta(x,\pm c)$ and $\phi(x,\pm c)$ cannot be simultaneously bounded as $x \to \pm\infty$, the last statement of the Lemma follows once the remaining constants are established. Consider $A_\phi(\lambda)$ for the moment. If $\alpha_0^\phi \neq 0$, then we can write

$$\begin{aligned} A_\phi(\lambda) - \alpha_0^\phi &= \int_0^\infty e^{iwt}[(Q_2(\phi(t,\lambda)) - Q_2(\phi(t,c)) - iw(\lambda+c)^{-1}Q_1(\phi(t,\lambda))]\,dt \\ &+ \int_0^\infty (e^{iwt}-1)Q_2(\phi(t,c))\,dt. \end{aligned} \tag{3.22}$$

Using arguments such as used in the proof of Lemma (4.1), below, we can show that the integrals in (3.22) are $o(1)$ as $\lambda \to c$. If $\alpha_0^\phi = 0$, then we write

$$\begin{aligned} &A_\phi(\lambda) - \tfrac{w}{2ic}\int_0^\infty dt\,[(Q_1(\phi(t,c)) - 2ctQ_2(\phi(t,c))] = \\ &\int_0^\infty dt\, e^{iwt}[Q_2(\phi(t,\lambda)) - Q_2(\phi(t,c))] + \int_0^\infty dt e^{iwt}[\Big(\tfrac{Q_1(\phi(t,c))}{2c} - \tfrac{Q_1(\phi(t,\lambda))}{\lambda+c}\Big)]+ \\ &\int_0^\infty (e^{iwt} - 1 - iwt)Q_2(\phi(t,c))\,dt + \int_0^\infty dt(1-e^{iwt})\tfrac{iw}{2c}Q_1(\phi(t,c)). \end{aligned} \tag{3.23}$$

Arguments as those in Lemma (4.1) again show that the right side of (3.23) is $o(w)$ as $\lambda \to c$.

Letting

$$\alpha_1^\phi \equiv \frac{1}{2ci}\int_0^\infty (Q_1(\phi(t,c)) - 2ctQ_2(\phi(t,c)))\,dt,$$

the Lemma is established for $A_\phi(\lambda)$. The other constants are defined to be

$$\beta_1^\phi = -\tfrac{1}{2ci}\int_{-\infty}^0 (Q_1(\phi) - 2ctQ_2(\phi))(t,c)\,dt,$$

$$\alpha_1^\theta = \tfrac{1}{2ci}(1 + \int_0^\infty (Q_1(\theta) - 2ctQ_2(\theta))(t,c)\,dt),$$

$$\beta_1^\theta = \tfrac{1}{2ci}(1 - \int_{-\infty}^0 (Q_1(\theta) - 2ctQ_2(\theta))(t,c)\,dt),$$

$$\alpha_{-0}^\phi = 1 + \int_0^\infty (Q_2(\phi) - 2ctQ_1(\phi))(t,c)\,dt,$$

$$\beta_{-0}^\phi = 1 - \int_{-\infty}^0 (Q_2(\phi) - 2ctQ_1(\phi))(t,c)\,dt,$$

$$\alpha_{-0}^\phi = \int_0^\infty (Q_2(\phi) - 2ctQ_1(\phi))(t,c)\,dt,$$

and

$$\beta_{-0}^\theta = -\int_{-\infty}^0 (Q_2(\theta) - 2ctQ_1(\theta))(t,c)\,dt. \blacksquare$$

Chapter IV

THE TITCHMARSH-WEYL $M(\lambda)$- COEFFICIENT AND SPECTRAL MATRIX

In this section we prove Theorem (2.2) and Corollary (2.3). Assumption (2.1) holds throughout this section.

Recall the definition (1.6) of $M(\lambda)$,

$$M(\lambda) = (m_- - m_+)^{-1} \begin{pmatrix} 1 & \frac{1}{2}(m_- + m_+) \\ \frac{1}{2}(m_- + m_+) & m_- m_+ \end{pmatrix}$$

where $m_\pm = m_\pm(\lambda)$ denote the Titchmarsh-Weyl $m(\lambda)$–coefficients at $\pm\infty$.

From the definitions (1.13) of $m_\pm(\lambda)$ and Lemma (3.6), if we write $M(\lambda) = (m_{ij})$, then we see that $m_{11}(\lambda) = [A_\phi(\lambda)B_\phi(\lambda)]F^{-1}(\lambda)$, $m_{22}(\lambda) = [A_\theta(\lambda)B_\theta(\lambda)]F^{-1}(\lambda)$, and $m_{12}(\lambda) = m_{21}(\lambda) = \frac{1}{2}[A_\theta(\lambda)B_\phi(\lambda) + A_\phi(\lambda)B_\theta(\lambda)]F^{-1}(\lambda)$, where we have defined $F(\lambda) \equiv A_\phi(\lambda)B_\theta(\lambda) - A_\theta(\lambda)B_\phi(\lambda)$.

The asymptotics of Lemma (3.9) give the following behavior (as $\lambda \to c$) for the numerators, $N(\cdot)$, of $m_{11}(\lambda)$, $m_{22}(\lambda)$ and $m_{12}(\lambda)$, respectively,

$$\begin{aligned} N(m_{11}) &= \alpha_0^\phi \beta_0^\phi + o(1) \\ N(m_{22}) &= \alpha_0^\theta \beta_0^\theta + o(1) \\ \text{and} \quad N(m_{12}) &= -(\alpha_0^\phi \beta_0^\theta + \alpha_0^\theta \beta_0^\phi) + o(1) \end{aligned} \tag{4.1}$$

Taking into account the last statement of Lemma (3.9), then Theorem (2.2) will follow once we prove the following

Lemma (4.1): *Let γ be as in Lemma (3.1). Then there exists a nonzero constant γ_1 such that the following behavior obtains as $\lambda \to c$:*

$$F(\lambda) = \begin{cases} \gamma + o(1) & \textit{if } \lambda = c \textit{ is not a } \text{HBS}, \\ \gamma_1 w + o(w) & \textit{if } \lambda = c \textit{ is a } \text{HBS}. \end{cases}$$

Assume the result of Lemma (4.1), and let $\lambda = c + iv$. Then, since $w = (2ci)^{\frac{1}{2}} + O(v)$ as $v \to 0$, if $\gamma \neq 0$, the relations (4.1) imply that $v^{\frac{1}{2}} m_{ij}(c + iv) \to 0$ as $v \to 0, i, j = 1, 2$. If $\gamma = 0$, then again (4.1) implies that, as $v \to 0, v^{\frac{1}{2}} m_{ij}(c + iv) \to s_{ij} + o(1)$, where $s_{11} = \alpha_0^{\phi}\beta_0^{\phi}\gamma_1^{-1}$, $s_{22} = \alpha_0^{\theta}\beta_0^{\theta}\gamma_1^{-1}$, and $s_{12} = s_{21} = -\frac{1}{2}(\alpha_0^{\phi}\beta_0^{\theta} + \alpha_0^{\theta}\beta_0^{\phi})\gamma_1^{-1}$, and we write $S = (s_{ij})$. This completes the proof of Theorem (2.2) for $\lambda = c$, taking account of Lemma(3.9).

For $\lambda = -c$, we similarly find a constant γ_{-1} such that

$$F(\lambda) = \begin{cases} \gamma_- + o(1) & \text{if } \lambda = c \text{ is not a HBS}, \\ \gamma_{-1} w + o(w) & \text{if } \lambda = c \text{ is a HBS}, \end{cases}$$

as $\lambda \to -c$. The matrix $S(-c) = (s_{ij}^-)$ is given by $s_{11} = \alpha_{-1}^{\phi}\beta_{-1}^{\phi}\gamma_{-1}^{-1}$, $s_{22} = \alpha_{-1}^{\theta}\beta_{-1}^{\theta}\gamma_{-1}^{-1}$ and $s_{12} = s_{21} = -\frac{1}{2}(\alpha_{-1}^{\phi}\beta_{-1}^{\theta} + \alpha_{-1}^{\theta}\beta_{-1}^{\phi})\gamma_{-1}^{-1}$.

Theorem (2.2) follows since $\lambda = \pm c$ are the only possible HBS's.

Corollary (2.3) also easily follows from the relations (4.1), with a similar argument for $\lambda = -c$. First, for $\lambda > c$, we can ([15]) pass to the limit under the integral sign and differentiate the Titchmarsh-Kodaira formula (1.7). This differentiation yields $\frac{d\tau(\lambda)}{d\lambda} = -\frac{1}{\pi}\Im M(\lambda)$. The corollary then follows from direct calculation using $\Im M(\lambda) = \frac{1}{2i}(M(\lambda) - M^*(\lambda))$, and similarly for $\lambda = -c$.

In order to establish Lemma(4.1), we will need to have some bounds on certain solution differences. These are given in the form of the following

Lemma (4.2)(cf. [20],Lemma (2.2)): *Let $x \geq 0$and let $y(x,\lambda)$be the solution of (2.1) defined by the initial condition $y(0,\lambda) = (a,b)^T, a, b$ arbitrary constants independent of λ. Suppose further that $y_0(x) \equiv y(x,c)$ is bounded as $x \to \infty$. Then the following bound holds:*

$$|y(x,\lambda) - y_0(x)| \leq K\left(\frac{|w|\,x}{1+|w|\,x}\right)^2, \; K = constant.$$

We only establish this bound for $x \geq 0$here, but we note that a similar bound can be obtained in similar fashion for $x \leq 0$. Namely one assumes that $y_0(x) \equiv y(x,c)$is bounded as $x \to -\infty$and obtains the bound

$$|y(x,\lambda) - y_0(x)| \leq K\left(\frac{|wx|}{1+|wx|}\right)^2.$$

Let us also observe that the same bounds also hold for $|y(x,\lambda) - y(x,-c)|$, if $y(x,-c)$is bounded, all with different K's.

Proof: By variation of parameters, we have

$$y(x,\lambda) = \begin{pmatrix} b\frac{\lambda+c}{w}\sin wx + a\cos wx \\ b\cos wx - a\frac{w}{\lambda+c}\sin wx \end{pmatrix} + \int_0^x E(x-t,\lambda)Q(y(t))\,dt, \tag{4.2}$$

$$y_0(x) = \begin{pmatrix} 2cbx + a \\ b \end{pmatrix} + \int_0^x E(x-t,c)Q(y_0(t))\,dt, \tag{4.3}$$

