PARAMETER ESTIMATION FOR SERIES OBSERVED WITH ROUND-OFF ERROR

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(ABSTRACT)

Time series data often is observed with measurement error. One type of measurement error almost always present is rounding error. A procedure is proposed for estimating parameters of a finite moving average time series which is observed only after rounding.

Method of moments estimators are proposed for estimation of parameters of time series observed with general measurement error, including error, ε_t, which is correlated with the series X_t being measured. This procedure requires knowledge of the autocovariance function (ACF) of ε_t and the cross covariances between X_t and ε_t.

For rounding error, the rounding error series is shown to approach uniform white noise as the rounding interval width, R, approaches zero, and the cross correlations between X_t and rounding error ε_t are shown to approach zero as $R \to 0$. For both small R and large R, the ACF of ε_t and the cross covariances between X_t and ε_t are approximated. These values are then used to estimate the parameters of the moving average model for X_t when X_t is observed after rounding.

DEDICATION

This dissertation is dedicated to my paternal grandmother, , who passed away in 1972, and to my father, . They are most responsible for my will to get a good education.

My grandmother received a masters degree in education in 1904 from Ohio State University and taught for fifteen or more years before marrying. She once told me, quoting her father, that an education is the only thing that can never be taken from you.

My father, a 1949 chemical engineering graduate of Case Institute of Technology and former School Board President, forced me to study as a child and has encouraged me in my studies up to the present.

In this last step of my formal education, I want to remember these two people who showed me the challenge of learning.

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Above all, I thank my wife, , who gave me the push to return to graduate school and who has made many sacrifices to help my studies. Without her love and encouragement, I likely would not have made it to this point in the program, or in our lives.

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Chapter I

INTRODUCTION

This thesis will answer several questions regarding the effects of measurement error on time series analysis. This chapter begins with basic definitions in time series, establishment of the notation that is to be used throughout this work, and an introduction to measurement error in time series models.

Time Series - Definitions and Notation

A time series is a set of observations indexed by the set of real numbers (continuous-time series) or by the set of integers (discrete-time series)

$$\{X_t : -\infty < t < \infty\}$$

or

$$\{X_t : t \in \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}\}.$$

The index, t, usually represents actual time in application, and we shall interpret t as time. However, t may represent another quantity, such as linear position.

Continuous-time series are (almost) always observed at discrete intervals and so can be treated for our purposes as discrete time series. Additionally, by the Sampling Theorem (see Koopmans, 1974, pg. 72), as long as the sampling interval, Δt, is chosen so that the spectrum (defined in Koopmans) is zero outside the interval

$$\left(\frac{-1}{\Delta t}, \frac{1}{\Delta t}\right),$$

the continuous-time series, X_t, can be exactly reconstructed from $\{X_{n\Delta t}\}_{n=-\infty}^{\infty}$. For these reasons, we shall restrict ourselves to discrete-time series in this work.

For the time series $\{X_t\}_{t=-\infty}^{\infty}$ we will simply write X_t and allow the context to dictate whether we refer to the entire series or a single observation in that series. The series X_t is said to be **weakly stationary** if

1. $E(X_t) = E(X_s)$ for all t,s

2. $Var(X_t) = Var(X_s)$ for all t,s and

3. $Cov(X_t, X_{t+k})$ depends only on k and not on t.

In this thesis, we shall mean 'weakly stationary' when we write 'stationary'. Provision 3 allows us to define the **autocovariance function (ACF)** of X_t

$$\gamma_x(k) = Cov(X_t, X_{t+k}) \quad \text{for all } k$$

as a function of the lag, k, only. When we divide through by $\gamma_x(0)$, we have the **autocorrelation function**,

$$\rho_X(k) = \frac{\gamma_X(k)}{\gamma_X(0)}$$

We shall assume stationarity in this thesis. Differencing can eliminate nonstationarity of a time series in the case of a polynomial trend, or in other cases, such as a random walk. If, for example, X_t has a linear trend:

$$X_t = a + bt + S_t$$

where S_t is stationary, then we consider

$$\widetilde{X}_t = \nabla X_t = X_t - X_{t-1}$$

$$= b + \nabla S_t$$

which is stationary. If the polynomial trend is of order d, then we take the d-th order difference

$$\widetilde{X}_t = \nabla^d X_t$$

Another assumption we shall make is that X_t has mean $\mu = 0$. If $\mu \neq 0$ then we subtract μ if μ is known, or subtract the sample mean $\overline{X}$ if μ is unknown. Thus we shall assume that X_t is a zero mean, stationary time series.

Another characteristic of a time series that is of importance is the **partial autocorrelation function** or **PACF**, defined by:

$$\phi_X(kk) = E\left[X_t X_{t-k} | X_{t-1}, X_{t-2}, ..., X_{t-k+1} \right] .$$

Partial autocorrelations may be computed from the autocorrelations by the formula

$$\phi_x(kk) = \frac{\begin{vmatrix} 1 & \rho_x(1) & \cdots & \rho_x(k-2) & \rho_x(1) \\ \rho_x(1) & 1 & \cdots & \rho_x(k-3) & \rho_x(2) \\ \rho_x(2) & \rho_x(1) & \cdots & \rho_x(k-4) & \rho_x(3) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho_x(k-1) & \rho_x(k-2) & \cdots & \rho_x(1) & \rho_x(k) \end{vmatrix}}{\begin{vmatrix} 1 & \rho_x(1) & \cdots & \rho_x(k-2) & \rho_x(k-1) \\ \rho_x(1) & 1 & \cdots & \rho_x(k-3) & \rho_x(k-2) \\ \rho_x(2) & \rho_x(1) & \cdots & \rho_x(k-4) & \rho_x(k-3) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho_x(k-1) & \rho_x(k-2) & \cdots & \rho_x(1) & 1 \end{vmatrix}} .$$

The importance of the PACF and ACF in model identification is a major thrust of Box and Jenkins (1976,Chapter 6).

Some special time series models will be used extensively in what follows, namely autoregressive (AR), moving average (MA) and the mixed autoregressive-moving average (ARMA) series. Definition and development of some properties of these models at this time will simplify matters later.

Autoregressive Time Series

Suppose $\{a_t\}_{t=-\infty}^{\infty}$ is a series of uncorrelated zero mean variates with common variance. Such a time series is called **white noise.** An **autoregressive process of order p, AR(p)** is a time series that can be represented in the form

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + a_t \qquad [1.1]$$

The real numbers $\phi_1, \phi_2, \ldots, \phi_p$ and σ_a^2 are the parameters of the model, with $\phi_p \neq 0$. The model [1.1] can be written as

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \ldots - \phi_p X_{t-p} = a_t$$

or

$$\Phi(B)\, X_t = a_t \qquad [1.2]$$

where B is the backshift operator ($B^s X_t = X_{t-s}$). The model given in [1.1] and [1.2] is stationary if and only if the roots of the equation

$$\Phi(B) = 0$$

all lie outside the unit circle $|B| = 1$. (A proof appears in Box and Jenkins, 1976, pg. 54.) An observation X_t may be expressed in terms of present and past values of the white noise process as

$$X_t = \Phi^{-1}(B)\, a_t.$$

This will be referred to as the **moving average form** of the model. In general, for stationary X_t, $\Phi^{-1}(B)$ is an infinite series in B.

The AR(p) process has an autocorrelation function which is dominated after lag p-1 by either a damped sine or damped exponential function, or a mixture of these, depending on the nature of the roots of the equation

$$\Phi(B) = 0 \quad .$$

The PACF of an AR(p) process vanishes after lag p,

$$\phi_x(kk) = 0 \quad \text{for } k > p \ .$$

A look at the AR(1) process will simplify later development.

The AR(1) Model: The AR(1) series is defined by

$$X_t = \phi X_{t-1} + a_t \ .$$

The moving average form is

$$X_t = (1 - \phi B)^{-1} a_t = \sum_{l=0}^{\infty} \phi^l a_{t-l}.$$

From this form, we can see that

$$\gamma_x(0) = \sum_{l=0}^{\infty} \phi^{2l} \sigma_a^2 = \frac{1}{1 - \phi^2} \sigma_a^2$$

$$\gamma_x(1) = \sum_{l=0}^{\infty} \phi^{2l+1} \sigma_a^2 = \frac{\phi}{1 - \phi^2} \sigma_a^2$$

$$\gamma_x(2) = \frac{\phi^2}{1 - \phi^2} \sigma_a^2$$

...

$$\gamma_x(k) = \frac{\phi^k}{1 - \phi^2} \sigma_a^2$$

...

Since $\rho_x(k) = \phi^k$, the ACF of an AR(1) process is an exponential function.

Moving Average Time Series

Let us now turn to moving average models, which will be the model used most heavily in this dissertation. We define a **moving average process of order q, MA(q)** as a series that can be represented in the form

$$X_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \ \ldots \ - \theta_q a_{t-q} \qquad [1.4]$$

where $q \geq 1$ and $\{a_t\}$ is a white noise process. The model [1.4] can also be written as

$$X_t = \Theta(B)\, a_t$$

using the backshift operator B. X_t is said to be **invertible** if the complex roots of

$$\Theta(B) = 0$$

are outside the unit circle. In that case, we may write the model in **inverted form**:

$$\Theta^{-1}(B)\, X_t = a_t \quad .$$

We note that moving average processes are always stationary, and autoregressive processes are always invertible, being defined in inverted form. Hereafter, we shall consider only stationary, invertible time series.

The ACF of a MA(q) process is given by:

$$\gamma_x(0) = (1 + \theta_1^2 + \theta_2^2 + \ ... \ + \theta_q^2)\,\sigma_a^2$$

$$\gamma_x(1) = (-\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \ ... \ + \theta_{q-1}\theta_q)\,\sigma_a^2$$

$$\gamma_x(2) = (-\theta_2 + \theta_1\theta_3 + \theta_2\theta_4 + \ ... \ + \theta_{q-2}\theta_q)\,\sigma_a^2$$

...

$$\gamma_x(q-1) = (-\theta_{q-1} + \theta_1\theta_q)\sigma_a^2$$

$$\gamma_x(q) = -\theta_q\,\sigma_a^2$$

$$\gamma_x(k) = 0 \qquad \text{for } k > q \ .$$

Thus the ACF of a MA(q) process cuts off after lag q.

The PACF of a MA(q) process is similar to the ACF of an autoregressive process in that the PACF is a mixture of damped sines and exponentials. In fact, the duality between the autoregressive and moving average models is extensive. Some of these dual properties are given in Table 1 on page 9.

Any stationary stochastic process with absolutely continuous spectral distribution can be approximated to any desired accuracy by a MA(q) process for some integer q. This fact is proved by Doob (1953, page 499).

Since most of this thesis deals with moving average time series, and in particular, MA(1) and MA(2) models, properties of these models are now considered.

Table 1. Properties of Autoregressive and Moving Average Time Series

$\Phi(B)\, X_t = \Theta(B)\, a_t$

	AR(p)	MA(q)
ACF	Dominated by damped sines and exponentials	Vanishes after lag q
PACF	Vanishes after lag q	Dominated by damped damped sines and exponentials
Stationarity	If all roots of $\Phi(B) = 0$ are outside the unit circle	Always
Invertibility	Always	If all roots of $\Theta(B) = 0$ are outside the unit circle

The MA(1) Model: The MA(1) process is defined by

$$X_t = a_t - \theta\, a_{t-1} \qquad [1.5]$$

where a_t is white noise. The ACF is given by:

$$\begin{aligned} \gamma_0 &= (1 + \theta^2)\,\sigma_a^2 \\ \gamma_1 &= -\theta\,\sigma_a^2 \\ \gamma_k &= 0 \qquad \textit{for } k > 1. \end{aligned} \qquad [1.6]$$

The PACF of the process [1.5] is

$$\phi_x(kk) = \frac{-\theta^k(1 - \theta^2)}{1 - \theta^2(k+1)}.$$

Since $|\phi_x(kk)| < \theta^k$, the PACF is dominated by a damped exponential.

The MA(2) Model: The MA(2) process is defined by the equation

$$X_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

where $\{a_t\}$ is white noise. The ACF of a MA(2) process is given by

$$\gamma_x(0) = \{1 + \theta_1^2 + \theta_2^2\}\,\sigma_a^2$$

$$\gamma_x(1) = \{-\theta_1 + \theta_1\theta_2\}\sigma_a^2$$

$$\gamma_x(2) = -\theta_2\,\sigma_a^2$$

$$\gamma_x(k) = 0 \text{ for } k > 2$$

The region of invertibility, i.e. where the values of θ_1 and θ_2 for which the roots of

$$1 - \theta_1 B - \theta_2 B^2 = 0$$

are outside the unit circle, is the region (in θ_1 and θ_2) graphed in Figure 1 on page 12. (This is also the region of stationarity in ϕ_1 and ϕ_2 for an AR(2) process.)

The PACF of a MA(2) process is dominated, after lag 2, by a damped sine or damped exponential function. The nature of the roots of the equation

$$1 - \theta_1 B - \theta_2 B^2 = 0$$

determine whether it is a damped sine or a damped exponential.

Mixed Models and Integrated Models

We may combine the AR(p) and MA(q) models into a more general model, the **mixed autoregressive-moving average (ARMA) process.** The ARMA(p,q) process is given by

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + a_t - \theta_1 a_{t-1} - \ldots \theta_q a_{t-q}$$

and can be written in the form

$$\Phi(B) X_t = \Theta(B) a_t$$

where $\Phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$, and $\Theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q$ are polynomials in the backshift operator B and where $\{a_t\}$ is white noise. Such a process is characterized by an ACF that is dominated by a damped sine or exponential after lag q , and a PACF that is dominated by a damped sine or exponential after lag p. An ARMA model is invertible if the roots of

$$\Theta(B) = 0$$

are outside the unit circle, and stationary if the roots of

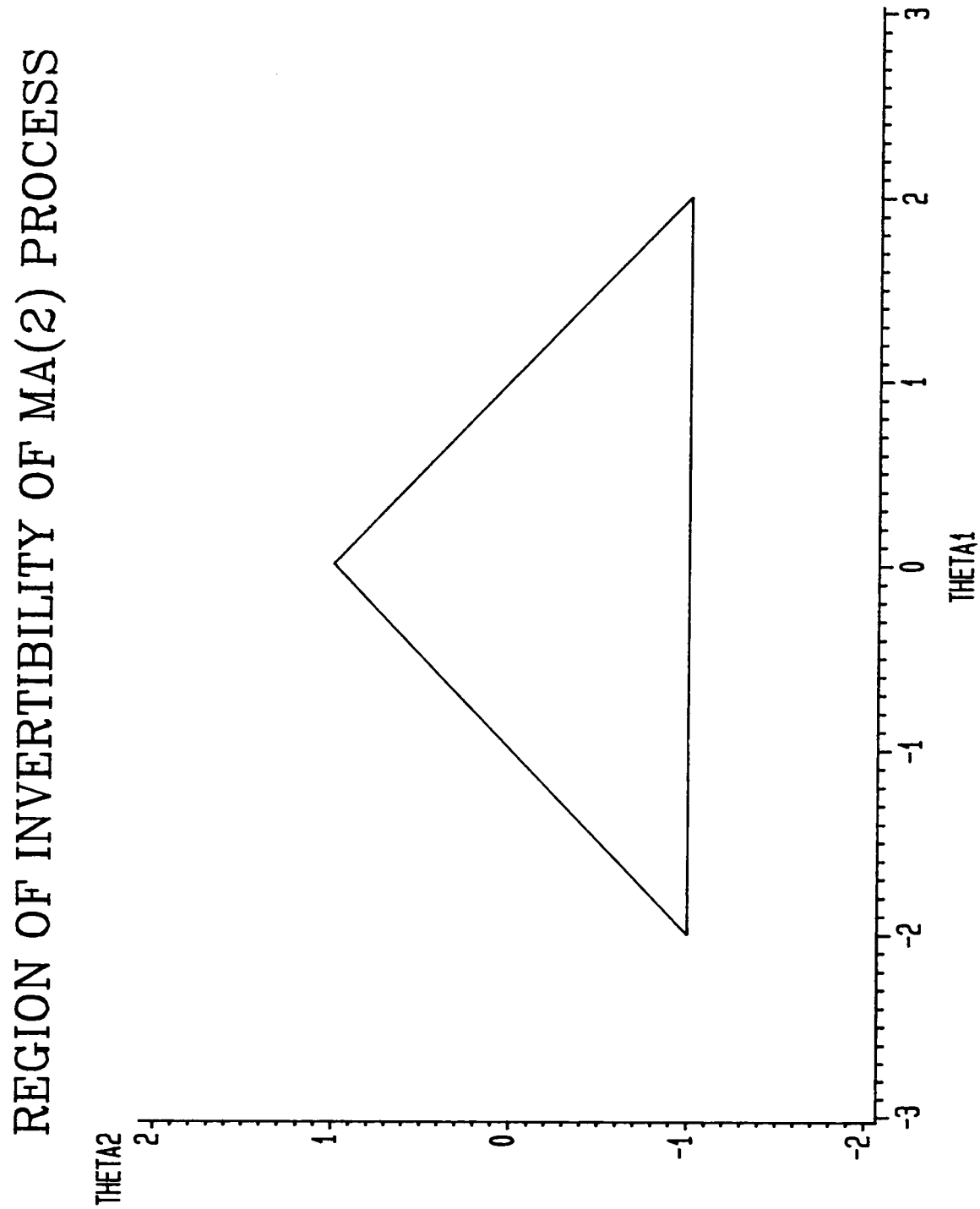

Figure 1. Region of Invertibility of a MA(2) Process

$$\Phi(B) = 0$$

are outside the unit circle.

As mentioned above, it may be necessary to difference to obtain stationarity. We may write the difference operator ∇ in terms of the backshift operator B, and so

$$\nabla^d X_t = (1 - B)^d X_t \ .$$

We then define the **autoregressive integrated moving average (ARIMA)** by the model

$$\Phi(B)\nabla^d X_t = \Theta(B)a_t \qquad [1.7]$$

If $\Phi(B)$ has order p and if $\Theta(B)$ has order q, then Equation [1.7] defines the ARIMA(p,d,q) model.

Relationships Between Series

Suppose two series, X_t and W_t are related. We would like to express their relationship in terms of the covariances between them, and also build a model for one in terms of the other.

We define the **cross covariance function,** $\gamma_{xw}(k)$ by

$$\gamma_{xw}(k) = Cov(X_t, W_{t+k}) \qquad \text{for k} = \ldots,-2,-1,0,1,2,\ldots$$

and the **cross correlation function** by

$$\rho_{xw}(k) = \frac{\gamma_{xw}(k)}{\sigma_x \sigma_w} \qquad \text{for k} = \ldots-2,-1,0,1,2,3,\ldots$$

In general, $\gamma_{xw}(k) \neq \gamma_{wx}(k)$. However, we need only consider $\gamma_{xw}(k)$ at all lags k since $\gamma_{xw}(k) = \gamma_{wx}(-k)$.

A second way to express the relationship between X_t and W_t is by way of a **discrete transfer function plus noise** model. We can express W_t in terms of X_t through the linear filter

$$W_t = \nu(B)X_t + N_t$$

where

$$\nu(B) = \ldots + \nu_{-2}B^{-2} + \nu_{-1}B^{-1} + \nu_o + \nu_1 B + \nu_2 B^2 + \ldots$$

$$B^{-k}X_t = X_{t+k} \qquad \text{for } k > 0$$

and where N_t is a noise series independent of the series X_t.

In many cases, W_t may respond to present and past values of X_t only. That is, the adjusted covariances

$$Cov(X_t, W_{t+k} | X_{t-1}, X_{t-2}, \ldots)$$

are all zero for negative values of k. (Even in the case where W_t is completely determined by present and past values of X_t, $\gamma_{wx}(k)$ may not be zero for some negative k.) In the case where W_t is determined by present and past X's, the transfer function $\nu(B)$ is a polynomial in the backshift operator B or an infinite series (convergent for $|B| < 1$) in B

$$\nu(B) = \nu_o + \nu_1 B + \nu_2 B^2 \ldots.$$

Then a parsimonious representation of $\nu(B)$ is

$$\nu(B) = \delta^{-1}(B)\,\omega(B)$$

where $\delta(B)$ and $\omega(B)$ are finite polynomials in B. Furthermore, if the cross covariances $\gamma_{xw}(k)$ between the series X_t and W_t vanish after lag r, then it can be shown that

$$\delta(B) = 1$$

and

$$v(B) = \omega(B) = v_o + v_1 B + \ldots v_r B^r \ .$$

It is assumed that the noise series N_t can be represented by an ARIMA model

$$N_t = \Phi^{-1}(B)\,\Theta(B)\,a_t$$

where a_t is white noise independent of X_t. Putting these representations together, we have

$$W_t = v(B)X_t + N_t$$

$$= \delta^{-1}(B)\omega(B)X_t + \Phi^{-1}(B)\Theta(B)a_t$$

with X_t and a_t independent.

Measurement Error in Time Series

As in most areas of statistics, a standard assumption in time series analysis is that the observations are made without error. That assumption is clearly violated in most observed time series. Very often time series observations are taken from surveys or other estimation processes in which the existence of error is obvious. Even when time series observations are measured from some physical process, observations almost always include some measurement error.

Measurement error in economic variables has been studied, but, as Griliches (1974) states, "We complain quite a bit but do very little about it. Much of the problem, I think, arises because of the separation in economics between data producers and data analyzers. By and large, we do not produce our own data and, hence, do not feel responsible for it."

Pierce (1981) classified different sources of error in economic data. His major classifications of error sources were

1. Conceptual error - Measuring a value that does not reflect fully what is intended to be measured. For example, to measure unemployment, one must define unemployment.

2. Transitory error - "Irregular, evanescent fluctuations in a data series, presumable due to causes extraneous to those related to our concept of the series."

3. Sampling error - When sample surveys are used, sampling error must be considered.

4. Seasonal adjustment error - Faulty removal of an unwanted trend.

5. Reporting error - Clerical errors, classification errors, etc.

Among the errors (probably best fitting into the reporting error category) is rounding error. Rounding error may enter into time series data in two ways. First, the figures collected from various agencies from which the series observation is compiled are often rounded, creating in their aggregate some noise in the final reported value. Secondly, the value reported by the reporting agency (such as the Bureau of Labor Statistics) is usually rounded when reported. This latter error is the rounding error about which this thesis will be primarily concerned. The former could be evaluated and modeled as part of general measurement error, perhaps using some of the ideas to be presented in Chapter 4 of this thesis.

Most of the work in time series measurement error has assumed that measurement error is independent of the series that is being measured. Additionally, much of the work until recently has assumed the error to be white noise, usually normally distributed. Recent work in survey methodology has used more complex models for error.

Rounding error, when considered at all, has been assumed without proof to be uniform white noise uncorrelated with the series being rounded. It is obviously dependent (completely) on the series being rounded. In Chapter 4, we shall evaluate these assumptions and show where they break down. We shall estimate the covariances when they exist.

Measurement Error Models and Notation

Suppose X_t is an ARIMA time series, and suppose X_t is not observed, but instead that Y_t is observed where

$$Y_t = X_t + \varepsilon_t \quad .$$

If no information about the error ε_t is available, then all one can do is pretend that the series Y_t is the series you intended to observe and estimate the parameters of Y_t as if it were X_t.

In some cases, the observed series Y_t is the series of primary interest. In many cases, the system to be studied (e.g. the U.S. economy) reacts to the reported value more strongly than to the value which it estimates. For example, the many individuals and agencies that influence economic activity react to the reported value of such indicators as Gross National Product (GNP). The actual value of GNP may be of secondary importance for some purposes. In the cases in which the reported value is of primary concern, we need to model the observed series.

In most cases, however, modelling and estimating the parameters of the unobserved series X_t is desired. We often want to estimate the present value of X_t (the signal extraction problem) and/or forecast future X values from the observed values of Y_t. The primary emphasis of this thesis will be on parameter estimation in the presence of measurment error.

This thesis has two main objectives. First, the case of parameter estimation in the presence of measurement error, ε_t, correlated with the series, X_t, which is being measured, will be examined

in Chapter 3. A method is developed for estimation of parameters of a series X_t when it is observed with error ε_t independent of the series X_t. A representation for ε_t will be suggested which allows the expansion to the case where ε_t is correlated with X_t. This technique will be fully developed for the case of MA(1) series X_t and MA(1) error ε_t. A simulation will show the improvement in parameter estimates.

