CONTROL CHART PROCEDURES BASED ON

CUMULATIVE GAUGING SCORES

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(ABSTRACT)

Control charts based on cumulative gauging scores rely on gauge scoring systems used for transforming actual observations into integer gauging scores. In some cases, the gauging scores are easy to obtain by using a mechanical device such as in the go-no-go inspection process. Thus, accurate measurements of selected quality characteristic are not necessary. Also, different control purposes can be achieved by using different scoring systems.

Cumulative gauging score charts based on two pairs of gauges are proposed to control the process mean or the standard deviation by either gauging one or several observations. Both random walk and cusum type cumulative gauging score charts are used. For controlling the process mean and standard deviation at the same time, a cusum type and a two-dimensional random walk type procedure are proposed. A gauging scheme can be applied to multivariate quality control by gauging either x^2 or T^2 statistics. A simple multivariate

control chart which is based on the multivariate sign score vector is also proposed.

The exact run length distribution of these cumulative gauging score charts can be obtained by formulating the procedures as Markov chain processes. For some procedures, the average run length (ARL) can be obtained in a closed form expression by solving a system of difference equations with appropriate boundary conditions.

Comparisons based on the ARL show that the cumulative gauging score charts can detect small shifts in the quality characteristic more quickly than the Shewhart type $\overline{X}$-chart. The efficiency of the cusum type gauging score chart is close to the regular CUSUM chart. The random walk type gauging score chart is more robust than the Shewhart and CUSUM charts to observations which have heavy a tailed distribution or which are serially correlated. For multivariate quality control. a procedure based on gauging the x^2 statistic has better performance than the x^2 chart. Also, a new multivariate control chart procedure which is more robust to the misspecification of the correlation than the x^2 chart is proposed.

ACKNOWLEDGEMENTS

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I. INTRODUCTION

1.1 PROCESS CONTROL AND CONTROL CHARTS

The quality of a product usually can be specified by
parameters such as the process mean, the process variance,
or the proportion of defective product. When the parameter
changes to unacceptable levels, the process is regarded as
out-of-control. The purpose of process control is to detect
changes as soon as possible so that rectifying action can be
taken in time to reset the parameter to the satisfactory
level. The control chart is a statistical device for moni-
toring the output of a continuous production process through
the sample measurement of a selected quality characteristic.
Typically, samples of fixed size n are taken at regular time
intervals and the quality characteristic is measured. Ap-
plying this information to a control chart, one can infer
whether or not a process is operating properly.

Very often, a process has a certain amount of inherent
random variation. This random variation cannot be eliminated
or avoided unless the process is completely revised. There-
fore, the quality characteristic can be treated as a random
variable (or vector) with a certain type of distribution.
There is another kind of variation which is attributed to
special causes such as differences in materials, machine

wear, etc. These causes are called assignable causes in the
quality control literature. It is usually possible to rec-
tify assignable causes as long as one can locate them. If
an assignable cause occurs, then a systematic variation may
show up in the observed samples and this variation results
in a change in the distribution of the quality character-
istic. If this happens, the process is deemed to be out of
control.

Let $\underline{X}_t = (X_{1t}, X_{2t}, \ldots, X_{nt})$ be the random sample taken
at time t, and $\underline{X}_1$, $\underline{X}_2, \ldots$ be a sequence of samples obtained
from a process. When the process is under control, all X_{ij}'s
are assumed to follow the same continuous distribution
$F(\underline{x}, \theta_o)$. Here θ_o is the parameter of the distribution F,
and could be a scalar or a vector. In quality control, θ_o
is referred to as the target value (or vector) or control
value. If an assignable cause occurs at time t*, then
$F(\underline{x}, \theta_o)$ changes to $G(\underline{x}, \theta)$. In this research, we assume
that F and G have the same parametric form. The only change
will be in θ. In general, whenever θ shifts to $\theta_o \pm \varepsilon$, the
process is regarded to be out-of-control even for arbitrary
small ε. However, in practice, a small shift in θ may not be
important. For this concern, Woodall (1985a) defined that a
process is in-control if $|\theta - \theta_o| < \varepsilon_a$ and out-of-control if
$|\theta - \theta_o| > \varepsilon_r$. Here ε_a is the largest shift from θ_o considered
to be of no importance and ε_r is the smallest shift from θ_o
considered to be detected quickly. This definition is more

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reasonable in practice. A control chart uses the samples $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_t$ at time point t to decide whether to signal that a change has occurred or to move to the next time point t+1 and take another sample.

The efficiency of a control chart's performance is determined by the run length distribution which is the distribution of the total number of samples (N) required to signal that a change in θ has occurred. In some control charts, however, the run length distribution is very difficult to derive. The average run length ARL(θ), which is the expected value of N, is used more frequently. When the process is in-control, a large $ARL(\theta_o)$ is preferred so that the frequency of false signals is low, whereas, when the process is out-of-control, a small ARL(θ) is preferred. Usually, comparisons between two control chart procedures can be made by setting the ARL's equal at the in-control situation ($ARL(\theta_o)$), and then comparing the ARL's at different out-of-control levels θ. The procedure which provides a smaller ARL(θ) is the better one. A more general criterion is to compare the profiles of the ARL curves (Woodall, 1985a). If the ARL profile of procedure A is above that of procedure B for $\theta < \theta_1$ and below that of B for $\theta > \theta_2$, then the procedure A is uniformly preferable to the procedure B. The ARL profiles A and B are not compared over the interval (θ_1, θ_2). For the sake of simplicity, the out-of-control ARL

is usually computed assuming that the change in θ occurs at the very beginning.

Although the process control problem and the sequential hypothesis testing problem both take observations sequentially and appear to be similar, they are not the same. First of all, in hypothesis testing it is assumed that we have either θ_0 or θ_1 with no changes over time. But in process control, the process starts at θ_0 then changes to θ_1 at a random time t^*. This θ_1 also may not be a constant over time. Secondly, in hypothesis testing we want small ASN which is the sample size times ARL at both θ_0 and θ_1 subject to restrictions on type I and type II errors. However, in process control we want a small ARL at θ_1 and a large ARL at θ_0.

1.2 OBJECTIVES

There are certain desirable properties that we look for in a control chart. First, it should be simple. A simple procedure that is easy to understand and to apply is more likely to be used by workers, and may reduce the chance of making errors. Second, it should have satisfactory efficiency in a wide range of changes in θ and also robustness to misspecification of the process distribution. Third, its properties should be easy to obtain. It is important for the user of the control chart to know the properties of a procedure and be able to compare it with the other possible pro-

cedures. Unfortunately, most control chart procedures cannot meet all three of these criteria. For example, the Shewhart chart meets only the first and the third. The cumulative sum chart (CUSUM) has satisfactory efficiency in a wide range of changes in θ, but it is not robust to the misspecification of the distribution. Although the properties of the CUSUM chart are difficult to compute, many properties are tabled for practical use. The main purpose of this research is to find a type of control chart that achieves these objectives without paying too high a price.

Usually we have to apply different kinds of control statistics to achieve different control purposes. For controlling the process mean we may, for example, use the sample means, but in controlling the variance we may use the sample ranges. Therefore, another objective of this research is to find a unified scheme which is more flexible in pursuing different control purposes. The unified procedure needs only one type of control statistic to control the mean, the standard deviation, or the mean and standard deviation. Also the ARL for the proposed procedure is easier to compute than the CUSUM procedure, but, in general, a computer is still needed.

The third objective of this research is to find a procedure which is more robust than the Shewhart chart and the CUSUM chart to observations which have a heavy tailed distribution or are serially correlated. Robustness is an im-

portant property of a procedure. This is because the distribution of the quality characteristic may not be normal, and successive observations may not be independent in a continuous production process. Most control charts are designed to deal only with normally distributed observations. The procedure we propose can be applied to any distribution as long as certain percentiles of the distribution can be determined. Also this proposed procedure using control limits based on the normality assumption may be less sensitive to departures from normality than the Shewhart chart. Furthermore, we believe the proposed procedure to be more resistant to serially correlated observations.

1.3 CONTROL CHARTS BASED ON GAUGING SCORES

From a review of the quality control literature, we found that control charts based on gauging scores have the potential to meet the goals mentioned in Section 1.2. First, the gauges are used to divide the possible measurement range of the quality characteristic into finite regions and assign each region an integer score called the gauging score. Then the cumulative gauging scores are used as a control statistic. For different control purposes we may only need to change the scoring system. This type of scheme depends on the probabilities of getting different scores so it can be applied to any known distribution. If the observations come

from a heavy tailed distribution, an extremely large observation will only have a small assigned integer score. Therefore the procedure based on gauging scores is expected to be more robust to heavy tailed distributions. Because the control statistic is a discrete random variable, the Markov chain method is the basic tool for obtaining the run length distribution and the ARL.

The gauging scores can be obtained very easily. For example, suppose a manufacturer wants to produce rods with diameter 10cm. He can use two gauging rings with diameters 10.05cm and 9.95cm. If a rod cannot go through the large ring, a score of one is assigned to this rod. If it can go through the large ring but not the small one, a score of zero is assigned. If a rod can pass through both rings, then a score of negative one is given. In practice, this type of inspection usually is called a go-no-go scheme. Since the gauging scheme does not need to measure the quality characteristic exactly, it can speed up the inspection process.

Control charts based on gauging scores were proposed by Page (1962) and revisited by Munford (1980). These authors used only one pair of gauges and gauged a single observation or the sample mean at each sampling time. Their results showed that the cumulative gauging score scheme is much better than the Shewhart chart and comparable with the CUSUM chart for small or moderate shifts in the mean. For large shifts in the mean, say above two units of the process

standard deviation when the process is in-control, the cumulative gauging score scheme is less sensitive than these two popular charts. The reason for this lower sensitivity to large shifts is that the gauging system gives the same scores for all observations outside the gauges and does not count the effect of the magnitude of the observations. Ncube and Woodall (1984) proposed a combined Shewhart-cumulative score control chart which can improve the performance for large shifts. An alternative way to improve the performance is to add more gauging pairs. If we add more gauging pairs and give larger scores to those observations with large observed values, then the magnitude effect will be reflected by higher scores. Hence we should expect that the ARL for larger shifts in the mean will be improved and the ARL at θ_o possibly still remain nearly the same as in using one pair of gauges. Based on Woodall's criterion, this modified procedure is uniformly better than Munford's cumulative score chart.

1.4 CHAPTER CONTENTS

This dissertation is arranged as follows. Chapter II is a review of some process control chart procedures. Many details such as the construction, advantages and disadvantages, and modifications are provided for the Shewhart chart and the CUSUM chart, because these two charts are the most

popular control charts. For the nonparametric and the
multivariate control charts only a brief review is given.

Chapter III presents a cumulative gauging score chart
based on two pairs of gauges for controlling the process mean
and the standard deviation. For controlling the process
mean, the two-sided control chart is a random walk process
between two absorbing barriers. For one-sided control for
the mean or the standard deviation, it is a random walk be-
tween an absorbing barrier and a reflecting barrier. The run
length distribution is derived by using the Markov chain
method. Closed form expressions for computing ARLs are ob-
tained by solving a system of difference equations. Some
rules of thumb based on empirical studies are provided for
choosing the chart parameters.

Chapter IV applies the gauging scores to a CUSUM type
procedure. Under certain restrictions on chart parameters,
the scoring system of this scheme is similar to that of the
two pairs of gauges scheme with a particular set of scores.
Therefore the discrete Markov chain method can be used in
obtaining the ARLs.

Chapter V compares the efficiency of the proposed pro-
cedures with the Shewhart chart, the CUSUM chart, Munford's
cumulative score chart, and Page's cusum scheme with gauging.
The effect of serially correlated observations on the ARL and
the robustness of the gauge's locations for heavy tailed
distributions are investigated through a Monte Carlo study.

Chapter VI uses two gauging-score systems to control the process mean and the standard deviation simultaneously.

The first part of Chapter VII presents a method to improve the performance of the x^2 chart by gauging the x^2 statistic. A random walk and a cusum scheme are employed for this gauging score approach. The second part of this chapter presents a proposed new multivariate control chart procedure based on scores. It is a multivariate sign test scheme.

Finally, Chapter VIII is the summary and discussion of directions for further study.

II. REVIEW

2.1 INTRODUCTION

The main purpose of this chapter is to introduce the most prevalent and well known control charts -- Shewhart charts and cumulative sum charts. Included are the structure, properties, drawbacks, and modifications of these charts.

Some nonparametric and multivariate control charts are also reviewed. The increased attention on nonparametric control chart procedures results from the features of nonparametric statistics which require less information and are resistant to the distribution of the observations. These properties are particularly useful at the early stages of a production process due to the lack of information about the process. The need for multivariate control charts is based on the fact that many products have at least two quality characteristics which specify the quality of the product. However, there appear to be relatively few nonparametric and multivariate procedures available in the literature on control charts.

2.2 SHEWHART CHART

The Shewhart chart is one of the simplest and most widely used control charts. The basic structure of this chart consists of a central line which is located at the control value θ_o and two control limits (C_l, C_u), the lower and upper action limits. The region between these two limits is referred to as the in-control region.

Suppose that $\underline{X}_t = (X_{1t}, X_{2t}, \ldots, X_{nt})$ is a random sample of size n observed at the t-th time interval and the control statistic $g(\underline{X}_t)$ is a function of $\underline{X}_t$. At each sampling time t, $g(\underline{X}_t)$ is plotted on the control chart in chronological order. As long as $g(\underline{X}_t)$ falls inside the in-control region, the process is regarded as in-control. Whenever $g(\underline{X}_t)$ falls outside the interval (C_l, C_u), an out-of-control signal is given.

Shewhart charts with appropriate control statistics can achieve different control purposes. For example, if $g(\underline{X}_t)$ is the sample mean, the sample range, or the fraction defective, the Shewhart chart can be used to control the process mean, the process standard deviation, or the proportion of defective items, respectively. Also, the corresponding names are $\bar{X}$-chart, R-chart, and P-chart. More details about these charts can be found in Duncan (1974).

When parameters of the distribution of a quality characteristic are known, the Shewhart chart is equivalent to a

sequence of independent tests applied at each sample. Therefore the run length distribution follows a geometric distribution with parameter $p = Pr[C_1 < g(\underline{X}_t) < C_u)$ when the process distribution does not change and the ARL is $1/p$. This procedure can be applied to observations with any distribution as long as one can compute the probability p accurately.

To design the $\overline{X}$-chart, one has to know the control value θ_o and the variance σ_o of the quality characteristic. For two-sided control, we usually choose $C_1, C_u = \theta_o \pm B\sigma_o/\sqrt{n}$, where n is typically four or five. The choice of B is determined by the desired value of ARL(θ_o). Woodall (1985a) proposed a simple method of choosing the sample size and the control limits. His purpose was to design an $\overline{X}$-chart to achieve the required performance when the process is in-control and at certain levels of out-of-control situations.

When the variance has to be estimated from initial sampled data, using small samples in estimating the variance will increase the ARL and the variance of the run length distribution. The properties of the $\overline{X}$-chart with estimated variance were investigated by Ghosh, Reynolds, and Hui (1981). Burr (1967) investigated the effects of non-normality on the control limits for $\overline{X}$ and R charts. He concluded that the ordinary normal control chart limits can be used unless the population is notably non-normal.

The Shewhart chart uses only information in $\underline{X}_t$ at sampling time t. For controlling the process mean, the Shewhart chart can detect large shifts more quickly. But it is not good for detecting small changes in the mean. To remedy this drawback, the Shewhart chart has been modified by adding two warning limits within the action limits (Page,1963). An out-of-control signal is given if any control statistic falls outside the action limits or r consecutive control statistics fall outside the warning limits.

Although a large sample size can alleviate the slow detection problem for small shifts in the mean, people still prefer to use a small sample size and to sample more frequently. Weiler (1953) suggested the use of runs to control the mean. The stopping rule for this control chart based on runs is to signal out-of-control whenever r consecutive sample means fall outside the control limits. This type of scheme allows us to narrow down the range between the control limits. As a result, it improves the sensitivity for small shifts in the mean with a small sample size. This type of control chart has been studied by several people, see for example Mosteller (1941), Wolfowitz (1943), and Weiler (1953).

2.3 CUSUM CHART

A more efficient procedure is the cumulative sum chart which was originally proposed by Page (1954). For detecting positive changes in the quality characteristic, we assume that the expected value of $g(\underline{X}_t)$ is equal to θ_o, then the control statistic at time t is

$$S_t^+ = \max\{S_{t-1}^+ + (g(\underline{X}_t) - \theta_o - K^+), \ 0\}$$

and $S_o^+ = 0$. For detecting negative changes in the quality characteristic, the control statistic at time t is

$$S_t^- = \min\{S_{t-1}^- + (g(\underline{X}_t) - \theta_o + K^-), \ 0\}$$

and $S_o^- = 0$. Here K^+ and K^- are parameters of the procedure called reference values. The stopping rule is to signal out-of-control whenever $S_t^+ \geq H^+$ for positive control or $S_t^- \leq H^-$ for negative control, where H^+ and H^- are called stopping boundaries. For two-sided control, one has to use S_t^+ and S_t^- simultaneously and signal out-of-control at the earliest time t for which either $S_t^+ > H^+$ or $S_t^- < H^-$.

For one-sided positive control, subtracting a positive reference value results in a downward trend in the CUSUM chart when the process is in-control. Therefore a low frequency of false signals can be achieve with a small value of

H^+. But if the expected values of $(g(\underline{X}_t)-\theta_0)$ exceed K^+, an upward trend will show up in the CUSUM chart. Also, whenever S_t^+ drops below zero, then S_t^+ will be reset at zero. Subtracting a reference value makes the path of S_t^+ have a higher chance to visit the point zero. Therefore the assumption that the quality changes at the very beginning is more reasonable for the computation of the ARL.

In designing a CUSUM chart, one needs to choose K^+ and H^+. If δ is the amount of shift from θ_0 which should be detected quickly, the most common choice of K^+ is $\delta/2$. The choice is independent to the values of $ARL(\theta_0)$. This result was proved by Reynolds (1975) for the case where the CUSUM is approximated by a Brownian motion process. The same formula was also provided by Ewan and Kemp (1960) and Johnson (1961), where it was assumed that the distribution of the observation was normal and using $\overline{X}$ to control the process mean. The choice of H^+ depends on the desired value of $ARL(\theta_0)$, but there is no clear rule how to pick the value H^+.

As for the Shewhart chart procedures, choosing different forms for $g(\underline{X}_t)$ can achieve different control purposes. The statistic $g(\underline{X}_t)$ may be the sample mean, sample range, or sample standard deviation. The properties calculated and procedures applied are usually based on the normality assumption. Johnson gave CUSUM type control charts for the folded normal and Weibull distributions in 1963 and 1966, respectively.

Computing the run length distribution or the ARL of a CUSUM chart procedure is not an easy task. For a one-sided CUSUM chart using $\overline{X}$ to control the process mean, Page (1954) and Ewan and Kemp (1960) showed that the run length distribution is approximately geometric when the $ARL(\theta_o)$ is large. Ewan and Kemp (1960) showed that the run length distribution can be approximated for large i by

$$P[N<i] \approx 1-e^{i/ARL(\theta_o)}.$$

A method for computing the exact ARL of the CUSUM procedure by solving an integral equation was described in chapter III of Van Dobben de Bruyn (1968). When the observations follow a normal distribution with known variance, a table of values of the ARL was also given. Goel and Wu (1971) provided a contour nomogram for CUSUM charts for controlling the normal mean. Using a Brownian Motion process on the interval $(0,\infty)$ to approximate the behavior of the CUSUM procedure was studied by Johnson and Bagshaw (1974) and Reynolds (1975). Using Wald's approximations to estimate the ARL was investigated by Khan (1978). Kahn (1984) also showed that both one-sided and two-sided CUSUM procedures are closely related to a sequential probability ratio test (SPRT). A Markov chain approach was used by Brook and Evans (1972), Lucas (1976), Lucas and Cosier (1982), and Woodall (1983,1984) among others.

For two-sided control, let ARL^+ and ARL^- be the corresponding average run lengths for the positive and negative control charts. Then Kemp (1961) showed that the ARL satisfies the relationship:

$$\frac{1}{ARL} = \frac{1}{ARL^+} + \frac{1}{ARL^-}$$

The CUSUM chart can pick up small changes or persistent trends in the mean very quickly. However, for large changes say above $3\sigma_o$, the CUSUM chart is slightly less sensitive than the Shewhart chart. A simple and reasonable modification is to add the Shewhart limits on the CUSUM chart. This combined Shewhart-CUSUM chart was proposed by Westgard et al. (1977) and Lucas (1982). In order to offset the effect of adding the Shewhart limits on $ARL(\theta_o)$, it is necessary to slightly increase H. The combined scheme can detect large shifts faster than the standard CUSUM scheme.

In practice, the observations from a continuous production process may not be independent. Therefore the assumption that $\underline{X}_t$ and $\underline{X}_{t'}$ are independent random samples may not be realistic. Goldsmith and Whitfield (1961) investigate the effect of serial correlation on the values of the ARL of CUSUM schemes through Monte Carlo simulation studies. Their results show that for large shifts in the mean the serial correlation has little effect on the ARL. But for small shifts in the mean, negative correlation in the process re-

sults in higher average run lengths. The effect of positive correlation in the process depends on the amount of shift' in the mean and the magnitude of the serial correlation. In most of cases, the positive correlation will reduce the ARL.

For using $\overline{X}$ to control the process mean, Johnson and Bagshaw (1974) and Bagshaw and Johnson (1975) studied the effect of autoregressive and moving average observations on the run length distribution and ARL of the CUSUM procedures. They mentioned that "relatively small correlations will lead to very differently shaped run length distributions." They also provided the approximation

$$ARL_{\phi}(\theta) = \frac{1-\phi}{1+\phi} \, ARL(\theta) \qquad \text{for AR(1) model}$$

and

$$ARL_{\psi}(\theta) = \frac{1+\psi^2}{(1-\psi)^2} \, ARL(\theta) \qquad \text{for MA(1) model}$$

to illustrate the influence of serial correlation on $ARL(\theta_o)$, where ϕ, ψ are the serial correlation of autoregressive and moving average models, respectively. Therefore the CUSUM procedure is not robust to dependence between observations. For Shewhart charts, we have not seen any results related to this dependence problem.

2.4 NONPARAMETRIC CHARTS

Several nonparametric control chart procedures were
proposed during the past two decades. Parent (1965) and
Reynolds (1972) developed some procedures based on signed
sequential ranks of the observations. But the exact proper-
ties of these procedures are difficult to evaluate. Bakir and
Reynolds (1979) developed some procedures using the Wilcoxon
signed rank statistic in the CUSUM chart. The ranking was
done within groups. The groups were either sampled at each
time interval or artifically grouped in a certain time pe-
riod. Bakir and Reynolds (1982) proposed two other Shewhart
type control charts which are based on the sign test or the
Wilcoxon signed rank statistic. For situations where the
control value is not specified, Park (1984) proposed some
Shewhart and CUSUM control charts based on two-sample
nonparametric statistics. The sequentially obtained samples
were compared with a standard sample and with these sequen-
tially obtained statistics, the usual process control proce-
dure can be applied.

2.5 MULTIVARIATE CHARTS

When the product requires more than one quality charac-
teristic for the specification of its quality, a multivariate
control chart procedure is needed. The procedure generally

employed is to use separate control charts for each quality characteristic. The process is said to be out-of-control at the earliest signal of one of these individual control charts. Instead of controlling each quality characteristic separately, a single chart to control the quality characteristic vector is desirable.

The use of the separate control chart will result in incorrect statements about the ARL unless specific allowance is made for using multiple charts. The design of the control region should consider the relationship among these quality characteristics (Jackson, 1959). This type of approach was suggested by Ewan and Kemp (1960) among others. Woodall and Ncube (1985b) proposed a method for determining the ARL for this type of scheme by using the Markov chain method when there are two quality characteristics. Their idea is to monitor continuously each quality characteristic individually, but the performance of this scheme is determined from the joint distribution of the observations.

A well known multivariate control chart is the T^2 chart designed for the situation where the observation vector has an unknown disperson matrix. This chart was described by Hotelling (1947), Jackson (1959), Ghare and Torgerson (1968), and Montgomery and Klatt (1972). A Chi-squared (x^2) chart has been proposed for the case of a known disperson matrix. These two charts are Shewhart type charts. For the attributes cases, Patel (1973) used a multivariate normal approximation

and a x^2 chart to deal with the case in which the quality characteristic has multivariate binomial and multivariate Poisson distributions. Marcucci (1982) generalized P-charts and C-charts for monitoring multivariate Bernoulli, Poisson, and multinomial observations. The drawback of this kind of control chart is in diagnosing the causes for rejection when the processes are found to be out-of-control. Hui (1980) proposed diagnostic charts using rectangular acceptance regions for the quality characteristic vectors. So far a multivariate CUSUM type chart has not appeared in the literature.

III. CONTROL CHARTS WITH TWO PAIRS OF GAUGES

3.1 INTRODUCTION

Control chart procedures based on cumulative gauging scores rely on a scoring system. The simplest scoring system with only one pair of gauges $\pm G_1$ was used by Page (1962) and Munford (1980). Without loss the generality, they assumed that $\theta_o = 0$. Their gauging score is based on the following function:

$$C(X_j) = \begin{cases} -1 & X_j \leq -G_1 \\ 0 & -G_1 < X_j < G_1 \\ 1 & G_1 \leq X_j, \end{cases}$$

here X_j can be either a single observation or a sample mean. The control statistic is $S_o = 0$ and $S_j = S_{j-1} + C(X_j)$, for $j=1,2,\ldots$. For controlling a two-sided shift in the mean, Page suggested gauging single observations and plotting S_j against j. Then test at each point with a V-mask. Therefore Page's scheme is equivalent to two one-sided CUSUM schemes. Munford replaced the V-mask with fixed absorbing barriers $S = \pm H$, where corrective action is taken as soon as $|S_j| \geq H$.

The performance of Munford's cumulative score chart is better than the Shewhart chart for small or moderate changes

in the mean (Munford, 1980). But for large changes in the mean, it has less sensitivity than the Shewhart chart. The biggest advantage of Munford's scheme is that the exact average run length can be computed from a closed form expression which is easier than the CUSUM chart.

There are several ways to improve the sensitivity of Munford's cumulative score chart for detecting a large shift in the mean. Ncube and Woodall (1984) proposed adding Shewhart limits to the cumulative score chart. The second alternative is to add more gauging pairs to the simple scoring system and assign larger scores to observations with large values in the quality characteristic. When the process is out-of-control, observations with higher scores will be more likely and large shifts can be detected more quickly. However, using too many gauging pairs will reduce the simplicity of computing the ARL. Also, it makes the scores harder to obtain. If scores are hard to get, it might be easier to use exact measurement and not lose any information. Therefore, we will concentrate on two pairs of gauges in this dissertation. The proposed procedure in this chapter is called the Two Pairs of Gauges scheme and abbreviated as the TPG scheme.

In this chapter, we first discuss the case when only one observation is taken at each sampling time. Then we will extend this to the case of taking n observations at each sampling time. If we decide to gauge the sample mean, the

formula for computing the ARL is still applicable. In practice, people usually prefer to take n observations at each sampling time. For this situation we gauge each individual observation separately.

3.2 SCORING SYSTEM AND CONTROL STATISTIC

Without loss of generality, we assume that μ is zero when the process is under control. The goal is to maintain the process mean at $\mu = 0$. The scoring system for the TPG is as follow:

$$
C(X_j) = \begin{cases}
-2 & X_j \leq -G_2 \\
-1 & -G_2 < X_j \leq -G_1 \\
0 & -G_1 < X_j < G_1 \\
1 & G_1 \leq X_j < G_2 \\
2 & X_j \geq G_2,
\end{cases}
\tag{3.1}
$$

where G_1, G_2 are the gauge locations and X_j is the observation taken at time j. A practical method of obtaining these scores is based on a sequence of go-no-go type inspection devices. For example, using the scoring system $C(X_j)$, we need two sets of go-no-go devices in order to obtain these scores. For gauging single observations, the $C(X_j)$ is equivalent to $g(X_j)$ mentioned in Section 2.2. After scoring, the score $C(X_j)$ is a discrete random variable with the following probability structure:

y	-2	-1	0	1	2
$p[C(X_j)=y]$	p_{-2}	p_{-1}	p_o	p_1	p_2.	(3.2)

Thus the mean and variance of $C(X_j)$ are

$$E[C(X_j)]=2(p_2-p_{-2})+(p_1-p_{-1})$$

(3.3)

$$var[C(X_j)]=4(p_2+p_{-2})+(p_{-1}+p_1)- E^2[C(X_j)].$$

For simplicity, the distribution of the observations is as-
sumed to be symmetric around $\mu_o = 0$. Therefore, when the
process is in-control, $p_{-2} = p_2$ and $p_{-1} = p_1$. When the dis-
tribution is not symmetric, than the upper and lower gauges
would have to be different in order to achieve these equal-
ities of the corresponding probabilites. The control sta-
tistic for the TPG chart is defined as

$$S_m= S_{m-1}+C(X_m) \qquad m=1,2,\ldots,$$

where $S_o=0$.

3.3 TWO-SIDED CONTROL

3.3.1 Procedure

For the problem of detecting both positive and negative changes in the mean, the proposed procedure is to signal if S_m falls outside the interval $(-H,H)$. The process is re-garded as in-control as long as S_m remains in the interval $(-H,H)$. Thus the TPG is essentially a generalized random walk process between two absorbing barriers H and -H.

3.3.2 Transition Matrix and Run Length

The properties of this discretized scheme can be easily derived by using a Markov chain approach. The Markov chain approach was originally used for control charts by Brook and Evans (1972). The essential part of this approach is to form the transition probability matrix.

For two-sided control, let E_i be the state that $S_m = i$, $i=-H+1,\ldots,H-1$. If $S_m = \pm H$, then we say the process is in the absorbing state E_H. Using the probability distribution of $C(X_j)$, the transition probability matrix P can be formed as follows:

$$P_{i+2,i} = P_{-2} \qquad i=-H+1,\ldots,H-3$$
$$P_{i+1,i} = P_{-1} \qquad i=-H+1,\ldots,H-2$$
$$P_{i,i} = P_0 \qquad i=-H+1,\ldots,H-1 \qquad (3.4)$$

$$P_{i,i+1} = P_1 \qquad i=-H+1,\ldots,H-2$$

$$P_{i,i+2} = P_2 \qquad i=-H+1,\ldots,H-3$$

$$P_{i,H} = 1-F_i \qquad i=-H+1,\ldots,H-1$$

$$P_{H,H} = 1$$

$$P_{i,j} = 0 \qquad \text{otherwise}$$

where $F_i = \Sigma_{j=-H+1}^{H-1} P_{i,j}$ and $P_{i,j}$ is the transition probability from E_i to E_j. Let R be the submatrix of P obtained by deleting the last row and column. The properties of the proposed scheme can be derived as a function of R.

