# BALANCED, CAPACITATED, LOCATION-ALLOCATION PROBLEMS ON NETWORKS WITH A CONTINUUM OF DEMAND

by

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(ABSTRACT)

Location-allocation problems can be described generically as follows: Given the location or distribution (perhaps, probabilistic) of a set of customers and their associated demands for a given product or service, determine the optimum location of a number of service facilities and the allocation of products or services from facilities to customers, so as to minimize total (expected) location and transportation costs.

This study is concerned with a particular subclass of location-allocation problems involving capacitated facilities and a continuum of demand. Specifically, two minisum, network-based location-allocation problems are analyzed in which facilities having known finite capacities are to be located so as to optimally supply/serve a known continuum of demand.

The first problem considered herein, is an absolute p-median problem in which $p > 1$ capacitated facilities are to be located on a chain graph having both nodal and link demands, the latter of which are defined by nonnegative, integrable demand functions. In addition, the problem is balanced, in that it is assumed the total demand equals the total supply. An exact solution procedure is developed, wherein the optimality of a certain location-allocation scheme (for any given

ordering of the facilities) is used to effect a branch and bound approach by which one can identify an optimal solution to the problem.

Results from the chain graph analysis are then used to develop an algorithm with which one can solve a dynamic, sequential location-allocation problem in which a single facility per period is required to be located on the chain.

Finally, an exact solution procedure is developed for locating a capacitated, absolute 2-median on a tree graph having both nodal and link demands and for which the total demand is again equal to the total supply. This procedure utilizes an algorithm to construct two subtrees, each of whose ends constitute a set of candidate optimal locations for one of the two elements of an absolute 2-median. Additional localization results are used to further reduce the number of candidate pairs (of ends) that need to be considered, and then a post-localization analysis provides efficient methods of comparing the relative costs of the remaining pairs.

To my wife,            , and
our children,            and

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My wife's confidence in me was always available when I most needed it, and she seemed to somehow understand, even better than myself, what this degree meant to me. Even our children, in spite of their young ages, sensed the importance to me of these past three years. I will always remember my son, and even my four-year-old daughter, telling me that "I better get busy and get some work done". I am certain that my wife and I will always consider these past years in Blacksburg as three of our happiest.

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# CHAPTER I

# INTRODUCTION AND A SELECTED REVIEW OF THE LITERATURE

## 1.1 Problem Description

A particularly fertile area of research within the field of Operations Research treats a class of mathematical programming problems which have collectively come to be called the Location-Allocation Problem. This problem can be described generically as follows: Given the location or distribution (perhaps, probabilistic) of a set of customers and their associated demands for a given product or service, determine the optimum location of a number of service facilities and the allocation of products or services from facilities to customers, so as to minimize total (expected) location and transportation costs.

Location-allocation problems lend themselves readily to the solution of real-world problems. Examples of such might include the location of warehouses, distribution centers, service and production facilities, and emergency service facilities (see Francis and White [1974], and Handler and Mirchandani [1979]).

The Location-Allocation Problem was originally formulated by Cooper [1963] (see also Cooper [1972]), but as suggested by both Cooper and by Kuhn [1973], its ancestry may be traced back to the General Fermat Problem, first formulated in the 17th century. As will be seen in the next section, the research and consequent literature that has evolved since Cooper's original paper [1963], varies quite extensively in both problem formulation and solution methodology. Consequently, there exist commonly used criteria and/or descriptors with which one can effect as detailed a partitioning of the set of all location-allocation problems as one so desires. In particular, any given location-allocation problem can be identified as belonging to some class/subclass of location-allocation problems. The particular problems analyzed herein for example, fall within the generic class of capacitated, network location-allocation problems, for they involve the location of capacitated facilities on a network in order to satisfy some known demand distribution on the network.

In order to formulate this class of problems, let G(N,A) be an n-vertex undirected (connected) network where the set N denotes the collection of nodes or vertices $v_k$, k=1, ..., n, and where the set A denotes the set of arcs or links $\ell \equiv (i,j)$ connecting certain designated node pairs $v_i$ and $v_j$ in N. For any two points P and Q on the network, let d(P,Q) denote the length of the shortest path between P and Q in the network. With each node $v_k \varepsilon N$, associate a nonnegative demand/weight $h_k$, k=1, ..., n. Note that by creating additional nodes if necessary, one may assume that all such discrete demands are confined to the nodes of G. Furthermore, with each link $\ell \varepsilon A$, associate a nonnegative, integrable demand function $f_\ell(x)$ defined for all points x on the link $\ell$. This function may be thought of as a continuum of demand or as a weighted probability density function, or as a weighted histogram such that the expected demand on an infinitesimal segment dx of the link $\ell$ at a point $x \varepsilon dx$ is given by $f_\ell(x)dx$. (For applications in which such demand densities arise, the reader is referred to Handler and Mirchandani [1979], Chiu [1982], Minieka [1977], and Denardo, et al [1982].) Let $y = \{y_1, \ldots, y_p\}$ represent the location decision variables, namely, the points on G at which the p facilities (or supply centers) are to be located. Let $s_i$ denote the supply available at facility i, i=1, ..., p. Accordingly, define the allocation decision variables as follows. Let $\omega_{ik}$ denote the quantity supplied to node $v_k$ from facility i, i=1, ..., p, k=1, ..., n and let $\Phi_{i\ell}(x)$, $x \in \ell$ denote the nonnegative function which gives the portion of the demand $f_\ell(x)$, $x \in \ell$ served by facility i, for i=1, ..., p and $\ell \in A$. Then, we formulate the capacitated (general) absolute p-median (or minisum) network location-allocation problem as follows:

$$\underline{\text{GAMNLAP}} : \underset{\substack{y \text{ on } G \\ \omega \text{ and } \Phi(\cdot) \geq 0}}{\text{minimize}} \quad \sum_{i=1}^{p} \sum_{k=1}^{n} \omega_{ik} d(y_i, v_k) + \sum_{i=1}^{p} \sum_{\ell \in A} \int_\ell \Phi_{i\ell}(x) d(x, y_i) dx$$

$$\text{subject to} \quad \sum_{k=1}^{n} \omega_{ik} + \sum_{\ell \in A} \int_\ell \Phi_{i\ell}(x) dx \leq s_i \text{ for each } i=1, \ldots, p$$

$$\sum_{i=1}^{p} \omega_{ik} = h_k \text{ for each } k=1, \ldots, n$$

$$\sum_{i=1}^{p} \Phi_{i\ell}(x) = f_\ell(x), \text{ for each } x \in \ell,\ \ell \in A .$$

We remark that in describing/classifying Problem GAMNLAP, the descriptor "capacitated" implies that $s_i < \infty$ for some or all i=1, ..., p, the descriptor "absolute" denotes that $y_i$ may lie anywhere on the network, i=1, ..., p, and the descriptor "(general)" is used to indicate that in addition, there exists discrete nodal demand as well as a continuum of link demands. Note that the constraints in the above formulation represent supply and demand constraints, and that the objective function attempts to minimize total (expected) costs, where costs are assumed to be directly proportional to the distance travelled in the network in order to satisfy a demand. A pertinent comment here is that the above mathematical formulation is more by way of precisely defining the problem as opposed to posing a mathematical program that needs to be solved directly. For the most part, solution algorithms for this class of problems rely heavily on the inherent graph theoretic nature of the problem.

In this study, we are primarily concerned with the exact solution of two important and insightful special cases of Problem GAMNLAP. The first of these involves the location of capacitated facilities on a chain graph, whereas the second involves locating two (2) such facilities on a tree network. Each of these problems assumes the respective graph/network to have both nodal and link demands. Abbreviated descriptions of these problems, their solutions and results, are given below. In addition to the chain graph problem, we provide an efficient algorithm for solving a period by period sequential location-allocation problem on the chain. We might add that the optimization techniques used in this study include linear and nonlinear programming, the Calculus, branch and bound concepts, and of course, graph theoretic procedures.

### 1.1.1 The Chain Graph Problem

The chain graph problem, denoted as Problem CP, is a special case of Problem GAMNLAP in which G(N,A) is a chain having $|N|$ nodes and $|A| = |N|-1$ links. However, by assuming that all discrete nodal demands have been continuously spread over some $\varepsilon$-length links as in Cavalier and Sherali [1983a], one can consider the problem to be that of locating an absolute p-median, involving $p > 1$ capacitated ($s_i < \infty$, $i=1, \ldots, p$) facilities, on a real line segment $[0,c]$ on which there are no points of positive (demand) mass, and on which the demand is defined by a single nonnegative, integrable (demand) function $f(\cdot)$. Specifically, the problem is to locate $p > 1$ facilities (having known finite capacities) on the interval $[0,c]$, and to determine exactly how each of them will allocate its supply to the demand on $[0,c]$, so as to minimize the total (expected) transportation cost as (appropriately) defined by the objective function of Problem GAMNLAP. Furthermore, it is important to note that Problem CP assumes that $\sum_{i=1}^{p} s_i = \int_0^c f(x)dx$. As such, we choose to describe Problem CP as being a capacitated, balanced, location-allocation problem on a chain graph with a continuum of link demands.

We begin our analysis of Problem CP by determining optimal solutions to both an associated location and allocation subproblem. The first of these requires only the use of the Calculus, whereas the latter requires that we make use of an approximating transportation problem and a corresponding optimal solution obtained via the Northwest Corner Rule. After determining how to solve these two subproblems, we are then able to solve the chain graph problem for any given permutation/ordering of the p facilities, and thereby "reduce" Problem CP to that of determining which of the possible orderings of the facilities results in a least cost solution.

The likelihood of there being a large number of different orderings (even for relatively small values of p) invoked our use of implicit enumeration (branch and bound) in trying to determine a least cost ordering. Specifically, we (algebraically) establish results that enable the efficient construction of an enumeration tree by which all orderings are implicitly (rather than explicitly) considered. The branching criteria/rules by which we construct such a tree are dictated by results such as the following: that ordering in which the facilities appear from left to right (in [0,c]) in nondecreasing order of capacity, is optimal in the case where the demand on [0,c] is defined by some nondecreasing function. In order to effect the second aspect of the branch and bound methodology, i.e. bounding the objective function at nodes of the tree so as to allow early fathoming, we establish results which enable us to compute a lower bound for any partial ordering which is often times tight enough to allow us to fathom the corresponding node and thereby avoid having to explicitly examine the completions of the associated partial ordering. We give several examples which illustrate the use of our branching criteria and bounding capability. To summarize our description of Problem CP, it can be stated that we obtain closed form solutions in some cases (i.e. for some f(•)), and obtain reduction theorems restricting the types of candidate solutions in other cases.

Finally, we present a simple algorithm to solve a dynamic, sequential location-allocation problem in which a single facility per period is required to be located. Specifically, the algorithm prescribes a reduced set of candidate optimal solutions from which to choose the best. We provide an illustrative example of this algorithm.

### 1.1.2 The Absolute 2-median Tree Problem

The second special case of Problem GAMNLAP considered herein, is denoted as Problem C2MTP, and involves the location of a capacitated absolute 2-median on a tree network. As was the case for the chain graph problem, our analysis of Problem C2MTP assumes that the problem is balanced, but unlike the chain graph problem, our analysis of Problem

C2MTP must (explicitly) contend with both nodal and link demands.

Our approach to solving Problem C2MTP is to develop an algorithm which can be used to construct two trees contained entirely within $G(N,A) \equiv T(N,A)$, each of whose ends constitute a set of candidate optimal locations for one of the two elements of an absolute 2-median. We provide results which address the geometry of these trees and which reduce the number of candidate pairs (of ends) that need to be considered. In addition, efficient methods of comparing the relative costs of candidate pairs are provided.

We would add, that once the locations of an absolute 2-median are identified, the results of our chain graph analysis can be used to determine corresponding optimal allocations for each of the two capacitated facilities. These allocations are such that no link of $T(N,A)$ will contain a subset having positive measure over which the demand is jointly supplied. Rather, our optimal solution is such that only nodal demands can be jointly supplied.

## 1.2 Literature Review

In this section, we present the reader with an overview of an area of analysis called Location Theory, of which the Location-Allocation Problem is one of two problems of interest. Specifically, Location Theory involves the analysis of two similar/related types of problems: the Location-Allocation Problem described above, and the (Pure) Location Problem. It is this author's opinion, that the literature is often times inconsistent in ascribing a particular problem formulation to one of the above types of problems. Consequently, we will consider a location-allocation problem to be one in which the exact allocation of supply to demand is unknown prior to locating the facilities, whereas, a location problem is one in which the allocations are known prior to the facility locations.

Note that our criterion for delineating between a location and location-allocation problem is all inclusive of any problem involving the location of $p > 1$ facilities (with respect to some set of points) so as to optimize some cost function, since any such problem necessarily has an associated allocation. A case in point, is the General Fermat Problem in which a single point is to be located on a plane so as to minimize the sum of the distances to three (3) fixed points on the plane. This problem has an associated allocation in which all allocations are unity. With respect to the issue at hand, we would classify the General Fermat Problem as a location problem, as does the literature. In general, our criterion and the literature are in agreement in classifying any single facility problem as a location problem, since all demand will necessarily be supplied by the single facility, regardless of its location.

The inconsistency to which we refer, usually does not involve such explicit (with respect to allocation) problem formulations as those of Francis and White [1974], in which they formulate a location-allocation problem to be one in which there are both location and allocation decision variables, and in which they formulate a location problem so that the amount of demand provided customer j by facility i, say, is a given data of the problem. Rather, the inconsistency usually concerns problems involving the location of uncapacitated facilities for which the entire demand of any customer is fully supplied by the nearest service facility. Some texts (see Handler and Mirchandani [1979]) as well as some of the literature, refer to such problems as location problems, whereas others refer to them as location-allocation problems. With respect to our criterion, such problems are considered to be location-allocation problems since one does not know which facility will (fully) supply/serve which customer until such time as the facility locations are determined.

The intent of the above discussion is not to confuse an issue that might otherwise have gone unnoticed, but rather we wish only to apply some rigor to what appears (to us) to be an issue lacking thereof. From time to time throughout our review of the literature, we will remind the reader of this issue by specifying the type of problem to which the literature pertains. Additionally, we would remark once again, that this study involves the analysis of two (2) special cases of Problem GAMNLAP, which are location-allocation problems.

We will begin our overview of Location Theory with a brief discussion of some of the criteria/descriptors commonly used to classify/categorize both location and location-allocation problems. This will in turn be followed by a selected review of the literature. (Note that for the most part, the following discussion applies equally to both location and location-allocation problems. However, for the sake of convenience and since this study treats the Location-Allocation Problem, we will only refer to the Location-Allocation Problem during the course of this discussion. At such time as necessary/appropriate, we will specify if in fact we are discussing a (pure) location problem.)

### 1.2.1 Classification of Location-Allocation Problems

One of the first criteria commonly used to classify a location-allocation problem concerns the solution space of the problem. Typically, one finds that most of the literature involves the location of facilities on either a plane (see references 19, 20, 22, 23, 69, 71, 72, 74, 86, 101, 105, 110) or a network (see references 4, 8, 10, 14, 18, 49, 50, 51, 57, 81, 82, 117, 118). This is not to say that one is discouraged from working on a more exotic topological space, but rather that practicality appears to mandate the choice of solution space. Additionally, the solution space can be either discrete (see references 32, 45, 48, 106, 113) or continuous (see references 19, 20, 22, 23, etc.). Typically, a discrete solution space is one in which the facility locations are restricted to a finite set of points, or at least to a countably infinite set (of points) of measure zero; whereas, a continuous solution space is one in which the facilities are to be located in a set having non-empty relative interior such as the entire plane or network, or in some specified regions of the plane or network. In either case, the solution space can be further defined/restricted by side constraints.

Obviously, any location-allocation problem will require that some measure of distance be used to determine the goodness of its solutions. As such, the distance measure/metric involved in location-allocation problems provides us with another basis for classifying these problems. For network-based location-allocation problems, the distance measure most often used is that in which the distance between two points on the network is defined to be the length of the shortest path connecting them (this assumes connectivity). For problems involving the location of facilities on a plane (or on any n-dimensional Euclidean Space), the most frequently used metrics are the rectilinear (see references 19, 20, 22, 23, 74, 105, 110), Euclidean/squared-Euclidean (see references 19, 20, 22, 23, 69, 71, 72, 86, 101), and the general $\ell_p$ norm (see references 27, 55, 84). Of course, the particular distance measure used in any given problem is usually (somewhat) dictated by the qualitative nature of the problem. For example, if one wished to locate some service facilities (i.e. fire houses, police stations, etc.) within a large city in which the street layout was of a "north-south-east-west" configuration, then the rectilinear metric would be the metric most likely to be used. We will define each of the above metrics during the course of our literature review.

A third category of classification concerns the <u>optimality criteria</u> used to evaluate alternative solutions. The most commonly used are the minisum and minimax criteria. The first of these usually involves the minimization of some total cost function, whereas the latter employs a conservative point of view and attempts to minimize the worst case behavior of the system being modeled. As is the case for the distance measure, the choice of optimality criterion is often influenced by the very nature of the problem itself. For example, if one were locating warehouses from which parts are to be shipped to automobile assembly plants say, then one would most likely want to minimize total transportation/shipment costs and would therefore use the minisum optimality criterion. On the other hand, if one were locating emergency medical facilities in a large metropolitan area, one may wish to minimize the maximum distance a person must travel to reach the closest facility and would thus opt for the mimimax criterion. Then too, a more appropriate choice of optimality criterion may be to utilize both the minisum and minimax criteria concurrently. For example, it may be best to minimize the average distance a person must travel to reach the closest facility as long as the maximum distance is less than a given threshold. Likewise, another criterion (utilizing both the minisum and minimax criteria) may be to minimize the maximum travel distance as long as the average travel distance is less than a specified value. Finally, we would remark that if one should find oneself tasked with locating "obnoxious facilities", then a maximin type of optimality criterion may be best suited for the task at hand. For example, if one were locating sewage treatment plants within a large suburban area, one may wish to maximize the minimum distance between any such plant and any township within the area, say.

Still another significant classification criterion concerns the distribution of customer/demand locations. A location-allocation problem is said to have either discrete or continuous demand, depending on whether its customer locations consist of a discrete set of points or a continuum of points. In the first case (see references 19, 20, 22, 49, 50, 80), the demand level at each customer location is usually specified by some positive real number, whereas for a problem having continuous demand (see references 11, 12, 14, 28, 79, 129, 131, 134), the demand requirements are usually specified by some nonnegative demand density function(s). One should be careful here to note the distinction between the criterion of discrete/continuous demand and that of a discrete/continuous solution space. For example, one could have a location-allocation problem for which there are area (continuous) demands (see references 12, 28) but a discrete solution space, such as would be the case if one were locating anti-ballistic missile installations to protect a large metropolitan area, say, from attack by inter-continental ballistic missiles. A vast majority of the literature assumes a discrete demand, mainly for mathematical ease. However, the two problems analyzed in this study assume a continuous demand.

In addition to the above criteria, location-allocation problems can be further classified according to the following problem characteristics. First, with regard to the service facilities themselves, problems are commonly referred to as being single facility or multi-facility location-allocation problems, depending on the number of facilities to be located. Still, some of the literature considers the number of facilities to be a decision variable rather than a parameter of the problem. Another variant that one might encounter involves the consideration of probabilistic behavior, in particular (but not exclusively), with regard to the demand levels and locations. Finally, the literature includes the analysis of dynamic (versus static) location-allocation problems which allow for the relocation and/or addition of facilities over a multiperiod horizon.

It is important that one realize that the above discussion is <u>not</u> all inclusive of the many (problem) variants one might encounter in Location Theory. In fact, our review of the literature will touch upon a few of these variations. Except for the implicit constraint of practicality, it almost seems as though formulating a location-allocation problem is limited only by one's imagination. This may explain in part, the high level of interest in Location Theory evidenced by the extensive amount of existing literature.

As a final thought regarding the classification of location-allocation problems, we would mention that Handler and Mirchandani [1979] adopt a queuing theory type of classification scheme to facilitate their presentation of the minimax network location-allocation problem. Such a scheme could in fact be used to classify all location-allocation problems. For example, one might choose to use a six-part "coded string", denoted by (*1/*2/*3/*4/*5/*6), where the components of this string could be defined as follows:

*1: solution space (medium), i.e. P $\equiv$ plane, N $\equiv$ general network, N-C $\equiv$ chain network, N-T $\equiv$ tree network, etc.

*2: solution space, i.e. $C_f \equiv$ continuous solution space, $D_f \equiv$ discrete solution space.

*3: distance measure, i.e. $\ell_p \equiv \ell_p$ norm, SP $\equiv$ shortest path (on network), RT $\equiv$ rectilinear, EC(EC2) $\equiv$ Euclidean (Squared Euclidean), etc.

*4: demand distribution, i.e. $C_c \equiv$ continuous demand, $D_c \equiv$ discrete demand.

*5: optimality criterion, i.e. MS $\equiv$ minisum, MMA $\equiv$ minimax, MAM $\equiv$ maximin, etc.

*6: number of facilities, i.e. p $\equiv$ p facilities.

Thus, any given string will denote a particular class of location-allocation problems. We would remark however, that unless ones goal is to harass ones readers (with long coded strings), any such classification scheme will most likely be such that there (still) exist variations within any particular class. We might add, that the two problems to be examined herein, i.e. the chain graph problem and the 2-median tree problem, fall within the classes described by the strings $(N\text{-}C/C_f/SP/C_c/MS/p)$ and $(N\text{-}T/C_f/SP/C_c/MS/p)$, respectively.

In light of the above remarks concerning the solution space of a location-allocation problem, we will present our review of the literature in two parts. Specifically, we begin by first reviewing some of the literature involving the location of facilities on a plane, with this to be followed by a review of some network-based research. The following two sections represent only a selected review of the vast amount of literature in Location Theory.

### 1.2.2 Location-Allocation Problems on Planes

Continuous, planar location-allocation problems having discrete demand (PLAP), address the problem of determining the location of p > 1 supply centers or facilities in a plane to serve n customers with fixed locations, and to simultaneously determine the allocation of services or products from the supply centers to the customers in order to minimize total transportation costs., Mathematically, this problem may be stated as follows.

$$
\begin{aligned}
\text{(PLAP) minimize} \quad & \sum_{i=1}^{p} \sum_{j=1}^{n} t_{ij} w_{ij} d(X_i, P_j) \\
\text{subject to} \quad & \sum_{j=1}^{n} w_{ij} < s_i, \quad i=1, \ldots, p \\
& \sum_{i=1}^{p} w_{ij} = d_j, \quad j=1, \ldots, n \\
& w_{ij} > 0 \quad i=1, \ldots, p;\ j=1, \ldots, n
\end{aligned}
$$

where the decision variables are:

$w_{ij}$ = annual number of units transported from supply center i to customer j,

$X_i$ = $(x_i,y_i) \equiv$ location of supply center i on the plane, $i=1, \ldots, p$,

and where the data is symbolized by:

$P_j$ = $(a_j,b_j) \equiv$ known location of customer j on the plane, $j=1, \ldots, n$,

$t_{ij}$ = transportation cost per unit shipped per unit distance from supply center i to customer j, $i=1, \ldots, p$; $j=1, \ldots, n$,

$s_i$ = annual capacity of supply center $i=1, \ldots, p$,

$d_j$ = annual demand of customer $j=1, \ldots, n$,

and where $d(X_i,P_j)$ is some appropriate distance measure between the locations $X_i$ of supply center i and $P_j$ of customer j, $i=1, \ldots, p$, $j=1, \ldots, n$.

The above formulation is generally referred to as the capacitated version of Problem PLAP since each supply center has a capacity restriction $s_i$. In the case when $s_i = \infty$ for all i, the problem is referred to as uncapacitated. It should be noted that in the uncapacitated case, the total demand of any given customer is entirely satisfied/supplied by a single facility. The interpretation of $s_i = \infty$ is that the capacity of supply center i is effectively determined by its total allocation in an optimal solution. Furthermore, note that a more general statement of the problem could consider multiple commodities along with interactions between new facilities as well. Such a version has been addressed by Shetty and Sherali [1977].

Observe also that Probelm PLAP is comprised of two major components, namely, a location and an allocation component. When the location variables $X_i$ are specified, $i=1, \ldots, p$, the problem reduces to the well known transportation problem. On the other hand, when the allocations $w_{ij}$ are specified (evidently feasible to (1)), the problem reduces to a (pure) location problem on the plane (recall our discussion preceding Section 1.2.1). Accordingly as $p=1$ or $p \geq 2$, this problem is known as a single facility location problem (SFLOC) or a multifacility location problem (MFLOC). Of course, if $p=1$, then SFLOC and PLAP are equivalent. On the other hand, if $p \geq 2$ and if there are no interactions between supply centers, then on fixing an allocation $w \equiv (w_{ij})$, Problem PLAP reduces to p SFLOC problems. Further, the structure of these location problems depends on the distance measure used. Since this feature lends an important structural attribute to PLAP, which in turn plays a significant role in the design of a solution method, we discuss some related literature below.

The (pure) rectilinear distance location problem is one in which the distance norm is defined by $d(X,P) = |x-a| + |y-b|$, where $X = (x,y)$ and $P = (a,b)$. The objective function in this case is separable and the determinination of the x and y new facility or supply center coordinates can be treated as two separate optimization problems. Francis [1963] is credited with first solving the rectilinear SFLOC problem. The rectilinear MFLOC problem was originally proposed by Francis [1963], who subsequently solved it in the special case of equal weightings (see Francis [1964]). Cabot, Francis, and Stary [1970] solved the general rectilinear MFLOC problem by decomposing it into two independent subproblems, each of which is equivalent to a linear program which is the dual of a minimum cost network flow problem. Fulkerson's out-of-kilter algorithm was then used to obtain an optimal solution. Wesolowsky and Love [1971b] also solved the general rectilinear MFLOC problem by using a linear programming formulation. A nonlinear approximation to the rectilinear MFLOC problem was developed by Wesolowsky and Love [1972] which was then solved using a gradient search procedure.

