APPROXIMATION AND CONTROL
OF A THERMOVISCOELASTIC SYSTEM

by

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(ABSTRACT)

In this paper consider the problem of controlling a thermoviscoelastic system. We present a semigroup setting for this system, and prove the well-posedness by applying a gerneral theorem which is given in this paper. We also study the stability of the system.

We give a finite element/averaging scheme to approximate the linear quadratic regulator problem governed by the system. We prove that yields faster convergence. We give a proof of convergence of the simulation problem for singular kernels and of the control problem for L_2 kernels.

We carry on the numerical computation to investigate the effect of heat transfer on damping and the closed-loop system.

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and .

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Chapter I Introduction

1.1 Problem Statement.

It is well known that the design of control systems for vibration suppression in flexible structures is highly dependent on the type and amount of internal damping present in the structure. Since the viscoelastic properties of the material contribute to the stability of the system similarly as internal damping, a considerable amount of work has been done on control, identification and numerical approximation to take advantage of the material properties [**3**][**4**][**5**][**6**][**12**][**13**].

In this paper, we consider the control and approximation of a thermoviscoelastic model of Boltzmann type. Our interests are to investigate the effect of heat dissipation on damping and on the closed loop control system, and to develop an appropriate approximation scheme.

We concentrate primarily on the axial vibrations of an elastic bar of length l whose motion is governed by the system

$$\begin{aligned} \rho\ddot{y}(t,x) = \frac{\partial}{\partial x}\left(\alpha\frac{\partial}{\partial x}y(t,x) + \int_{-r}^{0} g(s)\frac{\partial}{\partial x}y(t+s,x)ds\right) \\ - \gamma\frac{\partial}{\partial x}\theta(t,x) + b(x)u(t) \end{aligned} \tag{1.1.1}$$

$$\dot{\theta}(t,x) = -\gamma\theta_0\frac{\partial^2}{\partial x\partial t}y(t,x) + k\frac{\partial^2}{\partial x^2}\theta(t,x) \tag{1.1.2}$$

with the boundary conditions either

$$y(t,0) = y(t,l) = \theta(t,0) = \theta(t,l) = 0 \tag{1.1.3}$$

or

$$y(t,0) = y(t,l) = \theta_x(t,0) = \theta_x(t,l) = 0. \tag{1.1.4}$$

Here t represents time; x is the spatial varible; y is the axial displacement; $\rho > 0$ is the mass density; α is a positive constant; $\theta_0 > 0$ is the reference temperature; θ is the temperature deviation from θ_0; and $k > 0$ is the heat conductivity. The delay $r > 0$ is either infinite or finite, $b(\cdot) \in L_2(0,l)$, u is the control function, and $\gamma \geq 0$ is the coupling coefficient. Note that $\gamma = 0$ decouples (1.1.1)-(1.1.2) to a viscoelastic equation and a heat equation. We also assume that the function $g : [-r, 0) \to \mathcal{R}$ satisfies the following conditions:

(1) $g \in L_1(-r,0)$, $g(s) \leq 0$ on $[-r,0)$,

(2) g is absolutely continuous on $[-r,-\delta]$ for all $r > \delta > 0$ and $g'(s) \leq 0$ for $-r \leq s < 0$,

(3) There is a constant $\epsilon > 0$ such that $\epsilon = \alpha + \int_{-r}^{0} g(s)ds$.

The simplest case is when $g(s) \equiv 0$, which yields the classical thermoelastic system

$$\rho\ddot{y}(t,x) = \alpha\frac{\partial^2}{\partial x^2}y(t,x) - \gamma\frac{\partial}{\partial x}\theta(t,x) + b(x)u(t) \tag{1.1.5}$$

$$\dot{\theta}(t,x) = -\gamma\theta_0\frac{\partial^2}{\partial x\partial t}y(t,x) + k\frac{\partial^2}{\partial x^2}\theta(t,x). \tag{1.1.6}$$

Note that we allow the kernel $g(s)$ to have a singularity at $s = 0$. Although the singular kernels in viscoelasticity have been discussed at least since the 1970's, the current intense interest[**6**][**9**][**12**][**13**][**14**] in their qualitative properties is related to the need to understand internal damping when designing feedback control systems. It is known now that such a kernel gives faster energy dissipation in the system.

In Chapter 2, we formulate the system (1.1.1)-(1.1.3) as an abstract Cauchy problem on a Hilbert space Z, and use standard techniques from semigroup theory to prove well-posedness. Then we present a well-posedness theorem for a general class of abstract ordinary differential equations. We also discuss the stability of the system.

In Chapter 3, we develop a finite element/averaging approximation scheme. The convergence of the approximation systems is established by modifying the proofs given by Ito and Fabiano[**14**], and Miller[**15**]. Under additional assumptions on g, we establish the convergence of the approximation adjoint systems, which is crucial to the correct approximation of the feedback gain operator for the regulator problem.

In Chapter 4, we solve the regulator problem for the system (1.1.1)-(1.1.3) numerically by our approximation scheme. Some interesting phenomena are observed and discussed.

1.2 Thermoviscoelaticity.

The behavior of viscoelastic materials has received widespread attention and application in the last ten years . This is primarily due to the large scale development and utilization of polymeric materials.

As we know, elastic bodies react to a fixed deformation with stress which remains constant as time proceeds. In other words, they have a capacity to store mechanical energy without dissipation. On the other hand, viscoelastic bodies react to a fixed deformation with stress which decays as time proceeds. During that process, mechanical energy is converted to heat. Under isothermal condition, the stress is determined by the instantaneous strain and its past history. This is referred to as a memory effect.

The stress-strain constitutive law for the homogeneous, isothermal viscoelastic rod of Boltzmann type is given by

$$\sigma(t,x) = \alpha\varepsilon(t,x) + \int_0^\infty g_1(s)\varepsilon(t-s,x)ds, \tag{1.2.1}$$

where σ is the shear stress, ε is the shear strain, $\alpha > 0$. This leads to the equation

$$\ddot{y}(t,x) = \frac{\partial}{\partial x}\left(\alpha\frac{\partial}{\partial x}y(t,x) + \int_0^\infty g_1(s)\frac{\partial}{\partial x}y(t-s,x)ds\right) \tag{1.2.2}$$

where y is the displacement. We point out that (1.2.1) is a modification of a purely elastic system by adding a convolution type integral which describes the relation between the stress and the past strain history.

If the isothermal condition is not assumed, then we need to study heat flow and temperature states of the viscoelastic material, which leads to the theory of thermo-viscoelasticity. For the purpose of illustration, we concentrate on the one dimensional linear theory. The general theory can be found in [7], which is derived on two fundamental thermo-dynamical postulates, i.e, the balance of energy and the entropy production inequality. The following is a one-dimensional version of the derivation in [**16**]. Let's reconsider the homogeneous viscoelastic rod of Boltzmann type with zero initial stress and intial temperature θ_0. We omit the variables t, x whenever we can without causing confusion. The local statement of the energy balance law is given by

$$\rho r - \rho\left(\dot{\Psi} + \dot{\theta}\eta + T\dot{\eta}\right) + \sigma\dot{\varepsilon} - \frac{\partial}{\partial x}q = 0, \tag{1.2.3}$$

where σ is the stress; ε is the strain; ρ is the mass density; r is the specific heat supply; T is the current temperature; $\theta = T - \theta_0$, i.e, the temperature deviation from the reference tempreture θ_0; η is the specific entropy; q is the heat flux; and Ψ is the specific Helmholtz free energy which is given by.

$$\begin{aligned}
\rho\Psi(t) = {} & \frac{1}{2}G(\infty)\left(\frac{\partial y}{\partial x}\right)^2 - L(\infty)\frac{\partial y}{\partial x}\theta - \frac{1}{2}\rho C(\infty)\theta^2\theta_0^{-1} \\
& - \frac{1}{2}\int_{-\infty}^{t} \dot{G}(t-\tau)\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial x}y(\tau, x)\right)^2 d\tau \\
& + \int_{-\infty}^{t} \dot{L}(t-\tau)(\theta - \theta(\tau, x))\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial x}y(\tau, x)\right) d\tau \\
& + \frac{\rho}{2\theta_0}\int_{-\infty}^{t} \dot{C}(t-\tau)(\theta - \theta(\tau, x))^2\, d\tau,
\end{aligned}$$

where $G(s), L(s), C(s)$ are the material properties.

Notice that the free energy Ψ not only depends upon the strain history, but also depends upon the temperature history.

The local entropy production inequality is given by

$$\rho T\dot{\eta} - \rho r + \frac{\partial}{\partial x}q - \frac{1}{T}q\frac{\partial \theta}{\partial x} \geq 0. \tag{1.2.4}$$

Eliminating the heat supply function r from (1.2.3) and (1.2.4), we have

$$-\rho\dot{\Psi} - \rho\dot{\theta}\eta + \sigma\dot{\varepsilon} - \frac{1}{\theta_0}q\frac{\partial\theta}{\partial x} \geq 0, \tag{1.2.5}$$

where we retained only the first order term of $\frac{1}{T}\frac{\partial\theta}{\partial x}$. Inequality (1.2.5) must be satisfied for all values of $\dot{\varepsilon}(t)$ and $\dot{\theta}(t)$. Therefore, it is necessary that their coefficients equal zero. This gives

$$\sigma = G(0)\frac{\partial}{\partial x}y - L(0)\theta + \int_{-\infty}^{t} \dot{G}(t-\tau)\frac{\partial}{\partial x}y(\tau,x)d\tau - \int_{-\infty}^{t} \dot{L}(t-\tau)\theta(\tau,x)d\tau, \tag{1.2.6}$$

$$\eta = L(0)\frac{\partial}{\partial x}y + \rho\frac{C(0)}{\theta_0}\theta + \int_{-\infty}^{t} \dot{L}(t-\tau)\frac{\partial}{\partial x}y(\tau,x)d\tau + \frac{\rho}{\theta_0}\int_{-\infty}^{t} \dot{C}(t-\tau)\theta(\tau,x)d\tau. \tag{1.2.7}$$

From Fourier's law for the heat flux, we have the constitutive assumption

$$-q = k\frac{\partial}{\partial x}\theta$$

where constant $k > 0$ is the thermal conductivity.

Now, the local energy balance law (1.2.3) can be rewritten as

$$\rho r + \rho\delta - \rho T\dot{\eta} + k\frac{\partial^2}{\partial x^2}\theta = 0, \tag{1.2.8}$$

where δ, the internal dissipation, is given by

$$\rho\delta = \frac{1}{2}\int_{-\infty}^{t} \ddot{G}(t-\tau)\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial x}y(\tau,x)\right)^2 d\tau - \frac{\rho}{\theta_0}\int_{-\infty}^{t} \ddot{C}(t-\tau)\left(\theta - \theta(\tau,x)\right)^2 d\tau$$
$$- \int_{-\infty}^{t} \ddot{L}(t-\tau)\left(\theta - \theta(\tau,x)\right)\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial x}y(\tau,x)\right) d\tau.$$

In order to satisfy the local entropy production inequality (1.2.4), δ must be nonnegative. This can be obtained under certain assumptions on the material properties $G(s), L(s), C(s)$. Since δ is a second order term and $T = \theta_0 +$ first-order terms, for a consistent first-order theory, we must have

$$\rho r - \rho\theta_0\dot{\eta} + k\frac{\partial^2}{\partial x^2}\theta = 0. \tag{1.2.9}$$

Assuming that there are no external forces and heat supply, and using $\frac{\partial}{\partial x}\sigma = \rho\ddot{y}$ and (1.2.6), (1.2.9), we obtain the equations

$$\begin{aligned}\rho\ddot{y} = \frac{\partial}{\partial x}\Bigg(G(0)\frac{\partial}{\partial x}y - L(0)\theta + \int_{-\infty}^{t}\dot{G}(t-\tau)\frac{\partial}{\partial x}y(\tau,x)d\tau \\ - \int_{-\infty}^{t}\dot{L}(t-\tau)\theta(\tau,x)d\tau\Bigg)\end{aligned} \tag{1.2.10}$$

$$\begin{aligned}C(0)\dot{\theta} = -\frac{\partial}{\partial t}\Bigg(\theta_0 L(0)\frac{\partial}{\partial x}y + \theta_0\int_{-\infty}^{t}\dot{L}(t-\tau)\frac{\partial}{\partial x}y(\tau,x)d\tau \\ +\rho\int_{-\infty}^{t}\dot{C}(t-\tau)\theta(\tau,x)d\tau\Bigg) + k\frac{\partial^2\theta}{\partial x^2}.\end{aligned} \tag{1.2.11}$$

REMARK: Equations (1.1.1)-(1.1.2) are special cases of above equations, where $L(s), C(s)$ are constant functions.

1.3 The Linear Regulator Problem.

Let Z, Y and U be Hilbert spaces. Assume that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of bounded linear operators $T(t)$ on Z, $\mathcal{B} \in L(U,Z), \mathcal{C} \in L(Z,Y)$ and $\mathcal{R} \in L(U,U)$. Furthermore, let $\mathcal{Q}$ be the self-adjoint nonnegative operator defined by $\mathcal{Q} = \mathcal{C}^*\mathcal{C} \in L(Z,Z)$, and assume that $\mathcal{R}$ is self-adjoint and strictly positive.

The linear quadratic regulator problem is to find a control $u \in L_2(0,\infty;U)$ which minimizes a quadratic cost functional of the form

$$
\begin{aligned}
J(z,u) &= \int_0^\infty [\langle y,y\rangle_Y + \langle \mathcal{R}u,u\rangle_U]\,dt \\
&= \int_0^\infty [\langle \mathcal{Q}z,z\rangle_Z + \langle \mathcal{R}u,u\rangle_U]\,dt \qquad (1.3.1)
\end{aligned}
$$

subject to

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \qquad z(0) = z_0 \qquad (1.3.2)$$

$$y(t) = \mathcal{C}z(t). \qquad (1.3.3)$$

Note that the mild solution of (1.3.2) is

$$z(t) = T(t)z_0 + \int_0^t T(t-s)\mathcal{B}u(s)ds. \qquad (1.3.4)$$

The control space U is often taken to be R^m if we use m controllers. In that case, we assume that $\mathcal{R}$ is an $m \times m$ diagonal matrix where $r_{ii} > 0$ is the weight on the ith controller.

It is shown[**11**] that if the system (1.3.2)-(1.3.3) is stabilizable and detectable, then there is an unique control $u_{opt} \in L_2(0,\infty;U)$ such that

$$J(z,u_{opt}) = \min_{u \in L_2(0,\infty;U)} J(z,u).$$

This control can be written in a feedback form

$$u_{opt} = -\mathcal{R}^{-1}\mathcal{B}^*\Pi z(t), \qquad (1.3.5)$$

where Π is the solution of the algebraic Riccati equation

$$\mathcal{A}^*\Pi + \Pi\mathcal{A} - \Pi\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\Pi + \mathcal{Q} = 0. \tag{1.3.6}$$

In general, equation (1.3.6) is a non-linear partial-functional differential operator equation. A direct solution of such a system is complex. Therefore, it has become standard practice to approximate the entire control problem, thereby leading to an indirect approximation of (1.3.6). We shall take this approach. For $N = 1, 2, \ldots$, let Z^N be subspaces of Z, P^N be the orthogonal projections of Z onto Z^N, $\mathcal{A}^N$ be the generator of C_0-semigroup $T^N(t)$ on Z^N. Assume that $\mathcal{B}^N \in L(U, Z^N)$, and $\mathcal{Q}^N = \left[\mathcal{C}^{N*}\mathcal{C}^N\right] \in L(Z^N, Z^N)$ are uniformly bounded in N. The Nth approximation problem is to minimize

$$J^N(z^N, u) = \int_0^\infty \left[\left\langle \mathcal{Q}^N z^N, z^N\right\rangle_Z + \langle \mathcal{R}u, u\rangle_U\right] ds \tag{1.3.7}$$

subject to

$$\dot{z}^N(t) = \mathcal{A}^N z^N(t) + \mathcal{B}^N u(t), \qquad z^N(0) = P^N z_0 \tag{1.3.8}$$

$$y^N(t) = \mathcal{C}^N z^N(t). \tag{1.3.9}$$

Similarly, under the assumption that (1.3.8)-(1.3.9) is stabilizable and detectable, there is an unique optimal control $u^N_{opt} \in L_2(0, \infty; U)$ of the form

$$u^N_{opt}(t) = -\mathcal{R}^{-1}\mathcal{B}^{N*}\Pi^N z^N(t), \tag{1.3.10}$$

where Π^N is the unique solution of the algebraic (matrix) Riccati equation

$$\mathcal{A}^{N*}\Pi^N + \Pi^N\mathcal{A}^N - \Pi^N\mathcal{B}^N\mathcal{R}^{-1}\mathcal{B}^{N*}\Pi^N + \mathcal{Q}^N = 0. \tag{1.3.11}$$

The basic requirements are that the suboptimal control $u^N_{opt}(t)$ when applied to the infinite dimensional system (1.3.2)-(1.3.3) results in a stable closed-loop system whose response is "close" to optimal for any initial condition and that $u^N_{opt}(t)$ converges to $u_{opt}(t)$ in some appropriate sense.

To ensure these requirements, Gibson[11]] gave the following sufficient conditions:

(1) The problem must be well-posed.

(2) $\mathcal{B}^N u \longrightarrow \mathcal{B}u$, $\mathcal{B}^{N*}z \longrightarrow \mathcal{B}^*z$, $\mathcal{Q}^N P^N z \longrightarrow \mathcal{Q}z$, as $N \longrightarrow \infty$ for all $z \in Z$ and $u \in U$.

(3) $T^N(t), T^{N*}(t)$, the semigroups generated by $\mathcal{A}^N, \mathcal{A}^{N*}$, converge strongly to $T(t), T^*(t)$, the semigroups generated by $\mathcal{A}$ and $\mathcal{A}^*$, respectively.

(4) The approximation systems must preserve stabilizability and detectability uniformly in N.

Many of these basic requirements were developed in an attempt to apply the averaging approximation scheme to hereditary control problems. Banks and Burns[2] showed the strong convergence of the approximating semigroups to the semigroup representing the homogeneous solution of the hereditary system in 1978. Since they only considered the open loop problem, they did not raise the question of the strong convergence of the adjoint semigroups. Five years later, Gibson[11] established this convergence when he considered the regulator problem for delay-differential systems. In 1985, Salamon[22] proved that the approximating systems based on the averaging scheme are uniformly exponentially stable for sufficiently large N if the original system is stable. These ideas have since been refined and extended to other schemes developed for control design for infinite dimensional systems.

1.4 Notation.

The notation used in this paper is standard. If Y is a linear space, then $\|\cdot\|_Y$ and $\langle\cdot,\cdot\rangle_Y$ will denote the norm and inner product on Y. For a Hilbert space Z, the set of all square integrable functions defined on $[a,b]$ with values in Z will be denoted by $L_2(a,b;Z)$. The space of all absolutely continuous functions $f \in L_2(a,b;Z)$ with jth derivative $f^{(j)}$ absolutely continuous for $j = 1,2,\ldots,k-1$ and $f^{(k)} \in L_2(a,b;Z)$ is denoted by $H^k(a,b;Z)$. $H^1_L(a,b;Z)$ denotes the set of all H^1 functions which vanish at the left end-point of the interval; i.e., $H^1_L(a,b;Z) \equiv \{f \in H^1(a,b;Z) \mid f(a) = 0\}$. Similarly, $H^1_R(a,b;Z) \equiv \{f \in H^1(a,b;Z) \mid f(b) = 0\}$, and $H^1_0(a,b;Z) \equiv \{f \in H^1(a,b;Z) \mid f(a) = f(b) = 0\}$. Let $H^2_0(a,b;Z) = H^1_0(a,b;Z) \cap H^2(a,b;Z)$. $H^{-n}(0,l)$ is the dual space of $H^n_0(0,l)$. If $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $T(\cdot)$ on a Hilbert space Z satisfying $\|T(t)\|_Z \leq Me^{\beta t}$, then we write $\mathcal{A} \in G(M,\beta)$. Finally, the symbol $\xrightarrow{s}$ means converging strongly.

Chapter II Well-Posedness and Stability

In this chapter we formulate the linear thermoviscoelastic system (1.1.1)-(1.1.2) into the abstract Cauchy problem $\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t)$, $z(0) = z_0$ on certain Hilbert space Z. We then prove wellposedness by showing that $\mathcal{A}$ generates a C_0-semigroup on a Hilbert space Z. We also discuss the stability of this system.

2.1 State Space Formulation.

Finding an appropriate state space for an infinite demensional system is not trivial. One difficulty is in choosing the suitable topology or norm. Fortunately, the well-accepted state space choices for the linear viscoelatic system and the linear thermoelastic system [**23**] provide considerable insight.

Let $v(t,x) = \dot{y}(t,x)$, $w(t,s,x) = y(t,x) - y(t+s,x)$, then (1.1.1)-(1.1.2) can be recast into a first order system

$$\frac{d}{dt}\begin{pmatrix} y(t,x) \\ v(t,x) \\ \theta(t,x) \\ w(t,s,x) \end{pmatrix} = \begin{pmatrix} v(t,x) \\ \epsilon D_x^2\left(y(t,x) + \int_{-r}^0 g_\epsilon(s)w(t,s,x)ds\right) - \gamma\frac{\partial}{\partial x}\theta(t,x) + b(x)u(t) \\ -\gamma\frac{\partial}{\partial x}v(t,x) + kD_x^2\theta(t,x) \\ v(t,x) + \frac{\partial}{\partial s}w(t,s,x) \end{pmatrix} \tag{2.1.1}$$

where we assumed that $\theta_0 = 1$. $g_\epsilon(s) = -\frac{1}{\epsilon}g(s)$, and the constant ϵ is defined in section 1.1.

For simplicity of notation, we omit the variables t, x wherever we can without causing confusion. Define the spaces Y, V, Θ, W by

$$Y = H_0^1(0,l), \qquad \|y\|_Y^2 = \epsilon \int_0^l (D_x y)^2 dx;$$

$$V = L^2(0,l), \qquad \|v\|_V^2 = \int_0^l v^2 dx;$$

$$\Theta = L^2(0,l), \qquad \|\theta\|_\Theta^2 = \int_0^l \theta^2 dx;$$

$$W = L_g^2(-r,0;Y), \qquad \|w\|_W^2 = \int_{-r}^0 g_\epsilon(s)\|w\|_Y^2 ds.$$

We shall consider the state space

$$Z = Y \times V \times \Theta \times W$$

with the energy related norm

$$\|z\|_Z^2 = \|(y,v,\theta,w)^T\|_Z^2 = \|y\|_Y^2 + \|v\|_V^2 + \|\theta\|_\Theta^2 + \|w\|_W^2.$$

Define the operator $\mathcal{A}$ by

$$\mathcal{D}(\mathcal{A}) = \left\{ z \in Z \mid \begin{array}{c} y + \int_{-r}^0 g_\epsilon(s)w(s)ds \in H_0^2(0,l), \\ \theta \in H_0^2(0,l),\ v \in H_0^1(0,l),\ w \in H_{gR}^1(-r,0;H_0^1(0,l)) \end{array} \right\} \tag{2.1.2}$$

$$\mathcal{A}\begin{pmatrix} y(t,x) \\ v(t,x) \\ \theta(t,x) \\ w(t,s,x) \end{pmatrix} = \begin{pmatrix} v(t,x) \\ \epsilon D_x^2 \left(y(t,x) + \int_{-r}^0 g_\epsilon(s)w(t,s,x)ds \right) - \gamma \frac{\partial}{\partial x}\theta(t,x) \\ -\gamma \frac{\partial}{\partial x} v(t,x) + kD_x^2\theta(t,x) \\ v(t,x) + Dw(t,s,x) \end{pmatrix}, \tag{2.1.3}$$

where the operators D_x^2 and D are defined by

$$\mathcal{D}(D_x^2) = H_0^2(0,l), \qquad D_x^2 y = \frac{\partial^2 y}{\partial x^2}; \tag{2.1.4}$$

$$\mathcal{D}(D) = H_{gR}^1(-r,0;Y), \qquad Dw = \frac{\partial}{\partial s}w. \tag{2.1.5}$$

Define the operator $\mathcal{B} : U \to Z$ by

$$\mathcal{B}u(t) = (0, b(x)u(t), 0, 0)^T. \tag{2.1.6}$$

The linear thermoviscoelastic system (1.1.1)-(1.1.2) is thus formally transformed into the abstract Cauchy problem on the Hilbert Space Z

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \qquad z(0) = z_0 \tag{2.1.7}$$

where $z_0 = (y_0, \dot{y}_0, \theta_0, w_0)^T$.

We remark here that the state space formulation is not unique. Other choices may be appropriate if one considers different boundary conditions or if concentrated masses are added to to structure.

2.2 Well-Posedness.

The abstract Cauchy problem (2.1.7) has an unique solution if and only if the operator $\mathcal{A}$ generates a semigroup on the Hilbert space Z. In this case, we say that it is well-posed.

Navaro[**16**] proved the well-posedness for the general linear thermoviscoelastic system (1.2.10)-(1.2.11). But one of his assumptions is that the kernel $g(t) \in L^1(0,\infty) \cap C^1[0,\infty)$, which excluded the singular case. Recently, Ito and Fabiano[**14**] showed the well-posedness for a class of abstract integral-differential equations. Their result covers a special case of system (1.1.1)-(1.1.2), i.e., $\gamma = 0$..

