SPACES OF CONTINUOUS LINEAR FUNCTIONALS ON FUNCTION SPACES

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(ABSTRACT)

This thesis is a study of several spaces of continuous linear functionals on various function spaces with a natural norm inherited from a larger Banach space. The completeness of these normed linear spaces is studied in detail and several necessary and sufficient conditions are obtained in this regard. Since spaces of continuous linear functionals are inherently related to spaces of measures, their measure-theoretic counterparts are also studied. By using these counterparts, several necessary and sufficient conditions are obtained on the separability of these spaces of continuous linear functionals.

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This thesis is not only the fruit of my labor, it is also the result of unbelievable pain and suffering my mother, , went through to bring us to this point. I personally believe that no language in this world is rich enough to express my unbounded gratitude to her. So, I prefer to use my own existence as my burning acknowledgement and respect to her.

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Dedication

a la memoria de mi padre

y

a mi madre

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Introduction

By the late nineteenth century it was apparent that many domains of mathematics dealt with transformations or operators acting on functions. The idea that motivated the creation of functional analysis is that some of these operators could be considered under one abstract formulation of an operator acting on a class of functions. Moreover, these functions could be regarded as elements or points of a space. Then the operator transforms points into points and in this sense is a generalization of ordinary transformations such as rotations. Some of the above operators carry functions into real numbers, rather than functions. These operators that do yield real or complex numbers are today called functionals.

The abstract theory of functionals was initiated by Volterra in work concerned with the calculus of variations. Even before Volterra started his work, the notion that a collection of functions all defined on some common interval be regarded as points of a space had already been suggested. Riemann, in his thesis, spoke of a collection of functions forming a connected closed domain (of points of a space). Ascoli and Arzela sought to extend to sets of functions Cantor's theory of sets of points and so regarded functions as points of a space. Also around the same time, Hadamard was thinking of the family of all continuous functions defined over [0,1], a family that arose in his work on partial differential equations. He also undertook the study of functionals on behalf of the calculus of variations. The term functional is due to him. So it is clear that the idea of function spaces is as old as that of functionals. But the first major effort to build up an abstract theory of function spaces and functionals was made by Maurice Frěchet, a leading French professor of mathematics, in his doctoral thesis of 1906.

So far as abstract function spaces are concerned, one of the far-reaching results of the above-mentioned studies by different mathematicians is 'Riesz Representation Theorem' (RRT in short). Actually many different versions, though essentially same, of this theorem are given in different references. Here we give the most commonly known version of this theorem. But first we need to fix some notations. Let $\mathcal{C}(X)$ be the set of all real-valued continuous functions on a locally compact Hausdorff space X and let $\mathcal{C}_c(X) = \{f \in \mathcal{C}(X) : f$ vanishes off a compact subset of X $\}$. Note $\mathcal{C}_c(X)$ with sup-norm is a normed linear space where sup-norm $\|\cdot\|_\infty$ is defined as follows. For $f \in \mathcal{C}_c(X)$, define $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. Then RRT says that for every positive linear functional F on $\mathcal{C}_c(X)$, there exists a unique regular Borel measure μ such that $F(f) = \int f d\mu$ holds for every $f \in \mathcal{C}_c(X)$. Moreover the representing measure is the regular Borel measure induced by F on X. See [1] for details.

But note that RRT actually gives the conjugate space of a very particular function space in terms of a space of measures. But for a topologist, there are many different kinds of function spaces. So it is very natural for a topologist to ask what will be the measure-theoretic counterparts of the conjugate spaces of these function spaces (this term 'conjugate' here is slightly abused, because these function spaces, in general, are not normed linear spaces). Also an analyst, most of the time, deals with very nice topological spaces such as metric spaces, compact spaces or at best locally compact Hausdorff spaces. But a topologist would like to start working on a more general setting, i. e. , preferably with only one assumption on the space that it be a completely regular Hausdorff space just to ensure that there are 'enough' functions in $C(X)$ to separate points of X. This means that a topologist would like to start with a space on which less restrictions are imposed. But obviously it makes the study of conjugate spaces of different function spaces much harder. Also a topologist, contrary to the interest of an analyst, would like to investigate some topological properties of these conjugate spaces in terms of the topological properties of the completely regular Hausdorff space X. In this thesis, we try to do that.

In chapter one, we first introduce several function spaces. In fact, we also develop a new function space to answer a question of completeness (which comes up in chapter two) from a proper perspective. Then we introduce several normed linear spaces of continuous linear functionals over these function spaces. After that we develop some general properties and tools to be used in the next chapters.

In chapter two, we study the completeness of the normed linear spaces introduced in chapter one. But first we break up these normed linear spaces into two positive parts, more precisely, into two positive cones. After that several necessary and sufficient conditions are obtained for the completeness of these spaces, i. e. , for these spaces to be Banach spaces. Here in this chapter, we also get a clear idea why we have developed a new function space in chapter one.

Since spaces of continuous linear functionals are inherently related to spaces of measures, in chapter three, we study their measure-theoretic counterparts. We mainly do it in terms of regular Borel measures. But also the characterization of one counterpart is done in terms of regular Baire measures.

In chapter four, we study the separability of these spaces of continuous linear functionals. Here we use chapter three extensively to obtain some necessary and sufficient conditions for these spaces to be separable.

Results and definitions are numbered consecutively throughout the text. For instance, a reference of the type 'by Theorem 1.5.5' means that we refer to that theorem which is in chapter one. $\mathbb{R}$ will denote the set of all real numbers with the usual topology and $card(X)$ will denote the cardinality of X. And as mentioned before, we will require all our spaces X to be Tychonoff spaces (i. e. , completely regular Hausdorff spaces).■

Note: For the historical perspective given in this chapter, I would like to acknowledge Professor Morris Kline for his book 'Mathematical Thought from Ancient to Modern Times.'

Chapter I

Generalities

Let $\mathcal{C}(X)$ denote the set of all continuous real-valued functions on a completely regular Hausdorff space X and let $\mathcal{C}^*(X)$ be the set of bounded functions in $\mathcal{C}(X)$. Different topologies are used for $\mathcal{C}(X)$ and for $\mathcal{C}^*(X)$, and consequently we can have different spaces of continuous linear functionals (real-valued functions) over $\mathcal{C}(X)$ or $\mathcal{C}^*(X)$. So first we briefly study different topologies on $\mathcal{C}(X)$ and their relationship.

1 Topologies on $\mathcal{C}(X)$

Let us denote by $\mathcal{C}_k(X)$ (respectively by $\mathcal{C}_p(X)$) the set $\mathcal{C}(X)$ topologized with the compact-open (respectively the point-open) topology. A subbase for $\mathcal{C}_k(X)$ (respectively for $\mathcal{C}_p(X)$) is given by the collection $\{[A,V] : A$ is compact (respectively A is finite) in X and V is open in $\mathbb{R}\}$ where $[A,V] = \{f \in \mathcal{C}(X) : f(A) \subseteq V\}$. Also it can be shown that the compact-open (respectively point-open) topology on $\mathcal{C}(X)$ is the same as the topology of uniform convergence on compact sets (respectively on finite sets). See [9], Chapter I, for the details. This fact actually gives the following theorem. But first define $< f, A, \epsilon >=$

$\{g \in \mathcal{C}(X) : |f(x) - g(x)| < \epsilon$ for all $x \in A\}$ where $f \in \mathcal{C}(X), A$ is a subset of X and $\epsilon > 0$.

Theorem 1.1.1: *The collection* $\mathcal{B} = \{< f, A, \epsilon >: f \in \mathcal{C}(X),\ A$ *is compact (respectively finite) in* X *and* $\epsilon > 0\}$ *is a base for the compact-open (respectively point-open) topology on* $\mathcal{C}(X)$. ■

It is well-known that both $\mathcal{C}_k(X)$ and $\mathcal{C}_p(X)$ are locally convex spaces. The locally convex compact-open topology on $\mathcal{C}(X)$ is generated by the collection of seminorms $\{p_K :$ K is a compact subset of X $\}$ where $p_K(f) = \sup |f(x)| : x \in K\}$ for $f \in \mathcal{C}(X)$. Let $V_{p_A,\epsilon}$ $= \{f \in \mathcal{C}(X) : p_A(f) < \epsilon\}$ where A is a subset of X.

Let $\mathcal{U} = \{ V_{p_K,\epsilon}$: K is a compact subset of $X, \epsilon > 0\}$. Then $\mathcal{U}$ forms a neighborhood base at 0, where 0 denotes the zero-function in $\mathcal{C}(X)$. Consequently, for each $f \in C(X)$, $f + \mathcal{U} = \{f + V : V \in \mathcal{U}\}$ is a neighborhood base at f.

Similarly the locally convex point-open topology on $\mathcal{C}(X)$ is generated by the collection of seminorms $\{p_F : F$ is a finite subset of $X\}$ where $p_F(f) = \sup\{|f(x)| : x \in F\}$. As above, for each $f \in \mathcal{C}(X), f + \mathcal{U} = \{f + V : V \in \mathcal{U}\}$ is a neighborhood base at f where, in this case, $\mathcal{U} = \{ V_{p_F,\epsilon}$: F is a finite subset of $X, \epsilon > 0\}$.

Note that the compact-open topology on $\mathcal{C}(X)$ is finer than the point-open topology. The supremum norm on $\mathcal{C}^*(X)$ is defined as $\| f \|_\infty = \sup\{|f(x)| : x \in X\}$ for $f \in \mathcal{C}^*(X)$. This supremum norm generates a finer topology than the compact-open topology on $\mathcal{C}^*(X)$. We denote this normed linear space by $\mathcal{C}^*_\infty(X)$. Also $\mathcal{C}^*_\infty(X) = \mathcal{C}^*_k(X)$ if and only if X is compact (see [10], page 10).

Now we define a new topology on $\mathcal{C}^*(X)$. Let α be a collection of subsets of X which satisfies the following two conditions: (i) each member of α is C^*-imbedded and (ii) if $A, B \in \alpha$, then there exists $C \in \alpha$ such that $A \cup B \subseteq C$.

For each $A \in \alpha$, define a seminorm p_A on $C^*(X)$ as before. For $f \in C^*(X), p_A(f) = \sup\{|f(x)| : x \in A\}$. Consider the locally convex topology on $C^*(X)$ generated by the collection of seminorms $\{p_A : A \in \alpha\}$. Because of (ii), for each $f \in C^*(X), f + \mathcal{U} = \{f + V : V \in \mathcal{U}\}$ is a neighborhood base at f where $\mathcal{U} = \{V_{p_A,\epsilon} : A \in \alpha, \epsilon > 0\}$. We call this new locally convex topology on $C^*(X)$, α-topology and the corresponding topological space we denote by $C^*_\alpha(X)$.

Note when $\alpha = \mathcal{K}(X) = \{K \subseteq X : K$ is a compact subset of $X\}$ or $\alpha = \mathcal{F}(X) = \{F \subseteq X : F$ is a finite subset of $X\}$, we get compact-open or point-open topology on $C^*(X)$ respectively. The following theorem gives a relationship between $C^*_\alpha(X)$ and $C^*_\infty(X)$.

Theorem 1.1.3: *The sup-norm topology on $C^*(X)$ is finer than the α -topology on $C^*(X)$. We denote this fact by $C^*_\alpha(X) \leq C^*_\infty(X)$. Also $C^*_\alpha(X) = C^*_\infty(X)$ if and only if α contains X.*

Proof. Let $B(f, \epsilon) = \{g \in C^*(X) : \| f - g \|_\infty < \epsilon\}$ where $f \in C^*(X)$ and $\epsilon > 0$. Now if $A \in \alpha$, then $f \in B(f, \epsilon) \subseteq f + V_{p_A,\epsilon}$ for all $f \in C^*(X)$. But this implies that sup-norm topology on $C^*(X)$ is finer than α-topology on $C^*(X)$, i. e. , $C^*_\alpha(X) \leq C^*_\infty(X)$. Now suppose α contains X. Then $B(f, \epsilon) = f + V_{p_X,\epsilon}$ for all $\epsilon > 0$ and for all $f \in C^*(X)$. This means the sup-norm topology on $C^*(X)$ is weaker than the α-topology on $C^*(X)$. Hence $C^*_\alpha(X) = C^*_\infty(X)$.

Now let $C^*_\infty(X) = C^*_\alpha(X)$. Since $B(0, \epsilon)$ (choose $\epsilon < 1$) is open in $C^*_\infty(X)$ and $C^*_\infty(X) = C^*_\alpha(X)$, there exist an $A \in \alpha$ and a $\delta > 0$ such that $0 \in V_{p_A,\delta} \subseteq B(0, \epsilon)$. Note $\delta \leq \epsilon$. Now we claim that $A = X$. If not, then there exists x in $X \backslash A$ and consequently there exists g in $\mathcal{C}(X, I)$ ($\mathcal{C}(X, I)$ is the set of all continuous functions on X with range in $I = [0,1]$) such that $g(A) = 0$ and $g(x) = 1$. $g(A) = 0$ means $g \in V_{p_A, \delta}$. But $g(x) = 1$ means $\| g \|_\infty \geq 1$, i. e. , $g \notin B(0, \epsilon)$. Hence $X = A$. ■

2 Sets of Continuous Linear Functionals

Let $\Lambda_k(X)$ (respectively $\Lambda_p(X)$) be the set of all real-valued continuous linear functionals on $\mathcal{C}_k(X)$ (respectively on $\mathcal{C}_p(X)$).

Note since $\mathcal{C}_k^*(X)$ (respectively $\mathcal{C}_p^*(X)$) is a dense linear subspace of the locally convex space $\mathcal{C}_k(X)$ (respectively $\mathcal{C}_p(X)$) , the set of all continuous linear functionals on $\mathcal{C}_k(X)$ (respectively on $\mathcal{C}_p(X)$) is equal to the set of all continuous linear functionals on $\mathcal{C}_k^*(X)$ (respectively on $\mathcal{C}_p^*(X)$). (See [12], page 132.) In [7], we have used the notation $\Lambda(X)$ to denote $\Lambda_k(X)$.

Let $\Lambda_\alpha(X)$ be the set of all continuous linear functionals (real-valued) on $\mathcal{C}_\alpha^*(X)$. Note when $X \in \alpha$ we get $\Lambda_\alpha(X) = \Lambda_\infty(X)$, which is equal to the set of all (real-valued) continuous linear functionals on $\mathcal{C}_\infty^*(X)$. In [7], we have used the notation $\Lambda'(X)$ to denote $\Lambda_\infty(X)$.

A linear functional λ on $\mathcal{C}(X)$ (or on $\mathcal{C}^*(X)$) is positive provided that $\lambda(f) \geq 0$ for each $f \in \mathcal{C}(X)$ (or for each $f \in \mathcal{C}^*(X)$ respectively) such that $f \geq 0$ and we express this by the notation $\lambda \geq 0$. Now let $\Lambda_j^+(X) = \{\lambda \in \Lambda_j(X) : \lambda \geq 0\}$ where $j = \mathrm{p}, \mathrm{k}, \alpha$ and ∞. If λ is a linear functional on $\mathcal{C}(X)$ (or on $\mathcal{C}^*(X)$ respectively) and A is a subset of X , then λ is said to be supported on A provided that whenever $f \in \mathcal{C}(X)$ (or $f \in \mathcal{C}^*(X)$ respectively) with $f|_A = 0$, then $\lambda(f) = 0$. Since λ is linear, this is equivalent to saying that whenever $f, g \in C(X)$ (or $f, g \in C^*(X)$ respectively) with $f|_A = g|_A$ then $\lambda(f) = \lambda(g)$.

Needed in the following lemma is the fact that if $\lambda \in \Lambda_k^+(X)$ and $f, g \in C(X)$ with $f \leq g$ then $\lambda(f) \leq \lambda(g)$. This is true since $g - f \geq 0$ and $\lambda(g) - \lambda(f) = \lambda(g - f) \geq 0$. Also in the lemma constant functions on X are denoted by the constant to which they map.

Lemma 1.2.1: *For each $\lambda \in \Lambda_k(X)$, there exists a $K \in \mathcal{K}(X)$ such that λ is supported on K. Conversely, if λ is a positive linear functional on $C(X)$ which is supported on some $K \in \mathcal{K}(X)$, then $\lambda \in \Lambda_k^+(X)$.*

Proof. If $\lambda \in \Lambda_k(X)$, then since $\lambda : C_k(X) \to \mathbb{R}$ is continuous at 0, there exists a $K \in \mathcal{K}(X)$ and a $\delta > 0$ such that $\lambda(V_{p_K,\delta}) \subseteq (-1, 1)$. Let $f \in C(X)$ with $f|_K = 0$. It suffices to show that $|\lambda(f)| < \epsilon$ for every $\epsilon > 0$; so let $\epsilon > 0$. Now for each $x \in K$, $|\frac{1}{\epsilon}f(x)| = 0$, so $p_K(\frac{1}{\epsilon}f) = 0 < \delta$ and hence $\frac{1}{\epsilon}f \in V_{p_K,\delta}$. Then $\lambda(\frac{1}{\epsilon}f) \in (-1, 1)$, so that since λ is linear $\lambda(f) \in (-\epsilon, \epsilon)$ as desired.