A direct search algorithm for the rectilinear MFLOC problem was developed by Pritsker and Ghare [1970]. However, as demonstrated by Rao [1973], this was essentially a primal simplex based linear programming approach, and the stated optimality conditions were not sufficient in the presence of degeneracy. A complete set of necessary and sufficient optimality conditions have been derived by Juel and Love [1976].

An alternative, efficient, primal simplex based algorithm for this problem was developed by Sherali and Shetty [1978]. Perhaps the most efficient algorithm available for this problem is due to Picard and Ratliff [1978] who showed that the problem can be solved via at most (n-1) minimum cut problems on derived networks containing at most (p-2) vertices. Later, Kolen [1981] showed that the methods of Sherali and Shetty, and Picard and Ratliff are essentially equivalent, and differ only in the efficiency of computational implementation.

If $d(X,P) = (x-a)^2 + (y-b)^2$, then the resulting problem is called the squared-Euclidean distance problem, and is again separable in the x and y variables. In this case, both the SFLOC and MFLOC problems can be solved using standard calculus techniques. Determining the optimal solution in the MFLOC problem involves the solution of two systems of n linear equations in n variables.

For Euclidean distance problems, the distance norm is defined by $d(X,P) = \left[(x-a)^2 + (y-b)^2\right]^{1/2}$. Unlike the squared-Euclidean case there exist points of nondifferentiability in this problem, and even where the extremal equations are defined, they are non-linear and an exact closed-form solution cannot be obtained. However, a fixed-point iterative scheme was used by Weiszfeld [1937], Cooper [1963], and Kuhn and Kuenne [1962] to solve the SFLOC problem by finding a solution to the extremal equations when it exists. The scheme is commonly labeled the Weiszfeld procedure, and issues concerning its convergence have been investigated by Katz [1969], Kuhn [1973] and Morris [1981].

An extension of the Weiszfeld procedure used in solving the Euclidean distance problem was developed by Eyster, White and Wierwille [1973] for SFLOC and was extended to MFLOC. This procedure is based on approximating the objective with hyperboloids to eliminate indeterminancies in the extremal equations. The technique has been labeled the hyperboloid approximation procedure (HAP). Recently, Charalambous [1982] has developed a set of optimality conditions for the problem which lead to a far more efficient algorithm, devoid of the ill-conditioning of the Hessian matrix which afflicts the HAP method as the degree of accuracy is increased. Among other contributions for this problem are the convex programming approach of Love [1969], the necessary optimality conditions due to Francis and Cabot [1972], and the stopping criterion proposed by Love and Yeong [1981] and by Juel [1984].

The general $\ell_p$ distance problems, for which $d(X,P) = \left[|x-a|^p + |y-b|^p\right]^{1/p}$, with p usually in the interval (1,2], employ procedures similar to the Euclidean distance problem and are discussed in Drezner and Wesolowsky [1978], Morris and Verdini [1979] and Juel and Love [1981], among others.

Several variants and extensions of these problems have been considered in the literature. These include the consideration of minimal new-facility-separation constraints as in Schaefer and Hurter [1974], the consideration of stochastic demands and/or customer locations as in Cooper [1974], Katz and Cooper [1974, 1976], Seppala [1975], Aly and White [1978] and Wesolowsky [1977], the consideration of dynamically relocating facilities over a multiperiod horizon as in Wesolowsky [1973a] and Wesolowsky and Truscott [1976], the consideration of area distributed demands as in Wesolowsky and Love [1971a], Love [1972], Bennett and Mirakhor [1974], Drezner and Wesolowsky [1980] and Odoni and Sadiq [1982], and the consideration of special norms as in Ward and Wendell [1980] which lead to linear programming formulations of the location problem.

In contrast, there does not exist a significant body of literature for solving the joint location-allocation problem PLAP. The uncapacitated case, however, namely when $s_i = \infty$ is assumed for each $i=1, \ldots, p$, has been analyzed to a somewhat greater extent, although the solution methods do not readily extend to the capacitated versions. The related literature for both capacitated and uncapacitated versions is discussed below.

In the case of the rectilinear norm, Problem PLAP becomes a nonconvex bilinear programming problem (see Vaish [1974]). Love and Morris [1975] have developed an exact solution procedure for the uncapacitated version of this problem. The procedure consists of using a set reduction algorithm to reduce the possible solution set, and the problem is shown to be equivalent to the p-median problem on a weighted connected graph. Subsequently, Sherali and Shetty [1977] developed a convergent cutting plane algorithm to solve the capacitated version of PLAP. This algorithm was further improved upon by Shetty and Sherali [1977] who proposed deeper (negative-edge extension reverse polar) cutting planes, and provided a more efficient computational implementation. This procedure was actually developed for a more general problem which permitted interactions between new facilities and involved multiple commodities.

The uncapacitated version of problem PLAP with Euclidean distances was originally formulated by Cooper [1963], who proposed an enumerative procedure for its solution. However the method proved to be impractical on problems of any reasonable size. Several heuristic solution procedures were also proposed by Cooper [1964, 1967]. Subsequently, Eilon, et al [1971] developed a computationally tractable iterative solution procedure for this problem. An alternative heuristic as well as an exact branch and bound algorithm for this problem was later developed by Kuenne and Soland [1972]. For the special case when p=2, Ostresh [1975] provided an improved version of this algorithm. A variant of the uncapacitated version of PLAP which uses $\ell_p$ distances has been considered by Love and Juel [1982]. This problem has been shown to be equivalent to a concave minimization problem, and several perturbation type of strategies have been applied to the problem.

The capacitated version of the location-allocation problem was first considered by Cooper [1972] who provided an exact enumerative, combinatorial approach, and also developed two heuristic methods for the problem. Perhaps the most tractable, though not effective, solution method which exists for the capacitated Euclidean distance location-allocation problem is the cutting plane algorithm due to Selim [1979]. As far as a heuristic strategy for this problem is concerned, the method of Murtagh and Niwattisyawong [1982] which employs the commercial nonlinear programming package MINOS along with a reasonable starting solution, appears to be the most promising.

The Squared Euclidean Distance Location-Allocation problem has surprisingly not received any significant attention in the literature. The principal reason being that it shares a common structure with the Euclidean distance problem and the development of algorithms for the latter problem has evidently been thought of as being a natural precursor to the consideration of the squared Euclidean location-allocation problem.

Next, we would remark that there exists a class of location-allocation problems which require the supply centers to be located at only certain specified discrete sites. These problems are commonly known as fixed-charge or plant location problems with side constraints. One important case considered by Sherali and Adams [1982] seeks the simultaneous location of facilities on a set of potential sites in a one-to-one fashion, and the allocation of products to customers so as to minimize total (discounted) production, location and transportation costs. Other noteworthy papers in this context include those of Erlenkotter [1978], Guignard and Speilberg [1979], Geoffrion and McBride [1979], Bitran, et al. [1981] and Nagelhout and Thompson [1981].

Finally, we would mention three papers which treat location-allocation problems having area demands. Specifically, Leamer [1968] considers a problem in which the demand is continuously distributed over either a square, an equilateral triangle or a circle. Marucheck and Aly [1981] consider a location-allocation problem in which customer demands are characterized by bivariate uniform probability density functions over rectangular regions. A recent paper by Cavalier and Sherali [1983c] considers a problem in which the demand is continuously distributed over a convex polygon. They provide an algorithm for the single facility version of their problem which converges to a global optimal solution. Similarly, a convergent iterative heuristic is provided for the nonconvex multifacility version.

### 1.2.3 Location-Allocation Problems on Networks

The analytic treatment of location problems on a network can be traced back to the nineteenth-century mathematicians Jordan and Sylvester, and possibly to a seventeenth-century researcher by the name of Cavalieri. Much of the recent interest in this area has been stimulated by the seminal work of S. L. Hakimi [1964,1965]. A large part of this interest, and consequently the literature, concerns two particular types of problems which have come to be called the *p-center problem* and the *p-median problem*. As will be seen, the first of these problems employs a *minimax* optimality criterion, whereas the latter is concerned with optimizing some average behavior of the system being modeled and consequently, uses the *minisum* criterion. An excellent tutorial of the p-center and p-median problems is given in the text by Handler and Mirchandani [1979]. In addition, the text by Christofides [1975] devotes one chapter to each of theses problems. A text by Minieka [1978] also devotes a chapter to the discussion of network-based location problems, with most of its emphasis being directed towards the 1-center and 1-median problems. Finally, we remark that an outstanding review of network-based location theory exists in the form of a recent two-part survey by Tansel, Francis and Lowe [1983], the first part of which is devoted entirely to the p-center and p-median problems. Our review of the network-based literature will concern itself with (only) these two types of problems and will begin with the p-center problem.

1.2.3.1 The (absolute) p-center (minimax) Problem with Nodal Demands

Let G(N,A) denote an n-vertex undirected (connected) network having node set $N = \{v_1, \ldots, v_n\}$ and for which A denotes the set of arcs or links connecting certain designated node pairs. Define the distance between any two points x and y on G, denoted d(x,y), to be the length of the shortest path connecting them. In addition, to each node $v_i \varepsilon N$, associate a nonnegative weight/demand $h_i$, $i=1, \ldots, n$, and let the decision variables $X_p = (x_1, \ldots, x_p)$ denote the locations (on G) of p facilities from which the total demand on G is to be served/supplied. Then, defining $d(v_i,X_p) = \min\{d(v_i,x_1), \ldots, d(v_i,x_p)\}$, $i=1, \ldots, n$, a generic formulation of the absolute p-center problem can be given as follows:

$$\underset{X_p \text{ on } G}{\text{minimize}} \quad f(X_p) = \max_{1<i<n} h_i\, d(v_i,X_p) .$$

(Note: any $X_p^*$ which solves this problem is referred to as an "absolute p-center". The problem which would result from restricting $X_p$ to the nodes of G is known simply as the "p-center problem", and its solutions, as p-centers. Often times the literature refers to both of these problems as the p-center problem and then (of course) specifies which solution space is intended. In our review of the (absolute) p-center problem, we will take the time to distinguish between the two by use of the descriptor "absolute". Thus, we could make reference to any of the following: the p-center problem, the absolute p-center problem, or the (absolute) p-center problem, where the latter of these would be used to mean that the associated remark applies equally to both the p-center and the absolute p-center problem.)

Note also that the above formulation assumes that the facilities are uncapacitated and that demand can occur only on the nodes of G. More generally, we could have formulated the (absolute) p-center problem so as to allow for continuous link demands and/or capacitated facilities, However, the above formulation is the one used by Hakimi [1964,1965] and generically describes the problem most often treated in the literature. In fact, this author is not aware of any research that involves the location of capacitated facilities on a network and only very little involving problems having continuous demand. One final remark before we proceed with our review of the literature concerning the (absolute) p-center problem, is that the (absolute) p-center problem given above is a location-allocation problem (for $p \geq 2$) since the actual allocation of supply to demand is not known until such time as the facility locations have been determined. Of course, at such time as this, the demand at each node of G will be fully supplied by the nearest facility.

Hakimi [1964] was the first to formulate and solve both the 1-center and absolute 1-center problems. His solution to the first of these simply requires that one compute and examine an n x n distance matrix, and consequently results in an $O(n^3)$ algorithm. A recent O(n) algorithm by Hedetniemi, Cockayne, and Hedetniemi [1981] solves the 1-center problem in which all $h_i = 1$, i=1, ..., n, by utilizing an "efficient data structure for representing a tree called a canonical recursive representation". Rosenthal, Hersey, Pino, and Coulter [1978] introduced a generalized algorithm that solves a number of "eccentricity" problems on tree networks, one of which is the 1-center problem. In their algorithm, they define the eccentricity of a vertex to be the distance from that vertex to a farthest vertex. They prove that any vertex of minimum eccentricity is a 1-center. Other algorithms for solving this problem have been developed by Rosenthal [1981] and Slater [1981].

Hakimi [1964] solves the absolute 1-center problem by solving $|A|$ "simpler mini-max" problems in order to determine a local center on each of the $|A|$ arcs of G. The absolute 1-center is then selected as the best of these $|A|$ local centers. Additionally, computational refinements of Hakimi's method have resulted from the work of Hakimi, Schmeichel and Pierce [1978] as well as that of Kariv and Hakimi [1979]. Frank [1967] and Minieka [1977] proposed solution procedures to the continuous demand, absolute 1-center problem which are similar to Hakimi's method. Finally, Minieka [1980] considers "conditional" (absolute) 1-center problems in which a 1-center is to be located not only with respect to the nodal demands, but also with respect to previously located centers.

A commonly exploited type of network is that of a tree, i.e. a connected network having no cycles. The principal reason for this is that a (connected) tree satisfies the property that there exists a unique shortest path joining any two points on the tree. Goldman [1972] solved the unweighted (i.e. all $h_i=1$) alsolute 1-center problem via the repeated application of a "trichotomy theorem" which either determines the edge on which the center lies, or reduces the search to the two subtrees obtained by removing that edge. Handler [1973] also solved the unweighted problem by using an O(n) algorithm which determines a longest path in the tree and then locates the absolute 1-center at the midpoint of this path. Dearing and Francis [1974] were able to determine a lower bound (for any network) to the optimal objective value of the absolute 1-center problem and to prove that this bound is always attainable for the case in which G is a tree. In so doing, they identify two "critical" vertices such that the absolute 1-center is uniquely located on the unique path joining these vertices. Similarly, lower bounds were obtained by Dearing [1977] and by Francis [1977] to a nonlinear version of the absolute 1-center problem in which the weights $h_i$, i=1, ..., n are replaced by strictly increasing functions of the distances $d(v_i,X_p)$, i=1, ..., n. Both authors obtained a lower bound subsuming the one defined in Dearing and Francis [1974]. The bound is applicable to all networks and is always attainable for tree networks.

Minieka [1970] considered the unweighted problem on a general network for the case $p > 2$ and suggested a rudimentary algorithm that relies on solving a finite sequence of set covering problems. Christofides and Viola [1971] solved the weighted problem by first solving a sequence of r-cover problems with successively increasing values of r. Handler [1978] considered the continuous and absolute p-center problems on a tree network for the case $p = 2$. Finally, Chandrasekaran and Daughety [1981] gave a method to solve the continuous p-center problem on a tree network.

### 1.2.3.2 The (absolute) p-median (minisum) Problem with Nodal Demands

Let $G(N,A)$, $h_i$, $X_p$ and $d(v_i,X_p)$, $i=1, \ldots, n$, all be as defined in Section 1.2.3.1. Then a generic formulation of the absolute p-median problem can be given as follows:

$$\underset{X_p \text{ on } G}{\text{minimize}} \quad f(X_p) = \sum_{i=1}^{n} h_i d(v_i X_p) .$$

(Note: we could do just as we did for the minimax problem and obtain a variant of the absolute p-median problem, called the "p-median problem", by restricting $X_p$ to the nodes of G. However, due to a result by Hakimi [1964,1965] in which he proves that with only nodal demands, there exists an absolute p-median consisting entirely of nodes of G, there is no need to distinguish between the absolute p-median problem and the p-median problem in this case.)

We would remark that it is this problem rather than the minimax problem which is of most interest to our study, since Problem GAMNLAP is an absolute p-median problem. Of course, Problem GAMNLAP is a more general formulation than that of above in that it allows for link demands and/or capacitated facilities, whereas the above formulation restricts all demand to the nodes of G and assumes the facilities to be uncapacitated so that each node will be fully supplied/served by the facility nearest to it. Note however, that according to our criterion, both the above formulation and (of course) Problem GAMNLAP are location-allocation problems for any $p \geq 2$, and that both are (pure) location problems for the case $p = 1$. It is the above formulation that was first presented and solved by Hakimi [1964,1965], and it is this same formulation which is most often treated in the literature.

Hakimi [1964] was the first to define an absolute 1-median, and furthermore, he proved that such a point can always be found amongst the nodes of G. Consequently, he determines a median on G by summing the columns of a weighted (by the $h_i$) n x n distance matrix and then choosing any node which corresponds to a column having minimum sum.

For the case in which G is a tree network, more efficient procedures have been developed with which to determine a median location. In particular, Goldman [1971] developed an O(n) algorithm which reduces the search to successively smaller subtrees until such time as a median is found. Specifically, at each iteration of his algorithm, one chooses an arbitrary end/tip of the current tree and examines its weight. If the (modified) weight of the chosen end is $\geq$ half of the total weight (of the tree), then the end is a median location. Otherwise, add the weight of this end to that of the adjacent node and then delete both the end and arc incident to it. Now repeat this process with the new (reduced) tree. This algorithm is based on a "localization theorem" proved by Goldman and Witzgall [1970], which provides sufficient conditions for a subset of G to contain a median.

Note that Goldman's algorithm does not require the use of arc lengths, but uses only the incidence relationships and weights. Another Goldman-like algorithm for locating a 1-median is that of Kariv and Hakimi [1979]. Theirs is an O(n) algorithm which utilizes the concept of a "centroid" of a tree. They define such as follows. Let $T_1, \ldots, T_{k_i}$ be the subtrees obtained by removing $v_i$ from G. Let $W(T_j)$ be the sum of the weights of the vertices in $T_j$, and define $\bar{W}(v_i)$ to be the maximum of $W(T_j)$ for $1 < j < k_i$. A vertex $v_t$ is called a centroid of G if and only if it minimizes $\bar{W}(v_i)$ over all $v_i \varepsilon N$. They prove that a 1-median of a tree is identical to a centroid of the tree and then utilize this fact in a Goldman-like algorithm. In addition, Kariv and Hakimi [1979] showed that the p-median problem on a general network is NP-hard. For the case of tree networks, however, they provide a polynomial algorithm of order $O(n^2 p^2)$ for locating a p-median.

Two generalizations of the 1-median problem can be found in the papers by Minieka [1984] and Slater [1981]. In the first of these, Minieka considers various "conditional" 1-median problems, much as he did for the minimax problem. Slater [1981] formulates another generalization of the 1-median problem in which each "demand' is a collection of vertices and the problem is to find a vertex such that the sum of the distances from that vertex to a nearest element of each collection is a minimum. He showed that the set of vertices which solve this problem forms a connected set.

As we mentioned earlier, Hakimi [1965] proved that an absolute p-median can be found to lie entirely within the set of nodes of G. Levy [1972] proved that this is also true when the weights $h_i$, i=1,

..., n, are replaced by concave cost functions of the distance between $v_i$ and its nearest median. Goldman [1969] generalized Hakimi's result to include the case in which a vertex can be designated as either a source or destination and where the service facilities can be thought to have a "processing function". This allows for supply to flow from one vertex to another by first passing through a processing facility. Goldman proved that one need only consider the nodes of G with respect to optimally locating the processing facilities, and furthermore, conjectured that such would be the case for any multi-stage problem. Hakimi and Maheshwari [1972] proved Hakimi's vertex optimality result for the case of multiple commodities that go through multiple stages with the cost of transport from one stage to the next being given by a concave nondecreasing function of distance.

Finally, a number of probabilistic versions of the p-median problem have been considered. For example, Frank [1966,1967] considered a 1-median problem in which the $h_i$ were considered to be random variables and the arc lengths to be deterministic. Conversely, Mirchandani and Odoni [1979a,b] considered a p-median problem in which the arc lengths were random and the weights deterministic. Berman and Larson [1982] extended Hakimi's vertex optimality result to the case where the number of service facilities is a random variable. Mirchandani and Oudjit [1980] formulated and solved a 2-median problem on a tree network having deterministic weights and random arc lengths. Other probabilistic formulations include those of Berman and Larson [1980], Berman and Odoni [1982], Mirchandani, Oudjit and Wong [1981] and Chiu [1982].

1.2.3.3 Network Location-Allocation Problems with a Continuum of Link Demands

Recall from Section 1.1, that the two special cases of Problem GAMNLAP to be considered herein, involve the location of capacitated facilities on a network having both nodal and link demands. A survey of the literature indicates that no such location-allocation problem has yet been considered by any other researcher. In particular, very little of the literature treats network-based problems having both nodal and link demands. Perhaps this is due in part to the fact that Hakimi's [1965] vertex optimality result does not hold for such a demand structure. Furthermore, this author is unaware of any existing literature which involves capacitated network-based location-allocation problems.

Handler and Mirchandani [1979] formulate a location-allocation problem having continuous link demands, and then approximate it by replacing the demand on each link with a concentrated centroidal demand point. Alternatively, one may obtain a closer approximation by using several concentrated demand points to replace the continuous demand on each link. The first non-discretized approach to solving an absolute p-median problem having link demands is due to Minieka [1977]. Here, however, the demand on a link is said to be served by traveling to the furthest end of the link. Hence, the objective is to locate a facility which minimizes the sum of distances from the facility to the furthest point on each link. Such a location is called a general absolute median of the network.

A similar type of problem has been considered by Slater [1981].

Here a tree network is specified in which the demand is characterized by a collection of subtrees. A facility is said to serve a demand subtree by traveling in the network to the closest point in that subtree. The objective is therefore to find a set of single facility locations, each element of which minimizes the total sum of distances from the closest points in the demand subtrees. Slater calls such a set of points the branch centroid of the tree.

Perhaps the first contribution in which continuous link demands are actually served pointwise appears in a recent paper by Chiu [1982]. Here, a single facility is considered and the average distance function to be minimized is characterized as a function of the location of this facility. This characterization leads to exact as well as heuristic 1-median location algorithms on general networks and on trees. A different approach has been used by Cavalier and Sherali [1983a] in order to optimally locate p-medians on a chain graph, 2-medians on a tree and a 1-median on graphs with isolated cycles as defined by Goldman [1971]. Uniform demand densities were assumed on the links. Batta, Brandeau and Chiu [1983] have also developed specialized algorithms to localize the search for 2-medians on a tree graph with continuous link demands. We also mention that for minimax (or p-center) problems, as opposed to the minisum problems being considered herein, Tamir and Zemel [1982] have developed an algorithm for locating facilities on a tree graph with continuous link demads. Unlike our study, all of the above papers assume the facilities to be uncapacitated.

## 1.3 Research Tasks and Organization

In this study, we examine multifacility location-allocation problems in which capacitated service facilities are to be located on undirected networks having both nodal and link demands. Furthermore, the problems considered herein are balanced in the sense that the total available capacity/supply is equal to the total demand on the network. This study is organized as follows.

We begin, in Chapter II, by analyzing the simplest version of this problem type. Specifically, we consider a location-allocation problem in which p facilities having known finite capacities are to be located on a chain graph so as to optimally serve/supply the total demand on the graph. As such, the demand is specified by a given nonnegative, integrable (demand) function, and the task is to determine the locations and allocations of p capacitated facilities, so as to minimize some total (expected) transportation cost. Additionally, a period by period sequential location-allocation problem is considered in which a single facility per period is required to be located on the chain graph so as to optimally accommodate a known increase in the total demand on the graph.

In Chapter III, we utilize the results of the chain graph problem to analyze a location-allocation problem in which two facilities having known finite capacities are to be located on a tree network which has both nodal and link demands, the latter of which are specified by nonnegative, integrable (real-valued) functions.

Finally, in Chapter IV, we suggest (and briefly discuss) three extensions/variants of the problems analyzed herein, which one might want to consider with respect to future research. We would remark also, that a chronological summary of the theorems, corollaries and lemmas of Chapters II and III are given in Appendices A and B, respectively. The reader may find these useful as a means of reviewing the content of this study or possibly as a means of obtaining an initial (cursory) understanding of the same.

CHAPTER II

# A CAPACITATED, BALANCED, LOCATION-ALLOCATION PROBLEM ON A CHAIN GRAPH WITH A CONTINUUM OF LINK DEMANDS

## 2.1 Introduction

In this chapter, we consider for the first time, a network-based location-allocation problem in which the facilities to be located are assumed to have known finite capacities. Specifically, we will consider a case of Problem GAMNLAP in which a capacitated absolute p-median is to be located on a chain graph, or equivalently, on a closed real line segment $[0,c]$, say, in order to satisfy a probabilistic demand distribution over the interval $[0,c]$. Let $f:[0,c] \to R$ be a nonnegative, integrable function which characterizes this demand. For the most part, we will assume that $f(\cdot)$ is a union of the demand distributions $f_\ell(\cdot)$, for $\ell \varepsilon L$, the set of links of the chain graph. (Discrete nodal demands, if present, are assumed to be continuously spread over some $\varepsilon$-length links as in Cavalier and Sherali [1983a].) Additionally, we will assume that the total supply $\sum_{i=1}^{p} s_i$ equals the total demand $\int_0^c f(x)dx$. (We will also assume throughout that $s_i > 0$, $i=1, \ldots, p$.)

We remark that two somewhat different but useful interpretations may be given for the demand function $f(\cdot)$. First, one can consider $f(\cdot)$ to be a demand density function over $[0,c]$, so that for any infinitesimal length dx of $[0,c]$, $f(x)dx$ approximates the total demand on dx, all of which requires service. This interpretation lends itself to thinking in terms of each point on dx as being a customer who must travel to some facility for service. The second interpretation is to consider $f(\cdot)$ to be a weighted probability density function. As such, $f(x)$ could be a probability density function weighted by the total demand on $[0,c]$, or it could be a sum of such probability density functions as in Chiu [1982], or it could simply be some histogram describing the demand distribution. In any case, $f(x)dx$ would be the expected demand on an infinitesimal length dx at x, and as such, one can think in terms of the server leaving its facility location and traveling to the location of a demand incident in order to provide service to a customer. Similar interpretations apply to the functions $\Phi_{i\ell}(\cdot)$, $\ell \varepsilon L$. Then assuming (as we do) that the cost of service is directly proportional to the demand served times the distance between the serving facility and the customer, we see that our objective with respect to the first interpretation of $f(\cdot)$, is to minimize the total cost of service, whereas for the second interpretation, we seek to minimize the total expected cost of service.