Let $\underline{N} = [N_{-H+1}, \ldots, N_{H-1}]'$ be the run length random column vector whose components are the run length random variables corresponding to the initial state $S_o = i$, $i=-H+1,\ldots,H-1$. Then

$$Pr[\underline{N} = \underline{1}] = (I-R)\underline{1}$$

and $\hspace{8cm}$ (3.5)

$$Pr[\underline{N} = \underline{i}] = R^{i-1}Pr[\underline{N} = \underline{1}],$$

for $i=2,3,\ldots$, where $\underline{i}$ is a column vector with i in all components and I is an identity matrix. Also the mean and variance of $\underline{N}$ are

$$E[\underline{N}] = (I-R)^{-1}\underline{1}$$

and $\hspace{8cm}$ (3.6)

$$var[\underline{N}] = 2R(I-R)^{-2}\underline{1} + E[N] - E^2[\underline{N}],$$

respectively, where $E^2[\underline{N}] = [E^2(N_{-H+1}), \ldots, E^2(N_{H-1})]$. The expression for higher order factorial moments of $\underline{N}$ is also given by Brook and Evans (1972). For simplicity, usually we assume the process starts with $S_o=0$. Thus the run length N_o, ARL= $E[N_o]$, and var$[N_o]$ are

$$Pr[N_o = 1] = \underline{e}'Pr[\underline{N} = \underline{1}],$$
$$Pr[N_o = i] = \underline{e}'Pr[\underline{N} = \underline{i}],$$
$$ARL = \underline{e}'E[\underline{N}], \qquad\qquad (3.7)$$
$$var[N_o] = \underline{e}'var[\underline{N}],$$

respectively. Here $\underline{e}$ is a (2H-1)x1 column vector with one in the H-th posistion, and zero otherwise. A SAS program based on PROC MATRIX was written (Appendix B) for performing the computations.

Although the Markov chain approach does provide a method for calculating exact properties of the TPG scheme, for computing the ARL, the difference equations approach can simplify the calculations. Moreover, a closed form expression for ARL(0) can be derived by using the difference equations approach.

3.3.3 Average Run Length (ARL)

The average run length can be computed by using difference equations together with appropriate boundary conditions. When the current state is E_i, let D_i be the expected number of samples required to reach the absorbing state E_H. When the process is in control, the D_i's satisfy the difference equations

$$D_i = 1 + p_2 D_{i-2} + p_1 D_{i-1} + p_o D_i + p_1 D_{i+1} + p_2 D_{i+2},$$
$$i = -H+1, \ldots, H-1, \quad (3.8)$$

and

$$D_{-(H+1)} = D_{-H} = D_H = D_{H+1} = 0.$$

We need four boundary conditions to solve the fourth order difference equation (3.8). The absorbing barriers H and $-H$ imply $D_H = D_{-H} = 0$. Ideally, the states greater than H or less than $-H$ are possible states in this random walk type process. Because they are beyond the absorbing barriers H or $-H$, the $D_{-(H+1)}$ and D_{H+1} should be set to zero. To solve this system of difference equations, one has to find the homogeneous solutions and a particular solution (Haggerty, 1972). Let $D_i = \lambda^i$ be the solution of

$$p_2 D_{i+2} + p_1 D_{i+1} - p_o D_i + p_1 D_{i-1} + p_2 D_{i-2} = 0, \quad (3.9)$$
$$i = -H+1, \ldots, H-1.$$

Then $D_i = a + b\lambda_1^i + c\lambda_2^i + d\ i$ and

$$\lambda_{1,2} = -(1 + \frac{P_1}{2P_2}) \pm \sqrt{\frac{P_1}{P_2} + \frac{P_1}{4P_2^2}}$$

Here λ_1, λ_2 are the solutions of $p_2\lambda^2 + (2p_2 + p_1)\lambda + p_2 = 0$.
A particular solution for (3.9) is $D_i = -i^2/\text{var}[C(X)]$.
Therefore

$$D_i = a + b\lambda_1^i + c\lambda_2^i + d\ i - \frac{i^2}{\text{var}[C(X)]} \tag{3.10}$$

Substituting D_i from (3.10) into the boundary condi-
tions, one can obtain a,b,c, and d by solving the system of
equations

$$a + b\lambda_1^{-(H+1)} + c\lambda_2^{-(H+1)} - d(H+1) = \frac{(H+1)^2}{\text{var}[C(X)]}$$

$$a + b\lambda_1^{-H} + c\lambda_2^{-H} - dH = \frac{H^2}{\text{var}[C(X)]}$$

$$a + b\lambda_1^{H} + c\lambda_2^{H} + dH = \frac{(H+1)^2}{\text{var}[C(X)]}$$

$$a + b\lambda_1^{H+1} + c\lambda_2^{H+1} + d(H+1) = \frac{(H+1)^2}{\text{var}[C(X)]} .$$

Using Cramer's rule and $\lambda_1\lambda_2 = 1$, we get

$$a = \frac{H^2(\lambda_1^{H+1} + \lambda_2^{H+1}) - (H+1)^2(\lambda_1^{H} + \lambda_2^{H})}{(\lambda_1^{H+1} + \lambda_2^{H+1} - \lambda_1^{H} - \lambda_2^{H})\ \text{var}[C(X)]}$$

$$b = c = \frac{2H+1}{(\lambda_1^{H+1} + \lambda_2^{H+1} - \lambda_1^{H} - \lambda_2^{H})\ \text{var}[C(X)]}$$

$$d = 0.$$

Thus if the process is in-control,

$$ARL(0) = D_0 = \frac{1}{var[C(X)]}\left\{H^2 - \frac{(2H+1)\lambda_1(\lambda_1^H-1)^2}{(\lambda_1-1)(\lambda_1^{2H+1}-1)}\right\} \quad (3.11)$$

Similarly, when the process is out-of-control, one needs to solve equation (3.12)

$$D_i = 1 + p_{-2}D_{i-2} + p_{-1}D_{i-1} + p_oD + p_1D_{i+1} + p_2D_{i+2}$$
$$i = -H+1,\ldots,H-1 \quad (3.12)$$

with

$$D_{-(H+1)} = D_{-H} = D_H = D_{H+1} = 0.$$

The solution of (3.12) is

$$D_i = a' + b'\xi_1^i + c'\xi_2^i + d'\xi_3^i - \frac{i}{E[C(X)]}$$

where ξ_j, j=1,2,3, are the roots of

$$\xi^3 + (1+\frac{P_1}{P_2})\xi^2 - (\frac{P_{-2}+P_{-1}}{P_2})\xi - \frac{P_{-2}}{P_2} = 0,$$

and

$$a' = \frac{1}{detE[C(X)]}\left\{ H\begin{vmatrix} 1-\xi_1 & 1-\xi_2 & 1-\xi_3 \\ 1+\xi_1^{2H+1} & 1+\xi_2^{2H+1} & 1+\xi_3^{2H+1} \\ 1+\xi_1^{2H+2} & 1+\xi_2^{2H+2} & 1+\xi_3^{2H+2} \end{vmatrix} - \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_1^{2H+1} & \xi_2^{2H+1} & \xi_3^{2H+1} \\ 1+\xi_1^{2H+2} & 1+\xi_2^{2H+2} & 1+\xi_3^{2H+2} \end{vmatrix} \right\}$$

$$b'=\frac{\xi_1^{H+1}}{\det E[C(X)]}\left\{\begin{vmatrix} 1 & \xi_2 & \xi_3 \\ 1 & \xi_2^{2H+1} & \xi_3^{2H+1} \\ 2 & 1+\xi_2^{2H+2} & 1+\xi_3^{2H+2} \end{vmatrix} -2H \begin{vmatrix} 0 & 1-\xi_2 & 1-\xi_3 \\ 1 & 1+\xi_2^{2H+1} & 1+\xi_3^{2H+1} \\ 1 & 1+\xi_2^{2H+2} & 1+\xi_3^{2H+2} \end{vmatrix} \right\}$$

$$c'=\frac{\xi_2^{H+1}}{\det E[C(X)]}\left\{ - \begin{vmatrix} 1 & \xi_1 & \xi_3 \\ 1 & \xi_1^{2H+1} & \xi_3^{2H+1} \\ 2 & 1+\xi_1^{2H+2} & 1+\xi_3^{2H+2} \end{vmatrix} +2H \begin{vmatrix} 0 & 1-\xi_1 & 1-\xi_3 \\ 1 & 1+\xi_1^{2H+1} & 1+\xi_3^{2H+1} \\ 1 & 1+\xi_1^{2H+2} & 1+\xi_3^{2H+2} \end{vmatrix} \right\}$$

$$d'=\frac{\xi_3^{H+1}}{\det E[C(X)]}\left\{ \begin{vmatrix} 1 & \xi_1 & \xi_2 \\ 1 & \xi_1^{2H+1} & \xi_2^{2H+1} \\ 2 & 1+\xi_1^{2H+2} & 1+\xi_2^{2H+2} \end{vmatrix} -2H \begin{vmatrix} 0 & 1-\xi_1 & 1-\xi_2 \\ 1 & 1+\xi_1^{2H+1} & 1+\xi_2^{2H+1} \\ 1 & 1+\xi_1^{2H+2} & 1+\xi_2^{2H+2} \end{vmatrix} \right\}$$

where

$$\det= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & \xi_1 & \xi_2 & \xi_3 \\ 1 & \xi_1^{2H+1} & \xi_2^{2H+1} & \xi_3^{2H+1} \\ 1 & \xi_1^{2H+2} & \xi_2^{2H+2} & \xi_3^{2H+2} \end{vmatrix} \neq 0 .$$

Therefore

$$ARL(\mu)=\frac{1}{\det E[C(X)]}\left\{ H-(2H+1) \begin{vmatrix} 1 & \xi_1^{H+1} & \xi_2^{H+1} & \xi_3^{H+1} \\ 1 & \xi_1 & \xi_2 & \xi_3 \\ 1 & \xi_1^{2H+1} & \xi_2^{2H+1} & \xi_3^{2H+1} \\ 1 & \xi_1^{2H+2} & \xi_2^{2H+2} & \xi_3^{2H+2} \end{vmatrix} -(2H-1) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & \xi_1 & \xi_2 & \xi_3 \\ 1 & \xi_1^{2H+1} & \xi_2^{2H+1} & \xi_3^{2H+1} \\ 1 & \xi_1^{H+1} & \xi_2^{H+1} & \xi_3^{H+1} \end{vmatrix} \right\}$$

$$(3.14)$$

Using equation (3.14), one only needs to find the determinant of a 4x4 matrix instead of inverting a (2H-1)x(2H-1) matrix as in the Markov chain approach. The ξ's can be obtained by Cardan's formula (Uspensky 1946), see Appendix A. Also, a FORTRAN program for computing the ARL is in Appendix B.

3.3.4 Choices of Gauges and H

To design a cumulative gauging score chart, one has to choose the locations of the gauges G_1, G_2, and the absorbing barrier H (or equivalently, p_1, p_2, and H). These are called chart parameters.

The traditional way for choosing the parameters is to set ARL(0) equal to a constant, then pick the parameters which minimize the ARL(μ) for a certain value of μ. Examples of this approach can be found in Reynolds (1975) and Woodall (1985 a). Because both ARL(0) and ARL(μ) involve p_1, p_2, H in a complicated manner in the TPG scheme, the traditional method is difficult to apply here. To simplify the problem, we assume that $p_1 = Lp_2$ when $\mu = 0$. Later we shall see that the choice of L does not affect the average run length significantly. Given L, λ_1 and λ_2 are constants. Thus the ARL(0) from (3.11) is then a function of H and p_2 only.

Table 3.1 provides the values of ARL for two-sided TPG

chart and Munford's cumulative score chart when the observations are normally distributed. Empirical studies show that the TPG scheme is better than Munford's cumulative score chart, see Table 3.1. The result is what we expected. Because Munford's procedure is a special case of the TPG chart with $G_2 = \infty$. But these two procedures have some feature in common. For example, for fixed ARL(0) and L, the ARL(μ) decreases as H increases when μ is less than $1.5\sigma_o$. However, for μ greater than $1.5\sigma_o$, the ARL(μ) increases as H increases. Also the choice of L and H is not unique since different L,H combinations can provide similar performance.

From studies with $p_1 = 4p_2$ and H an integer, it was found that H = 3 + [ARL(0)/100] provides a good performance. Here [x] denotes the greatest integer which is less than or equal to x. Based on whether the primary interest is in small or large shifts, one may adjust H by increasing or decreasing the value from the above formula. For example, for L = 4, var[C(X)] = $16p_2$ and λ_1 = -5.828. Substituting var[C(X)] and λ_1 into equation (3.11), we get

$$p_2 = \frac{1}{16 \text{ARL}(0)} \{ H^2 + 0.8535(2H+1) \frac{[(-5.828)^H - 1]^2}{(5.828)^{2H+1} + 1} \}. \qquad (3.15)$$

Table 3.1 Values of the ARL for Two-sided TPG Charts and Munford's Cumulative Score Charts.

	G_1	G_2	H	μ 0.0	0.5	1.0	1.5	2.0	2.5	3.0
Munford	1.695		3	100.	29.3	12.5	7.1	4.8	3.8	3.3
	1.751		4	200.	42.9	17.9	10.0	6.7	5.2	4.5
	1.732		5	300.	52.0	21.8	12.3	8.3	6.4	5.6
	1.695		6	400.	58.9	25.0	14.2	9.7	7.6	6.6
	1.655		7	500.	64.0	27.7	16.0	11.0	8.7	7.7
TPG	2.18	2.756	4	100.	29.9	12.3	6.7	4.3	3.1	2.5
	1.733	2.393	5	200.	42.2	17.0	9.0	5.7	4.2	3.5
	1.756	2.39	6	300.	51.8	20.6	10.8	6.7	4.8	3.9
	1.756	2.41	7	400.	60.7	24.1	12.6	7.9	5.7	4.7
	1.733	2.395	8	500.	66.6	26.7	14.0	8.8	6.4	5.2

When H is given, we can use (3.15) and $p_1 = 4p_2$ to find the locations of G_1 and G_2. If we assume observations are taken from a normal distribution with variance one, then $G_1 = \Phi^{-1}(1-5p_2)$ and $G_2 = \Phi^{-1}(1-p_2)$, where Φ is the cumulative distribution function of the standard normal random variable.

For normal distributions, Tables 3.2-3.6 provide the values of ARL for L = 4, when ARL(0)= 100, 200, 300, 400, and 500. Table 3.7 is an example to illustrate that the choice of L does not significantly affect ARL(μ) whenever H is fixed.

3.4 ONE-SIDED CONTROL

3.4.1 Control Statistic

Without loss of generality, we shall consider the problem of detecting increased process means. For a one-sided control, an out-of-control signal is given if $S_m \geq H$. But if the S_m drops below -F, we shall reset S_m to $S_m = 0$. Therefore the one-sided control chart can be modelled as a generalized

Table 3.2 Values of the ARL(μ) for Two-sided TPG Charts with the ARL(0)=100 and $p_1=4p_2$ under $N(\mu,1)$ Distribution.

H	G_1	G_2	p_2	μ 0.5	1.0	1.5	2.0	2.5	3.0
2	2.1800	2.7555	.00293	38.9	14.1	6.4	3.4	2.2	1.6
3	1.8600	2.4967	.00627	32.1	12.6	6.4	4.0	2.9	2.3
4	1.6627	2.2967	.01082	29.9	12.3	6.7	4.3	3.1	2.5
5	1.3840	2.1290	.01663	25.3	11.2	6.6	4.6	3.7	3.2
6	1.1825	1.9829	.02369	23.5	10.9	6.8	4.9	4.0	3.5
7	0.9904	1.8516	.03200	22.3	10.8	7.0	5.3	4.5	4.2
8	0.8142	1.7329	.04156	21.7	10.8	7.3	5.7	4.8	4.4

Table 3.3 Values of the ARL(μ) for Two-sided TPG Charts with the ARL(O)=200 and $p_1=4p_2$ under N(μ,1) Distribution.

H	G_1	G_2	P_2	μ 0.5	1.0	1.5	2.0	2.5	3.0
3	2.157	2.736	.00314	56.1	19.9	9.1	5.1	3.4	2.6
4	1.926	2.550	.00541	47.5	18.0	8.9	5.3	3.6	2.8
5	1.733	2.393	.00832	42.2	17.0	9.0	5.7	4.2	3.5
6	1.561	2.265	.01184	38.7	16.3	9.1	6.0	4.5	3.7
7	1.405	2.1445	.01600	36.0	15.8	9.2	6.3	5.0	4.4
8	1.256	2.038	.02078	34.0	15.5	9.4	6.7	5.3	4.6

Table 3.4 Values of the ARL(μ) for Two-sided TPG Charts with the ARL(0)=300 and $p_1=4p_2$ under N(μ,1) Distribution.

H	G_1	G_2	p_2	μ 0.5	1.0	1.5	2.0	2.5	3.0
3	2.310	2.8640	.02090	77.3	25.9	11.2	5.9	3.7	2.7
4	2.096	2.7100	.00361	66.1	23.7	11.0	6.2	4.0	3.0
5	1.9155	2.5400	.00554	57.8	21.9	10.9	6.5	4.5	3.6
6	1.7565	2.3900	.00790	51.8	20.6	10.8	6.7	4.8	3.9
7	1.6134	2.3230	.01067	48.9	20.3	11.1	7.2	5.4	4.5
8	1.4814	2.2014	.01385	45.9	19.6	11.1	7.5	5.7	4.8

Table 3.5 Values of the ARL(μ) for Two-sided TPG Charts
with the ARL(0)=400 and $p_1=4p_2$ under N(μ,1) Distribution.

				μ					
H	G_1	G_2	p_2	0.5	1.0	1.5	2.0	2.5	3.0
3	2.4137	2.9540	.00157	98.0	31.5	13.1	6.6	4.0	2.9
4	2.2100	2.7810	.00271	82.3	28.3	12.7	6.8	4.4	3.1
4	2.0380	2.6400	.00416	72.5	26.4	12.6	7.2	4.9	3.8
6	1.8863	2.517	.00592	65.4	25.0	12.5	7.5	5.2	4.1
7	1.7565	2.4100	.00800	60.7	24.1	12.6	7.9	5.7	4.7
8	1.6263	2.3100	.01038	56.2	23.2	12.6	8.2	6.1	5.0

Table 3.6 Values of the ARL(μ) for Two-sided TPG Charts
with the ARL(0)=500 and $p_1=4p_2$ under N(μ,1) Distribution.

H	G_1	G_2	p_2	μ					
				0.5	1.0	1.5	2.0	2.5	3.0
4	2.2960	2.85400	.00216	98.7	32.9	14.2	7.4	4.6	3.3
5	2.1288	2.71700	.00333	86.6	30.6	14.1	7.8	5.1	3.9
6	1.9829	2.59400	.00474	77.9	28.9	14.0	8.1	5.5	4.2
7	1.8488	2.47630	.00640	70.8	27.4	13.9	8.4	5.9	4.8
8	1.7330	2.39500	.00831	66.6	26.7	14.0	8.8	6.4	5.2

Table 3.7 Values of the ARL(μ), λ ,and λ for Two-sided
TPG Charts under Different L When the ARL(0)=465 and H=7.

			μ					
L	0.5	1.0	1.5	2.0	2.5	3.0	λ_1	λ_2
2	71.91	27.38	13.70	8.27	5.85	4.57	-3.732	-0.2680
3	68.76	26.58	13.49	8.21	5.84	4.74	-4.791	-0.2087
4	67.65	26.38	13.50	8.25	5.87	4.75	-5.828	-0.1716
5	66.70	26.24	13.53	8.31	5.91	4.78	-6.854	-0.1459
6	65.56	25.96	13.47	8.31	5.93	4.78	-7.873	-0.1270
7	64.88	25.84	13.49	8.34	5.96	4.80	-8.887	-0.1125
8	64.38	25.77	13.51	8.38	5.99	4.83	-9.899	-0.0917

random walk process between an absorbing barrier at H and a reflecting barrier at -F. The control statistic S_m is defined as follows:

$$S_m = \begin{cases} 0 & S_{m-1} + C(X_m) \le -F \\ \\ S_{m-1} + C(X_m), & \text{otherwise } m=1,2,\ldots \end{cases} \quad (3.16)$$

and $S_o = 0$.

This is a generalization of one of Munford's procedures.

3.4.2 Average Run Length

The governing difference equation is the same as in the two-sided control case. But the boundary conditions change to $D_H = D_{H+1} = 0$ and $D_{-F} = D_{-(F+1)} = D_o$. So, the solution is

$$D_i = a^* + b^* i + c^* \lambda_1^i + d^* \lambda_2^i - \frac{i^2}{\text{var}[C(X)]}$$

where λ_1, λ_2 are the same as before and a^*, b^*, c^*, d^* satisfy the following system of equations:

$$a^* + Hb^* + \lambda_1^H c^* + \lambda_2^H d^* = \frac{H^2}{\text{var}[C(X)]}$$

$$a^* + (H+1)b^* + \lambda_1^{H+1} c^* + \lambda_2^{H+1} d^* = \frac{(H+1)^2}{\text{var}[C(X)]}$$

$$-Fb^* + (\lambda_1^{-F} - 1)c^* + (\lambda_2^{-F} - 1)d^* = \frac{F^2}{\text{var}[C(X)]}$$

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$$-(F+1)b^* + (\lambda_1^{-(F+1)}-1)c^* + (\lambda_2^{-(F+1)}-1)d^* = \frac{(F+1)^2}{\text{var}[C(X)]}$$

Using Cramer's rule, we get

$$a^* = \frac{1}{\det \text{var}[C(x)]}\begin{vmatrix} H^2 & H & \lambda_1^H & \lambda_2^H \\ 2H+1 & 1 & \lambda_1^{H+1}-\lambda_1^H & \lambda_2^{H+1}-\lambda_2^H \\ F^2 & -F & \lambda_1^{-F}-1 & \lambda_2^{-F}-1 \\ 2F+1 & -1 & \lambda_1^{-(F+1)}-\lambda_1^{-F} & \lambda_2^{-(F+1)}-\lambda_2^{-F} \end{vmatrix}$$

$$b^* = \frac{1}{\det \text{var}[C(x)]}\begin{vmatrix} 2H+1 & \lambda_1^{H+1}-\lambda_1^H & \lambda_2^{H+1}-\lambda_2^H \\ F^2 & \lambda_1^{-F}-1 & \lambda_2^{-F}-1 \\ 2F+1 & \lambda_1^{-(F+1)}-\lambda_1^{-F} & \lambda_2^{-(F+1)}-\lambda_2^{-F} \end{vmatrix}$$

$$c^* = \frac{1}{\det \text{var}[C(x)]}\begin{vmatrix} 1 & 2H+1 & \lambda_2^{H+1}-\lambda_2^H \\ -F & F^2 & \lambda_2^{-F}-1 \\ -1 & 2F+1 & \lambda_2^{-(F+1)}-\lambda_2^{-F} \end{vmatrix}$$

$$d^* = \frac{1}{\det \text{var}[C(x)]}\begin{vmatrix} 1 & \lambda_1^{H+1}-\lambda_1^H & 2H+1 \\ -F & \lambda_1^{-F}-1 & F^2 \\ -1 & \lambda_1^{-(F+1)}-\lambda_1^{-F} & 2F+1 \end{vmatrix}$$

where

$$\det = \begin{vmatrix} 1 & \lambda_1^{H+1}-\lambda_1^H & \lambda_2^{H+1}-\lambda_2^H \\ -F & \lambda_1^{-F}-1 & \lambda_2^{-F}-1 \\ -1 & \lambda_1^{-(F+1)}-\lambda_1^{-F} & \lambda_2^{-(F+1)}-\lambda_2^{-F} \end{vmatrix} \neq 0.$$

Therefore

$$ARL(0) = \frac{1}{var[C(X)]} \left\{ H + [(1-H),(1-\lambda_1^H),(1-\lambda_2^H)] \begin{bmatrix} 1 & \lambda_1^{H+1}-\lambda_1^H & \lambda_2^{H+1}-\lambda_2^H \\ -F & \lambda_1^{-F}-1 & \lambda_2^{-F}-1 \\ -1 & \lambda_1^{-(F+H)}-\lambda_1^{-F} & \lambda_2^{-(F+H)}-\lambda_2^{-F} \end{bmatrix}^{-1} \begin{bmatrix} 2H+1 \\ F^2 \\ 2F+1 \end{bmatrix} \right\}$$

$$(3.17)$$

Here, the identity $\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = |A_{22}||A_{11}-A_{12}A_{22}A_{21}|$ was used to obtain the final form (3.17). Similarly, when the process is out-of-control,

$$D_i = \bar{a} + \bar{b}\xi_1^i + \bar{c}\xi_2^i + \bar{d}\xi_3^i - \frac{i}{E[C(X)]}$$

and

$$\bar{a} = \frac{1}{detE[C(X)]} \begin{vmatrix} H & \xi_1^H & \xi_2^H & \xi_3^H \\ 1 & \xi_1^{H+1}-\xi_1^H & \xi_2^{H+1}-\xi_2^H & \xi_3^{H+1}-\xi_3^H \\ -F & \xi_1^{-F}-1 & \xi_2^{-F}-1 & \xi_3^{-F}-1 \\ -1 & \xi_1^{(F+H)}-\xi_1^{-F} & \xi_2^{(F+H)}-\xi_2^{-F} & \xi_3^{(F+H)}-\xi_3^{-F} \end{vmatrix}$$

$$\bar{b} = \frac{1}{detE[C(X)]} \begin{vmatrix} 1 & \xi_2^{H+1}-\xi_2^H & \xi_3^{H+1}-\xi_3^H \\ -F & \xi_2^{-F}-1 & \xi_3^{-F}-1 \\ -1 & \xi_2^{(F+1)}-\xi_2^{-F} & \xi_3^{(F+1)}-\xi_3^{-F} \end{vmatrix}$$

$$\bar{c} = \frac{1}{detE[C(X)]} \begin{vmatrix} \xi_1^{H+1}-\xi_1^H & 1 & \xi_3^{H+1}-\xi_3^H \\ \xi_1^{-F}-1 & -F & \xi_3^{-F}-1 \\ \xi_1^{(F+1)}-\xi_1^{-F} & -1 & \xi_3^{(F+1)}-\xi_3^{-F} \end{vmatrix}$$

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$$\bar{d} = \frac{1}{\det E[C(X)]} \begin{vmatrix} \xi_1^{H+1} - \xi_1^H & \xi_2^{H+1} - \xi_2^H & 1 \\ \xi_1^{-F} - 1 & \xi_2^{-F} - 1 & -F \\ \xi_1^{-(F+1)} - \xi_1^{-F} & \xi_2^{-(F+1)} - \xi_2^{-F} & -1 \end{vmatrix}$$

where

$$\det = \begin{vmatrix} \xi_1^{H+1} - \xi_1^H & \xi_2^{H+1} - \xi_2^H & \xi_3^{H+1} - \xi_3^H \\ \xi_1^{-F} - 1 & \xi_2^{-F} - 1 & \xi_3^{-F} - 1 \\ \xi_1^{-(F+1)} - \xi_1^{-F} & \xi_2^{-(F+1)} - \xi_2^{-F} & \xi_3^{-(F+1)} - \xi_3^{-F} \end{vmatrix} \neq 0 .$$

Thus

$$ARL(\mu) = \frac{1}{E[C(X)]} \left\{ H + [(1-\xi_1), (1-\xi_2), (1-\xi_3)] \begin{bmatrix} \xi_1^{H+1} - \xi_1^H & \xi_2^{H+1} - \xi_2^H & \xi_3^{H+1} - \xi_3^H \\ \xi_1^{-F} - 1 & \xi_2^{-F} - 1 & \xi_3^{-F} - 1 \\ \xi_1^{-(F+1)} - \xi_1^{-F} & \xi_2^{-(F+1)} - \xi_2^{-F} & \xi_3^{-(F+1)} - \xi_3^{-F} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -F \\ -1 \end{bmatrix} \right\}$$

(3.18)

3.4.3 Parameter Choice

As in the two-sided case, we shall assume $p_1 = Lp_2$ when $\mu = 0$. Furthermore, if we assume that the shift in the mean takes place at the beginning of the application of the procedure, then $F = H$ will give smaller ARL(μ) than the other choices. Table 3.8 presents values of ARL(μ) for one-sided TPG chart under different F values when the ARL(0) = 590 and $H = 8$. For fixed L, the response surface of ARL, with respect to μ and H, is a saddle and the saddle point is near the point

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$$H = \begin{cases} 1 + [\dfrac{ARL(0)}{100}] & \text{if } ARL(0) \geq 400 \\ \\ 2 + [\dfrac{ARL(0)}{100}] & \text{if } ARL(0) < 400, \end{cases}$$

and $\mu = 1.5$ for $L = 2,4,6$. When L increases, the ARLs drop
for small μ and increase for large μ. The change-over point
is also at $\mu = 1.5$.

The choice of L does not have a significant effect on
the values of ARL. Table 3.9 provides values of ARL(μ) for
one-sided TPG chart with different H and L combinations when
the ARL(0) = 590. For fixed H,F, and L the ARL(0) is a func-
tion of p_2 only, that is

$$p_2 *ARL(0) = \frac{1}{(8+2L)} \left\{ H^2 + [(1-H),(1-\lambda_1^H),(1-\lambda_2^H)] \begin{bmatrix} 1 & \lambda_1^{H+1} -\lambda_1^H & \lambda_2^{H+1} -\lambda_2^H \\ -F & \lambda_1^{-F} -1 & \lambda_2^{-F} -1 \\ -1 & \lambda_1^{-(F+1)} -\lambda_1^{-F} & \lambda_2^{-(F+1)} -\lambda_2^{-F} \end{bmatrix}^{-1} \begin{bmatrix} 2H+1 \\ F^2 \\ 2F+1 \end{bmatrix} \right\}$$

$$(3.19)$$

Therefore we can choose the gauges G_1 and G_2 in the same way
as we did for the two-sided control problem. Table 3.10
provides the values of $p_2 *ARL(0)$ and λ_1, λ_2 for different H,F,
and L combinations.