From (4.3), we have

$$y_{01}(x) = 2cx\,(b + \textstyle\int_0^x Q_2(y_0(t))\,dt) + a + \int_0^x (Q_1(y_0(t)) - Q_2(y_0(t)))\,dt,$$

$$y_{02}(x) = b + \textstyle\int_0^x Q_2(y_0(t))\,dt.$$

Therefore, standard asymptotic theory (see [5]) tells us that $y_0(x)$ is bounded as $x \to \infty$ if and only if $A = 0$, where

$$A \equiv b + \int_x^\infty Q_2(y_0(t))\,dt.$$

Let us consider $y(x,\lambda) - y_0(x)$. Recall that $|y(x,\lambda) - y_0(x)| = |y_1(x,\lambda) - y_{01}(x)| + |y_2(x,\lambda) - y_{02}(x)|$ and note that

$$|Q_i - Q_i^0| \le |P|\,|y - y_0|,\ |Q_i^0| \le |P|, i = 1, 2.$$

(4.2) and (4.3) therefore give us

$$\left.\begin{aligned}
|y(x,\lambda) - y_0(x)| &= |b\tfrac{\lambda+c}{w}\sin wx + a\cos wx - 2bcx - a + \textstyle\int_0^x \cos w(x-t)Q_1 \\
&\quad + \tfrac{\lambda+c}{w}\textstyle\int_0^x \sin w(x-t)Q_2 - \int_0^x (Q_1^0 - 2c(x-t)Q_2^0)| + |b\cos wx - a\tfrac{w}{\lambda+c}\sin wx \\
&\quad - b + \textstyle\int_0^x (\cos w(x-t)Q_2 - \tfrac{w}{\lambda+c}\sin w(x-t)Q_1) - \int_0^x Q_2^0| \\
&= |a(\cos wx - 1) + (\cos wx - 1)\textstyle\int_0^x Q_0^1 + \cos wx \int_0^x (\cos wt - 1)Q_1^0 \\
&\quad + (\sin wx - 1)\textstyle\int_0^x \sin wt Q_1^0 + \int_0^x \sin wt Q_1^0 + \int_0^x \cos w(x-t)(Q_1 - Q_1^0) \\
&\quad + 2cx\left(\tfrac{\lambda+c}{w}\tfrac{\sin wx}{2cx} - 1\right)(b + \textstyle\int_0^x Q_2^0) + \tfrac{\lambda+c}{w}\sin wx \int_0^x (\cos wt - 1)Q_2^0 \\
&\quad - \tfrac{\lambda+c}{w}(\cos wx - 1)\textstyle\int_0^x \sin wt Q_2^0 - \tfrac{\lambda+c}{w}\int_0^x \left(\sin wt - \tfrac{2cwt}{\lambda+c}\right)Q_2^0 \\
&\quad + \tfrac{\lambda+c}{w}\textstyle\int_0^x \sin(x-t)(Q_2 - Q_2^0)\Big| + |b(\cos wx - 1) - a\tfrac{w}{\lambda+c}\sin wx \\
&\quad + (\cos wx - 1)\textstyle\int_0^x Q_2^0 + \cos wx \int_0^x (\cos wt - 1)Q_2^0 + (\sin wx - 1)\int_0^x \sin wt Q_2^0 \\
&\quad + \textstyle\int_0^x \sin wt Q_2^0 + \int_0^x \cos w(x-t)(Q_2 - Q_2^0) - \tfrac{w}{\lambda+c}\sin wx \int_0^x Q_1^0 + \int_0^x \sin wt Q_1^0 \\
&\quad - \tfrac{w}{\lambda+c}\sin wx \textstyle\int_0^x (\cos wt - 1)Q_1^0 + \tfrac{w}{\lambda+c}(\cos wx - 1)\int_0^x \sin wt Q_1^0 \\
&\quad - \tfrac{w}{\lambda+c}\textstyle\int_0^x \sin w(x-t)(Q_1 - Q_1^0)\Big|
\end{aligned}\right\} \quad (4.4)$$

Since $A = 0, b + \int_0^x Q_2^0 = -\int_x^\infty Q_2^0$, and so we obtain

$$\left.\begin{aligned}
&\left|2cx\left(\tfrac{\lambda+c}{w}\tfrac{\sin wx}{2cx} - 1\right)(b + \textstyle\int_0^x Q_2^0)\right| \\
&\qquad \le \left|\tfrac{\lambda+c}{w}\tfrac{\sin wx}{2cx} - 1\right| \textstyle\int_x^\infty |tQ_2^0| \le K\left(\tfrac{wx}{1+wx}\right)^2,
\end{aligned}\right\} \quad (4.5)$$

where we have used the fact that $x \leq t$ and

$$\left|1 - \frac{\sin z}{z}\right| \leq K\left(\frac{|z|}{1+|z|}\right)^2, z \in \Re. \tag{4.6}$$

Next, using the estimates that for some $K \in \Re$,

$$|\sin z| \leq K\left(\frac{|z|}{1+|z|}\right), |1-\cos z| \leq K\left(\frac{|z|}{1+|z|}\right)^2,$$

the monotonicity of $K\left(\frac{|wx|}{1+|wx|}\right)$, the boundedness of $y_0(x)$, (4.5) and (4.6), we obtain from (4.4),

$$|y(x,\lambda) - y_0(x)| \leq K\left(\frac{|wx|}{1+|wx|}\right)^2 + K\left(\frac{|wx|}{1+|wx|}\right)\int_0^x |V(t)||y(t,\lambda) - y_0(t)|.$$

An application of the Gronwall Lemma and the monotonicity of $\frac{|x|}{1+|x|}$ therefore reveal that

$$|y(x,\lambda) - y_0(x)| \leq K\left(\frac{|wx|}{1+|wx|}\right)^2. \qquad \blacksquare$$

<u>Proof of Lemma (4.1)</u>: Let $y(x)$ and $z(x)$ be the solutions in Lemma (3.1) and let $\theta(x), \phi(x)$ be solutions to (3.3) such that $\theta(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \phi(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Straightforward computations yield, using (4.3),

$$\begin{aligned} y(x) &= \alpha_0^\phi \theta(x) + \alpha_0^\theta \phi(x), \\ z(x) &= \beta_0^\phi \theta(x) + \beta_0^\theta \phi(x). \end{aligned} \tag{4.7}$$

Therefore, $\gamma \equiv W[y,z] = \alpha_0^\phi \beta_0^\theta - \alpha_0^\theta \beta_0^\phi$. However, Lemma (3.9) yields $F(\lambda) = (\alpha_0^\phi \beta_0^\theta - \alpha_0^\theta \beta_0^\phi) + o(1)$, and therefore $F(\lambda) = \gamma + o(1)$ if $\gamma \neq 0$. Next, we exploit (4.7) to discover that

$$y_1(x) \to 2ci(\alpha_0^\phi \beta_1^\theta - \alpha_0^\theta \beta_1^\phi), \quad x \to -\infty,$$

$$z_1(x) \to 2ci(\beta_0^\theta \alpha_1^\phi - \beta_0^\phi \alpha_1^\theta), \quad x \to \infty.$$

Now if $\gamma = 0$, then there exists $\alpha \neq 0$ such that $y(x) = \alpha z(x)$.

Since $y_1(x) \to 1$ as $x \to \infty$, then we have $1 = \alpha \cdot 2ci(\alpha_0^{\phi}\beta_1^{\theta} - \alpha_0^{\theta}\beta_1^{\phi})$ and since $z_1(x) \to 1$, as $x \to -\infty$, $1 = \frac{1}{\alpha} \cdot 2ci(\alpha_0^{\phi}\beta_1^{\theta} - \alpha_0^{\theta}\beta_1^{\phi})$. Therefore, if $\gamma_1 \equiv \frac{1}{2ci}(\alpha + \frac{1}{\alpha})$, then the last statement of the Lemma is proved. Next, we establish that $\gamma_1 = \frac{1}{2ci}(\alpha + \frac{1}{\alpha})$ indeed. From the asymptotics given in Lemma (3.9), one can only conclude that

$$F(\lambda) = \gamma + o(1), \lambda \to c, \text{where } o(1) = \gamma_1 w + (\text{other } o(1) \text{ terms}).$$

Let $\psi(x, \lambda)$ be a solution of (1.1) satisfying

$$\psi(0, \lambda) = y_1(0), \quad \psi_2(0, \lambda) = y_2(0),$$

so that

$$\psi(x, \lambda) = y_1(0)\theta(x, \lambda) + y_2(0)\phi(x, \lambda). \tag{4.8}$$

Next, simple calculations show that

$$\begin{aligned} &y_1(0)F(\lambda) = \\ &\quad B_\phi(\lambda)\left\{-\tfrac{iw}{\lambda+c}y_1(0) + y_2(0) + y_1(0)\left(A_0(\lambda) + \tfrac{iw}{\lambda+c}\right) + y_2(0)(A_\phi(\lambda) - 1)\right\} \\ &\quad -A_\phi(\lambda)\left\{\tfrac{iw}{\lambda+c}y_1(0) + y_2(0) - y_1(0)\left(B_0(\lambda) + \tfrac{iw}{\lambda+c}\right) + y_2(0)(-B_\phi(\lambda) - 1)\right\}. \end{aligned} \tag{4.9}$$

Using (4.8) and (3.12)–(3.13), we may rewrite (4.9) as

$$\begin{aligned} &y_1(0)F(\lambda) = \\ &\quad -B_\phi(\lambda)\left\{\tfrac{iw}{\lambda+c}y_1(0) + y_2(0) + \int_0^\infty e^{iwt}\left(Q_2(\psi) - \tfrac{iw}{\lambda+c}Q_1(\psi)\right)(t,\lambda)\,dt\right\} \\ &\quad -A_\phi(\lambda)\left\{\tfrac{iw}{\lambda+c}y_1(0) + y_2(0) - \int_{-\infty}^0 e^{-iwt}\left(Q_2((\psi) + \tfrac{iw}{\lambda+c}Q_1(\psi)\right)(t,\lambda)\,dt\right\}. \end{aligned} \tag{4.10}$$

Using again the notation $Q_i^0 = Q_i(\psi(x, c))$ and $Q_i = Q_i(\psi(x, \lambda))$, we have

$$\int_0^\infty e^{iwt}(Q_2 - \frac{iw}{\lambda + c}Q_1)\,dt = \int_0^\infty \left(Q_2^0 - \frac{iw}{\lambda + c}Q_1^0\right)(e^{iwt} - 1)\,dt$$

$$+\int_0^\infty \left(Q_2^0-\frac{iw}{\lambda+c}Q_1^0\right)\,dt+\int_0^\infty[(Q_2-Q_2^0)-\frac{iw}{\lambda+c}(Q_1-Q_1^0)]e^{iwt}\,dt.$$

By Lemma (4.2) and the Dominated Convergence Theorem, we have that

$$\int_0^\infty[(Q_2-Q_2^0)-\frac{iw}{\lambda+c}(Q_1-Q_1^0)]e^{iwt}\,dt=o(w),\text{ as }\lambda\to c. \tag{4.11}$$

Since ψ is bounded ($\gamma=0$) as $x\to\infty$, $\int_0^\infty Q_1^0dt<\infty$, and by (4.3) we conclude that

$$\int_0^\infty\left(Q_2^0-\frac{iw}{\lambda+c}Q_1^0\right)\,dt=-\psi_{02}(o)+o(w)=-y_2(0)+o(w). \tag{4.12}$$

With $a=\psi_{01}(0)=y_1(0)$ in equation (4.3), we also find that

$$\int_0^\infty\left(Q_2^0-\frac{iw}{\lambda+c}\right)(e^{iwt}-1)\,dt=\frac{iw}{2c}(y_1(0)-1)+o(w). \tag{4.13}$$

Similar considerations reveal that as $\lambda\to c$,

$$\int_{-\infty}^0 e^{-iwt}\left(Q_2+\frac{iw}{\lambda+c}Q_1\right)\,dt=y_2(0)+\frac{iw}{2c}(y_1(0)+\alpha)+o(w), \tag{4.14}$$

where we have put $\alpha=\lim_{x\to-\infty}y_1(x)$.

From Lemma (3.9), we have $A_\phi(\lambda)=\alpha_0^\phi+o(1)$ and $B_\phi(\lambda)=\beta_0^\phi+o(1)$, as $\lambda\to c$, and from (4.7) we have that $\beta_0^\phi=z_1(0)=y_1(0)/\alpha$ and $\alpha_0^\phi=y_1(0)$. Therefore from (4.11) – (4.14), we see that (4.10) reveals that

$$F(\lambda)=\frac{1}{2ci}\left(\alpha+\frac{1}{\alpha}\right)w+o(w),\text{ as }\lambda\to c. \qquad\blacksquare$$

Chapter V

THE SCATTERING MATRIX AND LEVINSON'S THEOREM

Since we concern ourselves only with the continuous spectrum of H, our domain for λ in this chapter is $(-\infty, -c] \cup [c, \infty)$. To do this effectively, we let $w \in \Re$ and define λ in terms of w, i. e. , we set $\lambda = +\sqrt{w^2 + c^2}$. We emphasize that w is our basic spectral parameter by writing w for the spectral argument throughout this section, for example, where we wrote $y(x, \lambda)$ previously we now write $y(x, w)$.

In section 1 of this chapter, we prove Theorem (2.4). Levinson's Theorem is derived in Section 2. In Section 3, we study the specific problem of power law potentials. We are interested in deriving the leading correction behavior of the S-matrix, the leading order being predicted by Theorem (2.4). The results are the content of Theorem (5.1).