A three dimensional version of the system (1.1.1)-(1.1.2) was studied by Desch and Grimmer[**10**]. They assumed that the kernel g was completely monotonic. They provided the semigroup setting and a smoothing criterion. However, their state space and its norm have no intuitive physical meaning (such as energy), and more general kernels need to be considered.

We will use the following version of the Lumer-Phillips theorem[**17**].

THEOREM 2.2.1. *Let $\mathcal{A}$ be a closed densely defined linear operator on a Hilbert space H. If there exists $\beta \in R$ such that $\langle \mathcal{A}x, x\rangle \leq \beta \langle x, x\rangle$ for all $x \in \mathcal{D}(\mathcal{A})$, and $R(\lambda_0 I - A)$ is dense in H for some $\lambda_0 > \beta$, then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $T(t)$ on H satisfying $\|T(t)\| \leq e^{\beta t}$.*

In the next three lemmas, we check the conditions in Theorem 2.2.1 for the system (2.1.5)-(2.1.6).

LEMMA 2.2.2. *The domain of $\mathcal{A}, \mathcal{D}(\mathcal{A})$, is dense in Z and $\mathcal{A}$ is closed.*

PROOF: Clearly, $H_0^2(0,l) \times H_0^1(0,l) \times H_0^2(0,l) \times H_{gR}^1(-r,0;H_0^2(0,l))$ is a subset of $\mathcal{D}(\mathcal{A})$, and dense in Z. Thus, $\mathcal{D}(\mathcal{A})$ is also dense in Z.

Let $z_n = (y_n, v_n, \theta_n, w_n)^T \in \mathcal{D}(\mathcal{A})$. Suppose that

$$z_n \longrightarrow z = (y, v, \theta, w)^T$$

and

$$\mathcal{A}z_n = \begin{pmatrix} v_n \\ \epsilon D_x^2(y_n + \int_{-r}^0 g_\epsilon w_n(s)ds) - \gamma \frac{\partial}{\partial x}\theta_n \\ -\gamma \frac{\partial}{\partial x} v_n + kD_x^2\theta_n \\ v_n + Dw_n(s) \end{pmatrix} \rightarrow \begin{pmatrix} \phi \\ \psi \\ \xi \\ h \end{pmatrix}$$

Since $v_n \rightarrow \phi$ in $H_0^1(0,l)$ and $v_n \rightarrow v$ in $L^2(0,l)$, we have

$$\phi = v \in H_0^1(0,l).$$

Since $y_n \rightarrow y$ in $H_0^1(0,l)$, $w_n \rightarrow w$ in $L_g^2(-r,0;H_0^1(0,l))$, then

$$y_n + \int_{-r}^0 g_\epsilon(s)w_n(s)ds \rightarrow y + \int_{-r}^0 g_\epsilon(s)w(s)ds$$

in $H_0^1(0,l)$. This implies that

$$D_x^2\left(y_n + \int_{-r}^0 g_\epsilon(s)w_n(s)ds\right) \rightarrow D_x^2\left(y + \int_{-r}^0 g_\epsilon(s)w(s)ds\right)$$

in $H^{-1}(0,l)$. Similarly, we know that $\frac{\partial}{\partial x}\theta_n \rightarrow \frac{\partial}{\partial x}\theta$ in $H^{-1}(0,l)$. Thus

$$\epsilon D_x^2\left(y_n + \int_{-r}^0 g_\epsilon(s)w_n(s)ds\right) - \gamma\frac{\partial}{\partial x}\theta_n$$

$$\rightarrow \epsilon D_x^2\left(y + \int_{-r}^0 g_\epsilon(s)w(s)ds\right) - \gamma\frac{\partial}{\partial x}\theta$$

in $H^{-1}(0,l)$. But we already know that it converges to ψ in $L^2(0,l) \subset H^{-1}(0,l)$.

Therefore, it follows that

$$\psi = \epsilon D_x^2\left(y + \int_{-r}^0 g_\epsilon(s)w(s)ds\right) - \gamma\frac{\partial}{\partial x}\theta$$

and

$$y + \int_{-r}^0 g_\epsilon(s)w(s)ds \in H_0^2(0,l).$$

A similar argument shows that

$$\xi = -\gamma \frac{\partial}{\partial x} v + kD_x^2 \theta \qquad \textit{and} \qquad \theta \in H_0^2(0,l).$$

The closedness of D is established in [**14**], which leads to

$$h = v + Dw \qquad \textit{and} \qquad w \in H^1_{gR}(-r, 0; H_0^1(0,l)).$$

This completes the proof for the closedness of $\mathcal{A}$. ∎

LEMMA 2.2.3. *If the kernel g satisfies the conditions in section 1.1, then the operator $\mathcal{A}$ is dissipative.*

PROOF: For every $z \in \mathcal{D}(\mathcal{A})$,

$$\begin{aligned}
\langle \mathcal{A}z, z\rangle_Z &= \langle v, y\rangle_Y + \left\langle \epsilon D_x^2(y + \int_{-r}^0 g_\epsilon(s) w(s) ds) - \gamma \frac{\partial}{\partial x}\theta, v \right\rangle_V \\
&\qquad + \left\langle -\gamma \frac{\partial}{\partial x} v + kD_x^2 \theta, \theta \right\rangle_\Theta + \langle v + Dw, w\rangle_W \\
&= \langle v, y\rangle_Y - \epsilon \left\langle \frac{\partial}{\partial x} y, \frac{\partial}{\partial x} v \right\rangle_V - \langle w, v\rangle_W - \gamma \left\langle \frac{\partial}{\partial x}\theta, v \right\rangle_V + \gamma \left\langle v, \frac{\partial}{\partial x}\theta \right\rangle_\Theta \\
&\qquad - k \left\langle \frac{\partial}{\partial x}\theta, \frac{\partial}{\partial x}\theta \right\rangle_\Theta + \langle v, w\rangle_W + \langle Dw, w\rangle_W \\
&= -k \left\langle \frac{\partial}{\partial x}\theta, \frac{\partial}{\partial x}\theta \right\rangle_\Theta + \langle Dw, w\rangle_W \\
&\le \langle Dw, w\rangle_W \\
&\le \int_{-r}^0 \frac{g_\epsilon(s)}{2} \frac{\partial}{\partial s} \langle w, w\rangle_Y ds.
\end{aligned}$$

For $\delta > 0, R < r$, consider

$$
\begin{aligned}
I_{\delta,R} &= \int_{-R}^{-\delta} \frac{g_\epsilon(s)}{2} \frac{\partial}{\partial s} \langle w, w\rangle_Y ds \\
&= \frac{1}{2} g_\epsilon(-\delta)\langle w(-\delta), w(-\delta)\rangle_Y - \frac{1}{2} g_\epsilon(-R)\langle w(-R), w(-R)\rangle_Y - \frac{1}{2}\int_{-R}^{-\delta} g_\epsilon'(s)\langle w, w\rangle_Y ds \\
&\le \frac{1}{2} g_\epsilon(-\delta)\langle w(-\delta), w(-\delta)\rangle_Y .
\end{aligned}
$$

Since $w(-\delta) = w(0) - \int_{-\delta}^{0} Dw(s)ds = -\int_{-\delta}^{0} Dw(s)ds$, by Cauchy-Schwarz inequality,

$$\langle w(-\delta), w(-\delta)\rangle_Y \le \int_{-\delta}^{0} \frac{ds}{g_\epsilon(s)} \int_{-\delta}^{0} g_\epsilon(s)\langle Dw, Dw\rangle_Y ds.$$

Note that

$$g_\epsilon(-\delta)\int_{-\delta}^{0} \frac{ds}{g_\epsilon(s)} = \int_{-\delta}^{0} \frac{g_\epsilon(-\delta)}{g_\epsilon(s)} ds \le \delta.$$

It follows that

$$I_{\delta,R} \le \frac{\delta}{2}\int_{-\delta}^{0} g_\epsilon(s)\langle Dw, Dw\rangle_Y ds \le 0.$$

If $\delta \downarrow 0$ and $R \uparrow r$, then

$$\langle \mathcal{A}z, z\rangle_Z \le I_{\delta,R} \longrightarrow 0. \quad \blacksquare$$

LEMMA 2.2.4. *If the kernel g satisfies the conditions in section 1.1, then the range of $(\lambda_0 I - \mathcal{A})$ is all of Z for $\lambda_0 = 1$.*

PROOF: For $z = (y, v, \theta, w)^T \in Z$, consider the equation $(\lambda I - \mathcal{A})z = (\varphi, \psi, \xi, h)^T$, or equivalently,

$$\lambda y - v = \varphi, \tag{2.2.1}$$

$$\lambda v - \epsilon D_x^2 \left(y + \int_{-r}^{0} g_\epsilon(s) w(s) ds \right) + \gamma \frac{\partial}{\partial x} \theta = \psi, \tag{2.2.2}$$

$$\lambda \theta + \gamma \frac{\partial}{\partial x} v - k D_x^2 \theta = \xi \tag{2.2.3}$$

$$\lambda w - v - D w = h. \tag{2.2.4}$$

From (2.2.1),

$$v = \lambda y - \varphi. \tag{2.2.5}$$

Then from (2.2.4),

$$\begin{aligned} w(s) &= \int_s^0 e^{\lambda(s-\tau)} (v + h(\tau)) d\tau \\ &= \lambda \int_s^0 e^{\lambda(s-\tau)} y d\tau + \int_s^0 e^{\lambda(s-\tau)} (h(\tau) - \varphi) d\tau. \end{aligned} \tag{2.2.6}$$

From (2.2.3),

$$\begin{aligned} \theta &= (\lambda I - k D_x^2)^{-1} (\xi - \gamma \frac{\partial}{\partial x} v) \\ &= (\lambda I - k D_x^2)^{-1} (\xi - \gamma \frac{\partial}{\partial x} \varphi) + (\lambda I - k D_x^2)^{-1} (-\frac{\partial}{\partial x} y). \end{aligned} \tag{2.2.7}$$

Note that the inverse $(\lambda I - k D_x^2)^{-1}$ exists for all $\lambda > 0$ since D_x^2 is dissipative.

Substitute (2.2.5)-(2.2.7) into (2.2.2), we obtain the equation

$$\begin{aligned} &\left[\lambda^2 I - \epsilon D_x^2 \left(\frac{1}{\epsilon} - \int_{-r}^{0} g_\epsilon(s) e^{\lambda s} ds \right) - \lambda \gamma^2 \frac{\partial}{\partial x} (\lambda I - k D_x^2)^{-1} \frac{\partial}{\partial x} \right] y \\ &= \psi + \lambda \varphi - \epsilon D_x^2 \left[\int_{-r}^{0} g_\epsilon(s) \left(\int_s^0 e^{\lambda(s-\tau)} (\varphi - h(\tau) d\tau \right) ds \right] \\ &\qquad - \gamma \frac{\partial}{\partial x} (\lambda I - k D_x^2)^{-1} (\xi + \gamma \frac{\partial}{\partial x} \varphi), \end{aligned} \tag{2.2.8}$$

where we have used the identity $\int_s^0 \lambda e^{\lambda(s-\tau)}d\tau = 1 - e^{\lambda s}$. Note that the right side of (2.2.8) belongs to Y^*. Now, let $\Delta(\lambda)$ denote the expression

$$\Delta(\lambda) = \lambda^2 I - \epsilon D_x^2 \left(\frac{1}{\epsilon} - \int_{-r}^0 g_\epsilon(s)e^{\lambda s}ds\right) - \lambda\gamma^2 \frac{\partial}{\partial x}(\lambda I - kD_x^2)^{-1}\frac{\partial}{\partial x} \tag{2.2.9}$$

which belongs to $L(Y, Y^*)$. It follows that $\lambda \in \rho(\mathcal{A})$ if and only if $\Delta(\lambda)^{-1} \in L(Y^*, Y)$. Define the sesquilinear form μ on Y by

$$\begin{aligned} \mu(y_1, y_2) &= \langle \Delta(\lambda)y_1, y_2\rangle_{Y^*,Y} \\ &= \lambda^2 \langle y_1, y_2\rangle_V + \left(\frac{1}{\epsilon} - \int_{-r}^0 g_\epsilon(s)e^{\lambda s}ds\right)\langle y_1, y_2\rangle_Y \\ &\qquad \lambda\gamma^2\left\langle (\lambda I - kD_x^2)^{-1}\frac{\partial}{\partial x}y_1, \frac{\partial}{\partial x}y_2\right\rangle_V . \end{aligned} \tag{2.2.10}$$

Since $\|\cdot\|_V \leq c_1\|\cdot\|_Y$ for some constant $c_1 > 0$, and $(\lambda I - kD_x^2)^{-1}$ is a bounded operator on V for $\lambda > 0$, we have the estimate that for $\lambda = \alpha + i\beta, \alpha > 0$,

$$|\mu(y_1, y_2)| \leq C\|y_1\|_Y\|y_2\|_Y \tag{2.2.11}$$

where C is some positive constant. Moreover, we have

$$Re\mu(y, y) \geq (\alpha^2 - \beta^2)\|y\|_V + \frac{1}{\epsilon}\|y\|_Y, \tag{2.2.12}$$

where we have used

$$\begin{aligned} Re\left\langle (\lambda I - kD_x^2)^{-1}\frac{\partial}{\partial x}y_1, \frac{\partial}{\partial x}y_2\right\rangle_V &= Re\langle f, (\lambda I - kD_x^2)f\rangle_V \\ &= \alpha\langle f, f\rangle_V + k\left\langle \frac{\partial}{\partial x}f, \frac{\partial}{\partial x}f\right\rangle_V \\ &\geq 0. \end{aligned}$$

Thus it follows from [Tanabe, Lemma 3.6.1] that $\Delta^{-1}(\lambda) \in L(Y^*, Y)$ for $\lambda \geq 0$. In particular, it is true for $\lambda = 1$.

Now we are ready to state the main theorem of this section. This result follows immediately from the above three lemmas and the Lumer-Phillip theorem.

THEOREM 2.2.5. *If g satisfies the condition in section 1.1, then the operator $\mathcal{A}$ generates a C_0 semigroup $T(t)$ on the Hilbert space Z.*

REMARK: We want to point out that the well-posedness of (2.1.4)-(2.1.5) is proved in this section for both the finite delay ($r < \infty$) and the infinite delay ($r = \infty$).

2.3 A General Theorem on Well-Posedness.

In his Ph.D thesis, Miller[**15**] gave a general theorem on the well-posedness which can be applied to the viscoelastic system. However, that theorem does not apply to the thermoviscoelastic problem considered here. With slight modification, we can generalize Miller's result so that it can also be applied to the thermoviscoelastic systems.

Suppose that Y, V, Θ, and W are Hilbert spaces, and set $Z = Y \times V \times \Theta \times W$. Let

S be a subspace of V, and suppose we have the following linear operators:

$$A_0 : \mathcal{D}(A_0) \subseteq V \to V, \qquad A_1 : \mathcal{D}(A_1) \subseteq Y \to V,$$

$$G_1 : \mathcal{D}(G_1) \subseteq \Theta \to V, \qquad G_2 : \mathcal{D}(G_2) \subseteq V \to \Theta,$$

$$G_3 : \mathcal{D}(G_3) \subseteq \Theta \to \Theta, \qquad C_1 : \mathcal{D}(C_1) \subseteq W \to V,$$

$$D : \mathcal{D}(D) \subseteq W \to W, \qquad i : Y \to W, \qquad j : S \to Y.$$

Define A, C and G by $A = A_0A_1$, $C = A_0C_1$ and $G = A_0G_1$, and define F by

$$\mathcal{D}(F) = \left\{ \begin{pmatrix} y \\ \theta \\ w \end{pmatrix} \in Y \times \Theta \times W \;\middle|\; \begin{matrix} y \in \mathcal{D}(A_1), \theta \in \mathcal{D}(G_1), w \in \mathcal{D}(C_1), \\ A_1y + G_1\theta + C_1w \in \mathcal{D}(A_0) \end{matrix} \right\},$$

$$F\begin{pmatrix} y \\ \theta \\ w \end{pmatrix} = A_0(A_1y + G_1\theta + C_1w).$$

Define $\mathcal{A}$ by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} y \\ v \\ \theta \\ w \end{pmatrix} \in Z \;\middle|\; \begin{matrix} y \in S \cap \mathcal{D}(G_2), \theta \in \mathcal{D}(G_3), \\ w \in \mathcal{D}(D), \begin{pmatrix} y \\ \theta \\ w \end{pmatrix} \in \mathcal{D}(F) \end{matrix} \right\}, \tag{2.3.1}$$

$$\mathcal{A}\begin{pmatrix} y \\ v \\ \theta \\ w \end{pmatrix} = \begin{pmatrix} jy \\ F\begin{pmatrix} y \\ \theta \\ w \end{pmatrix} \\ G_2y + G_3\theta \\ ijy + Dw \end{pmatrix}. \tag{2.3.2}$$

Finally, suppose that j is injective and for $\lambda \in \rho(D) \cap \rho(G_3)$, define $L_\lambda : \mathcal{D}(L_\lambda) \subseteq Y \to V$ by

$$\mathcal{D}(L_\lambda) = \left\{ y \in \mathcal{R}(j) \mid (y, \lambda j^{-1}y, (\lambda I - G_3)^{-1}G_2\lambda j^{-1}y, (\lambda I - D)^{-1}i\lambda y)^T \in \mathcal{D}(\mathcal{A}) \right\},$$

$$L_\lambda y = \lambda^2 j^{-1}y - F\begin{pmatrix} y \\ (\lambda I - G_3)^{-1}G_2\lambda j^{-1}y \\ (\lambda I - D)^{-1}i\lambda y \end{pmatrix}.$$

We are now ready to state the main result of this chapter.

THEOREM 2.3.1. *Suppose that*

(1) i *and* j^{-1} *are continuous,*

(2) $\mathcal{D}(A)$ *is dense in* Y, $S \subseteq \mathcal{D}(G_2)$ *and* S *is dense in* V, $\mathcal{D}(G) \cap \mathcal{D}(G_3)$ *is dense in* Θ, *and* $\mathcal{D}(C) \cap \mathcal{D}(D)$ *is dense in* W,

(3) $j(S)$ *is closed in* Y,

(4) F, G_3 *and* D *are closed,*

(5) *for* $v \in S$, $\|G_2 v\|_\Theta \leq c\,\|jv\|_Y$ *for some* $c > 0$,

(6) *there exists* $\beta \in \mathbf{R}$ *such that* $\langle \mathcal{A}z, z\rangle_Z \leq \beta\,\langle z, z\rangle_Z$ *for all* $z \in \mathcal{D}(\mathcal{A})$,

(7) *there exists* $\lambda_0 > \beta$, $\lambda_0 \in \rho(D) \cap \rho(G_3)$, *such that* $\mathcal{R}(L_{\lambda_0})$ *is dense in* V, *and*

(8) $(\lambda_0 I - D)[\mathcal{D}(C) \cap \mathcal{D}(D)]$ *is dense in* W, *and* $(\lambda_0 I - D)[\mathcal{D}(G) \cap \mathcal{D}(G_3)]$ *is dense in* Θ.

Then $\mathcal{A}$ *is the infinitesimal generator of a* C_0 *semigroup* $S(t)$ *on* Z *satisfying* $\|S(t)\| \leq e^{\beta t}$.

PROOF: Set $\mathcal{D} = \mathcal{D}(A) \times S \times (\mathcal{D}(G) \cap \mathcal{D}(G_3)) \times (\mathcal{D}(C) \cap \mathcal{D}(D))$. Then $\mathcal{D} \subseteq \mathcal{D}(\mathcal{A})$ and $\mathcal{D}$ is dense in Z, so $\mathcal{D}(\mathcal{A})$ is dense in Z. For $n = 1, 2, \ldots$, let $\begin{pmatrix} y_n \\ v_n \\ \theta_n \\ w_n \end{pmatrix} \in \mathcal{D}(\mathcal{A})$, and suppose $\begin{pmatrix} y_n \\ v_n \\ \theta_n \\ w_n \end{pmatrix} \to \begin{pmatrix} y \\ v \\ \theta \\ w \end{pmatrix}$ and $\mathcal{A}\begin{pmatrix} y_n \\ v_n \\ \theta_n \\ w_n \end{pmatrix} = \begin{pmatrix} \varphi_n \\ \psi_n \\ \xi_n \\ h_n \end{pmatrix} \to \begin{pmatrix} \varphi \\ \psi \\ \xi \\ h \end{pmatrix}$ as $n \to \infty$. Then $v_n \in S$ and $jv_n = \varphi_n \to \varphi$. Since $j(S)$ is closed, there exists $\widehat{v} \in S$ such that $j\widehat{v} = \varphi$. But j^{-1} is bounded, so $\|\widehat{v} - v\| \leq \|j^{-1}\| \cdot \|\varphi - \varphi_n\| + \|v_n - v\| \to 0$ as $n \to \infty$. Therefore,

$v = \widehat{v}$; i.e., $v \in S$ and $jv = \varphi$. Now, $\begin{pmatrix} y_n \\ \theta_n \\ w_n \end{pmatrix} \in \mathcal{D}(F)$, $\begin{pmatrix} y_n \\ \theta_n \\ w_n \end{pmatrix} \to \begin{pmatrix} y \\ \theta \\ w \end{pmatrix}$, and $F\begin{pmatrix} y_n \\ \theta_n \\ w_n \end{pmatrix} = \psi_n \to \psi$ as $n \to \infty$. Since F is closed, $\begin{pmatrix} y \\ \theta \\ w \end{pmatrix} \in \mathcal{D}(F)$ and $F\begin{pmatrix} y \\ \theta \\ w \end{pmatrix} = \psi$. Since $jv_n \to jv$ and i is continuous, we have $ijv_n \to ijv$. We also have $ijv_n + Dw_n \to h$. Thus, $Dw_n \to h - ijv$. But D is closed, so $w \in \mathcal{D}(D)$ and $Dw = h - ijy$, which implies that $ijv + Dw = h$. Next, $\theta_n \to \theta$, $\theta_n \in \mathcal{D}(G_3)$ and $G_2v_n + G_3\theta_n = \xi_n \to \xi$. By (5), $\|G_2(v_n - v)\|_\Theta \le k\,\|j(v_n - v)\|_Y = k\,\|jv_n - jv\|_Y \to 0$, so $G_2v_n \to G_2v$, which implies that $G_3\theta_n \to \xi - G_2y$. Since G_3 is closed, $\theta \in \mathcal{D}(G_3)$ and $G_3\theta = \xi - G_2v$, or $\xi = G_2v + G_3\theta$. Therefore, $\mathcal{A}$ is closed. Finally, let $\begin{pmatrix} \varphi \\ \psi \\ \xi \\ h \end{pmatrix} \perp \mathcal{R}(\lambda_0 I - \mathcal{A})$; i.e.,

$$\langle \varphi, \lambda_0 y - jv\rangle_Y + \left\langle \psi, \lambda_0 v - F\begin{pmatrix} y \\ \theta \\ w \end{pmatrix}\right\rangle_V + \langle \xi, (\lambda_0 I - G_3)\theta - G_2 v\rangle_\Theta + \langle \theta, (\lambda_0 I - D)w - ijv\rangle_W = 0$$

for all $\begin{pmatrix} y \\ v \\ \theta \\ w \end{pmatrix} \in \mathcal{D}(\mathcal{A})$. Let $y \in \mathcal{D}(L_{\lambda_0})$. Then $\begin{pmatrix} y \\ \lambda_0 j^{-1} y \\ (\lambda_0 I - G_3)^{-1} G_2 \lambda_0 j^{-1} y \\ (\lambda_0 I - D)^{-1} i \lambda_0 y \end{pmatrix} \in \mathcal{D}(\mathcal{A})$, so

$$\left\langle \psi, \lambda_0^2 j^{-1} y - F\begin{pmatrix} y \\ (\lambda_0 I - G_3)^{-1} G_2 \lambda_0 j^{-1} y \\ (\lambda_0 I - D)^{-1} i \lambda_0 y \end{pmatrix}\right\rangle_V = \langle \psi, L_{\lambda_0} y\rangle_V = 0 \text{ for all } y \in \mathcal{D}(L_{\lambda_0}),$$

which implies $\psi = 0$ by (7). If $y = 0$, $v = 0$ and $\theta = 0$, then $\langle h, (\lambda_0 I - D)w\rangle_W = 0$ for all $w \in \mathcal{D}(C) \cap \mathcal{D}(D)$, and hence $h = 0$ by (8). Now for $y \in \mathcal{D}(A)$, $\begin{pmatrix} y \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{D}(\mathcal{A})$, so $\langle \varphi, \lambda_0 y\rangle_Y = 0$ for all $y \in \mathcal{D}(A)$. By (2) this implies that $\varphi = 0$. Finally, for

$\theta \in \mathcal{D}(G) \cap \mathcal{D}(G_3)$, $\begin{pmatrix} 0 \\ 0 \\ \theta \\ 0 \end{pmatrix} \in \mathcal{D}(\mathcal{A})$, so $\langle \xi, (\lambda_0 I - G_3)\theta \rangle_\Theta = 0$, which implies that $\xi = 0$ by (8). Therefore, $\mathcal{R}(\lambda_0 I - \mathcal{A})$ is dense in Z, and this completes the proof. ∎

We wish to apply this theorem to thermoviscoelastic systems (1.1.1) - (1.1.3). Using the state space formulation in section 2.1, we fit this problem into the frame of Theorem 2.3.1. Let $S \subseteq Y$ be given by $S = H_0^1(0,1)$. Define the following operators:

$$\mathcal{D}(A_0) = H^1(0,1), \qquad A_0 v = \epsilon v' \in V,$$

$$\mathcal{D}(A_1) = Y, \qquad A_1 y = y' \in V,$$

$$\mathcal{D}(G_1) = \Theta, \qquad G_1 \theta = -\frac{\gamma}{\epsilon}\theta \in V,$$

$$\mathcal{D}(G_2) = H^1(0,1), \qquad G_2 v = -\gamma v' \in \Theta,$$

$$\mathcal{D}(G_3) = H_0^1(0,1) \cap H^2(0,1), \qquad G_3 \theta = k\theta'' \in \Theta,$$

$$\mathcal{D}(C_1) = W, \qquad C_1 w = \int_{-r}^{0} g_\epsilon(s) w'(s) ds \in V,$$

$$[iy](s) \equiv y \in W, \qquad j : H_0^1 = S \to Y = H_0^1 \text{ is the identity operator.}$$

With the above definitions, we have

$$\mathcal{D}(A) = H_0^1 \cap H^2, \quad \mathcal{D}(C) = L_g^2(-r, 0; \mathcal{D}(A)), \quad \mathcal{D}(G) = H^1(0,1),$$

and therefore, the operator $\mathcal{A}$ defined by (2.1.2)-(2.1.3) is in the form (2.3.1)-(2.3.2). We now verify the conditions of Theorem 2.3.1.