In reality the shift in the mean seldom occurs at the
beginning. So, comparing values of ARL(μ) based on this as-
sumption is not appropriate. When the process is under con-
trol, S_m can be in any state between the reflecting and
absorbing barriers. If S_m reaches H, the absorbing barrier,
a check of the process will be made. If everything seems

Table 3.8 Values of the ARL(μ) for One-sided TPG Charts under Different Reflecting Barrier F When the ARL(0)=590 and H=8.

F	0.5	1.0	1.5	2.0	2.5	3.0
8	47.08	20.10	11.36	7.59	5.78	4.87
7	49.09	20.79	11.65	7.72	5.84	4.89
6	51.40	21.59	11.98	7.87	5.92	4.93
5	54.02	22.48	12.43	8.03	6.00	4.97
4	57.03	23.50	12.76	8.22	6.09	5.02
3	60.26	24.61	13.21	8.42	6.18	5.06
2	63.83	25.91	13.74	8.66	6.29	5.12
1	67.07	27.19	14.25	8.89	6.40	5.17

The column header μ spans over the value columns (0.5–3.0).

Table 3.9 Values of the ARL(μ) for One-sided TPG Charts
with Different H and L Combinations When ARL(0)=590.

H	L	0.5	1.0	μ 1.5	2.0	2.5	3.0
	6	45.87	19.88	11.40	7.68	5.87	4.94
8	4	47.08	20.10	11.36	7.59	5.78	4.87
	2	49.74	20.74	11.46	7.54	5.69	4.77
	6	49.13	20.52	11.29	7.34	5.47	4.58
7	4	50.52	20.79	11.28	7.28	5.42	4.55
	2	53.54	21.52	11.43	7.28	5.40	4.55
	6	53.58	21.44	11.26	7.01	5.02	4.02
6	4	55.18	21.76	11.26	6.94	4.95	3.94
	2	58.64	22.56	11.40	6.91	4.88	3.87
	6	59.90	22.79	11.32	6.72	4.66	3.68
5	4	61.74	23.15	11.35	6.69	4.62	3.66
	2	65.73	24.09	11.58	6.73	4.63	3.68
	6	69.33	24.80	11.52	6.42	4.18	3.09
4	4	71.54	25.22	11.55	6.36	4.11	3.03
	2	76.18	26.20	11.71	6.33	4.03	2.69
	6	85.22	28.23	12.10	6.29	3.87	2.80
3	4	87.89	28.77	12.19	6.29	3.88	2.82
	2	93.45	30.06	12.54	6.40	3.93	2.85

Table 3.10 Values of the $p_2 \cdot \text{ARL}(0)$ for One-sided TPG Charts with Different H, F, and L Combinations.

H	F	L=2 $\lambda_1 = -3.7321$ $\lambda_2 = -0.2679$	L=4 $\lambda_1 = -5.8284$ $\lambda_2 = -0.1716$	L=6 $\lambda_1 = -7.8730$ $\lambda_2 = -0.1270$
8	8	10.5794	7.8011	6.1853
	7	9.9802	7.3554	5.8303
	6	9.3814	6.9100	5.4755
	5	8.7840	6.4651	5.1210
	4	8.1861	6.0210	4.7671
	3	7.5978	5.5793	4.4144
	2	6.9921	5.1371	4.0629
	1	6.4950	4.7358	3.7329
7	7	8.0922	5.9518	4.7127
	6	7.5764	5.5688	4.4078
	5	7.0619	5.1862	4.1031
	4	6.5468	4.8042	3.7989
	3	6.0404	4.4243	3.4958
	2	5.5181	4.0438	2.9103
	1	5.0915	3.6992	2.9103
6	6	5.9377	4.3525	3.4401
	5	5.5060	4.0322	3.1852
	4	5.0738	3.7123	2.9308
	3	4.6491	3.3943	2.6773
	2	4.2101	3.0756	2.4246
	1	3.8537	2.7875	2.1878
5	5	4.1181	3.0033	2.3674
	4	3.7688	2.7455	2.1627
	3	3.4260	2.4893	1.9588
	2	3.0704	2.2324	1.7554
	1	2.7852	2.0010	1.5653
4	4	2.6265	1.9035	1.4945
	3	2.3654	1.7091	1.3402
	2	2.0929	1.5139	1.1862
	1	1.8763	1.3390	1.0426
3	3	1.4827	1.0551	0.8219
	2	1.2939	0.9217	0.7173
	1	1.1515	0.8039	0.6204
2	2	0.6331	0.4496	0.3470
	1	0.5555	0.3875	0.2964

to be normal, then S_m will be reset at zero. Therefore one can model this behavior of the procedure before an actual shift as a random walk process between two reflecting barriers at -F and H. Since this is an irreducible recurrent Markov chain, the steady state distribution exists. Let the steady state distribution be $Pr(E_i) = q_i$ for $i = -F+1,\ldots,$ H-1, when the process is in-control, and $ARL_i(\mu)$ be the average run length started at the state E_i. Thus the generalized $ARL(\mu)$ defined by

$$GARL(\mu) = \Sigma_{i=-F+1}^{H-1} q_i * ARL_i(\mu) \qquad (3.20)$$

is a more appropriate measure of the expected time required for a signal than is $ARL(\mu)$.

If P is the transition matrix, then the entry n_{ij} of $(I - P)^{-1}$ is the expected mean recurrent time from the state i to the state j. Using n_{ij}, $j = -F+1,\ldots,H-1$, q_i can be computed as follows:

$$q_i = \frac{N_{oi}}{\Sigma_{j=-F+1}^{H-1} N_{oj}}$$

As an example for H=8 and F=7, the transition probability matrix is

$$
\begin{array}{c}
\\
P = \\
\text{(state i)}
\end{array}
\begin{array}{c}
\\
\\
-6 \\ -5 \\ -4 \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{cccccccccccccc}
\multicolumn{14}{c}{\text{(state j)}} \\
-6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\left[\begin{array}{cccccccccccccc}
P_0 & P_1 & P_2 & 0 & 0 & 0 & P_1{+}P_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
P_{-1} & P_0 & P_1 & P_2 & 0 & 0 & P_{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
P_{-2} & P_{-1} & P_0 & P_1 & P_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & P_{-2} & P_{-1} & P_0 & P_1 & P_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & P_{-2} & P_{-1} & P_0 & P_1 & P_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & P_{-2} & P_{-1} & P_0 & P_1 & P_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_{-2} & P_{-1} & P_0 & P_1 & P_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & P_{-2} & P_{-1} & P_0 & P_1 & P_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & P_{-2} & P_{-1} & P_0 & P_1 & P_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{-2} & P_{-1} & P_0 & P_1 & P_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{-2} & P_{-1} & P_0 & P_1 & P_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{-2} & P_{-1} & P_0 & P_1 & P_2 \\
0 & 0 & 0 & 0 & 0 & 0 & P_2 & 0 & 0 & 0 & P_{-2} & P_{-1} & P_0 & P_1 \\
0 & 0 & 0 & 0 & 0 & 0 & P_1{+}P_2 & 0 & 0 & 0 & 0 & P_{-2} & P_{-1} & P_0
\end{array}\right]
\end{array}
$$

Based on the GARL(μ) the choice of F = H is better for
small shifts, say μ = 0.5. But for large shifts, the choice
of F does not cause any significant changes in the values of
GARL(μ). Also when choosing F = H the GARL(μ) is almost the
same as ARL(μ). Table 3.11 provides values of GARL(μ) and
ARL(μ) for one-sided TPG chart under different F values.
Therefore we recommend the use of F = H for the one-sided TPG
scheme. For the values of ARL(μ) for ARL(0) = 200,300,400,500
and L = 4 and L = 6, see Tables 3.12 and 3.13.

3.5 ONE PAIR OF GAUGES PLUS SHEWHART LIMITS

3.5.1 Introduction

A combined Shewhart-cumulative score quality control
chart has been done by Ncube and Woodall (1984). Although
we can add Shewhart limits on the TPG scheme, it results in
solving a polynomial equation of order four or five which is
not an easy task. Since adding Shewhart limits to Munford's
cumulative score chart essentially is using two pairs of
gauges, we follow Rule III of Ncube and Woodall (1984). Thus,
the scoring system is

Table 3.11 Values of the GARL(μ) and ARL(μ) for One-sided TPG Charts under Different F values when ARL(0)=590 $p_1=4p_2$, and H=8.

H	F	μ							
		0.5		1.0		1.5		2.0	
		GARL	ARL	GARL	ARL	GARL	ARL	GARL	AR1
8	8	46.87	47.08	20.10	20.10	11.37	11.36	7.60	7.59
	7	46.88	49.09	19.95	20.79	11.19	11.65	7.42	7.72
	6	47.00	51.40	19.84	21.59	11.02	11.98	7.25	7.87
	5	47.21	54.02	19.76	22.47	10.87	12.34	7.08	8.03
	4	47.54	57.03	19.72	23.50	10.73	12.76	6.92	8.23
	3	47.89	60.30	19.67	24.61	10.58	13.21	6.76	8.42
	2	48.41	63.83	19.68	25.91	10.45	13.73	6.61	8.66
	1	48.96	67.06	19.70	27.19	10.34	14.25	6.46	8.89

Table 3.12 The values of ARL(μ) for One-sided TPG Charts under Different ARL(0) When L=4.

| ARL(0) | H | μ | | | | | |
		0.5	1.0	1.5	2.0	2.5	3.0
	8	41.73	18.23	10.58	7.23	5.61	4.78
	7	44.66	18.81	10.47	6.91	5.25	4.47
500	6	48.64	19.63	10.42	6.57	4.77	3.87
	5	54.24	20.82	10.47	5.07	3.52	2.74
	4	62.68	22.61	10.61	5.98	3.94	2.95
	8	35.61	16.05	9.64	6.79	5.40	4.67
	7	37.92	16.48	9.50	6.47	5.04	4.38
400	6	41.11	17.12	9.41	6.12	4.56	3.76
	5	45.67	18.09	9.41	5.85	4.24	3.49
	4	52.55	19.56	9.49	5.51	3.72	2.84
	8	29.24	13.70	8.59	6.30	5.15	4.55
	7	30.90	13.97	8.42	5.97	4.80	4.28
300	6	33.26	14.42	8.29	5.61	4.31	3.63
	5	36.71	15.14	8.24	5.33	4.00	3.38
	4	41.97	16.27	8.25	4.98	3.47	2.72
	8	22.55	11.13	7.41	5.73	4.87	4.40
	7	23.48	11.21	7.19	5.39	4.54	4.17
200	6	24.95	11.44	7.00	5.01	4.01	3.48
	5	27.21	11.88	6.89	4.72	3.71	3.26
	4	30.76	12.63	6.81	4.35	3.17	2.57

Table 3.13 Values of the values of ARL(μ) for One-sided TPG Charts under Different ARL(0) and L=6.

ARL(0)	H	μ					
		0.5	1.0	1.5	2.0	2.5	3.0
	8	40.74	18.06	10.62	7.32	5.69	4.85
	7	43.49	18.59	10.49	6.98	5.30	4.50
500	6	47.26	19.36	10.42	6.64	4.85	3.93
	5	52.65	20.50	10.45	6.35	4.48	3.60
	4	60.78	22.25	10.60	6.04	4.00	3.00
	8	34.86	15.93	9.69	6.88	5.48	4.74
	7	37.02	16.32	9.53	6.54	5.09	4.40
400	6	40.03	16.92	9.43	6.19	4.63	3.82
	5	44.37	17.83	9.40	5.89	4.27	3.50
	4	50.98	19.25	9.48	5.57	3.79	2.90
	8	28.75	13.64	8.66	6.39	5.24	4.62
	7	30.22	13.86	8.46	6.04	4.85	4.30
300	6	32.47	14.28	8.32	5.68	4.38	3.69
	5	35.74	14.95	8.24	5.37	4.03	3.39
	4	40.80	16.04	8.25	5.04	3.54	2.78
	8	22.36	11.15	7.50	5.82	4.95	4.47
	7	23.15	11.19	7.26	5.46	4.58	4.18
200	6	24.48	11.37	7.05	5.08	4.08	3.54
	5	26.59	11.77	6.91	4.76	3.74	3.26
	4	29.99	12.48	6.83	4.41	3.24	2.62

$$
C(X) = \begin{cases} 2H-1 & X \geq G_2 \\ 1 & G_1 \leq X < G_2 \\ 0 & -G_1 < X < G_1 \\ -1 & -G_2 < X \leq -G_1 \\ -2H+1 & X \leq -G_2 \end{cases} \qquad (3.21)
$$

The purpose of including this scheme is to investigate the effect of scores which are assigned to observations with $|x| \geq G_2$. This scheme uses scores $\pm |2H-1|$ instead of ± 2 as in (3.1).

3.5.2 Two-sided Control

At each sampling time, if $|X_m| \geq G_2$ or $|S_m| \geq H$, then an out-of-control signal is given. The governing difference equations are slightly different from Munford's scheme, that is

$$
D_i = 1 + qD_{i-1} + (1-p-q-s_1-s_2)D_i + pD_{i+1}
$$
$$
i = -H+1, \ldots, H-1 \qquad (3.22)
$$

and $D_{-H} = D_H = 0$, where $p = \Pr[\ G_1 \leq X < G_2\ ]$, $q = \Pr[\ -G_2 < X \leq -G_1\ ]$, $s_1 = \Pr[\ X \geq G_2\ ]$, and $s_2 = \Pr[\ X \leq -G_2\ ]$.

When the process is in-control and the distribution of observation is symmetry, $s_1 = s_2$ and $p = q$. Let $2s = s_1 + s_2$. The solution of (3.22) is

$$D_i = A\Lambda_1^i + B\Lambda_2^i + \frac{1}{2s}$$

where

$$\Lambda_{1,2} = (1 + \frac{S}{P}) \pm \sqrt{\frac{S}{P}(2 + \frac{S}{P})},$$

$$A = B = -\frac{1}{2s(\Lambda_1^H + \Lambda_2^H)}.$$

Thus

$$ARL(0) = D_o = \frac{1}{2s}[\ 1 - \frac{2}{\Lambda_1^H + \Lambda_2^H}\ ]. \qquad (3.23)$$

When the process is out-of-control, let $s^* = s_1 + s_2$.
Then

$$D_i = A^*\Psi_1^i + B^i\Psi_2^i + \frac{1}{s^*}$$

where

$$\Psi_{1,2} = \frac{(p+q+s^*) \pm \sqrt{(p+q+s^*)^2 - 4pq}}{2P}$$

$$A^* = \frac{(q/P)^H \psi_2^H - \psi_1^H}{s^*(\psi_1^{2H} - \psi_2^{2H})}$$

and

$$B^* = \frac{\psi_2^H - (q/P)^H \psi_1^H}{s^*(\psi_1^{2H} - \psi_2^{2H})}.$$

Thus the ARL(μ) is

$$ARL(\mu) = \frac{1}{s^*}[\ 1 - \frac{(q/P)^H + 1}{\psi_1^H + \psi_2^H}\ ]. \qquad (3.24)$$

The scheme using One Pair of Gauges with Shewhart Limits is abbreviated as OPGSL. The only difference between the

OPGSL and TPG scheme is the scores $\pm$ (2H-1) for the OPGSL scheme when observations fall outside the gauging pair $\pm G_2$. If these two schemes are similar in performance, then the influence of assigning different scores outside $\pm G_2$ is not significant. Since the OPSGL chart allows one observation to signal, it is more sensitive to large shifts in the mean. Based on the results in Table 5.1, for small ARL(0) the TPG scheme is better than the OPGSL scheme in the sense of Woodall's comparison criterion. For ARL(0) = 465 (Table 5.2), the OPGSL scheme is roughly the same as the TPG scheme. Unless the shift in the mean is larger than 2.5 σ_o, the OPGSL is better than the TPG scheme.

3.5.3 One-sided Control

For one-sided control, the OPGSL scheme is also the same as Rule II of Ncube and Woodall (1984). The scoring system for controlling positive shifts is as follows:

$$C'(X) = \begin{cases} H+1 & X \geq G_2 \\ 1 & G_1 \leq X < G_2 \\ 0 & -G_1 < X < G_1 \\ -1 & X \leq -G_1 \end{cases}$$

and $s'=Pr[C'(X)=2H-1]$, $p'=Pr[C'(X)=1]$, $r'=Pr[C'(X)=0]$, and $q'=Pr[\ C'(X)=-1]$. As in the one-sided TPG chart, the control statistic is

$$
S'_m = \begin{cases} 0 & S'_{m-1}+C'(X_m) \leq -F, \\ \\ S'_{m-1} + C'(X_m) & \text{otherwise.} \end{cases}
$$

Therefore, whenever $X_m \geq G_2$ or $S'_m \geq H$, an out-of-control signal is given. The governing difference equations for D_i are

$$
D_i = 1 + q'D_{i-1} + r'D_i + p'D_{i+1}, \quad i=-F+1,\ldots, H-1, \tag{3.25}
$$

and $D_{-F} = D_0$, $D_H = 0$. Therefore the solution of (3.25) is

$$
D_i = A'\Lambda_1^i + B'\Lambda_2^i + \frac{1}{s'}
$$

where

$$
A' = \frac{1 - \Lambda_2^{-F}}{s'[\Lambda_1^H(\Lambda_2^{-F}-1) -\Lambda_2^H(\Lambda_1^{-F} -1)]^2}
$$

$$
B' = \frac{1}{s'[\Lambda_1^H(\Lambda_2^{-F}-1) -\Lambda_2^H(\Lambda_1^{-F} -1)]^2}
$$

and

$$
\Lambda_{1,2} = \frac{(1-r') \pm \sqrt{(1-r')^2-4p'q'}}{2P'}
$$

Thus,

$$
ARL(\mu) = \frac{1}{s'}[\ 1- \frac{\Lambda_1^F - \Lambda_2^F}{(\Lambda_1\Lambda_2)^F(\Lambda_2^H- \Lambda_1^H) + \Lambda_1^{H+F} -\Lambda_2^{H+F}}]. \tag{3.26}
$$

3.6 CONTROLLING THE STANDARD DEVIATION

So far relatively few procedures have been proposed to
control the standard deviation of the process. The most pop-
ular procedure is the R-chart, which uses the sample range
as the control statistic. Page (1962) proposed a cusum pro-
cedure to control the standard deviation by using a single
gauged observation at each time unit. He changed the score
-1 to +1 when $X \leq -G_1$ and then used these non-negative scores
in a cusum scheme.

In this section we will change the scores in our modi-
fied scoring system so that we can use the resulting statis-
tic to control the standard deviation. This illustrates the
versatility of the two pairs of gauges schemes. Usually only
the positive shift in the standard deviation is of concern,
and thus a one-sided control chart is appropriate for this
control purpose. Here we discuss a random walk type chart,
the Two Pairs of Gauges chart, for controlling the Standard
Deviation (TPG-SD). The CUSUM type chart will be discussed
later in Chapter IV.

The scoring system and the corresponding probabilities
for controlling the standard deviation are as follows:

$$
C(X) = \begin{cases} 2 & |X| > G_2 \\ 1 & G_1 \leq |X| < G_2 \\ -1 & |X| < G_1 \end{cases}
$$

and $r^* = \Pr[C(X)=-1]$, $q^* = \Pr[C(X)=1]$, and $p^* = \Pr[C(X) =2]$. Let $S_m^* = \max\{S_{m-1}^* + C(X_m), 0\}$ and $S_0^* = 0$. An out-of-control signal is given if $S_m^* \geq H$.

The ARL of this TPG-SD scheme can also be obtained by solving the difference equations:

$$D_i^* = 1 + r^* D_{i-1}^* + q^* D_{i+1}^* + p^* D_{i+2}', \qquad (3.27)$$

$$i = 0,\ldots,H-1,$$

$D_H^* = D_{H+1}^* = 0$, and $D_{-1}^* = D_0^*$. Following the same procedure as in Section 3.5 the solution of (3.27) is

$$D^* = A^* + B^* \eta_1^i + C^* \eta_2^i - \frac{i}{E[C(X)]}$$

with
$$A^* = \frac{H}{E[C(X)]} + \{(\eta_1 \eta_2)^{H+1}(\eta_2 - \eta_1) + \eta_2^{H+1}(\eta_1 - 1) - \eta_1^{H+1}(\eta_2 - 1)\}_{detEC(X)}$$

$$B^* = -\eta_1(\eta_2 - 1)(\eta_2^{H+1} - 1)/detE[C(X)]$$

$$C^* = \eta_2(\eta_1 - 1)(\eta_1^{H+1} - 1)/detE[C(X)]$$

and
$$det = (\eta_1 - 1)(\eta_2 - 1)(\eta_1^{H+1} - \eta_2^{H+1}).$$

Here $\eta_{1,2} = \{-(p^* + q^*) \pm \sqrt{(p^* + q^*) + 4p^* r^*}\}/2p^*$ and $E[C(X)] = 2p^* + q^* - r^*$. Thus

$$ARL(\mu) = \frac{1}{E[C(X)]}\left\{(H+1) + \frac{(\eta_2-1)(\eta_1^{H+1}-1)(\eta_2^{H+1}-1)}{(\eta_1-1)(\eta_2-1)(\eta_1^{H+1}-\eta_2^{H+1})}\right\}. \qquad (3.28)$$

3.7 GAUGING SCHEMES WITH SAMPLES OF SIZE n

In an implementation of control chart procedures, very often samples of size four or five are taken at each sampling time. When the sample size is larger than one, Munford (1980) applied the gauging scheme to the sample means. Thus using his procedure, the accurate measurement for each individual observation is needed. Usually a longer time and additional work are needed to get an accurate reading.

In order to speed up an inspection procedure, we may use some kind of mechanical device to gauge each observation separately. Therefore, the gauging scores can be obtained more easily and quickly. The proposed procedure, TPG(n), is to use the sum of gauging scores at each sampling time as a new score. Then this new score is added to update a cumulative score, $S_m^{(n)}$, which is used as the control statistic.

Let $\underline{X}_m = (X_{m1}, \ldots, X_{mn})$ be the n observations obtained at sampling time m. We gauge each X_{mj}, j=1,2,...,n, based on the scoring system $C(X_j)$ which is defined in equation (3.1), and let

$$U_m^{(n)} = \Sigma_{j=1}^{n} C(X_{mj}).$$

When the process is in-control, the expected value of $U_m^{(n)}$ is zero. Otherwise, the value of $E[U_m^{(n)}]$ is

$n[2(p_2-p_{-2})+(p_1-p_{-1}]$. Since $U_m^{(n)}$ is the sum of n independent identical discrete random variables $C(X_j)$, the distribution of $U_m^{(n)}$ can be obtained by expanding the moment generating function

$$M(t) = (e^{2t}p_2 + e^t p_1 + p_o + e^{-t}p_{-1} + e^{-2t}p_{-2})^n.$$

The distribution of $U_m^{(n)}$ can be tablulated as follows:

y	-2n	-2n+1	...	0	...	2n-1	2n
$p[U_m^{(n)} = y]$	P_{-2n}	P_{-2n+1}	$\cdots$	P_o	$\cdots$	P_{2n-1}	P_{2n}.

For two-sided control of the mean, the control statistic is

$$S_m^{(n)} = S_{m-1}^{(n)} + U_m^{(n)} \qquad m=1,2,\dots$$

and $S_o^{(n)} = 0$. An out-of-control signal is given if $S_m^{(n)} \notin (-H,H)$.

The average run length for this TPG(n) scheme can be obtained by using a Markov chain approach mentioned in Section 3.2. The possible transient states are E_i for $i = -H+1,\dots,H-1$. The transition probabilities from the state i to the state j are

$$P_{i,j} = \text{Pr}[\ U_m^{(n)} = j-i], \quad j=-H+1,\ldots,H-1,$$

and

$$P_{i,i} = \text{Pr}[\ U_m^{(n)} = 0].$$

For one-sided control, like the case of n=1, a reflecting barrier is set at -F. Therefore the control statistic for controlling positive shifts is

$$S_m^{(n)+} = \begin{cases} 0 & S_{m-1}^{(n)+} + U_m^{(n)} \leq -F \\ \\ S_{m-1}^{(n)+} + U_m^{(n)} & \text{otherwise.} \end{cases}$$

When gauging n observations at each sampling time, the gauge locations suggested before may not be a good choice. Tables 3.13 and 3.14 show that the choice of $G_1 = 1.0\sigma_o$ and $G_2 = 2.0\sigma_o$ gives better performance than does the choice of following the case of n = 1. They also show that the performance of the TPG(n) procedure is better than the $\overline{X}$ chart when the shift in the mean is small. In order to detect large shifts more quickly, Shewhart limits can be added to this scheme.

For controlling the standard deviation, we need to change all negative scores to positive scores. However, the non-negative scores result in a higher frequency of false signals in a random walk type procedure. Since a reference value will produce a downward trend when a process is in-

control, a one-sided cusum scheme may be better for this case. The cusum type scheme will be discussed later in Chapter IV.

Table 3.14 Values of ARL(μ) for the $\overline{X}$, CUSUM, and TPG(n) Charts When the Sample Size n=5 and ARL(0) = 130.

μ	$\overline{X}$ H=1.194	CUSUM K=0.4 H=1.0	TPG(n) G_1=1.732 G_2=2.395 H=9	TPG(n) G_1=1.0 G_2=2.0 H=17
0.0	131.9	133.6	129.7	134.3
0.2	71.1	39.6	40.5	29.6
0.4	26.2	11.7	19.5	14.9
0.6	10.8	5.3	12.6	9.8
0.8	5.3	3.3	8.8	7.4
1.0	3.0	2.4	6.4	5.9
1.2	2.0	1.8	4.9	4.9

Table 3.15 Values of ARL(μ) for the $\overline{X}$, CUSUM, and TPG(n) Charts When the Sample Size n=5 and ARL(0) = 465.

μ	$\overline{X}$ H=1.372	CUSUM K=0.4 H=1.322	TPG(n) G_1=1.82 G_2=2.462 H=16	TPG(n) G_1=1.0 G_2=2.0 H=32
0.0	467.3	465.	477.3	464.2
0.2	216.5	85.4	84.0	55.4
0.4	66.9	17.0	39.6	27.4
0.6	23.1	6.7	24.3	18.1
0.8	10.0	4.0	17.0	13.5
1.0	4.9	2.9	12.2	10.7
1.2	2.9	2.3	9.3	8.8

IV.CUSUM CHARTS BASED ON GAUGING SCORES

4.1 INTRODUCTION

Page (1962) applied the same gauging scores,-1,0, and 1, to the regular CUSUM scheme. Its performance is uniformly better than Munford's cumulative score chart in the sense that the ARL is always lower for shifts from zero. Two examples are in Table 5.1 and 5.2. However, this superiority is not retained if a zero reference value is used in the CUSUM scheme (Khan, 1984). Khan showed the ARL(0) for the CUSUM gauging scheme with zero reference value is much smaller than that for Munford's scheme. When we compare these two schemes by adjusting the stopping boundaries to make the ARL(0)s equal, then the superiority of the CUSUM scheme is washed out. Since the TPG chart is the same type as Munford's cumulative score chart, we expect that applying the modified gauging scores to the CUSUM scheme is also better than the TPG chart. Examples in Table 5.1 and 5.2 show that this expectation is true indeed.

In this chapter, we apply the modified scoring system to the CUSUM chart for controlling the process mean and the standard deviation.

4.2 CONTROLLING THE MEAN

For controlling the process mean, the CuSum chart based on Gauging Scores, denoted as CSGS hereafter, uses the same gauging system $C(X)$ as in the TPG scheme for each observation. For one-sided control, we will assume that the mean shifts to the positive direction. Therefore the control statistic is defined as

$$S'_m = \max\{S'_{m-1} + (C(X_m) - K^+), \ 0\} \qquad (4.1)$$

where $S'_o = 0$ and K^+ is the reference value. We will continue to take observations as long as $S'_m < H^+$.

In order to use the discrete Markov chain process to model this CSGS procedure, we let $K^+ = v/w$ and $H^+ = u/w$. Where u, v, w are integers and K^+ is a rational fraction with value less than one. Under these restrictions, the scoring system of the CSGS scheme is similar to that of the TPG chart except different scores are used. The scoring system for the CSGS chart is

$$C^+(X) = \begin{cases} 2w-v & X \geq G_2 \\ w-v & G_1 \leq X < G_2 \\ -v & -G_1 < X < G_1 \\ -(w+v) & -G_2 < X \leq -G_1 \\ -(2w+v) & X \leq -G_2. \end{cases} \qquad (4.2)$$

Thus the control statistic can be re-expressed as

$$S_m^+ = \max\{S_{m-1}^+ + C^+(x_m), \ 0\} \qquad (4.3)$$

and $S_o^+ = 0$, and S_m^+ is equivalent to wS'_m.

4.2.1 Transition Matrix

For this CSGS scheme, the D_i, as defined in chapter III, satisfies the following difference equations

$$D_i = 1 + P_{-2} D_{i-(2w+v)} + P_{-1} D_{i-(w+v)} \qquad (4.4)$$
$$+ P_o D_{i-v} + P_1 D_{i+w-v} + P_2 D_{i+2w-v}$$

for $i = 0, \ldots, u-1$ with the constrains $D_i = 0$ for $i \geq u$ and $D_i = D_o$ for $i \leq 0$. In order to solve for D_i, one has to find the roots of a polynomial equation with degree $4w$. Therefore it is too difficult to derive an explicit expression for the ARL. Thus the Markov chain method is used to obtain the ARL. The transition probability matrix has two different patterns which depend on the relationship among the parameters u, v, and w.

Let $r_{i,j}$ be the transition probabilities from the transient state i to the transient state j. The possible states are $0, 1, 2, \ldots, u-1$, and u is the absorbing state. At each state i, S_m^+ can move to state j which is either $i + (2w-v)$,

$i+(w-v)$, $i-v$, $i-(w+v)$, or $i-(2w+v)$ depending on the value of the gauging score $C^+(X_m)$. If j is larger than $u-1$, then S_m^+ is absorbed at u. However, if j is less than zero, S_m^+ is reset to zero.

