Zero Divisors, Group Von Neumann Algebras and Injective Modules

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(ABSTRACT)

In this thesis we discuss linear dependence of translations which is intimately related to the zero divisor conjecture. We also discuss the square integrable representations of the generalized Wyle-Heisenberg group in n^2 dimensions and its relations with Gabor's question from Gabor Analysis in the light of the time-frequency equation. We study the zero divisor conjecture in relation to the reduced C^*-algebras and operator norm C^*-algebras. For certain classes of groups we address the zero divisor conjecture by providing an isomorphism between the the reduced C^*-algebra and the operator norm C^*-algebra. We also provide an isomorphism between the algebra of weak closure and the von Neumann algebra under mild conditions. Finally, we prove some theorems about the injectivity of some spaces as $\mathbb{C}G$ modules for some groups G.

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Contents

CHAPTER 1

Introduction and Preliminaries

1. Group Von Neumann Algebra $\mathcal{N}(G)$

Definition 1.1. *Let G be a group and $p \in \mathbb{R}^+$. The ℓ^p-space of G, denoted $\ell^p(G)$, is the vector space of p-summable formal sums of complex numbers over G. Symbolically,*

$$\ell^p(G) := \left\{ \sum_{g \in G} a_g \cdot g \mid a_g \in \mathbb{C} \text{ and } \sum_{g \in G} |a_g|^p < \infty \right\}.$$

The normed space $\ell^p(G)$ for all $p \in \mathbb{N}$ is a Banach Space. In the special case where $p = 2$, $\ell^2(G)$ is equipped with an inner product making it a Hilbert space. Specifically, $\ell^2(G)$ with Hilbert basis G admits the following inner product:

$$\left\langle \sum_{g \in G} a_g \cdot g, \sum_{g \in G} b_g \cdot g = \sum_{g \in G} a_g \bar{b_g} \right\rangle.$$

If $p < 1$, then the triangle inequality is not satisfied and therefore $\ell^p(G)$ is not a Banach space. For all $p \geq 1 \in \mathbb{R}$, the complex group ring $\mathbb{C}G \subseteq \ell^p(G)$. It is natural to ask if the ring multiplication with which $\mathbb{C}G$ is endowed could be extended to $\ell^p(G)$ thus making it a ring as well. While the multiplication operation $\mathbb{C}G$ possesses is extend-able to $\ell^p(G)$, multiplication is not in general closed. In the case of $\ell^2(G)$, the multiplication is defined as follows:

$$\ell^2(G) \times \ell^2(G) \to \ell^\infty(G) = \left\{ \sum_{g \in G} a_g \cdot g \mid \sup_{g \in G} |a_g| < \infty \right\},$$

$$\sum_{h \in G} a_h \cdot h \sum_{g \in G} b_g \cdot g = \sum_{g,h \in G} a_h b_g \cdot hg = \sum_{g \in G} \big(\sum_{x \in G} a_{gx^{-1}} b_x \big) g.$$

It is natural to ask does there exist a subspace of $\ell^2(G)$ containing $\mathbb{C}G$ which admits a ring structure under the multiplication operation inherited from $\mathbb{C}G$? The group von Neumann algebra $\mathcal{N}(G)$ is the largest subspace of $\ell^2(G)$ which is closed under the multiplication defined above for $\mathbb{C}G$.

Definition 1.2. *The group von Neumann algebra $\mathcal{N}(G)$ of a group G is*

$$\mathcal{N}(G) = \left\{ \alpha \in \ell^2(G) \mid \alpha\beta \in \ell^2(G) \text{ for all } \beta \in \ell^2(G) \right\}$$

In order to define other algebras which will be essential to our discussions in next few chapters we define the concept of G-equivariant maps.

Definition 1.3. *Let $F : \ell^2(G) \to \ell^2(G)$, then F is G-equivariant if $F(x \cdot g) = F(x) \cdot g$ for all $g \in G$ and $x \in \ell^2(G)$ with respect to the natural right G-action on $\ell^2(G)$.*

It is noteworthy that an analytic definition of $\mathcal{N}(G)$ exists in terms of the bounded linear operators $\mathcal{B}(\ell^2(G))$. In particular $\mathcal{N}(G)$ is the algebra of G-equivariant bounded linear operators $\mathcal{B}(\ell^2(G))^G$ from $\ell^2(G)$ to $\ell^2(G)$.

2. Algebra of Affiliated Operators $\mathcal{U}(G)$

In this section the explore the Ore localization $\mathcal{U}(G)$ of $\mathcal{N}(G)$. This algebra $\mathcal{U}(G)$ which contains $\mathcal{N}(G)$ will be the center of our attention in the last chapter. We begin by defining the conditions which must be satisfied for an Ore localization to exist.

Definition 2.1. *let R be a ring, and let S be a multiplicatively closed subset of R. The pair (R, S) satisfies the (right) Ore condition if*

- *for all $(r, s) \in R \times S$, there exists a pair $(r', s') \in R \times S$ such that $rs' = sr'$, and*
- *for all $r \in R$ and $s \in S$ with $sr = 0$, there is $t \in S$ with $rt = 0$.*

Definition 2.2. *If (R, S) satisfies the Ore condition, define the (right) Ore localization to be the following ring RS^{-1}. Elements of RS^{-1} are elements of $R \times S/\sim$, with the following operations:*

- *Multiplication: $(r, s) \cdot (r', s') = (rc, s't)$, where $sc = r't$ for $t \in S$.*
- *Addition: $(r, s) + (r', s') = (rc + r'd, t)$, where $t = sc = s'd \in S$.*

Theorem 2.3. *Let G be a group, and let S be the set of nonzero divisors of $\mathcal{N}(G)$. Then the pair $(\mathcal{N}(G), S)$ satisfies the Ore condition.*

The definition of the algebra of affiliated operators $\mathcal{U}(G)$ is a direct consequence of the theorem above.

Definition 2.4. *The algebra of affiliated operators $\mathcal{U}(G)$ over the group G, as the Ore localization $\mathcal{N}(G)S^{-1}$, where S is the set of non-zero divisors in $\mathcal{N}(G)$.*

The analysis of $\mathcal{U}(G)$ is more apparent if one considers the following equivalent yet analytic definition of $\mathcal{U}(G)$ as compared to the algebraic definition provided above.

Definition 2.5. *An unbounded linear operator $f : \ell^2(G) \to \ell^2(G)$ over a group G is called affiliated to $\mathcal{N}(G)$ if f is densely defined, closed, and G-equivariant. The algebra of affiliated operators $\mathcal{U}(G)$ is the algebra of operators $f : \ell^2(G) \to \ell^2(G)$ affiliated to $\mathcal{N}(G)$.*

3. C^*-algebras $\mathrm{C}^*_{\mathrm{r}}(G)$ and $C^*(G)$

Another important class of operator algebras is the class of of C^*-algebras. For example the subset Γ of $\mathcal{B}(H)$ for some complex Hilbert space H, which is closed under all the algebraic operations on $\mathcal{B}(H)$, is closed with respect to the norm topology, and is closed the adjoint operation.

Definition 3.1. *A C^*-algebra is a complex Banach algebra (with a unit element I) with an involution that is closed under the norm topology.*

Von Neumann algebras are endowed with a weak topology which renders all von Neumann algebras as C^*-algebras. This is because the norm topology is stronger than the weak topology. The two relevant C^*-algebras, aside from von Neumann, associated with a group G are the reduced C^*-algebras denoted $\mathrm{C}^*_{\mathrm{r}}(G)$ and the "group C^*-algebra" denoted $C^*(G)$.

We begin with the definition of the reduced $C*$-algebras $\mathrm{C}^*_{\mathrm{r}}(G)$. Let $\lambda : G \to \mathcal{B}(\ell^2(G))$ be the left regular representation where for all $x \in G$, $\lambda(x) : \ell^2(G) \to \ell^2(G)$ sends $\sum_{g\in G} a_g \cdot g$ to $\sum_{g\in G} a_{x^{-1}g} \cdot g$. This induces a representation of $\ell^1(G)$, $\lambda : \ell^1(G) \to \mathcal{B}(\ell^2(G))$. For any element $K = \sum_{g\in G} a_g \cdot g$ we define $\lambda(K) = \sum_{g\in G} a_g \cdot \lambda(g) : \ell^2(G) \to \ell^2(G)$. The closure of such operators on $\ell^2(G)$ is the reduced C^*-algebra $\mathrm{C}^*_{\mathrm{r}}(G)$. We define the group C^*-algebra as the enveloping C^*-algebra of $\ell^1(G)$. If the group G is abelian then it is well known that $\mathrm{C}^*_{\mathrm{r}}(G) \cong C^*(G) \cong c_0(\Gamma)$ where Γ is pontryagin dual group of G and $c_0(\Gamma)$ is the space of continuous functions on Γ vanishing at infinity.

3.1. Connections between C^* and Physics. In quantum mechanics one usually employs C^*-algebras C with a unit in order to describe physical systems. The elements $x \in C$ which are invariant under the star-operation i.e. $x^* = x$ are considered as the observables of the physical systems. In order to make sense of the state of the physical system under consideration a $\mathbb{C}$-linear map $\psi : C \to \mathbb{C}$ with $\psi(x^*x) \geq 0$ for all $x \in C$ is needed. This map ψ defines a functional which takes on the value 1 on the 1 of C. The expectation value of x is $\psi(x)$ if the system happens to be in the state ψ. Other connection with local quantum field theories(local QFT (s)) exist as in the Haag-Kastler axiomatization of local QFT(s).

4. Brief Summary of Thesis

In chapter 2 we discuss linear dependence of translations which is intimately related to the zero divisor conjecture. We also discuss the square integrable representations of the generalized Wyle-Heisenberg group in n^2 dimensions and its relations with Gabor's questions from Gabor Analysis from the point of view of the time-frequency equation. In chapter three we study the zero divisor conjecture in relation to the reduced C^*-algebras and operator norm C^*-algebras. For certain classes of groups we address the zero divisor conjecture by providing an isomorphism between the the reduced C^*-algebra and the operator norm C^*-algebra. Finally in chapter four we prove some theorems about the injectivity of some spaces as $\mathbb{C}G$ modules for some groups G and then we conclude this thesis.

CHAPTER 2

Linear Dependency of Translations and Square Integrable Representations

1. Introduction

Let G be a locally compact Hausdorff group with left invariant Haar measure μ. Denote by $L^p(G)$ the set of complex-valued functions on G that are p-integrable with respect to μ, where $1 < p \in \mathbb{R}$. As usual, identify functions in $L^p(G)$ that differ only on a set of μ-measure zero. We shall write $\|\cdot\|_p$ to indicate the usual L^p-norm on $L^p(G)$. The *regular representation* of G on $L^p(G)$ is given by $L(g)f(x) = f(g^{-1}x)$, where $g, x \in G$ and $f \in L^p(G)$. The function $L(g)f$ is known as the left translation of f by g (many papers use the word "translate" instead of "translation"). In [**24**] Rosenblatt investigated the problem of determining when the left translations of a nonzero function f in $L^2(G)$ are linearly independent. In other words, when can there be a nonzero function $f \in L^2(G)$, some nonzero complex constants c_k, and distinct elements $g_k \in G$, where $1 \leq k \leq n, k \in \mathbb{N}$ (the positive integers) such that

$$\sum_{k=1}^{n} c_k L(g_k) f = 0? \tag{1.1}$$

It was shown in the introduction of [**24**] that if G has a nontrivial element of finite order, then there are nonzero elements in $L^2(G)$ that have a linear dependency among its left translations. Thus, when trying to find nontrivial functions that satisfy (1.1) it is more interesting to consider groups for which all nonidentity elements have infinite order. In the case of $G = \mathbb{R}^n$ it is known that every nonzero function in $L^2(\mathbb{R}^n)$ has no linear dependency among its left translations. Rosenblatt attacked (1.1) by trying to determine if there is a relationship between the linear independence of the translations of functions in $L^2(G)$ and the linear independence of an element and its images under the action of G in an irreducible representation of G. In order to gain insights into possible connections between these concepts, he computed examples for specific groups. The particular groups that he studied in [**24**] were the Heisenberg group and the affine group. What made these groups appealing is that they have irreducible representations that are intimately related to a time-frequency equation. Recall that an equation of the form

$$\sum_{k=1}^{n} c_k \exp(ib_k h(t)) f(a_k + t) = 0, \tag{1.2}$$

is a time-frequency equation, where $a_k, b_k \in \mathbb{R}, f \in L^2(\mathbb{R})$ and $h\colon \mathbb{R} \to \mathbb{R}$ is a nontrivial function. The case $h(t) = t$ corresponds to the Heisenberg group and $h(t) = e^t$ corresponds to the affine group.

Rosenblatt wondered if there existed a nontrivial $f \in L^2(\mathbb{R})$ that satisfied equation (1.2), then this f could be used to produce a nonzero $F \in L^2(G)$ with a linear dependency among its left translations. He then showed [**24**, Proposition 3.1] that there exists a nonzero $f \in L^2(\mathbb{R})$ that satisfies the following time-frequency equation

$$Cf(t) = f(t - \log 2) + \exp(-\frac{i}{2}e^t)f(t - \log 2), \tag{1.3}$$

where C is a constant. This time-frequency equation corresponds to the affine group A case since $h(t) = e^t$. This offers some hope that there might be a nonzero function in $L^2(A)$ that has a linear dependency among its left translations. However, there is no clear principle that can be used to show the existence of such a function given a nontrivial f that satisfies (1.3). Using the proof of the existence of f that satisfies (1.3) as a guide, a nonzero F in $L^2(A)$ with linearly dependent left translations was shown to exist [**24**, Proposition 3.2].