For the converse, let λ be a positive linear functional on $C(X)$ which is supported on $K \in \mathcal{K}(X)$. It suffices to check the continuity of λ at 0, so let $\epsilon > 0$. Define $\delta = \frac{\epsilon}{2\lambda(1)+1}$ and let $f \in V_{p_K,\delta}$. Then $f|_K : K \to [-\delta, \delta]$ has an extension $g \in C(X)$ which maps into $[-\delta, \delta]$. Note that $\lambda(g) = \lambda(f)$ and $|g(x)| \leq \delta$ for all $x \in X$. Let g^+ and g^- be defined by $g^+(x) = \max\{g(x), 0\}$ and $g^-(x) = \max\{-g(x), 0\}$ for each $x \in X$. Then $g = g^+ - g^-$, $0 \leq g^+ \leq \delta$ and $0 \leq g^- \leq \delta$ so that

$$|\lambda(f)| = |\lambda(g)| = |\lambda(g^+ - g^-| \leq |\lambda(g^+| + |\lambda(g^-)| \leq \lambda(\delta) + \lambda(\delta) = 2\delta\lambda(1) < \epsilon. \blacksquare$$

The proof of Lemma 1.2.1 can be modified to obtain the following two lemmas.

Lemma 1.2.2: *For each $\lambda \in \Lambda_p(X)$, there exists an $F \in \mathcal{F}(X)$ such that λ is supported on F. Conversely, if λ is a positive linear functional on $C(X)$ which is supported on some $F \in \mathcal{F}(X)$, then $\lambda \in \Lambda_p^+(X)$.*

Lemma 1.2.3: *For each $\lambda \in \Lambda_\alpha(X)$. there exists an element A in α such that λ is supported on A. Conversely, if λ is a positive linear functional on $C^*(X)$ which is supported on an element of α, then $\lambda \in \Lambda_\alpha^+(X)$.*

Also for $\Lambda_\infty^+(X)$, we have the following lemma.

Lemma 1.2.4: *Every positive linear functional on $C_\infty^*(X)$ is continuous.*

Proof. Suppose λ is a positive linear functional on the normed linear space $C_\infty^*(X)$. For $f \in C_\infty^*(X)$ with $\| f \|_\infty \leq 1$ we have $-1 \leq f \leq 1$ where 1 is the constant function on X mapping every element of X to 1. Since $\lambda \geq 0, -\lambda(1) \leq \lambda(f) \leq \lambda(1)$ which implies $\sup\{|\lambda(f)| : \|f\|_\infty \leq 1; f \in C^*(X)\} = \lambda(1) < \infty$. Hence λ is continuous on $C_\infty^*(X)$. ■

Now we are interested in knowing whether there exists a minimal compact (or finite respectively) support for a $\lambda \in \Lambda_k(X)$ (or $\in \Lambda_p(X)$ respectively). The answer to this question is affirmative, and the following series of Lemmas gives that answer.

Lemma 1.2.5: *Let A be a closed subset of X, let $K \in \mathcal{K}(X)$ and let $\lambda \in \Lambda_k(X)$. If λ is supported on each of A and K, then λ is supported on $A \cap K$.*

Proof. Think of X as a subset of its Stone-Čech compactification βX and let Z be the closure of A in βX. To show that $Z \cap K = A \cap K$, let $x \in Z \cap K$ and suppose $x \notin A$. Since A is closed in X, then there exists an open set W in βX such that $W \cap X = X \backslash A$. Since

$x \in Z$, then $W \cap A \neq \emptyset$. But $W \cap A \subset W \cap X = X \backslash A$ which is a contradiction. Hence $Z \cap K = A \cap K$.

To show that λ is supported on $A \cap K$, let $f \in \mathcal{C}(X)$ such that $f\,|_{A \cap K} = 0$. Define $g : Z \cup K \to \mathbb{R}$ by $g(x) = 0$ if $x \in Z$ and $g(x) = f(x)$ if $x \in K$. This is continuous and since $Z \cup K$ is closed in βX, there is a continuous extension $\hat{g}$ of g to βX. Then define $\hat{f} = \hat{g}\,|_X$. Since $\hat{f}\,|_K = f\,|_K$ and $\hat{f}\,|_A = 0$, then $\lambda(f) = \lambda(\hat{f}) = 0$. Therefore λ is supported on $A \cap K$. ∎

For each $\lambda \in \Lambda_k(X)$, define $\mathcal{K}_\lambda = \{K \in \mathcal{K}(X) : \lambda \text{ is supported on } K\}$ and let $K_\lambda = \bigcap \mathcal{K}_\lambda$. By lemma 1.2.5, $\mathcal{K}_\lambda$ has the finite intersection property. Then K_λ is a compact subset of X which is non-empty if and only if $\lambda \neq 0$.

Lemma 1.2.6: *For each $\lambda \in \Lambda_k(X)$, λ is supported on K_λ.*

Proof. Suppose, by way of contradiction, that there is an $f \in \mathcal{C}(X)$ such that $f\,|_{K_\lambda} = 0$ but $\lambda(f) \neq 0$; say $\lambda(f) > 0$. Since λ is continuous, there exists a $K_1 \in \mathcal{K}(X)$ and an $\epsilon > 0$ such that $\lambda(< f, K_1, \epsilon >) \subseteq (0, \infty)$. Define $U = f^{-1}(\frac{-\epsilon}{2}, \frac{\epsilon}{2})$. Since $K_\lambda \subseteq U$ and $\mathcal{K}_\lambda$ has the finite intersection property, then there exists a $K_2 \in \mathcal{K}_\lambda$ such that $K_2 \subseteq U$. Let δ and open V such that $0 < \delta < \frac{\epsilon}{2}, K_2 \subseteq V \subseteq \text{cl}V \subseteq U$ and $f(\text{cl}V) \subseteq [-\delta, \delta]$. Define $K = K_1 \cup K_2$ and continuous $g_1 : (K_2 \cup (K \cap \text{Bd}V)) \to [-\delta, \delta]$ by $g_1(x) = 0$ if $x \in K_2$ and $g_1(x) = f(x)$ if $x \in K \cap \text{Bd}V$. Then g_1 has a continuous extension $g_2 : K \cap \text{cl}V \to [-\delta, \delta]$. Define continuous $g_3 : K \to \mathbb{R}$ by $g_3(x) = g_2(x)$ if $x \in K \cap \text{cl}V$ and $g_3(x) = f(x)$ if $x \in K \backslash \text{cl}V$. Finally let g be a continuous extension of g_3 to X. Now $g \in < f, K, \epsilon >$, so that $\lambda(g) > 0$. But $g\,|_{K_2} = 0$, which implies that $\lambda(g) = 0$; a contradiction. ∎

Now the proof of Lemmas 1.2.5 and 1.2.6 can be modified to obtain the following two lemmas

Lemma 1.2.7: *Let A be a closed subset of X, let $F \in \mathcal{F}(X)$ and let $\lambda \in \Lambda_{\mathrm{p}}(X)$. If λ is supported on each of A and F, then λ is supported on $A \cap F$.*

For each $\lambda \in \Lambda_{\mathrm{p}}(X)$, define $\mathcal{F}_\lambda = \{F \in \mathcal{F}(X) : \lambda \text{ is supported on } F\}$ and let $F_\lambda = \bigcap \mathcal{F}_\lambda$. By Lemma 1.2.7, $\mathcal{F}_\lambda$ has the finite intersection property. F_λ is a finite subset of X which is non-empty if and only if $\lambda \neq 0$.

Lemma 1.2.8: *For each $\lambda \in \Lambda_{\mathrm{p}}(X), \lambda$ is supported on F_λ.*

If $\lambda \in \Lambda_{\mathrm{k}}(X)$ ($\in \Lambda_{\mathrm{p}}(X)$ respectively), then because of Lemma 1.2.5 (Lemma 1.2.7 respectively), K_λ (F_λ respectively) is contained in each closed subset of X on which λ is supported. So by Lemma 1.2.6 (Lemma 1.2.8 respectively) K_λ (F_λ respectively) can be called the *support* of λ.

Lemma 1.2.9: *For each $f \in C(X)$ and finite set $\{\lambda_1, \ldots, \lambda_n\} \subseteq \Lambda_j(X)$ $(j = p, k)$, there exists a bounded $g \in C(X)$ such that $\lambda_i(f) = \lambda_i(g)$ for each $i = 1, 2, \cdots, n$. Furthermore, if $f \geq 0$, then g can be chosen so that $g \geq 0$.*

Proof. Define $K = K_{\lambda_1} \cup \cdots \cup K_{\lambda_n}$ for $j = k$ and define $F = F_{\lambda_1} \cup \cdots \cup F_{\lambda_n}$ for $j = \mathrm{p}$. Since X is completely regular, the map $f\,|_K$ or the map $f\,|_F$ has a continuous extension g to X which maps into a bounded interval containing $f(K)$ or $f(F)$ respectively. Then $g \in C^*(X)$ and each $\lambda_i(g) = \lambda_i(f)$. ■

3 Algebraic dimensions of $\Lambda_j(X)$; $j = \{\mathrm{p}, \mathrm{k}, \alpha, \infty\}$

First we show that $\Lambda_{\mathrm{p}}(X) \subseteq \Lambda_{\mathrm{k}}(X) \subseteq \Lambda_\infty(X)$ and $\Lambda_\alpha(X) \subseteq \Lambda_\infty(X)$. Define $L_{\mathrm{p}} : \Lambda_{\mathrm{p}}(X) \to \Lambda_{\mathrm{k}}(X)$ as follows. Let $i : \mathcal{C}_{\mathrm{k}}(X) \to \mathcal{C}_{\mathrm{p}}(X)$ be the identity map. i is continuous and linear. Then for each $\lambda \in \Lambda_{\mathrm{p}}(X)$, define $L_{\mathrm{p}}(\lambda) = \lambda \circ i$ which is continuous and linear and is therefore an element of $\Lambda_{\mathrm{k}}(X)$. Clearly L_{p} is one-to-one and hence $\Lambda_{\mathrm{p}}(X) \subseteq \Lambda_{\mathrm{k}}(X)$. Now define $L_{\mathrm{k}} : \Lambda_{\mathrm{k}}(X) \to \Lambda_\infty(X)$ as follows. Let $i : \mathcal{C}^*_\infty(X) \to \mathcal{C}^*_{\mathrm{k}}(X)$ be the identity map and let $j : \mathcal{C}^*_{\mathrm{k}}(X) \to \mathcal{C}_{\mathrm{k}}(X)$ be the inclusion map; both maps are continuous and linear. Then for each $\lambda \in \Lambda_{\mathrm{k}}(X)$ define $L_{\mathrm{k}}(\lambda) = \lambda \circ j \circ i$ which is continuous and is therefore an element of $\Lambda_\infty(X)$. The map L_{k} thus defined is one-to-one. To see this, suppose that $\lambda, \mu \in \Lambda_{\mathrm{k}}(X)$ with $\lambda \neq \mu$. Then $\lambda(f) \neq \mu(f)$ for some $f \in \mathcal{C}_{\mathrm{k}}(X)$. By Lemma 1.2.9, there exists a $g \in \mathcal{C}^*_{\mathrm{k}}(X)$ such that $\lambda(g) = \lambda(f)$ and $\mu(g) = \mu(f)$. But then $L_{\mathrm{k}}(\lambda)(g) = \lambda(g) \neq \mu(g) = L_{\mathrm{k}}(\mu)(g)$ so that $L_{\mathrm{k}}(\lambda) \neq L_{\mathrm{k}}(\mu)$. Hence $\Lambda_{\mathrm{k}}(X) \subseteq \Lambda_\infty(X)$. So we have $\Lambda_{\mathrm{p}}(X) \subseteq \Lambda_{\mathrm{k}}(X) \subseteq \Lambda_\infty(X)$.

Now define $L_\alpha : \Lambda_\alpha(X) \to \Lambda_\infty(X)$ as follows. Let $i : \mathcal{C}^*_\infty(X) \to \mathcal{C}^*_\alpha(X)$ be the identity map. Then for each $\lambda \in \Lambda_\alpha(X)$ define $L_\alpha(\lambda) = \lambda \circ i$ which is continuous and linear and is therefore an element of $\Lambda_\infty(X)$. Clearly L_α is one-to-one. Hence $\Lambda_\alpha(X) \subseteq \Lambda_\infty(X)$.

Now for each $x \in X$, define $\phi_x : \mathcal{C}_{\mathrm{p}}(X) \to \mathbb{R}$ as follows: for each $f \in \mathcal{C}_{\mathrm{p}}(X)$, define $\phi_x(f) = f(x)$. Then ϕ_x is a positive linear functional on $\mathcal{C}_{\mathrm{p}}(X)$ supported on the finite set $\{x\}$. Hence by Lemma 1.2.2 $\lambda \in \Lambda^+_{\mathrm{p}}(X)$.

Theorem 1.3.1: *The set $\{\phi_x : x \in X\}$ is linearly independent and $\dim \Lambda_{\mathrm{p}}(X) = \mathit{card}(X)$.*

Proof. Let $\{\phi_{x_1}, \ldots, \phi_{x_n}\}$ be a finite subset of $\{\phi_x : x \in X\}$ and $a_1\phi_{x_1} + \cdots + a_n\phi_{x_n} = 0$ where $a_i \in \mathbb{R}$, $1 \leq i \leq n$. For each i $(1 \leq i \leq n)$, there exists an $f_i \in C(X, I)$ such that $f_i(x_i) = 1$ and $f_i(x_j) = 0$ for $j \neq i$; $1 \leq j \leq n$. Therefore $a_1\phi_{x_1}(f_i) + \cdots + a_i\phi_{x_i}(f_i) + \cdots + a_n\phi_{x_n}(f_i) = 0(f_i)$, i. e. , $a_i = 0; 1 \leq i \leq n$. Hence $\{\phi_x : x \in X\}$ is linearly independent. Now we show that $\{\phi_x : x \in X\}$ spans $\Lambda_{\mathrm{p}}(X)$.

Pick up any $\lambda \in \Lambda_{\mathrm{p}}(X)$ and suppose $F_\lambda = \{x_1, \ldots x_n\}$ is the support of λ. As before for each i $(1 \leq i \leq n)$, there exists an $f_i \in C(X, I)$ such that $f_i(x_j) = 1$ if $i = j$ and $f_i(x_j) = 0$ if $i \neq j, 1 \leq j \leq n$. Now given $g \in C(X)$ define $t = \sum_{j=1}^{n} g(x_j)f_j$. Then $t\,|_{F_\lambda} = g\,|_{F_\lambda}$ and hence

$$\begin{aligned} \lambda(g) = \lambda(t) &= \textstyle\sum_{j=1}^{n} g(x_j)\lambda(f_j) \\ &= \textstyle\sum_{j=1}^{n} \lambda(f_j)\phi_{x_j}(g) \\ &= \left(\textstyle\sum_{j=1}^{n} \lambda(f_j)\phi_{x_j}\right)(g). \end{aligned}$$

Therefore $\lambda = \sum_{j=1}^{n} \lambda(f_j)\phi_{x_j}$, i. e. , $\{\phi_x : x \in X\}$ spans $\Lambda_{\mathrm{p}}(X)$ and hence $\{\phi_x : x \in X\}$ is a basis for $\Lambda_{\mathrm{p}}(X)$.

Now we have the following injection $\Phi : X \rightarrow \Lambda_{\mathrm{p}}^{+}(X)$ defined as follows: for $x \in X, \Phi(x) = \phi_x$. Suppose $x_1 \neq x_2$. Then there exists an $f \in C(X, I)$ such that $f(x_1) = 1$ and $f(x_2) = 0$. So $\phi_{x_1}(f) = f(x_1) \neq f(x_2) = \phi_{x_2}(f)$, i. e. , $\phi_{x_1} \neq \phi_{x_2}$. Hence $card(\Phi(X) = card(X)$ and so $dim\Lambda_{\mathrm{p}}(X) = card(X)$. ■

Corollary 1.3.2: *$dim(\Lambda_j(X)) \geq card(X)$ for $j = \mathrm{k}, \infty$.*

Proof. $\Lambda_{\mathrm{p}}(X) \subseteq \Lambda_{\mathrm{k}}(X) \subseteq \Lambda_\infty(X)$ ■

Corollary 1.3.3: *Suppose $\mathcal{F}(X) \subseteq \alpha$. Then $dim(\Lambda_\alpha(X)) \geq card(X)$.*

Proof. If $\mathcal{F}(X) \subseteq \alpha$, then $C^*_{\mathrm{p}}(X) \leq C^*_\alpha(X)$ and consequently $\Lambda_{\mathrm{p}}(X) \subseteq \Lambda_\alpha(X)$. ■

4 Topologies on $\Lambda_j(X)$, $j = \mathrm{p}, \mathrm{k}, \alpha$ and ∞

Since the original problem for this dissertation started with the study of $\Lambda^+_{\mathrm{k}}(X)$, we begin by looking at the topologies of $\Lambda^+_{\mathrm{k}}(X)$ and $\Lambda_{\mathrm{k}}(X)$. Then we will consider topologies on $\Lambda_j(X)$, for $j = \mathrm{p}, \alpha$.