1. Continuity of the S-matrix.

Recall that we have the existence of the solutions of (1.1) which are defined by

$$y(x, w) = y_0(x, w) - \int_x^\infty E(x - t, w)P(t)y(t)\,dt, \tag{5.1)a}$$

$$z(x, w) = z_0(x, w) + \int_{-\infty}^x E(x - t, w)P(t)y(t)\,dt. \tag{5.1)b}$$

Let $w > 0$ and let

$$X_+ = \begin{pmatrix} 1 \\ \frac{iw}{\lambda+c} \end{pmatrix} e^{iwx}, \quad X_- = \begin{pmatrix} 1 \\ \frac{-iw}{\lambda+c} \end{pmatrix} e^{-iwx}.$$

Now, the Jost solutions $y^{\pm}(x, w), z^{\pm}(x, w)$ are solutions with asymptotic behaviors

$$\begin{gathered} y^+(x, w) \sim X_+, \\ y^-(x, w) \sim X_-, \quad \text{as } x \to +\infty \end{gathered} \tag{5.2}$$

and

$$
\begin{aligned}
z^+(x,w) &\sim X_-, \\
z^-(x,w) &\sim X_+, \quad \text{as } x \to -\infty.
\end{aligned}
\tag{5.3}
$$

In particular, $y^{\pm}(x,w)$ and $z^{\pm}(x,w)$ are defined by (5,1)a and (5.1)b, respectively. From (1.10), (1.11), (5.2), (5.3) and the comments about $u(x,w)$ in Chapter I §3, we obtain that

$$
y^+(x,w) = \frac{1}{\mathrm{T}} u(x,w) \text{ and } z^+(x,w) = \frac{1}{\mathrm{T}} u(x,-w). \tag{5.4}
$$

For $w < 0$ the same argument carries over with $v(x,w), v(x,-w)$ replacing $u(x,w), u(x,-w)$, respectively, and all that follows holds true as well. Next, let us write

$$
\begin{aligned}
z^+(x,w) &= ay^-(x,w) + by^+(x,w) \\
z^-(x,w) &= cy^-(x,w) + dy^+(x,w).
\end{aligned}
\tag{5.5}
$$

It is a matter of computation, using (5.2), (5.3), (5.5), to get

$$
\begin{aligned}
a &= \tfrac{\lambda+c}{2iw} W[z^+, y^+], \\
b &= -\tfrac{\lambda+c}{2iw} W[z^+, y^-], \\
c &= \tfrac{\lambda+c}{2iw} W[z^-, y^+], \\
d &= -\tfrac{\lambda+c}{2iw} W[z^-, y^-],
\end{aligned}
\tag{5.6}
$$

so that $ab - cd = 1$ since $W[z^+, z^-] = W[y^-, y^+] = \frac{2iw}{\lambda+c}$. However for real w, we see that $y^- = \overline{y^+}$, and $z^- = \overline{z^+}$. So since here we do assume real w, then we simply write y, z for y^+, z^+, respectively. In particular, (5.6) reads

$$
\begin{aligned}
a &= \tfrac{\lambda+c}{2iw} W[z, y], \\
b &= -\tfrac{\lambda+c}{2iw} W[z, \overline{y}], \\
c &= \tfrac{\lambda+c}{2iw} W[\overline{z}, y], \\
d &= -\tfrac{\lambda+c}{2iw} W[\overline{z}, \overline{y}].
\end{aligned}
\tag{5.6)'}
$$

Now we write, using (5.5) and the fact that $ad - bc = 1$,

$$
\begin{aligned}
y(x,w) &= a\overline{z}(x,w) - cz(x,w),\\
z(x,w) &= a\overline{y}(x,w) + by(x,w).
\end{aligned}
\tag{5.7}
$$

Then we have the following asymptotics holding:

$$
\begin{aligned}
y(x,w) &\sim aX_{+} - cX_{-} \ \text{ as } \to -\infty,\\
y(x,w) &\sim aX_{-} + bX_{+} \ \text{ as } \to +\infty,
\end{aligned}
\tag{5.8}
$$

by equations (5.2). However, (5.3), (5.4) and the relations (1.10), (1.11), reveal that

$$
\begin{aligned}
y(x,w) &\sim \frac{1}{\mathrm{T}}X_{+} + \frac{R_l}{\mathrm{T}}X_{-} \ \text{ as } x \to +\infty,\\
z(x,w) &\sim \frac{1}{\mathrm{T}}X_{-} + \frac{R_l}{\mathrm{T}}X_{+} \ \text{ as } x \to -\infty.
\end{aligned}
\tag{5.9}
$$

Comparison of (5.8) and (5.9) yields therefore

$$
a = \frac{1}{\mathrm{T}},\ b = \frac{R_r}{\mathrm{T}},\ \text{and } c = -\frac{R_l}{\mathrm{T}}. \tag{5.10}
$$

We turn once again to the functions

$$
\begin{aligned}
A_y(w) &= \left(\tfrac{w}{i(\lambda+c)}, 1\right) y(0,w) + \textstyle\int_0^\infty dt\, e^{iwt} \left(\tfrac{w}{i(\lambda+c)}, 1\right)\right) Q(y(t,w)),\\
B_y(w) &= \left(\tfrac{w}{i(\lambda+c)}, -1\right) y(0,w) - \textstyle\int_{-\infty}^0 dt\, e^{-iwt} \left(\tfrac{w}{i(\lambda+c)}, -1\right)\right) Q(y(t,w)).
\end{aligned}
\tag{5.11}
$$

We see from (5.11) that for $w \in \Re$, we have

$$
\overline{A_y(w)} = A_y(-w),\ \overline{B_y(w)} = -B_y(-w). \tag{5.12}
$$

Besides the solutions defined by (5.1), in particular $y(x,w)$ and $z(x,w)$ in this case, (1.1) also has a pair of solutions $\theta(x,w), \phi(x,w)$ defined for all w by

$$
[\theta(0,w),\ \phi(0,w)] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

By evaluating Wronskian determinants, one discovers that

$$\begin{aligned} y(x,w) &= A_\phi\theta(x,w) + A_\theta\phi(x,w), \\ z(x,w) &= B_\phi\theta(x,w) + B_\theta\phi(x,w). \end{aligned} \tag{5.13}$$

Combining (5.6), (5.10), (5.12), (5.13) and the fact that $W[\theta,\phi] = 1$, we arrive at

$$\begin{aligned} \frac{1}{\mathrm{T}} &= \frac{\lambda+c}{2iw}(A_\theta B_\phi - A_\phi B_\theta), \\ \frac{R_r}{\mathrm{T}} &= \frac{\lambda+c}{-2iw}(B_\phi \overline{A_\theta} - B_\theta \overline{A_\phi}), \\ \frac{R_l}{\mathrm{T}} &= \frac{\lambda+c}{2iw}(\overline{B_\phi} A_\theta - \overline{B_\theta} A_\phi). \end{aligned} \tag{5.14}$$

On account of Lemma (3.6), (5.14) shows that the S-matrix is continuous on $\Re - \{0\}$, i. e. for $\lambda \in (-\infty,-c) \cup (c,\infty)$.

We now turn towards the behavior of the S-matrix at the spectral gap endpoints $\pm c$. Recall the following asymptotics proven in Chapter III: Let τ denote either $\theta(x,c)$ or $\phi(x,c)$.

$$A_\tau(w) = \begin{cases} \alpha_0^\tau + o(1) & \text{if } \tau \text{ is unbounded as } x \to \infty, \\ \alpha_1^\tau w + o(1) & \text{if } \tau \text{ is bounded as } x \to -\infty, \end{cases} \tag{5.15a}$$

and

$$B_\tau(w) = \begin{cases} \beta_0^\tau + o(1) & \text{if } \tau \text{ is unbounded as } x \to -\infty, \\ \beta_1^\tau w + o(1) & \text{if } \tau \text{ is bounded as } x \to -\infty, \end{cases} \tag{5.16a}$$

for some constants $\alpha_i^\tau, \beta_i^\tau, i = 0,1$.

The corresponding situation at $\lambda = -c$ is

$$A_\tau(w) = \begin{cases} \alpha^{\tau}_{-0} w^{-1} + o(w^{-1}) & \text{if } \tau \text{ unbounded as } x \to \infty, \\ \alpha^{\tau}_{-1} + o(1) & \text{if } \tau \text{ bounded as } x \to \infty, \end{cases} \tag{5.15)b}$$

and

$$B_\tau(w) = \begin{cases} \beta^{\tau}_{-0} w^{-1} + o(w^{-1}) & \text{if } \tau \text{ unbounded as } x \to -\infty, \\ \beta^{\tau}_{-1} + o(1) & \text{if } \tau \text{ bounded as } x \to -\infty. \end{cases} \tag{5.16)b}$$

Below, we consider the limit $\lambda \to c$, the case $\lambda \to -c$ being similar. Recalling that $\lambda = c$ is a half bound state provided $\gamma \equiv W[y(x,c), z(x,c)] = 0$, we note that, by (5.13), $\gamma = \alpha_0^{\phi}\beta_0^{\theta} - \alpha_0^{\theta}\beta_0^{\phi}$. Since we may assume, without loss of generality (else choose another origin) that both $\theta(x,c)$ and $\phi(x,c)$ are unbounded as $x \to \pm\infty$, (5.15)a, (5.16)a and (5.14) complete the proof of Theorem (2.4) for $\gamma \neq 0$.

Next, assume that $\gamma = 0$ and let $y(x,c) = \alpha z(x,c)$. By straightforward computation one finds that $\alpha = 2ci(\alpha_0^{\phi}\beta_1^{\theta} - \alpha_0^{\theta}\beta_1^{\phi})$, and, in particular, $\alpha = \lim_{x \to -\infty} y_1(x,c)$.

The asymptotic limit

$$A_\phi B_\theta - A_\theta B_\phi = \frac{1}{2ci}\left(\alpha + \frac{1}{\alpha}\right) w + o(w), \text{ as } \lambda \to c, \tag{5.17}$$

was established in Lemma (4.1). Similar considerations reveal that

$$\begin{aligned} B_\phi \overline{A_\theta} - B_\theta \overline{A_\phi} &= \tfrac{1}{2ci}\left(\alpha - \tfrac{1}{\alpha}\right) w + o(w), \\ \overline{B_\phi} A_\theta - \overline{B_\theta} A_\phi &= \tfrac{1}{2ci}\left(\alpha - \tfrac{1}{\alpha}\right) w + o(w), \text{ as } \lambda \to c. \end{aligned} \tag{5.18}$$

Reference to (5.17), (5.18) and (5.14) completes the proof of Theorem (2.4). ■

2. Levinson's Theorem.

We recall that the eigenvalues of H are precisely the poles of the transmission coefficient $T(w)$, as can be seen from equations (5,7). Hence our proof of Corollary (2.5) consists of studying $T(w)$. In Lemma (3.8) the following asmptotics were derived, as $|\lambda| \to \infty$:

$$A_\phi(\lambda) \to e^{-i\int_0^\infty v},$$

$$B_\phi(\lambda) \to -e^{-i\int_{-\infty}^0 v},$$

$$A_\theta(\lambda) \to -ie^{-i\int_0^\infty v},$$

$$B_\theta(\lambda) \to -ie^{-i\int_{-\infty}^0 v},$$

where $v = \frac{1}{2}(v_1 + v_2)$. Hence $A_\theta B_\phi - B_\theta A_\phi \sim 2ie^{-i\int_{-\infty}^\infty v}$, and so, by (5.14), we conclude that

$$T(w) \sim e^{i\int_{-\infty}^\infty v}, \ |\lambda| \to \infty.$$

Thus, since $T(w)$ has a limit as $|\lambda| \to \infty$, we can define $\arg T(w)$ so as to be continuous for arbitrarily large radii, i. e. , as $|\lambda| \to \infty$. The following asymptotics table follows from Theorem (2.4), and its analogue for $\lambda = -c$ obtainable from (5.15)b, (5.16)b:

H has:	$T(w)$ (as $w \to 0$)
HBS at $\lambda = c$	$C + o(1)$
HBS at $\lambda = -c$	$C + o(1)$
no HBS at $\lambda = c$	$wC + o(w)$
no HBS at $\lambda = -c$	$wC + o(w)$

(5.19)

where we have used C for all constants $\neq 0$. In particular, $\lambda = \pm c$ cannot be cluster points of eigenvalues, i. e. , we have finitely many eigenvalues in $-c < \lambda < c$. Consider now the following contour, symmetric with respect to the real axis. The upper half consists of semicircles about $\pm c$ of radius ϵ, a semicircle about $\lambda = 0$ of radius R and the line segments $[-R, -c-\epsilon]$ and $[c+\epsilon, R]$. We assign a counterclockwise orientation to the contour, $\tilde{C}$. For ϵ small enough, we see that all the poles of $T(\lambda)$ in $(-c, c)$ are enclosed in $\tilde{C}$. Therefore we can apply the argument principle. Further, by symmetry, the variation of $\arg T(\lambda)$ over the upper half-plane is the same as that over the lower half-plane.

