APPLICATIONS OF THE CHINESE REMAINDER THEOREM TO THE CONSTRUCTION AND ANALYSIS OF CONFOUNDING SYSTEMS AND RANDOMIZED FRACTIONAL REPLICATES FOR MIXED FACTORIAL EXPERIMENTS

by

Won-Chin Liao Huang

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Statistics

APPROVED:

Marvin M. Lentner, Chairman

Jesse C. Arnold

Camilla A. Brooks

Klaus H. Hinkelmann

Marion R. Reynolds, Jr.

June, 1989

Blacksburg, Virginia

APPLICATIONS OF THE CHINESE REMAINDER THEOREM TO THE CONSTRUCTION AND ANALYSIS OF CONFOUNDING SYSTEMS AND RANDOMIZED FRACTIONAL REPLICATED FOR MIXED FACTORIAL EXPERIMENTS

by

Won-Chin Liao Huang

Committee Chairman: Marvin M. Lentner
Statistics

(ABSTRACT)

A well-known theorem in "Number Theory", the Chinese Remainder Theorem, was first utilized by Paul K. Lin in constructing confounding systems for mixed factorial experiments. This study extends the use of the theorem to cover cases when more than one component from some of the symmetrical factorials are confounded, and to include cases where the number of levels of factors are not all relative prime.

The second part of this study concerns the randomized fractional replicates, a procedure which selects confounded subsets with pre-assigned probabilities. This procedure provides full information on a specific set of parameters of interest while making no assumption of zero nuisance parameters. Estimation procedures in general symmetrical as well as asymmeterical factorial systems are studied under a "fully orthogonalized" model. The type-g estimator, investigated under the generalized inverse operator, and the class of linear estimators of parameters of interest and their variance-covariance matrices are given. The unbiasedness of these estimators can be obtained only under the condition that each subset of treatment combinations is selected with equal probability.

This work is concluded with simulation studies to compare the classical and the randomization procedures. The results indicate that when information about the nuisance parameters is not available, randomization procedure guards against a bad choice of design.

ACKNOWLEDGMENTS

I would like to extend my most sincere appreciation to my advisor Dr. Lentner for not only giving me the guidance for my research, but also being so patient and understanding as I tried to be a mother and a student at the same time. I would also like to thank Dr. Hinkelmann from whom I learned the essence of experimental design, Dr. Reynolds who taught me so much about doing research, and Dr. Arnold and Dr. Brooks for their valuable suggestions. I am also very grateful to Dr. Schulman for believing in my ability of getting the Ph.D. degree, for giving me so much help on the SAS programming, and the statistics department for the continuous financial support.

I would also like to thank my parents for their support and encouragements, and my husband for everything he has done to help me get through some tough time.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER 1

INTRODUCTION

Factorial experiments are frequently utilized by researchers of various disciplines. A special class of factorial experiments involves factors that all occur with the same number of levels. These are known as symmetrical factorial experiments. However, there are many more situations which involve factors that do not all occur with the same number of levels. These are known as asymmetrical factorial experiments or mixed factorial experiments. In any factorial experiment, as the number of factors increases, the number of treatment combinations increases rapidly. A researcher may have difficulties in obtaining such a large set of homogeneous experimental units. Thus, it is desirable to have some means of reducing the size of each block. For this purpose, the use of incomplete blocks has been found very useful.

The principle of confounding, introduced as a device for increasing precision by reducing block size, is well known in factorial experiments. It is also known that there exists a unified approach, based on Galois Fields, that solves in general the problems of confounding in symmetrical factorial designs, s^m, where s is a prime or power of a prime. Up to the present time, confounding in asymmetrical factorial experiments is a more complicated process than is confounding in the symmetrical factorial.

Various methods have been proposed for the construction of confounding plans in asymmetrical factorials. These methods can be categorized into the following groups:

1. By looking at each component symmetric system separately, then through a suitable arrangement, it may lead to an appropriate design.

2. By using the PBIB designs, in particular, the EGD-PBIB designs.
3. By using techniques for combining elements from distinct finite fields.
4. By introducing pseudo-factors where the levels of some factors are different powers of the same prime.

Among all the procedures that have been proposed, only the technique provided by Lin (1986) utilizes the Chinese Remainder Theorem (CRT) in constructing confounded plans for asymmetrical factorials. The procedure provides an easy way to map elements from distinct finite fields into subsets of another finite field. It is also shown that such designs possess a Kronecker product structure. This structure is used to identify a Principle of Generalized Interaction and establish orthogonal factorial structure for the corresponding class of designs. As a consequence, an experimenter can construct orthogonal designs with predetermined confounding patterns using only the field or modular arithmetic of the symmetric subexperiments.

Although Lin's technique is easy to apply, his procedure was to confound at most one effect or interaction component from each of the symmetrical factorials. To overcome this problem, a new rule based on CRT is derived. Another problem which arises in using such a technique is that the levels of the symmetrical factorials have to be relatively prime in pairs. This problem may be solved by utilizing the prime decompositions of the non-prime number and rewriting the factorial by combining the primes, i.e. by the usage of pseudo-factors.

In some cases, many factors are considered in experiments performed in the early stages of an investigation. The cost of conducting such "large" experiments is usually expensive, and it may not be practical to run the full factorial. Hence, reduction in the size of the experiments is necessary and it can be accomplished through a fractional replication.

In fractional replication, though the size of the experiment is reduced, information on certain parameters is sacrificed. These parameters, so-called nuisance parameters, are assumed to be zero in "classical" analysis. However, this assumption can be relaxed through randomized fractional replication in some cases, a procedure where all treatment combinations are divided into appropriate fractions with each fraction given a certain probability of being the experimental fraction.

Suppose that the set of all treatment combinations is partitioned into M subsets, say X_w for w=0,1, ... ,M-1. Let ϕ_w denote the probability of selecting subset X_w and $\phi' = (\phi_0, \phi_1, ..., \phi_{M-1})$ be the probability vector which completely determines a fractional experiment. A special class of fractional procedures are the non-randomized ones, those having a single ϕ_w equal to unity. Another special procedure is called Randomized Procedure I (RPI), which is specified by the probability vector $\phi^{*\prime} = (1/M)(1,1, ...,1)$.

Lentner (1967) studied the randomized fractional replicate under a "fully orthogonalized" model. Due to the limitation of the structure of the fully orthogonalized matrix, one has to restrict the use to only main effects in the defining contrast. After a modification of the orthogonalized form utilized by Lentner, we are able to extend the results given in Lentner (1967), without the restriction mentioned above, to more general s^m systems as well as the mixed factorial systems, where s is a prime or power of a prime number.

In this study, estimation problems are investigated in both symmetric and asymmetric systems under randomized fractional replication. The class of all linear unbiased estimators of the parameters of interest (α) is derived. Also, the class of estimators of α arising under generalized inverse theory is studied. It is shown that unbiased estimators of α can be obtained only under the Randomized Procedure I. Furthermore, a simulation

study is conducted to investigate the magnitude of variance of the unbiased estimators derived under Randomized Procedure I.

CHAPTER 2

LITERATURE REVIEW

The first purpose of this study is to extend the idea from some existent procedures in the construction of confounding plans to cover more general cases of mixed factorial experiments. Therefore, a review of the existing procedures is presented here.

It is known that there exists a unified approach (Kempthorne, 1973), based on Galois Fields, that solves in general the problem of confounding in symmetrical factorial designs, s^m, where s is a prime or the power of a prime. However, the analytical method, using Galois Fields, breaks down when mixed factorials are considered. One reason for this is that addition and multiplication of elements coming from distinct fields are not defined.

Several papers extend the use of analytical techniques commonly used for symmetrical factorials to include mixed factorial designs. White and Hultquist (1965) gave methods for the design and analysis of confounding plans for the $p^n \times q^m$ factorial experiment, where p and q are primes. They defined addition and multiplication of elements from distinct finite fields, by mapping these elements into a finite commutative ring containing sub-rings isomorphic to each of the fields in question. The addition and multiplication operations are described as follows:

If integers (mod p) and (mod q) are to be combined, define

$$h(p) + k(q) \equiv \phi[h(p)] + \phi[k(q)] \equiv b_1(pq)$$
$$h(p)k(q) \equiv \phi[h(p)]\phi[k(q)] \equiv b_2(pq)$$

where

$$\begin{aligned} \phi[h(p)] &\equiv h(pq)hq(pq) \\ \phi[k(q)] &\equiv k(pq)kp(pq) \\ h &= 0,1,\ldots,p-1 \\ k &= 0,1,\ldots,q-1 \end{aligned}$$

Raktoe (1969) published an equivalent theoretical basis by using Ideal Theory for the results obtained by White and Hultquist, and generalized his technique of combining elements from two fields to k fields.

More recently, Worthley and Banerjee (1974) developed a methodology which yields confounding plans which cover all possible numbers of factors at all possible levels for the mixed factorial designs. This is accomplished by a technique which maps elements from distinct finite rings into subsets of another finite ring. However, this procedure has its limitations. When the number of levels of a factor is not a prime or a power of a prime, there is a problem in establishing the orthogonality of effects. Hence some effects that are confounded may be mixed with those which are not confounded. Shihota and Banerjee (1981) also did an extension of White and Hultquist's method to cover mixed factorials where the numbers of factors are relatively prime, but not necessarily all prime. The method enables us to get mutually orthogonal effects.

It should be noted that there are some problems of using all the methods mentioned above. The first problem is that the confounding procedure derived by such techniques requires frameworks at a highly abstract mathematical level. The second problem is, for example, in a $2^2 \times 3^2$ factorial, if $ABCD^2$ is confounded, then the sum of squares due to

interaction components AB and CD^2 would remain inseparable from the block sum of squares.

Voss (1986) established a structure for the designs generated by the methods of White and Hultquist (1965), Raktoe (1969), Worthley and Banerjee (1974) and Shihota and Banerjee (1981), for experiments in which the numbers of levels of factors are either primes or powers of primes. The incidence matrix N of an $s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_n^{m_n}$ factorial is shown to have the Kronecker product form $\mathbf{N} = \mathbf{N}_1 \otimes \mathbf{N}_2 \otimes \ldots \otimes \mathbf{N}_n$, where N_i is the incidence matrix of an $s_i^{m_i}$ factorial design obtained by the classical method using either field or modular arithmetic. Then the Kronecker product structures was used to construct orthogonal factorial structures and to identify a Principle of Generalized Interaction.

The Chinese Remainder Theorem (CRT) was first introduced by Lin (1986) for the construction of confounded designs for mixed factorial experiments in which the numbers of levels of factors are relatively prime in pairs. The procedure applies the classical method separately to each symmetrical subexperiment, using either field or modular arithmetic for a subexperiment, then applies the CRT to set up the linear congruence to combine elements from those fields or residue systems. The procedure using CRT does not require defining isomorphisms as the others do, and it can be applied to more general cases without having the problem of nonorthogonality of effects. However, in setting up the linear congruence for assigning treatment combinations to blocks, the CRT procedure still involves constructing and solving n congruences. Lin (1987a,b) then derived another procedure which is as general as the CRT procedure, but is easier to apply. The new procedure is derived from an isomorphism between finite abelian groups as follows:

Let $s_1, s_2, ..., s_n$ denote n positive integers that are pairwise relatively prime, and let s be the product of these integers. Let Z_{s_i} be the abelian group of integers modulo s_i, for $i = 1, 2, ..., n$, *and let $Z_{s_1} \oplus Z_{s_2} \oplus ... \oplus Z_{s_n}$ the direct sum of these groups. Then the function from $Z_{s_1} \oplus Z_{s_2} \oplus ... \oplus Z_{s_n}$ to the abelian group Z_s of integers modulo s defined by*

$$(\frac{s}{s_1})z_1 + (\frac{s}{s_2})z_2 + ... + (\frac{s}{s_n})z_n \equiv z \pmod{s},$$

for all $(z_1, z_2, ..., z_n)$ in $Z_{s_1} \oplus Z_{s_2} \oplus ... \oplus Z_{s_n}$ is a group isomorphism. If we let

$$z_{i1} \equiv x_{i1} + \alpha_{i2}x_{i2} + ... + \alpha_{ik_i}x_{ik_i} \pmod{s_i},$$

where $(x_{i1}, x_{i2}, ..., x_{ik_i})$ is the treatment combination in i^{th} symmetrical factorial, then the element z in Z_s identifies a block in this confounding plan.

The second purpose of this study is to explore, under randomized fractional replication, the properties of estimators of parameters of interest in mixed factorials. Therefore, an understanding of the basic knowledge about fractional factorials is important. Also necessary in this study is a review of the related literature on randomized fractional replicate in symmetrical system.

Fractional replications are constructed so as to provide full information on a specific set of parameters of interest. The "classical" approach was first proposed by Finney in 1945. It is useful only when the parameters of interest are confounded with effects that are known with confidence to be negligible. The designs discussed by Kempthorne (1973), and Cochran and Cox (1957) are adapted for the situations where all interactions of higher order are negligible. Dey (1985) pointed out that there are three principal methods of obtaining orthogonal main-effect plans for mixed factorials.

The first method, due to Addelman (1962), is based on the procedures of collapsing and replacement and utilizing the condition of proportional frequencies. The second procedure uses Hadamard matrices. This method gives an alternative way of obtaining some plans given by Addelman. Furthermore, it also gives rise to a number of new plans, not derivable through the procedure of Addelman. The third procedure utilizes orthogonal arrays, due to Rao (1946), in obtaining the orthogonal main-effect plans.

The use of Hadamard matrices can also be employed to obtain some series of orthogonal resolution IV plans for certain mixed factorials. Plans for $t\,2^{n-1}$ experiments in tn runs, where t is even and n is a multiple of four, was derived by Margolin (1969) by using the Hadamard matrices. And Agrawal and Dey (1985) have also used the same method to derive orthogonal resolution IV for the following three series of designs:

(1)$4^2\,2^{2n-4}$ in 8n runs:

(2)$4^3\,2^{4n-12}$ in 16n runs:

(3)$8^2\,2^{4n-8}$ in 32n runs:

As mentioned in Dey (1985), orthogonal resolution-V designs for $t\,s^m$ factorials are also obtainable by using some special types of orthogonal arrays. Again, these three methods were studied all under the assumption of zero nuisance parameters.

Several authors, Satterthwaite (1959), Ehrenfeld and Zacks (1961), have considered the use of randomization with regard to the choice of treatment combinations. Satterthwaite's work emphasized looking at the interesting parameters from all the possible ones, while, Ehrenfeld and Zacks were more concerned with inference about certain "pre-assigned" parameters. The procedure that Ehrenfeld and Zacks utilized, suggested by Cochran and Cox, is to divide the set of all treatment combinations (total

of p^m treatment combinations) into p^{m-s} blocks (m > s) according to the usual fractional factorial schemes, and then randomly chooses a subset of these p^{m-s} blocks which will compose the entire experiment. These designs, called "Randomized Fractional Factorials", allow one to obtain unbiased estimates and valid tests for parameters of interest without the usual assumptions of zero nuisance parameters.

Intensive studies of using randomization procedures for symmetrical factorial experiments have been carried out by Ehrenfeld and Zacks (1961,1963), Zacks (1963,1964) and Lentner (1967). Ehrenfeld and Zacks (1961) studied primarily 2^m factorial systems under an orthogonalized form of statistical model. Letting α denote the $S = 2^r$ "pre-assigned" parameters of interest, they showed that an unbiased estimator of α and its variance-covariance could be derived. Also, under the condition that more than one block would be used, analysis of variance tables and various tests of hypotheses suggested by the usual F-like ratios were presented. Furthermore, a method was presented for testing whether the p^{m-s} parameters were significantly different from zero, provided that the number of repetitions of every chosen treatment combination was greater than two.

Zacks (1963) indicated that the estimator studied in the earlier paper (Ehrenfeld and Zacks, 1961) was the least squares estimator, and it was the uniformly best unbiased estimator only when the nuisance parameters are zero, or when each block was assigned equal probability to be chosen. In other words, if different blocks are assigned unequal probabilities, unbiased estimators can not be attained. Additionally, he derived the class of all linear unbiased estimator of α and proved that a subset of this class, the conditional least-squares estimators, (c.l.s.e.) was complete. The c.l.s.e. is an adjustment of the least squares estimators according to the block chosen and the information

available concerning the nuisance parameters. Generalized least-squares estimators for arbitrary sets of pre-assigned and nuisance parameters were also investigated.

In a later paper (Ehrenfeld and Zacks, 1963), minimax strategies relative to two loss functions for two different states of information concerning the nuisance parameters were studied. Suppose each randomization procedure is represented by a probability vector, ϕ, and each c.l.s.e. is represented by a vector γ in $2^m - 2^t$ dimensional Euclidean space, then (ϕ , γ) is the minimax and admissible strategy for two kinds of Risk functions: one based on the mean-square-error loss function and the other based on the closeness of $\alpha(\gamma)$ under ϕ, where $\phi' = (1/2^{m-s})(1,1, ...,1)$, and γ is the vector of nuisance parameters with known prior information.

Zacks (1964) considered the problem of estimating the entire vector of 2^m parameters on the basis of a fractional replication of size 2^t. The class of all generalized least-squares estimators(g.l.s.e) was studied. Zacks concluded that no randomized fractional replication procedure could give us an unbiased g.l.s.e. of the entire vector of parameters. A generalized inverse theorem, derived by Rao (1962), was applied to characterize the class of all estimable linear functions of the pre-assigned parameters having uniformly minimum variance.

Lentner (1967) extended the results of Ehrenfeld and Zacks to factorial systems with p^m treatments, for general prime p under a "fully orthogonalized" model. In this model, the expected value of a response of the v^{th} treatment combination, $E(y_v)$, is expressible as a linear function of the $N = p^m$ parameters:

$$E(y_v) = \sum_{f=0}^{N-1} c_N(v,f)\beta_f \quad \text{for all } v = 0,1,\ldots,N-1$$

And it follows that the statistical model for the full factorial can be written in matrix form as follows:

$$\mathbf{y} = \mathbf{C}_N\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

where

$$\mathbf{y}' = (y_0, y_1, \ldots, y_{N-1})$$
$$\mathbf{C}_N = \| c_N(v,f) \| \; ; \; v,f = 0,1,2,\ldots,N-1$$
$$\boldsymbol{\beta}' = (\beta_0, \beta_1, \ldots, \beta_{N-1})$$
$$\boldsymbol{\varepsilon}' = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{N-1})$$

$\varepsilon_v (v = 0,1,2,\ldots,N-1)$ are identically distributed independent random variable , with $E(\varepsilon_v) = 0$ and $E(\varepsilon_v^2) = \sigma^2$. These variables represent the experimental errors.

Let the set **B** of all N parameters be represented by

$$\mathbf{B} = \{(\lambda_0, \lambda_1, \ldots, \lambda_{m-1}); \; \lambda_i = 0, 1, \ldots, p-1\}$$

Every parameter β_f is denoted by an m-tuple $(\lambda_0, \lambda_1, \ldots, \lambda_{m-1})$ such that

$$f = \sum_{j=0}^{m-1} \lambda_j p^j$$

Parameter $\beta_0 \equiv (0, 0, ..., 0)$ designates the mean response of all treatment combinations in the full factorial. Parameter $\beta_3 \equiv (0, 0, 1, ..., 0)$ corresponds to the linear effect of the third factor. Parameters for which $\lambda_i = 0, 1$ for all i with at least two coordinates unity correspond to linear interaction, etc.

