Timber Supply in Dynamic General Equilibrium

by

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(ABSTRACT)

Given the neoclassical assumptions of optimizing economic agents, perfect information, perfect competition, and productive efficiency, timber supply is a dynamic process. Different discrete-time dynamic timber supply models and their solution methods are compared and their common elements derived. A continuous-time model is derived, but not solved. The discrete-time timber supply model is then incorporated into a dynamic multi-sector model and a dynamic general equilibrium model. In the multi-sector model, all household's utility functions are aggregated into a single community utility function which is maximized subject to the technology of the economy. The technology for the forest sector is the same as in the discrete-time dynamic timber supply models. Wood is treated as an intermediate input into the production of consumer goods. The technology of the consumer goods sectors is based on the technology used in computable general equilibrium models. The optimal steady state problem for this model is discussed, and the solution for an example problem is presented. Disaggregating the utilty function is necessary for modeling true general equilibrium. This greatly complicates the problem of finding numerical solutions, but enriches the model considerably. The formulation of the general equilibrium model as an optimization problem is described, but proved rather difficult to solve. The optimal steady state problem can be solved using an algorithm developed by Scarf (1967) for finding fixed points of continuous functions. The fixed-point approach provides a reliable solution method and appears to have more potential for modeling departures from perfect competition than the optimization approach. The equivalence of the two approaches is discussed.

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CHAPTER 1: INTRODUCTION

It is a bit surprising with all the thought and work that has gone into modeling timber supply that there is anything left to be done. There is still work to be done, however, and there probably will be for a long time to come. Nevertheless, progress is continually made in the area, and recently this progress has been significant. The recent dynamic models of timber supply are particularly interesting. An important theme of this dissertation is that timber supply is a dynamic process. What this means for how we think of and model timber supply will be considered at some length.

The scope of timber supply models has almost always been fairly narrow. The focus of these models has been primarily on extractive uses of the forest. The context of many forestry issues is much broader than that, especially today. For example, the modern forest planning process involves conflicts over jobs, recreation opportunities, wildland and wildlife preservation, and a general desire for maintaining a healthy level of environmental quality. These sometimes conflicting uses of the forest tend to become less compatible as populations and incomes grow.

Objectives and Justification

The objective of the research reported in this dissertation has been to extend the current dynamic models of timber supply to a general equilibrium framework. Since the timber supply models are dynamic, the general equilibrium model must also be dynamic. The primary emphasis of the work has been on establishing the theoretical foundations for such models. However, the usefulness of these general equilibrium models would be severely limited if they were unsolvable. Thus, much of the research effort has been directed toward the problem of computing solutions to the models developed.

The general equilibrium models presented below will give the broad purview which has been lacking in most of the timber supply models. These broader models will be useful for analyzing a variety of forestry issues including the income and employment effects of forest policies, trade in forest products, forest taxation, land use dynamics, and trade-offs between competing forest uses. They will also be useful for extending the vision of forest economists beyond the sideboards of their sectoral models, showing how the forest economy fits in and interacts with the rest of the economy.

The usefulness of dynamic general equilibrium models goes beyond the forestry sector. Other renewable resources and also nonrenewable resources can be incorporated in the framework of such models. They could be used for studying fisheries, energy, mining, and agricultural issues, to list only a few examples. Including natural resources in these models can have interesting implications. As will be discussed in Chapter 4, the models presented below have a strong resemblance to the economic growth models that appeared in the 1960's. The inclusion of a fixed factor such as land makes the perpetual growth paths of those models infeasible. Thus, the models presented below have a bit of a neo-Malthusian flavor.

Neoclassical Assumptions

The models in this dissertation are neoclassical. That is, the standard assumptions of neoclassical economic theory underlie each of the models. Briefly, these standard assumptions include: 1) agents are rational optimizers, that is, consumers have utility functions which they seek to maximize subject to their budget constraints, and producers seek to maximize profits or minimize costs, given the technology of production; 2) perfect information, which in a nonstochastic, dynamic context means not only that agents have full knowledge of the current values of all the relevant variables in the economy, but also that they have perfect foresight regarding future values of these variables; and 3) market clearing or equilibrium, which, in the models discussed below, occurs in the context of perfect competition where all agents are price takers. Models of imperfect competition with alternate rules for market clearing are not outside the neoclassical economics paradigm, but will not be considered in this dissertation. Note that assumption 1 also implies productive efficiency. That is, the maximum output possible is always obtained from whatever set of inputs is used, and the same output could not be obtained with less of any input without increasing some other input.

These assumptions are not made lightly. Each can easily be attacked as unrealistic. Almost anyone can cite an incident involving behavior that would be hard to characterize as rational. Productive efficiency is probably rare, rather than the norm. Few people would claim to have perfect foresight, let alone full knowledge of current affairs. Finally, many markets clearly do not meet the standards of perfect competition.

There are good reasons for making these assumptions, however. First, these assumptions have provided the foundation for a large body of research which has generated significant and sometimes profound insights into a variety of real-world phenomena. Empirical studies have demonstrated the usefulness of these assumptions by providing compelling explanations of the phenomena which generated the data sets to which the models were applied. In many ways, these assumptions appear to be useful abstractions of economic realities for the purposes of explanation and sometimes prediction. While they are obviously simplifications at best, they provide reasonable explanations for many real-world observations.

Another important reason for making these assumptions is that they make it possible to construct tractable models of extremely complex systems. Thus, while the models may have obvious weaknesses, they may still be the best that can be done at the present time. Without a clearly superior model, we generally cannot afford to throw out the models we have. Still, we should be careful to recognize the weaknesses of our models and not try to oversell the models. Models can be dangerous things in the hands of those who place too much confidence in them. For this reason, an attempt has been made to ensure that the assumptions and weaknesses of the models presented in this dissertation have been clearly stated. One of the strengths of these models is that the assumptions, and sometimes the weaknesses, tend to be fairly transparent.

A final reason for making the neoclassical assumptions in building these models is that these are the assumptions most economists carry around in the backs of their heads, if not directly at the forefront of their consciousnesses. Often economists are not aware of the problems inherent in making these assumptions. The weaknesses of the neoclassical assumptions become particularly apparent in the context of a dynamic, general equilibrium model. Exposing the weaknesses of our theories helps keep us open-minded, and indicates where future research efforts should be directed.

Overview

The dissertation begins with a review of the development of thinking about timber supply in Chapter 2. An early approach -- the so-called "gap models" -- is distinctly non-economic. The word "gap" reflects the fact that market clearing forces are ignored. Many of the later attempts to incorporate economic concepts into timber supply models have essentially been technology transfer, adapting concepts from traditional economics to forestry. By contrast, the dynamic models of timber supply reflect attempts to think about timber supply from the ground up, beginning with the fundamental technology of growing forests and a profit-maximization assumption. Interestingly, long-run timber supply models -- one of the oldest approaches to modeling timber supply -- are equivalent to steady-state formulations of dynamic models.

A detailed study of the dynamic timber supply models is presented in Chapter 3. Several different approaches are discussed and have a great deal in common. A simplified model of timber supply is formulated successively as a linear programming problem, a dynamic programming problem, a nonlinear programming problem, and a discrete-time optimal control problem. A continuous-time timber supply model is also formulated, but not solved. In looking at the different approaches, the process of constructing a dynamic model by starting with a fundamental description of the technology of timber growing is illustrated. This formulation of a dynamic supply problem is quite different from the usual static approach to supply theory.

In Chapter 4, the scope of the model is expanded to include the entire economy from production to consumption. This is achieved by imposing several additional simplifying assumptions -- most notably that individual utility functions can be aggregated to obtain a community utility function. The model is a maximization problem where the community utility function is maximized subject to the technological and resource constraints of the economy. The problem of finding optimal steady states of the model is discussed, and the solution to an example problem is presented.

The utility aggregation assumption is relaxed in Chapter 5. The resultant model is a maximization problem where the objective function is a weighted sum of the utility functions of individual households. The weights are determined endogenously, and their relative size is related to each household's income. The model is difficult to solve, and no solution was obtained for even a simple example problem using MINOS, a commercial nonlinear programming software package.[1]

The optimal steady state of the problem in Chapter 5 did prove to be obtainable, but not using an optimization approach. The problem can be formulated as an equivalent equilibrium problem and solved using Scarf's (1967) algorithm for approximating a fixed point of a function. A program was written to implement Scarf's algorithm for this particular problem. This approach is described in Chapter 6.

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[1] MINOS stands for Modular In-core Nonlinear Optimization System. Documentation can be found in Murtaugh and Saunders (1983).

A final chapter reviews the models developed and discusses what has been achieved and the major problems with the models. The problems of applying the model empirically are discussed and two alternate approaches considered. The final chapter also includes a discussion of the methodological issues raised by the work reported in the dissertation.

CHAPTER 2: A HISTORY OF TIMBER SUPPLY MODELS

Timber supply is a perennial issue in forest economics. In the United States, debates over timber supply go back to before 1891 when the original forest reserves were set aside. Fear that excessive harvesting would lead to wood shortages provided one of the impetuses for creating the reserves. Timber supply is an important issue in forest economics because it encompasses virtually all of the production economics of forestry: timber supply is the result of all the decisions that are made in forest management.

The problem of how to model timber supply is still unresolved. This is surprising in view of the fact that supply theory in general economics is quite firmly established. There are some important factors which complicate the modeling of timber supply, however. A key factor is that, because of its dynamic nature, timber supply is fundamentally different than the standard supply problems discussed in microeconomics textbooks. There are other complications as well. Among these are: the importance of spatial factors in forestry; the variety of forest owners, with their variety of management objectives; the biological and ecological complexity of many forest systems; and the multiple products of forests -- including not only sawtimber, pulpwood, and fuelwood, but wildlife, water, aesthetics, environmental quality, and many other noncommensurable, but highly valued natural amenities. The multi-dimensional nature of timber supply makes it hard to neatly categorize timber supply models. The discussion below focuses on the dynamic dimension of timber supply studies.

Duerr (1960) used the classical approach in economics of defining the length of run by the number of inputs that are fixed in the production process. Thus, he distinguished between stock, short-run, and long-run supply responses, ranging from holding virtually all inputs fixed in the stock response to the long-run response where almost all inputs are variable. Duerr recognized the arbitrary nature of these distinctions (Duerr 1960, p. 199).

A different classification is used below. In this chapter, the distinction is between dynamic models and their steady states, and static and pseudo-dynamic models. Dynamic models are defined in this dissertation as models incorporating an explicit intertemporal optimization process.[2] Most long run models of timber supply can be viewed as the steady-state versions of dynamic models. Static models are those where intertemporal optimization is either missing completely, not explicitly considered, or given only cursory attention. Also, static models are typically one-period models. Finally, pseudo-dynamic models are those that project market states over a multi-period time horizon by solving a static model for each period.

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[2] This is admittedly a narrow definition. It is, however, consistent with the neoclassical underpinnings of the model. A broader usage might include changing tastes and preferences, technology, and institutions, issues that are beyond the scope of the present work.

Static Models

The dynamics of timber supply ultimately boil down to future price expectations. Even if all other current variables were held constant, a change in expectations of future prices would lead to a change in current timber supply.[3] Most econometric models of timber supply do not include variables reflecting future price expectations. Thus, most econometric timber supply models are static.

Early econometric models of timber markets (eg. McKillop 1967, Robinson 1974, and Mills and Manthy 1974) focused on estimating demand and supply relationships for multiple market levels from raw material supply, through at least one processing stage, and including final demand for products. More recent models have focused more on raw material markets and have introduced production duality theory for model specification (eg. Newman 1987, and Brannlund, Johansson, and Lofgren 1985). Brannlund, Johansson, and Lofgren (1985) include a specific term for modeling price expectations, however, so theirs is not a strictly static model. Their expectations model is based on a distributed lag of prices, thus introducing lagged terms in the supply function. However, none of the lagged terms in their model have statistically significant coefficients except lagged price. This suggests that the dynamics are either unimportant or inadequately captured by their formulation.

Many of the pseudo-dynamic models discussed below use econometrically estimated static supply functions. For example, regional supply functions were estimated for each region modeled in TAMM (Adams and Haynes 1980, 1985). Several of the spatial price

3 At least, if this statement is false, then timber producers can hardly be profit maximizers, and we need a new theory.

equilibrium models simply combine an elasticity estimate with a single observed point to specify timber supply functions (eg. PAPYRUS and GTM (cited below)).

Current inventory is a common variable in econometrically estimated timber supply functions. Inventory has even been used in duality-based models (Newman, 1987) where the independent variables are usually limited to prices only. Daniels and Hyde (1986) do not refer to duality theory, but they include only price variables and inventory in their dependent variable set. Intuitively one would expect inventory to be an important determinant of timber supply. Experience with dynamic models suggests that perhaps more than just inventory levels should be included in these equations. This is because price expectations should be related to the current age-class distribution of the timber resource. None of the econometric models have included the age-class distribution of the forest.

The use of computable general equilibrium (CGE) models for analyzing forest sector issues is a recent development.[4] All of these (Boyd 1987, Boyd and Newman 1988, and Constantinio and Percy 1988) have been strictly static models, although the Constantinio and Percy (1988) model does distinguish between short-run and long-run scenarios. The CGE approach is especially useful for studying regional income and employment impacts of forest policies, and tax and trade issues. Given the potential usefulness of the CGE models, it is unfortunate that these models are not compatible with the dynamics of timber supply. An important reason for doing the research reported in this dissertation is this incompatibility of this static approach with forestry problems -- particularly timber supply modeling.

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4 Even in general economics, these models are relatively recent. See, for example, de Janvry and Sadoulet (1987), Dervis, de Melo, and Robinson (1985), Dixon et al. (1982), Ginsburgh and Waelbroeck (1981), Scarf and Shoven (1984), Shoven and Whalley (1984), and Srinivasan and Whalley (1986).

The household production timber supply models (Binkley 1981 and 1987) have a rather different emphasis from most timber supply models. These models focus on the behavior of private forest owners who own forests for income and for the personal satisfaction they derive from the forest through recreation, wildlife management, hunting, and for their aesthetic value. The nontimber objectives of these owners can significantly influence their propensity to harvest and thus the supply of timber to processors. Binkley's original (1981) model is strictly static. In his later (1987) discussion of the model, Binkley presents a dynamic version, but does not offer a solution. A dynamic, two-period version of the household production model is presented and solved in Max and Lehman (1988). This approach is also related to the steady state models discussed by Hartman (1976) and Strang (1983).

Pseudo-Dynamic Models

Many of the models of timber supply cannot be labeled as "static" because they project forest and market states over several periods. The word "dynamic", however, will be reserved for models which explicitly incorporate an intertemporal optimization process. The label "pseudo-dynamic" will therefore be used to describe a large class of models that falls somewhere in between the static and the dynamic categories. The defining characteristic of these pseudo-dynamic models is that they are multi-period, but they use a static model for each time period. The models project the state of the system over some time horizon in a stepwise fashion. That is, the model is solved for the first period, the state of the system updated appropriately, and then the model is solved for the next period, and so on.

Some of the earliest timber supply models were the Forest Service "gap models" (U.S.D.A. Forest Service 1958, 1965, and 1973). These models used a pseudo-dynamic approach. Harvest levels were extrapolated into the future based on then-current management practices. Future forest inventories were then projected given these harvest levels using biometric growth models. Consumption requirements were also forecast based on demographic trends. The differences between projected requirements and harvest levels (the "gaps") tended to be positive and increasing with time because demographic trends inevitably led to increasing consumption projections, while harvests were projected based on maintaining historical levels. These projected gaps provided an excellent political device for the Forest Service to obtain higher budgets from the Congress and maintain public support for their agenda.

The gap studies ignore the fundamental neoclassical economic concept of market clearing. The usual mechanism of market clearing is price changes. This argument was made by several forest economists in the 1950's (Vaux and Zivnuska 1952, Gregory 1955 and 1958, and Duerr 1960). Duerr (1960) also insisted that interest rates should play an important role in timber supply due to their importance in the intertemporal forestry optimization problem. None of these researchers proposed a workable alternative for modeling the intermediate-term dynamics of timber supply problems, however. Nevertheless, the Forest Service did attempt to estimate the price changes necessary to "close the gaps" in their 1973 study.

The first model integrating economic supply and demand functions for timber with a biological forest projection system was Leuschner's (1972a and 1972b) study of the aspen market in Wisconsin. Harvest levels were determined by the intersection of supply and demand curves for each period. Inventory levels were then projected using the TRAS (Timber Resource Analysis System) model (Larson and Goforth 1974).

The spatial dimension was added to timber supply models by Holley et al. (1975) and Haynes et al. (1978). In Holly et al.'s model, quantities required for each region and time period were given exogenously. The model was formulated as a linear programming transportation problem. The transportation problem was solved one period at a time. After the problem was solved for one period, inventories were updated and the model was solved again. Haynes et al. (1978) introduced demand curves for each region and period, and formulated their model as a reactive programming problem. Once again, the model was solved one period at a time, updating inventories between periods.

The work of Holley et al. (1975) and Haynes et al. (1978) provided the foundation for the collection of pseudo-dynamic, spatial price equilibrium models widely used today. This collection includes TAMM (Timber Assessment Market Model) developed by Adams and Haynes (1980, 1985), HAMM (Hardwood Assessment Market Model) developed by Binkley and Cardellichio (1985), and PAPYRUS developed by Gilless and Buongiorno (1987). These are the models currently used by the Forest Service (eg. U.S.D.A. Forest Service 1980, 1984, 1988) to project market conditions for softwood, hardwood, and pulpwood, respectively. A similar model, the Global Trade Model (GTM), was developed in a team effort by the International Institute for Applied Systems Analysis (IIASA) to model world timber markets (Kallio, Dykstra, and Binkley 1987). Adams et al. (1982) extend TAMM to include an endogenous timber investment module. Interestingly, in this model harvesting is determined in a pseudo-dynamic manner, but forestry investments are determined using a truly dynamic approach.

A variation on the spatial price equilibrium approach is the use of homogenous supply response cells rather than econometrically-estimated supply curves. In this approach, forest stands are aggregated into groups with similar characteristics such as productivity, accessibility, and ownership. Different supply responses are attributed to each cell based on the characteristics of the cell. All the stands in each cell are assumed to respond identically, except for possible marginal cells where producers are indifferent between harvesting and not harvesting. When supply response cells are differentiated by location, each cell is like a region in the spatial price equilibrium models.

The supply response cell approach was first developed by Vaux (1954), and is thus not limited to pseudo-dynamic models. Vaux's study will be discussed in the next section. The timber supply model developed by Clements (1987) and Clements et al. (1988) for Southwest Virginia is an example of a pseudo-dynamic supply response cell model. The TSAM (Timber Supply Assessment Model) of Rose et al. (1984) takes the response cell approach to the limit, modeling individual stands. The joint-product model developed by Greber (1983) and Greber and Wisdom (1985) is a one-period model only, but could easily be adapted to a pseudo-dynamic, multi-period framework. It also uses the response cell approach.

Long-Run Models

The oldest models of optimal timber supply behavior are the long-run models. Martin Faustmann (1849) solved the optimal steady-state rotation problem for forests one hundred and forty years ago. Samuelson (1976) and Johansson and Lofgren (1985) discuss the Faustmann formula and its history. The Faustmann formula gives the maximum present value of forest land under the assumption that the forest area is fixed and prices, costs, and forest growth relations are constant for all time. The maximum present value of bare forest land is known as the *soil expectation value* (SEV). The Faustmann formula has traditionally only been applied to forests that are managed for wood production. Hartman (1976) and Strang (1983) extend the Faustmann formula to account for the possibility that people may value the standing forest as well as the products that are obtained by harvesting the forest. A proof that the approach of the Faustmann formula is equivalent to solving for the optimal steady state of a dynamic age-class model of timber supply is given in Chapter 4 of this dissertation.

The Faustmann formula is the traditional tool for studying forest taxation. Examples include Gamponia and Mendelsohn (1987), Chang (1982, 1983), and Klemperer (1974, 1976, 1982). These are all partial equilibrium, steady-state analyses. Computable general equilibrium models have also been used for forest taxation analysis (Boyd and Newman 1988).

Vaux (1954, 1973) was one of the first to construct a long-run supply curve for a particular timber species and region. In his studies of California's timber supply, he grouped commercial timber-producing areas into response cells. A long-run average cost per unit of wood was calculated for each cell based on an assumption of optimal management practices. To construct a supply curve, the cells were then ranked in order of increasing cost.

Hyde (1980) provides the most complete discussion of the long-run timber supply problem. The dimensions of location, site productivity, and management intensity are all incorporated into the long-run timber supply model discussed in his book. In a steady state, all land with a positive maximum soil expectation value higher than the present value of the land in any other use will be used to grow forests. The annual timber supply is given by

$$S(\mathbf{z}) = \sum_{i \in L(\mathbf{z})} \frac{V_i^*(\mathbf{z}) A_i}{T_i^*(\mathbf{z})} \qquad 2.1$$

where $S(\mathbf{z})$ is the total timber volume supplied; $\mathbf{z}$ is a vector of parameters that affect timber supply; $L(\mathbf{z})$ is the set of land classes used for timber production, given $\mathbf{z}$; $V_i^*(\mathbf{z})$ is the volume produced over one rotation per acre of land class i when managed to maximize SEV, given $\mathbf{z}$; A_i is the area in land class i; $T_i^*(\mathbf{z})$ is the optimal rotation age for forests grown on land class i given $\mathbf{z}$. The vector $\mathbf{z}$ could include a large number of factors. A few of the most important would be steady-state wood prices, interest rates, land prices, taxes and technology.

Dynamic Models

As mentioned, dynamic models are defined here as those incorporating a specific intertemporal optimization process. A fundamental distinction should be made between two types of dynamic timber supply models -- the two-period and the age-class models. The majority of discrete-time dynamic models in general economics can be expressed in a two-period framework. Forestry problems, however, cannot be reduced to two periods because the age-class dynamics of forests cannot be modeled in a two-period framework. Yet, age-class dynamics are of fundamental interest in forestry. This distinction between two-period and age-class models carries over into continuous-time models. In the next chapter it will be shown how age-class dynamics can be modeled with a partial differential equation. Heaps (1984) also modeled age-class dynamics using a differential equation with a delay. Models where the forest dynamics are expressed with ordinary differential equations are analogous to two-period discrete-time models.

This is not to say that there is no role for two-period models in forestry. Two-period models are much simpler than age-class models. Many difficult problems that are currently intractable with age-class models can be reasonably approximated in a two-period framework. For example, Johansson and Lofgren (1985, Chapter 12) and Koskella (1989a, 1989b) introduce uncertainty into timber supply analysis using two-period models. Also, as mentioned, Max and Lehman (1988) present a dynamic, two-period version of the household production model.

The dynamic age-class models of timber supply will be discussed in great detail in the next chapter. Therefore, they are discussed relatively briefly here. The simplest models capturing the essential dynamics of timber supply are the harvest scheduling models (Navon 1971, Johnson and Scheurman 1977). In fact, it will become clear that the dynamic age-class models of timber supply are very closely related to harvest scheduling models. It is unfortunate that harvest scheduling models are not more commonly viewed as timber supply models since they capture the dependence of the quantity supplied on demand expectations, age structure, and other forest owner constraints within a maximization framework. The difference between forest-level and regional models is usually just a matter of the scope of the model and the degree of aggregation.

Optimal control theory is a powerful tool for optimizing dynamic systems. Thus it is an important tool for studying timber supply. Optimal control theory is an extension of the Lagrange multiplier method of optimization. It is also closely related to dynamic programming. Naslund (1969) and Anderson (1976) were among the first to apply optimal control theory to forestry. Their main results were to reproduce the Faustmann formula using the mathematics of optimal control.

Clark (1976) presents an optimal control model of forestry that borrows heavily from the fisheries literature. Berck (1976) also used the Clark model but noted that the technology of that model does not reflect forest biology very well. Berck tried, but failed, to construct a continuous model that more closely reflected even-aged forest management. Evidently, he did not understand the subtle differences between continuous- and discrete-time models well enough. (A continuous-time model will be formulated below, and these subtleties will become more apparent then.) Berck also constructed a linear discrete-time model of the even-aged forest management problem. This linear model is a harvest scheduling model and should perhaps be called "Model III" after Johnson and Scheurman's (1977) terminology. Johansson and Lofgren (1985, Chapter 6) repeat much of Berck's analysis.

Lyon and Sedjo (1982, 1983, 1985) present a model much like Berck's. Berck approached the solution of his model from a linear programming perspective. Lyon and Sedjo solved their model using discrete-time optimal control theory. Their approach can also be viewed as an application of nonlinear programming. However, their use of optimal control theory is important because it clarifies the dynamic structure of the problem. They present the *adjoint equations*[5] for the problem. The adjoint equations describe the equations of motion for the shadow prices in the model, and give important conditions required for intertemporal optimization. Understanding these equations is a significant step toward understanding timber supply.

[5] The words shown in italics in this dissertation are defined in the glossary in Appendix B.

In their later (1985) paper, Lyon and Sedjo discuss the interesting relationship between their model and Walker's (1971) binary search model. Walker's model is shown to be equivalent to the Lyon and Sedjo model when the land value and the endogenous regeneration inputs are ignored. Lyon and Sedjo show how Walker's ECHO algorithm can be used as a first iteration for solving their model.

Mitra and Wan (1985) also use "Model III" to study optimal timber management. They re-derive the Faustmann formula as the steady-state solution to the problem, but give an example where the optimal management program is inconsistent with the Faustmann formula. The solution to their example shows cycles, rather than settling down to a steady state.

Heaps (1984) provides the closest approach to an analytic solution of a continuous version of the timber management problem. Heaps simplifies the dynamics of the forest by first proving that for his model the oldest trees will always be harvested first. This allows him to express the dynamics as a differential equation with a delay (or a "process with a delay"). Heaps shows that if the process converges to a steady state, the steady state is consistent with the Faustmann formula. Of course, if the process does not converge to a steady state, the Faustmann formula may not be relevant. This later case corresponds to the results of Mitra and Wan (1985) in that their counterexample is one where the process does not converge to a steady state.

Discussion

This review of the timber supply literature is not comprehensive, but describes the most common approaches used in modeling timber supply. A more thorough review would only reveal an even greater variety of approaches. Part of the reason for this diversity is the range of problems addressed by the different studies. As was mentioned earlier, there are many different dimensions to the problem, and one purpose in modeling is to abstract from irrelevant dimensions.

It is probably safe to say that much of the diversity in the approaches is a reflection of the degree of confusion that has existed and still exists regarding the concept of timber supply. Arguably, much of this confusion is due to a lack of appreciation for the important dynamic nature of the timber supply problem. In many respects, the approach of Lyon and Sedjo (1982, 1983) seems to come the closest to fully capturing the dynamics of timber supply, but the cost seems be a less realistic model, especially with regard to the perfect foresight assumption.

Binkley (1987) criticizes dynamic models because they assume rational expectations. The problem is not so much due to the rational expectations assumption, but rather, it is due to the perfect foresight assumption.[6] A stochastic rational expectations model might prove to be an excellent approach. The perfect foresight assumption underlying the Lyon and Sedjo model is clearly unrealistic. Even the assumption that timber producers have full information regarding current conditions is unrealistic. Does the aver-

[6] While perfect foresight is one kind of rational expectations process, the degree of difference between perfect foresight models and stochastic models is tremendous -- analogous perhaps to the difference between advanced linear algebra and American high school algebra.

age nonindustrial private forest land owner know what the age-class distribution is for the forest in the local market area? Probably not. Given this, it is possible that the assumptions buried implicitly in many static econometric models about producers' knowledge bases and price expectations processes may more closely reflect reality than the perfect information assumption in the dynamic models.

The advantage of the dynamic models' approach is that critical assumptions, such as the price expectations process, are explicit. If we only need to know the current time, we never need to know anything about the inner workings of a clock. However, if we are interested in how the clock works, we need to be able to open it up and see what is inside. Models with deeply-buried implicit assumptions are like clocks that can never be opened. We can only judge whether they are fast or slow, but we cannot tell why.

It is clear that much of the lack of realism in the dynamic models is due to the perfect foresight assumption. Relaxing this assumption should therefore be an active area of research. Unfortunately, no progress will be made on that front in this dissertation. However, if there is any truth in the assumption that forest owners are utility -- or wealth -- maximizers, then there is much we can learn from the dynamic timber supply models about how things really work. One would expect that a close correspondence exists between a nonstochastic model and a stochastic, but otherwise equivalent, model.

CHAPTER 3: DYNAMIC TIMBER SUPPLY MODELS

The scope of an economic general equilibrium model includes all the activities of an economy. However, the primary focus in this dissertation is on the forestry sector -- in particular, timber supply. The forestry sector is modeled here using a dynamic, age-class structure approach. This approach was chosen because it is consistent with the fundamental assumptions of neoclassical economic theory: utility and profit maximization, full information, perfect competition, etc. Since this sector is the central focus of the model, the details of the dynamic, age-class models of timber supply will be explored in this chapter.

The age-structure models of timber supply are a relatively recent development in timber supply theory. The first model in this class was developed by Berck (1976, 1979). Lyon and Sedjo's (1982, 1983, 1985) discrete-time optimal control model of timber supply is the most complete of the dynamic age-class models. Binkley (1987), Mitra and Wan (1985) and Heaps (1984) also discuss models in this class. Finally, Johansson and Lofgren (1985) discuss and extend the model presented by Berck (1976). The models of

Berck (1976) and Lyon and Sedjo (1982, 1983) are considered in some detail here. The similarity between the two models is demonstrated, exposing the essential features of the dynamic, age-class models of timber supply.

The word "dynamic" has a very important and specific meaning in this context. Paraphrasing Kamien and Schwartz (1981, p. 5), a problem can be said to be *dynamic*[7] if an optimal decision for one period cannot be made without considering the effect of that decision on opportunities in future periods. From this perspective, it is evident that virtually all silvicultural decisions are made in a dynamic context. By contrast, the traditional supply theory found in microeconomics texts is derived from static optimization problems. Because of this difference, many of the results of static supply theory may not be appropriate for timber supply modeling. Extrapolation of a theory based on a static problem to a problem which is fundamentally dynamic is a loose kind of theorizing. Thus, in this chapter the analysis begins at a fundamental level: given the timber-producing technology of growing forests, what will be the timber supply behavior of profit-maximizing producers?

The first step is to describe mathematically the production technology of the forest. This is done in the first section of this chapter. Here, only an extremely simple forest is considered, and it is assumed that the forest is managed on an even-aged basis. The production technology for this problem does not look like a production function for a static production problem. Rather, the production technology is given by the *equations of motion* for the forest. The equations of motion are simply a mathematical representation of the way the state of the forest changes over time. Two versions of these equations of motion are discussed. In one version, cutting occurs at the end of the pe-

[7] The words shown in italics are defined in Appendix B.

riod; in the other, cutting occurs at the beginning of the period. Both types of equations have been used. Because different models use different versions of the equations of motion, those models may appear more different than they are. Neither the implications nor solution of the model should change, whichever version is used.

The equations of motion for the simple forest discussed below are linear. Thus, adding a linear objective function yields a linear programming problem. This problem was first discussed by Berck (1976). As Berck pointed out, the dual to this problem has a particularly useful structure which allows us to solve the linear program recursively, translating the problem into a dynamic programming framework. This dynamic programming problem is extremely easy to solve. This model is the simplest version of the dynamic age-class timber supply models and is discussed in the second section of this chapter.

The third section sets up the same problem in the framework of discrete time optimal control. This is the approach taken by Lyon and Sedjo (1982, 1983, 1985). In discrete-time, the optimal control problem is really just a nonlinear programming problem. The role of optimal control is to provide a framework for organizing the problem, emphasizing its recursive structure. The Lyon and Sedjo timber supply model is consistent with the fundamental neoclassical economic assumptions mentioned above and the dynamic representation of forests given by the equations of motion. A very similar approach is used to model the forestry sector in the multi-sector models discussed in later chapters.

Continuous-time models are often more convenient for theoretical discussions because they may often be solved analytically, rather than numerically. In the final section of this chapter, the equations of motion for the simple forest are expressed in a continuous-time formulation. The fundamental equation of motion turns out to be a partial differential equation. An objective function is added to show how timber supply can be formulated as a continuous-time optimal control problem. Such problems, especially when constrained by a partial differential equation, can be very difficult to solve.

Basic Dynamics

In this section the rather obvious dynamics of an even-aged forest are put into mathematical form. This is an important step since the equations of motion are the most fundamental constraints of the system. The equations of motion are translated into a more compact vector and matrix notation which will be used throughout the remainder of the dissertation. Similar equations and notation also appear in Berck (1976), Lyon and Sedjo (1982, 1983, 1985), Mitra and Wan (1985), Johansson and Lofgren (1985), and Binkley (1987).

The forest described by the equations presented here is an extremely simple one. It contains one site class only -- all acres have equal productivity. The forest comprises even-aged units. Only one species is modeled. No thinnings are performed, and regeneration occurs immediately or in the period following cutting, depending on whether cutting occurs at the beginning or end of the period. In order to bound the number of age classes which must be considered, it is assumed that when some age N is reached no growth or decay occurs. This allows all areas with forests at least N years old to be aggregated into a single class. Yields are also assumed to be known and are a function only of stand age. Finally, the forest area is constant over time. All these assumptions can be relaxed and the equations of motion suitably modified. They are made here for simplicity of exposition.

Notation will be defined as it is introduced. Appendix A also provides a summary of the notation. Let

$x_{t,i} \equiv$ the area (acres) of age class i at time t,

$c_{t,i} \equiv$ the area (in acres) cut from age class i at time t,

$q_i \equiv$ the yield of wood per unit area (acre) for age class i,

$Q_t \equiv$ the total volume of wood harvested in period t (quantity supplied).

To get the volume per acre for each age class, a yield equation is required:

$$y = q(a) \qquad 3.1$$

where y is the wood yield from an acre of forest of age a. Following Mitra and Wan (1985), the following assumptions will be made regarding the yield function:

Assumption 1) $q(a) = 0$ for $0 \leq a \leq \hat{a}$ for some $\hat{a} \geq 0$,

Assumption 2) q is continuous for $a \geq \hat{a}$ and $dq/da = 0$ for $a \geq N$.

The first assumption states that there may be some age $\hat{a}$ at which merchantable volume first becomes available. Before that age, merchantable volume is zero. The second assumption requires continuity for the function on $(\hat{a}, \infty)$ and states that growth and decay are zero after some age N.

Cutting at the End of the Period

The case where forests are cut at the end of the period is presented first. In this case, the dynamics of the forest are given by:

$$x_{0,i} = x_i(0) \qquad i = 1, \ldots, N \qquad 3.2a$$

$$x_{t+1,i+1} = x_{t,i} - c_{t,i} \qquad t = 0, \ldots, T-1;\ i = 1, \ldots, N-1 \qquad 3.2b$$

$$x_{t+1,N} = x_{t,N} + x_{t,N-1} - c_{t,N} - c_{t,N-1} \qquad t = 0, \ldots, T-1 \qquad 3.2c$$

$$x_{t+1,1} = \sum_{i=1}^{N} c_{t,i} \qquad t = 0, \ldots, T-1 \qquad 3.2d$$

$$Q_t = \sum_{i=1}^{N} q_i c_{t,i} \qquad t = 0, \ldots, T-1 \qquad 3.2e$$

$$x_{t,i} \geq c_{t,i} \geq 0 \qquad t = 0, \ldots, T-1;\ i = 1, \ldots, N-1 \qquad 3.2f$$

where T is the time horizon of the model and may be infinity. The $x_{t,i}$'s are the *state variables*, and the $c_{t,i}$'s are the *choice variables* or *decision variables*. The system can be influenced or controlled through the decision variables. Note that only Equations 3.2b - 3.2d are equations of motion. Equation 3.2b describes how the forest ages; those acres that are not cut are moved into the next age class at the end of each period. Equation 3.2c implements the assumption that areas supporting forests that are at least N years old can be aggregated. Equation 3.2d adds up the area that was cut in period t and places them in age class 1 in the next period.[8]

Equation 3.2a gives the initial age class distribution. The laws of motion only describe how states change, and there are many sequences of states the system could pass through which would be consistent with these equations. The initial conditions help pin

[8] Note that there is no zero age class. This is simply a notational convenience. If a zero age class were included, then there would be $N + 1$ age classes. It is simpler to refer to N age classes rather than $N + 1$.

down which sequences are possible. These initial conditions and a given set of decision variables uniquely determine the states of the system through the equations of motion. Equation 3.2e defines the total volume cut in each period, and Equation 3.2f requires that the area that is cut from each age class in period t be nonnegative and less than or equal to the area of forest in that age class.

These equations can be expressed in more compact notation. The use of this notation helps illustrate the large-scale structure of the problem -- allowing one to see the forest rather than the trees, so to speak.

Define the vectors

$$\mathbf{x}_t \equiv [x_{t,1},\ x_{t,2},\ \ldots,\ x_{t,N}]'$$

$$\mathbf{c}_t \equiv [c_{t,1},\ c_{t,2},\ \ldots,\ c_{t,N}]'$$

$$\mathbf{q} \equiv [q_1,\ q_2,\ \ldots,\ q_N]'.$$

The prime on each vector denotes a transpose operation. All the vectors used will be column vectors; their transposes are row vectors. Vectors and matrices will be denoted by bold letters to distinguish them from scalars. Greek letters are the one exception to this rule. Whether a Greek letter stands for a vector or a scalar must be determined from the context. (This is due to word processor user limitations.)

Also define the matrices

$$
\mathbf{G} \equiv \begin{bmatrix} 0\,0\,0 & & \\ 1\,0\,0 & & \\ 0\,1\,0 & & \\ & \ddots & \\ & & 1\,0\,0 \\ & & 0\,1\,1 \end{bmatrix}_{N\text{x}N}
$$

and

$$
\tilde{\mathbf{R}} \equiv \begin{bmatrix} 1\,1\,1 & \dots & 1 \\ 0\,0\,0 & & 0 \\ 0\,0\,0 & & 0 \\ \vdots\;\;\vdots & & \vdots \\ 0\,0\,0 & \dots & 0 \end{bmatrix}_{N\text{x}N.}
$$

When **G** premultiplies an $Nx1$ column vector it takes each element and moves it up one position; the last two elements are summed and placed in the last position of the new vector. Thus, the **G** matrix "grows" the forest. By premultiplying a vector, the matrix $\tilde{\mathbf{R}}$ sums up all the elements of the vector and places the sum in the first position of the new vector. The remaining elements of the new vector will be zero. Thus, the $\tilde{\mathbf{R}}$ matrix implements regeneration on all harvested acres.[9]

Equations 3.2b-d can now be written as

$$
\mathbf{x}_{t+1} = \mathbf{G}(\mathbf{x}_t - \mathbf{c}_t) + \tilde{\mathbf{R}}\mathbf{c}_t. \qquad 3.3a'
$$

Define

$$
\mathbf{R} \equiv \tilde{\mathbf{R}} - \mathbf{G} \equiv \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & 0 & 0 & & 0 \\ 0 & -1 & 0 & & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}_{N\text{x}N.}
$$

Now Equation 3.3′ can be written more compactly as

[9] These matrices are usually labeled **A** and **B** in the forestry literature. However, an input-output matrix which is also traditionally labelled with an **A** will be used later. Since the input-output literature has a longer history and a wider readership, the forestry matrices were renamed.

$$\mathbf{x}_{t+1} = \mathbf{G}\mathbf{x}_t + \mathbf{R}\mathbf{c}_t . \qquad 3.3a$$

The initial conditions are

$$\mathbf{x}_0 = \mathbf{x}(0). \qquad 3.3b$$

Equation 3.2e becomes

$$Q_t = \mathbf{q}'\mathbf{c}_t . \qquad 3.3c$$

The harvest area restriction (Equation 3.2f) can be written

$$\mathbf{x}_t \geq \mathbf{c}_t \geq \mathbf{0} . \qquad 3.3d$$

This requires that the elements of $\mathbf{x}_t$ must be greater than or equal to the corresponding elements of $\mathbf{c}_t$, which must be nonnegative. That is, the inequalities apply in an element-wise fashion.

This approach -- where cutting occurs at the end of the period -- is used by Lyon and Sedjo (1982, 1983, 1985) and Binkley (1987). Alternatively, cutting occurs at the beginning of the period in the models of Berck (1976) and Johansson and Lofgren (1985). One of the purposes of this chapter is to compare the approach of Berck (1976) with that of Lyon and Sedjo (1982, 1983). In order to do this, one must be able to work with both ways of writing the equations of motion of the forest.

Cutting at the Beginning of the Period

The equations of motion when cutting occurs at the beginning of the period are:

$$x_{0,i} = x_i(0) \qquad i = 1, \ldots, N \qquad 3.4a$$

$$x_{t+1,i+1} = x_{t,i} - c_{t+1,i+1} \qquad t = 0, \ldots, T-1;\ i = 1, \ldots, N-1 \qquad 3.4b$$

$$x_{t+1,N} = x_{t,N} + x_{t,N-1} - c_{t+1,N} \qquad t = 0, \ldots, T-1 \qquad 3.4c$$

$$x_{t,1} = \sum_{i=2}^{N} c_{t,i} \qquad t = 0, \ldots, T-1 \qquad 3.4d$$

$$Q_t = \sum_{i=2}^{N} q_i c_{t,i} \qquad t = 0, \ldots, T-1 \qquad 3.4e$$

$$x_{t,i} \geq 0,\ c_{t,i} \geq 0 \qquad t = 0, \ldots, T-1;\ i = 1, \ldots, N-1 \qquad 3.4f$$

Note that no harvest area restrictions are required in this case; the nonnegativity of $\mathbf{x}_t$ and $\mathbf{c}_t$ are enough to ensure this. Also, $c_{t,1}$ is never used and may be arbitrarily set to zero. Equations 3.4a-f are analogous to Equations 3.2a-f.

Now, define the matrix

$$\hat{\mathbf{G}} \equiv \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 0 & -1 & 0 & & 0 \\ 0 & 0 & -1 & & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & -1 \end{bmatrix}_{N\mathrm{x}N}$$

and add the restriction that $c_{t,1} = 0$ for all t. The equivalent versions of 3.3a-d are

$$\mathbf{x}_{t+1} = \hat{\mathbf{G}}\mathbf{x}_t + \mathbf{R}\mathbf{c}_{t+1} \tag{3.5a}$$

$$\mathbf{x}_0 = \mathbf{x}(0) \tag{3.5b}$$

$$Q_t = \mathbf{q}'\mathbf{c}_t \tag{3.5c}$$

$$\mathbf{x}_t \geq 0, \ \ \mathbf{c}_t \geq 0 \tag{3.5d}$$

Before moving on, this is a good time to define two more vectors. Let $\mathbf{e}_1^{n\prime} \equiv (1, 0, 0, \ldots, 0) \in \mathbf{R}^n$ and $\mathbf{d}^{n\prime} \equiv (1, 1, 1, \ldots, 1) \in \mathbf{R}^n$. The vector $\mathbf{e}_1^n$ is the first unit vector in $\mathbf{R}^n$. The vector $\mathbf{d}^n$ is useful for adding up the elements of a vector. That is, the dot product of $\mathbf{d}^n$ and any vector in $\mathbf{R}^n$ gives the sum of the elements of that vector.

Berck's Linear Maximization Model

At this point the model only describes the physical and biological process of producing wood in a simple forest. No objective or policy for choosing what to cut at what time has been introduced. In this section that step is taken. To do this, a price P_t per unit volume of wood *(stumpage)* in each period is introduced.

The assumption that this sequence of prices is known for the time horizon of the problem is significant because it is seldom true in reality. Rather than assume prices are known, one could assume that a demand curve is known for each period. The model would then have a nonlinear objective function, and prices would be a part of the solution to the model. This more sophisticated approach still requires the assumption that the demand curves can be projected with certainty, however. The perfect foresight assumption is very difficult to relax. For now, however, this assumption will be accepted and the linear problem where prices are given will be considered.

The objective is to maximize the present value of revenues from the sale of harvested wood from the forest. It is assumed that there are no planting or harvesting costs. With no costs, revenue maximization is equivalent to profit maximization. (Or, alternatively, if harvest costs were proportional to volume harvested, the given prices could be net prices.) Johansson and Lofgren (1985, pp. 2-4) show that, with the added assumption of a perfect capital market where individuals can borrow or lend all they wish at the given rate, present value maximization is consistent with utility maximization. (They also implicitly assume that forests are owned only for income.) It is assumed here for simplicity that the interest rate r is constant for all periods of the time horizon and the discount factor $\delta = (1 + r)^{-1}$ is used.

Cutting at the Beginning of the Period

Consider first the case where cutting occurs at the beginning of the period. The problem of maximizing the present value of revenues from the forest is a linear programming (LP) problem. The structure of this LP is illustrated in tableau form in Table 3.1a. Note that the initial conditions are incorporated in the first line of the model. This particular formulation of the timber supply problem was first presented in Berck (1976). The model is also discussed in Johansson and Lofgren (1985).

Table 3.1a. Model III Linear Programming Tableau; Cutting at the beginning of the period.

$$\underset{\mathbf{x}_t,\, \mathbf{c}_t}{\text{Maximize}} \sum_{t=1}^{T} \delta^t P_t \mathbf{q}' \mathbf{c}_t$$

Subject to:

$$\left[\begin{array}{cccc|cccc}
\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
-\hat{\mathbf{G}} & \mathbf{I} & \cdots & \mathbf{0} & -\mathbf{R} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & -\hat{\mathbf{G}} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{R} & \cdots & \mathbf{0} \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{R}
\end{array}\right]
\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \cdot \\ \mathbf{x}_T \\ - \\ \mathbf{c}_1 \\ \mathbf{c}_2 \\ \cdot \\ \mathbf{c}_T \end{bmatrix}
=
\begin{bmatrix} \mathbf{x}(0) \\ \mathbf{0} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{0} \end{bmatrix}$$

Table 3.1b. Dual Tableau for the Model III Linear Program; Cutting at the beginning of the period.

$$\underset{\delta^t \lambda_t}{\text{Minimize}}\ \lambda'_0 \mathbf{x}_0$$

Subject to:

$$\begin{bmatrix}
\mathbf{I} & -\hat{\mathbf{G}}' & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{I} & -\hat{\mathbf{G}}' & \cdots & \mathbf{0} \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -\hat{\mathbf{G}}' \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \\
\hline
\mathbf{0} & -\mathbf{R}' & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -\mathbf{R}' & \cdots & \mathbf{0} \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{R}'
\end{bmatrix}
\begin{bmatrix} \lambda_0 \\ \delta^1 \lambda_1 \\ \cdot \\ \cdot \\ \cdot \\ \delta^T \lambda_T \end{bmatrix}
\geq
\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{0} \\ - \\ \delta^1 P_1 \mathbf{q} \\ \delta^2 P_2 \mathbf{q} \\ \cdot \\ \cdot \\ \cdot \\ \delta^T P_T \mathbf{q} \end{bmatrix}$$

This model closely resembles the harvest scheduling models (eg. Navon 1971, Johnson and Scheurman 1977). However, it does not fit into either of Johnson and Scheurman's categories -- Model I or Model II. In Model I, individual rows track stands through the entire time horizon of the problem -- usually several rotations. Each row represents an alternate management strategy for a given type of stand for the entire time horizon. In Model II, stands are transferred to different rows only when they are harvested. Rows represent alternate management activities, including a harvest date, for all stands of a given type that were regenerated in a given period. In the LP presented here, all stands are transferred to a new row each period. This is a linear programming model for scheduling harvests, so it is clearly a harvest scheduling model. It has therefore been referred to in this dissertation as "Model III". As mentioned, it appears that the primary difference between harvest scheduling models and dynamic timber supply models is the problems to which they are applied and the degree of aggregation.