Now let $\Gamma_\pm$ denote the variation of $\arg T(w)$ around the semicircles about $\pm c$. From (5.19), we obtain the following chart:

H has	Γ_+	Γ_-
HBS at $\lambda = c$	0	—
HBS at $\lambda = -c$	—	0
no HBS at $\lambda = c$	$-\pi$	—
no HBS at $\lambda = -c$	—	$-\pi$

(5.20)

Letting N denote the number of eigenvalues, we have:

$$\begin{aligned} 2\pi N &= \text{Variation of } \arg T(w) \text{ over } C = var\Phi \text{ over } C \\ &= \Gamma_+ + 2[\Phi(\infty) - \Phi(c)] + 2[\Phi(-\infty) - \Phi(\infty)] + [\Phi(-c) - \Phi(-\infty)] + \Gamma_- \\ &= \Gamma_+ + \Gamma_- + 2[\Phi(-c) - \Phi(c)], \end{aligned}$$

where we write $\Phi(\pm\infty)$ for $\lim_{\lambda\to\pm\infty} \Phi(\lambda)$. Therefore, taking into account (5.20), Corollary (2.5) is fully demonstrated. ■

3. Asymptotics for Power-law Potentials.

In this section, we compute the leading order correction behavior for some specific potentials. In particular, we prove

Theorem (5.1): *Let $V(x)$ be such that*

$$p(x) \sim p_+ x^{-2-\epsilon},\ v_i(x) \sim v_i^+ x^{-2-\epsilon_i},\ x \to +\infty,$$

$$p(x) \sim p_- x^{-2-\delta},\ v_i(x) \sim v_i^- x^{-2-\delta_i},\ x \to +\infty,$$

where $0 < \epsilon, \epsilon_i, \delta_i < 1, i = 1, 2$. Then the asymptotics of $S(w)$ are as follows:

(a) If $\gamma \neq 0$ and $\epsilon_2 < \delta_2$, then

$$T(w) = \tfrac{iw}{\gamma c} + \tfrac{iv_2^+}{\gamma c} b_{\epsilon_2} w^{1+\epsilon_2} + o(w^{1+\epsilon_2}),$$

$$R_r(w) = -1 - \overline{b_{\epsilon_2}} v_2^+ w^{\epsilon_2} + o(w^{\epsilon_2}),$$

$$R_l(w) = 1 + v_2^+ b_{\epsilon_2} w^{\epsilon_2} + o(w^{\epsilon_2}),$$

where $b_s \doteq 2^{1+s} c e^{-\frac{1}{2}\pi s}(s(s+1))^{-1}\Gamma(1-s)$, and if $\epsilon_2 = \delta_2$ then

$$T(w) = \tfrac{iw}{\gamma c} + \tfrac{i}{\gamma c}(v_2^+ + v_2^-) b_{\epsilon_2} w^{1+\epsilon_2} + o(w^{1+\epsilon_2}),$$

$$R_r(w) = -1 - (2v_2^- b_{\epsilon_2} + v_2^+(b_{\epsilon_2} + \overline{b_{\epsilon_2}}))w^{\epsilon_2} + o(w^{\epsilon_2}),$$

$$R_l(w) = 1 + (2v_2^+ b_{\epsilon_2} + v_2^-(b_{\epsilon_2} + \overline{b_{\epsilon_2}}))w^{\epsilon_2} + o(w^{\epsilon_2}).$$

(b) If $\gamma = 0$ and $\epsilon_2 < \delta$, then, with α as in the proof of Theorem (2.4),

$$T(w) = \tfrac{2\alpha}{\alpha^2+1} + \tfrac{2\alpha}{\alpha^2+1)^2} v_2^+ b_{\epsilon_2} w^{\epsilon_2} + o(w^{\epsilon_2}),$$

$$R_r(w) = \tfrac{\alpha^2-1}{\alpha^2+1} + \left(\tfrac{(\alpha^2+1)\alpha^2\overline{b_{\epsilon_2}} + (\alpha^2-1)b_{\epsilon_2}}{(\alpha^2+1)^2}\right) w^{\epsilon_2} + o(w^{\epsilon_2}),$$

$$R_l(w) = \tfrac{1-\alpha^2}{\alpha^2+1} + \tfrac{\alpha^4+1}{(\alpha^2+1)^2} b_{\epsilon_2} w^{\epsilon_2} + o(w\epsilon_2),$$

with similar formulae holding for $\epsilon_2 = \delta_2$ and for $\epsilon_2 > \delta_2$.

Proof: Our proof uses the method used to prove the corresponding theorem for the Schrödinger equation on the line (Theorem (3.1), [21]). Our integrals are those for A_ϕ, A_θ, B_ϕ and B_θ, and so we deal only with the solutions $\theta(x, w)$ and $\phi(x, w)$.

We give details here only for A_ϕ, pointing out the relevant information for the other functions without proof. Recall,

$$A_\phi(w) = 1 + \int_0^\infty e^{iwt}(Q_2 - \frac{iw}{\lambda + c}Q_1)\,dt.$$

We rewrite this as

$$\begin{aligned} A_\phi(w) &= [1 + \textstyle\int_0^\infty Q_2^0] + \int_0^\infty (e^{iwt} - 1)Q_2^0 + \\ & \textstyle\int_0^\infty e^{iwt}(Q_2 - Q_2^0) + \frac{w}{i(\lambda+c)}\int_0^\infty e^{iwt}Q_1^0 + \frac{w}{i(\lambda+c)}\int_0^\infty e^{iwt}(Q_1 - Q_1^0) \qquad (5.21) \\ &= \alpha_0^\phi + I_1(w) + I_2(w) + I_3(w) + I_4(w) \end{aligned}$$

where α_0^ϕ is the same as in (5.15)a and $I_i(w), i = 1, \cdots, 4$ are the remaining integrals, in the order they appear. Consider $I_1(w)$. Without loss of generality, we assume that both $\theta(x,c)$ and $\phi(x,c)$ are unbounded as $x \to \pm\infty$, so that

$$\begin{aligned} \theta(x,c) &\sim \begin{pmatrix} 2cx \\ 1 \end{pmatrix} \alpha_0^\theta, \quad \phi(x,c) \sim \begin{pmatrix} 2cx \\ 1 \end{pmatrix} \alpha_0^\phi \quad as \quad x \to \infty, \\ \theta(x,c) &\sim \begin{pmatrix} 2cx \\ 1 \end{pmatrix} \beta_0^\theta, \quad \phi(x,c) \sim \begin{pmatrix} 2cx \\ 1 \end{pmatrix} \beta_0^\phi \quad as \quad x \to -\infty. \end{aligned} \qquad (5.22)$$

Thus we have that as $x \to \infty, -Q_2^0 = v_2\phi_1 + p\phi_2 \sim \alpha_0^\phi(2cv_2^+ x^{-1-\epsilon_2} + p_+ x^{-2-\epsilon}) = ax^{-\alpha} + o(x^{-\alpha})$, where $a = 2c\alpha_0^\phi v_2^+, \alpha = 1 + \epsilon_2$. Then

$$I_1(w) = a\int_0^\infty (1 - e^{iwt})t^{-\alpha} - \int_0^\infty (1 - e^{iwt})(Q_2^0(t) + at^{-\alpha}).$$

Let $\delta > 0$ and choose $R = R(\delta)$ so that $|\frac{1}{a}Q_2^0(t)t^\alpha + 1| < \delta$ for all $t > R$. Letting $s = wt$ we see that

$$\int_0^\infty (1 - e^{iwt})t^{-\alpha} = w^{\alpha-1}\int_0^\infty (1 - e^{is})s^{-\alpha}\,ds.$$

Also,

$$|\int_0^R (1-e^{iwt})(Q_2^0(t)+at^{-\alpha})\,dt| = O(w)$$

and

$$|\int_R^\infty (1-e^{iwt})(Q_2^0(t)+at^{-\alpha})\,dt| \le w^{\alpha-1}\delta \int_0^\infty |1-e^{is}|\; s^{-\alpha}\,ds = o(w^{\alpha-1}),$$

so that

$$I_1(w) = aw^{\alpha-1}\int_0^\infty (1-e^{is})s^{-\alpha}\,ds + o(w^{\alpha-1}),\ \lambda \to c. \tag{5.23}$$

It is clear that we have

$$I_3(w) = O(w) \text{as } \lambda \to c. \tag{5.24}$$

Next, from the equation for $\phi(x,w)$ and $\phi(x,c)$ we have, writing ϕ^c for $\phi(x,c)$,

$$\begin{aligned} \phi_1 - \phi_1^c &= 2cx\left(\tfrac{\lambda+c}{w}\tfrac{\sin wx}{2cx} - 1\right)\left(1 + \textstyle\int_0^x Q_2^0\right) + A_1(w), \\ \phi_2 - \phi_2^c &= (\cos wx - 1)\left(1 + \textstyle\int_0^x Q_2^0\right) + A_2(w), \end{aligned} \tag{5.25}$$

where $A_1(w)$ and $A_2(w)$ denote the remaining terms of $\phi_1 - \phi_1^c$ and $\phi_2 - \phi_2^c$, respectively (see (4.4)). Looking at the term $\frac{\lambda+c}{w}\sin wx\int_0^\infty(\cos wt - 1)Q_2^0$ of $A_1(w)$, we observe the following. Let $\tau_1 = \epsilon_2,\ \tau_2 = \min\{2\tau_1, 1\}$, and let $\sigma \in (\tau_1,\tau_2)$. Let $I(w) = \frac{\lambda+c}{w}\sin wx\int_0^\infty(\cos wt - 1)Q_2^0$. By estimates such as $|\sin wx| \le C\left|\frac{wx}{1+wx}\right| \le C\left|\frac{wx}{1+wx}\right|^\sigma$ we deduce that

$$|I(w)| \le Cw^\sigma x^{1+\sigma/2}\int_0^\infty t^{\sigma/2}Q_2^0 \le Cw^\sigma x^{1+\sigma/2}, \tag{5.26}$$

since $t^{\sigma/2}Q_2^0 \in L^1(0,\infty)$. In turn, the contribution of $I(w)$ to, say $I_2(w)$, is then such that

$$\left|\int_0^\infty dt\, e^{iwt} v_2 I(w)\right| \le Cw^\sigma, \tag{5.27}$$

with a similar contribution to $I_4(w)$.

The next result will allow us to estimate the terms involving $(Q_i - Q_i^0)$ in the decomposition (5.25), and hence their contribution to $I_2(w)$ and $I_4(w)$.

Lemma (5.2) (cf. [22], Lemma (2.1)(i)): *Let $x \geq 0$ and let $y(x,w)$ be the solution of (2.1) satisfying $y(0,w) = (\alpha,\beta)^T, \alpha,\beta$ arbitrary constants. Then, if $\gamma \neq 0$, $|\,y(x,w) - y_0(x)\,| \leq Cx\left[\frac{wx}{1+wx}\right]^2$.*

Notice that there is an x in the bound of Lemma (5.2) which would be absent if we assumed $\gamma = 0$ (see Lemma (4.2)).