Case 1 $u > w-v$. Based on (4.2) and (4.4), the transition probability matrix $R=(r_{i,j})$ is defined as

$$r_{i,j} = \begin{array}{ll} p_{-2}+p_{-1}+p_o & i=0,\ldots,v \\ p_{-2}+p_{-1} & i=v+1,\ldots,\ min(v+w,u-1) \\ p_o & i=v+w+1,\ldots,\ min(v+2w,u-1) \\ & if\ v+w+1 > min(v+2w,u-1) \end{array}$$

$$r_{i,i+w-v} = p_1 \qquad i=0,\ldots,u+v-w-1$$
$$r_{i,i+2w-v} = p_2 \qquad i=0,\ldots,\ min(u+v-2w-1,u-1)$$
$$\qquad\qquad\qquad\qquad if\ u+v-2w-1>0$$

$$r_{i+v,i} = p_o \qquad i=1,\ldots,u-v-1$$
$$r_{i+v+w,i} = p_1 \qquad i=1,\ldots,u-v-w-1,\ if\ u-v-w-1>1$$
$$r_{i+v+2w,i} = p_2 \qquad i=1,\ldots,u-v-2w-1,\ if\ u-v-2w>1$$
$$r_{i,j} = 0 \qquad otherwise$$

where $p_2 = Pr[C^+(x)=2w-v]$, $p_1 = Pr[C^+(x)=w-v]$, $p_o = Pr[C^+(x)=-v]$, $p_{-1} = Pr[C^+(x)=-(w+v)]$, and $p_{-2} =Pr[C^+(x)=-(2w+v)]$ Therefore, the average run length can be obtained by following equations (3.6) and (3.7). This gives

$$ARL^+ = \underline{e}(I - R)^{-1}\underline{1}, \tag{4.5}$$

where $\underline{e} = [1,0,\ldots,0]$ and $\underline{1}$ is a column vector with value one for each component.

As an example for $v=1$, $w=4$, and $u=12$, the transition probabilities matrix is

$R =$

	0	1	2	3	4	5	6	7	8	9	10	11
0	$p_{-2}+p_{-1}+p_0$	0	0	p_1	0	0	0	p_2	0	0	0	0
1	$p_{-2}+p_{-1}+p_0$	0	0	0	p_1	0	0	0	p_2	0	0	0
2	$p_{-2}+p_{-1}p_0$	0	0	0	0	p_1	0	0	0	p_2	0	0
3	$p_{-2}+p_{-1}$	0	p_0	0	0	0	p_1	0	0	0	p_2	0
4	$p_{-2}+p_{-1}$	0	0	p_0	0	0	0	p_1	0	0	0	p_2
5	$p_{-2}+p_{-1}$	0	0	0	p_0	0	0	0	p_1	0	0	0
6	p_{-2}	p_{-1}	0	0	0	p_0	0	0	0	p_1	0	0
7	p_{-2}	0	p_{-1}	0	0	0	p_0	0	0	0	p_1	0
8	p_{-2}	0	0	p_{-1}	0	0	0	p_0	0	0	0	p_1
9	p_{-2}	0	0	0	p_{-1}	0	0	0	p_0	0	0	0
10	0	p_{-2}	0	0	0	p_{-1}	0	0	0	p_0	0	0
11	0	0	p_{-2}	0	0	0	p_{-1}	0	0	0	p_0	0

<u>Case 2</u> $u < w-v$. The transistion probability matrix R is defined as

$$
\begin{aligned}
r_{i,0} &= p_{-2} + p_{-1} + p_0 & i&=0,\ldots,v \\
&= p_{-2} + p_{-1} & i&=v+1,\ldots,u-1 \\
r_{i+v,i} &= p_0 & i&=1,\ldots,u-v-1 \\
r_{i,j} &= 0 & &\text{otherwise}
\end{aligned}
$$

For this simple case the average run length is

$$
ARL^+ = \frac{1}{p_1 + p_2}. \tag{4.6}
$$

As an example, for $v = 2$, $w = 7$, and $u = 4$, R is

$$
R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\left[\begin{array}{cccc}
p_{-2} + p_{-1} + p_0 & 0 & 0 & 0 \\
p_{-2} + p_{-1} + p_0 & 0 & 0 & 0 \\
p_{-2} + p_{-1} + p_0 & 0 & 0 & 0 \\
p_{-2} + p_{-1} & p_0 & 0 & 0
\end{array}\right]
\end{array}
$$

4.2.2 Two-sided Control

For two-sided control, one has to use another one-sided control chart for detecting negative shifts, along with the chart for positive shifts. The chart for negative shifts uses the statistic

$$S_m^- = \min\{S_{m-1}^- + (C^-(x_m) + K^-), 0\}$$

and it signals out of control if $S_m^- < H^-$. The scores, $C^-(x)$, for the chart for negative shifts are $2w+v$, $w+v$, v, $-w+v$, and $-2w+v$ under the same restrictions on H^- and K^-. The run length N is defined as $N = \min\{N^+, N^-\}$, where N^+, N^- are the run lengths for the positive and negative charts, respectively. Also if ARL^+ and ARL^- are the average run lengths of the positive and negative chart, then the two-sided average run length is computed by

$$ARL = \frac{(ARL^+)(ARL^-)}{ARL^+ + ARL^-} \qquad (4.8)$$

(see Kemp (1961)). A SAS program for computing the ARL^+ and ARL^- for this scheme is provided in Appendix (B).

4.2.3 Choice of Parameters

Like the TPG scheme, the choice of chart parameters u, v, w, G_1, and G_2 for a desired ARL(0) is not unique. Table 4.1 is an example to illustrate this point. From this table, we can see that for a fixed value of G_1, smaller values of K

= v/w can detect small shifts more quickly than do larger values of K. But for detecting large shifts in the mean, larger K values are better. The effect of the gauge location G_1 on the ARL is similar to the effect of K. These phenomena are consistent with the properties of Page's CUSUM scheme with gauging.

Given a desired value of ARL(0), we have to rely on a trial-and-error approach to find appropriate values to meet the ARL(0) by picking the G_1 first then adjusting G_2 and u. In Table 4.1 schemes I and II(3) seem to be reasonable choices under the overall considerations about the range of the shifts. Since there is no clear rule for choosing the chart parameters, some other choices of parameters could be used for detecting particular shifts.

Table 4.1 Comparison of ARL's for CSGS schemes under different combination of chart parameters and gauges' locations.

μ	I $G_1=0.75$ $v=1$			II $G_1=1.2$ $v=1$			III $G_1=1.5$ $v=1$		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
	G_2			G_2			G_2		
	2.3	2.44	2.59	2.455	2.603	2.6	2.485	2.564	2.66
	w=3	w=4	w=5	w=3	w=4	w=5	w=3	w=4	w=5
	u=12	u=19	u=27	u=8	u=12	u=17	u=7	u=10	u=13
0.0	464.9	464.9	465.6	465.8	466.0	467.0	465.5	466.2	467.0
0.5	42.8	37.7	36.1	65.1	54.5	47.3	93.6	79.5	69.2
1.0	11.9	12.4	13.3	13.9	13.1	12.9	18.3	16.2	15.2
1.5	6.7	7.4	8.2	6.3	6.5	6.8	6.9	6.7	6.8
2.0	4.7	5.4	6.2	4.1	4.2	4.5	3.9	4.0	4.1
2.5	3.7	4.3	5.0	3.1	3.2	3.3	2.7	2.8	2.9
3.0	3.2	3.5	4.2	2.6	2.7	2.7	2.3	2.3	2.3

4.3 CONTROLLING THE STANDARD DEVIATION

The CUSUM type procedure for controlling the standard deviation is denoted by CSGS-SD. In this procedure all negative scores are changed to positive scores. Then with $K = v^*/w^*$ the control statistic is

$$S_m = \max\{S_{m-1} + C(X_m),\ 0\}.$$

where $S_0 = 0$, and

$$C(X) = \begin{cases} 2w^* - v^* & \text{if} \quad |X| \geq G_2 \\ w^* - v^* & \text{if} \quad G_1 \leq |X| < G_2 \\ -v^* & \text{if} \quad |X| < G_1 \end{cases}$$

Let $r = \Pr[C(X) = -v^*]$, $q = \Pr[C(X) = w^* - v^*]$, and $p = \Pr[C(X) = 2w^* - v^*]$. If $u^* > w^* - v^*$, the transition probability $r_{i,j}$ from the transient state i to the transient state j can be defined as follows:

$$r_{i,o} = r \qquad i = 0, \ldots, v^*$$

$$r_{i+v^*,i} = r \qquad i = 1, \ldots, u^* - v^* - 1$$

$$r_{i,i+w^*-v^*} = q \qquad i = 0, \ldots, u^* + v^* - w^* - 1$$

$$r_{i,i+2w^*-v^*} = p$$

$$i = 0, \ldots, \min(u^* + v^* - 2w^* - 1, u^* - 1)$$

$$\text{and} \quad \text{if} \ u^* + v^* - 2w^* - 1 \geq 0$$

$$r_{i,j} = 0 \qquad \text{otherwise}$$

For example if $v^*=2$, $w^*=3$, and $u^*=8$, then

$$
R = \begin{matrix}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & r & q & 0 & 0 & p & 0 & 0 & 0 \\
1 & r & 0 & q & 0 & 0 & p & 0 & 0 \\
2 & r & 0 & 0 & q & 0 & 0 & p & 0 \\
3 & 0 & r & 0 & 0 & q & 0 & 0 & p \\
4 & 0 & 0 & r & 0 & 0 & q & 0 & 0 \\
5 & 0 & 0 & 0 & r & 0 & 0 & q & 0 \\
6 & 0 & 0 & 0 & 0 & r & 0 & 0 & q \\
7 & 0 & 0 & 0 & 0 & 0 & r & 0 & 0
\end{matrix}
$$

Assuming observations are normal with mean zero, a numerical comparison based on the ARL between CSGS-SD and Page's scheme is given in Table 4.2.

4.4 SAMPLES OF SIZE

Extending the CSGS chart to the case of samples of size n is straightforward. One only needs to apply the score $U_m^{(n)}$ (see page 64) to a cusum type control chart. Since possible values of $U_m^{(n)}$ are integers, the reference value K is restricted to be an integer. Therefore a Markov chain approach is still applicable. A numerical example which compares the regular CUSUM chart with the CSGS(n) chart is in Table 4.3. The results show that the performance of the

CSGS(n) chart is close to the CUSUM chart except for a very small shifts, say $\mu = 0.2$. Again the choices of G_1, G_2, and K are not easy to make since they must be obtained by trial-and-error to achieve a specified ARL(0).

For controlling the standard deviation, one has to change all negative scores to positive scores in the gauging score system used for single observations (see (3.1)). Then, the moment generating function method can be used to find the probabilities for all possible non-negative integers $0,1,\ldots,2n$, and finally, these non-negative scores can be applied in a one-sided CUSUM scheme.

Zacks (1980) derived a recursive equation to compute the run length distribution for one-sided cusum charts with non-negative discrete random variables. His method is a special case of Brook and Evans (1972). Although his method can be used here, we still prefer to use a Markov chain approach. Since the SAS PROC MATRIX can handle matrix operation very easily, the Markov chain approach is easy to program.

The total numbers of observations required by various procedures for detecting shifts in the process standard deviation are given in Table 4.2. Results show that the CSGS-SD and CSGS(n)-SD charts are better than Page's gauging CUSUM scheme for this example. They are also better than the R chart when the change in σ is small.

Table 4.2 The Expected Number of Observations for Detecting Shifts in the Process Standard Deviation for the R, Page's Gauging CUSUM, CSGS-SD, and CSGS(n)-SD Charts.

	R	Page	CSGS-SD	CSGS(n)-SD
		$G = 1.2$	$G_1 = 1.2$	$G_1 = 1.0$
	$n=5$	$K = .25$	$G_2 = 2.46$	$G_2 = 2.03$
			$K = 0.25$	$K = 2.0$
σ	$H = 4.4$	$H = 5.5$	$H = 7.0$	$H = 6.0$
1.0	292.0	290.1	291.6	298.2
1.1	135.9	112.0	102.4	98.9
1.5	21.9	30.1	25.4	24.1
2.0	9.6	18.4	14.1	13.8
2.5	7.0	14.6	10.5	11.0
3.0	5.9	12.8	8.8	9.3

Table 4.3 Values of the ARL(μ) for the Two-sided CUSUM
and CSGS(n) Charts When the Sample Size Is Five

			μ						
			0.0	0.2	0.4	0.6	0.8	1.0	1.2
CUSUM	K=0.4	H=1.342	485.	93.8	17.7	6.8	4.1	3.0	2.3
CSGS(n)	G_1=1.0	G_2=1.96	485.	80.3	17.4	7.9	4.9	3.6	2.8
	k=1.0	H=7.0							

V. EFFICIENCY AND ROBUSTNESS OF CUMULATIVE GAUGING SCORE CHARTS

5.1 INTRODUCTION

In this chapter we compare the efficiencies of the three proposed cumulative gauging score charts: TPG, OPGSL, CSGS charts with the $\bar{X}$-chart, the CUSUM chart, Munford's cumulative score chart, and Page's cusum scheme using gauging. Comparisons are based on the average run length for different amounts of shift which range from 0.5 up to 3.0 standard deviations.

To study the robustness of TPG and CSGS schemes, we use Monte Carlo simulation. Here we focus on two different aspects of robustness. First, we are interested in the robustness to heavy tailed distributions when the chart parameters and gauge locations are chosen under the normality assumption. The heavy tailed distributions used in this study are the contaminated normal, double exponential, and Cauchy. We also study the robustness to serially correlated observations by generating autoregressive (AR) and moving average (MA) observations with order one. Here we only compare the TPG and CSGS charts with the $\bar{X}$-chart and the CUSUM chart.

5.2 EFFICIENCY UNDER NORMALITY

Two values of ARL(0), 130 and 465, are used to make comparisons among these procedures. The reason for choosing two values is the fact that the performance of the cumulative gauging score chart is not consistent with respect to ARL(0). We want to use a small ARL(0) like 130 and a moderate ARL(0) such as 465 to shed some light on this aspect. Tables 5.1 and 5.2 give the ARL(μ) for normal observations with ARL(0)=130 and ARL(0)=465 respectively.

Because the results show that the ARL is always smaller for the mean shifts from zero, the TPG and OPGSL charts are better than Munford's cumulative score charts. Similarly the CSGS chart is uniformly better than Page's cusum scheme. The CSGS chart is the best one among these cumulative gauging score charts. Its efficiency is also very close to the standard CUSUM chart.

When ARL(0)=130, the TPG chart is slightly better than the OPGSL chart if $\mu \leq 1.5$. For ARL(0)=465 the OPGSL chart is more sensitive than the TPG if $\mu \geq 2.5$. Based on these examples, the performances of the TPG and OPGSL charts are similar. Since the OPGSL chart is just an extreme case of the TPG scheme, the results indicate that assigning a larger score to the observations which fall outside the gauges $\pm$ G_2 will not significantly improve the performance. But the ARL(μ) for the OPGSL chart is easier to compute. In order

to design an OPGSL scheme, one only needs to increase the location of G_1 slightly and then use the same H as used in Munford's cumulative score chart.

Page's gauging cusum scheme and the CSGS scheme both using nonzero reference values are more sensitve than Munford's scheme and the TPG scheme. Thus a nonzero reference value is necessary whenever one decides to use a CUSUM type scheme. Although the CSGS chart seems better than the other gauging score charts, it is not easy to find the best combination of K, H and G_1, G_2. These parameters must be determined by a trial-and-error search in order to meet a certain desired value of ARL(0).

In general, if the observations are random samples from normal distributions, the CSGS scheme should be the first choice. The TPG scheme should be used if the observations are non-normal or serially correlated.

5.3 ROBUSTNESS

The effect of correlation on the average run length was investigated through simulation. First the chart parameters for each scheme were chosen such that the ARL(0) for each scheme is equal to 130 when the process is under control and the observations are independent normal. Then serially correlated observations were generated based on the following two models.

$$\text{AR}(1): \quad Y_t = \mu + \phi(Y_{t-1} - \mu) + \varepsilon_t$$

$$\text{MA}(1): \quad Y_t = \mu + \varepsilon_t - \psi\varepsilon_{t-1}$$

where ε_t are independent random variables (sometimes called white noise) with mean zero and ϕ and ψ are the serial correlations. The subscript t is the time unit.

The white noise ε_t is generated by using one of the following four random variables, normal, double exponential, contaminated normal, and Cauchy. If the variance exists, it is set equal to one. Here the contaminated normal is defined by

$$\text{CN} = (1-U)Z_1 + UZ_2$$

where U is a discrete random variable with $\Pr[U=1] = \alpha$ and $\Pr[U=0] = 1 - \alpha$, and Z_1, Z_2 are independent normal random variables with mean zero and variances σ_1^2, σ_2^2, respectively. U is independent of Z_1 and Z_2 also. The variance of CN is $(1 - \alpha)\sigma_1^2 + \alpha\sigma_2^2$. We choose $\sigma_1^2 = .714$, $\sigma_2^2 = 6.4285$, and $\alpha = 0.05$ to make the variance of CN equal to one. Although the variance of the Cauchy random variable does not exist, for comparison purposes we choose the scale parameter such that the probability for the Cauchy random variable within $\pm$

1.645 units of the median is .90.   This choice was suggested by Arnold (1965).

To investigate the effects of serial correlation on ARL(0), 1000 simulation runs limited by 600 observations per run were used for each scheme under different white noise distribution and correlation (ϕ or ψ) combinations. When the process mean is changed, 500 runs were used to estimate ARL(μ).

The simulated values of ARL(μ) for autoregressive and moving average models under different white noises are given in Tables 5.3-5.6 and 5.10-5.13, respectively. The figures within parentheses are estimated standard deviations of the estimateds of the ARL. The same results are also summarized in Figures 5.1-5.4 and 5.5-5.8. For the TPG, CUSUM, and CSGC charts, the values of ARL(μ) under different white noise, autocorrelation, and shift μ are provided in Tables 5.7-5.9.

When the observations are independent, (ϕ = 0 or ψ = 0), the TPG and CSGS charts are more robust to heavy tailed distributions than the $\overline{X}$ and CUSUM charts (see Figures 5.3 - 5.8). That is, the chart parameters for the TPG or CSGS chart have less effect on the values of ARL(0) when observations are heavy tailed distributions. On the other hand, the chart parameters for the $\overline{X}$-chart and CUSUM chart will reduce the values of ARL(0) significantly if the observations come from heavy tail distributions. For Cauchy and double exponential observations, the CSGS chart is more robust than the TPG

scheme. But for contaminated normal observation the result is just the opposite. The CUSUM chart is slightly better than $\overline{X}$-chart on this aspect.

Increasing positive serial correlation decreases the ARL(0) in all four schemes. For normal observations (Figure 5.1), decreasing negative correlation increases the ARL(0) up to $\phi=-0.3$ then the ARL(0) drops as ϕ decreases below -0.3. In Figures 5.2- 5.4, the $\overline{X}$-chart and CUSUM chart may seem to be more stable for varying ϕ, but this is not true in general. Because the ARL(0) is very small for the $\overline{X}$-chart and CUSUM chart when the observations have a heavy tail distribution, the effect of serial correlation on ARL(0) is disguised by the low value of ARL(0). In order to illustrate this point we set the ARL(0)=200 for the $\overline{X}$ and CUSUM charts when the observations generated from the double exponential distribution. Then the pattern is the same as for the normal distribution (see Figure 5.9). For the moving average model, the effect of negative values of ψ on the ARL(0) is similar to the autoregressive model with positive ϕ (see Figures 5.5 - 5.8). This similarity also was found by Bagshaw and Johnson (1974). In general, the TPG chart is more robust to heavy tailed distributions and more resistant to serially correlated observations.

When the process mean shifts, the values of ARL(μ) for large shifts are very close for all four distributions. Significant differences appear only when μ is less than or

equal to 0.5. The effect of small correlation is masked by large shifts in the mean. In conclusion, the heavy tailed and serially correlated observations can cause large changes in the ARL(0), but for large amounts of shift, these effects are masked due to the small value of ARL(μ).

One thing we should be aware of when the observations drift away from the control value, is that it is very difficult to say whether the serial correlation or the mean shift is the cause of the drift. Therefore further analysis such as using data smoothing techniques to detrend the data is necessary. Then a test of $\phi = 0$ can be done. Also when the correlation $|\phi|$ (or $|\psi|$) is large, we may not want to say that the process is under control. We have to be careful to define what we mean by out-of-control whenever there is a serious dependency among observations.

Table 5.1 Values of ARL(μ) for the $\overline{X}$, CUSUM, Munford's, Page's, OPGSL, TPG, and CSGS Charts When X is N(μ,1) and ARL(0)=130.

μ	$\overline{X}$ H=2.67	CUSUM H=6.06 K=0.25	MUNFORD H=4 G =1.54	PAGE H=2.2 K=0.2 G =1.24	OPGSL H=4 G_1=1.57 G_2=3.5	TPG H=4 G_1=1.732 G_2=2.395	CSGS H=2.6 K=0.2 G_1=1.2 G_2=2.5
0.0	130	130	130	131	130.88	130.03	129.08
0.5	63	21	31	31.68	31.86	34.02	29.14
1.0	21	8.8	14	10.26	13.85	13.73	9.56
1.5	8.26	5.5	8.28	5.65	7.89	7.25	5.15
2.0	3.97	4.1	5.90	4.03	5.16	4.54	3.43
2.5	2.31	3.3	4.70	3.38	3.62	3.27	2.58
3.0	1.59	2.8	4.31	3.13	2.60	2.62	2.20

Table 5.2 Values of ARL(μ) for the $\overline{X}$, CUSUM, Munford's, Page's, OPGSL, TPG, and CSGS Charts When X is N(μ,1) and ARL(0)=465.

μ	$\overline{X}$ H=3.07	CUSUM H=5 K=0.5	MUNFORD H=6 G =1.77	PAGE H=3 K=0.2 G =1.21	OPGSL H=6 G_1=1.80 G_2=3.72	TPG H=7 G_1=1.82 G_2=2.462	CSGS H=3.4 K=0.2 G_1=1.2 G_2=2.6
0.0	465	465	465	470	463.3	465.1	467
0.5	191.1	38	66.34	50.78	68.3	67.59	47.29
1.0	52	10.4	27.54	13.82	27.56	26.36	12.94
1.5	17.2	5.75	15.27	7.51	14.42	13.48	6.78
2.0	7.3	4.10	10.15	5.36	8.60	8.25	4.47
2.5	3.52	3.11	7.82	4.49	5.50	6.11	3.30
3.0	2.11	2.57	6.74	4.16	3.58	5.10	2.69

Table 5.3 Values of ARL(0) Obtained by Simulation for the TPG, $\bar{X}$, CUSUM, and CSGS Charts under the Autoregressive . Model of Order One with N(0,1) Noise.

ϕ	TPG	$\bar{X}$	CUSUM	CSGS
-0.5	92.4(2.48)	55.9(1.82)	111.3(3.41)	226.5(5.71)
-0.4	106.6(2.64)	72.3(2.36)	152.1(4.36)	221.5(5.78)
-0.3	121.4(3.18)	97.2(3.14)	185.0(5.25)	224.5(5.71)
-0.2	130.6(3.55)	110.5(3.33)	174.0(4.90)	194.1(5.32)
-0.1	134.6(3.35)	128.7(3.92)	168.3(4.81)	164.6(4.59)
0.0	129.9(3.35)	125.7(3.82)	128.9(3.96)	135.5(3.76)
0.1	127.3(3.33)	124.0(3.81)	92.4(2.84)	98.3(2.76)
0.2	100.2(2.59)	109.4(3.40)	66.3(2.06)	68.8(1.95)
0.3	85.1(2.19)	95.4(2.99)	48.9(1.49)	48.7(1.37)
0.4	73.1(1.92)	77.9(2.36)	33.9(1.00)	35.8(1.04)
0.5	49.9(1.36)	55.6(1.70)	23.5(0.67)	27.4(0.78)

Table 5.4 Values of ARL(0) Obtained by Simulation for the TPG, $\overline{X}$, CUSUM, and CSGS Charts under the Autoregressive Model of Order One with Double Exponential(0,1) Noise.

ϕ	TPG	$\overline{X}$	CUSUM	CSGS
-0.5	88.3(2.42)	34.3(1.05)	53.5(1.71)	176.7(5.14)
-0.4	91.0(2.35)	40.0(1.17)	56.7(1.82)	180.5(5.22)
-0.3	92.8(2.52)	41.0(1.25)	58.6(1.79)	173.2(5.08)
-0.2	93.6(2.47)	42.2(1.24)	57.3(1.79)	146.1(4.44)
-0.1	94.4(2.55)	41.2(1.29)	60.0(1.84)	131.5(3.87)
0.0	95.8(2.65)	40.8(1.23)	50.1(1.52)	114.4(3.47)
0.1	91.1(2.47)	44.1(1.44)	50.4(1.59)	83.1(2.43)
0.2	82.7(2.23)	44.4(1.41)	41.5(1.20)	71.9(2.17)
0.3	73.9(2.04)	41.7(1.30)	37.6(1.12)	51.0(1.59)
0.4	63.5(1.76)	38.5(1.25)	29.2(0.93)	35.8(1.01)
0.5	50.0(1.49)	35.0(1.02)	23.1(0.69)	27.4(0.79)

Table 5.5 Values of ARL(0) Obtained by Simulation for the TPG, $\bar{X}$, CUSUM, and CSGS Charts under the Autoregressive Model of Order One with Contaminated Normal(0,1) Noise.

ϕ	TPG	$\bar{X}$	CUSUM	CSGS
-0.5	120.3(3.05)	48.4(1.52)	53.6(1.63)	231.2(5.80)
-0.4	125.5(3.11)	56.6(1.73)	57.3(1.62)	235.6(5.99)
-0.3	132.5(3.34)	56.1(1.73)	56.2(1.74)	225.8(5.77)
-0.2	137.8(3.62)	62.0(2.01)	58.1(1.83)	204.3(5.38)
-0.1	148.9(3.92)	59.6(1.89)	57.6(1.85)	181.9(5.08)
0.0	137.3(3.57)	60.0(1.79)	54.1(1.72)	153.6(4.33)
0.1	135.7(3.69)	61.2(1.94)	48.6(1.53)	112.9(3.49)
0.2	127.9(3.57)	59.5(1.94)	41.6(1.26)	80.7(2.29)
0.3	102.1(2.69)	55.8(1.69)	36.4(1.15)	57.2(1.61)
0.4	78.3(2.41)	53.2(1.69)	30.4(1.15)	39.4(1.13)
0.5	58.6(1.67)	48.3(1.51)	24.5(0.77)	30.9(0.92)

Table 5.6 Values of ARL(0) Obtained by Simulation for the TPG, $\overline{X}$, CUSUM, and CSGS Charts under the Autoregressive Model of Order One with Cauchy Noise.

ϕ	TPG	$\overline{X}$	CUSUM	CSGS
-0.5	75.2(2.07)	14.0(0.43)	17.3(0.53)	107.8(3.25)
-0.4	74.2(1.96)	14.4(0.45)	17.6(0.53)	105.9(3.23)
-0.3	70.3(1.78)	14.5(0.44)	17.6(0.53)	91.6(2.73)
-0.2	67.2(1.76)	13.5(0.39)	18.4(0.57)	81.3(2.47)
-0.1	62.4(1.70)	14.1(0.42)	18.2(0.56)	80.5(2.35)
0.0	59.9(1.71)	14.4(0.45)	17.5(0.52)	68.5(2.16)
0.1	49.3(1.38)	13.7(0.42)	17.4(0.53)	38.9(1.23)
0.2	41.2(1.25)	13.9(0.42)	16.9(0.53)	28.0(0.83)
0.3	33.1(0.95)	14.5(0.46)	17.4(0.53)	21.9(0.62)
0.4	28.1(0.83)	14.4(0.43)	15.5(0.46)	19.1(0.56)
0.5	24.9(0.74)	14.0(0.41)	15.5(0.47)	16.5(0.48)

Table 5.7 Values of ARL(μ) Obtained by Simulation for
the TPG Chart with Noise N(0,1), Double Exponential,
Contaminated Normal, and Cauchy, When ϕ = -0.2, 0.0, and 0.2.

ϕ	μ	Normal	Contaminated Normal	Double Exponential	Cauchy
	0.0	130.6(3.55)	137.8(3.62)	93.6(2.47)	62.4(1.70)
	0.5	33.3(0.91)	46.2(1.44)	44.0(1.37)	48.7(1.89)
-.2	1.0	12.8(0.31)	16.9(0.42)	18.3(0.49)	32.2(1.02)
	1.5	7.3(0.15)	7.4(0.15)	8.4(0.20)	13.0(0.31)
	2.0	4.5(0.08)	4.4(0.08)	4.6(0.07)	4.6(0.07)
	2.5	3.3(0.05)	3.0(0.05)	3.0(0.04)	2.8(0.05)
	0.0	129.9(3.35)	137.3(3.57)	95.8(2.65)	59.9(1.71)
	0.5	36.0(1.01)	54.2(1.65)	45.2(1.40)	53.5(1.98)
0.0	1.0	14.2(0.38)	18.4(0.48)	19.1(0.51)	32.7(1.17)
	1.5	7.5(0.18)	8.2(0.20)	8.5(0.20)	15.8(0.46)
	2.0	4.7(0.10)	4.5(0.09)	4.6(0.09)	4.9(0.10)
	2.5	3.2(0.06)	3.1(0.05)	2.9(0.05)	2.7(0.04)
	0.0	100.2(2.59)	41.2(1.25)	82.7(2.23)	41.2(1.25)
	0.5	35.4(1.14)	48.6(1.53)	47.0(1.57)	33.8(1.33)
0.2	1.0	14.3(0.38)	18.3(0.54)	18.9(0.58)	27.1(1.07)
	1.5	7.4(0.19)	8.2(0.22)	9.3(0.25)	14.0(0.44)
	2.0	4.9(0.12)	4.6(0.10)	4.6(0.11)	5.0(0.14)
	2.5	3.3(0.06)	3.2(0.06)	3.0(0.05)	2.8(0.06)

Table 5.8 Values of ARL(μ) Obtained by Simulation for

the CUSUM chart with Noise N(0,1), Double Exponential,

Contaminated Normal, and Cauchy, When ϕ = -0.2, 0.0, and 0.2.