This chapter is organized as follows. We begin by formulating the location-allocation problem of interest, and then characterize the form of an optimal solution corresponding either to a fixed set of locations, or to a fixed allocation scheme. Thereafter, we discuss the monotone demand distribution case which admits a closed form solution, and which provides useful machinery for computing lower bounds in a partial enumeration algorithmic framework. Other important cases, namely, problems having symmetric or nonsymmetric unimodal demand distributions are also analyzed. The results of our analysis of these cases are then used in discussing the case in which the demand function is simply nonnegative and integrable. Finally, we provide an efficient algorithm for solving a period by period sequential location-allocation problem. The analysis of this chapter will be seen to lay the foundation for treating more general problems, such as the capacitated 2-median tree problem of Chapter III.

## 2.2 Formulation of a p-median Location-Allocation Problem on a Chain

In view of the above discussion, the problem of interest can be described as a capacitated, absolute p-median (or minisum) location-allocation problem in which a given number of supply centers/service facilities having known supplies/capacities, are to be located on the real line so as to optimally satisfy a continuum of demand over the interval $[0,c]$. The problem can be further classified as balanced, in that the total expected demand over $[0,c]$ is equal to the total supply.

Mathematically, the problem can be stated as follows:

Given a nonnegative, integrable, demand function $f:[0,c] \to R$, and positive capacities $s_1, \ldots, s_p$, of p facilities such that the total demand, $\int_0^c f(x)dx$, equals $\sum_{i=1}^{p} s_i$, determine $y_1, \ldots, y_p \in R$, and integrable functions $\Phi_i: [0,c] \to R$, i=1, ..., p, which solve

<u>CP</u>: minimize $\sum_{i=1}^{p} \int_0^c \Phi_i(x)|x-y_i|dx$

subject to

$$\sum_{i=1}^{p} \Phi_i(x) = f(x) \text{ for all } x\varepsilon[0,c] \quad (2.1)$$

$$\int_0^c \Phi_i(x)dx = s_i \text{ for } i=1, \ldots, p \quad (2.2)$$

$$\Phi_i(x) \geq 0 \text{ for all } x\varepsilon[0,c], \ i=1, \ldots, p \quad (2.3)$$

$$0 \leq y_i \leq c \text{ for } i=1, \ldots, p. \quad (2.4)$$

As explained earlier, Problem CP contains both location decision variables, namely the facility locations $y_1, \ldots, y_p$, and allocation decision "variables", $\Phi_i(\cdot)$, $i=1, \ldots, p$. We will refer to the $\Phi_i(\cdot)$ as allocation functions, a terminology quite compatible with their role, for they tell us where and how each facility expends or allocates its supply.

In keeping with the usual interpretation of location-allocation problems, we remark that the objective function of Problem CP represents either (depending on ones interpretation of $f(\cdot)$ and the $\Phi_i(\cdot)$, $i=1, \ldots, p$) a total transportation cost or a total expected transportation cost associated with any feasible set of facility locations $\{y_1, \ldots, y_p\}$, and any set of feasible allocation functions, $\{\Phi_1(\cdot), \ldots, \Phi_p(\cdot)\}$. In particular, this cost is given by the weighted sum of distances from the demand points to their respective serving facilities, the weights being the associated allocation functions corresponding to the respective facilities. Of course, the usual absolute value, $|x-y_i|$, is used to measure the shortest distance between points x and $y_i$ on the real line. Recalling that the Riemann integral is defined to be the limit of a sequence of approximating Riemann sums whose mesh goes to zero, a reasonable approximation to the cost of that service/supply provided to an infinitesimal length dx of $[0,c]$ from a facility located at point $y_i$, is $\Phi_i(x)|x-y_i|dx$. Hence the objective function of Problem CP.

In an effort to ensure a complete understanding of Problem CP, we make two additional remarks before proceeding with our treatment of the problem. First, we would ask the reader to note the requirements placed on the demand function, $f(\cdot)$. All that is required, is that it be non-negative and integrable. It need not be continuous, for example. It is felt that this generality in the demand function is one of the attractive features of the problem.

Finally, we remark that the formulation of Problem CP is such that it allows for the demand over a subset (of $[0,c]$) having positive measure, to be met/satisfied/served by more than one facility. However, we will show that an optimal solution exists, in which the subsets served by the facilities located at $y_1, \ldots, y_p$ have pairwise disjoint interiors, are all of the form $[\alpha,\beta]$, and appear from left to right in the same order as do the facilities themselves. This is an exceptionally nice result in that it gives us an optimal solution having a relatively simple allocation.

We emphasize here that the consideration of a chain graph network in GAMNLAP is an important starting point. The analysis of this case provides strong insights and lays the foundation for methodologies for more general network problems, including the case of locating medians on a tree. Several of the properties developed for Problem CP are readily seen to contribute to concepts for the more general cases. Nevertheless, we remark that Problem CP is itself an interesting nonconvex program which deserves attention in its own right.

We begin by characterizing in the next section, the solutions to the separate location and allocation subproblems inherent in Problem CP.

## 2.3 Characterizations of the Individual Location and Allocation Solution

Given a fixed set of allocation functions feasible to (2.1), (2.2) and (2.3), Problem CP is transformed into a location problem which itself separates into p location problems over the respective location variables $y_1, \ldots, y_p$. It is well known, and easy to show by differentiating the objective function of Problem CP with respect to each $y_i$, that the corresponding optimal values of $y_i$, $i=1, \ldots, p$, are determined as median locations with respect to the individual $\Phi_i(\cdot)$, $i=1, \ldots, p$ functions. Moreover, the following property holds.

<u>Lemma 2.1</u>. Let $y_1^*, \ldots, y_p^*, \Phi_1^*(\cdot), \ldots, \Phi_p^*(\cdot)$ represent an optimal solution to Problem CP. Then, there exists a reindexing of facilities such that $0 < y_1^* < \ldots < y_p^* < c$.

<u>Proof</u>. Let the facilities be reindexed so that $0 \leq y_1^* \leq \ldots \leq y_p^* \leq c$. Because of the median location property with respect to $\Phi_i^*(\cdot)$, $i=1, \ldots, p$, it follows that $y_1^* > 0$ and $y_p^* < c$. Further, if some $y_i^* = y_j^*$, $i \neq j$, then the objective function value remains the same, and feasibility is maintained if the functions $\Phi_i(\cdot)$ and $\Phi_j(\cdot)$ are redefined such that for some $0 < \alpha < c$, (2.2) holds with $\Phi_i(x) = \Phi_i^*(x) + \Phi_j^*(x)$ for $0 \leq x < \alpha$ and zero otherwise, and $\Phi_j(x) = \Phi_i^*(x) + \Phi_j^*(x)$ for $\alpha \leq x \leq c$, and 0 otherwise. But then, relocating $y_i^*$ and $y_j^*$ with respect to these newly defined allocation functions would result in $0 < y_i^* < \alpha < y_j^* < c$ with an improved objective value, contradicting optimality. This completes the proof. $\square$

Next, we consider the allocation problem which results when facility locations $y_1, \ldots, y_p$ are given and fixed. That is we consider the problem

$$CP(y_1, \ldots, y_p) : \text{Given } 0 < y_1 < \ldots < y_p < c,$$
$$\text{solve Problem CP.}$$

The decision "variables" of this problem are the allocation functions, $\Phi_i(\cdot)$, $i=1, \ldots, p$. We will show and later state as a theorem, that an optimal solution to Problem $CP(y_1, \ldots, y_p)$ exists, in which the allocation functions result in the facilities at $0 < y_1 < \ldots < y_p < c$ serving the intervals $I_1 = [0, \alpha_1]$, $I_2 = [\alpha_1, \alpha_2]$, $I_3 = [\alpha_2, \alpha_3]$, $\ldots$, $I_p = [\alpha_{p-1}, c]$, respectively, for some $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{p-1} < c$. Notice that these intervals have pairwise disjoint interiors, so that no set having positive measure is served by more than one facility. For obvious reasons, we descriptively refer to this particular optimal solution to Problem $CP(y_1, \ldots, y_p)$ as the "pack it from the left" solution. All future reference to this solution will be made via the abbreviation, PFL. We will not address (nor are we interested in) the question of the uniqueness of the optimal PFL solution.

In order to establish that the PFL solution is indeed an optimal solution to Problem $CP(y_1, \ldots, y_p)$, we first solve a problem, denoted by $CP(\Delta, y_1, \ldots, y_p)$, which is in fact a special case of Problem $CP(y_1, \ldots, y_p)$ on which an additional restriction has been imposed, and for which there exists an optimal solution that is "nearly PFL" in nature. Furthermore, this particular optimal solution to Problem $CP(\Delta, y_1, \ldots, y_p)$ is obtained by solving a corresponding transportion problem, denoted by $CPT(\Delta, y_1, \ldots, y_p)$. As will be shown, Problems $CP(\Delta, y_1, \ldots, y_p)$ and $CPT(\Delta, y_1, \ldots, y_p)$ are used in conjunction with a limiting type of argument in which a prescribed discretization becomes finer and finer, in order to obtain the desired PFL solution to Problem $CP(y_1, \ldots, y_p)$.

To be specific, let $\Delta$ equal the mesh of any partitioning of the interval $[0,c]$ for which the fixed points $y_1$, ..., $y_p$, with $0 < y_1 < \cdots < y_p < c$, are coincident with some p elements of the partition. Such a partition will be referred to as being "legitimate". It is important (for reasons that will be explained later) that only legitimate partitions be allowed. Now, given $\Delta > 0$ and any legitimate partition having mesh equal to $\Delta$, let $f_\Delta(\cdot)$ denote a step function whose steps occur at points coincident with elements of the partition. Note that $f_\Delta(\cdot)$ could, for example, have been obtained from some demand function $f(\cdot)$ by defining the step heights so as to be equal to $f(\cdot)$ evaluated at the midpoint of each of the increments. For the sake of completeness, let us define $f_\Delta(\cdot)$ so as to be right continuous at each element of the partition (similarly at zero) and to be left continuous at the point c. We then obtain Problem CP($\Delta$,$y_1$, ..., $y_p$) by affecting two changes to Problem CP($y_1$, ..., $y_p$). First, we replace the demand function $f(\cdot)$ by $f_\Delta(\cdot)$, ensuring as always that the positive supplies $s_i$, $i=1, \ldots, p$, satisfy $\int_0^c f_\Delta(\cdot)dx = \sum_{i=1}^{p} s_i$, so that we have a balanced problem. Observe that this yields only a special case of Problem CP($y_1$, ..., $y_p$). We remark that in case $f_\Delta(\cdot)$ has been obtained by discretizing $f(\cdot)$ for some problem, then the supplies $s_i$ may have to be adjusted to some (positive) values $s_i(\Delta)$, $i=1, \ldots, p$, in order to ensure a balanced problem. The second change imposes an additional restriction on Problem CP($\Delta$,$y_1$, ..., $y_p$), namely, that the allocation functions $\Phi_i(\cdot)$, $i=1, \ldots, p$, also be nonnegative step functions defined so that their steps occur at points coincident with elements of the partition, and so as to be right continuous on $[0,c)$ and left continuous at the point c. Everything else about Problem CP($\Delta, y_1$, ..., $y_p$) is just as it is in Problem CP($y_1$, ..., $y_p$). The known facility locations, $0 < y_1 < \dots < y_p < c$, are the same in both problems. Also, the form of the objective function and constraints remain the same. Clearly then, Problem CP($\Delta, y_1$, ..., $y_p$) can be considered to be a special case of Problem CP($y_1$, ..., $y_p$) on which the additional restriction on the form of the allocation functions has been imposed.

Before presenting the formulation of Problem CPT($\Delta, y_1$, ..., $y_p$), we introduce some notation regarding the partitioning of [0,c]. Specifically, for any legitimate partitioning of [0,c], we will denote its increments and their corresponding midpoints, by $\Delta_j$, j=1, ..., J and $z_j$, j=1, ..., J, respectively. Note that by our convention, each increment $\Delta_j$ corresponds to an interval which is closed at the left end point and open at the right end point, except for $\Delta_J$ which corresponds to a closed interval. In addition, $\bar{\Delta}_j$, j=1, ..., J will denote the lengths of the increments $\Delta_j$, j=1, ..., J. Thus, in terms of this notation, the total demand of Problem CP($\Delta, y_1$, ..., $y_p$) can be written as $\int_0^c f_\Delta(x)dx = \sum_{j=1}^{J} f(z_j)\bar{\Delta}_j$.

Corresponding to Problem CP($\Delta, y_1$, ..., $y_p$), we now construct a transportation problem, CPT($\Delta, y_1$, ..., $y_p$), by aggregating portions of the demand function $f_\Delta(\cdot)$. Specifically, the total demand $f(z_j)\bar{\Delta}_j$ on the $j^{th}$ increment is assumed to be lumped at its midpoint $z_j$, for each j=1, ..., J. Hence, the points $z_1$, ..., $z_J$ are being viewed as destination points with respective demands $f(z_1)\bar{\Delta}_1$, ..., $f(z_J)\bar{\Delta}_J$, and the facility locations $0 < y_1 < \dots < y_p < c$, as origin points with respective supplies $s_1, \ldots, s_p$. To complete our description of Problem CPT($\Delta, y_1, \ldots, y_p$), we let $x_{ij}$, $i=1, \ldots, p$, $j=1, \ldots, J$, represent the number of units of supply shipped from the facility located at $y_i$ to demand point $z_j$. The per unit shipment cost of $x_{ij}$ will be denoted by $d_{ij}$, and is equal to the distance between $y_i$ and the point $z_j$. Thus, Problem CPT($\Delta, y_1, \ldots, y_p$) can be written as the following linear transportation problem.

CPT($\Delta, y_1, \ldots, y_p$):

$$\text{minimize} \quad \sum_{i=1}^{p} \sum_{j=1}^{J} d_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{J} x_{ij} = s_i, \; i=1, \ldots, p$$

$$\sum_{i=1}^{p} x_{ij} = f(z_j)\bar{\Delta}_j, \; j=1, \ldots, J$$

$$\text{all } x_{ij} \geq 0.$$

The following result establishes a key relationship between Problems CP($\Delta, y_1, \ldots, y_p$) and CPT($\Delta, y_1, \ldots, y_p$).

Theorem 2.2. Problems CP($\Delta, y_1, \ldots, y_p$) and CPT($\Delta, y_1, \ldots, y_p$) are equivalent in the following sense. There exists a one-to-one correspondence between the sets of feasible solutions to Problems CP($\Delta, y_1, \ldots, y_p$) and CPT($\Delta, y_1, \ldots, y_p$), and more importantly, corresponding solutions have equal objective function values. It follows then, that any optimal solution to one problem will correspond to an optimal solution of the other.

Proof. In order to show the existence of such a one-to-one correspondence, we present a relationship which defines a one-to-one mapping from the set of feasible solutions to Problem CPT($\Delta, y_1, \ldots, y_p$) onto the set of feasible solutions to Problem CP($\Delta, y_1, \ldots, y_p$).

In particular, let $x=(x_{11}, \ldots, x_{1J}, \ldots, x_{p1}, \ldots, x_{pJ})$ be any feasible solution to Problem CPT($\Delta, y_1, \ldots, y_p$), and define the functions $\Phi_i(x)=x_{ij}/\bar{\Delta}_j$, $x\varepsilon\Delta_j$, $i=1, \ldots, p$, $j=1, \ldots, J$. Note that each $\Phi_i(\cdot)$ is a nonnegative step function in which the steps occur at points coincident with elements of the partition. In addition, we have that

$$\int_{\Delta_j} \Phi_i(x)dx = x_{ij}$$

so that

$$\int_0^c \Phi_i(x)dx = \sum_{j=1}^{J} x_{ij} = s_i, \quad i=1, \ldots, p.$$

Also, for any $x\varepsilon\Delta_j$, we can write

$$\begin{aligned}\sum_{i=1}^{p} \Phi_i(x) &= \sum_{i=1}^{p} (x_{ij}/\bar{\Delta}_j) = (1/\bar{\Delta}_j) \sum_{i=1}^{p} x_{ij} \\ &= (1/\bar{\Delta}_j)f(z_j)\bar{\Delta}_j = f(z_j) = f_\Delta(z_j) \\ &= f_\Delta(x), \quad j=1, \ldots, J, \quad \text{and so we have}\end{aligned}$$

$$\sum_{i=1}^{p} \Phi_i(x) = f_\Delta(x) \text{ on } [0,c].$$

Thus, we see that the $\Phi_i(\cdot)$ defined by the relationship $\Phi_i(x) = x_{ij}/\bar{\Delta}_j$, $x\varepsilon\Delta_j$, $i=1, \ldots, p$, $j=1, \ldots, J$, constitute a feasible solution to Problem CP($\Delta, y_1, \ldots, y_p$), and as such, we see that this relationship defines a <u>one-to-one</u> (obviously) mapping from the set of feasible solutions to Problem CPT($\Delta, y_1, \ldots, y_p$) <u>into</u> the set of feasible solutions to Problem CP($\Delta, y_1, \ldots, y_p$). It remains to show that this mapping is, in fact, onto.

Now let $\Phi = (\Phi_1, \ldots, \Phi_p)$ be any feasible solution to Problem CP($\Delta, y_1, \ldots, y_p$). As such, each $\Phi_i(\cdot)$ is a step function of the type described above, and therefore it must be the case that there exist some nonnegative values, call them $x_{ij}$, $j=1, \ldots, J$, such that $\Phi_i(x) = x_{ij}/\bar{\Delta}_j$ for $x \varepsilon \Delta_j$. If we can show that these $x_{ij}$ values, $i=1, \ldots, p$, $j=1, \ldots, J$, constitute a feasible solution to Problem CPT($\Delta, y_1, \ldots, y_p$), then we will have shown that the above mapping is onto. With this as our goal, we note that if $\Phi_i(x) = x_{ij}/\bar{\Delta}_j$ on $\Delta_j$, then $x_{ij} = \int_{\Delta_j} \Phi_i(x)dx$. Thus we can write

$$\sum_{j=1}^{J} x_{ij} = \sum_{j=1}^{J} \int_{\Delta_j} \Phi_i(x)dx = \int_0^c \Phi_i(x)dx = s_i, \quad i=1, \ldots, p$$

and

$$\sum_{i=1}^{p} x_{ij} = \sum_{i=1}^{p} \int_{\Delta_j} \Phi_i(x)dx = \int_{\Delta_j} \sum_{i=1}^{p} \Phi_i(x)dx$$

$$= \int_{\Delta_j} f_\Delta(x)dx = f(z_j)\bar{\Delta}_j, \quad j=1, \ldots, J .$$

Therefore, we see that the $x_{ij}$ do constitute a feasible solution to Problem CPT($\Delta, y_1, \ldots, y_p$), and hence we have established that the relationship $\Phi_i(x) = x_{ij}/\bar{\Delta}_j$, $x \varepsilon \Delta_j$, $i=1, \ldots, p$, $j=1, \ldots, J$, defines a one-to-one correspondence between the sets of feasible solutions to Problems CP($\Delta, y_1, \ldots, y_p$) and CPT($\Delta, y_1, \ldots, y_p$).

Next, we want to show that if $x$ and $\Phi$ are corresponding solutions to Problems CPT($\Delta, y_1, \ldots, y_p$) and CP($\Delta, y_1, \ldots, y_p$), respectively, then their objective function values are equal. We use the relationship, $\Phi_i(x) = x_{ij}/\bar{\Delta}_j$, $x \varepsilon \Delta_j$, $i=1, \ldots, p$, $j=1, \ldots, J$, to write

$$\sum_{i=1}^{p} \int_0^c \Phi_i(x)|x-y_i|dx = \sum_{i=1}^{p} \sum_{j=1}^{J} \int_{\Delta_j} (x_{ij}/\bar{\Delta}_j)|x-y_i|dx$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{J} (x_{ij}/\bar{\Delta}_j)\int_{\Delta_j} |x-y_i|dx. \qquad (2.5)$$

Now, because we have restricted ourselves to only working with legitimate partitions, we have that no $y_i$ lies in the interior of any $\Delta_j$, $i=1, \ldots, p$, $j=1, \ldots, J$. With this in mind, let us examine an arbitrarily chosen summand from (2.5), i.e. for any i and j, let us examine $(x_{ij}/\bar{\Delta}_j)\int_{\Delta_j} |x-y_i|dx$. Recalling that $\Delta_j$ is an increment/subinterval of $[0,c]$ created by the partitioning of $[0,c]$, let us denote the left and right end points of $\Delta_j$ as $\Delta_{jl}$ and $\Delta_{jr}$ respectively. Calling upon the legitimacy of our partition and without loss of generality, we will suppose that $y_i > \Delta_{jr}$. (The case $y_i < \Delta_{j\ell}$ is similarly handled.) We also note that $z_j = \Delta_{jl} + (\bar{\Delta}_j/2)$. With the notation as such, we can write

$$
\begin{aligned}
(x_{ij}/\bar{\Delta}_j)\int_{\Delta_j} |x-y_i|dx &= (x_{ij}/\bar{\Delta}_j)\int_{\Delta_{jl}}^{\Delta_{jr}}(y_i-x)dx \\
&= (x_{ij}/\bar{\Delta}_j)(y_i x-x^2/2)\Big|_{\Delta_{jl}}^{(\Delta_{jl}+\bar{\Delta}_j)} \\
&= (x_{ij}/\bar{\Delta}_j)\{[y_i(\Delta_{jl}+\bar{\Delta}_j)-(\Delta_{jl}+\bar{\Delta}_j)^2/2]-[y_i\Delta_{jl}-\Delta_{jl}^2/2]\} \\
&= (x_{ij}/\bar{\Delta}_j)\{y_i\bar{\Delta}_j-\bar{\Delta}_j^2/2-\Delta_{jl}\bar{\Delta}_j\} \\
&= x_{ij}\{y_i-(\Delta_{jl}+\bar{\Delta}_j/2)\} \\
&= x_{ij}\{d_{ij}\} = d_{ij}x_{ij} \;.
\end{aligned}
$$

Thus we have that (2.5) is equal to $\sum_{i=1}^{p} \sum_{j=1}^{J} d_{ij}x_{ij}$, which establishes the fact x and $\Phi$ have equal objective function values.

Finally, we remark that optimal solutions correspond and that an obvious proof by contradiction can be used to establish this fact. This completes the proof of Theorem 2.2. $\square$

Regarding the above theorem, we would remark that if some $y_i$ was to lie within any of the increments served by the facility located at $y_1$, then the corresponding x and $\Phi$ need not have equal objective function values. This is why we allow only legitimate partitions, which is fine in light of the fact that our analysis will require that $\Delta \to 0$ in any case.

At this point, we direct our efforts towards obtaining an optimal solution to Problem CPT($\Delta, y_1, \ldots, y_p$) which is "nearly PFL" in nature. Such a solution is one in which the facility located at $y_1$ fully serves some subset of the demand points $\{z_1, \ldots, z_J\}$, beginning at point $z_1$ and moving sequentially to the right, until such time as it exhausts its supply. The facility located at $y_2$ will then take over and continue serving from left to right until its supply is exhausted. At such time, the facility at $y_3$ will begin service, etc. The adverb "nearly" is used since such a solution may result in joint servicing of those demand points (switchpoints) at which the current servicing facility's supply is exhausted and the next facility's service begins. Depending on the size of $\Delta$, a switchpoint may require service/supply from as few as one (degenerate solution) up to as many as p facilities.

The thought may have already occurred to the reader, that such a solution would result if one were to use the Northwest Corner Rule to obtain a starting basic feasible solution to Problem CPT($\Delta, y_1, \ldots, y_p$). The reader may, however, be surprised to find that this solution is optimal. With this in mind, we now show that the Northwest Corner Rule will indeed give us such a solution to Problem CPT($\Delta, y_1, \ldots, y_p$), and that it is an optimal solution.

<u>Theorem 2.3</u>. The Northwest Corner Rule solution $x^*$ is optimal to Problem CPT($\Delta$,$y_1$, ..., $y_p$) for any $p > 2$.

<u>Proof</u>. Let $\hat{x}$ be any optimal solution to Problem CPT($\Delta$,$y_1$, ..., $y_p$), and suppose that $\hat{x} \neq x^*$. We will show that $\sum_{i=1}^{p} \sum_{j=1}^{J} d_{ij} x^*_{ij} < \sum_{i=1}^{p} \sum_{j=1}^{J} d_{ij} \hat{x}_{ij}$, and so equality must hold.

Since $\hat{x} \neq x^*$, there exist $r,s \in \{1, \ldots, p\}$ and $k,\ell \in \{1, \ldots, J\}$ with $r<s$ and $k<\ell$ such that $\delta = \min\{\hat{x}_{r\ell},\hat{x}_{sk}\} > 0$. Consider the solution $\bar{x}$ which is identical to $\hat{x}$ except that $\bar{x}_{rk} = \hat{x}_{rk} + \delta$, $\bar{x}_{r\ell} = \hat{x}_{r\ell} - \delta$, $\bar{x}_{sk} = \hat{x}_{sk} - \delta$ and $\bar{x}_{s\ell} = \hat{x}_{s\ell} + \delta$. Clearly, $\bar{x}$ is feasible to Problem CPT($\Delta$,$y_1$, ..., $y_p$).

Now, suppose we show that $Q \equiv \sum_{i=1}^{p} \sum_{j=1}^{J} d_{ij}\hat{x}_{ij} - \sum_{i=1}^{p} \sum_{j=1}^{J} d_{ij}\bar{x}_{ij} > 0$. Then we will have proven the theorem because evidently, by proceeding in this manner, $\hat{x}$ can be transformed into $x^*$ without increasing the objective function value.