According to the usual interpretation of the β's it can be shown that the coefficients $c_N(v,f)$ of the linear system are related to the coefficients of the orthogonal polynomials of order p, by the following relation:

$$\mathbf{C}_N = \mathbf{C}_p \otimes \mathbf{C}_p \otimes ... \otimes \mathbf{C}_p$$

where $\mathbf{C}_p$ is a matrix whose column vectors are the coefficients of the orthogonal polynomials of order p. The elements of $\mathbf{C}_p$ have the following properties:

1. all columns (except the first one) have zero-sum.
2. all columns are orthogonal.

For example, from tables of orthogonal polynomial coefficients, one finds

for $p = 2$

$$\mathbf{C}_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

for $q = 3$

$$\mathbf{C}_3 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

and for $q = 5$

$$\mathbf{C}_5 = \begin{bmatrix} 1 & -2 & 2 & -1 & 1 \\ 1 & -1 & -1 & 2 & -4 \\ 1 & 0 & -2 & 0 & 6 \\ 1 & 1 & -1 & -2 & -4 \\ 1 & 2 & 2 & 1 & 1 \end{bmatrix}$$

If a subset w of treatment combinations is selected at random, then the statistical model for the fractional replicate is

$$\mathbf{y}_w = (\mathbf{C} \mid \mathbf{H}_w) \begin{pmatrix} \alpha \\ \beta^* \end{pmatrix} + \varepsilon$$

Thus, the class of linear unbiased estimators of α has the form

$$\hat{\alpha}(y, \mathbf{F}_w) = \mathbf{C}^{-1}(\mathbf{y}_w - \mathbf{H}_w y) + \mathbf{F}_w \mathbf{y}_w$$

where y is any fixed vector. It is indicated that in the case of $p \geq 3$, the $\mathbf{C}$ matrix may be singular and some of the columns may be nonorthogonal. In order to avoid these structural problems, the pre-assigned parameters $S = p^r$ should be generated by the first

s main effects and the defining contrasts should be generated from the remaining m-s main effects. The class of all type-g (generalized inverse) solutions for the parameter of interest was presented and a decision theoretic framework was used to examine the properties of these estimators. The results show that, relative to certain risk functions, Bayes estimator, an admissible type-g estimator, and a minimax type-g estimator do exist.

CHAPTER 3

NOTATIONS AND STATISTICAL MODELS

3.1 NOTATIONS AND PRELIMINARIES FOR THE GENERAL SYMMETRIC FACTORIAL SYSTEM

3.1.1 EFFECTS AND INTERACTIONS

For an $N = s^m$ factorial with factors $F_1, F_2, ..., F_m$ where s is a prime number or a power of prime number, the set of all N treatment combinations are represented by

$$X = \{(x_1, x_2, ..., x_m);\quad x_j = 0,1, ..., s-1 \text{ for all j}\}$$

The $\frac{s^m - 1}{s - 1}$ symbols for effects and interactions (each with s-1 degrees of freedom) can be written as

$$E^{\alpha} = F_1^{\alpha_1} F_2^{\alpha_2} ... F_m^{\alpha_m}$$

where $\alpha' = (\alpha_1, \alpha_2, ..., \alpha_m)$ is the defining vector and α_j are integers such that $0 \le \alpha_j \le s-1$ with the first non-zero α_j equal to one. Furthermore, all x_j and α_j are elements from the Galois field of order s. Thus each main effect is represented by F_i, with all $\alpha_j = 0$ except $\alpha_i = 1$. The 2-factor interaction denoted by $F_i \times F_j$ between factors F_i and F_j consists of s-1 components $F_iF_j, F_iF_j^2, ..., F_iF_j^{s-1}$. The 3-factor interaction $F_i \times F_j \times F_k$ consists of (s-1)(s-1) components $F_iF_jF_k, F_iF_j^2F_k, ..., F_iF_j^{s-1}F_k^{s-1}$. And the other higher order interactions can be defined in a similar way. The set of s^m treatment combinations can be partitioned into s subsets of s^{m-1} treatment combinations through the equations

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m \equiv \delta \pmod{s} \tag{3.1}$$

if s is a prime number, and by

$$U_{\alpha_1} U_{x_1} + U_{\alpha_2} U_{x_2} + \ldots + U_{\alpha_m} U_{x_m} = U_\delta \tag{3.2}$$

where $U_{\alpha_i}, U_{x_i}, U_\delta \in GF(s)$ if s is the power of a prime number. Let E_δ^α, where $\delta = 0, 1, \ldots, s-1$, denote the true effects at level δ of E^α, then

$$E_\delta^\alpha = \bar{t} - \mu$$

where $\bar{t}$ is the true mean response of treatment combination satisfying equation 3.1 or 3.2, and μ is the overall mean. The s-1 degrees of freedom associated with main effect or interaction component E^α can be expressed by the set of contrasts

$$(E^\alpha)^i = c_0 E_0^\alpha + c_1 E_1^\alpha + \ldots + c_{s-1} E_{s-1}^\alpha \quad i = 1,2, \ldots, s-1$$

with $c_0 + c_1 + \ldots + c_{s-1} = 0$. In this study, only the set of orthogonal contrasts is considered. Thus, the coefficients c_i are from tables of coefficients of orthogonal contrasts. Such coefficients are not unique as shown by two sets of coefficients for three and four level factors in Table 3.1.

3.1.2 STATISTICAL MODEL

Define **D** as a matrix of order N×k which consists of rows correspond to treatment combinations in lexicographical order, and columns correspond to main effects and interaction components. Each entry d_{ij} is determined by

$$\alpha_{i1}x_1 + \alpha_{i2}x_2 + \ldots + \alpha_{im}x_m \equiv d_{ij} \pmod{s}$$

if s is a prime number, and by

$$U_{\alpha_{i1}}U_{x_1} + U_{\alpha_{i2}}U_{x_2} + \ldots + U_{\alpha_{im}}U_{x_m} = U_{d_{ij}}$$

if s is the power of a prime number.

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1k} \\ d_{21} & d_{22} & \ldots & d_{2k} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ d_{N1} & d_{N2} & \ldots & d_{Nk} \end{bmatrix}$$

where

$$0 \leq d_{ij} \leq s - 1$$

$$k = \frac{s^m - 1}{s - 1}$$

$$\sum_i d_{ij} \equiv 0 \pmod{s}$$

For example, consider a 3^2 factorial, with two factors F_1 and F_2, then the **D** matrix for this experiment can be expressed as follows:

$$\mathbf{D} = \begin{array}{c} \begin{matrix} F_1 & F_2 & F_1F_2 & F_1F_2^2 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

For each treatment combination the observed random variable is denoted by $y_{x_1x_2\cdots x_n}$. Then the statistical model of the full factorial can be expressed as

$$\mathbf{y} = \mathbf{C}_N\boldsymbol{\beta}_N + \boldsymbol{\varepsilon} \tag{3.3}$$

where

$\boldsymbol{\beta}_N' = (\beta_0, \beta_{11}, \ldots, \beta_{1(s-1)}, \ldots, \beta_{k1}, \ldots, \beta_{k(s-1)})$

$k = \dfrac{s^m - 1}{s - 1}$ = number of main effects and interaction components

β_0 : overall mean

β_{ij} : j^{th} contrast of treatment effects corresponding to effect i, where

$j = 1,2, \ldots, s-1$

$\boldsymbol{\varepsilon}' = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$

$\varepsilon_i (i = 1, 2, \ldots, N)$ are identically distributed independent random variables, with $E(\varepsilon_i) = 0$ and $E(\varepsilon_i^2) = \sigma^2$

The design matrix $\mathbf{C}_N$

is constructed from **D** and the table of coefficients of orthogonal contrasts. The columns j', $j'+1$, ..., $j'+(s-2)$ in $\mathbf{C}_N$ correspond to the j^{th} column in **D** where $j' = (s-1)j - (s-3)$. Thus, the elements of $\mathbf{C}_N$ have the following properties:

1. all columns (except the first one) have zero-sum.
2. all columns are orthogonal.

For example, the design matrix $\mathbf{C}_9$ and vector of parameters β_9 for the 3^2 factorial are:

$$\mathbf{C}_9 = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 & -2 & 0 & -2 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 & -2 \\ 1 & 0 & -2 & -1 & 1 & 0 & -2 & 0 & -2 \\ 1 & 0 & -2 & 0 & -2 & 1 & 1 & -1 & 1 \\ 1 & 0 & -2 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & -2 & -1 & 1 & 0 & -2 \\ 1 & 1 & 1 & 1 & 1 & 0 & -2 & -1 & 1 \end{bmatrix}$$

where

$$\beta_9 = \begin{bmatrix} M \\ \frac{1}{2}(F_1)^1 \\ \frac{1}{3}(F_1)^2 \\ \frac{1}{2}(F_2)^1 \\ \frac{1}{3}(F_2)^2 \\ \frac{1}{2}(F_1F_2)^1 \\ \frac{1}{3}(F_1F_2)^2 \\ \frac{1}{2}(F_1F_2^2)^1 \\ \frac{1}{3}(F_1F_2^2)^2 \end{bmatrix}$$

3.1.3 INDEPENDENT EFFECTS

Define an operation $\boxplus$ on $\alpha_i' = (\alpha_{i1}, \alpha_{i2}, ..., \alpha_{im})$ as follows:

$$\begin{aligned} \alpha_i' \boxplus \alpha_j' &= (\alpha_{i1}, \alpha_{i2}, , ..., \alpha_{im}) \boxplus (\alpha_{j1}, \alpha_{j2}, ..., \alpha_{jm}) \\ &= (\alpha_{i1} + \alpha_{j1}, \alpha_{i2} + \alpha_{j2}, ..., \alpha_{im} + \alpha_{jm}) \end{aligned}$$

where each $\alpha_{ik} + \alpha_{jk}$ is reduced modulo s when s is a prime number, and is an element from addition table of GF(s) when s is the power of a prime number.

Then any effect or interaction component $F_1^{\alpha 01}F_2^{\alpha 02}...F_m^{\alpha 0m}$ is said to be independent of $F_1^{\alpha 11}F_2^{\alpha 12}...F_m^{\alpha 1m}$, ..., $F_1^{\alpha n1}F_2^{\alpha n2}...F_m^{\alpha nm}$ if there are no n numbers $a_1, a_2,..., a_n$ such that

$$\alpha_0' = a_1\alpha_1' \boxplus ... \boxplus a_n\alpha_n'$$

3.1.4 FRACTIONATION OF DESIGN MATRIX

The usual scheme for constructing a fractional replication is first to divide the set X into M subsets each of size S , where $M = s^e$ and $S = s^{m-e}$, according to a chosen defining contrast. And then some of the subsets are chosen to form the fractional factorial. If a single subset is chosen at random, then only $\frac{s^{m-e}-1}{s-1}$ effects or interaction components can be estimated. For the purpose of our study, these effects or interaction components have to be generated by m-e independent effects or interactions other than the effects in the defining contrast and their generalized interactions.

Let X_w denote the treatments of the chosen fraction, where w ∈ 0,1,...,M-1 . Then the submatrix $\mathbf{D}_w$ for the fractional replicate utilizing fraction X_w *is*

$\mathbf{D}_w = [\mathbf{D}_{w0} \mid \mathbf{D}_{w1} \mid \mathbf{D}_{w2}]$

$\mathbf{D}_w$ is a submatrix of S rows of **D**

$\mathbf{D}_{w0}$ consists of columns corresponding to the effects or interaction components of interest.

$\mathbf{D}_{w1}$ consists of columns corresponding to the effects or interaction components which are aliased with effects or interaction components of interest.

$\mathbf{D}_{w2}$ consists of columns corresponding to the effects of the defining contrast and their generalized interactions.

$$\mathbf{D}_{wi} = \begin{bmatrix} (d_{wi})_{11} & (d_{wi})_{12} & \cdots & (d_{wi})_{1k_i} \\ (d_{wi})_{21} & (d_{wi})_{22} & \cdots & (d_{wi})_{2k_i} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (d_{wi})_{S1} & (d_{wi})_{S2} & \cdots & (d_{wi})_{Sk_i} \end{bmatrix}$$

where

$$k_0 = \frac{s^{m-e} - 1}{s - 1}$$

$$k_1 = \frac{s^m - 1}{s - 1} - k_0 - k_2$$

$$k_2 = \frac{s^e - 1}{s - 1}$$

For future study, we shall rearrange the rows in each $\mathbf{D}_w$ according to the following scheme:

1. Rearrange the rows according to the ascending order of the levels of the first effect or interaction component.
2. Within each level of the first effect or interaction component, rearrange the rows according to the ascending order of the levels of the second effect or interaction component.
3. Repeat the same procedure for the remaining independent effects and interaction components of interest.

We now illustrate the foregoing rearrangement with the following example.

Example A factorial experiment consists of three factors, F_1, F_2 and F_3, each having three levels. If we choose the interaction component $F_1F_2F_3$ as the defining contrast, we may generate the resulting aliased structure:

$$I = F_1F_2F_3 \quad F_1 = F_1F_2^2F_3^2 = F_2F_3 \quad F_2 = F_1F_2^2F_3 = F_1F_3 \quad F_3 = F_1F_2F_3^2 = F_1F_2$$

$$F_1F_2^2 = F_1F_3^2 = F_2F_3^2$$

Because of the restriction mentioned previously, we shall let the set of effects or interaction components of interest be generated by F_1 and F_2. Thus, $\mathbf{D}_u$ corresponding to each fraction will be:

$$\mathbf{D}_1 = \begin{array}{c} \begin{array}{ccccccccccccc} F_1 & F_2 & F_1F_2 & F_1F_2^2 & F_1F_2^2F_3^2 & F_2F_3 & F_1F_2^2F_3 & F_1F_3 & F_1F_2F_3^2 & F_3 & F_1F_3^2 & F_2F_3^2 & F_1F_2F_3 \end{array} \\ \left[\begin{array}{ccccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 & 0 & 1 & 2 & 2 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 & 2 & 1 & 1 & 2 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 & 2 & 2 & 0 & 0 & 2 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 2 & 0 \\ 2 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 2 & 0 \\ 2 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 2 & 1 & 0 \\ 2 & 2 & 2 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\mathbf{D}_2 = \begin{array}{c} \begin{matrix} F_1 & F_2 & F_1F_2 & F_1F_2^2 & F_1F_2^2F_3^2 & F_2F_3 & F_1F_2^2F_3 & F_1F_3 & F_1F_2F_3^2 & F_3 & F_1F_3^2 & F_2F_3^2 & F_1F_2F_3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 2 & 2 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 2 & 1 & 2 & 1 & 0 & 2 & 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 1 \\ 1 & 2 & 0 & 2 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 1 & 1 \\ 2 & 0 & 2 & 2 & 0 & 2 & 1 & 1 & 0 & 2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 & 2 & 2 & 0 & 2 & 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 0 & 2 & 2 & 1 \end{bmatrix} \end{array}$$

$$\mathbf{D}_3 = \begin{array}{c} \begin{matrix} F_1 & F_2 & F_1F_2 & F_1F_2^2 & F_1F_2^2F_3^2 & F_2F_3 & F_1F_2^2F_3 & F_1F_3 & F_1F_2F_3^2 & F_3 & F_1F_3^2 & F_2F_3^2 & F_1F_2F_3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 1 & 2 & 0 & 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 2 & 2 & 1 & 1 & 2 & 1 & 0 & 2 & 0 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 & 2 & 2 & 0 & 1 & 0 & 2 & 2 \\ 1 & 1 & 2 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 1 & 2 \\ 1 & 2 & 0 & 2 & 0 & 1 & 1 & 0 & 1 & 2 & 2 & 0 & 2 \\ 2 & 0 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 0 & 2 & 0 & 2 \\ 2 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 2 & 0 & 2 & 2 \\ 2 & 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

$\mathbf{D}_{w0}$ consists of columns corresponding to F_1, F_2, F_1F_2, $F_1F_2^2$

$\mathbf{D}_{w1}$ consists of columns corresponding to $F_1F_2^2F_3^2$, F_2F_3, $F_1F_2^2F_3$, F_1F_3, $F_1F_2F_3^2$, F_3, $F_1F_3^2$, $F_2F_3^2$

$\mathbf{D}_{w2}$ consists of columns corresponding to $F_1F_2F_3$

Correspondingly, the design matrix for the fractional replicate utilizing X_w is:

$$\begin{aligned}\mathbf{C}_{w,N} &= [\mathbf{C}_{w0,N} | \mathbf{C}_{w1,N} | \mathbf{C}_{w2,N}] \\ &= [\mathbf{C}_{w0,N} | \mathbf{H}_{w,N}] \\ &= [\mathbf{C} | \mathbf{H}_w]\end{aligned}$$

where $\mathbf{C}_{wi,N}$ is constructed based on $\mathbf{D}_{wi}$ and tables of coefficients of orthogonal contrasts.

$$\mathbf{C}_{wi} = \begin{bmatrix} (c_{wi})_{11} & (c_{wi})_{12} & \cdots & (c_{wi})_{1k_i^e} \\ (c_{wi})_{21} & (c_{wi})_{22} & \cdots & (c_{wi})_{2k_i^e} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (c_{wi})_{S1} & (c_{wi})_{S2} & \cdots & (c_{wi})_{Sk_i^e} \end{bmatrix}$$

where $k_i^e = k_i \times (s-1)$ with the $k_i's$ defined in section 3.1.4.

By the way $\mathbf{D}_{w0}$ is constructed, $\mathbf{C}$ is also a fully orthogonalized matrix. The statistical model for the fractional replicate is thus:

$$\mathbf{y}_w = \mathbf{C}_{w,N}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} + \varepsilon_w \qquad (3.4)$$

where

$\mathbf{y}_w$ is the subvector of observed random variables corresponding to the treatment combinations in X_w

α is the subvector of parameters corresponding to the chosen effects or interaction components

β is the subvector which consists of the remaining parameters, i.e. the nuisance parameters.

To estimate α, the method of least squares is used. From (3.4) one can get the least-square normal equation:

$$(\mathbf{C}\,|\,\mathbf{H}_w)'(\mathbf{C}\,|\,\mathbf{H}_w)\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} = (\mathbf{C}\,|\,\mathbf{H}_w)'\mathbf{y}_w \tag{3.5}$$

Since $(\mathbf{C}\,|\,\mathbf{H}_w)'$ is a rectangular $N \times S$ matrix of rank S , in which case $(\mathbf{C}\,|\,\mathbf{H}_w)(\mathbf{C}\,|\,\mathbf{H}_w)'$ is of order $S \times S$ and rank S, there exists a left inverse of $(\mathbf{C}\,|\,\mathbf{H}_w)'$ (Rao & Mitra, 1971). This means that the normal equation (3.5) can be converted into the following simpler form for each w=0,1, ... ,M-1:

$$(\mathbf{C}\,|\,\mathbf{H}_w)\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} = \mathbf{C}\hat{\alpha} + \mathbf{H}_w\hat{\beta} = \mathbf{y}_w \tag{3.6}$$

3.2 NOTATIONS AND PRELIMINARIES FOR THE ASYMMETRIC FACTORIAL SYSTEM

In 3.1, basic notions and statistical models for symmetric factorial systems have been described. Similar discussion for asymmetric factorial are considered in this section.

3.2.1 *EFFECTS AND INTERACTIONS*

For a factorial experiment of order $N = s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_n^{m_n}$ with factors $F_{11}, F_{12}, \ldots, F_{nm_n}$, where $s_i's$ are relatively prime numbers in pairs, the set of all N treatment combination are represented by

$$X = \{(X_1, X_2, \ldots, X_n)\}$$
$$= \{(x_{11}, \ldots, x_{1m_1}, x_{21}, \ldots, x_{2m_2}, \ldots, x_{n1}, \ldots, x_{nm_n}); x_{ij} = 0,1, \ldots, s_i - 1\}$$

where $X_i = (x_{i1}, x_{i2}, \ldots, x_{im_i})$. Following section 3.1.1, each main effect is represented by F_{ij} The r-factor interaction of the factors from the same symmetric factorial is defined in the same way as in section 3.1.1. However, for factors from different symmetric factorials, the r-factor interaction will be defined differently. We shall illustrate it with the following example.