The presentation of the model in matrix form allows one to see the large-scale structure of the problem. The structure of the dual of the Model III LP is particularly interesting (Table 3.1b). Note that the dual variables (the elements of the vectors λ_t) are multiplied by discount factors. In Berck (1976), the discount factors are subsumed in the dual variables. This modification of Berck's presentation is made to facilitate comparison with the discrete time optimal control approach of Lyon and Sedjo (1982, 1983) presented in the next section. (This technique will also be used again in later chapters.)

The dual problem is to minimize the value of the initial forest subject to a set of recursive restrictions on the values of the dual variables. As Berck has pointed out, this minimum value of the initial forest gives the competitive price of the forest (land with timber).

Of course, this claim is conditional on the assumption of full information, including, but not limited to, perfect foresight.

The recursive structure of the dual constraints makes the dual problem particularly interesting. The restrictions on the dual variables can be written in matrix form as

$$\delta^t \lambda_t - \delta^{t+1} \hat{\mathbf{G}}' \lambda_{t+1} \geq 0 \qquad t = 0, \ldots, T-1 \qquad 3.6a$$

$$\delta^T \lambda_T \geq 0 \qquad 3.6b$$

$$-\delta^t \mathbf{R}' \lambda_t \geq \delta^t P_t \mathbf{q} \qquad t = 1, \ldots, T \qquad 3.6c$$

In less compact notation, these are

$$\delta^t \lambda_{t,i} \geq \delta^{t+1} \lambda_{t+1, max(i+1, N)} \qquad t = 0, \ldots, T-1; \quad i = 1, \ldots, N \qquad 3.6a'$$

$$\delta^T \lambda_{T,i} \geq 0 \qquad i = 1, \ldots, N \qquad 3.6b'$$

$$\delta^t \lambda_{t,i} \geq \delta^t P_t q_i + \delta^t \lambda_{t,1} \qquad t = 1, \ldots, T; \quad i = 1, \ldots, N \qquad 3.6c'$$

Equation 3.6a requires that the discounted shadow price of an acre in age class i be greater than or equal to the discounted shadow price of that acre if it is allowed to age for one more period. Equation 3.6c requires that the discounted shadow price for a given acre of forest be at least as great as the value that would be obtained if the stand were cut. That is, it must be greater than or equal to the current value of the timber plus the value of the bare land that would remain. These are the options: to clearcut the stand or let it grow. The value of an acre of forest must be as large as the value of the better of these two options. Equations 3.6a and 3.6c, taken together, require that

$$\lambda_{t,i} \geq \max \{ \delta\lambda_{t+1,\max(i+1,N)}, \ P_t q_i + \lambda_{t,1} \}. \quad t = 1, \ldots, T-1; \ i = 1, \ldots, N \qquad 3.7$$

If the shadow prices of the forest in the final period are known, this relationship can be used to solve for the minimum values of the shadow prices for the second to last period. This process can be repeated, working backward until the minimum values of the shadow prices for the initial period are obtained. Since the problem is to minimize the value of the initial stand, these minimum values are the solution to the problem. The inequality in Equation 3.7 can therefore be replaced with an equality to give a *recursion relationship* for solving the problem as a dynamic programming (DP) problem.

What values should be given to the shadow prices of the forest in the final period? As the problem is currently stated, stands have no value in the final period unless they are cut; revenue is only obtained when stands are cut, and there will be no chance to cut them later. Thus all stands with any merchantable volume should be cut in the last period, and the appropriate dual prices for stands in the last period will be zero or the value of the merchantable volume, whichever is greater. This can be seen from Equations 3.6b-c. (Note that Equation 3.6a does not apply to the final period.) Equation 3.6b requires that the dual prices for the forest in the final period be positive. The zero age class will have no merchantable volume; thus,

$$\delta^T \lambda_{T,1} = O. \qquad 3.8$$

This condition, together with Equation 3.6c, requires that the dual variables in the final period be greater than or equal to the value of the merchantable timber (and positive); The bare land value does not matter because it is zero in the final period. That is, for the final period 3.6c becomes

$$\delta^T \lambda_{T,i} \geq \delta^T P_T q_i . \qquad i = 2, \ldots, N \qquad 3.9$$

Combining Equations 3.6b and 3.9,

$$\delta^T \lambda_{T,i} \geq \max \{0, \ \delta^T P_T q_i \} . \qquad i = 2, \ldots, N \qquad 3.10$$

Once again, this constraint will hold as an equality since the objective is to minimize the value of the forest in the initial time period. The dual constraints for this problem can be viewed as equations of motion for the dual variables, moving them backwards in time. Since these are greater-than-or-equal constraints, the smaller are the dual variables to begin with (that is, in the terminal time period) the smaller the dual variables will be when the initial time period is reached (going backward). These kinds of boundary conditions derived from optimality considerations are known as *transversality conditions* (Kamien and Schwartz 1981, p. 49).

The transversality conditions -- Equations 3.8 and 3.10 with the inequality changed to an equality -- give the shadow prices for the final period. The recursion relationship 3.7 -- with the inequality changed to an equality -- can then be used to solve for the shadow prices for all periods. Even for large problems, this is an extremely simple problem to solve. Solving the dual problem is equivalent to solving the primal problem in LP. In this case, the easiest way to find the solution to this primal problem from the solution to the dual problem is with a *forward pass* using the information from the dynamic programming solution. The shadow price for each age class in each period was determined by the value of the stand if let grow or if clearcut, whichever was higher. Thus the decision whether to leave or cut a stand is given by the rules:

If $\lambda_{t,i} = P_t q_i + \lambda_{t,1}$ then cut all the forest in age-class i in period t;

If $\lambda_{t,i} = \delta\lambda_{t+1,min(i+1,N)}$ then leave any forests in age class i in period t uncut.

Note that $\lambda_{t,i}$ may equal both $P_t q_i + \lambda_{t,1}$ and $\delta\lambda_{t+1,min(i+1,N)}$. In this case the area of forest in age class i in period t may be clearcut entirely, left entirely uncut, or partially harvested. The value of the objective function will be the same regardless of which of these choices is made.

Linear programming is just one way to solve this problem. The linear programming framework is used only to find the recursive relationship for setting the problem up as a dynamic programming problem. This recursion relation can also be found using the Kuhn-Tucker theorem.[10] To do this, form the Lagrangian[11]

$$\text{Max}\,\Lambda = \sum_{t=1}^{T} \delta^t \{P_t \mathbf{q}'\mathbf{c}_t - \delta\lambda_t[\mathbf{x}_t - \hat{\mathbf{G}}\mathbf{x}_{t-1} - \mathbf{R}\mathbf{c}_t]\} - \lambda_0(\mathbf{x}_0 - \mathbf{x}(0)). \qquad 3.11$$

The first-order conditions for this problem are:

$$\frac{\partial\Lambda}{\partial\mathbf{x}_t} = -\,\delta^t\lambda_t + \delta^{t+1}\hat{\mathbf{G}}'\lambda_{t+1} \le 0 \qquad t = 0,\ \ldots,\ T-1 \qquad 3.12a$$

$$\frac{\partial\Lambda}{\partial\mathbf{x}_T} = -\,\delta^T\lambda_T \le 0 \qquad 3.12b$$

$$\frac{\partial\Lambda}{\partial\mathbf{c}_t} = \delta^t P_t\mathbf{q} + \delta^t\mathbf{R}'\lambda_t \le 0 \qquad t = 1,\ \ldots,\ T \qquad 3.12c$$

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[10] There are many references that could be cited for the Kuhn-Tucker Theorem. Some examples are Takayama (1985), Bazaraa and Shetty (1979), or Silberberg (1978).

[11] A capital lambda will be used throughout the dissertation to indicate a Lagrangian function. A script L is usually used, but was not available with this word processor.

These are the same as the dual constraints to the LP formulation. It is perhaps worthwhile to see these derivatives without the matrix notation. Equation 3.11 can be written equivalently as

$$\text{Max}\,\Lambda = \sum_{t=1}^{T}\delta^t\{P_t\sum_{i=1}^{N}q_ic_{t,i} - \lambda_{t,1}[x_{t,1} - \sum_{i=2}^{N}c_{t,i}] - \sum_{i=2}^{N-1}\lambda_{t,i}[x_{t,i} + c_{t,i} - x_{t-1,i-1}]$$
$$- \lambda_{t,N}[x_{t,N} + c_{t,N} - x_{t-1,N-1} - x_{t-1,N}]\} - \sum_{i=1}^{N}\lambda_{0,i}(x_{0,i} - x_i(0)) \qquad 3.11'$$

Now, the first-order equations for this problem are:

$$\frac{\partial\Lambda}{\partial x_{t,i}} = -\delta^t\lambda_{t,i} + \delta^{t+1}\lambda_{t+1,i+1} \leq 0 \qquad t = 0, \ldots, T-1; \quad i = 1, \ldots, N-1 \qquad 3.12a'$$

$$\frac{\partial\Lambda}{\partial x_{t,N}} = -\delta^t\lambda_{t,N} + \delta^{t+1}\lambda_{t+1,N} \leq 0 \qquad t = 0, \ldots, T-1 \qquad 3.12a''$$

$$\frac{\partial\Lambda}{\partial x_{T,i}} = \delta^T\lambda_{T,i} \geq 0 \qquad i = 1, \ldots, N \qquad 3.12b'$$

$$\frac{\partial\Lambda}{\partial c_{t,i}} = \delta^t[P_tq_i + \lambda_{t,1} - \lambda_{t,i}] \leq 0 \qquad t = 1, \ldots, T; \quad i = 2, \ldots, N \qquad 3.12c'$$

The Kuhn-Tucker theorem also requires the complementary slackness conditions:

$$x_{t,i}\cdot\frac{\partial\Lambda}{\partial x_{t,i}} = 0 \text{ and } c_{t,i}\cdot\frac{\partial\Lambda}{\partial c_{t,i}} = 0. \qquad 3.13a-b$$

These complementary slackness conditions indicate when the inequalities in the first-order conditions will hold as equalities and provide a more formal justification for the forward pass algorithm discussed above. That is, for the area in any age class to be non-zero, the relevant version of Equation 3.12a must hold as an equality. Similarly, if a positive area is cut from a given age class, Equation 3.12c must hold as an equality for that age class. The forward pass algorithm states that if 3.12a holds as an equality the age class need not be cut, and if 3.12c holds as an equality, the age class may be cut. This follows directly from the complementary slackness conditions.

Cutting at the End of the Period

Now consider the case where cutting occurs at the end of the period. The linear programming tableau for this problem is presented in Table 3.2a. With the matrices appropriately redefined, the tableau for this problem is very similar to the tableau for the previous problem. Recall that the area restrictions for this problem do not follow from the nonnegativity of $\mathbf{x}_t$ and $\mathbf{c}_t$. Thus these restrictions must be added, making the tableau for this problem substantially larger than that of the previous problem. Also note that the time subscripts on the cut vectors for this problem vary from 0 to $T-1$ rather than from 1 to T as in the previous problem.

The dual problem is presented in Table 3.2b. Once again, the dual problem is to minimize the value of the initial forest subject to a set of recursive restrictions on the values of the dual variables. In matrix form, these restrictions are

$$\delta^t \lambda_t - \delta^{t+1} \mathbf{G}' \lambda_{t+1} - \delta^t \zeta_t \geq 0 \qquad t = 0, \ldots, T-1 \qquad 3.14a$$

$$\delta^T \lambda_T \geq 0 \qquad 3.14b$$

$$-\delta^{t+1} \mathbf{R}' \lambda_{t+1} + \delta^t \zeta_t \geq \delta^t P_t \mathbf{q} \qquad t = 0, \ldots, T-1 \qquad 3.14c$$

Or, without the matrix notation,

$$\delta^t \lambda_{t,i} \geq \delta^{t+1} \lambda_{t+1, max(i+1,N)} + \delta^t \zeta_{t,i} \qquad t = 0, \ldots, T-1;\ i = 1, \ldots, N \qquad 3.14a'$$

Table 3.2a. Model III Linear Programming Tableau; Cutting at end of period.

$$\underset{\mathbf{x}_t,\,\mathbf{c}_t}{\text{Maximize}} \sum_{t=0}^{T-1} \delta^t P_t \mathbf{q}' \mathbf{c}_t$$

Subject to:

$$\left[\begin{array}{cccc|cccc}
\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
-\mathbf{G} & \mathbf{I} & \cdots & \mathbf{0} & -\mathbf{R} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & -\mathbf{G} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{R} & \cdots & \mathbf{0} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{R} \\
\hline
-\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & -\mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}
\end{array}\right]
\left[\begin{array}{c}
\mathbf{x}_0 \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_T \\ \hline \mathbf{c}_0 \\ \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_{T-1}
\end{array}\right]
\begin{array}{c} \\ \\ = \\ \\ \\ \le \\ \\ \\ \end{array}
\left[\begin{array}{c}
\mathbf{x}(0) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0}
\end{array}\right]$$

Table 3.2b. Dual Tableau for the Model III Linear Program; Cutting at end of period.

$$\underset{\delta^t\lambda_t,\,\delta^t\zeta_t}{\text{Minimize}}\ \lambda'_0 x_0$$

Subject to:

$$\left[\begin{array}{ccccc|cccc}
\mathbf{I} & -\mathbf{G}' & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{I} & -\mathbf{G}' & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{G}' & \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{I} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\hline
\mathbf{0} & -\mathbf{R}' & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -\mathbf{R}' & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{R}' & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}
\end{array}\right]
\left[\begin{array}{c}
\lambda_0 \\ \delta^1\lambda_1 \\ \vdots \\ \delta^T\lambda_T \\ \hline \zeta_0 \\ \delta\zeta_1 \\ \vdots \\ \delta^T\zeta_T
\end{array}\right]
\ge
\left[\begin{array}{c}
\mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hline \delta^0 P_0 \mathbf{q} \\ \delta^1 P_1 \mathbf{q} \\ \vdots \\ \delta^T P_T \mathbf{q}
\end{array}\right]$$

$$\delta^T \lambda_{T,i} \geq 0 \qquad i = 1, \ldots, N \qquad 3.14b'$$

$$\delta^{t+1} \lambda_{t+1,max(i+1,N)} + \delta^t \zeta_{t,i} \geq \delta^t P_t q_i + \delta^{t+1} \lambda_{t+1,1} \quad t = 0, \ldots, T-1; \; i = 1, \ldots, N \quad 3.14c'$$

Equations 3.14a and 3.14c together imply that

$$\lambda_{t,i} \geq \zeta_{t,i} + \delta \lambda_{t+1,max(i+1,N)} \geq P_t q_i + \delta \lambda_{t+1,1}. \quad t = 0, \ldots, T-1; \; i = 1, \ldots, N \quad 3.15$$

Since the harvest area restrictions are inequalities, $\zeta_{t,i} \geq 0$ for all t and i. (Duality theory places no restrictions on the signs of the $\lambda_{t,i}$'s, but they turn out to be positive because of the dual constraints.) This means that $\lambda_{t,i} \geq \delta \lambda_{t+1,max(i+1,N)}$ and $\lambda_{t,i} \geq p_t q_i + \delta \lambda_{t+1,1}$, giving the recursive relation:

$$\lambda_{t,i} \geq \max \{ \delta \lambda_{t+1,max(i+1,N)}, \; P_t q_i + \delta \lambda_{t+1,1} \}. \quad t = 0, \ldots, T-1; \; i = 1, \ldots, N \quad 3.16$$

This is very similar to Equation 3.7. The only difference is that $\lambda_{t,i}$ is compared with $p_t q_i + \delta \lambda_{t+1,1}$, rather than $p_t q_i + \lambda_{t,1}$. This is because when harvesting occurs at the end of the period, regeneration does not occur until the next period; when harvesting occurs at the beginning of the period, cutting and regeneration occur in the same period. The interpretation of Equation 3.16 is virtually the same as that of Equation 3.7.

Note that $\zeta_{t,i}$ must be at least large enough to make the second inequality in Equation 3.15 true. Thus, when $\delta \lambda_{t+1,max(i+1,N)} < p_t q_i + \lambda_{t+1,1}$, then $\zeta_{t,i}$ must be strictly positive. These are the times when it is optimal to cut the stand if it has not already been cut. This is consistent with the interpretation of the $\zeta_{t,i}$'s as Lagrange multipliers corresponding to the harvest area constraints. The constraints are only binding when the entire area should be cut. At these times the multipliers can be positive. Alternatively, the right hand inequality in Equation 3.15 could be changed to an equality and the $\zeta_{t,i}$'s allowed to be positive or negative. In this case, the $\zeta_{t,i}$'s would measure the net benefit of cutting a stand versus letting it grow. The rule would be to cut the stand if the corresponding ζ-multiplier is positive; leave the stand if the multiplier is negative. The absolute value of the multiplier would be the cost of taking the wrong action.

It should be clear that this problem can also be set up using the Lagrange multiplier technique to obtain the same recursive relation as Equation 3.16. The Lagrangian in this case is

$$\text{Max}\,\Lambda = \sum_{t=0}^{T-1} \delta^t \{P_t \mathbf{q}'\mathbf{c}_t - \delta\lambda_{t+1}[\mathbf{x}_{t+1} - \mathbf{G}\mathbf{x}_t - \mathbf{R}\mathbf{c}_t] - \zeta_t[\mathbf{c}_t - \mathbf{x}_t]\} \\ - \lambda_0(x_0 - x(0)). \tag{3.17}$$

The first-order conditions are the same as Equations 3.14a-c.

The Lagrangian multiplier approach has the very important advantage that neither the objective function nor the constraints must be linear. For example, this would allow the introduction of demand curves rather than prices in the objective function. However, the introduction of a nonlinear objective function significantly complicates the solution of the problem. This is because the dual side of the nonlinear problem cannot be solved independently of the primal side. To see this, note that with demand curves prices depend on quantities. In the linear problem, quantities depend on prices but not vice-versa. This is what makes the dual of the linear problem so easy to solve. The recursive structure is still present in the nonlinear case, but it cannot be used to solve the problem as easily as in the linear case.

In the next section the discrete-time optimal control approach is considered. In discrete time, it turns out that optimal control theory is just another application of the Lagrange multiplier approach to optimization.

Discrete-Time Optimal Control

Lyon and Sedjo (1982, 1983, 1985) have used discrete-time optimal control (DTOC) theory to solve the dynamic timber supply problem. In this section, the linear models that were solved in the previous section are formulated in the DTOC framework. One purpose of this exercise is to show that the DTOC approach is essentially the same as the nonlinear programming approach, and for linear problems it is the same as the linear programming approach. This should not be too surprising since in each case it is the same problem that is being solved; the same solution should be obtained regardless of the solution method.

The Lyon and Sedjo model is important because it shows how the timber supply model can be solved with downward-sloping demand curves. Thus the dynamic production process of forests on the supply side is integrated with downward-sloping demand curves in a model of profit maximization. Downward-sloping demand curves will not be included in the formulation of the model in this section. The purpose here is to show how the linear model from the previous section can be formulated using DTOC. Generalization of the model to include downward-sloping demand curves will be discussed briefly at the end of the section.

The problem must still be solved for two versions of the equations of motion. This time, the problem where cutting occurs at the end of the period will be discussed first. This is the form of the equations of motion used by Lyon and Sedjo. There are some differences between the formulation here and that used by Lyon and Sedjo. They define the decision variables as the proportion of the area in an age class to be cut. The variables used here are the same as in the previous sections to facilitate comparison of the different methods.[12]

Cutting at the End of the Period.

Define $S_t(\mathbf{x}_t, \mathbf{C})$ as the present value at time t of the forest. The state of the forest is described by the state vector $\mathbf{x}_t$. The symbol $\mathbf{C}$ represents all future cutting decisions. That is,

$$\mathbf{C} \equiv \{\mathbf{c}_t,\ \mathbf{c}_{t+1},\ \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{T-1}\}.$$

The maximum present value of the forest at time t is obtained when all these future cutting decisions are chosen optimally. Denote this maximum present value by $S_t^*(\mathbf{x}_t)$. Note that $\mathbf{C}$ is not a variable in S_t^* since current and future cutting is fixed at optimal levels.

Now, let $s_t(\mathbf{x}_t, \mathbf{c}_t)$ be the current (in time t) contribution to the present value of the forest. For this problem,

$$s_t(\mathbf{x}_t, \mathbf{c}_t) = P_t \sum_{i=1}^{N} q_i\, c_{t,i} = P_t\, \mathbf{q}'\mathbf{c}_t. \qquad 3.18$$

If the problem has a terminal time period, let $S_T^*(\mathbf{x}_T^*)$ equal the present value of the forest in that period. The condition that the forest reach some ending state $\mathbf{x}_T^*$ is imposed. For now, the nature of this ending forest is left open. The issue of finding the best ending conditions is addressed in the next chapter.

[12] Equations 3.28 - 3.33 would have been much simpler if Lyon and Sedjo's notation had been used, however.

Now,

$$S_t(\mathbf{x}_t, \mathbf{C}) = s_t(\mathbf{x}_t, \mathbf{c}_t) + \delta s_{t+1}(\mathbf{x}_{t+1}, \mathbf{c}_{t+1}) + \dots \\ \dots + \delta^{T-t-1} s_{T-1}(\mathbf{x}_{T-1}, \mathbf{c}_{T-1}) + \delta^{T-t} S_T{}^*(\mathbf{x}_T{}^*). \qquad 3.19$$

Bellman's *principle of optimality,* states that an optimal policy has the property that, whatever the initial decision is, the remaining decisions must constitute an optimal policy with regard to the state resulting from the initial decision. When this principle is applied, the problem becomes:

$$S_t{}^*(\mathbf{x}_t) = \max_{\mathbf{c}_t} \{s_t(\mathbf{x}_t, \mathbf{c}_t) + \delta S_{t+1}{}^*(\mathbf{G}\mathbf{x}_t + \mathbf{R}\mathbf{c}_t)\}. \qquad 3.20$$

This is a dynamic programming recursion relation. It can be solved to obtain the recursion relation developed in the previous section.

In nonlinear programming format, the problem can be written:

$$\operatorname*{Max}_{\mathbf{c}_t} \; s_t(\mathbf{x}_t, \mathbf{c}_t) + \delta S_{t+1}{}^*(\mathbf{G}\mathbf{x}_t + \mathbf{R}\mathbf{c}_t) \qquad 3.21$$

subject to

$$\mathbf{x}_t \geq \mathbf{c}_t \geq \mathbf{0}.$$

Or

$$\operatorname*{Max}_{\mathbf{c}_t, \zeta_t} \; \Lambda_t = P_t \mathbf{q}'\mathbf{c}_t + \delta S_{t+1}{}^*(\mathbf{G}\mathbf{x}_t + \mathbf{R}\mathbf{c}_t) - \zeta_t(\mathbf{c}_t - \mathbf{x}_t) \qquad 3.22$$

The first-order and complementary slackness conditions for this problem are obtained by applying the Kuhn-Tucker theorem:

$$\frac{\partial \Lambda_t}{\partial \mathbf{c}_t} = P_t\mathbf{q} + \delta \mathbf{R}' \frac{dS_{t+1}{}^*}{d\mathbf{x}_{t+1}} - \zeta_t \leq 0 \qquad \mathbf{c}_t \cdot \frac{\partial \Lambda_t}{\partial \mathbf{c}_t} = 0 \qquad 3.23a$$

$$\frac{\partial \Lambda_t}{\partial \zeta_t} = \mathbf{c}_t - \mathbf{x}_t \leq 0 \qquad \zeta_t \cdot \frac{\partial \Lambda_t}{\partial \zeta_t} = 0 \qquad 3.23b$$

By the Envelope Theorem,[13]

$$\frac{dS_t^*}{d\mathbf{x}_t} = \delta \mathbf{G}' \frac{dS_{t+1}{}^*}{d\mathbf{x}_{t+1}} + \zeta_t. \qquad 3.24$$

Now, define

$$\lambda_t = \frac{dS_t^*}{d\mathbf{x}_t}. \qquad 3.25$$

Using this notation, Equations 3.23a and 3.24 may be rewritten as:

$$P_t\mathbf{q} + \delta \mathbf{R}'\lambda_{t+1} - \zeta_t \leq 0 \qquad 3.23a'$$

$$\lambda_t = \delta \mathbf{G}'\lambda_{t+1} + \zeta_t. \qquad 3.24'$$

Note the similarity between Equations 3.23a and 3.14c and between Equations 3.24 and 3.14a when this substitution is made. Equation 3.24′ is an equality, but 3.14a will always hold as an equality, too. Clearly, Equations 3.23a and 3.24 could be used to derive the same recursive relation as was obtained in the previous section (Equation 3.16). Thus the DTOC approach leads to the same recursive relation as the LP and Lagrangian methods.

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[13] See Silberberg (1978, pp. 168-171) for a discussion and a derivation of the Envelope Theorem.

The DTOC formulation can be viewed from a slightly different perspective. Equation 3.20 could have been written as:

$$S_t^*(\mathbf{x}_t) = P_t\mathbf{q}'\mathbf{c}_t^* + \delta S^*_{t+1}(\mathbf{G}\mathbf{x}_t + \mathbf{R}\mathbf{c}_t^*). \qquad 3.26$$

Recall that λ_t is defined as the derivative of S_t^* with respect to $\mathbf{x}_t$. A little care is required when taking the derivative of Equation 3.26 with respect to $\mathbf{x}_t$. This is because when the area cut equals the area available, $dc_{t,i}^* / dx_{t,i} = 1$. Otherwise, $dc_{t,i}^* / dx_{t,i} = 0$.[14] That is,

$$\frac{dc_{t,i}^*}{dx_{t,i}} = \begin{cases} 1 & c_{t,i}^* = x_{t,i} \\ 0 & c_{t,i}^* < x_{t,i} \end{cases} \qquad 3.27$$

With this in mind,

$$\begin{aligned} \frac{dS_t^*}{d\mathbf{x}_t} &= \frac{\partial S_t^*}{\partial \mathbf{x}_t} + \frac{\partial S_t^*}{\partial \mathbf{c}_t^*}\frac{d\mathbf{c}_t^*}{d\mathbf{x}_t} \\ &= \delta \mathbf{G}'\frac{dS_{t+1}^*}{d\mathbf{x}_{t+1}} + \left\{ P_t\mathbf{q} + \delta \mathbf{R}'\frac{dS_{t+1}^*}{d\mathbf{x}_{t+1}} \right\} \frac{d\mathbf{c}_t^*}{d\mathbf{x}_t} \end{aligned} \qquad 3.28$$

or

$$\frac{dS_t^*}{dx_{t,i}} = \delta\frac{dS_{t+1}^*}{dx_{t+1,max(i+1,N)}} + \left\{ P_t q_i + \delta\frac{dS_{t+1}^*}{dx_{t+1,1}} - \delta\frac{dS_{t+1}^*}{dx_{t+1,max(i+1,N)}} \right\} \frac{dc_t^*}{dx_t}. \qquad 3.28'$$

Rearranging 3.28′ and using 3.27,

$$\frac{dS_t^*}{dx_{t,i}} = \begin{cases} \delta\dfrac{dS_{t+1}^*}{dx_{t+1,max(i+1,N)}} & c_{t,i}^* < x_{t,i} \\ P_t q_i + \delta\dfrac{dS_{t+1}^*}{dx_{t+1,1}} & c_{t,i}^* = x_{t,i} \end{cases} \qquad 3.29$$

Once again, substitute in the definition of $\lambda_{t,i}$ (Equation 3.25) to obtain

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[14] This complication does not arise when the decision variables are defined as proportions, rather than areas.

$$\lambda_{t,l} = \begin{cases} \delta\lambda_{t+1,max(l+1,N)} & c_{t,l}^* < x_{t,l} \\ P_t q_l + \delta\lambda_{t+1,0} & c_{t,l}^* = x_{t,l} \end{cases} \qquad 3.30$$

This shows how the necessary conditions for the optimization problem give equations of motion for the dual variables which move backward through time.

Equations 3.28 and 3.24 imply that

$$\zeta_t = \left\{ P_t\mathbf{q} + \delta\mathbf{R}' \frac{dS_{t+1}^*}{d\mathbf{x}_{t+1}} \right\} \frac{d\mathbf{c}_t^*}{d\mathbf{x}_t}. \qquad 3.31$$

This is precisely what Equation 3.23a says, with the complementary slackness conditions. That is, the first-order conditions (Equation 3.23a) must hold as an equality if $c_{t,l} > 0$. They may also hold as an equality when $c_{t,l} = 0$, if $x_{t,l} = 0$ also. These are the times when $dc_{t,l}^*/dx_{t,l} = 1$. Otherwise, $\zeta_t = 0$, and the first-order conditions (Equations 3.23a-b) will not hold as strict equalities.

The necessary conditions for optimal control problems are usually stated in terms of maximizing a *Hamiltonian*. Two of the steps taken above are built into the Hamiltonian; thus the Hamiltonian can be viewed as a kind of shorthand for setting up the problem. The Hamiltonian is:

$$H_t = P_t\mathbf{q}'\mathbf{c}_t + \delta\lambda'_{t+1}(\mathbf{G}\mathbf{x}_t + \mathbf{R}\mathbf{c}_t) \qquad 3.32$$

where

$$\lambda_t = \frac{dS_t^*}{d\mathbf{x}_t}. \qquad 3.33$$

Thus, the Hamiltonian has built into it the step where the principle of optimality was used and the definition of the dual variables. In optimal control theory, these dual variables are usually called *adjoint variables.* Equation 3.30 gives a recursive definition of the adjoint variables and is called the *adjoint equation.* It can be verified rather easily that the maximization of the Hamiltonian subject to the harvest area restrictions and the nonnegativity of $\mathbf{x}_t$ and $\mathbf{c}_t$ gives the same first-order conditions as before.

Cutting at the Beginning of the Period

If the cut occurs at the beginning of the period, the DTOC solution procedure is changed only slightly. Define

$$s_t(\mathbf{x}_t, \mathbf{c}_{t+1}) = P_{t+1}\mathbf{q}'\mathbf{c}_{t+1} \qquad 3.34$$

and

$$S_t(\mathbf{x}_t, \mathbf{C}) = \sum_{\tau=t}^{T-1} \delta^{\tau-t} s_\tau(\mathbf{x}_\tau, c_{\tau+1}) + \delta^T S_T^*(\mathbf{x}_T^*). \qquad 3.35$$

Once again, applying Bellman's principle of optimality:

$$S_t^*(\mathbf{x}_t) = \max_{\mathbf{c}_{t+1}} \{\delta s_t(\mathbf{x}_t, c_{t+1}) + \delta S_{t+1}^*(x_{t+1})\}. \qquad 3.36$$

The Hamiltonian for this problem is

$$H_t = P_{t+1}\mathbf{q}'\mathbf{c}_{t+1} + \lambda'_{t+1}(\hat{\mathbf{G}}\mathbf{x}_t + \mathbf{R}\mathbf{c}_{t+1}) \qquad 3.37$$

where

$$\lambda_t = \frac{dS_t^*}{d\mathbf{x}_t}. \tag{3.38}$$

This is maximized subject only to nonnegativity constraints. The first-order condition is

$$\frac{\partial H_t}{\partial \mathbf{c}_{t+1}} = P_{t+1}\mathbf{q} + \mathbf{R}'\lambda_{t+1} \le 0. \tag{3.39}$$

Again, by the Envelope theorem,

$$\frac{dS_t^*}{d\mathbf{x}_t} = \delta\hat{\mathbf{G}}'\frac{dS_{t+1}^*}{d\mathbf{x}_t}. \tag{3.40}$$

Or, using Equation 3.38,

$$\lambda_t = \delta\hat{\mathbf{G}}'\lambda_{t+1}. \tag{3.40'}$$

Equations 3.39 and 3.40′ are the same as Equations 3.6a and 3.6c and can therefore be used to derive the recursive relation

$$\lambda_{t,l} = \max\ \{P_t q_l + \lambda_{t,1}, \delta\lambda_{t+1,\max(l+1,N)}\}. \tag{3.41}$$

Thus, it has been shown that the same necessary conditions and the same recursive relation can be derived for the linear timber supply model using three different approaches: linear programming, nonlinear programming, and discrete-time optimal control. Each of these different approaches leads to the same dynamic programming recursion relation. This has also been done for two different versions of the equations of motion: one where cutting occurs at the beginning of the period, and the other where cutting occurs at the end of the period.

The incorporation of downward-sloping demand curves changes the formulation and the first-order conditions of the problem only slightly. It does, however, complicate the computation of the solution somewhat. This is not to say that the downward-sloping demand problem is especially hard to solve; it just isn't as simple as the linear problem, which can be solved very easily. With downward-sloping demand curves, the problem is a linearly-constrained, nonlinear programming problem. It can be solved readily with many commercial nonlinear programming software packages.

A Continuous-Time Model

In this section, the equations of motion for the simple forest are converted to a continuous-time formulation. The continuous-time equations can be found by taking the limit of the discrete-time equations as the time interval goes to zero. Obviously, the distinction between cutting at the beginning of the period versus cutting at the end of the period is meaningless in continuous time. The continuous-time equations are derived here by considering *basic conservation laws*. These basic conservation laws give fundamental rules for looking at small changes in the variables. The continuous-time equations are derived by taking limits as these small changes go to zero.

The volume equation is unchanged in continuous time:

$$y = q(a)\,. \qquad 3.42$$

Define a function $c(a, t)$ analogous to the cut vectors $\mathbf{c}_t$ in the previous section. The rate at which forest area is cut in time t from age classes a to $a + \Delta a$ is given by

$$C(a + \Delta a, t) - C(a, t) = \int_a^{a+\Delta a} c(\alpha, t)\, d\alpha \qquad 3.43$$

where $C(a, t)$ denotes the rate at which forests less than age a are being cut at time t. The rate at which wood is supplied (in units of volume per unit time) to the market in time t is

$$w(t) = \int_0^{\infty} c(\alpha, t) q(\alpha)\, d\alpha\,. \qquad 3.44$$

Equation 3.44 is analogous to the inner product $\mathbf{q}'\mathbf{c}_t$ in the previous sections.

Let $M(a, t)$ be the area of forest of age less than or equal to age a at time t. Now define $m(a, t)$ such that

$$M(a, t) = \int_0^{a} m(\alpha, t)\, d\alpha\,. \qquad 3.45$$

The function $m(a, t)$ is analogous to the vector $\mathbf{x}_t$ in the previous section.

Consider the basic conservation law: in the absence of cutting, the area of forest less than or equal to a given age a is diminished in the period t to $t + \Delta t$ by the area of forest that becomes older than age a during that period. This is the area of forest between the ages $a - \Delta t$ and a at time t. This can be expressed as

$$M(a, t) - M(a, t + \Delta t) = -\left(M(a - \Delta t, t) - M(a, t)\right). \qquad 3.46$$

Dividing through by Δt and taking limits as Δt goes to 0, gives

$$\frac{\partial}{\partial t} M(a, t) = -\frac{\partial}{\partial a} M(a, t) \qquad 3.47$$

or

$$\frac{\partial}{\partial t} M(a, t) = -\, m(a, t)\,. \tag{3.48}$$

What if harvesting is considered? Now the basic conservation law states that the area of forest less than or equal to age a is reduced from time t to time $t + \Delta t$ by the area that grows older than a during that time, and is increased by the area that was older than $a - \Delta t$ but was cut. That is,

$$M(a, t) - M(a, t + \Delta t) = -\,(M(a - \Delta t, t) - M(a, t)) + \int_t^{t+\Delta t}\int_{a-\Delta t}^{\infty} c(\alpha, \tau)\, d\alpha\, d\tau\,. \tag{3.49}$$

Once again, dividing through by Δt and taking limits as Δt goes to zero,

$$\frac{\partial}{\partial t} M(a, t) = -\frac{\partial}{\partial a} M(a, t) + \int_a^{\infty} c(\alpha, t)\, d\alpha\,, \tag{3.50}$$

or

$$\frac{\partial}{\partial t} M(a, t) = -\frac{\partial}{\partial a} M(a, t) + C(\infty, t) - C(a, t)\,. \tag{3.50'}$$

Equation 3.50 is a partial differential equation defined on $a \geq 0$ and $t \geq 0$. Thus initial conditions at $t = 0$ and boundary conditions at $a = 0$ are needed. The initial conditions are given by the initial age-class distribution of the forest:

$$m(a, 0) = \phi(a)\,, \tag{3.51}$$

or

$$M(a, 0) = \int_0^a \phi(\alpha) d\alpha \, . \qquad 3.50'$$

To get the boundary conditions, consider a special case of the basic conservation law: The area of forest less than or equal to age Δt at time t is equal to the total area cut between $t - \Delta t$ and t. This means that

$$M(\Delta t, t) - M(0, t) = \int_t^{t+\Delta t} \int_0^\infty c(\alpha, \tau) \, d\alpha \, d\tau, \qquad 3.52$$

or

$$m(0, t) = \int_0^\infty c(\alpha, t) \, d\alpha \, . \qquad 3.52'$$

Compare this with Equation 3.50, and it is evident that this is equivalent to writing

$$\frac{\partial}{\partial t} M(0,t) = 0 \, . \qquad 3.53$$

From the initial conditions, $M(0, 0) = 0$. Thus, Equation 3.53 implies that

$$M(0, t) = 0 \, . \qquad 3.54$$

This just says that there are no acres with forests less than 0 years old. In order to solve a problem of this sort, it will probably also be necessary to put some upper bounds on the forest age class as was done in previous sections. That issue will not be considered further here.

Now, let $l(a)$ be the labor required to harvest a unit area of age a. The labor required at time t for harvesting activities can be expressed as:

$$L^h(t) = \int_0^{\infty} l(\alpha)c(\alpha, t)\, d\alpha\ , \tag{3.55}$$

where $L^h(t)$ is the total amount of labor employed in harvesting at time t.

Introduce an inverse demand function for harvested timber:

$$P(t) = f^d(w(t)). \tag{3.56}$$

Assume that $df^d/dt < 0$. Let η^l be the wage for harvesting labor. Now, an equilibrium price trajectory would be provided by the solution to the problem:

$$\underset{c(a,\,t)}{\text{maximize}} \qquad \int_{t_0}^{t_1} \left[\int_0^{w(\tau)} f^d(\theta)\, d\theta - \eta^l \int_0^{\infty} l(\alpha)c(\alpha,\ \tau) d\alpha \right] d\tau \tag{3.57}$$

subject to

$$w(t) = \int_0^{\infty} c(\alpha, t)\, q(\alpha)\, d\alpha$$

$$\frac{\partial}{\partial t} M(a, t) = -\frac{\partial}{\partial a} M(a, t) + C(\infty, t) - C(a, t)$$

$$M(0, t) = 0$$

$$M(a, 0) = \int_0^{a} \phi(\alpha)\, d\alpha$$

Peter Berck (1976, p. 53) was right; this is a difficult problem. The primary difficulty is the presence of the partial differential equation giving the dynamics of the age-class distribution. If a *maximum principle*[15] for problems with partial differential equations in the constraint set has been derived, I am not aware of it.

As mentioned, Heaps (1984) simplifies the problem by converting this partial differential equation into a process with a delay. In order to do this, he shows that the oldest age classes would be harvested first. The problem solved by Heaps is not the same as this one. In his problem, no demand curves are included, and revenue (price) is not a function of time. Rather, revenue is a function of the age class harvested. (Since the age class harvested varies over time, revenue is implicitly a function of time.) Heaps derives a necessary condition for this problem. It is not clear how one can use Heaps' result, however, and no discussion of the interpretation of the necessary condition is given. Possibly, a similar simplification can be used to solve 3.57.

Discussion

The primary accomplishment of this chapter has been to show the similarities between the different approaches used by Berck (1976) and Lyon and Sedjo (1982, 1983). Berck's linear programming approach is more limiting because it requires that the problem be entirely linear. However, his model is useful because it is the simplest of the dynamic age-class timber supply models, and it shows the relationship between these models and the harvest scheduling models. Berck also discussed the nonlinear programming approach. The nonlinear programming approach is mathematically equivalent to the

[15] The original *maximum principle* is due to Pontryagin et al. (1962). This was the first optimal control theorem. Since then, other theorems which can be applied to different problems than the one solved by Pontryagin have been published. These theorems are referred to collectively as *maximum principles*.

discrete-time optimal control approach used by Lyon and Sedjo. Lyon and Sedjo's model is actually a linearly-constrained nonlinear programming problem. The use of discrete-time optimal control emphasizes the recursive structure of the problem, however.

Lyon and Sedjo (1983) use a gradient method for solving their model in their first paper. In a very interesting later (1985) paper, they demonstrate the connection between their model and Walker's (1971) binary search model. They show that Walker ignored the land value in his model. They also describe a solution method for their model similar to Walker's binary search approach. In fact, Walker's ECHO algorithm provides the first iteration for their algorithm. They refer to that solution approach as a "shooting method" because the system is simulated forward in time iteratively until an optimal solution is obtained. I am not sure whether their shooting method is computationally more efficient for their model than commercial nonlinear programming software. Many commercial packages use a reduced-gradient algorithm for solving linearly-constrained nonlinear programming problems.

The continuous-time model remains unsolved. As mentioned, Heaps (1984) presented a necessary condition for a similar problem, but it is not clear how his necessary condition can be used to solve this problem. This is the first time, to my knowledge, that the equations of motion have been formulated as a partial differential equation. However, Binkley (1987) alludes to this formulation in a footnote (footnote [6]).

The work done in this chapter provides an important foundation for the multi-sector models presented in the next chapters. Hopefully, after dissecting the dynamic timber supply models thoroughly in this chapter, it will be easier to understand how the forestry sector model fits into the bigger picture presented by the following models.

CHAPTER 4: THE MULTI-SECTOR MODEL

If the Lyon and Sedjo (1982, 1983, 1985) model is accepted as a good representation of the economic interaction of timber producers and purchasers, how can this view of timber supply be integrated with a model of the economy as a whole? This is the problem which this dissertation addresses, and in this chapter the first step toward the construction of such a model is taken.

The model presented here represents a significant extension of the Lyon and Sedjo model. The model is expanded to include *m* sectors, and the objective function of the model becomes the maximization of a *community utility function*. The nature and interpretation of this aggregate utility function is discussed below, but the important point is that the scope of the model is expanded here to include the whole economy from production to consumption.

The problem of finding an optimal steady state for this model is solved, and some of the important ramifications of steady states in this context are discussed. A numerical example of the multi-sector model is presented, along with the solution for the optimal steady state of the example problem. The example was solved using MINOS, a commercial computer code for solving nonlinear programming problems. Finally, some relatively easy generalizations of the model are discussed.

While the scope of the model presented in this chapter includes the whole economy, many important aspects of an economy are left out. For example, the concept of a community utility function may be too restrictive. For policy purposes or welfare analysis, the welfare of separate consumer groups may be of special interest. For trade analysis, how can a single utility function for separate regions be justified-- particularly if some regions are different countries? It would clearly be useful to be able to disaggregate the community utility function.

Another important aspect of the economy that is not adequately addressed with this chapter's model is the treatment of public goods. Since in the model utility is maximized subject to the resource and technology constraints of the economy, any good that appears in the utility function will be produced as if a perfect market for that good existed. It is a first-best model of an economy. Note that this problem would apply to any kind of market imperfection that may exist such as externalities and concentrations of market power; it is hard to capture within the context of a maximization model the actual behavior of an economy when such problems exist.

Again, there is the perfect information/perfect foresight assumption of the model. This is perhaps the most unrealistic aspect of the model, but unfortunately, the most difficult to relax. This shortcoming has been recognized by others:

> The imperfections of the general equilibrium model as a description of economic reality are well known to economists and in a less informed way to the general public. . . . Investments and roundabout methods of production are poorly treated if the model is formulated in static terms, and any attempt to

rectify this by a dynamic model must find a replacement for the unrealistic assumption of perfect futures markets (Scarf 1984a, p. *xxi*).

These are, of course, only a few of the troubling simplifications of the model. Some of the problems with the multi-sector model are addressed with the model presented in the next chapter. Most will not be solved in this dissertation. The model presented here nevertheless has value for several reasons. It gives a broad view of the role of forests in society and provides an excellent analytical tool for studying the *(normative)* issues of how these forests should be managed to provide the greatest benefit for society. The implications of this picture are interesting and insightful. The model presented here provides a useful framework for planning and policy analysis. It should be useful for a range of purposes including the U.S. Forest Service planning process, regional and community development planning, and land use modeling. Also, the model provides the groundwork for future models which will address the issues described above more adequately. This model exposes these issues, whereas they are often obscured by other models.

Aggregate Utility Functions

One of the more heroic assumptions behind the model presented in this chapter is that a competitive economy can be treated as if it were a single entity acting to maximize a single objective function. This view, however, is the mathematical equivalent of a strict interpretation of Adam Smith's assertion that individuals pursuing their own gain are led by an "invisible hand" to increase the overall welfare of society. The "overall welfare of society" must be given a very specific -- and not generally very acceptable -- interpretation in this context, however. That is, that such welfare can be measured using the *community utility function.*

A *community utility function* is defined as "a utility function which, when maximized subject to an aggregate budget constraint, produces the same market demand functions obtained by summing consumer demand functions derived from individual utility maximization" (Katzner 1988, p. 241). The conditions under which such a utility function exists and for which the maximization of this utility function is equivalent to the equilibrium solution to a competitive economy are the same conditions as those required for a perfectly competitive economy, discussed in the introduction to this dissertation.[16] In a later chapter, where a general equilibrium model will be solved as an equilibrium problem, the existence and specific form of a community utility function for that problem will be demonstrated.

The community utility function can only be regarded as a measure of social welfare if the existing distribution of income in the economy is optimal and if the assumptions of a perfectly competitive economy are met. Only under these conditions is a community utility function equivalent to a *social welfare function.* A *social welfare function* is interpreted here as an objective function for an economy which, when maximized subject to the technology of an economy, yields an outcome that is *pareto optimal* and a socially optimal income distribution. *Pareto optimality* requires that no individual can be made better off without making another individual worse off.[17] What constitutes a socially

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[16] This is a fundamental result in welfare economics. For a discussion of this subject and the history of thought behind it, see Takayama (1985, Chapter 12).

[17] This includes allowing for the possibility of compensation. That is, the state of an economy is not pareto optimal if there is a change that will make some individuals worse off, but where the gainers can afford to compensate the losers so that all would have gained. For a more rigorous definition of pareto optimality, see Katzner (1988, p. 344).

optimal income distribution is a matter of judgement and is beyond the scope of this dissertation.[18]

The aggregate utility function is interpreted in this chapter as a community utility function. Thus it should not be interpreted as a measure of social welfare. However, if one has a social welfare function at hand, clearly it could serve the purpose of an objective function for the model. Of course, the existence of a community utility function is predicated on the existence of individual utility functions. For the purposes of this dissertation, the existence of individual utility functions is taken to be axiomatic.

The Model

In this section, the equations of the multi-sector model are presented and discussed. The label "multi-sector" is used to distinguish the model from the "true" general equilibrium model presented in the next chapter. This chapter's model is not a "true" general equilibrium model because more than one household cannot be modeled explicitly in this framework.

The model was constructed starting with the dynamic, age-class model of the forestry sector where cutting occurs at the beginning of the period. The wood produced in the forestry sector is treated as an intermediate product used as an input in the production of consumption goods. The other sectors of the economy are added assuming a technology similar to that used in static computable general equilibrium models.[19] That is,

[18] For a more thorough introduction to the issues of welfare economics, see Katzner (1988, Chapter 11).

[19] See, for example, Dervis, de Melo, and Robinson (1982) or Shoven and Whalley (1984).

gross production in each sector is given by neoclassical production functions combining the three primary factors of the economy: land, labor, and capital. Production in each of these sectors is used to meet consumption demand and demand for intermediate inputs. Intermediate-input technology is the same as in input-output models.