Toward this end, note that $Q/\delta = (d_{sk} - d_{rk}) - (d_{s\ell} - d_{r\ell})$ and observe that as the point $t$ varies from 0 to $c$, the function $d_{st} - d_{rt}$ is nonincreasing. Consequently, since $k<\ell$, $Q/\delta > 0$ and the proof is complete. □

So we now have that the "nearly PFL" solution, obtained via the Northwest Corner Rule, is an optimal solution to Problem CPT($\Delta$,$y_1$, ... , $y_p$) for any $p > 2$. Then, recalling Theorem 2.2 and the relationship $\Phi^*_i(x) = x^*_{ij}/\bar{\Delta}_j$, $x \in \Delta_j$, $i=1, \ldots, p$, $j=1, \ldots, J$, we have that $\Phi^* = (\Phi^*_1, \ldots, \Phi^*_p)$ is an optimal solution to Problem CP($\Delta$,$y_1$, ..., $y_p$), and as such, each $\Phi^*_i(\cdot)$, $i=1, \ldots, p$ is a step function in which the steps occur at points coincident with elements of the partition. Specifically,

$\Phi_i^*(\cdot) = f_\Delta(\cdot)$ on those $\Delta_j$ for which $x_{ij}^* = \int_{\Delta_j} f_\Delta(x)dx$, whereas $\Phi_i^*(\cdot) = 0$ on those $\Delta_j$ for which $x_{ij}^* = 0$. Finally, on any increment containing a switchpoint, we have for some $i\varepsilon\{1, \ldots, p\}$ and $\ell > 0$, $\Phi_i^*(\cdot) + \ldots + \Phi_{i+\ell}^*(\cdot) = f_\Delta(\cdot)$, with all allocation functions $\Phi_i^*(\cdot), \ldots, \Phi_{i+\ell}^*(\cdot)$ being positive on that increment. Again, we remark that all $\Phi_i^*(\cdot)$ are defined so as to be right continuous. Clearly then, as is the case for $x^*$, it is fitting that $\Phi^*$ also be described as "nearly PFL", in that just as $x_{i+1,j}^*$ begins service (assuming nondegeneracy) at the $i^{th}$ switchpoint which is the midpoint of some $\Delta_j$, $\Phi_{i+1}^*(\cdot)$ "takes over from" $\Phi_i^*(\cdot)$ on that $\Delta_j$ containing the $i^{th}$ switchpoint, and if necessary, "assists" in supplying the demand on that $\Delta_j$.

We are now in a position to utilize Theroem 2.2 and the optimal "nearly PFL" solution to Problem CP$(\Delta, y_1, \ldots, y_p)$ obtained via Problem CPT$(\Delta, y_1, \ldots, y_p)$ and the Northwest Corner Rule, to complete our analysis of Problem CP$(y_1, \ldots, y_p)$. We do so by presenting the following theorem.

Theorem 2.4. There exists an optimal solution $\Phi^* = (\Phi_1^*, \ldots, \Phi_p^*)$ to Problem CP$(y_1, \ldots, y_p)$ for which the allocation functions $\Phi_i^*(\cdot)$, $i=1, \ldots, p$, result in the facilities at $0 < y_1 < \ldots < y_p < c$ serving the intervals $I_1 = [0,\alpha_1]$, $I_2 = [\alpha_1,\alpha_2]$, $\ldots$, $I_p = [\alpha_{p-1},c]$, respectively, for some $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{p-1} < c$.

Proof. Let $\hat{\Phi} = (\hat{\Phi}_1, \ldots, \hat{\Phi}_p)$ be any optimal solution to Problem CP$(y_1, \ldots, y_p)$.

Construct any sequence of legitimate partitions of the interval $[0,c]$ where the $k^{th}$ partition has mesh $\Delta^k$ and such that $\{\Delta^k\} \to 0$.

For each $\Delta^k$ and the corresponding partition, define the step functions $f_{\Delta^k}(\cdot)$ and $\Phi_{ik}(\cdot)$, $i=1, \ldots, p$, to be equal on each increment of the partition to that value of $f(\cdot)$ and $\hat{\Phi}_i(\cdot)$, $i=1, \ldots, p$, respectively, evaluated at the midpoint of the increment. Then set $s_i(\Delta^k) = \int_0^c \Phi_{ik}(x)dx$ so that $\Phi_k \equiv (\Phi_{1k}, \ldots, \Phi_{pk})$ is a feasible solution to Problem $CP(\Delta^k, y_1, \ldots, y_p)$. (Note: this is easy to show.)

Now using the Northwest Corner Rule, solve Problem $CPT(\Delta^k, y_1, \ldots, y_p)$ and thereby (via Theorem 2.2) obtain the corresponding optimal "nearly PFL" solution to Problem $CP(\Delta^k, y_1, \ldots, y_p)$. Note of course, the objective function value of $\Phi_k$ must be $\geq$ that of the "nearly PFL" solution to Problem $CP(\Delta^k, y_1, \ldots, y_p)$. It follows then, that since this is true for all k, and since the $\Phi_{ik}(\cdot)$ and $s_i(\Delta^k)$ converge to the $\hat{\Phi}_i(\cdot)$ and $s_i$ (of Problem $CP(y_1, \ldots, y_p)$), we have that the objective function value of $\hat{\Phi}$ must be $\geq$ to that of the PFL solution $\Phi^*$ to Problem $CP(y_1, \ldots, y_p)$ described in the statement of the theorem, since this is the solution to which the optimal "nearly PFL" solutions (of the Problems $CP(\Delta^k, y_1, \ldots, y_p)$) converge. Therefore, the PFL solution must be optimal to Problem $CP(y_1, \ldots, y_p)$. This concludes the proof of Theorem 2.4. $\square$

To insure a complete understanding of the nature of the optimal PFL solution to Problem $CP(y_1, \ldots, y_p)$, we now give a more algebraic description/definition of $\Phi^*$ than that which is given in the above analysis and theorem. Specifically, the component (allocation) functions of $\Phi^* = (\Phi_1^*, \ldots, \Phi_p^*)$ satisfy the following:

$$\Phi_i^*(x) = \begin{cases} f(x), & x \in I_i = [\alpha_{i-1}\alpha_i),\ i=1,\ \ldots,\ p,\ \alpha_0 \equiv 0,\ \alpha_p \equiv c \\ 0, & \text{otherwise} \end{cases}$$

and $\Phi_p^*(c) = f(c)$,

where $\int_0^c \Phi_i^*(x)dx = \int_{I_i} f(x)dx = s_i$, $i=1, \ldots, p$.

(Note that because of Lemma 2.1, the statement and proof of Theorem 2.4 assumed the given facility locations to be distinct. It should be noted however, that one can easily show that the PFL solution is optimal to Problem CP($y_1$, ..., $y_p$) even when the $y_i$ are not all distinct.)

At this point, we raise a question that might ought to have been addressed immediately after stating Problem CP, but was delayed in anticipation of Theorem 2.4. Specifically, we now take a moment to consider the question of existence with regard to an optimal solution to Problem CP. Simply put, one would be justified in asking if Problem CP has an optimal solution. Clearly its objective function is bounded below by zero, but boundedness of an objective function does not guarantee the existence of an optimal solution. We argue for the existence of an optimal solution as follows. Let $\Lambda(y,\Phi)$ denote the objective function of Problem CP, where y represents the vector ($y_1$, ..., $y_p$) of facility locations and $\Phi$ the vector of allocation functions. Define the function $\lambda(y) \equiv \Lambda(y,\Phi^*)$, where $\Phi^*$ (i.e. the PFL allocation) solves Problem CP(y). Let $\Gamma \equiv \prod_{i=1}^{p} [0,c]$ denote the Cartesian product of $[0,c]$ with itself p times, and suppose that $\{y^k\} \to \bar{y}$ is a convergent sequence of points lying in $\Gamma$. Clearly then, since the PFL allocations to the Problems CP($y^k$) tend to the PFL allocation of Problem CP($\bar{y}$), it follows that $\{\lambda(y^k)\} \to \lambda(\bar{y})$, thereby establishing the continuity of

$\lambda$ on $\Gamma$. Then by recalling the fact that a continuous function defined on a compact set attains a minimum on that set, we have established the existence of an optimal solution to Problem CP, and thus can proceed with our analysis of this problem by stating the following theorem.

Theorem 2.5. There exists an optimal solution to Problem CP, for which

(a) each facility i serves an interval $[\alpha_i,\beta_i]$, with its location $y_i \varepsilon[\alpha_i,\beta_i]$, and such that

(b) $\int_{\alpha_i}^{y_i} f(x)dx = \int_{y_i}^{\beta_i} f(x)dx = s_i/2$ .

(Note that such $y_i$ are generally referred to as "median" locations.)

Proof. Given any optimal solution $(y^*,\Phi^*)$ to Problem CP, it follows that $(y^*,\Phi^{**})$ is also optimal to Problem CP, where $\Phi^{**}$ is the optimal PFL solution to Problem CP$(y_1^*, \ldots, y_p^*)$ obtained via Theorem 2.4. In addition, it is clear that for such an allocation as that given by $(y^*,\Phi^{**})$ to be optimal, each facility must be located at a median location. □

In the event that the last statement of the above proof is not immediately clear to the reader, we remark that a rigorous proof of this statement is given in Theorem 2.18.

Corollary 2.6. Given any ordering of the p facilities, their optimal locations are known readily, i.e. simply mark off the intervals to be served (from left to right) according to their capacities, and then place the facilities at their respective median locations. □

A consequence of the above theorems and corollary, is that the task of obtaining an optimal solution to Problem CP can be reduced to that of computing the minimum cost locations (and hence allocations) for each permutation/ordering of the p facilities, and then choosing an ordering and its associated $(y,\Phi)$ having a minimum cost. Ironically, our choice of the word "reduced" in the above sentence tends to be both correct and misleading. It is correct, in that rather than having to consider the uncountably infinite collection of all feasible $(y,\Phi)$, one need only consider $p!/n_1!n_2!\ldots n_r!$ feasible $(y,\Phi)$, where r is the number of distinct capacity values amongst $s_1, \ldots, s_p$, and $n_i$ is the number of facilities having the same $i^{th}$ capacity value. However, the word can be misleading if it is interpreted to mean that the task of obtaining an optimal solution to Problem CP has been rendered simple. To the contrary, a serious combinatorics problem is encountered as p is allowed to increase. It is this combinatorics problem on which we will now focus our attention, and to which we will refer as the "p! problem", a descriptor which is accurate only when $r = p$, but one which we will use (for convenience) in any case.

## 2.4 A Useful and Insightful Special Case

Knowing that an optimal solution to Problem CP can be found amongst the $p!/n_1!n_2!\ldots n_r!$ solutions of the above corollary, one's initial inclination may be to compute each of these solutions, and then to choose one having minimum cost. However, the cost prohibitiveness of explicit enumeration would soon become apparent, and would therefore be likely to foster the use of implicit enumeration. This being the case, we begin our investigation of the "p! problem" by first considering the case in which the demand function of Problem CP is monotone nondecreasing on $[0,c]$. (Without loss of generality, we will assume that $f(x) > 0$ for $x > 0$.) We will show that for such an f, an optimal ordering of the p facilities is one in which the facilities are ordered from left to right (in $[0,c]$) in order of nondecreasing capacity. In order to establish this result, we will present a lemma that will enable us to represent the cost of any ordering, in terms of the continuous (but not necessarily differentiable) function $F(x) = \int_0^x f(t)dt$, $x\varepsilon[0,c]$. Note that in the event f is a probability density function, F(x) would be a cumulative distribution function. We also remark that since $f > 0$ on $(0,c]$, F(x) is strictly increasing on $[0,c]$. Now, for f as considered herein, and for any ordering of the p facilities, the minimum cost locations/allocations of Corollary 2.6 are such that F(x) might look something like that of Figure 2.1. Noting the numbered regions in this figure, we now give a lemma which shows that their total area is equal to the sum of the transportation costs associated with the facilities located at $y_1$, $y_2$ and $y_3$, and serving $I_1 = [0,\alpha_1]$, $I_2 = [\alpha_1,\alpha_2]$ and $I_3 = [\alpha_2,c]$, respectively.

<u>Lemma 2.7.</u> Consider a facility i having supply $s_i$ which serves an interval $I_i \equiv [\alpha_{i-1},\alpha_i]$, and suppose that this facility is located at a median location $y_i$ with respect to $I_i$, i.e. $y_i$ satisfies

$$\int_{\alpha_{i-1}}^{y_i} f(x)dx = \int_{y_i}^{\alpha_i} f(x)dx = s_i/2. \quad \text{Then,} \tag{2.6}$$

$$\int_{I_i} f(x)|x-y_i|dx = A_{id} + A_{iu}\ , \text{ where} \tag{2.7}$$

$$A_{id} \equiv \int_{\alpha_{i-1}}^{y_i} f(x)(y_i-x)dx = y_i(s_i/2) - \int_{F(\alpha_{i-1})}^{F(y_i)} F^{-1}(y)dy \quad \text{and} \tag{2.8}$$

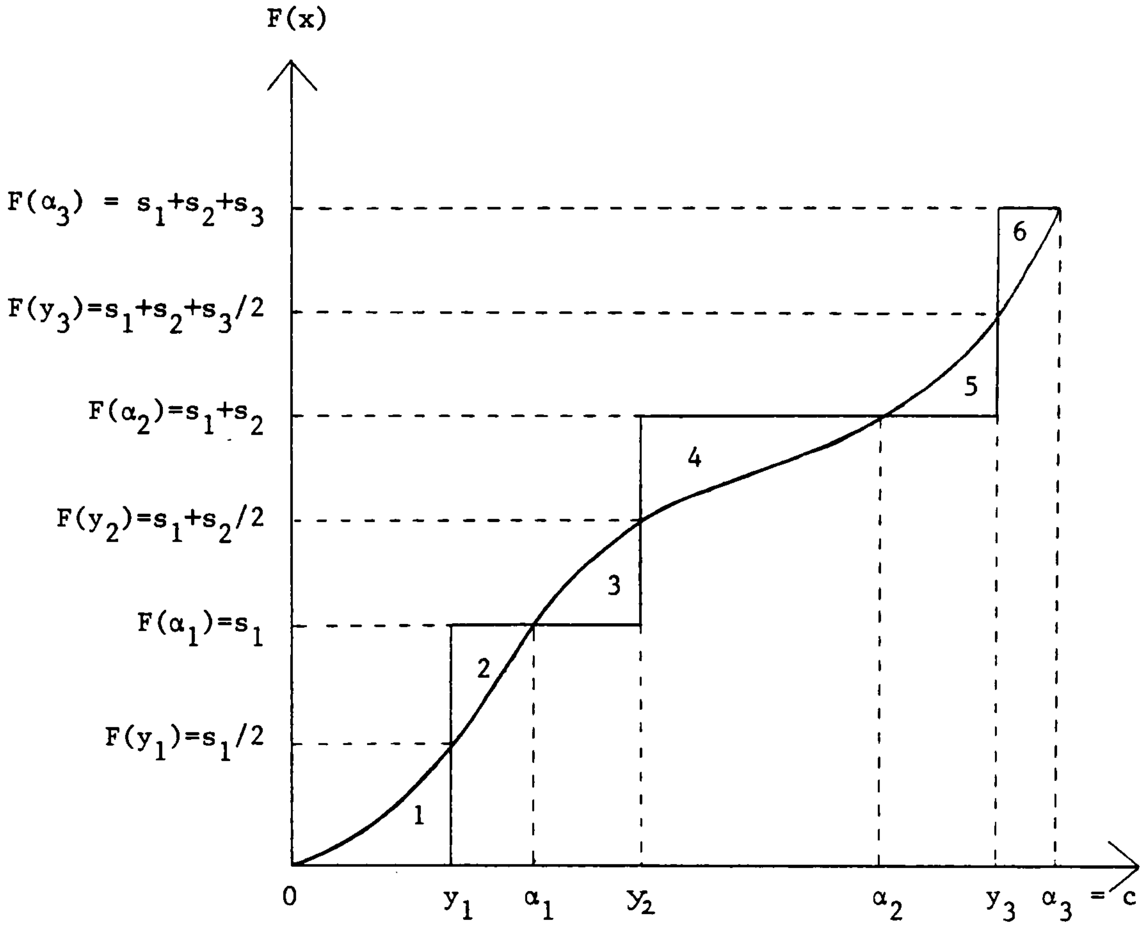



Figure 2.1. Geometric Representation of Transportation Costs

$$A_{iu} \equiv \int_{y_i}^{\alpha_i} f(x)(x-y_i)dx = \int_{F(y_i)}^{F(\alpha_i)} F^{-1}(y)dy - y_i(s_i/2). \tag{2.9}$$

(Note that $F^{-1}$ exists since F is strictly increasing under our current assumptions on f.)

Proof. Defining $A_{id}$ and $A_{iu}$ respectively as the first integrals in (2.8) and (2.9), it is easily seen that (2.7) holds. Hence, consider $A_{id}$. Integrating by parts, we get

$$A_{id} = -F(\alpha_{i-1})[y_i-\alpha_{i-1}] + \int_{\alpha_{i-1}}^{y_i} F(x)dx$$

$$= y_i[F(y_i)-F(\alpha_{i-1})] - [y_iF(y_i)-\alpha_{i-1}F(\alpha_{i-1})] + \int_{\alpha_{i-1}}^{y_i} F(x)dx. \tag{2.10}$$

But $F(y_i)-F(\alpha_{i-1}) = s_i/2$ from (2.6) and $[y_iF(y_i)-\alpha_{i-1}F(\alpha_{i-1})] = \int_{\alpha_{i-1}}^{y_i} F(x)dx + \int_{F(\alpha_{i-1})}^{F(y_i)} F^{-1}(y)dy$, since F is strictly increasing. Substituting this into (2.10), we get (2.8). Similarly, (2.9) holds and the proof is complete. □

Clearly, direct application of Lemma 2.7 to the situation depicted in Figure 2.1 shows that the total area of the regions numbered 1 through 6 is equal to the sum of the transportation costs incurred by the facilities shown. Thus, we now know how to obtain a geometric representation of the transportation cost of any ordering. This representation is used to establish the main result of this section, which we present via the following theorem.

Theorem 2.8. Given that the demand function f is nonnegative, integrable and nondecreasing, that ordering in which the p facilities appear from left to right (in $[0,c]$) in nondecreasing order of capacity, is optimal to Problem CP. That is, the associated $(y,\Phi)$ of Corollary 2.6 is an optimal solution to Problem CP.

(Note: our proof of Th. 2.8 requires the convexity of $F(x)$, and hence the following lemma.)

Lemma 2.9. Given that f is a nonnegative, integrable, nondecreasing function defined on $[0,c]$, the continuous function $F(x) = \int_0^x f(t)dt$, $x\varepsilon[0,c]$, is a convex function.

Proof. Let $x_1,x_2$ be any two points in $[0,c]$, and $\lambda\varepsilon[0,1]$. Setting $\bar{x} \equiv \lambda x_1 + (1-\lambda)x_2$, we have

$$\begin{aligned}
\lambda F(x_1) + (1-\lambda)F(x_2) - F(\bar{x}) &= (1-\lambda)[F(x_2) - F(\bar{x})] - \lambda[F(\bar{x}) - F(x_1)] \\
&= (1-\lambda)\int_{\bar{x}}^{x_2} f(t)dt - \lambda \int_{x_1}^{\bar{x}} f(t)dt \\
&\geq (1-\lambda)f(\bar{x})(x_2 - \bar{x}) - \lambda f(\bar{x})(\bar{x} - x_1) \\
&= f(\bar{x})[(1-\lambda)x_2 + \lambda x_1 - \bar{x}] \\
&= 0
\end{aligned}$$

□

Proof of Theorem 2.8. This theorem need only be proved for the case $p = 2$, since if we were given the locations of $p > 2$ facilities, we could apply the theorem to any pair of adjacent facilities and thus by a bubble-sort[1] type of argument, arrive at the desired ordering.

Letting $p = 2$ and assuming that the facilities are numbered

---

[1] The following string of six-tuples is a schematic showing the application of a bubble-sort on a given ordering of facilities having capacities, $s_1 = 2$, $s_2 = 4$, $s_3 = 5$, $s_4 = 7$, $s_5 = 8$, $s_6 = 9$, say. As can be seen, nine applications of this theorem (for $p = 2$) were required to obtain the desired ordering.
(7,5,8,4,9,2) → (5,7,8,4,9,2) → (5,7,4,8,9,2) → (5,4,7,8,9,2) → (4,5,7,8,9,2) → (4,5,7,8,2,9) → (4,5,7,2,8,9) → (4,5,2,7,8,9) → (4,2,5,7,8,9) → (2,4,5,7,8,9).

so that $s_1 < s_2$, we direct the readers' attention to Figure 2.2, in which are depicted the two possible orderings of the facilities. Notice that F(x) is convex and strictly increasing on $[0,c]$, and that Figure 2.2a depicts the ordering proclaimed optimal by this theorem. Recalling the geometric representation of the transportation cost, our objective is to show that $A_1 < A_2$.

(Note that Figure 2.2 implies that $f(x) > 0$ on $(0,c)$. We consider this to be a reasonable assumption in light of the monotonicity assumption on f.)

Since F(x) is convex and strictly increasing, the function $G = F^{-1}$ is well defined, concave and strictly increasing. Thus it is easy to see that $A_1$ can be written as,

$$A_1 = \left[(s_1/2)G(s_1/2) - \int_0^{(s_1/2)} G(y)dy\right] + \left[\int_{(s_1/2)}^{s_1} G(y)dy - (s_1/2)G(s_1/2)\right]$$

$$+ \left[(s_2/2)G(s_1 + s_2/2) - \int_{s_1}^{(s_1+s_2/2)} G(y)dy\right] + \left[\int_{(s_1+s_2/2)}^{(s_1+s_2)} G(y)dy - (s_2/2)G(s_1 + s_2/2)\right],$$

where each of the four summands is equal to one of the four regions in Figure 2.2a. Cancelling like terms, gives us

$$A_1 = -\int_0^{(s_1/2)} G(y)dy + \int_{(s_1/2)}^{s_1} G(y)dy - \int_{s_1}^{(s_1+s_2/2)} G(y)dy + \int_{(s_1+s_2/2)}^{(s_1+s_2)} G(y)dy \ .$$

Similarly,

$$A_2 = -\int_0^{(s_2/2)} G(y)dy + \int_{(s_2/2)}^{s_2} G(y)dy - \int_{s_2}^{(s_2+s_1/2)} G(y)dy + \int_{(s_2+s_1/2)}^{(s_1+s_2)} G(y)dy \ .$$

Recalling that $s_1 < s_2$, we consider two cases; Case (i), $s_1 < s_2/2$ and Case (ii), $s_1 > s_2/2$. Figure 2.3 provides a pictoral representation of each of these cases. In particular, note the relationships on

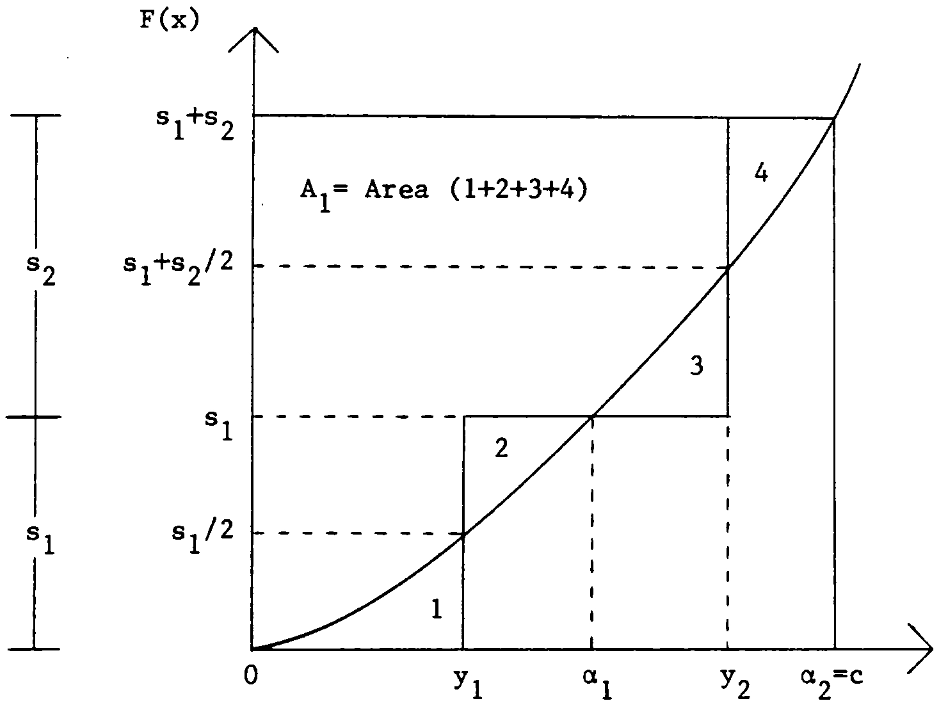



Figure 2.2a. (See Theorem 2.8)