Example Let us consider a three-way mixed factorial experiment with factors F_{11}, F_{12} having s_1 levels and F_{21} having s_2 levels. Then the 3-factor interaction $F_{11} \times F_{12} \times F_{21}$ can be partitioned into $(s_1 - 1)(s_1 - 1)(s_2 - 1)$ parts each having one degree of freedom :

$$\begin{array}{cccc}
(F_{11}F_{12})^1F_{21}^1 & (F_{11}F_{12})^1F_{21}^2 & \ldots & (F_{11}F_{12})^1F_{21}^{s_2-1} \\
(F_{11}F_{12})^2F_{21}^1 & (F_{11}F_{12})^2F_{21}^2 & \ldots & (F_{11}F_{12})^2F_{21}^{s_2-1} \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
(F_{11}F_{12}^{s_1-1})^{s_1-1}F_{21}^1 & (F_{11}F_{12}^{s_1-1})^{s_1-1}F_{21}^2 & \ldots & (F_{11}F_{12}^{s_1-1})^{s_1-1}F_{21}^{s_2-1}
\end{array}$$

3.2.2 STATISTICAL MODEL

The statistical model of the full asymmetric factorial can be expressed as:

$$\mathbf{y} = \mathbf{C}_N \boldsymbol{\beta}_N + \boldsymbol{\varepsilon}$$

where

$$\mathbf{C}_N = \mathbf{C}_{N_1} \otimes \mathbf{C}_{N_2} \otimes \ldots \otimes \mathbf{C}_{N_n}$$

$$\boldsymbol{\beta}_N = \boldsymbol{\beta}_{N_1} \otimes \boldsymbol{\beta}_{N_2} \otimes \ldots \otimes \boldsymbol{\beta}_{N_n}$$

$\mathbf{C}_{N_i}$ is the design matrix of the i^{th} symmetric factorial, and $N_i = s_i^{m_i}$.

$\boldsymbol{\beta}_{N_i}$ is the vector of the parameters of the i^{th} symmetric factorial.

To construct a randomized fractional replicate for a mixed factorial, we shall apply the Chinese Remainder Theorem to partition the set X into $M = s_1^{e_1} \times s_2^{e_2} \times \ldots \times s_r^{e_r}$ subsets first, where e_i is the number of independent effects in the defining contrast from the i^{th} symmetric factorial, where $1 \leq i \leq r \leq n$, and then randomly select one of the M subsets. The theorem and its application will be discussed in detail in the next chapter.

If a fraction X_w is chosen, then

$$X_w = \{(X_{1w}, \ldots, X_{rw}, X_{r+1}, \ldots, X_n)\}$$

And the corresponding $\mathbf{D}_w^i$ for the i^{th} symmetric factorial is

$$\mathbf{D}_w^i = [\mathbf{D}_{w0}^i \mid \mathbf{D}_{w1}^i \mid \mathbf{D}_{w2}^i]$$

where $\mathbf{D}_w^t$, $\mathbf{D}_{wj}^t$ are defined in a same manner as in section 3.1.4. Furthermore, the design matrix corresponding to each $\mathbf{D}_w^t$ is written as:

$$\begin{aligned}\mathbf{C}_{w,N_t} &= [\mathbf{C}_{w0,N_t} | \mathbf{H}_{w,N_t}] \\ &= [\mathbf{C}^t | \mathbf{H}_{w,N_t}]\end{aligned}$$

Therefore, the design matrix for the fractional replicate utilizing X_w is

$$\begin{aligned}\mathbf{C}_{w,N} &= \mathbf{C}_{w,N_1} \otimes \ldots \otimes \mathbf{C}_{w,N_r} \otimes \mathbf{C}_{N_{r+1}} \otimes \ldots \otimes \mathbf{C}_{N_n} \\ &= [\mathbf{C} | \mathbf{H}_{w,N}]\end{aligned}$$

where

$$\begin{aligned}\mathbf{C} &= \mathbf{C}^1 \otimes \ldots \otimes \mathbf{C}^r \otimes \mathbf{C}_{N_{r+1}} \otimes \ldots \otimes \mathbf{C}_{N_n} \\ \mathbf{H}_{w,N} &= [\mathbf{H}_{w,N}^1 | \mathbf{H}_{w,N}^2 | \ldots | \mathbf{H}_{w,N}^t] \otimes \mathbf{C}_{N_{r+1}} \otimes \ldots \otimes \mathbf{C}_{N_n}\end{aligned}$$

where

$$\begin{aligned}\mathbf{H}_{w,N}^1 &= \mathbf{C}^1 \otimes \ldots \otimes \mathbf{C}^{r-1} \otimes \mathbf{H}_{w,N_r} \\ \mathbf{H}_{w,N}^2 &= \mathbf{C}^1 \otimes \ldots \otimes \mathbf{H}_{w,N_{r-1}} \otimes \mathbf{C}^r \\ &\ldots\ldots \\ \mathbf{H}_{w,N}^r &= \mathbf{H}_{w,N_1} \otimes \ldots \otimes \mathbf{C}^{r-1} \otimes \mathbf{C}^r \\ &\ldots\ldots \\ \mathbf{H}_{w,N}^t &= \mathbf{H}_{w,N_1} \otimes \mathbf{H}_{w,N_2} \otimes \ldots \otimes \mathbf{H}_{w,N_r}\end{aligned}$$

Following the same argument as in section 3.1.3, the least-square normal equation for this mixed factorial can be derived as follows:

$$\mathbf{C}\hat{\boldsymbol{\alpha}} + \mathbf{H}_{w,N}\hat{\boldsymbol{\beta}} = \mathbf{y}_w \tag{3.7}$$

Table 3.1 Coefficients of orthogonal contrasts for three- and four-level factors

Three-level factor			Four-level factor			
level	$(E)^1$	$(E)^2$	level	$(E)^1$	$(E)^2$	$(E)^3$
c_0	2	0	c_0	1	1	1
c_1	-1	-1	c_1	1	-1	-1
c_2	-1	1	c_2	-1	-1	1
			c_3	-1	1	-1

Orthogonal polynomials for three- and four-level factors

Three-level factor			Four-level factor			
level	$(E)^1$	$(E)^2$	level	$(E)^1$	$(E)^2$	$(E)^3$
c_0	-1	1	c_0	-3	1	-1
c_1	0	-2	c_1	-1	-1	3
c_2	1	1	c_2	1	-1	-3
			c_3	3	1	1

If the factors are quantitative and equispaced, then $(E)^1$,$(E)^2$ and $(E)^3$ denote the linear, quadratic and cubic components of the effect E, respectively.

CHAPTER 4

CONSTRUCTION OF CONFOUNDED PLANS

4.1 CONFOUNDING IN SYMMETRIC FACTORIAL EXPERIMENTS

Since the material of this study extends the use of analytical techniques commonly used for "symmetrical" factorials to include "mixed" factorial designs, a brief review of constructing confounding plans in symmetric factorials is presented. In the case of an s^m factorial experiment, where s is a prime or power of a prime, the treatment combinations may be denoted by $(x_1, x_2, ..., x_m)$ where each x_j can take one of the values from 0 to s-1. If one wishes to construct a confounding system in blocks of size $s^t (t < m)$, and one can assume that $F_1^{*11} F_2^{*12} ... F_m^{*1m}$, ..., $F_1^{*(m-t)1} F_2^{*(m-t)2} ... F_m^{*(m-t)m}$ are negligible, then the system of confounding is determined by

$$\begin{aligned} \sum U_{\alpha_{1j}} U_{x_j} &= U_{\delta_1} \\ \sum U_{\alpha_{2j}} U_{x_j} &= U_{\delta_2} \\ &\cdots\cdots \\ \sum U_{\alpha_{(m-t)j}} U_{x_j} &= U_{\delta_{(m-t)}} \end{aligned} \tag{4.1}$$

where $U_{\alpha_{ij}}, U_{x_j}, U_{\delta_t} \in GF(s)$

When s is a prime number, i.e. s = p, the Galois field of p elements may be represented by $U_0 = 0, U_1 = 1, ..., U_{p-1} = p - 1$, in which addition and multiplication are the ordinary arithmetic operations, except that the resulting number is reduced modulo p. Thus, system 4.1 now can be rewritten as follows:

$$\begin{aligned}
\sum \alpha_{1j} x_j &\equiv a_1 \pmod{p} \\
\sum \alpha_{2j} x_j &\equiv a_2 \pmod{p} \\
&\cdots\cdots \\
\sum \alpha_{(m-t)j} x_j &\equiv a_{m-t} \pmod{p}
\end{aligned}$$

where $\alpha_{ij} \in \{0,1, ..., p-1\}$, and the first non-zero α_{ij} in each congruence equals one.

However, when s is a power of a prime , that is, $s = p^n$, the marks of the $GF(p^n)$ are not quite as easy to manipulate as the numbers 0,1, ...,$s-1$. For example, for the $(2^2)^2$ system: i.e., 2 factors each at 4 levels, the addition and multiplication tables of this field are shown in Table 4.1.

According to system 4.1, the s^m treatment combinations are partitioned into s^{m-t} sets of s^t treatment combinations each. The $(\frac{s^m-1}{s-1})$ sets of (s-1) degrees of freedom, which are partitioned by the use of Galois fields, are all orthogonal. Therefore, except those effects or interactions confounded with blocks, one can estimate all the other effects and interactions independently (Kempthorne, 1971).

4.2 CONFOUNDING IN ASYMMETRIC FACTORIAL EXPERIMENTS

In a $s_1^{m_1} \times s_2^{m_2} \times ... \times s_n^{m_n}$ factorial experiment, let F_{ij} denote the factors in the symmetrical factorial $s_i^{m_i}$, where $j = 1,2, ..., m_i$. The levels of the factor F_{ij} are indicated by x_{ij}, where $x_{ij} = 0,1, ..., s_i - 1$. For each symmetrical factorial $s_i^{m_i}$, let the component $(F_{i1}^{\alpha_{i1}} F_{i2}^{\alpha_{i2}} ... F_{im_i}^{\alpha_{im_i}})$ be one of the effects being confounded, where α_{il} are integers such that $0 \le \alpha_{il} \le s_i - 1$ with the first non-zero α_{il} equal to one, for $l = 1,2,3, ..., m_i$. In constructing the confounded plan for such an experiment, one wishes to obtain

$(y_1, y_2, ..., y_n)$ such that one can assign each treatment combination to an unique block through the following equation

$$\begin{aligned} & y_1(\alpha_{11}x_{11} + \alpha_{12}x_{12} + ... + \alpha_{1m_1}x_{1m_1}) \\ & + ... \\ & + y_n(\alpha_{n1}x_{n1} + \alpha_{n2}x_{n2} + ... + \alpha_{nm_n}x_{nm_n}) \equiv w(\text{ mod } M) \end{aligned} \tag{4.2}$$

where w indicates the block number. Then, one can partition the set of all treatment combinations into M blocks.

It is equivalent to say that one wishes to combine elements from different residue classes so that all elements of those residue classes are to be considered as members of the set of residue classes of integers modulo M. The Chinese Remainder Theorem, which is given in most textbook on number theory, is used in the present study since it provides a relatively simple method of combining elements from distinct Galois Fields.

4.2.1 THE CHINESE REMAINDER THEOREM

The Chinese Remainder Theorem provides the algorithm of finding an integer that yields certain remainders upon being divided by given integers. The description and the proof of the theorem are as follows:

Theorem 4.1 Chinese Remainder Theorem *Let $s_1, s_2, ..., s_r$ denote r positive integers that are relatively prime numbers in pairs, and let $a_1, a_2, ..., a_r$ denote any r integers. Then the congruences $y \equiv a_i(\text{ mod } s_i)$, $i = 1,2, ..., r$ have common solutions. Any two solutions are congruent modulo $s_1s_2...s_r$.*

Proof: Let $M = s_1 \times s_2 \times ... \times s_r$, then M/s_j is an integer and $(M/s_j, s_j) = 1$, i.e., M/s_j and s_j have no common divisor. There exist integers b_j such that $(M/s_j)b_j \equiv 1 \pmod{s_j}$, and $(M/s_j)b_j \equiv 0 \pmod{s_i}$ if i≠j. Define

$$y' = \sum_{j=1}^{r} (M/s_j)b_j a_j$$

where the a_j are any intergers. Then, according to the way b_j is defined,

$$y' \equiv (M/s_i)b_i a_i \equiv a_i \pmod{s_i}$$

Therefore, y′ is a common solution of $y \equiv a_i \pmod{s_i}$ for $i = 1,2, ..., r$.

If y′ and y″ are both common solutions of the congruences

$$y \equiv a_i \pmod{s_i}, \quad i = 1,2, ..., r$$

then

$$y' \equiv y'' \pmod{s_i}, \text{ for } i = 1,2, ..., r$$

Hence, y′-y″ is divisible by s_i, $i = 1,2, ..., r$, which implies that y′-y″ is divisible by $M = s_1 s_2 ... s_r$. Therefore, $y' \equiv y'' \pmod{M}$.

From section 4.1,

$$\alpha_{i1} x_{i1} + \alpha_{i2} x_{i2} + ... + \alpha_{im_i} x_{im_i} \equiv a_i \pmod{s_i}$$

when s_i is a prime number, and

$$U_{\alpha_{i1}} U_{x_{i1}} + U_{\alpha_{i2}} U_{x_{i2}} + \ldots + U_{\alpha_{im_i}} U_{x_{im_i}} = U_{a_i}$$

when s_i is a power of a prime. Therefore, by applying the Chinese Remainder Theorem, one can rewrite equation (4.2) as follows:

$$\sum_{j=1}^{r} \left(\frac{M}{s_j}\right) b_j a_j \equiv w (\bmod M) \tag{4.3}$$

For each treatment combinations, its numerical values x_{jl}, $j = 1,2, \ldots, n$, and $l = 1,2, \ldots, m_i$, are inserted into congruence (4.3) to compute the value w. This treatment combination is then assigned to Block w.

The following example illustrates the use of this theorem.

Example A factorial experiment consists of four factors, A,B,C,D, the first two having three levels each, and the last two having four levels each. The treatment combinations are denoted by $(x_{11}, x_{12}, x_{21}, x_{22})$ where $x_{1j} = 0,1,2$ and $x_{2j} = 0,1,2,3$. Consider the plan with AB, CD^3 and their generalized interaction $ABCD^3$ being confounded. The procedure to construct a confounded system of the $3^2 \times 4^2$ factorials is as follows:

1. Decompose the $3^2 \times 4^2$ factorial into two symmetrical factorials 3^2 and 4^2, then, according to Chinese Remainder Theorem, $r = 2$, $s_1 = 3$, and $s_2 = 4$.
2. Since AB is taken from the factorial 3^2, the corresponding remainder a_1 is from the congruence $x_{11} + x_{12} \equiv a_1 (\bmod 3)$. And since the component

CD^3 is from the factorial 4^2, the relationship between the treatment combinations (x_{21}, x_{22}) and the level a_2 is $U_{x_{21}} + U_3 U_{x_{22}} = U_{a_2}$, where the addition as well as multiplication are based on Table 4.1. A solution b_1 for $(M/s_1)b_1 \equiv 1 (\text{mod } 3)$ is 1, and a solution b_2 for $(M/s_2)b_2 \equiv 1 (\text{mod } 4)$ is 3. Thus, inserting the information given above into (4.3), one will get the equation

$$4a_1 + 9a_2 \equiv w (\text{mod } 12) \tag{4.4}$$

3. Finally, the numerical values of each of the treatment combinations are inserted to calculate w. This treatment combination is then assigned to Block w. The result is shown in Table 4.2.

Although the Chinese Remainder Theorem is relatively easy to apply, one can confound at most one effect from each of the symmetric factorials. The smallest block size one can get is $s_1^{t_1} s_2^{t_2} \ldots s_n^{t_n}$ where $t_i = m_i - 1$, which occurs when the number of blocks is $s_1 \times s_2 \times \ldots \times s_n$. This block size is not very practical if a large number of factors is considered. Furthermore, the number of applicable confounded plans are limited. For example, for a $3^3 \times 4^3 \times 5^2$ factorial, there are only seven possible confounded plans which can be constructed by Chinese Remainder Theorem. The description of each of the plans are listed in Table 4.3.

In practice, if the only information desired is that on main effects for some factors and low-order interactions for some of those factors, one can obtain that information from testing a lesser number of combinations than the block size for any of the above plans. Therefore, more components should be confounded to reduce the number of combinations.

4.2.2 CONFOUNDING MORE THAN ONE COMPONENT FROM SOME OF THE SYMMETRIC FACTORIALS

Next, we shall derive a theorem which will allow us to find the proper congruence if one wishes to confound more than one component from some symmetric factorials.

Definition 1: A set $x_1, ..., x_s$ is called a complete residue system modulo s if for every integer y there is one and only one x_j such that $y \equiv x_j \pmod{s}$. The set $\{0,1, ..., s-1\}$ is called the least non-negative complete residue system modulo s.

Theorem 4.2 *If s and z_i (for $i = 1, 2, ..., e$) are positive integers, then a set* $R = \{z_1 + sz_2 + ... + s^{e-1}z_e \ ; \ z_i = 0,1, ..., s-1 \text{ for } i = 1,2, ..., e\}$ *is a complete residue system modulo s^e.*

Proof: Since

$$\begin{aligned} 0 \le z_1 + sz_2 + ... + s^{e-1}z_e &\le (s-1) + s(s-1) + ... + s^{e-1}(s-1) \\ &= (s-1)\left(\frac{1-s^e}{1-s}\right) \\ &= s^e - 1 \end{aligned}$$

and let

$$\begin{aligned} f(s) &= z_1 + sz_2 + ... + s^{e-1}z_e \\ f'(s) &= z_1' + sz_2' + ... + s^{e-1}z_e' \end{aligned}$$

then $f(s) = f'(s)$ if and only if $(z_1, ..., z_e) = (z_1', ..., z_e')$. Therefore, each combination of $(z_1, ..., z_e)$ gives a distinct value for $f(s)$.

From the above, the set R is equal to $\{0,1, ..., s^e - 1\}$ which is a complete residue system modulo s^e.