The second objective is to estimate parameters of a series that has been rounded. A proof will show the asymptotic distribution of the rounding error as the rounding becomes finer. For heavy rounding, approximations of the covariances (autocovariances and cross covarinces) of the error series will be given, and can be applied to a MA(1) series or any other ARMA model. These covariances will be based on the degree of rounding. This is done in Chapter 4. Chapter 5 puts to use the results of Chapters 3 and 4 to show how parameters may be estimated for rounded time series. Chapter 6 will summarize, make additional conclusions, and indicate areas for future research.

Chapter II

LITERATURE REVIEW

The study of measurement error in linear statistical models is over a century old. While there is much literature in this area, most is only tangentially related to this thesis. Much has been written about regression in the presence of measurement errors. A review of this literature appear in Madansky (1959). We briefly consider the basic effect of measurement errors in regression and move on to time series measurement error.

Measurement Errors in Regression

Early work shows the asymptotic bias in the estimate of the parameter $\underline{\beta}$ in the model

$$\underline{Y} = X\underline{\beta} + \underline{U} \qquad [2.1]$$

when X is replaced by the observed

$$X^* = X + V \qquad [2.2]$$

where V is a matrix of measurement errors. Myers (1986, pg 211-213) contains a discussion of the problem. He points out that the covariance between model error and the regressor variables is not zero for the model [2.1], [2.2], and as a result, the Gauss-Markoff Theorem does not apply. Thus the least squares estimates are not, in general, unbiased. Derivations of the asymptotic bias appear in Johnston (1972, Sect 9-4) and elsewhere. Following the derivation in Johnston, we have

$$\begin{aligned} plim\,\hat{\underline{\beta}} &= \underline{\beta} + (X'X)^{-1}X'(\underline{U} - V\underline{\beta}) \\ &= \underline{\beta} - plim(\frac{1}{n}X'X)^{-1}\,plim\,(\frac{1}{n}V'V)^{-1}\,\underline{\beta}\ . \end{aligned}$$

For simple linear regression, this reduces to

$$plim\,\hat{\beta} = \beta - \frac{\sigma_v^2\beta}{\sigma_x^2 + \sigma_v^2}$$

and so β is biased toward zero.

Stefanski (1985) presents a general formulation of the problem of measurement error for linear and nonlinear models. He shows how to assess the asymptotic bias in this estimator when error is present. The model is assumed known and a method of moments estimator is assumed for the parameters of the model.

Measurement Error in Time Series

Recent attention has been given to problems of measurement error in time series. Much of this has been aimed at specific time series in application areas such as fisheries (for example Walters (1985) and Caputo (1988) who consider bias in the relationships between two stock-related series) and econometrics (for example Steckler (1967 and 1987) in looking at data revision and its effect on forecasts of Gross National Product, and Chen and Lee (1984) who look at measurement of the market rates of return.)

Ashley and Vaughan (1986) use a spectral decomposition method to obtain an upper bound on the amount of measurement error. They assume that the series being measured, X_t, and the error series, ε_t, are uncorrelated. They define a degree of corruption of X_t by ε_t as

$$R_e^2 = \frac{\sigma_\varepsilon^2}{\sigma_x^2}$$

and derive an upper bound, R_ω^2, for R_e^2 . The form of this upper bound is complex, and includes notions from spectral analysis not considered here, but when ε_t is white noise, R_ω^2 reduces to

$$R_\omega^2 = \frac{\min_\omega f_x(\omega)}{mean_\omega f_x(\omega)} \quad ,$$

where $f_x(\omega)$ is the spectral density function on X. This simply takes the smallest value of the spectral density (explaining the minimum activity at that frequency) as the upper bound on $2\pi\sigma_t^2$.

Much of the literature centers on the problem of signal detection. The model

$$Y_t = X_t + \varepsilon_t$$

is viewed as a signal (X_t) with added noise (ε_t). The objective of signal detection is to estimate a value of X_t from an observed record of values of the series Y_t .

Fuller (1976, pg. 166) derives a linear filter

$$\sum_{j=-L}^{M} a_j Y_{t-j} = \hat{X}_t$$

that minimizes the mean squared error of the estimate of X_t . Assuming that X_t and ε_t are independent, the linear filter is given by

$$a_j = \int_{-\pi}^{\pi} f_a(\omega) \exp(i\omega j)\, d\omega$$

where

$$f_a(\omega) = \frac{f_{YX}(\omega)}{2\pi f_{YY}(\omega)}$$

and $f_{YX}(\omega)$ and $f_{YY}(\omega)$ are, respectively, the cross spectral density between Y_t and X_t and the spectral density of Y_t. This assumes knowledge of these spectral densities, which might not be available in practice.

Much work has been published recently in the use of signal detection techniques in time series on the problems of estimation in repeated surveys. Articles developing these techniques include Scott and Smith (1974), Scott, Smith, and Jones (1977), Jones (1980), Miazaki (1985), Bell and Hilmer (1987), and Eltinge and Fuller (1989). These assume no correlation between survey error and X_t (Bell and Hilmer prove none exists under restricted conditions) and use various means to estimate X_t or to forecast future X's.

Several authors have addressed the problems of parameter estimation for time series ARIMA models. Most of the work has been restricted to autoregressive series with white noise measurement error.

Walker (1960) considers the estimation of parameters of an AR(p) process, X_t, observed with error. The error series is assumed to be white noise independent of ε_t . He showed that if $Y_t = X_t + \varepsilon_t$, then Y_t follows an ARMA(p,p) model. He develops methods of moments estimators using this representation and properties of the autocovariance function. The first $p+2$ sample autocovariances are equated to their expected values, yielding a system of equations in the unknown parameters of the AR(p) model and the variance, σ_ε^2, of ε_t. These estimators are shown to be con-

sistent, with asymptotic standard errors which are $O(n^{-1/2})$ as $n \to \infty$. The limiting joint distribution of the estimates is multivariate normal with finite covariance matrix.

Walker derives the asymptotic efficiencies of these estimates and discusses the effect of non-normality. The methods developed are shown to be asymptotically inefficient.

Pagano (1974) devised asymptotically efficient estimators for the model considered by Walker. Using the development of Walker for

$$Y_t = X_t + \varepsilon_t$$

$$\sum_{j=0}^{p} \beta_j X_{t-j} = \delta_t$$

where $\{\delta_t\}$ is white noise, $\beta_1 = 1$, and $\beta_p \neq 0$ (the AR(p) series), Pagano defines

$$\begin{aligned} Z_t &= \sum_{j=0}^{p} \beta_j Y_{t-j} \\ &= \sum_{j=0}^{p} \beta_j X_{t-j} - \sum_{j=0}^{p} \beta_j \varepsilon_{t-j} \\ &= \delta_t - \sum_{j=0}^{p} \beta_j \varepsilon_{t-j} \ . \end{aligned}$$

The ACF of Z_t can now be seen to vanish after lag p. By representing Z_t as an MA(p) process,

$$Z_t = \sum_{k=0}^{p} \alpha_k \eta_{t-k}$$

where $\alpha_0 = 1$, $\alpha_p \neq 0$, and $\{\eta_t\}$ is white noise, we then have

$$\sum_{j=0}^{p} \beta_j Y_{t-j} = \sum_{k=0}^{p} \alpha_k \eta_{t-k} \qquad [2.3]$$

which is an ARMA(p,p) process in the observed series Y_t involving the original parameters and the parameters α_1, α_2, ... , α_p and σ_η^2. The parameters of this model are then estimated using the method of Parzen (1971) for the parameters of an ARMA(p,q) process.

Pagano points out that this method may be extended to the case where X_t is ARMA(p,q) with $p > q$. In the case where $p \leq q$, the reparameterization in [2.3] is to a lower dimension, producing confounding of the parameter estimates. For this reason, his method cannot be applied directly to moving average models.

Miazaki (1985) extends the work of Pagano (1974) to the case where ε_t follows a MA(q) model. She shows that Y_t has an ARMA(p,p + q) model and she develops a nonlinear least squares estimate of the parameters of X_t, assuming knowledge of the parameters of the MA(q) model for ε_t . The estimates she derives are, under certain conditions, consistent and asymptotically normal.

Eltinge and Fuller (1989) outline the methods above and give in general form the maximum likelihood estimates (under normality) and an iterative least squares estimate for use in estimation of the population value (X_t in our notation) in the presence of a known survey error model.

Rounding Error Literature

The literature on rounding error is sparse. Rounding error in regression and its effect on coefficient estimation was considered by Swindel and Bower (1972). They derive bounds on the bias in coefficient estimates. In the model [2.1], where X is fixed, they represent the rounding by

$$X^* = X + U$$

where X^* and U are fixed matrices. (The assumption of fixed values is not reasonable for time series.) Then an arbitrary linear combination $L'\beta$ is estimated by

$$L'\hat{\beta}^* = L'(X^{*'}X^*)^{-1}X^{*'}\underline{y}$$

and the bias is

$$B(L'\hat{\beta}^*) = -L(X^{*'}X^*)^{-1}X^{*'}u\beta \quad .$$

From this, they show that the relative bias is given by

$$Rel\,Bias(L'\hat{\beta}^*) = \frac{B(L'\hat{\beta}^*)}{\sqrt{Var(L'\hat{\beta}^*)}}$$

$$= \sqrt{\frac{L^{*'}U\beta\beta'U'L^*}{L^{*'}L^*\sigma^2}}$$

where $L^* = X^*(X^{*'}X^*)^{-1}L$. Then

$$0 \leq RB(L'\hat{\beta}^*) < \frac{1}{\sigma}[B'U'UB]^{1/2} \quad .$$

Beaton, Rubin, and Barone (1976) considered the problem of computational accuracy in regression solutions by examining the Longley data (Longley (1967)). They show that parameter estimates are affected greatly by computational accuracy and rounding of regressors. By generating the rounding errors according to a uniform distribution to create plausible values of the regressors before rounding, they show that large variations in the estimated regression coefficients result from small rounding errors. When a subset of the regressors was used, eliminating the most damaging collinearity in the data, the swings in the coefficient estimates due to the probabilistic correction for rounding error were much smaller. They also found that the variations in the predictive ability of both models were affected less than were the coefficient estimates.

Dempster and Rubin (1983) compare the results of the algorithm of Beaton, Rubin, and Borone (1976) to Sheppard's correction. Applied to regression, Sheppard's correction reduces the

variances of the regressors and response, thus adjusting the parameter estimates. Dempster and Rubin find that as long as the rounding is not heavy, Sheppard's correction produces good results.

Machak and Rose (1984 and 1985) consider estimation of parameters of an ARMA(1,1) process

$$X_t + \phi X_{t-1} = \delta + a_t - \theta a_{t-1}$$

subjected to rounding error. They model, for purposes of simulation, the rounding error as uniform white noise. They assume that the rounding error and the series being rounded are independent. (An assumption that the series were uncorrelated would be plausible, but requires justification. This is discussed in Chapter 4.) In the 1984 article, they simulate 120 observations of an ARMA(1,1) series, repeated 100 times. In the 1985 article, they simulate 60 observations of an ARMA(1,1) series, repeated 100 times. They find that the moving average parameter estimates, $\hat{\theta}$, are biased toward zero, while the autoregressive parameter estimates, $\hat{\phi}$, are apparently not biased.

The fact that the autoregressive parameters (such as ϕ above) are unaffected by white noise measurement error has been noted by Box and Jenkins (1976, Appendix 4.4) and Walker (1960). The fact that the parameter θ is estimated with bias is similar to the results for regression coefficients and will be pointed out in Example 3.1 of Chapter 3.

Other studies of measurement error in time series models have considered frequency domain (spectral) effects. These will not be considered in this work.

Chapter III

PARAMETER ESTIMATION OF MOVING AVERAGE SERIES MEASURED WITH ERROR

In this chapter, a method is developed for estimation of parameters of a moving average time series observed with measurement error. The method is first developed for measurement error that is independent of the series being measured. This might be the case, for example, when errors are random fluctuations, such as the error that might occur from electronic instruments. Aided by a useful representation of the measurement error series, the method is extended to estimation of parameters of a series observed with error when that error is correlated with the series being measured. Error correlated with a series might arise from survey data, for example, when the respondent may answer with intent to deceive. Such might be the case in a survey on personal income, where the wealthy might attempt to hide their wealth, and the poor might attempt to hide their poverty. As will be discussed in Chapter 4, another example is rounding error, particularly error due to gross rounding.

Several examples will be given to illustrate this estimation procedure. A simulation will show the effect of its application to MA(1) series.

White Noise Measurement Error in Moving Average Time Series

Consider a time series X_t with ARIMA(p,d,q) model

$$\Phi(B)\nabla^d X_t = \Theta(B)a_t \quad . \qquad [3.1]$$

Box and Jenkins (1976, pg. 121) consider such a model where X_t is observed with added noise, such as measurement error, independent of X_t. In the case where ε_t is white noise, and X_t is an IMA(0,d,q) series (i.e. $\Phi(B) = 1$) they show that equations relating the true parameters of X_t and the true parameters of $Y_t = X_t + \varepsilon_t$ can be formed. This is done by writing Y_t in terms of the models for X_t and ε_t, and in terms of its own moving average model (see Theorem 3.1 below). The ACF of Y_t is written in both forms and the expressions for corresponding autocovariances are equated.

This idea can be applied to the estimation of parameters of X_t. We shall apply it to the case where X_t is a moving average series. The method is given first for the case where ε_t is white noise independent of X_t. We assume that the variance of ε_t is known. This assumption will be discussed later in this chapter. The procedure is

1. The representations of Y_t in terms of a) its own moving average model and b) the model for X_t plus ε_t , are written.

2. The lags of the ACF of Y_t up to the order of Y_t are expressed in terms of the elements of these models. For any lag of the ACF up to the order of the model for Y_t, that autocovariance is

expressed in terms of the parameters of the two models. This yields a system of equations in the parameters of X_t, Y_t and ε_t.

3. The observed values of Y_t are used to estimate the parameters of the moving average model for Y_t. Maximum likelihood estimates or least squares estimates may, for example, be used.

4. The estimates of the parameters of Y_t from Step 3 and the known variance of ε_t are substituted into the system of equations constructed in Step 2.

5. The system of equations is solved for the parameters of X_t. If among the solutions is one which corresponds to an invertible model for X_t, we take that solution for our estimate of the parameters of X_t.

In order to assure ourselves that the moving average model for Y_t, which is needed above in Step 1, exists, a theorem proved by Ansley, Spivey, and Wrobleski (1977) is needed.

Theorem 3.1 -The sum of two independent moving average time series of orders q and q′ has a moving average representation whose order is, at most, the larger of q and q′.

We now illustrate this procedure with an example. We take the simplest case.

Example 3.1 - MA(1) Series with White Noise Error

Let X_t have MA(1) model

$$X_t = a_t - \theta a_{t-1} \quad .$$

Suppose X_t is measured with white noise measurement error ε_t as $Y_t = X_t + \varepsilon_t$. Then

$$Y_t = a_t - \theta a_{t-1} + \varepsilon_t \ . \qquad [3.2]$$

If ε_t is independent of X_t, then by Theorem 3.1, Y_t has a moving average representation whose order is at most 1. Assume the order is 1 and let this representation be

$$Y_t = c_t - \eta c_{t-1} \ . \qquad [3.3]$$

The procedure above, carried out for this case is:

1. Write $c_t - \eta c_{t-1} = a_t - \theta a_{t-1} + \varepsilon_t$.

2. Express the variance and lag 1 autocovariance of Y_t in terms of the parameters of [3.2] and [3.3]. Write

$$\begin{aligned} (1 + \theta^2)\sigma_a^2 + \sigma_\varepsilon^2 &= \gamma_Y(0) = (1 + \eta^2)\sigma_c^2 \\ -\theta\sigma_a^2 &= \gamma_Y(1) = -\eta\sigma_c^2 \end{aligned} \qquad [3.4]$$

3. The observed values of Y_t may then be used to construct estimates $\hat{\eta}$ and $\hat{\sigma}_c^2$ of η and σ_c^2. These estimates may be made, for example, by maximum likelihood or least squares methods.

4. These estimates are then substituted into equations [3.4] above for η and σ_c^2. Equations [3.4] are equivalent to

$$\begin{aligned} \theta^2 + B\theta + 1 &= 0 \\ \sigma_a^2 &= \frac{\eta}{\theta}\sigma_c^2 \end{aligned} \qquad [3.5]$$

where

$$B = \frac{1}{\eta}\left[\frac{\sigma_\varepsilon^2}{\sigma_c^2} - (1 + \eta^2)\right].$$

The known value of the variance σ_ε^2 of ε_t is inserted into [3.5] along with the estimates $\hat{\eta}$ and $\hat{\sigma}_c^2$ to yield a quadratic equation in θ and an expression for σ_a^2 in terms of θ.

5. These equations may have a solution for θ and σ_a^2, which then may be used for estimates $\hat{\theta}$ and $\hat{\sigma}_a^2$. In this case, if a solution to [3.5] exists, then either a) there is a double solution at $\theta = \pm 1$, or b) exactly one solution lies in the region of invertibility $|\hat{\theta}| < 1$. We use that invertible solution if it exists.

From equation [3.5], we see that

$$\frac{\sigma_a^2}{\sigma_c^2} = \frac{\eta}{\theta} \quad .$$

It can be shown that $\sigma_c^2 > \sigma_a^2$, thus $|\theta| > |\eta|$, and so if an estimate $\hat{\eta}$ of η were used to estimate θ, it would be biased toward zero.

Moving Average Measurement Error Models

The method above is developed for white noise measurement error. It can easily be extended to the case where measurement error has a moving average model. Such measurement error is found, for example, in periodic surveys with overlapping samples, as noted by several authors, including Miazaki (1985), who states that "... in the case of repeated surveys, where the same units appear in the sample on more than one occasion, it is reasonable to assume that the errors form a sequence of correlated random variables." In order to extend the estimation procedure above, we must have knowledge of the autocovariance function (ACF) of the error series, ε_t. It will be assumed that the

ACF of ε_t vanishes beyond the lag equal to the order of the model for X_t, or possibly earlier. Then the order of the moving average model for Y_t can be taken as equal to that of X_t. (In practice, it would work in reverse. We would use the order of the model for Y_t as the order of the X_t model. The order of the best Y_t model can be estimated as outlined in Box and Jenkins (1976, Chapter 6)). In the (seemingly rare) event in which the error series has order greater than the order of the X_t model, the estimation procedure could proceed as given. Those autocovariances of Y_t at lags greater than the order of the X_t model could be ignored, being equal to the corresponding autocovariances of ε_t .

The estimation procedure described above for white noise error can be used as shown to estimate the parameters of a moving average model for X_t when ε_t has a moving average model of positive order. The only differences are that the additional nonzero autocovariances of ε_t must be known, and these autocovariances enter into the first $q+1$ equations, where ε has nonzero autocorrelations up to lag q. An example will illustrate.

Example 3.2 - MA(1) Series Observed with Independent MA(1) Error

Let X_t have MA(1) representation

$$X_t = a_t - \theta a_{t-1} \quad .$$

Suppose X_t is observed with MA(1) error ε_t, and that the autocovariances $\gamma_\varepsilon(0)$ and $\gamma_\varepsilon(1)$ (which are nonzero) are known and that ε_t is independent of X_t. Then

$$\begin{aligned} Y_t &= X_t + \varepsilon_t \\ &= a_t - \theta a_{t-1} + \varepsilon_t \end{aligned} \qquad [3.6]$$

has a moving average representation of order at most 1. Suppose the order is equal to 1 and denote its MA(1) representation by

$$Y_t = c_t - \eta c_{t-1} \ . \qquad [3.7]$$

1. The equations relating the parameters of these series are derived from equating the two representations [3.6] and [3.7] of Y_t

$$c_t - \eta c_{t-1} = a_t - \theta a_{t-1} + \varepsilon_t \ . \qquad [3.8]$$

2. Upon expressing the variance and lag one covariance of Y_t from [3.8], the equations obtained are

$$(1 + \eta^2)\sigma_c^2 = (1 + \theta^2)\sigma_a^2 + \gamma_\varepsilon(0)$$

$$-\eta\sigma_c^2 = -\theta\sigma_a^2 + \gamma_\varepsilon(1)$$

These equations are equivalent to

$$\theta^2 + B\theta + 1 = 0$$

$$B = \frac{\gamma_\varepsilon(0) - (1 + \eta^2)\sigma_c^2}{\gamma_\varepsilon(1) + \eta\sigma_c^2} \qquad [3.9]$$

$$\sigma_a^2 = \frac{\eta\sigma_c^2 + \gamma_\varepsilon(1)}{\theta}$$

3. The observed values of Y_t are used to compute estimates, $\hat{\eta}$ and $\hat{\sigma}_c^2$, of η and σ_c^2.

4. These estimates are substituted into equations [3.9] for the parameters they estimate. The known values of $\gamma_\varepsilon(0)$ and $\gamma_\varepsilon(1)$ are also substituted into the equations [3.9].

5. The quadratic equation in θ is then solved, yielding $\hat{\theta}$, and this value is substituted into the last equation in [3.9] to yield $\hat{\sigma}_a^2$.

As before, if a solution exists to the equations [3.9], then there is one solution in the invertibility region or on its boundary, and we would use that solution for our parameter estimates.

Error Correlated with the Series Being Measured

The procedure described above cannot be applied directly to the case where the series ε_t is cross correlated with the series X_t. Two difficulties arise. Through a representation of ε_t, both of these difficulties are overcome.

First, we must account for cross covariances between X_t and ε_t. We shall assume that these cross covariances are known, and that the ACF of ε_t is known. We shall further assume that the cross covariances vanish after some lag no greater than the order of the model for X_t. It seems reasonable that the cross covariances between

$$X_t = \Theta(B)a_t \qquad [3.10]$$

and ε_t, the error of measuring X_t, persist no longer than the autocovariances of X_t.

It is also assumed that ε_t cannot lead X_t. This assumption is not valid if, for example, there is a feedback loop. In that case, past observations, and therefore past errors ε_t, may influence future values of X_t. We assume this is not the case, and so only non-negative lags of $\gamma_{xa}(k)$ will be considered.

The cross covariances $\gamma_{xa}(k)$ will be incorporated into the equations formed. In modeling error cross correlated with X_t , it will be convenient to introduce additional parameters. The additional parameters, however, will simplify the problem. The additional equations needed to provide

a system which is capable of having a nonempty set of solutions will come from expressions for the ACF of ε_t and the cross covariances between X_t and ε_t.

The second difficulty is that we cannot apply Theorem 3.1 directly, since Y_t is not written as the sum of two *independent* moving average series. The sum of two correlated finite moving average series need not have a finite moving average representation. The lack of independence would not permit us to write a finite moving average model for Y_t. With our assumptions, however, we can represent Y_t as the sum of independent moving average series. This is the major motivation for our representation of ε_t.

As mentioned, to circumvent both of these problems, a new representation of ε_t is used here. Consider a transfer function plus noise model for the error ε_t in terms of the series X_t. Let

$$\varepsilon_t = \nu(B) X_t + N_t$$

where N_t is a noise series independent of X_t. If there is a moving average representation of N_t as

$$N_t = \Psi(B) b_t$$

where b_t is a white noise series and $\Psi(B) = 1 - \psi_1 B - \psi_2 B_2 - \ldots$, then

$$\varepsilon_t = \nu(B) X_t + \Psi(B) b_t \qquad [3.11]$$

and b_t is independent of X_t. Since the cross covariances between X_t and ε_t vanish after a finite number of positive lags, $\nu(B)$ may be taken to be a finite polynomial in the backshift operator B. Also, since ε_t is assumed to have an ACF that vanishes after finitely many lags, $\Psi(B)$ may be assumed to be a finite polynomial.