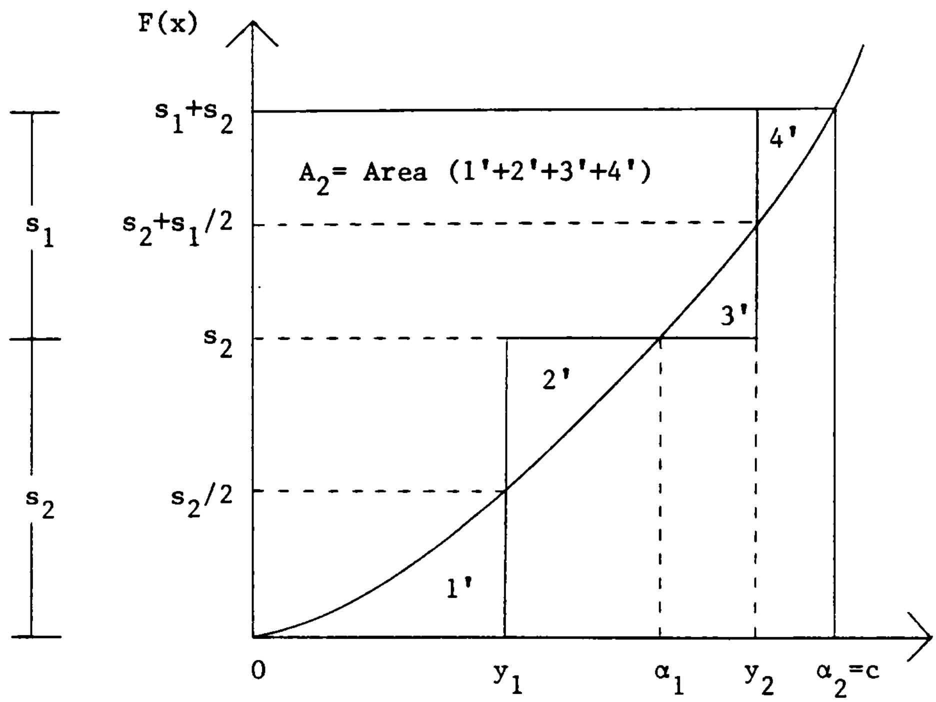
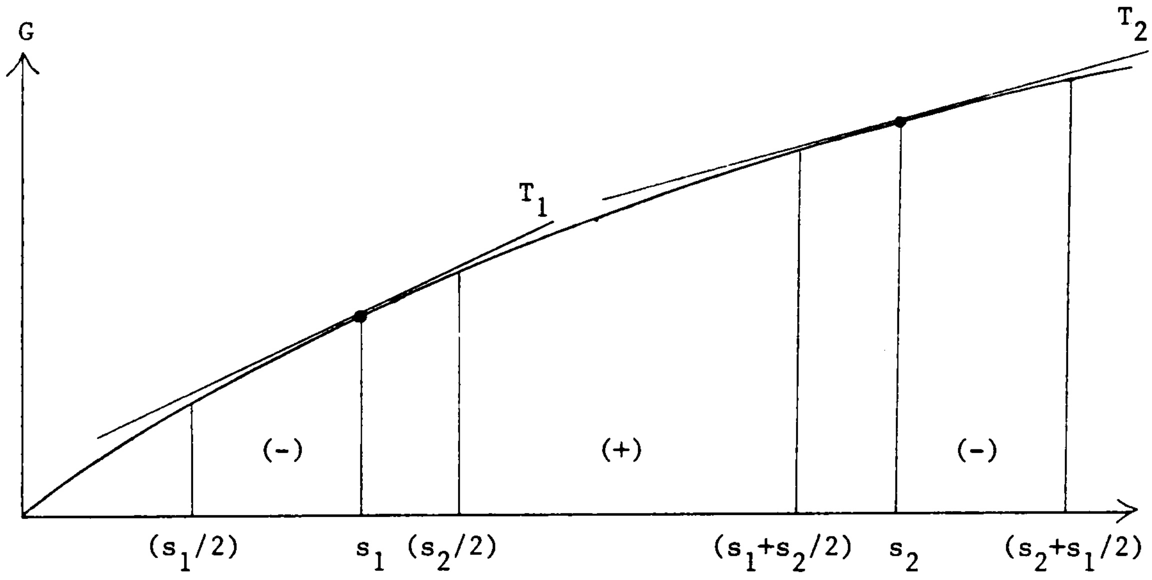
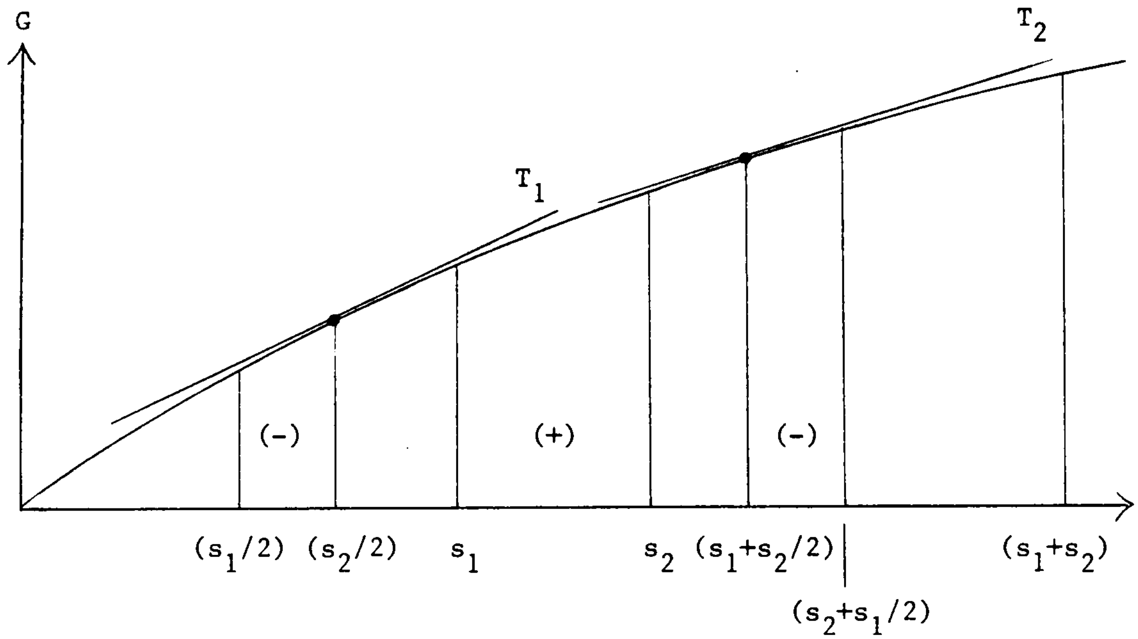



Figure 2.2b. (See Theorem 2.8)




Case(i), $s_1 < (s_2/2)$




Case(ii), $s_1 > (s_2/2)$

Figure 2.3. (See Theorem 2.8)

the horizontal axis between the various sums involving $s_1$ and $s_2$. The intent of the (+) and (−) signs will be understood very shortly.

Under Case (i), noting Figure 2.3, $A_1$ and $A_2$ can be written as,

$$A_1 = -\int_0^{(s_1/2)} + \int_{(s_1/2)}^{s_1} - \int_{s_1}^{(s_2/2)} - \int_{(s_2/2)}^{(s_1+s_2/2)} + \int_{(s_1+s_2/2)}^{s_2} + \int_{s_2}^{(s_2+s_1/2)} + \int_{(s_2+s_1/2)}^{(s_1+s_2)}$$

and

$$A_2 = -\int_0^{(s_1/2)} - \int_{(s_1/2)}^{s_1} - \int_{s_1}^{(s_2/2)} + \int_{(s_2/2)}^{(s_1+s_2/2)} + \int_{(s_1+s_2/2)}^{s_2} - \int_{s_2}^{(s_2+s_1/2)} + \int_{(s_2+s_1/2)}^{(s_1+s_2)},$$

where the integrand of each of these integrals is G(y)dy. It follows that

$$(A_2-A_1)/2 = -\int_{(s_1/2)}^{s_1} G(y)dy + \int_{(s_2/2)}^{(s_1+s_2/2)} G(y)dy - \int_{s_2}^{(s_2+s_1/2)} G(y)dy ,$$

and hence the (+) and (−) signs in Figure 2.3.

Recall that we want to show $A_2 > A_1$, i.e. $(A_2-A_1) > 0$. We do so by showing that the above expression for $(A_2-A_1)/2$ is nonnegative.

Due to the concavity of G, the area under G on both $[s_1/2, s_1]$ and $[s_2, s_2+s_1/2]$ can be overestimated by the area under the support functionals $T_1$ and $T_2$ shown in Figure 2.3. Doing so, we get that

$$(s_1/4)[2G(s_1)-(s_1/2)G^+(s_1)] > \int_{(s_1/2)}^{s_1} G(y)dy \quad \text{and}$$

$$(s_1/4)[2G(s_2)+(s_1/2)G^+(s_2)] > \int_{s_2}^{(s_2+s_1/2)} G(y)dy ,$$ where $G^+$

denotes the right hand derivative of G. (Recall that F is continuous but it need not be differentiable.)

Therefore we can write,

$$(A_2-A_1)/2 > \int_{(s_2/2)}^{(s_1+s_2/2)} G(y)dy - \{(s_1/4)[2G(s_1)-(s_1/2)G^+(s_1)] + (s_1/4)[2G(s_2)+(s_1/2)G^+(s_2)]\}$$

$$= \int_{(s_2/2)}^{(s_1+s_2/2)} G(y)dy - (s_1/2)[G(s_1)+G(s_2)]+(s_1^2/8)[G^+(s_1)-G^+(s_2)].$$

Now, G concave implies that $G^+(s_2) \leq G^+(s_1)$ and so $[G^+(s_1)-G^+(s_2)] \geq 0$. Thus we have,

$$(A_2-A_1)/2 \geq \int_{(s_2/2)}^{(s_1+s_2/2)} G(y)dy - (s_1/2)[G(s_1)+G(s_2)] \quad .$$

Next, we observe that

$$\int_{(s_2/2)}^{(s_1+s_2/2)} G(y)dy \geq (s_1/2)[G(s_2/2) + G(s_1+s_2/2)] \quad ,$$

where the right hand side is equal to the area of that polygon under G formed by the secant line drawn through the points $(s_2/2, G(s_2/2))$ and $(s_1+s_2/2, G(s_1+s_2/2))$. Using this underestimation, we have that

$$(A_2-A_1)/2 \geq (s_1/2)[G(s_2/2)+G(s_1+s_2/2)] - (s_1/2)[G(s_1)+G(s_2)] \quad .$$

Noting that $s_1 \leq s_2/2 < (s_1+s_2/2) \leq s_2$, and that $(s_2/2) - s_1 = s_2 - (s_1+s_2/2)$, we can choose $\lambda\varepsilon(0,1)$ so that $(s_2/2) = \lambda s_1 + (1-\lambda)s_2$, and thus $(s_1+s_2/2) = (1-\lambda)s_1 + \lambda s_2$ also. Then by the concavity of G,

$$G(s_2/2) = G(\lambda s_1+(1-\lambda)s_2) \geq \lambda G(s_1) + (1-\lambda)G(s_2) \quad \text{and}$$

$$G(s_1+s_2/2) = G((1-\lambda)s_1+\lambda s_2) \geq (1-\lambda)G(s_1) + \lambda G(s_2) \quad .$$

Adding the above two inequalities gives us,

$$G(s_2/2) + G(s_1+s_2/2) \geq G(s_1) + G(s_2) \quad .$$

Therefore, we have

$$(A_2-A_1)/2 \geq (s_1/2)[G(s_1)+G(s_2)] - (s_1/2)[G(s_1)+G(s_2)] = 0 \ ,$$

and thus have shown that $A_2 \geq A_1$. This proves the theorem for p=2 and under Case (i).

(Note: the proof for Case (ii) is identical to that of Case (i), and is included only for the sake of completeness.)

<u>Under Case (ii)</u>, $A_1$ and $A_2$ can be written as,

$$A_1 = -\int_0^{(s_1/2)} + \int_{(s_1/2)}^{(s_2/2)} + \int_{(s_2/2)}^{s_1} - \int_{s_1}^{s_2} - \int_{s_2}^{(s_1+s_2/2)} + \int_{(s_1+s_2/2)}^{(s_2+s_1/2)} + \int_{(s_2+s_1/2)}^{(s_1+s_2)}$$

and

$$A_2 = -\int_0^{(s_1/2)} - \int_{(s_1/2)}^{(s_2/2)} + \int_{(s_2/2)}^{s_1} + \int_{s_1}^{s_2} - \int_{s_2}^{(s_1+s_2/2)} - \int_{(s_1+s_2/2)}^{(s_2+s_1/2)} + \int_{(s_2+s_1/2)}^{(s_1+s_2)} .$$

Therefore,

$$(A_2-A_1)/2 = -\int_{(s_1/2)}^{(s_2/2)} G(y)dy + \int_{s_1}^{s_2} G(y)dy - \int_{(s_1+s_2/2)}^{(s_2+s_1/2)} G(y)dy .$$

Again, due to the concavity of G, the area under G on both $[s_1/2,s_2/2]$ and $[s_1+s_2/2,s_2+s_1/2]$ can be overestimated by the area under the support functionals $T_1$ and $T_2$ as shown in Figure 2.3. Doing so, and again denoting the right hand derivative of G by $G^+$, we get that

$$-(1/2)G^+(s_2/2)(s_1/2-s_2/2)^2 - G(s_2/2)(s_1/2-s_2/2) \geq \int_{(s_1/2)}^{(s_2/2)} G(y)dy$$

and

$$(1/2)G^+(s_1+s_2/2)(s_1/2-s_2/2)^2 + (s_2/2-s_1/2)G(s_1+s_2/2) \geq \int_{(s_1+s_2/2)}^{(s_2+s_1/2)} G(y)dy.$$

Therefore we can write,

$$(A_2-A_1)/2 \geq \int_{s_1}^{s_2} G(y)dy - (s_2/2-s_1/2)[G(s_1+s_2/2) + G(s_2/2)]$$
$$+ (1/2)(s_1/2-s_2/2)^2[G^+(s_2/2) - G^+(s_1+s_2/2)] .$$

Now, G concave implies that $[G^+(s_2/2)-G^+(s_1+s_2/2)] \geq 0$, and so

$$(A_2-A_1)/2 \geq \int_{s_1}^{s_2} G(y)dy - (s_2/2-s_1/2)[G(s_1+s_2/2) + G(s_2/2)] .$$

By underestimating the area under G on $[s_1,s_2]$ by the area under the secant through the points $(s_1,G(s_1))$, $(s_2,G(s_2))$, we get that

$$\int_{s_1}^{s_2} G(y)dy \geq (1/2)(s_2-s_1)[G(s_1)+G(s_2)] \text{ , giving us}$$

$$(A_2-A_1)/2 \geq (1/2)(s_2-s_1)[G(s_1)+G(s_2)] - (s_2/2-s_1/2)[G(s_1+s_2/2)+G(s_2/2)].$$

Since $s_2/2 < s_1 < s_2 < (s_1+s_2/2)$ and $(s_1-s_2/2) = (s_1+s_2/2) - s_2$, we can choose $\lambda\varepsilon(0,1)$ such that $s_1 = \lambda(s_2/2) + (1-\lambda)(s_1+s_2/2)$ and $s_2 = (1-\lambda)(s_2/2) + \lambda(s_1+s_2/2)$. Then by the concavity of G,

$$G(s_1) \geq \lambda G(s_2/2) + (1-\lambda)G(s_1+s_2/2)$$
$$G(s_2) \geq (1-\lambda)G(s_2/2) + \lambda G(s_1+s_2/2) \quad .$$

Adding these gives us,

$$G(s_1) + G(s_2) \geq G(s_2/2) + G(s_1+s_2/2), \quad \text{from which}$$

it follows, that $(A_2-A_1)/2 \geq 0$, i.e. $A_2 \geq A_1$. This completes the proof of Theorem 2.8. □

Of course if follows from Theorem 2.8 that if f is nonnegative, integrable and nonincreasing on $[0,c]$, then ordering the facilities from left to right according to nonincreasing capacity, would result in an optimal solution to Problem CP. This can be seen by defining $g(x) = f(c-x)$ and then applying Theorem 2.8 to $g(x)$.

Theorem 2.8 provides us with a (soon to be seen) useful sufficiency condition for optimality, and allows for the following generalization.

Corollary 2.10. Let f be any nonnegative, integrable, demand function for Problem CP. Then in determining an optimal solution to Problem CP, it is sufficient to restrict ones' attention to those orderings/ permutations of the p facilities which give solutions (per Corollary 2.6) satisfying the following condition: sets of facilities which use their entire supply to serve a connected subset of $[0,c]$ over which f is monotone nondecreasing (nonincreasing), must be arranged from left to right in nondecreasing (nonincreasing) order of capacity.

Proof. Follows directly from Theorem 2.8. □

Corollary 2.10 will be used later to construct an enumeration tree for which bounds will then be developed so as to enable early fathoming.

## 2.5 Analysis of the Symmetric, Unimodal Demand Distribution Case

Continuing in our effort to "chip away" at the "p! problem", we next consider the case in which f is nonnegative, integrable, unimodal and symmetric. (We will call $f:[0,c] \to R$ unimodal if there exists an $\alpha \epsilon [0,c]$ such that f is nondecreasing on $[0,\alpha]$ and nonincreasing on $[\alpha,c]$.) As before, and without loss of generality, we will assume that $f > 0$ on $(0,c)$, for otherwise we could redefine the domain of f. We also remark that $\alpha$ need not be uniquely determined for such an f as considered herein. This can be seen in Figure 2.4 which depicts a symmetric, unimodal, demand function for which $\alpha$ could be chosen to be any point in the interval $[15,20]$.

The term straddle is used to describe the situation in which some facility i serves an interval $I_i = [\alpha_{i-1},\alpha_i]$ over which f is not monotone. In such a case, the facility is said to straddle, or will be called a straddling facility. Figure 2.4 depicts a problem having a total demand of two hundred (200), and for which two different orderings of six (6) facilities having capacities $s_1 = 10$, $s_2 = 15$, $s_3 = 25$, $s_4 = 30$, $s_5 = 45$, $s_6 = 75$, say, are shown. The ordering shown in Figure 2.4a results in straddling, whereas that of Figure 2.4b does not. It should be noted however, that both orderings satisfy the condition of Corollary 2.10.

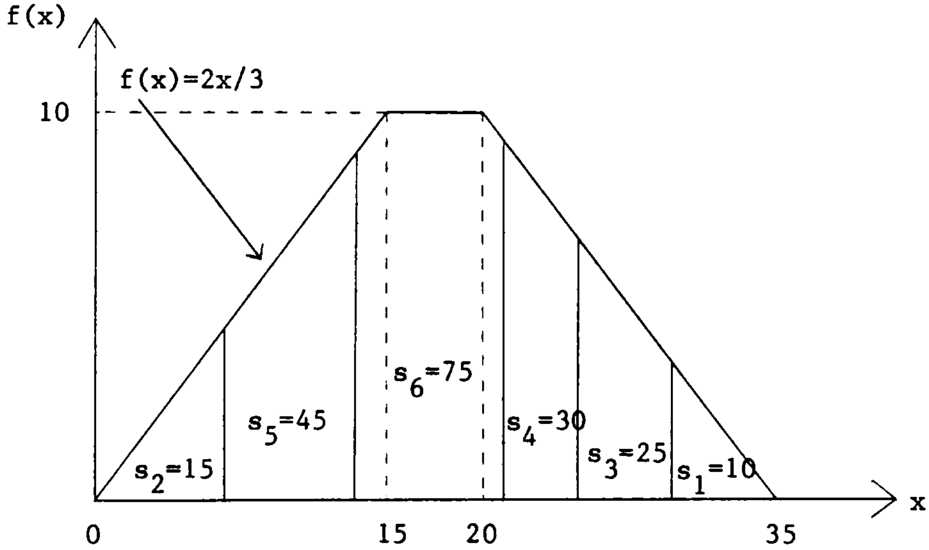



Figure 2.4a. (Straddling)

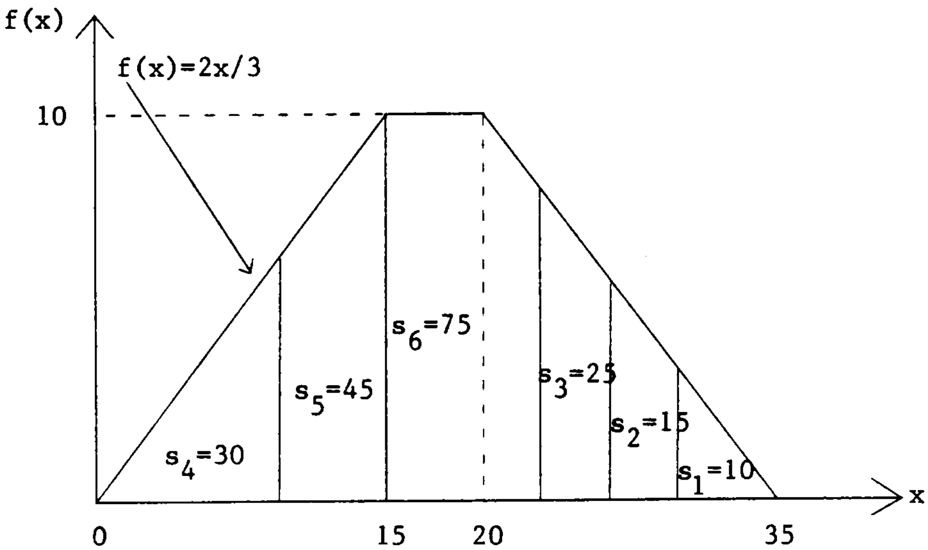



Figure 2.4b. (Non-Straddling)

Figure 2.4. Straddling vs. Non-Straddling

The following theorem presents another condition that will eventually be used in conjunction with Theorem 2.8/Corollary 2.10 to construct an enumeration tree for the "p! problem" in which f is symmetric and unimodal. Before proceeding with the theorem however, we would remark that the figures and examples used to illustrate the concepts and results of this section have been deliberately chosen with ease of construction and computation in mind. Specifically, we take f to be a function like that of Figure 2.5, and do so while understanding that with respect to the analysis presented herein, f need not be so convenient, but rather need only be nonnegative, integrable, unimodal and symmetric. Note for example, that the function in Figure 2.5 is such that $\alpha$ is unique and equal to c/2, but realize that this need not be so, as is clearly shown in Figure 2.4.

Theorem 2.11. Suppose that the demand function f for Problem CP is nonnegative, integrable, unimodal and symmetric on $[0,c]$. Then, in determining an optimal solution to Problem CP, it is sufficient to restrict attention to those orderings/permutations of the p facilities which give solutions (per Corollary 2.6) satisfying the following condition in addition to that of Corollary 2.10: if a facility M straddles, then it has maximum capacity.

Proof. Let $s_M$ denote the capacity of the straddling facility, and $s_L$, $s_R$ the capacities of those facilities to its immediate left and right respectively. By Theorem 2.8, we have that the facilities to the left (right) of the straddling facility are ordered according to nondecreasing (nonincreasing) capacity. Hence, if $s_M > \max\{s_L, s_R\}$,

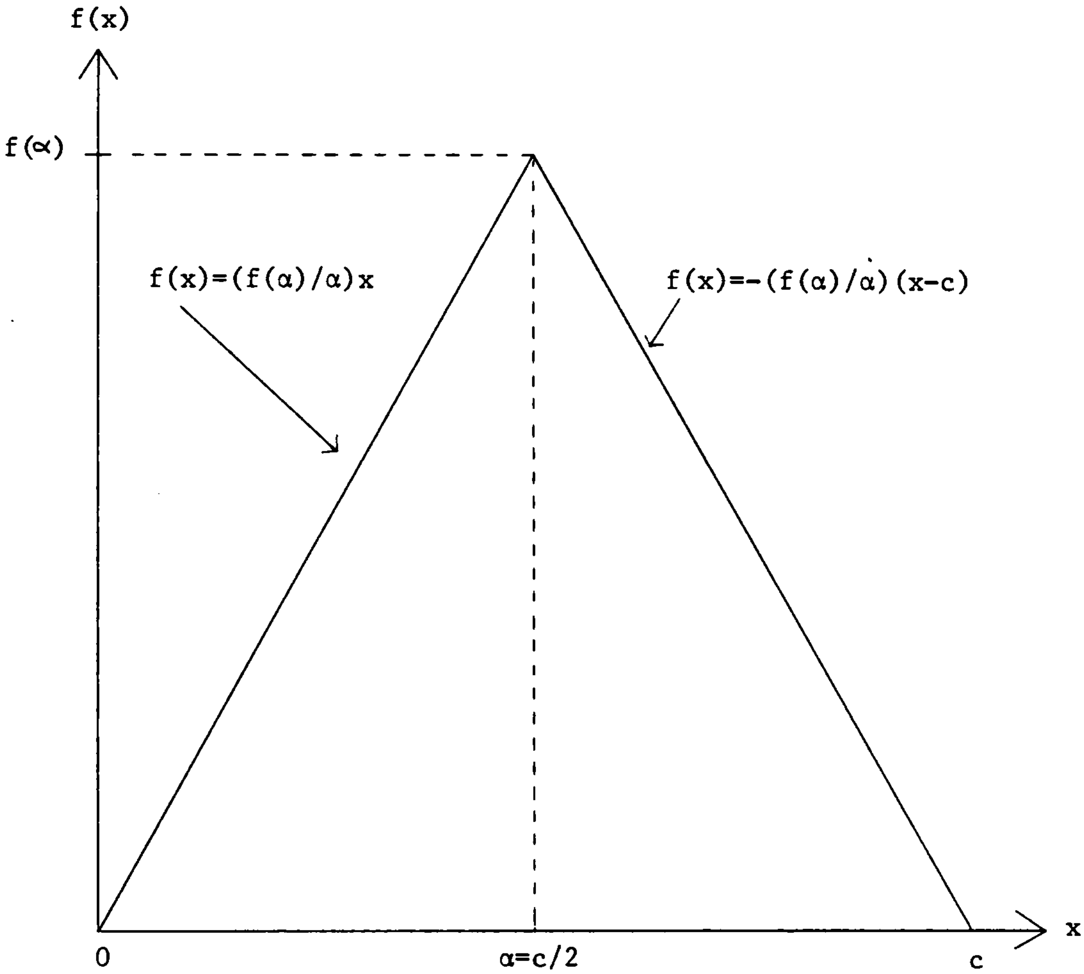



Figure 2.5. A Symmetric, Unimodal, Demand Function

then $s_M$ is the maximum capacity and we are done. Thus without loss of generality, let us assume that $s_M < s_R$.