The input-output technology assumption for intermediate inputs is made for three reasons. First, many people are familiar with input-output models. The use of the input-output technology for intermediate inputs in this model makes the relationship between this model and the more traditional input-output models easy to recognize. Second, input-output data are widely available. Thus the burden of specifying the model for empirical applications is greatly reduced by this assumption. Third, input-output technology can be expressed with linear functions, and the more linear the model, the easier it is to solve. The input-output assumption can be relaxed for some products and maintained for others. For those sectors where the assumption is relaxed, alternate functional forms will require additional data for their specification.

It is common practice in models of the sort discussed in this chapter to assume that individuals save some fixed proportion of their income. Rather than make such an assumption, savings -- or capital formation -- is determined endogenously in the model as a result of the utility maximization assumption. This approach seems more consistent with the overall neoclassical flavor of the model. In the next chapter, some of the interesting results that this approach to savings behavior leads to will be discussed.

With that overview, each of the equations of the model will now be presented. The community utility function provides the objective function of the model. Utility is a function of the quantity of each good consumed in each period. It is also assumed that utility is separable across periods. That is, utility in one period does not depend on how much is consumed in other periods. This assumption is perhaps unrealistic, but it allows the model to be formulated and solved recursively. Without this assumption, the model would be mathematically much more complicated.

The objective is therefore to maximize the discounted present value of utility:[20]

$$\Upsilon = \sum_{t=1}^{\infty} \rho^t U(X_{1,t}^c, X_{2,t}^c, ..., X_{m,t}^c) \qquad 4.1$$

where

$X_{i,t}^c \equiv$ the quantity of good i consumed in period t,

$\rho \equiv$ one over one plus the community discount rate.

Let **X** be the *commodity space*. That is, the consumption vectors $(X_{1,t}^c, X_{2,t}^c, ..., X_{m,t}^c)$ are elements of **X**. It is assumed that the utility function U satisfies the following classical properties (Katzner 1988, Section 2.3):

4.1a. U is continuous where finite on **X**, negatively infinite where not finite, and has continuous second-order derivatives everywhere in the interior of **X**.

4.1b. $\partial U / \partial X_{i,t}^c > 0$ for $i = 1, 2, ..., m$ for all $\mathbf{X}_t^c$ in the interior of **X**.

4.1c. U is strictly quasiconcave.

4.1d. For any $\mathbf{X}_t^{c\prime}$ and $\mathbf{X}_t^{c\prime\prime}$ in the nonnegative orthant of $\mathbf{R}^m$ such that $U(\mathbf{X}_t^{c\prime}) = U(\mathbf{X}_t^{c\prime\prime})$, if $\mathbf{X}_t^{c\prime} > \mathbf{0}$ then $\mathbf{X}_t^{c\prime\prime} > \mathbf{0}$.

Following Katzner (1988), these properties will be referred to as continuity, differentiability, differential increasingness, strict quasi-concavity, and the boundary

[20] The notation used here is summarized Appendix A.

condition, respectively. These technical restrictions are required in order to assure the existence of a utility maximum. It will be shown in the next chapter that, given this formulation of utility, the community discount rate is determined, in the steady state at least, by the lowest discount rate from all the individual utility functions in the economy. For non-steady-state solutions, the community discount rate will in general vary from period to period.

The production functions for the non-forestry sectors combine neoclassical production functions for primary inputs and input-output technology for intermediate inputs. These functions can be written:

$$X_{i,t} = \min\{f^i(K_{i,t},\ L_{i,t},\ M_{i,t}),\ \frac{1}{a_{i,j}} X_{i,j,t},\ \frac{1}{a_i^q} W_{i,t}\} \qquad 4.2$$

where

$X_{i,t} \equiv$ gross output of good i in period t,

$L_{i,t} \equiv$ the labor used in industry i in period t,

$K_{i,t} \equiv$ the capital used in industry i in period t,

$M_{i,t} \equiv$ the land used in industry i in period t,

$a_{i,j} \equiv$ Leontief input-output coefficients,

$X_{i,j,t} \equiv$ the amount of good j used as an intermediate input in industry i in period t,

$a_i^q \equiv$ input-output coefficients giving wood use in industry i,

$W_{i,t} \equiv$ the wood used in industry i in period t.

The functions f^i are neoclassical production functions, defined over the range Y of non-negative values of $L_{i,t}$, $K_{i,t}$, and $M_{i,t}$, allowing for smooth substitution of the primary factors of production in the production process. Again, following Katzner (1988, Section 4.1) assume that:

4.2a. $f^i(0, 0, 0) = 0$ and $f^i \geq 0$ for all values of $\mathbf{y} = [L_{i,t}, K_{i,t}, M_{i,t}]$ in $\mathbf{Y}$.

4.2b. The f^i are continuous on $\mathbf{Y}$ and twice, continuously differentiable in the interior of $\mathbf{Y}$.

4.2c. There exists an open, connected subset $\mathbf{D}_i$, of $\mathbf{Y}$ having the origin as a limit point, on which $f^i_L > 0$, $f^i_K > 0$, and $f^i_M > 0$.[21]

4.2d. The f^i are strictly quasi-concave.

Once again, these technical requirements are made to ensure the existence of a maximum for the profit-maximization problem of producers.

Equation 4.2 is rather cumbersome for use in nonlinear programming applications. It is possible to express the production technology for this economy more simply. First, by the assumption of productive efficiency (see p. 3), it must be true that

$$X_{i,t} = f^i(K_{i,t}, L_{i,t}, M_{i,t}). \qquad 4.3$$

Next, the input-output technology can be incorporated into the model with a dynamic input-output materials balance equation:

$$X^c_{i,t} = X_{i,t} - \sum_{j=1}^{m} a_{i,j} X_{j,t} - \sum_{j=1}^{m} \gamma_{i,j} \Delta K_{j,t+1} \qquad 4.4$$

where

[21] Partial derivatives of functions will be indicated by an appropriate subscript. Thus, $f^i_L = \partial f^i / \partial L_{i,t}$, $f^i_K = \partial f^i / \partial K_{i,t}$, and $f^i_M = \partial f^i / \partial M_{i,t}$.

$\Delta K_{j,t+1} \equiv$ new capital produced for sector k to be installed in period t + 1,

$\gamma_{i,j} \equiv$ input-output coefficients for new capital production,

and all other variables are previously defined. This equation introduces the technology for producing new capital. For similar reasons to those given earlier, this is also input-output technology. Equation 4.4 states that the amount that is required for intermediate inputs in each of the consumption sectors and for the production of new capital must be subtracted from the total (gross) production of each consumer good; that which remains is available for consumption.

Equation 4.4 can be written in vector form:

$$\mathbf{X}_t^c = (\mathbf{I} - \mathbf{A})\mathbf{X}_t - \Gamma\Delta\mathbf{K}_{t+1} \qquad 4.4'$$

where

$\mathbf{X}_t^c \equiv [X_{1,t}^c,\ X_{2,t}^c,\ \ldots,\ X_{m,t}^c]'$,

$\mathbf{X}_t \equiv [X_{1,t},\ X_{2,t},\ \ldots,\ X_{m,t}]'$,

$\Delta\mathbf{K}_{j,t+1} \equiv [\Delta K_{1,t+1},\ \Delta K_{2,t+1},\ \ldots,\ \Delta K_{m,t+1}]'$,

$\Gamma \equiv$ the matrix of capital-production coefficients,

$\mathbf{A} \equiv$ the matrix of Leontief input-output coefficients.

Note that capital produced for different sectors is different. The assumption here is that the capital produced for one sector cannot be used in another sector. Another way of putting this is to say that once capital is installed in one sector, it is "stuck" there. It is also assumed that capital depreciates at a particular rate $1 - \delta$ each period. A different depreciation rate for each sector could be specified easily. For simplicity, let δ be the same for all sectors. The capital stocks will evolve over time according to the relation:

$$K_{i,t+1} = \delta K_{i,t} + \Delta K_{i,t+1}. \qquad 4.5$$

Equation 4.5 states that the capital stock in each sector in period $t+1$ equals the capital remaining in that sector from the previous period after depreciation plus the newly installed capital (which was produced the period before). It is significant that Equation 4.5 gives each of the non-forestry sectors a dynamic element. That is, capital makes the non-forestry sectors' profit-maximization problems dynamic optimization problems as well.

The forest production technology is given by the equations of motion for the simple forest discussed in Chapter 3. First, total harvest volumes must equal or exceed wood requirements for production in the consumption goods sectors:

$$\mathbf{q}'\mathbf{c}_t \geq \sum_{j=1}^{m} a_j^q X_{j,t} \qquad 4.6$$

where

$\mathbf{q}' \equiv [q_1,\ q_2,\ \ldots\ q_N]'$,

$\mathbf{c}_t \equiv [c_{1,t},\ c_{2,t},\ \ldots\ c_{N,t}]'$,

$q_i \equiv$ the volume of wood per acre of forest of age i,

$c_{i,t} \equiv$ the number of acres cut from age-class i in period t,

and the other variables are defined previously. Note that productive efficiency does not require that all the wood that is harvested be used. This is because land must be cleared before it can be transferred out of forests and into other sectors. If too much land is forested, more wood than is needed must be harvested in order to clear the forest for other uses.

The equations of motion of the forest must be modified to allow for changes in the forest land base. As mentioned, land must be cleared before it is transferred out of forestry. Similarly, land entering forestry must begin in the initial age class. The net change in land use in all other sectors is accounted for by changes in the initial forest age class. This results in the following equations of motion for the forest:

$$\mathbf{M}_{F,t+1} = \mathbf{G}\mathbf{M}_{F,t} + \mathbf{R}\mathbf{c}_t - \sum_{j=1}^{m}(M_{j,t+1} - M_{j,t})\mathbf{e}_1 \qquad 4.7$$

where

$$\mathbf{M}_{F,t} \equiv [M_{1,F,t},\ M_{2,F,t},\ \dots\ M_{N,F,t}]',$$

$M_{i,F,t} \equiv$ the number of acres of forest in age class i in period t.

Recall that

$$\mathbf{G} \equiv \begin{bmatrix} 0\,0\,0 & & \\ 1\,0\,0 & & \\ 0\,1\,0 & & \\ & \ddots & \\ & & 1\,0\,0 \\ & & 0\,1\,1 \end{bmatrix}_{N\mathrm{x}N}$$

and

$$\mathbf{R} \equiv \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & 0 & 0 & & 0 \\ 0 & -1 & 0 & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}_{N\mathrm{x}N}$$

and $\mathbf{e}_1' \equiv (1, 0, 0, \dots, 0) \in \mathbf{R}^n$. Of course, the harvest area restrictions are still required:

$$\mathbf{M}_{F,t} \geq \mathbf{c}_t\,. \qquad 4.8$$

That is, the area cut from each age class must be less than or equal to the area of forest in that age class.

Finally, the amount of labor used in each time period must be less than or equal to the labor supply. Labor is used in each of the consumption goods sectors. It is also assumed that labor is required for planting and harvesting, but no capital is required for these activities. Thus, the labor supply constraint can be written as:

$$\sum_{i=1}^{m} L_{i,t} + l_p M_{1,F,t} + \mathbf{l}_c' \mathbf{c}_t \leq \overline{L}_t \qquad 4.9$$

where

$l_p \equiv$ the amount of labor required to plant an acre of forest,

$\mathbf{l}_c \equiv [l_{c,1},\ l_{c,2},\ \ldots\ l_{c,N}]'$,

$l_{c,i} \equiv$ the amount of labor required to harvest an acre of forest in age class i,

$\overline{L}_t \equiv$ the labor force in period t.

The size of the labor force in each time period is exogenous. Also, note that no equation of this type is required for land. The specification of Equation 4.7 ensures that the total land use in the economy never changes.

In addition to Equations 4.1-4.9, initial conditions and non-negativity constraints will also be needed. As stated now, the model has an infinite time horizon. This is, of course, impractical and some kind of time horizon and terminal conditions would be required to solve specific problems. Alternative specifications of the terminal conditions will be discussed in a later section.

Table 4.1 presents the equations of the model in a nonlinear programming format. The decision variables are $X^c_{i,t}$, $K_{i,t}$, $L_{i,t}$, $M_{i,t}$, $\Delta K_{i,t}$, $X_{i,t}$, $\mathbf{M}_{F,t}$ and $\mathbf{c}_t$ for $i = 1, 2, \ldots, m$ and $t = 1, 2, \ldots, T$. The only nonlinear functions are the objective function and the

Table 4.1. The multi-sector model.

$$\underset{\substack{X^c_{i,t},\, K_{i,t},\, L_{i,t},\, M_{i,t} \\ \Delta K_{i,t},\, X_{i,t},\, \mathbf{M}_{F,t},\, \mathbf{c}_t}}{\text{Maximize}} \quad \sum_{t=1}^{T-1} \rho^t U(X^c_{1,t}, X^c_{2,t}, \ldots, X^c_{m,t}) \tag{1}$$

subject to

$$X_{i,t} = f^i(K_{i,t},\ L_{i,t},\ M_{i,t}) \tag{2}$$

$$X^c_{i,t} = X_{i,t} - \sum_{j=1}^{m} a_{i,j} X_{j,t} - \sum_{j=1}^{m} \gamma_{i,j} \Delta K_{j,t+1} \tag{3}$$

$$K_{i,t+1} = \delta K_{i,t} + \Delta K_{i,t+1} \tag{4}$$

$$\mathbf{q}'\mathbf{c}_t \geq \sum_{j=1}^{m} a^q_j X_{j,t} \tag{5}$$

$$\mathbf{M}_{F,t+1} = \mathbf{G}\mathbf{M}_{F,t} + \mathbf{R}\mathbf{c}_t - \sum_{i=1}^{m} (M_{i,t+1} - M_{i,t})\mathbf{e}_1 \tag{6}$$

$$\mathbf{M}_{F,t} \geq \mathbf{c}_t \tag{7}$$

$$\sum_{i=1}^{m} L_{i,t} + l_p M_{1,F,t} + \mathbf{l}_c{}'\mathbf{c}_t \leq \overline{L}_t \tag{8}$$

Plus

- nonnegativity constraints
- initial conditions
- ending conditions

neoclassical production functions for each sector. Prices and costs do not appear explicitly in the formulation. In the next section, Lagrangian multipliers are introduced and the first-order conditions for maximization of the objective presented. These Lagrangian multipliers are the prices for the model. Thus, prices can be said to live on the dual side of the model. Initial conditions, ending conditions, the labor supply, preferences (utility function parameters), and the technological parameters are exogenous to the model.

First-Order Conditions

First-order conditions for the model are derived by constructing a Lagrangian function and taking partial derivatives of this Lagrangian with respect to each of the decision variables. To construct a Lagrangian function, Lagrangian multipliers are assigned to each constraint; the product of each constraint and its respective multiplier is subtracted from the objective function. Table 4.2 lists the Lagrangian multipliers and gives their interpretation. Table 4.3 presents the Lagrangian for the problem. As in Chapter 3, discount factors are specifically included with the dual variables. First-order conditions are also referred to as dual constraints.

The partial derivative of the Lagrangian with respect to $X^c_{i,t}$ gives the first dual constraint:[22]

$$\frac{\partial \Lambda}{\partial X^c_{i,t}} = \rho^t[U_{i,t} - \eta^d_{i,t}] \leq 0 \,. \tag{4.10}$$

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[22] Once again, the use of subscripts on function names indicates a partial derivative. Thus, $U_{i,t} = \partial U / \partial X^c_{i,t}$.

Table 4.2. The interpretation of the Lagrangian multipliers for the multi-sector model.

$\eta^{f}_{i,t} \equiv$ the price of gross output from industry i in period t.

$\eta^{d}_{i,t} \equiv$ the price of consumer good i in period t.

$\eta^{k}_{i,t} \equiv$ the price in period t of capital used in industry i.

$\eta^{l}_{t} \equiv$ the labor wage in period t.

$\eta^{q}_{t} \equiv$ the price of wood in period t.

$\lambda_t \equiv [\lambda_{1,t}, \lambda_{2,t}, \ldots, \lambda_{N,t}]'$.

$\lambda_{i,t} \equiv$ the price of an acre of forest in age class i in period t; $\lambda_{1,t}$ is the price of bare land.

$\zeta_t \equiv [\zeta_{1,t}, \zeta_{2,t}, \ldots, \zeta_{N,t}]'$.

$\zeta_{i,t} \equiv$ the cost of postponing harvesting age class i in period t (not really a price).

Table 4.3. The Lagrangian of the multi-sector model.

$$\underset{\substack{X^c_{i,t},\, K_{i,t},\, L_{i,t},\, M_{i,t} \\ \Delta K_{i,t},\, X_{i,t},\, \mathbf{M}_{F,t},\, \mathbf{c}_t}}{\text{Maximize}} \quad \Lambda = \sum_{t=1}^{T-1} \rho^t U(X^c_{1,t}, X^c_{2,t}, \ldots, X^c_{m,t}) -$$

$$\rho^t \eta^s_{i,t}[X_{i,t} - f^i(K_{i,t},\ L_{i,t},\ M_{i,t})] -$$

$$\rho^t \eta^d_{i,t}[X^c_{i,t} - X_{i,t} + \sum_{j=1}^{m} a_{i,j} X_{j,t} + \sum_{j=1}^{m} \gamma_{i,j} \Delta K_{j,t+1}] -$$

$$\rho^t \eta^k_{i,t}[K_{i,t+1} - \delta K_{i,t} - \Delta K_{i,t+1}] -$$

$$\rho^t \eta^q_t[\sum_{j=1}^{m} a^q_j X_{j,t} - \mathbf{q}'\mathbf{c}_t] -$$

$$\rho^{t+1} \lambda_{t+1}[\mathbf{M}_{F,t+1} - \mathbf{G}\mathbf{M}_{F,t} - \mathbf{R}\mathbf{c}_t + \sum_{i=1}^{m} (M_{i,t+1} - M_{i,t})\mathbf{e}_1] -$$

$$\rho^t \zeta_t[\mathbf{c}_t - \mathbf{M}_{F,t}] -$$

$$\rho^t \eta^l_t[\sum_{i=1}^{m} L_{i,t} + l_p M_{1,F,t} + \mathbf{l}_c{}'\mathbf{c}_t - \overline{L}_t]$$

Subject to

- nonnegativity constraints
- initial conditions
- ending conditions

This equation states the familiar condition that the demand price for each consumption good must equal the marginal utility of that good.[23]

The second dual constraint is given by the partial derivative of the Lagrangian with respect to gross outputs:

$$\frac{\partial \Lambda}{\partial X_{i,t}} = \rho^t[-\eta^s_{i,t} + \eta^d_{i,t} - \sum_{j=1}^{m} a_{j,i}\eta^d_{j,t} - a^q_i \eta^q_t] \leq 0 . \qquad 4.11$$

This equation defines a "gross output price". However, no market for "gross output" is assumed to exist; this is simply an accounting device. The technology is expressed as if production were a two-stage process: first, "gross output" is produced using labor, capital, and land; then "gross output" is recombined in the materials balance equation to produce capital and consumption goods. This is just a mathematically convenient way of expressing the production technology given in Equation 4.2. Equation 4.11 states that the accounting price of gross output of good i equals the demand price for good i minus the cost of the intermediate inputs required to convert gross output to final output. This gross output price is used to define the value marginal product (VMP) of primary inputs in the next three equations.

The partial derivative of the Lagrangian with respect to labor use in each industry and time period determines the wage rate:

$$\frac{\partial \Lambda}{\partial L_{i,t}} = \rho^t[\eta^s_{i,t} f^i_{L,t} - \eta^l_t] \leq 0 . \qquad 4.12$$

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[23] To simplify the exposition, the dual constraints have been discussed as if they held as equalities even though they are (correctly) written as inequalities. Precisely when they must hold as equalities is determined by the complementary slackness conditions given in Table 4.5.

Thus the VMP of labor in each industry in a given time period gives the wage rate for that time period.

The partial derivative of the Lagrangian with respect to capital use defines the rental rate for capital:

$$\frac{\partial \Lambda}{\partial K_{i,t}} = \rho^t[\eta^s_{i,t} f^i_{K,t} - \eta^k_{i,t} + \delta \rho \eta^k_{i,t+1}] \leq 0 \,. \tag{4.13}$$

The rental rate for capital in each industry equals the price of the capital in the current period minus the discounted price of the depreciated capital in the next period. The VMP of capital in each industry and time period must equal this rental rate. Alternatively, the price of capital must equal current earnings plus the discounted future value of the capital after depreciation.

A similar equation is derived for land used in the consumption goods sectors:

$$\frac{\partial \Lambda}{\partial M_{i,t}} = \rho^t[\eta^s_{i,t} f^i_{M,t} - \lambda_{1,t} + \rho \lambda_{i,t+1}] \leq 0 \,. \tag{4.14}$$

Equation 4.14 states that the VMP of land in each industry and time period must equal the value of the land in that time period minus the discounted value of the land in the next time period. That is, the VMP of land must equal the rental rate for land.

The partial derivative of the Lagrangian with respect to new capital determines the price of new capital:

$$\frac{\partial \Lambda}{\partial \Delta K_{i,t}} = \rho^t[\eta^k_{i,t} - \rho^{-1} \sum_{j=1}^{m} \gamma_{j,i} \eta^d_{j,t-1}] \leq 0 \,. \tag{4.15}$$

That is, the capital price equals the capitalized cost of the intermediate inputs required to produce the new capital.

The dual constraints relevant for the forestry sector are essentially the same as those discussed in the previous chapter. The partial derivative of the Lagrangian with respect to the forest age-class vector is:

$$\frac{\partial \Lambda}{\partial \mathbf{M}_{F,t}} = \rho^t[-\lambda_t + \rho \mathbf{G}'\lambda_{t+1} + \zeta_t - \eta_t^l l_p \mathbf{e}_1] \leq 0 . \qquad 4.16$$

Equation 4.16 is the same as Equation 3.12a except for the last term, which reflects planting costs. In less compact notation, Equation 4.16 becomes

$$\frac{\partial \Lambda}{\partial M_{1,F,t}} = \rho^t[-\lambda_{1,t} + \rho \lambda_{2,t+1} + \zeta_{1,t} - \eta_t^l l_p] \leq 0 \qquad 4.16'$$

and

$$\frac{\partial \Lambda}{\partial M_{k,F,t}} = \rho^t[-\lambda_{k,t} + \rho \lambda_{\max(k+1,N),t+1} + \zeta_{k,t}] \leq 0 . \qquad k = 2,\ 3,\ \ldots,\ N \qquad 4.16''$$

Thus Equation 4.16 states that the shadow price of an acre in a particular age class must be less than or equal to the compounded shadow price of that acre in the previous period. For the second youngest age class the shadow price must be less than or equal to the compounded shadow price of the youngest age class from the previous period plus the compounded planting costs from the previous period. These relationships hold as equalities as long as there are trees standing in that age class.

For the final age class k for which trees are standing, and for age classes older than that, $\zeta_{k,t}$ is potentially positive. This allows Equation 4.16 to hold as a strict equality for all age classes even though the shadow price for an acre in age class $k+1$ in period $t+1$ will probably not equal the compounded value of an acre in age-class k in period t for those k greater than or equal to the rotation age. If this is not crystal-clear at this stage, a graphical explanation is given below for these equations in the steady state.

The partial derivative of the Lagrangian with respect to the cut vector is:

$$\frac{\partial \Lambda}{\partial \mathbf{c}_t} = \rho^t[\eta_t^q \mathbf{q} + \rho \mathbf{R}' \lambda_{t+1} - \zeta_t - \eta_t^l \mathbf{l}_c] \leq 0 . \tag{4.17}$$

Equation 4.17 is the same as Equation 3.12c except for the term reflecting harvest costs. In less compact notation, Equation 4.17 becomes

$$\frac{\partial \Lambda}{\partial c_{k,t}} = \rho^t[\eta_t^q q_k + \rho(\lambda_{1,t+1} - \lambda_{\max(k+1,N),t+1}) - \zeta_{k,t} - \eta_t^l l_{c,k}] \leq 0 . \tag{4.17'}$$

From Equation 4.16, for all age classes with positive acreage except the first, $\lambda_{k,t} = \rho \lambda_{\max(k+1,N),t+1} + \zeta_{k,t}$. With this replacement, Equation 4.17 becomes

$$\lambda_{k,t} \geq \eta_t^q q_k + \rho \lambda_{1,t+1} - \eta_t^l l_{c,k} . \tag{4.17''}$$

This equation states that the shadow price of an acre of forest is always greater than or equal to the value of the wood minus harvest costs plus the discounted value of bare land in the next period. If any acres are cut from an age class, Equation 4.17″ will hold as an equality.

All of these first-order conditions are expressed as inequalities; when they will hold as equalities is determined by the complementary slackness conditions. The complementary slackness conditions require that when a variable is positive, the respective partial derivative must be strictly equal to zero. Table 4.4 lists the first-order conditions for the

Table 4.4. The first-order conditions for the multi-sector model.

$$\frac{\partial \Lambda}{\partial X_{i,t}^{c}} = \rho^{t}[U_{i,t} - \eta_{i,t}^{d}] \leq 0 \tag{1}$$

$$\frac{\partial \Lambda}{\partial X_{i,t}} = \rho^{t}[-\eta_{i,t}^{s} + \eta_{i,t}^{d} - \sum_{j=1}^{m} a_{j,i}\eta_{j,t}^{d} - a_{i}^{q}\eta_{t}^{q}] \leq 0 \tag{2}$$

$$\frac{\partial \Lambda}{\partial L_{i,t}} = \rho^{t}[\eta_{i,t}^{s} f_{L,t}^{i} - \eta_{t}^{l}] \leq 0 \tag{3}$$

$$\frac{\partial \Lambda}{\partial K_{i,t}} = \rho^{t}[\eta_{i,t}^{s} f_{K,t}^{i} - \eta_{i,t}^{k} + \delta\rho\eta_{i,t+1}^{k}] \leq 0 \tag{4}$$

$$\frac{\partial \Lambda}{\partial M_{i,t}} = \rho^{t}[\eta_{i,t}^{s} f_{M,t}^{i} - \lambda_{1,t} + \rho\lambda_{1,t+1}] \leq 0 \tag{5}$$

$$\frac{\partial \Lambda}{\partial \Delta K_{i,t}} = \rho^{t}[\eta_{i,t}^{k} - \rho^{-1}\sum_{j=1}^{m} \gamma_{j,i}\eta_{j,t-1}^{d}] \leq 0 \tag{6}$$

$$\frac{\partial \Lambda}{\partial \mathbf{M}_{F,t}} = \rho^{t}[-\lambda_{t} + \rho\mathbf{G}'\lambda_{t+1} + \zeta_{t} - \eta_{t}^{l} l_{p}\mathbf{e}_{1}] \leq 0 \tag{7}$$

$$\frac{\partial \Lambda}{\partial \mathbf{c}_{t}} = \rho^{t}[\eta_{t}^{q}\mathbf{q} + \rho\mathbf{R}'\lambda_{t+1} - \zeta_{t} - \eta_{t}^{l}\mathbf{l}_{c}] \leq 0 \tag{8}$$

Table 4.5. The complementary slackness conditions for the multi-sector model.

$$X^c_{i,t}[U_{i,t} - \eta^d_{i,t}] = 0 \tag{1}$$

$$X_{i,t}[-\eta^s_{i,t} + \eta^d_{i,t} - \sum_{j=1}^{m} a_{j,i}\eta^d_{j,t} - a^q_i\eta^q_t] = 0 \tag{2}$$

$$L_{i,t}[\eta^s_{i,t}f^i_{L,t} - \eta^l_t] = 0 \tag{3}$$

$$K_{i,t}[\eta^s_{i,t}f^i_{K,t} - \eta^k_{i,t} + \delta\rho\eta^k_{i,t+1}] = 0 \tag{4}$$

$$M_{i,t}[\eta^s_{i,t}f^i_{M,t} - \lambda_{1,t} + \rho\lambda_{1,t+1}] = 0 \tag{5}$$

$$\Delta K_{i,t}[\eta^k_{i,t} - \rho^{-1}\sum_{j=1}^{m}\gamma_{j,i}\eta^d_{j,t-1}] = 0 \tag{6}$$

$$\mathbf{M}_{F,t}'[-\lambda_t + \rho\mathbf{G}'\lambda_{t+1} + \zeta_t - \eta^l_t l_p\mathbf{e}_1] = 0 \tag{7}$$

$$\mathbf{c}_t'[\eta^q_t\mathbf{q} + \rho\mathbf{R}'\lambda_{t+1} - \zeta_t - \eta^l_t\mathbf{l}_c] = 0 \tag{8}$$

multi-sector model, and Table 4.5 gives the corresponding complementary slackness conditions.

The Optimal Steady State

Obviously, if the multi-sector model is to be solved numerically, a finite time horizon must be imposed. This requires that some way of specifying terminal conditions for the problem be given. There are a number of ways this can be done.[24]

First, since the discount factor ρ^t approaches zero for larger and larger values of t, if the time horizon is long enough, the terminal conditions will hardly matter. The primary drawbacks of this approach are that it is hard to say what "long enough" means, and long enough may be longer than one can afford to go given computing budgets and the limited capabilities of current solution algorithms. Regarding the first drawback, for age-class structure models, "long enough" will almost always be at least two rotations since the ending conditions directly affects the solution for one rotation prior to the terminal period. Regarding the second drawback, the equations for just one period of the multi-sector model may constitute a "large" model, and extending the time horizon unnecessarily can be costly.

A second way to specify ending conditions is to put a value on ending stocks. The obvious drawback to this approach is that the value that should be put on these ending stocks is probably unknown. Alternatively, a minimum level of ending stocks could be required. Once again, how should these minimum levels be determined?

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[24] For a discussion of ending conditions in a similar context, see Haight and Getz (1987).

A third alternative is to require that the system reach the optimal steady state by the terminal period. This approach is based on the twin assumptions that the optimal solution to the problem would approach this optimal steady state if given a very long time horizon, and that forcing the system to reach this steady state prematurely imposes only minor costs. The first assumption is not proven for this model, but all experience with the model suggests that it is true. The second assumption clearly depends on how far the initial state is from the optimal steady state. If these assumptions are satisfied, then this approach is probably the best way to give the model a finite time horizon. However, one must be able to solve for this optimal steady state if the approach is to be used. The purpose of this section is to show how this can be done.

The first step is to impose a steady state on the primal model formulation (Equations 4.1 and 4.3 - 4.9). This requires only a few minor changes. A strict interpretation of a steady state will be imposed where all the variables must be the same in each period. Less strict interpretations might allow cycles with regular periods and fluctuations.

First, consider the objective function. The utility function and its arguments are the same in each period. Therefore, the utility function can be factored out and the standard result for the sum of an infinite geometric series used to obtain:

$$\Upsilon = \frac{1}{1-\rho} U(X^c_{1,s}, X^c_{2,s}, \ldots, X^c_{m,s}) \qquad 4.18$$

where the subscript s indicates a steady state variable.

Capital and forest land are the only dynamic variables in the model. Imposing a steady state on Equation 4.5 requires that

$$\Delta K_{i,s} = (1 - \delta)K_{i,s}\,. \tag{4.19}$$

This equation can be eliminated by changing Equation 4.4 to

$$X^c_{i,s} = X_{i,s} - \sum_{j=1}^{m} a_{i,j}X_{j,s} - (1 - \delta)\sum_{j=1}^{m} \gamma_{i,j}K_{j,s}\,. \tag{4.20}$$

Imposing a steady state on the forest changes Equation 4.7 to read

$$(\mathbf{I} - \mathbf{G})\mathbf{M}_{F,s} = \mathbf{Rc}_s. \tag{4.21}$$

Equation 4.21 does not ensure that the land constraint is met, so the following constraint must be added:

$$\mathbf{d}^{n\prime}\mathbf{M}_{F,s} + \sum_{i=1}^{m} M_{i,s} \leq \overline{M}_s \tag{4.22}$$

where

$\overline{M}_s \equiv$ the land supply.

None of the other equations for the primal problem need changing for the steady state. Note that if one stopped here and solved this problem, the community rate of time preference would have no effect on the solution. This is because the term $1/(1 - \rho)$ is simply a constant by which the objective function is multiplied. If this term were removed completely, the solution of the problem would not change. In fact, the solution to this problem is the optimal steady state for the special case where the interest rate is zero ($\rho = 1,$), but not the optimal steady state in general.

The solution for the optimal steady state can be obtained by including the dual and the complementary slackness constraints in the model formulation explicitly. This dramatically complicates the problem by adding many additional constraints, many of which are nonlinear. However, it is possible to solve some problems of this type using commercial nonlinear programming packages. The numerical example in the next section was solved using MINOS.[25] Also, the general equilibrium model presented in the next section explicitly includes the dual and complimentary slackness constraints, so the solution of the optimal steady state problem provides a good bridge to that model.

Five of the dual constraints need modification for the steady state problem. Equation 4.13 becomes

$$\eta^{s}_{i,s} f^{i}_{K,s} - (1 - \delta\rho)\eta^{k}_{i,s} \leq 0 \,. \qquad 4.23$$

Thus, in the steady state the rental rate on capital is $(1 - \delta\rho)$ times the price of capital. The VMP of capital in each sector must equal this rental rate.

Equation 4.14 becomes

$$\eta^{s}_{i,s} f^{i}_{M,s} - (1 - \rho)\lambda_{i,s} \leq 0 \qquad 4.24$$

Thus, in the steady state the rental rate on land is $(1 - \rho)$ times the value of bare land. Note that the rental factor on capital is greater than that for land because capital depreciates but land does not.

Equation 4.15 becomes

[25] MINOS stands for Modular In-core Nonlinear Optimization System. Documentation can be found in Murtaugh and Saunders (1983).

$$\eta_{i,s}^{k} - \rho^{-1}\sum_{j=1}^{m}\gamma_{j,i}\eta_{j,s}^{d} \leq 0\,. \qquad 4.25$$

Note that the cost of new capital must be capitalized for one period even in the steady state.

Turning to the forestry sector, Equation 4.16 becomes

$$-\lambda_s + \rho\mathbf{G}'\lambda_s + \zeta_s - \eta_s^l l_p \mathbf{e}_1 \leq 0\,. \qquad 4.26$$

This is equivalent to the following N equations:

$$\begin{aligned}
\rho\lambda_{2,s} &\leq \lambda_{1,s} + \eta_s^l l_p - \zeta_{1,s} \\
\rho\lambda_{3,s} &\leq \lambda_{2,s} - \zeta_{2,s} \\
&\vdots \\
\rho\lambda_{N,s} &\leq \lambda_{N-1,s} - \zeta_{N-1,s} \\
\rho\lambda_{N,s} &\leq \lambda_{N,s} - \zeta_{N,s}
\end{aligned} \qquad 4.26'$$

Equation 4.17 becomes

$$\eta_s^q \mathbf{q} + \rho\mathbf{R}'\lambda_s - \zeta_s - \eta_s^l \mathbf{l}_c \leq 0\,. \qquad 4.27$$

which is equivalent to the N equations:

$$\begin{aligned}
&\rho\lambda_{2,s} + \zeta_{1,s} \geq \rho\lambda_{1,s} + \eta_s^q q_1 - \eta_s^l l_{c,1} \\
&\rho\lambda_{3,s} + \zeta_{2,s} \geq \rho\lambda_{1,s} + \eta_s^q q_2 - \eta_s^l l_{c,2} \\
&\quad\vdots \\
&\rho\lambda_{N-1,s} + \zeta_{N-1,s} \geq \rho\lambda_{1,s} + \eta_s^q q_{N-1} - \eta_s^l l_{c,N-1} \\
&\rho\lambda_{N,s} + \zeta_{N,s} \geq \rho\lambda_{1,s} + \eta_s^q q_N - \eta_s^l l_{c,N}
\end{aligned} \tag{4.27'}$$

These equations have been derived and discussed several times already from a number of different perspectives. It is perhaps easiest to understand their meaning in this steady-state context because they can be explained graphically in this form. Ignoring the ζ 's, Equations 4.26′ define an upper value for λ_2 through λ_N. This is depicted by the upper sequence of points and the dashed curve connecting them in Figure 4.1; given λ_1, $\lambda_2^{max} = \rho^{-1}(\lambda_1 + \eta^l l_p)$, $\lambda_3^{max} = \rho^{-1}(\lambda_2^{max})$, and so on. Upon harvesting, one obtains the value of the harvested wood minus harvest costs, and an acre of bare land is also available for use in the next period. Thus, the quantity $[\rho\lambda_{1,s} + \eta_s^q q_k - \eta_s^l l_{c,k}]$ provides a lower bound on the value of an acre of forest in age class k. This is the interpretation of Equation 4.27 and gives the points connected by the lower dashed curve in Figure 4.1. (The bottom dashed line is the present value of the bare land that will be available in the next period.) At the rotation age, the two curves are tangent, and the value of the forest at that age equals both the capitalized planting costs and land rent, and the net value of the harvested timber and the bare land available for the next period. This occurs in age class five in the figure. Given planting and harvesting costs and the price of bare land, the price of wood can be obtained by finding the price that gives this tangency.[26] Of course, in the multi-sector model, all of these prices and costs are solved for simultaneously.

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[26] See also the discussion of Subroutine FAUST in Chapter 6.

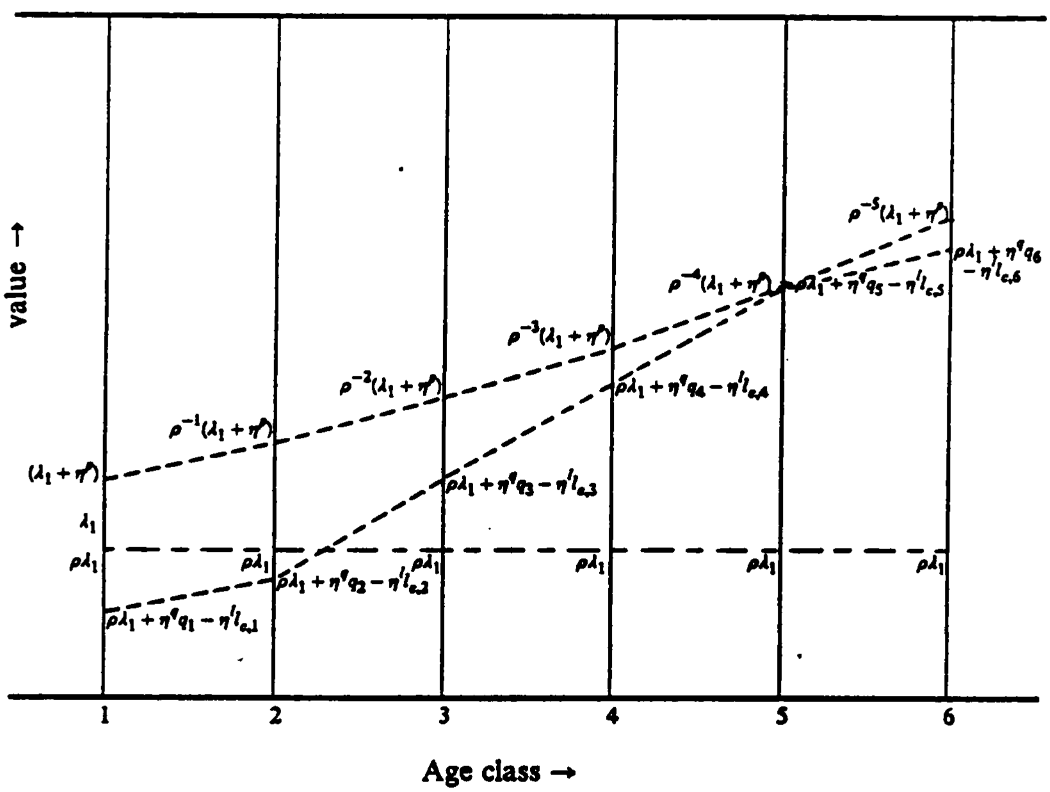

Figure 4.1. A graphical interpretation of the forestry dual constraints for the steady state problem.

The curves in Figure 4.1 should be familiar to most forest economists. The upper curve is the present value of the forest when managed on an optimal rotation. The lower curve is the liquidation value of the forest. It should be reiterated that these curves are derived from the special case of a steady state. The non-steady-state picture would have a similar appearance, but would always be in a state of flux.

The ζ-multipliers are also illustrated in to Figure 4.1. The first four elements of the ζ -vector (ζ_1 to ζ_4) are all equal to zero. The fifth element, ζ_5 is equal to the difference between $\rho^{-5}(\lambda_1 + \eta^p)$ and $\rho\lambda_1 + \eta^q q_6 - \eta^l l_{c,6}$. Note that this is pictured on the *sixth* age-class line. The sixth element of the ζ-vector is not pictured in the figure. However, from the last equation in 4.26′, we can see that $\zeta_6 = (1 - \rho)\lambda_{6,s}$.

The equivalence of these conditions to the Faustmann formula can be shown easily. At the optimal rotation, the capitalized value of the bare land and planting costs equals the net harvest value plus the value of bare land in the next period. Thus

$$\rho^{-(R-1)}(\lambda_{1,s} + \eta_s^l l_p) = \eta_s^q q_R - \eta_s^l l_{c,R} + \rho\lambda_{1,s}. \qquad 4.28$$

Solving for $\lambda_{1,s}$,

$$\lambda_{1,s} = [-\eta_s^l l_p + \rho^{R-1}(\eta_s^q q_R - \eta^l l_{c,R})](1 - \rho^R)^{-1}. \qquad 4.29$$

This is the Faustmann formula in discrete time.[27]

Summarizing this section, the optimal steady state can be found by maximizing the objective function 4.18 subject to the primal, dual, and complementary slackness con-

[27] As mentioned in Chapter 2, Mitra and Wan (1985) also show that the optimal steady state solution for a discrete-time dynamic timber supply model is equivalent to the Faustmann formula. The result derived here is much simpler than the one found in Mitra and Wan (1985). Also, this model is slightly different from theirs.

Table 4.6a. The optimal steady state problem for the multi-sector model: primal conditions.

$$\underset{\substack{X_{i,s}^c,\ K_{i,s},\ L_{i,s},\ M_{i,s} \\ X_{i,s},\ \mathbf{M}_{F,s},\ \mathbf{c}_s,\ \eta_{i,s}^s,\ \eta_{i,s}^d \\ \eta_{i,s}^k,\ \eta_s^q,\ \lambda_s,\ \zeta_s}}{\text{Maximize}} \qquad \Upsilon = U(X_{1,s}^c, X_{2,s}^c, \ldots, X_{m,s}^c) \tag{1}$$

subject to

$$X_{i,s} = f^i(K_{i,s},\ L_{i,s},\ M_{i,s}) \tag{2}$$

$$X_{i,s}^c = X_{i,s} - \sum_{j=1}^{m} a_{i,j} X_{j,s} - (1-\delta)\sum_{j=1}^{m} \gamma_{i,j} K_{j,s} \tag{3}$$

$$\Delta K_{i,s} = (1-\delta) K_{i,s}. \tag{4}$$

$$\mathbf{q}'\mathbf{c}_s = \sum_{j=1}^{m} a_j^q X_{j,s} \tag{5}$$

$$(\mathbf{I} - \mathbf{G})\mathbf{M}_{F,s} = \mathbf{R}\mathbf{c}_s. \tag{6}$$

$$\mathbf{M}_{F,s} \geq \mathbf{c}_s \tag{7}$$

$$\sum_{i=1}^{m} L_{i,s} + l_p M_{1,F,s} + \mathbf{l}_c'\mathbf{c}_s \leq \overline{L}_s \tag{8}$$

$$\mathbf{d}^{n\prime}\mathbf{M}_{F,s} + \sum_{i=1}^{m} M_{i,s} \leq \overline{M}_s \tag{9}$$

Table 4.6b. **The optimal steady state problem for the multi-sector model: dual and complementary slackness constraints.**

$[U_{i,s} - \eta^d_{i,s}] \leq 0$	$X^c_{i,s}[\cdot] = 0$	(10)
$[-\eta^s_{i,s} + \eta^d_{i,s} - \sum_{j=1}^{m} \eta^d_{j,s} - a^q_i \eta^q_s] \leq 0$	$X_{i,s}[\cdot] = 0$	(11)
$[\eta^s_{i,s} f^i_{L,s} - \eta^l_s] \leq 0$	$L_{i,s}[\cdot] = 0$	(12)
$[\eta^s_{i,s} f^i_{K,s} - (1 - \delta\rho)\eta^k_{i,s}] \leq 0$	$K_{i,s}[\cdot] = 0$	(13)
$[\eta^k_{i,s} - \rho^{-1} \sum_{j=1}^{m} \gamma_{j,i} \eta^d_{j,s}] \leq 0$	$(1 - \delta)K_{i,s}[\cdot] = 0$	(14)
$[\eta^s_{i,s} f^i_{M,s} - (1 - \rho)\lambda_{i,s}] \leq 0,$	$M_{i,s}[\cdot] = 0$	(15)
$[-\boldsymbol{\lambda}_s + \rho \mathbf{G}' \boldsymbol{\lambda}_s + \boldsymbol{\zeta}_s - \eta^l_s l_p \mathbf{e}_1] \leq 0$	$\mathbf{M}_{F,t}'[\cdot] = 0$	(16)
$[\eta^q_s \mathbf{q} + \rho \mathbf{R}' \boldsymbol{\lambda}_s - \boldsymbol{\zeta}_s - \eta^l_s \mathbf{l}_c] \leq 0$	$\mathbf{c}_t'[\cdot] = 0$	(17)

straints with the modifications discussed in this section. The entire steady state problem is shown in Tables 4.6a-b. One can test whether the solution obtained is indeed the optimal steady state. This is done by solving a T-period problem -- where T is reasonably large; for example, two times a typical rotation -- with the beginning and ending states constrained to be equal to the solution believed to be the optimal steady state, and solving for intermediate states. If one does indeed have an optimal steady state, and if it is a stable equilibrium, then the intermediate states should all be constant and equal to the initial and ending conditions. Of course, if the equilibrium is not stable, this will not work, but the optimal steady state would not be a suitable ending state for the model, anyway.

An Example

In this section a numerical example is presented along with its solution for the optimal steady state. A typical rotation for this example is five periods. An eleven-period version of the problem was also solved to establish that the solution found is the optimal steady state. The eleven-period version can be modified easily to solve for optimal eleven-period transitions from initial states other than the optimal steady state.

The purpose of the example is to show that the multi-sector problem can be solved numerically and to find out how difficult it is to solve such problems. Experience indicates that the model can be solved, but that it is difficult. The example presented here is extremely simple compared with a model with sufficient detail to produce credible results for policy analysis. However, Harberger's famous two-sector model (Harberger 1962) was vastly more simple and still provided considerable insight. Also, only one optimization routine (MINOS) has been tried, and other software may be available that will solve such problems more successfully.

The equations used in the example are presented in Table 4.7. Two consumption goods sectors are modeled; the first is constructed to resemble an agriculture sector; the second can be interpreted as a manufacturing sector. The oldest age class modeled is 6 ($N = 6$). One tenth of a unit of labor is required for each unit of forest planted in a period. Up to .3 units of labor are required for harvesting a unit of forest, depending on the age class. One hundred units of labor and one hundred units of land are available.

The optimal steady state for the example is presented in Table 4.8. Not surprisingly, given the model coefficients, more labor and land are employed in agriculture, and more capital is required for the industrial sector. The optimal rotation is five periods, and the forest is regulated (equal area in each age class), with 6.095 units of land in each age class. Gross production is about two times as large as consumption. As mentioned, this solution was used for starting and ending conditions for an eleven-period model. When this model was solved, the deviations from this steady-state solution were very small and can probably be attributed to round-off error.