By definition 1 and Theorem 4.2, one can see that for every integer y there is one and only one $(z_1, ..., z_e)$ such that $y \equiv z_1 + sz_2 + ... + s^{e-1}z_e (\mathrm{mod}\ s^e)$. If e_i components from $s_i^{m_i}$ are used in constructing a confounded factorial, the Chinese Remainder Theorem is applicable since $s_i^{e_i}$ is still relatively prime to s_j, $i \neq j$. The equation 4.3 now can be rewritten as

$$\sum_{j=1}^{r} \left(\frac{M}{s_j^{e_j}}\right) b_j a_j \equiv w (\mathrm{mod}\ M) \tag{4.5}$$

where

$$M = s_1^{e_1} \times s_2^{e_2} \times ... \times s_r^{e_r}, \quad 1 \leq e_i \leq m_i - 1$$

$$a_{j1} + s_j a_{j2} + ... + s_j^{e_j - 1} a_{je_j} \equiv a_j (\mathrm{mod}\ s_j^{e_j})$$

when s_j is a prime number,

$$\alpha_{jl1} x_{j1} + \alpha_{jl2} x_{j2} + ... + \alpha_{jlk_j} x_{jk_j} \equiv a_{jl} (\mathrm{mod}\ s_j)$$

when s_j is a power of a prime,

$$U_{\alpha_{jl1}} U_{x_{j1}} + U_{\alpha_{jl2}} U_{x_{j2}} + ... + U_{\alpha_{jlk_j}} U_{x_{jk_j}} = U_{a_{jl}}$$

$$1 \leq k_j \leq m_j, \quad 0 \leq \alpha_{j_{ll'}} \leq s_j - 1 \quad l = 1,2, ..., e_j, \ l' = 1,2, ..., k_j$$

Example To illustrate the foregoing theorem, consider again the $3^3 \times 4^3 \times 5^2$ factorial with the following effects being confounded: $F_{11}F_{12}^2$, $F_{21}F_{22}$, $F_{31}F_{32}$, $F_{11}F_{12}F_{13}$, $F_{21}F_{22}^2F_{23}$, and their generalized interactions. Then the linear congruence for assigning treatment combinations to blocks is:

$$640(a_{11} + 3a_{12}) + 225(a_{21} + 4a_{22}) + 576a_{31} \equiv w(\text{ mod } 720)$$

$$\begin{aligned} x_{11} + x_{12} + x_{13} &\equiv a_{11}(\text{ mod } 3) \\ x_{11} + 2x_{12} &\equiv a_{12}(\text{ mod } 3) \\ U_{x_{21}} + U_2U_{x_{22}} + U_{x_{23}} &= U_{a_{21}} \\ U_{x_{21}} + U_{x_{22}} &= U_{a_{22}} \\ x_{31} + x_{32} &\equiv a_{31}(\text{ mod } 5) \end{aligned}$$

The equation above allows one to get 720 blocks of 60 treatment combinations each, which is a more reasonable block size compared to the smallest size one can get by using the Chinese Remainder Theorem. The treatment combinations that belong to the intra-block subgroup are listed in Table 4.4.

A FORTRAN program, based on equation 4.5, for assigning all treatment combinations into blocks is presented in the Appendix.

This section concludes with a table that lists the number of blocks as well as block size for each of the following cases:

A. Designs with only main effects confounded.

1. confound one main effect from each of the symmetric factorials

2. confound one main effect from only some of the symmetric factorials
3. confound at least one main effect from each of the symmetric factorials
4. confound at least one main effect from only some of the symmetric factorials

B. Designs with no main effects confounded.

1. confound one component from each of the symmetric factorials
2. confound one component from only some of the symmetric factorials
3. confound at least one component from each of the symmetric factorials
4. confound at least one component from only some of the symmetric factorials

For case (A), except the main effects, the first order interactions, ... , and the (r-1)th order interactions of the confounded factors, one can estimate all the other main effects and interactions. As for case (B), all information, except those confounded interactions and their generalized interactions, is available.

4.3 ORTHOGONAL FACTORIAL STRUCTURE

In this section, the designs constructed by the CRT are shown to have orthogonal factorial structure. The fixed effects intra-block model with no block-treatment interaction is assumed. Such a design is said to be effect-wise orthogonal if best linear estimates of estimable treatment contrasts belonging to different factorial effects are orthogonal, i.e. uncorrelated. The following definitions and theorem (stated without proof), due to Mukerjee (1981), will be used in later work.

Definition 2: A *proper matrix* is a square matrix with all row sums and all column sums equal.

Definition 3: A $v \times v$ matrix A where $v = \prod_{j=1}^{m} s_j \, (s_j \geq 2)$ is said to have structure K if it be expressible as a linear combination of Kronecker products of proper matrices of orders $s_1, s_2, \ldots, s_m$ i.e., if

$$\mathbf{A} = \sum_{g=1}^{w} \xi_g (\mathbf{V}_{g1} \otimes \mathbf{V}_{g2} \otimes \ldots \otimes \mathbf{V}_{gm})$$

where w is a positive integer, $\xi_1, \xi_2, \ldots, \xi_w$ are some real numbers and for each g, $\mathbf{V}_{gj}$ is some proper matrix of order s_j, $1 \leq j \leq m$.

Theorem 4.3 *If* N *is the incidence matrix for an equireplicate factorial experiment in a block design with constant block size, a necessary and sufficient condition for the design to be effect-wise orthogonal is that the matrix* $\mathbf{NN}'$ *has structure K*

Let $\mathbf{N}_i$ be the incidence matrix of an $s_i^{m_i}$ factorial design obtained by the classical method using either field or modular arithmetic. Without loss of generality, let the r^{th} row of $\mathbf{N}_i$ correspond to the r^{th} lexicographically ordered value of $\mathbf{x}'_i = (x_{i1}, x_{i2}, ..., x_{ie_i})$ and let the c^{th} column correspond to the c^{th} lexicographically ordered value of $\mathbf{a}'_i = (a_{i1}, a_{i2}, ..., a_{ie_i})$ where

$$r_i = \sum_{j=0}^{m_i-1} x_{i(m_i-j)} s_i^j + 1$$

$$c_i = \sum_{j=1}^{e_i} a_{ij} s_i^{j-1} + 1$$

a_{ij}'s are as defined in section 4.2.2. If no effects or interaction components are confounded in $s_i^{m_i}$ symmetrical factorial, then $\mathbf{N}_i = 1_{(s_i^{m_i} \times 1)} = (1, 1, ..., 1)'$.

Let n_{jl} denote the element in $\mathbf{N}_i$, if a single replicate factorial experiment in a block design with constant block size is considered, then n_{jl} is either zero or one, where $1 \leq j \leq s_i^{m_i}$ and $1 \leq l \leq s_i^{e_i}$. Let n^*_{jl} denote the element in $\mathbf{N}_i\mathbf{N}_i'$, then

$$n^*_{jj'} = \sum_{l=1}^{s_i^{e_i}} n_{jl} n_{j'l}$$

$n^*_{jj'} = 1$ only when treatment combinations with lexicographically ordered value equal to j, j' occur in the same block. Thus

$$\sum_{j=1}^{s_i^{m_i}} n_{jj'}^{*} = \sum_{j'=1}^{s_i^{m_i}} n_{jj'}^{*} = \text{ block size } = s_i^{m_i - e_i}$$

By Definition 2, $\mathbf{N}_i\mathbf{N}'_i$ is a proper matrix.

Let

$$\mathbf{N}_{kp} = \mathbf{N}_1 \otimes \mathbf{N}_2 \otimes \ldots \otimes \mathbf{N}_n,$$

then $\mathbf{N}_{kp}$ is an N×M matrix with $N = s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_n^{m_n}$ and $M = s_1^{e_1} \times s_2^{e_2} \times \ldots \times s_r^{e_r}$. Column c' of $\mathbf{N}_{kp}$ is obtained as the Kronecker product of columns in $\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_n$, where

$$c' = \sum_{i=1}^{n-1} (c_i - 1) \prod_{j=i+1}^{n} s_j^{e_j} + c_n$$

If only effects from r symmetrical factorials are confounded, where $r \leq n$, without loss of generality, one can let the first $\mathbf{N}_r$ denote the incidence matrices of those r symmetrical factorials. Then

$$c' = \sum_{i=1}^{r-1} (c_i - 1) \prod_{j=i+1}^{r} s_j^{e_j} + c_r$$

Also, the row r' of N_{kp} corresponds to the r^{th} lexicographically ordered value of $x' = (x_{11}, ..., x_{1m_1}, ..., x_{n1}, ..., x_{nm_n})$

$$r' = \sum_{i=1}^{n} \sum_{j=1}^{m_i - 1} x_{i(m_i - j)} s_i^j \left[\prod_{k=2}^{n} (s_k^{m_k})^{\delta_k} \right] + 1$$

where $\delta_k = 1$ if i<k.

By the properties of Kronecker products,

$$\begin{aligned} N_{kp}N'_{kp} &= (N_1 \otimes N_2 \otimes ... \otimes N_n)(N_1 \otimes N_2 \otimes ... \otimes N_n)' \\ &= (N_1 \otimes N_2 \otimes ... \otimes N_n)(N'_1 \otimes N'_2 \otimes ... \otimes N'_n) \\ &= N_1N'_1 \otimes N_2N'_2 \otimes ... \otimes N_nN'_n \end{aligned}$$

Since each $N_iN'_i$ is a proper matrix of order $s_i^{m_i}$, by Definition 3, $N_{kp}N'_{kp}$ has structure K.

Theorem 4.4 *The designs constructed by CRT are effect-wise orthogonal*

Proof: Let N_e be the incidence matrix of $s_1^{m_1} \times s_2^{m_2} \times ... \times s_n^{m_n}$ factorial experiment with block given by (4.5). Let row r″ of N_e be defined as the same as r' in N_{kp}. Column w of N_e is obtained by equation (4.5), i.e.

$$\sum_{j=1}^{r} \left(\frac{M}{s_j^{e_j}} \right) b_j a_j \equiv w \pmod{M}$$

or equivalently,

$$\sum_{j=1}^{r} \left(\frac{M}{s_j^{e_j}}\right) b_j(c_j - 1) \equiv w \pmod{M}$$

By the way $\mathbf{N}_c$ and $\mathbf{N}_{kp}$ are constructed, we can obtain $\mathbf{N}_c$ by interchanging the column vectors of $\mathbf{N}_{kp}$. Therefore, $\mathbf{N}_c\mathbf{N}'_c = \mathbf{N}_{kp}\mathbf{N}'_{kp}$, which implies that $\mathbf{N}_c\mathbf{N}'_c$ also has structure K. By Theorem 4.3, the design is effect-wise orthogonal.

4.4 THE PRINCIPLE OF GENERALIZED INTERACTION

In this section, the Kronecker product structure obtained in section 4.3 is used to identify a Principle of Generalized Interaction. The following theorem, due to Voss (1986), provides the basis for identifying generalized interactions for the designs constructed by CRT.

Theorem 4.5 *Let* $\mathbf{N}$ and $\mathbf{N}_i$ *be the incidence matrix of confounded designs of* $s_1^{m_1}s_2^{m_2}\ldots s_n^{m_n}$ *and* $s_i^{m_i}$ *factorials, respectively. And let* d_i *denote the number of degrees of freedom confounded in* $\mathbf{N}_i$, $\mathbf{C}_i$ *denote a* $v_i \times d_i$ *matrix, where* v_i *is the number of treatment combinations in* $s_i^{m_i}$, *such that the columns of* $\mathbf{C}_i$ *form an orthogonal basis for the space of completely confounded contrasts for* $\mathbf{N}_i$. *Let*

$$(\mathbf{1}_v, \mathbf{C}) = (\mathbf{1}_{v_1}, \mathbf{C}_1) \otimes \ldots \otimes (\mathbf{1}_{v_n}, \mathbf{C}_n)$$

Then the columns of $\mathbf{C}$ *form an orthogonal basis for the space of completely confounded contrast vectors for* $\mathbf{N}$.

The following example illustrates the use of the theorem.

Example Consider a $2^3 \times 3^2 \times 5^2$ factorial in 60 blocks of size 30. Let F_{11}, F_{12} and F_{13} denote the factors occurring at two levels, F_{21} and F_{22} at three levels, and F_{31} and F_{32} at five levels. If one wishes to confound the interaction components $F_{11}F_{12}$, $F_{12}F_{13}$, $F_{21}F_{22}^2$ and $F_{31}F_{32}$, then, by Theorem 4.4, the orthogonal basis for the space of completely confounded contrast vectors for **N** is

$$(1_{1800}, \mathbf{C}) = (1_8, \mathbf{C}_1) \otimes (1_9, \mathbf{C}_2) \otimes (1_{25}, \mathbf{C}_3)$$

where $\mathbf{C}_i$ is a $v_i \times d_i$ matrix such that $\mathbf{C}_i$ is a submatrix of $\mathbf{C}_{N_i}$ and the columns of $\mathbf{C}_i$ span the vector space of contrast vectors of degrees of freedom completely confounded by $\mathbf{N}_i$.

$$\mathbf{C}'_1 = \begin{bmatrix} -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\mathbf{C}'_2 = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & -1 \\ 1 & 1 & -2 & -2 & 1 & 1 & 1 & -2 & 1 \end{bmatrix}$$

$$\mathbf{C}_3 = \begin{bmatrix} & & & \\ -2 & 2 & -1 & 1 \\ -1 & -1 & 2 & -4 \\ 0 & -2 & 0 & 6 \\ 1 & -1 & -2 & -4 \\ 2 & 2 & 1 & 1 \\ -1 & -1 & 2 & -4 \\ 0 & -2 & 0 & 6 \\ 1 & -1 & -2 & -4 \\ 2 & 2 & 1 & 1 \\ -2 & 2 & -1 & 1 \\ 0 & -2 & 0 & 6 \\ 1 & -1 & -2 & -4 \\ 2 & 2 & 1 & 1 \\ -2 & 2 & -1 & 1 \\ -1 & -1 & 2 & -4 \\ 1 & -1 & -2 & -4 \\ 2 & 2 & 1 & 1 \\ -2 & 2 & -1 & 1 \\ -1 & -1 & 2 & -4 \\ 0 & -2 & 0 & 6 \\ 2 & 2 & 1 & 1 \\ -2 & 2 & -1 & 1 \\ -1 & -1 & 2 & -4 \\ 0 & -2 & 0 & 6 \\ 1 & -1 & -2 & -4 \\ & & & \end{bmatrix}$$

Therefore, the degrees of freedom confounded in this design are $(d_1+1)(d_2+1)(d_3+1)-1=59$, and the generalized interactions being confounded are $F_{11}F_{13}$, $F_{11}F_{12}F_{21}F_{22}^2$, $F_{11}F_{12}F_{31}F_{32}$, $F_{12}F_{13}F_{21}F_{22}^2$, $F_{12}F_{13}F_{31}F_{32}$, $F_{11}F_{13}F_{21}F_{22}^2$, $F_{11}F_{13}F_{31}F_{32}$, $F_{21}F_{22}^2F_{31}F_{32}$, $F_{11}F_{12}F_{21}F_{22}^2F_{31}F_{32}$, $F_{12}F_{13}F_{21}F_{22}^2F_{31}F_{32}$, $F_{11}F_{13}F_{21}F_{22}^2F_{31}F_{32}$.

4.5 CONFOUNDING IN MIXED FACTORIALS WHEN THE NUMBER OF LEVELS ARE NOT RELATIVELY PRIME

So far, owing to the restriction of the Chinese Remainder Theorem, only the systems with number of levels being relatively prime in pairs are considered. In this section, we shall discuss two other important type of mixed factorial experiments:

1. When the number of levels for some factors are powers of the same prime number.
2. When the number of levels for some factors are product of different primes.

4.5.1 NUMBER OF LEVELS: POWERS OF THE SAME PRIME NUMBER

Consider a $s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_n^{m_n}$ factorial experiment, where $s_j = s_i^k (i \neq j)$, s_i being a prime and k being some integer greater than unity, Furthermore, $s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n$ are relatively prime in pairs. One can deal with this case by setting up a correspondence between the s_j levels of the one factor and the s_i^k combinations of k pseudofactors each at s_i levels. For example : for the case of $s_i = 2$ and $s_j = 8$

Level of factor	Levels of pseudofactors		
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1

Thus $s_i^{m_i} s_j^{m_j}$ can be treated as a symmetric factorial $s_i^{m_i + km_j}$ in $m_i + km_j$ pseudofactors each at s_i levels. Since all $s_l (l = 1, ..., j-1, j+1, ..., n)$ are now relatively prime in pairs, confounding can be done by applying the Chinese Remainder Theorem.

Using pseudofactors makes it easy to deal with this kind of experiment, however, one has to be more careful in choosing the components to be confounded. We shall illustrate the problem by the following example :

Example Consider a design for four factors, A at 2 levels, B at 3 levels, and C, D at 4 levels each. The 3 degrees of freedom for main effect of factor C may be represented by

$$(C)^1 = c_3 + c_2 - c_1 - c_0$$
$$(C)^2 = c_3 - c_2 - c_1 + c_0$$
$$(C)^3 = c_3 - c_2 + c_1 - c_0$$

Using pseudofactors, the design may be represented as an experiment on five 2-level factors, A,P,Q,R,S and one 3-level factor, B. The components of the main effect of the original factor C will appear as main effects and interaction of the new factors P and Q

$$(C)^1 = P$$
$$(C)^2 = PQ$$
$$(C)^3 = Q$$

The three degrees of freedom for C correspond to the main effects P and Q and the interaction PQ. The main effect of D are R, S and RS. The other nine interactions between the pseudofactors correspond to the interaction

C×D. Therefore, in order to estimate main effects C and D, one should avoid confounding P, Q, R, S, and iteractions PQ and RS. For example, if we wish to use blocks of 24, we can confound APR, QS, and APQRS so that no main effect comparisons will be confounded.

4.5.2 NUMBER OF LEVELS: A PRODUCT OF DIFFERENT PRIMES

This case is more complicated than the previous one. For simplicity, let us consider number of treatments given by $p_1^{m_1} \times p_2^{m_2} \times p_3^{m_3}$ where p_1, p_2 are primes and $p_3 = p_1 p_2$. A way to manipulate this special case is to regard each factor with p_3 levels as arising from two pseudofactors, one with p_1 levels and one with p_2 levels. Thus we could regard the experiment as a $p_1^{m_1+m_3} p_2^{m_2+m_3}$ experiment. For example, in the case of $p_3 = 6$, $p_1 = 2$ and $p_2 = 3$, then

Level of factor	Levels of pseudofactors
0	0 0
1	0 1
2	0 2
3	1 0
4	1 1
5	1 2

Again, the Chinese Remainder Theorem is applicable since the levels of new factors are both primes.

The possibility of confounding in this special case, however, is more restricted. For example, let us consider a $2 \times 3 \times 6^2$ factorial with factors A,

B, C and D. The five degrees of freedom for main effect of factor C may be represented by

$$(C)^1 = c_5 - c_3 + c_2 - c_0$$
$$(C)^2 = c_5 - 2c_4 + c_3 + c_2 - 2c_1 + c_0$$
$$(C)^3 = c_5 + c_4 + c_3 - c_2 - c_1 - c_0$$
$$(C)^4 = c_5 - c_3 - c_2 + c_0$$
$$(C)^5 = c_5 - 2c_4 + c_3 - c_2 + 2c_1 - c_0$$

We may replace each 6-level factor by two pseudofactors, one at two levels and the other at three levels, C by P and Q, D by R and S. The components of the main effect of the original factor C will appear as main effects and interaction of the new factor P and Q

$$(C)^1 = (Q)^1$$
$$(C)^2 = (Q)^2$$
$$(C)^3 = P$$
$$(C)^4 = P(Q)^1$$
$$(C)^5 = P(Q)^2$$

Therefore, all the main effects will be free of block effects if one does not confound P, Q, PQ, R, S, and RS.

4.6 ESTIMATION AND TEST OF HYPOTHESIS OF THE UNCONFOUNDED EFFECTS

It is proven in section 4.3 that the designs constructed by CRT is effect-wise orthogonal, therefore, a full-rank model $\mathbf{y} = \mathbf{C}\alpha + \varepsilon$, where α is the vector of orthogonal contrasts belonging to main effects and interactions, for the experiments can be derived. The property of uncorrelated estimates will be reflected in the fact that the variance-covariance matrix $(\mathbf{C}'\mathbf{C})^{-1}$ is a diagonal matrix.