ϕ	μ	Normal	Contaminated Normal	Double Exponential	Cauchy
	0.0	174.0(4.90)	58.1(1.83)	57.3(1.79)	18.4(0.57)
	0.5	48.6(2.18)	34.1(1.52)	39.4(1.66)	18.9(0.85)
-.2	1.0	10.3(0.38)	11.3(0.38)	11.4(0.45)	11.5(0.43)
	1.5	4.1(0.12)	4.5(0.11)	4.3(0.10)	4.2(0.08)
	2.0	2.7(0.06)	2.6(0.05)	2.6(0.05)	2.6(0.04)
	2.5	1.9(0.04)	1.9(0.03)	2.0(0.03)	1.9(0.03)
	0.0	128.9(3.96)	54.1(1.72)	50.1(1.52)	17.5(0.52)
	0.5	34.6(1.58)	34.9(1.48)	33.1(1.31)	16.6(0.71)
0.0	1.0	9.5(0.34)	11.9(0.47)	11.3(0.43)	10.9(0.40)
	1.5	4.5(0.12)	4.6(0.13)	4.6(0.13)	4.3(0.10)
	2.0	2.7(0.07)	2.7(0.05)	2.8(0.06)	2.5(0.04)
	2.5	2.0(0.04)	2.0(0.03)	2.0(0.04)	2.0(0.03)
	0.0	66.3(2.06)	41.6(1.26)	41.5(1.20)	16.9(0.53)
	0.5	27.1(1.07)	26.2(1.09)	26.5(1.12)	15.4(0.70)
0.2	1.0	9.8(0.34)	10.4(0.39)	11.2(0.40)	10.9(0.43)
	1.5	4.8(0.15)	5.0(0.17)	4.9(0.15)	4.4(0.11)
	2.0	3.1(0.08)	2.9(0.07)	2.8(0.07)	2.7(0.05)
	2.5	2.1(0.05)	2.1(0.04)	2.0(0.04)	2.0(0.05)

Table 5.9 Values of ARL(μ) Obtained by Simulation for
the CSGS Chart with Noise N(0,1), Double Exponential,
Contaminated Normal, and Cauchy, When ϕ = -0.2, 0.0, and 0.2.

ϕ	μ	Normal	Contaminated Normal	Double Exponential	Cauchy
	0.0	194.1(5.32)	58.1(1.83)	146.1(4.44)	81.3(2.47)
	0.5	31.5(1.17)	37.5(1.44)	41.6(1.57)	58.7(2.44)
-.2	1.0	9.6(0.23)	10.6(0.28)	11.0(0.29)	14.4(0.45)
	1.5	5.1(0.10)	5.1(0.10)	5.0(0.09)	4.8(0.08)
	2.0	3.4(0.06)	3.3(0.06)	3.5(0.06)	3.7(0.05)
	2.5	2.5(0.04)	2.5(0.04)	2.5(0.04)	2.5(0.04)
	0.0	135.5(3.76)	54.1(1.72)	114.4(3.47)	68.5(2.16)
	0.5	29.0(1.08)	34.3(1.20)	40.4(1.51)	56.1(2.49)
0.0	1.0	9.4(0.26)	10.6(0.30)	11.0(0.30)	16.7(0.63)
	1.5	5.2(0.12)	5.3(0.12)	5.0(0.10)	4.8(0.08)
	2.0	3.4(0.07)	3.3(0.06)	3.5(0.06)	3.6(0.06)
	2.5	2.6(0.04)	2.6(0.04)	2.5(0.04)	2.6(0.04)
	0.0	68.8(1.95)	41.6(1.26)	71.9(2.17)	28.0(0.82)
	0.5	25.1(0.95)	30.6(1.19)	32.6(1.39)	24.2(0.97)
0.2	1.0	9.7(0.30)	10.0(0.32)	11.5(0.37)	12.4(0.45)
	1.5	5.3(0.14)	5.3(0.12)	5.3(0.12)	4.9(0.11)
	2.0	3.5(0.08)	3.6(0.08)	3.6(0.08)	3.7(0.06)
	2.5	2.6(0.04)	2.6(0.05)	2.6(0.04)	2.6(0.04)
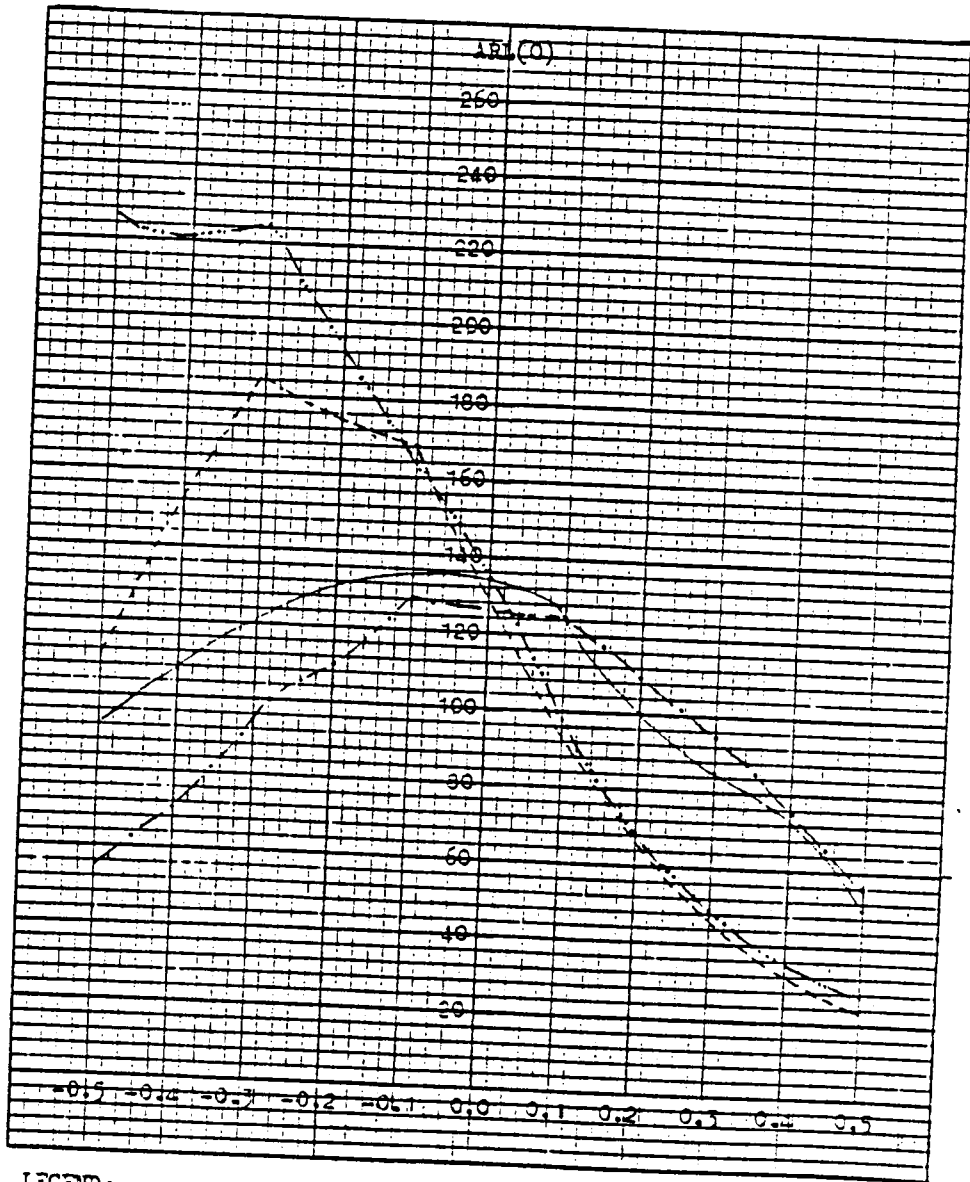
Figure 5.1 The Effect of Autoregressive Observations on the Values of ARL(0) for the X, CUSUM, TPG, and CSGS Charts When the White Noise Has a Normal Distribution.
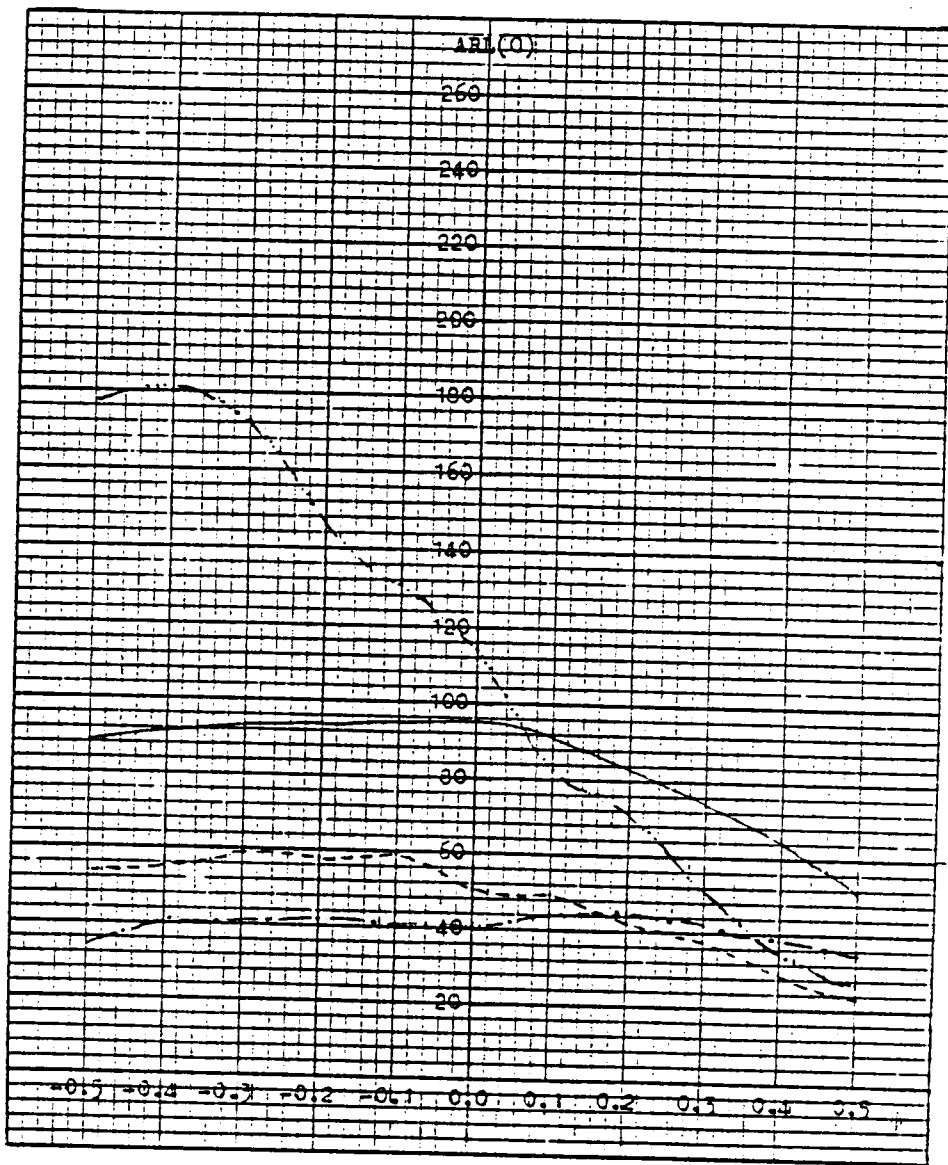
LEGEND: ————— TPG Chart   —·—·— X̄ Chart
        —···— CSGS Chart   — — — — CUSUM Chart

Figure 5.2 The Effect of Autoregressive Observations on the Values of ARL(0) for the X, CUSUM, TPG, and CSGS Charts When the White Noise Has a Double Exponential Distribution.

LEGEND: ———————— TPG Chart ——— · ——— · $\overline{X}$ Chart
———— · · · ———— CSGS Chart — — — — — CUSUM Chart
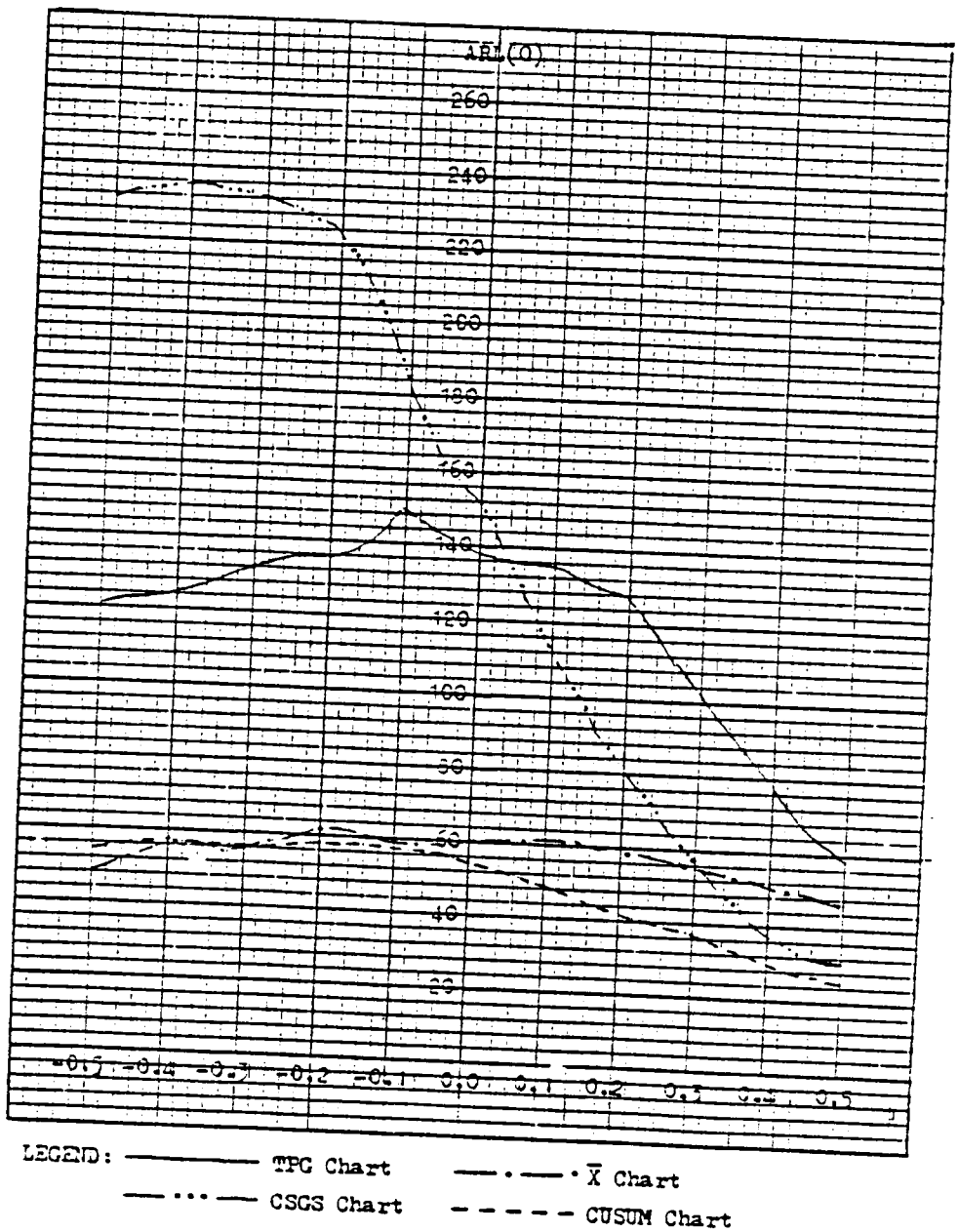
Figure 5.3 The Effect of Autoregressive Observations on the
Values of ARL(0) for the X, CUSUM, TPG, and CSGS Charts When
the White Noise Has a Contaminated Normal Distribution.
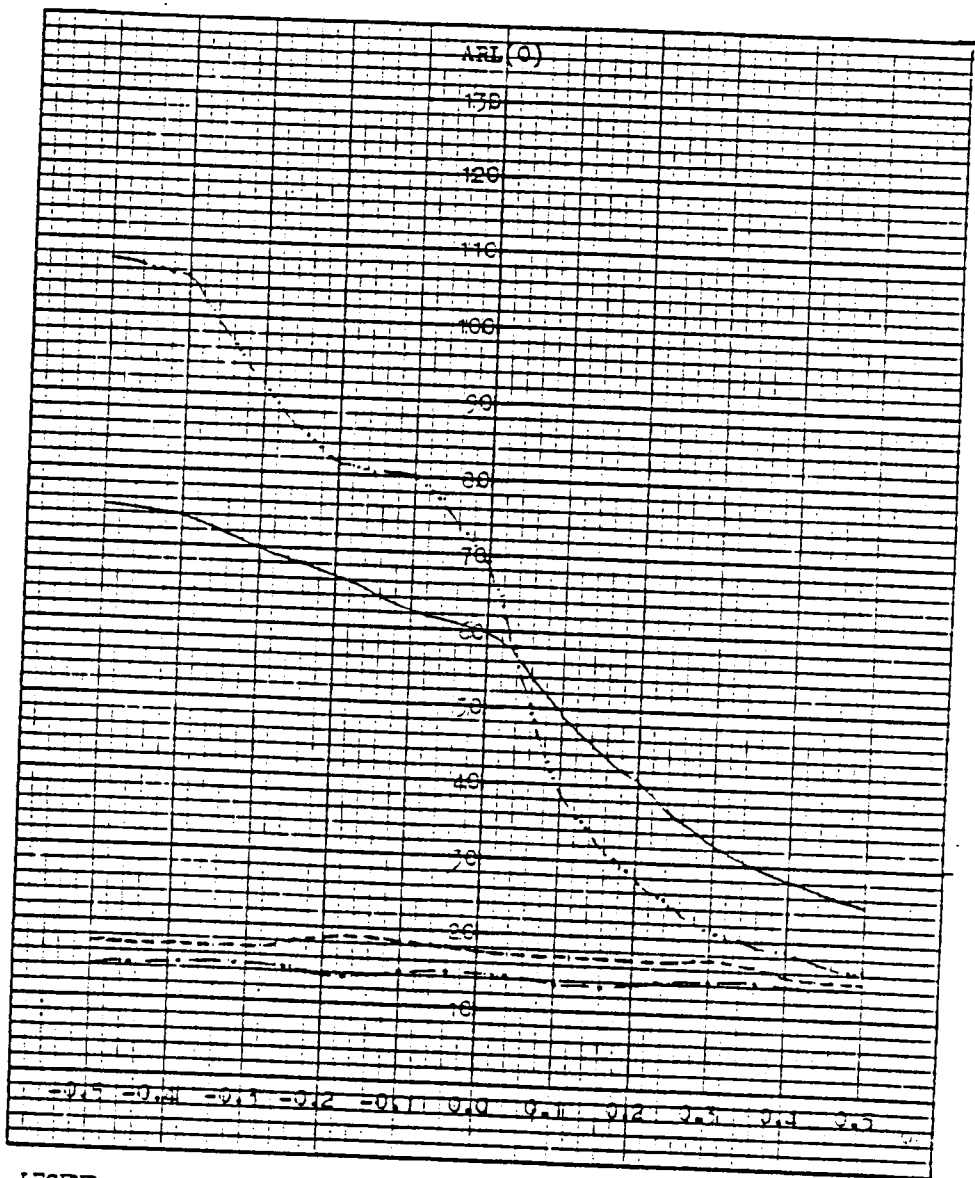
Figure 5.4 The Effect of Autoregressive Observations on the Values of ARL(0) for the X, CUSUM, TPG, and CSGS Charts When the White Noise Has a Cauchy Distribution.

Table 5.10 Values of ARL(0) Obtained by Simulation for the TPG, $\overline{X}$, CUSUM, and CSGS Charts under the Moving Average Model of Order One with N(0,1) Noise.

ψ	TPG	$\overline{X}$	CUSUM	CSGS
-0.5	67.8(1.88)	59.0(1.86)	24.9(0.73)	26.8(0.77)
-0.4	81.9(2.21)	75.5(2.47)	33.2(0.99)	35.9(1.06)
-0.3	96.9(2.60)	95.7(2.97)	51.4(1.60)	49.8(1.49)
-0.2	107.2(2.88)	114.3(3.57)	62.5(1.85)	70.2(1.99)
-0.1	122.8(3.13)	123.3(3.66)	92.3(2.98)	93.7(2.72)
0.0	134.0(3.52)	135.6(3.99)	125.2(3.63)	129.4(3.84)
0.1	127.8(3.24)	122.9(3.80)	167.8(4.68)	67.3(4.80)
0.2	132.5(3.34)	112.0(3.32)	176.6(4.99)	206.5(5.65)
0.3	121.8(3.23)	96.2(2.98)	189.7(5.30)	216.3(5.59)
0.4	117.3(2.96)	75.9(2.23)	151.7(4.45)	216.3(5.60)
0.5	102.8(2.73)	55.5(1.72)	117.3(3.58)	224.8(5.94)

Table 5.11 Values of ARL(0) Obtained by Simulation for the TPG, $\overline{X}$, CUSUM, and CSGS Charts under the Moving Average Model of Order One with Double Exponential(0,1) Noise.

ψ	TPG	$\overline{X}$	CUSUM	CSGS
-0.5	58.8(1.64)	37.3(1.17)	23.6(0.72)	28.7(0.78)
-0.4	66.0(1.84)	36.0(1.14)	29.9(0.88)	33.9(0.94)
-0.3	77.2(2.16)	41.5(1.40)	36.4(1.07)	47.7(1.41)
-0.2	84.3(2.34)	43.7(1.37)	44.9(1.43)	66.8(2.04)
-0.1	87.7(2.43)	44.3(1.41)	48.9(1.51)	86.7(2.62)
0.0	92.9(2.58)	42.0(1.25)	54.8(1.67)	104.9(3.11)
0.1	95.9(2.57)	44.0(1.42)	59.1(1.91)	126.5(3.82)
0.2	94.6(2.52)	46.6(1.51)	61.1(1.97)	141.1(4.21)
0.3	94.7(2.55)	40.2(1.32)	59.5(1.88)	162.6(4.49)
0.4	97.0(2.44)	36.4(1.11)	55.9(1.71)	175.8(4.81)
0.5	95.7(2.49)	34.1(1.05)	53.1(1.66)	178.5(5.05)

Table 5.12 Values of ARL(0) Obtained by Simulation for the TPG, $\overline{X}$, CUSUM, and CSGS Charts under the Moving Average Model of Order One with Contaminated Normal(0,1) Noise.

ψ	TPG	$\overline{X}$	CUSUM	CSGS
-0.5	73.3(2.01)	45.6(1.36)	26.7(0.79)	30.9(0.91)
-0.4	90.7(2.42)	54.2(1.59)	35.6(1.15)	42.0(1.26)
-0.3	107.2(2.84)	58.2(1.92)	45.8(1.37)	58.1(1.77)
-0.2	117.7(3.22)	59.3(1.83)	54.2(1.68)	80.5(2.43)
-0.1	131.3(3.40)	64.9(1.99)	64.5(2.00)	102.6(3.15)
0.0	144.6(3.74)	65.4(2.16)	68.3(2.08)	152.5(4.29)
0.1	146.3(3.73)	65.0(2.01)	72.7(2.33)	177.7(5.01)
0.2	143.7(3.66)	60.9(1.92)	75.3(2.39)	211.6(5.62)
0.3	141.2(3.51)	58.4(1.82)	74.4(2.30)	228.9(5.76)
0.4	136.7(3.38)	56.1(1.75)	72.9(2.32)	231.4(5.78)
0.5	135.3(3.49)	49.3(1.54)	68.2(2.16)	225.0(5.62)

Table 5.13 Values of ARL(0) Obtained by Simulation for the TPG, $\bar{X}$, CUSUM, and CSGS Charts under the Moving Average Model of Order One with Cauchy Noise.

ψ	TPG	$\bar{X}$	CUSUM	CSGS
-0.5	26.7(0.82)	15.8(0.46)	15.3(0.47)	17.2(0.48)
-0.4	31.7(0.94)	15.6(0.48)	15.6(0.46)	19.0(0.55)
-0.3	38.6(1.10)	16.9(0.53)	17.7(0.57)	21.9(0.63)
-0.2	42.0(1.19)	16.0(0.53)	17.7(0.54)	28.4(0.87)
-0.1	49.8(1.43)	16.0(0.49)	18.5(0.57)	40.4(1.20)
0.0	61.3(1.68)	16.7(0.50)	16.8(0.52)	70.3(2.00)
0.1	66.0(1.91)	15.9(0.52)	18.3(0.57)	83.8(2.51)
0.2	68.5(2.04)	15.1(0.46)	17.9(0.54)	87.0(2.60)
0.3	75.6(1.96)	15.8(0.47)	18.1(0.55)	97.1(2.99)
0.4	81.4(2.18)	16.3(0.52)	18.2(0.58)	103.6(3.09)
0.5	84.7(2.17)	16.6(0.51)	17.2(0.51)	104.3(3.05)
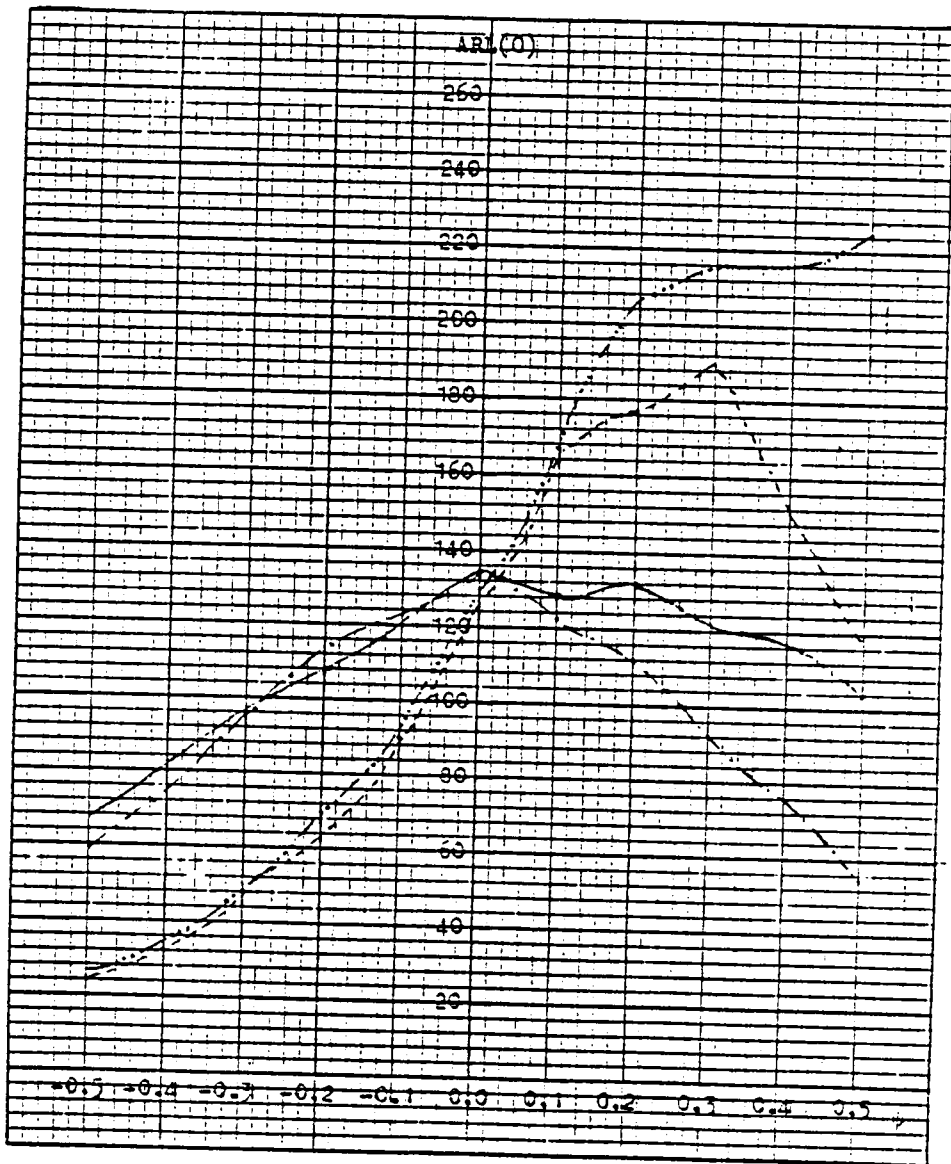
Figure 5.5 The Effect of Moving Average Observations on the Values of ARL(O) for the X, CUSUM, TPG, and CSGS Charts When the White Noise Has a Normal Distribution.
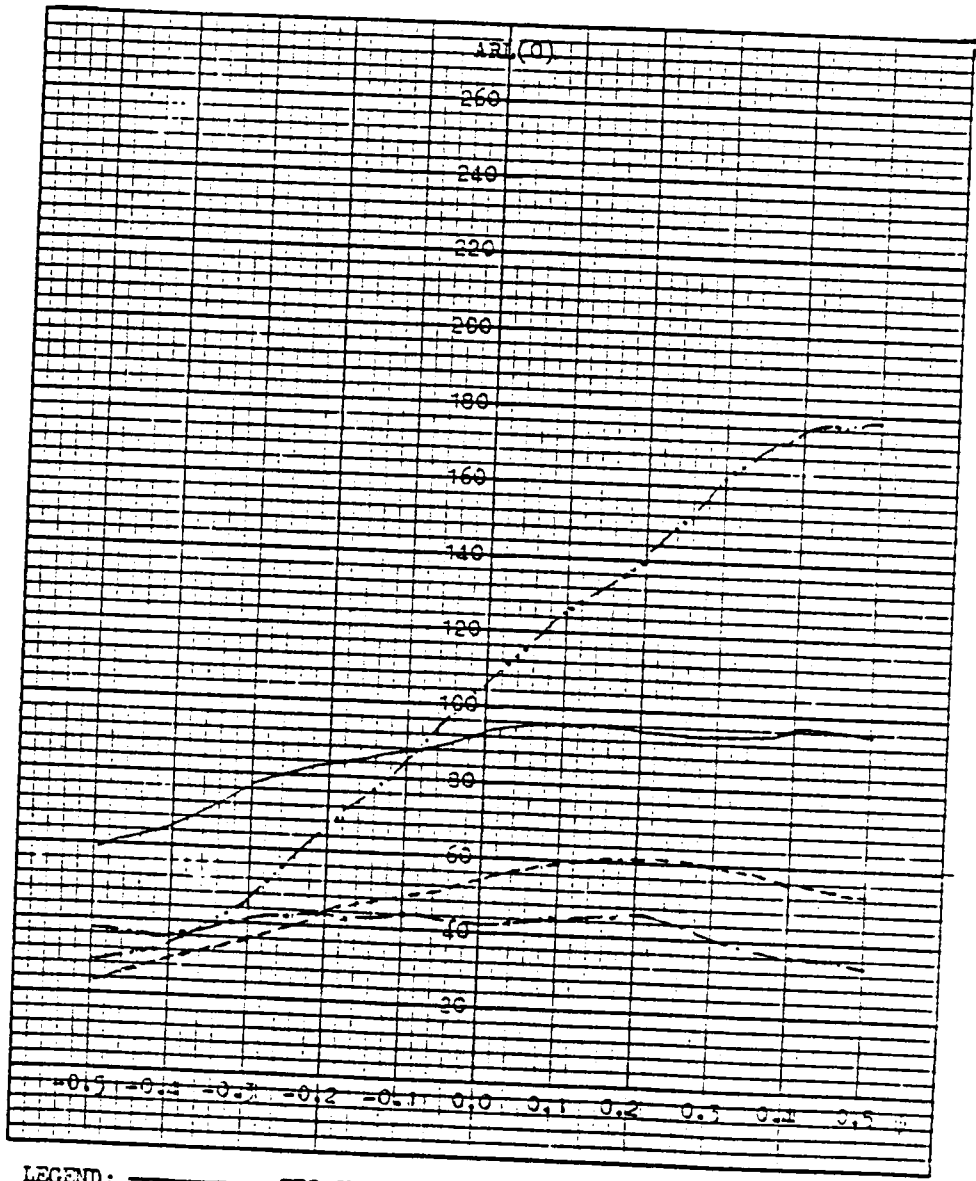
Figure 5.6 The Effect of Moving Average Observations on the Values of ARL(0) for the X, CUSUM, TPG, and CSGS Charts When the White Noise Has a Double Exponential Distribution.
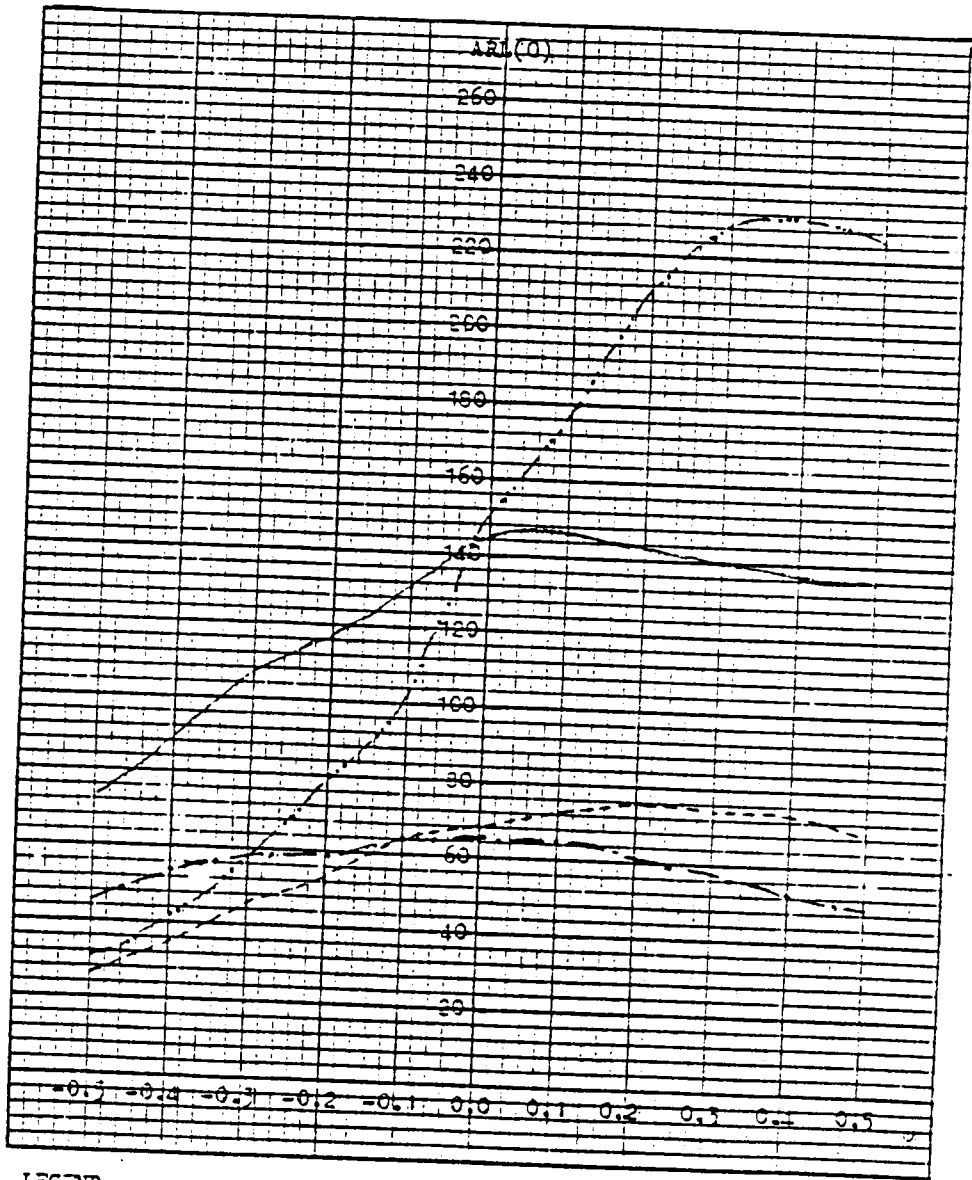
LEGEND: ———— TPG Chart   — · — · X̄ Chart
         — ··· — CSGS Chart — — — — CUSUM Chart