Proof: In the decomposition (4.4), $\gamma \neq 0$ implies that the term

$$J(x) = 2cx\left(\frac{\lambda + c}{w}\frac{\sin wx}{2cx} - 1\right)\left(\beta + \int_0^x Q_2^0\right)$$

has instead of the bound $C\left(\frac{wx}{1+wx}\right)^2$,

$$|\,J(x)| \leq Cx\left[\frac{wx}{1+wx}\right]^2. \tag{5.28}$$

Therefore, we obtain

$$|\,y(x,w) - y_0(x)| \leq Cx\left[\frac{wx}{1+wx}\right]^2 + C\left[\frac{wx}{1+wx}\right]\int_0^x P(t)\,|y(t,w) - y_0(t)|\;dt\,, \tag{5.29}$$

whence Gronwall's Lemma completes the proof of Lemma (5.2).■

Now consider the term $J(w) = \int_0^x \cos(x-t)(Q_1 - Q_1^0)$ of $A_1(w)$. With σ as in (5.26) and (5.27), we estimate this term by Lemma (5.2) and the fact that $|\,Q_i - Q_i^0\,| \leq (|p| + |v_i|)\;\;|y(x,w) - y_0(x)|$, to be

$$\left|\int_0^x \cos w(x-t)(Q_1 - Q_1^0)\right| \leq w^{\sigma} x^{1+\sigma/2} \tag{5.30}$$

and, in turn, we find its contribution to $I_2(w)$ to be

$$\left|\int_0^\infty dt e^{iwt} v_2 J(w)\right| \leq Cw^{\sigma}. \tag{5.31}$$

As all the integrals in $A_1(w)$ and $A_2(w)$ can be estimated in the same way as were $I(w)$ and $J(w)$ we therefore deduce that the contributions of $A_1(w)$ and $A_2(w)$ to the integrals $I_2(w)$ and $I_4(w)$ are $o(w^{\epsilon_2})$. Therefore, returning to (5.21) we have, by (5.25),

$$
\begin{aligned}
I_2(w) &= -\int_0^\infty dt e^{iwt}\left[v_2(\phi_1-\phi_1^c)+p(\phi_2-\phi_2^c)\right] \\
&= -\int_0^\infty dt\, e^{iwt}\left\{v_2\left(\frac{\lambda+c}{w}\frac{\sin wt}{2ct}-1\right)2\alpha_0^\phi ct+p(\cos wt-1)\alpha_0^\phi\right\}+o(w^{\epsilon_2}) \qquad (5.32)\\
&= 2\alpha_0^\phi cv_2^+ w^{\epsilon_2}\int_0^\infty du e^{iu}\left(1-(\lambda+c)\frac{\sin u}{2cu}\right)u^{-1-\epsilon_2}+o(w^{\epsilon_2}),\ \ \lambda\to c,
\end{aligned}
$$

where the last equality is obtained in the same manner as was (5.23). Similarly, we find that

$$I_4(w) = O(w^{1+\epsilon_1}) \ \ as\ \lambda \to c. \qquad (5.33)$$

Combining (5.23), (5.32), (5.24) and (5.33) hence results in, as $\lambda\to c$ (noting that $O(w^{1+\epsilon_1}) = o(w^{\epsilon_2})$)

$$A_\phi(w) = \alpha_0^\phi(1+v_2^+ b_{\epsilon_2} w^{\epsilon_2}) + o(w^{\epsilon_2}), \qquad (5.34)\text{a}$$

where b_{ϵ_2} is defined in the statement of the theorem. Similar considerations for A_θ, B_θ and B_ϕ lead to

$$
\begin{aligned}
A_\theta(w) &= \alpha_0^\theta(1+v_2^+ b_{\epsilon_2} w^{\epsilon_2}) + o(w^{\epsilon_2}),\\
B_\phi(w) &= \beta_0^\phi(1+v_2^- b_{\delta_2} w^{\delta_2}) + o(w^{\delta_2}), \qquad (5.34)\text{b}\\
B_\theta(w) &= \beta_0^\theta(1+v_2^- b_{\delta_2} w^{\delta_2}) + o(w^{\delta_2}).
\end{aligned}
$$

Taking into account the equations (5.14), the relations (5.34) complete the demonstration of Theorem (5.1)(a).

Remarks: Note the similarity to the corresponding asymptotics for the Schrödinger equation ([22]). It is interesting to note that the leading correction behavior for the Dirac case is completely determined by v_2 at $\lambda = c$. The leading behavior is completely determined by v_1 at $\lambda = -c$, i. e. , if we consider $\lambda = -\sqrt{w^2 + c^2}$.

For $\gamma = 0$, we use the identity (with $\psi(x, w)$ given by (4.8)),

$$\begin{aligned} y_1(A_\theta B_\phi - A_\phi B_\theta) &= -B_\phi(w)\left[-\tfrac{iw}{\lambda+c}y_1(0) + y_2(0) + \int_0^\infty e^{iwt}\left(Q_2(\psi) - \tfrac{iw}{\lambda+c}Q_1(\psi)\right)(t)\,dt\right] \\ &\quad -A_\phi(w)\left[\tfrac{iw}{\lambda+c}y_1(0) + y_2(0) - \int_0^\infty e^{iwt}\left(Q_2(\psi) + \tfrac{iw}{\lambda+c}Q_1(\psi)\right)(t)\,dt\right] \end{aligned}$$

to conclude that (5.34)a,b produce the correct result, as in (5.17). The corresponding relations used to evaluate R_r and R_l are, respectively,

$$\begin{aligned} &y_1(0)(\overline{A_\phi}B_\theta - \overline{A_\theta}B_\phi) \\ &\quad = -B_\phi\left[\tfrac{iw}{\lambda+c}y_1(0) + y_2(0) + \int_0^\infty e^{-iwt}\left(Q_2(\psi) + \tfrac{iw}{\lambda+c}Q_1(\psi)\right)\right] \\ &\quad -\overline{A_\phi}\left[\tfrac{iw}{\lambda+c}y_1(0) + y_2(0) - \int_{-\infty}^0 e^{-iwt}\left(Q_2(\psi) + \tfrac{iw}{\lambda+c}Q_1(\psi)\right)\right] \end{aligned} \tag{5.35}$$

and

$$\begin{aligned} &y_1(0)(A_\phi\overline{B_\theta} - A_\theta\overline{B_\phi}) \\ &\quad = -\overline{B_\phi}\left[\tfrac{-iw}{\lambda+c}y_1(0) + y_2(0) + \int_0^\infty e^{iwt}\left(Q_2(\psi) - \tfrac{iw}{\lambda+c}Q_1(\psi)\right)\right] \\ &\quad -A_\phi\left[-\tfrac{iw}{\lambda+c}y_1(0) + y_2(0) - \int_{-\infty}^0 e^{iwt}\left(Q_2(\psi) - \tfrac{iw}{\lambda+c}Q_1(\psi)\right)\right]. \end{aligned} \tag{5.36}$$

The resulting asymptotics are, as $w \to 0(\lambda \to c)$

$$\begin{aligned} \overline{A_\phi}B_\theta - \overline{A_\theta}B_\phi &= \tfrac{iw}{2c}\left[\tfrac{\alpha^2-1}{\alpha} + \tfrac{1}{\alpha}v_2^- b_{\delta_2}w^{\delta_2} + \alpha v_2^+\overline{b_{\epsilon_2}}w^{\epsilon_2} + o(w^{\epsilon_2})\right], \\ A_\phi\overline{B_\theta} - A_\theta\overline{B_\phi} &= -\tfrac{iw}{2c}\left[\tfrac{1-\alpha^2}{\alpha} + \tfrac{1}{\alpha}v_2^-\overline{b_{\delta_2}}w^{\delta_2}w^{\delta_2} + \alpha v_2^+ b_{\epsilon_2}w^{\epsilon_2} + o(w^{\epsilon_2})\right] \end{aligned} \tag{5.37}$$

and

$$T(w) = \tfrac{2\alpha}{\alpha^2+1} + \tfrac{2a}{(\alpha^2+1)^2}v_2^+ b_{\epsilon_2}w^{\epsilon_2} + \tfrac{2}{\alpha(\alpha^2+1)^2}v_2^- b_{\delta_2}w^{\delta_2} + o(w^{\epsilon_2}).$$

Theorem (5.1)(b) now follows by simply inserting equations (5.37) into (5.14). ■

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APPENDIX

In this section, we prove two theorems which link the Dirac Lippmann - Schwinger solutions, and hence the S-matrix, to the scattering operator. Theorem 1 is an expansion theorem by means of the Dirac Lippmann- Schwinger solutions. Theorem 2 establishes the S-matrix as a w-space kernel in the spectral representation of H_0, and hence the equivalence of the S-operator and the S-matrix in that representation. These results are well known for the Schrödinger equation ([19], [36], [39]). But there seems to be no such results in the literature for Dirac systems, and hence the inclusion of this appendix.

Theorem 1: *Suppose assumption (2.1) holds. Let H be the operator induced by (1.1) and let $H_0 = H - V$. Let $w \in \Re$ and write $\lambda = +\sqrt{w^2 + c^2}$. Then*

(a) There exist unique solutions $\Psi^{(1)}(\cdot, \lambda) = \Psi(\cdot, \lambda)$ and $\Psi^{(2)}(\cdot, \lambda) = \Psi(\cdot, -\lambda)$ of the Dirac Lippmann - Schwinger equation (1.8). We denote these solutions by $u(\cdot, w)$ and $v(\cdot, w)$ respectively.

(b) If $f \in L^2(\Re)$, then

$$f^{\#}(w) = \begin{pmatrix} f_+^{\#}(w) \\ f_-^{\#}(w) \end{pmatrix} = \frac{1}{\sqrt{2\pi}} lim \begin{pmatrix} \int dx (u(x,w))^* f(x) r(w) \\ \int dx (v(x,w))^* f(x) r(w) \end{pmatrix},$$

where $r(w) = \sqrt{\frac{\lambda + c}{2\lambda}}$, exists.

(c) If $f \in D(H)$, then

$$(Hf)^{\#}(w) = \lambda A f^{\#}(w), \text{ where } A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(d) *Ran*$^{\#} = L^2(\Re)$ *and for any Borel set* $B \subset (-\infty, c] \cup [c, \infty)$ *and* $f \in L^2(\Re)$,

$$\int_{B_+} |f_+^{\#}(w)|^2\, dw + \int_{B_-} |f_-^{\#}(w)|^2\, dw = \| P(B) f \|^2, \qquad \textit{(Parseval equality)}$$

where $B_\pm = \{w \in \Re \mid \pm\lambda(w) \in B\}$.

Also, if $g \in L^2(\Re)$, *then*

$$< f, g >= \int_{\Re} (f^{\#}(w))^* g^{\#}(w)\, dw. \qquad \textit{(generalized Parseval equality)}$$

(e) $(P(B)f)(x) = \frac{1}{\sqrt{2\pi}} \lim \left[\int_{B_+} dw\, u(w, w) f_+^{\#}(w) r(w) + \int_{B_-} dw v(x, w) f_-^{\#}(w) r(w) \right]$.

(f) For any $f \in L^2(\Re)$ *we have*

$$(\Omega^+ f)^{\#}(w) = \hat{f}(w),$$

where $\hat{f}$ *was defined in* (1.19).