Using representations [3.10] and [3.11], we may write

$$\begin{aligned} \varepsilon_t &= \nu(B)\,\Theta(B)\,a_t + \Psi(B)\,b_t \\ &= \pi(B)\,a_t + \Psi(B)\,b_t \end{aligned} \qquad [3.12]$$

where a_t and b_t are independent white noise processes. We now may express Y_t as

$$\begin{aligned} Y_t &= X_t + \varepsilon_t \\ &= \Theta(B)\,a_t + \pi(B)\,a_t + \Psi(B)\,b_t \qquad [3.13] \\ &= [\Theta(B) + \pi(B)]\,a_t + \Psi(B)b_t \ . \end{aligned}$$

In this last form, Y_t is written (except for a constant) as the sum of two independent finite moving average time series. We may now apply Theorem 3.1 and write Y_t in terms of its own moving average model

$$Y_t = \eta(B)\,c_t \ . \qquad [3.14]$$

The autocovariances of Y_t are now expressed in two ways using [3.13] and [3.14]. The ACF of ε_t and the cross covariances between X_t and ε_t are expressed in terms of the parameters of the models [3.10] and [3.12]. From these representations, a system of equations in the parameters of these models is formed.

The number of equations formed equals the number of parameters.

No. of Nonzero ACF Lags of Y_t = No. of Params for X_t

No. of Nonzero ACF Lags of ε_t = No. of Params in $\Psi(B)b_t$

No. of Nonzero Cross Covs = No. of Trans Fcn Params in $\nu(B)$

Therefore

No. equations = No. of Parameters to Estimate

and so there is hope for a solution. This system of equations is nonlinear and may be difficult to solve, may have no solutions, or may have more than one solution.

In order to calculate estimates of the parameters of X_t , as well as parameters of ε_t (about which we have only passing interest), we complete the procedure as before.

1. Write the models

$$Y_t = \eta(B)\, c_t = [\Theta(B) + \pi(B)]\, a_t + \Psi(B)\, b_t$$
$$\varepsilon_t = \pi(B)\, a_t + \Psi(B)\, b_t$$
$$X_t = \Theta(B)\, a_t$$

2. Construct equations from the ACF of Y_t, the ACF of ε_t, and the cross covariances of (X_t, ε_t).

3. Estimate the parameters of the model [3.14] using the observed values of Y_t.

4. Substitute the estimates from Step 3 and the known values of the $\gamma_\varepsilon(k)$ and $\gamma_{xx}(k)$, for $k = 0,1,2,...$ into the equations from Step 2.

5. Solve the system for the parameters of X_t and choose the solution(s) that fall in the invertibility region (if any) as estimates of the parameters of X_t.

An example will illustrate this method.

Example 3.3

Suppose that X_t is MA(1), that X_t is correlated with ε_t at lags 0 and 1 but not at lags higher than 1, and that ε_t is MA(1). We represent X_t as

$$X_t = a_t - \theta a_{t-1} \quad . \qquad [3.15]$$

Since the series X_t and ε_t are to be correlated at lag 0 and at lag 1, let

$$\pi(B) = \pi_o + \pi_1 B$$

in equation [3.12]. To allow any possible autocovariances of ε_t at lag 0 and at lag 1, let

$$\Psi(B) = 1 - \psi B$$

in [3.12]. We assume that $\gamma_\varepsilon(0)$, $\gamma_\varepsilon(1)$, $\gamma_{x\varepsilon}(0)$, and $\gamma_{x\varepsilon}(1)$ are known, and carry out the procedure.

1. Write

$$\varepsilon_t = \pi_0 a_t + \pi_1 a_{t-1} + b_t - \psi b_{t-1} \quad . \qquad [3.16]$$

$$Y_t = (1 + \pi_o)a_t + (\pi_1 - \theta)a_{t-1} + b_t - \psi b_{t-1}$$

and suppose the MA(1) model for Y_t is

$$Y_t = a_t - \eta a_{t-1}$$

2. The system of nonlinear equations relating parameters is

$$(1 + \eta^2)\sigma_c^2 = [(1 + \pi_0)^2 + (\pi_1 - \theta)^2]\sigma_a^2 + [1 + \psi^2]\sigma_b^2$$

$$-\eta\,\sigma_c^2 = (1 + \pi_0)(\pi_1 - \theta)\sigma_a^2 - \psi\sigma_b^2$$

$$\gamma_\varepsilon(0) = (\pi_0^2 + \pi_1^2)\sigma_a^2 + (1 + \psi^2)\sigma_b^2$$

$$\gamma_\varepsilon(1) = \pi_0\pi_1\sigma_a^2 - \psi\sigma_b^2 \qquad [3.17]$$

$$\gamma_{x\varepsilon}(0) = (\pi_0 - \pi_1\theta)\sigma_a^2$$

$$\gamma_{x\varepsilon}(1) = -\pi_0\,\theta\,\sigma_a^2 \quad .$$

3. The observations of Y_t are used to estimate η and σ_c^2

4. The estimates from Step 3, along with the values of $\gamma_\varepsilon(0)$, $\gamma_\varepsilon(1)$, $\gamma_{x\varepsilon}(0)$, and $\gamma_{x\varepsilon}(1)$ are substituted into the system of equations [3.17].

5. Equations [3.17] are solved for the unknown parameters. Since we are primarily interested in estimating θ and σ_a^2, the system of equations [3.17] reduces to solving the quartic equation

$$A\hat{\theta}^4 + B\hat{\theta}^3 + C\hat{\theta}^2 + D\hat{\theta} + E = 0 \qquad [3.18]$$

with

$$\hat{\sigma}_a^2 = \frac{[(1 + \hat{\eta}^2)\hat{\sigma}_c^2 - \gamma_\varepsilon(0) - 2\gamma_{x\varepsilon}(0)]}{\hat{\theta}^2 + 1}$$

where

$$A = \gamma_{x\varepsilon}(1) + \gamma_{\varepsilon}(1) + \hat{\eta}\hat{\sigma}_e^2$$

$$B = \gamma_{x\varepsilon}(0) + \gamma_{\varepsilon}(0) - (1 + \hat{\eta}^2)\hat{\sigma}_e^2$$

$$C = \gamma_{\varepsilon}(1) + \hat{\eta}\hat{\sigma}_e^2$$

$$D = \gamma_{x\varepsilon}(0)$$

$$E = \gamma_{x\varepsilon}(1)$$

If a solution to [3.18] exists which corresponds to an invertible model ($|\hat{\theta}| < 1$), then that solution is taken as $\hat{\theta}$.

Simulation Study

The estimates discussed above were compared using a simulation. The model chosen for simulation of X_t was the MA(1) model. MA(1) models were used for measurement error ε_t. These correspond to the situation covered by Example 3.2 and Example 3.3.

The purposes of this simulation were to compare the procedures described in Example 3.3 for error correlated with X_t against a) the estimate one would get if X_t were observed, b) the estimate one would get if one ignored measurement error, and c) the estimate one would get if one accounted for the ACF of the measurement error ε_t but ignored the cross correlations between X_t and ε_t.

To make these comparisons, a FORTRAN program, included as Appendix A, was written to

1. generate values for the series X_t and measurement error ε_t,

2. add X_t and ε_t to form Y_t,

3. estimate the parameters of a MA(1) model for the series Y_t,

4. use these parameter estimates for the Y_t model in equations [3.9] and [3.18], along with the ACF of ε_t and cross covariances between X_t and ε_t, to produce estimates of parameters for the model for X_t.

The time series X_t was generated using the IMSL subroutine RNARM, which also yields the white noise series a_t used to generate X_t. The measurement error ε_t was generated to be correlated with X_t using the model [3.16]. The values of a_t used to generate X_t were used as described by [3.16] to generate the portion of ε_t correlated with X_t. RNARM was also used to generate the portion of [3.16] that is not correlated with X_t (i.e., $b_t - \psi b_{t-1}$).

Parameter estimates were computed for the Y_t and X_t series using IMSL subroutine NSLSE, which uses an approximate least squares procedure.

The length of the observed record of Y_t used for estimating its parameters was set at 100 observations. In many economic time series, data is reported quarterly. In most of these, data has been collected for 25 to 30 years, thus 100 observations are available. The length of the series simulated was chosen to approximate this case. In many other time series (such as those for which monthly, weekly, or daily observations are available), a longer length of observed data is available.

The program simulated 5000 sets of series, thus creating 5000 sets of parameter estimates. In some cases, (nearly 2% of the observations), NSLSE failed to converge for one or more of the series. Those observations were eliminated and replaced to bring the total to 5000.

The estimates of θ (from model [3.10]) that were computed for each simulated series were

1. $\hat{\theta}_A$, the least squares estimate of θ calculated from the X_t values. (X_t is not observed in practice.)

2. $\hat{\theta}_B = \hat{\eta}$, a least squares estimate of the moving average parameter of the observed series, Y_t. In practice, if no information were available about the measurement error, $\hat{\theta}_B$ is the best estimator of θ that one could get.

3. $\hat{\theta}_C$, the estimator of θ if we use the ACF of ε_t, but ignore the cross correlation between X_t and ε_t. By considering $\hat{\theta}_C$, we may assess the effect of assuming independence of X_t and ε_t when they are cross correlated. Equations [3.5] were used to compute $\hat{\theta}_C$.

4. $\hat{\theta}_D$, the estimate of θ which takes into account the cross covariances between X_t and ε_t. Equations [3.18] were used to compute $\hat{\theta}_D$.

For each of the 5000 series X_t, three error series were generated. In this way, for each $\hat{\theta}_A$ value, three values were generated for each of the estimates $\hat{\theta}_B$, $\hat{\theta}_C$, and $\hat{\theta}_D$. The entire procedure was then repeated with three new sets of parameters for the error series ε_t, thus a total of six error series were used, and six sets of observed series, each of these simulated 5000 times. The error series parameters chosen are given in Table 2 on page 43. The parameters chosen for X_t were $\theta = -0.50$ and $\sigma_a^2 = 1.00$.

In some cases, where estimates of η and σ_ε^2 were poor, the estimation procedure using [3.9] to get $\hat{\theta}_C$, and/or the procedure using [3.18] to get $\hat{\theta}_D$, failed to yield a solution. (The values of $\hat{\eta}$ and $\hat{\sigma}_\varepsilon^2$ were inconsistent with any values of θ and σ_a^2.) The percentages of these cases are given in Table 3 on page 44 for each of the error series. For these series, none of the estimates was included when the means, variances, and mean squared errors of estimates were calculated.

The results of the simulation are summarized in Table 3 on page 44, where means, variances, and mean squared errors for each combination of error series and estimations procedure are given.

In this simulation, the procedure developed in this chapter for estimating θ (i.e. $\hat{\theta}_D$) is seen to work very well. The following comments summarize the results.

1. The least squares estimator $\hat{\theta}_A$ from X_t has no significant bias and a small mean squared error (MSE), as expected from standard time series estimation theory. The two runs (Run 1 - Error

Table 2. Parameters of Error Series for Simulation

	Error Series #1	Error Series #2	Error Series #3	Error Series #4	Error Series #5	Error Series #6
π_0	0.25	0.30	0.20	0.20	0.40	0.20
π_1	0.10	0.10	0.15	0.10	0.10	0.10
ψ	-0.15	0.20	-0.20	-0.20	-0.20	-0.20
σ_b^2	0.30	0.30	0.40	0.20	0.20	0.40
$\gamma_z(0)$	0.3793	0.412	0.4785	0.258	0.258	0.466
$\gamma_z(1)$	0.07	-0.03	0.11	0.06	0.06	0.10
$\gamma_{xz}(0)$	0.30	0.35	0.275	0.25	0.25	0.25
$\gamma_{xz}(1)$	0.125	0.15	0.10	0.10	0.10	0.10

Table 3. Simulation Means, Variances, and MSE's

	Error Series 1	Error Series 2	Error Series 3	Error Series 4	Error Series 5	Error Series 6
% With No Solution	2.26%	5.04%	5.54%	1.86%	1.16%	3.94%
$\hat{\theta}_A$ Mean	-.50100	-.50100	-.50100	-.50177	-.50177	-.50177
$\hat{\theta}_B$ Mean	-.41593	-.33190	-.44564	-.45656	-.40621	-.41937
$\hat{\theta}_C$ Mean	-.49031	-.40798	-.53669	-.50585	-.45484	-.50366
$\hat{\theta}_D$ Mean	-.50977	-.51245	-.50443	-.50781	-.50985	-.50851
$\hat{\theta}_A$ Standard Deviation	.09522	.09522	.09522	.09790	.09790	.09790
$\hat{\theta}_B$ Standard Deviation	.09108	.09019	.08647	.09223	.09413	.08910
$\hat{\theta}_C$ Standard Deviation	.13325	.15096	.14042	.12157	.12638	.14197
$\hat{\theta}_D$ Standard Deviation	.08906	.09809	.09716	.08492	.07649	.09852
$\hat{\theta}_A$ MSE	.00907	.00907	.00907	.00959	.00959	.00950
$\hat{\theta}_B$ MSE	.01536	.03639	.01043	.01039	.01766	.01444
$\hat{\theta}_C$ MSE	.01785	.02327	.02106	.01481	.01801	.02016
$\hat{\theta}_D$ MSE	.00803	.00978	.00946	.00727	.00595	.00978

Series 1,2,3; Run 2 - Error Series 4,5,6) yielded nearly equal means, variances, and MSE's for $\hat{\theta}_A$.

2. The least squared estimator $\hat{\theta}_B = \hat{\eta}$ from the observed (Y_t) series performs poorly as an estimator of θ. In general, $|\theta| > |\eta|$, and since $\hat{\eta}$ is the least squares estimator of η , it must be a poor estimator of θ in general. The large MSE is a result of the large bias in $\hat{\theta}_B$.

3. In the simulation, the estimator $\hat{\theta}_C$, computed by ignoring cross covariances between X_t and ε_t, had a bias that was smaller than the bias in $\hat{\theta}_B$. However, the $\hat{\theta}_C$ values had a large variance and hence a large MSE. In fact, in five of the six cases, the MSE of $\hat{\theta}_C$ was larger than the MSE of $\hat{\theta}_B$, where no information about the measurement error was used. The increased variance will be explained in Chapter 5.

4. The estimator $\hat{\theta}_D$ performed well. The bias in the estimates was small - between 4 and 8 standard errors of the mean. (The standard errors of the means of the $\hat{\theta}$ values were between 0.0011 and 0.0021 for all error series/estimation method combinations.) Variances were small, and so the MSE values were small. In fact, in 3 of the 6 cases, the MSE of $\hat{\theta}_D$ was smaller than the MSE of $\hat{\theta}_A$, which used the values of X_t directly.

The simulation study was a limited study. First, only a MA(1) model with $\theta = -0.50$ and $\sigma_a^2 = 1.00$ was considered. Only MA(1) measurement error series correlated with X_t at lags 0 and 1 were used. Finally, only six sets of parameters for the measurement error were used, and each of these measurement error series accounted for a fairly large proportion of the variation in Y_t. Despite these limitations, several conclusions are indicated.

1. Ignoring measurement error may lead to large bias in parameter estimates and therefore large mean squared error.

2. When measurement error is considered, but assumed falsely to be uncorrelated with the series being measured, we may get large MSE's. In some cases, ignoring the cross correlations be-

tween the series being measured and the measurment error series may be worse than ignoring the measurement error entirely.

3. The procedure outlined in this chapter works, at least for the model simulated. By taking into account the known cross covariances between error and X_t, and known ACF of the error series, estimates were obtained that were almost as good as those one could get if the X's were observed directly.

Measurement Error Autocovariances and Cross Covariances

The procedure developed in this chapter makes the assumption that the ACF of ε_t and the cross covariances between X_t and ε_t are known. Some information or data (in addition to the observed Y_t values) must be available to allow us to know or estimate these covariances (ACF and CCF). In some cases it may be difficult to assess these values, even when measurement error is obviously present. Some examples and suggestions may be of value.

One general suggestion may be helpful when control can be exercised over the amount of measurement error, i.e. when a measurement technique or device is available (perhaps at additional expense) that would allow very accurate measurements. In this case, it might be worthwhile to use the very accurate measurement device in addition to the "usual" device for a short period of time. Taking the accurate measurements as the "true" values, the measurement errors can be estimated for that period. These values could then be used to estimate the ACF of the measurement error and the cross correlations with the "true" (i.e. accurately measured) values.

In the area of sample survey techniques, some work is being done to use time series techniques in estimation of population values for repeated surveys. When rotation sampling techniques are used, the ACF of the sampling error can be estimated. Bell and Hilmer (1987) briefly discuss this possibility and its difficulties. They show that the sampling error, under mild assumptions, is

uncorrelated with the series of population values being estimated. They say "In principle, estimation of the sample error covariances, $\mathrm{Cov}(e_t, e_j)$, is the same problem as estimation of sample variances, $\mathrm{Var}(e_t)$, which is routinely done for many periodic surveys and for which many methods are available.... In practice, there may be difficulties in linking survey microdata over time to do this." Bell and Hilmer go on to discuss some of the problems associated with such estimation, and to use the estimated ACF of the sampling error to improve survey estimation. (Sampling error of the type discussed by Bell and Hilmer is not the only type of error encountered in survey estimation. Other error may be correlated with X_t.)

The principal focus of this dissertation is rounding error. In Chapter 4, estimation of the rounding error behavior is examined, and in particular, the autocovariance function of the rounding error and the cross covariance function between the series being rounded and the rounding error are estimated. That is, for rounding error we shall produce just the type of estimates of the error series ACF and cross covariances that are needed to apply the procedures of this chapter.

Chapter IV

ROUNDING ERROR DISTRIBUTIONS AND COVARIANCES

In this chapter, we consider the distribution and autocovariance function (ACF) of rounding error and the cross covariances between the series being rounded and its rounding error.

As in Chapter 3, we shall denote the series of interest by X_t , and we shall assume that X_t is a stationary, invertible ARMA(p,q) time series. We assume that its ARMA model is known, including the values of its parameters.

When data is rounded, the real line is partitioned into intervals of equal width, and any observation is rounded to a designated value in that interval in which it falls - usually the midpoint of the interval. Let us call the interval width R, a positive real number - the parameter of the rounding scheme. Any data value X is rounded to kR, where k is the (unique) integer such that

$$(k - .5)R \leq X < (k + .5)R \ .$$

Thus if R = 0.1 and X = 2.372, X is rounded to 2.4 = 24R since

$$(24 - .5)R = 2.35 \leq X < 2.45 = (24 + .5)R$$

Let $\varepsilon = Y - X$ be the rounding error. Large values of R correspond to what may be described as "heavy" rounding or "gross" rounding, and small values of R correspond to "light" rounding. As $R \to 0$, we approach no rounding, in which case Y = X. For any finite set of data, as R increases, a point must be reached beyond which all values would be rounded to zero. In this case, Y = k = 0 and $\varepsilon = -X$ for all observations.

The rounding error denoted by ε (or by ε_t when rounding a time series value X_t), is defined by $\varepsilon = kR - X$ where $(k - .5)R \leq X < (k + .5)R$. Since

$$k = \left[\frac{X + .5R}{R} \right]$$

where [•] denotes the greatest integer function,

$$\varepsilon = \left[\frac{X + .5R}{R} \right] R - X . \qquad [4.1]$$

If Y = kR is the observed value, we have

$$Y = X + \varepsilon$$

to conform to the notation for general measurement error introduced in Chapter 1.

The rounding parameter R takes on its significance relative to the variation in X. Whenever possible, we shall standardize (scale) by taking $\sigma_x^2 = 1$. Such scaling shall be noted when it takes place.

In Part I of Chapter 4, the limiting distribution of rounding errors as $R \to 0$ is given. Also in Part I, the limiting values of the cross covariances between X_t and ε_t are given. In Part II, the autocovariances and cross covariances of interest are computed for a range of values of R.

Part I - Limiting Distributions and Cross Covariances

In what follows, it will be convenient to use a slightly different form of rounding, commonly referred to as "rounding down" (or sometimes as truncation in the case where R is a power of 10). We define the rounding error ε^* by

$$\varepsilon^* = \left[\frac{X}{R}\right]R - X \ . \qquad [4.2]$$

The effect of this is that any value X is rounded to kR, where $kR \le X < (k+1)R$. This is done for simplicity of notation. It can easily be seen that

$$\varepsilon^* = \varepsilon + .5R \ .$$

It is clear that as $R \to 0$, $\varepsilon \to 0$ and $\varepsilon^* \to 0$. The limiting distribution of ε (and of ε^*) is degenerate - with all mass of its density at a single point. For this reason, we shall instead consider the distributions of

$$S = \frac{\varepsilon}{R} \quad \text{and} \quad S^* = \frac{\varepsilon^*}{R} \ . \qquad [4.3]$$

The values of S lie in the interval $[-.5, .5)$ for any positive value of R. The values of S^* lie in $[0, 1)$. The two lemmas that follow are presented for simplification of the proof of Theorem 4.1. Then in Theorem 4.1, the distribution of S^* is determined.

Lemma 4.1 - $\lim_{R\to 0} \sum_{k=-\infty}^{\infty} R \exp\left\{-\frac{(s+k)^2R^2}{2\sigma^2}\right\} = \sqrt{2\pi}\,\sigma$ for any s and any σ.

Proof - Consider the partition of the real line into $\{A_{k,R}\}_{k=-\infty}^{\infty}$ where

$$A_{k,R} = [(s+k)R, \ (s+k+1)R) \ .$$

The $A_{k,R}$ are intervals of the same width, with measure $m(A_{k,R}) = R$. Thus as $R \to 0$, $m(A_{k,R}) \to 0$ for all k. Now consider a Riemann integral based on this partition, letting $X_{k,R} = (s + k)R \in A_{k,R}$. Then

$$\begin{aligned}\sqrt{2\pi}\,\sigma &= \int_{-\infty}^{\infty} \exp\left(\frac{-X^2}{2\sigma^2}\right) dX \\ &= \lim_{m(A_{k,R})\to 0} \sum_{k=-\infty}^{\infty} \exp\left(\frac{-X_{k,R}^2}{2\sigma^2}\right) m(A_{k,R}) \\ &= \lim_{R\to 0} \sum_{k=-\infty}^{\infty} R \exp\left(\frac{-(s+k)^2R^2}{2\sigma^2}\right) \quad \blacksquare\end{aligned}$$

A lemma giving the density of S^* will be needed in the proof of Theorem 4.1.

Lemma 4.2 - If X has a $N(0,\sigma^2)$ distribution, if ε^* and S^* are as defined in [4.2] and [4.3], then the density of S^* is

$$f_{S^*}(s) = \sum_{k=-\infty}^{\infty} \frac{R}{\sqrt{2\pi}\,\sigma} \exp\left\{\frac{-(s+k)^2R^2}{2\sigma^2}\right\} \quad .$$

Proof -

$$
\begin{aligned}
F_{S^*}(s) &= P(S^* < s) \\
&= P(\varepsilon^* < Rs) \\
&= \int_0^{Rs} f_{\varepsilon^*}(\varepsilon)\, d\varepsilon \\
&= \int_0^{Rs} \sum_{k=-\infty}^{\infty} f_X(\varepsilon + kR)\, d\varepsilon \\
&= \int_0^{Rs} \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(\varepsilon + kR)^2}{2\sigma^2} \right\} d\varepsilon \\
&= \int_0^{s} \sum_{k=-\infty}^{\infty} \frac{R}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(s+k)^2 R^2}{2\sigma^2} \right\} ds
\end{aligned}
$$

The result follows immediately.■

Before presenting Theorem 4.1, which will give the limiting distribution of S^*, as $R \to 0$, a graphic motivation for the result may be helpful. If X is a normally distributed random variable, then the density of ε^* is the result of dividing the normal density into intervals [kR, (k + 1)R), and translating the function over each of these intervals to [0,R). The pieces are then added to yield the density of ε^*.

When these pieces are added, there is an "averaging" effect, and the density is much more level than most of the pieces, i.e., the density of ε^* is closer to the uniform density than are the pieces. This effect is illustrated in Figure 2 on page 53 to Figure 4 on page 55.

Theorem 4.1 - If X has a $N(0,\sigma^2)$ density, if ε^* is the error due to rounding X down to the next smaller multiple of R, and if $S^* = \varepsilon^*/R$, then as $R \to 0$, $S \xrightarrow{D} U[0,1]$.