For each $K \in \mathcal{K}(X)$, let $F_{\mathrm{K}} = \{f \in C(X) : \chi_{\mathrm{K}} \leq f \leq 1\}$ where χ_{K} is the characteristic function with value 1 on K. For any $K \in \mathcal{K}(X)$, and positive number ϵ, let $W(\lambda, F_{\mathrm{K}}, \epsilon) = \{\lambda' \in \Lambda^+_{\mathrm{k}}(X) : \mid \lambda(f) - \lambda'(f) \mid < \epsilon \text{ for all } f \in F_{\mathrm{K}}\}$ and let τ_λ be the topology on $\Lambda^+_{\mathrm{k}}(X)$ induced by the collection $\{W(\lambda, F_{\mathrm{K}}, \epsilon) : \lambda \in \Lambda^+_{\mathrm{k}}(X), K \in \mathcal{K}(X), \epsilon > 0\}$ as a subbase. The topology τ_λ on $\Lambda^+_{\mathrm{k}}(X)$ was first defined and studied by Okuyama in [11]. But then McCoy modified this topology a little and defined a new topology on $\Lambda^+_{\mathrm{k}}(X)$ as follows. For each $\lambda, \mu \in \Lambda^+_{\mathrm{k}}(X)$ and $K \in \mathcal{K}(X)$, let $e_{\mathrm{k}}(\lambda, \mu) = \sup\{\mid \lambda(f) - \mu(f) \mid : f \in F_K\}$. It can be easily shown that each e_{k} is a pseudometric on $\Lambda^+_{\mathrm{k}}(X)$. For each $\lambda, \mu \in \Lambda^+_{\mathrm{k}}(X)$ and $K \in \mathcal{K}(X)$, let $W(\lambda, K, \epsilon) = \{\mu \in \Lambda^+_{\mathrm{k}}(X) : e_{\mathrm{k}}(\lambda, \mu) < \epsilon\}$. Then the new topology on $\Lambda^+_{\mathrm{k}}(X)$ has $\{W(\lambda, K, \epsilon) : \lambda \in \Lambda^+_{\mathrm{k}}(X), K \in \mathcal{K}(X), \epsilon > 0\}$ as a subbase. Call this new topology on $\Lambda^+_{\mathrm{k}}(X)$ τ_e. Now given $W(\lambda, K, \epsilon)$ and $\mu \in W(\lambda, K, \epsilon)$, it can be easily shown that $\mu \in W(\mu, F_K, \frac{\delta}{2}) \subseteq W(\mu, K, \delta) \subseteq W(\lambda, K, \epsilon)$ where $\delta = \frac{1}{2}(\epsilon - e_{\mathrm{k}}(\lambda, \mu))$. Hence τ_e is weaker than τ_λ. From now onwards we will be interested only in the topology τ_e on $\Lambda^+_{\mathrm{k}}(X)$ and shortly we will see that $(\Lambda^+_{\mathrm{k}}(X), \tau_e)$ is a (topological) subspace of a certain normed linear space, i. e. , actually $(\Lambda_k^+(X), \tau_e)$ is a metrizable space.

On $\Lambda_\infty(X)$ we can define the following conjugate norm given by $\|\lambda\|_* = \sup\{|\lambda(f)| : f \in C^*(X); \|f\|_\infty \leq 1\}$ for each $\lambda \in \Lambda_\infty(X)$. $(\Lambda_\infty(X), \|\cdot\|_*)$, being the conjugate space of the normed linear space $C^*_\infty(X)$, is always complete, i. e. , $(\Lambda_\infty(X), \|\cdot\|_*)$ is a Banach space. We have already seen in §3 that $\Lambda_p(X) \subseteq \Lambda_k(X) \subseteq \Lambda_\infty(X)$ and $\Lambda_\alpha(X) \subseteq \Lambda_\infty(X)$. So on each of $\Lambda_j(X)$ $(j = p, k, \alpha)$ we can assign this $\|\cdot\|_*$-norm. From now onwards, by $\Lambda_j(X)$, we will mean the normed linear space $(\Lambda_j(X), \|\cdot\|_*)$.

Theorem 1.4.1: *The space $(\Lambda_k^+(X), \tau_e)$ is a (topological) subspace of the normed linear space $\Lambda_k(X)$.*

Proof. The basic open sets in the subspace topology that $\Lambda_k^+(X)$ inherits from $\Lambda_k(X)$ look like $B(\lambda, \epsilon) = \{\mu \in \Lambda_k^+(X) : \|\lambda - \mu\|_* < \epsilon\}$ for $\lambda \in \Lambda_k^+(X)$ and $\epsilon > 0$.

Let $\mu \in W(\lambda, K, \epsilon)$ where $\lambda \in \Lambda_k^+(X), K \in \mathcal{K}(X)$ and $\epsilon > 0$. Define $\delta = \frac{1}{2}(\epsilon - e_k(\lambda, \mu))$. To show that $B(\mu, \delta) \subseteq W(\lambda, K, \epsilon)$, let $\nu \in B(\mu, \delta)$ and let $f \in F_K$. Since $f \in C^*(X)$ and $\|f\|_\infty \leq 1$, then $|\mu(f) - \nu(f)| \leq \|\mu - \nu\|_* < \delta = \frac{1}{2}(\epsilon - e_k(\lambda, \mu))$. Also since $f \in F_K$, then $|\lambda(f) - \mu(f)| \leq e_k(\lambda, \mu)$. Thus $|\lambda(f) - \nu(f)| \leq |\lambda(f) - \mu(f)| + |\mu(f) - \nu(f)| < \frac{1}{2}(\epsilon - e_k(\lambda, \mu)) + e_k(\lambda, \mu) = \epsilon - \frac{1}{2}(\epsilon - e_k(\lambda, \mu))$. Since f is an arbitrary element of F_K, then $e_k(\lambda, \nu) \leq \epsilon - \frac{1}{2}(\epsilon - e_k(\lambda, \mu)) < \epsilon$. Therefore $\nu \in W(\lambda, K, \epsilon)$ so that $B(\mu, \delta) \subseteq W(\lambda, K, \epsilon)$. It follows that $W(\lambda, K, \epsilon)$ is open in the subspace topology that $\Lambda_k^+(X)$ inherits from $\Lambda_k(X)$. To establish the other direction, let $\lambda \in \Lambda_k^+(X)$ and $\epsilon > 0$. Define $K = \emptyset$ and let $\delta = \frac{1}{4}\epsilon$. To show that $W(\lambda, K, \delta) \subseteq B(\lambda, \epsilon)$, let $\mu \in W(\lambda, K, \delta)$ and let $f \in C^*(X)$ with $\|f\|_\infty \leq 1$. Define $g \in F_K$ by $g(x) = \frac{1}{2}f(x) + \frac{1}{2}$ for each $x \in X$. Then $|\lambda(g) - \mu(g)| < \delta$ so that $|\lambda(f) - \mu(f)| =$

$|2\lambda(g)-\lambda(1)-2\mu(g)+\mu(1)|\leq 2\,|\lambda(g)-\mu(g)|+|\lambda(1)-\mu(1)|$. Since both g and 1 (the constant 1 function) are in F_K, then $|\lambda(f)-\mu(f)|< 2\delta+\delta=3\delta=\frac{3}{4}\epsilon$. Hence $\mu\in B(\lambda,\epsilon)$ so that $W(\lambda,K,\delta)\subseteq B(\lambda,\epsilon)$. Therefore $B(\lambda,\epsilon)$ is open in $\Lambda_k^+(X)$ and it follows that $\Lambda_k^+(X)$ is a subspace of $\Lambda_k(X)$. ■

It follows from the proof of Theorem 1.4.1 that $\{W(\lambda,\emptyset,\epsilon):\lambda\in\Lambda_k^+(X),\epsilon>0\}$ is in fact a base for the topology τ_e on $\Lambda_k^+(X)$. It also follows from this theorem that $(\Lambda_k^+(X),\tau_e)$ is metrizable. In particular, if $d_*(\lambda,\mu)=\|\lambda-\mu\|_*$ for each $\lambda\in\Lambda_k^+(X)$, then d_* is a compatible metric on $\Lambda_k^+(X)$. From now onwards, while working on $\Lambda_k^+(X)$, we will be considering the metric space $(\Lambda_k^+(X),d_*)$. Also while working on the positive cones $\Lambda_j^+(X)$ of $\Lambda_j(X)$ $(j=\mathrm{p},\alpha,\infty)$, we will be considering the metric spaces $(\Lambda_j^+(X),d_*)$. In fact, from now onwards, by $\Lambda_j^+(X)$ $(j=\mathrm{p},\mathrm{k},\alpha,\infty)$ we will mean the metric spaces $(\Lambda_j^+(X),d_*)$.

Theorem 1.4.2: *$\Lambda_j^+(X)$ is a closed subspace of the normed linear space $\Lambda_j(X)$, $(j=\mathrm{p},\mathrm{k},$ $\alpha,\infty)$.*

Proof. We do it for $j=\mathrm{p},\mathrm{k}$. For $j=\alpha,\infty$, a slight obvious modification is needed. So let $\lambda\in\Lambda_j(X)\backslash\Lambda_j^+(X)$, $j=$p, k. Then there exists an $f\in C(X)$ such that $f\geq 0$ but $\lambda(f)<0$. By Lemma 1.2.9, there exists a $g\in C^*(X)$ such that $g\geq 0$ and $\lambda(g)=\lambda(f)$. Let r be a positive number such that $\|rg\|_\infty\leq 1$. Define $\epsilon=-\frac{r}{2}\lambda(f)$. Now suppose $\mu\in\Lambda_j(X)$ is such that $\|\mu-\lambda\|_*<\epsilon$. Then $|\mu(rg)-\lambda(rg)|<\epsilon$ so that $\mu(g)-\lambda(g)<\frac{\epsilon}{r}=-\frac{1}{2}\lambda(f)$. Therefore $\mu(g)<\frac{1}{2}\lambda(f)<0$ so that $\mu\in\Lambda_j(X)\backslash\Lambda_j^+(X)$. ■

5 Relation between $\Lambda_j(X)$ and $\Lambda_j(Y)$ ($j = \mathrm{p}, \mathrm{k}, \sigma$, and ∞) where X is a subspace of Y

First we deal with the case of $\Lambda_{\mathrm{p}}(X)$ and $\Lambda_{\mathrm{k}}(X)$ simultaneously. Then we will deal with $\Lambda_{\infty}(X)$ and $\Lambda_{\sigma}(X)$ where σ is a particular α. More generally, suppose that $f : X \to Y$ is a continuous map. This induces a continuous map $f^* : C_{\mathrm{j}}(Y) \to C_{\mathrm{j}}(X)$ $(j = \mathrm{p}, \mathrm{k})$ defined by $f^*(g) = g \circ f$. See [9], pages 18-22, for general topological properties of this induced function f^*. Again f^* induces a linear map $f^{**} : \Lambda_j(X) \to \Lambda_j(Y)$ defined by $f^{**}(\lambda) = \lambda \circ f^*$ where $\lambda \in \Lambda_j(X)\,(j = \mathrm{p}, \mathrm{k})$. Note that f^{**} takes $\Lambda_j^+(X)$ into $\Lambda_j^+(Y)$.

Theorem 1.5.1: *If $f : X \to Y$ is a continuous map, then $f^{**} : \Lambda_j(X) \to \Lambda_j(Y)$ $(j = \mathrm{p}, \mathrm{k})$ is continuous.*

Proof. In fact we show that given $\lambda \in \Lambda_j(X)$, we have $\| f^{**}(\lambda) \| \leq \| \lambda \|_*$. Note if $g \in C(Y)$ with $\| g \|_\infty \leq 1$, then $f^*(g) = g \circ f \in C(X)$ and $\| f^*(g) \|_\infty \leq 1$. Now

$$\begin{aligned}
\| f^{**}(\lambda) \|_* &= \sup\{| f^{**}(\lambda)(g) | : \| g \|_\infty \leq 1; g \in C(Y)\} \\
&= \sup\{| \lambda \circ f^*)(g) | : \| g \|_\infty \leq 1; g \in C(Y)\} \\
&= \sup\{| \lambda(f^*(g)) | : \| g \|_\infty \leq 1; g \in C(Y)\} \\
&\leq \sup\{| \lambda(h) | : \| h \|_\infty \leq 1; h \in C(X)\} \\
&= \| \lambda \|_* . \blacksquare
\end{aligned}$$

Theorem 1.5.2: *If $f^* : C_j(Y) \to C_j(X)$ $(j = \mathrm{p}, \mathrm{k})$ is almost onto, then f^{**} is one-to-one.*

Proof. Suppose $\lambda_1, \lambda_2 \in \Lambda_j(X)$ with $f^{**}(\lambda_1) = f^{**}(\lambda_2)$. Let $g \in f^*(C_j(Y))$. So for some

$h \in C_j(Y), g = f^*(h) = h \circ f$. Now $\lambda_1(g) = \lambda_1(f^*(h)) = (\lambda_1 \circ f^*)(h) = ((f^{**}(\lambda_1))(h) = (f^{**}(\lambda_2))(h) = (\lambda_2 \circ f^*)(h) = \lambda_2(f^*(h)) = \lambda_2(g)$. So $\lambda_1 = \lambda_2$ on $f^*(C_j(Y))$. But $f^*(C_j(Y))$ is dense in $C_j(X)$. So $\lambda_1 = \lambda_2$ and hence f^{**} is one-to-one.■

Note: f^* is almost onto if and only if f is one-to-one. See [9].

Theorem 1.5.3: *Suppose $f : X \to Y$ is continuous. Suppose $\lambda \in \Lambda_k(X)$* has compact support K_λ *and $f^* : C(f(K_\lambda), I_1) \to C(K_\lambda, I_1)$ is onto. Then $\|\lambda\|_* = \|f^{**}(\lambda)\|_*$ ($C(X, I_1)$ is the collection of all continuous functions on X mapping into I_1 where $I_1 = [-1, 1]$).*

Proof. Theorem 1.5.1 already gives us $\| f^{**}(\lambda) \| \leq \| \lambda \|_*$. So we only need to show that $\|\lambda\|_* \leq \|f^{**}(\lambda)\|_*$. Consider $g \in C(X)$ with $\|g\|_\infty \leq 1$. So $g|_{K_\lambda}$ is in $C(K_\lambda, I_1)$ and hence by hypothesis, there exists h in $C(f(K_\lambda), I_1)$ such that $f^*(h) = g|_{K_\lambda}$, i. e. , $h \circ (f|_{K_\lambda}) = g|_{K_\lambda}$. Since $f(K_\lambda)$ is compact, h has an extension h' in $C(Y, I_1)$. Now $x \in K_\lambda$ means $h'(f(x)) = h(f(x)) = g(x)$, i. e. , $f^*(h')|_{K_\lambda} = g|_{K_\lambda}$ and hence $\lambda(g) = \lambda(f^*(h')) = (\lambda \circ f^*)(h') = (f^{**}(\lambda))(h')$ where $h' \in C(Y)$ with $\| h' \|_\infty \leq 1$. Hence $\| \lambda \|_* = \sup\{|\lambda(g)| : \|g\|_\infty \leq 1, g \in C(X)\} \leq \sup\{|f^{**}(\lambda)(h')| : \|h'\|_\infty \leq 1, h' \in C(Y)\} = \|f^{**}(\lambda)\|_*$. ■

The proof of Theorem 1.5.3 can be modified to obtain the following theorem.

Theorem 1.5.4: *Suppose $f : X \to Y$ is continuous. Suppose $\lambda \in \Lambda_p(X)$ has finite support F_λ and $f^* : C(f(F_\lambda), I_1) \to C(F_\lambda, I_1)$ is onto. Then $\|\lambda\|_* = \|f^{**}(\lambda)\|_*$.*

Theorem 1.5.5: *If X is a subbspace of Y and $i : X \to Y$ is the inclusion map, then i^{**} is an isometry mapping $\Lambda_j(X)$ into $\Lambda_j(Y)$ ($j = p, k$).*

Proof. For i the hypotheses of Theorem 1.5.3 or of Theorem 1.5.4 are trivially satisfied.