Theorem 2: *Let* $\Omega^\pm$ *and* S *be the wave operators and scattering operator, respectively, defined in Chapter 1 §(p.7). Let the S-matrix as defined by (1.13) be written in the form*

$$S(w) = \begin{pmatrix} S(\lambda) & \underline{0} \\ \underline{0} & S(-\lambda) \end{pmatrix}, \; \textit{where}\; S(\pm\lambda) = \begin{pmatrix} T(\pm\lambda) & R_r(\pm\lambda) \\ R_l(\pm\lambda) & T(\pm\lambda) \end{pmatrix}.$$

Let $f, g \in S$ *the Schwartz class of* L^2, *be such that* $\hat{f}$ *and* $\hat{g}$ *have compact support in* $\{w \mid \lambda(w) \in \Re \setminus (-c, c)\}$. *Then*

$$< f, (S - \mathbb{I})\} >= \int_0^\infty (\hat{f}(\lambda))^* (S(w) - \mathbb{I}) \hat{g}(\lambda)\, dw,$$

where $\hat{f}(\lambda) = (\hat{f}_+(w) \hat{f}_+(-w) \hat{f}_-(w) \hat{f}_-(-w))^T$ *(similarly for* $\hat{g}(w)$*) and* $\mathbb{I}$ *denotes the identity operator.*

Let us begin by specifying that our basic spectral variable is taken to be w. In particular, we consider $w \in \Re$ and define $\lambda = \sqrt{w^2 + c^2}$, so that $= \lambda \in [c, \infty), -\lambda \in (-\infty, -c]$; and when we speak of complex spectral parameter, we mean $\lambda + i\delta$, where $\delta \in \Re$ and λ is as above.

First, we establish the existence of the Dirac Lippmann-Schwinger solutions for (1.1), i. e. , the solutions with representations, for $w \in \Re$,

$$
\begin{aligned}
u(x,w) &= u_0(x,w) + \int_{-\infty}^{\infty} W_+(w)e^{iw(x-t)}V(t)u(t,w)\,dt + \\
&\qquad \int_x^{\infty} W_+^T(w)e^{iw(x-t)}V(t)u(t,w)\,dt, \\
v(x,w) &= v_0(x,w) + \int_{-\infty}^{\infty} W_-(w)e^{iw(x-t)}V(t)v(t,w)\,dt + \\
&\qquad \int_x^{\infty} W_-^T(w)e^{iw(x-t)}V(t)v(t,w)\,dt,
\end{aligned}
\tag{A.1}
$$

where

$$
u_0(x,w) = \begin{pmatrix} 1 \\ \frac{iw}{\lambda+c} \end{pmatrix} e^{iwx}, \quad v_0(x,w) = \begin{pmatrix} 1 \\ \frac{\lambda+c}{iw} \end{pmatrix} e^{iwx},
$$

$$
W_+(w) = \frac{1}{2}\begin{pmatrix} \frac{\lambda+c}{iw} & -1 \\ 1 & -\frac{iw}{\lambda+c} \end{pmatrix} \text{ and } W_-(w) = \frac{1}{2}\begin{pmatrix} \frac{iw}{\lambda+c} & 1 \\ -1 & -\frac{\lambda+c}{iw} \end{pmatrix}.
$$

We begin with the solutions $y(x,\lambda), z(x,\lambda), \lambda$ complex, established in Chapter III. Let $\lambda \in [c,\infty)$ be as above. Then the pairs $y(x,\lambda), \overline{y(x,\lambda)}$ and $z(x,\lambda), \overline{z(x,\lambda)}$ form a fundamental system of solutions of (1.1). Denote by X_+, X_-, respectively, the vectors

$$
\begin{pmatrix} 1 \\ \frac{iw}{\lambda+c} \end{pmatrix} e^{iwx}, \quad \begin{pmatrix} 1 \\ -\frac{iw}{\lambda+c} \end{pmatrix} e^{-iwx}.
$$

Keep in mind the asymptotic behaviors

$$y \sim X_+, \quad \overline{y} \sim X_- \quad as\ x \to +\infty,$$
$$z \sim X_-, \quad \overline{z} \sim X_+ \quad as\ x \to -\infty, \tag{A.2}$$

where we have suppressed the (x, λ) dependence. Then we write

$$z = a\overline{y} + by,$$
$$\overline{z} = c\overline{y} + dy,$$

where $a = a(\lambda), b = b(\lambda), c = c(\lambda)$ and $d = d(\lambda)$. Simple Wronskian evaluations using (A.2) then reveal that $a = \frac{\lambda+c}{2iw}W[z, y], b = -\frac{\lambda+c}{2iw}W[z, \overline{y}], c = \frac{\lambda+c}{2iw}W[\overline{z}, y]$ and $ad - bc = 1$. This permits us to write

$$y = a\overline{z} - cz,$$
$$z = a\overline{y} + by. \tag{A.3}$$

We then divide (A.3) through by $a(\lambda)$ to obtain solutions of (1.1) defined by

$$\tilde{u}(x, w) = T(\lambda)y(x, \lambda) = \overline{z}(x, \lambda) + R_l(\lambda)z(x, \lambda),$$
$$\tilde{u}(x, -w) = T(\lambda)z(x, \lambda) = \overline{y}(x, \lambda) + R_r(\lambda)y(x, \lambda), \tag{A.4}$$

(which are defined for all complex numbers λ) where $T(\lambda) = 1/a(\lambda), R_l(\lambda) = -c(\lambda)/a(\lambda)$, $R_r(\lambda) = b(\lambda)/a(\lambda)$. Then the solution

$$u(x, w) = \begin{cases} \tilde{u}(x, w), & w \geq 0, \\ \tilde{u}(x, -w), & w \leq 0, \end{cases}$$

is seen, by substitution and use of (A.1), to satisfy the Dirac Lippmann-Schwinger equation for $u(x, w)$. The equation for $v(x, w)$ is similarly obtained by considering the solutions $y(x, -\lambda), \overline{y(x, -\lambda)}$, and $z(x, -\lambda), \overline{z(x, -\lambda)}$. The uniqueness follows from the uniqueness of the Jost solutions, hence establishing part (a) of Theorem 1.

From the representation (1.4) of Green's function and the asymptotic behavior of the solutions $\Psi_{\pm}(x,\lambda)$ of (1.4), which may be obtained from (3.10), we easily obtain

Lemma (A.1): *Let $G(x,y;\lambda)$denote the Green's function for (1.2) Let $\Im\alpha > 0, \Im\alpha^2 \neq 0$ and define $H(x,y;\alpha) = G(x,y;\sqrt{\alpha^2+c^2})$. Then $H(x,\cdot,\alpha) \in L^1(\Re)$ almost everywhere in x.*

Let us recall the integral representation (1.5) of the resolvent, $R_\lambda f(x) = \int G(x,y;\lambda)f(y)\,dy$. If we let $G_0(x,y;\lambda)$ denote the free Green's function, then the second resolvent equation, $R_\lambda - R_{0\lambda} = R_{0\lambda}VR_\lambda$, yields

$$G(x,y;\lambda) = G_0(x,y;\lambda) + \int G_0(x,z;\lambda)V(z)G(z,y;\lambda)\,dz. \tag{A.5}$$

The idea behind our proof is now to relate the Dirac Lippmann- Schwinger solutions $u(x,w)$ and $v(x,w)$ of (1.1) to the (componentwise) Fourier transform of the Green's function, which exists by Lemma (A.1). Let us introduce the function

$$\begin{aligned} p(x,w,\alpha) = &\int_{-\infty}^{x} W(w)e^{i\alpha(x-t)}V(t)[u_0(t,w),u_o(t,-w)]\,dt + \\ &\int_x^{\infty} W^T(w)e^{-i\alpha(x-t)}V(t)[u_0(t,w),u_0[t,-w)]\,dt. \end{aligned} \tag{A.6}$$

Let $g(x,w;\alpha)$ denote the (componentwise) Fourier transform of $H(x,y;\alpha)$, with $w \geq 0$, viz.

$$g(x,w;\alpha) = \frac{1}{\sqrt{2\pi}}\int_{\Re} e^{-iwt}H(x,t;\alpha)\,dt,\ w \geq 0. \tag{A.7}$$

Now let

$$h(x,w;\alpha) = \sqrt{2\pi}(|\lambda| - |\sqrt{\alpha^2+c^2}|)g(x,w;\alpha) = [u_0(x,w),u_0(x,-w)] + h^{(1)}(x,w;\alpha). \tag{A.8}$$

Then we have

Lemma (A.2): *For $w \in \Re^+, \Im\alpha \geq 0$ and $\Im\alpha^2 \neq 0, h^{(1)}(x,w;\alpha;$ satisfies the equation*

$$h^{(1)}(x,w;\alpha) = p(x,w;\alpha) + \int_{\Re} H_0(x,t;\alpha)V(t)h^{(1)}(t,w;\alpha)\,dt, \tag{A.9}$$

where $p(x,w,\alpha)$ is defined in (A.6). *Moreover, $h(x,w;\alpha)$ is uniformly continuous in all its arguments for $\Im\alpha \geq 0$, and in particular, $h(x,w;|w|) = [u(x,w), u(x,-w)]$.*

Proof: From the kernel equation (A.5), $H(x,y;\alpha)$ obeys

$$H(x,y;\alpha) = H_0(x,y;\alpha) + \int H_0(x,t;\alpha)V(t)H(t,y;\alpha)\,dt.$$

If we take the Fourier transforms with respect to y (all integrals involved are absolutely convergent), we obtain then

$$g(x,w;\alpha) = g_0(x,w;\alpha) + \int H_0(x,t;\alpha)V(t)g(t,w;\alpha)\,dt. \tag{A.10}$$

However, $g_0(x,w;\alpha) = \frac{1}{\sqrt{2\pi}}\left(|\lambda| - |\sqrt{\alpha^2+c^2}|\right)^{-1}[u_0(x,w), u_0(x,-w)]$, and hence (A.9) follows from (A.10), (A.8) and (A.6). Via standard arguments, one sees that $h(x,w;\alpha)$ is uniformly continuous in all its arguments. The last statement follows from (A.9) using (A.6) and (A.8). ∎

An immediate corollary to Lemma (A.2) is

Corollary (A.3): *Let $f \in C_0^\infty$. Then*

$$\Phi(w;\alpha) = \frac{1}{2\pi}\int_{\Re}(h(x,w;\alpha))^* f(x)r(w)\,dx$$

and

$$f_+^\#(w) = \frac{1}{\sqrt{2\pi}}\int_{\Re}(u(x,w))^* f(x)r(w)\,dx$$

exist. Furthermore, $\Phi(w;\alpha)$ has an extension to $\alpha \in \Re$ *with $\sqrt{\alpha^2+c^2} \in [c,\infty)$ and the extended Φ is uniformly continuous in w,α. In particular, $\Phi(w;|w|) = f_+^\#(w)$.*

Let us note that we have obtained $f_+^{\#}$ as $f_+^{\#}(w) = (f_+^{\#}(w), f^{\#}(-w))^T$, for $w \geq 0$. We shall use this decomposition again in proving Theorem 2. In what follows we shall, unless otherwise specified, assume that $w \in \Re$ and simply write $f_+^{\#}(w)$. The existence of $f_-^{\#}(w)$ for $f \in C_0^\infty$ follows similarly by replacing λ by $-\lambda\,(\lambda \geq c)$ in the above argument, i. e. , by considering the interval $(-\infty, -c]$ and hence the Dirac Lippmann-Schwinger equation for v.