Figures 2.6 and 2.7 are provided to help clarify the statement and proof of this theorem. In particular, Figures 2.6 and 2.7a depict the situation described above, whereas Figure 2.7b depicts the situa--tion after the straddling facility and the facility to its immediate right are interchanged. Note that F(x) is convex on $[0,\alpha=c/2]$ and concave on $[\alpha,c]$. This follows from Lemma 2.9 and the form of f(x) on $[0,\alpha]$ and $[\alpha,c]$.

The proof of this theorem is similiar to that of Theorem 2.8, in that the geometric representation of the transportation costs associated with the service provided by the facilities of interest, is the basis for the proof. However, the proof is a bit more difficult than that of Theorem 2.8, due to F(x) switching from being convex to concave at the "cc-point", $\alpha = c/2$. Under the assumption that $s_M < s_R$, the objective of this proof is to establish that $A_1 > A_2$, where $A_1$ and $A_2$ are as shown in Figure 2.7. From this figure, we get that

$$A_1 = \left[(s_M/2)G(\Pi+s_M/2) - \int_{\Pi}^{(\Pi+s_M/2)} G(y)dy\right] + \left[\int_{(\Pi+s_M/2)}^{(\Pi+s_M)} G(y)dy - (s_M/2)G(\Pi+s_M/2)\right]$$

$$+ \left[(s_R/2)G(\Pi+s_M+s_R/2) - \int_{(\Pi+s_M)}^{(\Pi+s_M+s_R/2)} G(y)dy\right] + \left[\int_{(\Pi+s_M+s_R/2)}^{(\Pi+s_M+s_R)} G(y)dy-(s_R/2)G(\Pi+s_M+s_R/2)\right],$$

where $G = F^{-1}$ and where each of the four summands is equal to the area of one of the four regions comprising $A_1$. Simplifying, we get

$$A_1 = -\int_{\Pi}^{(\Pi+s_M/2)} G(y)dy + \int_{(\Pi+s_M/2)}^{(\Pi+s_M)} G(y)dy - \int_{(\Pi+s_M)}^{(\Pi+s_M+s_R/2)} G(y)dy + \int_{(\Pi+s_M+s_R/2)}^{(\Pi+s_M+s_R)} G(y)dy.$$

Similiarly,

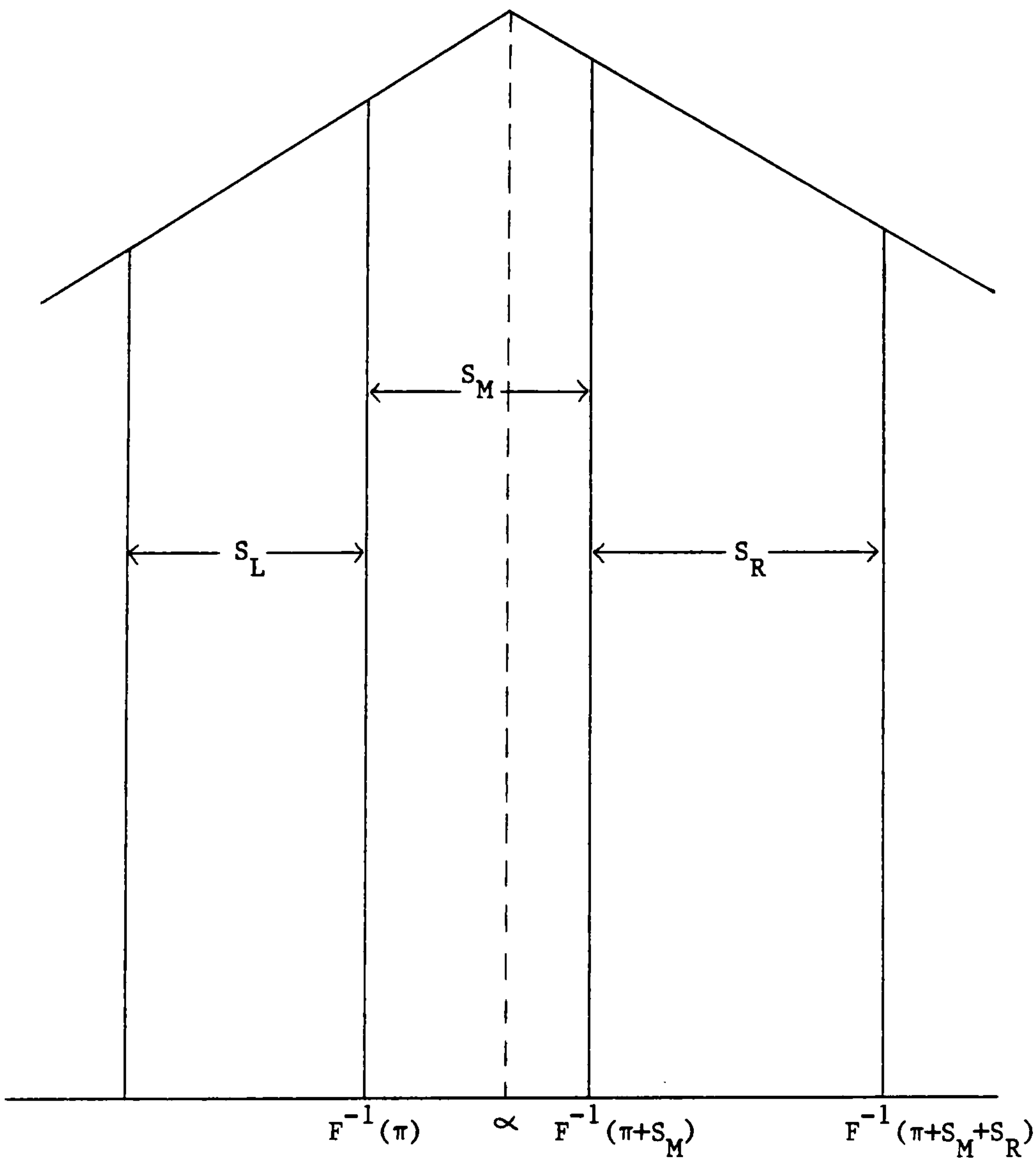



Figure 2.6. (See Theorem 2.11)

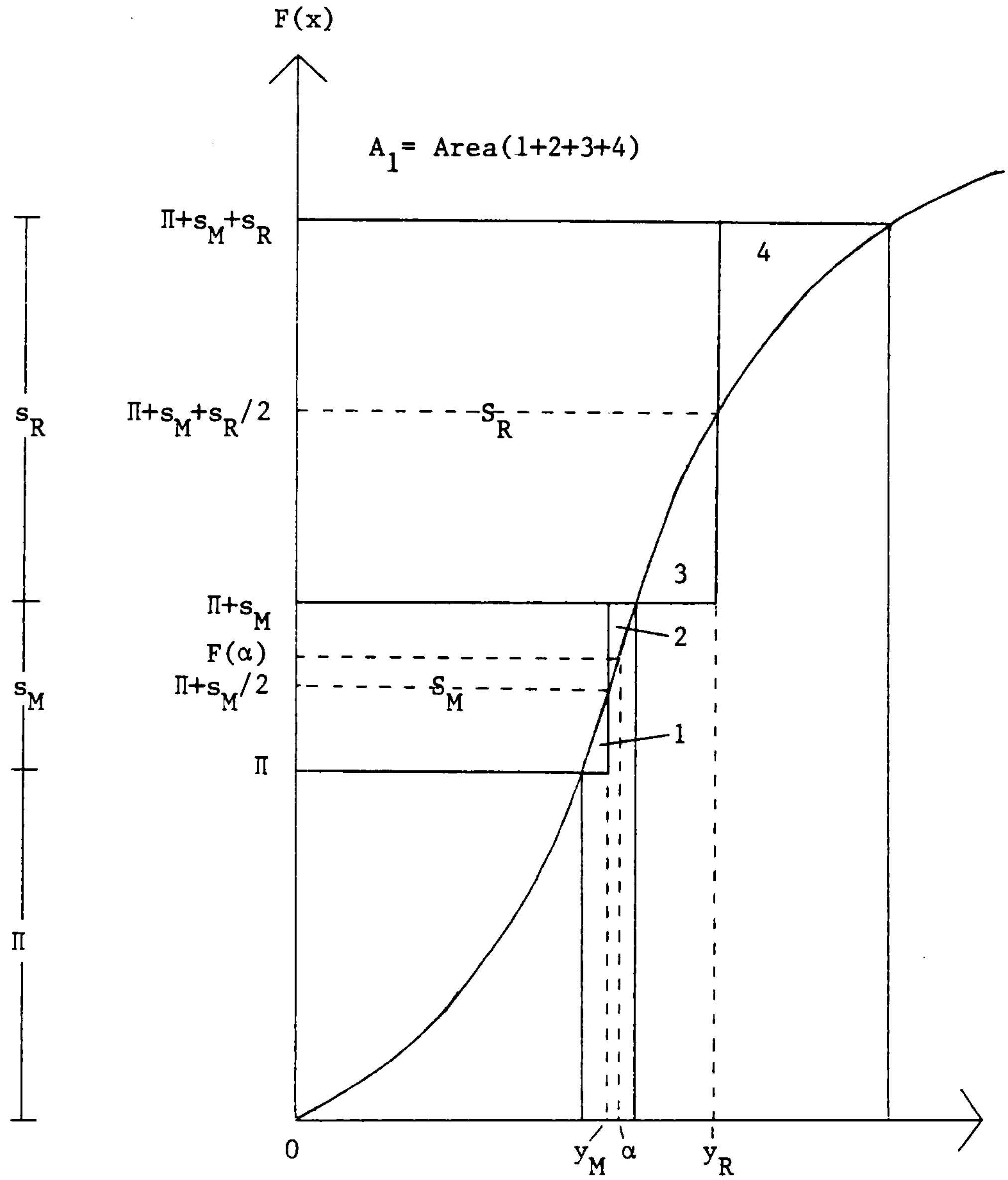



Figure 2.7a. (See Theorem 2.11)

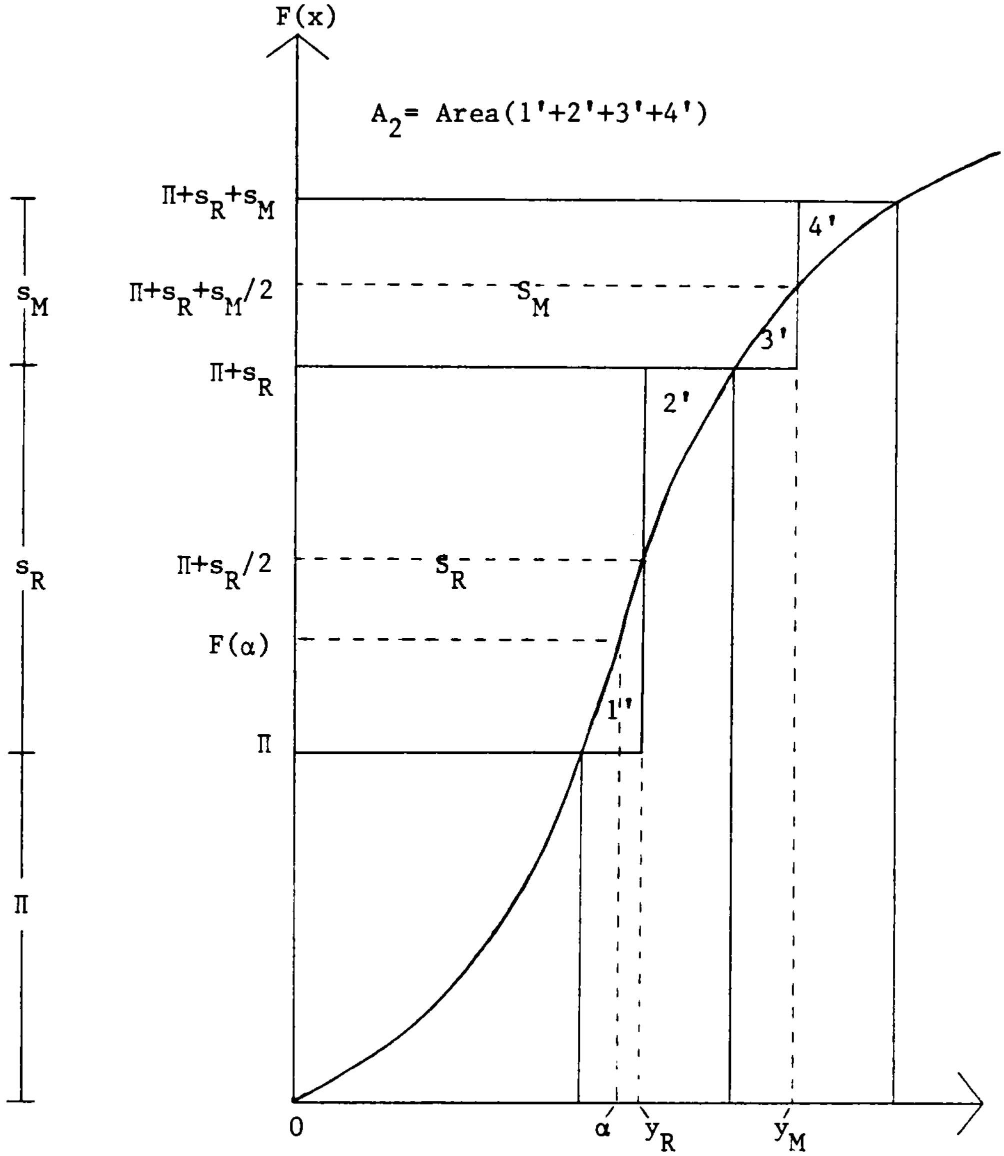



Figure 2.7b. (See Theorem 2.11)

$$A_2 = -\int_{\Pi}^{(\Pi+s_R/2)} G(y)dy + \int_{(\Pi+s_R/2)}^{(\Pi+s_R)} G(y)dy - \int_{(\Pi+s_R)}^{(\Pi+s_R+s_M/2)} G(y)dy + \int_{(\Pi+s_R+s_M/2)}^{(\Pi+s_R+s_M)} G(y)dy.$$

Recalling our assumption that $s_M < s_R$, we consider two cases; Case (i), $s_M < s_R/2$ and Case (ii), $s_M > s_R/2$.

<u>Case (i)</u>, $s_M < s_R/2$. Because $F(x)(G(y))$ has a cc-point at $x = \alpha = c/2$ $(y=F(\alpha))$, we need to consider two subcases; Case (ia), $\Pi < F(\alpha) < \Pi + s_M/2$ and Case (ib), $\Pi + s_M/2 < F(\alpha) < \Pi + s_M$. Figure 2.8 provides a pictorial representation of these subcases. The reader should note that Figure 2.8 assumes $s_M < s_R/2$, as does the following mathematics used in establishing that $A_1 > A_2$. However, the very same mathematics will also handle the case where $s_M = s_R/2$. We remark that Figure 2.6, and hence Figure 2.7a, just so happen to depict Case (ib). This should not bother the reader, since our current expressions for $A_1$ and $A_2$ are independent of whether $s_M < s_R/2$ or $s_M > s_R/2$. Only now do we incorporate the relationship between $s_M$ and $s_R$ into the expressions for $A_1$ and $A_2$.

<u>Under Case (ia)</u>, i.e. $s_M < s_R/2$ and $\Pi < F(\alpha) < \Pi + s_M/2$, $A_1$ and $A_2$ can be written as

$$A_1 = -\int_{\Pi}^{F(\alpha)} - \int_{F(\alpha)}^{(\Pi+s_M/2)} + \int_{(\Pi+s_M/2)}^{(\Pi+s_M)} - \int_{(\Pi+s_M)}^{(\alpha+s_R/2)} - \int_{(\Pi+s_R/2)}^{(\Pi+s_M+s_R/2)} + \int_{(\Pi+s_M+s_R/2)}^{(\Pi+s_R)} + \int_{(\Pi+s_R)}^{(\Pi+s_R+s_M/2)}$$
$$+ \int_{(\Pi+s_R+s_M/2)}^{(\Pi+s_M+s_R)}$$

and

$$A_2 = -\int_{\Pi}^{F(\alpha)} - \int_{F(\alpha)}^{(\Pi+s_M/2)} - \int_{(\Pi+s_M/2)}^{(\Pi+s_M)} - \int_{(\Pi+s_M)}^{(\Pi+s_R/2)} + \int_{(\Pi+s_R/2)}^{(\Pi+s_M+s_R/2)} + \int_{(\Pi+s_M+s_R/2)}^{(\Pi+s_R)} - \int_{(\Pi+s_R)}^{(\Pi+s_R+s_M/2)}$$

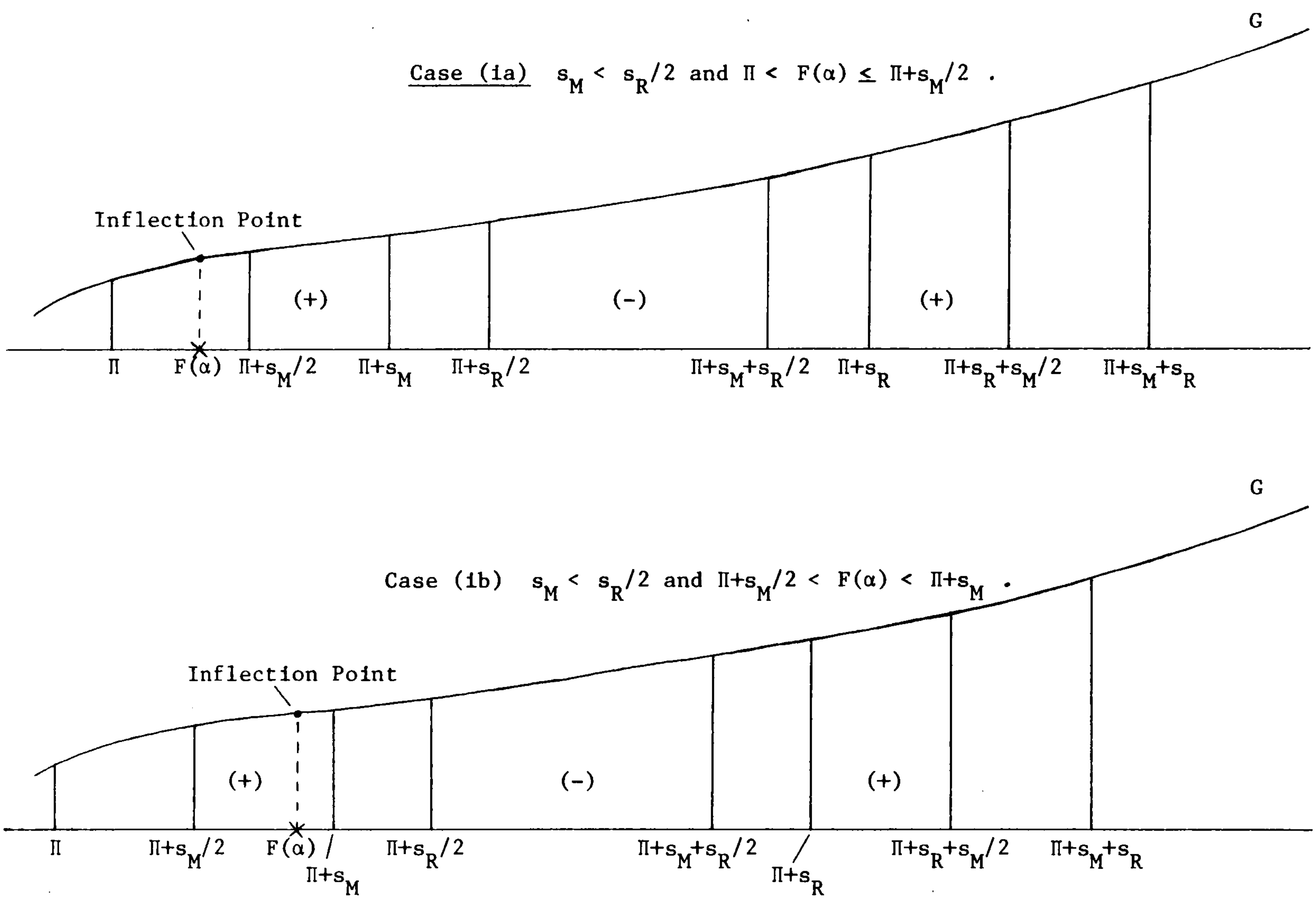



Figure 2.8 (See Theorem 2.11)

$$+ \int_{(\Pi+s_R+s_M/2)}^{(\Pi+s_M+s_R)} ,$$

where the integrand of each of these integrals is G(y)dy. It follows that

$$(A_1-A_2)/2 = \int_{(\Pi+s_M/2)}^{(\Pi+s_M)} G(y)dy - \int_{(\Pi+s_R/2)}^{(\Pi+s_M+s_R/2)} G(y)dy + \int_{(\Pi+s_R)}^{(\Pi+s_R+s_M/2)} G(y)dy .$$

We will show that $A_1 > A_2$ by showing that $(A_1-A_2)/2 > 0$, and this will be accomplished by using underestimates of the two positive signed integrals and an overestimate of the negative integral.

Notice that since $F(\alpha) < \Pi + s_M/2$, each of the three integrals in the above expression for $(A_1-A_2)/2$ is evaluated over an interval on which G(y) is convex. Therefore, we can underestimate the first and third integrals by the areas of the polygons formed using support functionals to G(y) at $(\Pi+s_M,G(\Pi+s_M))$ and $(\Pi+s_R,G(\Pi+s_R))$, respectively, as depicted in Figures 2.9a and 2.9b. Doing so, we get that

$$\int_{(\Pi+s_M/2)}^{(\Pi+s_M)} G(y)dy > -(s_M^2/8)G^+(\Pi+s_M) + (s_M/2)G(\Pi+s_M)$$

and

$$\int_{(\Pi+s_R)}^{(\Pi+s_R+s_M/2)} G(y)dy > (s_M^2/8)G^+(\Pi+s_R) + (s_M/2)G(\Pi+s_R) .$$

Similiarly, we obtain an overestimate to the second integral by constructing a secant line through the points $(\Pi+s_R/2,G(\Pi+s_R/2))$ and $(\Pi+s_M+s_R/2,G(\Pi+s_M+s_R/2))$, and then computing the area of the resulting polygon. This gives us that

$$-\int_{(\Pi+s_R/2)}^{(\Pi+s_M+s_R/2)} G(y)dy > -(s_M/2)G(\Pi+s_M+s_R/2) - (s_M/2)G(\Pi+s_R/2) .$$

Using these lower bounds, we can write

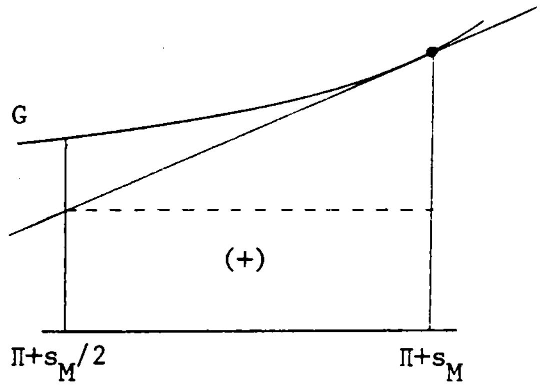



Figure 2.9a.

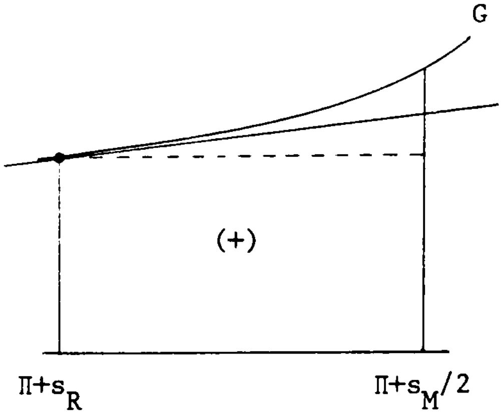



Figure 2.9b.

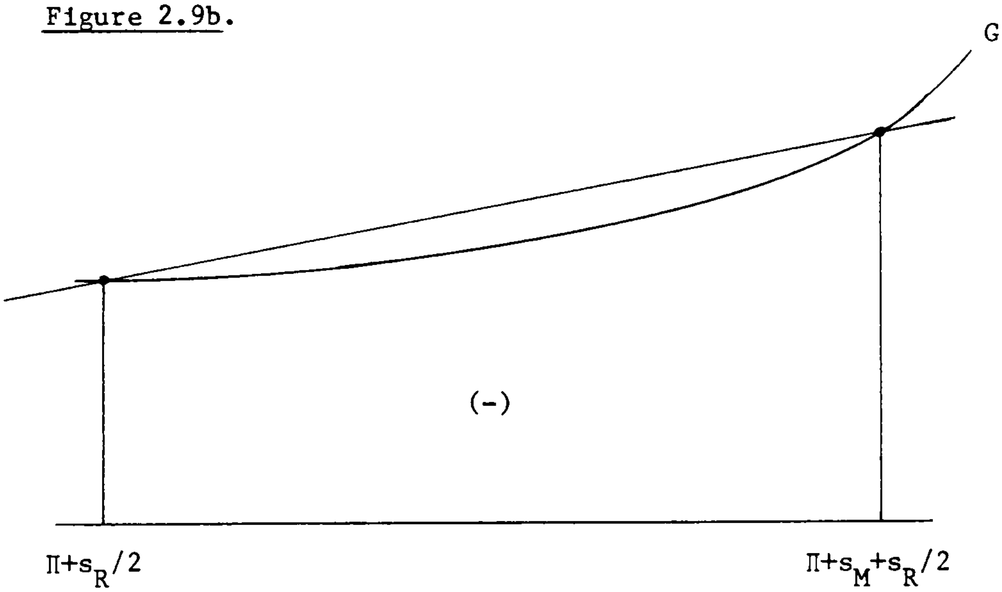



Figure 2.9c.