Hence $\| i^{**}(\lambda) \|_* = \| \lambda \|_*$ *for all* $\lambda \in \Lambda_j(X)$ $(j = \mathrm{p}, \mathrm{k})$. ∎

Theorem 1.5.6: *If X is a closed subspace of Y and $i : X \to Y$ is the inclusion map, then $i^{**}(\Lambda_j(X))$ is closed in $\Lambda_j(Y)$ and $i^{**}(\Lambda_j^+(X))$ is closed in $\Lambda_j^+(Y)$ where $j =$p, k.*

Proof. We prove it for $j = \mathrm{k}$. For $j = \mathrm{p}$, the proof is similar.

Let $\lambda \in \Lambda_{\mathrm{k}}(Y)$ and suppose that there is a sequence (λ_n) in $\Lambda_{\mathrm{k}}(X)$ such that $(i^{**}(\lambda_n))$ converges to λ. Then since $\| i^{**}(\lambda_n) - \lambda \| \to 0$ as $n \to \infty$, we have for each

$$f \in C^*(Y),\ \lambda(f) = \lim_{n\to\infty} i^{**}(\lambda_n)(f) = \lim_{n\to\infty} \lambda_n(f|_X).$$

Let $f \in C(Y)$ with $f|_X = 0$. By an argument similar to that in the proof of Lemma 1.2.5, there is an $\hat{f} \in C^*(Y)$ such that $\hat{f}|_{K_\lambda} = f|_{K_\lambda}$ and $\hat{f}|_X = 0$. Then $\lambda(f) = \lambda(\hat{f}) = \lim_{n\to\infty} \lambda_n(\hat{f}|_X) = 0$. Therefore λ is supported on X, so by Lemma 1.2.5 $K_\lambda \subseteq X$.

Finally, define $\mu \in \Lambda_{\mathrm{k}}(X)$ as follows. For each $g \in C(X)$, take $\mu(g) = \lambda(f)$ where f is any element of $C(Y)$ such that $f|_{K_\lambda} = g|_{K_\lambda}$. Then μ is well-defined since $K_\lambda \subseteq X$. Since λ is continuous and linear, it can be easily checked that μ is also continuous and linear, i. e. , $\mu \in \Lambda_{\mathrm{k}}(X)$. Also for each $f \in C(Y)$, $i^{**}(\mu)(f) = \mu(f|_X) = \lambda(f)$ so that $\lambda = i^{**}(\mu)$. This shows that $i^{**}(\Lambda_{\mathrm{k}}(X))$ is indeed closed in $\Lambda_{\mathrm{k}}(Y)$. A similar proof shows that $i^{**}(\Lambda_{\mathrm{k}}^+(X))$ is closed in $\Lambda_{\mathrm{k}}^+(Y)$. ∎

Now we deal with $\Lambda_\infty(X)$. If $f : X \to Y$ is a continuous map, then $f^* : C_\infty^*(Y) \to C_\infty^*(X)$ is a continuous linear map, because given $g \in C_\infty^*(Y)$, we have $\|f^*(g)\|_\infty \leq \| g \|_\infty$. So correspondingly we have a continuous linear map $f^{**} : \Lambda_\infty(X) \to \Lambda_\infty(Y)$. Note again f^{**} takes $\Lambda_\infty^+(X)$ into $\Lambda_\infty^+(Y)$.

Theorem 1.5.7: *Suppose $f : X \to Y$ is a continuous map. Then $f^{**} : \Lambda_\infty(X) \to \Lambda_\infty(Y)$ is continuous. In addition, if $f^* : C^*_\infty(Y, \mathrm{I}_1) \to C^*_\infty(X, \mathrm{I}_1)$ is onto, then f^{**} is an (into) isometry.*

Proof. See Theorems 1.5.1 and 1.5.3. ■

Corollary 1.5.8: *Suppose X is a C^*-imbedded subspace of Y and $i : X \to Y$ is the inclusion map. Then $i^{**} : \Lambda_\infty(X) \to \Lambda_\infty(Y)$ is an (into) isometry, i. e., $\Lambda_\infty(Y)$ contains a copy of $\Lambda_\infty(X)$ and $\Lambda^+_\infty(Y)$ contains a copy of $\Lambda^+_\infty(X)$.*

Remark 1.5.9: *In the next chapter, we will show that if X is a closed subspace of a normal Hausdorff space Y, then $i^{**}(\Lambda_\infty(X))$ is closed in $\Lambda_\infty(Y)$. Also the same result is true for $\Lambda^+_\infty(\cdot)$.*

At the end of this chapter, we discuss a $\Lambda_\alpha(X)$ where α is a particular family of subsets of X. For the rest of this chapter, every space will be assumed to be a normal Hausdorff space and let $\sigma = \{\mathrm{cl}_X A : A \text{ is a } \sigma\text{-compact subset of } X\}$. Note that σ is closed under finite union because $\bigcup_{n=1}^k \overline{A}_n = \overline{\bigcup_{n=1}^k A_n}$ where $\overline{A} = \mathrm{cl}_X A$. We denote the corresponding $\Lambda_\alpha(X)$ by $\Lambda_\sigma(X)$. Now let $f : X \to Y$ be a continuous map. Then as before f induces a linear map $f^* : C^*_\sigma(Y) \to C^*_\sigma(X)$ defined by $f^*(g) = g \circ f$ where $g \in C^*_\sigma(Y)$. Likewise, as before, a linear map $f^{**} : \Lambda_\sigma(X) \to \Lambda_\sigma(Y)$ is defined by $f^{**}(\lambda) = \lambda \circ f^*$. But for f^{**} to be well-defined, we need to ensure the continuity of f^*. On the continuity of f^*, we have the following theorem.

Theorem 1.5.10: *Suppose $f : X \to Y$ is a continuous map. Then $f^* : C^*_\sigma(Y) \to C^*_\sigma(X)$ is* *continuous.*

Proof. Let $g \in C_\sigma^*(Y)$ and $f^*(g)+V_{p_A,\epsilon}$ be a basic neighborhood of $f^*(g)$ in $C_\sigma^*(X)$ where $A \in \sigma$ (this σ is a subset of $\mathcal{P}(X)$ =the power set of X). Suppose $A = \overline{\bigcup_{n=1}^\infty K_n}$ where $K_n \in \mathcal{K}(X)$ for each n. So $f\left(\bigcup_{n=1}^\infty K_n\right) = \bigcup_{n=1}^\infty f(K_n)$ is a σ-compact subset of Y. Hence $\overline{f(\bigcup_{n=1}^\infty K_n)}$ belongs to σ (this σ is a subset of $\mathcal{P}(Y)$). Since $f(A) = f(\overline{\bigcup_{n=1}^\infty K_n}) \subseteq \overline{f(\bigcup_{n=1}^\infty K_n)} = B$ (say), it can be easily shown that $f^*(g+V_{p_B,\epsilon}) \subseteq f^*(g)+V_{p_A,\epsilon}$. Hence f^* is continuous. ∎

The above theorem says that the linear map $f^{**} : \Lambda_\sigma(X) \to \Lambda_\sigma(Y)$ is well-defined. Also note that, as before, f^{**} takes $\Lambda_\sigma^+(X)$ into $\Lambda_\sigma^+(Y)$. The next theorem can be proved in manners similar to Theorems 1.5.1 and 1.5.3.

Theorem 1.5.11: *Suppose $f : X \to Y$ is a continuous map. Then $f^{**} : \Lambda_\sigma(X) \to \Lambda_\sigma(Y)$ is continuous. In addition, if $f^* : C_\sigma(Y, \mathrm{I}_1) \to C_\sigma^*(X, \mathrm{I}_1)$ is onto, then f^{**} is an (into) isometry.*

Corollary 1.5.12: *Suppose X is a closed subspace of a space Y and $i : X \to Y$ is the inclusion map. Then $i^{**} : \Lambda_\sigma(X) \to \Lambda_\sigma(Y)$ is an (into) isometry,* i. e. , *$\Lambda_\sigma(Y)$ contains a copy of $\Lambda_\sigma(X)$ and $\Lambda_\sigma^+(Y)$ contains a copy of $\Lambda_\sigma^+(X)$.*

Remark 1.5.13: *In the next chapter, we will actually show that if X is a closed subspace of Y, then $i^{**}(\Lambda_\sigma(X))$ is closed in $\Lambda_\sigma(Y)$. The same result is also true for $\Lambda_\sigma^+(\cdot)$.*

Chapter II

Completeness

In this chapter, we study completeness of $\Lambda_j^+(X)$ and $\Lambda_j(X)$, j = p, k, α, ∞. But first we deal with $\Lambda_k(X)$ and $\Lambda_p(X)$ and their positive cones $\Lambda_j^+(X)$ (j = k, p). The space $\Lambda_j^+(X)$ is a metric space with metric d_*. This space is complete provided that if a sequence in $\Lambda_j^+(X)$ is a Cauchy sequence with respect to d_*, then it converges. Likewise, the normed linear space $\Lambda_j(X)$ is complete if it is complete with respect to its norm $\|\cdot\|_*$, i. e. , if it is a Banach space.

The following example is of some use for studying the completeness of $\Lambda_j^+(X)$ (j = p, k). Here we mention the useful fact that when compact subsets of X are finite, $\mathcal{C}_k(X) = \mathcal{C}_p(X)$ and consequently $\Lambda_k(X) = \Lambda_p(X)$. So in particular for the discrete space of natural numbers $\mathbb{N}$, $\mathcal{C}_k(\mathbb{N}) = \mathcal{C}_p(\mathbb{N})$ and $\Lambda_k(\mathbb{N}) = \Lambda_p(\mathbb{N})$.

Example 2.1: *If $\mathbb{N}$ is the discrete space of natural numbers, then $\Lambda_k(\mathbb{N})$ is topologically isomorphic to a linear subspace of the separable Banach space l_1. It follows that $\Lambda_k^+(\mathbb{N})$ is homeomorhic to a subspace of l_1. In addition, both $\Lambda_k(\mathbb{N})$ and $\Lambda_k^+(\mathbb{N})$ are of first category, that is, they can each be written as a countable union of nowhere dense subsets.*

Proof. Consider more generally a discrete space X. Let $\sigma(X)$ be the σ-product $\{t \in \mathbb{R}^X : t(x) = 0$ for all but finitely many $x \in X\}$ which is a subset of the product space $\mathbb{R}^X$. Now consider $\sigma(X)$ as a normed linear space with norm given by $\| t \|_1 = \Sigma\{| t(x) | : x \in X\}$. In particular, $\sigma(X)$ is a linear subspace of l_1. Define $\xi : \Lambda_k(X) \to \sigma(X)$ as follows. Let $\lambda \in \Lambda_k(X)$. For each $x \in X$, define $f_x \in C(X)$ to be the characteristic function taking x to 1 and other points to 0. Now define $\xi(\lambda)(x) = \lambda(f_x)$ for each $x \in X$. Note that $\xi(\lambda) \in \sigma(X)$ since K_λ is finite and since if $x \in X \backslash K_\lambda$ then $\lambda(f_x) = 0$. This clearly defines a linear function from $\Lambda_k(X)$ into $\sigma(X)$.

To show that ξ is one-to-one, let $\lambda \in \Lambda_k(X)$ be such that $\xi(\lambda) = 0$. Then $\lambda(f_x) = 0$ for every $x \in K_\lambda$. Now let $f \in C(X)$, Define $g = \sum\{f(x)f_x : x \in K_\lambda\}$. Then $g|_{K_\lambda} = f|_{K_\lambda}$, so that $\lambda(f) = \lambda(g) = \sum\{f(x)\lambda(f_x) : x \in K_\lambda\} = 0$. Therefore $\lambda = 0$ and ξ is one-to-one.

To show that ξ maps $\Lambda_k(X)$ onto $\sigma(X)$, let $t \in \sigma(X)$. Then there is a finite subset K of X such that $t(x) = 0$ whenever $x \in X \backslash K$. Define $\lambda : C_k(X) \to \mathbb{R}$ by $\lambda(f) = \sum\{f(x)t(x) : x \in K\}$. This is clearly a linear function. It is straightforward to check that $\xi(\lambda) = t$. It remains to show that λ is continuous at 0; so let $\epsilon > 0$. Define $\delta = \frac{\epsilon}{M+1}$ where $M = \sum\{| t(x) | : x \in K\}$. Then if $f \in < 0, K, \delta >$, $| \lambda(f) - \lambda(0) | \leq \sum\{| f(x) | | t(x) | : x \in K\} < \delta M < \epsilon$. Therefore λ is continuous, so that $\lambda \in \Lambda_k(X)$. It follows that ξ maps $\Lambda_k(X)$ onto $\sigma(X)$.

Next to establish that $\| \lambda \|_* \leq \| \xi(\lambda) \|_1$ for each λ, let $\lambda \in \Lambda_k(X)$ and let $f \in C^*(X)$ with $\| f \|_\infty \leq 1$. Then $| \lambda(f) | = | \lambda\, (\sum\{f(x)f_x : x \in K_\lambda\}) | \leq \sum\{| f(x) | | \lambda(f_x) | : x \in K_\lambda\} \leq \sum\{| \lambda(f_x) | : x \in K_\lambda\} = \sum\{| \xi(\lambda)(x) | : x \in X\} = \| \xi(\lambda) \|_1$. This shows that ξ^{-1} is continuous.

Finally, to show that ξ is continuous, let $\lambda \in \Lambda_{\rm k}(X)$. Let $P = \{x \in K_\lambda : \lambda(f_x) > 0\}$ and $N = \{x \in K_\lambda : \lambda(f_x) < 0\}$. Then $\| \lambda \|_* = \sup\{| \lambda(f) | : f \in C^*(X), \| f \|_\infty \leq 1\} \geq | \lambda(\sum\{f_x : x \in P\}) | = \sum\{\lambda(f_x) : x \in P\} = \sum\{| \lambda(f_x) | : x \in P\}$. Similarly $\| \lambda \|_* \geq \sum\{-\lambda(f_x) : x \in N\} = \sum\{| \lambda(f_x) | : x \in N\}$. Therefore $2 \, \| \lambda \|_* \geq \sum\{| \lambda(f_x) | : x \in K_\lambda\} = \| \xi(\lambda) \|_1$. It follows that ξ is continuous. This establishes that ξ is a topological isomorphism. Also note that ξ takes $\Lambda_{\rm k}^+(X)$ onto $\sigma^+(X) = \{t \in \sigma(X) : t(x) \geq 0 \text{ for all } x \in X\}$.

We show that $\Lambda_{\rm k}(\mathbb{N})$ is of first category. Let $T_n = \{t \in \sigma(\mathbb{N}) : t(m) = 0 \text{ for all } m > n\}$. Then $\sigma(\mathbb{N}) = \bigcup\{T_n : n \in \mathbb{N}\}$. Each T_n is closed in $\sigma(\mathbb{N})$. To show this, let $t \in \sigma(\mathbb{N}) \backslash T_n$. So there exists an m such that $m > n$ but $t(m) \neq 0$. For any $s \in T_n$, $\| s - t \|_1 = \sum_{i=1}^{\infty} | s(i) - t(i) | \geq | s(m) - t(m) | = | t(m) |$. Choose ϵ such that $0 < \epsilon < | t(m) |$. Then the ϵ-ball centered at t does not interesect T_n. Hence T_n is closed in $\sigma(\mathbb{N})$. Next we show that the interior of T_n is empty. If possible, let $t \in$ interior of T_n. Then there exists an $\epsilon > 0$ such that $t \in B(t, \epsilon) \subseteq$ interior of T_n where $B(t,\epsilon) = \{s \in \sigma(\mathbb{N}) : \| s - t \|_1 < \epsilon\}$. $t \in T_n$ means $t(m) = 0$ for all $m > n$. Define $s : \mathbb{N} \to \mathbb{R}$ as follows: $s(i) = t(i)$, for all i where $1 \leq i \leq n$, $s(n+1) = \frac{\epsilon}{2}$ and $s(m) = 0$ for all $m > n+1$. $\| s - t \|_1 = s(n+1) = \frac{\epsilon}{2} < \epsilon$. So $s \in B(t, \epsilon) \subseteq$ interior of T_n. But $s(n+1) = \frac{\epsilon}{2} \neq 0$ means $s \notin T_n$. So $s \notin$ interior of T_n. A contradiction, hence interior of T_n is empty. Hence $\sigma(\mathbb{N})$ is of first category and consequently $\Lambda_{\rm k}(\mathbb{N})$ is also of first category. The proof showing that $\sigma^+(\mathbb{N})$ is of first category is similar. Here we take $T_n = \{t \in \sigma^+(\mathbb{N}) : t(m) = 0 \text{ for all } m > n\}$. ∎

To establish that the completeness of $\Lambda_j^+(X)\,(j = {\rm p}, {\rm k}, \alpha, \infty)$ is equivalent to the completeness of $\Lambda_j(X)$, it is convenient to introduce vector lattices and to indicate how each member of $\Lambda_j(X)$ decomposes into the difference of its positive and negative parts from $\Lambda_j^+(X)$. A partially ordered vector space V is a real vector space equipped with an order relation $\leq$ that is compatible with the algebraic structure as follows:

1. If $u \leq v$, then $u + w \leq v + w$ holds for all $w \in V$

2, If $u \leq v$, then $au \leq av$ holds for all $a \geq 0$.