Proof -

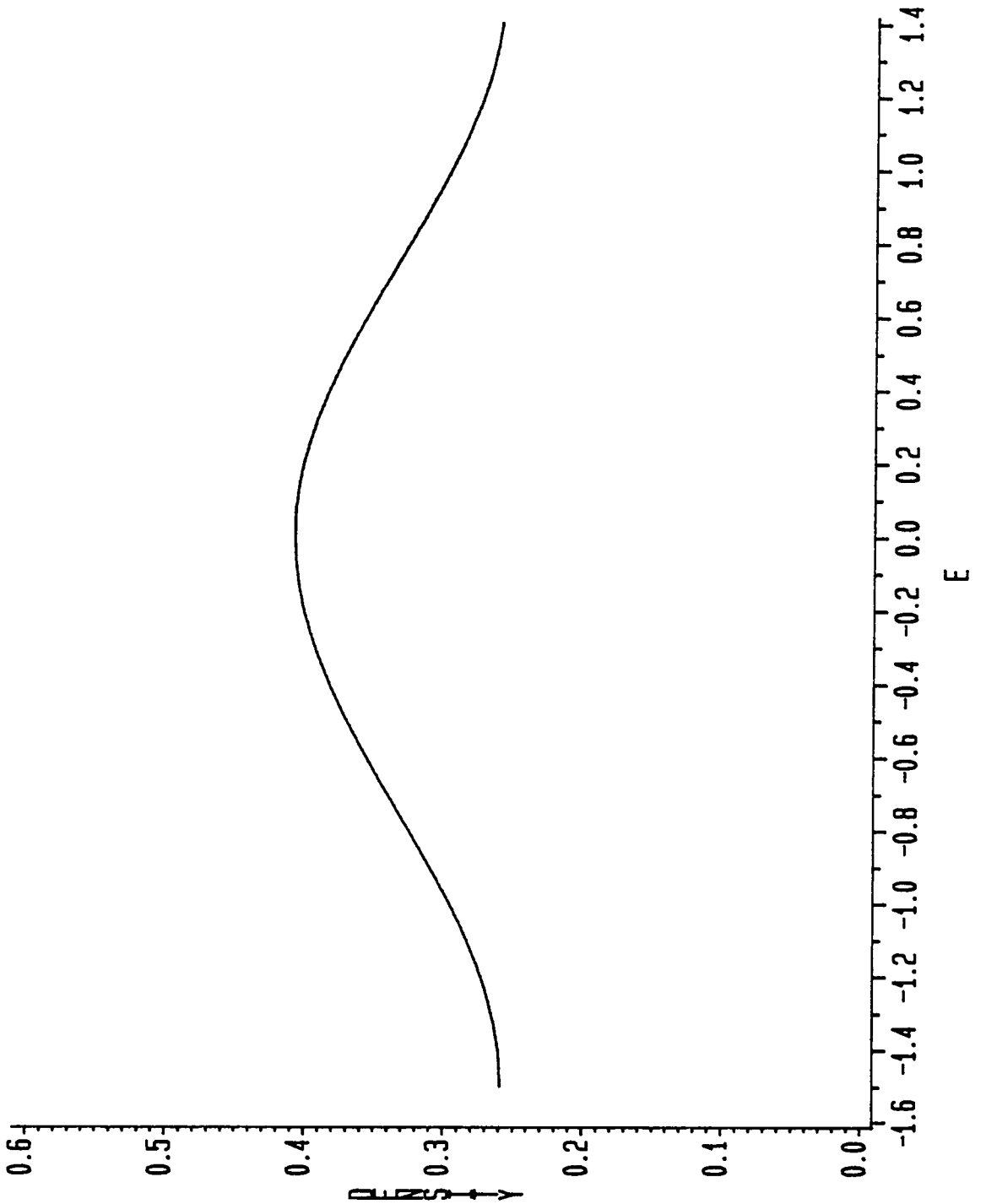

Figure 2. Normal distribution folded with R = 3.0

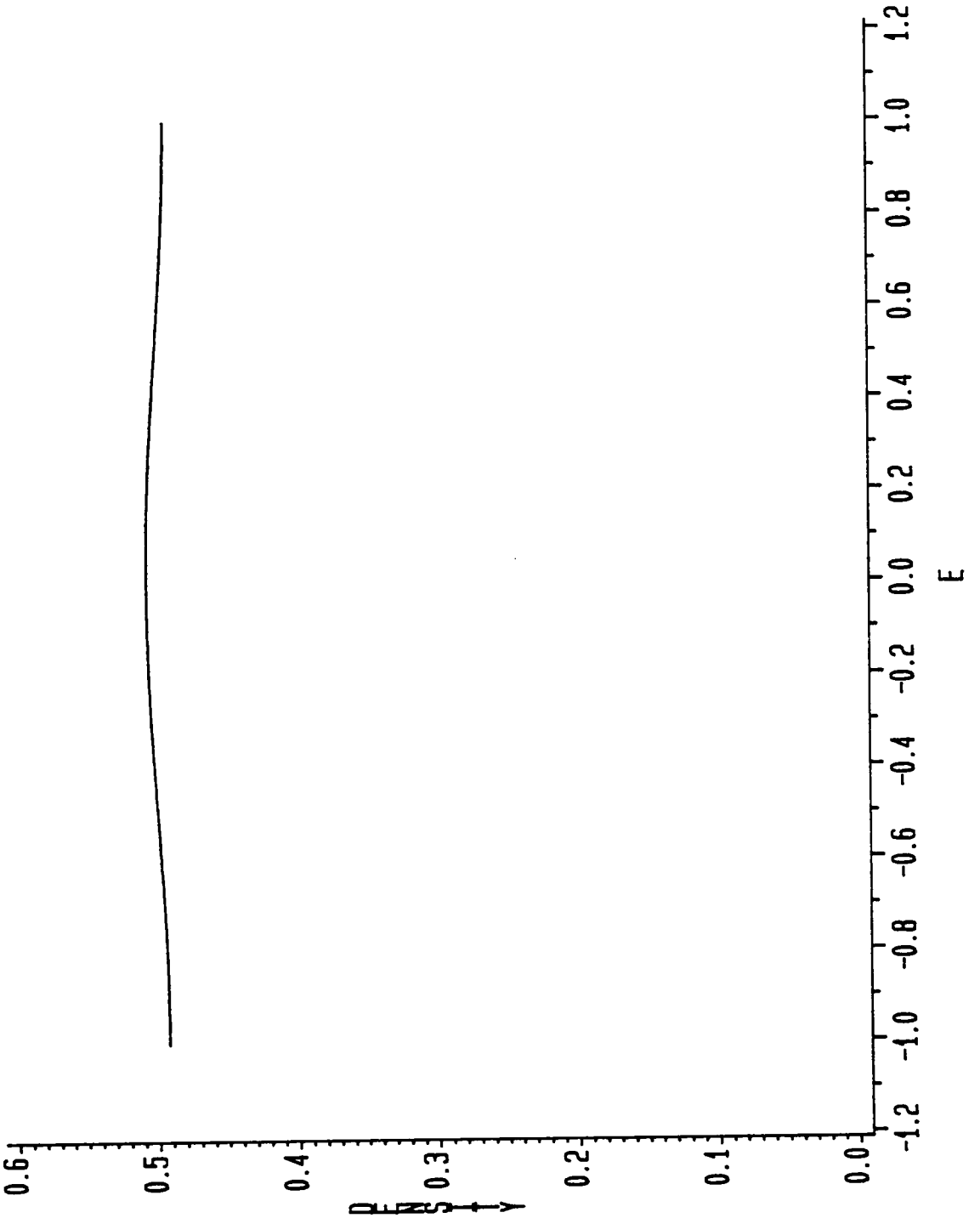

Figure 3. Normal distribution folded with R = 2.0

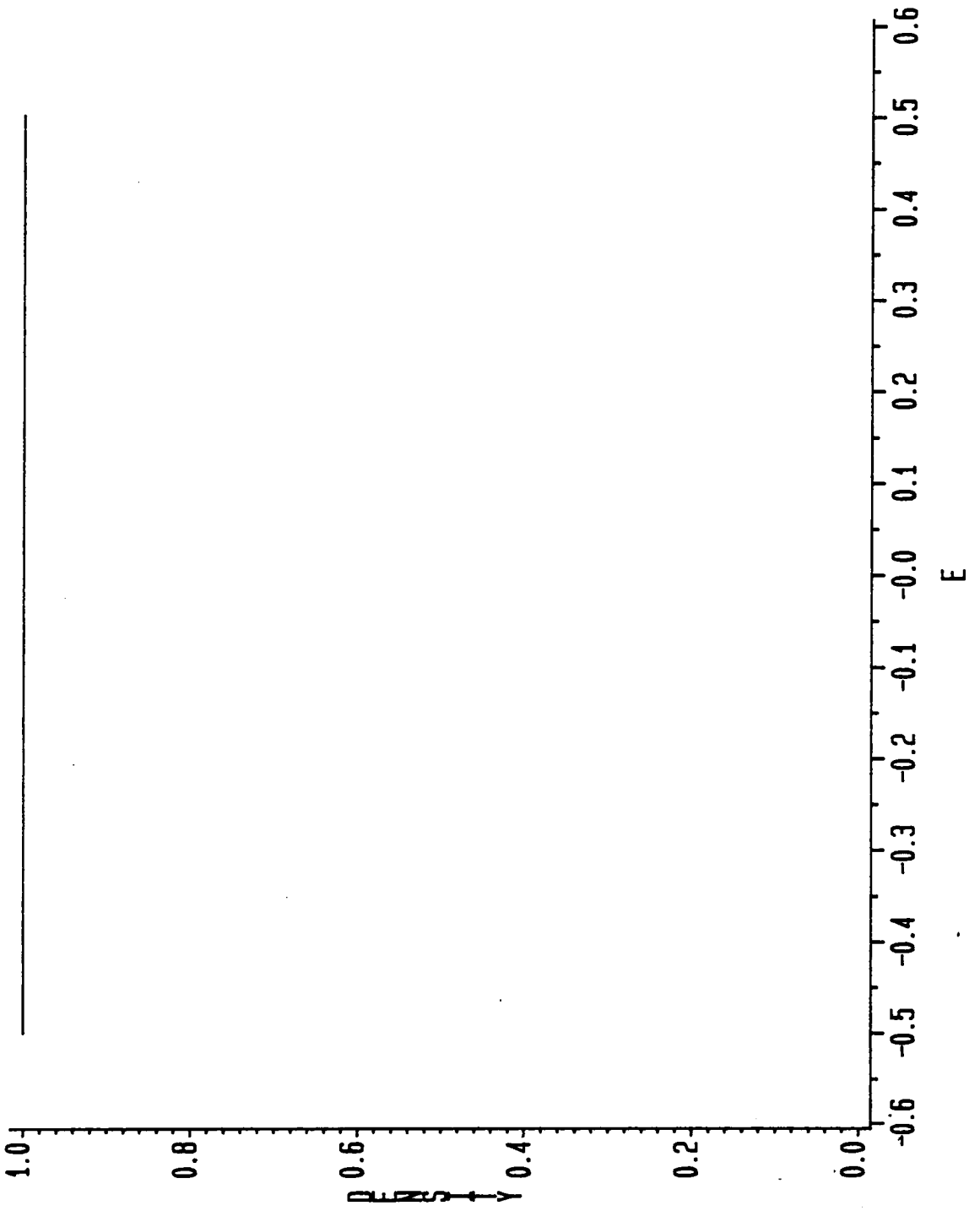

Figure 4. Normal distribution folded with R = 1.0

$$\lim_{R\to 0} F_{S^*}(s) = \lim_{R\to 0} \int_0^s \frac{1}{\sqrt{2\pi}\,\sigma} \sum_{k=-\infty}^{\infty} R \exp\left\{ -\frac{(S+k)^2R^2}{2\sigma^2} \right\} dS$$

$$= \int_0^s \frac{1}{\sqrt{2\pi}\,\sigma} \lim_{R\to 0} \sum_{k=-\infty}^{\infty} R \exp\left\{ -\frac{(S+k)^2R^2}{2\sigma^2} \right\} dS$$

$$= \int_o^s \frac{1}{\sqrt{2\pi}\,\sigma} (\sqrt{2\pi}\,\sigma)\, dS$$

$$= s \quad .$$

$F_Y(y) = y$ for $y \in [0,1]$ for a U[0,1] random variable, hence the result. The interchanging of the limit and the integral is justified by the Dominated Convergence Theorem, since for any $\alpha > 0$, an R can be found such that

$$\left| \sum_{k=-\infty}^{\infty} R \exp\left\{ -\frac{(s+k)^2R^2}{\sigma^2} \right\} \right| < \sqrt{2\pi}\,\sigma + \alpha \quad . \quad \blacksquare$$

Corollary 4.1 - If X has a $N(0,\sigma^2)$ distribution and if ε is the error due to usual rounding as defined in [4.1], then

$$S = \frac{\varepsilon}{R} \xrightarrow{D} U[-.5,.5] \text{ as } R \to 0 \ .$$

Proof - $\varepsilon = \varepsilon^* - .5R$ and so $S = S^* - .5$, and the corollary follows from Theorem 4.1. ■

Theorem 4.2 considers the bivariate distribution of the rounding errors due to rounding the two components of a bivariate normal distribution. Although our application to time series does not require it, the theorem will allow for different variances and rounding parameters for the two random variables being rounded. Lemma 4.3 will be stated to simplify the proof of Theorem 4.2.

The method of proof of Lemma 4.3 will only be indicated, the proof being very similar to that of Lemma 4.1.

Lemma 4.3 -

$$\lim_{\substack{R1\to 0\\ R2\to 0}} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} R_1 R_2 \exp\left\{ \frac{(s_1+k_1R_1)^2}{\sigma_1^2} + \frac{(s_2+k_2R_2)^2}{\sigma_2^2} - 2\rho \frac{\frac{(s_1+k_1R_1)(s_2+k_2R_2)}{\sigma_1\sigma_2}}{2(1-\rho^2)} \right\} \qquad [4.4]$$

$$= 2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}$$

Proof - The limit on the left side of equation [4.4] is converted into the form of a double integral through the Riemann integral formulation with regions

$$A_{k_1,k_2,R_1,R_2} = [(s_1+k_1)R_1,\ (s_1+k_1+1)R_1) \times [(s_2+k_2)R_2,\ (s_2+k_2+1)R_2) \ .$$

Then

$$m(A_{k_1,k_2,R_1,R_2}) = R_1 R_2$$

and as $R_1 \to 0$ and $R_2 \to 0$, the expression on the left side of [4.4] becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{ \frac{\frac{y_1^2}{\sigma_1^2} + \frac{y_2^2}{\sigma_2^2} - \frac{2\rho y_1 y_2}{\sigma_1\sigma_2}}{-2(1-\rho^2)} \right\} dy_2\, dy_1$$

Since $\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$ times the integrand above is the density of the bivariate normal distribution, the result follows from integrating that density over its range to get 1. ■

Theorem 4.2 - If (X_1, X_2) has a bivariate normal distribution with parameters $\mu_1 = 0$, $\mu_2 = 0$, σ_1^2, σ_2^2, and ρ, if ε_i^* and S_i^* are defined by [4.2] and [4.3] respectively, for i = 1,2 (ε_i^* is the error due to "rounding down") then $(S_1, S_2) \xrightarrow{R\to 0} U([0,\ 1]^2)$. That is, the limiting joint distribution of S_1 and S_2 is the distribution of independent random variables each with a uniform distribution over [0,1].

Proof - Let $C = \dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$. Then if $0 \le s_1 \le 1$ and $0 \le s_2 \le 1$,

$$\lim_{\substack{R_1 \to 0 \\ R_2 \to 0}} F_{(S_1^*, S_2^*)}(s_1, s_2) = F_{(\varepsilon_1^*, \varepsilon_2^*)}(s_1 R_1, s_2 R_2)$$

$$= \lim_{\substack{R_1 \to 0 \\ R_2 \to 0}} \int_0^{s_1 R_1} \int_0^{s_2 R_2} C \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} R_1 R_2 \exp\left\{ \frac{\dfrac{(\varepsilon_1 + k_1 R_1)^2}{\sigma_1^2} + \dfrac{(\varepsilon_2 + k_2 R_2)}{\sigma_2^2} - 2\rho \dfrac{(\varepsilon_1 + k_1 R_1)(\varepsilon_2 + k_2 R_2)}{\sigma_1 \sigma_2}}{-2(1-\rho^2)} \right\} ds_2\, ds_1$$

$$= \lim_{\substack{R_1 \to 0 \\ R_2 \to 0}} \int_0^{s_1 R_1} \int_0^{s_2 R_2} C \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} R_1 R_2 \exp\left\{ \frac{\dfrac{(s_1 + k_1)^2 R_1^2}{\sigma_1^2} + \dfrac{(s_2 + k_2)^2 R_2^2}{\sigma_2^2} - 2\rho \dfrac{(s_1 + k_1)(s_2 + k_2) R_1 R_2}{\sigma_1 \sigma_2}}{-2(1-\rho^2)} \right\} ds_2\, ds_1$$

$$= \int_0^{s_1} \int_0^{s_2} C \lim_{\substack{R_1 \to 0 \\ R_2 \to 0}} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} R_1 R_2 \exp\left\{ \frac{\dfrac{(s_1 + k_1)^2 R_1^2}{\sigma_1^2} + \dfrac{(s_2 + k_2)^2 R_2^2}{\sigma_2^2} - 2\rho \dfrac{(s_1 + k_1)(s_2 + k_2) R_1 R_2}{\sigma_1 \sigma_2}}{-2(1-\rho^2)} \right\} ds_2\, ds_1$$

$$= \int_0^{s_1} \int_0^{s_2} C\, 2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\; ds_2\, ds_1$$

$$= \int_0^{s_1} \int_0^{s_2} 1\; ds_1\, ds_2$$

$$= s_1 s_2$$

and $s_1 s_2$ is the joint distribution of independent uniform random variables when evaluated at s_1 and s_2. ■

Corollary 4.2 - If ε_1 and ε_2 are the errors from the usual rounding (defined in [4.1]) of X_1 and X_2 respectively, where (X_1, X_2) has a bivariate normal distribution with parameters $\mu_1 = 0$, $\mu_2 = 0$, σ_1^2, σ_2^2, and ρ, and if

$$S_1 = \frac{\varepsilon_1}{R_1} \quad \text{and} \quad S_2 = \frac{\varepsilon_2}{R_2}$$

then the limiting distribution of S_1 and S_2 is the distribution of independent uniform [-0.5 , 0.5] random variables.

Proof - S_1 and S_2 are translations of S_1^* and S_2^*

$$S_1 = S_1^* - 0.5 \qquad S_2 = S_2^* - 0.5 \ .$$

The result follows immediately from Theorem 4.2. ■

From Theorem 4.2 and Corollary 4.2, we can see that, as long as R is small, the joint distribution of the rounding error series $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ can be seen to be approximated well by the distribution of independent uniform random variables, with each ε_t approximately uniform over the range [-0.5 R, 0.5 R]. That is, the series ε_t is approximated well as uniform white noise with variance

$$\sigma_\varepsilon^2 = \frac{R^2}{12} \ .$$

We may therefore approximate the ACF of rounding error by

$$\gamma_\varepsilon(0) = \frac{R^2}{12}$$

$$\gamma_\varepsilon(1) = 0 \qquad \text{for } k \neq 0$$

if R is small. The determination of what values of R are considered "small" will be considered in Part II of this chapter.

In order to apply the procedure of Chapter 3 to the estimation of a time series X_t observed with rounding error ε_t, the cross covariance function between X_t and ε_t must be described. For small R, the cases are separated into concurrent cross covariance, $\gamma_{x\varepsilon}(0)$, and nonconcurrent cross covariance, $\gamma_{x\varepsilon}(k)$ for $k \neq 0$. The limits of these cross covariances as $R \to 0$ will be described in Theorems 4.3 and 4.4. Before stating Theorem 4.3, a lemma is stated to assist in the proof.

Lemma 4.3 -

$$\lim_{R \to 0} \sum_{k=-\infty}^{\infty} (k+s)R^2 \exp\left\{ \frac{-(k+s)^2 R^2}{2\sigma^2} \right\} = 0 \qquad [4.5]$$

Proof - The proof is again similar to the proof of Lemma 4.1 and so will be abbreviated here. The real line can again be broken up as in the proof of Lemma 4.1 in order to show that the limiting sum in [4.5] equals the Riemann integral

$$\lim_{R\to 0} \sum_{k=-\infty}^{\infty} (k+s)R^2 \exp\left\{ \frac{(k+s)^2R^2}{-2\sigma^2} \right\}$$

$$= \int_{-\infty}^{\infty} y \exp\left\{ -\frac{y^2}{2\sigma^2} \right\} dy$$

$$= \sqrt{2\pi}\,\sigma\, E(Y)$$

$$= 0$$

where Y has a $N(0,\sigma^2)$ distribution. ■

Theorem 4.3 - Let X be a random variable with a $N(0,\sigma^2)$ distribution. Let

$$S^* = \left[\frac{X}{R} \right] - \frac{X}{R} = \frac{\varepsilon^*}{R}$$

where ε^* is the error due to "rounding down" of X. Then

$$\lim_{R\to 0} Cov(X,S^*) = 0$$

Proof - Since ε^*, and therefore S^*, is completely determined by X, the joint distribution of X and S^* is simply $f_X(x)$. Then

$$Cov(X,\ S^*) = \int_{-\infty}^{\infty} xs^* f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} xs^* \exp\left\{ -\frac{x^2}{2\sigma^2} \right\} dx$$

$$= \int_0^1 \sum_{k=-\infty}^{\infty} (k+s^*)R\, s^* \exp\left\{ -\frac{(k+s^*)^2R^2}{2\sigma^2} \right\} R\, ds^*$$

By using the Dominated Convergence Theorem, we obtain

$$\lim_{R\to 0} Cov(X,S^*) = \lim_{R\to 0} \int_0^1 s^* \sum_{-\infty}^{\infty} (k+s^*)R^2 \exp\left\{-\frac{(k+s^*)^2R^2}{2\sigma^2}\right\} ds^*$$

$$\overset{D.C.T.}{=} \int_0^1 s^* \lim_{R\to 0} \sum_{-\infty}^{\infty} (k+s^*)R^2 \exp\left\{-\frac{(k+s^*)^2R^2}{2\sigma^2}\right\} ds^*$$

$$\overset{Lemma\ 4.4}{=} \int_0^1 s^* \cdot 0\, ds^*$$

$$= 0$$

Corollary 4.3 - If ε is the error due to "usual" rounding of a $N(0,\sigma^2)$ random variable X, if R is the rounding interval width, and if $S = \frac{\varepsilon}{R}$ then $Cov(X, S) \overset{R\to 0}{\to} 0$.

Proof - $S = S^* - 0.5$, so

$$Cov(X, S) = Cov(X S^*) = 0 \quad \blacksquare$$

From this corollary we can conclude that $\gamma_{x\varepsilon}(0)$ may be taken to be zero as long as R is small.

Following Theorem 4.3, it would be suprising if the cross correlations at nonzero lags did not converge to zero. Theorem 4.4 and its corollary show that the cross correlations do approach zero. First, two lemmas are stated to simplify the proof of Theorem 4.4. Their proofs are very similar to the proofs of Lemma 4.1 and Lemma 4.2 and are just outlined here.

Lemma 4.4 -

$$\lim_{R\to 0} \sum_{k=-\infty}^{\infty} \frac{R}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left\{\frac{X_1^2}{\sigma_1^2} + \frac{(S_2+k)^2R^2}{\sigma_2^2} - 2\rho\frac{X_1(S_2+k)R}{\sigma_1\sigma_2}\right\}\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{X_1^2}{2\sigma^2}\right\} \qquad [4.6]$$

Proof - The expression on the left side of [4.6] is the Riemann sum which converges to

$$\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left\{ \frac{X_1^2}{\sigma_1^2} + \frac{X_2^2}{\sigma_2^2} - 2\rho \frac{X_1 X_2}{\sigma_1\sigma_2} \right\} \right\} dX_2$$

which, being the integral of the joint density of X_1 and X_2 (random variables with a bivariate normal distribution) over all values of X_2, is the density of the normally distributed random variable X_1.