Even less is known about the Heisenberg group $H_n, n \in \mathbb{N}$. The relevant time-frequency equation, which has been intensely studied in the context of Gabor analysis, is

$$\sum_{k=1}^{m} c_k e^{2\pi i b_k \cdot t} f(t + a_k) = 0, \tag{1.4}$$

where c_k are nonzero constants, $a_k, b_k \in \mathbb{R}^n$, and $f \in L^2(\mathbb{R}^n)$. Linnell showed that $f = 0$ is the only solution to (1.4) when the subgroup generated by (a_k, b_k), where $k = 1, \dots, m$, is discrete. This gave a partial answer to a conjecture posed by Heil, Ramanathan and Topiwala on page 2790 of [**12**] that $f = 0$ is the only solution to (1.4) when $n = 1$. As far as we know the conjecture is still open.

The motivation for this paper is to give a clearer picture of the link between the linear independence of an element and its images under the action of G in an irreducible representation of G and the linear independence of the left translations of a function in $L^2(G)$. We will prove the following:

Proposition 1.5. *Let G be a locally compact group and suppose π is an irreducible, unitary, square integrable representation of G on a Hilbert space $\mathcal{H}_\pi$. If there exists a nonzero v in $\mathcal{H}_\pi$ such that*

$$\sum_{k=1}^{n} c_k \pi(g_k)v = 0$$

for some nonzero constants c_k and g_k in G, then there exists a nonzero $F \in L^2(G)$ that satisfies

$$\sum_{k=1}^{n} c_k L(g_k)F = 0.$$

In particular if there exists a nonzero v in $\mathcal{H}_\pi$ with linearly dependent translations, then there exists a nonzero F in $L^2(G)$ with linearly dependent translations.

We will use Proposition 1.5 to construct explicit examples of nontrivial functions in $L^2(A)$, where A is the affine group, that have a linear dependency among their left translations.

We then move on to the case where G is a discrete group. We will see that the problem of determining if the left translations of a nonzero function in $\ell^2(G)$ forms a linearly independent set is related to the Atiyah conjecture.

Let K be a subgroup of a group G. If $k \in K, x \in G$, and $f \in L^2(G)$, then we shall say that

$$L(k)f(x) = f(k^{-1}x)$$

is a *left K-translate* of f. We shall prove

THEOREM 1.6. *Let G be a locally compact, σ-compact group. If K is a torsion-free discrete subgroup of G that satisfies the Atiyah conjecture, then each nonzero function in $L^2(G)$ has linearly independent K-translations.*

Denote by $\tilde{H}_n$ the Weyl-Heisenberg group, a variant of the Heisenberg group, H_n. The group $\tilde{H}_n$ is of interest to us because it has an irreducible unitary representation on $L^2(\mathbb{R}^n)$, the Schrödinger representation, which is square integrable. Furthermore, the time-frequency equation (1.4) is related to the Schrödinger representation. Now if K is a torsion-free discrete subgroup of $\tilde{H}_n$, then by Theorem 1.6 every nonzero element in $L^2(\tilde{H}_n)$ has linearly independent left K-translations. It will then follow from Proposition 1.5 that if the subgroup of $\mathbb{R}^{2n}$ generated by $(a_k, b_k), 1 \leq k \leq m$, is discrete and for some $r \in \mathbb{N}$ the product $ra_k \cdot b_k \in \mathbb{Z}$ for all k, then $f = 0$ is the only solution to (1.4), see Proposition 6.3. This gives a new proof of a special case of [**18**, Proposition 1.3] and sheds new insights on the problem.

In the last section of the paper we investigate the linear independence of left translations of functions in $L^p(G)$ for abelian-by-finite groups G with no nontrivial compact subgroups.

2. Proof of Proposition 1.5

In this section we will prove Proposition 1.5. Before we give our proof we will give some necessary definitions. A *unitary representation* of G is a homomorphism π from G into the group $U(\mathcal{H}_\pi)$ of unitary operators on a nonzero Hilbert space $\mathcal{H}_\pi$ that is continuous with respect to the strong operator topology. This means that $\pi\colon G \to U(\mathcal{H}_\pi)$ satisfies $\pi(xy) = \pi(x)\pi(y), \pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$, and $x \to \pi(x)u$ is continuous from G to $\mathcal{H}_\pi$ for each $u \in \mathcal{H}_\pi$. A closed subspace W of $\mathcal{H}_\pi$ is said to be *invariant* if $\pi(x)W \subseteq W$ for all $x \in G$. If the only invariant subspaces of $\mathcal{H}_\pi$ are $\mathcal{H}_\pi$ and 0, then π is said to be an *irreducible representation* of G. A representation that is not irreducible is defined to be a *reducible representation*. We will assume throughout this paper that the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}_\pi$ is conjugate linear in the second component. If $u, v \in \mathcal{H}_\pi$, a *matrix coefficient* of π is the function $F_{v,u}\colon G \to \mathbb{C}$ defined by

$$F_{v,u}(x) = \langle v, \pi(x)u \rangle.$$

We will indicate the $F_{u,u}$ case by F_u. An irreducible representation π is said to be *square integrable* if there exists a nonzero $u \in \mathcal{H}_\pi$ such that $F_u \in L^2(G)$. We shall say that $u \in \mathcal{H}_\pi$ is *admissible* if $F_u \in L^2(G)$. The set of admissible elements in $\mathcal{H}_\pi$ will be denoted by $\mathrm{Ad}(\mathcal{H}_\pi)$. A consequence of π being irreducible is that if there is a nonzero admissible element in $\mathcal{H}_\pi$, then $\mathrm{Ad}(\mathcal{H}_\pi)$ is dense in $\mathcal{H}_\pi$. In fact, $\mathrm{Ad}(\mathcal{H}_\pi) = \mathcal{H}_\pi$ if G is unimodular, in addition to $\mathrm{Ad}(\mathcal{H}_\pi)$ containing a nonzero element. By [**11**, Theorem 3.1] there exists a self adjoint positive operator $C\colon \mathrm{Ad}(\mathcal{H}_\pi) \to \mathcal{H}_\pi$ such that if $u \in \mathrm{Ad}(\mathcal{H}_\pi)$ and $v \in \mathcal{H}_\pi$, then

$$\begin{aligned}\int_G |\langle v, \pi(x)u\rangle|^2 \, d\mu &= \int_G \langle v, \pi(x)u\rangle \overline{\langle v, \pi(x)u\rangle} \, d\mu \\ &= \|Cu\|^2 \|v\|^2,\end{aligned}$$

where $\|\cdot\|$ denotes the $\mathcal{H}_\pi$-norm. Thus if $u \in \mathrm{Ad}(\mathcal{H}_\pi)$, then $F_{v,u} \in L^2(G)$ for all $v \in \mathcal{H}_\pi$.

We now prove Proposition 1.5. Suppose there exists a nonzero $v \in \mathcal{H}_\pi$ for which there exists a linear dependency among some of the elements $\pi(g)v$, where $g \in G$. So there exists nonzero constants $c_1, c_2, \ldots, c_n$ and elements $g_1, g_2, \ldots, g_n$ from G with $\pi(g_j) \neq \pi(g_k)$ if $j \neq k$ such that

$$\sum_{k=1}^{n} c_k \pi(g_k)v = 0.$$

Let $u \in \mathrm{Ad}(\mathcal{H}_\pi)$. Then $0 \neq F_{v,u} \in L^2(G)$. We now demonstrate that $F_{v,u}$ has linearly dependent left translations. For $x \in G$,

$$\begin{aligned}\left(\sum_{k=1}^{n} c_k L(g_k) F_{v,u}\right)(x) &= \sum_{k=1}^{n} c_k F_{v,u}(g_k^{-1}x) \\ &= \sum_{k=1}^{n} c_k \langle v, \pi(g_k^{-1}x)u\rangle \\ &= \sum_{k=1}^{n} c_k \langle v, \pi(g_k^{-1})\pi(x)u\rangle \\ &= \sum_{k=1}^{n} \langle c_k \pi(g_k)v, \pi(x)u\rangle \\ &= \langle \sum_{k=1}^{n} c_k \pi(g_k)v, \pi(x)u\rangle \\ &= \langle 0, \pi(x)u\rangle \\ &= 0.\end{aligned}$$

Therefore, $\sum_{k=1}^{n} c_k L(g_k) F_{v,u} = 0$. The proof of Proposition 1.5 is now complete.

3. The Affine Group

In this section we give examples of nonzero functions in $L^2(G)$, where G is the affine group, that have a linear dependency among some of its left translations. Let $\mathbb{R}$ denote the real numbers and let $\mathbb{R}^*$ be the set $\mathbb{R} \setminus \{0\}$. Recall that $\mathbb{R}$ is a group under addition and $\mathbb{R}^*$ is a group with respect to multiplication. The affine group, also known as the $ax+b$ group, is defined to be the semidirect product of $\mathbb{R}^*$ and $\mathbb{R}$. That is,

$$G = \mathbb{R}^* \rtimes \mathbb{R}.$$

Let (a,b) and (c,d) be elements of G. The group operation on G is given by $(a,b)(c,d) = (ac, b+ad)$. The identity element of G is $(1,0)$ and $(a,b)^{-1} = (a^{-1}, -a^{-1}b)$. The left Haar measure on G is $d\mu = \frac{dadb}{a^2}$ and the right Haar measure is $d\mu = \frac{dadb}{|a|}$. Thus G is a nonunimodular group because the right and left Haar measures do not agree. So $f \in L^2(G)$ if and only if

$$\int_{\mathbb{R}} \int_{\mathbb{R}^*} |f(a,b)|^2 \frac{dadb}{a^2} < \infty.$$

An irreducible unitary representation of G can be defined on $L^2(\mathbb{R})$ by

$$\pi(a,b)f(x) = |a|^{-1/2} f\left(\frac{x-b}{a}\right),$$

where $(a,b) \in G$ and $f \in L^2(\mathbb{R})$. Before we show that π is square integrable we recall some facts from Fourier analysis.

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the Fourier transform of f is defined to be

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x}\, dx,$$

where $\xi \in \mathbb{R}$. The Fourier transform can be extended to a unitary operator on $L^2(\mathbb{R})$. For $y \in \mathbb{R}$ we also have the following unitary operators on $L^2(\mathbb{R})$,

$$T_y f(x) = f(x-y), E_y f(x) = e^{2\pi i y x} f(x)$$

$$D_y f(x) = |y|^{-1/2} f\left(\frac{x}{y}\right), (y \neq 0).$$

Given $f, g \in L^2(\mathbb{R})$ the following relations are also true $\langle f, T_y g\rangle = \langle T_{-y} f, g\rangle, \langle f, E_y g\rangle = \langle E_{-y} f, g\rangle$ and $\langle f, D_y g\rangle = \langle D_{y^{-1}} f, g\rangle$. Furthermore, $\widehat{T_y f} = E_{-y}\widehat{f}$ and $\widehat{D_y f} = D_{y^{-1}}\widehat{f}$. Observe that for $(a,b) \in G$ and $f \in L^2(\mathbb{R})$,

$$\pi(a,b)f(x) = T_b D_a f(x) = |a|^{-1/2} f\left(\frac{x-b}{a}\right).$$

Using the above relations it can be shown that for $f \in L^2(\mathbb{R})$,

$$\int_G \langle f, \pi(a,b)f\rangle\, d\mu = \int_{\mathbb{R}} \int_{\mathbb{R}^*} |\langle f, T_b D_a f\rangle|^2 \frac{dadb}{a^2} = \|f\|_2 \int_{\mathbb{R}^*} \frac{|\widehat{f}(\xi)|^2}{|\xi|} d\xi,$$

see [**12**, Theorem 3.3.5]. Thus $f \in L^2(\mathbb{R})$ is admissible if $\int_{\mathbb{R}^*} \frac{|\widehat{f}(\xi)|^2}{|\xi|} d\xi < \infty$. The function $f(x) = \sqrt{2\pi} x e^{-\pi x^2}$ satisfies this criterion since $\widehat{f}(\xi) = -\sqrt{2\pi} i \xi e^{-i\xi^2}$, which we obtained by combining [**23**, Proposition 2.2.5] with [**23**, Example 2.2.7]. Hence π is a square integrable, irreducible unitary representation of the affine group G. We are now ready to construct a nonzero function in $L^2(G)$ that has linearly dependent left translations.