The set $V^+ = \{v \in V : v \geq 0\}$ is called the positive cone of V and its members are called the positive elements of V. Clearly, the sum of two positive elements is again a positive element.

A partially ordered vector space V is called a vector lattice (or a Riesz space) if for every pair of elements $u, v \in V$ both $\sup\{u, v\}$ and $\inf\{u, v\}$ exist. As usual, $\sup\{u, v\}$ is denoted by $u \vee v$ and $\inf\{u, v\}$ by $u \wedge v$. That is $u \vee v = \sup\{u, v\}$ and $u \wedge v = \inf\{u, v\}$. In a vector lattice, the positive part, the negative part and the absolute value of an element v are defined by $v^+ = v \vee 0$, $v^- = (-v) \vee 0$ and $|v| = v \vee (-v)$. A number of useful inequalities are now derived.

Theorem 2.2: *If u, v and w are elements of a vector lattice, then the following inequalities hold:*

1. $|u + v| \leq |u| + |v|$;

2. $||u| - |v|| \leq |u - v|$;

3. $|u^+ - v^+| \leq |u - v|$ *and* $|u^- - v^-| \leq |u - v|$.

Proof. 1. From $u \leq |u|$ and $v \leq |v|$, it follows that $u + v \leq |u| + |v|$. Similarly $-(u + v) \leq$

$|u| + |v|$ holds and thus $|u+v| = (u+v) \vee [-(u+v)] \leq |u| + |v|$.

2. By (1) we see that $|u| = |u-v+v| \leq |u-v| + |v|$ and so $|u| - |v| \leq |u-v|$. Similarly $|v| - |u| \leq |u-v|$ and hence $||u| - |v|| \leq |u-v|$.

3. Note that $u^+ = \frac{1}{2}(u + |u|)$. Then to establish the required inequality, use (1) and (2) as follows: $|u^+ - v^+| = |\frac{1}{2}(u+|u|) - \frac{1}{2}(v+|v|)| = \frac{1}{2}|(u-v) + (|u| - |v|)| \leq \frac{1}{2}|u-v| + \frac{1}{2}||u| - |v|| \leq |u-v|$. Similarly $|u^- - v^-| \leq |u-v|$. ∎

A subset A of V is called *order bounded* if there exists an element $v \in V$ such that $|u| \leq v$ holds for all $u \in A$. A linear fuctional λ on a vector lattice V is said to be order bounded if it carries order bounded subsets of V onto bounded subsets of $\mathbb{R}$. That is, a linear functional λ on V is order bounded if for every $v \in V^+$ there exists some $M > 0$ such that $|\lambda(u)| \leq M$ holds for all $u \in V$ with $|u| \leq v$. A linear functioal λ on V is called *positive* if $\lambda(v) \geq 0$ holds for each $v \in V^+$. Clearly every positive linear functional is order bounded. Now let $V^\sim$ be the set of all order bounded linear functionals on V. $V^\sim$ is called the order dual of V. For each $\lambda, \mu \in V^\sim$, define $\lambda \leq \mu$ provided that $\lambda(v) \leq \mu(v)$ for all $v \in V^+$. Then $V^\sim$ becomes a partially ordered vector space, which is in fact a vector lattice by the following Reisz Theorem (see[2], page 13).

Theorem 2.3: (F. Reisz). *If V is a vector lattice, then its order dual $V^\sim$ is likewise a vector lattice. Moreover,*

$$\begin{aligned} \lambda^+(v) &= \sup\{\lambda(w) : 0 \leq w \leq v\} \\ \lambda^-(v) &= \sup\{-\lambda(w) : 0 \leq w \leq v\}, \textit{ and} \\ |\lambda|(v) &= \sup\{\lambda(w) : |w| \leq v\} = \sup\{|\lambda(w)| : |w| \leq v\} \end{aligned}$$

hold for each $\lambda \in V^{\sim}$ *and* $v \in V^{+}$.

The function space $C(X)$ or $C^*(X)$ is a vector lattice under the ordinary partial order defined by: $f \leq g$ provided that $f(x) \leq g(x)$ for all $x \in X$. Now $C(X)^{\sim}$ contains $\Lambda_k(X)$ or $\Lambda_p(X)$, as shown by the the following argument. We show the argument for $\Lambda_k(X)$. The argument for $\Lambda_p(X)$ is similar. Let $\lambda \in \Lambda_k(X)$ and let $g \in C(X)$ with $g \geq 0$. Since λ is continuous on $C_k(X)$, there is some constant M and a compact subset K of X such that for every $f \in C(X)$ $\quad |\lambda(f)| \leq Mp_K(f)$. So if $|f| \leq g$, then $|\lambda(f)| \leq Mp_K(g)$. That is, λ carries order bounded subsets of $C(X)$ into bounded subsets of $\mathbb{R}$. Therefore λ is order bounded and an element of $C(X)^{\sim}$, as desired. By using the same argument as used above, we can show that if λ is an element of $\Lambda_\alpha(X)$, then λ belongs to $C^*(X)^{\sim}$. Now we deal with the case of $\Lambda_\infty(X)$. Let $\lambda \in \Lambda_\infty(X)$ and let $g \in C^*(X)$ with $g \geq 0$. Now if $|f| \leq g$ and $f \in C^*(X)$, then $|\lambda(f)| \leq \|\lambda\|_* \|f\|_\infty \leq \|\lambda\|_* \|g\|_\infty$. So $\lambda \in C^*(X)^{\sim}$. It follows from the discussion above that each $\lambda \in \Lambda_j(X)$ ($j =$ p, k, α and ∞) can be written as $\lambda = \lambda^+ - \lambda^-$ where λ^+ and λ^- are positive linear functionals on $C(X)$ or $C^*(X)$ depending on j. Now by Lemmas 1.2.1–1.2.4 both λ^+ and λ^- are members of $\Lambda_j^+(X)$ ($j = p, k, \alpha, \infty$). This establishes the first part of the following theorem

Theorem 2.4: *Each* $\lambda \in \Lambda_j(X)$ $(j = p, k, \alpha, \infty)$ *can be written as* $\lambda = \lambda^+ - \lambda^-$, *where* λ^+ *and* λ^- *are members of* $\Lambda_j^+(X)$. *Furthermore, if* $\lambda, \mu \in \Lambda_j(X)$, *then* $\|\lambda^+ - \mu^+\|_* \leq \|\lambda - \mu\|_*$ *and* $\|\lambda^- - \mu^-\|_* \leq \|\lambda - \mu\|_*$.

Proof. For the second part, it suffices to show that if $\lambda, \mu \in \Lambda_j(X)$, and $|\lambda| \leq |\mu|$, then $\|\lambda\|_* \leq \|\mu\|_*$. Now for $f \in C^*(X)$, $|\lambda|(|f|) = \sup\{\lambda(g) : |g| \leq |f|\}$ so that for all

$f \in C^*(X), |\lambda(f)| \leq |\lambda| (|f|) \leq |\mu| (|f|) = \sup\{\mu(g) : |g| \leq |f|\}$. Therefore $\|\lambda\|_* =$ $\sup\{|\lambda(f)| : \|f\|_\infty \leq 1\} \leq \sup\{|\mu(g)| : \|g\|_\infty \leq 1\} = \|\mu\|_*$. ■

Theorem 2.5: *The metric space $\Lambda_j^+(X)$ $(j = p, k, \alpha, \infty)$ is complete if and only if the normed linear space $\Lambda_j(X)$ is complete.*

Proof. The sufficiency follows from Theorem 1.4.2. For the necessity, suppose that (λ_n) is a Cauchy sequence in $\Lambda_j(X)$. From Theorem 2.4 it follows that for each m and n, $\|\lambda_m^+ - \lambda_n^+\|_* \leq \|\lambda_m - \lambda_n\|_*$ and $\|\lambda_m^- - \lambda_n^-\|_* \leq \|\lambda_m - \lambda_n\|_*$. Therefore (λ_n^+) and (λ_n^-) are Cauchy sequences in $\Lambda_j^+(X)$. Since $\Lambda_j^+(X)$ is complete, then (λ_n^+) and (λ_n^-) converge to some μ and ν in $\Lambda_j^+(X)$. But since each $\lambda_n = \lambda_n^+ - \lambda_n^-$, then (λ_n) converges to $\mu - \nu$. Therefore $\Lambda_j(X)$ is complete.■

Because of Theorems 1.4.2 and 2.5, each of the following theorems about $\Lambda_j^+(X)$ is also true for $\Lambda_j(X)$ (of course for appropriate j). The first result is a necessary condition for the topological completeness of $\Lambda_j(X)$ $(j = p, k)$.

Theorem 2.6: *If $\Lambda_j^+(X)$ $(j = p, k)$ is completely metrizable, then X is countably compact.*

Proof. Suppose X is not countably compact. Then X contains a closed copy of the discrete space $\mathbb{N}$ of natural numbers. Therefore by Theorem 1.5.6, $\Lambda_j^+(X)$ $(j = p, k)$ contains a closed copy of $\Lambda_j^+(\mathbb{N})$. But in example 2.1, it is shown that $\Lambda_j^+(\mathbb{N})$ is of first category (Note $\Lambda_{\mathrm{p}}^+(\mathbb{N}) = \Lambda_{\mathrm{k}}^+(\mathbb{N})$). Then from the Baire Category Theorem it follows that $\Lambda_j^+(X)$ cannot be completely metrizable. ■

Note that while considering $\Lambda_\alpha(X)$, we considered $C^*(X)$, not $C(X)$. But by Theorem 2.6, we have when $\Lambda_j^+(X)$ $(j = p, k)$ is completely metrizable, $C^*(X) = C(X)$. So while answering the question of the completeness of $\Lambda_j^+(X)$ or $\Lambda_j(X)$ $(j = p, k)$. we need to consider only $C^*(X)$, i. e. , in this case, we can consider $\Lambda_j(X)$ $(j = p, k)$ as particular cases of $\Lambda_\alpha(X)$. So in the remainder of this chapter we assume $C(X) = C^*(X)$ and deduce all the following resullts on the completness on a more general setting on $\Lambda_\alpha(X)$.

Theorem 2.7: *Suppose X is infinite and $\mathcal{F}(X) \subseteq \alpha$. Now if $\Lambda_\alpha^+(X)$ is complete, then every countable subset of X is contained in some members of α.*

Proof. Let $A = \{x_n : n \in \mathbb{N}\}$ be any countable subset of X. For each $m \in \mathbb{N}$, define $\lambda_m : C_\alpha^*(X) \to \mathbb{R}$ as follows. For each $f \in C_\alpha^*(X)$, take $\lambda_m(f) = \sum_{n=1}^{m} \frac{1}{2^n} f(x_n)$. Each λ_m is a positive linear functional on $C_\alpha^*(X)$ supported on the finite set $\{x_1, \ldots, x_m\}$. Then by Lemma 1.2.3, λ_m is continuous. Now for each k and m with $k < m$, $d_*(\lambda_k, \lambda_m) = \| \lambda_k - \lambda_m \|_* \leq \sum_{n=k+1}^{m} \frac{1}{2^n}$. Therefore (λ_m) is a Cauchy sequence in $\Lambda_\alpha^+(X)$. Since $\Lambda_\alpha^+(X)$ is complete, then (λ_m) converges to some λ in $\Lambda_\alpha^+(X)$. Also $\lambda_m \to \lambda$ implies $\lambda(f) = \lim_{m\to\infty} \lambda_m(f) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x_n)$ for all $f \in C_\alpha^*(X)$.

Now suppose λ has a support Y which belongs to α. We show that $A \subseteq Y$. Suppose not, then there is some m such that $x_m \notin Y$. Since X is completely regular, there is some continuous function f on X with values in the unit interval I such that $f(x_m) = 1$ and $f(Y) = \{0\}$. Since λ is supported on Y, $\lambda(f) = 0$. But $\lambda(f) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x_n) \geq \frac{1}{2^m} f(x_m) = \frac{1}{2^m} > 0$. With this contradiction, it follows that $A \subseteq Y$. ■

Corollary 2.8: *If X is infinite, then $\Lambda_{\mathrm{p}}^+(X)$ and $\Lambda_{\mathrm{p}}(X)$ are not complete.*

Proof. All the members of α in this case are finite. So any countably infinite subset of X cannot be contained in a member of α. ■

Corollary 2.9: *If $\Lambda_k^+(X)$ is complete, then the closure of each countable subset of X is compact.*

Proof. Suppose A is a countably infinite subset of X. Then by Theorem 2.7, there exists a compact subset K of X such that $A \subseteq K$. So $\mathrm{cl}_X A$ is compact. ∎

The Corollary 2.9 shows that countable compactness is not sufficient for the metric d_* on $\Lambda_k^+(X)$ to be complete. The following counter-example confirms that fact. But to explain this counterexample, we need the following results (see [5], page 262).

(i) A Hausdorff space X is countably compact if and only if every countably infinite subset of X has an accumulation point.

(ii) Every infinite closed set $F \subseteq \beta\mathbb{N}$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$, contains a subset homeomorphic to $\beta\mathbb{N}$; in particular F has cardinality 2^c where $c =$ the cardinality of the set of real numbers.

Now let $X = \beta\mathbb{N} \setminus \{x\}$ where $x \in \beta\mathbb{N} \setminus \mathbb{N}$. Since X is a proper dense subset of $\beta\mathbb{N}$, the space X is not compact. On the other hand, for every countably infinite subset A of $X \subseteq \beta\mathbb{N}$, we have by (ii), the cardinality of the set $\mathrm{cl}_{\beta\mathbb{N}} A = 2^c$ which implies $A^d \cap X \neq \emptyset$ where $A^d =$ the set of all accumulation points of A in $\beta\mathbb{N}$. Thus the set A has an accumulation point in X, which shows that the space X is countably compact.

Now note that $\mathbb{N}$ is dense in X. Since $\mathrm{cl}_X\mathbb{N} = X$ is not compact, $\Lambda_k^+(X)$ cannot be complete.

The next result gives a sufficient condition for completeness of $\Lambda_\alpha^+(X)$.

Theorem 2.10: *If the closure of each countable union of elements of α belongs to α, then*

$\Lambda^+_\alpha(X)$ *is complete.*

Proof. Let (λ_n) be a Cauchy sequence in $\Lambda^+_\alpha(X)$. Consider $\Lambda^+_\alpha(X)$ as a subspace of the complete metric space $\Lambda^+_\infty(X)$. Then (λ_n) is a Cauchy sequence in $\Lambda^+_\infty(X)$ and hence converges to some λ in $\Lambda^+_\infty(X)$. Suppose each λ_n is supported on A_n where $A_n \in \alpha$. We show that λ is supported on $A = \overline{\bigcup_{n=1}^\infty A_n}$. Let $f \in C^*(X)$ with $f|_A = 0$. Since each λ_n is supported on $A_n \subseteq A$, then each $\lambda_n(f) = 0$ and consequently $\lambda(f) = \lim_{n\to\infty} \lambda_n(f) = 0$. Therefore λ has support A. But by hypothesis $A \in \alpha$. Hence by Lemma 1.2.3 $\lambda \in \Lambda^+_\alpha(X)$. So $\Lambda^+_\alpha(X)$ is complete.■

Corollary 2.11: *Suppose X is a normal Hausdorff space. Then $\Lambda^+_\sigma(X)$ is always complete.*

Proof. Suppose for each n, A_n is a σ-compact subset of X. Then $\overline{\bigcup_{n=1}^\infty \overline{A_n}} = \overline{\bigcup_{n=1}^\infty A_n} \in \sigma$. ■.

Corollary 2.12: *If the closure of each σ -compact subset of X is compact, then $\Lambda^+_k(X)$ is complete.*

It follows from Corollary 2.12, for example, that if X is the space of countable ordinals, then $\Lambda^+_k(X)$ is complete.

The sufficient condition in Corollary 2.12 can be expressed in terms of nets in $\Lambda_k(X)$. The next theorem (found in [8]) shows that X is compact if and only if every convergent net in $C_k(X)$ has a bounded subnet (i. e. , there is an element of $C_k(X)$ which is greater than or equal to each member of some subnet). To prove this theorem, we need the following lemma.