■

Lemma 4.5 - If X_1 and X_2 have a bivariate normal distribution, and if

$$S_2 = \frac{\varepsilon_2}{R} = \left[\frac{X_2}{R} \right] - \frac{X_2}{R}$$

then

$$f_{X_1,S_2}(x_1, s_2) = \sum_{k=-\infty}^{\infty} \frac{R}{\sqrt{2\pi}\,\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left\{ \frac{x_1^2}{\sigma_1^2} + \frac{(s_2+k)^2R^2}{\sigma_2^2} - 2\rho \frac{x_1(s_2+k)R}{\sigma_1\sigma_2} \right\} \right\}$$

Proof - Let $C = \dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$.

$$F_{X_1,S_2}(x_1, s_2) = P(X_1 < x_1,\ S_2 < s_2)$$

$$= \sum_{k=-\infty}^{\infty} P(X_1 < x_1,\ kR < X_2 < (s_2+k)R)$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{x_1} \int_{kR}^{(s_2+k)R} C \exp\left\{ \frac{-1}{2(1-\rho^2)} \left\{ \frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2} - 2\rho \frac{x_1 x_2}{\sigma_1\sigma_2} \right\} \right\} dx_2\, dx_1$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{x_1} \int_{0}^{s_2} C \exp\left\{ \frac{-1}{2(1-\rho^2)} \left\{ \frac{x_1^2}{\sigma_1^2} + \frac{(s_2+k)^2R^2}{\sigma_2^2} - 2\rho \frac{x_1(s_2+k)R}{\sigma_1\sigma_2} \right\} \right\} R\, ds_2\, dx_1$$

$$\overset{Fubini}{=} \int_{-\infty}^{x_1} \int_{0}^{s_2} \sum_{k=-\infty}^{\infty} C R \exp\left\{ \frac{-1}{2(1-\rho^2)} \left\{ \frac{x_1^2}{\sigma_1^2} + \frac{(s_2+k)^2R^2}{\sigma_2^2} - 2\rho \frac{x_1(s_2+k)R}{\sigma_1\sigma_2} \right\} \right\} ds_2\, dx_1$$

From the last line, the result follows immediately. ■

Theorem 4.4 - If (X_1, X_2) has a bivariate normal distribution with parameters $\mu_1 = \mu_2 = 0$, σ_1^2, σ_2^2, and ρ, and if $S_2^* = \dfrac{\varepsilon_2^*}{R} = \left[\dfrac{X_2}{R}\right] - \dfrac{X_2}{R}$ then $\lim\limits_{R\to 0} Cov(X_1, S_2^*) = 0$

Proof - Let $C = \dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$. Then

$$\lim_{R\to 0} Cov(X_1, S_2^*) = \lim_{R\to 0} E[X_1 S_2^*] - .5\,E[X_1]$$

$$= \lim_{R\to 0} \int_{-\infty}^{\infty}\int_0^1 x_1 s_2 f_{(X_1, S_2^*)}(x_1, s_2)\, ds_2\, dx_1$$

$$\overset{\text{Lemma 4.5}}{=} \lim_{R\to 0} \int_{-\infty}^{\infty}\int_0^1 x_1 s_2 \sum_{k=-\infty}^{\infty} R\,C\,\exp\left\{\frac{-1}{2(1-\rho^2)}\left\{\frac{x_1^2}{\sigma_1^2} + \frac{(s_2+k)^2R^2}{\sigma_2^2} - 2\rho\frac{x_1(s_2+k)R}{\sigma_1\sigma_2}\right\}\right\} ds_2\, dx_1$$

$$\overset{D.C.T.}{=} \int_{-\infty}^{\infty}\int_0^1 x_1 s_2 \lim_{R\to 0}\sum_{k=-\infty}^{\infty} R\,C\,\exp\left\{\frac{-1}{2(1-\rho^2)}\left\{\frac{x_1^2}{\sigma_1^2} + \frac{(s_2+k)^2R^2}{\sigma_2^2} - 2\rho\frac{x_1(s_2+k)R}{\sigma_1\sigma_2}\right\}\right\} ds_2\, dx_1$$

$$\overset{\text{Lemma 4.4}}{=} \int_{-\infty}^{\infty}\int_0^1 x_1 s_2 \exp\left\{-\frac{x_1^2}{2\sigma_2^2}\right\} ds_2\, dx_1$$

$$= \int_{-\infty}^{\infty} x_1 \exp\left\{-\frac{x_1^2}{2\sigma_1^2}\right\}\int_0^1 s_2\, ds_2\, dx_1$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x_1 \exp\left\{-\frac{x_1^2}{2\sigma_1^2}\right\} dx_1$$

$$= \frac{1}{2}\cdot 0$$

$$= 0$$

Corollary 4.4 - If ε_2 is the error due to the usual rounding of X_2, and if $S_2 = \dfrac{\varepsilon_2}{R}$, then $Cov(X_1, S_2) \to 0$ as $R \to 0$.

Proof - $S_2 = S_2^* - 0.5$ so $Cov(X_1, S_2) = Cov(X_1, S_2^*) = 0$. ■

Theorem 4.1 - Theorem 4.4 and their corollaries show that for small values of R, that the rounding error series ε_t can be approximated by uniform white noise uncorrelated with the series X_t being rounded. In Chapter 5, these results shall be used to estimate parameters of X_t when X_t is a moving average series which is observed only in rounded form.

It should be noted that Theorems 4.1 - 4.4 are not restricted to time series applications. For any bivariate normal random variables X_1 and X_2, these theorems hold and yield information about the rounding errors.

Part II - Autocovariances and Cross Covariances for General R

In Part I, we see that for "small" values of R, the rounding error series, ε_t, acts like uniform white noise. Two questions that now arise are answered in Part II.

1. What values of R are considered small?

2. What happens to the ACF of ε_t and the cross covariance function between X_t and ε_t when rounding error is not small?

These questions shall be answered by computing the relevant variance and covariances for random variables X_1 and X_2 with a bivariate normal distribution and their rounding errors. The means of X_1 and X_2 will be assumed to be zero for simplicity. Their variances will be assumed to be 1. The correlation, ρ , will be used as a parameter.

As in Part I, the results can be generalized to random variables X_1 and X_2 outside the time series setting. A few adjustments would be needed to allow for different variances or different rounding parameters for X_1 and X_2.

In assuming that $\sigma_1^2 = \sigma_2^2 = \sigma^2 = 1$, we have scaled the time series. In application, one would need to scale R by σ in order to use the values given below. The determination of the definition of a "small" value of R in order to use the results of Part I of this chapter would have to be relative to the variance of X. Once scaled, the conclusions can be stated for this scaled R.

Variance of ε_t - For small R, we have shown that ε_t is approximately uniform over [-0.5R,0.5R]. From this, we know that for small R,

$$\sigma_\varepsilon^2 \simeq \frac{R^2}{12}$$

increasing approximately quadratically as R increases.

The variance of ε, denoted by σ_ε^2 or by $\gamma_\varepsilon(0)$, may be calculated as follows. (Recall that σ^2 = Var(X) = 1)

$$\begin{aligned}
\sigma_\varepsilon^2 &= \int_{-.5R}^{.5R} \varepsilon^2 f_\varepsilon(\varepsilon)\, d\varepsilon \\
&= \int_{-.5R}^{.5R} \varepsilon^2 \sum_{k=-\infty}^{\infty} f_X(kR+\varepsilon)\, d\varepsilon \qquad [4.7] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-.5R}^{.5R} \varepsilon^2 \sum_{k=-\infty}^{\infty} \exp\left\{ -\frac{(kR+\varepsilon)^2}{2} \right\} d\varepsilon
\end{aligned}$$

The value of σ_ε^2 may be approximated as closely as desired by replacing the integral by a Riemann sum with a sufficiently fine partition of the real line, and truncating the infinite sum after sufficiently many terms in each direction from k = 0.

$$\sigma_\varepsilon^2 \simeq \frac{1}{\sqrt{2\pi}} \sum_{i=-M}^{M} \left(\frac{iR}{2M} \right)^2 \sum_{k=-N}^{N} \exp\left[\frac{(k + \frac{i}{2M})^2 R^2}{2} \right] \frac{R}{2M}$$

for sufficiently large M and N.

A FORTRAN program written to approximate the last integral in [4.7] is included as Appendix B. The results of executing that program are plotted in Figure 5 on page 67 and Figure 6 on page 68. These show clearly that $\gamma_\varepsilon(0)$ rises as the square of R until approximately R = 2.0. The standard deviation of ε (Figure 6) rises approximately linearly with R until R = 2.0. Beyond that point, the distribution of ε is deviating significantly from a uniform distribution, and so the variance is not that of the uniform distribution. As R increases, the distribution of ε is being transformed to a unimodal distribution - eventually appearing like a truncated normal distribution. Thus as R increases, the distribution of ε has more of its area concentrated around the mean, and so smaller variance than a uniform distribution over the same interval. As $R \to 0$, $\sigma_\varepsilon \to 1 = Var(X)$, since for large R, most of the values are rounded to zero and hence $\varepsilon = -X$.

ACF of ε_t at Positive Lags - The Autocovariances of the rounding errors at all nonzero lags can be calculated from the autocorrelations of the series X_t. $\gamma_\varepsilon(k)$ depends only on $\rho_X(k)$ and the rounding parameter R. If $\sigma_1 = \sigma_2 = 1$, we calculate $\gamma_\varepsilon(k)$ as $\mathrm{Cov}(\varepsilon_1, \varepsilon_2)$ by

$$Cov(\varepsilon_1, \varepsilon_2) = \int_{-.5R}^{.5R} \int_{-.5R}^{.5R} \varepsilon_1 \varepsilon_2 f_{\varepsilon_1 \varepsilon_2}(\varepsilon_1, \varepsilon_2)\, d\varepsilon_2\, d\varepsilon_1 .$$

$$\text{where } f_{\varepsilon_1 \varepsilon_2}(\varepsilon_1, \varepsilon_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left\{ (\varepsilon_1 + k_1 R)^2 + (\varepsilon_2 + k_2 R)^2 - 2\rho(\varepsilon_1 + k_1 R)(\varepsilon_2 + k_2 R) \right\} \right\} \qquad [4.9]$$

where ε_1 and ε_2 may be any two measurement errors at different times.

Integral [4.9] can be approximated by truncating the infinite sums and replacing the integrals by a Riemann sum, as done for σ_ε^2. A FORTRAN program to calculate $\gamma_\varepsilon(k)$ appears as Appendix C. The results are plotted in Figure 7 on page 70. In Figure 8 on page 71 are the corresponding autocorrelations, found by dividing the autocovariance by σ_ε^2 (because $\sigma_x^2 = 1$).

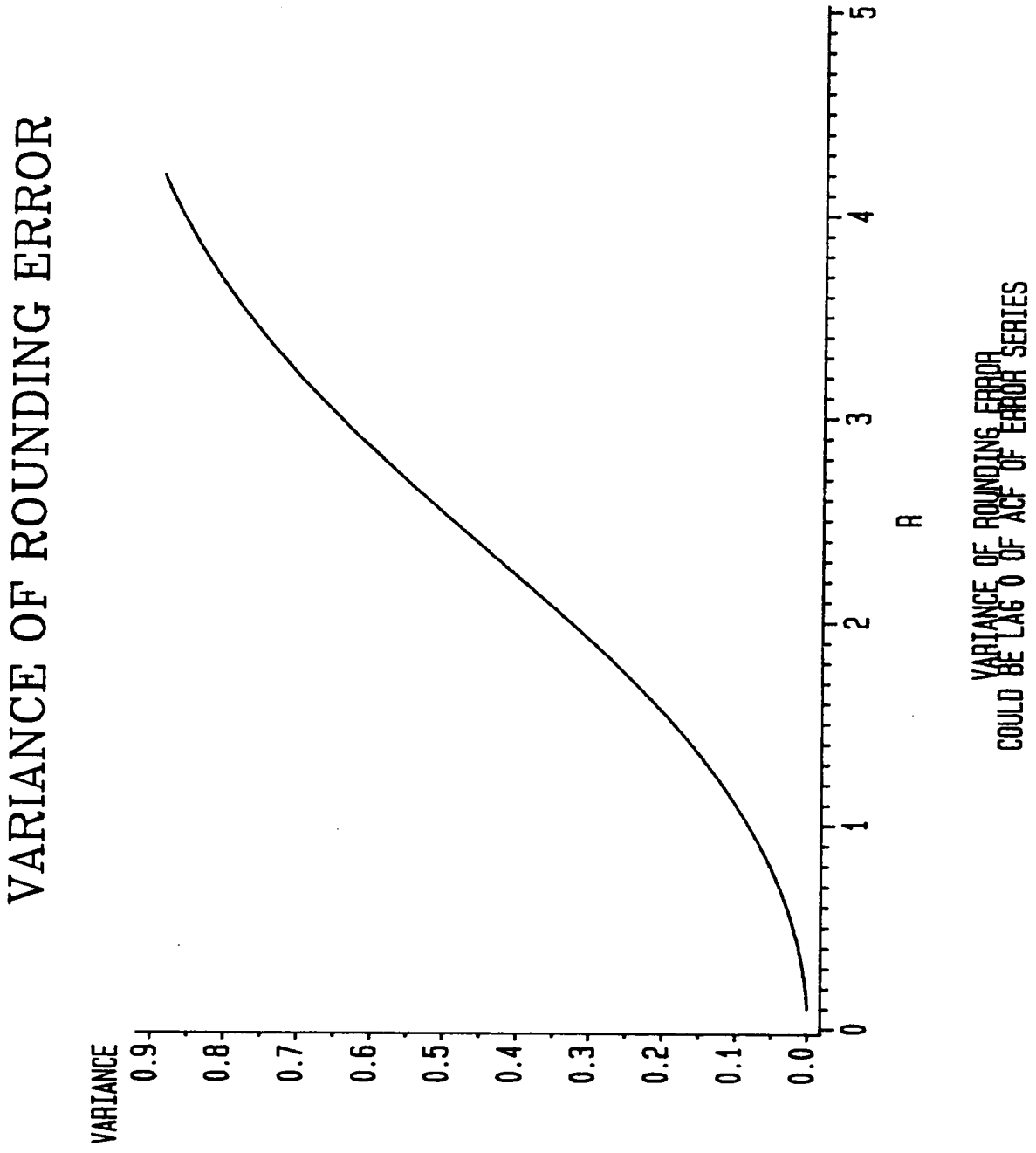

Figure 5. Variance of Rounding Error ε_t

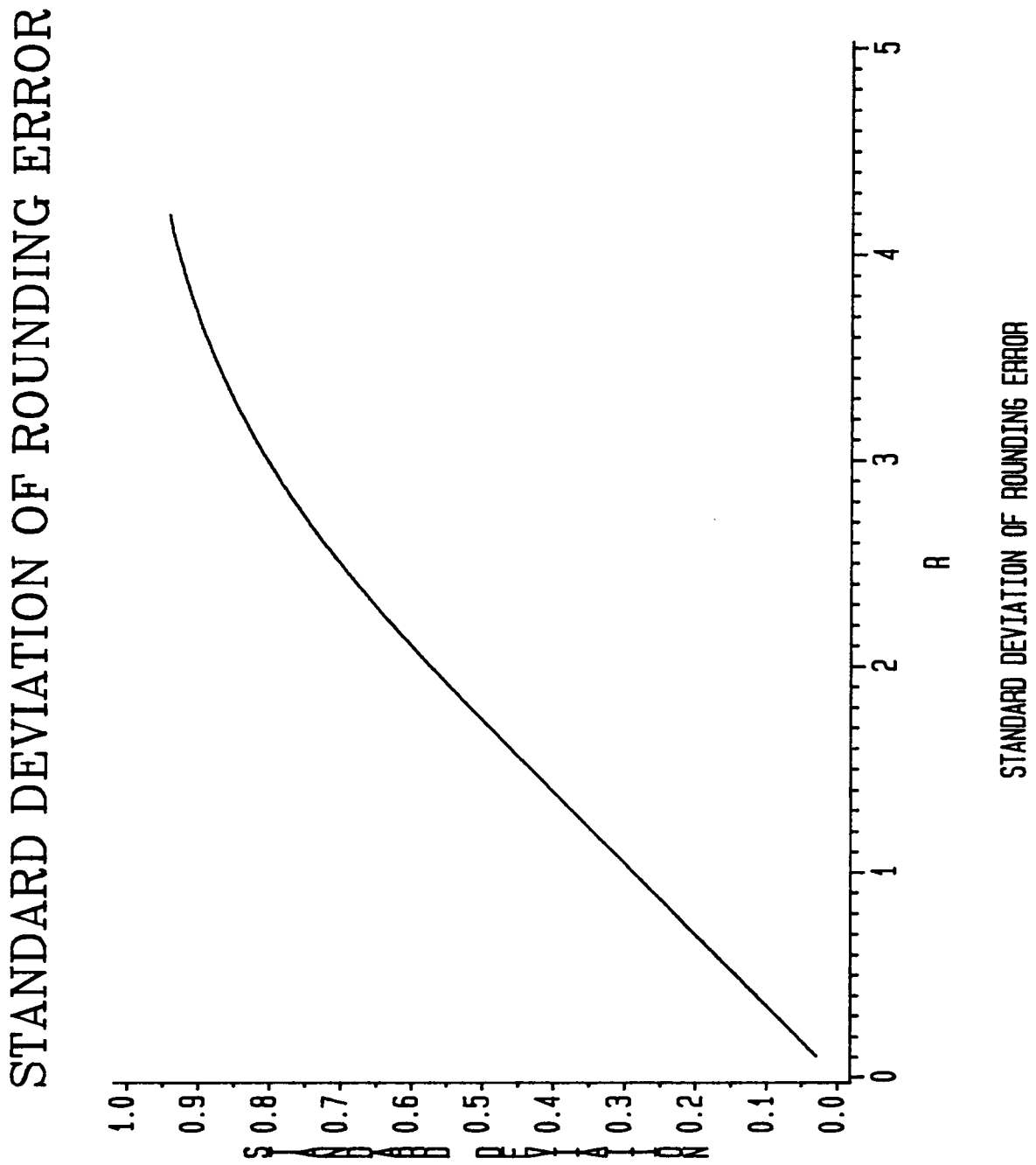

Figure 6. Standard Deviation of Rounding Error ε_t

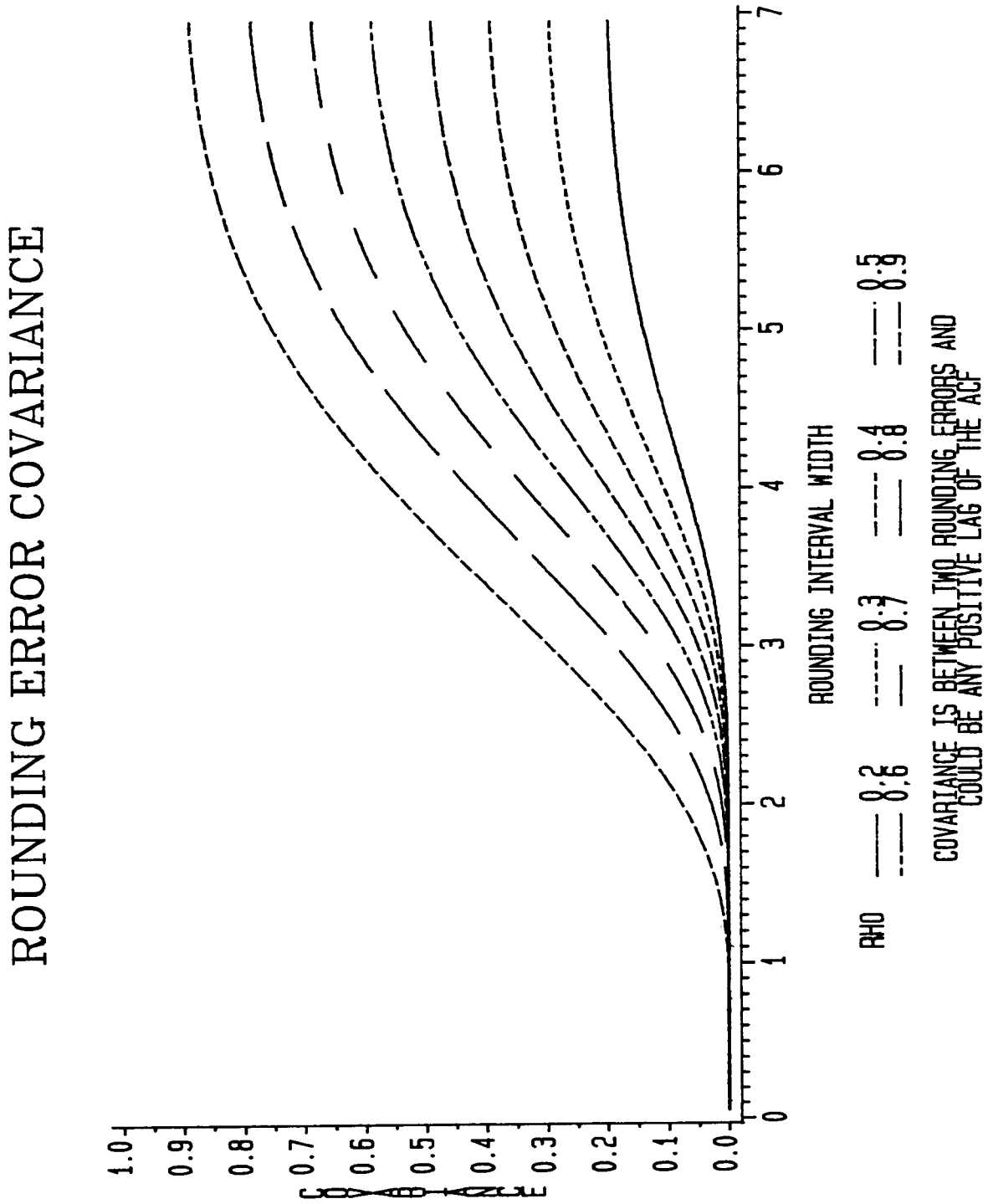

Figure 7. Autocovariance of Rounding Error ε_t at Nonzero Lags

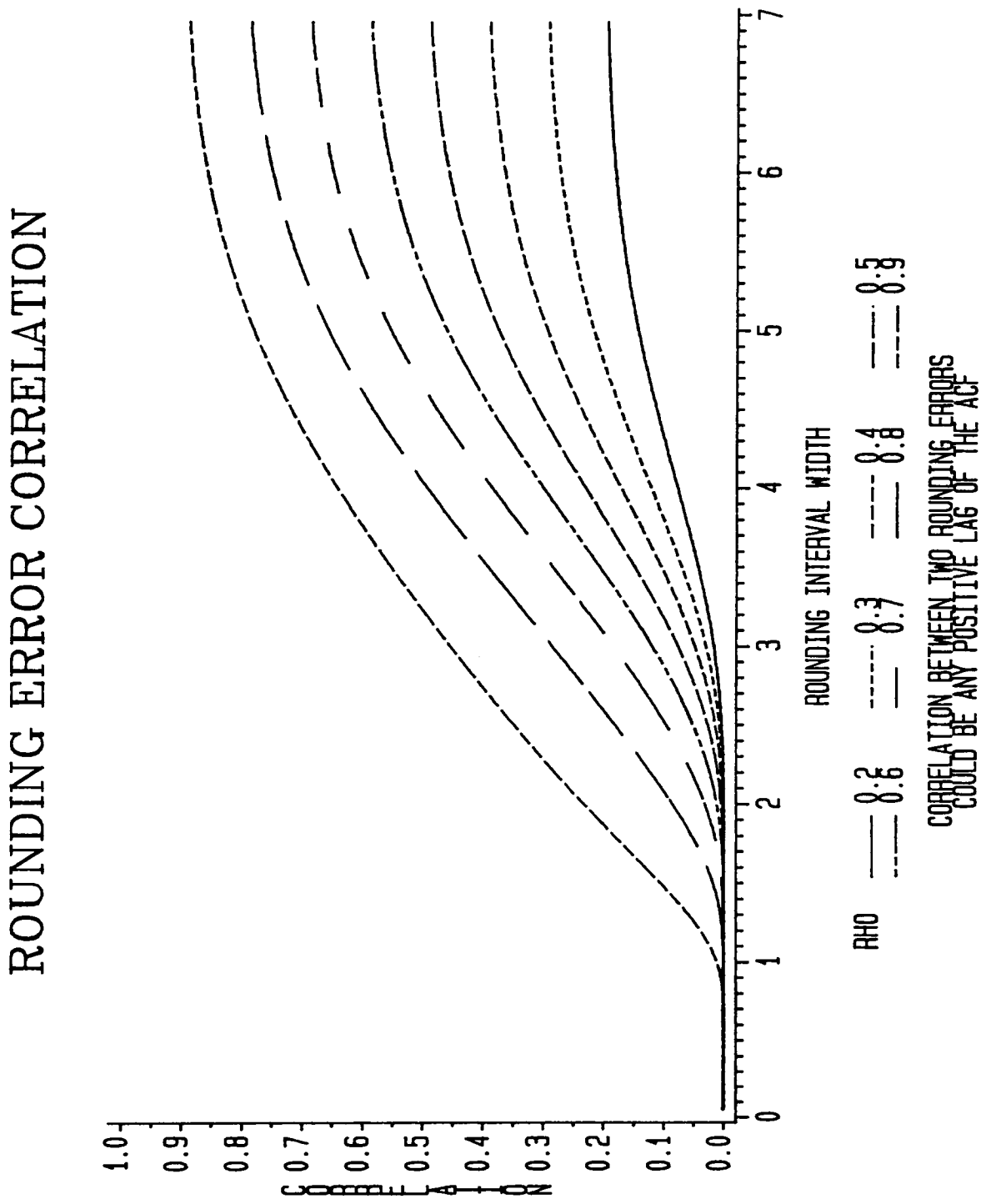

Figure 8. Autocorrelation of Rounding Error ε_t at Nonzero Lags

The use of Figure 7 and Figure 8 for a time series whose autocovariance funtion is known would require one to

1. Scale the series X_t to have unit length (or at least calculate the scaling factor).

2. Determine the effective rounding parameter R by scaling the rounding interval width by the standard deviation of X_t.

3. Determine the autocorrelations of X_t.

4. Use these autocorrelations and Figure 7 (Figure 8) to find the autocovariances (autocorrelations) for ε_t. (Similar figures could be used for values of ρ not adequately approximated by Figure 7 and Figure 8, or the program in Appendix 3b could be altered to produce the autocovariances (autocorrelations) for given values of R and ρ.