Let $\chi_{[0,1)}$ be the characteristic function on the interval $[0,1)$. It follows from the following *refinement equation*

$$\chi_{[0,1)}(x) = \chi_{[0,1)}(2x) + \chi_{[0,1)}(2x-1)$$

that

$$\pi(1,0)\chi_{[0,1)}(x) = 2^{-1/2}\pi\left(2^{-1}, 0\right)\chi_{[0,1)}(x) + 2^{-1/2}\pi\left(2^{-1}, 2^{-1}\right)\chi_{[0,1)}(x). \tag{3.1}$$

Thus $\chi_{[0,1)}$ has a linear dependency among the $\pi(a,b)\chi_{[0,1)}$, where $(a,b) \in G$. We now use $\chi_{[0,1)}$ to construct a nontrivial function in $L^2(G)$ that has linearly dependent left translations. Let $f \in L^2(\mathbb{R})$ be an admissible function for π and let $(a,b) \in G$. Then the function

$$F(a,b) = \langle \chi_{[0,1)}, \pi(a,b)f \rangle = \int_0^1 |a|^{-1/2} \overline{f\left(\frac{x-b}{a}\right)}\, dx$$

belongs to $L^2(G)$. By Proposition 1.5, $F(a,b)$ has linearly dependent left translations. More specifically,

$$L(1,0)F(a,b) = 2^{-1/2}L(2^{-1},0)F(a,b) + 2^{-1/2}L(2^{-1},2^{-1})F(a,b),$$

which translations to

$$\int_0^1 |a|^{-1/2} \overline{f\left(\frac{x-b}{a}\right)}\, dx = \int_0^{1/2} |a|^{-1/2} \overline{f\left(\frac{x-b}{a}\right)}\, dx + \int_{1/2}^1 |a|^{-1/2} \overline{f\left(\frac{x-b}{a}\right)}\, dx.$$

Equation (3.1) above was used in the proof of [**24**, Proposition 3.1] to show that the time-frequency equation (1.3) with $C = \sqrt{2}$ has a nonzero solution. Basically, equation (1.3) is a reinterpretation of the above refinement equation where the representation π is replaced by an equivalent representation. See [**24**, Section 3] for the details.

We now turn our attention to the subgroup K of the affine group G which consists of all $(a,b) \in G$ for which $a > 0$. This was the version of the affine group considered in [**24**]. The left Haar measure for K is the same as the left Haar measure for G. Up to unitary equivalence there are two irreducible unitary infinite dimensional representations of K, see [**9**, Section 6.7] for the details. One of these representations is given by

$$\pi^+(a,b)f(x) = a^{1/2}e^{2\pi ibx}f(ax) = E_bD_{a^{-1}}f(x),$$

where $(a,b) \in K$ and $f \in L^2(0,\infty)$. The representation π^+ is square integrable. We are now ready to produce a nontrivial function in $L^2(K)$ that has linearly dependent left translations. From (3.1) we have

$$\chi_{[0,1)} = 2^{-1/2}D_{2^{-1}}\chi_{[0,1)} + 2^{-1/2}T_{2^{-1}}D_{2^{-1}}\chi_{[0,1)}.$$

By taking Fourier transforms we obtain

$$\begin{aligned}\widehat{\chi}_{[0,1)}(\xi) &= 2^{-1/2}D_2\widehat{\chi}_{[0,1)}(\xi) + 2^{-1/2}E_{-2^{-1}}D_2\widehat{\chi}_{[0,1)}(\xi)\\ &= 2^{-1/2}\pi^+(2^{-1},0)\widehat{\chi}_{[0,1)}(\xi) + 2^{-1/2}\pi^+(2^{-1},-2^{-1})\widehat{\chi}_{[0,1)}(\xi).\end{aligned}$$

Hence, there is a linear dependency among the $\pi^+(a,b)\widehat{\chi}_{[0,1)}$, where $(a,b) \in K$. It follows from

$$\widehat{\chi}_{[0,1)}(\xi) = \frac{e^{-2\pi i\xi} - 1}{-2\pi i\xi}$$

that $\widehat{\chi}_{[0,1)} \in L^2(0,\infty)$. Pick an admissible function $f \in L^2(0,\infty)$ for π^+. Then the function

$$F(a,b) = \langle \widehat{\chi}_{[0,1)}, \pi^+(a,b)f \rangle = \int_0^\infty \widehat{\chi}_{[0,1)}(\xi)a^{1/2}e^{-2\pi ib\xi}\overline{f(\xi)}d\xi$$

is a member of $L^2(K)$. Proposition 1.5 yields the following linear dependency in $L^2(K)$ among the left translations of $F(a,b)$.

$$F(a,b) = 2^{-1/2}L(2^{-1},0)F(a,b) + 2^{-1/2}L(2^{-1},-2^{-1})F(a,b).$$

This equation can easily be verified by using the relations

$$\widehat{\chi}_{[0,1)}(\frac{\xi}{2})(1+e^{-\pi i\xi}) = \frac{(e^{-\pi i\xi}-1)(1+e^{-\pi i\xi})}{-2\pi i\xi} = \widehat{\chi}_{[0,1)}(\xi).$$

4. Discrete groups and the Atiyah conjecture

In this section we connect the problem of linear independence of left translations of a function to the Atiyah conjecture. Unless otherwise stated we make the assumption that all groups in this section are discrete. For discrete groups the Haar measure is counting measure. Let f be a complex-valued function on a group G. We will represent f as a formal sum $\sum_{g\in G} a_g g$, where $a_g \in \mathbb{C}$ and $f(g) = a_g$. Denote by $\ell^2(G)$ those formal sums for which $\sum_{g\in G} |a_g|^2 < \infty$, and $\mathbb{C}G$, the group ring of G over $\mathbb{C}$ will consist of all formal sums that satisfy $a_g = 0$ for all but finitely many g. The group ring $\mathbb{C}G$ can also be thought of as the set of all functions on G with compact support. If $g \in G$ and $f = \sum_{x\in G} a_x x \in \ell^2(G)$, then the left translation of f by $g, L_g f$ is represented by the formal sum $\sum_{x\in G} a_{g^{-1}x} x$ since $L_g f(x) = f(g^{-1}x)$. Suppose $\alpha = \sum_{g\in G} a_g g \in \mathbb{C}G$ and $f = \sum_{g\in G} b_g g \in \ell^2(G)$. We define a multiplication, known as convolution, $\mathbb{C}G \times \ell^2(G) \to \ell^2(G)$ by

$$\alpha * f = \sum_{g,h\in G} a_g b_h gh = \sum_{g\in G} \left(\sum_{h\in G} a_{gh^{-1}} b_h \right) g.$$

Left multiplication by an element of $\mathbb{C}G$ is a bounded linear operator on $\ell^2(G)$. So $\mathbb{C}G$ can be considered as a subring of $\mathcal{B}(\ell^2(G))$, the space of bounded linear operators on $\ell^2(G)$. We now define two subalgebras of $\mathcal{B}(\ell^2(G))$ that will be used in Section 5. Let $\mathrm{C}^*_\mathrm{r}(G)$ be the operator norm closure of $\mathbb{C}G$ in $\mathcal{B}(\ell^2(G))$. The space $C^*_r(G)$ is known as *reduced group* C^*-*algebra* of G. The *group von Neumann algebra* of G, which is denoted by $\mathcal{N}(G)$, is the weak closure of $\mathbb{C}G$ in $\mathcal{B}(\ell^2(G))$. We shall say that G is torsion-free if the only element of finite order in G is the identity element of G. We now state a version of the Atiyah conjecture, which can also be considered an analytic version of the zero divisor conjecture:

CONJECTURE 4.1. *Let G be a torsion-free group, if $0 \neq \alpha \in \mathbb{C}G$ and if $0 \neq f \in \ell^2(G)$, then $\alpha * f \neq 0$.*

The hypothesis that G is torsion-free is essential. Indeed, let 1 be the identity element of G and let $g \in G$ such that $g \neq 1$ and $g^n = 1$ for some $n \in \mathbb{N}$. Then $(1+g+\cdots+g^{n-1})*(1-g) = 0$. The Atiyah conjecture is important in the study of von Neumann dimension. For further information see [**5, 16, 17**] and [**20**, Section 10].

The following proposition gives the link between zero divisors and the linear independence of left translations of a function.

PROPOSITION 4.2. *Let G be a discrete group and let $f \in \ell^2(G)$. Then f has linearly independent left translations if and only if $\alpha * f \neq 0$ for all nonzero $\alpha \in \mathbb{C}G$.*

Proof. Let $g \in G$ and let $f = \sum_{x \in G} a_x x \in \ell^2(G)$. Then

$$g * f = \sum_{x \in G} a_x gx = \sum_{x \in G} a_{g^{-1}x} x = L_g(f).$$

Consequently, if $g_1, \ldots, g_n \in G$ are distinct and $c_1, \ldots, c_n$ are constants, then

$$\sum_{k=1}^{n} c_k L(g_k) f = \sum_{k=1}^{n} c_k g_k * f = \left(\sum_{k=1}^{n} c_k g_k \right) * f.$$

The proposition now follows since $\sum_{k=1}^{n} c_k g_k \in \mathbb{C}G$. $\square$

As we saw in Section 3 there are nontrivial, square integrable functions on the affine group that have a linear dependency among their left translations. Since all nonidentity elements of the affine group have infinite order, it seems reasonable by taking a discrete subgroup D of the affine group, such as $1 \rtimes \mathbb{Z}$, we might be able to construct a nontrivial function in $\ell^2(D)$ that has a linear dependency among its left translations. It would then be an immediate consequence of Proposition 4.2 that Conjecture 4.1 is false. However, this is not the case. The affine group is a solvable Lie group, and all discrete subgroups of solvable Lie groups are polycyclic. By [**16**, Theorem 2] Conjecture 4.1 is true for torsion-free elementary amenable groups, a class of groups that contain all torsion-free polycyclic groups.

5. Proof of Theorem 1.6

In this section we prove Theorem 1.6. We start by giving some necessary background and definitions. Recall that our standing assumptions on the group G is that it is locally compact, Hausdorff with invariant left Haar measure μ. Let $g_1, \ldots, g_n$ be elements of G and let $c_1, \ldots, c_n$ be some constants. Set $\theta = \sum_{k=1}^{n} c_k L_{g_k}$ and $\theta^* = \sum_{k=1}^{n} \overline{c}_k L_{g_k^{-1}}$. So $\theta \in \mathcal{B}(L^2(G))$, the set of bounded linear operators on $L^2(G)$. Define

$$\mathbb{C}G = \{ \sum_{g \in G} a_g L_g \mid a_g = 0 \text{ for all but finitely many } g \in G \}.$$

The set $\mathbb{C}G$ is a $*$-subalgebra of $\mathcal{B}(L^2(G))$ because $L_g L_h = L_{gh}$. Note that if there exists a nonzero $f \in L^2(G)$ with linearly dependent left translations, then there exists a nonzero $\theta \in \mathbb{C}G$ with $\theta f = 0$. Let $f_1, f_2 \in L^2(G)$ and $g \in G$. It follows from

$$\langle L_g f_1, f_2 \rangle = \langle f_1, L_{g^{-1}} f_2 \rangle$$

that θ^* is indeed the adjoint of θ.

For the rest of this section H will denote a discrete subgroup of G. We will also assume that G is σ-compact in addition to our standing assumptions on G. The subgroup H acts of G by left multiplication. By [**1**, Proposition B.2.4] there exists a Borel fundamental domain for this action of H on G. More precisely, there exists a Borel subset B of G such that $hB \cap B = \emptyset$ for all $h \in H \setminus 1$ and $G = HB$ (thus B is a system of right coset representatives of H in G which is also a Borel subset). If X is a Borel subset of G, then we will identify $L^2(X)$ with the subspace of $L^2(G)$ consisting of all functions on G whose support is contained in X.

Let $\{q_i \mid i \in \mathcal{I}\}$ be a Hilbert basis for $L^2(B)$. We claim that $S := \{hq_i \mid h \in H,\ i \in \mathcal{I}\}$ is a Hilbert basis for $L^2(G)$. First we show that S is orthonormal. If $h \neq k$, then $\langle hq_i, kq_j\rangle = 0$, because the supports of hq_i and hq_j are contained in hB and kB respectively, which are disjoint subsets. On the other hand if $h = k$, then $\langle hq_i, hq_j\rangle = \langle q_i, q_j\rangle$, because the Haar measure is left invariant. This proves that S is orthonormal. Finally we show that the closure of the linear span $\overline{S}$ of S is $L^2(G)$. It will be sufficient to show that $L^2(hB) \subseteq \overline{S}$. Denote by χ_{hB} the characteristic function on hB. If $f \in L^2(G)$, then we may write $f = \sum_{h\in H} f_h$, where $f_h = \chi_{hB} f$ (so f_h has support contained in hB). Thus it will be sufficient to show that $L^2(hB) \subseteq \overline{S}$. Since $\overline{S}$ is invariant under H, it will be sufficient to show that $L^2(B) \subseteq \overline{S}$, which is obvious because the q_i form a Hilbert basis for $L^2(B)$.