Lemma 2.13: *If X is psuedocompact but not compact, then there exists a convergent net* *in $C_k(X)$ which has no bounded subnet.*

Proof. Since X is regular, there exists an open filter $\mathcal{U}$ on X which has no cluster point in X. Define $D = \mathcal{U} \times \mathbb{N}$, and direct D by: $(U,m) \leq (V,n)$ if and only if $V \subseteq U$ and $m \leq n$. Now define net $\phi : D \to C(X)$ as follows. First suppose that X and $C(X)$ have been well-ordered. Let $d = (U,m) \in D$. Let x be the first element of U. Then take $\phi(d)$ to be the first element of $C(X)$ such that $\phi(d)(x) = m$ and $\phi(d)(X \setminus U) = \{0\}$.

To show that ϕ converges to 0 in $C_k(X)$, let K be a compact subset of X. Then there is some $U \in \mathcal{U}$ such that $U \cap K = \emptyset$. Let $m \in \mathbb{N}$, and let $d = (U,m)$. Now let $e = (V,n) \in D$ be such that $e \geq d$. Then $K \subseteq X \setminus V$, so that $\phi(e)(x) = 0$ for each $x \in K$. Then ϕ is eventually 0 on K, so that ϕ converges to 0 in $C_k(X)$.

To show that every subnet of ϕ is not bounded in $C(X)$, let $\psi : E \to D$ be cofinal for some directed set E, and let $g \in C(X)$. Since X is pseudocompact, there is some $m \in \mathbb{N}$ so that $g(x) < m$ for all $x \in X$. Let $U \in \mathcal{U}$, and let $d = (U,m)$. Since ψ is cofinal, there exists an $e \in E$ such that $\psi(e) \geq d$; say $\psi(e) = (V,n)$. Let x be the first element of V. Then $(\phi \circ \psi)(e)(x) = n \geq m > g(x)$. So the subnet $\phi \circ \psi$ of ϕ is not bounded by g. ■

Theorem 2.14: *Every convergent net in $C_k(X)$ has a bounded subset if and only if X is compact.*

Proof. First suppose that X is compact. Then the supremum metric on $C(X)$ generates the topology of $C_k(X)$. Let $\phi : D \to C_k(X)$ be a net which converges to some f in $C_k(X)$. Then there is some $d \in D$ such that for all $e \in D$ with $e \geq d$, $|\phi(e)(x) - f(x)| < 1$ for all $x \in X$. Let $E = \{e \in D : e \geq d\}$ with the order inherited from D, and let $\psi : E \to D$ be the inclusion map. Then ψ is cofinal, so that $\phi \circ \psi$ is a subnet of ϕ. Define $g = f + 1$. To show that $\phi \circ \psi$ is bounded by g, let $e \in E$ and $x \in X$. Then $e \geq d$, so that $(\phi \circ \psi)(e)(x) = \phi(e)(x) < f(x) + 1 = g(x)$.

For the converse, suppose that every convergent net in $C_k(X)$ has a bounded subnet. Because of the Lemma 2.13, it suffices to prove that X is pseudocompact. Suppose that X is not pseudocompact. Then there exists a positive unbounded f in $C(X)$. Choose a sequence (x_n) in X such that for each n, $n \leq f(x_n) < f(x_{n+1})$. Define $C = \{x_n : n \in \mathbb{N}\}$. Let $\mathcal{F}$ be a free ultrafilter on C, directed by inclusion. Define net $\phi : \mathcal{F} \to C(X)$ as follows. Let $A \in \mathcal{F}$. Let n be the smallest element of $\mathbb{N}$ greater than 1 such that $x_n \in A$. Let α be the piecewise linear map from $\mathbb{R}$ to $\mathbb{R}$ which takes $(-\infty, f(x_{n-1})]$ to 0 and, for each $i \geq n$, takes $f(x_i)$ to the smallest element m on $\mathbb{N}$ greater than i such that $x_m \in A$. Now define $\phi(A) = \alpha \circ f$. This defines the net ϕ.

To show that ϕ converges to 0 in $C_k(X)$, let K be a compact subset of X. There exists an $m \in \mathbb{N}$ such that $K \subseteq f^{-1}((-\infty, f(x_{m-1})))$. Let $A = C \setminus \{x_1, ..., x_m\}$, which is in $\mathcal{F}$ since $\mathcal{F}$ is free. Now let $B \in \mathcal{F}$ with $B \subseteq A$. Let n be the smallest element of $\mathbb{N}$ such that $x_n \in B$. Then since $n > m$, $K \subseteq f^{-1}((-\infty, f(x_{n-1})])$, and thus $\phi(B)(x) = 0$ for every $x \in K$. Hence ϕ converges to 0 in $C_k(X)$.

To show that every subnet of ϕ is not bounded in $C(X)$, let $\psi : D \to \mathcal{F}$ be cofinal for some directed set D, and let $g \in C(X)$. Define sequence $\{n_i : i = 0, 1, 2, ...\}$ by induction as follows. Let $n_0 = 0$ and $n_1 = 1$. Suppose the n_k have been defined for $k \geq 1$. Then define n_{k+1} to be the smallest element of $\mathbb{N}$ which is greater than $\max\{n_k, g(x_1), ..., g(x_{n_k})\}$. Now define $A = \{x_n : n_{2i} < n \leq n_{2i+1}$ for some $i = 0, 1, 2, \ldots, \}$ and $B = C \setminus A$. Since $\mathcal{F}$ is an ultrafilter, either $A \in \mathcal{F}$ or $B \in \mathcal{F}$. If $A \in \mathcal{F}$, then since ψ is cofinal, there exists a $d \in D$ such that $\psi(d) \subseteq A$. Let $x_n \in \psi(d)$ for some $n > 1$. Then $n_{2i} < n \leq n_{2i+1}$ for some $i = 0, 1, 2, \ldots$. Let m be the largest element of $\mathbb{N}$ such that $m \leq n_{2i+1}$ and $x_m \in \psi(d)$. Then $(\phi \circ \psi)(d)(x_m) > n_{2i+2} > g(x_m)$. On the other hand, if $B \in \mathcal{F}$, there exists a $d \in D$ such that $\psi(d) \subseteq B$. Let $x_n \in \psi(d)$ for some $n > 1$. In this case $n_{2i-1} < n \leq n_{2i}$ for some $i = 1, 2, \ldots$. Let m be the largest element of $\mathbb{N}$ such that $m \leq n_{2i}$ and $x_m \in \psi(d)$. Then $(\phi \circ \psi)(d)(x_m) > n_{2i+1} > g(x_m)$. So in any case, the subnet $\phi \circ \psi$ of ϕ is not bounded by g. With this contradiction, it follows that X is pseudocompact. ∎

Now we have the last theorem of this chapter.

Theorem 2.15: *Suppose for each σ-*compact *subset A of X, every convergent net in $C_k(\overline{A})$ has a bounded subnet. Then $\Lambda_k^+(X)$ is complete.*

Proof. Use Corollary 2.12 and Theorem 2.14. ∎

At the end of this chapter, we prove what we have said in the Remarks 1.5.9 and 1.5.13. We do it for the Remark 1.5.9. The proof for the Remark 1.5.13 is similar. Now suppose X is a closed subspace of a normal Hausdorff space Y. So X is C^*-imbedded in Y. Hence by Corollary 1.5.8 $i^{**} : \Lambda_\infty(X) \to \Lambda_\infty(Y)$ is an (into) isometry where $i : X \to Y$ is the inclusion map. We already know that $\Lambda_\infty(X)$ and $\Lambda_\infty(Y)$ are complete. Hence $i^{**}(\Lambda_\infty(X))$ is a complete (normed linear) subspace of $\Lambda_\infty(Y)$. Hence $i^{**}(\Lambda_\infty(X))$ is closed in $\Lambda_\infty(Y)$. The proof for $\Lambda_\infty^+(\cdot)$ is similar.

Chapter III

Measure-Theoretic Counterparts

In this chapter we will talk about the measure-theoretic counterparts of $\Lambda_j(X)$ ($j =$ p, k, α, and ∞). In case of $\Lambda_\alpha(X)$ and $\Lambda_\infty(X)$, we will need to impose some extra conditions on α and X. So we first introduce some ideas from measure theory.

Definition 3.1: *A set function is a function defined on a family of subsets having values in the set of real numbers. A positive set function is a real-valued set function which has no negative values.*

Definition 3.2: *A set function μ defined on a family τ of sets is said to be additive or finitely additive if*

(i) $\mu(\emptyset) = 0$ *and*

(ii) $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ *for every finite family $\{A_1, \ldots, A_n\}$ of pairwise disjoint members of τ whose union is τ.*

But we will be interested only in particular families τ of sets. Suppose X is a completely regular Hausdorff space. The algebras generated by the zero sets and the closed sets of X are denoted by $\mathcal{A}_z$ and $\mathcal{A}_c$ respectively while the σ-algebras they generate are denoted by $\mathcal{B}_a$ and $\mathcal{B}$, called the Baire and Borel sets respectively.

For us, a finitely additive measure (also called a signed measure) on $\mathcal{A}_z$ or $\mathcal{A}_c$ is a real-valued finitely additive set function defined on $\mathcal{A}_z$ or $\mathcal{A}_c$ respectively. The total variation $|\mu|$ of a finitely additive set function μ defined on any of the above algebras is the set function defined at each set A in the algebra to be the supremum of the sums $\sum |\mu(E_i)|$ taken over all finite pairwise disjoint collections (E_i) of subsets of A taken from the algebra.

In the event that μ is a positive finitely additive set function defined on an algebra of sets, it is bounded. Thus a difference of two such positive finitely additive set functions is also a bounded set function. Conversely, a bounded finitely additive set function μ defined on an algebra of sets has a finite-valued total variation $|\mu|$ and can be decomposed into the difference of positive additive set functions μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$ (see [4], pages 97-99).

A finitely additive measure μ defined on $\mathcal{A}_z$ or $\mathcal{B}_a$ is regular if for each A on which μ is defined and $\epsilon > 0$, there are $Z, Z' \in \mathcal{Z}$ = the collection of zero sets of X, such that $Z \subset A \subset X \setminus Z'$ and $|\mu|\ ((X \setminus Z') \setminus Z) < \epsilon$. Similarly if μ is defined on $\mathcal{A}_c$ or $\mathcal{B}$, μ is regular whenever A is in the domain of definition of μ and $\epsilon > 0$, there are closed and open sets C and U such that $C \subset A \subset U$ and $|\mu|\,(U \setminus C) < \epsilon$.

A finitely additive measure μ defined on an algebra $\mathcal{A}$ is called a countably additive measure or σ-additive measure or a signed measure or simply a measure provided that the following holds:

$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for all pairwise disjoint sequences $(A_n)_{n=1}^{\infty}$ such that $A_n \in \mathcal{A}$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

When a measure μ is defined on $\mathcal{B}$, we call it a Borel measure. Similarly when a measure μ is defined on $\mathcal{B}_a$, we call it a Baire measure. A measure μ defined on $\mathcal{B}$ (or on $\mathcal{B}_a$) has support A where $A \subseteq X$ and $A \in \mathcal{B}$ (or $\in \mathcal{B}_a$ respectively) if $|\mu|(X \setminus A) = 0$. Note when μ is a Borel measure with compact support, the definition of regularity of a Borel measure given here coincides with the one usually given in the books on measure theory. For more information on measure theory see [4] and [6].

Now we fix some notations.

A (signed) measure μ defined on $\mathcal{B}$ (or on $\mathcal{B}_a$) is said to be a finite (signed) measure if $|\mu(A)| < \infty$ holds for each $A \in \mathcal{B}$ (or $A \in \mathcal{B}_a$ respectively). It can be shown that this is the case if and only if $|\mu(X)| < \infty$ holds. Note that a signed measure μ is finite if and only if $|\mu|(X) < \infty$. If $|\mu|(X) < \infty$, then since $|\mu(A)| \leq |\mu|(A) \leq |\mu|(X)$ holds for each $A \in \mathcal{B}$ (or $A \in \mathcal{B}_a$), μ is finite. On the other hand if μ is a finite signed measure, the Jordan decomposition $\mu = \mu^+ - \mu^-$ with at least one of μ^+ and μ^- finite shows that μ^+ and μ^- are both finite and so $|\mu|(X) = \mu^+(X) + \mu^-(X) < \infty$. It also shows that a finite signed measure defined on $\mathcal{B}$ or $\mathcal{B}_a$, has finite total variation. For details on the above, see [1], §26.

Now let $M_b(X)$ be the set of all finite (signed) regular Borel measures on X. Let $M_b^+(X) = \{\mu \in M_b(X) : \mu \geq 0$, i. e. , μ is a positive measure$\}$.

Throughout this and the next chapter we will assume the following extra condition on

α : The members of α are closed.

Now define $M_{b,\alpha}(X) = \{\mu \in M_b(X) : \mu$ has a support $A(\subseteq X)$ such that $A \in \alpha\}$. Let $M^+_{b,\alpha}(X) = \{\mu \in M_{b,\alpha}(X) : \mu \geq 0\}$. When $\alpha = \mathcal{K}(X)$ or $\mathcal{F}(X)$, we write $M_{b,k}(X)$ or $M_{b,p}(X)$ respectively.

The next thing to observe is that given $\mu \in M_b(X)$, $\| \mu \| = | \mu | (X)$ defines a norm on $M_b(X)$. So $(M_b(X), \| \cdot \|)$ is actually a normed linear space. Also $M_b^+(X)$ is a metric space when equipped with the metric ρ given by $\rho(\mu_1, \mu_2) = \| \mu_1 - \mu_2 \|$ for every $\mu_1, \mu_2 \in M_b^+(X)$. Note $(M_{b,\alpha}(X), \| \cdot \|)$ is a normed linear space while $(M^+_{b,\alpha}(X), \rho)$ is a metric space.

Before having our first important theorem in this chapter, we need the following two lemmas.

Lemma 3.3: *Suppose Y is a Borel subset of a completely regular Hausdorff space X. Let $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ be the σ-algebras of Borel subsets of X and Y respectively. Then $\mathcal{B}(X) \cap Y = \mathcal{B}(Y)$ where $\mathcal{B}(X) \cap Y = \{B \cap Y : B \in \mathcal{B}(X)\}$.*

Proof. Define $\mathcal{D} = \{A \in \mathcal{P}(X) : A = E \cup (B \setminus Y);\ E \in \mathcal{B}(Y)$ and $B \in \mathcal{B}(X)\}$ where $\mathcal{P}(X)$ is the power set of X. Note $X \setminus (E \cup (B \setminus Y) = (Y \setminus E) \cup ((X \setminus (B \setminus Y)) \setminus Y)$. Now it can be easily shown that $\mathcal{D}$ is a σ-algebra on X containing all the closed subsets of X. Hence $\mathcal{B}(X) \subseteq \mathcal{D}$. So $\mathcal{B}(X) \cap Y \subseteq \mathcal{D} \cap Y$. But $\mathcal{D} \cap Y = \mathcal{B}(Y)$. So $\mathcal{B}(X) \cap Y \subseteq \mathcal{B}(Y)$. Note $\mathcal{B}(X) \cap Y$ is a σ-algebra on Y and if C is a closed subset of Y, then $C = C' \cap Y$ for some closed subset C' of X which means $C \in \mathcal{B}(X) \cap Y$. Hence $\mathcal{B}(Y) \subseteq \mathcal{B}(X) \cap Y$. Therefore $\mathcal{B}(X) \cap Y = \mathcal{B}(Y)$. ∎

Lemma 3.4: *If A is a compact subset of a Tychonoff space X, then for every closed set $B \subset X \setminus A$, there exists a continuous function $f : X \to \mathrm{I}$ such that $f(x) = 0$ for $x \in A$ and*

$f(x) = 1$ *for* $x \in B$.

Proof. See [5], page 168. ■

Theorem 3.5: $(M_{b,j}(X), \|\cdot\|)$ $(j = \mathrm{p}, \mathrm{k})$ *is isometrically isomorphic to* $(\Lambda_j(X), \|\cdot\|_*)$ *via the map* $F : \mu \to \int \cdot\, d\mu$ *where* $\mu \in M_{b,j}(X)$. *Also* $M^+_{b,j}(X)$ *is identified with* $\Lambda^+_j(X)$ *under this isometric isomorphism.*

Proof. We do it for $j = k$. For $j = p$, the proof is similar.

Define $F : M_{b,k}(X) \to \Lambda_{\mathrm{k}}(X)$ by $F(\mu)(f) = \int f d\mu$ for each $\mu \in M_{b,k}(X)$ and $f \in C_{\mathrm{k}}(X)$. Let K be a compact support of μ, i. e. , $|\mu|(X \setminus K) = 0$. Then for each $f \in C_{\mathrm{k}}(X)$, $|F(\mu)(f)| = |\int f d\mu| = |\int_K f d\mu| \leq \int_K |f|\, d|\mu| \leq |\mu|(K) p_K(f)$ and so $F(\mu)$ is continuous. Clearly $F(\mu)$ is linear. Hence $F(\mu) \in \Lambda_{\mathrm{k}}(X)$. Also $\|F(\mu)\|_* = \sup\{|F(\mu)(f)| : f \in C^*(X);\ \|f\|_\infty \leq 1\} \leq \sup\{|\mu|(K) p_K(f) : f \in C^*(X),\ \|f\|_\infty \leq 1\} = |\mu|(K) = |\mu|(X) = \|\mu\|$.