Inspecting Figure 7 and Figure 8, one can see that the covariance between ε_1 and ε_2 is very close to zero for small values of R. The point at which ε_1 and ε_2 have significant positive correlation depends on $\rho = Corr(X_1, X_2)$. This point occurs between R = 1 and R = 2 for the values of ρ shown. For negative values of ρ, the corresponding positive value can be used to find the absolute value of the covariance or correlation desired. That is, for $\rho = -0.5$, use $\rho = 0.5$ and find Cov $(\varepsilon_1, \varepsilon_2)$. Then take the negative of this value for the desired value.

As $R \to \infty$, we have $\varepsilon_1 = -X_1$ and $\varepsilon_2 = -X_2$, and so

$$Corr(\varepsilon_1, \varepsilon_2) = Corr(X_1, X_2) = \rho \quad .$$

Cross Covariances Between X_t and ε_t - The cross covariance $\gamma_{x\varepsilon}(0)$ is calculated as follows.

$$\gamma_{x\varepsilon} = \int_{-\infty}^{\infty} x\varepsilon f_X(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \int_0^1 (\varepsilon + kR)\varepsilon \exp\left\{ -\frac{(\varepsilon + kR)^2}{2} \right\} d\varepsilon$$

This integral can be approximated by truncating the sum and replacing the integral by a Riemann sum, as done previously. A FORTRAN program to compute values of this integral appears in Appendix D. Plots of the cross covariance $\gamma_{xx}(0)$ and the cross correlation

$$\rho_{x\varepsilon}(0) = \frac{\gamma_{x\varepsilon}(0)}{\sigma_\varepsilon \sigma_X} = \frac{\gamma_{x\varepsilon}(0)}{\sigma_\varepsilon}$$

appear in Figure 9 on page 73 and Figure 10 on page 74, respectively.

To use these plots to find $\gamma_{xx}(0)$ or $\rho_{xx}(0)$ for a series with known autocovariance function, one need only scale the rounding parameter R as suggested before, and read the covariance (correlation) corresponding to this scaled R from Figure 9 (Figure 10). For the covariance, one would need to return to the original scale by multiplying by σ_x.

The cross covariance is very close to zero until $R \simeq 2$. Past R = 2, the covariance (correlation) falls steadily until R = 5. For R > 5, $\gamma_{xx}(0)$ turns to approach -1.0 as $R \to \infty$, ($\rho_{xx} \to -1.0$ as $R \to \infty$)

Cross Covariances Between X_t and ε_t at Positive Lags - First, note that for rounding error, as opposed to general measurement error, the cross covariance function must be symmetric about the zero lag ($\gamma_{xx}(k) = \gamma_{xx}(-k)$). This occurs because ε_t depends entirely upon X_t and the rounding parameter R. Since the ACF of X_t is symmetric about the zero lag, so must the CCF for (X_t, ε_t). It is therefore only necessary to consider positive lags of $\gamma_{xx}(k)$.

We consider $\gamma_{xx}(k)$ by considering

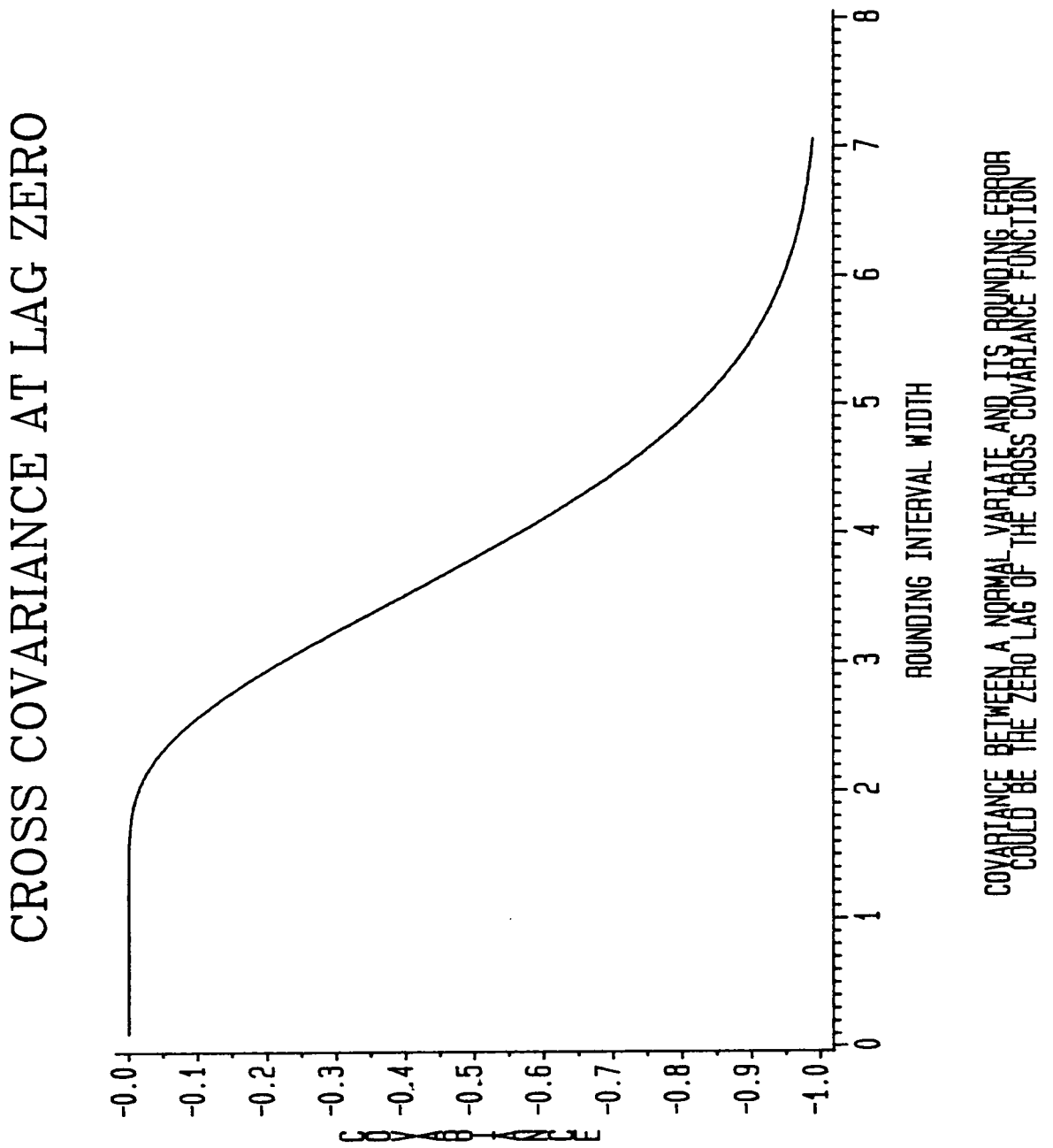

Figure 9. Cross Covarance of a Moving Average Series with Its Rounding Error at Lag Zero

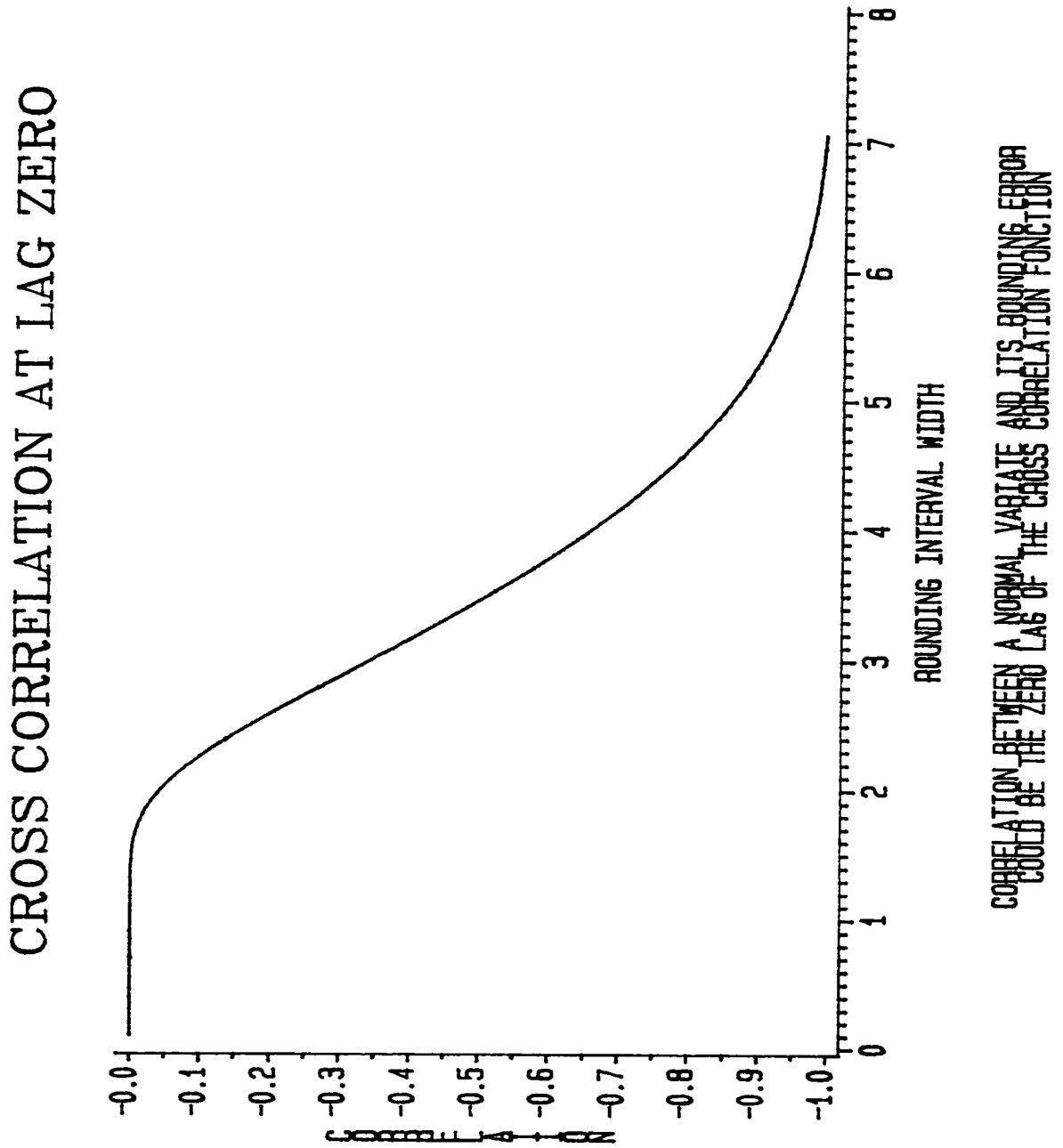

Figure 10. Cross Correlation of a Moving Average Series with Its Rounding Error at Lag Zero

$$Cov(X_1, \varepsilon_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \int_{0}^{1} \sum_{-\infty}^{\infty} x_1 \varepsilon_2 \exp\left\{ \frac{-1}{2(1-\rho^2)} \left\{ x_1^2 + (\varepsilon_2 + kR)^2 - 2\rho x_1 (\varepsilon_2 + kR) \right\} \right\}$$

where $\rho = Corr(X_1, X_2)$ and ε_2 is the error from rounding X_2 with rounding parameter R. A FORTRAN program used to approximate this integral, as for the previous covariances, appears as Appendix E. The results are plotted in Figure 11 on page 76. In Figure 12 on page 77, the corresponding cross correlations are plotted.

For small values of R (at least up to r = 1.5), the cross correlations are not significantly different from zero. They decline from zero until they approach $-\rho$ as R increases.

Taken together, Figure 5 on page 67 to Figure 12 on page 77 indicate that "small" values of the rounding parameter R, as discussed in Part I of this chapter, should be taken to be defined by $R \leq 1.0$. For values of R not larger than 1.0 (scaled by σ_x), it is reasonable to model the rounding error series, ε_t, as uniformly distributed white noise. The variance of ε_t is approximately $R^2/12$. It is also reasonable to assume that ε_t is not cross correlated with the series, X_t being rounded.

For values of R that are not small, Figure 5 to Figure 12 may be used to find the ACF of ε_t and the cross covariances between ε_t and X_t . Whether R is small or not small, the theorems and figures from this chapter provide all the information required to estimate the parameters of X_t using the procedures of Chapter 3. This application will be considered in Chapter 5.

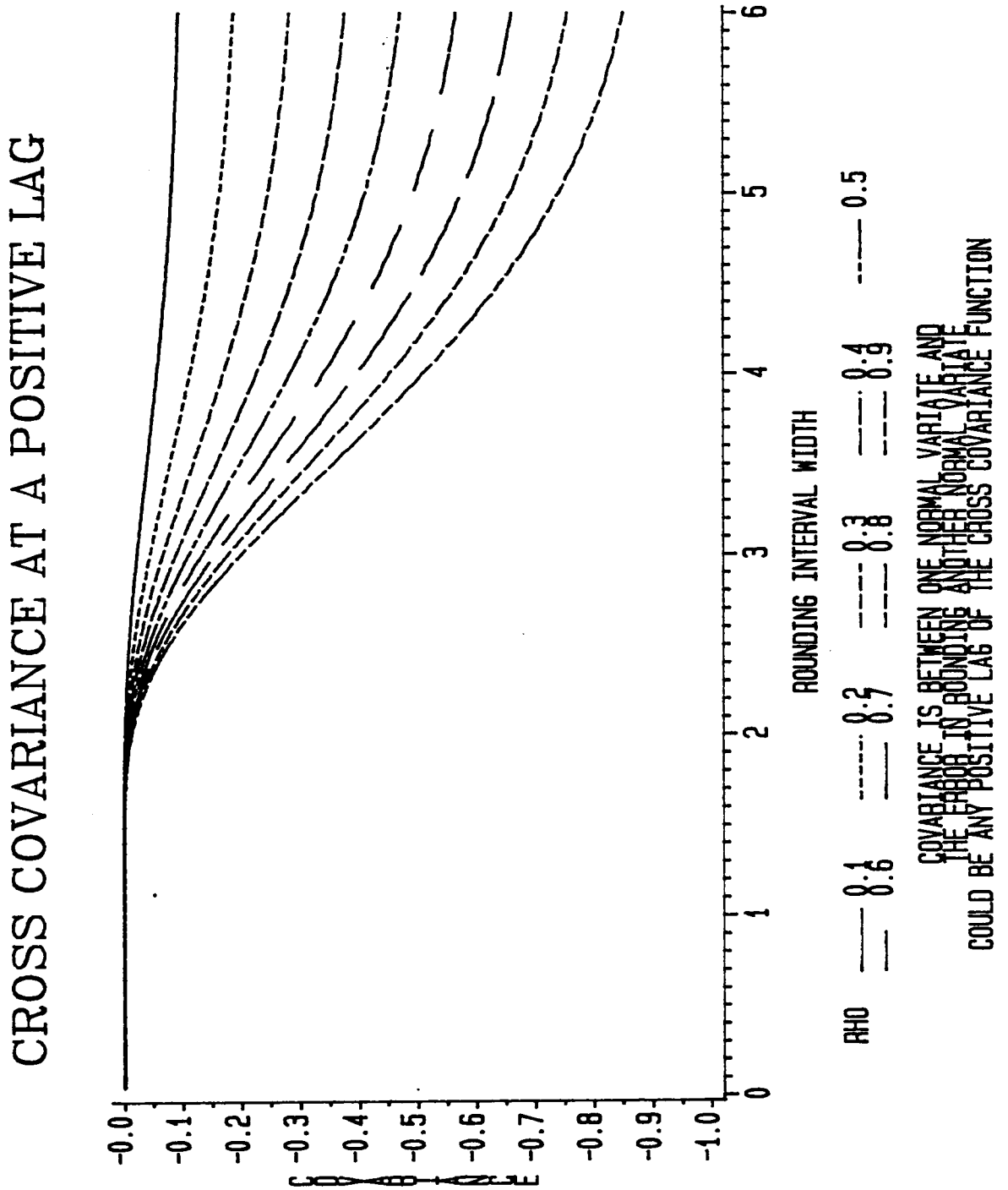

Figure 11. Cross Covarance of Moving Average Series with Its Rounding Error at Nonzero Lags.

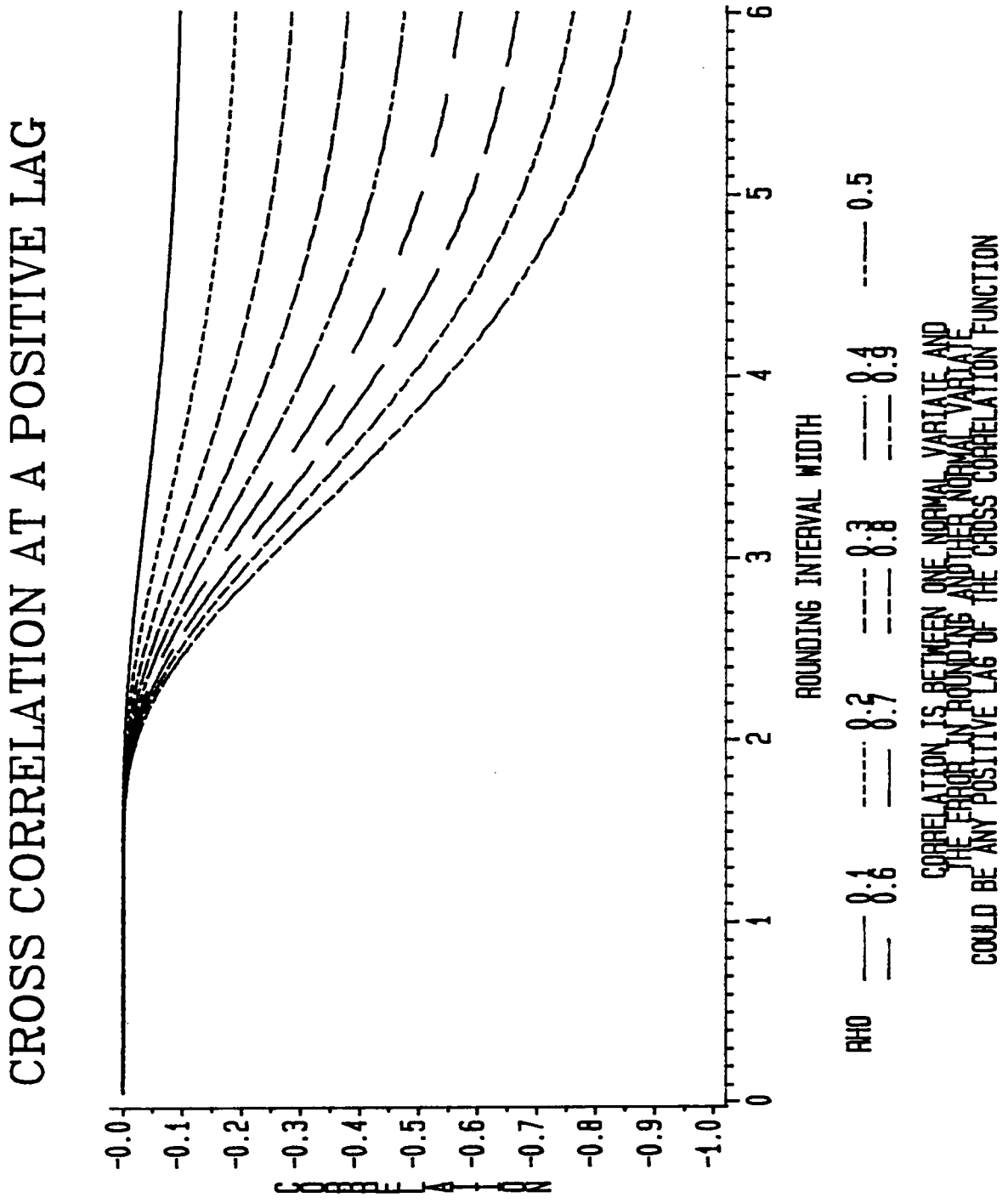

Figure 12. Cross Correlation of Moving Average Series with Its Rounding Error at Nonzero Lags.

Chapter V

MOVING AVERAGE PARAMETER ESTIMATION AFTER ROUNDING

In this chapter, elements of Chapter 3 and Chapter 4 are combined to estimate parameters of moving average time series which are observed after rounding. The procedure of Chapter 3 for general measurement error shall be used for parameter estimation. The autocovariances and cross covariances that this procedure requires are provided by the results of Chapter 4.

The two cases - small R (light rounding) and large R (heavy rounding) - shall be considered separately. The case of light rounding ($R \leq 1.0$ after scaling) shall be considered first. The more complicated, and more unusual, case of gross or heavy rounding ($R > 1.0$) shall be considered last.

Light Rounding - The Small R Case

The results developed in Chapter 4 show that if, after scaling, $R \leq 1.0$, then the rounding error series is approximated well by white noise with a uniform distribution over its range, $-0.5\,R \leq \varepsilon_t \leq 0.5\,R$, and that the series ε_t is uncorrelated with the series, X_t, being rounded. The parameter estimation procedure developed in Chapter 3 requires knowledge of the autocovariance function of ε_t and its cross covariances with X_t. These are known from Chapter 4 to be approximated well by

$$\gamma_\varepsilon(0) = \frac{R^2}{12}$$

$$\gamma_\varepsilon(k) = 0 \qquad \text{for all } k \neq 0 \qquad [5.1]$$

$$\gamma_{x\varepsilon}(k) = 0 \qquad \text{for all k .}$$

We can then apply the procedure of Chapter 3.

1. Represent the observed series Y_t in terms of its own moving average model and the model for X_t plus ε_t.

2. Use the ACF of Y_t, written in the two forms, to create a system of equations in the parameters of these models.

3. Use the observed Y_t to estimate the parameters of the moving average model for Y_t.

4. Substitute the estimates from Step 3 and $\gamma_\varepsilon(0) = R^2/12$ into the system of equations from Step 2.

5. Solve the system of equations (if possible) for the parameters of the original model for X_t, selecting a solution corresponding to an invertible model for X_t.