4.6.1 EXPERIMENTAL DESIGN MODEL

Consider a factorial design $s_1 \times s_2 \times ....s_n$ involving n factors $F_1 \times F_2 \times ...F_n$. Any treatment effects of these factors occurring at levels $(x_1, x_2, ..., x_n)$ respectively will be denoted by $\tau_{x_1 x_2 ... x_n}$. For each treatment combination the observed random variable is denoted by $y_{x_1 x_2 ... x_n}$. Let $N = s_1 s_2 ... s_n$ and r equal the number of observed random variables for each treatment combination, then the experimental design model is

$$\mathbf{y} = \mathbf{X}\beta^* + \varepsilon$$

$$= [\vartheta_{rN} \,|\, \mathbf{X}_\tau \,|\, \mathbf{X}_\beta] \begin{bmatrix} \mu \\ \tau \\ \beta \end{bmatrix} + \varepsilon$$

where

$$\mathbf{y} = \begin{bmatrix} y_{00\ldots0,1} \\ y_{00\ldots0,2} \\ \cdot \\ \cdot \\ \cdot \\ y_{00\ldots0,r} \\ \cdot \\ \cdot \\ \cdot \\ y_{(s_1-1,\ldots,s_n-1),r} \end{bmatrix}$$

$$\vartheta = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

$$\mathbf{X}_\tau = \begin{bmatrix} \vartheta_r & & & \\ & \vartheta_r & & 0 \\ & & \cdot & \\ & & & \cdot \\ & 0 & & \cdot \\ & & & & \vartheta_r \end{bmatrix}$$

$$\mathbf{X}_\beta = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_b]$$

$\mathbf{x}_i$ is a $rN \times 1$ vector of ones and zeros.

$$\tau = \begin{bmatrix} \tau_{00\ldots0,1} \\ \tau_{00\ldots0,2} \\ \cdot \\ \cdot \\ \cdot \\ \tau_{00\ldots0,r} \\ \cdot \\ \cdot \\ \cdot \\ \tau_{(s_1-1,\ldots,s_n-1),r} \end{bmatrix}$$

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_b \end{bmatrix}$$

X is thus a rN×(N+b+1) matrix with rank N. By Definition 13.2.5, page 493 of Graybill(1976), the experimental design model $\mathbf{y} = \mathbf{X}\beta^* + \varepsilon$, can be orthogonally reparameterized to a full-column rank model $\mathbf{y} = \mathbf{C}\alpha + \varepsilon$, where **C** is a rN×N matrix with rank N and **C'C** is a diagonal matrix. The relationship between β^* and α is $\alpha = \mathbf{U}\beta^*$, where each element of α is a linear contrast of β_i^*'s. For asymmetrical factorials, the model can be constructed through Kronecker product of the design matrices discussed in section 3.2.2.

4.6.2 POINT ESTIMATION

In this section, we shall first discuss point estimates of a linear function of β_i^*, $1 \leq i \leq N + b + 1$ i.e. the orthogonal contrast α_j, belonging to some effect or interaction. Under the case that $E(\varepsilon) = \mathbf{0}$ and $Cov(\varepsilon) = \sigma^2\mathbf{I}$, the estimate of α is given by $\hat{\alpha} = (\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}'\mathbf{y}$. Furthermore, it is proven (Graybill, 1976) that $\hat{\alpha}$ is the best linear unbiased estimate of α.

To illustrate the estimation procedure, we consider the plan for one two-level factors, A, and two three-level factors, B and C. With the component BC being confounded, the experiment design model is as follows :

$$
\begin{bmatrix} y_{000} \\ y_{001} \\ y_{002} \\ y_{010} \\ y_{011} \\ y_{012} \\ y_{020} \\ y_{021} \\ y_{022} \\ y_{100} \\ y_{101} \\ y_{102} \\ y_{110} \\ y_{111} \\ y_{112} \\ y_{120} \\ y_{121} \\ y_{122} \end{bmatrix} =
\begin{bmatrix}
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0
\end{bmatrix}
\begin{bmatrix} \mu \\ \tau_{000} \\ \tau_{001} \\ \tau_{002} \\ \tau_{010} \\ \tau_{011} \\ \tau_{012} \\ \tau_{020} \\ \tau_{021} \\ \tau_{022} \\ \tau_{100} \\ \tau_{101} \\ \tau_{102} \\ \tau_{110} \\ \tau_{111} \\ \tau_{112} \\ \tau_{120} \\ \tau_{121} \\ \tau_{122} \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} +
\begin{bmatrix} \varepsilon_{000} \\ \varepsilon_{001} \\ \varepsilon_{002} \\ \varepsilon_{010} \\ \varepsilon_{011} \\ \varepsilon_{012} \\ \varepsilon_{020} \\ \varepsilon_{021} \\ \varepsilon_{022} \\ \varepsilon_{100} \\ \varepsilon_{101} \\ \varepsilon_{102} \\ \varepsilon_{110} \\ \varepsilon_{111} \\ \varepsilon_{112} \\ \varepsilon_{120} \\ \varepsilon_{121} \\ \varepsilon_{122} \end{bmatrix}
$$

Let $E^{(1)}$ and $E^{(2)}$ represent the first and the second contrasts belonging to effect or interaction E, then the reparameterized model for this experiment can be expressed as

$$
\begin{bmatrix} y_{000} \\ y_{001} \\ y_{002} \\ y_{010} \\ y_{011} \\ y_{012} \\ y_{020} \\ y_{021} \\ y_{022} \\ y_{100} \\ y_{101} \\ y_{102} \\ y_{110} \\ y_{111} \\ y_{112} \\ y_{120} \\ y_{121} \\ y_{122} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 0 & -2 & -1 & 1 & 1 & 1 & -1 & 0 & 2 & 1 & -1 & 0 & 2 & -1 & -1 & 0 & -2 \\ 1 & 1 & 1 & -1 & 1 & 0 & -2 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 0 & 2 & 1 & 1 \\ 1 & -1 & 1 & 0 & -2 & 0 & -2 & -1 & 1 & -1 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 & -2 & -1 & 1 & -1 & 0 & 2 & 0 & 2 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 0 & -2 & 1 & 1 & -1 & -1 & -1 & 0 & 2 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 0 & -2 & 1 & 1 & 0 & -2 & -1 & 0 & 2 & -1 & -1 & 1 & -1 & 0 & 2 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & 2 & 1 & -1 & 0 & -2 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -2 & -1 & 1 & 1 & 1 & 1 & 0 & -2 & -1 & 1 & 0 & -2 & 1 & 1 & 0 & -2 \\ 1 & 1 & 1 & -1 & 1 & 0 & -2 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 0 & -2 & 1 & 1 \\ 1 & -1 & 1 & 0 & -2 & 0 & -2 & 1 & -1 & 1 & 0 & -2 & 0 & -2 & 0 & -2 & 0 & -2 \\ 1 & 0 & -2 & 0 & -2 & -1 & 1 & 1 & 0 & -2 & 0 & -2 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & -2 & 1 & 1 & 1 & 1 & 1 & 0 & -2 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 & 1 & 0 & -2 & 1 & 0 & -2 & 1 & 1 & -1 & 1 & 0 & -2 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & -2 & -1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 6M \\ 3(C)^1 \\ 2(C)^2 \\ 3(B)^1 \\ 2(B)^2 \\ 3(BC^2)^1 \\ 2(BC^2)^2 \\ 3A \\ 3A(C)^1 \\ 2A(C)^2 \\ 3A(B)^1 \\ 2A(B)^2 \\ 3A(BC)^1 \\ 2A(BC)^2 \\ 3A(BC^2)^1 \\ 2A(BC^2)^2 \\ 3((BC)^1 + \beta_3 - \beta_1) \\ 2((BC)^2 + \beta_3 - 2\beta_2 + \beta_1) \end{bmatrix} + \begin{bmatrix} \varepsilon_{000} \\ \varepsilon_{001} \\ \varepsilon_{002} \\ \varepsilon_{010} \\ \varepsilon_{011} \\ \varepsilon_{012} \\ \varepsilon_{020} \\ \varepsilon_{021} \\ \varepsilon_{022} \\ \varepsilon_{100} \\ \varepsilon_{101} \\ \varepsilon_{102} \\ \varepsilon_{110} \\ \varepsilon_{111} \\ \varepsilon_{112} \\ \varepsilon_{120} \\ \varepsilon_{121} \\ \varepsilon_{122} \end{bmatrix}
$$

Thus, the estimates of unconfounded effects and interactions are

$$
\begin{bmatrix}
\hat{6M} \\
3(\hat{C})^1 \\
2(\hat{C})^2 \\
3(\hat{B})^1 \\
2(\hat{B})^2 \\
3(\hat{BC^2})^1 \\
2(\hat{BC^2})^2 \\
3\hat{A} \\
3\hat{A}(\hat{C})^1 \\
2\hat{A}(\hat{C})^2 \\
3\hat{A}(\hat{B})^1 \\
2\hat{A}(\hat{B})^2 \\
3\hat{A}(\hat{BC})^1 \\
2\hat{A}(\hat{BC})^2 \\
3\hat{A}(\hat{BC^2})^1 \\
2\hat{A}(\hat{BC^2})^2
\end{bmatrix}
=
\frac{1}{6}
\begin{bmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
-3 & 0 & 3 & -3 & 0 & 3 & -3 & 0 & 3 & -3 & 0 & 3 & -3 & 0 & 3 & 0 \\
1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 \\
-3 & -3 & -3 & 0 & 0 & 0 & 3 & 3 & 3 & -3 & -3 & -3 & 0 & 0 & 0 & 3 \\
1 & 1 & 1 & -2 & -2 & -2 & 1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & -2 & 1 \\
-3 & 3 & 0 & 0 & -3 & 3 & 3 & 0 & -3 & -3 & 3 & 0 & 0 & -3 & 3 & 3 \\
1 & 1 & -2 & -2 & 1 & 1 & 1 & -2 & 1 & 1 & 1 & -2 & -2 & 1 & 1 & 1 \\
-2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 0 & -3 & 3 & 0 & -3 & 3 & 0 & -3 & -3 & 0 & 3 & -3 & 0 & 3 & 0 \\
-1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 & 1 & -2 & 1 & 1 & -2 & 1 & 1 \\
3 & 3 & 3 & 0 & 0 & 0 & -3 & -3 & -3 & -3 & -3 & -3 & 0 & 0 & 0 & 3 \\
-1 & -1 & -1 & 2 & 2 & 2 & -1 & -1 & -1 & 1 & 1 & 1 & -2 & -2 & -2 & 1 \\
3 & 0 & -3 & 0 & -3 & 3 & -3 & 3 & 0 & -3 & 0 & 3 & 0 & 3 & -3 & 3 \\
-1 & 2 & -1 & 2 & -1 & -1 & -1 & -1 & 2 & 1 & -2 & 1 & -2 & 1 & -1 & -1 \\
3 & -3 & 0 & -3 & 3 & 0 & 3 & -3 & -3 & 3 & 0 & 0 & 3 & -3 & 3 & 0 \\
-1 & -1 & 2 & 2 & -1 & -1 & 2 & 1 & -1 & -1 & 2 & -2 & 1 & 1 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
y_{000} \\
y_{001} \\
y_{002} \\
y_{010} \\
y_{011} \\
y_{012} \\
y_{020} \\
y_{021} \\
y_{022} \\
y_{100} \\
y_{101} \\
y_{102} \\
y_{110} \\
y_{111} \\
y_{112} \\
y_{120} \\
y_{121} \\
y_{122}
\end{bmatrix}
$$

4.6.3 TEST OF HYPOTHESIS

In order to test the hypothesis about each unconfounded main effects and interactions, which are determined by the method mentioned in section 4.4, we shall partition α into subvector $\alpha_1, \alpha_2, ..., \alpha_k$, where each α_i represent the group of r_i linear contrasts belonging to main effect or interaction component $(E)_i$ However, if $(E)_i$ is confounded, then α_i represent linear combinations of treatment effects as well as block effects. Thus the full-rank linear model can be rewritten as

$$\mathbf{y} = \mathbf{C}_1\alpha_1 + \mathbf{C}_2\alpha_2 + ... + \mathbf{C}_k\alpha_k + \varepsilon$$

And due to the property of the $\mathbf{C}$ matrix, $\mathbf{C}_i'\mathbf{C}_j = \mathbf{C}_j'\mathbf{C}_i = \mathbf{0}$ for all $1 \le i,j \le k,\ i \ne j$.

Under the case that $\varepsilon \sim N(\mathbf{0}, \sigma^2\mathbf{I})$, the maximum likelihood estimators of α and σ^2 are

$$\hat{\alpha} = (\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}'\mathbf{y}$$
$$\hat{\sigma}^2 = \frac{\mathbf{y}'[\mathbf{I} - \mathbf{C}(\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}']\mathbf{y}}{rN - N}$$

By Theorem 6.2.1, page 176 of Graybill (1976), the estimators $\hat{\alpha}$ and $\hat{\sigma}^2$ have the following properties:

(1) $\hat{\alpha} \sim N(\alpha, \sigma^2(\mathbf{C}'\mathbf{C})^{-1})$

(2) $\hat{\sigma}^2 \sim \sigma^2\chi^2_{rN-N}/(rN - N)$

(3) $\hat{\alpha}, \hat{\sigma}^2$ are UMVUE

(4) $\hat{\alpha}, \hat{\sigma}^2$ are independent

Let $SS(\alpha_i)$ denote the sum of squares due to the set of contrasts α_i where

$$SS(\alpha_i) = \hat{\alpha}_i' \mathbf{C}_i' \mathbf{y} = \mathbf{y}' \mathbf{C}_i (\mathbf{C}_i' \mathbf{C}_i)^{-1} \mathbf{C}_i' \mathbf{y}.$$

Graybill further proved that

$$SS(\alpha_1, \alpha_2, ..., \alpha_k) = SS(\alpha_1) + SS(\alpha_2) + ... + SS(\alpha_k)$$

and all SS and $\hat{\sigma}^2$ are mutually stochastically independent.

The ANOVA table for testing $H_0: \alpha_i = \mathbf{0}$ for the unconfounded main effects or interaction components $(E)_i$, $i < k$, is shown in Table 4.6.

Table 4.1 The addition and multiplication tables for $GF(2^2)$

	Addition				Multiplication			
	U_0	U_1	U_2	U_3	U_0	U_1	U_2	U_3
U_0	U_0	U_1	U_2	U_3	U_0	U_0	U_0	U_0
U_1		U_0	U_3	U_2		U_1	U_2	U_3
U_2			U_0	U_1			U_3	U_1
U_3				U_0				U_2

Table 4.2 Allocation of treatment combinations to blocks for confounding of AB and CD^3 in $3^2 \times 4^2$ factorial.

Block	Treatment combination					
0	0000	0012	0023	0031	1200	1212
	1223	1231	2100	2112	2123	2131
1	0102	0110	0121	0133	1002	1010
	1021	1033	2202	2210	2221	2233
2	0203	0211	0220	0232	1103	1111
	1120	1132	2003	2011	2020	2032
3	0001	0013	0022	0030	1201	1213
	1222	1230	2101	2113	2122	2130
4	0100	0112	0123	0131	1000	1012
	1023	1031	2200	2212	2223	2231
5	0202	0210	0221	0233	1102	1110
	1121	1133	2002	2010	2021	2033
6	0003	0011	0020	0032	1203	1211
	1220	1232	2103	2111	2120	2132
7	0101	0113	0122	0130	1001	1013
	1022	1030	2201	2213	2222	2230
8	0200	0212	0223	0231	1100	1112
	1123	1131	2000	2012	2023	2031
9	0002	0010	0021	0033	1202	1210
	1221	1233	2102	2110	2121	2133
10	0103	0111	0120	0132	1003	1011
	1020	1032	2203	2211	2220	2232
11	0201	0213	0222	0230	1101	1113
	1122	1130	2001	2013	2022	2030

Table 4.3 Possible confounding plans for a $3^3 \times 4^3 \times 5^2$ factorial constructed by the Chinese Remainder Theorem.

number of components confounded from each of the symmetric factorial			number of blocks	block size
3^3	4^3	5^2		
1	1	1	3×4×5	720
1	1	0	3×4	3600
1	0	1	3×5	2880
0	1	1	4×5	2160
1	0	0	3	14400
0	1	0	4	10800
0	0	1	5	8640

Table 4.4 The treatment combinations in the intra-block subgroup of a $3^3 \times 4^3 \times 5^2$ factorial with $F_{11}F_{12}^2$, $F_{21}F_{22}$, $F_{31}F_{32}$, $F_{11}F_{12}F_{13}$, $F_{21}F_{22}^2F_{13}$ confounded.

0 0 0 0 0 0 0 0	1 1 1 0 0 0 0 0	2 2 2 0 0 0 0 0
0 0 0 0 0 0 4 1	1 1 1 0 0 0 4 1	2 2 2 0 0 0 4 1
0 0 0 0 0 0 3 2	1 1 1 0 0 0 3 2	2 2 2 0 0 0 3 2
0 0 0 0 0 0 2 3	1 1 1 0 0 0 2 3	2 2 2 0 0 0 2 3
0 0 0 0 0 0 1 4	1 1 1 0 0 0 1 4	2 2 2 0 0 0 1 4
0 0 0 1 1 3 0 0	1 1 1 1 1 3 0 0	2 2 2 1 1 3 0 0
0 0 0 1 1 3 4 1	1 1 1 1 1 3 4 1	2 2 2 1 1 3 4 1
0 0 0 1 1 3 3 2	1 1 1 1 1 3 3 2	2 2 2 1 1 3 3 2
0 0 0 1 1 3 2 3	1 1 1 1 1 3 2 3	2 2 2 1 1 3 2 3
0 0 0 1 1 3 1 4	1 1 1 1 1 3 1 4	2 2 2 1 1 3 1 4
0 0 0 2 2 1 0 0	1 1 1 2 2 1 0 0	2 2 2 2 2 1 0 0
0 0 0 2 2 1 4 1	1 1 1 2 2 1 4 1	2 2 2 2 2 1 4 1
0 0 0 2 2 1 3 2	1 1 1 2 2 1 3 2	2 2 2 2 2 1 3 2
0 0 0 2 2 1 2 3	1 1 1 2 2 1 2 3	2 2 2 2 2 1 2 3
0 0 0 2 2 1 1 4	1 1 1 2 2 1 1 4	2 2 2 2 2 1 1 4
0 0 0 3 3 2 0 0	1 1 1 3 3 2 0 0	2 2 2 3 3 2 0 0
0 0 0 3 3 2 4 1	1 1 1 3 3 2 4 1	2 2 2 3 3 2 4 1
0 0 0 3 3 2 3 2	1 1 1 3 3 2 3 2	2 2 2 3 3 2 3 2
0 0 0 3 3 2 2 3	1 1 1 3 3 2 2 3	2 2 2 3 3 2 2 3
0 0 0 3 3 2 1 4	1 1 1 3 3 2 1 4	2 2 2 3 3 2 1 4

Table 4.5 Number of blocks and block size for different confounding plans

Case	No. of blocks	block size	
$(A)a,(B)a$	$M=\prod_{i=1}^{n} s_i$	$\prod_{i=1}^{n} s_i^{t_i}$	$(t_i=m_i-1)$
$(A)b,(B)b$	$M=\prod_{i=1}^{r} s_i$	$(\prod_{i=1}^{r} s_i^{t_i})(\prod_{i=r+1}^{n} s_i^{m_i})$	$(t_i=m_i-1)$
$(A)c,(B)c$	$M=\prod_{i=1}^{n} s_i^{e_i}$	$\prod_{i=1}^{n} s_i^{t_i}$	$(t_i=m_i-e_i)$
$(A)d,(B)d$	$M=\prod_{i=1}^{r} s_i^{e_i}$	$(\prod_{i=1}^{r} s_i^{t_i})(\prod_{i=r+1}^{n} s_i^{m_i})$	$(t_i=m_i-e_i)$

Table 4.6 Analysis of variance for testing unconfounded effects

Source	df	SS	F
Due to α_i	$(df)_i$	$\hat{\boldsymbol{\alpha}}_i'\mathbf{C}_i'\mathbf{y}$	$\dfrac{MS(\alpha_i)}{\text{MSE}}$
Error	$rN - N$	$\mathbf{y}'[\mathbf{I} - \mathbf{C}(\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}']\mathbf{y}$	
Total	rN	$\mathbf{y}'\mathbf{y}$	

CHAPTER 5

RANDOMIZED FRACTIONAL FACTORIALS

In chapter 3, it was stated that the design matrix $\mathbf{C}_{w,N}$ can be constructed with no limitation on the selection of a defining contrast. Therefore, we shall extend some results on randomized fractional replicates in the general s^m system, given in Lentner (1967) which used only main effects as the defining contrast, to more general cases for both symmetrical and asymmetrical factorials.