(1) Clearly, i and j^{-1} are continuous.

(2) Clearly, $\mathcal{D}(A)$ is dense in Y, $S \subseteq \mathcal{D}(G_2)$ and S is dense in V, $\mathcal{D}(G) \cap \mathcal{D}(G_3) = \mathcal{D}(G_3)$ is dense in Θ, and $\mathcal{D}(C) \cap \mathcal{D}(D) = H_R^1(-r,0;\mathcal{D}(A))$ is dense in W.

(3) Since $j(S) = Y$, $j(S)$ is closed.

(4) We already know that D is closed. It is easy to see that G_3 is densely defined and dissipative, and for any $\varphi \in \Theta$, there exists $\theta \in \mathcal{D}(G_3)$ such that $(I - G_3)\theta = \varphi$ (see [**8**, p. 147]). Thus, G_3 is closed. Let $\begin{pmatrix} y_n \\ \theta_n \\ w_n \end{pmatrix} \in \mathcal{D}(F)$, $\begin{pmatrix} y_n \\ \theta_n \\ w_n \end{pmatrix} \to \begin{pmatrix} y \\ \theta \\ w \end{pmatrix}$, and $F\begin{pmatrix} y_n \\ \theta_n \\ w_n \end{pmatrix} = \psi_n \to \psi$. Observe that $F\begin{pmatrix} y_n \\ \theta_n \\ w_n \end{pmatrix} = \epsilon\frac{d}{dx}\left(y_n' - \frac{\gamma}{\epsilon}\theta_n + \int_{-r}^0 g_\epsilon(s)w_n'(s)ds\right)$. Now, $y_n \to y$ in Y implies that $y_n' \to y'$ in V, $\theta_n \to \theta$ in Θ implies that $\theta_n \to \theta$ in V, and $w_n \to w$ in W implies that $\int_{-r}^0 g_\epsilon(s)w_n'(s)ds \to \int_{-r}^0 g_\epsilon(s)w'(s)ds$ in V. Thus, $y_n' - \frac{\gamma}{\epsilon}\theta_n + \int_{-r}^0 g_\epsilon(s)w_n'(s)ds \to y' - \frac{\gamma}{\epsilon}\theta + \int_{-r}^0 g_\epsilon(s)w'(s)ds$ in V. Since A_0 is closed, F is closed.

(5) It is easy to check that $\|G_2v\|_\Theta^2 = \frac{\gamma^2}{\epsilon}\|jv\|_Y^2$. Set $c = \sqrt{\frac{\gamma^2}{\epsilon}} > 0$.

(6) For $z = (y,v,\theta,w)^T \in \mathcal{D}(\mathcal{A})$,

$$\begin{aligned}\langle \mathcal{A}z, z\rangle &= \epsilon\int_0^1 v'y' + \int_0^1 \epsilon\frac{d}{dx}\left(y' - \frac{\gamma}{\epsilon}\theta + \int_{-r}^0 g_\epsilon(s)w'(s)ds\right)v \\ &\quad + \int_0^1 (-\gamma y' + k\theta'')\theta + \int_{-r}^0 g_\epsilon(s)\int_0^1 v'w' + \langle Dw, w\rangle_W \\ &= k\langle G_3\theta,\theta\rangle_\Theta + \langle Dw, w\rangle_W \le 0\end{aligned}$$

since G_3 and D are dissipative.

(7) We will take $\lambda_0 = 1$. Let $y \in \mathcal{D}(L_1)$. If we set $w(s) = (1 - e^s)iy$, then

$(I-D)w = iy$, so $(I-D)^{-1}iy = (1-e^s)iy = (1-e^s)y$. Note that

$$\begin{aligned} L_1 y &= y - \epsilon \frac{d}{dx}\left(y' + \frac{\gamma^2}{\epsilon}(I-G_3)^{-1}y' + \int_{-r}^{0} g_\epsilon(s)(1-e^s)y\,ds\right) \\ &= y - \epsilon \frac{d}{dx}\left(\epsilon_1 y' + \frac{\gamma^2}{\epsilon}(I-G_3)^{-1}y'\right), \end{aligned}$$

where $\epsilon_\lambda = \frac{1}{\epsilon}\left[\epsilon - \int_{-r}^{0} g(s)(1-e^{\lambda s})ds\right] = \frac{1}{\epsilon}\left[\alpha + \int_{-r}^{0} g(s)e^{\lambda s}ds\right] > 0$ for $\lambda > 0$. Since $Y \subseteq V$, we can think of L_1 as being defined on a subspace of V. Observe that $\mathcal{D}(L_1) = \left\{v \in H_0^1(0,1) \mid \epsilon_1 v' + \frac{\gamma^2}{\epsilon}(I-G_3)^{-1}v' \in H^1\right\}$ is dense in V. Define the operators T_1 and T_2 as follows:

$$\mathcal{D}(T_1) = H^1(0,1), \qquad T_1 v = v',$$

$$\mathcal{D}(T_2) = H_0^1(0,1), \qquad T_2 v = v'.$$

Note that T_1 and T_2 are adjoint to each other. With this notation we can write L_1 as follows:

$$L_1 = I - \epsilon T_1 \left[\epsilon_1 I + \frac{\gamma^2}{\epsilon}(I-G_3)^{-1}\right] T_2.$$

Since $G_3^* = G_3$, we have $[(I-G_3)^{-1}]^* = (I-G_3)^{-1}$. Thus,

$$\begin{aligned} L_1^* &= I - \frac{\alpha}{\sigma} T_2^* \left[\epsilon_1 I + \frac{\gamma^2}{\epsilon}(I-G_3)^{-1}\right]^* T_1^* \\ &= I - \epsilon T_1 \left[\epsilon_1 I + \frac{\gamma^2}{\epsilon}(I-G_3)^{-1}\right] T_2 = L_1; \end{aligned}$$

that is, L_1 is self-adjoint. Now, for $v \in \mathcal{D}(L_1)$,

$$\begin{aligned}
\langle L_1 v, v\rangle_V &= \langle v, v\rangle_V - \epsilon \int_0^1 \frac{d}{dx}\left(\epsilon_1 v' + \frac{\gamma^2}{\epsilon}(I - G_3)^{-1} v'\right) v \\
&= \|v\|_V^2 + \epsilon \int_0^1 \left(\epsilon_1 v' + \frac{\gamma^2}{\epsilon}(I - G_3)^{-1} v'\right) v' \\
&= \|v\|_V^2 + \epsilon_1 \|jv\|_Y^2 + \gamma^2 \left\langle (I - G_3)^{-1} v', v'\right\rangle_V \geq \|v\|_V^2
\end{aligned}$$

since $I - G_3 \geq 0$ implies that $(I - G_3)^{-1} \geq 0$. Thus, L_1 is one-to-one. Hence by Theorem 13.11 in [**21**], $\mathcal{R}(L_1)$ is dense in Y.

(8) Since $\mathcal{D}(G_3) \subseteq \mathcal{D}(G)$, we have $(I - G_3)[\mathcal{D}(G) \cap \mathcal{D}(G_3)] = (I - G_3)\mathcal{D}(G_3) = \Theta$ from (4). Next, for $h \in \mathcal{D}(C)$, if we set $w(s) = e^s \int_s^0 e^{-\sigma} h(\sigma) d\sigma$, then $w \in H_R^1(-r, 0; \mathcal{D}(A)) = \mathcal{D}(C) \cap \mathcal{D}(D)$ and $(I - D)w = h$. Thus, $\mathcal{D}(C) \subseteq (I - D)[\mathcal{D}(C) \cap \mathcal{D}(D)]$, and $\mathcal{D}(C)$ is dense in W.

Since (1) - (8) hold, $\mathcal{A}$ generates a C_0 semigroup on Z.

REMARK: This theorem can also be applied to the case of boundary condition (1.1.4). The only thing that needs to be changed in the above proof is $\mathcal{D}(G_3)$. Now we define the operator G_3 by

$$\mathcal{D}(G_3) = \left\{\theta \mid \theta \in H^2(0, l), \frac{\partial}{\partial x}\theta(t, 0) = \frac{\partial}{\partial x}\theta(t, 1)\right\}$$

$$G_3(\theta) = \frac{\partial^2}{\partial x^2}\theta.$$

It is easy to see that G_3 is self-adjoint and dissipative. Therefore, the proof of the well-posedness of the system (1.1.1)-(1.1.2) with the boundary condition (1.1.4) goes through directly by following the above proof.

2.4 Stability.

In this section, we study the asymptotic behavior of the thermoviscoelastic system (1.1.1)-(1.1.2) with the boundary condition (1.1.3) or (1.1.4). Since the existence of a genuine memory and the heat dissipation induce a damping mechanism, asymptotic stability is to be expected.

A general discussion of stability for infinite dimensional system can be found in[**18**]. Suppose that we have a linear system

$$\frac{d}{dt}z(t) = \mathcal{L}z(t), \qquad z(0) = z_0 \in Z \tag{2.4.1}$$

where Z is a Hilbert space and $\mathcal{L}$ is the infinitesimal generator of a C_0-semigroup $S(t), t > 0$.

DEFINITION 2.4.1. *System (2.4.1) is said to be asymptotically stable if, for every initial condition $z_0 \in Z$, the corresponding mild solution $z(t) \to 0$ as $t \to \infty$.*

DEFINITION 2.4.2. *System (2.4.1) is said to be exponentially stable if, for every initial condition $z_0 \in Z$, the corresponding mild solution $z(t) \to 0$ exponentially as $t \to \infty$.*

The next theorem is proved by S. Hansen[**to appear**].

THEOREM 2.4.3. *The thermoelastic system (1.1.5)-(1.1.6) (i.e., $g \equiv 0$ in (1.1.1)-(1.1.2)), with the boundary condition (1.1.4), is exponentially stable, and the real part of its eigenvalues corresponding to (1.1.1) tends to $-\frac{\gamma^2}{2}$ asymptotically.*

The coupling coefficient γ is small for existing materials. For example, $\gamma \simeq 0.01$ for aluminum. Therefore, the mild solution of above thermoelastic system tends to zero very slowly. Physically, this means that the pure thermal damping is weak, although it leads to an exponential decay rate of the vibration in the elastic bar. The system is energy conservative, and all mechnical energy will be transfered to heat energy eventually. At the equilibrium, $y = \frac{\partial}{\partial t}y = 0$, $\theta = \theta_0 + C$, where $C > 0$ is a constant. If the boundary condition (1.1.3) is considered, the energy will be absorbed through the end of the bar. Thus a better decay rate is expected. However, this has not been confirmed yet.

THEOREM 2.4.4. *The viscoelastic system ($\gamma = 0, r = \infty$ in (1.1.1)), with the boundary condition (1.1.3) is asymptotically stable if the kernel g satisfies*

(1) $g \leq 0 \quad$ *on* $\quad (-\infty, 0)$,

(2) $g' \leq 0 \quad$ *on* $\quad (-\infty, 0)$,

(3) $\alpha + \int_{-\infty}^{0} g(s)ds > 0$.

PROOF: See [**8**].

Hannsgen and Wheeler noted in [**12**] that exponential stability can be achieved under the hypotheses of above theorem and the additional condition that g is concave.

Now, we state our main theorem for this section.

THEOREM 2.4.5. *The thermoviscoelastic system (1.1.1)-(1.1.2) with boundary con-*

dition (1.1.3) or (1.1.4) is asymptotically stable if in addition to the conditions in section 1.1, the kernel g also satisfies

(1) $\lim_{s\to\infty} s^2 g(s) \leq \infty$;

(2) $g'(s) \neq 0$ *in* $-\sigma < s < 0$ *for some* $\sigma > 0$.

We will prove this theorem later. First, assume the initial displacement and temperature $(y(s), \theta(s)) \in C^3([-r,0];Y)\times C^3([-r,0];\Theta)$. Then by [**16**, Theorem 4.3], the mild solution (y,θ) of system (1.1.1)-(1.1.2) belongs to $C^3([0,T];Y) \times C^3([0,T];V)$ for any $T > 0$.

Define the functions

$$F_l(t) = \int_0^1 \left\{ \epsilon\left(\frac{\partial^{l+1}}{\partial t^l \partial x} y\right)^2 + \left(\frac{\partial^l}{\partial t^l}\theta\right)^2 + \left(\frac{\partial^{l+1}}{\partial t^{l+1}} y\right)^2 \right\} dx \tag{2.4.2}$$
$$+ \int_{-r}^0 \int_0^1 g_\epsilon(s) \left[\frac{\partial^{l+1}}{\partial t^l \partial x} y(t) - \frac{\partial^{l+1}}{\partial t^l \partial x} y(t+s)\right]^2 dx ds, \qquad l = 0,1,2.$$

Note that $F_0(t) = \left\|(y, \frac{d}{dt}y, \theta, y(t) - y(t+s))^T\right\|_Z^2$.

LEMMA 2.4.6. *The functions $F_l(t), l = 0,1,2$, are nonnegative and nonincreasing.*

PROOF:: The nonnegativeness of $F_l(t), l = 0,1,2$ is clear. Define

$$z^{(l)} = \left(\frac{\partial^l}{\partial t^l} y, \frac{\partial^{l+1}}{\partial t^{l+1}} y, \frac{\partial^l}{\partial t^l}\theta, \frac{\partial^l}{\partial t^l} y(t) - \frac{\partial^l}{\partial t^l} y(t+s)\right)^T$$

for $l = 0,1,2$. Note that they satisfy (2.1.1). Therefore, by Lemma 2.2.3, we obtain

$$\dot{F}_l(t) = 2\left\langle \dot{z}^{(l)}, z^{(l)}\right\rangle_Z = 2\left\langle \mathcal{A} z^{(l)}, z^{(l)}\right\rangle_Z \leq 0, \qquad l = 0,1,2.$$

PROOF OF THEOREM 2.4.5: First, we consider the boundary condition (1.1.3). We prove that $\lim_{t\to\infty} \dot{F}_0(t)$ exists and equals zero. By a straightforward computation, we have

$$\ddot{F}_0(t) = 2\int_0^1 \left\{ \epsilon \left[\frac{\partial}{\partial t}(\frac{\partial}{\partial x}y)\right]^2 + \left(\frac{\partial^2}{\partial t^2}y\right)^2 + \left(\frac{\partial}{\partial t}\theta\right)^2 \right\} dx$$
$$2\int_{-r}^0 \int_0^1 g_\epsilon(s) \left[\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}y(t) - \frac{\partial}{\partial x}y(t+s)\right)\right]^2 dxds$$
$$2\int_0^1 \left\{ \epsilon \frac{\partial^2}{\partial t^2}(\frac{\partial}{\partial x}y)\frac{\partial}{\partial x}y + (\frac{\partial^3}{\partial t^3}y)\frac{\partial}{\partial t}y + (\frac{\partial^2}{\partial t^2}\theta)\theta \right\} dx$$
$$+2\int_{-r}^0 \int_0^1 g_\epsilon(s)\frac{\partial^2}{\partial t^2}\left[\frac{\partial}{\partial x}y(t) - \frac{\partial}{\partial x}y(t+s)\right]\left[\frac{\partial}{\partial x}y(t) - \frac{\partial}{\partial x}y(t+s)\right] dxds.$$

Applying Hölder's inequality to the last two integrals yields

$$|\ddot{F}_0(t)| \le 2\left[F_1(t) + F_0(t) + F_2(t)\right] \qquad t \ge 0.$$

By Lemma 2.4.6, it follows

$$\ddot{F}_0(t) \le 2\left[F_1(0) + F_0(0) + F_2(0)\right] \le C \tag{2.4.3}$$

for some $C > 0$. Then we have

$$\dot{F}_0(t)\ddot{F}_0(t) \le |\dot{F}_0(t)||\ddot{F}_0(t)| \le -C\dot{F}_0(t)$$

and

$$\frac{d}{dt}\left[\frac{1}{2}\left(\dot{F}_0(t)\right)^2 + CF_0(t)\right] = \dot{F}_0(t)\ddot{F}_0(t) + C\dot{F}_0(t) \le 0, \tag{2.4.4}$$
$$\frac{1}{2}\left(\dot{F}_0(t)\right)^2 + CF_0(t) \ge 0. \tag{2.4.5}$$

Inequalities (2.4.4)-(2.4.5) imply the existence of $\lim_{t\to\infty} \frac{1}{2}\left(\dot{F}_0(t)\right)^2 + CF_0(t)$. Similarly, $\lim_{t\to\infty} F_0(t)$ exists. Thus we obtain the existence of $\lim_{t\to\infty} \dot{F}_0(t)$. The last limit must be 0 since $F_0(t) \geq 0$ and $\dot{F}_0(t) \leq 0$.

In the proof of Lemma 2.2.3, we can integrate by parts to obtain

$$\langle \mathcal{A}z, z\rangle_Z = -\int_{-r}^{0}\int_{0}^{1} g'_\epsilon(s)\left[\frac{\partial}{\partial x}y(t) - \frac{\partial}{\partial x}y(t+s)\right]^2 dxds - k\int_0^1 (\frac{\partial}{\partial x}\theta)^2 dx \quad (2.4.6)$$

if the integrals exist. This is guaranteed by our additional condition (1) on g. Note that both terms on the right side of (2.4.6) are nonpositive. Thus $\lim_{t\to\infty} \dot{F}_0(t) = 0$ implies

$$\lim_{t\to\infty}\int_{-r}^{0}\int_{0}^{1} g'_\epsilon(s)\left[\frac{\partial}{\partial x}y(t) - \frac{\partial}{\partial x}y(t+s)\right]^2 dxds = 0 \quad (2.4.7)$$

and

$$\lim_{t\to\infty}\int_0^1 (\frac{\partial}{\partial x}\theta)^2 dx = 0 \quad (2.4.8)$$

By the Poincaré's inequality, there is a $\beta > 0$ such that

$$\beta\int_0^1 (\frac{\partial}{\partial x}\theta)^2 dx \geq \int_0^1 \theta^2 dx.$$

Thus (2.4.8) yields

$$\|\theta\|_\Theta \longrightarrow 0 \qquad \text{as} \qquad t \to \infty. \quad (2.4.9)$$

To prove that $\|y\|_Y \to 0$, and $\left\|\frac{\partial}{\partial t}y\right\|_V \to 0$ as $t \to \infty$, we use (2.4.7) and the condition (2) on g, and follow [**16**] exactly. By a denseness argument, the proof is completed.

For the boundary condition (1.1.4), the limit in (2.4.8) implies $\|\theta\|_\Theta \to \theta_0 + C$ for some $C > 0$.

Chapter III Approximation

In this chapter we present an approximation scheme for the thermoviscoelastic system. We prove the strong convergence of the approximating semigroups $T^N(t)$. We also prove the strong convergence of the approximating adjoint semigroups $T^{N*}(t)$ under the restriction that the kernel $g(s)$ is a L_2 function.

3.1 An Approximation Scheme.

In order to approximate the system (1.1.1)-(1.1.3) by a sequence of ordinary differential equations which can be solved numerically, we need to discretize the spatial variable x and the delay variable s. This suggests a two-stage approximation, see [**5**].

First, for each positive integer N, let Y^N, V^N, Θ^N be the finite dimensional subspaces of Y, V, Θ, respectively. We shall use spaces of spline functions. Let $W^N = L_g^2(-r, 0; Y^N) \subset W$, and set $Z^N = Y^N \times V^N \times \Theta^N \times W^N$ with the norm induced from the $Z-$norm. We define $\mathcal{A}^N : \mathcal{D}(\mathcal{A}^N) \subseteq Z^N \to Z^N$ by

$$\mathcal{D}(\mathcal{A}^N) \equiv Y^N \times V^N \times \Theta^N \times H^1_{gR}(-r, 0; Y^N), \tag{3.1.1}$$

$$\mathcal{A}^N \begin{pmatrix} y^N \\ v^N \\ \theta^N \\ w^N \end{pmatrix} \equiv \begin{pmatrix} v^N \\ \epsilon A_1 [y^N + \int_{-r}^0 g_\epsilon(s) w^N(s) ds] - \gamma D_1^N \theta^N \\ -\gamma D_2 v^N + k A_2 \theta^N \\ v^N + D_S w^N \end{pmatrix}, \tag{3.1.2}$$

where $A_1 : Y \to V$, $\quad A_2 : \Theta \to \Theta$, $\quad D_1 : \Theta \to V$, and $D_2 : V \to \Theta$ are defined by

$$\langle A_1 y, v\rangle_V = -\left\langle \frac{\partial}{\partial x} y, \frac{\partial}{\partial x} v \right\rangle_V \qquad \text{for all} \quad v \in V \tag{3.1.3}$$

$$\langle A_2 \theta, \theta_1\rangle_\Theta = -\left\langle \frac{\partial}{\partial x} \theta, \frac{\partial}{\partial x} \theta_1 \right\rangle_\Theta \qquad \text{for all} \quad \theta_1 \in \Theta. \tag{3.1.4}$$

$$\langle D_1 \theta, v\rangle_V = -\left\langle \theta, \frac{\partial}{\partial x} v \right\rangle_V \qquad \text{for all} \quad v \in V \tag{3.1.5}$$

$$\langle D_2 v, \theta\rangle_\Theta = -\left\langle v, \frac{\partial}{\partial x} \theta \right\rangle_\Theta \qquad \text{for all} \quad \theta \in \Theta. \tag{3.1.6}$$

Thus, we have the Cauchy problem for a differential-delay equation

$$\dot{z}^N(t) = \mathcal{A}^N z^N(t). \tag{3.1.7}$$

For each positive integer M partition the interval $[-r, 0]$ into subintervals $[t_j^M, t_{j-1}^M]$, $j = 1, 2, \ldots, M$, where

$$-r = t_M^M < t_{M-1}^M < \cdots < t_1^M < t_0^M = 0.$$

There are several approximation schemes for differential-delay equations in the literature. Here we will use the averaging scheme, which approximates the history function by a finite number of piecewise constant functions. The uniform-mesh AVE scheme, i.e., $t_j^M = -\frac{jr}{M}$ for $j = 1, 2, \ldots, M$, does not show a fast convergence. Recently, Ito and Fabiano[**14**] suggested a new averaging scheme using nonuniform mesh, which is numerically superior to the old one. Later, Miller[**15**] applied this to the approximation of the viscoelastic system, and proved the strong convergence of the approximating semigroups. We will show that it also works in our thermoviscoelastic case.