Figure 5.7 The Effect of Moving average Observations on the
Values of ARL(0) for the X, CUSUM, TPG, and CSGS Charts When
the White Noise Has a Contaminated Normal Distribution.

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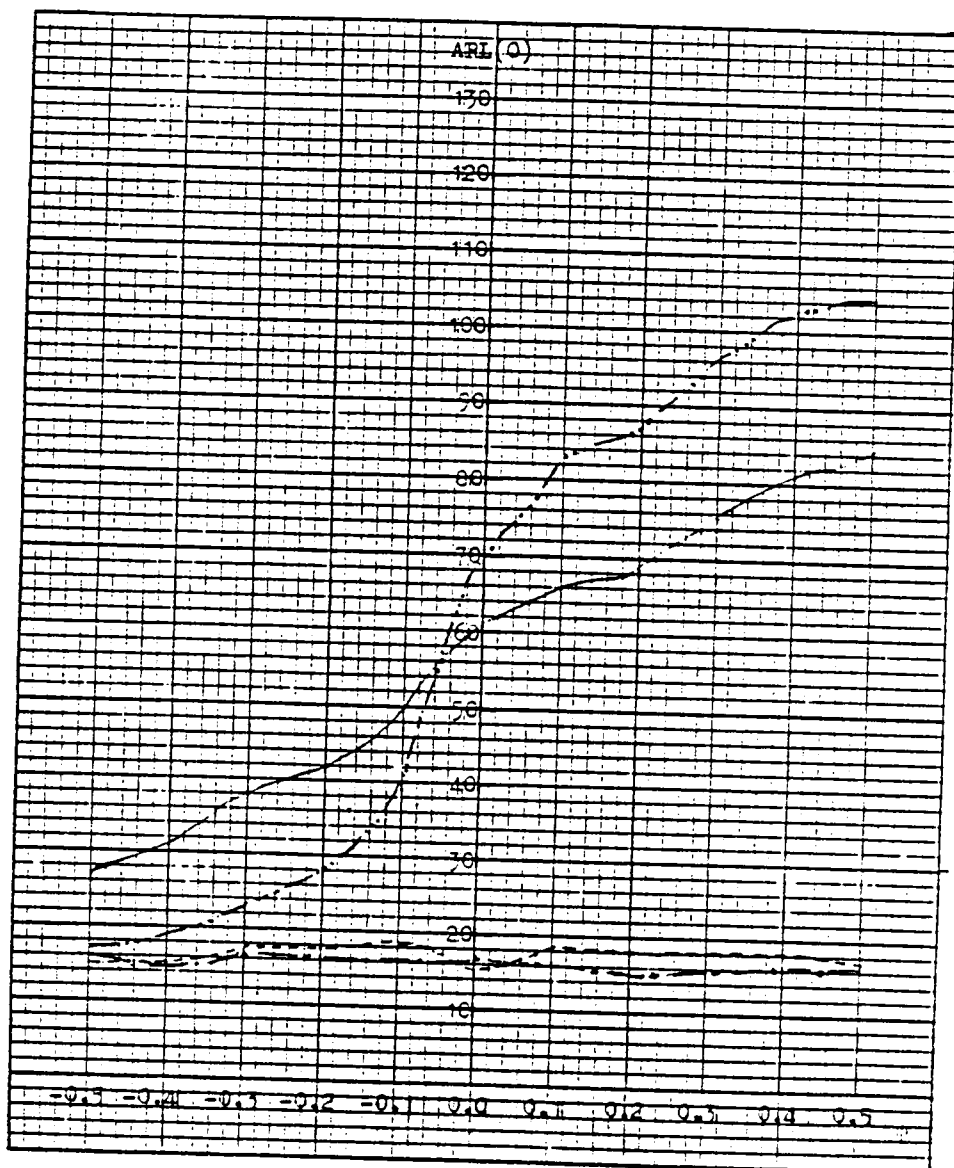
Figure 5.8 The Effect of Moving Average Observations on the Values of ARL(0) for the X, CUSUM, TPG, and CSGS Charts When the White Noise Has a Cauchy Distribution.
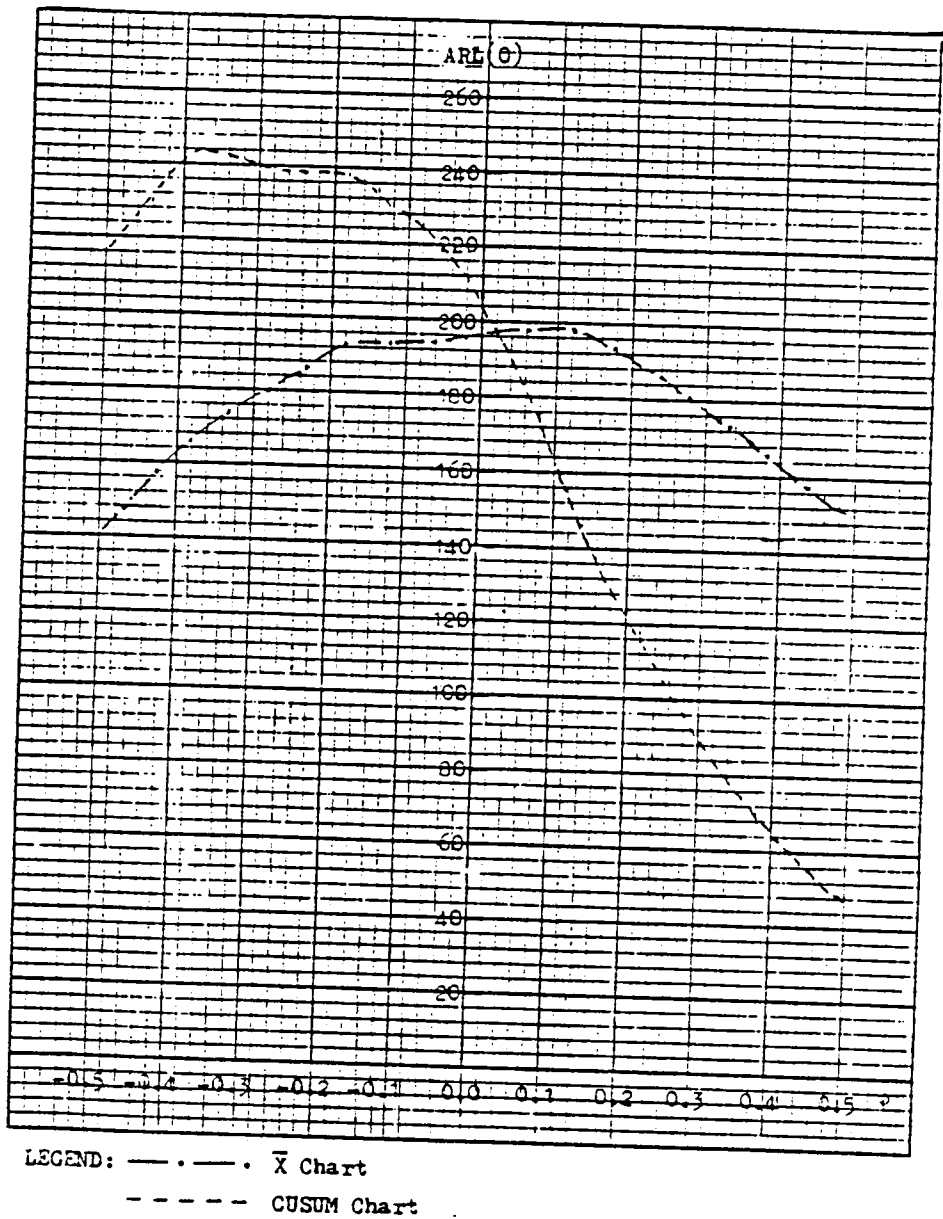
Figure 5.9 The Effect of Autoregressive Observations on the Values of ARL(0) for the X and CUSUM When the White Noise Has a Double Exponential Distribution.

VI. CONTROLLING THE PROCESS MEAN AND THE STANDARD DEVIATION

6.1 INTRODUCTION

During the operation of a process, the process mean and the standard deviation both may change. Usually, separate control charts, one for the mean and one for the standard deviation, are used simultaneouly for detecting these changes. When either one of these two charts signals, the process is regarded as out-of-control. For example, one can plot the sample mean and the sample range in an $\overline{X}$-chart and a R-chart, respectively, to control the mean and the standard deviation at the same time. However, the dependence between the sample mean and the sample range results in difficulties in obtaining the average run length for the combined procedure. The alternative is using the sample mean and the sample standard deviation, which are independent under the normality assumption, at each sampling time. The run length of this approach is the minimum of two independent geometric random variables which is itself a geometric random variable. However, this result does not hold if the samples of observations are not normal random samples. Since these two-chart procedures need n observations at each time, sometimes it is not convenient in practice.

In this chapter, we introduce two different control chart schemes to control the process mean and the standard deviation simultaneously by using only one chart and a single observation at each sampling time. The exact average run length can be obtained by formulating the process as a random walk process.

6.2 PROPOSED CONTROL CHARTS

The first scheme borrows from the CSGS-SD scheme. Since the score at each sampling time tends to be large when either the mean changes or the standard deviation increases, the CSGS-SD scheme is a reasonable choice for this purpose.

The second scheme assigns a score pair to each observation by using the same gauging pairs as in the TPG scheme. This scheme is called the Two-Score Gauging System (TSGS) scheme. At time m the observation X_m receives a score pair $\underline{S}(X_m) = (S_1(X_m), S_2(X_m))$. The possible scores for $S_1(X_m)$ are ± 2, ± 1, and 0 which are the same as in the TPG chart for controlling the process mean. The possible scores for $S_2(X_m)$ are -1, 1, and 2. The score $S_2(X_m)$ is used to control the process standard deviation. The joint probability density function for this random score pair $\underline{S}(X_m)$ is as follows:

$$S_1(X)$$

	-2	-1	0	1	2
-1	0	0	P_0	0	0
1	0	P_{-1}	0	P_1	0
2	P_{-2}	0	0	0	P_2

$S_2(X)$ is shown at the left of the rows, and the equation is labeled (6.1).

For each observation X, assuming symmetry, only five score pairs have nonzero probabilites. When the process is in control $P_{-1} = P_1 = q$ and $P_{-2} = P_2 = P$.

Define

$$C_m^\mu = \Sigma_{i=1}^m \ S_1(X_i)$$

and   (6.2)

$$C_m^\sigma = \Sigma_{i=1}^m \ S_2(X_i),$$

and the control statistic is $\underline{C}_m = (C_m^\mu, C_m^\sigma)$. The control chart will give a signal to indicate the process is out-of-control whenever $|C_m^\mu| \geq H_1$ or $C_m^\sigma \geq H_2$, where $\pm H_1$, and H_2 are absorbing barriers. Also, whenever C_m^σ is below zero, we reset $C_m^\sigma = 0$.

This procedure can be modelled by a two-dimensional random walk process with three absorbing barriers at $C_m^\mu = \pm H_1$, and $C_m^\sigma = H_2$ and one reflecting barrier at $C_m^\sigma = -1$. The possible transient states $E_2(i,j)$ are $i=-H_1+1,\ldots,H_1-1$, and $j=-1,0,1,\ldots,H_2-1$. If the path of $\underline{C}_m$ reaches the states $(\pm H_1,j)$ first, then the process mean is deemed to be out-of-control at time m. If $\underline{C}_m$ reaches the states (i,H_2), then the process standard deviation is deemed to have increased.

Whenever the path of $\underline{C}_m$ moves below the state $(i,0)$ we reset $\underline{C}_m$ to $(i,0)$. The possible movements at state (i,j) are illustrated as follows:
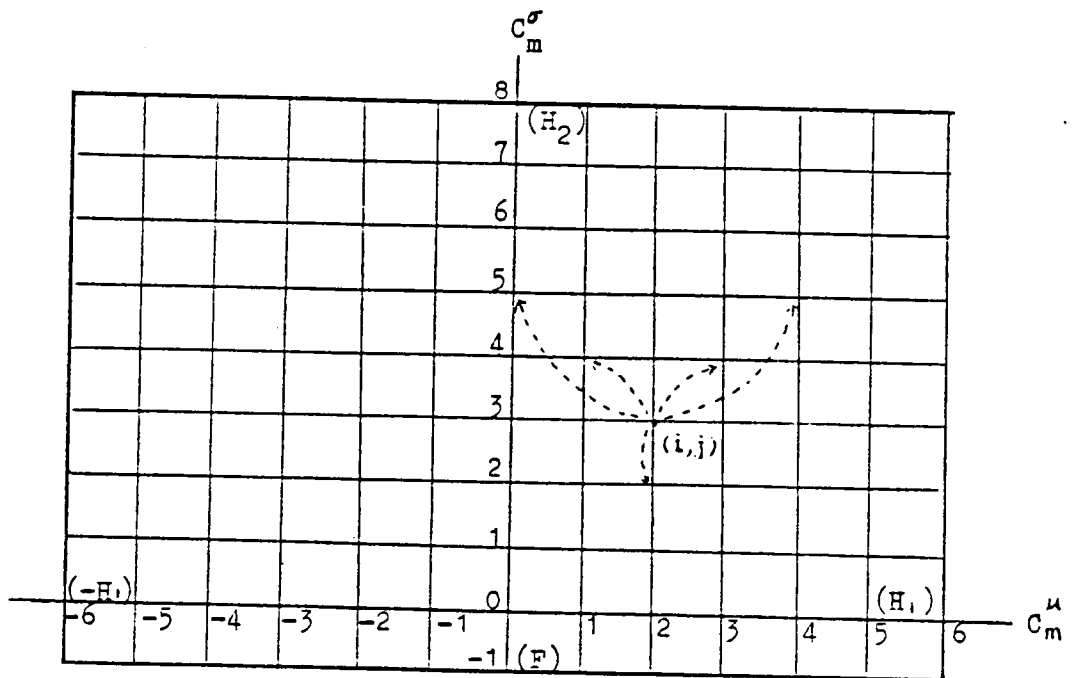
6.3 AVERAGE RUN LENGTH OF THE TSGS CHART

Let $D(i,j)$ be the expected number of samples needed for the remaining path of $\underline{C}_m$, starting at $\underline{C}_m = (i,j)$, to reach one of the absorbing barriers. The governing difference equations for $D(i,j)$ are

$$D(i,j)=1+P_{-2}D(i-2,j+2)+P_{-1}D(i-1,j+1)+P_0D(i,j-1)$$
$$+P_1D(i+1,j+1)+P_2D(i+2,j+2) \quad (6.3)$$
$$i=-H_1+1,\ldots,H_1-1 \text{ and } j=0,1,\ldots,H_2-1,$$

and the boundary conditions are

$$D(i,j)=0 \quad \text{for} \quad |i| \geq H_1 \text{ and for all } j,$$
$$D(i,j)=0 \quad \text{for} \quad j \geq H_2 \text{ and for all } i,$$

and

$$D(i,j)=D(i,0) \quad \text{for } j \leq -1 \text{ and for all } i.$$

In order to compute the run length distribution and the ARL, we transform the two-dimensional random walk process to a one-dimensional random walk process. The two-dimensional transient states $E_2(i,j)$ have to be relabeled as $E_1(k)$, $k = j+[i+(H_1-1)]xH_2+1$.

If $D(k)$ is the expected number of samples needed for the one-dimensional random walk, starting at k, to be absorbed, then

$$D(k)=1+P_{-2}D(k-2(H_2-1))+P_{-1}D(k-(H_2-1))$$

$$+p_0D(\delta(k)-1)+P_1(k+H_2+1)+P_2D(k+2(H_2+1)],$$

where $\delta(k)=k+1$ if $k=1+iH_2$ $i=0,1,2,\ldots,2(H_1-1)$ and $\delta(k)=k$ otherwise.

Since equation (6.4) is a high order difference equation, it is very difficult to solve. Therefore the Markov chain approach will be used. The transition probability matrix $R = (r_{k,j})$ from the transient state k to the transient state j can be defined as:

$$r_{k,k} = p_0 \qquad\qquad k=1+iH_2$$

$$r_{k,k-1} = p_0 \qquad\qquad k\ne1+iH_2$$

$$r_{k,k-2(H_1-1)}=P_{-2} \quad k-2(H_1-1)\ge2H_2+1 \text{ and } k \ne iH_2 \text{ or } iH_2-1$$

$$r_{k,k-(H_2-1)} =P_{-1} \quad k-(H_2-1)\ge H_2+1 \text{ and } k \ne iH_2$$

$$r_{k,k+(H_2+1)} =p_1 \quad k+(H_2+1)\le(2H_1-2)H_2 \text{ and } k\ne iH_2$$

$$r_{k,k+2(H_2+1)}=P_2 \quad k+2(H_1+1)\le(2H_1-3)H_2 \text{ and } k\ne iH_2 \text{ or } iH_2-1$$

$$r_{k,j}=0 \qquad\qquad \text{otherwise,}$$

where $i=0,1,\ldots,2(H_1-1)$. Thus

$$ARL = \underline{e}_{H_1}(I-R)^{-1}\underline{1}$$

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where $\underline{e}_{H_1}$ is a $1 \times (2H_1-1)$ row vector with one in the H_1-th component and zero elsewhere.

6.4 COMPARISONS

To compare the CSGS-SD and TSGS schemes, we compare the values of the ARL when the process mean, the process standard deviation, or both the mean and the standard deviation are changed. The observations are assumed to be normal. Since current practice is to use either the $\overline{X}$-chart and R-chart or the $\overline{X}$-chart and S-chart simultaneously, we compare the CSGS-SD and TSGS schemes with these two-chart schemes. Comparisons are based on the expected number of observations which is the ARL multiplied by the sample size. For computing the values of the ARL for the $\overline{X}$-R schemes, we use simulation to obtain the values of the ARL for this scheme.

In the TSGS scheme, if we are only interested in one-sided shifts in the mean, we use a one-sided TPG scheme instead of a two-sided scheme for the control statistic C_m^μ. Table 6.1 and Table 6.2 give the total number of observations for the CSGS-SD, TSGS, $\overline{X}$-R, and $\overline{X}$-S charts for detecting two-sided and one-sided shifts in the mean with possible changes in the standard deviation respectively.

If only the process mean is changed, the performance of the TSGS scheme is better than the CSGS-SD scheme, but if only the standard deviation is changed, the CSGS-SD chart is

better. The TSGS chart is slightly better than the CSGS-SD
chart when both the mean and the standard deviation are
varied.

When the target values μ_0 and σ_0 both are known, the
efficiencies of the $\overline{X}$-R and $\overline{X}$-S schemes are better than the
CSGS-SD and TSGS schemes (see Table 6.1 and Table 6.2). The
CSGS-SD chart can detect only small shifts in the standard
deviation and the TSGS chart can detect only small shifts in
the mean more quickly than the others.

The CSGS-SD and TSGS charts can be applied to a situ-
ation when only one observation is available at each sampling
time. Their exact average run lengths are obtainable. A
practical advantage of using the CSGS-SD procedure is that
only one chart needed to be plotted whereas the separate
charts procedure requires two charts. Also, if observations
come from a heavy tailed distribution, we would expect the
TSGS and CSGS-SD charts to be more robust than the $\overline{X}$-R and
$\overline{X}$-S charts.

When an out-of-control signal is given, the next step
is to find out whether the mean or the standard deviation
or both is changed. Although knowing the possible changes
in the process may help to locate the assignable causes,
finding the causes is not an easy task. Experience with the
process along with plots of the control statistics are usu-
ally required.

Table 6.1 Expected Number of Observations Needed for Detecting Changes in the Process Mean or Standard Deviation Under Normality.

μ	σ	CSGS-SD	TSGS(1)	TSGS(2)	$\overline{X}$-R	$\overline{X}$-S
		$G_1=1.2$	$G_1=1.983$	$G_1=1.733$	$H_{\overline{X}}=2.81$	$H_{\overline{X}}=2.81$
		$G_2=2.47$	$G_2=2.594$	$G_2=2.395$	$H_R=4.886$	$H_S=1.9274$
		$H=9$	$H_1=6$	$H_1=8$		
		$K=0.25$	$H_2=5$	$H_2=5$	$n=5$	$n=5$
0.0	1.0	498.6	497.6	481.8	481.2	491.8
0.5	1.0	120.8	77.8	66.2	94.3	103.5
1.0	1.0	35.0	28.8	26.1	18.0	17.0
1.5	1.0	17.3	13.8	12.9	7.2	6.9
2.0	1.0	11.0	7.8	7.1	5.2	5.2
0.0	1.1	142.2	297.3	296.4	216.0	194.1
0.0	1.5	32.9	70.3	58.8	26.8	24.2
0.0	2.0	18.0	22.7	18.2	10.0	9.2
0.5	1.5	28.6	32.8	30.8	19.7	19.5
1.0	2.0	15.1	12.4	11.7	7.7	7.5
1.5	1.5	14.9	10.4	10.3	7.2	7.0
2.0	2.0	10.8	6.9	6.8	5.7	5.6

Table 6.2 Expected Number of Observations for Detecting One-sided Shifts in the Mean or in the Standard Deviation Under Normality.

μ	σ	CSGS-SD	TSGS	$\overline{X}$-R	$\overline{X}$-S
		$G_1=1.2$	$G_1=1.733$	$H_{\overline{X}}=1.194$	$H_{\overline{X}}=1.194$
		$G_2=2.47$	$G_2=2.395$	$H_R=4.7$	$H_S=1.86$
		$H=9$	$H_1=8$ $F_1=4$		
		$K=0.25$	$H_2=-8$ $F_2=-2$	$n=5$	$n=5$
0.0	1.0	443.3	430.9	448.5	428.6
0.5	1.0	113.7	64.4	69.8	78.5
1.0	1.0	33.8	24.4	14.4	14.5
1.5	1.0	16.8	11.9	6.7	6.7
2.0	1.0	10.7	6.3	5.2	5.2
0.0	1.1	133.1	320.8	172.0	174.3
0.0	1.5	31.8	50.6	24.2	23.1
0.0	2.0	17.6	15.1	10.0	9.4
0.5	1.5	27.7	26.6	16.3	16.6
1.0	2.0	14.8	10.1	7.5	7.4
1.5	1.5	14.6	9.3	6.9	6.9
2.0	2.0	10.6	5.8	5.7	5.5

VII. MULTIVARIATE CONTROL CHART PROCEDURES

7.1 INTRODUCTION

When a product requires more than one characteristic to specify its quality, a multivariate control chart procedure is needed. The procedure frequently employed is to use separate control charts for each quality characteristic. A process is regarded as out-of-control as soon as one of these charts gives an out-of-control signal. The procedure proposed by Woodall and Ncube (1985b) is a multivariate CUSUM procedure and is an example of this approach. Due to the dependency among quality characteristics, it is usually difficult to obtain the exact average run length of this procedure. Also, the practice of using separate charts for controlling multivariate observations may not be the most efficient method for detecting changes in the process that involve some or all of these quality characteristics. Another approach is to use either a x^2 or T^2 chart depending on whether the variance-covariance matrix of the observations is known or unknown. The x^2 and T^2 charts are Shewhart type charts. Their properties have been investigated by Jackson (1959), Ghare and Torgersen (1968), and Hui (1980). So far, cusum type schemes using one statistic that is a function of

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the multivariate observations has not been developed for multivariate quality control.

Two types of control chart procedures are proposed in this chapter. The first procedure is a direct application of the univariate gauging score scheme. We first compute an appropriate univariate statistic such as the x^2 or T^2 which is a function of the multivariate observations. Then we apply the gauging scoring system to these univariate statistics. This procedure is called a Gauging CHI-Square (GCHIS) or Gauging T-SQuare (GTSQ) scheme. After gauging, we can choose either a generalized random walk type or a cusum type scheme to control the process. The second procedure is called the Multivariate Sign-Test Score (MSTS) scheme and assigns a score vector to each of the n multivariate observations at each sampling time. The score for each quality characteristic is either positive one or negative one depending on whether the magnitude of the measured quality characteristic is above or below the corresponding control value. Then the n score vectors are combined together into a univariate score. Finally a cumulative sum of these univariate scores is used as a control statistic.

7.2 GAUGING CHI-SQUARE SCHEMES

Suppose n multivariate observations with m quality characteristics are obtained at each sampling time. Let $\underline{x}_i^t$

$= (X_{i1}, \ldots, X_{im})'$ be the i-th observed column vector at time t and $\underline{\mu}_o = (\mu_{o1}, \ldots, \mu_{om})'$ be the target mean-vector. The n vector observations at time t are presented in terms of a matrix of the form $\underset{\sim}{X}^t = [\underline{X}_1^t, \underline{X}_2^t, \ldots, \underline{X}_n^t]$. The gauging scheme is applied to the values of

$$x_t^2 = n(\underline{\bar{X}}^t - \underline{\mu}_o)' \Sigma^{-1}(\underline{\bar{X}}^t - \underline{\mu}_o)$$

if the variance-covariance matrix Σ is known, or to

$$T_t^2 = n(\underline{\bar{X}}^t - \underline{\mu}_o)' S^{-1}(\underline{\bar{X}}^t - \underline{\mu}_o)$$

if Σ is unknown, where $\underline{\bar{X}}^t = \Sigma_{i=1}^n \underline{X}_i^t / n$ and

$$S = \frac{1}{N_o - 1} \Sigma_{i=1}^{N_o} (\underline{X}_i - \underline{\bar{X}})(\underline{X}_i - \underline{\bar{X}})'$$

is an estimate of Σ based on N_o vector observations. Therefore, when Σ is known, the scoring system we choose is

$$C(\underset{\sim}{X}^t) = \begin{cases} -1 & x_t^2 < G_1 \\ 1 & G_1 \leq x_t^2 < G_2 \\ 2 & x_t^2 \geq G_2 \end{cases}$$

with $p^o = Pr[C(\underset{\sim}{X}^t) = 2]$, $q^o = Pr[C(\underset{\sim}{X}^t) = 1]$, and $r^o = Pr[C(\underset{\sim}{X}^t) = -1]$.