5.1 GENERAL SYMMETRIC SYSTEM

In this section, two lemmas are developed which will be used to establish later results.

Lemma 5.1 $E_\phi(\mathbf{H}_w) = \mathbf{0}$ *if and only if* $\phi' = \phi^{*'} = \frac{1}{M}(1,1,...,1)$

Proof: Since $\mathbf{H}_w = [\mathbf{C}_{w1} | \mathbf{C}_{w2}]$ is constructed based on the matrix $[\mathbf{D}_{w1} | \mathbf{D}_{w2}]$, we know that if $\sum_i d_{ij} = k[0 + 1 + ... + (s-1)]$, then $\sum_i c_{ij'} = k(c_0 + c_1 + ... + c_{s-1}) = 0$ for all j, j' , where $c_i's$ are defined in section 3.1 and $j' = (s-1)j - (s-2),\ (s-1)j - (s-3),\ ...,\ (s-1)j$.

(1) $E_\phi(\mathbf{C}_{w2}) = \mathbf{0}$ if and only if $\phi = \phi^*$

(1.1) *Sufficiency:* Since $\mathbf{D}_{w2}$ consists of columns corresponding to levels of the effects and their generalized interactions in the defining contrast, each subset w contains only one level of these effects, i.e.

$$(\mathbf{d}_{w2})_l = 0 \text{ or } 1 \text{ or } \ ... \ \text{ or } s\text{-}1$$

Correspondingly,

$$(\mathbf{c}_{w2})_{l'} = \mathbf{c}_0 \text{ or } \mathbf{c}_1 \text{ or } \ldots \text{ or } \mathbf{c}_{s-1}$$

where $l' = (s-1)l-(s-2), (s-1)l-(s-3), \ldots, (s-1)l$. Furthermore, levels of each effects are distributed equally among subsets. Therefore, if a subset is chosen under a randomization procedure with equal probability, i.e. $\phi = \phi^*$ then

$$\begin{aligned} E_\phi((\mathbf{d}_{w2})_l) &= \sum_{w=0}^{M-1} \phi_w^*(\mathbf{d}_{w2})_l \\ &= \frac{1}{M} \sum_{w=0}^{M-1} (\mathbf{d}_{w2})_l \\ &= (\frac{1}{M})(\frac{M}{s})(0+1+\ldots+s\text{-}1) \quad \text{for all } l = 1,2,\ldots,k_2 \end{aligned}$$

which implies that

$$E_\phi((\mathbf{c}_{w2})_{l'}) = (\frac{1}{s})(\mathbf{c}_0+\mathbf{c}_1+\ldots+\mathbf{c}_{s-1}) = \mathbf{0} \quad \text{for all } l' = 1,2,\ldots,k_2^c$$

Therefore, $E_\phi(\mathbf{C}_{w2}) = \mathbf{0}$.

(1.2) Necessity: If $E_\phi(\mathbf{C}_{w2}) = \mathbf{0}$, then $E_\phi((\mathbf{c}_{w2})_{l'}) = \sum_{w=0}^{M-1} \phi_w(\mathbf{c}_{w2})_{l'} = \mathbf{0}$.

If $\phi \neq \phi^*$, then

$$\sum_{w=0}^{M-1} \phi_w(\mathbf{c}_{w2})_{l'} = \rho_0\mathbf{c}_0 + \rho_1\mathbf{c}_1 + \ldots + \rho_{s-1}\mathbf{c}_{s-1} \neq \mathbf{0},$$

since $\rho_i's$ are not all equal, which is a contradiction. Thus $E_\phi(\mathbf{C}_{w2}) = 0$ implies $\phi = \phi^*$.

(2) $E_\phi(\mathbf{C}_{w1}) = 0$ if and only if $\phi = \phi^*$

(2.1) *Sufficiency:* Since

$$(d_{w1})_{ij'} \equiv k(d_{w0})_{ij} + k'(d_{w2})_{ij} \pmod{s},$$

where $j' = (s-1)j - (s-2), (s-1)j - (s-3), \ldots, (s-1)j$, $1 \le k, k' < s$, and $(d_{00})_{ij} = (d_{10})_{ij} = \ldots = (d_{(M-1)0})_{ij}$, the value of $(d_{w1})_{ij'}$ depends totally on the value of $(d_{w2})_{ij}$ for each w = 0,1, ... , M-1. From (1.1),

$$E_{\phi^*}((\mathbf{d}_{w2})_j) = \frac{1}{s}(0 + 1 + \ldots + \mathbf{s\text{-}1}) \text{ for all } j,$$

It is equivalent to state that

$$E_{\phi^*}((d_{w2})_{ij}) = \frac{1}{s}(0 + 1 + \ldots + (s-1)) \text{ for all } ij.$$

Since all effects are constructed to be orthogonal to one another, the previous equation implies that

$$E_{\phi^*}((d_{w1})_{ij'}) = \frac{1}{s}(0 + 1 + \ldots + (s-1)) \text{ for all ij'}.$$

Thus,

$$E_{\phi^*}((c_{w1})_{il}) = \frac{1}{p}(c_0 + c_1 + \ldots + c_{s-1}) = 0 \text{ for all } il.$$

(2.2) Necessity: The proof follows immediately from (1.2) and (1.1).

Lemma 5.2 *For $w = 0,1,2, \ldots, M-1$, let $\mathbf{y}_w$ be the random vector of observations associated with the treatment combinations X_w, where the subset X_w is chosen with probability vector $\phi' = (\phi_0, \phi_1, \ldots, \phi_{M-1})$. Then $E_\phi(\mathbf{y}_w)$ is independent of the vector of the nuisance parameters, β, if and only if $\phi = \phi^*$.*

Proof:

$$\begin{aligned} E_\phi(\mathbf{y}_w) &= E(\mathbf{C}\alpha + \mathbf{H}_w\beta + \varepsilon_w) \\ &= \mathbf{C}\alpha + E_\phi(\mathbf{H}_w)\beta + E_\phi(\varepsilon_w) \end{aligned}$$

Since $E_\phi(\varepsilon_w) = \mathbf{0}$ and by Lemma 5.1, it follows that $E_\phi(\mathbf{y}_w) = \mathbf{C}\alpha$ if and only if $\phi = \phi^*$. Therefore $E_\phi(\mathbf{y}_w)$ is independent of β if and only if $\phi = \phi^*$.

5.1.1 LINEAR UNBIASED ESTIMATOR

From equation (3.4), one can derive an estimate for the parameters of interest α as follows:

$$\hat{\alpha} = \mathbf{C}^{-1}(\mathbf{y}_w - \mathbf{H}_w\hat{\beta}) \tag{5.1}$$

We shall replace $\hat{\beta}$ by any fixed vector γ so that a general structure of all linear estimators of α can be derived as:

$$\hat{\alpha}(\gamma, \mathbf{F}) = \mathbf{C}^{-1}(\mathbf{y}_w - \mathbf{H}_w\gamma) + \mathbf{F}\mathbf{y}_w \tag{5.2}$$

where $\mathbf{F}$ is any S×S matrix. Next we shall prove that all linear unbiased estimators of α can be obtained only under a randomization procedure with equal probability, i.e. $\phi = \phi^*$

Theorem 5.3 *The necessary and sufficient conditions for a linear estimator to be unbiased are*

$$(i)\ \phi = \phi^*$$
$$(ii)\ \mathbf{F} = \mathbf{0}$$

Proof:

$$\begin{aligned} E[\hat{\alpha}(y, \mathbf{F})] &= E[\mathbf{C}^{-1}(\mathbf{y}_w - \mathbf{H}_w y) + \mathbf{F}\mathbf{Y}_w] \\ &= \mathbf{C}^{-1}E(\mathbf{y}_w - \mathbf{H}_w y) + \mathbf{F}E(\mathbf{y}_w) \\ &= (\mathbf{I} + \mathbf{FC})\alpha + [\mathbf{C}^{-1}(\beta - \gamma) + \mathbf{F}\beta]E(\mathbf{H}_w) + (\mathbf{C}^{-1} + \mathbf{F})E(\varepsilon_w) \end{aligned}$$

By Lemma 5.1, $E(\mathbf{H}_w) = \mathbf{0}$ if and only if $\phi = \phi^*$. Thus, $E[\hat{\alpha}(y, \mathbf{F})] = (\mathbf{I} + \mathbf{FC})\alpha$ if and only if $\phi = \phi^*$. Furthermore, it is obvious that $E[\hat{\alpha}(y, \mathbf{F})] = \alpha$ if and only if $\mathbf{FC} = \mathbf{0}$, i.e. if and only if $\mathbf{F} = \mathbf{0}$ since $\mathbf{C}$ is nonsingular. This completes the proof of the theorem.

Now, the variance-covariance matrix of any linear unbiased estimator will be derived.

Theorem 5.4 *Let* $\hat{\alpha}(y, \mathbf{0}) = \mathbf{C}^{-1}(\mathbf{y}_w + \mathbf{H}_w y)$ *be a linear unbiased estimator of* α *under randomization procedure, then the variance-covariance matrix of* $\hat{\alpha}(y)$ *has the following form:*

$$V(\hat{\alpha}(y)) = \mathbf{C}^{-1}E[\mathbf{H}_w \eta\eta' \mathbf{H}'_w](\mathbf{C}')^{-1} + \sigma^2(\mathbf{C}'\mathbf{C})^{-1}$$

where $\eta = \beta - \gamma$

Proof:

$$
\begin{aligned}
V(\hat{\alpha}(\gamma)) &= E\left[(\hat{\alpha} - E(\hat{\alpha}))(\hat{\alpha} - E(\hat{\alpha}))'\right] \\
&= E\left[(\mathbf{C}^{-1}\mathbf{y}_w - \mathbf{C}^{-1}\mathbf{H}_w\gamma - \alpha)(\mathbf{C}^{-1}\mathbf{y}_w - \mathbf{C}^{-1}\mathbf{H}_w\gamma - \alpha)'\right] \\
&= E\left[(\mathbf{C}^{-1}\mathbf{H}_w(\beta - \gamma) + \mathbf{C}^{-1}\varepsilon_w)(\mathbf{C}^{-1}\mathbf{H}_w(\beta - \gamma) + \mathbf{C}^{-1}\varepsilon_w)'\right] \\
&= \mathbf{C}^{-1}E\left[(\mathbf{H}_w\eta + \varepsilon_w)(\eta'\mathbf{H}_w' + \varepsilon_w')\right](\mathbf{C}')^{-1} \\
&= \mathbf{C}^{-1}E\left[(\mathbf{H}_w\eta\eta'\mathbf{H}_w' + \mathbf{H}_w\eta\varepsilon_w' + \varepsilon_w\eta'\mathbf{H}_w' + \varepsilon_w\varepsilon_w')\right](\mathbf{C}')^{-1}
\end{aligned}
$$

Since $\mathbf{H}_w$ is independent of ε_w, $E[\mathbf{H}_w\eta\varepsilon_w'] = E[\varepsilon_w\eta'\mathbf{H}_w] = \mathbf{0}$, thus

$$
V(\hat{\alpha}(\gamma)) = \mathbf{C}^{-1}E\left[\mathbf{H}_w\eta\eta'\mathbf{H}_w'\right](\mathbf{C}')^{-1} + \sigma^2(\mathbf{C}'\mathbf{C})^{-1} \qquad (5.3)
$$

In (5.3), σ^2 and $\mathbf{C}$ are constants, so $Var(\hat{\alpha}_i(\gamma))$ is minimized when the diagonal elements of $E(\mathbf{H}_w\eta\eta'\mathbf{H}'_w)$ are minimized. Since

$$\mathbf{H}_w\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{H}'_w = \begin{bmatrix} (\sum_{j=1}^{N-S} h_{1j}\eta_j)^2 & (\sum_{j=1}^{N-S} h_{1j}\eta_j)(\sum_{j=1}^{N-S} h_{2j}\eta_j) & \cdots & (\sum_{j=1}^{N-S} h_{1j}\eta_j)(\sum_{j=1}^{N-S} h_{Sj}\eta_j) \\ \cdot & (\sum_{j=1}^{N-S} h_{2j}\eta_j)^2 & \cdot & (\sum_{j=1}^{N-S} h_{2j}\eta_j)(\sum_{j=1}^{N-S} h_{Sj}\eta_j) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & (\sum_{j=1}^{N-S} h_{Sj}\eta_j)^2 \end{bmatrix}$$

$$(\sum_{j=1}^{N-S} h_{ij}\eta_j)^2 \geq 0 \quad \text{for all } 1 \leq i \leq S$$

$$E\left[(\sum_{j=1}^{N-S} h_{ij}\eta_j)^2\right] \geq 0 \quad \text{for all } 1 \leq i \leq S$$

Thus, $Var(\alpha_i(\gamma))$ is minimized iff $(\sum_{j=1}^{N-S} h_{ij}\eta_j)^2 = 0$, iff $(\sum_{j=1}^{N-S} h_{ij}\eta_j) = 0$ for all $1 \leq i \leq S$. But $S < N - S$ and h_{ij} are not all zero, therefore, all η_j have to be zero for all j, which implies $\boldsymbol{\eta} = \boldsymbol{\beta} - \boldsymbol{\gamma} = \mathbf{0}$. Now, we can conclude that in the class of all linear unbiased estimators, $\hat{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = \mathbf{C}^{-1}(\mathbf{y}_w - \mathbf{H}_w\boldsymbol{\beta})$ is the best linear unbiased estimator (BLUE) of $\boldsymbol{\alpha}$.

5.1.2 ESTIMATION UNDER GENERALIZED INVERSE THEORY

A. Some results on generalized inverse

Consider a general matrix **A** of size m×n and rank k which is less than or equal to min(m,n), the solution of the linear equation $\mathbf{Ax} = \mathbf{y}$, with **y** being a m×1 column vector, is of the form $\mathbf{x} = \mathbf{Gy}$, where **G** is called a generalized inverse (g-inverse) of **A** . The definitions of a g-inverse are as follows:

> ***Definition 1:*** Let **A** be an m×n matrix of arbitrary rank. A generalized inverse of **A** is an n×m matrix **G** such that $\mathbf{x} = \mathbf{Gy}$ is a solution of $\mathbf{Ax} = \mathbf{y}$ for any **y** which makes the equation consistent.

> ***Definition 2:*** A g-inverse of **A** of size m×n is a matrix $\mathbf{A}^-$ of size n×m such that

$$\mathbf{AA}^-\mathbf{A} = \mathbf{A}$$

Let $\mathcal{g}(\mathbf{A})$ denote the class of all g-inverses of matrix **A** under definition 2. A theorem about g-inverse, derived by Rao (1971), that will be used in later work is stated here without proof.

> Theorem 5.5 *If* $(\mathbf{A}'\mathbf{A})^- \in \mathcal{g}(\mathbf{A}'\mathbf{A})$, *then* $(\mathbf{A}'\mathbf{A})^-\mathbf{A}' \in \mathcal{g}(\mathbf{A})$.

If a matrix is in a partitioned form, the following theorem derived by Rohde(1965) will be used; it is stated here without proof.

> Theorem 5.6 *If* **P** *is a positive semi-definite matrix in partitioned form:*

$$\mathbf{P} = \left[\begin{array}{c:c} \mathbf{P}_1 & \mathbf{P}_2 \\ \hdashline \mathbf{P}_2' & \mathbf{P}_3 \end{array}\right]$$

then a generalized inverse of $\mathbf{P}$ *is*

$$\mathbf{P}^- = \left[\begin{array}{c:c} \mathbf{P}_1^- + \mathbf{P}_1^-\mathbf{P}_2\mathbf{W}^-\mathbf{P}_2'\mathbf{P}_1^- & -\mathbf{P}_1^-\mathbf{P}_2\mathbf{W}^- \\ \hdashline -\mathbf{W}^-\mathbf{P}_2'\mathbf{P}_1^- & \mathbf{W}^- \end{array}\right]$$

where

$$\mathbf{P}_1^- \in \mathcal{g}(\mathbf{P}_1)$$
$$\mathbf{W} = \mathbf{P}_3 - \mathbf{P}_2'\mathbf{P}_1^-\mathbf{P}_2$$
$$\mathbf{W}^- \in \mathcal{g}(\mathbf{W})$$

B. Type-g estimator under randomized fractional replicates

For each w = 0,1, ... , M-1, the normal equations for a fractional replicate utilizing block $\mathbf{X}_w$ are given by equation (3.4). The system (3.4) is consistent if and only if $\mathbf{C}_{w,N}\mathbf{C}_{w,N}^-\mathbf{y} = \mathbf{y}$. Since the rank of $\mathbf{C}_{w,N}$ equals the number of its row, $\mathbf{C}_{w,N}\mathbf{C}_{w,N}^- = \mathbf{I}_S$ (Rao & Mitra, 1971). Hence, the normal equations (3.4) are indeed consistent. In order to solve equations (3.4), a generalized inverse of $\mathbf{C}_{w,N}'\mathbf{C}_{w,N}$ has to be derived. Since

$$\mathbf{C}_{w,N}'\mathbf{C}_{w,N} = (\mathbf{C}\,|\,\mathbf{H}_w)'(\mathbf{C}\,|\,\mathbf{H}_w) = \left[\begin{array}{c:c} \mathbf{C}'\mathbf{C} & \mathbf{C}'\mathbf{H}_w \\ \hdashline \mathbf{H}_w'\mathbf{C} & \mathbf{H}_w'\mathbf{H}_w \end{array}\right]$$

it is a singular square matrix in partitioned form. Let $\mathbf{P} = \mathbf{C}_{w,N}'\mathbf{C}_{w,N}$, then $\mathbf{W} = \mathbf{H}_w'\mathbf{H}_w - \mathbf{H}_w'\mathbf{C}(\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}'\mathbf{H}_w = \mathbf{0}$. Thus, by Theorem 5.6

$$\mathbf{P}^{-} = \begin{bmatrix} (\mathbf{C}'\mathbf{C})^{-1} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \mathbf{0} \end{bmatrix}$$

Using Theorem 5.5, a g-inverse of $\mathbf{C}_{w,N}$ is

$$\mathbf{C}_{w,N}^{-} = \mathbf{P}^{-}\mathbf{C}_{w,N}' = \begin{bmatrix} \mathbf{C}^{-1} \\ \cdots \\ \mathbf{0} \end{bmatrix}$$

Consequently, an estimator of α is as follows:

$$\hat{\alpha}_g = \mathbf{C}_{w,N}^{-}\mathbf{y}_w = \mathbf{C}^{-1}\mathbf{y}_w \tag{5.4}$$

Since $E(\hat{\alpha}_g) = \alpha$, the type-g estimator $\hat{\alpha}$ is an unbiased estimator. This particular estimator coincides with the estimator obtained under the classical analysis where nuisance parameters are assumed zero.

This section is concluded with a comparison about the variance-covariance matrices of the type-g estimator $\hat{\alpha}_g$ and the classical estimator of α. We shall first derive the variance-covariance matrix for $\hat{\alpha}_g$ in the following theorem.