Set $\alpha_j^M \equiv t_{j-1}^M - t_j^M$ for $j = 1, 2, \ldots, M$, let E_j^M denote the characteristic function of $[t_j^M, t_{j-1}^M)$ for $j = 2, \ldots, M$, and let E_1^M denote the characteristic function of $[t_1^M, 0]$. Let $B_i^M(t)$, $i = 0, 1, \ldots, M$ be the usual linear spline functions satisfying $B_i^M(t_j^M) = \delta_{ij}$. Define the finite dimensional subspaces $W^{N,M}$ and $\widetilde{W}^{N,M}$ of W by

$$\begin{aligned} W^{N,M} &\equiv \left\{ w \in W \;\middle|\; w = \sum_{i=1}^{M} a_i^M E_i^M,\ a_i^M \in X^N \right\}, \\ \widetilde{W}^{N,M} &\equiv \left\{ w \in W \;\middle|\; w = \sum_{i=1}^{M} b_i^M B_i^M,\ b_i^M \in X^N \right\}. \end{aligned} \tag{3.1.8}$$

Define the operator $\widetilde{D}^{N,M} : \widetilde{W}^{N,M} \to W^{N,M}$ by

$$\widetilde{D}^{N,M} w^{N,M} \equiv \sum_{i=1}^{M} \frac{1}{\alpha_i^M} \left(b_{i-1}^M - b_i^M \right) E_i^M, \tag{3.1.9}$$

where $w^{N,M} = \sum_{i=1}^{M} b_i^M B_i^M$ and $b_0^M = 0$. Consider the isomorphism $\imath^{N,M} : \widetilde{W}^{N,M} \to W^{N,M}$ defined by

$$\imath^{N,M} w^{N,M} \equiv \sum_{i=1}^{M} b_i^M E_i^M.$$

Now define $D^{N,M} : W^{N,M} \to W^{N,M}$ by $D^{N,M} \equiv \widetilde{D}^{N,M} \left(\imath^{N,M} \right)^{-1}$. To complete the approximation, set $Z^{N,M} = Y^N \times V^N \times \Theta^N \times W^{N,M}$ with the norm induced from the Z norm, and for $z^{N,M} = \left(y^N, v^N, \theta^N, w^{N,M} \right)^T \in Z^{N,M}$, define

$$\mathcal{A}^{N,M} z^{N,M} \equiv \begin{pmatrix} v^N \\ \epsilon A_1 [y^N + \int_{-r}^{0} g_\epsilon(s) w^{N,M}(s) ds] - \gamma D_1 \theta^N \\ -\gamma D_2 v^N + k A_2 \theta^N \\ v^N + D^{N,M} w^{N,M} \end{pmatrix}. \tag{3.1.10}$$

If $w^{N,M} = \sum_{i=1}^{M} w_i^M E_i^M$, then

$$\mathcal{A}^{N,M} z^{N,M} = \begin{pmatrix} v^N \\ \epsilon A_1(y^N + \sum_{i=1}^{M}(g_\epsilon)_i^M w_i^M - \gamma D_1 \theta^N \\ -\gamma D_2 v^N + k A_2 \theta^N \\ v^N + \sum_{i=1}^{M} \frac{1}{\alpha_i^M}(w_{i-1}^M - w_i^M) E_i^M \end{pmatrix}, \tag{3.1.11}$$

where $(g_\epsilon)_i^M \equiv \int_{t_i^M}^{t_{i-1}^M} g_\epsilon(s)ds$.

Note that

$$\dot{z}^{N,M}(t) = \mathcal{A}^{N,M} z^{N,M}(t) \tag{3.1.12}$$

is a linear system of ODE and $\{\mathcal{A}^{N,M}\}$ forms an approximation sequence for $\mathcal{A}$.

3.2 Convergence of the semigroup.

In this section we show that the operator $\mathcal{A}^{N,M}$ generates a C_0- semigroup $T^{N,M}(t)$, which converges to $T(t)$. Consequently, the solution of (3.1.10) converges to the solution of (2.1.1) with the appropriate initial data. The main idea in our proof comes from [**14**],[**15**].

First, we state the following version of the Trotter-Kato theorem[**17**]:

THEOREM 3.2.1. *Let $A \in G(M, \beta)$ be the infinitesimal generator of a C_0 semigroup $T(t)$ on a Hilbert space Z. For $n = 1, 2, \ldots$, let Z^n be the finite dimensional subspace of Z such that $P^n \xrightarrow{s} I$ as $n \to \infty$ where P^n is the orthogonal projection of Z onto*

Z^n. *Suppose*

H1. $A^n \in G(M, \beta)$ *is the infinitesimal generator of a* C_0 *semigroup* $T^n(t)$ *on* Z^n *for* $n = 1, 2, \ldots$.

H2. *For all* $z \in Z$, *there is a* λ *with* $Re\lambda > \beta$ *such that* $(\lambda I - A^n)^{-1} P^n z \to (\lambda I - A)^{-1} z$ *as* $n \to \infty$.

Then for all $z \in Z$, $T^n(t) P^n z \to T(t) z$ *as* $n \to \infty$, *and the convergence is uniform on bounded t-intervals.*

We also use Miller's assumption[**15**] on the partitions of $[-r, 0]$:

A) For each positive integer M let $P^M = \{t_j^M | j = 0, 1, \ldots, M\}$ be a partition of $[-r, 0]$ satisfying (3.1.3), and set $\Lambda^M \equiv \{1, 2, \ldots, M, \}$. Then there exist positive constants ϵ_1, ϵ_2 and C independent of M such that $\Lambda^M = \Lambda_1^M \cup \Lambda_2^M$, where

$$\Lambda_1^M = \left\{ j \in \Lambda^M \;\middle|\; \alpha_j^M \leq r M^{-(1+\epsilon_1)/2} \right\}.$$

If $j \in \Lambda_2^M$, then $(g_\alpha)_j^M \leq \dfrac{C}{M}$, and Λ_2^M contains at most $M^{1-\epsilon_2}$ elements of Λ^M. Furthermore, $\alpha_{j-1}^M (g_\alpha)_j^M \leq (g_\alpha)_{j-1}^M \alpha_j^M$ for $j = 2, 3, \ldots, M$, and if $j \in \Lambda_1^M$, then $1, 2, \ldots, j-1 \in \Lambda_1^M$.

For the remainder of this section we will assume that we have a partition which satisfies A). The next lemma shows that our approximation sequence $\mathcal{A}^{N,M}$ satisfies the condition **H1**.

LEMMA 3.2.2. *The operators* $\mathcal{A}^{N,M}$ *belongs to* $G(1, 0)$ *for all* N, M.

PROOF: It is sufficient to show that $\mathcal{A}^{N,M}$ is dissipative in $Z^{N,M}$. Let $z^{N,M} = (y^N, v^N, \theta^N, w^{N,M})^T$ with $w^{N,M} = \sum_{i=1}^{M} w_i^M E_i^M$. Then

$$\begin{aligned}
\langle \mathcal{A}^{N,M} z^{N,M}, z^{N,M} \rangle_Z &= \langle v^N, y^N \rangle_Y + \left\langle \epsilon A_1(y^N + \int_{-r}^{0} g_\epsilon(s) w^{N,M} ds), v^N \right\rangle_V \\
&\quad - \gamma \langle D_1 \theta^N, v^N \rangle_V + \langle -\gamma D_2 v^N + k A_2 \theta^N, \theta^N \rangle_\Theta + \langle v^N + D^{N,M} w^{N,M}, w^{N,M} \rangle_W \\
&= \langle v^N, y^N \rangle_Y - \epsilon \left\langle \frac{\partial}{\partial x} y^N, \frac{\partial}{\partial x} v^N \right\rangle_V - \langle w^{N,M}, v^N \rangle_W - \gamma \left\langle \frac{\partial}{\partial x} \theta^N, v^N \right\rangle_V \\
&+ \gamma \left\langle v^N, \frac{\partial}{\partial x} \theta^N \right\rangle_\Theta - k \left\langle \frac{\partial}{\partial x} \theta^N, \frac{\partial}{\partial x} \theta^N \right\rangle_\Theta + \langle v^N, w^{N,M} \rangle_W + \langle D^{N,M} w^{N,M}, w^{N,M} \rangle_W \\
&= -k \left\langle \frac{\partial}{\partial x} \theta^N, \frac{\partial}{\partial x} \theta^N \right\rangle_\Theta + \langle D^{N,M} w^{N,M}, w^{N,M} \rangle_W \\
&\leq \langle D^{N,M} w^{N,M}, w^{N,M} \rangle_W \\
&\leq \sum_{i=1}^{M} \frac{1}{\alpha_i^M} (g_\epsilon)_i^M \langle w_{i-1}^M - w_i^M, w_i^M \rangle_Y \\
&\leq \sum_{i=1}^{M} \frac{(g_\epsilon)_i^M}{\alpha_i^M} \left[\|w_{i-1}^M\|_Y \cdot \|w_i^M\|_Y - \|w_i^M\|_Y^2 \right] \\
&\leq \frac{1}{2} \sum_{i=1}^{M} \frac{(g_\epsilon)_i^M}{\alpha_i^M} \left[\|w_{i-1}^M\|_Y^2 - \|w_i^M\|_Y^2 \right] \\
&= \frac{1}{2} \left[\sum_{i=1}^{M-1} \|w_i^M\|_Y^2 \left(\frac{(g_\epsilon)_{i+1}^M}{\alpha_{i+1}^M} - \frac{(g_\epsilon)_i^M}{\alpha_i^M} \right) - \|w_M^M\|_Y^2 \frac{(g_\epsilon)_M^M}{\alpha_M^M} \right] \leq 0
\end{aligned}$$

where we used the Cauchy-Schwarz Inequality and the inequality $2ab \leq a^2 + b^2$, and (from A) the fact that $(g_\alpha)_{i+1}^M / \alpha_{i+1}^M \leq (g_\alpha)_i^M / \alpha_i^M$ for $i = 1, 2, \ldots, M-1$. ∎

Let P_Y^N, P_V^N, P_Θ^N be the orthogonal projections of Y, V, Θ, respectively. Assume that each of them converges to the corresponding identity operator. Let $P_W^{N,M}$ be the orthogonal projection of W onto $W^{N,M}$. By Lemma 3.1.5 in [**15**], $P_W^{N,M} h \to h$

for all $h \in W$ as $N, M \to \infty$. If $P_Z^{N,M}$ denotes the orthogonal projection of Z onto $Z^{N,M}$, then for $z = (y, v, \theta, w)^T \in Z$, $P_Z^{N,M} z = \left(P_Y^N y, P_V^N v, P_\Theta^N \theta, P_W^{N,M} w\right)^T$. Thus, $P_Z^{N,M} \xrightarrow{s} I$ as $N, M \to \infty$.

Now, for $z^{N,M} = \left(y^N, v^N, \theta^N, w^{N,M}\right)^T \in Z^{N,M}$ and $\lambda = 1$, consider the equation $\left(\lambda I - \mathcal{A}^{N,M}\right) z^{N,M} = \left(\varphi^N, \psi^N, \xi^N, h^{N,M}\right)^T$, Or equivalently,

$$y^N - v^N = \varphi^N, \tag{3.2.1}$$

$$v^N - \epsilon A_1(y^N + \int_{-r}^{0} (g_\epsilon(s) w_i^{N,M} ds) + \gamma D_1 \theta^N = \psi^N, \tag{3.2.2}$$

$$\theta^N + \gamma D_2 v^N - k A_2 \theta^N = \xi^N \tag{3.2.3}$$

$$w^{N,M} - v^N - D^{N,M} w^{N,M} = h^{N,M}. \tag{3.2.4}$$

From (3.2.4), $w^{N,M} = (I - D^{N,M})^{-1}(h^{N,M} + v^N)$. The inverse exists because the operator $D^{N,M}$ is dissipative. From (3.2.3), $\theta^N = (I - kA_2)^{-1}(\xi^N - \gamma D_2 v^N)$. The existence of the inverse is also guaranteed by the dissipativeness of the operator A_2. From (3.2.1), $v^N = y^N - \varphi^N$. Substitute $v^N, \theta^N, w^{N,M}$ into (3.2.2), we obtain

$$\begin{aligned} &\left[I - \epsilon A_1 \left(1 + \int_{-r}^{0} g_\epsilon(s)(I - D^{N,M})^{-1} ds\right) - \gamma^2 D_1 (I - kA_2)^{-1} D_2\right] y^N \\ &= \psi^N + \varphi^N - \epsilon A_1 \left(\int_{-r}^{0} g_\epsilon(s)(I - D^{N,M})^{-1}(\varphi^N - h^{N,M}(s)) ds\right) \\ &\quad - \gamma D_1 (I - kA_2)^{-1}(\xi^N + \gamma D_2 \varphi^N). \end{aligned} \tag{3.2.5}$$

On the other hand, we compute an explicit expression for $\int_{-r}^{0} g_\epsilon(s)(I - D^{N,M})^{-1} ds$.

Let $w^{N,M} = \sum_{i=1}^{M} w_i^M E_i^M$ and $h^{N,M} = \sum_{i=1}^{M} h_i^M E_i^M$. Then (3.2.2) and (3.2.4) become

$$v^N - \epsilon A_1 \left(y^N + \sum_{i=1} (g_\epsilon)_i^M w_i^M \right) + \gamma \frac{\partial}{\partial x} \theta^N = \psi^N, \tag{3.2.6}$$

$$w_i^M - v^N - \frac{1}{\alpha_i^M}(w_{i-1}^M - w_i^M) = h_i^M \quad \text{for} \quad i = 1, 2, \ldots, M. \tag{3.2.7}$$

From (3.2.7)

$$\left(1 + \frac{1}{\alpha_i^M}\right) w_i^M = \frac{1}{\alpha_i^M} w_{i-1}^M + v^N + h_i^M,$$

or

$$w_i^M = \left(1 + \alpha_i^M\right)^{-1} \left[w_{i-1}^M + \alpha_i^M \left(v^N + h_i^M\right)\right] \text{ for } i = 1, 2, \ldots, M,$$

where $w_0^M = 0$. By induction, $w_i^M = \sum_{k=1}^{i} \left[\prod_{l=k}^{i} \left(1 + \alpha_l^M\right)^{-1}\right] \alpha_k^M \left(v^N + h_k^M\right)$. Substituting into (3.2.6) and comparing with (3.2.5), we get

$$\int_{-r}^{0} g_\epsilon(s)(I - D^{N,M})^{-1} ds = 1 + \sum_{i=1}^{M} (g_\epsilon)_i^M \left(1 - \prod_{k=1}^{i} \left(1 + \alpha_k^M\right)^{-1}\right).$$

Here we used $\sum_{k=1}^{i} \left[\prod_{l=k}^{i} \left(1 + \alpha_l^M\right)^{-1}\right] \alpha_k^M = 1 - \prod_{k=1}^{i} \left(1 + \alpha_k^M\right)^{-1}$. Then, since $\sum_{i=1}^{M} (g_\epsilon)_i^M = -\frac{1}{\epsilon} \int_{-r}^{0} g(s) ds = \frac{1}{\epsilon} - 1$, we define

$$\Delta^{N,M}(1) = I - \epsilon A_1 \left[\frac{1}{\epsilon} - \int_{-r}^{0} g(s) e^M(1, s) ds\right] - \gamma^2 D_1 (I - k A_2)^{-1} D_2, \tag{3.2.8}$$

where

$$e^M(1, s) \equiv \sum_{i=1}^{M} \left(\prod_{k=1}^{i} \left(1 + \alpha_k^M\right)^{-1}\right) E_i^M(s). \tag{3.2.9}$$

Now, (3.2.5) becomes

$$\Delta^{N,M}(1)y^N = \psi^N + \varphi^N - \epsilon A_1\left(\int_{-r}^{0} g_\epsilon(s)(I - D^{N,M})^{-1}(\varphi^N - h^{N,M})ds\right)$$
$$- \gamma^2 D_1(I - kA_2)^{-1}(\xi^N + D_2\varphi^N) \tag{3.2.10}$$

The next two lemmas can be found in[**15**].

LEMMA 3.2.3. *For $\lambda > 0$, $\left(\lambda I - D^{N,M}\right)^{-1} P_W^{N,M} h \to (\lambda I - D)^{-1}h$ for all $h \in W$.*

LEMMA 3.2.4. *For $\lambda > 0$, $\int_{-r}^{0} g_\epsilon(s)|e^{\lambda s} - e^M(\lambda, s)|ds \to 0$ as $M \to \infty$.*

LEMMA 3.2.5. *For all $\psi \in V$, $\left\|(I - kD_x^2)^{-1}\psi - (I - kA_2)^{-1}\widehat{\psi}\right\|_V \longrightarrow 0$ as $N \to 0$, where $\widehat{\psi} = P_V^N \psi$.*

PROOF: Let $(I - kD_x^2)^{-1}\psi = v, \quad (I - kA_2)^{-1}\widehat{\psi} = v^N$. Then

$$\langle v, r\rangle_V - k\langle D_x^2 v, r\rangle_V = \langle \psi, r\rangle_V \qquad \textit{forall} \quad r \in V \tag{3.2.11}$$

$$\langle v^N, r^N\rangle_{V^N} + k\left\langle \frac{\partial}{\partial x}v^N, \frac{\partial}{\partial x}r^N\right\rangle_{V^N} = \left\langle \widehat{\psi}, r^N\right\rangle_{V^N} \qquad \textit{forall} \quad r^N \in V^N. \tag{3.2.12}$$

Choosing $r = r^N$ in (3.2.11), and using the fact that $\left\langle \psi - \widehat{\psi}, r^N\right\rangle_V = 0$, we have

$$\langle \widehat{v}^N - v^N, r^N\rangle_V - k\langle A_2(\widehat{v}^N - v^N), r^N\rangle_V$$
$$= k\langle D_x^2 v, r^N\rangle_V - k\langle A_2\widehat{v}, r^N\rangle_V + \langle \widehat{v} - v, r^N\rangle_V \tag{3.2.13}$$

for all $r^N \in V^N$, where $\widehat{v}^N = P_V^N v$. Pick $r^N = \widehat{v}^N - v^N$in (3.2.13), then

$$\begin{aligned}
&\left\|\widehat{v}^N - v^N\right\|_V^2 + \frac{k}{\epsilon}\left\|\widehat{v}^N - v^N\right\|_Y^2 \\
&\quad = k\left\langle \frac{\partial}{\partial x}(\widehat{v}^N - v), \frac{\partial}{\partial x}(\widehat{v}^N - v^N)\right\rangle_V + \left\langle \widehat{v}^N - v, \widehat{v}^N - v^N\right\rangle_V \\
&\quad \le \frac{k}{\epsilon}\left\|\widehat{v}^N - v\right\|_Y \left\|\widehat{v}^N - v^N\right\|_Y + \left\|\widehat{v}^N - v\right\|_V \left\|\widehat{v}^N - v^N\right\|_V \\
&\quad \le \left(\frac{k}{\epsilon}\left\|\widehat{v}^N - v\right\|_Y + c_1\left\|\widehat{v}^N - v\right\|_V\right)\left\|\widehat{v}^N - v^N\right\|_Y.
\end{aligned}$$

Now it is easy to see that $\left\|\widehat{v}^N - v^N\right\|_Y \to 0$ as $N \to \infty$. Therefore, $\left\|v - v^N\right\|_Y \le \left\|v - \widehat{v}^N\right\|_Y + \left\|\widehat{v}^N - v^N\right\|_Y \to 0$ as well. Since $\|\cdot\|_V \le c_1 \|\cdot\|_Y$ for some $c_1 > 0$, we also have $\left\|v - v^N\right\|_V \to 0$ as $N \to \infty$. ∎

We are now ready to prove that our approximation scheme satisfies the condition **H2)** of the Trotter-Kato theorem.

LEMMA 3.2.6. *For all $z \in Z$, $\left(I - \mathcal{A}^{N,M}\right)^{-1} P_Z^{N,M} z \longrightarrow (I - \mathcal{A})^{-1} z$ as $N, M \to \infty$.*

PROOF: Define the bilinear forms $\mu(\cdot,\cdot)$ and $\mu^M(\cdot,\cdot)$ on Y by

$$\mu(y_1, y_2) \equiv \langle \Delta(1) y_1, y_2\rangle_V\,,$$

$$\mu^M(y_1, y_2) \equiv \left\langle \Delta^{N,M}(1) y_1, y_2\right\rangle_V\,,$$

where $\Delta(1)$ is defined by (2.2.9) with $\lambda = 1$. Then for $y_1, y_2 \in Y$,

$$\begin{aligned}
\mu(y_1, y_2) = \langle y_1, y_2\rangle_V &+ \left(\frac{1}{\epsilon} - \int_{-r}^{0} e^s g_\epsilon(s)\,ds\right)\langle y_1, y_2\rangle_Y \\
&+ \gamma^2\left\langle \left(I - kD_x^2\right)^{-1}\frac{\partial}{\partial x} y_1, \frac{\partial}{\partial x} y_2\right\rangle_V \qquad (3.2.14)
\end{aligned}$$

and

$$\mu^M(y_1,y_2) = \langle y_1,y_2\rangle_Y + \left(\frac{1}{\epsilon} - \int_{-r}^0 e^M(1,s)g_\epsilon(s)ds\right)\langle y_1,y_2\rangle_Y$$
$$+\gamma^2\left\langle (I-A_2)^{-1}D_2y_1, \frac{\partial}{\partial x}y_2\right\rangle_V. \tag{3.2.15}$$

Thus, we have

$$\left|\mu^M(y_1,y_2)-\mu(y_1,y_2)\right| \le \left(\int_{-r}^0 g_\epsilon(s)\left|e^s - e^M(1,s)\right| ds\right)\|y_1\|_Y\cdot\|y_2)\|_Y$$
$$+\frac{\gamma^2}{\epsilon^{1/2}}\left\|\left(I-D_x^2\right)^{-1}\frac{\partial}{\partial x}y_1 - (I-A_2)^{-1}D_2y_1\right\|_V \|y_2\|_Y. \tag{3.2.16}$$

Moreover,

$$\mu^M(y,y) = \|y\|_V^2 + \left(\frac{1}{\epsilon} - \int_{-r}^0 g_\epsilon(s)e^M(1,s)ds\right)\|y\|_Y^2 + \gamma^2\left\langle (I-A_2)^{-1}D_2y, \frac{\partial}{\partial x}y\right\rangle_V$$

and if $(I-A_2)^{-1}D_2y = f$, then

$$\left\langle (I-A_2)^{-1}D_2y, \frac{\partial}{\partial x}y\right\rangle_V = \langle f,f\rangle_V - \langle f, A_2 f\rangle_V$$
$$= \|f\|_V^2 + \left\|\frac{\partial}{\partial x}f\right\|_V^2 \ge 0.$$

Since $\omega = \left(\frac{1}{\epsilon} - \int_{-r}^0 g_\epsilon(s)e^M(1,s)ds\right) > 0$, we get

$$\mu^M(y,y) \ge \|y\|_V^2 + \omega\,\|y\|_Y^2. \tag{3.2.17}$$

For $z=(\varphi,\psi,\xi,h)^T \in Z$, let $(I-\mathcal{A})^{-1}z = (y,v,\theta,w)^T$, and $(I-\mathcal{A}^{N,M})^{-1}P_Z^{N,M}z = (y^N,v^N,\theta^N,w^{N,M})^T$. Then, by (2.1.4), for all $u\in Y$,

$$\mu(y,u) = \langle \psi+\varphi,u\rangle_V + \epsilon\left\langle\frac{\partial}{\partial x}\int_{-r}^0 g_\epsilon(s)\,(I-D)^{-1}(\varphi - h(s))\,ds, \frac{\partial}{\partial x}u\right\rangle_V$$
$$+\gamma\left\langle\left(I-kD_x^2\right)^{-1}\left(\xi+\gamma\frac{\partial}{\partial x}\varphi\right), \frac{\partial}{\partial x}u\right\rangle_V \tag{3.2.18}$$

and by (3.2.15), for all $u^N \in Y^N$,

$$\mu^M(y^N, u^N) = \langle \psi^N + \varphi^N, u^N \rangle_V$$
$$+ \epsilon \left\langle \frac{\partial}{\partial x} \int_{-r}^{0} g_\epsilon(s) \left(I - D^{N,M}\right)^{-1} \left(\varphi^N - h^{N,M}(s)\right) ds, \frac{\partial}{\partial x} u^N \right\rangle_V$$
$$+ \gamma \left\langle \left(I - kA_2\right)^{-1} \left(\xi^N + \gamma D_2 \varphi^N\right), \frac{\partial}{\partial x} u^N \right\rangle_V, \quad (3.2.19)$$

where $\left(\varphi^N, \psi^N, \xi^N, h^{N,M}\right)^T = P_Z^{N,M} z$. Let $\widehat{y}^N = P_Y^N y$. Taking $u = u^N$ in (3.2.18), it follows that

$$\mu^M\left(\widehat{y}^N - y^N, u^N\right) = \mu^M\left(\widehat{y}^N, u^N\right) - \mu^M\left(y, u^N\right) + \mu^M\left(y, u^N\right)$$
$$- \mu\left(y, u^N\right) + \langle \psi - \psi^N, u^N \rangle_V + \langle \left(\varphi - \varphi^N\right), u^N \rangle_V$$
$$+ \epsilon \left\langle \frac{\partial}{\partial x} \int_{-r}^{0} g_\epsilon(s) \left[(I - D)^{-1}(\varphi - h(s)) - (I - D^{N,M})^{-1}(\varphi^N - h^{N,M}(s))\right] ds, \frac{\partial}{\partial x} u^N \right\rangle_V$$
$$+ \gamma \left\langle \left(I - kD_x^2\right)^{-1} \left(\xi + \gamma \frac{\partial}{\partial x} \varphi\right) - \left(I - kA_2\right)^{-1} \left(\xi^N + \gamma D_2 \varphi^N\right), \frac{\partial}{\partial x} u^N \right\rangle_V. \quad (3.2.20)$$