Generalizations of the Model

Many simplifying assumptions have been made in constructing the multi-sector model presented above. The most important assumptions are often the ones that are the most difficult to relax. However, there are many ways the model can be generalized relatively

Table 4.7. **The primal equations for the example steady state problem for the multi-sector model.**

$$\underset{\substack{X^c_{i,s},\, K_{i,s},\, L_{i,s},\, M_{i,s} \\ X_{i,s},\, \mathbf{M}_{F,s},\, \mathbf{c}_s,\, \eta^s_{i,s},\, \eta^d_{i,s} \\ \eta^k_{i,s},\, \eta^l_s,\, \eta^q_s,\, \lambda_s,\, \zeta_s}}{\text{Maximize}} \quad \Upsilon = 1.0 {X^c_{1,s}}^{0.5} {X^c_{2,s}}^{0.3} \tag{1}$$

subject to

$$X_{1,s} = 1.5(K_{1,s})^{0.2}(L_{1,s})^{0.4}(M_{1,s})^{0.4} \tag{2}$$

$$X_{2,s} = 2.0(K_{2,s})^{0.5}(L_{2,s})^{0.4}(M_{2,s})^{0.1} \tag{3}$$

$$X^c_{1,s} = (1.0 - 0.2)X_{1,s} - 0.3X_{2,s} - (1.0 - 0.9)[0.3K_{1,s} + 0.25K_{2,s}] \tag{4}$$

$$X^c_{2,s} = -\,0.3X_{1,s} + (1.0 - 0.1)X_{2,s} - (1.0 - 0.9)[0.7K_{1,s} + 0.75K_{2,s}] \tag{5}$$

$$0.1c_2 + 2.0c_3 + 5.0c_4 + 7.0c_5 + 7.5c_6 \geq 0.1X_{1,s} + 0.5X_{2,s} \tag{6}$$

$$(\mathbf{I} - \mathbf{G})\mathbf{M}_{F,s} = \mathbf{R}\mathbf{c}_s. \tag{7}$$

$$\mathbf{M}_{F,s} \geq \mathbf{c}_s \tag{8}$$

$$\sum_{i=1}^{m} L_{i,s} + 0.1M_{1,F,s} + 0.05c_1 + 0.1c_2 + 0.2c_3 + 0.25c_4 + 0.29c_5 + 0.3c_6 \leq 100 \tag{9}$$

$$\mathbf{d}^{n\prime}\mathbf{M}_{F,s} + \sum_{i=1}^{m} M_{i,s} \leq 100 \tag{10}$$

Table 4.8. The solution to the example multi-sector steady state problem.

$X_{1,s} = 83.69$	$\eta^c_{1,s} = 0.1074$
$X_{2,s} = 68.59$	$\eta^c_{2,s} = 0.0690$
$X^c_{1,s} = 44.16$	$\eta^d_{1,s} = 0.2102$
$X^c_{2,s} = 30.69$	$\eta^d_{2,s} = 0.1815$
$K_{1,s} = 35.02$	$\eta^k_{1,s} = 0.2223$
$K_{2,s} = 46.46$	$\eta^k_{2,s} = 0.2206$
$L_{1,s} = 63.95$	$\eta^l_s \equiv 0.0562$
$L_{2,s} = 33.67$	
$M_{1,s} = 61.44$	
$M_{2,s} = 8.09$	
$M_{1,F,s} = 6.095$	$\lambda_{1,s} = 0.4037$
$M_{2,F,s} = 6.095$	$\lambda_{2,s} = 0.4788$
$M_{3,F,s} = 6.095$	$\lambda_{3,s} = 0.5600$
$M_{4,F,s} = 6.095$	$\lambda_{4,s} = 0.6550$
$M_{5,F,s} = 6.095$	$\lambda_{5,s} = 0.7660$
$M_{6,F,s} = 0.0$	$\lambda_{6,s} = 0.7966$
$c_{5,s} = 6.095$	$\eta^q_s = 0.0625$
$c_{1,s} = c_{2,s} = c_{3,s} = c_{4,s} = c_{6,s} = 0.0$	

easily. "Relatively easy" means that the changes are easy to specify and would only complicate the model insofar as the model size is increased.

Two generalizations can be transferred directly from the work of Lyon and Sedjo (1982, 1983). Their model allows the regeneration input to be determined endogenously. The amount of silvicultural effort then influences the wood yield at each age. To introduce this option into the multi-sector model, a new state vector would be introduced giving the regeneration input corresponding to each age class. The equations of motion for this new state vector are given in Lyon and Sedjo (1983). Also, an equation describing the production technology for regeneration would have to be introduced. This equation would describe how labor and capital (or just labor) combine to produce the regeneration input. Finally, Equation 4.6 would become nonlinear since the vector **q** would have to be a function of the regeneration input.

Another generalization, discussed by Lyon and Sedjo (1983), is the incorporation of different land classes. Land classes could be differentiated on the basis of soil quality, location, accessibility, etc. In the multi-sector model, this would be accomplished by including an age-class vector and cut vector for each land class and each time period. That is, the forestry vectors would each be given an additional subscript reflecting the land class. Thus the notation would be:

> $\mathbf{M}_{F,z,t} \equiv$ the forest age class vector for land class z in period t,

and

> $\mathbf{c}_{z,t} \equiv$ a vector giving the number of acres cut from each age class of land class z in period t.

Aside from the subscript changes, Equations 4.7 and 4.8 would be unchanged. Equation 4.6 would be changed to

$$\sum_{z \in Z} \mathbf{q}_z{}'\mathbf{c}_{z,t} \geq \sum_{j=1}^{m} a_j^q X_{j,t} \qquad 4.30$$

where

> $\mathbf{q}_z \equiv$ a vector giving the yield of wood that can be harvested from an acre of forest growing in land class z for each age class.

and Z is the set of land classes. Similarly, Equation 4.9 would be changed to

$$\sum_{i=1}^{m} L_{i,t} + \sum_{z \in Z} [l_{p,z} M_{1,F,z,t} + \mathbf{l}_{c,z}{}'\mathbf{c}_{z,t}] \leq \overline{L}_t \qquad 4.31$$

where the subscript changes on the labor requirements allow for possibly different planting and harvesting costs for different land classes. Finally, the production functions in the consumption goods sectors would have to be modified to account for the variety of land classes available. This could be done directly or through a land aggregation function for each sector.[28]

Another relatively easy generalization of the model is the introduction of different cutting activities. The result of different cutting activities would be different products such as pulpwood or firewood versus sawtimber. Equation 4.7 would be changed to read

$$\mathbf{M}_{F,t+1} = \mathbf{G}\mathbf{M}_{F,t} + \mathbf{R} \sum_{w \in W} \mathbf{c}_{w,t} - \sum_{i=1}^{m} (M_{i,t+1} - M_{i,t}) \mathbf{e}_1 \qquad 4.32$$

where

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[28] For a discussion of factor aggregation functions, see Dervis, de Melo, and Robinson (1982, pp. 139-141).

$\mathbf{c}_{w,t} \equiv$ a vector giving the number of acres cut from forests in each age class in period t using cutting activity w.

and W is the set of cutting activities. Equation 4.6 would become

$$\mathbf{q}_w{}'\mathbf{c}_{w,t} \geq \sum_{j=1}^{m} a_{j,w}^{q} X_{j,t}. \qquad 4.33$$

where

$\mathbf{q}_w \equiv$ a vector giving the volume of wood that can be harvested for each age class from an acre of forest using cutting activity w,

$a_{j,w}^{q} \equiv$ an input-output coefficient giving the requirements for industry j for the particular type of wood product obtained with cutting activity w.

Thus the volume vector would be different for different cutting activities and each industry would have separate requirements for each type of product. There would be an equation of the form 4.33 for each time period and type of forest product (corresponding to each cutting activity).

The ideas of different cutting activities and/or regeneration inputs can be generalized to introduce thinning in the model. This would significantly complicate the explanation and interpretation of the model as well as greatly increase model size. Thus, the details of how this would be done will not be discussed here. Generalization of the model to allow thinning will provide an important area for future research. Another potential area for generalizing the model is allowing for uneven age management of forests. The forestry model used here bears a striking resemblance to the uneven age management model discussed by Haight (1985) and Haight, Brodie, and Adams (1985). Their models can probably be incorporated directly into the multi-sector model, just as the Lyon and

Sedjo model was. Thus, it is probably reasonable to conjecture that the inclusion of uneven aged forests in the multi-sector model would be relatively straightforward.

It should also be relatively straightforward to allow for greater substitution possibilities in the model. The realism of the model would be greatly increased if more substitution were allowed. In particular, substitution between forest and non-forest products would be of significant interest to foresters. However, input-output coefficients are much easier to obtain than production functions, and production functions introduce more nonlinearities into the model.

There are, of course, many generalizations that would be useful but much harder to incorporate. As mentioned, the introduction of uncertainty into the model would probably increase the realism of the model. The multi-sector model can also be stated in continuous time, as was done for the forestry model in the previous chapter. If the forestry model could be solved in continuous time, it would probably be relatively easy to extend the solution to the multi-sector model.

As the model is currently formulated, the utility function is identical in each period and technology is constant. Different utility and production functions could be specified for different periods, but to do this, a model of how technology and tastes and preferences change over time is required. A changing labor force over time has already been included in the model specification, but demographics are relatively easy to forecast. To some extent, new technologies that will become operational within the next few decades can be anticipated and built into the model. Changes in tastes and preferences, on the other hand, are very difficult to predict. Nevertheless, these would be interesting areas for future research.

Many of the computable general equilibrium models developed have been used for trade and tax policy analysis. Unfortunately, the multi-sector model is not well-suited for such questions. The problem with using an aggregate utility function for trade models has already been mentioned. Similarly, the impact of tax policies on different income groups is often a central issue in tax policy analysis. More important is the way taxes and trade policy instruments are usually imposed on an economy. All income taxes and most tariffs, for example, are imposed on values. The computation of values depends on explicitly modeling prices. Except in the optimal steady-state model, prices are not explicit in the multi-sector model. Of course, the dual constraints can also be added to the multi-period model, and this is precisely what will be done in the next chapter. Thus, the general equilibrium model presented in the next chapter is the generalization of the multi-sector model that will make it useful for trade and tax policy analysis.

Discussion

The multi-sector model presented in this chapter is closely related to the family of growth models that have their roots in the Harrod-Domar growth model.[29] A fundamental difference between this model and growth models is that here, one primary factor -- land -- is fixed and cannot grow. The steady-state growth paths for growth models require that all factors increase at some constant rate. For land, the only feasible growth rate is zero. Thus, the optimal steady state discussed here corresponds to the optimal growth path in growth models.

[29] For a relatively readable introduction to growth models, see Branson (1979, Chapter 21-25). More complete and also demanding references include Burmeister and Dobell (1970) and Takayama (1985, Chapters 5-8).

Both optimal growth paths and optimal steady states are fictions. Land *is* a fixed factor. Land-augmenting technology increases the effective supply of land, but this is a distinctly different process than what is modeled in growth models. In those models, technology is generally constant, and growth occurs through increases in the labor force and capital stock. In fact, the basic notion that perpetual growth is optimal is debatable. For example, growth models usually assume that population will grow at a constant rate. Many people would not agree that in the long term a steadily increasing population is desirable. is.

On the other hand, economies are seldom, if ever, observed in steady states, and if anything, change occurs more rapidly every year. The steady state is a mathematical convenience. As illustrated with the forestry sector, the steady state is relatively easy to understand. A steady state is also one context where a one-period model is consistent with a dynamic model.

When steady states are globally stable,[30] they also have an important influence on dynamic systems. Such globally stable steady states exert a "pull" on the system state. That is, the state of the unperturbed system will approach this steady state as time passes. Thus, even though the system is not at a steady state, the steady state reveals where the system is headed. If the parameters of the model are constantly changing, the optimal steady state will also be moving. The actual state of the system will follow along, possibly never quite catching up.

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[30] The stability of the steady states of dynamic systems is a complex subject. A *globally assymptotically stable* steady state is one where all trajectories of the system approach the steady state. For discussions of the stability of dynamic systems, see Boyce and Diprima (1986, Chapter 9) and Brauer and Nohel (1968, Chapter 4-5).

Returning to the similarities between this model and economic growth models, the multi-sector model appears to have a *turnpike property*. The word "turnpike" refers to the optimality of using a turnpike to go from one large city to another, even though the turnpike does not necessarily follow the shortest path between the cities. In economic growth models, the optimal growth path is like a turnpike. A growth model is said to have a turnpike property if the optimal path between two points not on the optimal growth path spends a greater proportion of the time in the neighborhood of the optimal growth path as the time required to reach the ending point is increased.[31]

Since the optimal steady state for the multi-sector model presented here is analogous to the optimal growth path, the optimal steady state is the "turnpike" for this model. (The analogy is not terribly appropriate here.) In this case, a turnpike property would require that, if the initial conditions and ending conditions were not equal to the optimal steady state, the optimal states for intermediate periods would approach the optimal steady state in early periods and diverge from the path only in later periods to reach the required ending conditions. For longer time horizons, more time will be spent near the optimal steady state. Experience with the eleven-period model indicates that for this model this is the case (although this does not prove that the model has a turnpike property).

In the household production models (Binkley 1987), people are assumed to derive utility from standing forests. This idea is also considered in Hartman (1976). The utility function in this model could also be made to include standing forests. However, this is difficult to interpret in the context of a community utility function. It implies that there is a *market* for standing timber -- i.e. individuals agree (in a market, as opposed to a

[31] Turnpikes are related to globally assymptotic steady states. For a discussion of turnpike theorems, see Turnovsky (1970) or Takyama (1985, Chapter 7).

political context) to pay forest owners a fee for each acre of forest they maintain; the fee would presumably vary over age classes. This is not likely to happen since standing forests have many of the qualities of public goods: many people can enjoy them without decreasing the enjoyment of others, and it is difficult to exclude people from enjoying them if they derive enjoyment from just looking at them.

It is difficult, given the formulation of the model as a maximization problem, to model how most economies actually determine the appropriate level of production for public goods. With this formulation, any good that appears in the utility function is treated as if a perfect market existed for that good. However, while the solution to the multi-sector model may not reflect how markets actually work in supplying public goods, the solution is useful because it indicates what the ideal production levels of these goods should be to maximize the satisfaction of the members of society. (Of course, this assumes that the aggregate utility function accurately reflects the desires of the members of society and the production technology is accurately modeled.) Thus, the multi-sector model seems to have more potential for *normative* rather than for *positive* analysis. This issue will be discussed at greater length in the next two chapters. In Chapter 7 an alternative formulation -- an equilibrium model -- will be presented. Under the assumptions of perfect competition, both approaches are equivalent. However, the equilibrium approach provides a more flexible framework for modeling the actual outcomes of market imperfections.

CHAPTER 5: THE GENERAL EQUILIBRIUM MODEL

The problem of disaggregating the community utility function is discussed in this chapter. The resultant model can be considered a true dynamic general equilibrium model because it represents an equilibrium between *n* consumers and *n* producers. Some level of aggregation will always be necessary, however, and it is unlikely that the model will ever be applied with disaggregation to the level of individuals. Thus, utility functions for *households*[32] will be referred to rather than utility functions for individuals. In the discussion below, households are differentiated by labor skill categories. Labor from each household is treated as a separate primary factor of production.

The model is formulated here as a maximization problem. The objective function is a weighted sum of the utility functions of the different households for each period. These weights are variables, determined in the solution of the problem, and are not necessarily "optimal" in any social sense. They are determined largely by the constraints of the

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[32] The words shown in italics are defined in Appendix B.

problem, and tend to reflect income. Their interpretation is discussed at greater length below.

Once again, the problem of finding the optimal steady state for the model is discussed. The next chapter shows how this optimal steady state problem can be solved as an equilibrium problem, rather than as a maximization problem. The equivalence of the two problems depends on meeting the necessary assumptions for a perfectly competitive economy.[33] No attempt was made to solve the non-steady-state problem as an equilibrium problem. That problem has been relegated to the category of future research.

The determination of rates of return on assets is much more complicated in this model than in the multi-sector model. Equilibrium in the market for investments requires that all assets earn the same rate of return.[34] However, each household may have a different rate of time preference. It is likely that at least some households will not have the same rate of time preference as the equilibrium rate of return on assets. The chapter begins with a look at the individual household's savings decision. This provides a basis for understanding the market for assets.

The disaggregation of the consumption side of the model requires that budget constraints be modeled explicitly. For the multi-sector model, the budget constraint for the economy is automatically satisfied. When he utility function is disaggregated, however, it becomes necessary to ensure that each household lives within its budget constraint. Budget constraints are functions of prices and quantities. Thus, prices must be modeled explicitly. This is done by including the first-order conditions, or "dual constraints", and

[33] Again, see Takayama (1985, Chapter 12) for a discussion on the equivalence of pareto optimality and the outcome of equilibrium in a perfectly competitive economy.

[34] This would not be required in a stochastic model with some risk-averse households.

the complementary slackness conditions explicitly in the model formulation, as was done in the optimal steady state problem in the previous chapter.

Since the dual constraints are part of the model formulation, the distinction between "dual" and "primal" constraints becomes blurred. This nomenclature is used here, however, because it is a useful distinction. Thus, equations that determine real quantities are referred to as "primal constraints", and those determining prices are referred to as "dual constraints". The budget constraints do not fit neatly into either of these categories, so they are placed in a category of their own.

It is assumed that primary factors earn the value of their marginal products (VMP). Given this assumption, matters are greatly simplified by assuming that all production functions are *constant returns to scale.*[35] This ensures that factor payments equal revenues. The assumption of a perfectly competitive economy is also much more plausible with constant-returns-to-scale production functions. These assumptions greatly simplify the problem, but considerable work has been done on relaxing them in similar contexts. Relaxing these assumptions in this context would be an interesting area for future research.

Modeling budget constraints requires accounting for the ownership of the factors of production. Of course, each household's income includes the return to its corresponding labor type. Modeling the ownership and income from assets is a bit more complicated. There are $m + 2$ assets in this economy: capital for each of the m consumption-goods sectors, land, and trees. Since all assets must earn the same rate of return and because

[35] A production function exhibits *constant returns to scale* if and only if it is homogeneous of degree one in the inputs. That is, if and only if when inputs are all increased or decreased by a constant factor, output is also increased or decreased by that amount.

in this model households invest only to earn future income, households will be indifferent between alternative allocations of their investments. That is, they will not care whether they own a hundred dollars worth of capital in sector one or one hundred dollars worth of forests (land plus trees). Since individual asset portfolios are indeterminate, all assets are grouped together and households own shares of the total asset portfolio.

In the optimal steady state, only the household or group of households with the lowest rate of time preference will own positive shares in the asset portfolio. If more than one household belongs to this group, the division of the asset stock between them is indeterminate without reference to the historical trajectory of the system as it approaches the optimal steady state. If these results seem unrealistic, recall that it has already been recognized that even the concept of a steady state is a fiction. The results are useful if they generate insights into the nature of the true system. Obvious departures from observation reflect on the validity of the assumptions of the model and may indicate which assumptions should be relaxed or abandoned in future research.

The general equilibrium model specified in this chapter is a much more complicated nonlinear programming problem than the multi-sector model discussed in Chapter 4. Many of the dual constraints and all the complementary slackness conditions are nonlinear. The budget constraints also are nonlinear. Solving for the weights in the objective function is an additional complication. Even a simple two-household, two-sector steady state example problem proved to be unsolvable using MINOS. Obtaining solutions to nonlinear programming problems is as much an art as a science, given current technology, and perhaps a more knowledgeable MINOS user would have had greater success. It is also possible that other nonlinear optimization software packages will prove to be more robust in finding solutions to problems of this type. It is clear, however, that problems of this type are difficult to solve.

In the next chapter, a computer program will be discussed which finds solutions to the optimal steady state problem using an equilibrium approach. The equilibrium approach is more promising than the optimization approach for a variety of reasons. However, it is useful to understand both perspectives and the relationship between them.

A Single Household's Consumption Problem

In this section, a single household's savings decision will be considered in a two-period model where investments may not earn the same rate of return as the household's rate of time preference. The results from this section help clarify how the investment market works in the general equilibrium model.

Consider the problem[36]

$$\underset{X^c_{i,t},\ S}{\text{maximize}}\ U(X^c_{1,1},\ X^c_{2,1}) + \rho_h U(X^c_{2,1},\ X^c_{2,2}) \tag{5.1}$$

subject to

$$\sum_{i=1}^{2} P_{i,1} X^c_{i,1} \le Y_1 - S$$
$$\sum_{i=1}^{2} P_{i,2} X^c_{i,2} \le Y_2 + \rho^{-1} S$$

[36] The notation used here is summarized in Appendix A.

where

$X^c_{i,t} \equiv$ the quantity of good i consumed in period t,

$P_{i,t} \equiv$ the the price of the i^{th} good in period t,

$S \equiv$ savings from period 1,

$Y_t \equiv$ exogenous income from period t,

$\rho_h \equiv$ one over one plus the household rate of time preference,

$\rho \equiv$ one over one plus the rate of return for investments (savings).

The U's denote utility functions with the classical properties discussed in Chapter 4. Assume that savings must be nonnegative. The Lagrangian for this problem is .

$$\begin{aligned}\Lambda = U(X^c_{1,1},\ X^c_{2,1}) + \rho_h U(X^c_{2,1},\ X^c_{2,2}) - \mu_1[\sum_{i=1}^{2} P_{i,1}X^c_{i,1} - Y_1 + S] \\ - \mu_2[\sum_{i=1}^{2} P_{i,2}X^c_{i,2} - Y_2 - \rho^{-1}S]\end{aligned} \qquad 5.2$$

where the μ_t's are the multipliers corresponding to the budget constraints.

The first-order and complementary slackness conditions for this problem are:

$$\frac{\partial\Lambda}{\partial X^c_{i,1}} = U_{i,1} - \mu_1 P_{i,1} \leq 0 \qquad X^c_{i,1}\,\frac{\partial\Lambda}{\partial X^c_{i,1}} = 0 \qquad 5.3$$

$$\frac{\partial\Lambda}{\partial X^c_{i,2}} = \rho_h U_{i,2} - \mu_2 P_{i,2} \leq 0 \qquad X^c_{i,2}\,\frac{\partial\Lambda}{\partial X^c_{i,2}} = 0 \qquad 5.4$$

$$\frac{\partial\Lambda}{\partial S} = -\mu_1 + \rho^{-1}\mu_2 \leq 0 \qquad S\,\frac{\partial\Lambda}{\partial S} = 0 \qquad 5.5$$

Equations 5.3 - 5.5 imply that, when savings and consumption of both goods in both periods are all positive, then

$$\mu_1 = \frac{U_{i,1}}{P_{i,1}} = \rho^{-1}\mu_2 = \rho_h\rho^{-1}\frac{U_{i,2}}{P_{i,2}}. \tag{5.6}$$

This gives a relationship between the ratios of the marginal utilities of goods to their prices for consecutive periods. That is, for those goods that are consumed, and if savings are positive:

$$\frac{U_{i,t}}{P_{i,t}} = \rho_h\rho^{-1}\frac{U_{i,t+1}}{P_{i,t+1}}. \tag{5.6'}$$

The size of the factor $\rho_h\rho^{-1}$ is determined by the relationship between the rate of return on savings and the rate of time preference. Let r denote the rate of return on savings and r_h the rate of time preference. Then

$$\rho_h\rho^{-1}\begin{cases} >1 & \text{if } r > r_h \\ =1 & \text{if } r = r_h \\ <1 & \text{if } r < r_h. \end{cases} \tag{5.7}$$

If prices are constant and savings are assumed to be positive, some inferences can be drawn about trends in consumption for each household from Equations 5.6 and 5.7. When the term $\rho_h\rho^{-1}$ is greater than one, then Equation 5.6′ implies that marginal utility is declining for all goods consumed in positive quantities by this household. Similarly, when $\rho_h\rho^{-1}$ is less than one, marginal utility must be increasing. From the assumptions about utility, marginal utility declines when consumption increases, and vice versa. Thus, when $\rho_h\rho^{-1}$ is greater than one, overall consumption must increase, and when $\rho_h\rho^{-1}$ is less than one, overall consumption must decrease. With constant prices and positive savings, therefore, households with rates of time preference below the equilibrium rate of return on assets will be increasing their consumption over time, and households with rates of time preference greater than the equilibrium rate of return will be consuming decreasing quantities.

If an additional assumption of stable income from sources other than investment is added, this implies that savings will only be positive for those households with rates of time preference lower than the equilibrium rate of return. This is because with stable incomes from other sources and increased incomes from investments, consumption must increase. Households cannot save and decrease their consumption at the same time under these assumptions, so households with high rates of time preference cannot be saving.

In models with depreciation, such as the models in this dissertation, some household's savings must be positive to maintain even a constant level of consumption in the economy. In the steady state for these models, the equilibrium rate of return must equal the minimum rate of return for all households. If some household's rate of time preference were lower than the equilibrium rate of return, the first-order conditions would require increasing consumption for that household. This would be a contradiction of the assumption that the economy is in a steady state. Also, as discussed above, no households with rates of time preference greater than this equilibrium rate could be saving, because that would imply that their consumption must be increasing. Again, this would contradict the steady state assumption. Thus, in the steady state, the equilibrium rate of return will equal the minimum rate of time preference, and only those households with this rate of time preference will save or own any assets. They will invest at the rate necessary to maintain the steady state capital stock.

If negative savings (borrowing) are allowed, then the first-order condition 5.5 will always hold as a strict equality. This would condemn those households whose rates of time preference are greater than the prevailing rate of return to perpetually declining consumption levels. The only steady state for those households would be zero consumption. For this reason, it is assumed that households can invest only positive amounts or divest their savings, but they cannot borrow against future income.

What do these results mean for the general equilibrium model? First, consider the determination of the prevailing rate of return. The relationships discussed in this section determine a supply-of-savings function. Equation 5.6′ is relevant for all households with positive investments or with asset stocks to divest. Increasing interest rates would induce more and more households to invest as the interest rate moves above their rate of time preference. For those households who invest, higher interest rates induce them to shift greater amounts of consumption to future periods. Thus higher interest rates will induce greater savings and a greater investment supply.

There is also a demand for investment resources. Capital in each industry earns the value of its marginal product. This return must equal the cost of maintaining the capital or the capital will be allowed to depreciate away (no new investment). But the cost of maintaining the capital depends on the interest rate. Higher interest rates raise the cost of maintaining capital and thus reduce the demand for new investments.

The prevailing interest rate, investment rate, and capital stocks in this economy are determined by equilibrium between the supply of savings and the demand for investment resources. If the initial equilibrium rate of return is above the minimum rate of time preference, then this rate must fall as the economy approaches the steady state. In order for this to happen, the total capital stock in the economy must increase. As the capital stock grows, the interest rate will decline because of the diminishing marginal productivity of the capital. In this situation, some households will amass fortunes while the interest rate is relatively high and then divest these fortunes when the interest rate falls below their rate of time preference. Eventually, the interest rate will fall to equal the lowest rate of time preference for all households. When this occurs, the household or group of households with this lowest rate will continue to invest only enough to replace depreciated capital. This will be characteristic of the steady state.

The Model

Considering the length of the discussion leading up to this section, the formulation of the general equilibrium model is surprisingly similar to the multi-sector model. The four primary changes are the changes in the objective function, the addition of labor aggregation functions, the addition of the budget constraints, and the addition of equations tracking the ownership of the capital stock. The notation changes very little from the previous chapter; subscript and superscript *h*'s differentiate among households. Again, the notation is summarized in Appendix A. There are *H* households.

The Objective Function

The objective function for the general equilibrium model is:

$$\Upsilon = \sum_{t=1}^{\infty}\sum_{h=1}^{H}\omega_{h,t}\rho_h^t U^h(X_{1,h,t}^c, X_{2,h,t}^c, \dots, X_{m,h,t}^c) \qquad 5.8$$

where

$X^c_{i,h,t} \equiv$ the quantity of good i consumed by the h^{th} household in period t,

$\rho_h \equiv$ one over one plus the rate of time preference for the h^{th} household,

$\omega_{h,t} \equiv$ the weight given to the h^{th} household in period t.

The utility function for each household is assumed to have the classical properties of continuity and differentiability, differential increasingness, strict quasi-concavity, and the boundary condition, as discussed in the previous chapter and in Katzner (1988, p.44).

The weights reflect two requirements. First, for a given household, the ratio of the price of each good to the marginal utility of that good must be equal for all goods consumed. The weight for a given household and time period equals this ratio with an adjustment for the differences between the historical rates of return and the rate of time preference for that household. That is,

$$\omega_{h,t} = \frac{\eta^d_{i,t}}{U^h_{i,t}} \frac{\rho^t_h}{\prod_{\tau=1}^{t} \rho_\tau} \qquad 5.9$$

where

$\eta^d_{i,t} \equiv$ the competitive price of consumer good i in period t.

$\rho_t \equiv$ one over one plus the equilibrium rate of return on assets for the economy in period t.

Equation 5.9 must hold for all $i \in C_h$, where C_h is the set of goods that is consumed in positive amounts by the h^{th} household in period t. This relationship follows directly from the dual constraints that are discussed below. The second condition that is reflected in the weights is income. Note that households with higher income will be able in general to consume more than lower-income households. This means that these weights will tend to be larger for higher-income households.[37]

Normalization

The weights in Equation 5.8 are variables, to be solved for in the solution of the problem. However, the objective function can take on arbitrarily large or small values for any given consumption vector by simply increasing or decreasing all the weights by the same factor. Thus, some kind of normalization is required.

Two obvious normalizations are to constrain the sum of the weights to equal one, or to set one weight equal to one. Also, since Equation 5.9 must hold for all households and products, the prices of consumer goods can be normalized rather than the weights. This is the more traditional normalization, and is accomplished with the restriction that

$$\sum_{i=1}^{m} \eta_{i,t}^{d} = 1 \qquad 5.10$$

for $t = 1, 2, 3, \ldots, T$.

[37] This need not be precisely true since utility functions are invariant under monotonically increasing transformations. Still, the tendency is for larger weights to accompany larger incomes, and this interpretation of the weights is useful. For the purposes of the models here, once a utility function is determined, applying a transformation changes the problem in the sense that the weights that solve the problem are changed. Nothing else will be changed. Note that this is different from the assumption that tastes and preferences do not change over time.

The Primal Constraints

The gross production functions are unchanged except for the interpretation of the labor input. Thus gross production is given by

$$X_{i,t} = f^i(K_{i,t},\ L_{i,t},\ M_{i,t}) \tag{5.11}$$

where

$X_{i,t} \equiv$ gross output of good i in period t,

$L_{i,t} \equiv$ the aggregate labor input for industry i in period t,

$K_{i,t} \equiv$ the capital used in industry i in period t,

$M_{i,t} \equiv$ the land (acres) used in industry i in period t.

Here, $L_{i,t}$ is an aggregate labor input. This aggregate is given by

$$L_{i,t} = g^i(L_{i,1,t},\ L_{i,2,t},\ \ldots,\ L_{i,H,t}) \tag{5.12}$$

where

$L_{i,h,t} \equiv$ the amount of labor input from household h employed in the i^{th} consumer-good industry in period t.

The labor aggregation functions are neoclassical production functions with the same properties as the gross production functions -- including constant returns to scale.

The materials balance equation for the general equilibrium model changes only slightly:

$$\sum_{h=1}^{H} X^c_{i,h,t} = X_{i,t} - \sum_{j=1}^{m} a_{i,j} X_{j,t} - \sum_{j=1}^{m} \gamma_{i,j} \Delta K_{j,t+1} \tag{5.13}$$

where

$a_{i,j} \equiv$ Leontief input-output coefficients,

$\gamma_{i,j} \equiv$ input-output coefficients for new capital production,

$\Delta K_{j,t+1} \equiv$ new capital produced for sector k to be installed in period $t+1$,

and all other variables are previously defined.

The capital dynamics equation is unchanged:

$$K_{i,t+1} = \delta K_{i,t} + \Delta K_{i,t+1} \tag{5.14}$$

where

$\delta \equiv$ the proportion of capital remaining after one period of depreciation,

and all other variables are previously defined.

Also, the forest sector equations are unchanged:

$$\mathbf{q}'\mathbf{c}_t \geq \sum_{j=1}^{m} a_j^q X_{j,t} \tag{5.15}$$

where

$a_j^q \equiv$ input-output coefficient for wood use in industry j,

$\mathbf{q}' \equiv [q_1,\ q_2,\ \ldots\ q_N]'$,

$\mathbf{c}_t \equiv [c_{1,t},\ c_{2,t},\ \ldots\ c_{N,t}]'$,

$q_i \equiv$ the volume of wood per acre of forest of age i,

$c_{i,t} \equiv$ the number of acres cut from age-class i in period t.

Also,

$$\mathbf{M}_{F,t+1} = \mathbf{G}\mathbf{M}_{F,t} + \mathbf{R}\mathbf{c}_t - \sum_{l=1}^{m}(M_{l,t+1} - M_{l,t})\mathbf{e}_1 \tag{5.16}$$

where

$$\mathbf{M}_{F,t} \equiv [M_{1,F,t},\ M_{2,F,t},\ \dots\ M_{N,F,t}]',$$

$M_{i,F,t} \equiv$ the number of acres of forest in age class i in period t.

Again, recall that

$$\mathbf{G} \equiv \begin{bmatrix} 0\,0\,0 & & \\ 1\,0\,0 & & \\ 0\,1\,0 & & \\ & \ddots & \\ & & 1\,0\,0 \\ & & 0\,1\,1 \end{bmatrix}_{N\mathrm{x}N}$$

and

$$\mathbf{R} \equiv \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & 0 & 0 & & 0 \\ 0 & -1 & 0 & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}_{N\mathrm{x}N}$$

and $\mathbf{e}_1^{n\prime} \equiv (1, 0, 0, \dots, 0) \in \mathbf{R}^n$. Of course,

$$\mathbf{M}_{F,t} \geq \mathbf{c}_t\,. \qquad 5.17$$

Aggregation functions are needed for planting and harvesting labor. For planting labor,

$$l_p M_{1,F,t} = g^P(L_{p,1,t},\ L_{p,2,t},\ \dots,\ L_{p,3,t}) \qquad 5.18$$

where

$L_{p,h,t} \equiv$ the amount of labor input from household h employed in planting trees in period t,

$l_p \equiv$ the amount of aggregate labor services required to plant an acre of trees.

For harvesting labor,

$$\mathbf{l}_c'\mathbf{c}_t = g^c(L_{c,1,t},\ L_{c,2,t},\ \ldots,\ L_{c,H,t}) \tag{5.19}$$

where

$L_{c,h,t} \equiv$ the amount of labor input from household h employed in harvesting trees in period t,

$\mathbf{l}_c' \equiv [l_{1,c},\ l_{2,c},\ \ldots\ l_{N,c}]'$,

$l_{i,c} \equiv$ the amount of aggregate labor services required to harvest an acre of forest of age class i.

There is also a labor supply constraint for each household:

$$\sum_{l=1}^{m} L_{l,h,t} + L_{p,h,t} + L_{c,h,t} \leq \overline{L}_{h,t} \tag{5.20}$$

where

$\overline{L}_t \equiv$ the labor force in period t.

The Dual Constraints

The dual constraints are constructed as if Equation 5.8 were maximized subject to the constraints 5.11 - 5.20, and as if the weights were given as parameters, rather than as variables. The Lagrangian for this problem is given in Table 5.1, and the descriptions of the dual variables are given in Table 5.2. Once again, the dual variables are presented with the implicit discount factors explicitly factored out. This aids in interpreting the equations. Note that the equilibrium rate of return may vary from period to period. (As discussed earlier, it will decline if assets are being accumulated in the economy.)

Table 5.1. The Lagrangian function used to identify the dual constraints for the general equilibrium model.

$$\underset{\substack{X^c_{i,h,t},\ L_{i,h,t},\ L_{p,h,t},\ L_{c,h,t} M_{i,t} \\ K_{i,t},\ \Delta K_{i,t},\ X_{i,t},\ \mathbf{M}_{F,t},\ \mathbf{c}_t}}{\text{Maximize}} \quad \Lambda = \sum_{t=1}^{\infty}\sum_{h=1}^{H}\omega_{h,t}\rho_h^t U^h(X^c_{1,h,t}, X^c_{2,h,t}, \ldots, X^c_{m,h,t})$$

$$\eta^s_{i,t}\prod_{\tau=1}^{t}\rho_\tau[X_{i,t} - f^i(K_{i,t},\ L_{i,t},\ M_{i,t})] -$$

$$\eta^l_{h,t}\prod_{\tau=1}^{t}\rho_\tau[L_{i,t} - g^i(L_{i,1,t},\ L_{i,2,t},\ \ldots,\ L_{i,H,t})] -$$

$$\eta^d_{i,t}\prod_{\tau=1}^{t}\rho_\tau[\sum_{h=1}^{H}X^c_{i,h,t} - X_{i,t} + \sum_{j=1}^{m}a_{i,j}X_{j,t} + \sum_{j=1}^{m}\gamma_{i,j}\Delta K_{j,t+1}] -$$

$$\eta^k_{i,t}\prod_{\tau=1}^{t}\rho_\tau[K_{i,t+1} - \delta K_{i,t} - \Delta K_{i,t+1}] -$$

$$\eta^q_t\prod_{\tau=1}^{t}\rho_\tau[\sum_{j=1}^{m}a^q_j X_{j,t} - \mathbf{q}'\mathbf{c}_t] -$$

$$\lambda_{t+1}\prod_{\tau=2}^{t+1}\rho_\tau[\mathbf{M}_{F,t+1} - \mathbf{G}\mathbf{M}_{F,t} - \mathbf{R}\mathbf{c}_t + \sum_{i=1}^{m}(M_{i,t+1} - M_{i,t})\mathbf{e}_1] -$$

$$\zeta_t\prod_{\tau=1}^{t}\rho_\tau[\mathbf{c}_t - \mathbf{M}_{F,t}] -$$

$$\eta^p_t\prod_{\tau=1}^{t}\rho_\tau[l_p M_{1,F,t} - g^p(L_{p,1,t},\ L_{p,2,t},\ \ldots,\ L_{p,H,t})] -$$

$$\eta^c_t\prod_{\tau=1}^{t}\rho_\tau[\mathbf{l}_c'\mathbf{c}_t - g^c(L_{c,1,t},\ L_{c,2,t},\ \ldots,\ L_{c,H,t})] -$$

$$\eta^l_{h,t}\prod_{\tau=1}^{t}\rho_\tau[\sum_{i=1}^{m}L_{i,h,t} + L_{p,h,t} + L_{c,h,t} - \overline{L}_{h,t}]$$

Subject to
- nonnegativity constraints
- initial conditions
- ending conditions

Table 5.2. The interpretation of the Lagrangian multipliers for the general equilibrium model.

$\eta^s_{i,t} \equiv$ the price of gross output from industry i in period t.

$\eta^d_{i,t} \equiv$ the price of consumer good i in period t.

$\eta^k_{i,t} \equiv$ the price in period t of capital used in industry i.

$\eta^l_{i,t} \equiv$ the cost per unit of the aggregate labor input in period t for industry i.

$\eta^l_{h,t} \equiv$ the labor wage for labor from household h in period t.

$\eta^q_t \equiv$ the price of wood in period t.

$\eta^p_t \equiv$ the cost for a unit of planting services in period t.

$\eta^c_t \equiv$ the cost for a unit of harvesting (cutting) services in period t.

$\lambda_t \equiv [\lambda_{1,t}, \lambda_{2,t}, \ldots, \lambda_{N,t}]'$.

$\lambda_{i,t} \equiv$ the price of an acre of forest in age class i in period t. $\lambda_{1,t}$ is the price of bare land.

$\zeta_t \equiv [\zeta_{1,t}, \zeta_{2,t}, \ldots, \zeta_{N,t}]'$.

$\zeta_{i,t} \equiv$ the cost of postponing harvesting age class i in period t (not really a price).

The problem shown in Table 5.1 is not the general equilibrium model. However, the solution to the general equilibrium problem would also solve the problem in Table 5.1 if the weights given were the correct weights for the general equilibrium problem.

The first dual constraint is obtained by taking the partial of the Lagrangian in Table 5.1 with respect to the consumption variables:

$$\frac{\partial \Lambda}{\partial X^{c}_{i,h,t}} = \omega_{h,t}\rho^{t}_{h}U^{h}_{i,t} - \prod_{\tau=1}^{t}\rho_{\tau}\eta^{d}_{i,t} \leq 0\,. \tag{5.21}$$

This equation (together with complementary slackness requirements) demonstrates the result given in Equation 5.9.

Once again, the partial of the Lagrangian with respect to gross output levels defines the price of gross output. This equation is basically unchanged from Chapter 4:

$$\frac{\partial \Lambda}{\partial X_{i,t}} = \prod_{\tau=1}^{t}\rho_{\tau}[-\eta^{s}_{i,t} + \eta^{d}_{i,t} - \sum_{j=1}^{m}a_{j,i}\eta^{d}_{j,t} - a^{q}_{i}\eta^{q}_{t}] \leq 0\,. \tag{5.22}$$

The partial of the Lagrangian with respect to the aggregate labor input variables defines the wage for the aggregate labor input for each industry and time period:

$$\frac{\partial \Lambda}{\partial L_{i,t}} = \prod_{\tau=1}^{t}\rho_{\tau}[\eta^{s}_{i,t}f^{i}_{L,t} - \eta^{l}_{i,t}] \leq 0\,. \tag{5.23}$$

The partial of the Lagrangian with respect to the labor input from each household in each industry and time period defines the wage for that household's labor:

$$\frac{\partial\Lambda}{\partial L_{i,h,t}} = \prod_{\tau=1}^{t}\rho_\tau[\eta^{l}_{i,t}g^{l}_{h,t} - \eta^{l}_{h,t}] \leq 0\,. \qquad 5.24$$

Similarly, the marginal value products of labor used in planting and harvesting must equal the wage rate for that type of labor:

$$\frac{\partial\Lambda}{\partial L_{p,t}} = \prod_{\tau=1}^{t}\rho_\tau[\eta^{p}_{t}g^{p}_{h,t} - \eta^{l}_{h,t}] \leq 0 \qquad 5.25$$

and

$$\frac{\partial\Lambda}{\partial L_{c,t}} = \prod_{\tau=1}^{t}\rho_\tau[\eta^{c}_{t}g^{c}_{h,t} - \eta^{l}_{h,t}] \leq 0\,. \qquad 5.26$$

The rental rate on capital is given by the partial of the Lagrangian with respect to the capital stock in each sector:

$$\frac{\partial\Lambda}{\partial K_{i,t}} = \prod_{\tau=1}^{t}\rho_\tau[\eta^{s}_{i,t}f^{i}_{K,t} - \eta^{k}_{i,t} + \delta\rho_{t+1}\eta^{k}_{i,t+1}] \leq 0\,. \qquad 5.27$$

The price of new capital is given by the partial of the Lagrangian with respect to new capital construction:

$$\frac{\partial\Lambda}{\partial \Delta K_{i,t}} = \prod_{\tau=1}^{t}\rho_\tau[\eta^{k}_{i,t} - \rho_t^{-1}\sum_{j=1}^{m}\gamma_{j,i}\eta^{d}_{j,t-1}] \leq 0\,. \qquad 5.28$$

The rental rate for land is given by the partial of the Lagrangian with respect to land use in each of the consumer-goods industries:

$$\frac{\partial \Lambda}{\partial M_{i,t}} = \prod_{\tau=1}^{t} \rho_\tau [\eta_{i,t}^{s} f_{M,t}^{i} - \lambda_{1,t} + \rho_{t+1}\lambda_{i,t+1}] \leq 0. \qquad 5.29$$

The last two dual constraints should be familiar by now. They determine the optimal harvest age and give the value of forest land in each age class:

$$\frac{\partial \Lambda}{\partial \mathbf{M}_{F,t}} = \prod_{\tau=1}^{t} \rho_\tau [-\lambda_t + \rho_{t+1}\mathbf{G}'\lambda_{t+1} + \zeta_t - \eta_t^{p} l_p \mathbf{e}_1] \leq 0\,, \qquad 5.30$$

and

$$\frac{\partial \Lambda}{\partial \mathbf{c}_t} = \prod_{\tau=1}^{t} \rho_\tau [\eta_t^{q}\mathbf{q} + \rho_{t+1}\mathbf{R}'\lambda_{t+1} - \zeta_t - \eta_t^{c}\mathbf{l}_c] \leq 0\,. \qquad 5.31$$

The Budget Constraint

Before presenting the budget constraint, it is convenient to first derive expressions for the income from the total asset portfolio, the value of the asset portfolio, and the way shares for each household are calculated for each period.

Capital and land used in the consumption goods industries earn their marginal value products. Income from forest rent equals the value of wood sold minus the cost of harvesting and planting for that period. Thus, the income from all assets for a given period is given by:

$$\pi_t = \sum_{i=1}^{m} \eta_{i,t}^{s}[f_{m,t}^{i} M_{i,t} + f_{k,t}^{i} K_{i,t}] + \eta_t^q \mathbf{q}'\mathbf{c}_t - \eta_t^p l_p M_{1,F,t} - \eta_t^c \mathbf{l}_c{}'\mathbf{c}_t \qquad 5.32$$

where

> $\pi_t \equiv$ the total factor returns for all assets: capital, land, and forests in period t.

Note that the quantity of wood harvested may be less than the quantity sold, but if this happens, η_t^q will equal zero, and the equation will still be correct.

The value of the portfolio is equal to the sum over assets of the price times the quantity for each asset. That is:

$$V_t = \lambda_t{}'\mathbf{M}_{F,t} + \lambda_{1,t} \sum_{i=1}^{m} M_{i,t} + \sum_{i=1}^{m} \eta_{i,t}^{k} K_{i,t} \qquad 5.33$$

where

> $V_t \equiv$ the value of the total asset portfolio of capital, land, and forests in period t.

Household's shares in the total asset portfolio change according to the rule

$$s_{h,t+1} = s_{h,t}\left(1 - \frac{\sum_{h=1}^{H} I_{h,t}}{V_{t+1}}\right) + \frac{I_{h,t}}{V_{t+1}} \qquad 5.34$$

where

> $s_{h,t} \equiv$ the share of the total asset portfolio belonging to household h in period t,
>
> $I_{h,t} \equiv$ the value of new investment from household h in period t.

Equation 5.34 can be explained as follows: Each household's share is first reduced by the proportion of total new investment to the value of the portfolio; then the household's share in the new investment is added back into its share of the portfolio. Note that investment in time period t is used to pay for capital that will be installed in period $t+1$. Thus, investment in period t only affects shares in returns for period $t+1$. It is easy to verify that if the sum of the shares is one in period t, it must still equal one in period $t+1$. Also, note that if no new investment is made in time period t, the shares remain the same in time period $t+1$.

New investment funds are invested in new capital to be installed in the next period. Thus,

$$\sum_{h=1}^{H} I_{h,t} = \rho_{t+1} \sum_{l=1}^{m} \eta_{l,t+1} \Delta K_{l,t+1}. \qquad 5.35$$

The discount factor can be explained with reference to Equation 5.28. New investment pays for the intermediate goods used to produce the new capital. The cost of these intermediate goods equals the discount factor for the next period times the price of the capital in the next period.

With these auxiliary equations, the budget constraint can now be written:

$$\eta^{l}_{h,t} \overline{L}_{h,t} + s_{h,t} \pi_t = \sum_{l=1}^{m} \eta^{d}_{l,t} X^{c}_{l,h,t} + I_{h,t} \,. \qquad 5.36$$

The budget constraint requires that income must equal expenditures. Income comes from labor and shares in the asset stock. Expenditures are made on consumption and investment.