Theorem 5.7 *The variance-covariance matrix of the type-g estimator of α is given by*

$$V(\hat{\alpha}_g) = \mathbf{C}^{-1}E\left[\mathbf{H}_w\beta\beta'\mathbf{H}_w'\right](\mathbf{C}')^{-1} + \sigma^2(\mathbf{C}'\mathbf{C})^{-1}$$

Proof:

$$
\begin{aligned}
V(\hat{\alpha}_g) &= E\left[(\hat{\alpha}_g - E(\hat{\alpha}_g))(\hat{\alpha}_g - E(\hat{\alpha}_g))'\right] \\
&= E\left[(\mathbf{C}^{-1}\mathbf{y}_w - \alpha)(\mathbf{C}^{-1}\mathbf{y}_w - \alpha)'\right] \\
&= \mathbf{C}^{-1}E\left[(\mathbf{H}_w\beta + \varepsilon)(\beta'\mathbf{H}_w' + \varepsilon')\right](\mathbf{C}')^{-1} \\
&= \mathbf{C}^{-1}E\left[\mathbf{H}_w\beta\beta'\mathbf{H}_w'\right](\mathbf{C}')^{-1} + \sigma^2(\mathbf{C}'\mathbf{C})^{-1}
\end{aligned}
$$

Now let $\hat{\alpha} = \mathbf{C}^{-1}\mathbf{y}$ be the estimator obtained under the classical analysis, then the variance-covariance matrix of $\hat{\alpha}$ is

$$\mathbf{V}(\hat{\alpha}) = \sigma^2(\mathbf{C}'\mathbf{C})^{-1}$$

Since $(\mathbf{C}'\mathbf{C})^{-1}$ is a diagonal matrix $\text{cov}(\hat{\alpha}_i, \hat{\alpha}_j) = 0$, and $\text{var}(\hat{\alpha}_i) = \sigma^2\sum_j c_{ij}^2$.

Let $\mathbf{z} = \mathbf{C}^{-1}\mathbf{H}_w\beta$, where $\mathbf{z}' = (z_1, z_2, \ldots, z_S)$, then

$$(\mathbf{C}^{-1}\mathbf{H}_w\beta)(\mathbf{C}^{-1}\mathbf{H}_w\beta)' = \mathbf{z}\mathbf{z}' = \begin{bmatrix} z_1^2 & z_1z_2 & \cdots & z_1z_S \\ z_2z_1 & z_2^2 & \cdots & z_2z_S \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ z_Sz_1 & z_Sz_2 & \cdots & z_S^2 \end{bmatrix}$$

Thus,

$$\text{var}((\hat{\alpha}_l)_g) = \sigma^2\sum_j c_{ij}^2 + E(z_i^2) \geq \sigma^2\sum_j c_{ij}^2 = \text{var}(\hat{\alpha}_l), \text{ and}$$

$$\text{cov}((\hat{\alpha}_i)_g,\ (\hat{\alpha}_j)_g) = E(z_i z_j)$$

We now can conclude that although $\hat{\alpha}_g$ is unbiased, problems such as larger variance and correlated estimators may arise by using the type-g estimator. We shall further investigate these problem by a simulation study in a later section.

5.2 GENERAL ASYMMETRIC SYSTEM

In this section, results of section in 5.1 are extended to the general asymmetric factorial system. Following immediately from Lemma 5.1, a lemma which will be used to derive linear unbiased estimator as well as type-g unbiased estimator is now developed:

Lemma 5.8 $E_\phi(\mathbf{H}_{w,N}) = \mathbf{0}$ *if and only if* $\phi = \phi^*$.

Proof: From section 3.2, the matrix $\mathbf{H}_{w,N}$ is defined as

$$\mathbf{H}_{w,N} = \left[\mathbf{H}^1_{w,N} | \mathbf{H}^2_{w,N} | \ldots | \mathbf{H}^t_{w,N}\right] \otimes \mathbf{C}_{N_{r+1}} \otimes \ldots \otimes \mathbf{C}_{N_n}$$

where

$$\begin{aligned}
\mathbf{H}^1_{w,N} &= \mathbf{C}^1 \otimes \ldots \otimes \mathbf{C}^{r-1} \otimes \mathbf{H}_{w,N_r} \\
\mathbf{H}^2_{w,N} &= \mathbf{C}^1 \otimes \ldots \otimes \mathbf{H}_{w,N_{r-1}} \otimes \mathbf{C}^r \\
&\cdots\cdots \\
\mathbf{H}^r_{w,N} &= \mathbf{H}_{w,N_1} \otimes \ldots \otimes \mathbf{C}^{r-1} \otimes \mathbf{C}^r \\
&\cdots\cdots \\
\mathbf{H}^t_{w,N} &= \mathbf{H}_{w,N_1} \otimes \mathbf{H}_{w,N_2} \otimes \ldots \otimes \mathbf{H}_{w,N_r}
\end{aligned}$$

By Lemma 5.1

$$
\begin{aligned}
E[\mathbf{H}^1_{w,N}] &= \mathbf{C}^1 \otimes \ldots \otimes \mathbf{C}^{r-1} \otimes E[\mathbf{H}_{w,N_r}] &= \mathbf{0} \\
E[\mathbf{H}^2_{w,N}] &= \mathbf{C}^1 \otimes \ldots \otimes E[\mathbf{H}_{w,Nr-1}] \otimes \mathbf{C}^r &= \mathbf{0} \\
&\cdot \qquad\qquad\qquad\qquad \cdot \\
&\cdot \qquad\qquad\qquad\qquad \cdot \\
E[\mathbf{H}^r_{w,N}] &= E[\mathbf{H}_{w,N_1}] \otimes \ldots \otimes \mathbf{C}^{r-1} \otimes \mathbf{C}^r &= \mathbf{0} \\
&\cdot \qquad\qquad\qquad\qquad \cdot \\
&\cdot \qquad\qquad\qquad\qquad \cdot \\
E[\mathbf{H}^t_{w,N}] &= E[\mathbf{H}_{w,N_1}] \otimes \mathbf{H}_{w,N_2} \otimes \ldots \otimes \mathbf{H}_{w,N_r} &= \mathbf{0}
\end{aligned}
$$

if and only if $\phi = \phi^*$. Therefore, $E[\mathbf{H}_{w,N}] = \mathbf{0}$ if and only if $\phi = \phi^*$.

By Lemma 5.8, one can easily prove that every linear unbiased estimator of α under $\phi = \phi^*$ has the form of

$$\hat{\alpha}(y) = \mathbf{C}^{-1}(\mathbf{y}_w - \mathbf{H}_{w,N}\gamma)$$

and the type-g unbiased estimator has the form of

$$\hat{\alpha}_g = \mathbf{C}^{-1}\mathbf{y}_w$$

And the variance-covariance matrices for $\hat{\alpha}(y)$ and $\hat{\alpha}_g$ are

$$V(\hat{\alpha}(y)) = \mathbf{C}^{-1}E[\mathbf{H}_{w,N}\eta\eta'\mathbf{H}'_{w,N}](\mathbf{C}')^{-1} + \sigma^2(\mathbf{C}'\mathbf{C})^{-1}$$

$$V(\hat{\alpha}_g) = \mathbf{C}^{-1}E[\mathbf{H}_{w,N}\beta\beta'\mathbf{H}'_{w,N}](\mathbf{C}')^{-1} + \sigma^2(\mathbf{C}'\mathbf{C})^{-1}$$

5.3 A MONTE CARLO COMPARISON OF THE TWO PROCEDURES

One of the relevant aspects in the comparison between the randomization procedure and the "classical" fractional replication designs is that the classical(or fixed) design may give biased estimates. However, the randomization procedure might remove bias at the expense of increased variance. Monte carlo techniques are applied here to examine the variance and the bias from those two procedures.

Two approaches are used to make the comparison. One criterion is to look at $V_p + B_p^2$, where V_p is the variance using procedure p, and B_p is the bias using procedure p . Another approach is to adopt a "closeness" criterion. We shall look at

$$\text{Closeness} = \text{Prob}\{|\hat{\alpha} - \alpha| \leq \lambda |\alpha|\},$$

where $\hat{\alpha}$ is the estimator and $0 \leq \lambda \leq 1$. The parameter λ measures how important it is to be close to α.

A simulation bases on univariate normality was conducted for the $3^2 \times 4^2$ factorial with 2 factors A and B at three levels and 2 factors C and D at four levels. The choice of the fraction is determined by the defining contrast:

$$I = AB = CD^3 = ABCD^3$$

Using only twelve treatment combinations, we can estimate twelve parameters. The twelve parameters that are chosen in this simulation study belong to main effects A, C and interaction A×C. The true values for the parameters of interest (or chosen parameters) are listed in Table 5.1. The standard deviation of $y_{x_{11}x_{12}x_{21}x_{22}}$ for any treatment

combination is $\sigma = 5.0$. An IMSL subroutine RNNOA is used to generate 1,500 random samples for each set of treatment combinations.

In the case of fixed fractional replication designs, the biases of the estimators of the chosen parameters, expressed by

$$\text{Relative bias} = \left| \frac{\hat{\alpha} - \alpha}{\alpha} \right|,$$

are given in Table 5.2. And the mean-squared-errors of $\alpha's$ from both procedures are given in Tables 5.3 and 5.4. As indicated in the tables, none of the fixed designs gives uniformly smaller mean-squared-error of the estimates for all chosen parameters when compared to those from the randomization procedure. Furthermore, the use of randomization procedure has resulted in moderate mean-squared-errors for all estimates, but has introduced correlations between the estimates. These correlations, as shown in Table 5.5, will render tests of significance somewhat tedious.

Some calculations of the probability, for $\lambda = 0.2$, are shown in Table 5.6. This table illustrates the same results as given in Table 5.3. Another criteria, $\sum MSE(\hat{\alpha}_i)$, is introduced here to look at the overall performance of each of the designs. The results are listed as follows:

Design	$\sum MSE(\hat{\alpha}_i)$
Fixed 0	110.001
Fixed 3	143.933
Fixed 9	144.336
Fixed 8	150.248
Fixed 11	153.218
Fixed 6	154.372
Rand. Proc.	158.734
Fixed 4	159.918
Fixed 7	161.078
Fixed 5	162.710
Fixed 2	163.694
Fixed 10	166.372
Fixed 1	202.202

As one can see from the table above, extreme value of sum of mean square error occurs while fixed procedure were utilized. Therefore, if information about the nuisance parameters is not available, there is no way to decide which fixed design is a good one and which is a bad one. Thus, the randomization procedure guards against a bad choice of a design when the nuisance parameters are unknown.

5.4 k-in-M FRACTIONAL DESIGN

So far, we have considered a fractional replicate of an $s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_n^{m_n}$ experiment which contains $S = s_1^{m_1 - e_1} \times \ldots \times s_r^{m_r - e_r} \times s_{r+1}^{m_{r+1}} \times \ldots \times s_n^{m_n}$ treatment combinations. It is not necessary that we restrict ourselves in this way, in fact, we can make up a k-in-M fractional design, in which $M = s_1^{e_1} \times s_2^{e_2} \times \ldots \times s_r^{e_r}$ and k is any number between 1 and $\max(s_1, s_2, \ldots, s_r)$, and $k \neq s_i$. That is, we shall choose at random, with or without replacement, subsets of treatment combinations, out of M constructed by the CRT according to a given defining contrast.

If k subsets are chosen, then

$$\mathbf{C}_{w^*, N} = \left[\mathbf{C}^* \mid \mathbf{H}_w^*\right] = \begin{bmatrix} \mathbf{C} & \mathbf{H}_{w_1} \\ \mathbf{C} & \mathbf{H}_{w_2} \\ . & . \\ . & . \\ \mathbf{C} & \mathbf{H}_{w_k} \end{bmatrix}$$

where $\mathbf{C}$ and $\mathbf{H}_{w_i}$ are defined as in section 3.2.2, where only one subset was selected.

A. Without replacement:

If k subsets are chosen without replacement, $\mathbf{H}_{w_i} \neq \mathbf{H}_{w_j}$ for $i \neq j$. Under $\phi = \phi^*$, $E(\mathbf{H}_{w_i}) \neq \mathbf{0}$ for $2 \leq i \leq k$, which implies that $E[\mathbf{H}_w^*] \neq \mathbf{0}$. Therefore, no unbiased estimator can be found under $\phi = \phi^*$ when subsets are chosen without replacement.

B. With replacement:

If k subsets are selected at random with replacement, then $E[\mathbf{H}_{w_i}] = \mathbf{0}$. for all i. Therefore, $E[\mathbf{H}_w^*] = \mathbf{0}$.

The least-squares normal equations can be written as

$$\mathbf{C}^*\hat{\boldsymbol{\alpha}} + \mathbf{H}_w^*\hat{\boldsymbol{\beta}} = \mathbf{y}_w$$

Thus, the class of linear estimators of $\boldsymbol{\alpha}$ is

$$\boldsymbol{\alpha} = (\mathbf{C}^*)^-[\mathbf{y}_w - \mathbf{H}^*\gamma]$$

where γ is any fixed vector. Since $\mathbf{C}^*$ is an kS×S matrix of rank S, by Rao and Mitra (1971), $(\mathbf{C}^*)^- = [(\mathbf{C}^*)'\mathbf{C}^*]^{-1}(\mathbf{C}^*)'$. Therefore,

$$E(\hat{\boldsymbol{\alpha}}(\gamma)) = \boldsymbol{\alpha}$$

$$V(\hat{\boldsymbol{\alpha}}(\gamma)) = (\mathbf{C}^*)^- E[\mathbf{H}_w^*\eta\eta'(\mathbf{H}_w^*)'][((\mathbf{C}^*)^-)' + \sigma^2[(\mathbf{C}^*)'\mathbf{C}^*]^{-1}$$

A simulation study on a 2-in-12 fractional factorial was conducted for the same $3^2 \times 4^2$ factorial as mentioned in Section 5.3, and the results are shown in Table 5.7 and 5.8. As indicated in Table 5.7, the unbiased estimators of α_i , derived under randomization procedure, only give a slight increase in the magnitude of variance for some i. Table 5.8 shows that, similar to that in the 1-in-12 case, the randomization procedure always gives moderate sum of the mean-squared-errors of the estimators.

Table 5.1 True values of the chosen parameters

Chosen Parameters		True values
α_0	M	2.00
α_1	$(A)^1$	-3.53
α_2	$(A)^2$	9.20
α_3	$(C)^1$	-4.45
α_4	$(C)^2$	9.01
α_5	$(C)^3$	5.02
α_6	$(A)^1(C)^1$	2.37
α_7	$(A)^2(C)^1$	0.86
α_8	$(A)^1(C)^2$	3.83
α_9	$(A)^2(C)^2$	0.92
α_{10}	$(A)^1(C)^3$	2.06
α_{11}	$(A)^2(C)^3$	0.82

Table 5.2 Relative bias of the estimators of the chosen parameters

	Fixed design											
Parameters	0	1	2	3	4	5	6	7	8	9	10	11
α_0	0.522	1.385	0.649	0.420	0.287	0.421	0.487	0.392	0.681	0.589	0.326	0.576
α_1	0.469	0.196	0.626	0.428	0.090	0.618	0.456	0.144	0.621	0.448	0.148	0.588
α_2	0.276	0.033	0.238	0.279	0.053	0.243	0.275	0.043	0.241	0.272	0.031	0.244
α_3	0.609	0.168	0.281	0.171	0.637	0.161	0.252	0.194	0.617	0.138	0.250	0.167
α_4	0.173	0.053	0.386	0.151	0.194	0.054	0.372	0.148	0.174	0.035	0.395	0.160
α_5	0.187	0.163	0.318	0.649	0.209	0.168	0.321	0.650	0.195	0.142	0.305	0.643
α_6	0.699	0.292	0.933	0.637	0.134	0.920	0.679	0.215	0.926	0.667	0.221	0.876
α_7	2.955	0.358	2.542	2.981	0.565	2.602	2.941	0.460	2.576	0.912	0.327	2.615
α_8	0.207	1.246	0.703	0.225	0.123	0.711	0.084	0.131	0.262	0.094	1.213	0.280
α_9	1.001	2.218	3.028	0.883	0.465	2.893	3.471	0.435	0.340	0.577	2.205	0.426
α_{10}	0.797	2.273	0.139	0.609	1.204	0.776	0.866	2.374	0.189	0.560	1.128	0.861
α_{11}	0.076	3.139	2.020	2.511	2.131	3.499	0.007	3.006	1.859	0.495	2.156	3.471

Table 5.3 Mean-squared-error of the estimators of the chosen parameters

	Fixed design											
Parameters	0	1	2	3	4	5	6	7	8	9	10	11
α_0	3.243	9.675	3.807	3.769	2.487	2.798	3.070	2.651	3.853	3.451	2.469	3.433
α_1	14.685	13.142	17.607	14.621	13.073	17.652	14.688	12.686	16.920	14.248	12.653	17.028
α_2	15.911	9.863	14.005	15.801	10.002	14.339	16.087	9.647	13.954	15.510	9.545	14.680
α_3	15.646	8.634	10.231	9.099	16.515	9.050	9.989	9.090	15.409	8.718	9.152	8.457
α_4	10.867	8.591	20.294	10.465	11.336	8.897	19.756	9.585	10.948	8.267	20.971	10.559
α_5	9.600	8.899	10.552	19.139	9.533	9.293	10.745	18.866	9.629	8.619	11.273	18.399
α_6	11.668	30.280	14.941	11.534	30.453	16.014	12.468	14.317	16.941	13.094	13.538	14.243
α_7	14.603	13.285	11.305	10.229	13.659	15.398	9.752	9.691	15.464	12.902	9.320	10.733
α_8	12.718	34.961	19.452	13.332	12.591	19.955	12.973	12.498	13.768	12.336	34.541	13.305
α_9	10.097	13.519	17.130	9.862	9.804	16.639	19.979	9.844	9.853	20.462	13.727	9.185
α_{10}	15.269	35.029	12.369	13.391	18.012	14.985	15.517	36.346	11.999	13.676	16.847	16.189
α_{11}	9.533	16.324	12.001	13.691	12.433	17.690	9.348	15.857	11.510	13.083	12.336	17.007

Table 5.4 Comparison of the mean-squared-error of the estimators from the fixed design and the randomization procedure

Parameters	Max.(Fixed Design)	Rand. Procedure	Min.(Fixed Design)
α_0	9.675 (1)	3.572	2.469(10)
α_1	17.652 (5)	14.901	12.653(10)
α_2	16.087 (6)	12.890	9.545(10)
α_3	16.515 (4)	10.889	8.457(11)
α_4	20.971(10)	12.723	8.591 (1)
α_5	19.139 (3)	12.112	8.899 (1)
α_6	30.453 (4)	16.639	11.534 (3)
α_7	15.464 (8)	12.141	9.320(10)
α_8	34.961 (1)	17.693	12.336 (9)
α_9	20.462 (9)	13.323	9.804 (4)
α_{10}	36.346 (7)	18.874	11.999 (8)
α_{11}	17.690 (5)	12.984	9.748 (6)

Table 5.5 Variance-covariance matrix of the estimators derived from the randomization procedure

Parameters	α_0	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}
α_0	3.572	-0.657	-0.027	0.415	0.279	0.102	1.666	0.673	1.854	0.532	1.620	0.759
α_1		14.896	2.753	0.396	0.006	-0.295	-1.189	0.839	-1.207	1.593	-1.104	0.722
α_2			14.005	-0.013	-0.243	0.135	0.745	-1.267	0.983	-3.175	0.847	-1.611
α_3				10.889	-1.783	-0.735	-1.045	-1.147	0.322	0.702	1.650	0.067
α_4					12.715	-1.894	1.980	0.905	-1.560	-0.643	1.834	1.290
α_5						12.112	0.959	1.078	0.074	0.201	-1.669	-1.506
α_6							16.637	0.764	1.947	-0.363	1.468	0.294
α_7								12.141	0.340	0.962	-0.174	0.555
α_8									17.665	0.305	1.606	-0.065
α_9										13.323	-0.512	0.553
α_{10}											18.868	1.748
α_{11}												12.986