For $\theta \in \mathcal{B}(L^2(G))$ or $\mathcal{B}(\ell^2(H))$, let $\|\theta\|$ or $\|\theta\|'$ denote the corresponding operator norms respectively. For $i \in \mathcal{I}$, let $S_i = \{hq_i \mid h \in H\}$ and let $\overline{S}_i$ denote the closure of S_i. Observe that we have a natural isomorphism $\phi_i \colon \mathcal{B}(\overline{S}_i) \to \mathcal{B}(\ell^2(H))$ induced by $hq_i \mapsto h \colon \overline{S}_i \to \ell^2(H)$. Now $L^2(G) = \bigoplus_{i\in\mathcal{I}} \overline{S}_i$ (where $\bigoplus$ indicates the Hilbert direct sum). We will need the following

Lemma 5.1. *Let $\theta \in \mathbb{C}H$. Then $\|\theta\| = \|\theta\|'$*

Proof. Note that θ can be considered as an operator on $L^2(G)$ or $\ell^2(H)$. Let $u \in L^2(G)$, we may write $u = \sum_{i\in\mathcal{I}} u_i$ with $u_i \in \overline{S}_i$. So

$$\begin{aligned}\|\theta\| = \sup_{u\in L^2(G),\ \|u\|_2=1} \|\theta u\|_2 &= \sup_{u\in L^2(G),\ \|u\|_2=1} \|\theta \sum_{i\in\mathcal{I}} u_i\|_2 \\ &\le \sup_{u\in L^2(G),\ \|u\|_2=1} \sum_{i\in\mathcal{I}} \sqrt{\|\theta\|'^2\|u_i\|_2^2} = \|\theta\|'.\end{aligned}$$

Fix $\iota \in \mathcal{I}$. Then

$$\|\theta\|' = \sup_{u\in\overline{S}_\iota,\ \|u\|_2=1} \|\theta u\|_2 \le \sup_{u\in L^2(G),\ \|u\|_2=1} \|\theta u\|_2 \le \|\theta\|.$$

Therefore, $\|\theta\| = \|\theta\|'$. □

Denote by $\mathcal{O}(H)$ the operator norm closure, and $\mathcal{W}(H)$ the weak closure of $\mathbb{C}H$ in $\mathcal{B}(L^2(H))$. The space $\mathcal{W}(H)$ is a von Neumann algebra and by the double commutant theorem is equal to the strong closure of $\mathbb{C}H$ in $\mathcal{B}(L^2(G))$. Recall that in Section 4 we defined the discrete group versions, $C_r^*(H)$ and $\mathcal{N}(H)$ in $\mathcal{B}(\ell^2(H))$, of the above algebras. Note $\mathcal{O}(H) \subseteq \mathcal{W}(H)$ and $C_r^*(H) \subseteq \mathcal{N}(H)$. We now relate these various algebras:

Proposition 5.2. *There is a $*$-isomorphism $\alpha \colon \mathcal{W}(H) \to \mathcal{N}(H)$. Moreover, α preserves the operator norm and maps $\mathcal{O}(H)$ onto $\mathrm{C}^*_\mathrm{r}(H)$.*

Proof. Recall that for $u \in L^2(G)$ we can uniquely write $u = \sum_{i\in\mathcal{I}} u_i$ with $u_i \in \overline{S}_i$. Let $\theta \in \mathcal{W}(H)$. Then there exists a net (θ_i) in $\mathbb{C}H$ which converges strongly to θ. Therefore for every $u \in L^2(G)$, the net $(\theta_i u)$ is convergent in $L^2(G)$, consequently the net $(\theta_i u_j)$ is convergent for every j, in particular $(\theta_i f)$ is a Cauchy net in $\ell^2(H)$ for every $f \in \ell^2(H)$. We deduce that (θ_i) is a Cauchy net in $\mathcal{B}(\ell^2(H))$ (in the strong operator topology) and hence converges to an operator $\theta' \in \mathcal{N}(H)$. We note that θ' doesn't depend on the choice of the net (θ_i) and therefore we have a well-defined map $\alpha\colon \mathcal{W}(H) \to \mathcal{N}(H)$, where $\alpha(\theta) = \theta'$ and α is the identity on $\mathbb{C}H$.

We now construct the inverse to α by reversing the above steps. Let $\phi \in \mathcal{N}(H)$. By the Kaplansky density theorem there exists a net (θ_i) in $\mathbb{C}H$ which converges strongly to ϕ and $\|\theta_i\|'$ bounded. Thus $\|\theta_i\|$ is bounded because $\|\theta_i\| = \|\theta_i\|'$ for each i by Lemma 5.1. Now let $u \in L^2(G)$. If $\mathcal{J}$ is a finite subset of $\mathcal{I}$, set $v_{\mathcal{J}} = \sum_{j \in \mathcal{J}} u_j$. Then $(\theta_i v_j)$ converges in $L^2(G)$ for every $\mathcal{J}$. Since $\|\theta_i\|$ is bounded, it follows that $(\theta_i u)$ is convergent in $L^2(G)$ and we conclude that (θ_i) converges strongly to an operator $\tilde{\phi} \in \mathcal{B}(L^2(G))$. It follows that we have a well-defined map $\phi \to \tilde{\phi}\colon \mathcal{N}(H) \to \mathcal{W}(H)$, which is the inverse to α.

It is easily checked that α is a $*$-isomorphism and therefore is an isomorphism of C^*-algebras, in particular it preserves the operator norm. We deduce that α maps $\mathcal{O}(H)$ onto $\mathrm{C}^*_{\mathrm{r}}(H)$. $\square$

Pick a $\iota \in \mathcal{I}$ and set $T = \bigoplus_{i \in \mathcal{I} \setminus \{\iota\}} \overline{S}_i$, so $L^2(G) = \overline{S}_\iota \oplus T$ and $\mathcal{B}(\overline{S}_\iota) \oplus \mathcal{B}(T) \subset \mathcal{B}(L^2(G))$. Observe that there is a natural projection from $\mathcal{B}(\overline{S}_\iota) \oplus \mathcal{B}(T) \to \mathcal{B}(\ell^2(H))$. Combining this observation with Proposition 5.2 we obtain the following commutative diagram.

$$\begin{tikzcd}
 & & & \mathcal{B}(\overline{S}_\iota) \oplus \mathcal{B}(T) \subset \mathcal{B}(L^2(G)) \arrow[dddddd] \\
 & & \mathcal{W}(H) \arrow[ur] \arrow[dddd, "\alpha"] & \\
 & \mathcal{O}(H) \arrow[ur] \arrow[dd, "\alpha"] & & \\
\mathbb{C}H \arrow[ur] \arrow[dr] & & & \\
 & \mathrm{C}^*_{\mathrm{r}}(H) \arrow[dr] & & \\
 & & \mathcal{N}(H) \arrow[dr] & \\
 & & & \mathcal{B}(\ell^2(H))
\end{tikzcd}$$

Now suppose that $\theta \in \mathbb{C}H$ and $\theta * f = 0$ for some nonzero $f \in L^2(G)$. Let $\pi \in \mathcal{B}(L^2(G))$ denote the projection of $L^2(G)$ onto $\ker \theta$. Then $0 \neq \pi \in \mathcal{W}(H)$ and hence $0 \neq \alpha(\pi) \in \mathcal{N}(H)$. It follows that there exists $0 \neq f \in \mathcal{B}(\ell^2(H))$ such that $\theta * f = 0$. We can summarize some of the above as follows.

Proposition 5.3. *Let H be a discrete subgroup of the σ-compact locally compact group G and let $\theta \in \mathbb{C}H$. If $\theta * f = 0$ for some non-zero $f \in L^2(G)$, then $\theta k = 0$ for some non-zero $k \in \ell^2(H)$.*

Now let H be a torsion-free group which satisfies the strong Atiyah conjecture, e.g. a torsion-free elementary amenable group. Then for $0 \neq \theta \in \mathbb{C}H$, we know that $\theta k \neq 0$ for all non-zero $k \in \ell^2(H)$. It follows from Proposition 5.3 that $\theta * f \neq 0$ for all non-zero $f \in L^2(G)$, in other words, any nonzero element of $L^2(G)$ has linearly independent H-translations. The proof of Theorem 1.6 is now complete.

In a similar fashion, we can prove

Theorem 5.4. *Let G be a locally compact σ-compact group and let H be an amenable discrete subgroup of G. If α is a non-zerodivisor in $\mathbb{C}H$, then $\alpha * f \neq 0$ for all nonzero $f \in L^2(G)$.*

Proof. Since $\alpha\beta \neq 0$ for all nonzero $\beta \in \mathbb{C}H$, it follows that $\alpha\beta \neq 0$ for all nonzero $\beta \in \ell^2(H)$ by [**8**, Theorem] (or use [**20**, Theorem 6.37]). The result now follows form Proposition 5.3. $\square$

We saw in Section 3 that for the affine group A there exists nonzero f in $L^2(A)$ with linearly dependent left translations. However, $\mathbb{Z}$ can be identified with the discrete subgroup $1 \rtimes \mathbb{Z}$ of A. A direct consequence of Theorem 1.6 is

Corollary 5.5. *Let A be the affine group. Then every nonzero f in $L^2(A)$ has linearly independent left $\mathbb{Z}$-translations.*

6. The Weyl-Heisenberg group

In this section we use techniques developed in this paper to determine when $f = 0$ is the only solution to the time-frequency equation (1.4). The relevant group here is the Weyl-Heisenberg group since it has an irreducible representation that is square integrable.

Let $n \in \mathbb{N}$. The *Heisenberg group* H_n is the set of $(n+2) \times (n+2)$ matrices of the form

$$\begin{pmatrix} 1 & a & z \\ 0 & 1_n & b \\ 0 & 0 & 1 \end{pmatrix}$$

where a is a $1 \times n$ matrix, b is a $n \times 1$ matrix, the zero in the $(2,1)$ position is the $n \times 1$ zero matrix, the zero in the $(3,2)$ position is the $1 \times n$ zero matrix, and the 1_n in the $(2,2)$ position is the $n \times n$ identity matrix. Another way to represent H_n is as the product $\mathbb{R} \times \widehat{\mathbb{R}^n} \times \mathbb{R}^n$. Here we view $\mathbb{R}^n$ as $n \times 1$ column matrices and $\widehat{\mathbb{R}^n}$ as $1 \times n$ row matrices. For $(z_1, a_1, b_1), (z_2, a_2, b_2) \in H_n$ the group law becomes $(z_1, a_1, b_1)(z_2, a_2, b_2) = (z_1 + z_2 + a_1 \cdot b_2, a_1 + a_2, b_1 + b_2)$. Thus the identity element in H_n is $(0,0,0)$ and $(z, a, b)^{-1} = (a \cdot b - z, -a, -b)$. For $f \in L^2(\mathbb{R}^n)$ and $(z, a, b) \in H_n$ define

$$\pi(z, a, b) f(x) = e^{2\pi i z} e^{-2\pi i a \cdot b} e^{2\pi i a \cdot x} f(x - b).$$

It turns out that π is a representation of H_n on $L^2(\mathbb{R}^n)$. Indeed, let (z_1, a_1, b_1), $(z_2, a_2, b_2) \in H_n$. Then

$$\begin{aligned} \pi(z_1, a_1, b_1)\big(\pi(z_2, a_2, b_2) f(x)\big) &= \pi(z_1, a_1, b_1)\big(e^{2\pi i z_2} e^{-2\pi i a_2 \cdot b_2} e^{2\pi i a_2 \cdot x} f(x - b_2)\big) \\ &= e^{2\pi i z_1} e^{2\pi i z_2} e^{-2\pi i a_1 \cdot b_1} e^{-2\pi i a_2 \cdot b_2} e^{2\pi i a_1 \cdot x} e^{2\pi i a_2 \cdot (x - b_1)} f(x - b_2 - b_1) \\ &= e^{2\pi i (z_1 + z_2)} e^{-2\pi i (a_1 \cdot b_1 + a_2 \cdot b_2)} e^{-2\pi i a_2 \cdot b_1} e^{2\pi i a_2 \cdot x} f(x - (b_1 + b_2)) \\ &= e^{2\pi i (z_1 + z_2 + a_1 \cdot b_2)} e^{-2\pi i (a_1 + a_2) \cdot (b_1 + b_2)} e^{2\pi i (a_1 + a_2) \cdot x} f(x - (b_1 + b_2)) \\ &= \big(\pi(z_1, a_1, b_1)\pi(z_2, a_2, b_2)\big) f(x). \end{aligned}$$

Let $Z = \langle (2\pi, 0, 0) \rangle$, the subgroup of H_n generated by $(2\pi, 0, 0)$. Set $\tilde{H}_n = H_n / Z$. The group $\tilde{H}_n$ is known as the *Weyl-Heisenberg group*. Clearly $Z = \ker \pi$ and so π induces a representation $\tilde{\pi}$ on $\tilde{H}_n$. Observe that $\tilde{H}_n = \{(t, a, b) \mid t \in \mathbb{T}, a, b \in \mathbb{R}^n\}$ (here $\mathbb{T}$ is the unit circle $\{z \in \mathbb{Z} \mid |z| = 1\}$). The Lebesgue measure on $H_n = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ is left and right invariant Haar measure on H_n. Similarly, Lebesgue measure on $\mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n$ is left and right invariant Haar measure on $\tilde{H}_n$ (here Lebesgue measure on $\mathbb{T}$ is normalized so that $\int_{\mathbb{T}} dt = 1$). The next result was proved in [**13**, Proposition 3.2.4] for the special case $n = 1$.

By interchanging the roles of a and b the proof given there carries through verbatim to our case.