Figure 2.9. (See Theorem 2.11)

$$(A_1-A_2)/2 > (s_M^2/8)[G^+(\Pi+s_R) - G^+(\Pi+s_M)]$$

$$+ (s_M/2)[G(\Pi+s_R) - G(\Pi+s_M+s_R/2) + G(\Pi+s_M) - G(\Pi+s_R/2)] \ .$$

Now G convex implies that $G^+(\Pi+s_R) > G^+(\Pi+s_M)$, i.e. the term $(s_M^2/8)[G^+(\Pi+s_R) - G^+(\Pi+s_M)] > 0$ .

Concentrating on the second summand of the above expression, we note that because of the convexity of G, the slope of the secant line through the points $(\Pi+s_M, G(\Pi+s_M))$ and $(\Pi+s_R/2, G(\Pi+s_R/2))$ is less than or equal to the slope of the secant line through the points $(\Pi+s_M+s_R/2, G(\Pi+s_M+s_R/2))$ and $(\Pi+s_R, G(\Pi+s_R))$, i.e.

$$[G(\Pi+s_R/2)-G(\Pi+s_M)]/(s_R/2-s_M) < [G(\Pi+s_R)-G(\Pi+s_M+s_R/2)]/(s_R/2-s_M) \ .$$

But since $s_M < s_R/2$, $(s_R/2-s_M) > 0$ and so $G(\Pi+s_R/2) - G(\Pi+s_M) < G(\Pi+s_R) - G(\Pi+s_M+s_R/2)$ .

Therefore, $0 < G(\Pi+s_R) - G(\Pi+s_M+s_R/2) + G(\Pi+s_M) - G(\Pi+s_R/2)$ and thus we have that $(A_1-A_2)/2 > 0$, thereby establishing that $A_1 > A_2$. This proves Theorem 2.11 for Case (ia).

<u>Under Case (ib)</u>, i.e. $s_M < s_R/2$ and $\Pi + s_M/2 < F(\alpha) < \Pi + s_M$, $A_1$ and $A_2$ can be written as

$$A_1 = -\int_{\Pi}^{(\Pi+s_M/2)} + \int_{(\Pi+s_M/2)}^{F(\alpha)} + \int_{F(\alpha)}^{(\Pi+s_M)} - \int_{\Pi+s_M}^{(\Pi+s_R/2)} - \int_{(\Pi+s_R/2)}^{(\Pi+s_M+s_R/2)} + \int_{(\Pi+s_M+s_R/2)}^{(\Pi+s_R)} + \int_{\Pi+s_R}^{(\Pi+s_R+s_M/2)}$$

$$+ \int_{(\Pi+s_R+s_M/2)}^{(\Pi+s_M+s_R)}$$

and

$$A_2 = -\int_{\Pi}^{(\Pi+s_M/2)} - \int_{(\Pi+s_M/2)}^{F(\alpha)} - \int_{F(\alpha)}^{(\Pi+s_M)} - \int_{(\Pi+s_M)}^{(\Pi+s_R/2)} + \int_{(\Pi+s_R/2)}^{(\Pi+s_M+s_R/2)} + \int_{(\Pi+s_M+s_R/2)}^{(\Pi+s_R)} - \int_{(\Pi+s_R)}^{(\Pi+s_R+s_M/2)}$$

$$+ \int_{(\Pi+s_R+s_M/2)}^{(\Pi+s_M+s_R)} .$$

Therefore,

$$(A_1-A_2)/2 = \int_{(\Pi+s_M/2)}^{F(\alpha)} G(y)dy + \int_{F(\alpha)}^{(\Pi+s_M)} G(y)dy - \int_{(\Pi+s_R/2)}^{(\Pi+s_M+s_R/2)} G(y)dy + \int_{\Pi+s_R}^{(\Pi+s_R+s_M/2)} G(y)dy \,.$$

Unlike Case (ia), G is not convex on each of the intervals of integration of the above four integrals. In particular, G is concave on $[\Pi+s_M/2,F(\alpha)]$, and convex on the other three intervals of integration. Thus we consider two cases under Case (ib): $F(\alpha)\varepsilon[\Pi+(3/4)s_M,\Pi+s_M)$ and $F(\alpha)\varepsilon(\Pi+s_M/2,\Pi+(3/4)s_M)$.

<u>Suppose $F(\alpha)\varepsilon[\Pi+(3/4)s_M,\Pi+s_M)$.</u>

Because of the symmetry of f, the function $G = F^{-1}$ possesses a sort of negative symmetry of its own. Specifically, $G(F(\alpha)+\theta) - G(F(\alpha)) = G(F(\alpha)) - G(F(\alpha)-\theta)$ for all $\theta\varepsilon[0,F(\alpha)]$.

Because of $F(\alpha)$'s assumed location and the symmetry of G, the area under the secant line through the points $(\Pi+s_M/2,G(\Pi+s_M/2))$ and $(\Pi+s_M,G(\Pi+s_M))$ is an underestimate of $\int_{(\Pi+s_M/2)}^{(\Pi+s_M)} G(y)dy$. Thus we can write

$$\int_{(\Pi+s_M/2)}^{(\Pi+s_M)} G(y)dy = \int_{(\Pi+s_M/2)}^{F(\alpha)} G(y)dy + \int_{F(\alpha)}^{(\Pi+s_M)} G(y)dy > (s_M/4)G(\Pi+s_M/2) + (s_M/4)G(\Pi+s_M).$$

Now since G is convex on $[\Pi+s_R/2,\Pi+s_M+s_R/2]$ and $[\Pi+s_R,\Pi+s_R+s_M/2]$, we can use the same overestimate and underestimate of $\int_{(\Pi+s_R/2)}^{(\Pi+s_M+s_R/2)} G(y)dy$ and $\int_{\Pi+s_R}^{(\Pi+s_R+s_M/2)} G(y)dy$, respectively, that was used in Case (ia). Doing so, we have that

$$(A_1-A_2)/2 > \left[(s_M/4)G(\Pi+s_M/2)+(s_M/4)G(\Pi+s_M)\right] + \left[-(s_M/2)G(\Pi+s_M+s_R/2) - (s_M/2)G(\Pi+s_R/2)\right] + \left[(s_M^2/8)G^+(\Pi+s_R)+(s_M/2)G(\Pi+s_R)\right] \,.$$

Now since $G^+$ is non-decreasing to the right of $y = F(\alpha)$, and because of the symmetry of G about $y = F(\alpha)$, we have

$$G^{+}(\Pi+s_R) > [G(\Pi+s_M) - G(\Pi+s_M/2)]/(s_M/2) \ .$$

Therefore, $(s_M^2/8)G^{+}(\Pi+s_R) + (s_M/4)G(\Pi+s_M/2) > (s_M/4)G(\Pi+s_M)$, and thus, $(A_1-A_2)/2 > (s_M/2)G(\Pi+s_M) + (s_M/2)G(\Pi+s_R) + [-(s_M/2)G(\Pi+s_M+s_R/2) - (s_M/2)G(\Pi+s_R/2)]$.

By the convexity of G to the right of $y = F(\alpha)$,

$$G(\Pi+s_R) - G(\Pi+s_M+s_R/2) > G(\Pi+s_R/2) - G(\Pi+s_M) > 0 \ ,$$

and so

$$G(\Pi+s_R) - G(\Pi+s_M+s_R/2) - G(\Pi+s_R/2) + G(\Pi+s_M) > 0 \ .$$

Therefore,

$$(A_1-A_2)/2 > 0 \quad \text{and so} \quad A_1 > A_2 \ .$$

Next, suppose $F(\alpha)\varepsilon(\Pi+s_M/2,\Pi+(3/4)s_M)$.

Because of the location of $y = F(\alpha)$ and the symmetry of G, the area of the polygon formed by the support functional to G at $(\Pi+s_M,G(\Pi+s_M))$ can be used as an underestimate of $\int_{(\Pi+s_M/2)}^{(\Pi+s_M)} G(y)dy$. That is,

$$\int_{(\Pi+s_M/2)}^{(\Pi+s_M)} G(y)dy = \int_{(\Pi+s_M/2)}^{F(\alpha)} G(y)dy + \int_{F(\alpha)}^{(\Pi+s_M)} G(y)dy > (s_M/2)G(\Pi+s_M) - (s_M^2/8)G^{+}(\Pi+s_M).$$

Again, because G is convex on $[\Pi+s_R/2,\Pi+s_M+s_R/2]$ and $[\Pi+s_R,\Pi+s_R+s_M/2]$, we can use the same overestimate and underestimate of $\int_{(\Pi+s_R/2)}^{(\Pi+s_M+s_R/2)} G(y)dy$ and $\int_{\Pi+s_R}^{(\Pi+s_R+s_M/2)} G(y)dy$, respectively, that was used in Case (ia).

Doing so, we have that

$$(A_1-A_2)/2 > [(s_M/2)G(\Pi+s_M) - (s_M^2/8)G^{+}(\Pi+s_M)] + [-(s_M/2) G(\Pi+s_M+s_R/2)-(s_M/2)G(\Pi+s_R/2)] + [(s_M^2/8)G^{+}(\Pi+s_R) + (s_M/2)G(\Pi+s_R)] \ .$$

But $G^{+}$ nondecreasing gives us that

$(s_M^2/8)G^{+}(\Pi+s_R) - (s_M^2/8)G^{+}(\Pi+s_M) > 0$ , and also that

$$[G(\Pi+s_R) - G(\Pi+s_M+s_R/2)]/(s_R/2-s_M) > [G(\Pi+s_R/2)-G(\Pi+s_M)]/(s_R/2-s_M).$$

From these it follows that $(A_1-A_2)/2 > 0$ and so $A_1 > A_2$, thereby proving the theorem for Case (ib).

This proves Theorem 2.11 for Case (i).

Because the proof of Case (ii) is very similar to that of Case (i) in both approach and scope, it will not be given here and we declare Theorem 2.11, proven. $\square$

We remark that the condition/property established in the above theorem does not hold for non-symmetric f. The following example shows this.

<u>Example I</u>: Let $f(x) = \begin{cases} x & , x\varepsilon[0,10] \\ -0.0001x + 10.001 & , x\varepsilon[10,20] \end{cases}$

and p = 3 with capacities $s_1$=10, $s_2$=50 and $s_3$=89.995. Note that

$$\text{Total Demand} = \int_0^{20} f(x)dx = s_1+s_2+s_3 \quad .$$

Figure 2.10 shows the locations and allocations (per Corollary 2.6) of two different orderings of the facilities, each of which satisfies Corollary 2.10. In addition, straddling occurs in both orderings, with Figure 2.10a showing the straddling facility to be the one having maximum capacity, and Figure 2.10b showing otherwise. Contrary to Theorem 2.11, the ordering of Figure 2.10b is the least costly (and is the unique optimum), having a total cost of 288.91000, whereas that of Figure 2.10a costs 290.20633.

The conditions/properties of Theorem 2.8/Corollary 2.10 and Theorem 2.11 lead us to investigate the question of whether an optimal ordering for a problem whose demand function satisfies the requirements of Theorem 2.11 (i.e. symmetric, unimodal, etc.), would be obtained if the

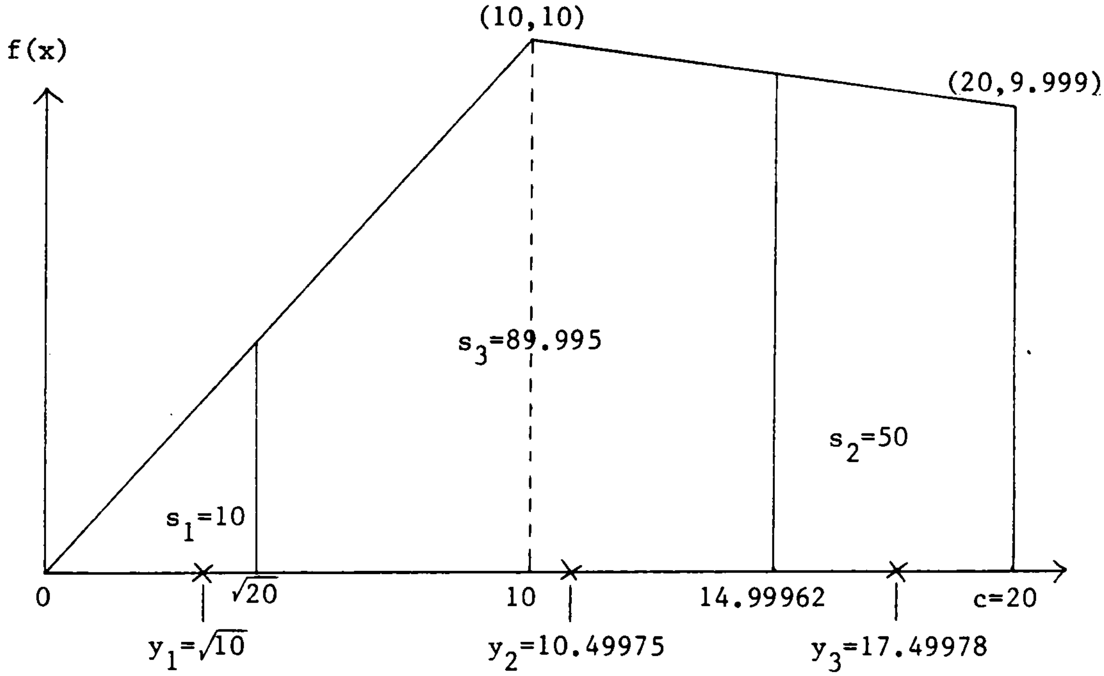



Figure 2.10a.

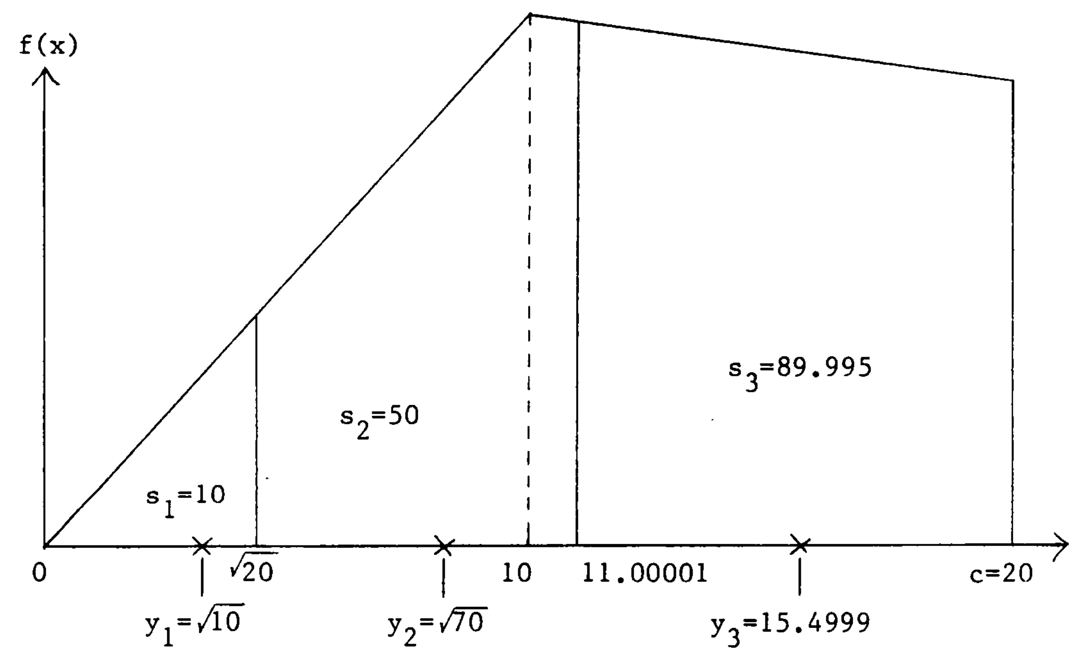



Figure 2.10b.

Figure 2.10. (See Example I)

facilities were located (according to Corollary 2.6) in an alternating manner left and right of $\alpha = c/2$, say, in order of increasing capacity, and so as to satisfy Corollary 2.10. Figure 2.11 is provided to further explain what is meant. It depicts a problem having a total demand of one-hundred (100), and having six facilities whose capacities have been numbered so that $s_1=5 < s_2=10 < s_3=15 < s_4=20 < s_5=24 < s_6=26$. Notice that in both Figure 2.11a and 2.11b, the orderings satisfy Corollary 2.10 and straddling occurs. As an aside, we remark that since the ordering of Figure 2.11b does not satisfy Theorem 2.11, we would not bother to consider such an ordering. With respect to the issue at hand, we note that the ordering of Figure 2.11a is an alternating ordering, whereas that of Figure 2.11b is not.

The outcome of our investigation of the alternating facility question can best be summarized by the following theorem.

Theorem 2.12. Given a demand function f satisfying the requirements of Theorem 2.11, and $p = 3$ facilities having capacities $s_1 < s_2 < s_3$, the alternating ordering is optimal. For $p = 4$, such an ordering need not be optimal.

Proof. A complete proof of the $p = 3$ part of this theorem is quite lengthy since it requires arguments exactly like those used in the proof of Theorems 2.8 and 2.11. Therefore, we simply outline what must be done to prove this part of the theorem. The $p = 4$ portion is established by an example showing that the alternating ordering is non-optimal.

Let $p = 3$ and $s_1 < s_2 < s_3$. A complete treatment of the problem would have to consider each of the cases; (1) $s_1 = s_2 = s_3$, (2) $s_1 < s_2 < s_3$,

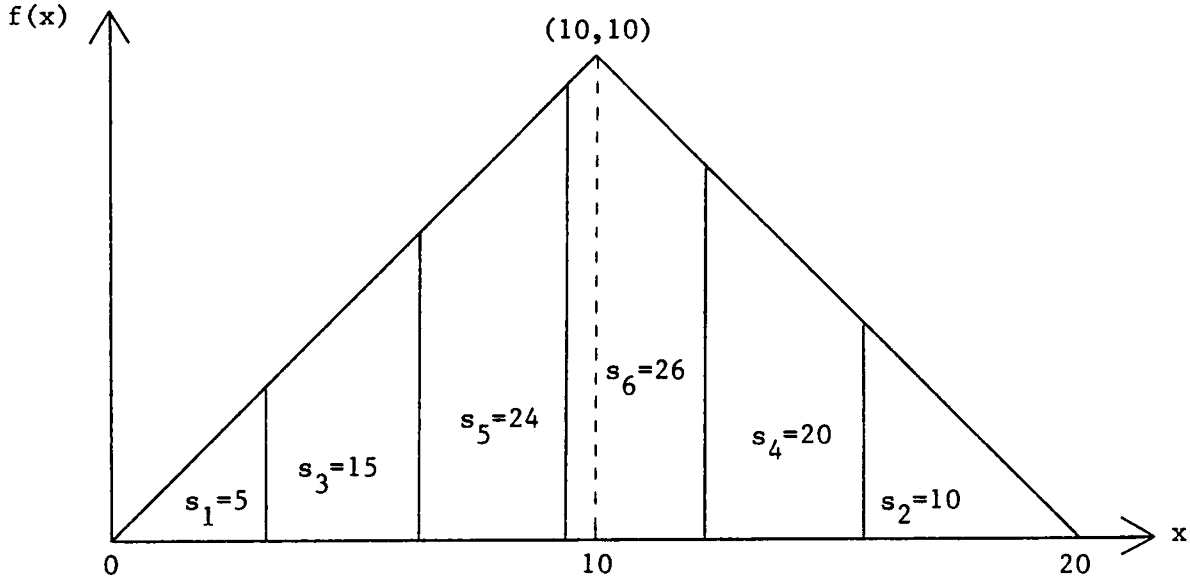



Figure 2.11a. (Alternating)

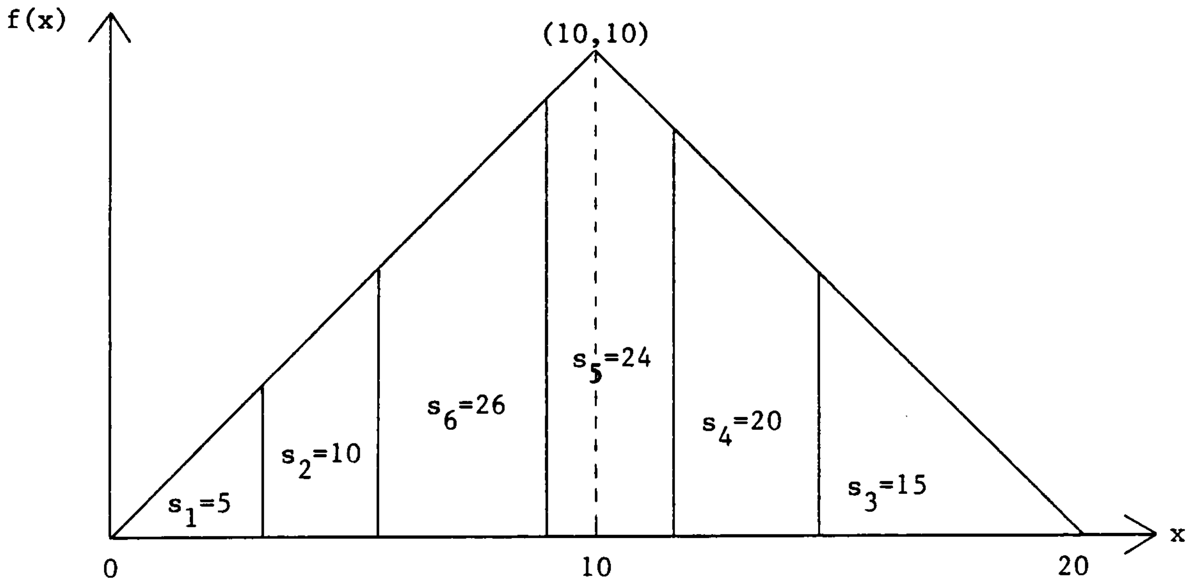



Figure 2.11b. (Non-Alternating)

Figure 2.11. Alternating/Non-Alternating Orderings

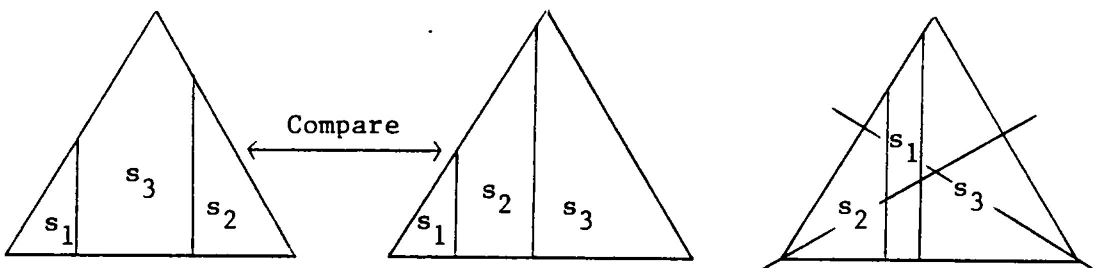
(3) $s_1 = s_2 < s_3$, and (4) $s_1 < s_2 = s_3$. Clearly, the theorem holds vacuously for Case (1) and thus only the remaining three cases would need to be examined.

(Note: a schematic is used to show the different orderings within each case. The letter D stands for the total demand.)

We demonstrate the approach taken and work required in examining Case (2).

<u>Case (2)</u> i.e. $s_1 < s_2 < s_3$.

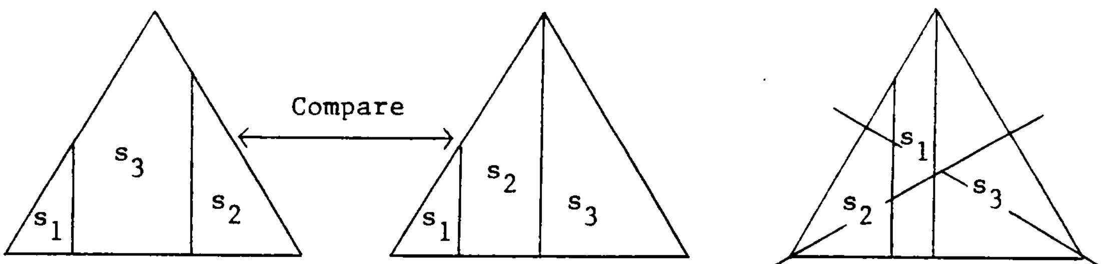
<u>(2a)</u> $s_1 + s_2 < D/2 \rightarrow s_1 < D/2,\ s_2 < D/2,\ s_3 > D/2$




2a1 (Alternating) 2a2 2a3

Remarks: i) discard ordering 2a3 via Corollary 2.10.
ii) need to compare 2a1 and 2a2 to determine if 2a1 is less costly.

<u>(2b)</u> $s_1 + s_2 = D/2 \rightarrow s_1 < D/2,\ s_2 < D/2,\ s_3 = D/2$

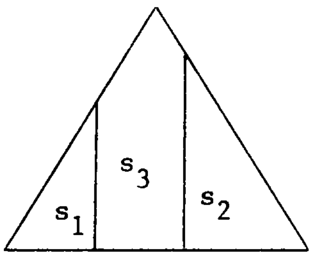



2b1 2b2 2b3

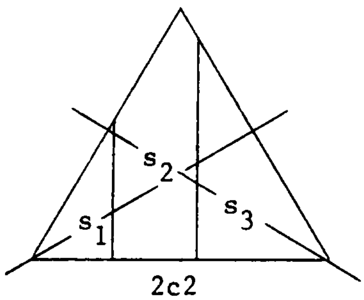
(Alternating)

Remarks: i) discard ordering 2b3 via Corollary 2.10.
ii) need to compare 2b1 and 2b2 to determine if 2b1 is less costly.