This completes the specification of the dynamic general equilibrium model. All the equations of the model are presented together in Tables 5.3a-c.

The Optimal Steady State

The dynamic general equilibrium model can be converted into a steady state problem with changes analogous to the changes that were made with the multi-sector model. Only the changes in the budget constraint and related equations are new, and only the changes in those equations are discussed here. The entire steady state general equilibrium model is presented in Tables 5.4a-b.

The budget constraint equations can be simplified dramatically for the steady state problem. First, shares do not change in the steady state, so no equation like Equation 5.34 is required. Equations 5.33 and 5.35 are really just accounting equations to simplify Equation 5.34, so they too can be dropped.

In the steady state, investments occur at just the level required to offset depreciation of the capital stock. Each household's investment must equal its share times the total necessary investment. Thus, the budget constraint can be written as

Table 5.3a. The general equilibrium model: primal constraints.

$$\underset{\substack{X^c_{i,h,t},\, L_{i,h,t},\, L_{p,h,t},\, L_{c,h,t} M_{i,t} \\ K_{i,t},\, \Delta K_{i,t},\, X_{i,t},\, \mathbf{M}_{F,t},\, \mathbf{c}_t \\ \eta^s_{i,t},\, \eta^d_{i,t},\, \eta^l_{h,t},\, \eta^p_t,\, \eta^c_t,\, \eta^q_t \\ \eta^k_{i,t},\, \lambda_t,\, \zeta_t,\, \omega_{h,t},\, \rho_t}}{\text{Maximize}} \quad \Upsilon = \sum_{t=1}^{\infty}\sum_{h=1}^{H} \omega_{h,t}\rho_h^t U^h(X^c_{1,h,t}, X^c_{2,h,t}, \ldots, X^c_{m,h,t}) \tag{1}$$

subject to

$$X_{i,t} = f^i(K_{i,t},\ L_{i,t},\ M_{i,t}) \tag{2}$$

$$L_{i,t} = g^i(L_{1,i,t},\ L_{2,i,t},\ \ldots,\ L_{H,i,t}) \tag{3}$$

$$\sum_{h=1}^{H} X^c_{i,h,t} = X_{i,t} - \sum_{j=1}^{m} a_{i,j} X_{j,t} - \sum_{j=1}^{m} \gamma_{i,j} \Delta K_{j,t+1} \tag{4}$$

$$K_{i,t+1} = \delta K_{i,t} + \Delta K_{i,t+1} \tag{5}$$

$$\mathbf{q}'\mathbf{c}_t \geq \sum_{j=1}^{m} a^q_j X_{j,t} \tag{6}$$

$$\mathbf{M}_{F,t+1} = \mathbf{G}\mathbf{M}_{F,t} + \mathbf{R}\mathbf{c}_t - \sum_{i=1}^{m} (M_{i,t+1} - M_{i,t})\mathbf{e}_1 \tag{7}$$

$$\mathbf{M}_{F,t} \geq \mathbf{c}_t \tag{8}$$

$$l_p M_{1,F,t} = g^p(L_{p,1,t},\ L_{p,2,t},\ \ldots,\ L_{p,H,t}) \tag{9}$$

$$\mathbf{l}_c'\mathbf{c}_t = g^c(L_{c,1,t},\ L_{c,2,t},\ \ldots,\ L_{c,H,t}) \tag{10}$$

$$\sum_{i=1}^{m} L_{i,h,t} + L_{p,h,t} + L_{c,h,t} \leq \overline{L}_{h,t} \tag{11}$$

Plus

- nonnegativity constraints
- initial conditions
- ending conditions

Table 5.3b. The general equilibrium model: budget and related constraints.

$$\pi_t = \sum_{i=1}^{m} \eta_{i,t}^{s} [f_{m,t}^{i} M_{i,t} + f_{k,t}^{i} K_{i,t}] + \eta_t^{q} \mathbf{q}'\mathbf{c}_t - \eta_t^{p} M_{1,F,t} - \eta_t^{c}\, \mathbf{l}_c{}'\mathbf{c}_t \tag{12}$$

$$V_t = \lambda_t{}'\mathbf{M}_{F,t} + \lambda_{1,t} \sum_{i=1}^{m} M_{i,t} + \sum_{i=1}^{m} \eta_{i,t}^{k} K_{i,t} \tag{13}$$

$$s_{h,t+1} = s_{h,t}\left(1 - \frac{\sum_{h=1}^{H} I_{h,t}}{V_{t+1}}\right) + \frac{I_{h,t}}{V_{t+1}} \tag{14}$$

$$\sum_{h=1}^{H} I_{h,t} = \rho_t \sum_{i=1}^{m} \eta_{i,t+1} \Delta K_{i,t+1}. \tag{15}$$

$$\eta_{h,t}^{l} \overline{L}_{h,t} + s_{h,t}\pi_t = \sum_{i=1}^{m} \eta_{i,t}^{d} X_{i,h,t}^{c} + I_{h,t} \tag{16}$$

Table 5.3c. The general equilibrium model: dual constraints.

$$\sum_{i=1}^{m} \eta_{i,t}^{d} = 1 \tag{17}$$

$$\omega_{h,t} \rho_h^t U_{i,t}^h - \prod_{\tau=1}^{t} \rho_\tau \eta_{i,t}^{d} \leq 0 \tag{18}$$

$$\prod_{\tau=1}^{t} \rho_\tau [-\eta_{i,t}^{s} + \eta_{i,t}^{d} - \sum_{j=1}^{m} a_{j,i} \eta_{j,t}^{d} - a_{i,w} \eta_t^{q}] \leq 0 \tag{19}$$

$$\prod_{\tau=1}^{t} \rho_\tau [\eta_{i,t}^{s} f_{L,t}^{i} - \eta_{i,t}^{l}] \leq 0 \tag{20}$$

$$\prod_{\tau=1}^{t} \rho_\tau [\eta_{i,t}^{l} g_{h,t}^{i} - \eta_{h,t}^{l}] \leq 0 \tag{21}$$

$$\prod_{\tau=1}^{t} \rho_\tau [\eta_t^{p} g_{h,t}^{p} - \eta_{h,t}^{l}] \leq 0 \tag{22}$$

$$\prod_{\tau=1}^{t} \rho_\tau [\eta_t^{c} g_{h,t}^{c} - \eta_{h,t}^{l}] \leq 0 \tag{23}$$

$$\prod_{\tau=1}^{t} \rho_\tau [\eta_{i,t}^{s} f_{K,t}^{i} - \eta_{i,t}^{k} + \delta \rho_{t+1} \eta_{i,t+1}^{k}] \leq 0 \tag{24}$$

$$\prod_{\tau=1}^{t} \rho_\tau [\eta_{i,t}^{k} - \rho_t^{-1} \sum_{j=1}^{m} \gamma_{j,i} \eta_{j,t-1}^{d}] \leq 0 \tag{25}$$

$$\prod_{\tau=1}^{t} \rho_\tau [\eta_{i,t}^{s} f_{M,t}^{i} - \lambda_{1,t} + \rho_{t+1} \lambda_{1,t+1}] \leq 0 \tag{26}$$

$$\prod_{\tau=1}^{t} \rho_\tau [-\lambda_t + \rho_{t+1} \mathbf{G}' \lambda_{t+1} + \zeta_t - \eta_t^{p} l_p \mathbf{e}_1] \leq 0 \tag{27}$$

$$\prod_{\tau=1}^{t} \rho_\tau [\eta_t^{q} \mathbf{q} + \rho_{t+1} \mathbf{R}' \lambda_{t+1} - \zeta_t - \eta_t^{c} \mathbf{l}_c] \leq 0 \tag{28}$$

Plus
- complementary slackness constraints

Table 5.4a. The optimal steady state for the general equilibrium model: primal constraints.

$$\underset{\substack{X^c_{i,h,s},\ L_{i,h,s},\ L_{p,h,s},\ L_{c,h,s} M_{i,s} \\ K_{i,s},\ \Delta K_{i,s},\ X_{i,s},\ \mathbf{M}_{F,s},\ \mathbf{c}_s \\ \eta^{\zeta}_{i,s},\ \eta^{d}_{i,s},\ \eta^{l}_{h,s},\ \eta^{p}_{s},\ \eta^{c}_{s} \\ \eta^{k}_{i,s},\ \eta^{q}_{s},\ \lambda_s,\ \zeta_s,\ \omega_{h,s}}}{\text{Maximize}} \qquad \Upsilon = \sum_{h=1}^{H} \omega_{h,s} U^h(X^c_{1,h,s}, X^c_{2,h,s}, \ldots, X^c_{m,h,s}) \tag{1}$$

subject to

$$X_{i,s} = f^i(K_{i,s},\ L_{i,s},\ M_{i,s}) \tag{2}$$

$$L_{i,s} = g^i(L_{i,1,s},\ L_{i,2,s},\ \ldots,\ L_{i,H,s}) \tag{3}$$

$$\sum_{h=1}^{H} X^c_{i,h,s} = X_{i,s} - \sum_{j=1}^{m} a_{i,j} X_{j,s} - (1-\delta) \sum_{j=1}^{m} \gamma_{i,j} K_{j,s} \tag{4}$$

$$\Delta K_{i,s} = (1-\delta) K_{i,s} \tag{5}$$

$$\mathbf{q}'\mathbf{c}_s = \sum_{j=1}^{m} a^q_j X_{j,s} \tag{6}$$

$$(\mathbf{I} - \mathbf{G})\mathbf{M}_{F,s} = \mathbf{R}\mathbf{c}_s \tag{7}$$

$$\mathbf{M}_{F,s} \geq \mathbf{c}_s \tag{8}$$

$$l_p M_{1,F,s} = g^p(L_{p,1,s},\ L_{p,2,s},\ \ldots,\ L_{p,H,s}) \tag{9}$$

$$\mathbf{l}_c'\mathbf{c}_s = g^c(L_{c,1,s},\ L_{c,2,s},\ \ldots,\ L_{c,H,s}) \tag{10}$$

$$\sum_{i=1}^{m} L_{i,h,s} + L_{p,h,s} + L_{c,h,s} \leq \overline{L}_{s,t} \tag{11}$$

$$\mathbf{d}^{n\prime}\mathbf{M}_{F,s} + \sum_{i=1}^{m} M_{i,s} \leq \overline{M}_s \tag{12}$$

Plus

- nonnegativity constraints

Table 5.4b. The optimal steady state for the general equilibrium model: dual and budget constraints.

$$\eta^l_{h,s}\bar{L}_{h,s} + s_{h,s}\pi^n_s = \sum_{i=1}^{m}\eta^d_{i,s}X^c_{i,h,s} \tag{13}$$

$$\pi^n_s = (1-\rho_s)[\lambda_s' \mathbf{M}_{F,s} + \lambda_{1,s}\sum_{i=1}^{m}M_{i,s} + \delta\sum_{i=1}^{m}\eta^k_{i,s}K_{i,s}] \tag{14}$$

$$\sum_{i=1}^{m}\eta^d_{i,s} = 1 \tag{15}$$

$$\omega_{h,s}U^h_{i,s} = \eta^d_{i,t} \tag{16}$$

$$-\eta^s_{i,s} + \eta^d_{i,s} - \sum_{j=1}^{m}a_{j,i}\eta^d_{j,s} - a^q_i\eta^q_s \leq 0 \tag{17}$$

$$\eta^s_{i,s}f^i_{L,s} - \eta^l_{i,s} \leq 0 \tag{18}$$

$$\eta^l_{i,s}g^l_{h,s} - \eta^l_{h,s} \leq 0 \tag{19}$$

$$\eta^p_s g^p_{h,t} - \eta^l_{h,s} \leq 0 \tag{20}$$

$$\eta^c_s g^c_{h,s} - \eta^l_{h,s} \leq 0 \tag{21}$$

$$\eta^s_{i,s}f^i_{K,t} - (1-\delta\rho_s)\eta^k_{i,s} \leq 0 \tag{22}$$

$$\eta^k_{i,s} - \rho_s^{-1}\sum_{j=1}^{m}\gamma_{j,i}\eta^d_{j,s} \leq 0 \tag{23}$$

$$\eta^s_{i,s}f^i_{M,s} - (1-\rho_s)\lambda_{1,s} \leq 0 \tag{24}$$

$$-\lambda_s + \rho_s\mathbf{G}'\lambda_s + \zeta_s - \eta^p_s\mathbf{e}_1 \leq 0 \tag{25}$$

$$\eta^q_s\mathbf{q} + \rho_s\mathbf{R}'\lambda_s - \zeta_s - \eta^c_s\mathbf{l}_c \leq 0 \tag{26}$$

$$\eta_{h,s}^{l}\overline{L}_{h,s} + s_{h,s}\pi_s = \sum_{i=1}^{m}\eta_{i,s}^{d}X_{i,h,s}^{c} + s_{h,s}\overline{I}_s \qquad 5.37$$

where

$\overline{I}_s \equiv$ the total investment necessary to replace depreciated capital.

The terms in Equation 5.37 can be combined to obtain:

$$\eta_{h,s}^{l}\overline{L}_{h,s} + s_{h,s}\pi_s^{n} = \sum_{i=1}^{m}\eta_{i,s}^{d}X_{i,h,s}^{c} \qquad 5.37'$$

where

$\pi_s^n \equiv$ the net income in the steady state from the total asset stock after depreciation.

The equation for net income from all assets can be greatly simplified in the steady state, but it takes some work. From Equation 5.32 and the definition of π_s^n given implicitly by Equations 5.37 and 5.37′,

$$\begin{aligned}\pi_s^n = \sum_{i=1}^{m}\eta_{i,s}^{s}f_{m,s}^{i}M_{i,s} + \sum_{i=1}^{m}\eta_{i,s}^{s}f_{k,s}^{i}K_{i,s} - (1-\delta)\sum_{i=1}^{m}\eta_{i,s}^{k}K_{i,s} \\ + \eta_s^q\mathbf{q}'\mathbf{c}_s - \eta_s^p l_p M_{1,F,s} - \eta_s^c\mathbf{l}_c'\mathbf{c}_s\,.\end{aligned} \qquad 5.38$$

The first term is the land rental; the second is the capital rental; the third is the necessary investment level; and the last three terms are the net income from forest management.

From Equation 5.29,

$$\sum_{i=1}^{m}\eta_{i,s}^{s}f_{m,s}^{d}M_{i,s}=(1-\rho_s)\lambda_{1,s}\sum_{i=1}^{m}M_{i,s}. \tag{5.39}$$

From Equation 5.27,

$$\sum_{i=1}^{m}\eta_{i,s}^{s}f_{k,s}^{d}K_{i,s}=(1-\delta\rho_s)\sum_{i=1}^{m}\eta_{i,s}^{k}K_{i,s}. \tag{5.40}$$

This result can be used to combine the terms giving the return to capital and the required capital investment. Thus,

$$\begin{aligned}\sum_{i=1}^{m}\eta_{i,s}^{s}f_{k,s}^{d}K_{i,s}-(1-\delta)\sum_{i=1}^{m}\eta_{i,s}^{k}K_{i,s}&=[(1-\delta\rho_s)-(1-\delta)]\sum_{i=1}^{m}\eta_{i,s}^{k}K_{i,s}\\&=\delta(1-\rho_s)\sum_{i=1}^{m}\eta_{i,s}^{k}K_{i,s}\,.\end{aligned} \tag{5.41}$$

Finally, the income from the forest must equal the necessary rent from the forest. That is,

$$\eta_s^q\mathbf{q}'\mathbf{c}_s-\eta_s^p l_p M_{1,F,s}-\eta_s^c\mathbf{l}_c'\mathbf{c}_s=[(1-\rho_s)\sum_{j=1}^{R}\lambda_{j,s}]c_{R,s} \tag{5.42}$$

where subscript and superscript R' s indicate the optimal rotation age. Assume that in the steady state only one age-class is harvested. It is possible that the second-oldest age class could be partially harvested, but this is very unlikely. A similar version of Equation 5.42 could be proven for that case.

To show that Equation 5.42 is true, recall that[38]

$$\begin{aligned}
\lambda_{1,s} &= \lambda_{1,s} \\
\lambda_{2,s} &= \rho_s^{-1}[\eta_s^p l_p + \lambda_{1,s}] \\
\lambda_{3,s} &= \rho_s^{-2}[\eta_s^p l_p + \lambda_{1,s}] \\
&\ \vdots \\
\lambda_{R,s} &= \rho_s^{1-R}[\eta_s^p l_p + \lambda_{1,s}]
\end{aligned} \tag{5.43.1}$$

Adding these equations together,

$$\sum_{i=1}^{R} \lambda_{i,s} = \lambda_{1,s} \sum_{i=0}^{R-1} \rho_s^{-i} + \eta_s^p l_p \sum_{i=1}^{R-1} \rho_s^{-i}. \tag{5.43.2}$$

Substituting the expression for $\lambda_{1,s}$ given by the Faustmann formula (Equation 4.29) into Equation 5.43.2 gives:

$$\sum_{i=1}^{R} \lambda_{i,s} = \frac{\left[\sum_{i=0}^{R-1} \rho_s^{-i}\right]}{(1-\rho_s^R)} [\rho_s^{R-1}(\eta_s^q q_R - \eta_s^c l_{R,c}) - \eta_s^p l_p] + \eta_s^p l_p \sum_{i=1}^{R-1} \rho_s^{-i}. \tag{5.43.3}$$

Multiply through by $1-\rho_s$, and note that[39]

$$\frac{1-\rho_s}{1-\rho_s^R}\left[\sum_{i=0}^{1-R} \rho_s^{-i}\right] = \frac{\sum_{i=0}^{1-R} \rho_s^{-i} - \sum_{i=-1}^{-R} \rho_s^{-i}}{1-\rho_s^R} = \frac{\rho_s^{1-R} - \rho_s}{1-\rho_s^R} = \rho_s^{1-R}\frac{1-\rho_s^R}{1-\rho_s^R} = \rho^{1-R}. \tag{5.43.4}$$

[38] Or, see Equations 4.26′ and the discussion that follows.

[39] The key to understanding this expression and Equation 5.43.6 is to note that many of the terms in the two summations cancel.

Substitute Equation 5.43.4 into Equation 5.43.3. Note that the terms ρ_s^{R-1} and ρ_s^{1-R} cancel. Also, collect the terms including $\eta_s^p l_p$. This yields the expression:

$$(1-\rho_s)\sum_{i=1}^{R}\lambda_{i,s} = \eta_s^q q_R - \eta_s^c l_{R,c} + \left[(1-\rho_s)\sum_{i=1}^{1-R}\rho_s^{-i} - \rho^{1-R}\right]\eta_s^p l_p\,. \qquad 5.43.5$$

Now, note that

$$(1-\rho_s)\sum_{i=1}^{R-1}\rho_s^{-i} = \sum_{i=1}^{R-1}\rho_s^{-i} - \sum_{i=0}^{R}\rho_s^{-i} = \rho_s^{1-R} - 1. \qquad 5.43.6$$

Thus, the terms in the bracket in Equation 5.43.5 collapse, allowing that expression to be simplified to:

$$(1-\rho_s)\sum_{i=1}^{R}\lambda_{i,s} = \eta_s^q q_R - \eta_s^c l_{R,c} - \eta_s^p l_p\,. \qquad 5.43.7$$

To get Equation 5.42 using Equation 5.43.7, note that $\mathbf{q}'\mathbf{c}_s = q_R c_{R,s}$, and similarly, $\mathbf{l}_c'\mathbf{c}_s = l_{c,R} c_{R,s}$. Also, in the steady state the area harvested each period equals the area planted. Since harvesting only occurs in the R^{th} age class, this means that $M_{1,F,s} = c_{R,s}$. Finally, since the area in each age class less than or equal to the rotation age is the same, and since the area in each age class greater than the rotation age is zero,

$$c_{R,s}[\sum_{i=1}^{R}\lambda_{i,s}] = \lambda_s'\mathbf{M}_{F,s}$$

Putting all this together, net asset earnings in the steady state are given by:

$$\pi_s^n = (1 - \rho_s)[\lambda_s'\mathbf{M}_{F,s} + \lambda_{1,s}\sum_{l=1}^{m} M_{l,s} + \delta\sum_{l=1}^{m}\eta_{l,s}^k K_{l,s}]. \qquad 5.44$$

As mentioned above, the complete steady state general equilibrium model is presented in Tables 5.4a-b.

Discussion

It is clear that disaggregating the utility function complicates the model significantly. However, the gains probably outweigh the costs. As mentioned at the end of Chapter 4, it would be difficult to model trade and tax issues with the multi-sector model. The model presented in this chapter is ideally suited for such problems. This model also shows how differences among households determines each household's role in the economy.

Of course, the model's usefulness will be limited unless solutions to the model can be obtained with more reliability. The problem of finding solutions to models like this is a technical problem, however, and very likely will be overcome before long. Some progress in this direction is reported in the next chapter. The problems of empirically specifying a model like this and dealing with some of the methodological issues raised will be harder to overcome.

The model provides useful insight with just the analysis presented here. The most interesting results are those pertaining to investment behavior. The model provides one possible explanation of why it is often observed that "the rich get richer and the poor get poorer". Of course, the specification of separable utility and the invariance of households' rates of time preference are probably gross simplifications of reality. Alternative specifications of utility could reverse these results or reinforce them.

Consider, for example, what would happen if a household's rate of time preference was a function of its income. If the rate of time preference decreased with income, then that would reinforce the tendency for the rich to get richer. In that case, a household's initial wealth would also influence whether it belonged to the class that gets richer or the class that gets poorer. On the other hand, if the rate of time preference increased with income, each household would have an equilibrium level of investments for each interest rate. This equilibrium level would increase with increases in the interest rate. Combining this with the downward-sloping demand for capital in the economy implies that there would be one stable steady state market interest rate where total investment supply would equal investment demand. Not all households would necessarily own investments, but the rate of time preference for those households who do would equal this market interest rate.[40]

In Chapter 4, the possibility of including forest ownership in the utility function was also discussed. This would make the model a general equilibrium version of Binkley's (1987) household production model of timber supply. Such a change would make forest ownership a different kind of investment than owning capital or bare land and ownership of these assets would have to be modeled separately. Of course, it would be plausible to include the ownership of bare land in the utility function. Perhaps capital ownership even provides utility beyond its investment value for some individuals. Such changes

[40] These comments seem plausible. They are not proven, however, and should be viewed as conjectures.

would complicate the model. Whether such changes are worth making depends on whether the answers obtained from the model would be improved sufficiently to warrant the cost of the increased complexity.

The fact that the results of the model can be changed so readily by simply changing the specification of the utility function illustrates two very important points. First, the concept of utility and how it is specified is a key component in the model. In forestry economics, the focus tends to be more on the production side of forests and less on the consumption side. Perhaps this emphasis is misplaced.

Second, the model is virtually unfalsifiable. Whenever the model fails to behave in conformity with reality, there will probably be some way to "fix it up" and bring the results back into line by modifying the utility functions. Since these utility functions are unobservable, one is free to choose whatever utility functions are necessary in order to make the model conform better to reality. This is a serious problem, and not one to be taken lightly. In this regard, the model is indefensible. The focus here will be on the ability of the model to explain and to provide insight into reality. It is also possible that the model will enable analysts to make predictions that are better than those from other models. On this issue, the jury is still out.

CHAPTER 6: AN EQUIVALENT EQUILIBRIUM MODEL

Equilibrium means a balancing of forces. In economics, equilibrium usually means a balancing of the forces of supply and demand. General equilibrium means a simultaneous balancing of supply and demand in all markets. Demand and supply functions give quantities as functions of prices and income. But income is also a function of prices. A solution to a general equilibrium problem can therefore be characterized by a set of prices.

In this chapter an equilibrium approach is used to solve for the prices and quantities that characterize the optimal steady state for the dynamic general equilibrium model described in the previous chapter. The algorithm proceeds by choosing a vector of prices, checking to see if that vector of prices yields an approximate equilibrium, and if not, choosing a new vector of prices. The difficulty, of course, comes in determining how the next vector of prices should be chosen.

The algorithm discussed in this chapter is based on an algorithm developed by Scarf (1967) for approximating *fixed points*[41] of functions. A *fixed point* is one that is mapped back into itself by the function. Solving a static general equilibrium problem is equivalent to finding a fixed point of a mapping. The optimal steady state problem for the dynamic general equilibrium model, while not truly static, can also be solved using a fixed-point algorithm. At this time, it is not clear whether the approach can be applied to the non-steady-state problem; probably it can.

The equilibrium approach seems to be more flexible than the optimization approach for modeling a variety of departures from perfect competition. With the equilibrium approach, the behavior of each agent in the economy and the structure of each market can be modeled uniquely. Non-existent markets can also be modeled as such. The problem is then to find a set of prices that yields an equilibrium in those markets that do exist. The equilibrium approach allows each market to be modeled *as it is*, rather than *as it ought to be* under perfect competition. Thus, the equilibrium model could be characterized as a *positive* model and the optimization model as a *normative* model.

Fixed Points and Scarf's Algorithm

This section provides a brief introduction to Scarf's algorithm. The algorithm provides an approximation of a fixed point for a continuous function which maps a closed, bounded convex set in n-dimensional Euclidian space back onto itself. The discussion below follows closely the discussion in Scarf (1984b).

---

41 The words shown in italics are defined in Appendix B.

Let S be a closed, bounded convex set in $\mathbf{R}^n$ which is mapped onto itself by the continuous mapping $\mathrm{x} \rightarrow f(\mathrm{x})$. A *fixed point* of this mapping is a point $\hat{\mathrm{x}}$ such that $\hat{\mathrm{x}} = f(\hat{\mathrm{x}})$. Brouwer's fixed-point theorem guarantees the existence of a fixed point for the mapping f. However, Brouwer's theorem is nonconstructive. That is, it does not tell us how such a point can be found. Scarf's algorithm provides a constructive proof by providing an algorithm which yields as close an approximation as is desired for the fixed point.

The closed, bounded convex that is of central concern in this chapter is the *unit simplex.* Scarf (1984b, p.10) defines a *simplex* as

> the convex hull of its n vertices $v^1, ..., v^n$, that is, the set of points of the form
>
> $$\mathrm{x} = \sum_{j=1}^{n} \alpha_j v_j \quad \text{with} \quad \alpha_j \geq 0 \quad \text{and} \quad \sum_{j=1}^{n} \alpha_j = 1.$$
>
> The vertices are assumed to be linearly independent in the sense that each vector in the simplex has one and only one representation in the above form.

A simplex with n vertices has dimension $n-1$. The ***unit simplex*** in $\mathbf{R}^n$ has n vertices given by the unit vectors in $\mathbf{R}^n$. Each vector in the unit simplex comprises nonnegative elements whose sum equals one. Thus, for example, the unit simplex in $\mathbf{R}^2$ equals the line segment joining the points (1,0) and (0,1); the unit simplex in $\mathbf{R}^3$ is the set of points in the equilateral triangle joining the points (1,0,0), (0,1,0) and (0,0,1). (See Figure 6.2).

Scarf's algorithm divides the unit simplex into a set of smaller simplices which form a *simplicial subdivision.* Again, quoting Scarf (1984b, p.11):

> A collection of simplices $S^1, ..., S^k$ is called a ***simplicial subdivision*** of S if 1) S is contained in the union of the simplices $S^1, ..., S^k$, and 2) the intersection of any two simplices is either empty or a full face of both of them.

The algorithm identifies one of these smaller simplices as an approximation of the fixed point. Any of the vertices of this simplex provide an approximation of the fixed point, but the fixed point need not lie in the simplex chosen. If one constructs a sequence of simplicial subdivisions, each having a smaller *mesh* -- determined by the size of the largest simplex in the simplicial subdivision -- then any of the vertices of these simplices chosen by the algorithm can be used to construct a sequence of vectors that converges to the fixed point in the limit as the mesh goes to zero. The closeness of the approximation is therefore determined by the mesh of the simplicial subdivision of the unit simplex.

In order to implement Scarf's algorithm, a label must be assigned to each vertex in the simplicial subdivision. These labels are integers from the set $\{1, 2, ..., n\}$, where n is the number of vertices defining the simplices. Each of the n vertices of the unit simplex must have a unique label, and vertices located on a *face* of the unit simplex must have the same label as one of the vertices that defines that face. A *face* of a simplex is a lower-dimensioned simplex defined by a proper subset of the vertices of the original simplex. (Alternatively, one could stipulate that one or more of the $\alpha_j \equiv 0$ for all vectors on the face.) For example, one face of the unit simplex in $\mathbf{R}^3$ is defined by linear combinations of the two vertices: (1,0,0) and (0,0,1). If these vertices are labeled 1 and 3, then any vertex in the simplicial subdivision lying on this face must also be labeled either 1 or 3.

The specific labeling scheme in Scarf's algorithm requires finding in the vector $\mathbf{v}$ locating each vertex an element v_i such that $v_i > f^i(\mathbf{v})$. That is, f maps the vector $\mathbf{v}$ to a vector whose i^{th} element is greater than the i^{th} element of $\mathbf{v}$. Unless $\mathbf{v}$ is the fixed point, such an i exists. The label i is then assigned to that vertex. If $v_i > f^i(\mathbf{v})$ for more than one i, then any one of these i may be chosen for the label. Given this labeling scheme, any of the vertices of a *completely labeled simplex* -- i.e., a simplex whose n vertices each have unique labels -- is an approximation of the fixed point. The fact that a completely labeled simplex does exist is proven in Sperner's lemma (Scarf 1984b, p.16). Scarf's algorithm is a constructive proof of Sperner's lemma. In the limit, as the mesh of the simplicial subdivision is made smaller, Sperner's lemma is equivalent to Brouwer's fixed-point theorem.

Scarf's algorithm starts on the boundary of the original simplex with a (smaller) simplex in the simplicial subdivision with $n-1$ unique labels. Two vertices of this smaller simplex share the same label, so one of these vertices must be eliminated. When the removed vertex is replaced with an appropriately chosen new vertex, this effects a move into the adjacent simplex sharing the face defined by the vertices that were removed. Thus, movement is accomplished by dropping one vertex and replacing it with another. If the label of the new vertex is unique -- i.e., not equal to the labels on any of the vertices shared with the previous simplex -- then the algorithm is done; the new simplex is completely labeled. Otherwise, the old vertex with the same label as the new vertex is eliminated and the algorithm continues. Scarf shows that the algorithm can never return to the same simplex. Since the total number of simplices is finite, a completely labeled simplex will be reached in a finite number of steps.

Before showing how Scarf's algorithm can be applied to general equilibrium problems, two examples may help illustrate the concept of a fixed point and how Scarf's algorithm works. Similar examples are given in Scarf (1984b).

Figure 6.1 shows a continuous function that maps the unit interval (a simplex with vertices (0,0) and (0,1) -- but not a unit simplex) into numbers on the unit interval. Any point where the function crosses the 45 degree line is a fixed point of the function. A

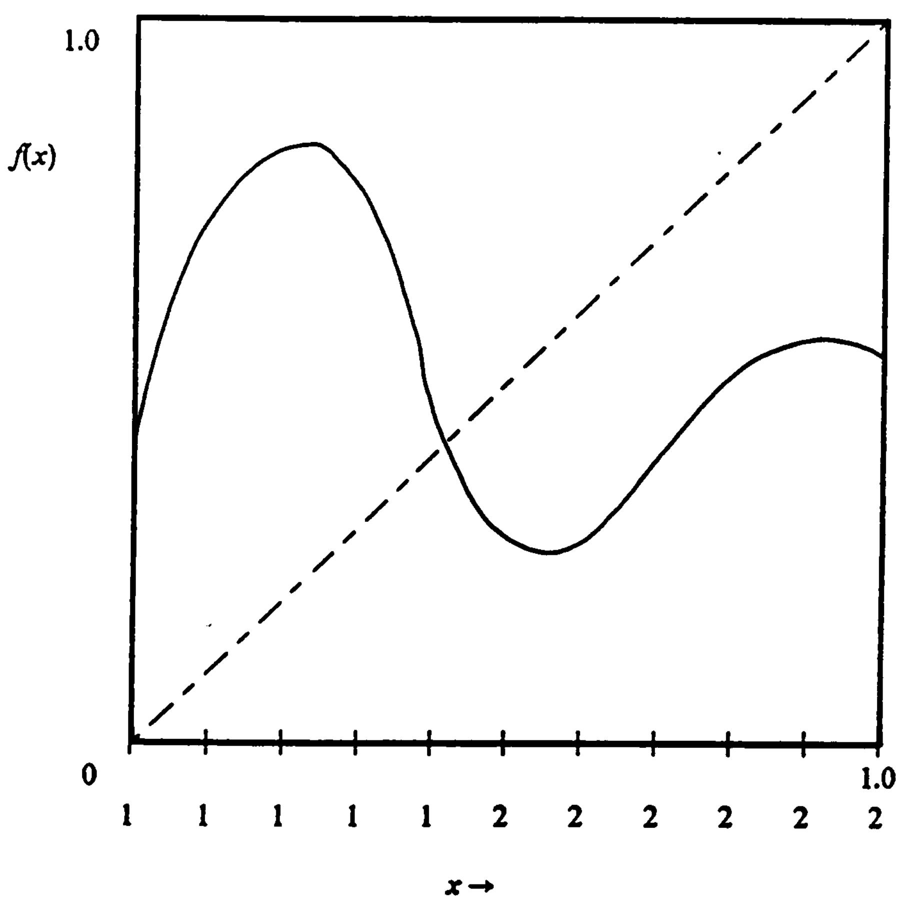

Figure 6.1. Approximating a fixed point of a continuous function from (0,1) to (0,1).: An example of Scarf's algorithm. (A similar figure appears in Scarf (1984b).)

simplicial subdivision of the unit interval is obtained by partitioning the interval into a set of shorter intervals. The endpoints of these shorter intervals are the vertices of the simplices making up the simplicial subdivision. Unless one of these vertices is a fixed point, each can be assigned a label 1 or 2 depending on whether the function at that point lies respectively above or below the 45 degree line. Any simplex that is not completely labeled has $n-1$ distinct labels ($n-1=1$), so the algorithm can begin at either end (the faces) of the unit interval. The algorithm would choose consecutive subintervals along the unit interval until a completely labeled simplex (subinterval) is reached. Clearly, in this example, a completely labeled simplex must contain a fixed point.

Figure 6.2 shows the unit simplex in $\mathbf{R}^3$. The function

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} \qquad \text{where} \qquad \mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

maps the unit simplex back to itself.[42] The labels in Figure 2 are assigned according to the rule:

$$l(\mathbf{v}) = \max_{i} \; \{v_i - f^i(\mathbf{v})\}$$

Figure 6.2 shows one possible path that Scarf's algorithm would follow. This example also illustrates an important fact about Scarf's algorithm: the completely labeled simplex

[42] The fixed point of this function is an eigenvector of the matrix A corresponding to the eigenvalue 1.

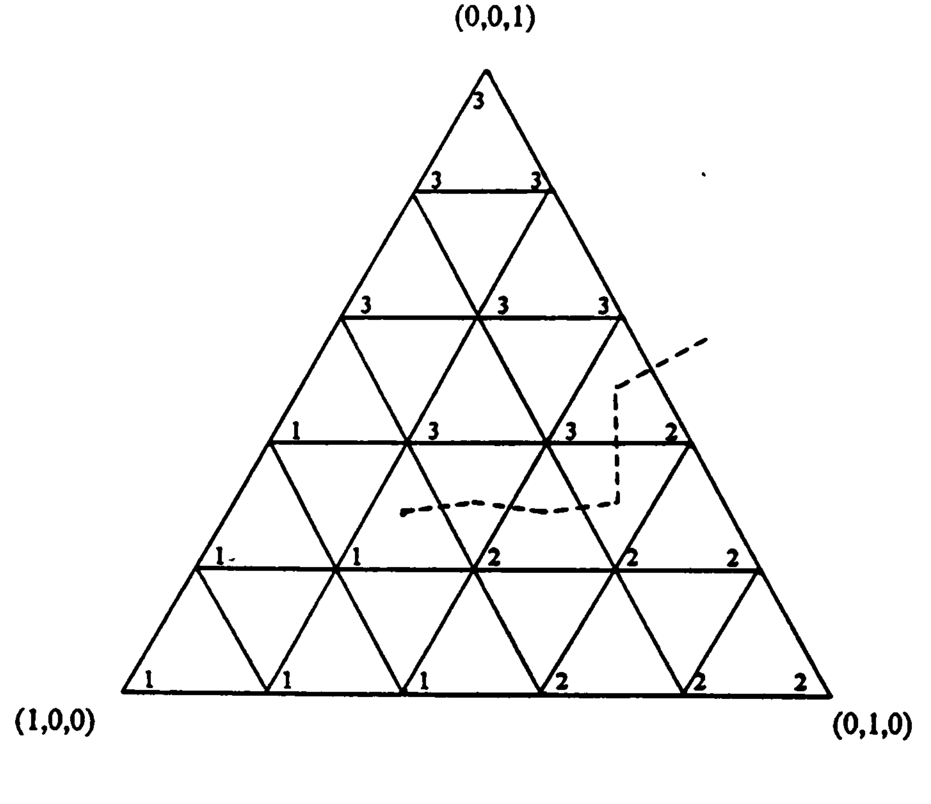

Figure 6.2. Approximating a fixed point of a linear function on the unit simplex in three-dimensional Euclidian space.

need not contain the fixed point. In this case, the actual fixed point is the vector $x = [1/3, 1/3, 1/3]$. It is only in the limit as the subdivision becomes increasingly fine that the algorithm can be guaranteed to give the exact answer. (In this case, the approximation improves dramatically when the mesh size is decreased from 0.2 to 0.01.)

How can the solution of a general equilibrium problem be converted into a fixed point problem? Assume that an economy can be characterized by a set of continuous *excess demand functions* giving the quantity demanded minus the quantity supplied in each market as a function of prices.[43] That is, for each market, there exists a single-valued function of the form[44]

$$\varepsilon^i(\eta) = d^i(\eta) - s^i(\eta) \qquad 6.1$$

where η is a vector of prices and ε^i, d^i, and s^i denote excess demand, demand, and supply functions for the i^{th} market. A nonzero price vector η^* for which excess demand is nonpositive in each market and for which excess demand equals zero for those markets with positive prices is an *equilibrium price vector* (Scarf 1984b). Scarf (1984b) demonstrates that an equilibrium price vector for the economy characterized by the excess demand functions 6.1 is a fixed point of the mapping

$$h^i(\eta) = \frac{\eta_i + \max[0, \varepsilon^i(\eta)]}{1 + \sum_{j \in J} \max[0, \varepsilon^j(\eta)]} \qquad 6.2$$

where $J \equiv$ the set of markets in the economy.

[43] One set of conditions under which this would be true is described in Scarf (1984b). The assumption is made here only to illustrate the relationship between fixed points and equilibrium in an economy.

[44] The notation used here is summarized Appendix A. The notation for the model is the same as in Chapter 4.

An Algorithm for Finding the Optimal Steady State

With some modification, Scarf's algorithm can be used to solve for an optimal steady state of the dynamic general equilibrium model presented in Chapter 5. The algorithm is described in this section. A FORTRAN program developed to implement the algorithm for a simple example problem is listed in Appendix C. The solution to the example is discussed in the next section.

There are two consumer goods sectors in the example problem, two households, and one land class. There are twelve prices to be determined. They are: the demand prices of the two consumer goods, the prices of gross output in the two consumer goods sectors, the price of capital for the two sectors, the wages of the two types of labor, the cost of planting and harvesting services, the price of bare land, and the price of wood. Rather than search for the equilibrium price vector in the unit simplex in $\mathbf{R}^{12}$, the problem can be reduced to one of finding five equilibrium prices. All of the other prices can be found as functions of these five prices. Thus, the problem is reduced to searching through the unit simplex in $\mathbf{R}^5$ -- a much smaller problem.

If the equilibrium demand prices for consumer goods, the equilibrium wages for the two types of labor, and the equilibrium price of land are all known, it is possible to solve for the equilibrium values of all the other variables in the economy. The price of capital can be determined from the demand prices of consumer goods by the equation:

$$\eta_{i,s}^{k} = \rho_s^{-1} \sum_{j=1}^{m} \gamma_{j,i} \eta_{j,s}^{d} \qquad 6.3$$

This comes from Equation 5.28. One can assume that $\Delta K_{i,s} > 0$, since $\Delta K_{i,s} = (1 - \delta)K_{i,s}$ in the steady state, and capital must be used in positive amounts in order to have positive output (given the Cobb-Douglas form of the production functions). The complementary slackness conditions ensure that Equation 5.28 holds as an equality in this case.

To find the cost of planting or harvesting services, the cost minimizing demand for each type of labor in these activities is found. Since the production functions for planting and harvesting services exhibit constant returns to scale, the demand for factors given some fixed level of output must be derived. This is done by solving the problem:

$$\begin{aligned}&\underset{L_{1,i,s},\ L_{2,i,s}}{\text{minimize}} \quad \eta^{i}_{1,s} L_{1,i,s} + \eta^{i}_{2,s} L_{2,i,s} \\ &\text{subject to} \\ &\quad \bar{x}_i = g^i(L_{1,i,s},\ L_{2,i,s}) = \theta_{0,i}(L_{1,i,s})^{\theta_{1,i}}(L_{2,i,s})^{\theta_{2,i}}\end{aligned} \qquad 6.4$$

where

$\eta^i_{h,s} \equiv$ the price of labor from the h^{th} household in the steady state,

$L_{h,i,s} \equiv$ the input of labor from the h^{th} household in activity i in the steady state,

$\bar{x}_i \equiv$ the fixed amount of output from activity i in the steady state,

$\theta_{j,i} \equiv$ production function parameters for activity i,

and g^i is the labor aggregation function for the respective activity. In this case, the labor aggregation functions are constant-returns-to-scale Cobb-Douglas production functions. (That is, $\theta_{1,i} + \theta_{2,i} = 1$.)

Solving problem 6.4 using standard comparative statics techniques yields the demand for each type of labor and the marginal cost (which equals average cost in this case) as functions of the labor prices and the level of output. For the labor aggregation functions used in the example, these demand functions are:

$$L_{h,l}{}^{*} = \frac{\bar{y}_l}{\theta_{0,l}}\left(\frac{\eta^l_{j,s}}{\theta_{j,l}}\right)^{\theta_{j,l}}\left(\frac{\eta^l_{k,s}}{\theta_{k,l}}\right)^{\theta_{k,l}-1}. \qquad 6.5$$

The average and the marginal cost are given by

$$\eta^{l}_{s}{}^{*} = \frac{1}{\theta_{0,l}}\left(\frac{\eta^l_{1,s}}{\theta_{1,l}}\right)^{\theta_{1,l}}\left(\frac{\eta^l_{2,s}}{\theta_{2,l}}\right)^{\theta_{2,l}}. \qquad 6.6$$

Equation 6.5 gives the quantity of each labor input required as a function of wages and the output level. Note that the amount required is proportional to output. (Homogenous functions are homothetic). Also note that the average cost is not a function of output. This is a result of the constant-returns-to-scale assumption.

From Equation 6.6, it is clear that the cost of harvesting and planting services is a function only of the labor wages (and technology, which is fixed). As mentioned in Chapter 4, the price of wood can be solved for as a function of harvesting and planting costs and the price of bare land. The price of bare land is the price for the initial age class of the forest ($\lambda_{1,s}$). Setting the ζ's equal to zero and using Equations 4.26′, a sequence of points giving the maximum values of acres in each age class of the forest can be constructed. That is,

$$\begin{aligned}
\lambda^m_{2,s} &= \rho_s^{-1}(\lambda_{1,s} + \eta^p_s l_p)\\
\lambda^m_{3,s} &= \rho_s^{-1}(\lambda_{2,s})\\
&\vdots\\
\lambda^m_{N,s} &= \rho_s^{-1}(\lambda_{N-1,s})
\end{aligned} \qquad 6.7$$

where $\lambda_{i,s} \leq \lambda_{i,s}^m$. These upper values for the age-class shadow prices are functions only of the bare land value and the cost of planting.

Using Equations 4.26′ and 4.27′, one can also construct a sequence of lower values for the forest age-class shadow prices. Thus,

$$\begin{aligned}
\lambda_{1,s}^l &= \rho_s\lambda_{1,s} + \eta_s^q q_1 - \eta_s^h l_{1,h} \\
\lambda_{2,s}^l &= \rho_s\lambda_{1,s} + \eta_s^q q_2 - \eta_s^h l_{2,h} \\
&\vdots \\
\lambda_{N,s}^l &= \rho_s\lambda_{1,s} + \eta_s^q q_N - \eta_s^h l_{N,h}
\end{aligned} \tag{6.8}$$

where $\lambda_{i,s} \geq \lambda_{i,s}^l$. Note that these lower bounds depend on the price of wood. The price of wood can be adjusted until

$$\max_i \; \{\lambda_{i,s}^l - \lambda_{i,s}^m\} = 0. \tag{6.9}$$

The i for which $\lambda_{i,s}^l = \lambda_{i,s}^m$ is the optimal rotation age of the forest in the steady state. Most important, Equations 6.7 - 6.9 define the price of wood as a function of the price of bare land and the costs of harvesting and planting.

The gross output prices of consumer goods and the price of wood can be determined using Equation 5.22 and the assumption that gross output is positive in all industries.[45] Thus,

$$\eta_{i,s}^s = \eta_{i,s}^d - \sum_{j=1}^m a_{j,i}\eta_{j,s}^d - a_i^q \eta_s^q. \tag{6.10}$$

[45] Again, this will be true because of the Cobb-Douglas form of the utility functions.

At this point, all twelve prices in the economy can be found assuming only five equilibrium prices are known: the prices of of the two consumer goods, the wages earned by each type of labor in equilibrium, and the equilibrium price of bare land. Given this complete set of prices, it should be possible to solve for the values of the real variables in the system -- the quantities. This is not a trivial problem, however.

Assume each household's net income is known. First, consider what would have to be known in order for this to be true. From Equation 5.37′, each household's net income (after savings) is given by:

$$Y^n_{h,s} = \eta^l_{h,s}\overline{L}_{h,s} + s_{h,s}\pi^n_s \,. \qquad 6.11$$

where

$Y^n_{h,s} \equiv$ the net (after savings) income of the h^{th} household.

Several of the terms of this equation are known. The wage and the labor supply for each household are known. If a household's rate of time preference is not equal to the minimum rate of time preference in the economy, its share will equal zero. If a household's rate of time preference equals the minimum rate, and it is the only such household, its share equals one. If a household is one of many households with this minimum rate, then its share is indeterminate. It is assumed that shares for each household are exogenous, but consistent with these restrictions.

Recall from Equation 5.44 that total net income from assets in the economy is given by

$$\pi^n_s = (1-\rho_s)[\lambda_s{}'\mathbf{M}_{F,s} + \lambda_{1,s}\sum_{l=1}^{m} M_{l,s} + \delta\sum_{l=1}^{m}\eta^k_{l,s}K_{l,s}] \qquad 6.12$$

In this equation, only prices and the discount rate are known. The total amount of land is known, but not its distribution between bare-land uses and forest use. The amount of capital in each sector is also unknown. Clearly, the data are not available at this stage to identify each household's income. Nevertheless, assume that net income is known for each household.