Example 5.1 -Suppose X_t is similar to the model simulated in Chapter 3,

$$\begin{aligned} X_t &= a_t - \theta\, a_{t-1} \\ &= a_t + .5a_{t-1} \end{aligned}$$

with $\theta = -0.5$ and $\sigma_a^2 = 20$. (We used $\sigma_a^2 = 1$ before.) Suppose X_t is rounded to the nearest unit, that is,

$$Y_t = [X_t + .5] \quad \text{and} \quad \varepsilon_t = Y_t - X_t$$

where [•] indicates the greatest integer function. For our example, the rounding parameter R is

$$\begin{aligned} R &= \frac{\text{Rounding Interval Width}}{\text{Standard Deviation of } X_t} \\ &= \frac{1.0}{\sqrt{(1 + (-0.5)^2)20}} \\ &= 0.2 \end{aligned}$$

Since R is considered small, as defined in Chapter 4, the ACF of the rounding error series can be approximated by

$$\begin{aligned} \gamma_\varepsilon(0) &= \frac{R^2}{12}(VarX_t) \\ &= \frac{0.04}{12}(25) \\ &= \frac{1}{12} \\ &= \frac{[\text{Rounding Interval Width}]^2}{12} \end{aligned}$$

$$\gamma_\varepsilon(k) = 0 \quad \text{for } k \neq 0$$

It is worth noting that the representation

$$\gamma_\varepsilon(0) = \frac{[\text{Rounding Interval Width}]^2}{12}$$

does not depend on the parameters of the model for X_t. One only needs to know $\text{Var}(X_t)$ approximately in order to decide whether R is "large" or "small." In practice, the sample variance of Y_t could be used as a rough approximation. If the estimated R exceeds 1.0, it would be conservative to assume R is "large."

For our example, $\gamma_\varepsilon(0) = \frac{1}{12}$. The procedure is carried out as follows.

1. $c_t - \eta c_{t-1} = a_t - \theta a_{t-1} + \varepsilon_t$

2. Representing the variance and lag 1 covariance of the models on the two sides of the equation in Step 1, we have

$$(1 + \eta^2) = (1 + \theta^2)\sigma_a^2 + \gamma_\varepsilon(0)$$

$$-\eta \sigma_c^2 = -\theta \sigma_a^2$$

These equations are equivalent to

$$\theta^2 + B\theta + 1 = 0$$

$$\sigma_a^2 = \frac{\eta}{\theta}\sigma_c^2 \qquad [5.2]$$

$$B = \frac{1}{\eta}\left[\frac{\gamma_\varepsilon(0)}{\sigma_c^2} - (1 + \eta^2)\right]$$

3. The estimates $\hat{\eta}$ and $\hat{\sigma}_c^2$ are computed using the observed values of Y_t.

4. The estimates from Step 3 are substituted into [5.2] along with $\gamma_x(0) = 1/12$ to yield

$$\theta^2 + B\theta + 1 = 0$$

$$\sigma_a^2 = \frac{\hat{\eta}}{\theta}\,\hat{\sigma}_c^2$$

$$B = \frac{1}{\hat{\eta}}\left[\frac{1.0}{12\,\hat{\sigma}_c^2} - (1 + \hat{\eta}^2)\right]$$

5. The equations above are solved for θ and σ_a^2 to yield estimates $\hat{\theta}$ and $\hat{\sigma}_a^2$.

It should be noted here that the estimation of the variance of X_t by subtracting $R^2/12$ from the variance of Y_t, is similar to Sheppard's correction for the second moment of a random variable when observed after grouping. A description of Sheppard's correction can be found in Kendall (1948, pg. 64 - 81).

Simulation -The procedure above was simulated using the model

$$X_t = a_t + .5\,a_{t-1} \qquad (\theta = -0.5)$$

with $\sigma_a^2 = 20$. Then the variance of X_t is $\sigma_x^2 = 25$ and the "scale factor" is $\sigma_x = 5$. Rounding interval widths of 1.0, 3.0, and 5.0 were used. The scaled values of the rounding parameter, R, were R = 0.2, R = 0.6, and R = 1.0 respectively. All of these are in the "small R" case.

Because of some interesting results for series of length N = 100, the results to be noted below, each of the values of R was used in a simulation with N = 500 and N = 1000 as well. The results appear in Table 4 on page 83.

The simulation shows that if mean squared error is the criterion for choosing a procedure, our choice must depend on the amount of rounding (R), and the variation among the estimates $\hat{\eta}$

Table 4. Results of MA(1) Rounding Simulation for Parameter Estimates.

R	N	$\hat{\theta}$ Mean	$\hat{\eta}$ Mean	$\hat{\theta}$ MSE	$\hat{\eta}$ MSE for θ	$\hat{\sigma}_a^2$ Mean	$\hat{\sigma}_c^2$ Mean
0.2	100	-0.4997	-0.4966	0.00913	0.00892	19.47	19.58
0.2	500	-0.5015	-0.4986	0.00164	0.00161	19.95	20.06
0.2	1000	-0.5001	-0.4973	0.00074	0.00073	19.96	20.08
0.6	100	-0.5029	-0.4765	0.01075	0.00931	19.60	20.62
0.6	500	-0.4998	-0.4759	0.00176	0.00207	19.91	20.91
0.6	1000	-0.5024	-0.4785	0.00096	0.00128	19.85	20.94
1.0	100	-0.5143	-0.4463	0.01410	0.00117	19.62	22.42
1.0	500	-0.5007	-0.4403	0.00214	0.00498	19.81	22.50
1.0	1000	-0.5000	-0.4405	0.00130	0.00440	19.98	22.60

and $\hat{\sigma}_c^2$, as determined by N. Specifically, the following observations can be made from the simulation results.

1. The estimates $\hat{\theta}$ have very little bias for estimating θ, while the estimates $\hat{\eta}$ as estimates of θ have bias that depends on the rounding parameter R.

2. For R = 0.2, the mean squared errors for $\hat{\theta}$ and $\hat{\eta}$ are very nearly equal, with slight advantage to $\hat{\eta}$. In no case would this difference be significant at $\alpha < .10$.

3. For R = 0.6 and R = 1.0 the estimate ($\hat{\theta}$ or $\hat{\eta}$) producing the smaller MSE depends upon the length of the series. $\hat{\theta}$ performed better (smaller MSE) for longer series (N = 500 and N = 1000) while $\hat{\eta}$ is better for a short series (N = 100).

The second and third observations above may be explained by the variance in the estimates. For a fixed R (and therefore fixed σ_ϵ^2), the variance among the $\hat{\eta}$ values decreases as N increases. The values of $\hat{\theta}$ must have larger variance than the $\hat{\eta}$ values as may be seen in Figure 13 on page 85 in which the values of $\hat{\eta}$ and the corresponding values of $\hat{\theta}$ are plotted, for some fixed σ_ϵ^2. The slope of the curve is less than 1.0 everywhere. Since $\hat{\theta}$ is a function of $\hat{\eta}$, we have

$$Var(\hat{\theta}) > Var(\hat{\eta}) \quad .$$

For R = 0.2, very little adjustment is made, since $\sigma_\epsilon^2 = .003\overline{3}...$ in standardized units. The bias in $\hat{\eta}$ values is slight, and the variances of $\hat{\theta}$ and $\hat{\eta}$ are close, so $MSE(\hat{\theta}) \simeq MSE(\hat{\eta})$. This explains Conclusion 2 above.

For larger values of R, the adjustments to $\hat{\eta}$ which produce $\hat{\theta}$ increase the variance of the estimates, but decrease the bias. If the bias is small to moderate, and the variance among the $\hat{\eta}$ values (and therefore among the $\hat{\theta}$ values) is large, the increased variance in $\hat{\theta}$ overwhelms the decreasing bias, and the MSE increases. This was the case for N = 100.

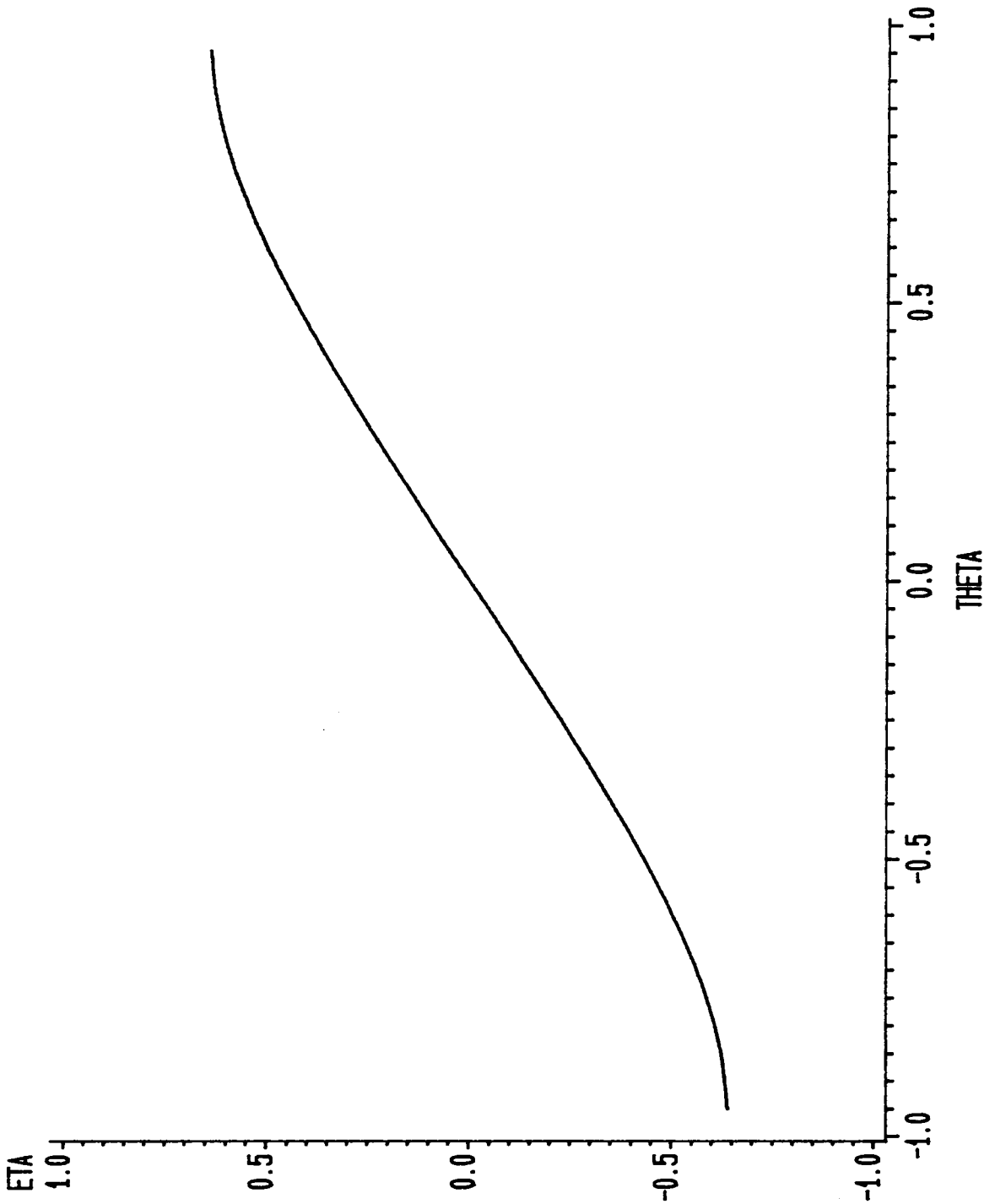

Figure 13. Correspondence Between the Parameters of Observed Series and the Parameters of the Series Being Rounded

If the variance in $\hat{\eta}$ values is small and the bias of $\hat{\eta}$ is moderate to large, the reverse occurs. The reduction in bias of $\hat{\theta}$ from the bias of $\hat{\eta}$ reduces the MSE by enough to overcome the increased variance. For N = 500 and N = 1000 in this simulation, the variances are relatively small, and so the increase in variance of $\hat{\theta}$ over the variance of $\hat{\eta}$ is small. The reduction in bias need not be very large, then, to produce a reduced MSE for $\hat{\theta}$.

Heavy Rounding - The Large R Case

For heavy rounding, the rounding error series is no longer uniform white noise. Autocovariances of ε_t at positive lages are not necessarily zero, nor are the cross covariances between X_t and ε_t necessarily zero. The variance of ε_t drops off from the variance of a uniform variate for very large R.

Figure 5 on page 67 to Figure 12 on page 77 can be used to find the ACF of ε_t and the cross covariances between X_t and ε_t for series X_t with a known autocovariance function. However, X_t is a moving average series with unknown parameters, and unknown ACF. Because of this, these charts cannot be used directly to estimate parameters of X_t.

To circumvent this difficulty, the following procedure is suggested.

1. Estimate the moving average model parameters of the observed series Y_t and its autocovariance function.

2. Use the ACF of Y_t as an initial estimate of the ACF of the series X_t.

3. Use the estimated ACF of X_t to estimate the covariance of interest. (Note that even the value of R must be estimated because of the scaling by the standard deviation of X_t .) Figure 5 to Figure 12, or programs similar to those in Appendix 3, can be used to estimate the ACF of ε_t and the cross covariance function for X_t and ε_t.

4. The ACF and CCF estimates from Step 3 can be used, along with the parameter estimates from Step 1 for Y_t, in the procedure from Chapter 3 to estimate the parameters of X_t.

5. The estimates from Step 4 can be used to refine estimates of the ACF of ε_t and cross covariance function of X_t and ε_t. Steps 3 - 5 can be repeated until some convergence criterion is met, updating the estimated ACF of X_t each time.

Example -The procedure above will be illustrated by an example in which we shall use the iterative procedure to find estimates of the parameters of X_t. Again, we shall use a MA(1) series with $\theta = -0.5$ and $\sigma_a^2 = 20.0$ One such series was generated and rounded with a rounding interval width of 10, so that

$$R = \frac{10}{\sqrt{Var\, X_t}} = \frac{10}{5} = 2.0 \quad .$$

This is heavy rounding. The values of $\hat{\gamma}_x(0)$, $\hat{\gamma}_x(1)$, $\hat{\gamma}_\varepsilon(0)$, $\hat{\gamma}_\varepsilon(1)$, $\gamma_{x\varepsilon}(0)$, $\gamma_{x\varepsilon}(1)$, $\hat{\theta}$, and $\hat{\sigma}_a^2$ are given in Table 5 on page 88 for each iteration. Note that $\hat{\eta} = -0.3007$ and $\sigma_\varepsilon^2 = 31.3367$ are the estimates from the observed series. They are also used to compute the initial estimates of $\gamma_x(0)$ and $\gamma_x(1)$.

If convergence is taken to mean having $\hat{\theta}$ to within 0.001 units, then convergence is obtained in three iterations. In two other trials, convergence was obtained within 4 iterations. Other convergence criteria, on $\hat{\sigma}_a^2$ values or on the ACF or CCF values, can be used.

Chapter Summary

In this chapter, the procedure is given for estimating the parameters of a moving average series under rounding. For light rounding, the error series is approximated by uniform white noise uncorrelated with the series being rounded. The variance is determined by the rounding interval width. The parameter estimation procedure outlined in Chapter 3 for error uncorrelated with the series of

Table 5. Example - Iterative Procedure for Heavy Rounding

Iteration	1	2	3
$\hat{\theta}$	-0.3007	-0.4027	-0.419
$\hat{\sigma}_a^2$	31.3367	24.0032	22.5549
$\hat{\gamma}_x(0)$	34.1702	26.0150	26.5146
$\hat{\gamma}_x(1)$	9.4229	9.3954	9.4505
$\hat{\rho}_x(1)$	0.2758	0.3612	0.3564
$\hat{R}$	1.7107	1.9606	1.9420
$\hat{\gamma}_z(0)$	8.3161	8.2684	8.2740
$\hat{\gamma}_z(1)$	.00029	.0072	0.6046
$\hat{\gamma}_{xz}(0)$	-0.0805	-0.3035	-0.2830
$\hat{\gamma}_{xz}(1)$	-0.0020	-0.1094	-0.1003
$\hat{\theta}$	-0.427	-0.419	-0.420
$\hat{\sigma}_a^2$	24.0032	22.5549	22.4942

interest is used. For heavy rounding, an iterative procedure was suggested to estimate the parameters. This procedure calculates the ACF of ε_t and the cross covariances between X_t and ε_t. The procedure in Chapter 3 for cross correlated error is used to estimate the parameters in each iteration, and the procedure repeats until a convergence criterion is met.

Chapter VI

CONCLUSIONS, COMMENTS AND AREAS OF FUTURE RESEARCH

In this thesis, a method has been developed for estimation of parameters of a moving average time series which is observed after rounding. In developing this method, several ideas have been introduced. These are summarized here, in the order in which they occurred in previous chapters.

First, a method of moments estimator was developed for estimation of the parameters of a moving average time series X_t observed with measurement error ε_t. The relationships between the parameters of X_t and the parameters of the observed series $Y_t = X_t + \varepsilon_t$ are used, and the parameters of Y_t are replaced by their estimates (from the observed record of Y_t). The resulting system of equations is solved for the parameters of the model for X_t.

The method of moments estimator above was extended to the case of measurement error correlated with X_t, a case usually ignored in the literature. In the case where X_t and ε_t are cross correlated, the estimation technique used a transfer function plus noise model for ε_t in terms of X_t. New parameters were introduced in order to establish a system of equations involving the param-

eters of X_t, and the method proceeds as in the case where the ε's are not correlated with the X's. This method proved to produce small MSE for simulated series with cross correlated error.

In Chapter 4, four theorems established the limiting distribution of rounding error and its covariance with the random variables being rounded. These theorems proved that the limiting joint distribution of

$$S_1 = \frac{\varepsilon_1}{R_1} \quad \text{and} \quad S_2 = \frac{\varepsilon_2}{R_2} .$$

approaches the distribution of independent uniform random variables as $R_1 \to 0$ and $R_2 \to 0$. Also, S_1 and S_2 are asymptotically uncorrelated with X_1 as $R_1 \to 0$ and $R_2 \to 0$. Using these theorems, we can approximate the variance of ε_t by $Var(\varepsilon_t) = \frac{R^2}{12}$, and the other autocovariances of ε_t and cross covariances between X_t and ε_t by zero, as long as the rounding parameter, R, is small. These values were shown to be good approximations as long as $R \leq \sigma_x$. In Chapter 5, the values of these autocovariances and cross covariances were approximated in the case of heavy rounding.

Finally, the method of moments estimators from Chapter 3 were combined with the information on rounding error ACF and cross covariances from Chapter 4 to estimate the parameters of a moving average series observed with rounding. For heavy rounding, an iterative procedure was needed because of the dependence of the parameter estimation technique and the estimation of the ACF of ε_t and the cross covariances between X_t and ε_t upon each other.

Therefore, for both light rounding and heavy rounding, a method has been described to estimate parameters of a moving average series in the presence of that rounding.

Conclusions

Several additional conclusions are appropriate at this time.

1. In some cases, some aspects of data collection can be controlled. If so, the results of this dissertation can be used to suggest the best combination of precision and length of observational record.

   Rounding of data can be looked upon as precision of measurement, except that for rounding, one usually thinks of giving up already existing precision for convenience of recording. High precision (corresponding to light rounding) has a cost. This cost is incurred, for example, for better measuring equiptment, larger samples, etc. If costs for precision can be cut with little damage to the desired estimates, an overall savings may occur.

   The data collector may also have control over the length of the observational record. This control may occur in the frequence of observation or in the length of time during which observations are taken. A cost is usually incurred for additional observations.

   If the data collector has control over both precision and the number of observations taken, a cost analysis might be undertaken to determine whether some precision can be sacrificed productively in order to increase the number of observations. The procedures in this dissertation could be used to decrease bias in parameter estimates, while the added observations would reduce the variances of the observed series parameter estimates, and therefore the estimates of the parameters of the series X_t. A simulation like that included in Part I of Chapter 5 might be used to establish the balance between observations and precision, and a cost analysis would be needed for the cost of precision vs. the number of observations.

   Much of the time the analyst has no control over the data collection process. In these cases, the techniques described here may help reduce the damage due to a lack of precision.

2. The assumption has been made by Machak and Rose (1984, 1985) and by others that rounding error is uniformly distributed and uncorrelated with the random variable being rounded. Theorems 4.1-4.4 of Chapter 4 show that this assumption is a good approximation, even for rounding that is more severe than one would expect to yield uniform rounding errors. These assumptions are a reasonable approximation if $R \leq \sigma_x$.

3. The exact procedure one should use to estimate parameters in the presence of rounding depends on R, the rounding interval width, and N, the number of observations. There are three cases.

   a. If R is very small and N is small, the bias incurred by using the rounded values is small. Use of the rounding error variance as described in Chapter 5 to estimate the parameters of X_t would increase the variance and not decrease the bias sufficiently, and so one should use the observed series parameters (e.g. $\hat{\eta}$) to estimate the parameters of X_t (e.g.θ).

   b. If R is moderate (but $R \leq \sigma_x$) and/or N is large, then the rounding error should be approximated by uniform white noise, and the procedure from Part I of Chapter 5 should be used to estimate the parameters of X_t.

   c. If R is large ($R > \sigma_x$), the iterative procedure of Part II of Chapter 5 should be used to estimate the parameters of X_t.

Further Research

Additional research is needed to answer several questions that arise from the results of this dissertation. Some of these areas are noted and discussed very briefly here.

1. Method of moments estimators for moving average parameters were constructed in Chapter 3. No attempt has been made here to obtain properties of these estimators. These should be examined, as was done by Walker (1960) for estimators of parameters of autoregressive series. He notes that the consistency and asymptotic standard error ($O(n^{-1/2})$) follow from the properties of the estimated ACF of Y_t. Similar properies may be expected for our estimators. The asymptotic distributions of the estimates as $n \to \infty$ and the asymptotic efficiencies might also be derived. The complexities from using moving average models (as opposed to autoregressive series), and especially the complexities added by cross correlations considered here, make this a difficult problem.

2. The procedures of Chapter 3 increased the variance of parameter estimates, resulting in increased MSE in the case of very light rounding. Improvement of this procedure or use of a better procedure might allow for reduced MSE estimates in the case of very light rounding and small sample size. One possibility is to develop a dual for the approach of Pagano (1974) and convert the MA(q) model in X_t to an ARMA(q,q) process in Y_t or a function of the Y's, and possibly making use of the PACF rather than the ACF.

3. In Chapter 4, the limiting distribution was found for rounding error and the covariances between rounding error and the random variable being rounded were given. These could be applied to other problems.

For autoregressive series observed after rounding, the theorems of Chapter 4 could be used to estimate the ACF of the rounding error. Parameter estimates in the case of light rounding could be constructed using the methods of Miazaki (1985). For ARMA(p,q) processes with $p > q$, the methods of Pagano (1974) might be used. For gross rounding, the cross covariances between the series being rounded and the rounding error series would need to be used, and changes in the procedures of Miazaki and Pagano would be required.

Since the results in Chapter 4 are not specific to time series observations, applications might be made to multivariate analyses in the presence of rounding, and in other areas of statistics.

4. Chapter 4 assumes that the rounded random variables have a multivariate normal distribution. It seems reasonable to expect Theorems 4.1 - 4.4 and their corollaries to hold for a larger class of distributions. The exact class of distributions for which they hold should be determined.

5. This dissertation examines parameter estimation in the presence of rounding. Forecasting presents additional challenges, and the results of this dissertation may not improve forecasts. A study should be undertaken to determine the best forecasting procedure in the presence of rounding. A related problem is signal extraction in the presence of rounding. These problems should be examined.

These areas of additional research point to the fact that this area of research has been neglected in the past. This dissertation makes a start to research in the analysis of time series observed with rounding error.