Now, by (3.2.17), $\omega \left\|\widehat{y}^N - y^N\right\|_Y^2 \le \mu^M(\widehat{y}^N - y^N, \widehat{y}^N - y^N)$ and setting $u^N = \widehat{y}^N - y^N$ in (3.2.20) we obtain the estimate

$$\mu^M(\widehat{y}^N - y^N, \widehat{y}^N - y^N) \le \left|\mu^M(\widehat{y}^N - y, \widehat{y}^N - y^N)\right|$$
$$+ \left|\mu^M(y, \widehat{y}^N - y^N) - \mu(y, \widehat{y}^N - y^N)\right|$$
$$+ c_1 \left\|\psi - \psi^N\right\|_V \left\|\widehat{y}^N - y^N\right\|_Y + \left\|\varphi - \varphi^N\right\|_V \left\|\widehat{y}^N - y^N\right\|_Y \quad (3.2.21)$$
$$+ \frac{1}{\epsilon} \int_{-r}^{0} g_\epsilon(s) \left\|(I - D)^{-1}(\varphi - h) - (I - D^{N,M})^{-1} P_W^{N,M}(\varphi - h)\right\|_Y ds \left\|\widehat{y}^N - y^N\right\|_Y$$
$$+ \frac{\gamma}{\epsilon^{1/2}} \left\|(I - kD_x^2)^{-1} \left(\xi + \gamma \frac{\partial}{\partial x} \varphi\right) - \left(I - kA_2\right)^{-1} \left(\xi^N + \gamma D_2 \varphi^N\right)\right\|_V \left\|\widehat{y}^N - y^N\right\|_Y$$

where we have used $\|\cdot\|_V \leq c_1 \|\cdot\|_Y$. From (3.2.14), the first term on the right-hand side can be estimated as

$$\left|\mu^M(\widehat{y}^N - y, \widehat{y}^N - y^N)\right| \leq c_2 \left\|\widehat{y}^N - y\right\|_Y \cdot \left\|\widehat{y}^N - y^N\right\|_Y \qquad \text{for some} \qquad c_2 > 0.$$

From (3.2.16) we have

$$\begin{aligned}\left|\mu^M(y, \widehat{y}^N - y^N) - \mu(y, \widehat{y}^N - y^N)\right| \leq & \int_{-r}^0 g_\epsilon(s) \left|e^s - e^M(1,s)\right| ds \cdot \|y\|_Y \cdot \left\|\widehat{y}^N - y^N\right\|_Y \\ & + \frac{\gamma}{\epsilon} \left\| \left(I - \frac{\partial}{\partial x}^2\right)^{-1} D_x y - (I - A_2)^{-1} D_2 y \right\|_V \cdot \left\|\widehat{y}^N - y^N\right\|_Y .\end{aligned}$$

By Hölder's Inequality it follows that

$$\begin{aligned}&\int_{-r}^0 g_\epsilon(s) \left\|(I-D)^{-1}(\varphi - h) - (I - D^{N,M})^{-1} P_W^{N,M}(\varphi - h)\right\|_Y ds \\ &\quad \leq \left(\int_{-r}^0 g_\epsilon(s) ds\right)^{1/2} \left\|(I-D)^{-1}(\varphi - h) - (I - D^{N,M})^{-1} P_W^{N,M}(\varphi - h)\right\|_W .\end{aligned}$$

Hence, we obtain the estimate

$$\begin{aligned}\left\|\widehat{y}^N - y^N\right\|_Y \leq & \frac{1}{\omega}\Bigg[c_2 \left\|\widehat{y}^N - y\right\|_Y + \left(\int_{-r}^0 g_\epsilon(s) \left|e^s - e^M(1,s)\right| ds\right) \|y\|_Y \\ & + \frac{\gamma^2}{\epsilon} \left\|(I - D_x^2)^{-1} \frac{\partial}{\partial x} y - (I - A_2)^{-1} D_2 y\right\|_V + c_1 \left(\left\|\psi - \psi^N\right\|_V + \left\|\varphi - \varphi^N\right\|_V\right) \\ & + \left(\int_{-r}^0 g_\epsilon(s) ds\right)^{1/2} \left\|(I-D)^{-1}(\varphi - h) - (I - D^{N,M})^{-1} P_W^{N,M}(\varphi - h)\right\|_W \\ & + \frac{\gamma}{\epsilon^{1/2}} \left\|(I - kD_x^2)^{-1}(\xi + \gamma \frac{\partial}{\partial x}\varphi) - (I - kA_2)^{-1}(\xi^N + \gamma D_2 \varphi^N)\right\|_V\Bigg] .\end{aligned} \tag{3.2.22}$$

Inside the bracket of the right-handside of (3.2.22), as $N, M \to 0$, the first term $\to 0$; the second term $\to 0$ by Lemma 3.2.4; the third term $\to 0$ by Lemma 3.2.5; the fourth term $\to 0$; the fifth term $\to 0$ by Lemma 3.2.3; the last term $\to 0$ also by Lemma 3.2.5. We conclude that $\|\widehat{y}^N - y^N\|_Y \to 0$. Therefore, $\|y - y^N\|_Y + \|\widehat{y}^N - y^N\|_Y \to 0$ as $N, M \to \infty$. Now, if $v = y - \varphi$ and $v^N = y^N - \varphi^N$, then

$$\begin{aligned}\|v - v^N\|_V &\le \|x - x^N\|_V + \|\varphi - \varphi^N\|_V \\ &\le c_1\left(\|y - y^N\|_Y + \|\varphi - \varphi^N\|_Y\right) \to 0 \quad \text{as} \quad N, M \to \infty.\end{aligned}$$

Next, $\theta = (I - kD_x^2)^{-1}(\xi - \gamma\frac{\partial}{\partial x}v)$ and $\theta^N = (I - kA_2)^{-1}(\xi^N - \gamma D_2 v^N)$ implies

$$\begin{aligned}&\|\theta - \theta^N\|_\Theta \le \|(I - kD_x^2)^{-1}\xi - (I - kA_2)^{-1}\xi^N\|_\Theta \\ &+\gamma\left\|(I - kD_x^2)^{-1}\frac{\partial}{\partial x}v - (I - kA_2)^{-1}\frac{\partial}{\partial x}\widehat{v}\right\|_\Theta + \frac{\gamma}{\epsilon}\|(I - kA_2)^{-1}\| \cdot \|\widehat{v} - v^N\|_Y \to 0\end{aligned}$$

as $N, M \to \infty$, where $\widehat{v} = P_\Theta^N v$. Finally, $(I - D)w = h + v = h + y - \varphi$, and $(I - D^{N,M})w^{N,M} = h^{N,M} + y^N = P_W^{N,M}(h - \varphi) + y^N$. Thus,

$$\begin{aligned}\|w - w^{N,M}\|_W &\le \left\|(I - D)^{-1}(h - \varphi) - (I - D^{N,M})^{-1}P_W^{N,M}(h - \varphi)\right\|_W \\ &+ \left(\int_{-r}^0 g_\epsilon(s)ds\right)^{1/2}\|(I - D)^{-1}\| \cdot \|y - y^N\|_Y \\ &+ \left\|(I - D)^{-1}y^N - (I - D^{N,M})^{-1}P_W^{N,M}y^N\right\|_W \to 0\end{aligned}$$

as $N, M \to \infty$. ∎

The main theorem of this section, Theorem 3.2.7, is a direct consequence of Lemma 3.2.2, Lemma 3.2.6 and the Trotter-Kato theorem.

THEOREM 3.2.7. *For all $z \in Z$, $e^{A^{N,M}t}P_Z^{N,M}z \to T(t)z$ as $N, M \to \infty$, uniformly on bounded t-intervals.*

3.3 Convergence of the Adjoint Semigroup.

As we have mentioned before, for the approximation of the regulator problem the strong convergence of the adjoint semigroup generated by $\mathcal{A}^*$ is crucial to the right approximation of the gain operator. In this section, we show that our approximation scheme also gives the desired convergence $(T^{N,M})^*(t) \xrightarrow{s} T^*(t)$. This is obtained under the additional assumption $g(\cdot) \in L_2(-r,0)$.

First, we compute the form of the adjoint operators $\mathcal{A}^*$ and $(\mathcal{A}^{N,M})^*$.

THEOREM 3.2.1. *The adjoint of $\mathcal{A}$ with respect to the norm $\|\cdot\|_{Z_e}$ is given by*

$$\mathcal{D}(\mathcal{A}^*) \equiv \left\{ \begin{pmatrix} y \\ v \\ \theta \\ w \end{pmatrix} \in Z \;\middle|\; \begin{array}{c} y + \int_{-r}^{0} w(s)ds \in H_0^2(0,l), \quad v \in H_0^1(0,l) \\ \theta \in H_0^2(0,l), \quad w \in H_{gL}^1(-r,0;Y) \end{array} \right\},$$

$$\mathcal{A}^* \begin{pmatrix} y \\ v \\ \theta \\ w \end{pmatrix} \equiv \begin{pmatrix} -v \\ -\epsilon D_x^2 \left(y + \int_{-r}^{0} w(s)ds \right) + \gamma \frac{\partial}{\partial x}\theta \\ \gamma \frac{\partial}{\partial x} v + k D_x^2 \theta \\ -g_\epsilon(s) v - \frac{\partial}{\partial s} w \end{pmatrix}.$$

PROOF: Let $(y,v,\theta,w)^T \in \mathcal{D}(\mathcal{A}^*)$. Then there exists $(\varphi,\psi,\xi,h)^T \in Z$ such that

$$\left\langle \mathcal{A} \begin{pmatrix} \bar{y} \\ \bar{v} \\ \bar{\theta} \\ \bar{w} \end{pmatrix}, \begin{pmatrix} y \\ v \\ \theta \\ w \end{pmatrix} \right\rangle_{Z_e} = \left\langle \begin{pmatrix} \bar{y} \\ \bar{v} \\ \bar{\theta} \\ \bar{w} \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \\ \xi \\ h \end{pmatrix} \right\rangle_{Z_e}$$

for all $\bar{z} = (\bar{y},\bar{v},\bar{\theta},\widehat{w})^T \in \mathcal{D}(\mathcal{A})$, and this implies $\mathcal{A}^* \begin{pmatrix} y \\ v \\ \theta \\ w \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ \xi \\ h \end{pmatrix}$. Now, we write

(φ, ψ, ξ, h) in terms of (y, v, θ, w). From (3.3.1), we have

$$\langle \bar{v}, y\rangle_Y + \left\langle \epsilon D_x^2 \left(\bar{y} + \int_{-r}^0 g_\epsilon(s)\bar{w}(s)ds\right), v\right\rangle_Y - \left\langle \gamma \frac{\partial}{\partial x}\bar{\theta}, v\right\rangle_V$$
$$- \left\langle \gamma \frac{\partial}{\partial x}\bar{v}, \theta\right\rangle_\Theta + \int_{-r}^0 \left\langle \bar{v} + \frac{\partial}{\partial s}\bar{w}(s), w(s)\right\rangle_Y ds$$
$$= \langle \bar{y}, \varphi\rangle_Y + \langle \bar{v}, \psi\rangle_V + \langle \bar{\theta}, \xi\rangle_\Theta + \int_{-r}^0 \langle \bar{w}(s), h(s)\rangle_Y\, ds. \tag{3.3.2}$$

Let $\bar{v} = \bar{\theta} = \bar{w} = 0$. Then $\langle \epsilon D_x^2 \bar{y}, v\rangle_V = \langle \bar{y}, \varphi\rangle_Y$ for all $\bar{y} \in Y_1 = \{y \mid (y, v, \theta, w) \in \mathcal{D}(\mathcal{A})\}$. But, $\langle \bar{y}, \varphi\rangle_Y = \epsilon \left\langle \frac{\partial}{\partial x}\bar{y}, \frac{\partial}{\partial x}\varphi\right\rangle_V = -\left\langle \epsilon \frac{\partial^2}{\partial x^2}\bar{y}, \varphi\right\rangle_V$, so $\left\langle \epsilon \frac{\partial^2}{\partial x^2}\bar{y}, v + \varphi\right\rangle_V = 0$ for all $\bar{y} \in Y_1$. Since $\left\{\frac{\partial^2}{\partial x^2}\bar{y} \mid \bar{y} \in Y_1\right\}$ is dense in V, we get

$$\varphi = -v \in H_0^1(0, l) \tag{3.3.3}$$

Set $\bar{y} = \bar{v} = \bar{w} = 0$ in (3.3.2), then $\left\langle -\gamma \frac{\partial}{\partial x}\bar{\theta}, v\right\rangle_V + \left\langle k D_x^2 \bar{\theta}, \theta\right\rangle_\Theta = \left\langle \bar{\theta}, \xi\right\rangle_\Theta$ for all $\bar{\theta} \in H_0^2(0, l)$. Since $\left\langle -\gamma \frac{\partial}{\partial x}\bar{\theta}, v\right\rangle_V = \left\langle \bar{\theta}, \gamma \frac{\partial}{\partial x} v\right\rangle_\Theta$, and $\left\langle k D_x^2 \bar{\theta}, \theta\right\rangle_\Theta = \left\langle \bar{\theta}, k \frac{\partial^2}{\partial x^2}\theta\right\rangle_\Theta + k(\frac{\partial}{\partial x}\bar{\theta})\theta|_0^l$ for all $\bar{\theta} \in H_0^2(0, l)$, which is dense in Θ, we have

$$\theta(0) = \theta(l) = 0, \qquad \xi = \gamma \frac{\partial}{\partial x} v + k D_x^2 \theta. \tag{3.3.4}$$

By (3.3.3)

$$\left\langle \epsilon D_x^2 \left(\bar{y} + \int_{-r}^0 g_\epsilon(s)\bar{w}(s)ds\right), v\right\rangle_V = \left\langle \epsilon \frac{\partial}{\partial x}\left(\bar{y} + \int_{-r}^0 g_\epsilon(s)\bar{w}(s)ds\right), \frac{\partial}{\partial x}\varphi\right\rangle_V$$
$$= \left\langle \bar{y} + \int_{-r}^0 g_\epsilon(s)\bar{w}(s), \varphi\right\rangle_Y.$$

Substituting into (3.3.2) we get

$$\langle \bar{v}, y\rangle_Y - \int_{-r}^{0} g_\epsilon(s)\,\langle \bar{w}(s), v\rangle_Y \, ds + \left\langle \gamma \frac{\partial}{\partial x}\bar{\theta}, v \right\rangle_V$$

$$+ \left\langle -\gamma \frac{\partial}{\partial x}\bar{v}, \theta \right\rangle_\Theta + \langle k D_x^2 \bar{\theta}, \theta\rangle_\Theta + \int_{-r}^{0} \left\langle \bar{v} + \frac{\partial}{\partial s}\bar{w}(s), w(s) \right\rangle_Y ds$$

$$= \langle \bar{v}, \psi\rangle_V + \langle \bar{\theta}, \xi\rangle_\Theta + \int_{-r}^{0} \langle \bar{w}(s), h(s)\rangle_Y \, ds.$$

Let $\bar{\theta} = \bar{w} = 0$. Then $\langle \bar{v}, y\rangle_Y + \left\langle -\gamma \frac{\partial}{\partial x}\bar{v}, \theta\right\rangle_V + \int_{-r}^{0} \langle \bar{v}, w(s)\rangle_Y \, ds = \langle \bar{v}, \psi\rangle_V$ for all $\bar{v} \in H_0^1(0,l)$, which implies

$$\left\langle \bar{v}, -\epsilon D_x^2 \left(y + \int_{-r}^{0} w(s) ds \right) + \gamma \frac{\partial}{\partial x}\theta \right\rangle_V = \langle \bar{v}, \psi\rangle_V, \qquad \textit{forall} \quad \bar{v} \in H_0^1(0,l).$$

Since $H_0^1(0,l)$ is dense in V, we obtain

$$\psi = -\epsilon D_x^2 \left(y + \int_{-r}^{0} w(s) ds \right) + \gamma \frac{\partial}{\partial x}\theta. \tag{3.3.5}$$

Finally, set $\bar{y} = \bar{v} = \bar{\theta} = 0$ in (3.3.2) we obtain

$$\int_{-r}^{0} [\langle \bar{w}(s), g_\epsilon(s) v + h(s)\rangle_Y - \left\langle \frac{\partial}{\partial s}\bar{w}(s), w(s) \right\rangle_Y] ds = 0$$

for all $\bar{w} \in H_{gR}(-r, 0; Y)$. By the Fundamental Lemma of the Calculus of Variations,

$$w \in H_{gL}^1(-r, 0; Y), \qquad \textit{and} \qquad h(s) = -g_\epsilon(s) v - \frac{\partial}{\partial s} w(s) \tag{3.3.6}$$

Combining (3.3.3)-(3.3.6), the proof is complete.

THEOREM 3.2.2. *For* $z^{N,M} = (y^N, v^N, \theta^N, w^{N,M})^T \in Z^{N,M}$

$$(\mathcal{A}^{N,M})^* z^{N,M} = \begin{pmatrix} -v^N \\ -\epsilon A_1 \left(y^N + \sum_{i=1}^{M} \alpha_i^M w_i^M \right) + \gamma D_1 \theta^N \\ \gamma D_2 v^N + k A_2 \theta^N \\ -g_\epsilon v^N - \sum_{i=1}^{M} \frac{1}{\alpha_i^M} (w_i^M - w_{i+1}^M) E_i^M \end{pmatrix}.$$

PROOF: The proof is analogous to the proof of last theorem. Let $(y^N, v^N, \theta^N, w^{N,M})^T \in Z^{N,M}$. Then, by Rietz theorem, there exists $(\varphi^N, \psi^N, \xi^N, h^{N,M})^T \in Z^{N,M}$ such that

$$\left\langle \mathcal{A}^{N,M}\begin{pmatrix} \bar{y}^N \\ \bar{v}^N \\ \bar{\theta}^N \\ \bar{w}^{N,M} \end{pmatrix}, \begin{pmatrix} y^N \\ v^N \\ \theta^N \\ w^{N,M} \end{pmatrix}\right\rangle_{Z_e} = \left\langle \begin{pmatrix} \bar{y}^N \\ \bar{v}^N \\ \bar{\theta}^N \\ \bar{w}^{N,M} \end{pmatrix}, \begin{pmatrix} \varphi^N \\ \psi^N \\ \xi^N \\ h^{N,M} \end{pmatrix}\right\rangle_{Z_e} \tag{3.3.7}$$

for all $\bar{z} = (\bar{y}, \bar{v}, \bar{\theta}, \widehat{w})^T \in \mathcal{D}(\mathcal{A})$, and this implies $\mathcal{A}^* \begin{pmatrix} y \\ v \\ \theta \\ w \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ \xi \\ h \end{pmatrix}$, for all $\bar{z}^{N,M} = (\bar{y}^N, \bar{v}^N, \bar{\theta}^N, \bar{w}^{N,M})^T \in Z^{N,M}$. It follows that $(\mathcal{A}^{N,M})^*(y^N, v^N, \theta^N, w^{N,M})^T = (\varphi^N, \psi^N, \xi^N, h^{N,M})^T$. From (3.3.7), we have

$$\begin{aligned} &\langle \bar{v}^N, y^N\rangle_Y + \left\langle \epsilon A_1\left(\bar{y}^N + \int_{-r}^0 g_\epsilon(s)\bar{w}^{N,M}(s)ds\right), v^N\right\rangle_Y - \langle \gamma D_1\bar{\theta}^N, v^N\rangle_V \\ &\quad - \langle \gamma D_2\bar{v}^N, \theta^N\rangle_\Theta + \int_{-r}^0 \langle \bar{v}^N + D^{N,M}\bar{w}^{N,M}(s), w^{N,M}(s)\rangle_Y\, ds \\ &= \langle \bar{y}^N, \varphi^N\rangle_Y + \langle \bar{v}^N, \psi^N\rangle_V + \langle \bar{\theta}^N, \xi^N\rangle_\Theta, + \int_{-r}^0 \langle \bar{w}^{N,M}(s), h^{N,M}(s)\rangle_Y\, ds. \end{aligned} \tag{3.3.8}$$

If $\bar{v}^N = \bar{\theta}^N = \bar{w}^{N,M} = 0$. Then $\langle \epsilon A_1\bar{y}^N, v^N\rangle_V = \langle \bar{y}^N, \varphi^N\rangle_Y$ for all $\bar{y}^N \in Y^N$, and

$$\varphi^N = -v^N \in Y^N \tag{3.3.9}$$

Set $\bar{y}^N = \bar{v}^N = \bar{w}^{N,M} = 0$ in (3.3.8), then $\langle -\gamma D_1\bar{\theta}^N, v^N\rangle_V + \langle kA_2\bar{\theta}^N, \theta^N\rangle_\Theta = \langle \bar{\theta}^N, \xi^N\rangle_\Theta$ for all $\bar{\theta}^N \in \Theta^N$. By the definitions of D_1 and A_2, we have

$$\xi^N = \gamma D_2 v^N + kA_2\theta^N. \tag{3.3.10}$$

Identity (3.3.9) yields

$$\begin{aligned} \left\langle \epsilon A_1\left(\bar{y}^N + \int_{-r}^0 g_\epsilon(s)\bar{w}^{N,M}(s)ds\right), v^N\right\rangle_V &= \left\langle \epsilon\frac{\partial}{\partial x}\left(\bar{y}^N + \int_{-r}^0 g_\epsilon(s)\bar{w}^{N,M}(s)ds\right), \frac{\partial}{\partial x}\varphi^N\right\rangle_V \\ &= \left\langle \bar{y}^N + \int_{-r}^0 g_\epsilon(s)\bar{w}^{N,M}(s), \varphi^N\right\rangle_Y. \end{aligned}$$

Substituting into (3.3.8) we obtain

$$\langle \bar{v}^N, y^N \rangle_Y - \int_{-r}^{0} g_\epsilon(s) \langle \bar{w}^{N,M}(s), v^N \rangle_Y \, ds + \langle \gamma D_1 \bar{\theta}^N, v^N \rangle_V$$

$$+ \langle -\gamma D_2 \bar{v}^N, \theta^N \rangle_\Theta + \langle k A_2 \bar{\theta}^N, \theta^N \rangle_\Theta + \int_{-r}^{0} \langle \bar{v}^N + D^{N,M} \bar{w}^{N,M}(s), w^{N,M}(s) \rangle_Y \, ds$$

$$= \langle \bar{v}^N, \psi^N \rangle_V + \langle \bar{\theta}^N, \xi^N \rangle_\Theta + \int_{-r}^{0} \langle \bar{w}^{N,M}(s), h^{N,M}(s) \rangle_Y \, ds.$$

If $\bar{\theta}^N = \bar{w}^{N,M} = 0$, then $\langle \bar{v}^N, y^N \rangle_Y + \langle -\gamma D_2 \bar{v}^N, \theta^N \rangle_V + \int_{-r}^{0} \langle \bar{v}^N, w^{N,M}(s) \rangle_Y \, ds =$ $\langle \bar{v}^N, \psi^N \rangle_V$ for all $\bar{v} \in V^N$, which implies that $\left\langle \bar{v}^N, -\epsilon A_1 \left(y^N + \int_{-r}^{0} w^{N,M}(s) ds \right) + \gamma D_1 \theta^N \right\rangle_V$ $= \langle \bar{v}^N, \psi^N \rangle_V$, for all $\bar{v} \in H_0^1(0,l)$. Therefore,

$$\psi^N = -\epsilon A_1 \left(y^N + \int_{-r}^{0} w^{N,M}(s) ds \right) + \gamma D_2 \theta^N.$$

Since $w^{N,M} = \sum_{i=1}^{M} E_i^M$, it follows that

$$\psi^N = -\epsilon A_1 \left(y^N + \sum_{i=1}^{M} \alpha_i^M w_i^M \right) + \gamma D_1 \theta^N. \tag{3.3.11}$$

Finally, let $\bar{y}^N = \bar{v}^N = \bar{\theta}^N = 0$ in (3.3.8) to obtain

$$\int_{-r}^{0} \left[\langle \bar{w}^{N,M}(s), g_\epsilon(s) v^N + h^{N,M}(s) \rangle_Y - \langle D^{N,M} \bar{w}^{N,M}(s), w^{N,M}(s) \rangle_Y \right] ds = 0$$

for all $\bar{w}^{N,M} \in W^{N,M}$. The definition of $D^{N,M}$ conbined with straightforward algebraic munipulations, leads to the result

$$h^{N,M}(s) = -(g_\epsilon^M)(s) v^N - \sum_{i=1}^{M} \frac{1}{\alpha_i^M} (w_i^M - w_{i+1}^M) E_i^M. \tag{3.3.12}$$

Combining (3.3.9)-(3.3.12), completes the proof.