Lemma (A.4) (Parseval equality): *Let* $f \in C_0^\infty$ *and let* $[a,b] \subset [c,\infty)$. *Then*

$$\|P_{[a,b]}f\|^2 = \int_{-b\leq-\lambda\leq-a} |f_-^{\#}(w)|^2 \; dw + \int_{a\leq\lambda\leq b} |f_+^{\#}(w)|^2 \; dw.$$

Proof: The Parseval equality for ordinary Fourier transforms implies

$$\int H(t,x;\alpha)\overline{H(t,y;\alpha)}\,dt = \int g(x,w;\alpha)\overline{g(y,w;\alpha)}\,dw$$

a. e. in x, y and $\Im\alpha > 0, \Im\alpha^2 \neq 0$. From (A.8) and writing $\beta = \sqrt{\alpha^2 + c^2} = \mu + i\epsilon$, this becomes

$$(\beta - \overline{\beta}) \int H(t,x;\alpha)\overline{H(t,y;\alpha)}\,dt = \int \frac{2i\epsilon}{(\lambda-\mu)^2+\epsilon^2} h(x,w;\alpha)\overline{h(x,w;\alpha)}. \qquad \text{(A.11)}$$

Multiplying both sides of (A.11) by $(f(x))^* f(y)$ and integrating with respect to x and y, the left side gives

$$(\beta - \overline{\beta}) < R_\beta f, R_{\overline{\beta}} f > = (\beta - \overline{\beta}) < R_\beta R_{\overline{\beta}} f, f > = < (R_\beta - R_{\overline{\beta}}) f, f >, \qquad \text{(A.12)}$$

where we have the first resolvent formula, having noted the absolute convergence of the integrals considered and freely interchanged the order of integrations. Multiplying the right side of (A.11) by the same factor yields

$$\int \frac{2i\epsilon}{(\lambda-\mu)^2+\epsilon^2}\,|\Phi(w;\sqrt{\beta})|^2, \tag{A.13}$$

where once again $\Phi(w;\cdot)$ is valid for all $\lambda \in \Re$. Using the fact that a, b are not eigenvalues and a well known property of the function $k(x) = \int_a^b \frac{2\epsilon}{(x-\tau)^2+\epsilon^2}\,d\tau$ one obtains from Stone's formula ([35])

$$< f, P_{[a,b]} f > = \frac{1}{2\pi i} \lim_{\epsilon \downarrow 0} \int_a^b < f, (R_\beta - R_{\overline{\beta}}) f > \, d\mu. \tag{A.14}$$

Therefore, combining (A.14), (A.13) and (A.11) we obtain

$$\| P_{[a,b]} f \|^2 = \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_a^b d\mu \int dw \frac{\epsilon}{(\lambda-\mu)^2+\epsilon^2}\,|\Phi(w;\sqrt{\beta})|^2 \,. \tag{A.15}$$

By the boundedness of $\Phi(w;\sqrt{\beta})$ and a short argument using dominated convergence, we can interchange the μ and w integrations and take the ϵ-limit inside the w integral. It is a standard fact that for g cont0inuous,

$$lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_a^b dx \frac{\epsilon}{(x-\mu)^2+\epsilon^2} g(x) = \begin{cases} g(\mu), & a < \mu < b, \\ 0, & \mu \notin [a,b]. \end{cases}$$

Hence (A.15) now becomes

$$\| P_{[a,b]} \|^2 = \int_{a<|w|<b} dw \;|\; \Phi(w;|w|)\,|^2 \,. \tag{A.16}$$

However, $\Phi(w;|w|) = f^{\#}(w)$, and so the lemma follows, taking into account the comments following Corollary (A.3). ∎

We have so far proved (a) of Theorem 1. Part (b) has been proven only for $f \in C_0^\infty(\Re)$. However, standard approximation arguments establish (b) for arbitrary f since $C_0^\infty(\Re)$ is dense in $L^2(\Re)$. Part(c) is self evident from the definition of $\#$, once $\#$ is shown to be surjective. The Parseval equality will follow from Lemma (A.4) by standard approximation (and the generalized equality follows by polarization) once we establish the surjectivity of $\#$.

Now, we have that

$$\|P_{[a,b]}f\|^2 = \int_{a\leq\lambda\leq b}[|f_-^{\#}(w)|^2 + |f_+^{\#}(w)|^2]\,dw,\ f\in C_0^{\infty}. \tag{A.17}$$

Let $g\in C_0^{\infty}$. Since (A.17) holds, it follows by polarization that

$$< g, P_{[a,b]}f > = \int_{a\leq\lambda\leq b}[g^{\#}(w)]^{*}f^{\#}(w)r(w)\,dw =$$

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx[g(x)]^{*}\int_{a\leq\lambda\leq b} dw[u(x,w), v(x,w)]f^{\#}(w)r(w),$$

so that

$$(P_{[a,b]}f)(x) = \frac{1}{\sqrt{2\pi}}\int_{a\leq\lambda\leq b} dw[u(x,w), v(x,w)]f^{\#}(w)r(w). \tag{A.18}$$

Hence if $\#$ is surjective, part (c) of Theorem 1 follows from (A.18).

We now aim to show that the operator S is equivalent to multiplication by the S-matrix in the spectral representation of H_0. In this direction, Lemma (A.7) proves to be the main link between the two quantities. In particular, this lemma provides the first link between the Dirac Lippmann-Schwinger solutions of Theorem 1 and the wave operators, hence the operator S of Theorem 2.

Before proving Lemma (A.7), we need a well-known preliminary result on Abelian limits, whose proof we only include for completeness.

Lemma (A.5): *Let* $f(x)$ be a bounded measurable function such that $\lim_{t\to\infty}\int_0^t f(s)\,ds = C$. *Then* $\lim_{\epsilon\downarrow 0}\int_0^{\infty} e^{-\epsilon s}f(s)\,ds = C$.

<u>Proof</u>: Let $f(x)$ be as prescribed. Define $g(x) = \int_0^x f(s)\,ds$ and $h(\epsilon) = \int_0^{\infty} e^{-\epsilon s}f(s)\,ds$. We have that $g(x)$ is continuous, $g(0) = 0$ and $\lim_{x\to\infty} g(x) = C$, whence g is bounded. Also, $g'(x) = f(x)$ a. e, .

Therefore,

$$h(\epsilon) = \int_0^\infty e^{-\epsilon s} f(s)\, ds$$

$$= \lim_{x\to\infty} \int_0^x e^{-\epsilon s} g'(s)\, ds$$

$$= \lim_{x\to\infty} \left[\int_0^x \epsilon e^{-\epsilon s} g(s)\, ds + e^{-\epsilon x} g(x)\right].$$

Thus,

$$h(\epsilon) = \int_0^\infty \epsilon e^{-\epsilon s} g(s)\, ds, \tag{A.19}$$

and so

$$h(\epsilon) - C = \int_0^\infty \epsilon e^{-\epsilon s}[g(s) - C]\, ds.$$

Thus given $\delta > 0$, choose R such that $|g(x) - C| < \delta$, for all $x > R$, to obtain $|h(\epsilon) - C| \leq \epsilon R(C + \|g\|_\infty) + \delta$, and the lemma follows. ∎

Corollary (A.6): *Let $M = \{f \in H_{ac} | f^\#$ has compact support in some set $\{w | a < |\lambda| < b\}\}$. If $f \in M$ and $g \in C_0^\infty$, then*

$$< f, (\Omega^\pm - \mathbb{I})g >= \lim_{\epsilon\downarrow 0} \int_0^{\mp\infty} i < f, e^{iHt} V e^{-iH_0 t} g > e^{\pm\epsilon t}\, dt. \tag{A.20}$$

<u>Proof</u>: From the definition of $\Omega^\pm$ we have that

$$\Omega^\pm g = \lim_{\epsilon\downarrow 0} \int_0^{\mp\infty} ds \epsilon e^{\pm\epsilon s} e^{iHs} e^{-iH_0 s} g. \tag{A.21}$$

However, it is easy to show that

$$\frac{d}{dt} < f, e^{iHt} e^{-iH_0 t} g >= i < f, e^{iHt} V e^{-iH_0 t} g >,$$

so that (A.21) implies that

$$< f, (\Omega^\pm - \mathbb{I})g >= \lim_{t\to\mp\infty} \int_0^t i < f, e^{iHt} V e^{-iH_0 t} g >\, dt.$$

The corollary now follows from the Lemma. ∎

Lemma (A.7): $((\Omega^+)^* f)^\wedge(w) = f^\#(w)$.

<u>Proof</u>: Suppose f, g satisfy the hypothesis of Corollary (A.6). By Theorem 1 (c) and (d), we have

$$
\begin{aligned}
< f, e^{iHt} V e^{-iH_0 t} g > &= \int_{\mathbb{R}} dw (f^\#(w))^* e^{i\lambda A t} (V e^{-iH_0 t} g)^\#(w) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dw (f^\#(w))^* e^{i\lambda A t} \int dx \begin{pmatrix} (u(x,w))^* \\ \\ (v(x,w))^* \end{pmatrix} V e^{-iH_0 t} g(x) r(w)).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_0^{-\infty} dt < f, e^{iHt} V e^{-iH_0 t} g > e^{\epsilon t} = \\
&\frac{1}{\sqrt{2\pi}} \int_0^{-\infty} dt \int_{\mathbb{R}} dw \int_{\mathbb{R}} dx\, r(w)(f^\#(w))^* e^{i\lambda A t} \begin{pmatrix} (u(x,w)^* \\ \\ (v(x,w))^* \end{pmatrix} V e^{-iH_0 t} g(x) e^{\epsilon t} \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{-\infty} dt \int dw \int dx \left(\overline{f_+^\#(w)} e^{i\lambda t} (u(x,w))^* V(x) e^{-iH_0 t} g(x) + \right. \\
&\left. \overline{f_-^\#(w)} e^{-i\lambda t} (v(x,w))^* V(x) e^{-iH_0 t} g(x) \right) e^{\epsilon t} r(w) \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{-\infty} dt \int dw \int dx \left(\overline{f_+^\#(w)} (u(x,w))^* V(x) [e^{-it(H_0 - \lambda + i\epsilon)} g](x) + \right. \\
&\left. \overline{f_-^\#(w)} (v(x,w))^* V(x) [e^{-it(H_0 + \lambda + i\epsilon)} g](x) \right) r(w).
\end{aligned}
\tag{A.22}
$$

The integrand is bounded independent of t, x, ϵ, and so we may interchange the order of integration. The t -integration gives then

$$
\begin{aligned}
&\int_0^{-\infty} dt (e^{-it(H_0 - \lambda + i\epsilon)} g)(x) = -i[(H_0 - \lambda + i\epsilon)^{-1} g](x) \\
&= i \left[\int_{-\infty}^x W_+(\tilde{w}) e^{i\tilde{w}(x-s)} g(s)\, ds + \int_x^\infty W_+(\tilde{w}) e^{-i\tilde{w}(x-s)} g(s)\, ds \right]
\end{aligned}
\tag{A.23}
$$

and

$$
\begin{aligned}
&\int_0^{-\infty} dt (e^{-it(H_0 + \lambda + i\epsilon)} g)(x) = -i[(H_0 + \lambda + i\epsilon)^{-1} g](x) \\
&= -i \left[\int_{-\infty}^x W_-(\tilde{\tilde{w}}) e^{i\tilde{\tilde{w}}(x-s)} g(s)\, ds + \int_x^\infty W_-^T(\tilde{\tilde{w}}) e^{-i\tilde{\tilde{w}}(x-s)} g(s)\, ds \right],
\end{aligned}
\tag{A.24}
$$

where $\tilde{w} = \sqrt{(\lambda - i\epsilon)^2 - c^2}$ and $\tilde{\tilde{w}} = \sqrt{(\lambda + i\epsilon)^2 - c^2}$. Inserting (A.23) and (A.24) into (A.22) yields

$$\int_0^{-\infty} dt e^{\epsilon t} < f, e^{iHt} V e^{-iH_0 t} g >=$$

$$\frac{-i}{\sqrt{2\pi}} \int dw \int dx \int_{-\infty}^{x} ds \left[\overline{f_+^{\#}(w)} (u(x,w))^* V(x) W_+(\tilde{w}) e^{i\tilde{w}(x-s)} g(s) + \right.$$

$$\left. \overline{f_-^{\#}(w)} (v(x,w))^* V(x) W_-(\tilde{\tilde{w}}) e^{i\tilde{\tilde{w}}(x-s)} g(s) \right] r(w) -$$

$$\frac{i}{\sqrt{2\pi}} \int dw \int dx \int_x^{\infty} \left[\overline{f_+^{\#}(w)} (u(x,w))^* V(x) W_+^T(\tilde{w}) e^{-i\tilde{w}(x-s)} g(s) + \right.$$

$$\left. \overline{f_-^{\#}(w)} (v(x,w))^* V(x) W_-^T(\tilde{\tilde{w}}) e^{-i\tilde{\tilde{w}}(x-s)} g(s) \right] r(w).$$

Again, the integrands on the right hand side are bounded independent of ϵ. So we can interchange the x and s integration. Inserting the result in (A.20) and taking the limit inside the integral results in

$$< f, \Omega^+ g) - < f, g >= -\frac{1}{\sqrt{2\pi}} \int dw \; ds \left[\overline{f_+^{\#}(w)} [u(s,w) - u_0(s,w)]^* \right.$$
$$\left. + \overline{f_-^{\#}(w)} [v(s,w) - v_0(s,w)]^* \right] g(s) r(w).$$

One more appeal to the generalized Parseval equality completes the proof of the lemma for $f \in M, g \in C_0^\infty$. Since C_0^∞ is dense in $L^2(\Re)$, and M is dense in $H_{a.c.}$, the lemma is proven for arbitrary f.■

As a corollary to Lemma (A.7), we obtain part(f) of Theorem 1 by the simple calculation

$$(\Omega^+ f)^{\#} = [(\Omega^+)^* \Omega^+ f]^\wedge = \hat{f}.$$

Another corollary is the surjectivity of $\#$, which we reason as follows: $(\Omega^+)^*\Omega^+ = \mathbb{I} \Rightarrow (\Omega^+)^*$ is surjective. Therefore, $\{(\Omega^+)^* f \mid f \in L^2(dx)\} = L^2(dx)$. But $\wedge$ is surjective, so that $\{[(\Omega^+)^* f]^\wedge \mid f \in L^2(x)\} = L^2(w)$. And hence $\#$ is surjective by Lemma (A.7).