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Appendix A

APPENDIX

APPENDIX A - Program for Chapter 3 Simulation

```
C*****************************************************************
C       Program simulates estimation of the parameters of
C          a MA(1) series. Four different estimations are
C          generated.
C
C       Base series of length 100 are generated by RNARM
C          For each, three error series correlated with
C          the base series are generated and added to the
C          base series.
C       Estimators for error series E2 are:
C          PMA - The base series estimate.
C          PMA2 - The observed (base + error) series estimate.
C          ADJ2 - The estimate using the ACF of the error series.
C          CADJ2 - The estimate using both the ACF of the
C                  error series and the Cross covariances
C                  with the base series.
C
C          PMA3,PMA4,ADJ3,ADJ4,CADJ3,CADJ4 are the corresponding
C            estimates for error series E3 and E4.
C*****************************************************************
    INTEGER LAGAR(1), LAGMA(1)
    REAL A(200), A1(105), B(200), MAE2(1), MAE3(1),
   $  MAE4(1), MAE(1), PMA(1), PMA2(1), PMA3(1),
   $  W(100), Y2(100), Y3(100), Y4(100),
   $  E2(100), E3(100), E4(100), WI(1),
```

```
     $    PRM(5300), PRM2(5300), PRM3(5300),
     $    PRM4(5300), RE(1), PAR(1), PMA4(1), X(6), P(6),
     $    ADJ2(5300), ADJ3(5300), ADJ4(5300),
     $    CADJ2(5300), CADJ3(5300), CADJ4(5300)
      EXTERNAL RNARM, RSET, SOLVE, NSPE, NSLSE
      ISEED = 0
      CALL RNSET (ISEED)
      DO 134 J = 1,5300
C*************************************************************************
C          Set the parameters for RNARM
C*************************************************************************
      PMA(1) = 0.0
      PMA2(1) = 0.0
      PMA3(1) = 0.0
      PMA4(1) = 0.0
      NPAR = 0
      NPMA = 1
      NW = 100
      CONST = 0.0
      NPMA = 1
      MAE(1) = -0.5
      LAGMA(1) = 1
      IADIST = 0
      AVAR = 1.0
C*************************************************************************
C       Generate the base series with MA(1) model
C            X(t) = a(t) - (-0.5)*a(t-1)
```

```
C            where VAR(a) = 1.0
C*****************************************************************
     CALL RNARM(NW, CONST, NPAR, PAR, LAGAR, NPMA, MAE, LAGMA,
    $        IADIST, AVAR, A, WI, W)
C*****************************************************************
C       Generate error series E2
C*****************************************************************
     LAGMA(1) = 1
     MAE2(1) = -0.20000000000
     NPMA2 = 1
     AVAR2 = 0.20000000000
     CALL RNARM (NW, CONST, NPAR, PAR, LAGAR, NPMA2, MAE2, LAGMA,
    $  IADIST, AVAR2, A1, WI, E2)
     DO 888 IJ = 1, 100
       E2(IJ) = E2(IJ) + 0.20 * A(IJ + 1) + 0.1 * A(IJ)
 888 CONTINUE
C*****************************************************************
C       Generate error series E3
C*****************************************************************
     LAGMA(1) = 1
     MAE3(1) = -0.20000000000
     NPMA3 = 1
     AVAR3 = 0.2000000000
     CALL RNARM (NW, CONST, NPAR, PAR, LAGAR, NPMA3, MAE3, LAGMA,
    $  IADIST, AVAR3, A1, WI, E3)
     DO 887 IJ = 1, 100
       E3(IJ) = E3(IJ) + 0.4 * A(IJ + 1) + 0.1 * A(IJ)
```

```
 887 CONTINUE
C*****************************************************************
C       Generate error series E4
C*****************************************************************
    LAGMA(1) = 1
    MAE4(1) = -0.20000000000
    NPMA4 = 1
    AVAR4 = 0.40000000000
    CALL RNARM (NW, CONST, NPAR, PAR, LAGAR, NPMA4, MAE4, LAGMA,
   $  IADIST, AVAR4, A1, WI, E4)
    DO 886 IJ = 1, 100
      E4(IJ) = E4(IJ) + 0.2 * A(IJ + 1) + 0.10 * A(IJ)
 886 CONTINUE
C*****************************************************************
C       Add the error series to base series W(k) to
C            get observed series Y2,Y3,Y4.
C*****************************************************************
    DO 111 K=1,100
      Y2(K) = W(K) + E2(K)
      Y3(K) = W(K) + E3(K)
      Y4(K) = W(K) + E4(K)
 111 CONTINUE
C*****************************************************************
C       Set the parameters for parameter estimation routines.
C*****************************************************************
    NOBS = 100
    IPRINT = 0
    IMEAN = 0
```

```
      WMEAN = 0.0
      NPAR = 0
      PAR(1) = 0.0
      LAGAR(1) = 0
      NPMA = 1
      LAGMA(1) = 1
      MAXBC = 100
      TOLBC = 0.1
      TOLSS = 0.01
      CONST = 0.0
      COV = 1.0
      LDCOV = 1
      NA = 200
C*************************************************************************
C       NSPE constructs preliminary estimates which are passed
C          to NSLSE, which constructs approximate least squares
C          estimates of the base series W(t).
C*************************************************************************
      CALL NSPE ( 100, W, 0, 0, WMEAN, 0, 1, 0.1, 9, C, PAR, PMA, AVAR)
      CALL NSLSE (100, W, IPRINT, IMEAN, WMEAN, NPAR, PAR, LAGAR, NPMA,
     $ PMA, LAGMA, MAXBC, TOLBC, TOLSS, CONST, COV, LDCOV, NA, A, AVAR)
C*************************************************************************
C       Estimation of the parameters of the observed series Y2.
C
C       SOLVEA  estimates theta using the ACF of error.
C
C       SOLVEB  estimates theta using the ACF of error and the CCF
```

```
C           between W(t) and error.
C*************************************************************************
      LAGMA(1) = 1
      CALL NSPE ( 100,Y2, 0, 0, WMEAN, 0, 1, 0.1, 9, C, PAR,PMA2,AVAR)
      CALL NSLSE (100,Y2, IPRINT, IMEAN, WMEAN, NPAR, PAR, LAGAR, NPMA,
     $ PMA2,LAGMA, MAXBC, TOLBC, TOLSS, CONST, COV, LDCOV, NA, A, AVAR)
      P(1) = PMA2(1)
      P(2) = AVAR
      P(3) = 0.258
      P(4) = 0.06
      P(5) = 0.00
      P(6) = 0.00
      CALL SOLVEA (X,P)
      ADJ2(J) = X(1)
      P(5) = 0.25
      P(6) = 0.100
      CALL SOLVEB (X,P)
      CADJ2(J) = X(1)
C*************************************************************************
C     Estimation of the parameters of the observed series Y3.
C
C     SOLVEA  estimates theta using the ACF of error.
C
C     SOLVEB  estimates theta using the ACF of error and the CCF
C           between W(t) and error.
C*************************************************************************
      LAGMA(1) = 1
      CALL NSPE ( 100,Y3, 0, 0, WMEAN, 0, 1, 0.1, 9, C, PAR,PMA3,AVAR)
```

```
      CALL NSLSE (100,Y3, IPRINT, IMEAN, WMEAN, NPAR, PAR, LAGAR, NPMA,
     $ PMA3,LAGMA, MAXBC, TOLBC, TOLSS, CONST, COV, LDCOV, NA, A, AVAR)
      P(1) = PMA3(1)
      P(2) = AVAR
      P(3) = 0.378
      P(4) = 0.08
      P(5) = 0.0
      P(6) = 0.0
      CALL SOLVEA (X,P)
      ADJ3(J) = X(1)
      P(5) = 0.45
      P(6) = 0.20
      CALL SOLVEB (X,P)
      CADJ3(J) = X(1)
C*************************************************************************
C       Estimation of the parameters of the observed series Y4.
C
C       SOLVEA  estimates theta using the ACF of error.
C
C       SOLVEB  estimates theta using the ACF of error and the CCF
C            between W(t) and error.
C*************************************************************************
      LAGMA(1) = 1
      CALL NSPE ( 100,Y4, 0, 0, WMEAN, 0, 1, 0.1, 9, C, PAR,PMA4,AVAR)
      CALL NSLSE (100,Y4, IPRINT, IMEAN, WMEAN, NPAR, PAR, LAGAR, NPMA,
     $ PMA4,LAGMA, MAXBC, TOLBC, TOLSS, CONST, COV, LDCOV, NA, A, AVAR)
      P(1) = PMA4(1)
      P(2) = AVAR
```

```
      P(3) = 0.466
      P(4) = 0.10
      P(5) = 0.000
      P(6) = 0.000
      CALL SOLVEA (X,P)
      ADJ4(J) = X(1)
      P(5) = 0.250000
      P(6) = 0.100000
      CALL SOLVEB (X,P)
      CADJ4(J) = X(1)
C*************************************************************************
      PRM(J) = PMA(1)
      PRM2(J) = PMA2(1)
      PRM3(J) = PMA3(1)
      PRM4(J) = PMA4(1)
C*********************************************************************
C     Output parameter estimates.
C*********************************************************************
      WRITE(6,432)PMA(1),PMA2(1),ADJ2(J),CADJ2(J),PMA3(1),ADJ3(J),
     $   CADJ3(J),PMA4(1),ADJ4(J),CADJ4(J)
  432 FORMAT(1X,10(F6.3, 1X))
  134 CONTINUE
      STOP
      END
C*************************************************************************
C       SOLVEA solves a quadratic equation to estimate theta
C            using the ACF of epsilon.
C*************************************************************************
```

```
      SUBROUTINE SOLVEA (X,P)
      REAL X(6), P(6)
      B = ((1.0 + P(1)**2) * P(2) - P(3))/(P(1)*P(2) + P(4))
      CALL ZERO3 (A,B)
      X(1) = A
      X(2) = (P(1) * P(2) + P(4))/X(1)
      X(3) = 0.0
      X(4) = 0.0
      CALL ZERO2 (X,P)
      X(6) = - P(4)/X(5)
      RETURN
      END
C*******************************************************
C       ZERO3 solves a quadratic equation.
C*******************************************************
      SUBROUTINE ZERO3 (X,B)
      A = 0.0000
  333 A = A - 0.1000
      D = A**2 - B*A + 1.0
      IF( (D .GT. 0.000) .AND. (A .GT. -1.0)) GO TO 333
      C = A + 0.10
  334 C = C - 0.010
      D = C**2 - B*C + 1.0
      IF( (D .GT. 0.000) .AND. (C .GT. A)) GO TO 334
      E = C + 0.01000
  335 E = E - 0.00100
      D = E**2 - B*E + 1.0
      IF( (D .GT. 0.000) .AND. (E .GT. C)) GO TO 335
```

```
      X = E
      RETURN
      END
C*************************************************************
C       SOLVEB solves a quartic equation in order to
C            estimate theta using the ACF of epsilon and
C            the CCF of W and epsilon.
C*************************************************************
      SUBROUTINE SOLVEB(X,P)
      REAL X(6), P(6), A(6)
      A(1) = (1.0 + P(1)**2) * P(2)
      A(2) = - P(1)*P(2)
      A(3) = P(3)
      A(4) = P(4)
      A(5) = P(5)
      A(6) = P(6)
      CALL ZERO1(X(1),A)
      X(2) = (A(1) - A(3) - 2.0*A(5))/(X(1)**2 + 1.0)
      X(3) = -A(6)/(X(1)*X(2))
      X(4) = -(A(5)/X(2) - X(3))/(- X(1))
      CALL ZERO2(X,P)
      X(6) = ( X(3)*X(4)*X(2) - A(4))/X(5)
      RETURN
      END
C*************************************************************
C       ZERO1 solves a finds a zero of a function.
C*************************************************************
      SUBROUTINE ZERO1 (X,P)
```

```
      REAL P(6)
      A = 0.0
 2222 CONTINUE
        A = A - 0.10
        D = F(A,P)
        IF( (D. LT. -0.000) .AND. (A .GT. -1.0) ) GO TO 2222
      B = A + 0.10
 2225 CONTINUE
        B = B - 0.0100
        D = F(B,P)
        IF( (D. LT. -0.000) .AND. (B .GT. A) ) GO TO 2225
      C = B + 0.0100
 2224 CONTINUE
        C = C - 0.0010
        D = F(C,P)
        IF( (D. LT. -0.000) .AND. (C .GT. C) ) GO TO 2224
      X = C
      RETURN
      END
C*************************************************************
C       Function F is the quartic expression needed
C         to use ZERO1 to estimate theta.
C*************************************************************
      FUNCTION F(X,P)
      REAL P(6)
      F = ( P(6) + P(4) - P(2) ) * X**4 + ( P(5) + P(3) - P(1) ) * X**3
     1  + ( P(4) - P(2) ) * X**2 - P(5) * X - P(6)
      RETURN
```

```
      END
      SUBROUTINE ZERO2 (X,P)
      REAL P(6), X(6)
      A = 1.0
      B = (P(3)-(X(3)**2 + X(4)**2) * X(2) )/(P(4)-X(3) * X(4) * X(2))
3333 CONTINUE
       A = A - 0.10
       D = G(A,B)
       IF ((D .GT. 0.000) .AND. (A .GT. -1.000)) GO TO 3333
      C = A + 0.10
3334 CONTINUE
       C = C - 0.010
       D = G(C,B)
       IF ((D .GT. 0.000) .AND. (C .GT. A)) GO TO 3334
      E = C + 0.0100
3335 CONTINUE
       E = E - 0.0010
       D = G(E,B)
       IF ((D .GT. 0.000) .AND. (E .GT. C)) GO TO 3335
      X(5) = E
      RETURN
      END
      FUNCTION G(A,B)
      G = A**2 - B*A + 1.0
      RETURN
      END
```

APPENDIX B - Error Series Variance Program

```
C*****************************************************************
C  This FORTRAN program calculates the variance of rounding
C  error ε_t as a function of the rounding
C  interval width R.
C*****************************************************************
C   R decriments from 4.2 to 0.10 by 0.02.
C*****************************************************************
      R = 4.200000
      DR = 0.0200000
      M = 40
   50 R = R - DR
      VAR = 0.00000
      PI = 4.0000*ATAN(1.0)
C*****************************************************************
C         IMAX insures that underflow does not occur when the
C         argument in exp( ) gets small.
C*****************************************************************
      IMAX = INT(6.0/R)*2 + 1
      IF(IMAX.LT.13) IMAX = 13
C*****************************************************************
C         KR is the integer multiple of R in the
C         rounding process. K goes from -(IMAX-1)/2 to (IMAX-1)/2
C         to replace the infinite sum in the variance formula.
C*****************************************************************
      DO 200 I = 1,IMAX
```

```
      K = I - (IMAX-1)/2
C*********************************************************************
C         EPS = ((J-.5)/M -.5)R  goes from -.5 R to .5 R
C         This replaces the integral in the exact variance.
C*********************************************************************
      DO 100 J=1,M
        EPS = ((FLOAT(J)-0.5)/FLOAT(M) - .5000000)*R
        X = EPS + FLOAT(K) * R
        VAR = VAR + EPS**2 *
     1   (1.0/SQRT(2.0*PI))*EXP((-X**2)/2.00)*(R/FLOAT(M))
  100  CONTINUE
  200 CONTINUE
      WRITE(6,300)R,VAR
  300 FORMAT(1X,F8.4,5X,F15.8)
      IF(R.GT.0.1)GO TO 50
      STOP
      END
```

APPENDIX C - Error Series Autocovariance Program

```
C*********************************************************
C       Program to calculate the covariance and
C       the correlations between two
C       rounding errors - could be the ACF of
C       rounding error series.
C
C       RHO is the correlation between the normal
C       variates being rounded.
C
C       R is the rounding interval width. R decriments
C       from 7.0 to 0.05 by 0.05.
C*********************************************************
      M = 20
      PI = 4.00000*ATAN(1.00)
      RHO = 0.10000000
   30 RHO = RHO + 0.100000000
      R = 7.0
   50 R = R - 0.0500000
      COV = 0.0000000
      L1 = INT(5.0/R)*2 + 1
      IF(L1.LT.5)L1=5
      C = 1.000/(2.0000000*PI*SQRT(1-RHO**2))
C*********************************************************
C          E1 = ((I-.5)/M -.5)*R
C          E2 = ((J-.5)/M -.5)*R
```

```
C          E1 and E2 run from -.5R to .5R in increments
C               of R/M
C
C          K and N are the integral multiples of R to which
C               X1 and X2, respectively, are rounded. The
C               infinite sum is truncated to run from
C               -(L1-1)/2 to (L1-1)/2, where L1 is chosen
C               to avoid underflow in exp( ).
C*********************************************************
      DO 400 I = 1,M
       DO 300 J = 1,M
        DO 200 K1 = 1,L1
         K = K1 - (L1 + 1)/2
         DO 100 N1 = 1,L1
          N = N1 - (L1 + 1)/2
          E1 = ((FLOAT(I)-0.5)/FLOAT(M) - 0.50000)*R
          E2 = ((FLOAT(J)-0.5)/FLOAT(M) - 0.50000)*R
          X1 = FLOAT(K)*R - E1
          X2 = FLOAT(N)*R - E2
          B = -(X1**2 + X2**2 - 2.0*RHO*X1*X2)/(2.00*(1.00 - RHO**2))
          IF(ABS(B).LT.12.0) ADD = E1*E2*(R/FLOAT(M))**2*C*EXP(B)
          IF(ABS(B).LT.12.0) COV = COV + ADD
  100    CONTINUE
  200   CONTINUE
  300  CONTINUE
  400 CONTINUE
      VAREPS = VAR(R)
      CORR = COV/VAREPS
```

```
      WRITE(6,500)RHO,R,COV,VAREPS,CORR
  500 FORMAT(1X,F5.2,5X,F5.2,5X,F12.8,5X,F12.8,5X,F12.8)
      IF(R.GT.0.07) GO TO 50
      IF(RHO.LT..85) GO TO 30
      STOP
      END
C*********************************************************
C       Function VAR is used in order to compute the correlation
C          by dividing the covariance by VAR(E1).
C*********************************************************
      FUNCTION VAR(R)
      M = 40
      VAR = 0.00000
      PI = 4.0000*ATAN(1.0)
      IMAX = INT(6.0/R)*2 + 1
      IF(IMAX.LT.5) IMAX=5
      DO 200 I=1,IMAX
        K = I - (IMAX-1)/2
        DO 100 J=1,M
          EPS = ((FLOAT(J)-0.5)/FLOAT(M) - .5000000)*R
          X = EPS + FLOAT(K) * R
          VAR = VAR + EPS**2 *
     1    (1.0/SQRT(2.0*PI))*EXP((-X**2)/2.00)*(R/FLOAT(M))
  100   CONTINUE
  200 CONTINUE
      RETURN
      END
```

APPENDIX D - Cross Covariance, Lag 0 Program

```
C*************************************************************************
C       Program to compute the covariance and the
C          correlation between a normal
C          random variable and its rounding error.
C          Could be lag zero of the cross covariance
C          function of a time series.
C*************************************************************************
    PI = 4.0*ATAN(1.0)
    M = 50
    R = 7.1
 80 R = R - 0.05
    J = 2*INT(8.00/R)+1
    COV = 0.00000
C*************************************************************************
C       R, the rounding interval width, decriments from
C       7.05 to 0.05 by 0.05 .
C
C       The variance of X, the normally distributed random
C          variable being rounded, is standardized to 1.0.
C
C       EPS=((n-.5)/M -.5)R goes from (approx.) -0.5R to .05R
C
C       K is the multiple of R to which X is rounded.
C          The infinite sum in K in the formula is truncated
C          using J, to avoid underflow in the function eps.
```

```
C
C*********************************************************************
      DO 100 N=1,M
       DO 99 L=1,J
        K = L - (J+1)/2
        EPS = ((FLOAT(N)-0.5)/FLOAT(M) - 0.5)*R
        X = FLOAT(K)*R - EPS
        COV = COV + X*EPS*R/FLOAT(M)*EXP((-X**2)/2.00)
 99    CONTINUE
 100  CONTINUE
      COV = COV/SQRT(2.0*PI)
      VAREPS = VAR(R)
      CORR = COV/SQRT(VAREPS)
      WRITE(6,120)R,COV,VAREPS,CORR
 120  FORMAT(1X,F5.2,3(5X,F14.10))
     IF(R.GT.0.1)GO TO 80
     STOP
     END
C*************************************************************
C      Function VAR calculates the variance of EPS
C         in order to compute the correlation
C         from the covariance
C*************************************************************
     FUNCTION VAR(R)
     M = 40
     VAR = 0.00000
     PI = 4.0000*ATAN(1.0)
     IMAX = INT(6.0/R)*2 + 1
```

```
      IF(IMAX.LT.5) IMAX = 5
      DO 200 I = 1,IMAX
        K = I - (IMAX-1)/2
        DO 100 J = 1,M
          EPS = ((FLOAT(J)-0.5)/FLOAT(M) - .5000000)*R
          X = EPS + FLOAT(K) * R
          VAR = VAR + EPS**2 *
     1    (1.0/SQRT(2.0*PI))*EXP((-X**2)/2.00)*(R/FLOAT(M))
  100   CONTINUE
  200 CONTINUE
      RETURN
      END
```

APPENDIX E - Cross Covariance Program

```
C*******************************************************
C     Program to calculate the covariance between
C       one normally distributed random variable
C       and the error of rounding another normally
C       distributed random variable.
C       Both random variables are assumed to have
C       variance 1.0.
C       Could be a lag of the cross covariance function
C       between a time series when rounded and the
C       rounding error series.
C
C    RHO is the correlation between the normally distributed
C       random variables being rounded. Rho decriments from
C       0.9 to 0.1 by 0.1.
C
C    R is the rounding interval width. R decriments from 7.0
C       to 0.1 by 0.1.
C*******************************************************
      DX = 0.0200000
      PI = 4.00000*ATAN(1.0000000)
      RHO = 1.0
   10 RHO = RHO - 0.1
      R = 7.1
   20 R = R - 0.1
      C1 = 1.00/(2.000*PI*SQRT(1.0000 - RHO**2))
```

```
      C2 = -1.000/(2.00*(1.000 - RHO**2))
       DEP = 0.02000*R
       COV = 0.00000
       X = -5.0000 - DX
   60  X = X + DX
C*************************************************************
C      One normal random variables is X. Its values run
C          from -5.0 to 5.0 by DX = 0.02 in order to integrate
C          over X.
C      The other random variable is X2 and is defined as
C          X2 = EP + K*R    where K takes on integral values
C          defined to prevent underflow in function exp( ).
C          EP runs from -.5*R to .5*R .
C*************************************************************
       IEND = INT(8.000/R)*2+1
       IF(IEND.LT.7)IEND=7
        DO 100 I=1,IEND
          K = I - (IEND+1)/2
          EP = -0.50000*R - 0.5*DEP
   70     EP = EP + DEP
          X2 = FLOAT(K)*R - EP
           A=C2*(X**2+X2**2-2.0*RHO*X*X2)
           IF(ABS(A).LT.10.00) COV=COV+X*EP*C1*EXP(A)*DX*DEP
          IF(EP.LT.0.500*R - 0.5*DEP)GO TO 70
  100    CONTINUE
       IF(X.LT.5.0000)GO TO 60
       VAREPS = VAR(R)
       CORR = COV/SQRT(VAREPS)
```

```
      WRITE(6,120)RHO,R,COV,VAREPS,CORR
  120 FORMAT(1X,F5.2,5X,F5.2,5X,F12.9,5X,F12.9,5X,F12.9)
      IF(R.GT.0.1)GO TO 20
      IF(RHO.GT.0.1)GO TO 10
      STOP
      END
C*********************************************************************
C       Function VAR computes the variance of epsilon, used to
C           compute the correlations.
C*********************************************************************
      FUNCTION VAR(R)
      M = 40
      VAR = 0.00000
      PI = 4.0000*ATAN(1.0)
      IMAX = INT(6.0/R)*2 + 1
      IF(IMAX.LT.5) IMAX=5
      DO 200 I=1,IMAX
        K = I - (IMAX-1)/2
        DO 100 J=1,M
          EPS = ((FLOAT(J)-0.5)/FLOAT(M) - .5000000)*R
          X = EPS + FLOAT(K) * R
          VAR = VAR + EPS**2 *
     1    (1.0/SQRT(2.0*PI))*EXP((-X**2)/2.00)*(R/FLOAT(M))
  100   CONTINUE
  200 CONTINUE
      RETURN
      END
```

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