Let $W^{N,M}$ be as in (3.1.6) and define $\widetilde{W}_*^{N,M} \subseteq W$ by

$$\widetilde{W}_*^{N,M} \equiv \left\{ w \in W \;\middle|\; w = \sum_{i=0}^{M-1} b_i^M B_i^M,\ b_i^M \in X^N \right\}.$$

Define $\widetilde{D}_*^{N,M} : \widetilde{W}_*^{N,M} \to W^{N,M}$ by

$$\widetilde{D}_*^{N,M} w^{N,M} \equiv \sum_{i=1}^{M} \frac{1}{\alpha_i^M} \left(b_{i-1}^M - b_i^M \right) E_i^M,$$

where $w^{N,M} = \sum_{i=0}^{M-1} b_i^M B_i^M$ and $b_M^M = 0$. Also define the isomorphism $i_*^{N,M} : \widetilde{W}_*^{N,M} \to W^{N,M}$ by

$$i_*^{N,M} w^{N,M} \equiv \sum_{i=1}^{M} b_{i-1}^M \chi_i^M,$$

and define $D_*^{N,M} : W^{N,M} \to W^{N,M}$ by $D_*^{N,M} \equiv \widetilde{D}_*^{N,M} \left(i_*^{N,M} \right)^{-1}$. We can now write $\left(\mathcal{A}^{N,M} \right)^*$ in the form

$$\left(\mathcal{A}^{N,M} \right)^* \begin{pmatrix} y^N \\ v^N \\ \theta^N \\ w^{N,M} \end{pmatrix} = \begin{pmatrix} -v^N \\ -A_1 \left(y^N + \int_{-r}^{0} w^{N,M}(s) ds \right) + \gamma D_1 \theta^N \\ \gamma D_2 v^N + k A_2 \theta^N \\ -g_\epsilon^M v^N - D_*^{N,M} w^{N,M} \end{pmatrix}. \tag{3.3.13}$$

The proof of the convergence of the adjoint semigroup is very similar to the proof in the previous section. Let Z_ϵ be the space Z equipped with the norm $\|\cdot\|_{Z_\epsilon}$, and let $P_{Z_\epsilon}^{N,M}$ be the orthogonal projections of Z_ϵ onto $Z^{N,M}$. Since and $\|\cdot\|_{Z_\epsilon}$ and $\|\cdot\|_Z$ are equivalent norms, we have

LEMMA 3.3.3. *The projections $P_{Z_\epsilon}^{N,M}$ converge strongly to I with respect to the norm $\|\cdot\|_{Z_\epsilon}$.*

Let D_* be defined by $\mathcal{D}(D_*) \equiv H^1_{gL}(-r,0;X)$, $D_* \equiv \dfrac{\partial}{\partial s}$. For $z = (y,v,\theta,w)^T \in \mathcal{D}(\mathcal{A}^*)$ and $\lambda = 1$, consider the equation $(\lambda I - \mathcal{A}^*)\, z = (\varphi,\psi,\xi,h)^T$, or equivalently

$$y + v = \varphi, \tag{3.3.14}$$

$$v + \epsilon D_x^2 \left(y + \int_{-r}^{0} w(s)ds \right) - \gamma \frac{\partial}{\partial x}\theta = \psi, \tag{3.3.15}$$

$$\theta - \gamma \frac{\partial}{\partial x} v - k D_x^2 \theta = \xi, \tag{3.3.16}$$

$$w + g_\epsilon v + D_* w = h. \tag{3.3.17}$$

From (3.3.17), $w(s) = (I + D_*)^{-1}(-g_\epsilon v + h) = -\int_{-r}^{s} e^{\lambda(\tau - s)}\left(g_\epsilon(\tau)v - h(\tau)\right)d\tau$ and from (3.3.14), $v = -y + \varphi$. Also, (3.3.16) implies that $\theta = (I - kD_x^2)^{-1}(\xi + \gamma\frac{\partial}{\partial x}v)$. Substituting into (3.3.15) yields

$$\begin{aligned}\Delta_*(1)y = -\psi + \varphi - \epsilon D_x^2 \left(\int_{-r}^{0} (I + D_*)^{-1} \left[g_\epsilon(s)\varphi - h(s) \right] ds \right) \\ + \gamma \frac{\partial}{\partial x}(I - kD_x^2)^{-1}(\xi + \gamma \frac{\partial}{\partial x}\varphi),\end{aligned} \tag{3.3.18}$$

where the expression $\Delta_*(1)$ is defined by

$$\begin{aligned}\Delta_*(1)y = y^2 + \epsilon A_1 \left(y + \int_{-r}^{0} (I + D_*)^{-1} g_\epsilon(s) y ds \right) \\ - \gamma^2 \frac{\partial}{\partial x}(I - kD_x^2)^{-1} \frac{\partial}{\partial x} y.\end{aligned} \tag{3.3.19}$$

For $z^{N,M} = \left(y^N, v^N, \theta^N, w^{N,M}\right)^T \in Z^{N,M}$ and $\lambda = 1$, consider the equation

$\left(\lambda I - \left(\mathcal{A}^{N,M}\right)^*\right) z^{N,M} = \left(\varphi^N, \psi^N, \xi^N, h^{N,M}\right)^T$. We have

$$y^N + v^N = \varphi^N, \tag{3.3.20}$$

$$v^N + \epsilon A_1 \left(y^N + \int_{-r}^0 w^{N,M}(s)ds \right) - \gamma D_1 \theta^N = \psi^N, \tag{3.3.21}$$

$$\theta^N - \gamma D_2 v^N - k A_2 \theta^N = \xi^N \tag{3.3.22}$$

$$w^{N,M} + g_\epsilon v^N + D_*^{N,M} w^{N,M} = h^{N,M}, \tag{3.3.23}$$

where $w^{N,M} = \sum_{i=1}^{M} w_i^M E_i^M$ and $h^{N,M} = \sum_{i=1}^{M} h_i^M E_i^M$. From (3.3.20),(3.3.22) and (3.3.23),

$$v^N = -y^N + \varphi^N$$

$$\theta^N = (I - kA_2)^{-1}(\xi^N + \gamma D_2 v^N)$$

$$w^{N,M} = (I + D_*^{N,M})^{-1}(h^{N,M} - g_\epsilon v^N).$$

Substituting into (3.3.21), we have

$$\begin{aligned}\Delta_*^{N,M}(1) y^N = -\psi^N + \varphi^N - A_1 \int_{-r}^0 (I + D_*^{N,M})^{-1}(g_\epsilon \varphi^N - h^{N,M})ds \\ + \gamma D_1 (I - kA_2)^{-1}(\xi^N + \gamma D_2 \varphi^N),\end{aligned} \tag{3.3.24}$$

where $\Delta_*^{N,M}(1)$ is defined by

$$\begin{aligned}\Delta_*^{N,M}(1) y^N = y^N + \epsilon A_1 \int_{-r}^0 (y^N + (I + D_*^{N,M})^{-1} g_\epsilon(s) y^N ds) \\ - \gamma^2 D_1 (I - kA_2)^{-1} D_2 y^N.\end{aligned} \tag{3.3.25}$$

The next lemma is found in [**15**].

LEMMA 3.3.4. *For $\lambda > 0$, $\left(\lambda I + D_*^{N,M}\right)^{-1} P_{W_c}^{N,M} h \to \left(\lambda I + D_*\right)^{-1} h$ for all $h \in W$.*

Now we are ready to prove the main result in this section.

THEOREM 3.3.5.. *For all* $z \in Z$, $e^{(\mathcal{A}^{N,M})^* t} P_{Z_e}^{N,M} \to T^*(t)z$ *as* $N, M \to \infty$, *uniformly on bounded* t*-intervals.*

PROOF:: Define the bilinear forms $\mu_*^M(\cdot,\cdot)$ on Y by

$$\mu_*(y_1, y_2) \equiv \langle \Delta_*(1) y_1, y_2 \rangle\rangle_V$$

$$\mu_*^M(y_1, y_2) \equiv \langle \Delta_*^{N,M}(1) y_1, y_2 \rangle\rangle_V \, .$$

For $z = (\varphi, \psi, \xi, h)^T \in Z$, let $(I - \mathcal{A}^*)^{-1} z = (y, v, \theta, w)^T$, and $(I - (\mathcal{A}^{N,M})^*)^{-1} P_{Z_e}^{N,M} z = (y^N, v^N, \theta^N, w^{N,M})^T$. Then by (3.3.18), for all $u \in Y$,

$$\mu_*(y, u) = \langle -\psi + \varphi, u \rangle_V + \epsilon \left\langle \frac{\partial}{\partial x} \int_{-r}^{0} (I + D_*)^{-1} [g_\epsilon I s)\varphi - h(s)] ds, \frac{\partial}{\partial x} u \right\rangle_V$$
$$- \gamma \left\langle (I - k D_x^2)^{-1} (\xi + \gamma \frac{\partial}{\partial x} \varphi), \frac{\partial}{\partial x} u \right\rangle_V$$

and by (3.3.24), for all $u^N \in Y^N$,

$$\begin{aligned} \mu_*^M(y^N, u^N) = {} & \langle -\psi^N + \varphi^N, u^N \rangle_V \\ & + \epsilon \left\langle \frac{\partial}{\partial x} \int_{-r}^{0} (I + D_*^{N,M})^{-1} [g_\epsilon^M(s) \varphi^N - h^{N,M}(s)] ds, \frac{\partial}{\partial x} u^N \right\rangle_V \\ & - \gamma \left\langle (I - k A_2)^{-1} (\xi^N + \gamma D_2 \varphi^N), \frac{\partial}{\partial x} u^N \right\rangle_V , \end{aligned}$$

where $P_{Z_e}^{N,M} z = (\varphi^N, \psi^N, \xi^N, h^{N,M})^{-1}$. The rest of the proof is analogous to the proof of Lemma 3.2.6 and Theorem 3.2.7 by observing that $P_{W_e}^{N,M}(g_\epsilon(s)\varphi) = g_\epsilon^M(s)\varphi^N$ and applying Lemma 3.2.4, Lemma 3.2.5, and Lemma 3.3.4.

Chapter IV Numerical Results

In this chapter we discuss the implementation of the Finite Element/Averaging approximation scheme for our thermoviscoelastic system. We describe how to construct the matrices for the open-loop problem and the closed-loop control problem. We compare and discuss the numerical results which show the the effects of heat dissipation and temperature state on damping and control. The computer codes were implemented on a Vax 8800.

4.1 Finite Dimensional Construction.

Recall that the first stage of approximation is to choose the finite dimensional subspaces $Y^N \subset Y, \quad V^N \subset V, \quad \Theta^N \subset \Theta$. We will use both linear splines and cubic splines . Here we provide the details for linear splines. The case of cubic splines is completely analogous.

Divide the interval $[0, l]$ into N subintervals with equal length. Choose the linear hat functions

$$h_0^N(x) = \begin{cases} 1 - \frac{N}{l}x & 0 \le x \le \frac{1}{N}l, \\ 0, & otherwise \end{cases}$$

$$h_i^N(x) = \begin{cases} -(i-1) + \frac{N}{l}x, & \frac{(i-1)}{N}l \le x \le \frac{i}{N}l \\ (i+1) - \frac{N}{l}x, & \frac{i}{N}l \le x \le \frac{(i+1)}{N}l \\ 0, & otherwise. \end{cases} \qquad i = 1, 2, \ldots, N-1$$

$$h_N^N(x) = \begin{cases} -(N-1) + \frac{N}{l}x & \frac{(N-1)}{N}l \le x \le l, \\ 0, & otherwise \end{cases}$$

Let Y^N, V^N, Θ^N be $\text{span}\{h_i^N \mid i = 1, \ldots, N-1\}$. The functions h_0^N, h_N^N are discarded because of the boundary condition (1.1.3). Choose the basis for the product space $Y^N \times V^N \times \Theta^N$ to be

$$e_i^N = \begin{pmatrix} h_i^N \\ 0 \\ 0 \end{pmatrix}, \qquad e_{(N-1)+i}^N = \begin{pmatrix} 0 \\ h_i^N \\ 0 \end{pmatrix} \qquad \text{and} \qquad e_{2(N-1)+i}^N = \begin{pmatrix} 0 \\ 0 \\ h_i^N \end{pmatrix}$$

for $i = 1, 2, \ldots, N-1$. Thus, $Y^N \times V^N \times \Theta^N \equiv \text{span}\{e_i^N \mid i = 1, 2, \ldots, 3(N-1)\}$ with the norm inherited from $Y \times V \times \Theta$. Any $z^N \in Y^N \times V^N \times \Theta^N$ can be written as $z^N = \sum_{i=1}^{3(N-1)} \alpha_i^N e_i^N$. First, we assume that $g(s) \equiv 0$, and let $\bar{z}^N = \mathcal{A}^N z^N$ where $\bar{z}^N = \sum_{i=1}^{3(N-1)} \beta_i^N e_i^N$. For $i = 1, 2, \ldots, 3(N-1)$, it follows that

$$\langle \bar{z}^N, e_i^N \rangle_{Y \times V \times \Theta} = \langle \mathcal{A}^N z^N, e_i^N \rangle_{Y \times V \times \Theta},$$

or equivalently,

$$\sum_{j=1}^{3(N-1)} \beta_j^N \langle e_j^N, e_i^N \rangle_{Y \times V \times \Theta} = \sum_{j=1}^{3(N-1)} \alpha_j^N \langle \mathcal{A}^N e_j^N, e_i^N \rangle_{Y \times V \times \Theta}. \tag{4.1.1}$$

Define the $(N-1) \times (N-1)$ matrices D, H and E by

$$D_{ij} = \alpha \int_0^l h_i'(x) h_j'(x) dx$$

$$H_{ij} = \int_0^l h_j(x) h_i(x) dx$$

$$E_{ij} = \int_0^l h_j'(x) h_i(x) dx$$

for $i,j = 1,2,\ldots,N-1$. We rewrite (4.1.1) as the $3(N-1)\times 3(N-1)$ system $M\vec{\beta} = K\vec{\alpha}$ where

$$M = \begin{bmatrix} D & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & H \end{bmatrix} \qquad \text{and} \qquad F = \begin{bmatrix} 0 & D & 0 \\ -D & 0 & -\gamma E \\ 0 & -\gamma E^T & -kD \end{bmatrix}.$$

is the mass matrix and the stiff matrix respectively. Thus, A^N, the matrix representation of $\mathcal{A}^N$, is given by

$$A^N = \begin{bmatrix} 0 & I & 0 \\ -H^{-1}D & 0 & -\gamma H^{-1}E \\ 0 & -\gamma H^{-1}E^T & -kH^{-1}D \end{bmatrix}. \tag{4.1.2}$$

Note that $\dot{\alpha}^N = A^N\alpha^N$ is the Nth approximation ODE for the thermoelastic system (1.1.4) – (1.1.5) with the boundary condition (1.1.3).

Now we drop the assumption that $g(s) \equiv 0$ and continue to our second stage of approximation. We divide the interval $[-r,0]$ into M subintervals. Following the above steps in the derivation of the matrix A^N, we find that the matrix representation of the operator $\mathcal{A}^{N,M}$ is given by

$$A^{N,M} = \begin{bmatrix} 0 & I & 0 & 0 & \cdots & & & 0 \\ -H^{-1}D & 0 & H_\gamma^{-1}E & -H^{-1}D_1 & \cdots & & & -H^{-1}D_M \\ 0 & H_\gamma^{-1}E^T & H_k^{-1}D & 0 & \cdots & & & 0 \\ 0 & I & 0 & -\frac{1}{\alpha_1^M}I & \cdots & & & 0 \\ 0 & I & 0 & \frac{1}{\alpha_2^M}I & & & & \cdot \\ \cdot & \cdot & \cdot & & \ddots & & & \cdot \\ \cdot & \cdot & \cdot & & & \ddots & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & -\frac{1}{\alpha_{M-1}^M}I & 0 \\ 0 & I & 0 & \cdots & & 0 & \frac{1}{\alpha_M^M}I & -\frac{1}{\alpha_M^M}I \end{bmatrix}$$

where $D_i = (g_\epsilon)_i^M D$ for $i = 1, 2, \ldots, M, H_\gamma^{-1} = -\gamma H^{-1}, H_k^{-1} = -kH^{-1}$. Notice that $A^{N,M}$ is a square matrix of order $(M+3)(N-1)$. The dimension of the approximating system increases quickly as N, M increase. For example, if we take 16 linear splines and use 16 step functions to approximate the history function, we end up with a system of dimension 304. Therefore, the nonuniform mesh scheme greatly reduces the computational problem.

4.2 Open-loop Problem.

As mentioned in the beginning of this paper, our main interests are the natural modes and frequencies of the vibrating bar described by equation (1.1.1). Throughout this and the next section, we always concentrate on these eigenvalues without specific explanation. For the simplicity, we use the system parameter values of $\rho = \alpha = \theta_0 = k = 1$, delay $r = 1$. We also choose $g(s) = -[e^{-5s}/5\sqrt{-s}]$. Note that $g \in L_1(-r, 0)$ and has a singularity at zero. We compare the eigenvalues of matrix $A^{N,M}$ for the thermoviscoelastic system ($\gamma \neq 0$) and the viscoelastic system ($\gamma = 0$). We also do the same comparison for the thermoelastic system ($g = 0, \gamma \neq 0$) and the purely elastic system ($g = 0, \gamma = 0$).

To show the convergence of our approximation numerically, we use $N - 1 = 15$ linear spline functions for both uniform mesh and nonuniform mesh. Figure 4.2.1 shows the eigenvalues of $A^{16,M}$ using the uniform mesh with $M = 16, 32, 64, 128$.

Observe that curves connecting the eigenvalues for each value of M are getting closer as M increases, and moving towards the right after $M = 32$. Figure 4.2.2 shows the eigenvalues obtained by using the non-uniform mesh with $M = 8, 16, 32$. This time, the curves nearly coincide, and are all on the right side of the ones in Figure 4.1.1. This implies that with $M = 8$ for the non-uniform mesh, we have a better approximation than $M = 128$ for the uniform mesh. Notice that the system dimension for these two cases are 165 and 1966 respectively. Therefore, the non-uniform mesh method reduces computational work dramatically. Ito and Fabiano[**14**], Miller[**15**] also observed this behavior in their models.

Figure 4.2.3 compares the eigenvalues of the thermoviscoelastic system (1.1.1)-(1.1.3) and the viscoelastic system($\gamma = 0$), using $N - 1 = 7$ linear splines with $M = 8, 32$. We see that, as to be expected from the physics point of view, the thermal damping is very light. Thus both models predict almost the same behavior. Table 4.2.1 lists these eigenvalues for $M = 32$. For the first seven modes, the real part of the eigenvalues ($Re\lambda_{TVEj}$) of the thermoviscoelastic system are smaller than the one ($Re\lambda_{VEj}$) of the viscoelastic system, but only by an amout of 0.0005 to 0.0038 which is very small comparing to $Re\lambda_{VEj}$. This shows that the structural damping(due to the Boltzmann damping) is much greater than the thermal damping.

Figure 4.2.4 shows the location of the eigenvalues of thermoelastic system and the purely elastic system. We also list them in Table 4.1.2. Notice that the differences

$(Re\lambda_{TEj} - Re\lambda_{PEj})$ of the real parts of the corresponding eigenvalues for the thermoelastic and purly elastic systems are almost equal to the ones $(Re\lambda_{TVEj} - Re\lambda_{VEj})$ for the thermoviscoelastic and the viscoelastic systems. This means that the damping added to the system due to the heat loss at the ends of the bar is the same both with the the presence of viscoelastic damping or without viscoelastic damping. We also observe that this damping is stronger in the lower frequency modes than in the higher frequency modes.

It is interesting to note that the numerical scheme (i.e. finite elements) predicts that the real part of the eigenvalues of the thermoelastic system (1.1.5)-(1.1.6) with the boundary condition (1.1.3) goes to zero as N increases. Figure 4.2.5 shows the location of these eigenvalues, using the cubic spline for $N = 32$. It is unknown if this is true or is only an numerical antifact of the finite element scheme.

We change the boundary condition (1.1.3) to (1.1.4) which means the end of the bar is insulated. Recall that Theorem 2.4.3 implies that the real part of the eigenvalues of this system goes to $-\frac{\gamma^2}{2}$ asymptotically. We use the same finite element scheme, and note that the numerical results turn out to be a contradiction of the theorem. Figure 4.2.6 shows the location of the eigenvalues using the linear splines for $N = 16, 32, 64$. We see that the real parts of the eigenvalues reach approximately -0.005 (for our choice of $\gamma = 0.1$) after the first several modes, and then start to go back to zero monotonically. Figure 4.2.7 shows the location of the eigenvalues using cubic splines for $N = 128$. Again, the real parts are close to -0.005 after the first several modes. But this time they remain close to -0.005 for the next 80 modes, then tend to zero. We observed the same behavior as N increases. Comparing Figure 4.2.6 with Figure 4.2.7, we know that the cubic spline gives much more acuracy than the linear spline. The above observations indicate that our finite element scheme does not appear to give uniformly stable approximating systems. However, for the thermoviscoelastic system and the kernel g we used, the numerical result shows that the real part of the eigenvalues goes to $-\infty$. Hence, the unifrom stability of the approximating systems by our finite element/averaging scheme seems to be possible. This is still an open question which we shall study in the future.