Since the x_t^2 is a non-negative valued random variable, for controlling the process mean-vector a one-sided control

chart should be an appropriate choice. The score of negative
one for $x_t^2 < G_1$ can produce a downward trend when the process
is under control. Therefore, we can model the control proc-
ess as a generalized random walk process between an absorbing
and a reflecting barrier.

Although the statistic x_t^2 (or T_t^2) still needs to be
computed before the gauging is applied, the purpose of the
gauging approach is to improve the performance for detecting
changes in the mean-vector by applying either a generalized
random walk type or a cusum type procedure to the gauging
scores. In some cases it might be more efficient to use the
actual values of the x_t^2 (or T_t^2) directly in a cusum chart.
But gauging the values of the statistic has certain advan-
tages such as the ease of obtaining the exact average run
length and possible robustness to non-normal distributions.
Therefore we shall investigate the gauging score charts for
multivariate quality control.

If only two quality characteristics need to be measured,
then the computation for x_t^2 (or T_t^2) is not necessary.
The gauging can be done by using a transparent sheet on which
are drawn two concentric ellipses with the center at μ_o.
After plotting $\overline{\underline{X}}^t$ on a data sheet, the gauging scores can
be obtained by laying the transparent gauging sheet on top
of the data sheet. The gauging score is 2, 1, or -1, if the
vector $\overline{\underline{X}}^t$ falls outside, between, or within these two el-

lipses, respectively. Therefore for this special case, only $\bar{\underline{X}}^t$ needs to be determined.

For a generalized random walk procedure, the control statistic at each time is defined as follows:

$$S_t = \begin{cases} 0 & S_{t-1}+C(\underline{X}^t) \le -F \\ \\ S_{t-1}+C(\underline{X}^t) & \text{otherwise,} \end{cases}$$

where $S_o=0$. An out-of-control signal is given if $S_t \ge H$. When $F=1$, this procedure is the same as the procedure discussed in Section 3.6 for controlling the process standard deviation except that $C(\underline{X}^t)$ is used instead of $C(X)$.

The average run length can be obtained by solving difference equations

$$D_i = 1+r^o D_{i-1} + q^o D_{i+1} + p^o D_{i+2},$$
$$i=-F+1,\ldots,H-1, \qquad (7.1)$$

with boundary conditions $D_H = D_{H+1} = 0$ and $D_{-F} = D_o$. The solution of equation (7.1) is

$$D_i = a^o + b^o \lambda_1^i + c^o \lambda_2^i - \frac{i}{E}$$

where

$$a^o = \frac{H+1}{E} - \frac{1}{\Delta E}[\lambda_1^{H+1}b^o + \lambda_2^{H+1}c],$$

$$b^o = \frac{1}{\Delta E} [F(\lambda_1 \lambda_2)^F \lambda_2^H (1-\lambda_2) - \lambda_1^F (1-\lambda_2^F)],$$

$$c^o = \frac{-1}{\Delta E} [F(\lambda_1 \lambda_2)^F \lambda_1^H (1-\lambda_1) - \lambda_2^F (1-\lambda_1^F)],$$

$$\lambda_{1,2} = \{-(p^o+q^o) \pm \sqrt{(p^o+q^o)^2 + 4p^o r^o} /2p^o,$$

$$E = 2p^o + q^o - r^o,$$

and

$$\Delta = \lambda_1^{H+F}(1-\lambda_1)(1-\lambda_2^F) - \lambda_2^{H+F}(1-\lambda_2)(1-\lambda_1^F).$$

Then the average run length is $a^o + b^o + c^o$.

We can also use the gauging scores in a one-sided cusum scheme. Since the reference value K will produce a downward trend when the process is in-control, we use score zero for $x_t^2 < G_1$ instead of negative one. The control statistic for the cusum GCHIS scheme is

$$S_t = \max\{S_{t-1} + (C(\underset{\sim}{X}^t) - K), 0\}$$

where $S_o = 0$. Signal out-of-control, if $S_t \geq H$.

7.3 MULTIVARIATE SIGN-TEST SCORE SCHEME

In order to avoid measuring each vector observation, a score vector can be assigned by comparing each quality characteristic with its target value. Suppose n observations are

obtained at time t. The MSTL scheme assigns to each vector $\underline{X}_i^t$ a score vector which is obtained by using

$$C(\underline{X}_i^t) = \begin{Bmatrix} \text{sign}(X_{i1}^t - \mu_{o1}) \\ \text{sign}(X_{i2}^t - \mu_{o2}) \\ . \\ . \\ . \\ \text{sign}(X_{im}^t - \mu_{om}) \end{Bmatrix}$$

where sign(X)=1 or -1 depending on whether $X \geq 0$ or $X < 0$. There are 2^m possible vector scores denoted by $\underline{C}_1, \underline{C}_2, \ldots, \underline{C}_{2^m}$. Two score vectors $\underline{C}_i, \underline{C}_j$ are said to be a conjugate pair if $\underline{C}_j = -\underline{C}_i$. There are 2^{m-1} conjugate pairs.

At each sampling time t, define N_i^t to be the total number of observations with score vector $\underline{C}_i$ and N_{-i}^t to be the total number of observations with score vector $\underline{C}_j = -\underline{C}_i$, for $i=1,2,\ldots,2^{m-1}$. Define a univariate score, PS_t, as

$$PS_t(\underline{X}^t) = \Sigma_{i=1}^{2^{m-1}} |N_i^t - N_{-i}^t|.$$

As an example, consider the case where m=2, n=6, $\mu_{o1} = \mu_{o2} = 0$, and the observations spread out as follows:

Then $PS_t(\underline{X}^t) = |N_1^t - N_{-1}^t| + |N_2^t - N_{-2}^t| = |3-1| + |1-1| = 2.$

Chatterjee (1966) proposed a bivariate sign test for testing a location vector for random variables symmetrically distributed about a specified median vector. He mentioned a reasonable test for this purpose would be obtained by comparing N_1 with N_{-1} and N_2 with N_{-2}. However, in order to obtain the distribution free property, his test statistic involves only N_1 and N_2 with a condition that N_1+N_{-1} equal a given constant.

For this MSTS scheme, we also use the values of N_i and N_{-i}. Because we do not care about the direction of shift in the mean vector, we use $|N_1 - N_{-1}|+ |N_2 - N_{-2}|$. Intuitively, when the process is under control, $PS_t(\underset{\sim}{X}^t)$ will tend to be small. But, if the process is out-of-control the diagonal symmetric property in which N_i tends to be close to N_{-i} will be destroyed, and as a result $PS_t(\underset{\sim}{X}^t)$ will tend to be large. The MSTS scheme is not distribution-free because the probability for each score vector depends on the distribution of the multivariate observations.

The control statistic for the MSTS scheme is

$$
S_t = \begin{cases} 0 & S_{t-1}+(PS_t(\underset{\sim}{X}^t)- K) \le -F \\ \\ S_{t-1}+(PS_t(\underset{\sim}{X}^t) -K) & \text{otherwise} \end{cases}
$$

where $S_o=0$. If $S_t \ge H$, then an out-of-control signal is given. The reference value K is choosen to produce a down-

ward trend when the process is in-control. Therefore this scheme is a random walk type scheme. The computation of the average run length is based on a Markov chain approach. The transition probability matrix can be formed by using the distribution of PS_t.

Although the MSTS scheme can be applied to control more than two quality characteristics, the computation is more difficult. For the purpose of illustration, we consider the case of m=2 only.

The statistic PS_t is a discrete random variable with only even or odd integer values depending on whether n is even or odd. Its distribution can be calculated by knowing all possible combinations of (N_1^t, N_2^t, N_{-1}^t, N_{-2}^t) and the corresponding probability for each score vector.

Assume $\underline{X}_i^t$ to be a bivariate normal random vector with density function

$$f(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\{\frac{-1}{2(1-\rho^2)}[(\frac{x_1-\mu_1}{\sigma_1})^2-2\rho(\frac{x_1-\mu_1}{\sigma_1})(\frac{x_2-\mu_2}{\sigma_2})+(\frac{x_2-\mu_2}{\sigma_2})^2]\}.$$

Let $p_3 = Pr[C(\underline{X}^t)=\begin{bmatrix}-1\\-1\end{bmatrix}]=\int_{-\infty}^{\mu_1}\int_{-\infty}^{\mu_2} f(x_1,x_2)dx_2dx_1$. Then it can be shown that

$$p_2 = Pr[C(\underline{X}^t) = \begin{bmatrix}-1\\1\end{bmatrix}] = \Phi(\mu_1)-p_3$$

$$p_4 = Pr[C(\underline{X}^t) = \begin{bmatrix}1\\-1\end{bmatrix}] = \Phi(\mu_2)-p_3 .$$

and

$$p_1 = \Pr[C(\underline{x}^t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}] = 1-p_2-p_3-p_4.$$

Numerical examples for bivariate normal random vectors with $\sigma_1=1$, $\sigma_2=1$, $\rho=0.5$ and $\mu_o = (0,0)'$ are given in Table 7.1. For n=6, the corresponding distributions of PS_t are given in Table 7.2.

7.4 COMPUTATIONS AND COMPARISONS

When the process mean-vector shifts, the test statistic for the x^2 chart has a noncentral chi-square distribution with noncentrality parameter $n(\underline{\mu}-\underline{\mu}_o)'\Sigma^{-1}(\underline{\mu}-\underline{\mu}_o)$. Without loss of generality, we assume that n=1 to compute the ARLs for the x^2 and GCHIS charts. The IMSL subroutine MDCHN is used for computing probabilities related to the noncentral chi-square distribution in each chart. The values of the ARL for the x^2 chart, a random-walk GCHIS chart, and a cusum GCHIS chart for different (μ_1, μ_2, ρ) combinations are listed in Table 7.3.

In this numerical example, the results show that the GCHIS charts using a random walk type or a cusum type stopping rule are more efficient than the x^2 chart when the process has moderate shifts in the mean vector. The cusum GCHIS chart is better than the random-walk chart for small shifts in the mean-vector. But for large shifts, the random-walk scheme can detect changes quicker than the cusum GCHIS scheme.

Table 7.4 lists the values of the ARL for the x^2 and two MSTS charts for bivariate normal observations with sample size six. The MSTS-AD chart is a modification of the MSTS chart. The MSTS-AD chart allows a signal triggered when all six observations fall into one of the four quadrants. According to this example, the MSTS chart is less efficient than the x^2 chart. However, when the process mean vector is changed, the MSTS-AD chart seems to be more robust to the misspecification of the correlation. Therefore, in practice, we don't need a very accurate estimate of the correlation coefficient. Also, for a given correlation, there are no significant changes in the values of the ARL when the amounts of shift in each component are the same but in different directions. For example, when $\rho=-.75$ the ARL of the x^2 chart for $(\mu_1,\mu_2)=(0.25,0.25)$ is 8.9 and for $(\mu_1,\mu_2)=(0.25,-0.25)$ is 66.3. But using the MSLS-AD the ARLs are roughly equal to 44 in both cases.

Using the MSTS chart, we don't need to compute x^2 or T^2 values. We only need to count the numbers of observations for each score vector. Therefore we can apply the procedure more easily and have less chance of making errors. The statistic $PS_t(\underset{\sim}{X}^t)$ is also invariant with respect to the scale of measurement used and the permutation of axes. For example, when m=2, switching the two axes results in the exchange of N_2 and N_{-2}, but the probabilties p_2 and p_4 are also exchanged. Therefore the statistic PS_t is unchanged. Since the

MSTS chart has the spirit of the sign test statistic, we expect the MSTS chart may be more robust to multivariate heavy tail distributions. If we use the principal components instead of the original axes, the probabilities that each observation falls into one of the quadrants are 0.25. Therefore, the MSTS chart is a distribution free procedure with known Σ when the process is in-control.

Table 7.1 Probabilities of Score Vectors under the Assumption of a Bivariate Normal Distribution with $\sigma_1=1$, $\sigma_1=1$ and $\rho=0.5$.

case	μ_1	μ_2	p_1	p_2	p_3	p_4
1	0.00	0.00	0.333333	0.166667	0.333333	0.166667
2	0.25	0.25	0.437684	0.161022	0.240272	0.161022
3	0.25	-.25	0.316629	0.084665	0.316629	0.282077
4	0.50	0.00	0.418340	0.081660	0.226878	0.273122
5	0.50	-.50	0.272239	0.036299	0.272239	0.419223
6	1.00	0.00	0.468743	0.031257	0.127398	0.372602
7	1.00	1.00	0.745204	0.096141	0.062514	0.096141
8	1.00	-1.0	0.154873	0.003783	0.154873	0.686472

Table 7.2 Distributions and Mean Values of PS_t When n=6 and ρ =0.5

case	μ_1	μ_2	PS_t 0	2	4	6	$E[PS_t]$
1	0.00	0.00	0.105024	0.52019	0.31507	0.059714	2.65895
2	0.25	0.25	0.087949	0.46796	0.35089	0.093206	2.89871
3	0.25	-.25	0.073924	0.45034	0.37779	0.097944	2.99951
4	0.50	0.00	0.061985	0.40319	0.39933	0.135498	3.21668
5	0.50	-.50	0.026348	0.28301	0.47656	0.214078	3.75674
6	1.00	0.00	0.013225	0.17563	0.43848	0.372665	4.34117
7	1.00	1.00	0.006365	0.08905	0.36644	0.538145	4.87273
8	1.00	-1.0	0.000574	0.03864	0.35607	0.604728	5.12991

Table 7.3 ARL Values for the GCHIS and x^2 Charts.

chart	μ_1	μ_2	$-.75$	$-.50$	$-.25$	ρ 0.00	0.25	0.50	0.75
x^2	0.0	0.0	125	125	125	125	125	125	125
	0.5	0.0	46.9	65.6	73.2	75.2	73.1	65.6	46.9
H=9.65	0.5	-.5	70.8	65.6	59.3	51.5	41.7	29.2	13.6
	1.0	0.0	11.5	21.8	27.5	29.2	27.5	21.8	11.5
	1.0	1.0	2.2	5.4	9.4	13.6	17.8	21.8	25.6
	1.0	-1.	25.6	21.8	17.8	13.6	9.4	5.4	2.2
	2.0	0.0	1.9	3.7	5.0	5.4	5.0	3.7	1.9
	2.0	-2.	4.5	3.7	2.9	2.2	1.6	1.2	1.0
RANDOM	0.0	0.0	123.7	123.7	123.7	123.7	123.7	123.7	123.7
-WALK	0.5	0.0	41.1	60.6	68.6	70.9	68.6	60.6	41.4
GCHIS	0.5	-.5	66.2	60.6	54.0	45.8	35.9	24.0	10.7
G_1=4.0	1.0	0.0	9.1	17.4	22.4	24.0	22.4	17.4	9.1
G_2=5.10	1.0	1.0	2.7	4.9	7.6	10.7	14.0	17.4	20.7
H=4.0	1.0	-1.	20.7	17.4	14.0	10.7	7.6	4.9	2.7
F=1.0	2.0	0.0	2.5	3.7	4.5	4.9	4.5	3.7	2.5
	2.0	-2.	4.2	3.7	3.2	2.7	2.3	2.1	2.0
CUSUM	0.0	0.0	126.4	126.4	126.4	126.4	126.4	126.4	126.4
GCHIS	0.5	0.0	35.9	54.1	62.1	64.5	62.1	54.1	35.9
G_1=4.2	0.5	-.5	59.6	54.1	47.6	40.1	31.4	21.4	10.8
G_2=5.55	1.0	0.0	9.5	16.1	20.1	21.4	20.1	16.1	9.5
H=4	1.0	1.0	3.6	5.7	8.2	10.8	13.5	16.1	18.8
K=.25	1.0	-1.	18.8	16.1	13.5	10.8	8.2	5.7	3.6
	2.0	0.0	3.4	4.6	5.4	5.7	5.4	4.6	3.4
	2.0	-2.	5.2	4.6	4.1	3.6	3.3	3.1	3.0

Table 7.4 ARL Values for the MSTS and x^2 Bivariate Control Charts with Sample Size Six.

	μ_1 . μ_2		ρ						
			-.75	-.50	-.25	0.0	.25	.50	.75
MSTS	0.00	0.00	250.2	177.8	154.6	148.5	154.6	177.8	250.2
K=3	0.25	0.25	49.7	46.5	46.3	48.2	52.7	62.0	86.6
H=7	0.25	-.25	86.6	62.0	52.7	48.2	46.3	46.5	49.7
F=7	0.50	0.00	27.3	25.0	24.0	23.7	24.0	25.0	27.3
	0.50	-.50	17.8	14.1	12.4	11.3	10.5	9.9	9.4
	1.00	0.00	5.7	5.7	5.7	5.7	5.7	5.7	5.7
	1.00	1.00	3.7	3.7	3.7	3.8	4.0	4.2	4.5
	1.00	-1.0	4.5	4.2	4.0	3.8	3.7	3.7	3.7
χ^2	0.00	0.00	148.9	148.9	148.9	148.9	148.9	148.9	148.9
H=10.0	0.25	0.25	8.9	21.7	33.5	43.7	52.4	59.9	66.3
	0.25	-.25	66.3	59.9	52.4	43.7	33.5	21.7	8.9
	0.50	0.00	7.4	15.3	20.1	21.7	20.1	15.3	7.4
	0.50	-.50	18.5	15.3	12.1	8.9	5.9	3.4	1.5
	1.00	0.00	1.3	2.3	3.1	3.4	3.1	2.3	1.3
	1.00	1.00	1.0	1.0	1.2	1.5	1.9	2.3	2.8
	1.00	-1.0	2.8	1.0	1.9	1.5	1.2	1.0	1.0
MSTS-	0.00	0.00	102.5	125.3	131.6	132.6	131.6	125.3	102.5
AD	0.25	0.25	43.2	43.1	43.6	44.6	45.9	46.8	44.3
K=3	0.25	-.25	44.3	46.6	45.9	44.6	43.6	43.1	43.2
H=7	0.50	0.00	23.6	23.1	22.8	22.7	22.8	23.1	23.6
F=7	0.50	-.50	12.7	12.0	11.3	10.7	10.2	9.7	9.3
	1.00	0.00	5.6	5.6	5.6	5.6	5.6	5.6	5.6
	1.00	1.00	3.3	3.3	3.3	3.3	3.3	3.3	3.3
	1.00	-1.0	3.3	3.3	3.3	3.3	3.3	3.3	3.3

VIII. SUMMARY AND CONCLUSIONS

In the original cumulative gauging score charts proposed by Page and Munford, only one pair of gauges is used. Using a single pair of gauges is similar to the use of the go-no-go inspection procedure in practice. The advantage of using a single pair of gauges is that the exact average run length is easy to obtain. Since accurate measurements are not necessary and the gauging may be done by a mechnical device, the cumulative gauging score chart can accelerate the inspection process. Although this type of procedure is very simple to use, it still has decent performance. The disadvantage in using one pair of gauges is less sensitivity for detecting large shift in the quality characteristic. Some modifications have been discussed in this dissertation. In order to keep the simplicity in application and in the computation of the exact average run length, two pairs of gauges are used.

In this dissertation, we also illustrated the flexibility of the cumulative gauging score charts. Besides controlling the process mean, it can also be used to control the standard deviation, the mean and standard deviation, and the vector mean by using appropriate scoring systems. The gauging can be applied to either a single observation or the sample mean. Also, this type of procedure is not restricted to normal random variables.

For controlling the process mean, the TPG and OPGSL charts are modifications of Munford's cumulative score chart. The CSGS chart is similar to Page's gauging cusum scheme. These three procedure essentially are of the same type except that the scores are different. The TPG and OPGSL charts are better in general than Munford's chart. The CSGS chart is also better than Page's chart. This shows that the modification of using two pairs of gauges improves the method of using one pair of gauges only.

The CSGS procedure is better than the TPG and OPGSL procedures. When observations are normal and the range of shift in the mean is up to three standard deviations, the performance of the CSGS chart is close to the regular CUSUM chart. Both the CSGS and TPG charts are more resistant to observations whose distribution has heavy tails than are the CUSUM and X charts. If observations are serially correlated, the TPG chart is more robust than the other charts.

If $\mu \geq 1.5\sigma_o$ and ARL(0) is small, the TPG chart is better than the OPGSL chart. If ARL(0) is large, the OPGSL chart can detect a large shift in the mean more quickly. The only difference between these two charts is the scores for $|X_i| \geq G_2$. The values of ARL(μ) show that assigning a larger score for observations with $|X_i| \geq G_2$ does not improve the performance significantly.

Procedures with either one pair or two pairs of gauges have some properties in common. The rule of thumb for

choosing H for the TPG chart is similar to that for Munford's procedure. The choice of K and G_1 in the CSGS procedure follows the same rule as in Page's gauging cusum scheme. For one pair of gauges, the narrower the gauges' range $(-G_1,G_1)$, the smaller the value of the ARL for small μ, but the larger the ARL for large μ. The performance changes at the point $\mu = 1.5\sigma_0$ for both one and two pairs of gauging schemes. The response surface of the ARL is a saddle with respect to H and μ, where the saddle point occurs at $\mu=1.5\sigma_0$.

There are many combinations of H, G_1, and G_2 which have similar performance. The choice depends on whether the primary concern is to control large or small amounts of shift. If we want to control small shifts in the mean, a large H value is recommend. Since a large H value will induce a narrower gauge range $(-G_1,G_1)$, it is more sensitive to small changes.

For controlling both the mean and the standard deviation, the $\overline{X}$-R and the $\overline{X}$-S charts are still better than the CSGS-SD or the TSGS chart. However, the latter two procedures can be used if only a single observation is available. In addition the exact average run length is obtainable.

If n observations are taken at each time, we can either gauge the sample mean or each individual observation. If we choose gauging the sample mean for a CUSUM scheme, Shewhart limits can be added to improve the performance of detecting large shifts. The choice of the reference value K is the same

as used in the regular CUSUM scheme. This approach has been used by Ncube and Woodall (1984).

The gauging scheme can be applied to multivariate quality control by gauging the x^2 (or T^2) statistic. Applying either a random-walk or CUSUM scheme to the gauging scores is better than the x^2 chart in most cases. The cusum GCHIS chart is a new cusum procedure which uses only one chart to control multivariate observations. The MSTS chart is more resistant to the misspecification of the correlation and easy to apply.

In conclusion, the gauging score chart is a large class of control charts. It is very simple in use and has decent performance. The average run length can be obtained by solving a system of difference equations or by using a Markov chain approach. The gauging score chart is also flexible. Using different scoring systems can achieve different control purposes. It can be used for any known distribution and is more robust to heavy tailed distributions and serially correlated observations.

Although several numerical examples show that the performance of gauging score charts is close to the regular CUSUM chart, there is very little theoretical justification to back up these findings. Also the optimal choice for the number of gauging pairs and the chart parameters are still not clear. Additional theoretical and numerical results are needed. For multivariate quality control, especially for the

bivariate case, the extension of the MSTS procedure to a distribution-free procedure such as using Hodges' bivariate sign test statistic will be very helpful. These are possible areas for further studies.

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APPENDIX A. CARDAN'S METHOD

1.CARDAN'S FORMULA

One of the methods to find roots for a polynomial equation with degree three is Cardan's formula (Uspensky, 1948). In order to solve the equation

$$x^3 + a\,x^2 - b\,x - c = 0 \qquad\qquad (A.1)$$

where $a = (p_1 + p_2)/p_2$, $b = (p_{-2} + p_{-1})/p_2$, and $c = p_{-2}/p_2$, let $y = x+k$ and subsitute it in equation (A.1). After simplication, the equation (A.1) changes to

$$y^3 + (3k+a)\,y^2 + (3k^2 + 2ka - b)y + (k^3 + ak^2 - bk - c) = 0.$$

If $k = -a/3$ is chosen, then the y^2 term is dropped. Thus

$$y^3 - (b + a^2/3)y + (-c + ab/3 + 2a^3/27) = 0 \qquad\qquad (A.3)$$

Let $s = (b + a^2/3)$ and $t = (-c + ab/3 + 2a^3/27)$, then the equation (A.3) is simplified to

$$y^3 - sy + t = 0. \qquad\qquad (A.4)$$

Furthermore, letting y = u+v and uv = s/3, the equation (A.4) becomes .

$$u^3 + v^3 + (u+v)(3uv-s) + t = 0 \qquad (A.5)$$

If $u^3 + v^3 = -t$ and $u^3 v^3 = s^3/27$, then the u^3 and v^3 are the roots of the equation

$$y^2 + ty + s^3/27 = 0.$$

Therefore,

$$u^3 = -t/2 + \sqrt{g}$$
$$v^3 = -t/2 - \sqrt{g} \qquad (A.6)$$

where $g = t^2/4 - s^3/27$.

(1) $g < 0$.

We re-express u^3 in polar form, that is

$$u^3 = \sqrt{s^3/27}\ e^{i(\theta+2\pi m)}$$

where $\theta = \cos^{-1}(-t/2\sqrt{27/s^3})$ and $m = 0,1,2$. Therefore

$$u = \sqrt{s/3}\ e^{i(\theta+2\pi m)/3} \qquad m = 0,1,2.$$

Similarly,

$$v = \sqrt{s/3}\ e^{-i(\theta+2\pi m)/3} \qquad m = 0,1,2.$$

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In order to satisfy equation (A.6), the three roots x1, x2, and x3 are

$$x1 = 2\sqrt{s/3}\,\cos(\theta/3) - a/3$$

$$x2 = -2\sqrt{s/3}\,\cos(\pi/3 - \theta/3) - a/3$$

$$x3 = -2\sqrt{s/3}\,\cos(\pi/3 + \theta/3) - a/3.$$

(2) $g > 0$.

The u and v are

$$u = \sqrt[3]{-t/2 + \sqrt{g}}\; e^{2\pi m i/3} \qquad m = 0,1,2$$

and

$$v = \sqrt[3]{-t/3 - \sqrt{g}}\; e^{2\pi m i/3} \qquad m = 0,1,2.$$

Therefore, the roots are

$$x1 = \sqrt[3]{-t/2 + \sqrt{g}} + \sqrt[3]{-t/2 - \sqrt{g}} - a$$

$$x2 = (\sqrt[3]{-t/2 + \sqrt{g}} + \sqrt[3]{-t/2 - \sqrt{g}})e^{2\pi i/3} - a$$

$$x3 = (\sqrt[3]{-t/2 + \sqrt{g}} + \sqrt[3]{-t/2 - \sqrt{g}})e^{4\pi i/3} - a.$$

(3) $g = 0$.

The roots are

$$x1 = 2\sqrt{-t/2} - a$$

$$x2 = \sqrt{t/2} - a$$

$$x3 = \sqrt{t/2} - a.$$

2. THEOREM

If $p_{-1} > p_{-2}$, then $g < 0$.

proof.

$$g = \frac{t^3}{4} - \frac{s^3}{27} = \frac{1}{4}\left(\frac{P_{-2}}{P_2}\right)^2 - \frac{1}{108}\left(1 + \frac{P_1}{P_2}\right)^2\left(\frac{P_{-2}+P_{-1}}{P_2}\right)^2$$

$$- \frac{1}{6}\frac{P_{-2}}{P_2}\left(1 + \frac{P_{-1}}{P_2}\right)\left(\frac{P_{-2}+P_{-1}}{P_2}\right)$$

$$- \frac{1}{27}\frac{P_{-2}}{P_2}\left(1 + \frac{P_{-1}}{P_2}\right)^3 - \frac{1}{27}\left(\frac{P_{-2}+P_{-1}}{P_2}\right)^3$$

$$< \frac{1}{4}\left(\frac{P_{-2}}{P_2}\right)^2 - \frac{1}{108}\left(1 + \frac{P_1}{P_2}\right)^2\frac{4P_{-2}^2}{P_2^2} - \frac{1}{3}\left(\frac{P_{-2}}{P_2}\right)^2$$

$$= -\frac{1}{12}\left(\frac{P_{-2}}{P_2}\right)^2 - \frac{1}{27}\left(1 + \frac{P_1}{P_2}\right)^2\left(\frac{P_{-2}}{P_2}\right)^2$$

$$< 0.$$

APPENDIX B. COMPUTER PROGRAMS

Some major computer programs used in this dissertation are included in this appendix. A brief summary of the programs' name, usage, and program type is listed in the following table. The program type S or F represents SAS or FORTRAN.