Proposition 6.1. *If $f, g \in L^2(\mathbb{R}^n)$, then*

$$\int_{\widehat{\mathbb{R}^n}} \int_{\mathbb{R}^n} \int_{\mathbb{T}} |\langle f, \tilde{\pi}(t,a,b)g\rangle|^2 \, dtdbda = \|f\|_2^2 \|g\|_2^2.$$

Corollary 6.2. *The representation $\tilde{\pi}$ of $\tilde{H}_n$ on $L^2(\mathbb{R}^n)$ is unitary and square integrable, and every $g \in L^2(\mathbb{R}^n)$ is admissible.*

Proof. By taking $f = g$ in the above proposition we see immediately that every element of $L^2(\mathbb{R}^n)$ is admissible. Suppose $g \in L^2(\mathbb{R}^n) \setminus \{0\}$ is fixed and assume $f \in L^2(\mathbb{R}^n)$ satisfies $\langle f, \tilde{\pi}(t,a,b)g\rangle = 0$ for all $(t,a,b) \in \tilde{H}_n$. Then $\|f\|_2 \|g\|_2 = 0$ and it follows that $f = 0$. Hence $\tilde{\pi}$ is irreducible as desired. $\square$

Proposition 6.3. *Let $n \in \mathbb{N}$, let $(a_k, b_k) \in \mathbb{R}^{2n}$ such that (a_k, b_k) generate a discrete subgroup of $\mathbb{R}^{2n}$ and $a_k \cdot b_k \in \mathbb{Q}$ for all k. If*

$$\sum_{k=1}^{r} c_k e^{2\pi i b_k \cdot t} f(t + a_k) = 0$$

with $0 \neq c_k \in \mathbb{C}$ constants, then $f = 0$.

Proof. We have $\mathbb{R}^{2n} = \tilde{H}_n/\mathbb{T}$. Lift the (a_k, b_k) to elements $g_k \in \tilde{H}_n$. Note that $\langle g_1, \dots, g_k\rangle$ is a discrete subgroup of $\tilde{H}_n$. We claim that $\alpha := \sum_k g_k$ is a non-zerodivisor in $\mathbb{C}\tilde{H}_n$. Indeed if $0 \neq \beta \in \mathbb{C}\tilde{H}_n$ and $\alpha\beta = 0$, let T be a transversal for $\mathbb{T}$ in $\tilde{H}_n$ containing T and write $\beta = \sum_{t\in T} \beta_t t$ where $\beta_t \in \mathbb{C}\mathbb{T}$. Since $\mathbb{R}^{2n}$ is an ordered group, we can apply a leading term argument: let k be such that $g_k \in T$ is largest and let $s \in T$ be the largest element such that $\beta_s \neq 0$. Then by considering $g_k s$, we see that $\beta_s \neq 0$ because $\alpha\beta = 0$, which is a contradiction. The result now follows from Proposition 1.5, Corollary 6.2 and Theorem 5.4. $\square$

7. Virtually abelian groups

In this section we consider virtually abelian groups, that is groups with an abelian subgroup of finite index.

Proposition 7.1. *Let G be a locally compact group which has an abelian closed subgroup A of finite index, and let $1 \leq p \in \mathbb{R}$. Assume that if $0 \neq \phi \in \mathbb{C}A$ and $0 \neq f \in L^p(A)$, then $\phi f \neq 0$. Let $0 \neq f \in L^p(G)$, let $H \leqslant G$ and let $\theta \in \mathbb{C}H$.*

(a) *If θ is a nonzero divisor in $\mathbb{C}H$, then $\theta f \neq 0$.*
(b) *If H is torsion free and $\theta \neq 0$, then $\theta f \neq 0$.*

Proof. Note that $\mathbb{C}A$ is an integral domain. Let B be the intersection of the conjugates of A in G, a closed abelian normal subgroup of finite index in G. Let $\{a_1, \dots, a_m\}$ be a set of coset representatives for B in A. Then $L^p(A) = \bigoplus_{i=1}^m L^p(B)a_i$ and we see that if $0 \neq \phi \in \mathbb{C}B$ and $0 \neq f \in L^p(B)$, then $\phi f \neq 0$. Let $\{g_1, \dots, g_n\}$ be a set of coset representatives for B in G. Then $L^p(G) = \bigoplus_{i=1}^n L^p(B)g_i$. We may view this as an isomorphism of $\mathbb{C}B$-modules. Set

$S = \mathbb{C}B \setminus \{0\}$. Then we may form the ring of fractions $S^{-1}\mathbb{C}G$. Since every element of S is a non-zerodivisor in $\mathbb{C}G$, it follows that $S^{-1}\mathbb{C}G$ is a ring containing $\mathbb{C}G$. Furthermore $S^{-1}\mathbb{C}B$ is a field, and $S^{-1}\mathbb{C}G$ has dimension n over this field. Therefore $S^{-1}\mathbb{C}G$ is an artinian ring, and since $S^{-1}\mathbb{C}B$ is a field of characteristic zero, we see that $S^{-1}\mathbb{C}G$ is a semisimple artinian ring, by Maschke's theorem. We deduce that non-zerodivisors in $S^{-1}\mathbb{C}G$ are invertible. Using [**10**, Theorem 10.8], we may form the $S^{-1}\mathbb{C}G$-module $S^{-1}L^p(G)$.

(a) If θ is non-zerodivisor in $\mathbb{C}H$, then θ is a non-zerodivisor in $\mathbb{C}G$ and hence is invertible in $S^{-1}\mathbb{C}G$, so θ^{-1} exists. We may regard f as an element of $S^{-1}L^p(G)$, because $S^{-1}L^p(G)$ contains $L^p(G)$. So if $\theta f = 0$, then $\theta^{-1}\theta f = 0$, consequently $f = 0$ and we have a contradiction.

(b) If H is torsion free, then we know that every non-zero element of $\mathbb{C}H$ is a non-zerodivisor in $\mathbb{C}H$; this was first proved by K. A. Brown [**4**]. Thus the result follows from (a). □

Theorem 7.2. *Let G be a locally compact group with no nontrivial compact subgroups and let $1 \leq p \leq 2$. Suppose G has an abelian closed subgroup of finite index. Then every nonzero element of $L^p(G)$ has linearly independent translations.*

Proof. Let A be an abelian closed subgroup of finite index. Since G has no nontrivial compact subgroups, the same is true for A and in particular A is torsion free. Furthermore if $0 \neq \phi \in \mathbb{C}A$ and $0 \neq f \in L^p(A)$, then $\phi f \neq 0$ by [**7**, Theorem 1.2]. The result now follows from Proposition 7.1(b). □

Theorem 7.3. *Let G be a locally compact abelian group, let $n \in \mathbb{N}$, and let $1 \leq p \in \mathbb{R}$. Assume that $p \leq 2n/(n-1)$. Suppose G has a closed subgroup of finite index isomorphic to $\mathbb{R}^n$ or $\mathbb{Z}^n$ as a locally compact abelian group. Let $H \leqslant G$, let $\theta \in \mathbb{C}H$, and let $0 \neq f \in L^p(G)$.*

(a) If θ is a nonzero divisor in $\mathbb{C}H$, then $\theta f \neq 0$.
(b) If H is torsion free and $\theta \neq 0$, then $\theta f \neq 0$.

Proof. We apply Proposition 7.1 with $A = \mathbb{R}^n$ and $\mathbb{Z}^n$. We need to check the hypothesis that if $0 \neq \phi \in \mathbb{C}A$ and $0 \neq f \in L^p(A)$, then $\phi f \neq 0$. For the case $A = \mathbb{R}^n$, this follows from [**24**, Theorem 3], while for the case $A = \mathbb{Z}^n$, this follows from [**19**, Theorem 2.1]. □

CHAPTER 3

Isomorphisms of C^*-Algebras

1. Introduction

The classical zero divisor conjecture states

Conjecture 1.1. *If H be a torsion-free group, then the group ring $\mathbb{C}H$ is a domain.*

There are various other ways of stating this conjecture. Let L be a vector space over $\mathbb{C}$ and suppose H acts on the left of L, equivalently L is a left $\mathbb{C}H$-module. Suppose $\beta \in L$. Then we say that β has linearly independent translates (or translations) if given distinct elements $g_1, \dots, g_n \in H$, the set $\{g_1\beta, \dots, g_n\beta\}$ is linearly independent over $\mathbb{C}$. Then Conjecture 1.1 can be stated as

Conjecture 1.2. *Let H be a torsion-free group and let $L = \mathbb{C}H$. If $0 \neq \beta \in \mathbb{C}H$, then β has linearly independent translations.*

Yet another way of stating Conjecture 1.1 is

Conjecture 1.3. *Let H be a torsion-free group, let $L = \mathbb{C}H$ and let $0 \neq \alpha \in L$. Then α is a non-zerodivisor in L, i.e. if $0 \neq \beta \in L$, then $\alpha\beta \neq 0$.*

In a general ring R, the element α is a left non-zerodivisor if $\alpha\beta \neq 0$ for all $\beta \in R \setminus 0$, whereas α is a right non-zerodivisor if $\beta\alpha \neq 0$ for all $\beta \in R \setminus 0$. Finally α is a non-zerodivisor means that α is both a left and right non-zerodivisor in R, that is $\alpha\beta \neq 0 \neq \beta\alpha$ for all $\beta \in R \setminus 0$. Thus in Conjecture 1.3, if L happens to be a ring and $\mathbb{C}H$ is a subring of L and the left module structure is given by left multiplication, then α is a non-zerodivisor in L really means α is a left non-zerodivisor in L.

In this paper we will consider other choices for L, which all have $\mathbb{C}H$ naturally embedded. Let G be a locally compact group with a left Haar measure (all topological groups considered will be Hausdorff), and let $L^2(G)$ denote the square integrable functions on G (where two functions are considered equal if and only if they disagree only on a set of measure zero). In the special case G is discrete, we shall write $\ell^2(G)$ for $L^2(G)$. Then H acts on $L^2(G)$ according to the rule $h(f(g)) = f(h^{-1}g)$. The case $H = G$ and $L = L^2(G)$ was considered in [**24**] for various torsion-free groups G. In particular it was shown there [**24**, Proposition 3.2] that for G the affine group, there exists $0 \neq f \in L^2(G)$ which has linearly dependent translations. The same paper leaves the case G is the Heisenberg group open. Also it was pointed out there [**24**, p. 464] that if G has an element of finite order, then there always exists a nonzero $f \in L^2(G)$ with linearly dependent translations. On the other hand, if G is a LCA (locally compact abelian) group, without nontrivial compact subgroups, then every nonzero $f \in L^2(G)$ has linearly independent translations.

The special case G is discrete is closely related to what is often termed the Atiyah conjecture [**21**, §10]. It is not difficult to see that if $H \leq G$ and H is a torsion-free group which satisfies the strong Atiyah conjecture [**21**, Conjecture 10.2], then every nonzero element of $\mathbb{C}H$ is a non-zerodivisor in $\ell^2(G)$. The strong Atiyah conjecture holds for torsion-free groups that have a normal free subgroup with elementary amenable quotient [**21**, Theorem 10.19] and many other groups.

Let $\mathbb{K}$ be a field, e.g. $\mathbb{R}$ or $\mathbb{C}$, and let $n \in \mathbb{N}$, the positive integers. $\mathrm{M}(n, \mathbb{K})$ denote the $n \times n$ matrices with entries from $\mathbb{K}$, and let I_n denote the identity matrix of $\mathrm{M}(n, \mathbb{K})$. Define $\mathrm{Tr}(n, \mathbb{K}) = \{(a_{ij}) \mid a_{ij} = 0 \text{ if } i \leq j\}$, the strictly upper triangular matrices, and let $\mathrm{Tr}_1(n, \mathbb{K}) = I_n + \mathrm{Tr}(n, \mathbb{K})$, the upper unitriangular matrices. Then $\mathrm{Tr}_1(n, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a locally compact group. Let $\mathcal{B}(V)$ indicate the bounded linear operators on the Hilbert space V and define $\mathcal{O}(H, G)$ to be the operator norm closure of $\mathbb{C}H$ in $\mathcal{B}(L^2(G))$. We shall prove

Theorem 1.4. *Let G be a LCA group or $\mathrm{Tr}_1(n, \mathbb{R})$ or $\mathrm{Tr}_1(n, \mathbb{C})$, and let H be an abelian subgroup of G. Then every nonzero $f \in \mathcal{O}(H, G)$ has linearly independent H-translations.*

The method of proof will be to identify $\mathcal{O}(H, G)$ with the reduced group C^*-algebra of H, and then the proof is easy.

The concept of having linearly independent translations is closely related to whether the left $\mathbb{C}H$-module L is flat as a left $\mathbb{C}H$-module. Indeed suppose L is a $\mathbb{C}H$-bimodule, which is the case in many natural situations. For example the case $L = L^2(G)$; then H acts on both the left and right of L, so for $f \in L^2(G)$ and $h \in H$, we have $(fh)(g) = f(gh^{-1})$. Of course if H is abelian, then L is always naturally a $\mathbb{C}H$-module. Let us suppose that $\mathbb{C}H$ is a domain; this will certainly be the case if H is torsion-free abelian. Then it is not difficult to see that if $L^2(G)$ is a flat right $\mathbb{C}H$-module, then every $f \in L$ has linearly independent left translations.

One could consider closely related problems with Hopf algebras. For example let L be a Hopf algebra over the field k and let H be a subgroup of the group-like elements of L [**6**, Definition 1.4.13]. Then kH (the k-linear span of H in L) is a k-subalgebra of A which is isomorphic to the group algebra of the group H over the field k [**6**, Proposition 1.4.14, Remark 4.2.9]. Let us assume that kH is a domain (thus H is torsion free). Then it follows [**22**, Theorem 6] that L is a flat right (and left) kH-module. If $0 \neq \alpha \in kH$, then left multiplication on kH is an injective right kH-module homomorphism. It now follows from the right flatness of L over kH that left multiplication by α on L is also an injective right kH-module homomorphism and hence α is a left non-zerodivisor in L. Similarly α is a right non-zerodivisor in L and therefore α is a non-zerodivisor in L. Of course in the previous paragraphs, we could have replaced $\mathbb{C}$ with an arbitrary field k. Thus we obtain that if $0 \neq f \in L$, then f has linearly independent translations.