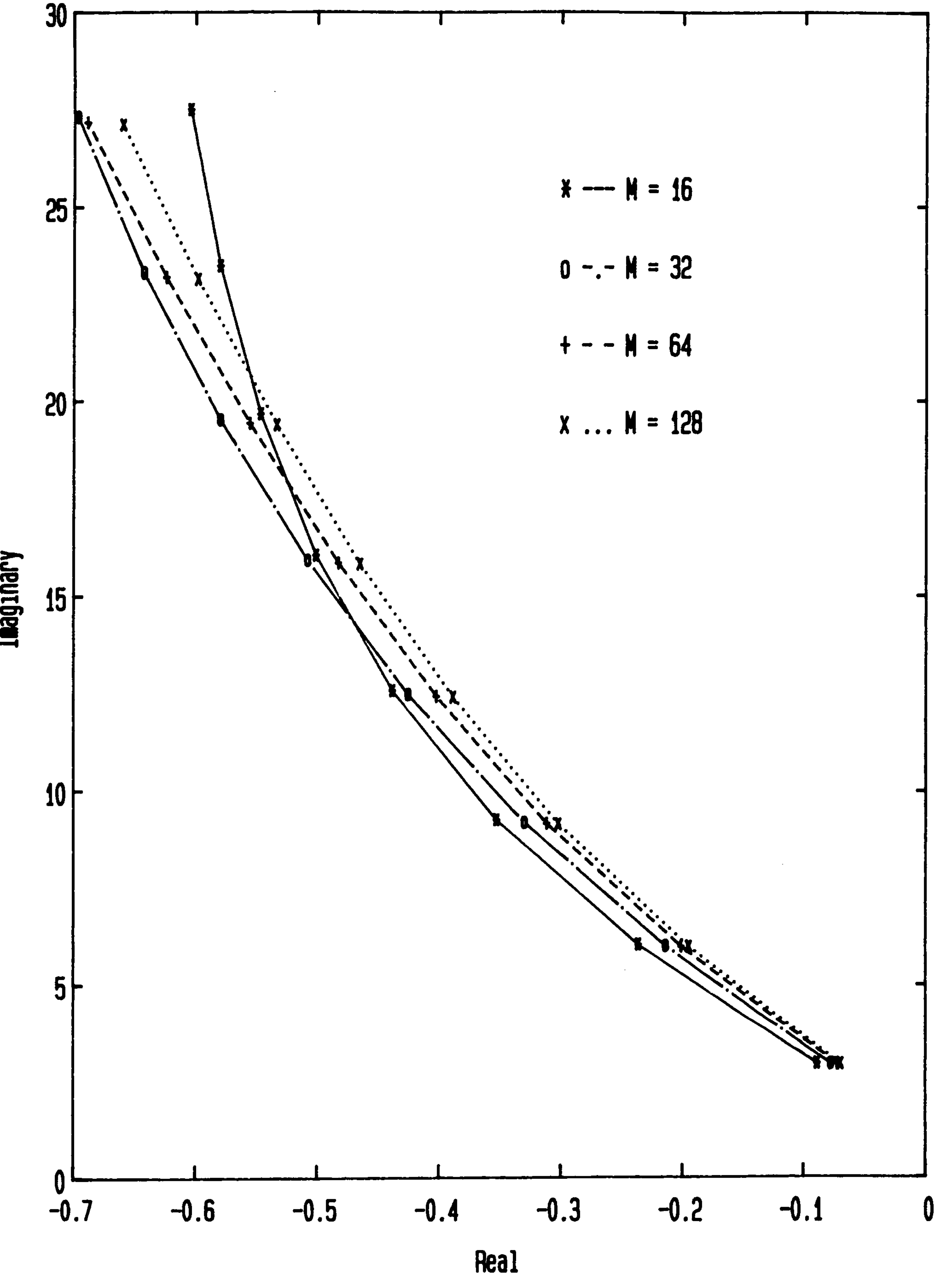

Figure 4.2.1 Uniform Mesh, Linear Spline N = 16

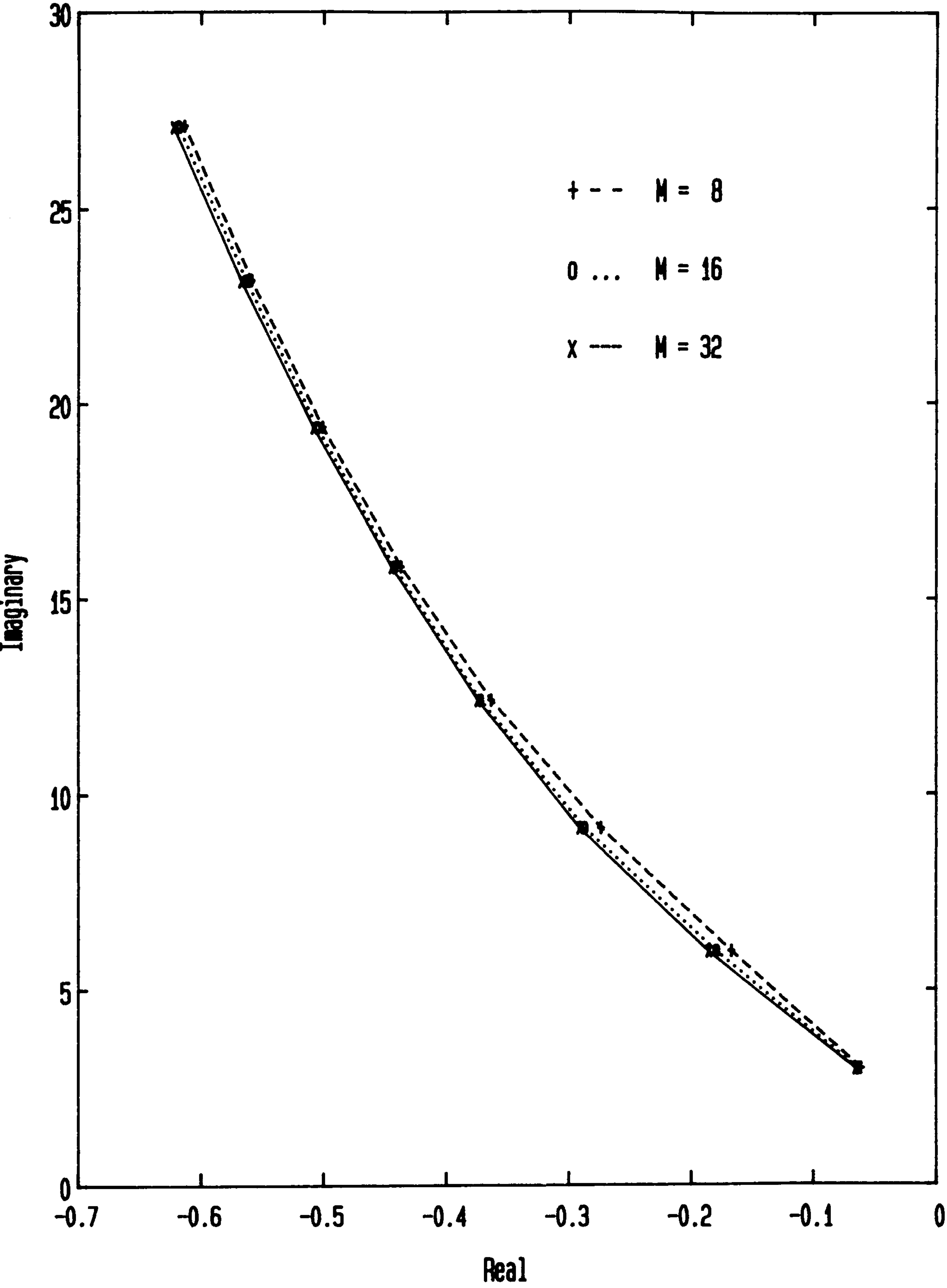

Figure 4.2.2 Nonuniform Mesh, Linear Spline N = 16

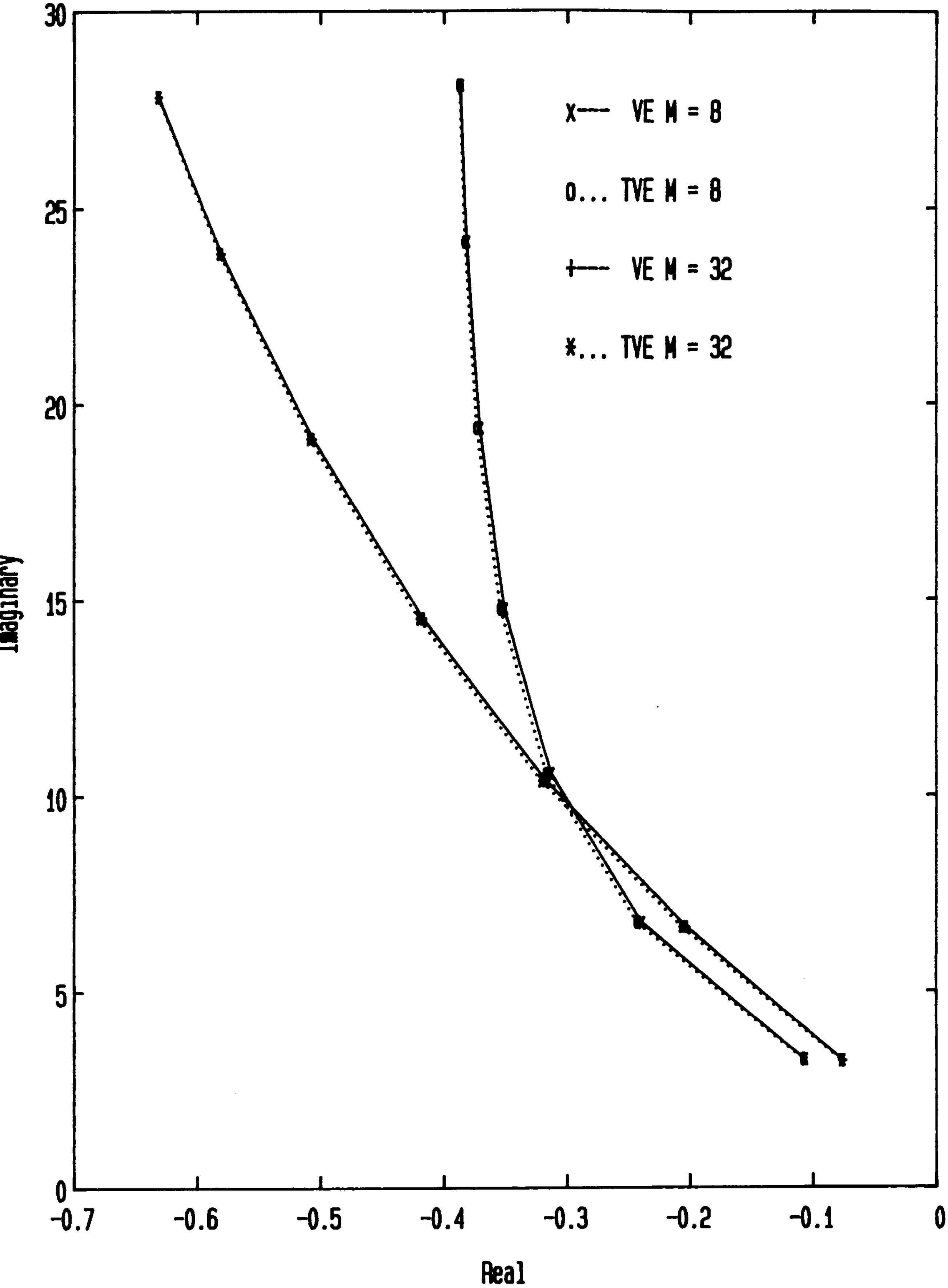

Figure 4.2.3 Nonuniform Mesh, Linear Spline N = 8

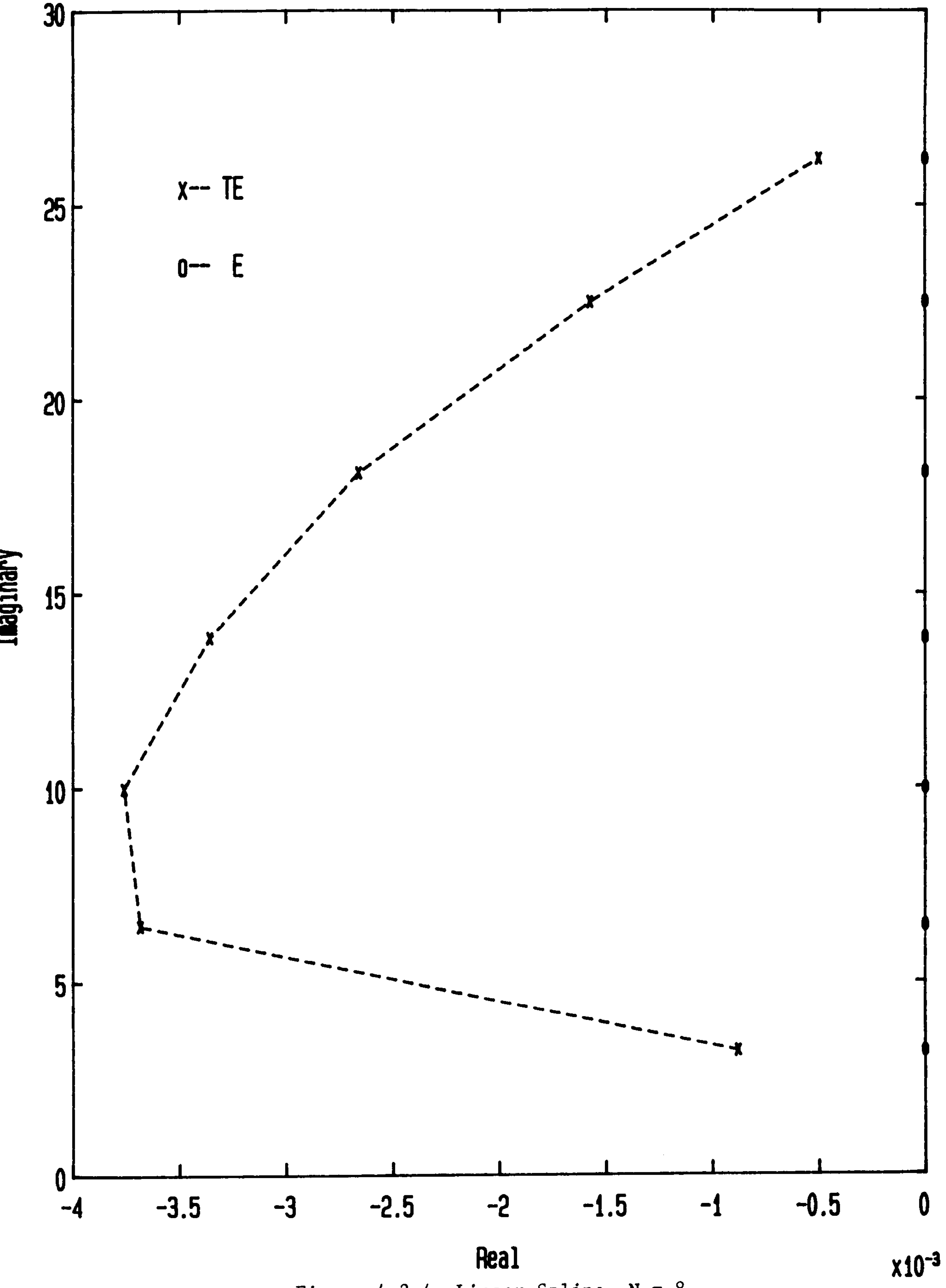

Figure 4.2.4 Linear Spline, N = 8

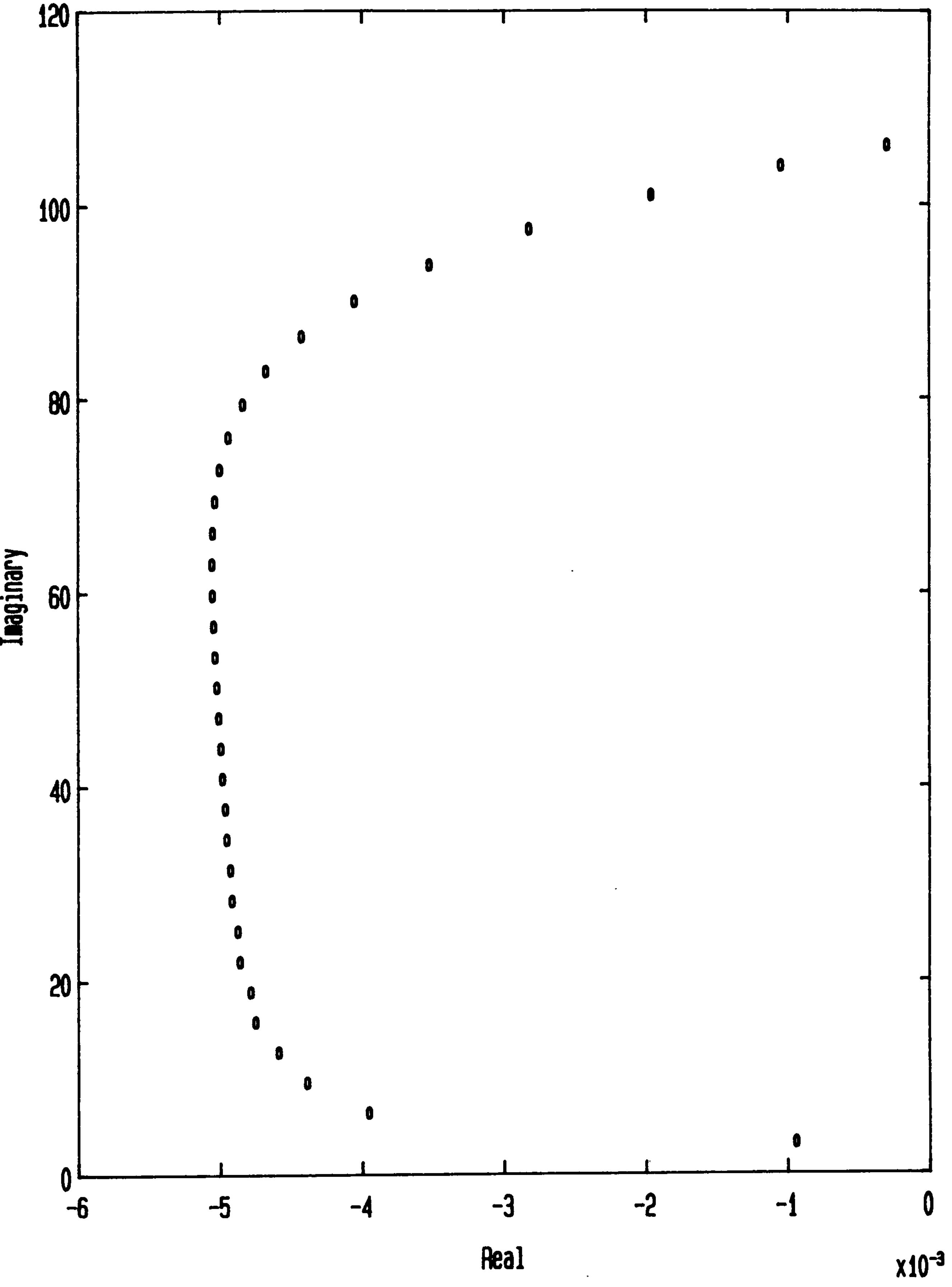

Figure 4.2.5 Cubic Spline N = 32

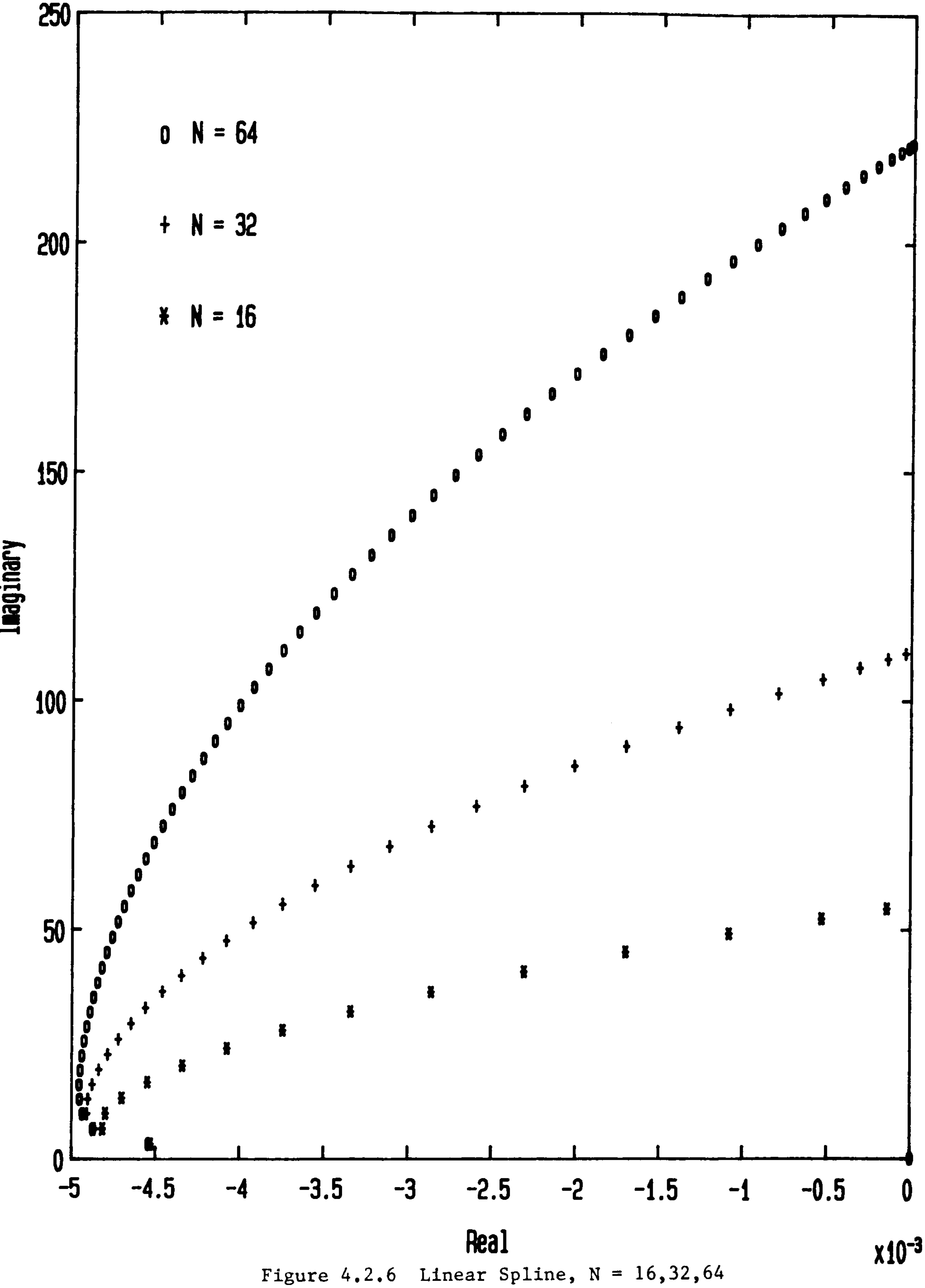

Figure 4.2.6 Linear Spline, N = 16,32,64

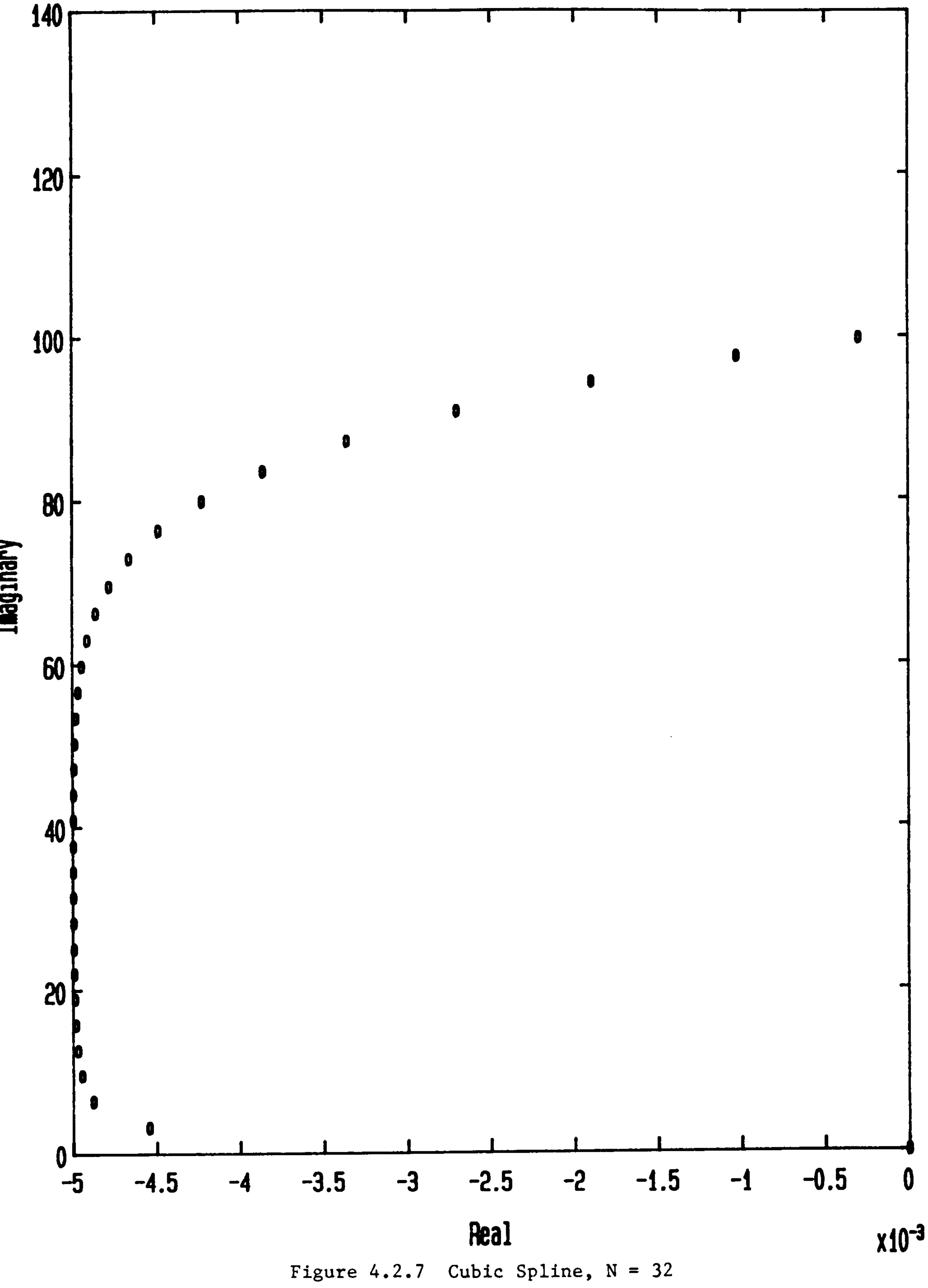

Figure 4.2.7 Cubic Spline, N = 32

Table 4.1.1 The Eigenvalues of $A^{8,32}$ for TVE, VE Systems

j	$Re\lambda_{TVEj}$	$Im\lambda_{TVEj}$	$Re\lambda_{VEj}$	$Im\lambda_{VEj}$
1	−.076590	3.196795	−.075694	3.196740
2	−.206360	6.628444	−.202587	6.626654
3	−.320412	10.375504	−.316549	10.374760
4	−.420027	14.529327	−.416604	14.528763
5	−.508636	19.109730	−.505924	19.109368
6	−.581964	23.849238	−.580385	23.849035
7	−.631847	27.825029	−.631340	27.824970

Table 4.1.2 The Eigenvalues of A^{8} for TE, PE Systems

j	$Re\lambda_{TVEj}$	$Im\lambda_{TVEj}$	$Re\lambda_{VEj}$	$Im\lambda_{VEj}$
1	−.000878	3.161881	−.000000	3.161816
2	−.003685	6.447351	−.000000	6.445663
3	−.003761	9.975069	−.000000	9.974391
4	−.003359	13.857001	−.000000	13.856406
5	−.002661	18.119114	−.000000	18.118802
6	−.001574	22.517388	−.000000	22.517212
7	−.000500	26.201425	−.000000	26.201376

4.3 Closed-loop Control Problem.

We turn now to a quadratic regulator problem for the system (1.1.1)-(1.1.3). Let $0 < x_1 < x_2 \cdots < x_l < 1$ be a partition of the bar. Our goal is to drive the average displacements and velocities at x_i to zero. Thus, for $\delta > 0$ define the operators $\mathcal{M}_i^\delta : L_2(0,1) \to \mathbf{R}$ by

$$\mathcal{M}_i^\delta(\varphi) = \frac{1}{2\delta} \int_{x_i-\delta}^{x_i+\delta} \varphi(x)dx \tag{4.3.1}$$

and let $\mathcal{M}^\delta : L_2(0,1) \to \mathbf{R}^l$ be defined by

$$\mathcal{M}^\delta(\varphi) = \left(\mathcal{M}_1^\delta(\varphi), \mathcal{M}_2^\delta(f), \cdots, \mathcal{M}_l^\delta(\varphi)\right)^T. \tag{4.3.2}$$

Let $o_i, p_i > 0, i = 1, 2, \cdots, l$, and denote by O and P the $l \times l$ diagonal matrices $O = diag(o_1, o_2, \cdots, o_l)$ and $P = diag(p_1, p_2, \cdots, p_l)$, respectively. Given $z_0 \in Z$ we wish to minimize

$$J^\delta = \int_0^\infty \left[\left\|OM^\delta(y(t,\cdot))\right\|^2 + \left\|PM^\delta\left(\frac{\partial}{\partial t}y(t,\cdot)\right)\right\|^2 + R|u(t)|^2 \right] dt$$

where $z(t) = (y(t,\cdot), \frac{\partial}{\partial t}y(t,\cdot), \theta(t,\cdot), w(t,s,\cdot))^T$ is the solution to (1.1.1)-(1.1.3) with initial data $z(0) = z_0$. It is well known that ifJ^δ has a minimizer $u^*(t)$, then it is given by a state feedback; i.e., there is a bounded gain operator $\mathcal{K} : Z \to \mathbf{R}$ such that $u^*(t) = -\mathcal{K}z^*(t)$. We are interested in the closed-loop system

$$\dot{z}(t) = (\mathcal{A} - \mathcal{B}\mathcal{K})\, z(t). \tag{4.3.3}$$

In particular, we wish to compare the closed-loop system (4.3.3), $\gamma > 0$, to the strictly viscoelastic case, $\gamma = 0$.