Table 5.6 Comparison of the closeness of the estimators from the fixed design and the randomization procedure

Parameters	Max.(Fixed Design)	Rand. Procedure	Min.(Fixed Design)
α_0	0.207(10)	0.171	0.029 (1)
α_1	0.172 (7)	0.144	0.115 (8)
α_2	0.456 (4)	0.391	0.333 (3)
α_3	0.247 (3)	0.226	0.144 (4)
α_4	0.460 (5)	0.379	0.234(10)
α_5	0.264 (0)	0.222	0.152 (3)
α_6	0.113 (5)	0.101	0.039 (1)
α_7	0.048(10)	0.042	0.033 (5)
α_8	0.183 (7)	0.143	0.069(10)
α_9	0.053 (0)	0.032	0.025 (2)
α_{10}	0.097 (8)	0.073	0.037 (1)
α_{11}	0.043 (4)	0.036	0.027 (1)

Table 5.7 Results from randomization procedure for a 2-in-12 case

Parameters	Closeness	Variance(Fixed Design)		Relative Bias
α_0	0.240	1.788	(1.041)	0.0138
α_1	0.182	7.150	(6.250)	0.0076
α_2	0.536	6.457	(4.687)	0.0105
α_3	0.250	5.749	(4.167)	0.0003
α_4	0.552	5.541	(4.167)	0.0094
α_5	0.324	5.827	(4.167)	0.0206
α_6	0.092	7.736	(6.250)	0.0009
α_7	0.044	5.813	(4.687)	0.0173
α_8	0.228	7.900	(6.250)	0.0081
α_9	0.102	6.791	(4.687)	0.0301
α_{10}	0.108	8.252	(6.250)	0.0263
α_{11}	0.036	6.666	(4.687)	0.0443

Table 5.8 $\sum MSE(\hat{\alpha}_i)$ of the fixed designs and randomization procedure for a 2-in-12 fractional factorial (listed in ascending order)

Design	Sets	$\sum MSE(\hat{\alpha}_i)$	Design	Sets	$\sum MSE(\hat{\alpha}_i)$
Fixed 49	5,9	54.617	Fixed 39	4,5	71.310
Fixed 65	9,11	59.754	Fixed 64	9,10	71.365
Fixed 41	4,7	59.789	Fixed 44	4,10	71.532
Fixed 32	3,5	60.094	Fixed 6	0,6	71.844
Fixed 61	8,9	60.135	Fixed 50	5,10	72.108
Fixed 53	6,8	61.421	Fixed 18	1,8	72.188
Fixed 5	0,5	61.957	Fixed 25	2,6	72.263
Fixed 35	3,8	62.531	Fixed 12	1,2	72.318
Fixed 46	5,6	62.553	Fixed 16	1,6	73.055
Fixed 2	0,2	63.486	Fixed 9	0,9	73.651
Fixed 37	3,10	63.717	Fixed 21	1,11	73.686
Fixed 59	7,10	63.728	Fixed 36	3,9	74.309
Fixed 28	2,9	63.385	Fixed 13	1,3	74.919
Fixed 7	0,7	63.865	Fixed 47	5,7	75.022
Fixed 52	6,7	64.167	Fixed 4	0,4	75.525
Fixed 22	2,3	64.291	Rand. Proc.		75.670
Fixed 56	6,11	64.942	Fixed 30	2,11	76.057
Fixed 45	4,11	64.965	Fixed 27	2,8	76.138
Fixed 11	0,11	64.984	Fixed 42	4,8	76.564
Fixed 31	3,4	65.868	Fixed 54	6,9	78.095
Fixed 8	0,8	66.340	Fixed 60	7,11	78.776
Fixed 66	10,11	66.713	Fixed 34	3,7	79.524
Fixed 57	7,8	66.969	Fixed 48	5,8	79.696
Fixed 10	0,10	67.078	Fixed 19	1,9	80.205
Fixed 23	2,4	67.125	Fixed 15	1,5	80.872
Fixed 40	4,6	67.799	Fixed 29	2,10	82.412
Fixed 43	4,9	67.804	Fixed 55	6,10	82.443
Fixed 33	3,6	68.096	Fixed 51	5,11	82.823
Fixed 38	3,11	68.290	Fixed 14	1,4	84.901
Fixed 26	2,7	68.500	Fixed 24	2,5	85.203
Fixed 3	0,3	69.249	Fixed 62	8,10	87.931
Fixed 1	0,1	69.484	Fixed 20	1,10	89.117
Fixed 58	7,9	70.235	Fixed 17	1,7	96.192
Fixed 63	8,11	70.430			

CHAPTER 6

SUMMARY AND DISCUSSION

In the first part of this dissertation, the Chinese Remainder Theorem has been shown to be both simpler and more general than most of the existent methods for construction of confounding plans of mixed factorial experiments. Using the procedure described in this study as well as using pseudofactors, confounding plans may be found for any number of levels and any number of factors, and the effects are all mutually orthogonal. However, one still has to work within groups of factors each at the same number of levels. Therefore, there appears to be room for further exploration and study in the area of constructing confounded designs for mixed factorials when one wishes to confound some interaction components composed of factors with different number of levels.

The second goal of this research has been to study the randomization procedure for mixed factorial experiment without the classical assumption that nuisance parameters are all negligible. It is apparent from the development in section 3.1 that the randomization procedure can readily be applied using standard confounding methods.

Also presented was a procedure, based on tables of coefficients of orthogonal contrasts and Kronecker product, for writing down the design matrix for any desired set of factor effect estimates broken into the the usual single degree of freedom components, given any specified incomplete set of treatment combinations. Then the class of linear unbiased estimators and the type-g unbiased estimators of the parameters of interest were derived. Nevertheless, without the usual assumption of negligible nuisance parameters, one does not have freedom in choosing the set of parameters of interest. It

has to be generated by the effects or interaction components which are independent of those effects or interaction components in the defining contrast.

For comparison of the randomization procedure and the classical one (fixed procedure), simulation studies were conducted to compare the mean-squared-errors obtained under the two methods. If no information is available concerning nuisance parameters, the randomization procedure seems preferable since it not only gives us unbiased estimates but also guards against extremes in the mean-squared-error. This is particular true if one is equally interested in the parameters of interest.

Under certain conditions, for example, with the number of repetitions of every chosen treatment combination greater than one, one might be able to conduct a significance test of the nuisance parameters. In the case where fractional factorial experiments are used, for exploratory purposes, it seems a definite advantage to be able to test the nuisance parmeters. However, before conducting the test, one should first investigate the properties of these tests and distribution problems carefully.

REFERENCES

Addelman, S. (1962), "Orthogonal Main-Effect Plans for Asymmetrical Factorial Experiments", ***Technometrics***, 4(1):21-46.

Cochran, W. G. and Cox, G. M. (1957), ***Experimental Design***, 2nd. ed., John Wiley and Sons, New York.

Dey, A. (1985), ***Orthogonal Fractional Factorial Designs***, John Wiley and Sons, New York.

Ehrenfeld, S. and Zacks S. (1961), "Randomization and Factorial Experiments", ***Ann. Math. Stat.***, 32:270-297.

----- (1963). "Optimal Strategies in Factorial Experiments", ***Ann. Math. Stat.***, 34:780-791.

Graybill, F. A. (1976), ***Theory and Application of the Linear Model***, Duxbury Press, Massachusetts.

Kempthorne, O. (1973), ***Design and Analysis of Experiments***, John Wiley and sons, Inc., New York.

Lentner, M. M. (1967), "Application of Generalized Inverses for the Optimal Design and Analysis of Randomized Fractional Replicates", Unpublished doctoral dissertation, Kansas State University, Manhattan, Kansas.

----- (1970), "A Characterization of All Generalized Inverses of an Arbitrary Real Matrix" Statistical Laboratory Technical Report Series No. 19 , New Mexico State University.

Lin, P. K. H. (1986), "Using the Chinese Remainder Theorem in Constructing Confounded Designs for Mixed Factorial Experiments", ***Comm. Stat-Theor. Meth.***, 15(4):1389-1398.

----- (1987a), "A Simple Method for Constructing Confounded Designs for Mixed Factorial Experiments", ***Comm. Stat-Theor. Meth.***, 16(2):407-420.

----- (1987b), "Confounding in Mixed Factorial Experiments through Isomorphisms" *Comm. Stat-Theor. Meth.*, 16(2):421-429.

Margolin, B. H. (1969), "Resolution-IV Fractional Factorial Designs", *J. R. Stat. Soc. B.*, 31:514-523.

Mukerjee, R. (1981), "Construction of Effect-Wise Orthogonal Factorial Designs" *J. of Stat. Plan. Inf.*, 5:221-229.

Niven, I. (1960), *An Introduction of the Theory of Numbers*, John Wiley and Sons, New York.

Raktoe, B.L. (1969), "Combining Elements from Distinct Finite Fields in Mixed Factorials", *Ann. Math. Stat.*, 40:498-504.

Rao, C. R. (1962), "A note on a generalized inverse", *J. R. Stat. Soc. B.*, 24:152-158.

Rao, C. R. and Mitra, S. K. (1971), *Generalized Inverse of Matrices and its Applications*, John Wiley and Sons, New York.

Rohde, C. A. (1965), "Generalized Inverses of Partitioned Matrices", *J. SIAM*, 13(4):1033-1035.

Satterthwaite, F. E. (1959), "Random Balance Experiment", *Technometrics*, 1:111-138.

Sihota, S. S. and Banerjee, K. S. (1981), "On the Algebraic Structures in the Construction of Confounding Plans in Mixed Factorial Designs on the Lines of White and Hultiquist", *J. Amer. Stat. Assoc.*, 76:996-1001.

Voss, D.T. (1986), "On Generalizations of the Classical Method of Confounding to Aymmetric Factorial Experiments", *Comm. Stat-Theor. Meth.*, 15(4):1299-1314.

White, D. and Hultquist, R. A. (1965), "Construction of Confounding Plans for Mixed Factorials", *Ann. Math. Stat.*, 36:1256-1271.

Worthley, R. and Banerjee, K. S. (1974), "A General Approach to Confounding Plans in Mixed Factorial Experiments when the Number of Levels of a Factor is any Positive Integer", *Ann. Stat.*, 2:579-585.

Zacks S. (1963), "On a Complete Class of Linear Unbiased Estimators for Randomized Factorial Experiments", ***Ann. Math. Stat.***, 34:769-779.

----- (1964), "Generalized Least Squares Estimators for Randomized Factorial Experiments", ***Ann. Math. Stat.***, 35:696-704.

APPENDIX

```
      DIMENSION A4(4,4),A8(8,8),A9(9,9),B(10),C(10),D1(5),D2(5),
     *  E(10),M(10),M4(4,4),M8(8,8),M9(9,9),P(10),X(10,10,10)
      INTEGER A4,A8,A9,B,C,C1,D,D1,D2,D3,E,P,R,SD,SUM,SUM1,SUM2,SUM3,
     *  T,TP,W,X,Z1,Z2,Z3
C
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C     * N IS THE NUMBER OF SYMMETRIC FACTORIAL                       *
C     * R IS THE NUMBER OF SYMMETRIC FACTORIAL UTILIZED IN           *
C     *   DETERMINING THE CONGRUENCE                                 *
C     * P(I): NUMBER OF LEVEL OF THE I(TH) SYMMETRIC FACTORIAL       *
C     * M(I): NUMBER OF FACTORS IN THE I(TH) SYMMETRIC FACTORIAL     *
C     * E(I): NUMBER OF COMPONENTS BEING CONFOUNDED IN THE           *
C     *       I(TH) SYMMETRIC FACTORIAL                              *
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C
C    DATA (P(I),I=1,N)/     /
C    DATA (M(I),I=1,N)/     /
C    DATA (E(I),I=1,R)/     /
C
C     ** EXAMPLE: 3*3*3*4*4*4*5*5 FACTORIAL **
C
      PARAMETER (N=3)
      PARAMETER (R=3)
```

```
      DATA (P(I),I=1,N)/3,4,5/
      DATA (M(I),I=1,N)/3,3,2/
      DATA (E(I),I=1,R)/2,2,1/
C
C     * * * * * * * * * * * * * * * * * * * * * * * *
C     *   ADDITION AND MULTIPLICATION TABLES FOR    *
C     *   GALOIS FIELDS (4),(8), AND (9)            *
C     * * * * * * * * * * * * * * * * * * * * * * * *
C
      DATA ((A4(I,J),J=1,4),I=1,4)/0,1,2,3,1,0,3,2,2,3,0,1,3,2,1,0/
      DATA ((M4(I,J),J=1,4),I=1,4)/0,0,0,0,0,1,2,3,0,2,3,1,0,3,1,2/
      DATA ((A8(I,J),J=1,8),I=1,8)/0,1,2,3,4,5,6,7,1,0,3,2,5,4,7,6,
     *                             2,3,0,1,6,7,4,5,3,2,1,0,7,6,5,4,
     *                             4,5,6,7,0,1,2,3,5,4,7,6,1,0,3,2,
     *                             6,7,4,5,2,3,0,1,7,6,5,4,3,2,1,0/
      DATA ((M8(I,J),J=1,8),I=1,8)/0,0,0,0,0,0,0,0,0,1,2,3,4,5,6,7,
     *                             0,2,4,6,5,7,1,3,0,3,6,5,1,2,7,4,
     *                             0,4,5,1,7,3,2,6,0,5,7,2,3,6,4,1,
     *                             0,6,1,7,2,4,3,5,0,7,3,4,6,1,5,2/
      DATA ((A9(I,J),J=1,9),I=1,9)/0,1,2,3,4,5,6,7,8,1,2,0,4,5,3,7,8,6,
     *                             2,0,1,5,3,4,8,6,7,3,4,5,6,7,8,0,1,2,
     *                             4,5,3,7,8,6,1,2,0,5,3,4,8,6,7,2,0,1,
     *                             6,7,8,0,1,2,3,4,5,7,8,6,1,2,0,4,5,3,
     *                             8,6,7,2,0,1,5,3,4/
      DATA ((M9(I,J),J=1,9),I=1,9)/0,0,0,0,0,0,0,0,0,0,1,2,3,4,5,6,7,8,
```

```
     *                              0,2,1,6,8,7,3,5,4,0,3,6,2,5,8,1,4,7,
     *                              0,4,8,5,6,1,7,2,3,0,5,7,8,1,3,4,6,2,
     *                              0,6,3,1,7,4,2,8,5,0,7,5,4,2,6,8,3,1,
     *                              0,8,4,7,3,2,5,1,6/
C
C      ** NB IS THE NUMBER OF BLOCK **
C
      NB=1
      DO 5 I=1,R
    5 NB=NB*(P(I)**E(I))
C
C       *** TO SOLVE B(I) FOR I=1,2,...,R IN THE CONGRUENCE ***
C
      DO 10 I=1,R
      DO 20 I1=1,NB
      B(I)=I1
      TP=P(I)**E(I)
      C(I)=B(I)*NB/TP
      C1=C(I)-1
      T=C1-C1/TP*TP
      IF(T.EQ.0) GO TO 10
      GO TO 20
   20 CONTINUE
   10 CONTINUE
C
```

```
C       *** TO GENERATE TREATMENT COMBINATION ***
C
      DO 30 J=1,N
      K1=M(J)
      K2=P(J)
      DO 40 J1=1,K1
      DO 50 J2=1,K2
        X(J,J1,J2)=J2-1
   50 CONTINUE
   40 CONTINUE
   30 CONTINUE
C
C    *** TO ASSIGN TREATMENT COMBINATIONS TO BLOCKS ***
C
C    Z1=P(1)
C     :=:
C    ZN=P(N)
C    DO 61 K11=1,Z1
C    DO  :    :=1,Z1
C    DO 6(Z1) K1(Z1)=1,Z1
C    DO 71 K21=1,Z2
C    DO  :    :=1,Z2
C    DO 7(Z2) K2(Z2)=1,Z2
C        :
C    DO  : KN(ZN)=1,ZN
```

```
C   DO  : KN(ZN)=1,ZN
C
C   *** INPUT CONGRUENCE(DERIVED FROM MODIFIED CRT) ***
C
C       SUM=C(1)*SUM1+...+C(R)*SUMR
C       W=SUM-SUM/NB*NB
C   CONTINUE
C
C  ***                    EXAMPLE                    ***
C
C       CONFOUND 2 COMPONENTS FROM THE FIRST SYMMETRIC FACTORIAL
C                  (X(1,1)+X(1,2)+X(1,3)),(X(1,1)+2*X(1,2))
C                2 COMPONENTS FROM THE SECOND SYMMETRIC FACTORIAL
C                  (X(2,1)+2*X(2,2)+X(2,3)),(X(2,1)+(2,2))
C                1 COMPONENT FROM THE THIRD SYMMETRIC FACTORIAL
C                  (X(3,1)+X(3,2))
C
C       IF P(I) IS A PRIME NUMBER
C          SUMI=0
C          DO # K=1,E(I)
C          DI(K)=X(I,1,K11)+...+X(I, ,  )
C          DI(K)=DI(K)-DI(K)/P(I)*P(I)
C          SUMI=SUMI+CONSTANT*DI(K)
C      # CONTINUE
C
```

```
C       IF P(I) IS A PRIME-POWER
C       GO TO DATA STATEMENT
C
      Z1=P(1)
      Z2=P(2)
      Z3=P(3)
      DO 60 K11=1,Z1
      DO 61 K12=1,Z1
      DO 62 K13=1,Z1
      DO 70 K21=1,Z2
      DO 71 K22=1,Z2
      DO 72 K23=1,Z2
      DO 80 K31=1,Z3
      DO 81 K32=1,Z3
         D1(1)=X(1,1,K11)+X(1,2,K12)+X(1,3,K13)
         D1(1)=D1(1)-D1(1)/3*3
         D1(2)=X(1,1,K11)+2*X(1,2,K12)
         D1(2)=D1(2)-D1(2)/3*3
         SUM1=D1(1)+3*D1(2)
C
         L1=2+1
         L2=X(2,2,K22)+1
         D2(1)=M4(L1,L2)
         L1=X(2,1,K21)+1
         L2=D2(1)+1
```

```
      D2(1)=A4(L1,L2)
      L1=D2(1)+1
      L2=X(2,3,K23)+1
      D2(1)=A4(L1,L2)
      L1=X(2,1,K21)+1
      L2=X(2,2,K22)+1
      D2(2)=A4(L1,L2)
      SUM2=D2(1)+4*D2(2)
C
      D3=X(3,1,K31)+X(3,2,K32)
      SUM3=D3-D3/5*5
      SUM=C(1)*SUM1+C(2)*SUM2+C(3)*SUM3
      W=SUM-SUM/720*720
C
C   ** TO PRINT OUT INTRA-BLOCK SUBGROUP ***
C
  IF(W.EQ.0)GO TO 900
    GO TO 81
900 WRITE(6,100)W,X(1,1,K11),X(1,2,K12),X(1,3,K13),
   *            X(2,1,K21),X(2,2,K22),X(2,3,K23),
   *            X(3,1,K31),X(3,2,K32)
 81 CONTINUE
 80 CONTINUE
 72 CONTINUE
 71 CONTINUE
```

```
 70 CONTINUE
 62 CONTINUE
 61 CONTINUE
 60 CONTINUE
100 FORMAT(5I2,3X,I5)
    STOP
    END
```

The vita has been removed from the scanned document