Demand functions giving the quantities of each good demanded by each household as functions of prices and net income are found by solving the utility maximization problem for each household:

$$\begin{aligned} &\underset{X_{i,h,s}}{\text{maximize}} \quad U^h(X^c_{1,h,s}, X^c_{2,h,s}, \ldots, X^c_{m,h,s}) \\ &\text{subject to} \\ &\sum_{i=1}^{m} \eta^d_{i,s} X^c_{i,h,s} = Y^n_{h,s} \end{aligned} \qquad 6.13$$

Given the Cobb-Douglas functional form of the utility functions in the example problem, these demand functions have the form:

$$X^c_{i,h,s} = \frac{Y^n_{h,s}}{\eta^d_{i,s}} \frac{\beta_{i,h}}{\sum_{j=1}^{m} \beta_{j,h}} \qquad 6.14$$

where

> $\beta_{i,h} \equiv$ the power on $X^c_{i,h,s}$ in the Cobb-Douglas utility function for the h^{th} household.

Summing up over households gives the total consumption demand for each consumer good:

$$X^c_{i,s} = \sum_{h=1}^{H} X^c_{i,h,s}. \qquad 6.15$$

Using the vector form of Equation 4.4, the gross production in each sector necessary to allow these consumption levels can be computed:

$$\mathbf{X}_s = (\mathbf{I} - \mathbf{A})^{-1}[\mathbf{X}^c_s + (1 - \delta)\Gamma\mathbf{K}_s] \qquad 6.16$$

where

$\mathbf{X}^c_t \equiv [X^c_{1,t}, X^c_{2,t}, \ldots, X^c_{m,t}]'$,

$\mathbf{X}_t \equiv [X_{1,t}, X_{2,t}, \ldots, X_{m,t}]'$,

$\Delta\mathbf{K}_{j,t+1} \equiv [\Delta K_{1,t+1}, \Delta K_{2,t+1}, \ldots, \Delta K_{m,t+1}]'$,

$\Gamma \equiv$ the matrix of capital-production coefficients,

$\mathbf{A} \equiv$ the matrix of Leontief input-output coefficients.

Recall that $\Delta K_{i,s} = (1 - \delta)K_{i,s}$. Recall, too, that the capital stock is assumed to be known in order to give the net income of each household.

The demands for primary inputs in the consumption-goods industries are given by the cost minimization problem for each industry.[46] That is, each industry tries to

[46] Again, the constant-returns-to-scale assumption forces us to use cost minimization rather than profit maximization.

$$\underset{L_{i,h,s},\, K_{i,s},\, M_{i,s}}{\text{minimize}} \quad \sum_{h=1}^{H} \eta^{l}_{h,s} L_{i,h,s} + (1-\rho_s)\lambda_{1,s} M_{i,s} + (1-\delta\rho_s)\eta^{k}_{i,s} K_{i,s}$$

subject to 6.17

$$\overline{X}_{i,s} = f^{i}(g^{i}(L_{i,1,s},\ L_{i,2,s},\ \ldots,\ L_{i,H,s}),\ K_{i,s},\ M_{i,s})$$

Once again, given the constant-returns-to-scale Cobb-Douglas production functions, and assuming only two types of labor, the demand functions for the primary inputs are:

$$L_{i,1,s} = \frac{\overline{X}_{i,s}}{\alpha_{i,0}} \left(\frac{\eta^{l}_{1,s}}{\alpha_{i,1}}\right)^{\alpha_{i,1}-1} \left(\frac{\eta^{l}_{2,s}}{\alpha_{i,2}}\right)^{\alpha_{i,2}} \left(\frac{\eta^{k}_{i,s}}{\alpha_{i,k}}\right)^{\alpha_{i,k}} \left(\frac{\eta^{m}_{i,s}}{\alpha_{i,m}}\right)^{\alpha_{i,m}} \qquad 6.18$$

$$L_{i,2,s} = \frac{\overline{X}_{i,s}}{\alpha_{i,0}} \left(\frac{\eta^{l}_{1,s}}{\alpha_{i,1}}\right)^{\alpha_{i,1}} \left(\frac{\eta^{l}_{2,s}}{\alpha_{i,2}}\right)^{\alpha_{i,2}-1} \left(\frac{\eta^{k}_{i,s}}{\alpha_{i,k}}\right)^{\alpha_{i,k}} \left(\frac{\eta^{m}_{i,s}}{\alpha_{i,m}}\right)^{\alpha_{i,m}} \qquad 6.19$$

$$K_{i,s} = \frac{\overline{X}_{i,s}}{\alpha_{i,0}} \left(\frac{\eta^{l}_{1,s}}{\alpha_{i,1}}\right)^{\alpha_{i,1}} \left(\frac{\eta^{l}_{2,s}}{\alpha_{i,2}}\right)^{\alpha_{i,2}} \left(\frac{\eta^{k}_{i,s}}{\alpha_{i,k}}\right)^{\alpha_{i,k}-1} \left(\frac{\eta^{m}_{i,s}}{\alpha_{i,m}}\right)^{\alpha_{i,m}} \qquad 6.20$$

$$M_{i,s} = \frac{\overline{X}_{i,s}}{\alpha_{i,0}} \left(\frac{\eta^{l}_{1,s}}{\alpha_{i,1}}\right)^{\alpha_{i,1}} \left(\frac{\eta^{l}_{2,s}}{\alpha_{i,2}}\right)^{\alpha_{i,2}} \left(\frac{\eta^{k}_{i,s}}{\alpha_{i,k}}\right)^{\alpha_{i,k}} \left(\frac{\eta^{m}_{i,s}}{\alpha_{i,m}}\right)^{\alpha_{i,m}-1} \qquad 6.21$$

The notation is derived from the specific functional form of the production function:

$$f^{i}(g^{i}(L_{i,1,s},\ L_{i,2,s},\ \ldots,\ L_{i,H,s}),\ K_{i,s},\ M_{i,s}) = \alpha_{i,0}(L_{i,1,s})^{\alpha_{i,1}}(L_{i,2,s})^{\alpha_{i,2}}(K_{i,s})^{\alpha_{i,k}}(M_{i,s})^{\alpha_{i,m}}.$$

That is, the $\alpha_{i,j}$ are production function coefficients for the i^{th} industry.

The demand for wood in each industry is proportional to output. That is, each industry uses a^{q}_{i} units of wood per unit of gross output. In the steady state, each acre of forest produces q_R/R units of wood per year on average, where R is the rotation age. Thus, the total forest size is given by:

$$\mathbf{d}^{N\prime}\mathbf{M}_{F,s} = \frac{R}{q_R}\sum_{l=1}^{m} a_l^q X_{l,s}. \tag{6.22}$$

Or, since there are R age classes with equal area:

$$M_{j,F,s} = \frac{1}{q_R}\sum_{l=1}^{m} a_l^q X_{l,s}, \qquad j = 1,\ 2,\ \ldots,\ R. \tag{6.22'}$$

If the initial assumptions about capital stocks and the division of land between forests and bare land were correct, Equations 6.20, 6.21, and 6.22 would verify this. (These assumptions were made implicitly when it was assumed that the net income of each household was known.) If the initial values for demand prices, wages and land price were indeed equilibrium prices, the complete solution for the optimal steady state would have been found. If the initial price vector was not an equilibrium price vector, a label would be assigned to this price vector. Scarf's algorithm would continue by moving on to a new simplex.

At this stage, the remaining steps of the equilibrium steady state problem can be reduced to four sub-problems: 1) how to label a vector, 2) how to move from one simplex to the next, 3) how to find the land use pattern and capital stocks that are consistent with a particular price vector, and 4) how to adjust the wood price to achieve the necessary "tangency condition" between the upper and lower sequences of shadow prices for each forest-age class. Each of these sub-problems is handled by a separate subroutine in the

FORTRAN program that was used to solve the example problem. Each is discussed below in reverse order, beginning with the forestry subproblem.

Subroutine FAUST

The problem of finding the equilibrium price of wood is relatively simple. As has already been discussed, Equations 6.7 define a set of upper values for the shadow prices for forest acres in each age class. This upper sequence is fixed, given the prices for labor (and thus harvesting and planting) and land. Equations 6.8 define lower values for the forest age-class shadow prices. These lower values can be shifted up or down by increasing or decreasing the price of wood.

Consider the function

$$f(\hat{\eta}^q) = \max_i \ \{\lambda^l_{i,s}(\hat{\eta}^q) - \lambda^m_{i,s}\} \,. \tag{6.23}$$

A zero of this function gives the equilibrium price of wood. A variant of Newton's method can be used to find a zero of this function. More specifically, an estimate of the rate of change of Equation 6.23 is found by evaluating the function at two guesses at the price of wood and then used to find a new guess at the price of wood. The guess for iteration $k + 1$ is given by

$$\hat{\eta}^q_{k+1} = \hat{\eta}^q_k - f(\hat{\eta}^q_k)\left[\frac{\hat{\eta}^q_k - \hat{\eta}^q_{k-1}}{f(\hat{\eta}^q_k) - f(\hat{\eta}^q_{k-1})}\right]. \tag{6.24}$$

Because the function f is monotonic, the algorithm converges rapidly to a very precise estimate of the equilibrium price of wood.

Subroutine KAPBAL

There is a circularity in the discussion of how the equilibrium output levels can be obtained once the equilibrium price vector is known. This circularity is due to the fact that the demand for factors cannot be calculated without knowing the quantity that must be produced. Because of the constant-returns-to-scale in production, the quantity produced must be determined by the quantity demanded. But, to calculate the quantity demanded, each household's income must be known. The necessary quantities of each factor must be known to know income, which completes the circle.

This problem can be thought of as a fixed point problem. The chain of equations from Equation 6.11 to Equation 6.22 is a composite function from an initial guess at the capital stock in each sector and the division of land between bare land uses and forests to a new vector of capital demands, and bare land and forest requirements. The appropriate capital stock and land-use allocation is a fixed point of this function. This suggests that this problem of the appropriate asset configuration for an equilibrium price vector could be solved using a fixed point algorithm.

The appropriate capital level and land-use pattern could also be solved using Newton's method. Here a discrete approximation of the gradient of this vector function need not be used because the explicit functional form is available. This explicit functional form is obtained by combining the relevant equations from Equation 6.11 to Equation 6.22. This is a complicated process, but when it is done, it becomes clear that there is an even more straightforward approach. For the particular functional forms of the equations given in the example, the appropriate capital stocks and land use pattern can be solved for explicitly. The derivation of this formula is tedious and has therefore been relegated to Appendix D.

The equation derived in the appendix essentially allows one to solve Equations 6.11 to 6.22 simultaneously. If the chosen price vector is an equilibrium price vector, the quantities of labor and land demanded at these prices will equal supplies. For a general price vector, nonzero excess demands will be generated for some of these factors.

Actually, it does not make much sense in this context to speak of excess demands for consumer goods. This is because consumer-good supply functions are perfectly elastic due to the constant returns to scale. Thus, the excess demand functions are not single-valued -- or even finitely-valued. It makes more sense to use *excess profit functions* which give the difference between the price of the good and the marginal cost of producing it. Note that excess profit functions may return positive or negative values.

Subroutine LABELER

The information from subroutine KAPBAL on excess demands for primary factors and excess profits for consumer goods is used to assign a label to any vector of prices. It is also possible that the gross output prices determined from Equation 6.10 are negative. Given the potential outcomes, a label is assigned to identify one of the original five prices -- demand prices for two consumer goods, wages for two types of labor, and bare land prices -- that is too high.

The labeling scheme is as follows: Once the price of wood has been calculated, gross output prices are calculated. A label is assigned immediately without going further if one or both of the gross output prices is negative. If only one of these prices is negative, the label is assigned identifying as too high the demand price of the good with the positive gross output price. If both gross output prices are negative, then neither of the consumer good prices can be too high. In this case, the primary factor most recently identified as having the most negative excess demand is identified as having too high a price, and the label is assigned accordingly.

If both gross output prices are positive, excess demands for labor and land and excess profits for the two consumer-goods industries are calculated. Priority is given to the two consumer goods; if either industry is earning a positive excess profit then the label is assigned identifying the industry with the highest excess profits. If both excess profits are negative, then the label is assigned to identify the primary factor with the most negative excess demand. These rules are illustrated with a flow chart of Subroutine LABELER shown in Figure 6.3.

Subroutine CHANGER

The simplicial subdivision used in the program and the method of moving from one simplex to the next are the same as the one described by Scarf (1984b, p. 25). Each simplex in the subdivision is defined by a set of vertices $\{\mathbf{k}^0/D, \mathbf{k}^1/D, ..., \mathbf{k}^n/D\}$, where D is a large positive integer whose inverse determines the mesh size of the simplicial subdivision. The $\mathbf{k}^i$'s are vectors[47] in $\mathbf{R}^{n+1}$ whose elements sum to D. The vectors $\mathbf{k}^1$ through $\mathbf{k}^n$ are generated from the vector $\mathbf{k}^0$ (called a *base point)* according to the following rules:

Define the vectors $\mathbf{e}^i$ as

[47] Scarf (1984b) identifies these k's as integers, but his notation is confusing, so it is not used in precisely the same manner here. Carefully distinguish between subscripts and superscripts when reading Scarf (1984b).

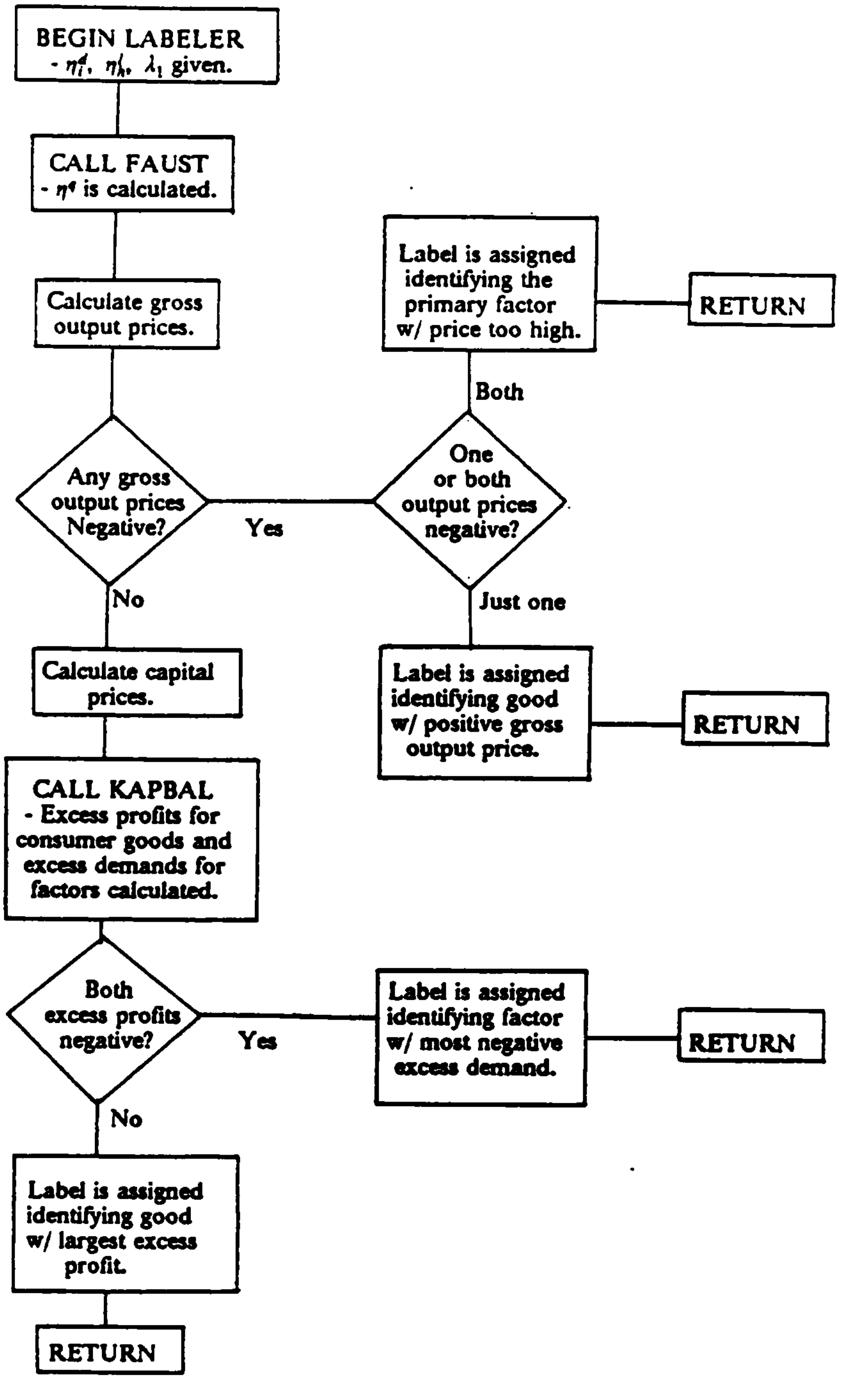

Figure 6.3. Flowchart for Subroutine LABELER.

$$\begin{aligned} e^1 &\equiv (1,\ -1,\ 0,\ \ldots,\ 0)' \\ e^2 &\equiv (0,\ 1,\ -1,\ \ldots,\ 0)' \\ &\ \ \vdots \\ e^n &\equiv (0,\ 0,\ \ldots,\ 1,\ -1)'. \end{aligned}$$

Let k^0 be some vector whose elements are integers that sum to D and let $(\phi_1, \phi_2, \ldots, \phi_n)$ be a permutation of the integers 1 through n. Then the vectors k^1 through k^n are defined by

$$\begin{aligned} k^1 &= k^0 + e^{\phi_1} \\ k^2 &= k^1 + e^{\phi_2} \\ &\ \ \vdots \\ k^n &= k^{n-1} + e^{\phi_n}. \end{aligned} \tag{6.25}$$

Scarf (1984b, p. 24) gives the following theorem -- adapted to the notation here -- for a *replacement operation* which effects a movement from one simplex to a neighboring simplex:

> **Theorem:** Let the vertices of a simplex in the simplicial subdivision be given by k^0, k^1, ..., k^n according to 6.25. The replacement for an arbitrary vertex k is given by
>
> $$k^{j-1} + k^{j+1} - k^j$$
>
> with superscripts interpreted modulo n.

Thus, the current simplex can be represented (and stored in computer memory) as an array whose columns give the vectors defining the vertices of the simplex. To move from one simplex to the next, the vertex (column) that should be removed is chosen. It is replaced with another vertex by summing the elements of the two adjacent columns and subtracting the elements of the column to be replaced. The first and last columns are considered to be adjacent to each other.

The Main Program

One problem that has not yet been discussed is finding an initial simplex for Scarf's algorithm. Recall that the algorithm must begin with a simplex on a face of the main simplex with $n-1$ distinct labels. Another minor difficulty that has not been addressed is the fact that the input demand functions from the Cobb-Douglas production functions are not defined on the faces (boundary points) of the unit simplex. That is, they are not defined when certain prices equal zero.

The problem of finding a simplex with $n-1$ distinct labels is solved by beginning with a one-dimensional problem. Recall that along any face of the simplex, each vertex of the simplicial subdivision can only be given a label from the set of labels assigned to the vertices defining that face. The line segment joining the two vectors (1,0,0,0,0) and (0,1,0,0,0) is a one-dimensional face of the unit simplex in $\mathbf{R}^5$. These vertices will have the labels 1 and 2 respectively. Any of the vertices which lie on this line segment must also be labeled 1 or 2. Since the first two prices are the only ones that can be positive along this line segment, and since the labels are assigned to identify prices that are too high, this will be the case.

This line segment is also the unit simplex in $\mathbf{R}^2$. Thus, Scarf's algorithm may be applied to just those vertices that lie on this line segment and the simplicial subdivision of the unit simplex in $\mathbf{R}^2$ defined by them. When a completely labeled simplex in this unit simplex is found, this will give two vertices with different labels, and the search can be expanded to the unit simplex in $\mathbf{R}^3$. This would mean searching through simplices with three vertices lying in the triangle whose corners are (1,0,0,0,0), (0,1,0,0,0) and (0,0,1,0,0). When the search through this face yields a completely labeled (triangular) simplex, the search begins through the unit simplex in $\mathbf{R}^4$, and so on.

Of course, many of the functions used to determine the labels are not defined if some prices are zero. Thus the faces of the unit simplex cannot be searched. Instead, the algorithm must be confined to faces which are obtained by "stripping off" at least one "layer" of smaller simplices from the original unit simplex. The number of "layers" that are "stripped off" is determined by the starting point for the algorithm and the mesh size of the simplicial subdivision. In the program listed in Appendix C the initial point is (.92,.02,.02,.02,.02). The algorithm is constructed so that, given a fine enough mesh size, no simplex from the layers that were "stripped away" will ever be entered.

This means that the search never leaves the simplex defined by the vertices

$$\mathbf{v}^1 \equiv (.92,\ .02,\ .02,\ .02,\ .02)'$$
$$\mathbf{v}^2 \equiv (.02,\ .92,\ .02,\ .02,\ .02)'$$
$$\mathbf{v}^3 \equiv (.02,\ .02,\ .92,\ .02,\ .02)'$$
$$\mathbf{v}^4 \equiv (.02,\ .02,\ .02,\ .92,\ .02)'$$
$$\mathbf{v}^5 \equiv (.02,\ .02,\ .02,\ .02,\ .92)'.$$

There is a danger in removing too many layers of the simplex. If too many layers are removed, it can no longer be guaranteed that all the vertices located on the faces of this smaller simplex will be properly labeled. That is, there may be vertices on a face that have labels different from the labels on the vertices on the corners of that face (the vertices that define the face). In practice, this has never been a problem. However, it does mean that the algorithm cannot be started from an arbitrary simplex.

A flow chart of the main program is given in Figure 6.4. The flow chart illustrates the process of searching through increasingly higher-dimensioned simplices. When the final completely-labeled simplex is found, the values of the variables for each of the five vertices are averaged to give a single final solution. The program also provides a short routine to check the accuracy of the result. Finally, the results are reported. A subroutine for reporting intermediate iterations is included. This subroutine is useful mostly for debugging.

An Example

The program described in the previous section can be used to find the optimal steady state for two-sector, two-household general equilibrium problems having the structure described in Chapter 5. It would be relatively straightforward to generalize the program to solve for problems with more sectors or more households. Changing the functional forms of the utility and/or production functions might require significant changes in the way the asset stocks are determined (the problem solved here by the subroutine KAPBAL). In this section, the equations and optimal steady state solution of an example problem are discussed.

The objective functions and the primal constraints contain almost all the data necessary to obtain the dual constraints. Thus, Table 6.1 presents only the objective functions and the primal constraints. Note that the utility functions for each household are presented

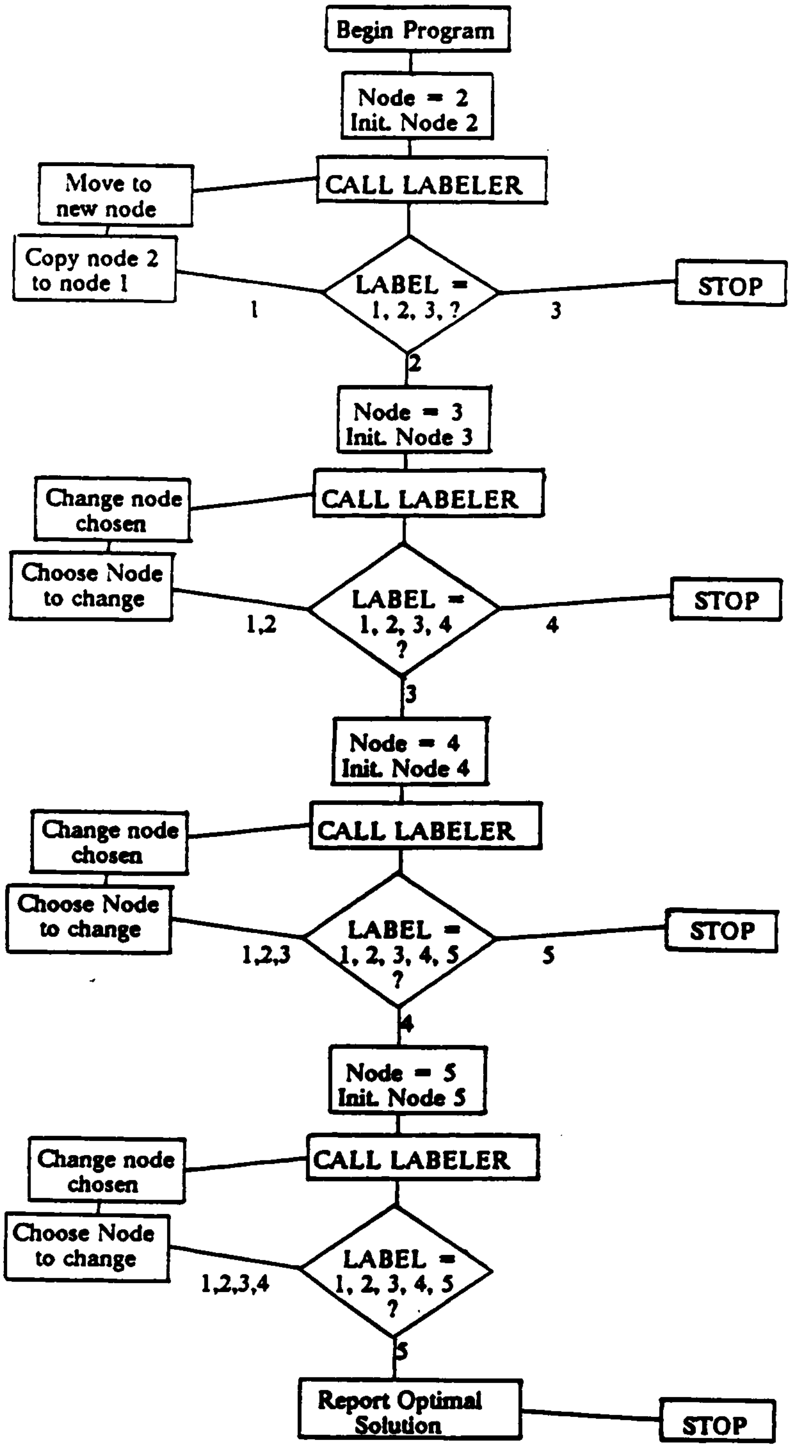

Figure 6.4. Flowchart for the main program.

as separate objective functions. Each household is assumed to be independently maximizing a utility function, subject to the technology given and the structure of the markets for consumer goods and primary factors. These markets are assumed to be perfectly competitive. The only additional information needed is the shares of the asset stock owned by each household and the rate of time preference for each household. Each household in the example has the same rate of time preference: $\rho = (1.17)^{-1}$. The shares are therefore indeterminate and were set exogenously at one half for each household. Household two is larger in the sense that it supplies sixty units of labor compared with forty units from household one. There are one hundred units of land available.

The technology in this example is very similar to the technology in the example in Chapter 4. The only significant change is the inclusion of labor aggregation functions. The labor aggregation functions have been specified to make the labor from household two generally more productive -- especially in industry two. However, both types of labor are equally productive in planting and harvesting activities. Once again, sector one can be interpreted as an agriculture sector, and sector two can be viewed as a manufacturing sector. Note that household two has a relatively stronger preference for consumption good one than household one.

The solution for the optimal steady state is reported in Table 6.2. None of the results are particularly surprising. Household one earns more, even though it has fewer labor units, because of the high productivity and relative scarcity of labor from that household. Thus, household one consumes more of both consumer goods -- especially when viewed from a per-unit-of-labor perspective. As should be expected, the consumption bundle for household two includes a larger proportion of good one than the consumption bundle from household one.

Table 6.1. **The primal equations for the example steady-state, general equilibrium problem.**

$$\begin{aligned}
&\text{Maximize} \quad && \Upsilon_1 = 1.0 {X^c_{1,1,s}}^{0.45} {X^c_{2,1,s}}^{0.35} \\
&\quad\text{and} && \Upsilon_2 = 1.0 {X^c_{1,2,s}}^{0.5} {X^c_{2,2,s}}^{0.3}
\end{aligned}$$

$$X^c_{i,h,s},\ K_{i,s},\ L_{i,h,s},\ L_{p,h,s},\ L_{c,h,s} \tag{1}$$

$M_{i,s},\ X_{i,s},\ \mathbf{M}_{F,s},\ \mathbf{c}_s,\ \eta^s_{i,s},\ \eta^d_{i,s}$

$\eta^k_{i,s},\ \eta^l_{h,s},\ \eta^q_s,\ \lambda_s,\ \zeta_s$

subject to

$$X_{1,s} = 1.5(K_{1,s})^{0.2}[(L_{1,1,s})^{0.6}(L_{1,2,s})^{0.4}]^{0.4}(M_{1,s})^{0.4} \tag{2}$$

$$X_{2,s} = 2.0(K_{2,s})^{0.5}[(L_{2,1,s})^{0.8}(L_{2,2,s})^{0.2}]^{0.4}(M_{2,s})^{0.1} \tag{3}$$

$$X^c_{1,s} = (1.0-0.2)X_{1,s} - 0.3X_{2,s} - (1.0-0.9)[0.3K_{1,s} + 0.25K_{2,s}] \tag{4}$$

$$X^c_{2,s} = -0.3X_{1,s} + (1.0-0.1)X_{2,s} - (1.0-0.9)[0.7K_{1,s} + 0.75K_{2,s}] \tag{5}$$

$$0.1c_2 + 2.0c_3 + 5.0c_4 + 7.0c_5 + 7.5c_6 \geq 0.1X_{1,s} + 0.5X_{2,s} \tag{6}$$

$$(\mathbf{I}-\mathbf{G})\mathbf{M}_{F,s} = \mathbf{R}\mathbf{c}_s. \tag{7}$$

$$\mathbf{M}_{F,s} \geq \mathbf{c}_s \tag{8}$$

$$0.1M_{1,F,s} = (L_{p,1,s})^{0.5}(L_{p,2,s})^{0.5} \tag{9}$$

$$0.05c_1 + 0.1c_2 + 0.2c_3 + 0.25c_4 + 0.29c_5 + 0.3c_6 = (L_{c,1,s})^{0.5}(L_{c,2,s})^{0.5} \tag{10}$$

$$\sum_{i=1}^{m} L_{i,1,s} + L_{p,1,t} + L_{c,1,t} \leq 40 \tag{11}$$

$$\sum_{i=1}^{m} L_{i,2,s} + L_{p,2,t} + L_{c,2,t} \leq 60 \tag{12}$$

$$\mathbf{d}^{n\prime}\mathbf{M}_{F,s} + \sum_{i=1}^{m} M_{i,s} \leq 100 \tag{13}$$

Table 6.2. The solution to the example steady-state, general equilibrium problem.

$X_{1,s} = 58.07$	$\eta^{s}_{1,s} = 0.1452$
$X_{2,s} = 48.37$	$\eta^{s}_{2,s} = 0.1095$
$X^{c}_{1,1,s} = 15.86$	$\eta^{d}_{1,s} = 0.2814$
$X^{c}_{1,2,s} = 13.25$	
$X^{c}_{2,1,s} = 17.62$	$\eta^{d}_{2,s} = 0.2474$
$X^{c}_{2,2,s} = 13.53$	
$K_{1,s} = 24.25$	$\eta^{k}_{1,s} = 0.3014$
$K_{2,s} = 38.33$	$\eta^{k}_{2,s} = 0.2994$
$L_{1,1,s} = 21.04$	$\eta^{l}_{1,s} = 0.0962$
$L_{2,1,s} = 17.62$	
$L_{1,2,s} = 43.10$	$\eta^{l}_{2,s} = 0.0313$
$L_{2,2,s} = 13.53$	
$L_{p,1,s} = 0.24$	$\eta^{p}_{s} = 0.1098$
$L_{p,2,s} = 0.75$	
$L_{c,1,s} = 0.71$	$\eta^{c}_{s} = 0.1098$
$L_{c,2,s} = 2.18$	
$M_{1,s} = 67.58$	
$M_{2,s} = 10.61$	
$M_{1,F,s} = 4.285$	$\lambda_{1,s} = 0.3436$
$M_{2,F,s} = 4.285$	$\lambda_{2,s} = 0.4149$
$M_{3,F,s} = 4.285$	$\lambda_{3,s} = 0.4854$
$M_{4,F,s} = 4.285$	$\lambda_{4,s} = 0.5679$
$M_{5,F,s} = 4.285$	$\lambda_{5,s} = 0.6645$
$M_{6,F,s} = 0.0$	$\lambda_{6,s} = 0.6921$
$c_{5,s} = 4.285$	$\eta^{q}_{s} = 0.0575$
$c_{1,s} = c_{2,s} = c_{3,s} = c_{4,s} = c_{6,s} = 0.0$	

As in Chapter 4, more capital is employed in sector two than in sector one. The overall capital stock is lower here than in that example, however. Also, over two thirds of the labor from household two is employed in the agriculture sector, while only a little more than half of the labor from household one is employed in this sector. Three quarters of the labor used in harvesting and planting is from household two. Most of the labor employed in manufacturing is from household one. Over two thirds of the land base is used for agriculture. About one fifth of the land base is in forestry. About one tenth of the land is used in the manufacturing sector.

There is some error in the results. The total amount of land used and the labor used from each sector are less than the quantities that are available. The error is always less than one percent, but it is a bit disturbing.[48] Also, the program checks whether several of the dual constraints are met. These constraints are always met at the level of precision determined by the mesh size of the simplices. However, the "round-off error" is always negative. Since these errors are all fairly small, no great amount of time was spent looking for their source. Potential sources for these errors are: the algorithm, round-off error, errors in the specification of the equations, or a typographical or logical error in the program implementing the algorithm ("bugs").

The community utility function for this example can also be derived. This is done by calculating the weights that are implicitly placed on the utility function for each household. These weights can be obtained using the steady-state version of Equation 5.9. That is,

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[48] Similar errors can be found in the "optimal solution" obtained with MINOS for the problem in Chapter 4.

$$\omega_{h,t} = \frac{\eta^{d}_{i,s} \frac{1}{1-\rho_h}}{U^{h}_{i,s} \frac{1}{1-\rho_s}} \qquad 6.26$$

In this example $\rho_h = \rho_s$ for each household, so the discount rate terms cancel. For this example, the weight on household one is 0.1649; for the second household, the weight is 0.1538. Thus, an equivalent objective function for solving this problem as a maximization problem is:

$$\text{maximize } 0.1649\, U^1 + 0.1538\, U^2. \qquad 6.27$$

This is the community utility function for this example economy. Once again, there is no implication here that these weights are somehow socially optimal. Rather, they are the *de facto* weights that would be placed on the utilities of these households in a perfectly competitive economy. These weights are arbitrary in the sense that if the utility function of one household were multiplied by, say, two in the formulation of the problem, absolutely nothing in the solution would change except the value of the weight for that household; the weight for that household would be halved if this were done.

Discussion

The version of Scarf's algorithm discussed in this chapter is relatively primitive. Since 1967 when the algorithm was first published, more efficient ways to solve fixed point problems have been developed. Scarf (1984) describes improved algorithms introduced by Merrill (1972), van der Laan and Talman (1979a, 1979b), and Eaves (1972). Todd (1984) also discusses an improved version of the algorithm. These improvements were not incorporated here because the primary goal was to simply solve the problem, and time limitations did not permit considerations of alternative algorithms.

Faster algorithms will probably be necessary for solving realistic problems with realistic computing budgets. The example problem was solved with a mesh-size of 0.0001 (D = 10,000) in about 42 minutes on a Micro VAX II. A total of 48,062 iterations were performed. With only minor improvements in the algorithm, run-time could have been reduced to less than ten minutes. On the other hand, computation time can be expected to increase rapidly as the dimensions of the problem are increased.

This chapter demonstrates the solution of a general equilibrium problem that is consistent with a dynamic view of an economy. The optimal steady state problem is a one-period model, but it is consistent with this dynamic view. Undoubtedly, the non-steady-state problem will also be solvable soon.

The solutions to dynamic general equilibrium models will be useful for a variety of policy analysis problems. Most types of policy analysis will require that a government sector be explicitly included in the model. One way to do this is to define a set of government activities. Usually, these would correspond to those activities in which the particular government under consideration is already involved. Likely government activities include the production of public goods such as national parks, national defense, infrastructure, and a legal system. The government may also transfer income between households.

In an equilibrium model, the levels of government activities should be considered exogenous (unless a model of political equilibrium is also included). For those government products that are sold in markets, government supply would be perfectly inelastic unless actual government policies are known to be price-sensitive. While the government might raise some revenue from sales of products, much of the cost of government production is paid for through taxes. Taxes must also be considered exogenous in an economic equilibrium model. However, different tax systems could be imposed to determine the impacts on income levels and distribution and on production from different tax systems.

Recall that solving a general equilibrium model as an optimization problem tends to treat each good as if it were traded in a perfect market. In such a model, when government activities are included as endogenous variables, the solution will indicate the optimal levels of these activities, given the model specification. Perhaps even the tax system itself could be made endogenous. The solution of such a model would indicate the optimal role for the government in the economy depicted by the model. This could be compared with equilibrium models of how the government actually does behave. Given that the model is not the real world, such results should not be taken *too* seriously. The insights gained from such an exercise might be worth the effort, however.

CHAPTER 7: DISCUSSION AND CONCLUSIONS

Katzner (1988) remarks that the singular success of economics is that "economists are able to assert coherence rather than chaos, at least in theory ... in an economy motivated solely by the self-interest of individuals and firms". This dissertation represents an attempt to show how the economic models of the forest sector can be integrated into this grand scheme -- at least in theory. This has been a worthwhile undertaking for several reasons.

First, an understanding of how forest sector models fit into the big picture gives a deeper understanding of models with a narrower scope. Having a big picture helps analysts see what lies beyond the sideboards of their sectoral models. It helps identify the limitations of these sideboards and provides a framework for determining how significant these limitations might be. For example, when might the feedback missing from the unmodeled parts of the economy be too important to ignore? The sideboards of models often include implicit assumptions in their construction. Having the big picture helps one identify and scrutinize these assumptions.

The big picture is also useful for understanding how forestry fits into society and how society is influenced by forestry. For example, the discussion above showed how including a forestry sector in a growth model throws a wrench into the concept of constant growth. Forests depend on a primary factor which is fundamentally limited in supply and cannot grow at a constant rate forever. The recognition of such limits to growth changes dramatically the picture presented by the growth models.

The general equilibrium framework provides a useful setting for modeling conflicting uses of the forest. The importance of nonconsumptive uses of the forest probably increases with income for most people. Throughout history, income and population have tended to increase together, however. This portends increasing conflict over different uses of the forest. While one does not need a model to foresee these changes, a model provides a more rigorous framework for analysis. How such conflicts will be resolved depends on whether it is easier to find substitutes for industrial forest products or for forest recreation, wildlife values, and environmental quality. It also depends on the relative strength of people's preferences for differing uses of forest lands. All of these factors -- preferences, substitution, and income -- can be included in the framework of the general equilibrium models discussed in this dissertation.

In fact, the framework of these models could be useful for analyzing a large variety of issues. The basic approach could be used for modeling other natural resource issues outside forestry such as fisheries regulation and the optimal depletion of nonrenewable resources. A general equilibrium approach is particularly useful for analyzing tax issues because taxes are often levied on many sectors simultaneously. This has been a common application of general equilibrium modeling from Harberger (1962) in general economics, to Boyd and Newman (1988) in forestry. General equilibrium models are also used in trade analysis for similar reasons; tariffs and quotas are often applied to groups of sectors.

Finally, general equilibrium models provide a means for analyzing the income and employment impacts of government policies. Such models are always in demand. The standard current approach for analyzing these issues is input-output analysis. The computable general equilibrium approach represents a significant generalization of input-output models. Unfortunately, the data requirements for computable general equilibrium models and the difficulty of solving them will probably help maintain the popularity of the input-output approach for a long time.

Approaches to Empirical Application

If models such as those developed in this dissertation are to be used for policy analysis, the parameters of the equations in the model must be estimated or calibrated so that they correspond to observed current and past states of the economy. Note that the purpose of such an exercise is not likely to be a test of the model. Rather, the purpose is to construct a model that can be used to simulate the responses in the economy to changes in policies. Even when the term "validation" is used in this context, it usually only implies testing the fit of the model, rather than the model itself. Methodological issues related to the model are discussed in the next section.

The first step in constructing an empirical application of the model is choosing the particular functional forms to be used for the utility and production functions. Cobb-Douglas functions were used in the examples here, but other functional forms would probably be more useful for serious empirical application. Specifying and estimating the consumption side of the economy will be the most difficult part of the problem. Utility is fundamentally unobservable. Thus, demand functions are usually estimated. This raises the issue of *integrability;* that is, can these demand functions be integrated simultaneously to obtain some utility function with the implied properties of continuity and differentiability, differential increasingness, strict quasi-concavity, and the boundary condition, discussed in Chapter 4?[49] In practice, this issue is often ignored, but the demand functions should at least be homogeneous of degree zero in prices and income. Dervis, deMelo and Robinson (1982, pp. 145-150, 475-485) discuss practical approaches to the specification of demand in empirical models.

On the production side, a common practice is to use constant elasticity of substitution (CES) or nested CES production functions. Elasticity estimates are often obtained independently (for example, from the literature) allowing these production functions to be "calibrated" with just one observation. Mansur and Whalley (1984) discuss a "calibration" procedure for specifying an entire model. This is the most common approach to numerical specification of computable general equilibrium models. Jorgenson (1984) has estimated an entire model using econometric simultaneous-equation techniques. His approach is exceptional, however.

It should be clear from this brief discussion that the empirical specification of computable general equilibrium models is a large task. Thus, these issues have been placed largely outside the scope of this dissertation and left for future research.

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[49] Katzner (1988 and 1970) provides discussions of integrability.

Methodological Issues

There are many methodological issues raised by the models discussed in the dissertation. These problems are not unique to this work, but are common to many neoclassical economics models. As discussed in the introduction, neoclassical economics has been a useful theoretical framework for understanding many aspects of economic systems in spite of these problems. Nevertheless, it is advisable to be aware of the failings as well as the successes of our theories.

Falsifiability

Perhaps the most important methodological issue relevant to this work is falsifiability. Blaug (1980, pp. 187-192) has charged that general equilibrium theory cannot be falsified and that it contains so many restrictive assumptions that the entire approach is irrelevant. Weintraub (1985, pp. 119-122) counters that falsifiability is an inappropriate criterion for appraising general equilibrium theory. He argues that general equilibrium theory is a part of the "hard core" of neoclassical economics and is therefore exempt from falsification. Rather, theories in the "hard core" are to be appraised by "criteria appropriate for gauging mathematical progress". These criteria are essentially logical consistency and rigor. Weintraub does say that theories in the "protective belt" of neoclassical economics should be subject to falsification, however.

Blaug is certainly correct on the impossibility of falsifying general equilibrium theory. However, general equilibrium theory provides insight into the way economic systems work and is clearly not irrelevant. The insights generated by general equilibrium theory are an important source of theories in the "protective belt" of neoclassical economics. The proliferation and success of many theories in the "protective belt" of neoclassical economics speaks to the relevance usefulness of the core theories.

Most of the work reported in this dissertation can claim Weintraub's immunity from falsification. Note that no claims such as "this is how the real world is" or "this is how it ought to be" have been made. What has been claimed is that the model generates insight into how things are or perhaps ought to be. More insights will come with empirical applications of the model, and these should be susceptible to falsification. However, at the present stage of this research, the important tests of the models are that they are logically and mathematically consistent.

It is not completely satisfactory to allow a large portion of the theory to be above testing. Arguably, the aim of science is not only to provide *an* explanation, but to provide *the* explanation. Thus, ways of testing the explanations given by particular must be considered. An alternate sense in which the overall framework can be tested is to use the model to make predictions about future states of the economy and compare these predictions with the actual states that evolve. This is not an absolute test like falsification. In contrast to using the criterion of falsification, when a model is tested by its predictions, it may not rejected if its predictions are not exactly in accordance with observations. This criterion allows one to use the "best" available model in a world that we will never be able to predict perfectly. Even in physics, the idea of a deterministic universe has been out of vogue for three quarters of a century.

Utility

The concept of utility also deserves comment. The utility function is a mathematical construct which has been used by economists to formalize a consistent theory of behavior. This approach has been useful for many purposes, but it has its drawbacks. The most important of these is the inability of anyone to observe utility. Much can undoubtedly be learned in this area from psychologists and sociologists, and economists should be open to alternatives to the utility function.

One of the more interesting implications of the work reported here is that utility is an important aspect of forestry economics. This has been a neglected area in forest economics where the tendency has usually been to "assume we have a demand curve". Anyone who has ever estimated a demand curve knows how fickle this procedure can be. Estimating the demand for nonmarket forest amenities is an even more difficult task. The results in Chapter 5 show how easily the answers given by these models can change when the specification of the utility functions is changed. It is important, therefore, that more attention be given to these problems.

Uncertainty

It has been argued repeatedly that the perfect foresight assumption is a poor one, but, for now, a necessary one. Blaug (1980) would probably claim that this is merely an attempt to immunize these models from falsification. On the contrary, these comments on the problem of uncertainty are only a recognition that we do not have any answers to this problem. For now, we have to do what we can, recognizing the problems with our approach. This is clearly an area where more research is needed.

Constant Technology and Tastes and Preferences

Two of the assumptions of the models presented in this dissertation are that technology is constant and tastes and preferences are constant. Changes in tastes and preferences are usually considered to be outside the scope of economics. Perhaps this is too narrow a view. At any rate, if a model of how tastes and preferences evolve were available, it could probably be incorporated into the framework of the dynamic general equilibrium model. On the other hand, perhaps at the level of aggregation of a computable general equilibrium model, the assumption of constant tastes and preferences is not too bad. Surely fads come and go, but do the basic needs and wants of humans change much over time?

The assumption of fixed technology is relatively easy to relax. Much work has been done in economics on technological change. Also, changes in the technology that will be made over the next two or three decades are more predictable than one might initially believe. Many of the new technologies that will become operational in the next two or three decades are already in the theoretical stage of development. Changes in the regulatory environment will be influential, but are also sometimes predictable. For example, it is likely that the current trend towards increased voluntary and mandatory recycling will continue. Again, if a model of technical change were available, it could be incorporated into the dynamic general equilibrium model. Anticipated changes in technology could be specified exogenously. Such issues were not considered in the dissertation in order to limit the scope of the study.

Conclusions

Significant progress has been made in this dissertation in integrating a dynamic model of the timber sector into a general equilibrium framework. The work presented in this study allows forest economists to see how their models fit into the grand scheme of neoclassical economics and to obtain a broader perspective of the forest sector.

The general equilibrium models discussed here will provide a powerful analytical tool for understanding the role of forests in society. They will be useful for analyzing both positive and normative issues. On the normative side, the models will be useful for understanding how forests should be managed for the best use of society. On the positive side, the models are probably the best way we have to predict such things as the impact of alternative policies on regional economic conditions or forest states. The models should also be useful for predicting how tax changes or trade policies will influence production patterns and income distributions.

This dissertation has also explored in great detail the dynamic nature of timber supply. In so doing, the common ground shared by several of the dynamic timber supply models was identified and clarified. The Faustmann formula was also derived from the optimal steady state for the dynamic model, and the connection between long-run models of timber supply and the steady states of dynamic models was demonstrated.

Because of the dynamic general equilibrium model's potential usefulness for analyzing policy issues, the problem of computing solutions to these models was addressed. The models are easier to solve when all households' utility functions are aggregated into a single community utility function. However, the range of issues that the multi-sector model would be useful for is relatively limited. Disaggregating the utility function makes the model much harder to solve, but greatly increases the richness of the model. Some progress was made toward solving the dynamic general equilibrium model. When perfect competition is assumed, the steady-state version of the model can be formulated as an equilibrium problem or as a maximization problem. The equilibrium approach can be used to solve the steady state problem reliably. The non-steady-state problem was not, however, formulated as an equilibrium problem here. Nevertheless, the steady state model is *the* one-period model that is consistent with a dynamic model. If timber supply is a dynamic problem, then the steady state model is the most appropriate one-period model to use (rather than a static model, for example).