2. Notation, terminology and assumed results

In this section we collect together some notation, terminology and well-known results.

Let $\mathbb{N}$ denote the positive integers $\{1, 2, \dots\}$, and let $\mathbb{T} = \{t \in \mathbb{C} \mid |t| = 1\}$. The order of an element g in a group will be denoted $\mathrm{o}(g)$. For $r \in \mathbb{N}$, let $\mathbb{T}_r = \{t \in \mathbb{T} \mid t^r = 1\}$, the rth roots of unity in $\mathbb{T}$, indicate the Pontryagin dual of $\mathbb{Z}/r\mathbb{Z}$. Suppose H is a finitely generated abelian group. Then $H \cong \mathbb{Z}^d \times \mathbb{Z}/r_1\mathbb{Z} \times \cdots \times \mathbb{Z}/r_e\mathbb{Z}$, where $0 \leq d, e \in \mathbb{Z}$ and $1 < r_i \in \mathbb{N}$ for all i. Therefore we may choose $x_1, \ldots, x_d, y_1, \ldots, y_e \in H$ such that $\langle x_i \rangle \cong \mathbb{Z}$ and $\langle y_i \rangle \cong \mathbb{Z}/r_i\mathbb{Z}$ for all i, and

$$H = \langle x_1 \rangle \times \cdots \times \langle x_d \rangle \times \langle y_1 \rangle \times \cdots \times \langle y_e \rangle. \tag{2.1}$$

If we set $n = d + e$ and $x_i = y_{i-d}$ for $d + 1 \leq i \leq n$, then equation 2.1 becomes

$$H = \langle x_1 \rangle \times \cdots \times \langle x_n \rangle. \tag{2.2}$$

When dealing with finitely generated abelian groups, we will often write H in one the above forms.

Let G be a locally compact group and let H be a subgroup of G. Then we have two actions of $\mathbb{C}H$ on a Hilbert space, namely the action on $L^2(G)$ and the action on $\ell^2(H)$; when we write $\ell^2(H)$ (small ℓ), we are always considering H as a discrete group. Let us recall how these actions are defined. If $f \in L^2(G)$, then $f\colon G \to \mathbb{C}$ is a square integrable function, and for $h \in H$ and $g \in G$, we define $hf(g) = f(h^{-1}g)$, and this extends by linearity to an action of $\mathbb{C}G$ on $L^2(G)$. The same formula works for the action of H on $\ell^2(H)$. We may also consider $\ell^2(H)$ as all formal sums $\{\sum_{g\in G} a_g g \mid \sum_g |a_g|^2 < \infty\}$.

For $\alpha \in \mathbb{C}G$, let $\operatorname{supp}\alpha$ indicate the support of α, so if we write $\alpha = \sum_{g\in G} a_g g$ where $a_g \in \mathbb{C}$, then $\operatorname{supp}\alpha = \{g \in G \mid a_g \neq 0\}$. Also denote the operator norm closure of $\mathbb{C}H$ in $\mathcal{B}(\ell^2(H))$ by $\mathrm{C}^*_{\mathrm{r}}(H)$. Note that $\mathrm{C}^*_{\mathrm{r}}(H)$ is the reduced group C^*-algebra of the discrete group H. In the case H is abelian, it is well-known that $\mathrm{C}^*_{\mathrm{r}}(H) \cong C(\hat{H})$, the ring of continuous functions on the Pontryagin dual $\hat{H}$ of H [**?**, Proposition VII.1.1]. We note if H was not discrete, we would have to replace $C(\hat{H})$ with $C_0(\hat{H})$, the continuous functions vanishing at infinity, but when H is discrete, then $\hat{H}$ is compact. Furthermore in this isomorphism, the operator norm is the sup norm on $C_0(\hat{H})$. In the special case $H \cong \mathbb{Z}^n$, then $\hat{H} \cong \mathbb{T}^n$ and $\mathbb{C}H$ corresponds to the Laurent polynomial functions on $\mathbb{T}^n$. Since the zero set of such a nonzero polynomial is finite, it has measure zero and so is a non-zerodivisor in $\mathrm{C}^*_{\mathrm{r}}(H)$. Thus nonzero elements of $\mathbb{C}H$ are non-zerodivisors in $\mathrm{C}^*_{\mathrm{r}}(H)$.

For H a discrete locally compact group, let $c_0(H)$ denote the functions on H which vanish at ∞, that is $\{f\colon H \to \mathbb{C} \mid \text{ given } \epsilon > 0, \text{ there exists a finite subset } F \text{ of } H \text{ such that } |f(h)| < \epsilon \text{ for all } h \in H \setminus F\}$. Thus $\ell^2(H) \subset c_0(H)$ and we define a multiplication, often called convolution, $\ell^2(H) \times \ell^2(H) \to c_0(H)$ as follows: if $\alpha = \sum_{g\in H} a_g g$, $\beta = \sum_{h\in H} a_h h$, then

$$\alpha\beta = \sum_{g\in H,\, h\in H} a_g b_h gh = \sum_{x\in H} (\sum_{g\in H} a_g b_{g^{-1}x}) x.$$

We can view $\mathrm{C}^*_{\mathrm{r}}(H)$ as a subspace of $\ell^2(H)$ by sending $\theta \in \mathrm{C}^*_{\mathrm{r}}(H)$ to $\theta(1) \in \ell^2(H)$, where 1 indicates the element of $\ell^2(H)$ which is 1 on the identity element of H and zero on the other elements of H. Then the multiplication is given by convolution.

3. Preliminary Lemmas

Lemma 3.1. *Let G be a LCA group and let H be a finitely generated subgroup of G. Write H as in equation 2.2. Given $\xi_1, \cdots, \xi_n \in \mathbb{T}$ where $\xi_i \in \mathbb{T}_{r_i}$ if $o(x_i) = r_i$, then there exists a maximal ideal in $\mathcal{O}(H, G)$ containing the ideal $(x_1 - \xi_1, \cdots, x_n - \xi_n)$.*

Proof. Let $\chi \in \hat{G}$ i.e. $\chi : G \to \mathbb{T}$ with $\|\chi\| = 1$ and let $\epsilon > 0$. Since $\{(\chi(x_1), \cdots, \chi(x_n)) \mid \chi \in \hat{G}\}$ is dense in $\mathbb{T}^d \times \mathbb{T}_{r_1} \times \cdots \times \mathbb{T}_{r_e}$ by Lemma 4.5, we may choose $x_i \in H$ such that $|\chi(x_i) - \xi_i| < \epsilon$ for all i. Then

$$\begin{aligned} |(x_i - \xi_i)\chi(t)| = |\chi(tx_i) - \xi_i\chi(t)| &= |\chi(t)(\chi(x_i) - \xi_i)| \\ &\leq |\chi(t)||(\chi(x_i) - \xi_i)| \\ &= |\chi(x_i) - \xi_i| < \epsilon. \end{aligned}$$

By the Følner condition let $U \subseteq G$ measurable such that $\frac{m(U\Delta x_i^{-1}U)}{m(U)} < \epsilon/n$ where Δ denotes the symmetric difference and $m(\cdot)$ is the measure. Define $\tilde{\chi} = \frac{\chi}{n\sqrt{m(U)}}$ on U and zero otherwise. For $t \in (U \cap x_i^{-1}U)$ we have

$$n|(x_i - \xi_i)\tilde{\chi}(t)| = |(x_i - \xi_i)\frac{\chi(t)}{\sqrt{m(U)}}| < \sqrt{m(U)\left(\frac{\epsilon}{\sqrt{m(U)}}\right)^2} = \epsilon.$$

For $t \in (U\Delta x_i^{-1}U)$ we know that

$$|(x_i - \xi_i)\tilde{\chi}(t)| \leq \frac{2}{n\sqrt{m(U)}}.$$

Since $m(U\Delta x_i^{-1}U) < \epsilon\sqrt{m(U)}$ we conclude that

$$|(x_i - \xi_i)\tilde{\chi}(t)| \leq \sqrt{\left(\frac{2}{\sqrt{m(U)}}\right)^2 m(U\Delta x_i^{-1}U)} < 2\sqrt{\epsilon}/n.$$

Finally for $t \in (U^c - x_i^{-1}U)$ by assumption $\chi(t) = 0$ and hence $\tilde{\chi}(t) = 0$. It follows that

$$n\|(x_i - \xi_i)\tilde{\chi}(t)\|_2 < \epsilon + 2\sqrt{\epsilon}.$$

Now if there does not exist a maximal ideal in A containing $(x_1 - \xi_1, \cdots, x_n - \xi_n)$, then 1 must be in the closure of $(x_1 - \xi_1, \cdots, x_n - \xi_n)$. Then $\exists\ \alpha_i \in \mathbb{C}G$ such that

$$\|1 - (x_1 - \xi_1)\alpha_1 - \cdots - (x_n - \xi_n)\alpha_n\| < \frac{1}{2}.$$

This implies that for $u \in L^2(G)$ with $\|u\|_2 = 1$

$$\|(1 - (x_1 - \xi_1)\alpha_1 - \cdots - (x_n - \xi_n)\alpha_n)u\|_2 < \frac{1}{2}.$$

Using the triangle inequality we obtain,

$$\|u\|_2 \leq \|((x_1 - \xi_1)\alpha_1 u + \cdots + (x_n - \xi_n)\alpha_n u)\|_2 + \frac{1}{2}.$$

Set $u = \tilde{\chi}$. Then

$$\|((x_1 - \xi_1)\alpha_1 u + \cdots + (x_n - \xi_n)\alpha_n u)\|_2 < \epsilon + 2\sqrt{\epsilon}$$

by the above calculation which is absurd since $\|u\|_2 = 1$. ∎

Let $\mathbb{K}$ be a field, e.g. $\mathbb{R}$ or $\mathbb{C}$. Let $n \in \mathbb{N}$ and let $m \in \mathbb{Z}$ with $0 \le m \le n$. Let $\mathrm{Tr}(n,\mathbb{K}) = \{(a_{ij}) \mid a_{ij} = 0 \text{ if } i \le j\}$, the strictly upper triangular matrices. Let $\mathrm{Tr}(m,n,\mathbb{K}) \subseteq \mathrm{M}(n,\mathbb{K})$ denote $\{(a_{ij}) \mid a_{ij} = 0 \text{ if } i \le j, \text{ or } i > m \text{ and } j < n+1-m\}$ and set $\mathrm{Tr}_1(m,n,\mathbb{K}) = I + \mathrm{Tr}(m,n,\mathbb{K})$. Note that $\mathrm{Tr}(m,n,\mathbb{K})$ is a $\mathbb{K}$-subspace $\mathrm{M}(n,\mathbb{K})$, $\mathrm{Tr}(n,n,\mathbb{K}) = \mathrm{Tr}(n,\mathbb{K})$ and $\mathrm{Tr}_1(m,n,\mathbb{K})$ is a subgroup of $\mathrm{GL}(n,\mathbb{K})$. Furthermore $\mathrm{Tr}(0,n,\mathbb{K}) = 0$, $\mathrm{Tr}_1(0,n,\mathbb{K}) = 1$, and $\mathrm{Tr}_1(1,n+2,\mathbb{K})$ is often called the n-dimensional Heisenberg group and denoted $\mathbf{H}(n,\mathbb{K})$.

Lemma 3.2. *Let U be a $\mathbb{K}$-subspace of $\mathrm{M}(n,\mathbb{K})$ and set $G = I + U := \{I + u \mid u \in U\}$. Suppose $G \leqslant \mathrm{GL}(n,\mathbb{K})$. Then*

(a) *G is nilpotent.*
(b) *Let A be a maximal abelian subgroup of G. The $A = I + V$ for some $\mathbb{K}$-subspace of U.*

Proof. Suppose $u \in U$ has a nonzero eigenvalue λ with an associated eigenvector v, then $uv = \lambda v$. By using a basis of the form $\{u, \dots\}$, we may assume that

$$u = \begin{pmatrix} \lambda & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \in U.$$

Since U is a subspace, we see that $-\lambda^{-1}u \in U$ and hence

$$-\lambda^{-1}u = \begin{pmatrix} -1 & * & \cdots & * \\ 0 & \vdots & \cdots & \vdots \\ 0 & * & \ddots & * \\ 0 & * & \cdots & * \end{pmatrix} \in U.$$

Since $G = I + U$, we deduce that

$$I - \lambda^{-1}u = \begin{pmatrix} 0 & * & \cdots & * \\ 0 & \vdots & \vdots & * \\ 0 & * & \ddots & * \\ 0 & * & \cdots & * \end{pmatrix} \in G$$

has a zero eigenvalue. But $G \leqslant \mathrm{GL}(n,\mathbb{K})$, whose elements have no zero eigenvalues and now we have a contradiction. Therefore all the eigenvalues of u are zero. Thus G is a unipotent group and it follows that G is nilpotent [**26**, Corollary 1.21]. This proves (a).