We are now in a position to complete the proof of Theorem 2. Let S denote the Schwartz space of functions in $L^2(\Re)$. Let $f, g \in S$ be such that $\hat{f}(w)$, $\hat{g}(w)$ have support respectively in $\{w \mid a_i < |\lambda| < b_i\}, i = 1,2$. Then

$$< f,(S-\mathbf{I})\, g > =< f,(\Omega^- - \Omega^+)^* \Omega^+ g >=< (\Omega^- - \Omega^+) f, \Omega^+ g >=$$

$$\lim_{T\to\infty} \int_{-T}^{T} < e^{iHt}(iV)e^{-iH_0t}f, \Omega^+ g > dt =$$

$$\lim_{\epsilon\downarrow 0}(-i)\int_{\Re} e^{\epsilon|t|} < e^{iHt}Ve{-}iH_0tf, \Omega^+ g > \; dt,$$

and so

$$< f,(S-\mathbf{I})g >= \lim_{\epsilon\downarrow 0}(-i)\int_{\Re} dt e^{-\epsilon|t|}\int_{\Re} dw \left[\left(e^{iHt}Ve^{-iH_0t}f\right)^{\#}(w)\right](\Omega^+ g)^{\#}(w), \quad \text{(A.25)}$$

where we have used Lemma (A.5) and the generalized Parseval equality. By part (f) of Theorem 1, we have $(\Omega^+ g)^{\#}(w) = \hat{g}(w)$, and, by part (c),

$$\begin{aligned}(e^{iHt}Ve^{-iH_0t}f)^{\#}(w) &= e^{\lambda At}(Ve^{-iH_0t}f)^{\#}(w)\\ &= \frac{1}{\sqrt{2\pi}}\int_{\Re} dx e^{i\lambda At}\begin{pmatrix}(u(x,w))^* \\ (v(x,w))^*\end{pmatrix} V(x)e^{-iH_0t}f(x)r(w).\end{aligned} \quad \text{(A.26)}$$

By choice of f, we have

$$(e^{-iH_0t}f)(x) = \frac{1}{\sqrt{2\pi}}\int_{\Re}\left[e^{-i\lambda' t}\hat{f}_+(w')u_0(x,w) + e^{i\lambda' t}\hat{f}_-(w')v_0(x,w)\right] r(w')\,dw. \quad \text{(A.27)}$$

Hence (A.25) becomes, using (A.26) and (A.27), suppressing the ϵ-limit ,

$$< f,(S-\mathbf{I})g >= \frac{(-i)}{2\pi}\int_{\Re} dt\int_{\Re} dw\int_{\Re} dx\int_{\Re} dw' r(w)r(w')$$

$$\left\{ e^{\epsilon|t|}e^{i\lambda At}\begin{pmatrix} u(x,w))^* \\ (v(x,w))^*\end{pmatrix} V(x) \right.$$

$$\left.\left[e^{-i\lambda t}\hat{f}_+(w')u_0(x,w') + e^{i\lambda t}\hat{f}_-(w')v_0(x,w'\right] \right\}^* \hat{g}(w)$$

$$= \frac{(-i)}{2\pi} \int_{\Re} dt \int_{\Re} dw \int_{\Re} dx \int_{\Re} w' \left\{ e^{-\epsilon|t|} \begin{pmatrix} e^{i\lambda t}(u(x,w))^* \\ e^{-i\lambda t}(v(x,w))^* \end{pmatrix} V(x) \right.$$

$$\left. \left[e^{-i\lambda' t}\hat{f}_+(w')u_0(x,w') + e^{i\lambda' t}\hat{f}_-(w')v_0(x,w')\right] \right\}^* \hat{g}(w)r(w)r(w')$$

$$= \frac{(-i)}{2\pi} \int \Re dt \int_{\Re} dw \int_{\Re} dx \int_{\Re} dw' r(w)r(w')$$

$$\left\{ e^{-\epsilon|t|} \begin{pmatrix} (u(x,w))^*V(x)u_0(x,w')e^{i(\lambda-\lambda')t}\hat{f}_+(w') + \\ (v(x,w))^*V(x)u_0(x,w')e^{i(\lambda+\lambda')t}\hat{f}_+(w') + \\ (u(x,w))^*V(x)v_0(x,w')e^{i(\lambda+\lambda')t}\hat{f}_-(w') \\ (v(x,w))^*V(x)v_0(x,w')e^{-i(\lambda-\lambda')t}\hat{f}_-(w') \end{pmatrix} \right\}^* \hat{g}(w).$$

By our choice of f and the hypothesis on V, the integrand above is absolutely integrable. Looking at the t-integration and bearing in mind that both λ and λ' are positive, we see that the terms involving $e^{\pm i(\lambda+\lambda')}$ will vanish as we let $\epsilon \to 0$. Thus what remains is

$$< f, (S - \mathbb{I})g >= \frac{(-i)}{2\pi} \int_{\Re} dw' \int_{\Re} dw \int_{\Re} dx \int_{\Re} dt$$

$$\begin{pmatrix} (u(x,w))^*V(x)u_0(x,w')e^{i(\lambda-\lambda')t-\epsilon|t|}\hat{f}_+(w') \\ (v(x,w))^*V(x)v_0(x,w')e^{-i(\lambda-\lambda')t-\epsilon|t|}\hat{f}_-(w') \end{pmatrix}^* \hat{g}(w)r(w)r(w')$$

which, upon performing the t-integration, becomes

$$< f, (S - \mathbb{I})g >= \frac{(-i)}{2\pi} \int_{\Re} dw' \int_{\Re} dw \int_{\Re} dx$$

$$\begin{pmatrix} (u(x,w))^*V(x)u_0(x,w')\hat{f}_+(w') \\ (v(x,w))^*V(x)v_0(x,w')\hat{f}_-(w') \end{pmatrix}^* \frac{2\epsilon}{(\lambda-\lambda')^2+\epsilon^2}\hat{g}(w)r(w)r(w'). \tag{A.28}$$

Noting once again that the integral is absolutely convergent, we take the ϵ-limit inside the integral to obtain, from (A.28),

$$< f,(S-\mathbb{I})g >= -i\int_{\Re} dw' \int_{\Re} dw \int_{\Re} dx$$
$$\begin{pmatrix} (u(x,w))^* V(x) u_0(x,w')\hat{f}_+(w') \\ (v(x,w))^* V(x) v_0(x,w')\hat{f}_-(w') \end{pmatrix}^* \delta\left(\frac{(w-w')(w+w')}{\sqrt{w^2+c^2}\sqrt{(w')^2+c^2}}\right)\hat{g}(w)r(w)r(w').$$

Splitting the $w-$ and $w'-$integration in the fashion

$$\int_{\Re} dw \int_{\Re} dw'(\cdot) =$$
$$\int_{-\infty}^{0} dw \int_{-\infty}^{0} dw'(\cdot) + \int_{-\infty}^{0} dw \int_{0}^{\infty} dw'(\cdot) + \int_{0}^{\infty} dw \int_{-\infty}^{0} dw'(\cdot) + \int_{0}^{\infty} dw \int_{0}^{\infty} dw'(\cdot),$$

arranging the integrals in the order $dw\, dx\, dw'$ and performing the w' integral yields

$$\begin{aligned}
< f,(S-\mathbb{I})g >= (-i)\int_0^\infty dw \int_{\Re} dx \sqrt{\tfrac{\lambda^2+c^2}{w^2}}\, r^2(w) \\
\Big((u_0(x,w))^* V(x) u(x,w)\overline{\hat{f}_+(w)}\hat{g}_+(w)+ \\
(v_0(x,w))^* V(x) v(x,w)\overline{\hat{f}_-(w)}\hat{g}_-(w)+ \\
(u_0(x,w))^* V(x) u(x,-w)\overline{\hat{f}_+(-w)}g_+(w) + (v_0(x,w))^* V(x) v(x,-w)\overline{\hat{f}_-(-w)}\hat{g}_-(w)\Big) \quad \text{(A.29)} \\
+(i)\int_0^\infty dw \int_{\Re} dx \sqrt{\tfrac{\lambda^2+c^2}{w^2}}\Big((v_0(x,w))^* V(x) u(x,-w)\overline{\hat{f}_+(-w)}\hat{g}_+(w) + \\
(v_0(x,w))^* V(x) v(x,-w)\overline{\hat{f}_+(-w)}\hat{g}_+(w) \\
+(u_0(w,w))^* V(x) u(x,w)\overline{\hat{f}_+(w)}\hat{g}_+(w) + (v_0(x,w))^* V(x) v(x,w)\overline{\hat{f}_-(w)}\hat{g}_-(w)\Big)\, r^2(w)
\end{aligned}$$

where we have used the fact that

$$\overline{(u(x,w))^* V(x) u_0(x,w)} = (u_0(x,w))^* V(x) u(x,w),$$

and similarly for other terms.

Recalling that $S = \mathbb{I} + A$, where A was defined in (1.12) as

$$A(\lambda) = \frac{\lambda + c}{2iw}\int_{\Re} dt [u_0(t,w), u_0(t,-w)]^* V(t)[u(t,w), u(t,-w)], \; \lambda > 0, w > 0,$$

and similarly for $A(-\lambda)$, (A.29) then becomes for $w \in \Re, \lambda = +\sqrt{w^2+c^2}, r(w) = \sqrt{\frac{\lambda \pm c}{2\lambda}}$,

$$< f,(S-\mathbf{I})g >= \int_0^\infty dw \Big(\tilde{T}_l(\lambda)\overline{\hat{f}_+(w)}\hat{g}_+(w) + \tilde{T}_l(-\lambda)\overline{\hat{f}_-(w)}\hat{g}_-(w) +$$

$$R_l(\lambda)\overline{\hat{f}_+(-w)}\hat{g}_+(w) + R_l(-\lambda)\overline{\hat{f}_-(-w)}\hat{g}_-(w) + \tilde{T}_r(\lambda)\overline{\hat{f}_+(-w)}\hat{g}_+(-w) -$$

$$\tilde{T}_r(-\lambda)\overline{\hat{f}_-(-w)}\hat{g}_-(-w) - R_r(\lambda)\overline{\hat{f}_+(w)}\hat{g}_-(-w) - R_r(-\lambda)\overline{\hat{f}_-(w)}\hat{g}_-(-w)\Big)$$

where $\tilde{T}_{r/l}(\pm\lambda) = T_{r/l}(\pm\lambda) - 1$, with the coefficients $T_{r/l}(\pm\lambda)$ and $R_{r/l}(\pm\lambda)$ given by (1.9).

This concludes the demonstration of Theorem 2, and hence establishes equivalence of the $S-$ matrix and the operator S in the spectral representation of H_0.■

QED

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