The comparison of the maximization and equilibrium formulations of these models is an interesting aspect of the research reported in this dissertation. Chapters 5 and 6 provide a demonstration of the fundamental welfare economics results that, in a competitive economy, an optimal solution is an equilibrium solution and an equilibrium solution is an optimal solution. ("Optimal" is not used here to mean "socially optimal", but rather in the sense that one has a solution to an optimization problem. The solutions to the models given in the dissertation are "pareto optimal", given the model formulations.) When the perfect competition assumptions are not met, the optimization formulation can be viewed as a normative model, and the equilibrium formulation can be viewed as a positive model.

Repeated references to the assumptions of the model and related problems may seem negative, but this is actually one of the strong points of this research. I believe that it is desirable to have a model's problems out where they can be seen. It can be be dangerous to place too much confidence in the results, or to allow others to place too much confidence in them. A good model's problems need not be viewed negatively, rather, they should be viewed as research opportunities.

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Appendix A. NOTATION

This appendix summarizes the notation used in Chapters 3-6. The symbols are listed in approximately the order in which they are introduced in the text. Subscripts may occasionally be different. Thus, for example, j's and i's are sometimes used in similar contexts. Also, t's are replaced with s's, when referring to steady-state models.

Chapter 3

$\mathbf{R}^n \equiv$ Euclidian n-space.

$N \equiv$ the forest age class at which growth stops; after this age it is assumed that no growth or decay occurs.

$T \equiv$ the planning horizon of the problem.

$x_{t,i} \equiv$ the area (acres) of age-class i at time t.

$c_{t,i} \equiv$ the area cut from age-class i at time t.

$q_i \equiv$ the yield of wood per unit area (acres) for age-class i.

$Q_t \equiv$ the volume harvested in period t (quantity supplied).

$y = q(a) \equiv$ an equation giving the volume of wood per unit area (acres) at age a.

$\hat{a} \equiv$ the age at which merchantable volume first becomes available from a stand of trees.

$\mathbf{x}_t \equiv (x_{t,1},\ x_{t,2},\ \ldots,\ x_{t,N})'$.

$\mathbf{c}_t \equiv (c_{t,1},\ c_{t,2},\ \ldots,\ c_{t,N})'$.

$\mathbf{q} \equiv (q_1,\ q_2,\ \ldots,\ q_N)'$.

$$\mathbf{G} \equiv \begin{bmatrix} 0\,0\,0 & & \\ 1\,0\,0 & & \\ 0\,1\,0 & & \\ & \ddots & \\ & & 1\,0\,0 \\ & & 0\,1\,1 \end{bmatrix}_{N\mathrm{x}N}.$$

$$\tilde{\mathbf{R}} \equiv \begin{bmatrix} 1\,1\,1 & \ldots & 1 \\ 0\,0\,0 & & 0 \\ 0\,0\,0 & & 0 \\ \vdots\ \ \vdots & & \vdots \\ 0\,0\,0 & \ldots & 0 \end{bmatrix}_{N\mathrm{x}N}.$$

$$\mathbf{R} \equiv \tilde{\mathbf{R}} - \mathbf{G} \equiv \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ -1 & 0 & 0 & & 0 \\ 0 & -1 & 0 & & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & -1 \end{bmatrix}_{N\mathrm{x}N}.$$

$$\hat{\mathbf{G}} \equiv \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 0 & -1 & 0 & & 0 \\ 0 & 0 & -1 & & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & -1 \end{bmatrix}_{N\mathrm{x}N}.$$

$P_t \equiv$ the price per unit volume of wood in period t (stumpage price).

$r \equiv$ the interest rate.

$\delta \equiv$ one over one plus the interest rate: a one-period discount factor.

$\lambda_t \equiv (\lambda_{t,1},\ \lambda_{t,2},\ \ldots,\ \lambda_{t,N})'$.

$\lambda_{t,i} \equiv$ a Lagrangian multiplier giving the shadow price of an acre of forest in age-class i at time t.

$\zeta_t \equiv (\zeta_{t,1}, \zeta_{t,2}, \ldots, \zeta_{t,N})'$.

$\zeta_{t,i} \equiv$ a Lagrangian multiplier giving the cost of postponing harvesting an acre from age-class i from time t to time t+1.

$\Lambda \equiv$ the symbol for a Lagrangian function.

$s_t(\mathbf{x}_t, \mathbf{c}_t) \equiv$ the current (in time t) contribution to the present value of the forest.

$S_t(\mathbf{x}_t, \mathbf{C}) \equiv$ the present value at time t of the forest.

$\mathbf{C} \equiv \{\mathbf{c}_t, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \ldots\}$.

$\mathbf{x}_t^* \equiv$ the state the forest much reach in the terminal time period (determined exogenously).

$S_t^*(\mathbf{x}_t^*) \equiv$ the present value of the forest in period t if an optimal policy is followed for the remaining time horizon.

$S_T^*(\mathbf{x}_T^*) \equiv$ the present value of the forest in the terminal time period.

$H_T \equiv$ the Hamiltonian for the discrete-time optimal control problem for period t.

$c(a, t) \equiv$ a function which, when integrated over a time period and range of age classes, gives the area cut from that range of age classes during that period.

$w(t) \equiv$ the amount of wood cut in time t.

$M(a,t) \equiv$ the area of forest of age less than or equal to age a at time t.

$m(a, t) \equiv \dfrac{\partial M}{\partial a}$.

$l(a) \equiv$ the labor required to harvest a unit area of age a.

$L^h(a) \equiv$ the total amount of labor employed in harvesting at time t.

$P(t) = f^d(w(t)) \equiv$ the inverse demand function for harvested timber.

$\eta' \equiv$ the wage for harvesting labor.

Chapter 4

$\mathbf{R}^n \equiv$ Euclidian n-space.

$N \equiv$ the forest age class at which growth stops; after this age it is assumed that no growth or decay occurs.

$T \equiv$ the planning horizon of the problem.

$\mathbf{X} \equiv$ the commodity space; the set of possible consumption vectors.

$\Upsilon \equiv$ the symbol for a utility-maximization objective function.

$U \equiv$ the symbol for a utility function; Chapter 4 gives the properties of these functions.

$f^i \equiv$ the symbol for a production function for the i^{th} industry.

$X^c_{i,t} \equiv$ the quantity of good i consumed in period t.

$\rho \equiv$ one over one plus the community discount rate.

$X_{i,t} \equiv$ gross output of good i in period t.

$L_{i,t} \equiv$ the labor used in industry i in period t.

$K_{i,t} \equiv$ the capital used in industry i in period t.

$M_{i,t} \equiv$ the land used in industry i in period t (acres).

$a_{i,j} \equiv$ input-output coefficients.

$X_{i,j,t} \equiv$ the amount of good j used as an intermediate input in industry i in period t.

$a^q_i \equiv$ input-output coefficient for wood use in industry i.

$W_{i,t} \equiv$ the wood used in industry i in period t.

$\Delta K_{j,t+1} \equiv$ new capital produced for sector k to be installed in period t+1.

$\gamma_{i,j} \equiv$ input-output coefficients for new capital production.

$\mathbf{X}^c_t \equiv [X^c_{1,t}, X^c_{2,t}, \ldots, X^c_{m,t}]'$.

$\mathbf{X}_t \equiv [X_{1,t}, X_{2,t}, \ldots, X_{m,t}]'$.

$\Delta \mathbf{K}_{j,t+1} \equiv [\Delta K_{1,t+1}, \ \Delta K_{2,t+1}, \ \ldots, \ \Delta K_{m,t+1}]'$.

$\Gamma \equiv$ the matrix of capital-production coefficients.

$\mathbf{A} \equiv$ the matrix of input-output coefficients.

$\delta \equiv$ the proportion of capital remaining after one period of depreciation.

$\mathbf{q}' \equiv [q_1, \ q_2, \ \ldots \ q_N]'$.

$q_i \equiv$ the yield of wood per unit area (acre) of forest of age i,

$\mathbf{c}_t \equiv [c_{1,t}, \ c_{2,t}, \ \ldots \ c_{N,t}]'$.

$c_{i,t} \equiv$ the number of acres cut from age-class i in period t.

$\mathbf{M}_{F,t} \equiv [M_{1,F,t}, \ M_{2,F,t}, \ \ldots \ M_{N,F,t}]'$.

$M_{i,F,t} \equiv$ the number of acres of forest in age class i in period t. (The subscript F is used to distinguish forest land from land used in other sectors.)

$$\mathbf{G} \equiv \begin{bmatrix} 0\,0\,0 & & \\ 1\,0\,0 & & \\ 0\,1\,0 & & \\ & \ddots & \\ & & 1\,0\,0 \\ & & 0\,1\,1 \end{bmatrix}_{N \mathrm{x} N.}$$

$$\mathbf{R} \equiv \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ -1 & 0 & 0 & & 0 \\ 0 & -1 & 0 & & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & -1 \end{bmatrix}_{N \mathrm{x} N.}$$

$\mathbf{e}_1' \equiv (1, 0, 0, \ldots, 0) \in \mathbf{R}^N$.

$\mathbf{d}_1^{n\prime} \equiv (1, 1, 1, \ldots, 1) \in \mathbf{R}^n$.

$l_p \equiv$ the amount of labor required to plant an acre of forest.

$\mathbf{l}_c' \equiv [l_{c1}, \ l_{c2}, \ \ldots \ l_{cN}]'$.

$l_{c,i} \equiv$ the amount of labor required to harvest an acre of forest of age class i.

$\overline{L}_t \equiv$ the labor force in period t.

$\overline{M} \equiv$ the supply of land.

$\Lambda \equiv$ the symbol for a Lagrangian function.

$\eta^s_{i,t} \equiv$ the price of gross output from industry i in period t.

$\eta^d_{i,t} \equiv$ the price of consumer good i in period t.

$\eta^k_{i,t} \equiv$ the price in period t of capital used in industry i.

$\eta^l_t \equiv$ the labor wage in period t.

$\eta^q_t \equiv$ the price of wood in period t.

$\lambda_t \equiv [\lambda_{1,t}, \lambda_{2,t}, \ldots, \lambda_{N,t}]'$.

$\lambda_{i,t} \equiv$ the price of an acre of forest in age class i in period t. $\lambda_{1,t}$ is the price of bare land.

$\zeta_t \equiv [\zeta_{1,t}, \zeta_{2,t}, \ldots, \zeta_{N,t}]'$.

$\zeta_{i,t} \equiv$ the cost of postponing harvesting age class i in period t.

$\mathbf{M}_{F,z,t} \equiv$ the forest age class vector for land class z in period t.

$\mathbf{c}_{z,t} \equiv$ a vector giving the number of acres cut from each age class of land class z in period t.

$\mathbf{q}_z \equiv$ a vector giving the volume of wood that can be harvested from an acre of forest growing on land class z for each age class.

$l_{p,z} \equiv$ the amount of labor required to plant an acre of forest in land-class z.

$\mathbf{l}_{c,z} \equiv$ a vector giving the amount of labor required to harvest an acre of forest in each age class of forest in land-class z.

$\mathbf{c}_{w,t} \equiv$ a vector giving the number of acres cut from forests in each age-class according to cutting activity w in period t.

$\mathbf{q}_w \equiv$ a vector giving the volume of wood that can be harvested for each age class from an acre of forest with cutting activity w.

$a^q_{j,w} \equiv$ an input-output coefficient giving the requirements for industry i for the particular type of wood product obtained with cutting activity w.

Chapters 5 and 6

$\mathbf{R}^n \equiv$ Euclidian n-space.

$N \equiv$ the forest age class at which growth stops; after this age it is assumed that no growth or decay occurs.

$T \equiv$ the planning horizon of the problem.

$\mathbf{X} \equiv$ the commodity space; the set of possible consumption vectors.

$\Upsilon \equiv$ the symbol for a utility-maximization objective function.

$U \equiv$ the symbol for a utility function; Chapter 4 gives the properties of these functions.

$f^i \equiv$ the symbol for a production function for the i^{th} industry.

$g^i \equiv$ the symbol for a labor aggregation function for industry i (or, if i = c or p, for harvesting or planting).

$X^c_{i,t} \equiv$ the quantity of good i consumed in period t.

$P_{i,t} \equiv$ the the price of the i^{th} good in period t.

$S \equiv$ savings from period 1.

$Y_t \equiv$ exogenous income from period t.

$r \equiv$ the rate of return for investments or savings.

$\rho \equiv$ one over one plus the rate of return for investments (savings).

$\mu \equiv$ Lagrangian multipliers on income constraints.

$X^c_{i,h,t} \equiv$ the quantity of good i consumed by the h^{th} household in period t.

$r_h \equiv$ the rate of time preference for the h^{th} household.

$\rho_h \equiv$ one over one plus the rate of time preference for the h^{th} household.

$\omega_{h,t} \equiv$ the weight given to the h^{th} household in period t.

$\rho_t \equiv$ one over one plus the equilibrium rate of return on assets for the economy in period t.

$X_{i,t} \equiv$ gross output of good i in period t.

$L_{i,t} \equiv$ the aggregate labor input for industry i in period t.

$K_{i,t} \equiv$ the capital used in industry i in period t.

$M_{i,t} \equiv$ the land used in industry i in period t (acres).

$L_{i,h,t} \equiv$ the amount of labor input from household h employed in the i^{th} consumer-good industry in period t.

$a_{i,j} \equiv$ input-output coefficients.

$\gamma_{i,j} \equiv$ input-output coefficients for new capital production.

$\Delta K_{j,t+1} \equiv$ new capital produced for sector k to be installed in period t + 1.

$\delta \equiv$ the proportion of capital remaining after one period of depreciation.

$a_i^q \equiv$ input-output coefficient for wood use in industry i.

$\mathbf{q}' \equiv [q_1,\ q_2,\ \ldots\ q_N]'$.

$\mathbf{c}_t \equiv [c_{1,t},\ c_{2,t},\ \ldots\ c_{N,t}]'$.

$q_i \equiv$ the yield of wood per unit area (acres) of forest of age i,

$c_{i,t} \equiv$ the number of acres cut from age-class i in period t.

$\mathbf{M}_{F,t} \equiv [M_{1,F,t},\ M_{2,F,t},\ \ldots\ M_{N,F,t}]'$.

$M_{i,F,t} \equiv$ the number of acres of forest in age class i in period t.

$$\mathbf{G} \equiv \begin{bmatrix} 0\,0\,0 & & \\ 1\,0\,0 & & \\ 0\,1\,0 & & \\ & \ddots & \\ & & 1\,0\,0 \\ & & 0\,1\,1 \end{bmatrix}_{N\text{x}N}.$$

$$\mathbf{R} \equiv \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & 0 & 0 & & 0 \\ 0 & -1 & 0 & & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}_{N \mathrm{x} N}.$$

$\mathbf{e}_1' \equiv (1, 0, 0, \dots, 0) \in \mathbf{R}^N$.

$\mathbf{d}_1^{n\prime} \equiv (1, 1, 1, \dots, 1) \in \mathbf{R}^n$.

$L_{p,h,t} \equiv$ the amount of labor input from household h employed in planting trees in period t.

$l_p \equiv$ the amount of aggregate labor services required to plant an acre of trees.

$L_{c,h,t} \equiv$ the amount of labor input from household h employed in harvesting trees in period t.

$\mathbf{l}_c' \equiv [l_{1,c},\ l_{2,c},\ \dots\ l_{N,c}]'$.

$l_{i,c} \equiv$ the amount of aggregate labor services required to harvest an acre of forest of age class i.

$\overline{L}_t \equiv$ the labor force in period t.

$\overline{M} \equiv$ the supply of land.

$\Lambda \equiv$ the symbol for a Lagrangian function.

$\eta^s_{i,t} \equiv$ the price of gross output from industry i in period t.

$\eta^d_{i,t} \equiv$ the price of consumer good i in period t.

$\eta^k_{i,t} \equiv$ the price in period t of capital used in industry i.

$\eta^l_{i,t} \equiv$ the cost per unit of the aggregate labor input in period t for industry i.

$\eta^l_{h,t} \equiv$ the labor wage for labor from household h in period t.

$\eta^q_t \equiv$ the price of wood in period t.

$\eta^p_t \equiv$ the cost for a unit of planting services in period t.

$\eta^c_t \equiv$ the cost for a unit of harvesting (cutting) services in period t.

$\lambda_t \equiv [\lambda_{1,t}, \lambda_{2,t}, \ldots, \lambda_{N,t}]'$.

$\lambda_{i,t} \equiv$ the price of an acre of forest in age class i in period t. $\lambda_{1,t}$ is the price of bare land.

$\zeta_t \equiv [\zeta_{1,t}, \zeta_{2,t}, \ldots, \zeta_{N,t}]'$.

$\zeta_{i,t} \equiv$ the cost of postponing harvesting age class i in period t (not really a price).

$\pi_t \equiv$ the total factor returns for all assets: capital, land, and forests in period t.

$V_t \equiv$ the value of the total asset portfolio of capital, land, and forests in period t.

$s_{h,t} \equiv$ the share of the total asset portfolio belonging to household h in period t.

$I_{h,t} \equiv$ the value of new investment from household h in period t.

$\bar{I}_t \equiv$ the total investment necessary to replace depreciated capital.

$\pi^n_s \equiv$ the net income in the steady state from the total asset stock after depreciation.

$\eta^l_{h,s} \equiv$ the price of labor from the h^{th} household.

$L_{h,i} \equiv$ the input of labor from the h^{th} household in activity i.

$\eta \equiv$ a vector of prices for an economy.

$\varepsilon^i(\eta) \equiv$ an excess demand function for market i.

$d^i(\eta) \equiv$ a demand function for market i.

$s^i(\eta) \equiv$ a supply function for market i.

$\eta^l_{h,s} \equiv$ the price of labor from the h^{th} household in the steady state.

$L_{h,i,s} \equiv$ the input of labor from the h^{th} household in activity i in the steady state.

$\bar{x}_{i,s} \equiv$ the fixed amount of output from industry i in the steady state.

$Y^n_{h,s} \equiv$ the net (after savings) income of the h^{th} household.

$\theta_{j,i} \equiv$ production function parameters for the i^{th} activity.

$\eta_t^{i*} \equiv$ the average or marginal cost of activity i in the steady state.

$\lambda_{i,s}^{m} \equiv$ an upper bound on the value of an acre of forest in the i^{th} age class.

$\lambda_{i,s}^{l} \equiv$ a lower bound on the value of an acre of forest in the i^{th} age class.

$\beta_{i,h} \equiv$ the power on $X_{i,h,s}^{c}$ in the Cobb-Douglas utility function.

$\mathbf{X}_t^c \equiv [X_{1,t}^c,\ X_{2,t}^c,\ \ldots,\ X_{m,t}^c]'$.

$\mathbf{X}_t \equiv [X_{1,t},\ X_{2,t},\ \ldots,\ X_{m,t}]'$.

$\Delta\mathbf{K}_{j,t+1} \equiv [\Delta K_{1,t+1},\ \Delta K_{2,t+1},\ \ldots,\ \Delta K_{m,t+1}]'$.

$\Gamma \equiv$ the matrix of capital-production coefficients.

$\mathbf{A} \equiv$ the matrix of input-output coefficients.

$\alpha_{j,i} \equiv$ production function parameters for the i^{th} industry.

$\hat{\eta}_t^k \equiv$ the guess at the equilibrium price of wood on the k^{th} iteration.

$\mathbf{k}^i \equiv$ the i^{th} vertex of a simplex in the simplicial subdivision given by Scarf (1987b).

$D \equiv$ a large positive integer whose inverse determines the mesh size of the simplicial subdivision given in Scarf (1987b).

$\mathbf{e}^{i'} \equiv$ a vector in $\mathbf{R}^n$ of zeros excepting two consecutive elements which are 1 and -1.

$\phi_i \equiv$ a positive integer in the set $\{1, 2, \ldots, n\}$

Appendix B. GLOSSARY OF SELECTED TERMS

Adjoint Equations equations of motion for the adjoint variables in an optimal control model, derived from optimality conditions.

Adjoint Variables dual variables which reflect the cost, as defined by the objective function, imposed by the equations of motion as constraints on an optimal control problem; a generalization of Lagrangian multipliers in nonlinear programming.

Base Point a vertex of a simplex in a simplicial subdivision from which the other vertices of the simplex are located.

Basic Conservation Laws fundamental rules for the relationships between small changes in independent variables and a dependent variable; used in deriving differential equations.

Choice Variables variables that influence the state of a system and that are subject to control outside the system.

Commodity Space a subset of the positive orthant of $\mathbf{R}^n$ whose elements are potential consumption vectors.

Community Utility Function "a utility function which, when maximized subject to an aggregate budget constraint, produces the same market demand functions derived from individual utility maximization" (Katzner 1988, p 241).

Completely Labeled Simplex a simplex whose n vertices each have unique labels.

Constant Returns to Scale a property of a production function where a proportional increase or decrease in all inputs yields the same proportional increase or decrease in output; homogeneity of degree one.

Decision Variables variables that influence the state of a system and that are subject to control outside the system.

Dynamic in this dissertation, having the property of interdependence across periods; as in a dynamic problem, where the solution in one period depends on the solution in other (particularly future) periods.

Equations of Motion a mathematical description of the way the states of a system evolve over time.

Equilibrium Price Vector a nonzero price vector for which excess demand is nonpositive in each market and for which excess demand equals zero for those markets with positive prices (Scarf 1984b).

Excess Demand Functions a function giving the quantity demanded minus the quantity supplied in a given market as a function of (potentially all) prices.

Excess Profit Functions functions giving the difference between the price of the good and the marginal cost of producing it; note that excess profit functions may return positive or negative values.

Face a subset of a simplex that is a lower-dimensioned simplex defined by a proper subset of the vertices of the original simplex; alternatively, a subset of the elements of a simplex given by all the vectors for which $\alpha_j \equiv 0$ for j in some particular nonempty, proper subset of the integers 1, 2, ..., n.

Fixed Point a point which is mapped back onto itself by a function; i.e., a point $\hat{x}$ such that $\hat{x} = f(\hat{x})$ for some particular function f.

Forward Pass a procedure in dynamic programming where, once the optimal decision at each decision point is determined by a backward recursion through the time horizon of a problem, the optimal primal variable values are determined from the equations of motion and the optimal decisions for each state encountered.

Globally Assymptotically Stable a characteristic of a steady state of a dynamic systems where the all the trajectories of the system terminate at the steady state.

Hamiltonian a device for stating the optimality conditions for optimal control problems which incorporates Bellman's principle of optimality and the definition of the adjoint variables.

Household in this context, an aggregation of consumers with similar characteristics; especially with similar labor skills.

Integrability a property of a set of consumer demand functions where they can be simultaneously integrated to obtain a utility function with the standard neoclassical properties.

Market in this context, an institution of exchange where traders find the price at which the maximum quantity of a particular good will be voluntarily exchanged.

Maximum Principle a theorem giving optimality conditions for optimal control problems; usually expressed in terms of a Hamiltonian and equations of motion for adjoint variables.

Mesh a measure of the fineness of a simplicial subdivision given by some measure of the size of the largest simplex in the simplicial subdivision.

Normative in this context, a characteristic of a model where the model is constructed to describe things as they should be, given some norm or criterion for making such a judgement (contrast with "positive").

Pareto Optimality a state where no individual can be made better off without making another individual worse off.

Positive in this context, a characteristic of a model where the model is constructed to describe things as they actually are (contrast with "normative").

Principle of Optimality a theorem, due to Bellman (1957), which is the basis of dynamic programming; the theorem states that an optimal policy has the property that, whatever the initial decision is, the remaining decisions must constitute an optimal policy with regard to the state resulting from the initial decision; while not obvious, the theorem is trivially true. (Bellman's proof is only a few sentences long.)

Production Function a function giving the maximum output that can be obtained for a given set of inputs.

Recursion Relation an expression which is optimized in dynamic programming that gives the value of the objective function or of a dual variable as a function of the current decision, assuming that future decisions are made optimally.

Replacement Operation the operation of dropping one vertex of a simplex and replacing it with a new vertex which results in a new set of vertices defining a simplex adjacent to the original simplex.

Simplex the convex hull of n vertices $v^1, ..., v^n$; that is, the set of points of the form $x = \sum_j \alpha_j v_j$ with $\alpha_j \geq 0$ and $\sum_j \alpha_j = 1$; the vertices are assumed to be linearly independent in the sense that each vector in the simplex has one and only one representation in the above form; a simplex with n vertices has dimension $n - 1$ (Scarf 1984b, p.10).

Simplicial Subdivision a collection of simplices S^1, ..., S^k contained in a simplex S having the properties 1) S is contained in the union of the simplices $S^1, ..., S^k$, and 2) the intersection of any two simplices is either empty or a full face of both of them.

Social Welfare Function an objective function for an economy which, when maximized subject to the technology of the economy, yields an outcome that is pareto optimal and a socially optimal income distribution.

Stumpage wood sold "on the stump"; that is, before harvest.

State Variables variables that describe and quantify the state of a system at a particular time.

Time the system of sequential relations that any event has to any other, as past, present, or future; indefinite continuous duration regarded as that in which events succeed one another.

Transversality Condition boundary conditions derived from optimality considerations.

Turnpike Property a property of some models of optimal economic growth where the optimal path between two states spends a greater proportion of the time in the neighborhood of the optimal growth path as the time required (or allowed) for making the transition is increased.

Unit Simplex the simplex in $\mathbf{R}^n$ having n vertices given by the unit vectors in $\mathbf{R}^n$; Each vector in the unit simplex comprises nonnegative elements whose sum equals one.