Each element of A can be written uniquely as $I + N$ where $N \in U$, consequently we may write $A = I + V$ where $V \subseteq U$. We show that V is a $\mathbb{K}$-subspace of U. Let $\mathbb{K}V$ denote all $\mathbb{K}$-linear combinations of elements of V, i.e. the $\mathbb{K}$-subspace generated by V. It is easily checked that the elements of $\mathbb{K}V$ commute, because the elements of V commute, and that $\mathbb{K}V \subseteq U$. Set $A_1 = \langle I + \mathbb{K}V\rangle$. Since the elements of $I + \mathbb{K}N$ commute, A_1 is an abelian subgroup of G. Since A is a maximal abelian subgroup of G it follows that $A = A_1$. We deduce that $V = \mathbb{K}V$ is a $\mathbb{K}$-subspace of U, and (b) is proven. □

Now specialize to the case $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then G is nilpotent by Lemma 3.2(a), and a maximal abelian subgroup A of G is a simply connected closed subgroup of G by Lemma

3.2(b). Let $\mathfrak{g}$ denote the Lie algebra of G. Then G is a simply connected nilpotent closed linear group and the exponential map $\exp\colon \mathfrak{g} \to G$ is a homeomorphism. [**14**, pp. 2, 6].

Let $\mathfrak{a} = \{X \in \mathfrak{g} \mid \exp(sX) \in A\}$ for all $s \in \mathbb{R}$, a Lie subalgebra of $\mathfrak{g}$. Since A is a closed linear group, $\mathfrak{a}$ is the Lie algebra of A [**14**, Proposition 0.14]. Let $\mathfrak{b}$ be a vector space direct complement of $\mathfrak{a}$ in $\mathfrak{g}$ and set $B = \exp(\mathfrak{b})$. Since A is a simply connected nilpotent analytic group, the exponential map $\exp\colon \mathfrak{a} \to A$ is a homeomorphism [**14**, Theorem 1.127]. Therefore $G = \exp(\mathfrak{a}) \times \exp(\mathfrak{b})$, and every element h of $\mathfrak{g}$ is uniquely of the form (a, b) where $a \in A$ and $b \in B$. Furthermore if $p \in A$, then the unique form for ph is (pa, b), because A is subgroup of G.

Now if χ is a character on A, then the way we want to extend χ to χ' on G is by the formula $\chi'(ab) = \chi(a)$. Suppose $x \in A$ and $t \in G$. Write $t = yb$ where $y \in A$ and $b \in B$. Then

$$\begin{aligned} x\chi'(t) &= \chi'(x^{-1}t) = \chi'(x^{-1}yb) = \chi(x^{-1}y) \\ &= \chi(x^{-1})\chi(y) = \chi(x^{-1})\chi'(yb) \\ &= \chi(x^{-1})\chi'(t). \end{aligned}$$

Let G be defined as in Lemma 3.2.

Lemma 3.3. *Let H be a finitely generated abelian subgroup of G. Write $H = \langle x_1, \dots, x_n \rangle$. Given $\xi_1, \cdots, \xi_n \in \mathbb{T}$ where $o(\xi_i) = \infty$, then there exists a maximal ideal in $\mathcal{O}(H, G)$ containing the ideal $(x_1 - \xi_1, \cdots, x_n - \xi_n)$.*

Proof. Let A be a maximal abelian subgroup of G containing H. Let $\chi \in \hat{A}$ i.e. $\chi : A \to \mathbb{T}$ with $\|\chi\| = 1$ and let $\epsilon > 0$. Since $\{(\chi(x_1), \cdots, \chi(x_n)) \mid \chi \in \hat{A}\}$ is dense by Theorem 4.5 in $\mathbb{T}^n$, we may choose $\chi \in \hat{A}$ such that $|\chi(x_i^{-1}) - \xi_i| < \epsilon$ for all i. Then

$$\begin{aligned} |(x_i - \xi_i)\chi'(t)| = |\chi(x_i^{-1})\chi'(t) - \xi_i\chi'(t)| = |\chi'(t)(\chi(x_i^{-1}) - \xi_i)| &\le |\chi'(t)||(\chi(x_i^{-1}) - \xi_i)| \\ &= |\chi(x_i^{-1}) - \xi_i| < \epsilon. \end{aligned}$$

By the Følner condition let $U \subseteq G$ measurable such that $\frac{m(U\Delta x_i^{-1}U)}{m(U)} < \epsilon/n$ where Δ denotes the symmetric difference and $m(\cdot)$ is the measure. Define $\tilde{\chi} = \frac{\chi'}{n\sqrt{m(U)}}$ on U and zero otherwise. For $t \in (U \cap x_i^{-1}U)$ we have

$$n|(x_i - \xi_i)\tilde{\chi}(t)| = |(x_i - \xi_i)\frac{\chi'(t)}{\sqrt{m(U)}}| < \sqrt{m(U)\left(\frac{\epsilon}{\sqrt{m(U)}}\right)^2} = \epsilon.$$

For $t \in (U\Delta x_i^{-1}U)$ we know that

$$|(x_i - \xi_i)\tilde{\chi}(t)| \le \frac{2}{n\sqrt{m(U)}}.$$

Since $m(U\Delta x_i^{-1}U) < \epsilon\sqrt{m(U)}$ we conclude that

$$|(x_i - \xi_i)\tilde{\chi}(t)| \le \sqrt{\left(\frac{2}{\sqrt{m(U)}}\right)^2 m(U\Delta x_i^{-1}U)} < 2\sqrt{\epsilon}/n.$$

Finally for $t \in (U^c - x_i^{-1}U)$ by assumption $\tilde{\chi}(t) = 0$. It follows that

$$n\|(x_i - \xi_i)\tilde{\chi}(t)\|_2 < \epsilon + 2\sqrt{\epsilon}.$$

Now if there does not exist a maximal ideal in A containing $(x_1 - \xi_1, \cdots, x_n - \xi_n)$, then 1 must be in the closure of $(x_1 - \xi_1, \cdots, x_n - \xi_n)$. Then $\exists\ \alpha_i \in \mathbb{C}G$ such that

$$\|1 - (x_1 - \xi_1)\alpha_1 - \cdots - (x_n - \xi_n)\alpha_n\| < \frac{1}{2}.$$

This implies that for $u \in L^2(G)$ with $\|u\|_2 = 1$

$$\|(1 - (x_1 - \xi_1)\alpha_1 - \cdots - (x_n - \xi_n)\alpha_n)u\|_2 < \frac{1}{2}.$$

Using the triangle inequality we obtain,

$$\|u\|_2 \le \|((x_1 - \xi_1)\alpha_1 u + \cdots + (x_n - \xi_n)\alpha_n u)\|_2 + \frac{1}{2}.$$

Set $u = \tilde{\chi}$. Then

$$\|((x_1 - \xi_1)\alpha_1 u + \cdots + (x_n - \xi_n)\alpha_n u)\|_2 < \epsilon + 2\sqrt{\epsilon}$$

by the above calculation which is absurd since $\|u\|_2 = 1$. □

4. Theorems and Proofs

Theorem 4.1. *Let G be either LCA or of the form $G = I + U := \{I + u \mid u \in U\}$ where U is a $\mathbb{K}$-subspace of $\mathrm{M}(n, \mathbb{K})$. Let H be a finitely generated abelian subgroup of G, then the identity map on $\mathbb{C}H$ extends to a C^*-algebra isomorphism from $\mathcal{O}(H, G)$ to $\mathrm{C}^*_{\mathrm{r}}(H)$.*

Proof. Write $H = \overbrace{\mathbb{Z} \times \cdots \times \mathbb{Z} \times \mathbb{Z}/r_1\mathbb{Z} \times \cdots \times \mathbb{Z}/r_e\mathbb{Z}}^{n \text{ factors}} = \langle x_1, \cdots, x_n\rangle$ where x_i is the generator of the ith factor. Then

$$\mathrm{C}^*_{\mathrm{r}}(H) \cong C(\hat{H}) = C(\overbrace{\mathbb{T} \times \cdots \times \mathbb{T} \times \mathbb{T}_{r_1} \times \cdots \times \mathbb{T}_{r_e}}^{n \text{ factors}}).$$

Recall that if C is an abelian C^*-algebra with a 1, then every maximal ideal M of C has codimension 1 in C, and if X is the space of maximal ideals (with the appropriate topology), then $C \cong C(X)$. From this, we see that if $\alpha \in C$, then α is invertible in C if and only is α is nonzero on X, equivalently, $\alpha \notin M$ for all $M \in X$.

Now let $\alpha \in \mathbb{C}H$. Let M be a maximal ideal of either $\mathcal{O}(H, G)$ or $\mathrm{C}^*_{\mathrm{r}}(H)$. Then $M \cap \mathbb{C}H$ is a maximal ideal of $\mathbb{C}H$, and will have the form $(x_1 - \xi_1, \cdots, x_n - \xi_n)$, where $|\xi_i| = 1$ for $1 \le i \le n$ and $\mathrm{o}(\xi_i) \mid \mathrm{o}(x_i)$. Furthermore by Lemma 3.1 and Lemma 3.3 for each $\xi_i \in \mathbb{C}$ with $|\xi_i| = 1$, there exist maximal ideals in $\mathcal{O}(H, G)$ containing $(x_1 - \xi_1, \cdots, x_n - \xi_n)$ provided $\mathrm{o}(\xi_i) \mid \mathrm{o}(x_i)$. The same holds for $\mathrm{C}^*_{\mathrm{r}}(H)$. This has the following consequence: if $\alpha \in \mathbb{C}H$, then α is invertible in $\mathcal{O}(H, G)$ if and only if α is invertible in $\mathrm{C}^*_{\mathrm{r}}(H)$. Now $\|\alpha\| = \sqrt{\|\alpha\alpha^*\|}$. Furthermore, if $\beta = \alpha\alpha^*$, then the norm of β is the spectral radius of β. Since $\beta - \lambda I$ is invertible in $\mathcal{O}(H, G)$ if and only if $\beta - \lambda I$ is invertible in $\mathrm{C}^*_{\mathrm{r}}(H)$, we see that $\|\beta\|$ is the same whether we consider it as an element of $\mathcal{O}(H, G)$ or an element of $\mathrm{C}^*_{\mathrm{r}}(H)$. The same can be said for the general element α. We conclude that the identity map on $\mathbb{C}H$ extends to an isomorphism from $\mathcal{O}(H, G)$ onto $\mathrm{C}^*_{\mathrm{r}}(H)$. □

Corollary 4.2. *Let H be any subgroup of the LCA group G. Then the identity map on $\mathbb{C}H$ extends to a C^*-algebra isomorphism from $\mathcal{O}(H, G)$ to $\mathrm{C}^*_{\mathrm{r}}(H)$.*

Proof. Let $\|\cdot\|$ denote the norm on $\mathcal{O}(H, G)$, and let $\|\cdot\|_r$ denote the norm on $\mathrm{C}^*_{\mathrm{r}}(G)$. If $\theta\colon \mathbb{C}G \to \mathbb{C}G$ denotes the identity map on $\mathbb{C}G$ and $\alpha \in \mathbb{C}G$, then $\|\alpha\| = \|\theta\|_r$ by Theorem 4.1. It follows that θ extends to a C^*-algebra isomorphism from $\mathcal{O}(H, G)$ onto $\mathrm{C}^*_{\mathrm{r}}(H)$. □

Remark 4.3. *Note that if H is a group and $\alpha \in \mathbb{C}H$, then $\alpha \in \mathbb{C}K$ for some finitely generated subgroup K of H.*

Theorem 4.4. *Let H be a discrete group and let $\alpha \in \mathbb{C}H$. Assume that $\alpha \in \mathbb{C}K$ for some finitely generated subgroup K of H by remark 4.3. Suppose $0 \neq \beta \in \mathrm{C}^*_{\mathrm{r}}(H)$ and $\alpha\beta = 0$. Then there exists $0 \neq \gamma \in \mathrm{C}^*_{\mathrm{r}}(K)$ such that $\alpha\gamma = 0$.*

Proof. We view $\mathrm{C}^*_{\mathrm{r}}(H)$ as a subspace of $\ell^2(H)$. Then the multiplication is given by convolution. Let $g \in \operatorname{supp}(\beta)$. Then $\alpha(\beta g^{-1}) = 0$ and $1 \in \operatorname{supp}(\beta g^{-1})$, so we may assume that $1 \in \operatorname{supp}(\beta)$. Write $\beta = \sum_{h \in H} b_h h$. Set $\beta_1 = \sum_{k \in K} b_k k$ and $\beta_2 = \sum_{g \in H \setminus K} b_g g$. Then $\alpha\beta_1 + \alpha\beta_2 = \alpha\beta = 0$. Since $\operatorname{supp}(\alpha\beta_1) \subseteq K$ and $\operatorname{supp}(\alpha\beta_2) \subseteq H \setminus K$, we deduce that $\alpha\beta_1 = 0$. Also $1 \in \operatorname{supp}(\beta_1)$, so $\beta_1 \neq 0$ and the result follows. □

Theorem 4.5. *Let G be a locally compact Hausdorff abelian group and let H be a finitely generated subgroup of G. Write $H = \langle x_1 \rangle \times \cdots \times \langle x_d \rangle \times \langle y_1 \rangle \times \cdots \times \langle y_e \rangle$, as in equation 2.1. Then $\{\chi(x_1), \dots, \chi(y_e) \mid \chi \in \hat{G}\}$ is dense in $\mathbb{T}^d \times \mathbb{T}_{r_1} \times \cdots \times \mathbb{T}_{r_e}$.*

Proof. Let $Y = \mathbb{T}^d \times \mathbb{T}_{r_1} \times \cdots \times \mathbb{T}_{r_e}$, so $Y \cong \hat{H}$ and $\hat{Y} \cong H$. We may view Y as elements $(t_1, \dots, t_d, s_1, \dots, s_e)$, where $t_i \in \mathbb{T}$ and $s_i \in \mathbb{T}_{r_i}$ for all i. Set $X := \{(\chi(x_1), \dots, \chi(y_e)) \mid \chi \in \hat{G}\}$ and $Y := \mathbb{T}^d \times \mathbb{T}_{r_1} \times \cdots \times \mathbb{T}_{r_e}$. Then $X \leqslant Y$. Suppose X is not dense in Y, i.e. $\overline{X} \neq Y$. Then there exists a nontrivial character $\psi\colon Y \to \mathbb{T}$ such $\overline{X} \subseteq \ker \psi$, by [**25**, Theorem 2.1.2].