The numerical results below were computed using parameter values given in the last section. The sensor locations are at $x_1 = 0.25$, $x_2 = 0.32$, $x_3 = 0.50$, $x_4 = 0.67$ with $\delta = 0.01$. The output weights are $q_i = p_i = 1, i = 1, 2, 3, 4$, and control weight is set at $R = 0.01$. Since the theory does not guarantee convergence of the adjoint for an L_1 kernel, we replace g by the function g_p defined by

$$g_p(s) \equiv \begin{cases} g(s), & \text{for } -r \le s \le -p, \\ g(-p) + g'(-p)(s+p), & \text{for } -p \le s \le 0. \end{cases}$$

We will take $p = 2^{-10}$.

The operator $\mathcal{B} : \mathbf{R} \to Z$ is given by

$$\mathcal{B} = \begin{pmatrix} 0 \\ x^2 \\ 0 \\ 0 \end{pmatrix}.$$

Let $\mathcal{B}u = \sum \alpha_j f_j$ where $\{f_i\}$ is a basis for $Z^{N,M}$. Then $\sum \alpha_j \langle f_j, f_i \rangle_Z = \langle \mathcal{B}u, f_i \rangle_Z$ for all i, which implies $\sum \alpha_j \langle e_j, e_i \rangle_{Y \times V \times \Theta} = \left\langle \begin{pmatrix} 0 \\ x^2 u \\ 0 \end{pmatrix}, e_i \right\rangle_{Y \times V \times \Theta}$ where $\{e_i\}$ is a basis for $Y^N \times V^N \times \Theta^N$. Thus, $\vec{\alpha} = \begin{pmatrix} 0 \\ uH^{-1}B_1 \\ 0 \end{pmatrix}$ where $(B_1)_i = \int_0^l b(x) h_i(x) dx$. Thus,

$$B^{N,M} = \begin{pmatrix} 0 \\ H^{-1}B_1 \\ 0 \end{pmatrix}.$$

Take $\mathcal{Q} = \mathcal{C}^*\mathcal{C}$ where

$$\mathcal{C} \begin{pmatrix} y \\ v \\ \theta \\ w \end{pmatrix} = \begin{pmatrix} O\mathcal{M}^\delta y \\ P\mathcal{M}^\delta v \\ 0 \\ 0 \end{pmatrix}$$

In order to construct the approximation to C, we merely have to integrate the "approximate delta functions" δ_i^ϵ defined by

$$\delta_i^\epsilon = \frac{1}{2\epsilon}\chi_{[x_i-\epsilon, x_i+\epsilon]}, \qquad i = 1, 2, \ldots, n$$

against each of the basis functions.

We will use Potter's method (see [**21**]) to solve the finite dimensional algebraic Riccati equation (1.3.6). The first step in Potter's method is to form the matrix

$$P^{N,M} = \begin{bmatrix} A^{N,M^T} & Q^{N,M} \\ B^{N,M}R^{-1}B^{N,M^T} & -A^{N,M} \end{bmatrix}.$$

Next, find the eigenvalues and eigenvectors of $P^{N,M}$ and form the matrix $Z = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ where the columns of Z are the eigenvectors of $P^{N,M}$ corresponding to the eigenvalues with positive real part. When eigenvalues occur in complex conjugate pairs, so do the eigenvectors. In this case, the real and the imaginary part of the eigenvector each forms a column of Z. Finally, the solution to the Riccati equation is given by $\Pi = X_1 X_2^{-1}$. Once we have found Π, we can compute the "gain" matrix $K^{N,M} = -R^{-1}B^{N,M^T}\Pi$ and the "closed-loop" matrix $A^{N,M} + B^{N,M}K^{N,M}$. We will only consider the boundary condition (1.1.3) in the numerical computation in this section.

Figure 4.3.1 compares the closed-loop eigenvalues for the thermoviscoelastic system and the viscoelastic system, using $N + 1 = 9$ cubic splines and the nonuniform mesh with $M = 16$. Similar to the case of open-loop eigenvalues, we again observe that the effect of the thermal damping is almost negligible.

Figure 4.3.2 compares the closed-loop and open-loop eigenvalues for the thermoviscoelastic system, also using $N+1=9$ cubic splines and the nonuniform mesh with $M=16$. It is interesting to note that the thermal damping combined with active control enhances the damping in lower modes much more than in the higher modes and damping at higher frequencies is due almost entirely to the viscoelastic damping.

Since $u^*(t) = \mathcal{K}z(t)$, by Riesz-Representation Theorem, there exist $\hat{z} \in Z$ such that $\mathcal{K}z(t) = \langle \hat{z}, z(t) \rangle_Z$. Let $\hat{z} = (K_1(\cdot), K_2(\cdot), K_3(\cdot), K_4(\cdot,\cdot))^T$. We wish to find approximations to the "functional gains" K_1, K_2, K_3 and K_4 (See [**15**] for the details). In Figure 4.3.3 - 4.3.5, we show the plots of K_1, K_2 and K_3, using nonuniform mesh and cubic spline with $N=8$. Observe that the curves for $M=8$ and $M=16$ are nearly indistinguishable, which implies that to get a good approximation of the gain functional we do not need a large M for the nonuniform mesh.

We would like to comment at this point that we haven't proved the convergence for the approximation of the functional gains primarily because we are unable to get the uniform stabilizability of the final dimensional approximation systems (see [**5**]) by now, although our numerical results strongly support it.

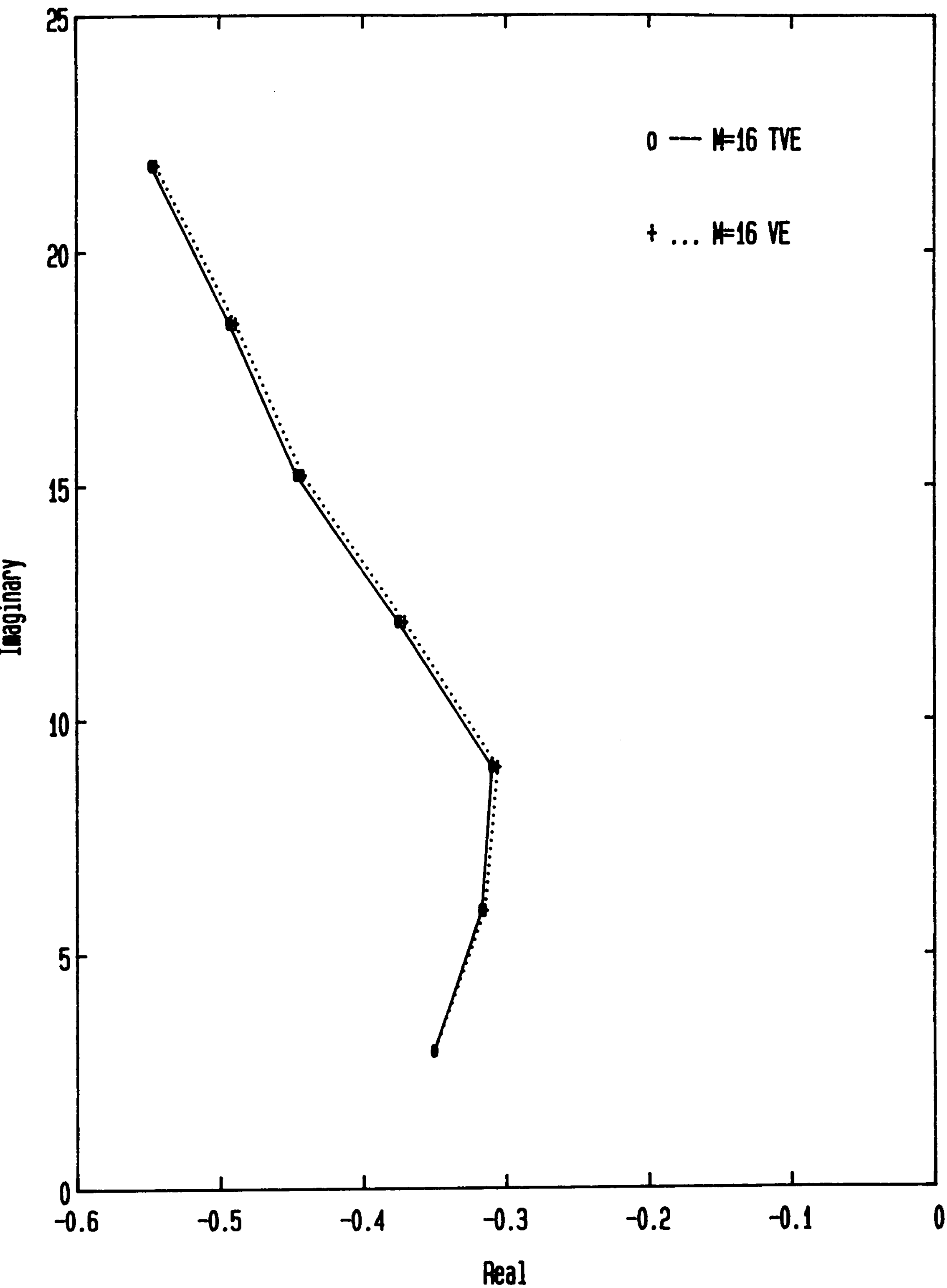

Figure 4.3.1 Nonuniform Mesh, Cubic Spline N = 16

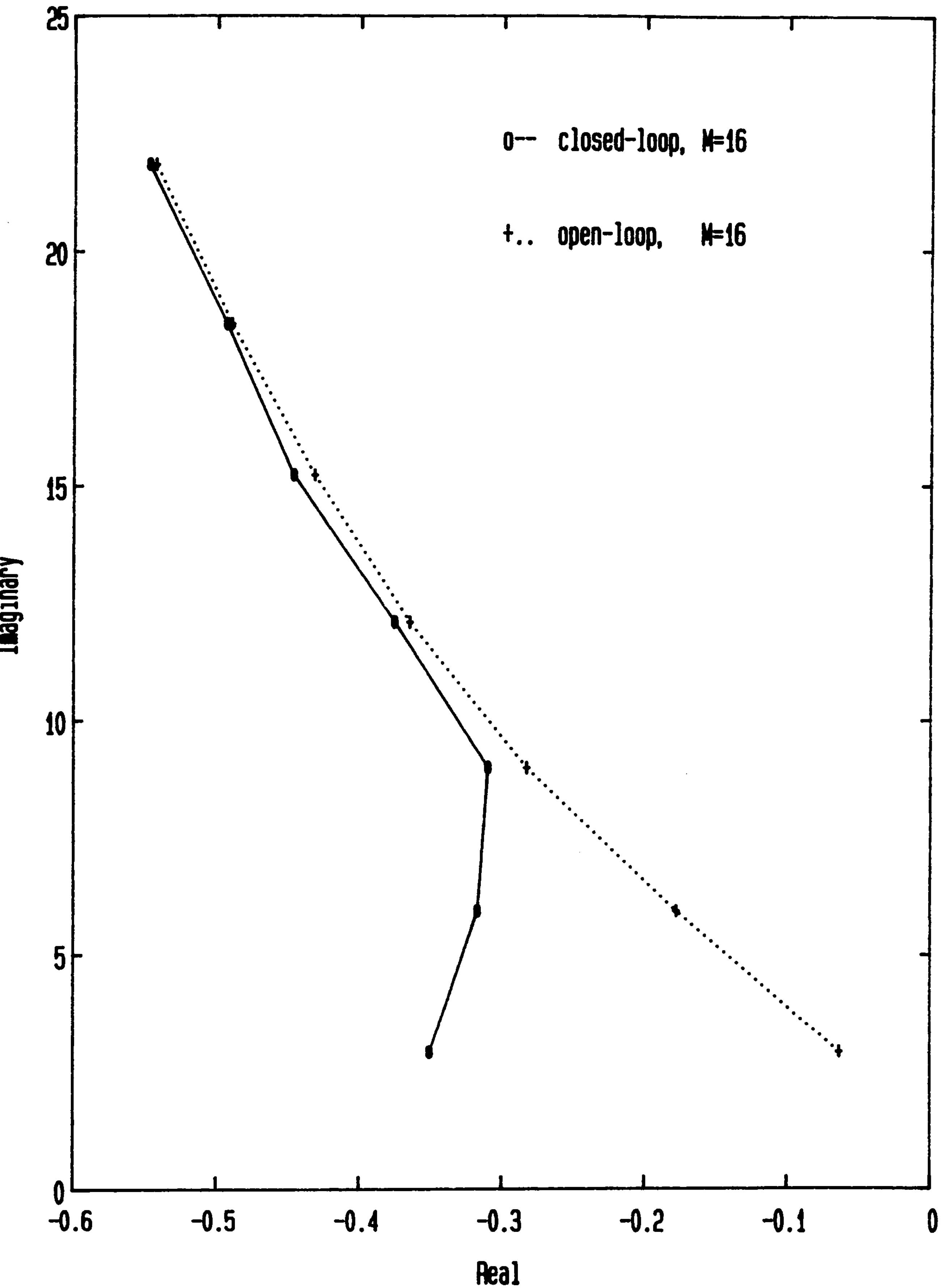

Figure 4.3.2 Nonuniform Mesh, Cubic Spline N = 16

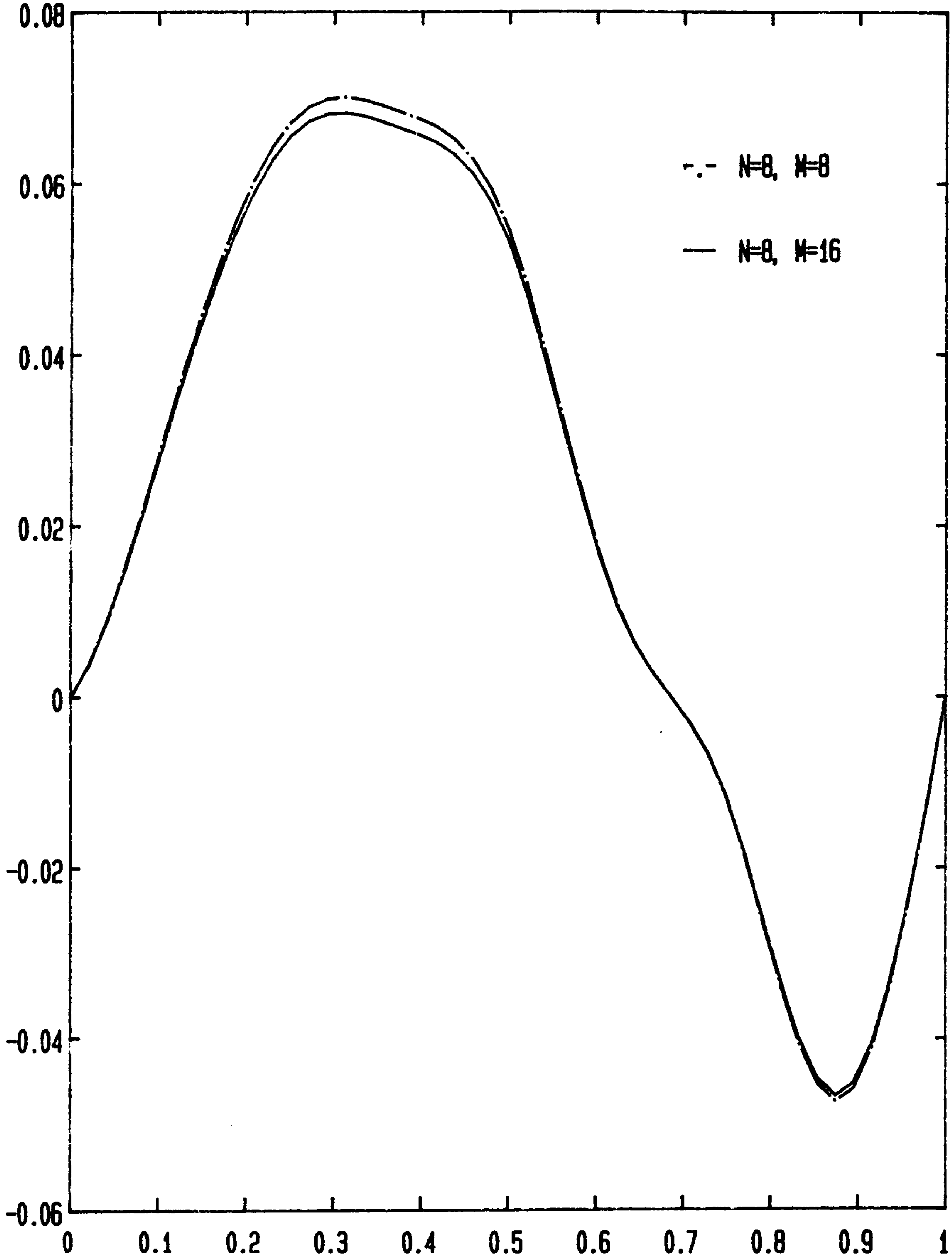

Figure 4.3.3 Kl(x), Nonuniform Mesh, Cubic Spline

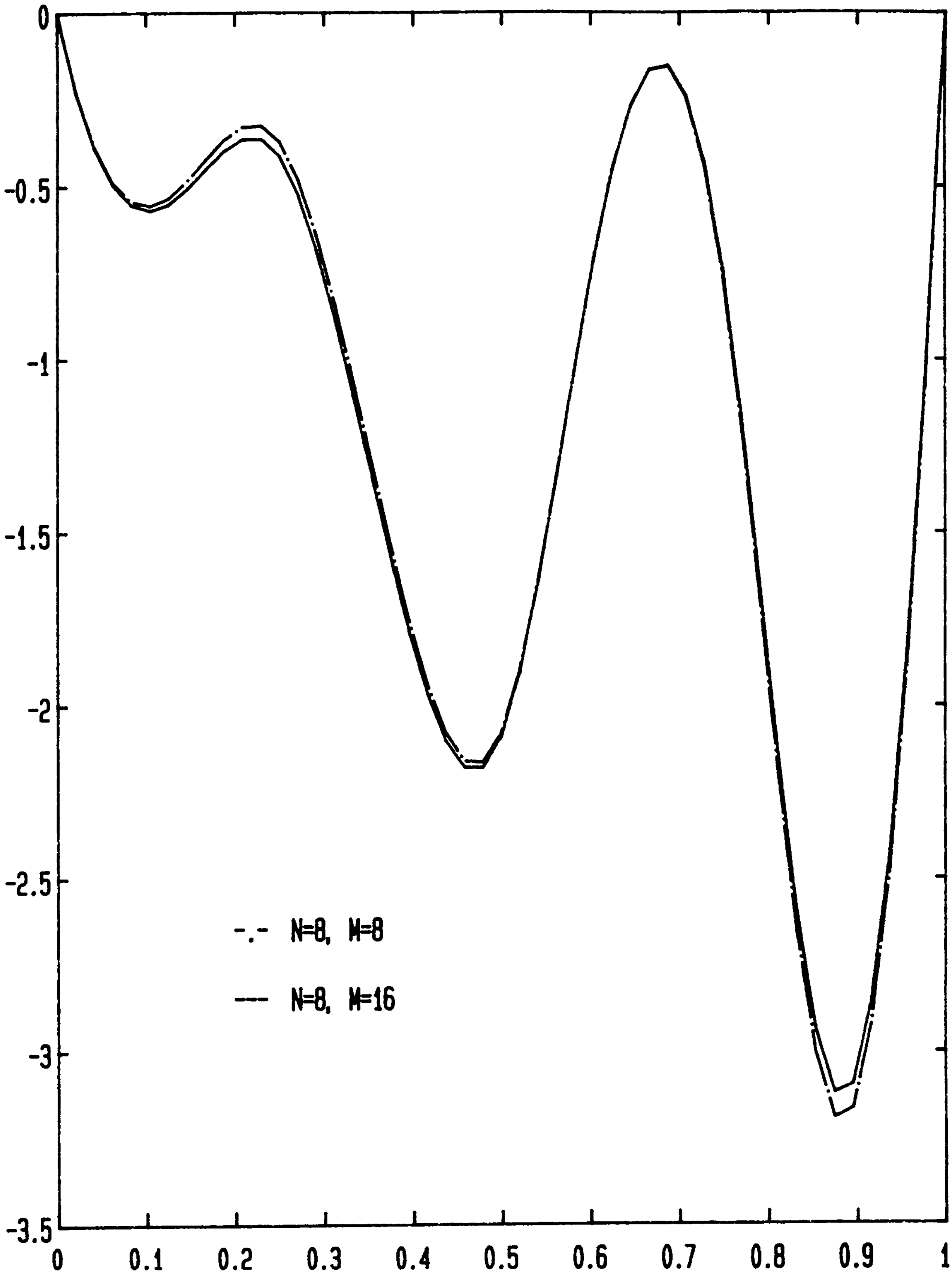

Figure 4.3.4 K2(x), Nonuniform Mesh, Cubic Spline

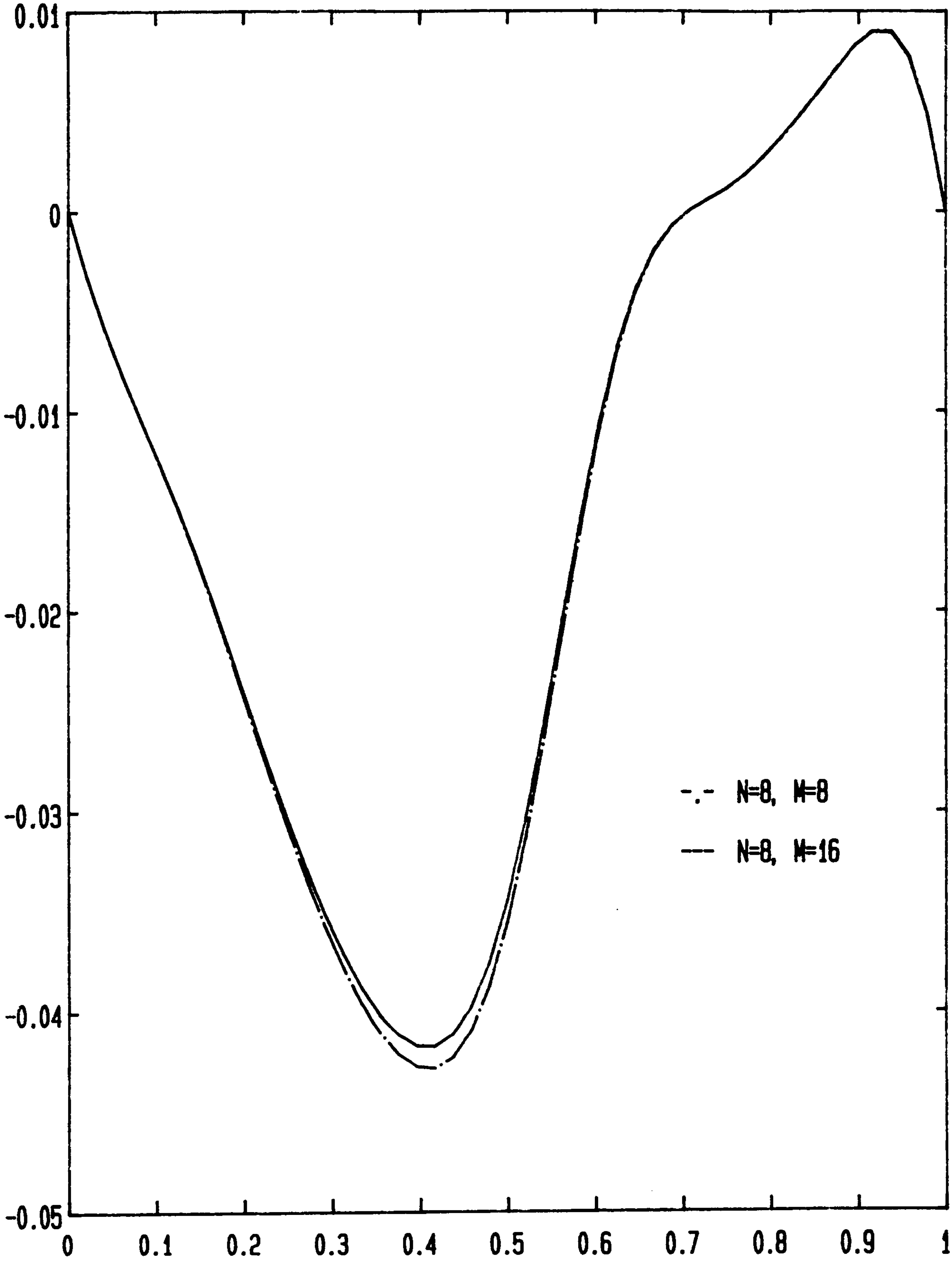

Figure 4.3.5 K3(x), Nonuniform Mesh, Cubic Spline

CONCLUSION

In this paper we have studied a thermoviscoelastic model with a singular kernel in the strain memory term. We provided a semigroup setting, and then proved a generalized result on well-posedness which can be applied to this model with various boundary conditions we considered. We also gave a proof for the asymptotic stability. We applied a finite element/averaging approximation scheme to carry on the numerical computation. We showed the convergence of the approximating systems, and the convergence of the approximating adjoint systems for bounded kernels. Finally, we estimated the eigenvalues for the open-loop and closed loop problem to investigate the effect of heat dissipation on damping and control.

There are still many related works need to be considered in the future. In particular

(1) investigating the convergence of the approximating adjoint systems for singular kernels;

(2) investigating the uniform stabilizability of the approximating systems;

(3) considering the case when the stress is also depend on the temperature history.

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