Now we prove the reverse inequality, i. e. , $\|\mu\| \leq \|F(\mu)\|_*$.

Note $|\mu|(K) = \sup\{\sum |\mu(A_i)| : \{A_i\}$ is a disjoint finite collection of $\mathcal{B}$ with $\bigcup A_i \subseteq K\}$. So given $\epsilon > 0$, there exist $A_1, \ldots A_n \in \mathcal{B}$ such that $A_i's$ are pairwise disjoint and $\sum_{i=1}^n |\mu(A_i)| > |\mu|(K) - \epsilon$. Since μ is regular, there exist compact sets C_i and open sets U_i such that $C_i \subseteq A_i \subseteq U_i$ and $|\mu|(U_i \setminus C_i) < \frac{\epsilon}{n}$ for $1 \leq i \leq n$. Since $A_i's$ are pairwise disjoint, the compact subsets $C_i's$ are also pairwise disjoint and hence pairwise disjoint open sets V_i exist such that $C_i \subseteq V_i$. Now let $W_i = U_i \cap V_i$. W_i is open. Now $W_i \subseteq U_i$ implies $W_i \setminus C_i \subseteq U_i \setminus C_i$ which in turn implies $|\mu|(W_i \setminus C_i) \leq |\mu|(U_i \setminus C_i) < \epsilon/n$. $C_i \subseteq U_i$ and $C_i \subseteq V_i$ imply $C_i \subseteq U_i \cap V_i = W_i$. So $C_i \cap (X \setminus W_i) = \emptyset$. Hence by Lemma 3.4, there exists an $f_i \in C(X, \mathrm{I})$ such that $f_i(C_i) = \{1\}$ and $f_i(X \setminus W_i) = \{0\}$. Let $a_i = \frac{|\mu(A_i)|}{\mu(A_i)}$ *if* $\mu(A_i) \neq 0$ and if $\mu(A_i) = 0$, let $a_i = 0$. Note $|a_i| = 1$ or 0. Let $f = \sum_{i=1}^n a_i f_i$. Since $W_i's$ are pairwise disjoint, $\|f\|_\infty \leq 1$.

Now

$$\begin{aligned}
\left|\int f d\mu - \sum_{i=1}^n |\mu(A_i)|\right| &= \left||\sum_{i=1}^n a_i \int f_i d\mu - \sum_{i=1}^n |\mu(A_i)|\right| \\
&= \left|\sum_{i=1}^n a_i \int_{W_i} f d\mu - \sum_{i=1}^n |\mu(A_i)|\right| \text{ (since } f_i(X \setminus W_i) = \{0\}) \\
&= \left|\sum_{i=1}^n \left[a_i \int_{C_i} f_i d\mu - |\mu(A_i)|\right] + \sum_{i=1}^n a_i \int_{W_i \setminus C_i} f_i d\mu|\right| \\
&= \left|\sum_{i=1}^n \left[a_i \mu(C_i) - a_i \mu(A_i)\right] + \sum_{i=1}^n a_i \int_{W_i \setminus C_i} f_i d\mu\right| \\
&\leq \sum_{i=1}^n |a_i||\mu(C_i) - \mu(A_i)| + \sum_{i=1}^n |a_i| \int_{W_i \setminus C_i} |f_i|\, d\,|\mu| \\
&\leq \sum_{i=1}^n |\mu(A_i \setminus C_i)| + \sum_{i=1}^n |\mu|\,(W_i \setminus C_i) \\
&\leq \sum_{i=1}^n |\mu|\,(A_i \setminus C_i) + \sum_{i=1}^n |\mu|\,(W_i \setminus C_i) \\
&< n \cdot \tfrac{\epsilon}{n} + n \cdot \tfrac{\epsilon}{n} = 2\epsilon.
\end{aligned}$$

So $\sum_{i=1}^n |\mu(A_i)| - 2\epsilon < |\int f d\mu| < \sum_{i=1}^n |\mu(A_i)| + 2\epsilon$. But $\|F(\mu)\|_* = \sup\{|\int g d\mu| : g \in C^*(X); \|g\|_\infty \leq 1\} \geq |\int f d\mu|$ which implies $\|F(\mu)\|_* > \sum_{i=1}^n |\mu(A_i)| - 2\epsilon > |\mu|\,(K) - \epsilon - 2\epsilon = \|\mu\| - 3\epsilon$. Therefore $\|\mu\| - 3\epsilon < \|F(\mu)\| \leq \|\mu\|$. Since $\epsilon > 0$ is arbitrary $\|F(\mu)\|_* = \|\mu\|$. So F is an isometry.

Next we show that F is onto. Suppose $\lambda \in \mathcal{C}_k(X)$. Then λ can be written as $\lambda = \lambda^+ - \lambda^-$ where $\lambda^+, \lambda^- \in \Lambda_k(X)$. Note if λ has a compact support K then both λ^+ and λ^- have compact support K. To show F is onto, we try to get $\mu_1, \mu_2 \in M_{b,k}^+(X)$ such that $\lambda^+ = F(\mu_1)$ and $\lambda^- = F(\mu_2)$. So $\lambda = \lambda^+ - \lambda^- = F(\mu_1) - F(\mu_2) = F(\mu_1 - \mu_2) = F(\mu)$ where $\mu = \mu_1 - \mu_2 \in M_{b,k}(X)$. So we just need to consider λ^+. Define $\lambda_K^+ : \mathcal{C}_k(K) \to \mathbb{R}$ as follows. For $f \in \mathcal{C}_k(K)$, let $\hat{f}$ be in $\mathcal{C}_k(X)$ such that $\hat{f}|_K = f$. Define $\lambda_K^+(f) = \lambda^+(\hat{f})$. Since λ^+ has (compact) support K, this definition of λ_K^+ is well-defined. Clearly λ_K^+ is linear since λ^+ is linear. Finally λ_K^+ is continuous since $\sup\{|\lambda_K^+(f)| : f \in C^*(K), \| f \|_\infty \leq 1\} = \sup\{|\lambda^+(f)| : f \in C^*(X), \|f\|_\infty \leq 1\} = \| \lambda^+ \|_* < \infty$. By the Riesz Representation Theorem (see [1], page 242), there exists a $\mu_K \in M_b^+(K)$ such that $\lambda_K^+(f) = \int_K f d\mu_K$ for all $f \in C(K)$.

The thing that is now needed is a $\mu_1 \in M_b^+(X)$ such that $\mu_1(B) = \mu_K(B \cap K)$ for all $B \in \mathrm{B}$. Then μ_1 would be supported on K so that μ_1 would be in $M_{b,k}^+(X)$ and thus $\lambda^+(f) = \lambda_K^+(f\,|_K) = \int_K f\,|_K\, d\mu_K = \int f d\mu_1 = F(\mu_1)(f)$ which shows that F maps $M_{b,k}^+(X)$ onto $\Lambda_k^+(X)$.

First observe that because of Lemma 3.3, μ_1 is well-defined on $\mathcal{B}$. So we only need to show that μ_1 is regular. Let $B \in \mathcal{B}$ and let $\epsilon > 0$. Since μ_K is regular, there exists a compact subset C of K and an open subset U of K such that $C \subseteq B \cap K \subseteq U$ and $\mu_K(U \backslash C) < \epsilon$. Let $V = U \cup (X \setminus K)$ which is open in X. Then $C \subseteq B \subseteq V$ and $\mu_1(V \setminus C) = \mu_K((V \setminus C) \cap K) = \mu_K(U \setminus C) < \epsilon$. Therefore μ_1 is regular. Similarly for λ^-, we get a $\mu_2 \in M_{b,k}^+(X)$ such that $\lambda^- = F(\mu_2)$. So $\lambda = F(\mu)$ where $\mu = \mu_1 - \mu_2 \in M_{b,k}(X)$. Hence F is onto. Also note that the proof shows F maps $M_{b,k}^+(X)$ onto $\Lambda_k^+(X)$, i. e. , under F, $M_{b,k}^+(X)$ is identified with $\Lambda_k^+(X)$. ■.

Also the proof of the above theorem can be modified to obtain the following theorem.

Theorem 3.6 *Suppose $\alpha \subseteq \mathcal{K}(X)$,* i. e. , *the members of α are compact. Then $M_{b,\alpha}(X)$ is isometrically isomorphic to $\Lambda_\alpha(X)$ while $M_{b,\alpha}^+(X)$ is identified with $\Lambda_\alpha^+(X)$ under this isometric isomorphism.*

Remark 3.7 *Note $M_{b,p}(X)$ is actually the normed linear space over $\mathbb{R}$ generated by the set of Dirac's measures δ_x ($x \in X$) on X where δ_x is defined as follows. For each $A \subseteq X$, $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ if $x \notin A$. This fact explains why $\Lambda_p(X)$ or $\Lambda_p^+(X)$ can not be complete because a limit of a Cauchy sequence in $M_{b,p}(X)$ or in $M_{b,p}^+(X)$ may converge to a regular Borel measure on X with infinite support.*

Now what are the measure-theoretic counterparts of $\Lambda_\infty(X)$ and $\Lambda_\alpha(X)$? The remainder of this chapter except the last theorem and its corollaries answers that question. Let $M_c(X)$ be the set of all bounded finitely additive regular measures defined on $\mathcal{A}_c$. Again $M_c(X)$ is a normed linear space with total variation norm. Similarly let $M_z(X)$ be the set of all bounded finitely additive regular measures defined on $\mathcal{A}_z$. Also $M_z(X)$ is a normed linear space with total variation norm.

Theorem 3.8 *If X is normal and Hausdorff, then $M_c(X)$ is isometrically isomorphic to $\Lambda_\infty(X)$ while $M_c^+(X)$ is identified with $\Lambda_\infty^+(X)$ under this isometric isomorphism. ($M_c^+(X) = \{\mu \in M_c(X) : \mu \geq 0\}$).*

Proof. We give only a partial sketch of the proof. For details see [3], pages 78-83.

Define the map $F : M_c(X) \to \Lambda_\infty(X)$ by $F(\mu)(f) = \int f d\mu$ for each $\mu \in M_c(X)$ and $f \in C_\infty^*(X)$. Note $|F(\mu)(f)| = |\int f d\mu| \leq \int |f| \, d|\mu| \leq \|f\|_\infty |\mu| (X) = \|\mu\| \, \|f\|_\infty$ which gives us $\|F(\mu)\|_* \leq \|\mu\| < \infty$. So $F(\mu) \in \Lambda_\infty(X)$, i. e. , F is well defined.

To obtain the reverse inequality $\|\mu\| \leq \|F(\mu)\|_*$, we need to imitate the proof given in Theorem 3.5. So we have $\|F(\mu)\|_* = \|\mu\|$ which means F is an isometry. Now we need to show that F is onto. So let $\lambda \in \Lambda_\infty(X)$. This λ can be written as $\lambda = \lambda^+ - \lambda^-$ where

$\lambda^+, \lambda^- \in \Lambda^+_\infty(X)$. So to show F is onto, we can assume that $\lambda \in \Lambda^+_\infty(X)$. So our problem is to find a finite non-negative finitely additive regular measure μ defined on $\mathcal{A}_c$ such that $\lambda(f) = \int f d\mu$ for all $f \in C^*_\infty(X)$. We do it by defining a real-valued set function μ on the class of all subsets of X, i. e. , on $\mathcal{P}(X)$ which can be proved to subadditive. Subsequently it can be shown that this set function when restricted to $\mathcal{A}_c$ is, in fact, regular and additive and it can be finally established that $\lambda(f) = \int f d\mu$ for all $f \in C^*_\infty(X)$. Now we finish the sketch of the proof by giving the definition of μ on $\mathcal{P}(X)$. It is defined as follows. If U is an open subset of X, we define $\mu(U) = \sup\{\lambda(f) : f \in C^*_\infty(X);\ 0 \le f \le \chi_U\}$ where χ_U is the characteristic function of U. If A is an arbitrary subset of X, we define $\mu(A) = \inf\{\mu(U) : U$ is open in X and $A \subseteq U\}$. ■

Remark 3.9: *Suppose $\lambda \in \Lambda^+_\infty(X)$ has support C where C is a closed subset of X. Then $X \setminus C$ is open in X. If $f \in C^*_\infty(X)$ with $0 \le f \le \chi_{(X-C)}$, then $f\mid_C = 0$. Consequently $\lambda(f) = 0$. So $\mu(X \setminus C) = 0$,* i. e. , *μ has also support C.*

But if X is just completely regular and Hausdorff, but *not* normal, then what will be the measure-theoretic counterpart of $\Lambda_\infty(X)$. For this we have the following answer.

Theorem 3.10: *If X is completely regular and Hausdorff, then $M_z(X)$ is isometrically isomorphic to $\Lambda_\infty(X)$ while $M_z^+(X)$ is identified with $\Lambda^+_\infty(X)$ under this isometric isomorphism ($M_z^+(X) = \{\mu \in M_z(X) : \mu \ge 0\}$.*

Proof. The proof is very similar to the one given in Theorem 3.8. Only the terms "zero set" and "complement of zero set" are needed to be substituted for "closed" and "open" respectively in the argument given in Theorem 3.8■

But what about the countable additiveness of elements of $M_c(X)$ or $M_z(X)$? When X is countably compact, we have the following answer.

Theorem 3.11: *If X is countably compact and if μ is a bounded regular finitely additive measure defined on $\mathcal{A}_c$, then μ is countably additive on $\mathcal{A}_c$, i. e. , $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ whenever (A_n) is a countable family of pairwise disjoint sets from $\mathcal{A}_c$ with union in $\mathcal{A}_c$. Moreover μ has a regular countably additive extension to the σ-algebra $\mathcal{B}$ of Borel subsets of X.*

Remark 3.12: *As in Theorems 3.8 and 3.10, Theorem 3.11 remains valid if the closed sets are replaced by the zero sets, $\mathcal{A}_c$ by $\mathcal{A}_z$ and $\mathcal{B}$ by $\mathcal{B}_a$.*

Proof. First we show that the total variation $|\mu|$ is countably additive on $\mathcal{A}_c$. Let $\epsilon > 0$ and let (A_n) be a disjoint sequence of sets in $\mathcal{A}_c$ with union A in $\mathcal{A}_c$. Since μ is regular, there exists a closed set E in X such that $E \subseteq A$ and $|\mu|\,(A \backslash E) < \epsilon$ and for each $n \geq 1$ let U_n be an open set of X containing A_n such that $|\mu|\,(U_n \setminus A_n) < \frac{\epsilon}{2^n}$. So $E \subseteq A = \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} U_n$. E, being a closed subset of X, is also countably compact. Now (U_n) is a countable open cover of E, so there are $U_1, \ldots, U_m$ such that $E \subseteq \bigcup_{n=1}^{m} U_n$. $|\mu|\,(U_n \backslash A_n) < \frac{\epsilon}{2^n}$ means $|\mu|\,(U_n) - |\mu|\,(A_n) < \frac{\epsilon}{2^n}$, i. e. , $|\mu|\,(A_n) > |\mu|\,(U_n) - \frac{\epsilon}{2^n}$. Hence $\sum_{n=1}^{\infty} |\mu|\,(A_n) \geq \sum_{n=1}^{\infty} |\mu|\,(U_n) - \epsilon \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} |\mu|\,(U_n) - \epsilon \geq \sum_{n=1}^{m} |\mu|\,(U_n) - \epsilon$. Since $|\mu|$ is a non-negative finitely additive set function, it is monotone and finitely subadditive and it follows that $\sum_{n=1}^{\infty} |\mu|\,(A_n) \geq \sum_{n=1}^{m} |\mu|\,(U_n) - \epsilon \geq |\mu|\,(\bigcup_{n=1}^{m} U_n) - \epsilon \geq |\mu|\,(E) - \epsilon \geq$

$| \mu | (A) - 2\epsilon$. Since $\epsilon > 0$ is arbitrary $\sum_{n=1}^{\infty} | \mu | (A_n) \geq | \mu | (A)$. On the other hand $\sum_{n=1}^{k} |\mu| (A_n) = |\mu| (\bigcup_{n=1}^{k} A_n) \leq |\mu| (A)$ for any k so $\sum_{n=1}^{\infty} |\mu| (A_n) \leq |\mu| (A)$. Hence $\sum_{n=1}^{\infty} |\mu| (A_n) = |\mu| (A)$, i. e. , $|\mu|$ is countably additive.