It follows that $\psi(t_1, \dots, t_d, s_1, \dots, s_e) = t_1^{n_1} \dots t_d^{n_d} s_1^{m_1} \dots s_e^{m_e}$, where $n_i, m_i \in \mathbb{Z}$, $0 \leq m_i < r_i$ for all i and not all the n_i, m_i zero. Therefore

$$\chi(x_1^{n_1} \dots x_d^{n_d} y_1^{m_1} \dots y_e^{m_e}) = 1 \text{ for all } \chi \in \hat{G},$$

and it follows that $x_1^{n_1} \dots y_e^{m_e} = 1$, which contradicts the hypothesis that $H = \langle x_1, \dots, y_e \rangle$. □

Theorem 4.6. *Let G be a locally compact group with an abelian closed subgroup of finite index. Then the identity map on $\mathbb{C}G$ extends to a C^*-algebra isomorphism between the operator norm closure $\mathcal{O}(G, G)$ in $\mathcal{B}(L^2(G))$ to the reduced C^*-algebra $\mathrm{C}^*_{\mathrm{r}}(G)$.*

Proof. Since G has an abelian closed subgroup of finite index, it has a closed abelian normal subgroup of finite index; let this index be n. Let A be a closed abelian normal subgroup of finite index n in G. Consider operator norm closure $\mathcal{O}(G,G)$ in $\mathcal{B}(L^2(G))$. Let $g_1,\dots,g_n$ be coset representatives of A in G. Then as Hilbert spaces, $L^2(G)=\bigoplus_{i=1}^n L^2(A)g_i$. It follows that $\mathcal{O}(G,G)=\bigoplus_{i=1}^n \mathcal{O}(A,G)g_i$. By Corollary 4.2 we have an isomorphism $\theta\colon \mathcal{O}(A,G)\to \mathrm{C}^*_{\mathrm{r}}(A)$. Then we have a well-defined isomorphism $\hat{\theta}\colon \mathcal{O}(G,G)\to \mathrm{C}^*_{\mathrm{r}}(G)$ given by $\hat{\theta}(x_1g_1+\cdots+x_ng_n)=\theta(x_1)g_1+\cdots+\theta(x_n)g_n$ where $x_1,\dots,x_n\in\mathcal{O}(A,G)$. $\square$

CHAPTER 4

Injective Modules

1. Introduction

Let G be a group and let $\mathcal{U}(G)$ denote the unbounded operators affiliated to $\mathcal{N}(G)$. Let $\mathcal{D}(G)$ denote the division closure of $\mathbb{C}G$ in $\mathcal{U}(G)$ (the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{C}G$ which is closed under taking inverses). Module will mean right module.

Let R be a ring and let S denote the set of non-zerodivisors of R. A right R-module M is *torsion free* if $0 \neq m \in M$ and $s \in S$ implies $ms \neq 0$. Also M is called divisible if $Ms = M$ for all $s \in S$.

2. Theorems and Proofs

THEOREM 2.1. *Let G be an elementary amenable group and assume that the finite subgroups have bounded order.*

(a) *$\mathbb{C}G$ has a classical ring of quotients Q which is a semisimple artinian ring.*
(b) *$\mathcal{U}(G)$ is a divisible $\mathbb{C}G$-module.*
(c) *Q and $\mathcal{U}(G)$ are injective $\mathbb{C}G$-modules.*
(d) *Q and $\mathcal{U}(G)$ are self injective.*

PROOF. (a) This follows from [**16**, Theorem 7]
(b) This follows from [**16**, Theorem 7], because $\mathcal{U}(G)$ is precisely U in that theorem.
(c) This follows from (b) and [**15**, Theorem 3.3].
(d) Q is self injective by (a), because semisimple artinian rings are self injective. Also $\mathcal{U}(G)$ is self injective, by [**2**, Theorem 2]. $\square$

PROPOSITION 2.2. *Let G be a group which contains a nonabelian free subgroup, let k be a field, and let M be a nonzero kG-module containing kG. Assume that if $0 \neq m \in M$ and $g \in G$ has infinite order, then $m(1-g) \neq 0$. Then M is not an injective kG-module.*

PROOF. Let H be a free subgroup of rank 2 of G, and write $H = \langle x, y\rangle$ where x, y are free generators for H. Then the augmentation ideal $\omega(kH) := (x-1)kH + (y-1)kH$ is a free kH-module of rank 2 on $\{x-1, y-1\}$. Therefore $\omega(kH)kG$ is a free kG-module on $\{x-1, y-1\}$. Let $0 \neq m \in M$. Then we may define a right kG-homomorphism $\theta\colon \omega(kH)kG \to kG$ by sending $x-1$ and $y-1$ to m, explicitly $\theta\big((x-1)\alpha + (y-1)\beta\big) = m(x-1)\alpha + m(y-1)\beta$ for $\alpha, \beta \in kG$. If M was injective, then we could extend θ to the whole of kG, say $\theta(1) = n$ where $n \in M$. Then $\theta(x-1) = n(x-1)$ and $\theta(y-1) = n(y-1)$. We deduce that $n(x-1) = m = n(y-1)$, hence $n \neq 0$ and $n(x-y) = 0$, and we have a contradiction. We conclude that M is not an injective kG-module. $\square$

Theorem 2.3. *Let G be a group which contains a nonabelian free subgroup. Then all nonzero $\mathbb{C}G$-submodules of $\mathcal{U}(G)$ are not injective.*

Proof. If $g \in G$ has infinite order, then $1-g$ is a non-zerodivisor in $\mathcal{U}(G)$. Thus the result follows from Proposition 2.2. $\square$

Proposition 2.4. *Let G be a group that contains an element of infinite order. Then $\ell^\infty(G)$ is not an injective $\mathbb{C}G$-module.*

Proof. Let $x \in G$ be an element of infinite order, and write $X = \langle x \rangle$ and $Y = G \setminus X$. Set $\sigma = \sum_{g \in X} g \in \ell^\infty(G)$. Define a $\mathbb{C}G$-module map $\theta\colon (x-1)\mathbb{C}G \to \ell^\infty(G)$ by $\theta(x-1) = \sigma$, specifically $\theta\big((x-1)\big)\alpha = \sigma\alpha$ for $\alpha \in \mathbb{C}G$. Note that θ is well defined, because $x-1$ is a non-zerodivisor in $\mathbb{C}G$. If we could extend θ to $\mathbb{C}G$, then $\theta(1) = \tau$ for some $\tau \in \ell^\infty(G)$ and we have $\tau(x-1) = \sigma$. Write $\tau = \sum_{g \in G} \tau_g$. Then $\tau_{x^{n-1}} - \tau_{x^n} = 1$ for all $n \in \mathbb{Z}$. This shows that τ_{x^n} is unbounded for $n \in \mathbb{Z}$ and we deduce that $\tau \notin \ell^\infty(G)$. $\square$

Proposition 2.5. *Let G be a group that contains an element of infinite order. Then $\mathcal{N}(G)$ is not an injective $\mathbb{C}G$-module.*

Proof. Let $x \in G$ be an element of infinite order. Let ϕ be the map sending $1_{\mathbb{C}G}$ to the $1_{\mathcal{N}(G)}$. Further let $1-x$ be the map from $\mathbb{C}G$ to itself acting by left multiplication with $1-x$. If $\mathcal{N}(G)$ is an injective $\mathbb{C}G$-module, then there exists a map θ sending $1_{\mathbb{C}G}$ to $\alpha \in \mathcal{N}(G)$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}G \arrow[r, "1-x"] \arrow[d, "\phi"'] & \mathbb{C}G \arrow[dl, "\theta"] \\
& \mathcal{N}(G) &
\end{tikzcd}$$

Since the diagram commutes it follows that $1_{\mathcal{N}(G)} = \phi(1) = \theta(1-x) = \alpha(1-x)$. But $1-x$ is not invertible in $\mathcal{N}(G)$ since x is of infinite order. Thus $1_{\mathcal{N}(G)} \neq \alpha(1-x)$ which presents a contradiction. The result follows. $\square$

Define a Banach-$\star$ G-bi-module B to be a Banach space B which is a bi-module over $\mathbb{C}G$ together with a star operation (an involution) such that for $b \in B$ and $g \in G$ we have $(gb)^* = b^* g^*$. In what follows we assume the action of the element x of infinite order in G acts on B is continuous.

Theorem 2.6. *Let G be a group that contains an element of infinite order. Then a Banach-$\star$ G-bi-module B is not an injective left (right) $\mathbb{C}G$-module.*

Proof. Let $x \in G$ be an element of infinite order and define $y = x + x^{-1}$. Let ϕ be the map sending $1_{\mathbb{C}G}$ to the $\alpha \in B$. Further let $\xi - y$ be the map from $\mathbb{C}G$ to itself acting by left multiplication with $\xi - y$ where $\xi \in \mathbb{C}$. If B is an injective left (right) $\mathbb{C}G$-module,then there exists a map θ sending $1_{\mathbb{C}G}$ to $\beta \in B$ such that the following diagram commutes.

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{C}G \arrow[r, "\xi - y"] \arrow[d, "\phi"'] & \mathbb{C}G \arrow[dl, "\theta"] \\
 & B &
\end{tikzcd}
$$

Since the diagram commutes it follows that $\alpha = \phi(1) = \theta(\xi - y) = \beta(\xi - y)$. This implies $L^2(G)(\xi - y) = L^2(G)$ and therefore $\xi - y$ is left invertible for all $\xi \in \mathbb{C}$. Applying the $\star$ operation we obtain $(\xi - y)^* L^2(G) = L^2(G)$ which is equivalent to $(\bar{\xi} - y)L^2(G) = L^2(G)$ since $y^* = y$. It follows that $\bar{\xi} - y$ is right invertible for all $\bar{\xi} \in \mathbb{C}$. But it is well known that the spectrum of y, $Spec(y)$, is not empty. Therefore there exists an element $\xi' \in Spec(y) \subset \mathbb{C}$ such that $\xi' - y$ is not invertible which presents a contradiction. This proves the result. $\square$

It is note worthy that the above result is very general. The set of Banach spaces for which the above theorem is applicable includes but not limited to the von Neumann algebra $\mathcal{N}(G)$, the reduced C^*-algebra $\mathrm{C}^*_{\mathrm{r}}(G)$, $L^p(G)$, $\ell^p(G)$, $\ell^\infty(G)$, $\mathcal{B}(L^p(G))$ and the operator norm closure algebra $\mathcal{O}(H, G)$ where H would replace the group G in the theorem above.

We need the following result which is proved in [**3**, (c) on p. 2].

Proposition 2.7. *Let $\gamma\colon \Gamma \to \Lambda$ be a unital ring homomorphism. Let A be a left Λ-module and let C be a left Γ-module. If Λ is flat as a right Γ-module via γ, then $\mathrm{Ext}^k_\Gamma(C, A) \cong \mathrm{Ext}^k_\Lambda(\Lambda \otimes_\Gamma C, A)$ for all $k \in \mathbb{N}$.*

We apply the right module version of this with $\Gamma = \mathbb{C}H$, $\Lambda = \mathbb{C}G$, where $H \leq G$ and $\gamma\colon \mathbb{C}H \to \mathbb{C}G$ is the natural inclusion. Let A be a right $\mathbb{C}G$-module and let M be a right $\mathbb{C}H$-module. Note that $\mathbb{C}G$ is a free right $\mathbb{C}H$-module and so certainly flat. Then

$$\mathrm{Ext}^k_{\mathbb{C}H}(M, A) \cong \mathrm{Ext}_{\mathbb{C}G}(M \otimes_{\mathbb{C}H} \mathbb{C}G, A). \tag{2.8}$$

If K is a group, let $A(K)$ denote one of $\mathcal{U}(K)$, $\mathcal{N}(K)$, $\ell^p(K)$ $(1 \leq p \leq \infty)$, $c_0(K)$, $\mathrm{C}^*_{\mathrm{r}}(K)$. Note that $A(H)$ is a direct summand of $A(G)$ as right $\mathbb{C}H$-modules, consequently if $A(H)$ is not an injective $\mathbb{C}G$-module, then $A(G)$ is not an injective $\mathbb{C}H$-module. Thus $\mathrm{Ext}_{\mathbb{C}H}(M, A(G)) \neq 0$ for some right $\mathbb{C}H$-module M and hence $\mathrm{Ext}_{\mathbb{C}G}(M \otimes_{\mathbb{C}H} \mathbb{C}G, A(G)) \neq 0$ by (2.8). We deduce that $A(G)$ is not an injective $\mathbb{C}G$-module.

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