Appendix C. PROGRAM FOR FINDING THE GENERAL EQUILIBRIUM STEADY STATE

```
C
C      This program finds general equilibrium prices
C       using a modification of Scarf's fixed point algorithm.
C
       REAL*8 VAR(39,5), AVG(39), RESO,
     ¤   WEIGHT(2,2), KCHEK(2), LCHEK(2,2), MCHEK(2),
     ¤   PLCHEK(2), HLCHEK(2), PLANCO, HARCO, LANDUZ, LABUZ(2),
     ¤   RATE, RHO, DELT, WORK1, WORK2, LAND, SHARE(2),
     ¤   ALPH01, B1C1, B2C1, ALPH02, B1C2, B2C2,
     ¤   ALPH1, B1K, B1L, B1L1, B1L2, B1M,
     ¤   ALPH2, B2K, B2L, B2L1, B2L2, B2M,
     ¤   ALPHP, BP1, BP2, ALPHC, BC1, BC2,
     ¤   A(2,3), GAM(2,2), VOL(6), LH(6), IAINV(2,2),
     ¤   LABRAT, LABLAN, TOL
       INTEGER I, J, NODE, LABEL(5), MXEPSI, ROTA(5), COUNT,
     ¤   STAGE
       DATA  RATE/0.17/ DELT/0.90/ WORK1/40.0/ WORK2/60.0/
     ¤   LAND/100.0/ SHARE/0.5,0.5/ TOL/0.00000001/ RESO/0.0001/
     ¤   ALPH01/1.0/ B1C1/0.45/ B2C1/0.35/
     ¤   ALPH02/1.0/ B1C2/0.50/ B2C2/0.30/
     ¤   ALPH1/1.5/ B1K/0.2/ B1L/0.4/ B1L1/0.6/ B1L2/0.4/ B1M/0.4/
     ¤   ALPH2/2.0/ B2K/0.5/ B2L/0.4/ B2L1/0.8/ B2L2/0.2/ B2M/0.1/
     ¤   ALPHP/10.0/ BP1/0.5/ BP2/0.5/
     ¤   ALPHC/1.0/ BC1/0.5/ BC2/0.5/
     ¤   A/0.2,0.3,0.3,0.1,0.1,0.5/
     ¤   GAM/0.3,0.25,0.7,0.75/
     ¤   VOL/0.0,0.1,2.0,5.0,7.0,7.5/
     ¤   LH/0.05,0.1,0.2,0.25,0.29,0.30/
       COMMON // VAR, MXEPSI, ROTA
     ¤   /DAT/ RATE, RHO, DELT, WORK1, WORK2,
     ¤   ALPH01, B1C1, B2C1, ALPH02, B1C2, B2C2,
     ¤   ALPH1, B1K, B1L, B1L1, B1L2, B1M,
     ¤   ALPH2, B2K, B2L, B2L1, B2L2, B2M,
     ¤   ALPHP, BP1, BP2, ALPHC, BC1, BC2,
     ¤   A, GAM, VOL, LH, IAINV
     ¤   /FAUST/ LABRAT, LABLAN, TOL, PLANCO, HARCO
     ¤   /KAP/ SHARE, LAND
C
C      Open a file for writing results to: PR.DAT.
C
       OPEN(UNIT=1,FILE='PR.DAT',STATUS='NEW',CARRIAGECONTROL='LIST')
C
       IAINV(1,1) = 10.0/7.0
       IAINV(2,1) = 10.0/21.0
       IAINV(1,2) = 10.0/21.0
       IAINV(2,2) = 80.0/63.0
       RHO = 1.0/(1.0+RATE)
       LABRAT = 0.0
       LABLAN = 0.0
       COUNT = 0
C
C      Initialize VAR, NODE, MXEPSI, ROTA, and LABEL
C
       NODE = 2
       MXEPSI = 3
       DO 5 I = 1, 5
         ROTA(I) = 9
 5     CONTINUE
```

```
      DO 20 I = 1, 39
        DO 10 J = 1, 5
          VAR(I,J) = 0.0
 10     CONTINUE
 20   CONTINUE
      DO 25 I = 1, 5
        LABEL(I) = 0
 25   CONTINUE
C
      VAR(1,NODE) = 0.92
      VAR(2,NODE) = 0.02
      VAR(3,NODE) = 0.02
      VAR(4,NODE) = 0.02
      VAR(5,NODE) = 0.02
      VAR(16,NODE) = 0.01
C
C     Assign a label.
C
 30     CALL LABELER(NODE,LABEL)
C
C     If the label is 2, we have a completely labeled
C       simplex for this stage, and we can go on to
C       stage 3.  Otherwise, we write an iteration
C       report, pivot, and continue.
C
      COUNT = COUNT + 1
      IF(LABEL(NODE) .EQ. 1) THEN
        LABEL(1) = 1
C
        CALL REPORT(COUNT,LABEL)
C
        DO 40 I = 1, 39
          VAR(I,1) = VAR(I,NODE)
 40     CONTINUE
        VAR(1,NODE) = VAR(1,NODE) - RESO
        VAR(2,NODE) = VAR(2,NODE) + RESO
        GO TO 30
      ELSE IF(LABEL(NODE) .EQ. 2) THEN
        NODE = 3
        STAGE = 3
C
C     Initialize node 3.
C
        VAR(1,NODE) = MIN(VAR(1,1),VAR(1,2))
        VAR(2,NODE) = MIN(VAR(2,1),VAR(2,2))
        VAR(3,NODE) = VAR(3,2) + RESO
        VAR(4,NODE) = VAR(4,2)
        VAR(5,NODE) = VAR(5,2)
        VAR(16,NODE) = 0.5*(VAR(16,1)+VAR(16,2))
C
C     Assign a label.
C
 50     CALL LABELER(NODE,LABEL)
C
C     If the label is 3, we have a completely labeled
C       simplex for this stage, and we can go on to
C       stage 4.  Otherwise, we write an iteration
C       report, pivot, and continue.
C
```

```
          COUNT = COUNT + 1
          IF ((LABEL(NODE) .EQ. 1) .OR. (LABEL(NODE) .EQ. 2)) THEN
C
            CALL REPORT(COUNT,LABEL)
C
            CALL CHANGER(NODE, LABEL, STAGE)
            GO TO 50
          ELSE IF(LABEL(NODE) .EQ. 3) THEN
            NODE = 4
            STAGE = 4
C
C       Initialize node 4.
C
            VAR(1,NODE) = MIN(VAR(1,1),VAR(1,2),VAR(1,3))
            VAR(2,NODE) = MIN(VAR(2,1),VAR(2,2),VAR(2,3))
            VAR(3,NODE) = MIN(VAR(3,1),VAR(3,2),VAR(3,3))
            VAR(4,NODE) = VAR(4,3) + RESO
            VAR(5,NODE) = VAR(5,3)
            VAR(16,NODE) = (VAR(16,1)+VAR(16,2)+VAR(16,3))/3.0
C
C       Assign a label.
C
 60         CALL LABELER(NODE,LABEL)
C
C       If the label is 4, we have a completely labeled
C         simplex for this stage, and we can go on to
C         stage 5.  Otherwise, we write an iteration
C         report, pivot, and continue.
C
            COUNT = COUNT + 1
            IF ((LABEL(NODE) .EQ. 1) .OR. (LABEL(NODE) .EQ. 2)
     x       .OR. (LABEL(NODE) .EQ. 3)) THEN
C
              CALL REPORT(COUNT,LABEL)
C
              CALL CHANGER(NODE, LABEL, STAGE)
              GO TO 60
            ELSE IF(LABEL(NODE) .EQ. 4) THEN
              NODE = 5
              STAGE = 5
C
C       Initialize node 5.
C
              VAR(1,NODE) = MIN(VAR(1,1),VAR(1,2),VAR(1,3),VAR(1,4))
              VAR(2,NODE) = MIN(VAR(2,1),VAR(2,2),VAR(2,3),VAR(2,4))
              VAR(3,NODE) = MIN(VAR(3,1),VAR(3,2),VAR(3,3),VAR(3,4))
              VAR(4,NODE) = MIN(VAR(4,1),VAR(4,2),VAR(4,3),VAR(4,4))
              VAR(5,NODE) = VAR(5,4) + RESO
              VAR(16,NODE) =
     x          (VAR(16,1)+VAR(16,2)+VAR(16,3)+VAR(16,4))/4.0
C
C       Assign a label.
C
 70           CALL LABELER(NODE,LABEL)
C
C       If the label is 5, we have a completely labeled
C         simplex for this stage, and we are done!
C         Otherwise, we write an iteration report,
C         pivot, and continue.
C
```

```
                 COUNT = COUNT + 1
                IF((LABEL(NODE) .GE. 1) .AND. (LABEL(NODE) .LE. 4))THEN
C
                 CALL REPORT(COUNT,LABEL)
C
                 CALL CHANGER(NODE, LABEL, STAGE)
                 GO TO 70
                ELSE IF (LABEL(NODE) .EQ. 5) THEN
                 GO TO 100
                END IF
              ELSE IF (LABEL(NODE) .EQ. 5) THEN
                GO TO 100
              END IF
            ELSE IF (LABEL(NODE) .GE. 4) THEN
              GO TO 100
            END IF
          ELSE IF(LABEL(NODE) .GE. 3) THEN
            GO TO 100
          END IF
 100      DO 110 I = 1, 39
            AVG(I) = (VAR(I,1)+VAR(I,2)+VAR(I,3)+VAR(I,4)+VAR(I,5))/5.0
 110      CONTINUE
C
C         Calculate the first-order conditions for the equivalent
C           maximization problem as a check on the results.
C
          WEIGHT(1,1) = AVG(1)*(1.0-RHO)/(ALPH01*B1C1
     #        *(AVG(23)**(B1C1-1.0))*(AVG(25)**B2C1))
          WEIGHT(1,2) = AVG(2)*(1.0-RHO)/(ALPH01*B2C1
     #        *(AVG(23)**(B1C1))*(AVG(25)**(B2C1-1.0)))
          WEIGHT(2,1) = AVG(1)*(1.0-RHO)/(ALPH02*B1C2
     #        *(AVG(24)**(B1C2-1.0))*(AVG(26)**B2C2))
          WEIGHT(2,2) = AVG(2)*(1.0-RHO)/(ALPH02*B2C2
     #        *(AVG(24)**(B1C2))*(AVG(26)**(B2C2-1.0)))
C
          KCHEK(1) = AVG(19)*ALPH1*B1K*(AVG(21)**(B1K-1.0))
     #        *(AVG(29)**(B1L*B1L1))*(AVG(30)**(B1L*B1L2))
     #        *(AVG(33)**(B1M)) - (1.0-DELT*RHO)*AVG(17)
          KCHEK(2) = AVG(20)*ALPH2*B2K*(AVG(22)**(B2K-1.0))
     #        *(AVG(31)**(B2L*B2L1))*(AVG(32)**(B2L*B2L2))
     #        *(AVG(34)**(B2M)) - (1.0-DELT*RHO)*AVG(18)
C
          LCHEK(1,1) = AVG(19)*ALPH1*B1L*B1L1*(AVG(21)**(B1K))
     #        *(AVG(29)**(B1L*B1L1-1.0))*(AVG(30)**(B1L*B1L2))
     #        *(AVG(33)**(B1M)) - AVG(3)
          LCHEK(1,2) = AVG(19)*ALPH1*B1L*B1L2*(AVG(21)**(B1K))
     #        *(AVG(29)**(B1L*B1L1))*(AVG(30)**(B1L*B1L2-1.0))
     #        *(AVG(33)**(B1M)) - AVG(4)
          LCHEK(2,1) = AVG(20)*ALPH2*B2L*B2L1*(AVG(22)**(B2K))
     #        *(AVG(31)**(B2L*B2L1-1.0))*(AVG(32)**(B2L*B2L2))
     #        *(AVG(34)**(B2M)) - AVG(3)
          LCHEK(2,2) = AVG(20)*ALPH2*B2L*B2L2*(AVG(22)**(B2K))
     #        *(AVG(31)**(B2L*B2L1))*(AVG(32)**(B2L*B2L2-1.0))
     #        *(AVG(34)**(B2M)) - AVG(4)
C
          MCHEK(1) = AVG(19)*ALPH1*B1M*(AVG(21)**(B1K))
     #        *(AVG(29)**(B1L*B1L1))*(AVG(30)**(B1L*B1L2))
     #        *(AVG(33)**(B1M-1.0)) - (1.0-RHO)*AVG(5)
```

```
      MCHEK(2) = AVG(20)*ALPH2*B2M*(AVG(22)**(B2K))
     ¤     *(AVG(31)**(B2L*B2L1))*(AVG(32)**(B2L*B2L2))
     ¤     *(AVG(34)**(B2M-1.0)) - (1.0-RHO)*AVG(5)
C
      PLCHEK(1) = PLANCO*ALPHP*BP1*(AVG(6)**(BP1-1.0))
     ¤     *(AVG(7)**(BP2)) - AVG(3)
      PLCHEK(2) = PLANCO*ALPHP*BP2*(AVG(6)**(BP1))
     ¤     *(AVG(7)**(BP2-1.0)) - AVG(4)
      HLCHEK(1) = HARCO*ALPHC*BC1*(AVG(8)**(BC1-1.0))
     ¤     *(AVG(9)**(BC2)) - AVG(3)
      HLCHEK(2) = HARCO*ALPHC*BC2*(AVG(8)**(BC1))
     ¤     *(AVG(9)**(BC2-1.0)) - AVG(4)
C
C     Total up the primary factor requirements.
C
      LANDUZ = AVG(33) + AVG(34) + AVG(35)
      LABUZ(1) = AVG(29) + AVG(31) + AVG(36) + AVG(38)
      LABUZ(2) = AVG(30) + AVG(32) + AVG(37) + AVG(39)
C
C     Print the results.
C
      WRITE(1,600) 'Variable List for Last Iteration.', COUNT
      WRITE(1,510) 'Label', LABEL(1), LABEL(2), LABEL(3),
     ¤          LABEL(4), LABEL(5)
      WRITE(1,500) 'N1d', VAR(1,1), VAR(1,2), VAR(1,3),
     ¤          VAR(1,4), VAR(1,5)
      WRITE(1,500) 'N2d', VAR(2,1), VAR(2,2), VAR(2,3),
     ¤          VAR(2,4), VAR(2,5)
      WRITE(1,500) 'N1L', VAR(3,1), VAR(3,2), VAR(3,3),
     ¤          VAR(3,4), VAR(3,5)
      WRITE(1,500) 'N2L', VAR(4,1), VAR(4,2), VAR(4,3),
     ¤          VAR(4,4), VAR(4,5)
      WRITE(1,500) 'Nm', VAR(5,1), VAR(5,2), VAR(5,3),
     ¤          VAR(5,4), VAR(5,5)
      WRITE(1,500) 'LP1', VAR(6,1), VAR(6,2), VAR(6,3),
     ¤          VAR(6,4), VAR(6,5)
      WRITE(1,500) 'LP2', VAR(7,1), VAR(7,2), VAR(7,3),
     ¤          VAR(7,4), VAR(7,5)
      WRITE(1,500) 'LC1', VAR(8,1), VAR(8,2), VAR(8,3),
     ¤          VAR(8,4), VAR(8,5)
      WRITE(1,500) 'LC2', VAR(9,1), VAR(9,2), VAR(9,3),
     ¤          VAR(9,4), VAR(9,5)
      WRITE(1,500) 'LA1', VAR(10,1), VAR(10,2), VAR(10,3),
     ¤          VAR(10,4), VAR(10,5)
      WRITE(1,500) 'LA2', VAR(11,1), VAR(11,2), VAR(11,3),
     ¤          VAR(11,4), VAR(11,5)
      WRITE(1,500) 'LA3', VAR(12,1), VAR(12,2), VAR(12,3),
     ¤          VAR(12,4), VAR(12,5)
      WRITE(1,500) 'LA4', VAR(13,1), VAR(13,2), VAR(13,3),
     ¤          VAR(13,4), VAR(13,5)
      WRITE(1,500) 'LA5', VAR(14,1), VAR(14,2), VAR(14,3),
     ¤          VAR(14,4), VAR(14,5)
      WRITE(1,500) 'LA6', VAR(15,1), VAR(15,2), VAR(15,3),
     ¤          VAR(15,4), VAR(15,5)
      WRITE(1,500) 'NQ', VAR(16,1), VAR(16,2), VAR(16,3),
     ¤          VAR(16,4), VAR(16,5)
      WRITE(1,500) 'N1k', VAR(17,1), VAR(17,2), VAR(17,3),
     ¤          VAR(17,4), VAR(17,5)
      WRITE(1,500) 'N2k', VAR(18,1), VAR(18,2), VAR(18,3),
     ¤          VAR(18,4), VAR(18,5)
```

```
      WRITE(1,500) 'N1s', VAR(19,1), VAR(19,2), VAR(19,3),
¤            VAR(19,4), VAR(19,5)
      WRITE(1,500) 'N2s', VAR(20,1), VAR(20,2), VAR(20,3),
¤            VAR(20,4), VAR(20,5)
      WRITE(1,500) 'K1', VAR(21,1), VAR(21,2), VAR(21,3),
¤            VAR(21,4), VAR(21,5)
      WRITE(1,500) 'K2', VAR(22,1), VAR(22,2), VAR(22,3),
¤            VAR(22,4), VAR(22,5)
      WRITE(1,500) 'X1C1', VAR(23,1), VAR(23,2), VAR(23,3),
¤            VAR(23,4), VAR(23,5)
      WRITE(1,500) 'X1C2', VAR(24,1), VAR(24,2), VAR(24,3),
¤            VAR(24,4), VAR(24,5)
      WRITE(1,500) 'X2C1', VAR(25,1), VAR(25,2), VAR(25,3),
¤            VAR(25,4), VAR(25,5)
      WRITE(1,500) 'X2C2', VAR(26,1), VAR(26,2), VAR(26,3),
¤            VAR(26,4), VAR(26,5)
      WRITE(1,500) 'X1s', VAR(27,1), VAR(27,2), VAR(27,3),
¤            VAR(27,4), VAR(27,5)
      WRITE(1,500) 'X2s', VAR(28,1), VAR(28,2), VAR(28,3),
¤            VAR(28,4), VAR(28,5)
      WRITE(1,500) 'L11', VAR(29,1), VAR(29,2), VAR(29,3),
¤            VAR(29,4), VAR(29,5)
      WRITE(1,500) 'L12', VAR(30,1), VAR(30,2), VAR(30,3),
¤            VAR(30,4), VAR(30,5)
      WRITE(1,500) 'L21', VAR(31,1), VAR(31,2), VAR(31,3),
¤            VAR(31,4), VAR(31,5)
      WRITE(1,500) 'L22', VAR(32,1), VAR(32,2), VAR(32,3),
¤            VAR(32,4), VAR(32,5)
      WRITE(1,500) 'M1', VAR(33,1), VAR(33,2), VAR(33,3),
¤            VAR(33,4), VAR(33,5)
      WRITE(1,500) 'M2', VAR(34,1), VAR(34,2), VAR(34,3),
¤            VAR(34,4), VAR(34,5)
      WRITE(1,500) 'Mf', VAR(35,1), VAR(35,2), VAR(35,3),
¤            VAR(35,4), VAR(35,5)
      WRITE(1,500) 'LP1', VAR(36,1), VAR(36,2), VAR(36,3),
¤            VAR(36,4), VAR(36,5)
      WRITE(1,500) 'LP2', VAR(37,1), VAR(37,2), VAR(37,3),
¤            VAR(37,4), VAR(37,5)
      WRITE(1,500) 'LC1', VAR(38,1), VAR(38,2), VAR(38,3),
¤            VAR(38,4), VAR(38,5)
      WRITE(1,500) 'LC2', VAR(39,1), VAR(39,2), VAR(39,3),
¤            VAR(39,4), VAR(39,5)
      WRITE(1,610) 'Average Values of Variables.'
      WRITE(1,400) 'N1d', AVG(1)
      WRITE(1,400) 'N2d', AVG(2)
      WRITE(1,400) 'N1L', AVG(3)
      WRITE(1,400) 'N2L', AVG(4)
      WRITE(1,400) 'Nm', AVG(5)
      WRITE(1,400) 'LP1', AVG(6)
      WRITE(1,400) 'LP2', AVG(7)
      WRITE(1,400) 'LC1', AVG(8)
      WRITE(1,400) 'LC2', AVG(9)
      WRITE(1,400) 'LA1', AVG(10)
      WRITE(1,400) 'LA2', AVG(11)
      WRITE(1,400) 'LA3', AVG(12)
      WRITE(1,400) 'LA4', AVG(13)
      WRITE(1,400) 'LA5', AVG(14)
      WRITE(1,400) 'LA6', AVG(15)
      WRITE(1,400) 'NQ', AVG(16)
      WRITE(1,400) 'N1k', AVG(17)
```

```
      WRITE(1,400) 'N2k', AVG(18)
      WRITE(1,400) 'N1s', AVG(19)
      WRITE(1,400) 'N2s', AVG(20)
      WRITE(1,400) 'K1', AVG(21)
      WRITE(1,400) 'K2', AVG(22)
      WRITE(1,400) 'X1C1', AVG(23)
      WRITE(1,400) 'X1C2', AVG(24)
      WRITE(1,400) 'X2C1', AVG(25)
      WRITE(1,400) 'X2C2', AVG(26)
      WRITE(1,400) 'X1s', AVG(27)
      WRITE(1,400) 'X2s', AVG(28)
      WRITE(1,400) 'L11', AVG(29)
      WRITE(1,400) 'L12', AVG(30)
      WRITE(1,400) 'L21', AVG(31)
      WRITE(1,400) 'L22', AVG(32)
      WRITE(1,400) 'M1', AVG(33)
      WRITE(1,400) 'M2', AVG(34)
      WRITE(1,400) 'Mf', AVG(35)
      WRITE(1,400) 'LP1', AVG(36)
      WRITE(1,400) 'LP2', AVG(37)
      WRITE(1,400) 'LC1', AVG(38)
      WRITE(1,400) 'LC2', AVG(39)
      WRITE(1,610) 'Optimality Check:'
      WRITE(1,730) 'Land use:', LANDUZ
      WRITE(1,710) '1', '2'
      WRITE(1,720) 'Labor use:', LABUZ(1), LABUZ(2)
      WRITE(1,720) 'Weight 1:', WEIGHT(1,1), WEIGHT(1,2)
      WRITE(1,720) 'Weight 2:', WEIGHT(2,1), WEIGHT(2,2)
      WRITE(1,700) 'Capital Check 1:', KCHEK(1)
      WRITE(1,700) 'Capital Check 2:', KCHEK(2)
      WRITE(1,720) 'Labor Check 1:', LCHEK(1,1), LCHEK(1,2)
      WRITE(1,720) 'Labor Check 2:', LCHEK(2,1), LCHEK(2,2)
      WRITE(1,700) 'Land Check 1:', MCHEK(1)
      WRITE(1,700) 'Land Check 2:', MCHEK(2)
      WRITE(1,720) 'Harvest Labor:', HLCHEK(1), HLCHEK(2)
      WRITE(1,720) 'Planting Labor:', PLCHEK(1), PLCHEK(2)
400   FORMAT (2X,A6,2X,F15.9)
500   FORMAT (2X,A6,2X,5(F11.5,1X))
510   FORMAT (2X,A6,2X,5(I4,8X))
600   FORMAT (2X,A,2X,I7)
610   FORMAT (2X,A)
700   FORMAT (2X,A,3X,F10.7)
710   FORMAT (20X,A,15X,A)
720   FORMAT (2X,A,5X,F10.7,3X,F10.7)
730   FORMAT (2X,A,3X,F12.7)
      END
C
```

```
      SUBROUTINE LABELER(NODE, LABEL)
      REAL*8 VAR(39,5), MINNSI, MAXNSI, EPSI(3),
     &   MAXEP, PI(2), MC(2), MAXPI,
     &   RATE, RHO, DELT, WORK1, WORK2,
     &   ALPH01, B1C1, B2C1, ALPH02, B1C2, B2C2,
     &   ALPH1, B1K, B1L, B1L1, B1L2, B1M,
     &   ALPH2, B2K, B2L, B2L1, B2L2, B2M,
     &   ALPHP, BP1, BP2, ALPHC, BC1, BC2,
     &   A(2,3), GAM(2,2), VOL(6), LH(6), IAINV(2,2)
      INTEGER I, J, NODE, LABEL(5), MXEPSI, ROTA(5)
      COMMON // VAR, MXEPSI, ROTA
     &   /DAT/ RATE, RHO, DELT, WORK1, WORK2,
     &   ALPH01, B1C1, B2C1, ALPH02, B1C2, B2C2,
     &   ALPH1, B1K, B1L, B1L1, B1L2, B1M,
     &   ALPH2, B2K, B2L, B2L1, B2L2, B2M,
     &   ALPHP, BP1, BP2, ALPHC, BC1, BC2,
     &   A, GAM, VOL, LH, IAINV
C
C     This subroutine assigns a label to a node.
C
      CALL FAUST(NODE)
C
C     FAUST calculates the price of wood (VAR(16,*)).
C
C     Calculate supply prices.
C
      VAR(19,NODE) = (1.0-A(1,1))*VAR(1,NODE)
     &                 - A(1,2)*VAR(2,NODE) - A(1,3)*VAR(16,NODE)
      VAR(20,NODE) = (1.0-A(2,2))*VAR(2,NODE)
     &                 - A(2,1)*VAR(1,NODE) - A(2,3)*VAR(16,NODE)
C
C     If any supply prices are negative, no need to go further.
C     Assign a label according to the rule:
C       i) If Max(nsi) le 0 , then label = Max(epsi).
C      ii) If Max(nsi) ge 0 , then label = Max(nsi).
C
      MINNSI = MIN(VAR(19,NODE),VAR(20,NODE))
      MAXNSI = MAX(VAR(19,NODE),VAR(20,NODE))
      IF (MINNSI .LE. 0.0) THEN
        IF (MAXNSI .GE. 0.0) THEN
          DO 10 I = 19, 20
            IF (MAXNSI .EQ. VAR(I,NODE)) THEN
               LABEL(NODE) = I - 18
            END IF
 10       CONTINUE
        ELSE
          LABEL(NODE) = MXEPSI
        END IF
        RETURN
      END IF
C
C     Calculate capital prices.
C
      VAR(17,NODE) = (GAM(1,1)*VAR(1,NODE)
     &                 + GAM(1,2)*VAR(2,NODE))/RHO
      VAR(18,NODE) = (GAM(2,1)*VAR(1,NODE)
     &                 + GAM(2,2)*VAR(2,NODE))/RHO
C
```

```
      CALL KAPBAL(NODE, EPSI, PI)
C
C     KAPBAL finds the capital for each sector that is
C       consistent with current prices (and therefore
C       income).
C
C     Now find the primary factor with the greatest
C       excess supply.
C
      MAXEP = MAX(EPSI(1), EPSI(2), EPSI(3))
      DO 20 I = 3, 1, -1
        IF (MAXEP .EQ. EPSI(I)) THEN
          MXEPSI = 2+I
        END IF
 20   CONTINUE
C
C     Assign a label:
C      i) Priority is given to consumption goods.
C           -If profits are positive for either good,
C             the label is assigned to the sector with
C             the greatest profits.
C     ii) Otherwise, the label is assigned to the primary
C           factor sector with the greatest excess supply.
C
      MAXPI = MAX(PI(1), PI(2))
      IF (MAXPI .GT. 0.0) THEN
        DO 30 I = 1, 2
          IF (MAXPI .EQ. PI(I)) THEN
            LABEL(NODE) = I
          END IF
 30     CONTINUE
      ELSE
        LABEL(NODE) = MXEPSI
      END IF
      RETURN
      END
C
```

```
      SUBROUTINE FAUST(NODE)
      REAL*8 VAR(39,5), PI(6), UP(6), LABRAT, LABLAN,
     ¤   TEMPRAT(2), PLANCO, HARCO, TOL, NQ, NQL, HOLD(7),
     ¤   PIMAX, CUVA(6), TEMP, LABKEEP, BLOWUP,
     ¤   RATE, RHO, DELT, WORK1, WORK2,
     ¤   ALPH01, B1C1, B2C1, ALPH02, B1C2, B2C2,
     ¤   ALPH1, B1K, B1L, B1L1, B1L2, B1M,
     ¤   ALPH2, B2K, B2L, B2L1, B2L2, B2M,
     ¤   ALPHP, BP1, BP2, ALPHC, BC1, BC2,
     ¤   A(2,3), GAM(2,2), VOL(6), LH(6), IAINV(2,2)
      INTEGER I, J, NODE, MXEPSI, ROTA(5), OLDROT
      COMMON // VAR, MXEPSI, ROTA
     ¤   /DAT/ RATE, RHO, DELT, WORK1, WORK2,
     ¤   ALPH01, B1C1, B2C1, ALPH02, B1C2, B2C2,
     ¤   ALPH1, B1K, B1L, B1L1, B1L2, B1M,
     ¤   ALPH2, B2K, B2L, B2L1, B2L2, B2M,
     ¤   ALPHP, BP1, BP2, ALPHC, BC1, BC2,
     ¤   A, GAM, VOL, LH, IAINV
     ¤   /FAUST/ LABRAT, LABLAN, TOL, PLANCO, HARCO
C
C     This subroutine calculates the price of wood: VAR(16,*).
C
C     If the price ratios are the same as the last pass, the
C      price of wood need not be recalculated.
C
      TEMPRAT(1) = VAR(4,NODE)/VAR(3,NODE)
      TEMPRAT(2) = VAR(5,NODE)/VAR(3,NODE)
      IF ((TEMPRAT(1) .NE. LABRAT)
     ¤  .OR. (TEMPRAT(2) .NE. LABLAN)) THEN
        LABRAT = TEMPRAT(1)
        LABLAN = TEMPRAT(2)
        LABKEEP = VAR(3,NODE)
C
C     Calculate marginal cost per acre of planting, marginal
C       cost of harvest services, and labor requirements for
C       planting and harvesting.
C
        VAR(6,NODE) = (1.0/ALPHP)*((BP1/VAR(3,NODE))**(BP2))
     ¤        *((VAR(4,NODE)/BP2)**(BP2))
        VAR(7,NODE) = (1.0/ALPHP)*((VAR(3,NODE)/BP1)**(BP1))
     ¤        *((BP2/VAR(4,NODE))**(BP1))
        PLANCO = VAR(3,NODE)*VAR(6,NODE) + VAR(4,NODE)*VAR(7,NODE)
        VAR(8,NODE) = ((BC1/VAR(3,NODE))**(BC2))
     ¤        *((VAR(4,NODE)/BC2)**(BC2))
        VAR(9,NODE) = ((VAR(3,NODE)/BC1)**(BC1))
     ¤        *((BC2/VAR(4,NODE))**(BC1))
        HARCO = VAR(3,NODE)*VAR(8,NODE) + VAR(4,NODE)*VAR(9,NODE)
C
C     Do upper curve.
C
      VAR(10,NODE) = VAR(5,NODE) + PLANCO
      VAR(11,NODE) = (1.0+RATE)*VAR(10,NODE)
      VAR(12,NODE) = (1.0+RATE)*VAR(11,NODE)
      VAR(13,NODE) = (1.0+RATE)*VAR(12,NODE)
      VAR(14,NODE) = (1.0+RATE)*VAR(13,NODE)
      VAR(15,NODE) = (1.0+RATE)*VAR(14,NODE)
      UP(1) = VAR(10,NODE) - RHO*VAR(5,NODE)
      UP(2) = VAR(11,NODE) - RHO*VAR(5,NODE)
      UP(3) = VAR(12,NODE) - RHO*VAR(5,NODE)
      UP(4) = VAR(13,NODE) - RHO*VAR(5,NODE)
```

```
      UP(5) = VAR(14,NODE) - RHO*VAR(5,NODE)
      UP(6) = VAR(15,NODE) - RHO*VAR(5,NODE)
C
C     Perturb initial wood price and calculate PIMAX.
C       We will need two points to estimate differences.
C       The estimated differences are used in a modified
C       Newton's method to find the price of wood.
C
      NQL = VAR(16,NODE) + 0.01
      DO 20 I = 1, 6
        CUVA(I) = NQL*VOL(I) - HARCO*LH(I)
        PI(I) = CUVA(I) - UP(I)
20    CONTINUE
      PIMAX = MAX(PI(1), PI(2), PI(3), PI(4), PI(5), PI(6))
C
C     Check to see if this price is, by chance, correct.
C       When PIMAX equals zero, the price is correct.
C
      IF ((PIMAX .GT. (-TOL)) .AND. (PIMAX .LT. TOL)) THEN
        DO 30 I = 1, 6
          IF (PIMAX .EQ. PI(I)) THEN
            ROTA(NODE) = I
      PRINT *, 'Rotation:', ROTA(NODE), 'First'
          END IF
30      CONTINUE
        IF (ROTA(NODE) .LT. 6) THEN
          DO 40 I = (ROTA(NODE)+1), 6
            VAR(9+I,NODE) = CUVA(I) + RHO*VAR(5,NODE)
40        CONTINUE
        END IF
        VAR(16,NODE) = NQL
        DO 45 I = 1, 6
          HOLD(I) = VAR(9+I,NODE)
45      CONTINUE
        HOLD(7) = VAR(16,NODE)
        OLDROT = ROTA(NODE)
        RETURN
      ELSE
        PIMAXL = PIMAX
      END IF
C
C     If the perturbed price wasn't correct, go back to
C       the original price and calculate PIMAX. Use
C       the modified Newton's method to update the
C       price until a close enough approximation is
C       found.
C
      NQ = VAR(16,NODE)
50    DO 60 I = 1, 6
        CUVA(I) = NQ*VOL(I) - HARCO*LH(I)
        PI(I) = CUVA(I) - UP(I)
60    CONTINUE
      PIMAX = MAX(PI(1), PI(2), PI(3), PI(4), PI(5), PI(6))
```

```
C
C       Check to see if this price is correct.
C
        IF ((PIMAX .GT. (-TOL)) .AND. (PIMAX .LT. TOL)) THEN
          DO 70 I = 1, 6
            IF (PIMAX .EQ. PI(I)) THEN
              ROTA(NODE) = I
        PRINT *, 'Rotation:', ROTA(NODE)
            END IF
 70       CONTINUE
          IF (ROTA(NODE) .LT. 6) THEN
            DO 80 I = (ROTA(NODE)+1), 6
              VAR(9+I,NODE) = CUVA(I) + RHO*VAR(5,NODE)
 80         CONTINUE
          END IF
          VAR(16,NODE) = NQ
          DO 85 I = 1, 6
            HOLD(I) = VAR(9+I,NODE)
 85       CONTINUE
          HOLD(7) = VAR(16,NODE)
          OLDROT = ROTA(NODE)
          RETURN
        ELSE
C
C       This is where we get a new price to try using
C         the modified Newton's method.
C
          TEMP = NQ
          NQ = NQ - ((PIMAX*(NQ-NQL))/(PIMAX-PIMAXL))
            IF (NQ .LT. 0.0) THEN
              PRINT *, 'ERROR: PRICE OF WOOD NEGATIVE'
            END IF
          NQL = TEMP
          PIMAXL = PIMAX
          GO TO 50
        END IF
        ELSE
          BLOWUP = VAR(3,NODE)/LABKEEP
          LABKEEP = VAR(3,NODE)
          DO 90 I = 10, 16
            VAR(I,NODE) = BLOWUP*HOLD(I-9)
 90       CONTINUE
          ROTA(NODE) = OLDROT
          DO 95 I = 1, 6
            HOLD(I) = VAR(9+I,NODE)
 95       CONTINUE
          HOLD(7) = VAR(16,NODE)
          RETURN
        END IF
        END
C
```

```
      SUBROUTINE KAPBAL(NODE, EPSI, PI)
      REAL*8 VAR(39,5), PI(2), EPSI(3), KINC, INC(2),
     &  WOOD, WPA, Z(10), DET(2), E(4), INV(4), B(4),
     &  DEMPROP(2,2), KAPROP(2), MPROP(2), SUM,
     &  SHARE(2), MC(2), LAND, KRENT(2), MRENT,
     &  RATE, RHO, DELT, WORK1, WORK2,
     &  ALPH01, B1C1, B2C1, ALPH02, B1C2, B2C2,
     &  ALPH1, B1K, B1L, B1L1, B1L2, B1M,
     &  ALPH2, B2K, B2L, B2L1, B2L2, B2M,
     &  ALPHP, BP1, BP2, ALPHC, BC1, BC2,
     &  A(2,3), GAM(2,2), VOL(6), LH(6), IAINV(2,2)
      INTEGER I, J, NODE, MXEPSI, ROTA(5)
      COMMON // VAR, MXEPSI, ROTA
     &  /DAT/ RATE, RHO, DELT, WORK1, WORK2,
     &  ALPH01, B1C1, B2C1, ALPH02, B1C2, B2C2,
     &  ALPH1, B1K, B1L, B1L1, B1L2, B1M,
     &  ALPH2, B2K, B2L, B2L1, B2L2, B2M,
     &  ALPHP, BP1, BP2, ALPHC, BC1, BC2,
     &  A, GAM, VOL, LH, IAINV
     &  /KAP/ SHARE, LAND
C
C     Preliminaries
C
      KRENT(1) = (1.0-DELT*RHO)*VAR(17,NODE)
      KRENT(2) = (1.0-DELT*RHO)*VAR(18,NODE)
      MRENT = (1.0-RHO)*VAR(5,NODE)
      DEMPROP(1,1) = B1C1/(VAR(1,NODE)*(B1C1+B2C1))
      DEMPROP(1,2) = B1C2/(VAR(1,NODE)*(B1C2+B2C2))
      DEMPROP(2,1) = B2C1/(VAR(2,NODE)*(B1C1+B2C1))
      DEMPROP(2,2) = B2C2/(VAR(2,NODE)*(B1C2+B2C2))
      KAPROP(1) = (1.0/ALPH1)*((KRENT(1)/B1K)**(B1K-1.0))
     &     *((VAR(3,NODE)/(B1L*B1L1))**(B1L*B1L1))
     &     *((VAR(4,NODE)/(B1L*B1L2))**(B1L*B1L2))
     &     *((MRENT/(B1M))**(B1M))
      KAPROP(2) = (1.0/ALPH2)*((KRENT(2)/B2K)**(B2K-1.0))
     &     *((VAR(3,NODE)/(B2L*B2L1))**(B2L*B2L1))
     &     *((VAR(4,NODE)/(B2L*B2L2))**(B2L*B2L2))
     &     *((MRENT/(B2M))**(B2M))
      MPROP(1) = (1.0/ALPH1)*((KRENT(1)/B1K)**(B1K))
     &     *((VAR(3,NODE)/(B1L*B1L1))**(B1L*B1L1))
     &     *((VAR(4,NODE)/(B1L*B1L2))**(B1L*B1L2))
     &     *((MRENT/(B1M))**(B1M-1.0))
      MPROP(2) = (1.0/ALPH2)*((KRENT(2)/B2K)**(B2K))
     &     *((VAR(3,NODE)/(B2L*B2L1))**(B2L*B2L1))
     &     *((VAR(4,NODE)/(B2L*B2L2))**(B2L*B2L2))
     &     *((MRENT/(B2M))**(B2M-1.0))
      WPA = VOL(ROTA(NODE))/ROTA(NODE)
      SUM = VAR(5,NODE)
      DO 10 I = 2, ROTA(NODE)
        SUM = SUM + VAR(9+I,NODE)
 10   CONTINUE
      SUM = SUM/ROTA(NODE)
C
```

```
C        Here, we try to solve for the appropriate amount of capital
C          directly:
C                       K = -((d((I-A)Inv)B-I)Inv)d((I-A)Inv)c
C
C        Calculate z's.
C
         Z(1)  =(1.0-DELT)*(IAINV(1,1)*GAM(1,1)+IAINV(1,2)*GAM(2,1))
         Z(2)  =(1.0-DELT)*(IAINV(1,1)*GAM(1,2)+IAINV(1,2)*GAM(2,2))
         Z(5)  =(1.0-DELT)*(IAINV(2,1)*GAM(1,1)+IAINV(2,2)*GAM(2,1))
         Z(6)  =(1.0-DELT)*(IAINV(2,1)*GAM(1,2)+IAINV(2,2)*GAM(2,2))
         Z(3)  = (IAINV(1,1)*DEMPROP(1,1)+IAINV(1,2)*DEMPROP(2,1))
     ¤         *VAR(3,NODE)*WORK1
     ¤         + (IAINV(1,1)*DEMPROP(1,2)+IAINV(1,2)*DEMPROP(2,2))
     ¤         *VAR(4,NODE)*WORK2
         Z(7)  = (IAINV(2,1)*DEMPROP(1,1)+IAINV(2,2)*DEMPROP(2,1))
     ¤         *VAR(3,NODE)*WORK1
     ¤         + (IAINV(2,1)*DEMPROP(1,2)+IAINV(2,2)*DEMPROP(2,2))
     ¤         *VAR(4,NODE)*WORK2
         Z(4)  = (IAINV(1,1)*DEMPROP(1,1)+IAINV(1,2)*DEMPROP(2,1))
     ¤         *SHARE(1)
     ¤         + (IAINV(1,1)*DEMPROP(1,2)+IAINV(1,2)*DEMPROP(2,2))
     ¤         *SHARE(2)
         Z(8)  = (IAINV(2,1)*DEMPROP(1,1)+IAINV(2,2)*DEMPROP(2,1))
     ¤         *SHARE(1)
     ¤         + (IAINV(2,1)*DEMPROP(1,2)+IAINV(2,2)*DEMPROP(2,2))
     ¤         *SHARE(2)
         Z(9)   = (((SUM*A(1,3))/WPA)+VAR(5,NODE)*MPROP(1))*(1.0-RHO)
         Z(10)  = (((SUM*A(2,3))/WPA)+VAR(5,NODE)*MPROP(2))*(1.0-RHO)
C
C        Do d((I-A) Inverse): E(1) E(2)
C                             E(3) E(4)
C
         DET(1) = 1.0 - Z(4)*Z(9) - Z(8)*Z(10)
         E(1) = ((1.0 - Z(8)*Z(10))/DET(1))*KAPROP(1)
         E(2) = ((Z(4)*Z(10))/DET(1))*KAPROP(1)
         E(3) = ((Z(8)*Z(9))/DET(1))*KAPROP(2)
         E(4) = ((1.0 - Z(4)*Z(9))/DET(1))*KAPROP(2)
C
C        Do B: B(1) B(2)
C              B(3) B(4)
C
         B(1) = DELT*(1.0-RHO)*VAR(17,NODE)*Z(4)  + Z(1)
         B(2) = DELT*(1.0-RHO)*VAR(18,NODE)*Z(4)  + Z(2)
         B(3) = DELT*(1.0-RHO)*VAR(17,NODE)*Z(8)  + Z(5)
         B(4) = DELT*(1.0-RHO)*VAR(18,NODE)*Z(8)  + Z(6)
C
C        Do (EB-I) Inverse: INV(1) INV(2)
C                           INV(3) INV(4)
C
         DET(2) = (E(1)*B(1)+E(2)*B(3)-1.0)*(E(3)*B(2)+E(4)*B(4)-1.0)
     ¤        - (E(3)*B(1)+E(4)*B(3))*(E(1)*B(2)+E(2)*B(4))
         INV(1) = (E(3)*B(2)+E(4)*B(4)-1.0)/DET(2)
         INV(2) = -(E(1)*B(2)+E(2)*B(4))/DET(2)
         INV(3) = -(E(3)*B(1)+E(4)*B(3))/DET(2)
         INV(4) = (E(1)*B(1)+E(2)*B(3)-1.0)/DET(2)
C
```

```
C       Now, we get the right amount of capital.
C
        VAR(21,NODE) = - Z(3)*(INV(1)*E(1)+INV(2)*E(3))
     ¤          - Z(7)*(INV(1)*E(2)+INV(2)*E(4))
        VAR(22,NODE) = - Z(3)*(INV(3)*E(1)+INV(4)*E(3))
     ¤          - Z(7)*(INV(3)*E(2)+INV(4)*E(4))
C
C       Now, we can calculate everything else: Gross production.
C
        VAR(27,NODE) = VAR(21,NODE)/KAPROP(1)
        VAR(28,NODE) = VAR(22,NODE)/KAPROP(2)
C
C       Wood, forest, and land factor demands.
C
        VAR(33,NODE) = VAR(27,NODE)*MPROP(1)
        VAR(34,NODE) = VAR(28,NODE)*MPROP(2)
        WOOD = A(1,3)*VAR(27,NODE) + A(2,3)*VAR(28,NODE)
        VAR(35,NODE) = WOOD/WPA
C
C       Capital income.
C
          KINC = (1.0-RHO)*((SUM*VAR(35,NODE)
     ¤      + VAR(5,NODE)*(VAR(33,NODE)+VAR(34,NODE)))
     ¤      + (VAR(17,NODE)*VAR(21,NODE) + VAR(18,NODE)*VAR(22,NODE))
     ¤       *DELT)
C
C       Income.
C
        INC(1) = VAR(3,NODE)*WORK1 + SHARE(1)*KINC
        INC(2) = VAR(4,NODE)*WORK2 + SHARE(2)*KINC
C
C       Consumption demands.
C
        VAR(23,NODE) = INC(1)*DEMPROP(1,1)
        VAR(24,NODE) = INC(2)*DEMPROP(1,2)
        VAR(25,NODE) = INC(1)*DEMPROP(2,1)
        VAR(26,NODE) = INC(2)*DEMPROP(2,2)
C
C       Labor demands.
C
          VAR(29,NODE) = (VAR(27,NODE)/ALPH1)
     ¤      *((KRENT(1)/B1K)**(B1K))
     ¤      *((VAR(3,NODE)/(B1L*B1L1))**(B1L*B1L1-1.0))
     ¤      *((VAR(4,NODE)/(B1L*B1L2))**(B1L*B1L2))
     ¤      *((MRENT/(B1M))**(B1M))
          VAR(30,NODE) = (VAR(27,NODE)/ALPH1)
     ¤      *((KRENT(1)/B1K)**(B1K))
     ¤      *((VAR(3,NODE)/(B1L*B1L1))**(B1L*B1L1))
     ¤      *((VAR(4,NODE)/(B1L*B1L2))**(B1L*B1L2-1.0))
     ¤      *((MRENT/(B1M))**(B1M))
          VAR(31,NODE) = (VAR(28,NODE)/ALPH2)
     ¤      *((KRENT(2)/B2K)**(B2K))
     ¤      *((VAR(3,NODE)/(B2L*B2L1))**(B2L*B2L1-1.0))
     ¤      *((VAR(4,NODE)/(B2L*B2L2))**(B2L*B2L2))
     ¤      *((MRENT/(B2M))**(B2M))
          VAR(32,NODE) = (VAR(28,NODE)/ALPH2)
     ¤      *((KRENT(2)/B2K)**(B2K))
     ¤      *((VAR(3,NODE)/(B2L*B2L1))**(B2L*B2L1))
     ¤      *((VAR(4,NODE)/(B2L*B2L2))**(B2L*B2L2-1.0))
     ¤      *((MRENT/(B2M))**(B2M))
```

```
C       Marginal costs of production.
C
          MC(1) = (KRENT(1)*VAR(21,NODE)
     &        + VAR(3,NODE)*VAR(29,NODE) + VAR(4,NODE)*VAR(30,NODE)
     &        + MRENT*VAR(33,NODE))/VAR(27,NODE)
          MC(2) = (KRENT(2)*VAR(22,NODE)
     &        + VAR(3,NODE)*VAR(31,NODE) + VAR(4,NODE)*VAR(32,NODE)
     &        + MRENT*VAR(34,NODE))/VAR(28,NODE)
C
C       Convert planting and harvesting labor quantities to
C         totals, rather than per-acre quantities.
C
          VAR(36,NODE) = VAR(6,NODE)*VAR(35,NODE)/ROTA(NODE)
          VAR(37,NODE) = VAR(7,NODE)*VAR(35,NODE)/ROTA(NODE)
          VAR(38,NODE) = VAR(8,NODE)*VAR(35,NODE)
     &        *LH(ROTA(NODE))/ROTA(NODE)
          VAR(39,NODE) = VAR(9,NODE)*VAR(35,NODE)
     &        *LH(ROTA(NODE))/ROTA(NODE)
C
C       Calculate excess profits and excess supplies.
C
          PI(1) = VAR(19,NODE) - MC(1)
          PI(2) = VAR(20,NODE) - MC(2)
          EPSI(1) = WORK1 - VAR(29,NODE) - VAR(31,NODE)
     &        - VAR(36,NODE) - VAR(38,NODE)
          EPSI(2) = WORK2 - VAR(30,NODE) - VAR(32,NODE)
     &        - VAR(37,NODE) - VAR(39,NODE)
          EPSI(3) = LAND - VAR(33,NODE) - VAR(34,NODE) - VAR(35,NODE)
          RETURN
        END
```

```
      SUBROUTINE CHANGER(NODE, LABEL, STAGE)
      REAL*8 VAR(39,5)
      INTEGER I, J, K, NODE, LABEL(5), STAGE, OLDLAB,
     a  OLDNODE, MXEPSI, ROTA(5)
      COMMON // VAR, MXEPSI, ROTA
C
C     This subroutine implements a pivot operation.
C       The node with the same label as the last label
C       returned is eliminated and a new node is added.
C
C     Choose node to change.
C
      OLDLAB = LABEL(NODE)
      OLDNODE = NODE
      K = STAGE - 1
      DO 10 I = 1, K
        J = MOD((OLDNODE+I-1+STAGE),STAGE)+1
        IF (LABEL(J) .EQ. OLDLAB) THEN
          NODE = J
        END IF
 10   CONTINUE
C
C     Change node chosen.
C
      K = MOD((NODE+STAGE),STAGE)+1
      J = MOD((NODE-2+STAGE),STAGE)+1
      DO 20 I = 1, 5
        VAR(I,NODE) = VAR(I,J) + VAR(I,K) - VAR(I,NODE)
 20   CONTINUE
      VAR(16,NODE) = VAR(16,J) + VAR(16,K) - VAR(16,NODE)
      RETURN
      END
```

```
      SUBROUTINE REPORT(COUNT,LABEL)
      REAL*8 VAR(39,5)
      INTEGER LABEL(5), MXEPSI, ROTA(5), COUNT
      COMMON // VAR, MXEPSI, ROTA
C
C     WRITE(1,600) 'Prices for Iteration:', COUNT
C     WRITE(1,510) 'Label', LABEL(1), LABEL(2), LABEL(3),
C    ¤           LABEL(4), LABEL(5)
C     WRITE(1,500) 'N1d', VAR(1,1), VAR(1,2), VAR(1,3),
C    ¤           VAR(1,4), VAR(1,5)
C     WRITE(1,500) 'N2d', VAR(2,1), VAR(2,2), VAR(2,3),
C    ¤           VAR(2,4), VAR(2,5)
C     WRITE(1,500) 'N1L', VAR(3,1), VAR(3,2), VAR(3,3),
C    ¤           VAR(3,4), VAR(3,5)
C     WRITE(1,500) 'N2L', VAR(4,1), VAR(4,2), VAR(4,3),
C    ¤           VAR(4,4), VAR(4,5)
C     WRITE(1,500) 'Nm', VAR(5,1), VAR(5,2), VAR(5,3),
C    ¤           VAR(5,4), VAR(5,5)
C     WRITE(1,500) 'NQ', VAR(16,1), VAR(16,2), VAR(16,3),
C    ¤           VAR(16,4), VAR(16,5)
C     WRITE(1,500) 'X1s', VAR(27,1), VAR(27,2), VAR(27,3),
C    ¤           VAR(27,4), VAR(27,5)
C     WRITE(1,500) 'X2s', VAR(28,1), VAR(28,2), VAR(28,3),
C    ¤           VAR(28,4), VAR(28,5)
 500  FORMAT (2X,A6,2X,5(F11.5,1X))
 510  FORMAT (2X,A6,2X,5(I4,8X))
 600  FORMAT (2X,A,2X,I4)
      RETURN
      END
```

Variable List for Last Iteration. 48062

Label	2	3	1	5	4
N1d	0.28140	0.28140	0.28150	0.28140	0.28140
N2d	0.24750	0.24740	0.24740	0.24740	0.24740
N1L	0.09620	0.09630	0.09620	0.09620	0.09620
N2L	0.03130	0.03130	0.03130	0.03130	0.03140
Nm	0.34360	0.34360	0.34360	0.34370	0.34360
LP1	0.05704	0.05701	0.05704	0.05704	0.05713
LP2	0.17531	0.17540	0.17531	0.17531	0.17503
LC1	0.57041	0.57011	0.57041	0.57041	0.57132
LC2	1.75314	1.75405	1.75314	1.75314	1.75034
LA1	0.35457	0.35458	0.35457	0.35467	0.35459
LA2	0.41485	0.41486	0.41485	0.41497	0.41487
LA3	0.48538	0.48539	0.48538	0.48551	0.48540
LA4	0.56789	0.56790	0.56789	0.56805	0.56792
LA5	0.66443	0.66444	0.66443	0.66462	0.66447
LA6	0.69209	0.69210	0.69209	0.69229	0.69213
NQ	0.05751	0.05752	0.05751	0.05753	0.05752
N1k	0.30147	0.30139	0.30143	0.30139	0.30139
N2k	0.29949	0.29940	0.29943	0.29940	0.29940
N1s	0.14512	0.14515	0.14523	0.14515	0.14515
N2s	0.10957	0.10948	0.10945	0.10948	0.10948
K1	24.22745	24.27273	24.22472	24.24729	24.29213
K2	38.28732	38.36629	38.29419	38.31906	38.38228
X1C1	15.58280	15.60369	15.57611	15.58765	15.60082
X1C2	12.93878	12.95311	12.93290	12.94416	12.97213
X2C1	13.78002	13.80407	13.78457	13.78988	13.80154
X2C2	8.82660	8.83995	8.82929	8.83384	8.85293
X1s	58.03305	58.11437	58.01931	58.06120	58.14565
X2s	48.32642	48.40291	48.33037	48.35799	48.42682
L11	21.02527	21.03697	21.01964	21.03677	21.07567
L12	43.08053	43.14931	43.06899	43.10410	43.04628
L21	17.60440	17.61723	17.60412	17.61383	17.64289
L22	13.52670	13.55063	13.52649	13.53395	13.51310
M1	67.52260	67.63040	67.50452	67.53988	67.68447
M2	10.60059	10.61934	10.60042	10.60318	10.62377
Mf	21.40465	21.43778	21.40508	21.41794	21.44855
LP1	0.24419	0.24444	0.24419	0.24434	0.24508
LP2	0.75050	0.75206	0.75052	0.75097	0.75085
LC1	0.70814	0.70887	0.70816	0.70858	0.71073
LC2	2.17646	2.18097	2.17651	2.17782	2.17745

Average Values of Variables.

N1d	0.281420000
N2d	0.247420000
N1L	0.096220000
N2L	0.031320000
Nm	0.343620000
LP1	0.057052956
LP2	0.175275845
LC1	0.570529565
LC2	1.752758451
LA1	0.354599269
LA2	0.414881145
LA3	0.485410940
LA4	0.567930800
LA5	0.664479035
LA6	0.692140152
NQ	0.057518087
N1k	0.301415400
N2k	0.299426400
N1s	0.145158191
N2s	0.109492957
K1	24.252865980
K2	38.329827837
X1C1	15.590213464
X1C2	12.948214479
X2C1	13.792015958
X2C2	8.836520596
X1s	58.074715842
X2s	48.368900855
L11	21.038863316
L12	43.089840290
L21	17.616492423
L22	13.530174657
M1	67.576374287
M2	10.609461499
Mf	21.422801437
LP1	0.244446908
LP2	0.750979692
LC1	0.708896032
LC2	2.177841107

```
Optimality Check:
Land use:       99.6086372
                     1              2
Labor use:       39.6086987     59.5488357
Weight 1:        0.1642875      0.1642875
Weight 2:        0.1530624      0.1530624
Capital Check 1:    -0.0000397
Capital Check 2:    -0.0000131
Labor Check 1:      -0.0000549     -0.0000179
Labor Check 2:      -0.0000182     -0.0000060
Land Check 1:    -0.0000285
Land Check 2:    -0.0000095
Harvest Labor:      -0.0000407     -0.0000133
Planting Labor:      -0.0000407     -0.0000133
```

Appendix D. DERIVATION OF THE EQUILIBRIUM CAPITAL STOCK

This appendix describes the derivation of an expression for the capital stocks that are consistent with a price vector in the general equilibrium steady-state problem. It is assumed that the utility and production functions are Cobb-Douglas functions exhibiting constant returns to scale. This is done by combining Equations 6.11, 6.12, 6.14 - 6.16, and 6.20 - 6.21. The result is a simultaneous solution of all these equations. The capital stocks given by these equations are those demanded in each sector, given prices and the quantity of output demanded. The quantity of output demanded depends on income, which depends on the capital stocks, hence the need for a simultaneous solution. The results are derived for a two-household, two-sector problem, but the argument generalizes naturally to higher-dimensioned problems.

Consider first the required gross output levels, given by Equation 6.16:

$$\mathbf{X}_s = (\mathbf{I} - \mathbf{A})^{-1}[\mathbf{X}_s^c + (1 - \delta)\Gamma\mathbf{K}_s]\,. \qquad D.1$$

or

$$\mathbf{X}_s = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{X}_s^c + (1 - \delta)(\mathbf{I} - \mathbf{A})^{-1}\mathbf{\Gamma}\mathbf{K}_s \qquad D.1'$$

The quantities of each good consumed by each household is given by the demand functions (Equation 6.14):

$$X_{1,1,s}^c = \frac{Y_1^n}{\eta_{1,s}^d} \frac{\beta_{1,1}}{\sum_{j=1}^{m} \beta_{j,1}} \qquad D.2.a$$

$$X_{1,2,s}^c = \frac{Y_2^n}{\eta_{1,s}^d} \frac{\beta_{1,2}}{\sum_{j=1}^{m} \beta_{j,2}} \qquad D.2.b$$

$$X_{2,1,s}^c = \frac{Y_1^n}{\eta_{2,s}^d} \frac{\beta_{2,1}}{\sum_{j=1}^{m} \beta_{j,1}} \qquad D.2.c$$

$$X_{2,2,s}^c = \frac{Y_2^n}{\eta_{2,s}^d} \frac{\beta_{2,2}}{\sum_{j=1}^{m} \beta_{j,2}} \qquad D.2.d$$

Summing over households, gives the total consumption of each good (Equation 6.15):

$$X_{i,s}^c = \sum_{h=1}^{H} X_{i,h,s}^c. \qquad D.3$$

Now define the matrix

$$\mathbf{D} = \begin{bmatrix} \dfrac{\beta_{1,1}}{(\eta^d_{1,s}\sum_{j=1}^{2}\beta_{j,1})} & \dfrac{\beta_{1,2}}{(\eta^d_{1,s}\sum_{j=1}^{2}\beta_{j,2})} \\ \dfrac{\beta_{2,1}}{(\eta^d_{2,s}\sum_{j=1}^{2}\beta_{j,1})} & \dfrac{\beta_{2,2}}{(\eta^d_{2,s}\sum_{j=1}^{2}\beta_{j,2})} \end{bmatrix}$$

Equations D.2.a-d and D.3 can be combined in the matrix equation:

$$\mathbf{X}^c_s = \mathbf{D}\mathbf{Y}^n_s \,. \qquad D.4$$

Combining this with Equation D.1′, yields:

$$\mathbf{X}_s = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D}\mathbf{Y}^n_s + (1 - \delta)(\mathbf{I} - \mathbf{A})^{-1}\Gamma\mathbf{K}_s \,. \qquad D.5$$

Now, consider net income. From Equation 6.11,

$$Y^n_{h,s} = \eta^l_{h,s}\overline{L}_{h,s} + s_{h,s}\pi^n_s \,. \qquad D.6$$

Define

$$\mathbf{L} = \begin{bmatrix} \eta^l_{1,s}\overline{L}_{1,s} \\ \eta^l_{2,s}\overline{L}_{2,s} \end{bmatrix}$$

and

$$\mathbf{S} = \begin{bmatrix} s_{1,s} \\ s_{2,s} \end{bmatrix}.$$

Now, in matrix form, Equation D.6 becomes

$$\mathbf{Y}_s^n = \mathbf{L} + \pi_s^n \mathbf{S} . \qquad D.6'$$

Combining this with Equation D.5, we have:

$$\mathbf{X}_s = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{DL} + \pi_s^n(\mathbf{I} - \mathbf{A})^{-1}\mathbf{DS} + (1 - \delta)(\mathbf{I} - \mathbf{A})^{-1}\Gamma\mathbf{K}_s . \qquad D.7$$

Net asset income is given by Equation 6.12:

$$\pi_s^n = (1 - \rho_s)[.\lambda_s'\mathbf{M}_{F,s} + \lambda_{1,s}\sum_{l=1}^{m} M_{l,s} + \delta\sum_{l=1}^{m}\eta_{l,s}^k K_{l,s}] \qquad D.8$$

Since the forest is regulated in the steady state with equal acreage in each age class,

$$\lambda_s'\mathbf{M}_{F,s} = c_R\sum_{l=1}^{R}\lambda_{l,s} . \qquad D.9$$

Also,

$$\sum_{l=1}^{R}\lambda_{1,s} = \lambda_{l,s} + (\lambda_{1,s} + \eta_s^p)\sum_{l=1}^{R-1}\rho_s^{-l}. \qquad D.10$$

Note that these prices and the interest rate are all given. Define

$$\eta^f = \sum_{l=1}^{R}\lambda_{l,s}$$

and

$$\mathbf{R}_K{}' = \left[(1-\rho_s)\delta\eta_{1,s}^{k},\ (1-\rho_s)\delta\eta_{2,s}^{k} \right].$$

Now, Equation D.8 can be simplified to:

$$\pi_s^n = (1-\rho_s)[\eta^f c_r + + \lambda_{1,s}\sum_{i=1}^{m} M_{i,s}] + \mathbf{R}_K{}'\mathbf{K}_s\,. \qquad D.8'$$

The demand for land in each industry is given by Equation 6.21:

$$M_{i,s} = \frac{\overline{X}_{i,s}}{\alpha_{i,0}}\left(\frac{\eta_{1,s}^{l}}{\alpha_{i,1}}\right)^{\alpha_{i,1}}\left(\frac{\eta_{2,s}^{l}}{\alpha_{i,2}}\right)^{\alpha_{i,2}}\left(\frac{\eta_{i,s}^{k}}{\alpha_{i,k}}\right)^{\alpha_{i,k}}\left(\frac{\eta_{i,s}^{m}}{\alpha_{i,m}}\right)^{\alpha_{i,m}-1}.$$

Let

$$\mathbf{M} = \begin{bmatrix} \frac{1}{\alpha_{1,0}}\left(\frac{\eta_{1,s}^{l}}{\alpha_{1,1}}\right)^{\alpha_{1,1}}\left(\frac{\eta_{2,s}^{l}}{\alpha_{1,2}}\right)^{\alpha_{1,2}}\left(\frac{\eta_{1,s}^{k}}{\alpha_{1,k}}\right)^{\alpha_{1,k}}\left(\frac{\eta_{1,s}^{m}}{\alpha_{1,m}}\right)^{\alpha_{1,m}-1} \\ \frac{1}{\alpha_{2,0}}\left(\frac{\eta_{1,s}^{l}}{\alpha_{2,1}}\right)^{\alpha_{2,1}}\left(\frac{\eta_{2,s}^{l}}{\alpha_{2,2}}\right)^{\alpha_{2,2}}\left(\frac{\eta_{2,s}^{k}}{\alpha_{2,k}}\right)^{\alpha_{2,k}}\left(\frac{\eta_{2,s}^{m}}{\alpha_{2,m}}\right)^{\alpha_{2,m}-1} \end{bmatrix}.$$

Thus,

$$\sum_{i=1}^{m} M_{i,s} = \mathbf{M}'\mathbf{X}_s. \qquad D.12$$

The total acres that must be cut each year is given by Equations 6.22 and 6.22′ (also note that $M_{j,F,s} = c_{R,s}$ in the steady state):

$$c_{R,s} = \frac{\sum_{i=1}^{m} a_i^q X_{i,s}}{q_R}. \qquad D.13$$

Let

$$\mathbf{Q}' = \left[\frac{a_1^q}{q_R}, \frac{a_2^q}{q_R} \right].$$

Thus,

$$c_R = \mathbf{Q}'\mathbf{X}_s. \qquad D.13'$$

Now, Equation D.8′ can be written as:

$$\pi_s^n = (1 - \rho_s)[\eta^f \mathbf{Q}' + + \lambda_{1,s}\mathbf{M}']\mathbf{X}_s + \mathbf{R}_K{}'\mathbf{K}_s. \qquad D.8''$$

Let

$$\mathbf{R}_M{}' = (1 - \rho_s)[\eta^f \mathbf{Q}' + \lambda_{1,s}\mathbf{M}'].$$

Thus,

$$\pi_s^n = \mathbf{R}_M{}'\mathbf{X}_s + \mathbf{R}_K{}'\mathbf{K}_s. \qquad D.8'''$$

Combining Equations D.7 and D.8‴, yields

$$\mathbf{X}_s = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{DL} + (\mathbf{I} - \mathbf{A})^{-1}\mathbf{DSR}_M{}'\mathbf{X}_s + (\mathbf{I} - \mathbf{A})^{-1}[\mathbf{DSR}_K{}' + (1 - \delta)\mathbf{\Gamma}]\mathbf{K}_s. \qquad D.14$$

Solving for $\mathbf{X}_s$ yields:

$$\mathbf{X}_s = (\mathbf{I} - (\mathbf{I} - \mathbf{A})^{-1}\mathbf{DSR}_{M}')^{-1}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{DL} + (\mathbf{I} - (\mathbf{I} - \mathbf{A})^{-1}\mathbf{DSR}_{M}')^{-1}(\mathbf{I} - \mathbf{A})^{-1}[\mathbf{DSR}_{K}' + (1 - \delta)\Gamma]\mathbf{K}_s. \qquad D.15$$

The demand for capital in each sector is given by Equation 6.20:

$$K_{i,s} = \frac{\overline{X}_{i,s}}{\alpha_{i,0}}\left(\frac{\eta^{l}_{1,s}}{\alpha_{i,1}}\right)^{\alpha_{i,1}}\left(\frac{\eta^{l}_{2,s}}{\alpha_{i,2}}\right)^{\alpha_{i,2}}\left(\frac{\eta^{k}_{i,s}}{\alpha_{i,k}}\right)^{\alpha_{i,k} - 1}\left(\frac{\eta^{m}_{i,s}}{\alpha_{i,m}}\right)^{\alpha_{i,m}}. \qquad D.16$$

Define the diagonal matrix

$$\mathbf{C} = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$$

where

$$c_1 = \frac{1}{\alpha_{1,0}}\left(\frac{\eta^{l}_{1,s}}{\alpha_{1,1}}\right)^{\alpha_{1,1}}\left(\frac{\eta^{l}_{2,s}}{\alpha_{1,2}}\right)^{\alpha_{1,2}}\left(\frac{\eta^{k}_{1,s}}{\alpha_{1,k}}\right)^{\alpha_{1,k} - 1}\left(\frac{\eta^{m}_{1,s}}{\alpha_{1,m}}\right)^{\alpha_{1,m}}$$

$$c_2 = \frac{1}{\alpha_{2,0}}\left(\frac{\eta^{l}_{1,s}}{\alpha_{2,1}}\right)^{\alpha_{2,1}}\left(\frac{\eta^{l}_{2,s}}{\alpha_{2,2}}\right)^{\alpha_{2,2}}\left(\frac{\eta^{k}_{2,s}}{\alpha_{2,k}}\right)^{\alpha_{2,k} - 1}\left(\frac{\eta^{m}_{2,s}}{\alpha_{2,m}}\right)^{\alpha_{2,m}}$$

Now,

$$\mathbf{K}_s = \mathbf{C}\mathbf{X}_s. \qquad D.17$$

Substituting these into Equations D.15, we obtain

$$\mathbf{K}_s = \mathbf{C}^{-1}(\mathbf{I} - (\mathbf{I} - \mathbf{A})^{-1}\mathbf{DSR}_{M}')^{-1}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{DL} + \mathbf{C}^{-1}(\mathbf{I} - (\mathbf{I} - \mathbf{A})^{-1}\mathbf{DSR}_{M}')^{-1}(\mathbf{I} - \mathbf{A})^{-1}[\mathbf{DSR}_{K}' + (1 - \delta)\Gamma]\mathbf{K}_s. \qquad D.18$$

Solving for $\mathbf{K}_s$ gives the capital stocks that are consistent with the original set of prices. That is,

$$\mathbf{K}_s = [\mathbf{C}^{-1}(\mathbf{I} - (\mathbf{I} - \mathbf{A})^{-1}\mathbf{DSR}_M')^{-1}(\mathbf{I} - \mathbf{A})^{-1}[\mathbf{DSR}_K' + (1 - \delta)\Gamma]\mathbf{K}_s]^{-1} \cdot \mathbf{C}^{-1}(\mathbf{I} - (\mathbf{I} - \mathbf{A})^{-1}\mathbf{DSR}_M')^{-1}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{DL}\,. \quad D.19$$

Now, one can go back and calculate the equilibrium values of all the other variables. The gross production vector is given by

$$\mathbf{X}_s = \mathbf{C}^{-1}\mathbf{K}_s\,. \quad D.20$$

Land use in each industry is given by Equation D.11. The number of acres in each age class of forest is given by Equation D.13. Net asset income is given by Equation D.8. Net income for each household is given by Equation D.6. Consumption of each good by each household is given by Equation D.2. Finally, labor demands are given by Equations 6.18 and 6.19.

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