As $| \mu |$ is finite, $| \mu | (A) = \sum_{n=1}^{\infty} | \mu | (An) < \infty$ and so $| \mu | (\bigcup_{n \geq k} A_n) = \sum_{n \geq k} | \mu | (A_n) \to 0$ as $k \to \infty$. Thus $| \mu(A) - \sum_{n=1}^{k-1} \mu(A_n) | = | \mu(\bigcup_{n \geq k} A_k) | \leq | \mu | (\bigcup_{n \geq k} A_k) \to 0$ as $k \to \infty$. So $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$, i. e. , μ is countably additive on $\mathcal{A}_c$.

Since μ can be written as $\mu = \mu^+ - \mu^-$ where $\mu^+, \mu^- \in M_c^+(X)$, we may reduce the problem of existence of extension of μ to $\mathcal{B}$ to the case where μ is a regular non-negative measure on $\mathcal{A}_c$. First we extend μ to an outer measure (i. e. , a non-negative monotone countably subadditive set function which takes $\emptyset$ into 0) μ^* on $\mathcal{P}(X)$. To this end let $\mu^*(A) = \inf \{\sum_{n=1}^{\infty} \mu(An) : A_n \in \mathcal{A}_c ; A \subseteq \bigcup_{n=1}^{\infty} A_n\}$ for each $A \subseteq X$. It can be routinely shown that μ^* is an outer measure defined on $\mathcal{P}(X)$. Now define a subset E of X to be μ^*-measurable if $\mu(A) = \mu(E \cap A) + \mu(A \cap (X \setminus E))$ holds for all $A \subseteq X$. The collection of all μ^*-measurable sets is denoted by Σ. Now the following things can be shown.

(i) If $A \in \mathcal{A}_c$ $\mu^*(A) = \mu(A)$ holds.

(ii) Σ is a σ-algebra.

(iii) $\mathcal{A}_c \subseteq \Sigma$. Hence $\mathcal{B} \subseteq \Sigma$. So μ^* is an extension of μ to $\mathcal{B}$.

(iv) μ^* is the only extension of μ to Σ. So μ^* is the unique extension of μ to $\mathcal{B}$.

The details of the proofs of the above results can be found in §11, 12, 13 in [1].

Finally then it only remains to show that μ^* is regular on $\mathcal{B}$. Let $B \in \mathcal{B}$ and choose a sequence (A_n) from $\mathcal{A}_c$ such that $B \subseteq \bigcup_{n=1}^{\infty} A_n$ and $\mu^*(B) \geq \sum_{n=1}^{\infty} \mu(A_n) - \epsilon$. As μ is regular on $\mathcal{A}_c$ there exists an open set $U_n \supset A_n$ for each n such that $\mu(A_n) > \mu(U_n) - \frac{\epsilon}{2^n}$ for each n. Thus $\mu^*(B) \geq \sum_{n=1}^{\infty} \mu(U_n) - \epsilon \geq \mu(\bigcup_{n=1}^{\infty} U_n) - 2\epsilon$ where the last inequality follows by countable subadditivity of μ on $\mathcal{A}_c$. Note $B \subseteq \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} U_n = U$ (say). Then $\mu^*(U \setminus B) \leq 2\epsilon$. Similarly for $X \setminus B$, we can find an open set V such that $X \setminus B \subseteq V$ and $\mu^*(V \setminus (X \setminus B)) \leq 2\epsilon$. So $B \supseteq X \setminus V = C$ (say). C is a closed set. Now $U \setminus C = U \cap V = (U \setminus B) \cup (V \setminus (X \setminus B))$. So $\mu^*(U \setminus C) \leq \mu^*(U \setminus B) + \mu^*(V \setminus (X \setminus B)) \leq 4\epsilon$. Hence μ^* is regular on $\mathcal{B}$.■

Now with the help of Theorem 3.11 and Remark 3.12, Theorems 3.8 and 3.10 can be improved to the following versions.

Theorem 3.13: *If X is countably compact, normal and Hausdorff, then $M_b(X)$ is isometrically isomorphic to $\Lambda_\infty(X)$ while $M_b^+(X)$ is identified with $\Lambda_\infty^+(X)$ under this isometric isomorphism.*

Theorem 3.14: *If X is completely regular, Hausdorff and countably compact, then the normed linear space $M_a(X)$ of all regular finite (signed) Baire measures on $\mathcal{B}_a$ equipped with the total variation norm is isometrically isomorphic to $\Lambda_\infty(X)$. Furthermore, $M_a^+(X) = \{\mu \in M_a(X) : \mu \geq 0\}$ is identified with $\Lambda_\infty^+(X)$ under this isometric isomorphism.*

Since $\Lambda_\alpha(X) \subseteq \Lambda_\infty(X)$, the Remark 3.9 and the proof of Theorem 3.8 give the following theorem.

Theorem 3.15: *If X is countably compact, normal and Hausdorff, then $M_{b,\alpha}(X)$* is isometrically isomorphic to $\Lambda_\alpha(X)$ *while $M_{b,\alpha}^+(X)$ is identified with $\Lambda_\alpha^+(X)$ under this isometric* *isomorphism.*

Now we close this chapter by proving the following theorem which will be needed in the next chapter.

Theorem 3.16: *If X is countable, then $M_b(X)$* is separable.

Proof. For each $x \in X$, let δ_x be the Dirac's measure defined on $\mathcal{P}(X)$. It is easy to check that $\delta_x \in M_b(X)$. Let $\{x_n : n \in \mathbb{N}\}$ be an ordering of X and let $\mathbb{Q}$ be the set of rational numbers. Then define $S = \{\sum_{i=1}^{n} q_i\delta_{x_i} : n \in \mathbb{N}\}$ and $q_1, \ldots, q_n \in \mathbb{Q}$ which is a countable set.

Now we show that S is dense in $M_b(X)$. To this end, first choose $\mu \in M_b(X)$ such that $\mu \geq 0$, i. e. , $\mu \in M_b^+(X)$ and let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} \mu(x_m) = \mu(X) < \infty$, there exists an $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} \mu(x_n) < \frac{\epsilon}{2}$. For each $n = 1, 2, \ldots, N$ choose a $q_n \in \mathbb{Q}$ such that $0 \leq \mu(x_n) - q_n < \frac{\epsilon}{2N}$. Define $\nu \in S$ by $\nu = \sum_{n=1}^{N} q_n\delta_{x_n}$. Note that $\nu(x_n) = q_n$ if $1 \leq n \leq N$ and $\nu(x_n) = 0$ if $n > N$. Then $(\mu - \nu)(x_n) = \mu(x_n) - q_n$ if $1 \leq n \leq N$ and $(\mu - \nu)(x_n) = \mu(x_n)$ if $n > N$, so that in any case $\mu - \nu \geq 0$. Now $\| \mu - \nu \| = | \mu - \nu | (X) = (\mu - \nu)(X) = \sum_{n=1}^{\infty} (\mu - \nu)(x_n) = \sum_{n=1}^{N} (\mu - \nu)(x_n) + \sum_{n=N+1}^{\infty} (\mu - \nu)(x_n) = \sum_{n=1}^{N} (\mu(x_n) - q_n) + \sum_{n=N+1}^{\infty} \mu(x_n) < N \cdot \frac{\epsilon}{2N} + \frac{\epsilon}{2} = \epsilon$. Now if $\mu = \mu^+ - \mu^-$, then as above choose ν_1 and ν_2 from S such that $\| \mu^+ - \nu_1 \| < \frac{\epsilon}{2}$ and $\| \mu^- - \nu_2 \| < \frac{\epsilon}{2}$. Therefore $\| \mu - (\nu_1 - \nu_2) \| = \| (\mu^+ - \mu^-) - (\nu_1 - \nu_2) \| = \| (\mu^+ - \nu_1) + (\nu_2 - \mu^-) \| \leq \| \mu^+ - \nu_1 \| + \| \nu_2 - \mu^- \| < \epsilon$. But $\nu_1, \nu_2 \in S$ implies $\nu_1 - \nu_2 \in S$. Hence S is dense in $M_b(X)$ and so $M_b(X)$ is separable.■

Corollary 3.17: *If X is countable, then $M_{b,\alpha}(X)$ and $M_{b,\alpha}^+(X)$ are separable.*

Proof. A subspace of a separable metric space is separable.■

Chapter IV

Separability

In this chapter we study the separability of $\Lambda_j^+(X)$ and $\Lambda_j(X)$ ($j = p, k,\alpha, \infty$).

The *density*, $d(X)$, of a space X is the smallest infinite cardinal number m such that X has a dense subset which has cardinality less than or equal to m. Now a space X is separable if and only if $d(X) = \aleph_0$.

If X is a subspace of a metrizable space, then $d(X) \le d(Y)$. So by Theorem 1.5.5, if X is a subspace of Y, then $d(\Lambda_j(X)) \le d(\Lambda_j(Y))$ and $d(\Lambda_j^+(X)) \le d(\Lambda_j^+(Y))$ for $j = p, k$. When Y is normal Hausdorff and X is a closed subspace of Y, by Corollaries 1.5.8 and 1.5.11, $d(\Lambda_j(X)) \le d(\Lambda_j(Y))$ and $d(\Lambda_j^+(X)) \le d(\Lambda_j^+(Y))$ for $j = \sigma, \infty$.

Theorem 4.1: *$d(\Lambda_j^+(X)) = d(\Lambda_j^+(Y))$ for $j = p, k, \alpha, \infty$.*

Proof. Since $\Lambda_j(X)$ is metrizable, it suffices to show that $d(\Lambda_j(X)) \le d(\Lambda_j^+(X))$. Let Δ be a dense subset of $\Lambda_j^+(X)$ having cardinality $d(\Lambda_j^+(X))$. Define $E = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \Delta\}$ which is a subset of $\Lambda_j(X)$ having cardinality $d(\Lambda_j^+(X))$. It remains to show that E is dense in $\Lambda_j(X)$. Let $\lambda \in \Lambda_j(X)$ and let $\epsilon > 0$. Then since Δ is dense in $\Lambda_j^+(X)$, there exists $\mu_1, \mu_2 \in \Delta$ such that $d_*(\lambda^+, \mu_1) < \frac{\epsilon}{2}$ and $d_*(\lambda^-, \mu_2) < \frac{\epsilon}{2}$. Therefore $\| \lambda - (\mu_1 - \mu_2) \|_* =$

$\|(\lambda^+ - \lambda^-) - (\mu_1 - \mu_2)\|_* \leq \|\lambda^+ - \mu_1\|_* + \|\lambda^- - \mu_2\|_* = d_*(\lambda^+, \mu_1) + d_*(\lambda^-, \mu_2) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.■

Corollary 4.2 : *$\Lambda_j^+(X)$ is separable if and only if $\Lambda_j(X)$ is separable ($j = p, k, \alpha, \infty$).*

First we deal with the separability of $\Lambda_j(X)$ where $j = p, k$.

We have seen in I.3, the evaluation function defined at x, $\phi_x : C_p(X) \to \mathbb{R}$, which takes an element f of $C_p(X)$ to $f(x)$, is in $\Lambda_p^+(X)$. So $\phi_x \in \Lambda_p^+(X) \subseteq \Lambda_k^+(X)$. Again in I.3, we have seen that the evaluation function $\Phi : X \to \Lambda_p^+(X)$ which takes x to ϕ_x is one-to-one. Hence the $card(X) = card\,\Phi(X)$.

It is also true that $\Phi(X)$ is a discrete subset of $\Lambda_j^+(X)$. To see this, let x and y be distinct points of X. Then $d_*(\phi_x, \phi_y) = \|\phi_x - \phi_y\|_* = \sup\{|\phi_x(f) - \phi_y(f)| : f \in C(X), \|f\|_\infty \leq 1\} = \sup\{|f(x) - f(y)| : f \in C(X), \|f\|_\infty \leq 1\} \geq 1$.

For the remainder of this section, the notation $|X|$ stands for the greater of $\aleph_0$ or the cardinality of X. Now since Φ is one-to-one and $\Phi(X)$ is a discrete subset of $\Lambda_j^+(X)$, $d(\Phi(X)) = |X|$. Therefore since $d(\Phi(X)) \leq d(\Lambda_j^+(X))$, the following is true.

Theorem 4.3: *For each space X, $|X| \leq d(\Lambda_j^+(X))\,(j = p, k)$.*

Corollary 4.4: *If $\Lambda_j^+(X)\,(j = p, k)$ is separable, then X* is countable.

Proof. $\Lambda_j^+(X)$ is separable implies $d(\Lambda_j^+(X)) = \aleph_0$.■

We also have the converse of the Corollary 4.4.

Theorem 4.5: *If X is countable. then $\Lambda_j^+(X)\,(j = p, k)$ is separable.*

Proof. By Theorem 3.5 $d(M_{b,j}^+(X)) = d(\Lambda_j^+(X))$. But by Corollary 3.17, $d(M_{b,j}^+(X)) =$

$\aleph_0$ when X is countable. So when X is countable, $\Lambda_j^+(X)$ is separable.■

Now the Corollary 4.4 and Theorem 4.5 give the following theorem.

Theorem 4.6: *$\Lambda_j^+(X)$ $(j = p, k)$ is separable if and only if X is countable.*

Now we deal with the separability of $\Lambda_j(X)$ $(j = \alpha, \infty)$. As before, for each $x \in X$, define the evaluation function at x, $\phi_x : C_j^*(X) \to \mathbb{R}$, $(j = \alpha, \infty)$ by taking $\phi_x(f) = f(x)$ for each $f \in C_j^*(X)$. Now ϕ_x is a positive linear functional on $C_j^*(X)$ which is supported on $\{x\}$. Now if $\{x\} \in \alpha$, by the Lemma 1.2.3, $\phi_x \in \Lambda_\alpha^+(X)$. Since ϕ_x is positive, by the Lemma 1.2.4, $\phi_x \in \Lambda_\infty^+(X)$. Now the cases of $\Lambda_j^+(X)$ $(j = p, k)$ can be modified to obtain the next two theorems.

Theorem 4.7: *Suppose $\mathcal{F}(X) \subseteq \alpha$. Then $\Phi(X)$ is a discrete subset of $\Lambda_\alpha^+(X)$ and $|X| \leq d(\Lambda_\alpha^+(X))$.*

Corollary 4.8: *Suppose $\mathcal{F}(X) \subseteq \alpha$ and $\Lambda_\alpha^+(X)$ is separable. Then X is countable.*

Corollary 4.9: *Suppose X is normal and Hausdorff. If $\Lambda_\sigma^+(X)$ is separable, then X is countable.*

Theorem 4.10: *For each space X, $|X| \leq d(\Lambda_\infty^+(X))$.*

The next question is if we can have the similar version of Theorem 4.6 for $\Lambda_\alpha^+(X)$. The following theorem gives a partial answer to that question.

Theorem 4.11: *Suppose $\mathcal{F}(X) \subseteq \alpha \subseteq \mathcal{K}(X)$. Then $\Lambda_\alpha^+(X)$ is separable if and only if X is countable.*

Proof. Suppose $\Lambda_\alpha^+(X)$ is separable. Then by the Corollary 4.8, X is countable. Conversely, let X be countable. Now since $\alpha \subseteq \mathcal{K}(X)$, by the Theorem 3.6 $\Lambda_\alpha^+(X)$ is isometric to $M_{b,\alpha}^+(X)$. So $d(\Lambda_\alpha^+(X)) = d(M_{b,\alpha}^+(X))$. But by Corollary 3.17, $d(M_{b,\alpha}^+(X)) = \aleph_0$. Hence $\Lambda_\alpha^+(X)$ is separable. ■

But what about the converse of the Corollary 4.9? Note in this case we cannot appeal successfully to Theorem 3.15. Because a countably compact countable space is compact and consequently $C_\sigma^*(X) = C_k^*(X) = C_k(X)$ which in turn implies $\Lambda_\sigma(X) = \Lambda_k(X)$. So we get nothing new.

Lastly, we talk about the separability of $\Lambda_\infty(X)$. Note that, by Theorem 4.1, $\Lambda_\infty(X)$ is separable if and only if $\Lambda_\infty^+(X)$ is separable.

Theorem 4.12: $\Lambda_\infty(X)$ *is separable if and only if* X *is compact and countable.*

Proof. If $\Lambda_\infty(X)$ is separable, then $\Lambda_p(X)$ is separable. Hence by the Corollary 4.2 and Theorem 4.6, X is countable. Again, since $\Lambda_\infty(X)$ is the conjugate space of the normed linear space $C_\infty^*(X)$, $C_\infty^*(X)$ is separable. But this implies that X is compact (see [9], page 54).

Conversely, let X be countable and compact. Since X is compact, $C_k(X) = C_k^*(X) = C_\infty^*(X)$ and consequently $\Lambda_\infty(X) = \Lambda_k(X)$. But since X is countable, by Theorem 4.6, $\Lambda_k(X)$ is separable. Hence $\Lambda_\infty(X)$ is separable. ■

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