1.PROGRAM INFORMATION

PROGRAM NAME	USAGE	Type
1. RUN LENGTH	Compute the run length distribution for two-sided TPG charts	S
2. ARL TPG	Compute the ARL for two-sided TPG charts by solving a system of difference equations	F
3. TPG SHEWHART	Compute the ARL for one-sided OPGSL charts	F
4. CSGS MEAN	Compute the ARL for one-sided CSGS charts for controlling process mean	S
5. CSGS STD	Compute the ARL for the CSGS-SD chart	S
6. MEAN STD	Compute the ARL for the TSGS chart	S
7. CHISQ RWALK	Compute the ARL for the GCHIS chart by using a random walk type stopping rule	F
8. CHISQ CUSUM	Compute the ARL(μ) for the GCHIS chart by using the CUSUM type stopping rule	S
9. CHISQ CUSUMO	Compute the ARL(μ) for the GCHIS chart by using the CUSUM type stopping rule	S
10. BIVAR MSTS	Compute the ARL for the MSTS chart with sample of size six	S

2.PROGRAM LISTING

* PROGRAM # 1-- RUN LENGTH *

```
c********************************************************c
c This program computes the run length distribution,     c
c the ARL, the variance of the run length random variable c
c for two-sided TPG charts.                              c
c p2=Pr[ X ≤ -G2 ], q2=Pr[ -G2 < X ≤ -G1],               c
c q1=Pr[ G1 ≤ X < G2], and p1=Pr[ X ≥ G2].               c
c H is the absorbing barrier. NO is truncated point.      c
c********************************************************c

DATA INFORM;
INPUT P2 Q2 Q1 P1 H NO;
CARDS;
*THIS PROG CALCULATES RUN LENGTH DIST OF GAUGING CONTROL;
*Here NO=30;
PROC MATRIX;
PRUN=J(30,6,0); TAILP=J(30,5,0);
FETCH X; KK=NROW(X); KK1=KK+1;
ARL=J(1,KK);VARIANCE=J(1,KK);
DO K=1 TO KK; K1=K+1;
P2=X(K,1);Q2=X(K,2);   Q1=X(K,3); P1=X(K,4);
H=X(K,5); NO=X(K,6);
R=1-P2-P1-Q1-Q2;
D=(2*H)-1; DH=H;
P=J(D,D,0);
*FORM THE TRANSITION PROB MATRIX;
DO I1=1 TO D;
P(I1,I1)=R;END;
D1=D-1; D2=D-2;
DO I2=1 TO D1;
IN=I2+1; IM=I2+2;
P(I2,IN)=Q1; P(IN,I2)=Q2;
IF I2 >D2 THEN GO TO Z;
P(I2,IM)=P1; P(IM,I2)=P2;END;
Z: ONE=J(D,1,1); ID=I(D);
EO=J(1,D,0,); EO(1,DH)=1;
*COMPUTE THE ARL AND THE PROB OF N;
L=ID-P;
INVIP=INV(L);
ARL(1,K)=EO*INVIP*ONE; L1=L*ONE;
SM=P*(INVIP**2)*ONE;
VARIANCE(1,K)=EO*SM#2+ARL(1,K)-ARL(1,K)**2;
PRUN(1,K1)=EO*L1; PRUN(1,1)=1; TAILP(1,K)=1-PRUN(1,K1);
DO JJ=2 TO NO;
L1=P*L1;   JJ1=JJ-1;
PRUN(JJ,K1)=EO*L1; TAILP(JJ,K)=TAILP(JJ1,K)-PRUN(JJ,K1);
PRUN(JJ,1)=JJ;
END;        END;
PRUNT=PRUN';
PRINT TAILP; PRINT ARL  VARIANCE ;
OUTPUT PRUN OUT=DENSITY
```

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```
(RENAME=(COL1=N   ));
PROC PLOT DATA=DENSITY;
 PLOT COL2*N='.'   COL4*N=','COL6*N='+'/OVERLAY
 HPOS=120 VPOS=50 VSPACE=5;



              * PROGRAM # 2-- ARL TPG *

C*************************************************************C
C This program computes the ARL for the TPG chart by         C
C solving a system of difference equations.                  C
C M is the absorbing barrier. R1 and R2 are λ1 and           C
C λ2 L=4.                                                     C
C*************************************************************C

C THE 1ST PART OF THIS PROG. IS TO CALCULATE ARL0
      DIMENSION Y(4), FN(4),RT(3),RT2M1(3),RTM1(3),RTM(3),
     1 RT2M2(3),W(4,4),WA(4,4),WB(4,4), WC(4,4)
C THE 31 IS THE NUMBER OF CASES TO BE RUN.
      DO 450 MM=1,31
      READ(5,10) M, P, Q, G1, G2 ,R1, R2
      DMU=0.
  10  FORMAT(I2, 6F10.5)
      VAR=8.0*P+2.0*Q
      R1M=R1**M
      R2M=R2**M
      R1M1=R1M*R1
      R2M1=R2M*R2
      ARL0=M*M/VAR+(2*M+1)*(R1M+R2M-2)/VAR/(R1M+R2M-R1M1-R2M1)
      WRITE(6,20) M, P, Q, G1, G2, ARL0
  20  FORMAT(1X, I3,2X,5F10.5)
C THIS PART IS TO CALCULATE P1,P2,Q1,Q2 WHEN DMU#0
      DO 400 II=1,6
      DMU=II*0.5
      Y(1)=G2-DMU
      Y(2)=G1-DMU
      Y(3)=G2+DMU
      Y(4)=G1+DMU
      DO 30 I=1,4
      CALL CDF(Y(I),FN(I))
  30  CONTINUE
      P1=1.0-FN(1)
      Q1=FN(1)-FN(2)
      Q2=FN(3)-FN(4)
      P2=1.0-FN(3)
C THE 3RD PART OF THIS PROG. IS TO CALCULATE THE ROOTS OF
C X3+(P1+Q1)/P1*X2-(P2+Q2)/P1*X-P2/P1=0.
      PQ1=P1+Q1
      PQ2=P2+Q2
      PQP1=PQ1/P1
      PQP2=PQ2/P1
```

```
        T=-P2/P1+PQP1*PQP2/3.+PQP1**3/13.5
        S=PQP2+PQP1*PQP1/3.0
C IF T2/4-S3/27<0,THEN THE EQUATION HAS 3 REAL ROOTS
C IN THIS PROG,WE ASSUME Q=LP, SO ALL ROOTS ARE REAL.
        IF( T*T/4-S*S*S/27 .GT. 0.) GO TO 450
        SS=SQRT(S/3)
        PHI=ARCOS(-T*1.5/S/SS)
        PHID3=PHI/3.0
        RT(1)=2*SS*COS(PHID3)-PQP1/3.0
        RT(2)=-2*SS*COS(1.0472-PHID3)-PQP1/3.0
        RT(3)=-2*SS*COS(1.0472+PHID3)-PQP1/3.0
        XMEAN=2*(P1-P2)+(Q1-Q2)
        DO 40 J=1,3
        RTM(J)=RT(J)**M
        RTM1(J)=RTM(J)*RT(J)
        RT2M1(J)=RTM(J)*RTM1(J)
        RT2M2(J)=RTM1(J)*RTM1(J)
   40   CONTINUE
        DO 50 K=1,4
        W(1,K)=1.0
        W(K,1)=1.0
   50   CONTINUE
        DO 60 K1=2,4
        W(2,K1)=RT(K1-1)
        W(3,K1)=RT2M1(K1-1)
        W(4,K1)=RT2M2(K1-1)
   60   CONTINUE
        WA(1,1)=1.0
        WB(4,1)=1.0
        WC(2,1)=1.0
        DO 70 IO=2,4
        WA(1,IO)=RTM1(IO-1)
        WB(4,IO)=RTM1(IO-1)
        WC(2,IO)=RTM1(IO-1)
   70   CONTINUE
        DO 80 I1=2,4
        DO 80 I2=1,4
        WA(I1,I2)=W(I1,I2)
   80   CONTINUE
        DO 90 L1=1,4
        IF(L1 .EQ. 2) GO TO 90
        DO 90 L2=1,4
        WC(L1,L2)=W(L1,L2)
   90   CONTINUE
        DO 100 J1=1,3
        DO 100 J2=1,4
        WB(J1,J2)=W(J1,J2)
  100   CONTINUE
        CALL DET(W,U)
        DETD=U
        CALL DET(WA,U)
```

```
      DETA=U
      CALL DET(WB,U)
      DETB=U
      CALL DET(WC,U)
      DETC=U
      AA=(DETB-2*M*DETC)/DETD/XMEAN
      ARL=(M-(2*M+1)*DETA/DETD)/XMEAN+AA
      WRITE(6,120)DMU, ARL, P2,Q2,Q1,P1
 120  FORMAT(1X, 6F10.5)
 400  CONTINUE
 450  CONTINUE
 500  STOP
      END
C THIS SUBROUTINE IS TO CALCULATE THE CDF OF N(0,1)
      SUBROUTINE CDF(X, PROB)
      DIMENSION A(6)
      A(1)=0.0498673470
      A(2)=0.0211410062
      A(3)=0.0032776263
      A(4)=0.380036
      A(5)=0.488906
      A(6)=0.053830
      SIGN=1.0
      IF (X .GE. 0.0) GO TO 20
      SIGN=-1.0
      X=-X
  20  X2=X*X
      X3=X2*X
      F11=1.0+A(1)*X+A(2)*X2+A(3)*X3
      F12=X3*(A(4)*X+A(5)*X2+A(6)*X3)/10000.
      F1=F11+F12
      F2=F1*F1
      F4=F2*F2
      F8=F4*F4
      F=F8*F8
      PROB=1.0-0.5/F
      IF (SIGN .GE. 0.0) GO TO 40
      PROB=1-PROB
  40  CONTINUE
      RETURN
      END
C THIS SUBROUTINE IS TO CALCULATE THE DETERMINANT OF 4X4 MATRIX
C THE FIRST COLUMN IS THE ONE'S COLUMN
      SUBROUTINE DET(Z,U)
      DIMENSION Z(4,4)
      U1=(Z(3,3)*Z(4,2)-Z(3,2)*Z(4,3))*(Z(1,4)-Z(2,4))
      U2=(Z(2,3)*Z(3,2)-Z(2,2)*Z(3,3))*(Z(1,4)-Z(4,4))
      U3=(Z(1,3)*Z(2,2)-Z(1,2)*Z(2,3))*(Z(3,4)-Z(4,4))
      U4=(Z(1,3)*Z(4,2)-Z(1,2)*Z(4,3))*(Z(2,4)-Z(3,4))
      U5=(Z(2,2)*Z(4,3)-Z(2,3)*Z(4,2))*(Z(1,4)-Z(3,4))
      U6=(Z(1,2)*Z(3,3)-Z(1,3)*Z(3,2))*(Z(2,4)-Z(4,4))
```

```
      U=U1+U2+U3+U4+U5+U6
      RETURN
      END


              * PROGRAM # 3-- TPG SHEWHART *

C*****************************************************************C
c             This program computes the ARL             c
c             for one-sided OPGSL charts.               c
C*****************************************************************C

C THE 1ST PART OF THIS PROG. IS TO CALCULATE ARLO
      DIMENSION Y(4), FN(4), E(6)
      INTEGER H,F
      READ(5,10) H,F,G1, G2,K,NR
  10  FORMAT(I2,2F10.6,2I2)
C NR IS THE TOTAL NUMBER OF CASES
C K=1 IS ONE-SIDED CONTROL, K=2 IS TWO-SIDED CONTROL
C IF K=2, THEN F=0
      DO 100 MM=1,NR
      DMU=0.
      CALL CDF(G2,W2)
      CALL CDF(G1,W1)
      S=1.-W2
      P=W2-W1
      SP=S/P
      SPSQ=SQRT(SP*(2.+SP))
      X1=1.+SP+SPSQ
      X2=1.+SP-SPSQ
      X1H=X1**H
      X2H=X2**H
      IF (K .EQ. 1)GO TO 20
      EO=0.5*(1.-2./(X1H+X2H))/S
      GO TO 25
  20  X1F=X1**F
      X2F=X2**F
      EO=(1.-(X1F-X2F)/(X1F*X1H-X2F*X2H-X1H+X2H))/2./S
  25  WRITE(6,28) H,F,K,G1,G2,P,S,EO
  28  FORMAT(1X, 3I2,2X,4F10.6,2X,F10.3)
      DO  50 I=1,6
      DMU=I*0.5
      Y(1)=G2-DMU
      Y(2)=G1-DMU
      Y(3)=G2+DMU
      Y(4)=G1+DMU
      DO 30 J=1,4
      CALL CDF(Y(J),FN(J))
  30  CONTINUE
      S1=1.-FN(1)
      P1=FN(1)-FN(2)
```

```fortran
      Q1=FN(3)-FN(4)
      S2=1.-FN(3)
      SSUM=S1+S2
      Q1P1=Q1/P1
      PQSS=P1+Q1+SSUM
      ZZ=SQRT(PQSS**2-4.*P1*Q1)
      Z1=0.5*(PQSS+ZZ)/P1
      Z2=0.5*(PQSS-ZZ)/P1
      Z1H=Z1**H
      Z2H=Z2**H
      IF(K .EQ. 1) GO TO 35
      E(I)=(1.-(Q1P1**H+1.)/(Z1**H+Z2**H))/SSUM
      GO TO 40
      Z1F=Z1**F
      Z2F=Z2**F
      P1Q1F=(P1/Q1)**F
      ZFH=Z1F*Z1H-Z2f*Z2H
      E(I)=(1.-P1Q1F*(Z1F-Z2F)/P1Q1F*(ZFH-Z1H+Z2H))/SSUM
      WRITE(6,40) P1,Q1,S1,S2,E(I)
   40 FORMAT(1X,4F10.6,2X, F10.3)
   50 CONTINUE
   60 FORMAT(1X,I3,2X,4F10.6,2X,F10.3)
  100 CONTINUE
      STOP
      END
C THIS SUBROUTINE IS TO CALCULATE THE CDF OF N(0,1)
      SUBROUTINE CDF(X, PROB)
      DIMENSION A(6)
      A(1)=0.0498673470
      A(2)=0.0211410062
      A(3)=0.0032776263
      A(4)=0.380036
      A(5)=0.488906
      A(6)=0.053830
      SIGN=1.0
      IF (X .GE. 0.0) GO TO 20
      SIGN=-1.0
      X=-X
   20 X2=X*X
      X3=X2*X
      F11=1.0+A(1)*X+A(2)*X2+A(3)*X3
      F12=X3*(A(4)*X+A(5)*X2+A(6)*X3)/10000.
      F1=F11+F12
      F2=F1*F1
      F4=F2*F2
      F8=F4*F4
      F=F8*F8
      PROB=1.0-0.5/F
      IF (SIGN .GE. 0.0) GO TO 40
      PROB=1-PROB
   40 CONTINUE
```

```
        RETURN
        END


              * PROGRAM # 4-- CSGS MEAM *

C************************************************************C
c This program computes the ARL for one-sided CSGS charts  c
c controlling the process mean                             c
c H=u/w. K=v/w. mu=mean. STD=standard deviation.           c
C************************************************************C

OPTION LS=80;
DATA CUSUM;
INPUT G1 G2 MU STD   U V W;
CARDS;
PROC MATRIX FUZZ;
FETCH X DATA=CUSUM; NOBS=NROW(X);
PRINT X;
DO K=1 TO NOBS;
G1=X(K,1); G2=X(K,2); DMU=X(K,3);STD=X(K,4);
U=X(K,5); V=X(K,6); W=X(K,7);MU=DMU;
*COMPUTE P2,Q2,Q1,P1;
IF MU>=0 THEN GO TO MM;
DMU=-MU;
MM: NG2=(G2+DMU)#/STD#/1.414; NG1=(G1+DMU)#/STD#/1.414;
PG1=(G1-DMU)#/STD#/1.414; PG2=(G2-DMU)#/STD#/1.414;
 PP2=ERF(NG2)#/2;
 PQ2=ERF(NG1)#/2;
P2=0.5-PP2; Q2=PP2-PQ2;
IF PG1<0 THEN GO TO WA;
 PP1=ERF(PG2)#/2; PQ1=ERF(PG1)#/2;
 P1=0.5-PP1; Q1=PP1-PQ1;
WA: PG1=-PG1; PQ1=ERF(PG1)#/2;
 IF PG2<0 THEN GO TO WB;
 PP1=ERF(PG2)#/2; P1=0.5-PP1; Q1=PP1+PQ1;
WB: PG2=-PG2;PP1=ERF(PG2)#/2;
P1=0.5+PP1; Q1=PQ1-PP1;
IF MU>=0 THEN GO TO WC;
    NP1=P1; NQ1=Q1; NQ2=Q2; NP2=P2;
    P1=NP2; Q1=NQ2; Q2=NQ1; P2=NP1;
WC: R=1-P1-P2-Q1-Q2;
P=J(U,U,0); ONE=J(U,1,1); E=J(1,U,0); E(1,1)=1;
*FORM TRANSITION MATRIX;
PQR=P2+Q2+R; PQ=P2+Q2;V1=V+1; VW=V+W; V2W=V+2*W;
 DO I1=1 TO V1;
  P(I1,1)=PQR; END;
 V2=V+2; VW1=V+W+1;
 Z1=J(1,2,0); Z1(1,1)=VW1; Z1(1,2)=U; G1=MIN(Z1);
 DO I2=V2 TO G1;
  P(I2,1)=PQ;   END;
```

```
VW2=VW+2;  V2W1=V2W+1;
Z2=J(1,2,0);Z2(1,1)=V2W1; Z2(1,2)=U; G2=MIN(Z2);
IF VW2 > G2 THEN GO TO WW;
DO I3=VW2 TO G2;
 P(I3,1)=P2; END;
WW: UVMW=U+V-W; WMV=W-V;
DO I4=1 TO UVMW;
 P(I4,I4+WMV)=Q1; END; Z=J(1,2,0);
UVM2W=U+V-2*W; Z(1,1)=UVM2W; Z(1,2)=U; G=MIN(Z);
 IF UVM2W < 0 THEN GO TO S1;
DO J1=1 TO G;
 P(J1,J1+2*W-V)=P1; END;
S1: UMV=U-V;
DO J2=2 TO UMV;
 P(J2+V,J2)=R; END;
UMVW=U-V-W;
   IF UMVW < 2 THEN GO TO SS;
DO J3=2 TO UMVW;
 P(J3+VW,J3)=Q2; END;
UMV2W=U-V-2*W;
IF UMV2W-2<1 THEN GO TO SS;
DO J4=2 TO UMV2W;
 P(J4+V2W,J4)=P2; END;
SS: ID=I(U); IMP=ID-P; IMPI=INV(IMP);
 ARL=E*IMPI*ONE;
 PRINT  ARL P;  END;
```

```
                 * PROGRAM # 5-- CGSG STD *

c************************************************************c
c     This program computes the ARL for the CSGS-SD chart    c
c************************************************************c

DATA CUSUM;
INPUT G1 G2 MU STD  U V W;
CARDS;
PROC MATRIX FUZZ;
FETCH X DATA=CUSUM; NOBS=NROW(X);
PRINT X;
DO K=1 TO NOBS;
G1=X(K,1); G2=X(K,2); DMU=X(K,3);STD=X(K,4);
U=X(K,5); V=X(K,6); W=X(K,7);MU=DMU;
IF MU>=0 THEN GO TO MM;
DMU=-MU;
MM: NG2=(G2+DMU)#/STD#/1.414; NG1=(G1+DMU)#/STD#/1.414;
PG1=(G1-DMU)#/STD#/1.414; PG2=(G2-DMU)#/STD#/1.414;
 PP2=ERF(NG2)#/2;
 PQ2=ERF(NG1)#/2;
P2=0.5-PP2; Q2=PP2-PQ2;
IF PG1<0 THEN GO TO WA;
```

```
 PP1=ERF(PG2)#/2; PQ1=ERF(PG1)#/2;
 P1=0.5-PP1; Q1=PP1-PQ1;                        .
WA: PG1=-PG1; PQ1=ERF(PG1)#/2;
 IF PG2<O THEN GO TO WB;
 PP1=ERF(PG2)#/2; P1=0.5-PP1; Q1=PP1+PQ1;
WB: PG2=-PG2;PP1=ERF(PG2)#/2;
P1=0.5+PP1; Q1=PQ1-PP1;
IF MU>=O THEN GO TO WC;
    NP1=P1; NQ1=Q1; NQ2=Q2; NP2=P2;
    P1=NP2; Q1=NQ2; Q2=NQ1; P2=NP1;
WC: R=1-P1-P2-Q1-Q2;
P=J(U,U,0); ONE=J(U,1,1); E=J(1,U,0); E(1,1)=1;
V1=V+1; WMV=W-V;
PP=P2+P1; QQ=Q1+Q2;  RR=1-PP-QQ;
DO I1=1 TO V1;
    P(I1,1)=RR;  END;
 V2=V1+1; UMV=U-V+1;
DO I2 =V2 TO UMV;
    IMV=I2-V; P(I2,IMV)=RR;  END;
 UVMW=U+V-W;
DO I3=1 TO UVMW;
    IWMV=I3+WMV; IF IWMV > U THEN GO TO TT;
P(I3,IWMV)=QQ;  END;
 TT: Z=J(1,2,0); UVM2W=U+V-2*W;
 Z(1,1)=UVM2W; Z(1,2)=U; G=MIN(Z);  W2MV=2*W-V;
DO I4=1 TO G;
    II=I4+W2MV;
    IF II > U THEN GO TO SS;
    P(I4,II)=PP;  END;
SS: ID=I(U); IMP=ID-P;
IMPI=INV(IMP);
ARL=E*IMPI*ONE;
PRINT ARL; END;


           * PROGRAM # 6-- MEAN STD *

c*****************************************************************c
c This program computes the ARL for the TSGS chart.           c
c ABSORBM is the absorbing barrier for controlling the mean.c
c ABSORBS is the absorbing barrier for controlling the        c
c standard deviation.                                          c
c*****************************************************************c

DATA INFORM;
INPUT ABSOBM ABSOBS PP2 QQ2 QQ1 PP1;
CARDS;
PROC MATRIX;
FETCH X DATA=INFORM; NC=NROW(X);
ARL=J(NC,1,0);
DO M=1 TO NC;
```

```
H1=X(M,1); H2=X(M,2); P2=X(M,3);
Q2=X(M,4); Q1=X(M,5); P1=X(M,6);
R=1-P2-P1-Q1-Q2;
H1D=H1*2; H11=H1+1; H11D=H11*2; H1DM1=H1D-1; H1DM3=H1D-3;
H2M1=H2-1; H2M2=H2-2; H1DM2=H1D-2; NS=H1DM1*H2;
H2P1=H2+1; H2P1D=H2P1*2;
P=J(NS,NS,0);
  DO D0=1 TO H1DM1;
     K=H2*(D0-1)+1; P(K,K)=R; END;
  DO D1=1 TO H1DM1;
     DO  R2=2 TO H2;
      K1=H2*(D1-1)+R2; K11=K1-1;
      P(K1,K11)=R; END; END;
  DO UR1=1 TO H1DM2;
     DO R3=1 TO H2M1;
        K2=H2*(UR1-1)+R3;  K2R1=K2+H2P1;
        P(K2,K2R1)=Q1; END; END;
  DO UL1=2 TO H1DM1;
     DO R4=1 TO H2M1;
        K3=H2*(UL1-1)+R4; K3L1=K3-H2M1;
        P(K3, K3L1)=Q2; END; END;
  DO UR2=1 TO H1DM3;
     DO R5=1 TO H2M2;
        K4=H2*(UR2-1)+R5; K4R2= K4+H2P1D;
        P(K4,K4R2)=P1; END; END;
  DO UL2=3 TO H1DM1;
     DO R6=1 TO H2M2;
        K5=H2*(UL2-1)+R6; K5L2=K5-H2M1*2;
        P(K5,K5L2)=P2; END; END;
 ID=I(NS); E=J(NS,1,1); U=J(1,NS,0);
 H12=(H1-1)*H2+1; U(1,H12)=1;
 IMP=ID-P; IMPIV=INV(IMP);
 ARL(M,1)=U*IMPIV*E; END;
 XX=X||ARL; PRINT XX ;
```

                    * PROGRAM # 7-- CHISQ RWALK *

```
c*************************************************************c
c This program computes the ARL for the GCHIS chart          c
c by using a random walk type stopping rule.                 c
c*************************************************************c

        INTEGER H,F
C       N IS THE NO. OF CASES
        DF=2.0
        DO 100 I=1,8
        READ(5,10) G2,G1,XMU1,XMU2,RHO,H,F
   10   FORMAT(2F6.3,3F5.2,2I2)
C       USING NON-CENTRAL CHISQ TO FIND
C       GAUGING SCORE PROBABILITY
```

```
        PNONC=XMU1*XMU1-2.*XMU1*XMU2*RHO+XMU2*XMU2
        PNONC=PNONC/(1.-RHO*RHO)
        CALL MDCHN(G2,DF,PNONC,P1,IER)
        CALL MDCHN(G1,DF,PNONC,R,IER)
C       CALL MDCH(G2,2,P1,IER)
C       CALL MDCH(G1,2,R,IER)
        P=1-P1
        Q=P1-R
C       COMPUTE THE ROOTS
        PQ=P+Q
        PR4=4*P*R
        Z=SQRT(PQ*PQ+PR4)
        X=(-PQ+Z)*0.5/P
        Y=-(PQ+Z)*0.5/P
        E=2.*P+Q-R
C       COMPUTE THE AVERAGE RUN LENGTH
        XH=X**H
        YH=Y**H
        XH1=XH*X
        YH1=YH*Y
        XF=X**F
        YF=Y**F
        YHF=YH*YF
        XHF=XH*XF
        XYF=XF*YF
        XYH=XH*YH
        D1=XHF*(1-X)*(1-YF)
        D2=YHF*(1-Y)*(1-XF)
        D=D1-D2
        B=F*XYF*YH*(1-Y)-XF*(1-YF)
        C=-F*XYF*XH*(1-X)+YF*(1-XF)
        ARL=(H+1)/E+((1-XH1)*B+(1-YH1)*C)/D/E
        WRITE(6,20)G1,G2,XMU1,XMU2,RHO,H,F,P,Q,R,ARL
   20   FORMAT(1X,5F6.3,2I2,3F7.5,1X,F10.4)
  100   CONTINUE
        STOP
        END



            * PROGRAM # 8-- CHISQ CUSUM *

c*****************************************************************c
c This program computes the ARL(μ) for the GCHIS chart          c
c by using a random walk type stopping rule.                    c
c*****************************************************************c

DATA CUSUM;
INPUT G1 G2  U V W;
CARDS;
PROC MATRIX FUZZ;
FETCH X DATA=CUSUM; NOBS=NROW(X);
```

```
DO K=1 TO NOBS;
G1=X(K,1); G2=X(K,2);
U=X(K,3); V=X(K,4); W=X(K,5);
PP=1-PROBCHI(G2,2); RR=PROBCHI(G1,2);
P=J(U,U,0); ONE=J(U,1,1); E=J(1,U,0); E(1,1)=1;
V1=V+1; WMV=W-V;
QQ=1-PP-RR;
DO I1=1 TO V1;
    P(I1,1)=RR;   END;
 V2=V1+1; UMV=U-V+1;
DO I2 =V2 TO UMV;
    IMV=I2-V; P(I2,IMV)=RR;   END;
 UVMW=U+V-W;
DO I3=1 TO UVMW;
    IWMV=I3+WMV; IF IWMV > U THEN GO TO TT;
  P(I3,IWMV)=QQ;   END;
 TT: Z=J(1,2,0); UVM2W=U+V-2*W;
 Z(1,1)=UVM2W; Z(1,2)=U; G=MIN(Z);   W2MV=2*W-V;
DO I4=1 TO G;
    II=I4+W2MV;
    IF II > U THEN GO TO SS;
    P(I4,II)=PP;   END;
SS: ID=I(U); IMP=ID-P;
IMPI=INV(IMP);
ARL=E*IMPI*ONE;
PRINT ARL;END;
```

```
c****************************************************************c
c This program computes the ARL for the GCHIS chart            c
c by using the CUSUM type stopping rule.                       c
c****************************************************************c

DATA CUSUM;
INPUT P Q R  ;
U=16;V=1; W=4;
CARDS;
PROC MATRIX FUZZ;
FETCH X DATA=CUSUM; NOBS=NROW(X);
DO K=1 TO NOBS;
PP=X(K,1); QQ=X(K,2);RR=X(K,3);
U=X(K,4); V=X(K,5); W=X(K,6);
P=J(U,U,0); ONE=J(U,1,1); E=J(1,U,0); E(1,1)=1;
V1=V+1; WMV=W-V;
DO I1=1 TO V1;
    P(I1,1)=RR;   END;
 V2=V1+1; UMV=U-V+1;
DO I2 =V2 TO UMV;
    IMV=I2-V; P(I2,IMV)=RR;   END;
```

```
  UVMW=U+V-W;
DO I3=1 TO UVMW;
     IWMV=I3+WMV;  IF IWMV > U THEN GO TO TT;
P(I3,IWMV)=QQ;   END;
 TT: Z=J(1,2,0); UVM2W=U+V-2*W;
 Z(1,1)=UVM2W; Z(1,2)=U; G=MIN(Z);   W2MV=2*W-V;
DO I4=1 TO G;
     II=I4+W2MV;
     IF II > U THEN GO TO SS;
     P(I4,II)=PP;   END;
SS: ID=I(U);  IMP=ID-P;
IMPI=INV(IMP);
ARL=E*IMPI*ONE;
PRINT ARL;  END;


                * PROGRAM # 10-- BIVAR MSTS *

c*************************************************************c
c This program computes the ARL for the MSTS chart          c
c with sample of size six.                                  c
c P0, P2, P4, and P6 are the probabilities for PS = 0, 2,   c
c 4, and 6 respectively.                                    c
c The reference value K is restricted to 1, 2, and 3.       c
c*************************************************************c

OPTION LS=80;
DATA PST;
INPUT P0 P2 P4 P6  ;
K=3; H= 7;F=7;
CARDS;
PROC MATRIX;
FETCH X; N=NROW(X);  ARL=J(N,1,0);
DO II=1 TO N;
Q0=X(II,1); Q2=X(II,2); Q4=X(II,3); Q6=X(II,4);
K=X(II,6); H=X(II,7);F=X(II,8); M=H+F-1;
P=J(M,M,0); ID=I(M); ONE=J(M,1,1);
UNIT=J(1,M,0); UNIT(1,F)=1;
*FORM THE TRANSITION MATRIX BASED ON THE VALUE K;
HM1=M-1; HM2=M-2; HM3=M-3; HM4=M-4; HM5=M-5;
IF K >1 THEN GO TO Z1;
DO I1=1 TO HM1; J1=I1+1; P(I1,J1)=Q2; P(J1,I1)=Q0;END;
DO I2=1 TO HM3; J2=I2+3; P(I2,J2)=Q4; END;
DO I3=1 TO HM5; J3=I3+5; P(I3,J3)=Q6; END;
P(1,1)=Q0;
GO TO Z3;
Z1: IF K>2 THEN GO TO Z2;
DO II1=1 TO HM1; P(II1,II1)=Q2; END;
DO II2=1 TO HM2; JJ2=II2+2; P(II2,JJ2)=Q4;P(JJ2,II2)=Q0; END;
DO II3=1 TO HM4; JJ3=II3+4;P(II3,JJ3)=Q6; END;
P(1,F)=P(1,F)+Q0;P(2,F)=P(2,F)+Q0;
```

```
GO TO Z3;
Z2: DO IJ1=1 TO HM1; IK1=IJ1+1;P(IJ1,IK1)=Q4;
P(IK1,IJ1)=Q2; END;
DO IJ2=1 TO HM3; IK2=IJ2+3;  P(IJ2,IK2)=Q6; P(IK2,IJ2)=Q0;END;
P(1,F)=Q2+Q0; P(2,F)=Q0; P(3,F)=P(3,F)+Q0;;
Z3: IMP=ID-P; IMPI=INV(IMP);
ARL(II,1)=UNIT*IMPI*ONE; END;PRINT ARL;
```