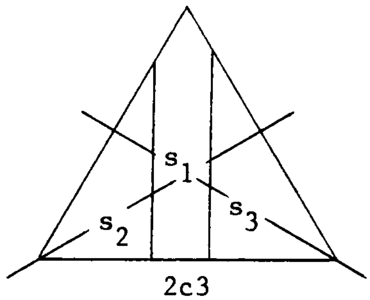
<u>(2c)</u> $s_1 + s_2 > D/2 \rightarrow s_3 < D/2 \rightarrow s_1 < D/2,\ s_2 < D/2$




2c1
(Alternating)




2c2




2c3

Remarks: i) discard both 2c2 and 2c3 via Theorem 2.11, i.e. max-capacity straddles.
ii) thus under case (2c), the alternating is optimal and the theorem holds.

Now, to complete Case (2), one needs to determine the least costly ordering of each of the pairs (2a1,2a2) and (2b1,2b2). To do so requires an argument like that used to prove Theorem 2.8, and hence the reader can appreciate why we only outline the proof of this part of the theorem. The writer has, however, drudged through the necessary arguments, and thus can state that the alternating ordering is optimal for each of the cases 2a, 2b and 2c. The same approach was used to establish the theorem for cases (3) and (4).

The following example shows that for p = 4, the alternating ordering need not be optimal.

<u>Example II</u>: Let $f(x) = \begin{cases} 10 & , x\varepsilon[0,3]\cup[4,7] \\ 180x - 530 & , x\varepsilon[3,3.5] \\ -180x + 730 & , x\varepsilon[3.5,4] \end{cases}$

and p = 4 with capacities $s_1$=10, $s_2$=20, $s_3$=30, and $s_4$=55.

Figure 2.12 shows the locations and allocations (per Corollary 2.6) of two different orderings of the facilities, each of which satisfies Corollary 2.10 and Theorem 2.11. Note that the ordering of Figure 2.12a is alternating, whereas that of 2.12b is not. The costs of the orderings depicted in Figures 2.12a and 2.12b are 34.92939 and 32.50000, respectively, thus establishing the p = 4 part of this theorem. $\square$

As was mentioned earlier, our purpose in presenting results such as those of Theorem 2.8/Corollary 2.10 and Theorem 2.11, is to enable the efficient construction of an enumeration tree that can be used in tackling the "p! problem". This being the case, we are now in a position to demonstrate how these results can be used to reduce the number of orderings that need to be considered in searching for an optimal solution to Problem CP, where f satisfies the requirements of Theorem 2.11.

Given such an f as that of Theorem 2.11, and p facilities having known distinct capacities, we remark that there are p! different orderings of the facilities, but only p!/2 of them need to be considered, due to the symmetry of f. We begin construction of an enumeration tree by numbering/labeling the facilities so that $s_1 < s_2 < \dots < s_p$. Then by employing a "left-right" branching criterion, where "left" refers to $[0,\alpha \equiv c/2]$ and "right" to $[\alpha,c]$, we arbitrarily (due to symmetry of f) branch at node $\theta$ (Level 0) according to placement of the facilitiy having capacity $s_1$, to the left of $\alpha$, and per Corollary 2.10, so as to serve the interval $I_1 = [0,\alpha_1]$. The resulting node constitutes Level 1 of the tree. Level 2 is obtained by branching at the single

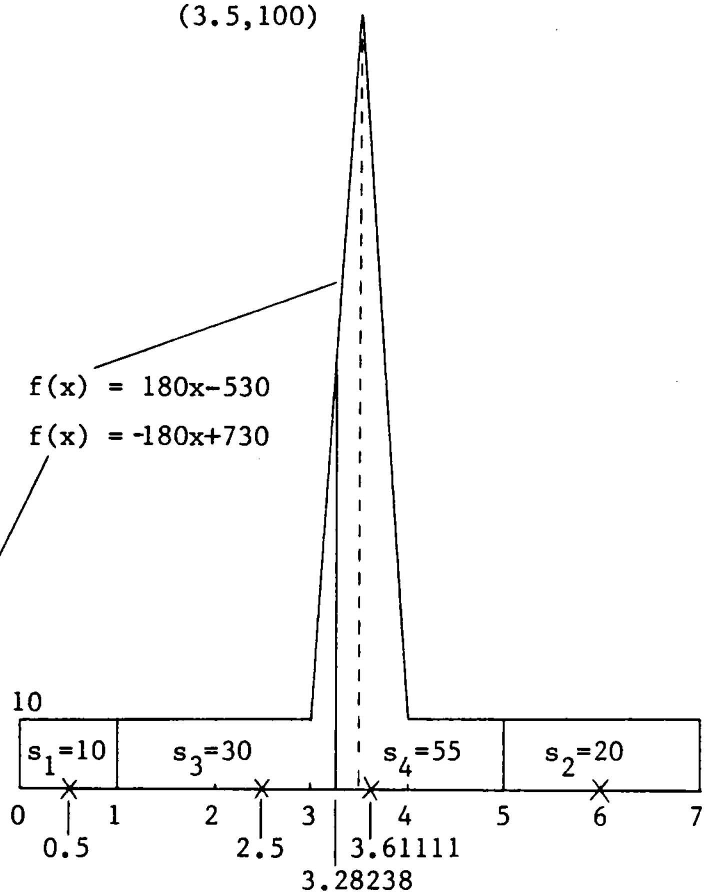



Figure 2.12a.

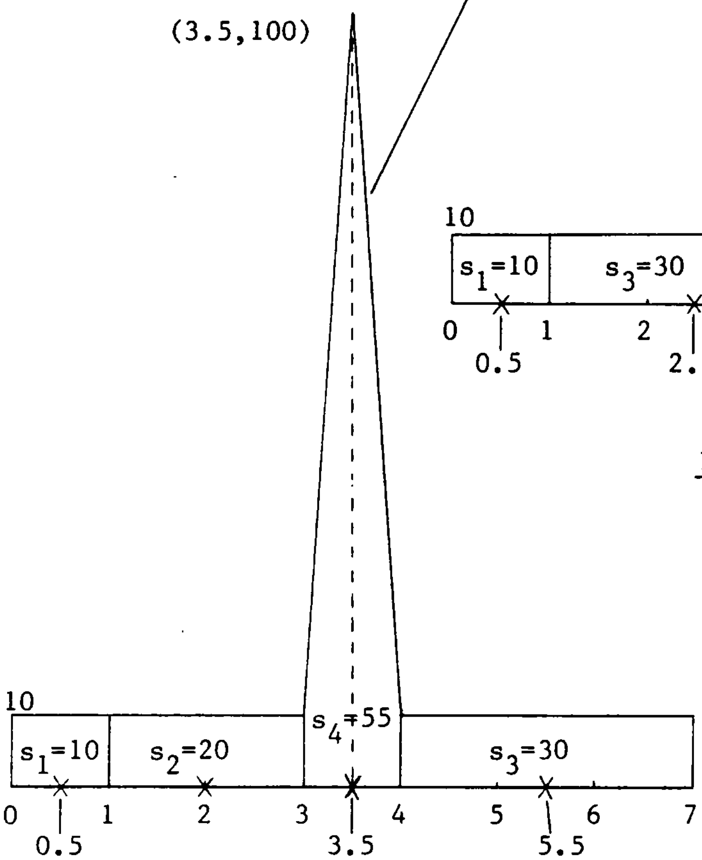



Figure 2.12b.

Figure 2.12. (See Example II)

node of Level 1 according to whether the facility having capacity $s_2$ is to be located to the left or right of $\alpha$, and so as to satisfy Corollary 2.10. At each of the two nodes in Level 2, branching is initiated with respect to $s_3$, etc. Construction of the tree continues as such along each branch of the tree until such time as a node is reached at which branching with respect to some $s_i$, $i < p$, would result in straddling, thereby violating Theorem 2.11. Such a node would not be branched, for its completion is unique and readily known. Of course, if at some intermediate stage/level of the tree one were to encounter a node for which the remaining unserved demand was symmetric, then he/she would only need to branch to the left or the right (but not both) as was the case at node $\theta$. Additionally, if at such a node there remained only three facilities still to be located, then the completion of that node is known immediately from Theorem 2.12.

The following example illustrates how Corollary 2.10 and Theorems 2.11 and 2.12 are used to construct an enumeration tree.

<u>Example (Alpha)</u>: Let $f(x) = \begin{cases} x & , x\varepsilon[0,12] \\ 24- x & , x\varepsilon[12,24] \end{cases}$

and $p = 6$ with capacities $s_1=10$, $s_2=12$, $s_3=22$, $s_4=24$, $s_5=28$ and $s_6=48$. Note that f satisfies the requirements of Theorem 2.11 and that Total Demand $= \int_0^{24} f(x)dx = \sum_{i=1}^{6} s_i = 144$.

Keeping in mind that $\alpha = c/2 = 12$ and that $\int_0^{\alpha} f(x)dx = 72$, Corollary 2.10 and Theorem 2.11 are used via the "left-right" branching criterion, to obtain the tree shown in Figure 2.13. Notice that only nine (9) of the $6! = 720$ different orderings need to be considered. In particular, note that further branching at node E is not necessary due to Theorem

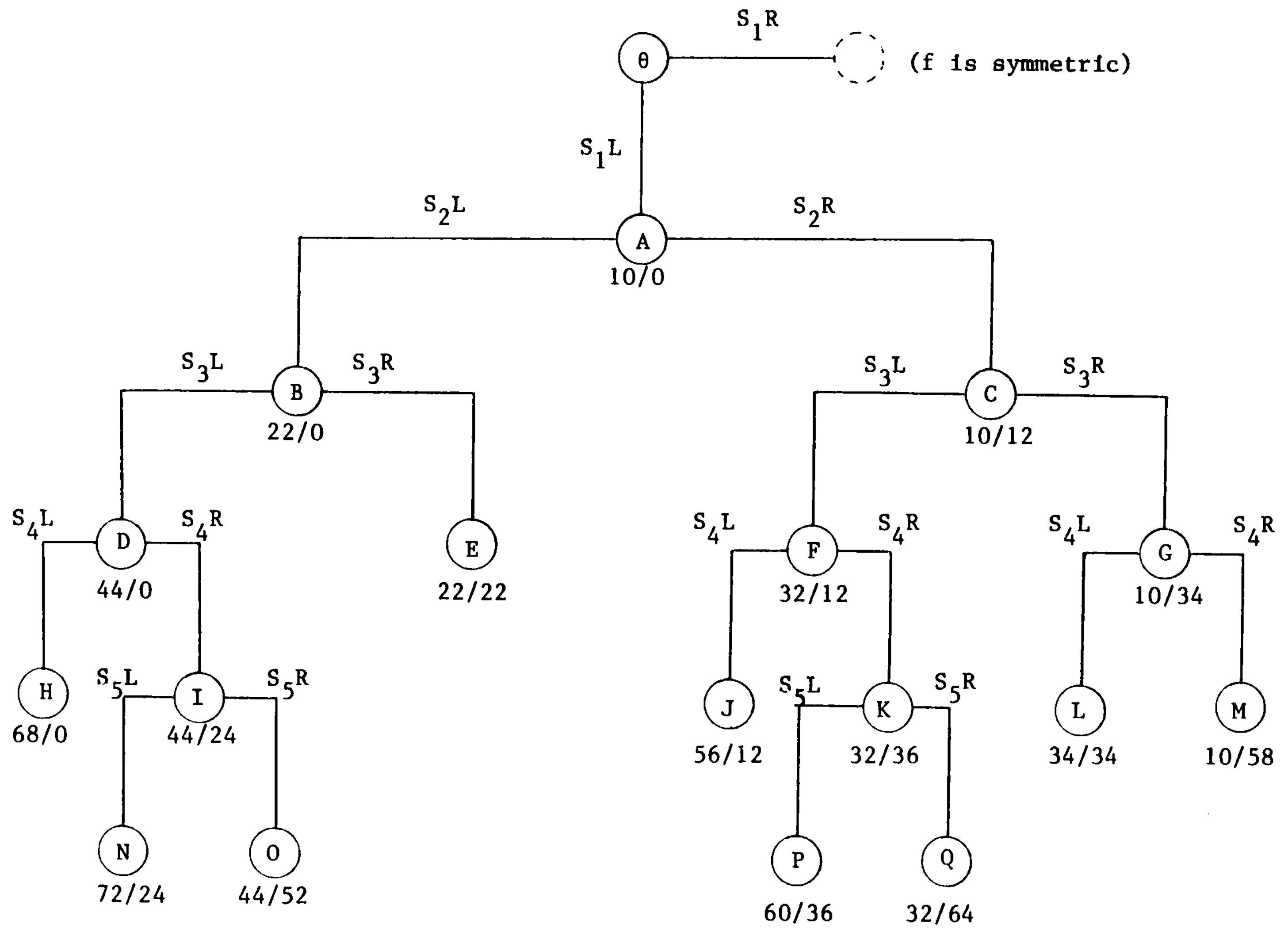



Figure 2.13. Example Alpha

2.12, and similarly at node L, since the remaining unserved demand is symmetric and three or less facilities remain to be located. Corollary 2.10, Theorem 2.11 and Theorem 2.12 assure us that an optimal solution to Problem CP can be found amongst the orderings/permutations corresponding to nodes E, H, J, L, M, N, O, P, AND Q.

Obviously, Theorem 2.11 contributes towards lessening the size of the enumeration tree (as does Corollary 2.10). However, without Theorem 2.11, such orderings as $(s_2,s_3,s_5,s_4,s_6,s_1)$ would have to be represented in the tree, since this ordering satisfies Corollary 2.10. It is Theorem 2.11 which validates the procedure of branching from Level (i-1) to Level i according to left-right placement of the facility having capacity $s_i$.

## 2.6 Analysis of the Non-Symmetric, Unimodal Demand Distribution Case

Recalling that Theorem 2.11 need not hold for non-symmetric f (see Figure 2.10), one might conclude that the method presented in section 2.5 for constructing an enumeration tree is useless/invalid for problems involving non-symmetric demand functions. In fact, the method remains valid with the exception that the straddling facility is only required to have maximum capacity with respect to the facilities to the left _or_ right of itself. The following theorem validates our branching criterion for the case in which f is non-symmetric.

_Theorem 2.13._ Consider Problem CP and suppose that the demand function f is nonnegative, integrable and unimodal on $[0,c]$. (Recall that being unimodal means f is nondecreasing on $[0,\alpha]$ and nonincreasing on $[\alpha,c]$, for some $\alpha\varepsilon[0,c]$.) Then in determining an optimal solution to Problem CP, it is sufficient to restrict attention to those orderings/

permutations of the p facilities which give solutions (per Corollary 2.6) satisfying the following condition in addition to that of Corollary 2.10: if facility M straddles, then $s_M$ must be a maximum with respect to the capacities of those facilities located on at least one side of it.

Proof. If f is symmetric, then this theorem follows immediately from Theorem 2.11. Suppose f is not symmetric. To facilitate understanding of this proof, we make use of Figure 2.14 which depicts a demand function satisfying all of the properties required by this theorem. Without loss of generality, let us suppose that the straddling facility (having capacity $s_M$) is located at a point $y_M < \alpha$ and that it serves the interval $[\gamma,\beta]$. Let $A = \int_\alpha^\beta f(x)dx$ and let $s_L$ denote the capacity of the facility to the immediate left of the straddling facility (if no such facility exists the theorem holds vacuously).

Now, if $s_M > s_L$ the theorem follows from Corollary 2.10. Thus, let us suppose that $s_M < s_L$. After interchanging the facilities currently located at points $y_L < y_M < \alpha$, the new facility locations, denoted $y'_L$, $y'_M$, are such that $y'_L < y_M$, since all locations are median locations.

Let $\theta = y_M - y'_L$. We have that the cost of supplying/serving the demand over $[\alpha,\beta]$ from $y'_L$ is given by

$$\int_\alpha^\beta f(x)|x-y'_L|dx = \int_\alpha^\beta f(x)(x-y'_L)dx$$

$$= \int_\alpha^\beta f(x)(x-y_M+\theta)dx$$

$$= \int_\alpha^\beta f(x)(x-y_M)dx + \theta\int_\alpha^\beta f(x)dx$$

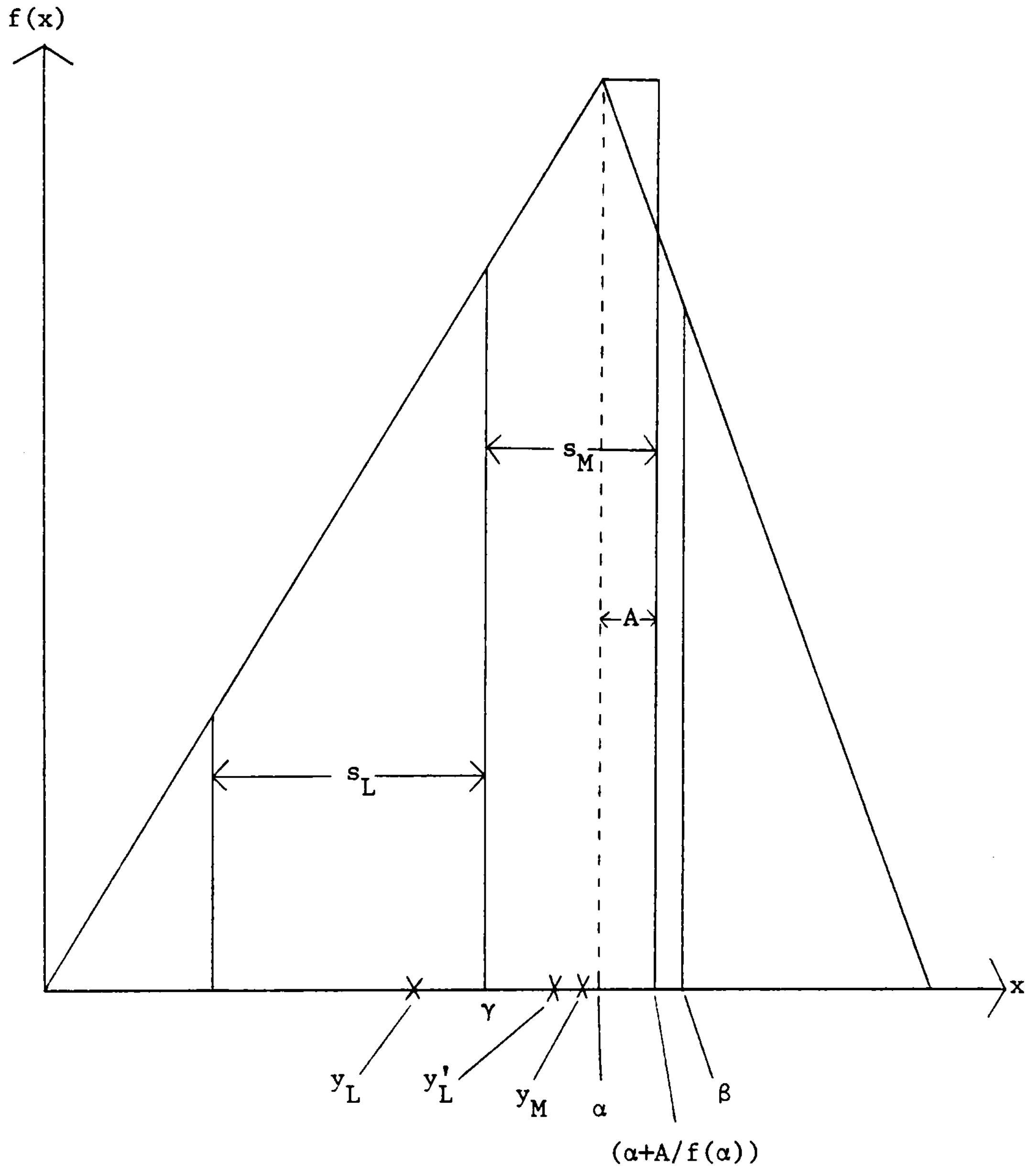



Figure 2.14. (See Theorem 2.13)

$$= \int_{\alpha}^{\beta} f(x)|x-y_M|dx + \theta A \ .$$

Thus we see that the difference in costs between serving the demand over $[\alpha,\beta]$ from the points $y_L'$ and $y_M$ is simply a multiple of A, the area under f and over $[\alpha,\beta]$. The cost differential is independent of the geometry/shape of this area, and so we could "square off" this area by redefining f to equal $f(\alpha)$ on $[\alpha,\alpha+A/f(\alpha)]$ and 0 on $(\alpha+A/f(\alpha),\beta)$. However, in doing so we would then have a non-decreasing demand function on $[0,\alpha+A/f(\alpha)]$, and thus by Corollary 2.10, it follows that $s_M > s_L$. $\square$

Figure 2.15 shows the enumeration tree obtained using Corollary 2.10 and Theorem 2.13 for the following example problem.

<u>Example (Bravo)</u>: Let $f(x) = \begin{cases} x & , x\epsilon[0,10] \\ -2x + 30 & , x\epsilon[10,15] \end{cases}$

and p = 5 with capacities $s_1=5$, $s_2=10$, $s_3=15$, $s_4=20$ and $s_5=25$. Note that $\alpha = 10$ and Total Demand $= \int_0^{15} f(x)dx = 75 = \sum_{i=1}^{5} s$ .

As shown in Figure 2.15, Corollary 2.10 and Theorem 2.13 have reduced the number of orderings that need to be considered from 5! = 120 down to fourteen (14). Notice that node Q, for instance, satisfies Corollary 2.10 and has a straddling facility of maximum capacity, whereas node z satisfies Corollary 2.10 and Theorem 2.13. Also note that straddling does not even occur at nodes O,J, and V.

Given that we now have an acceptable branching criterion by which we are able to construct enumeration trees for problems having demand functions such as those of Theorems 2.11 and 2.13, we now direct our

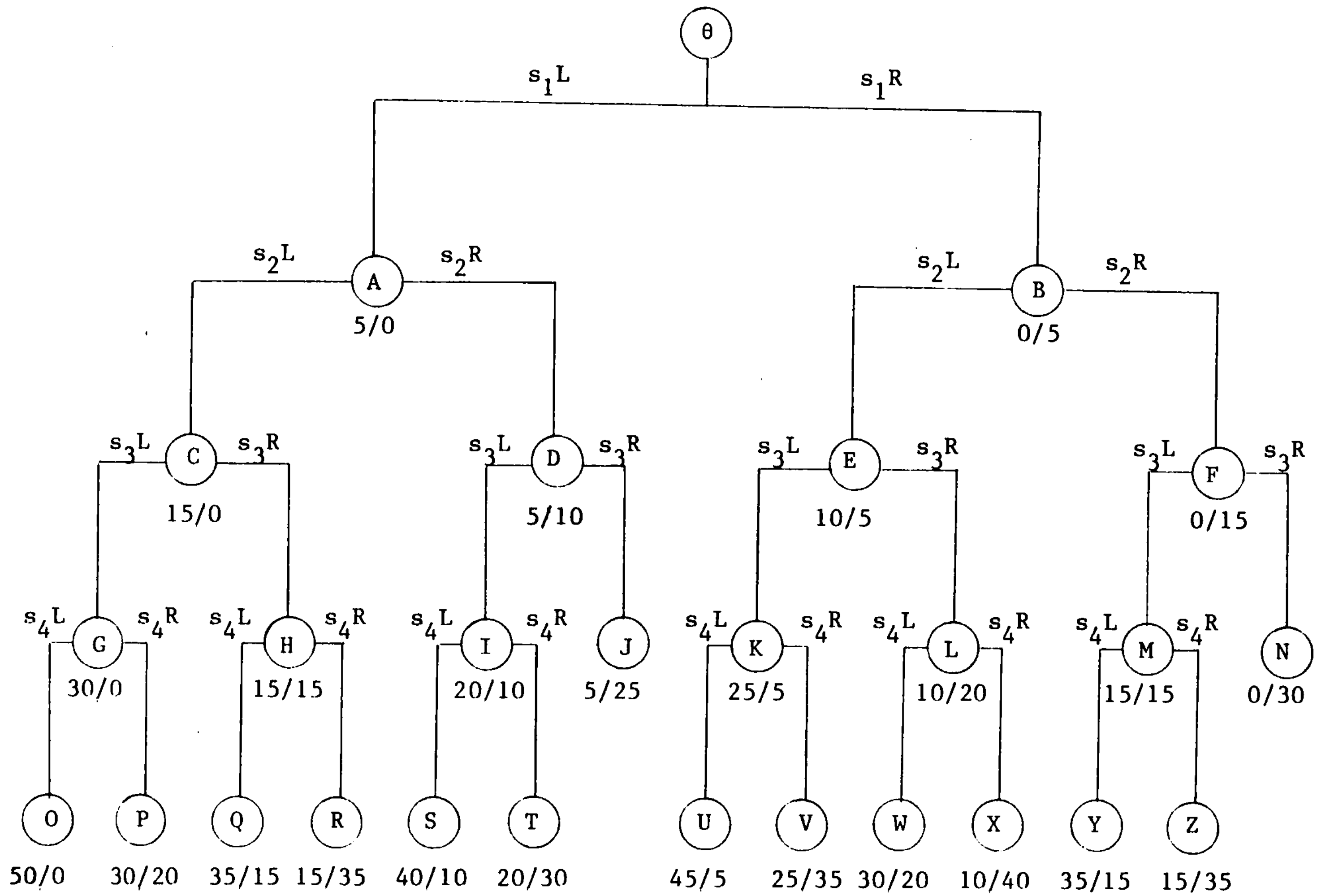



Figure 2.15. Example Bravo

efforts towards obtaining lower bounds on partial completions/orderings so as to avail ourselves to the time and effort saving value of fathoming.

## 2.7 Using Lower Bounds to Fathom Partial Orderings

In an effort to further reduce the number of orderings for which actual costs must be computed, methods of computing lower bounds on the costs of partial orderings were sought. The intent of bounding is to enable one to fathom a partial ordering, and therefore forgo the need to determine the costs of its completions.

Methods of computing lower bounds on the cost associated with the locations/allocations (per Corollary 2.6) of an ordering of p facilities, almost seem to be limited only by ones' imagination. However, for a lower bound to be useful in fathoming, it must exceed the cost of the incumbent ordering/solution. Of the bounds investigated in our analysis, most failed to result in fathoming. The question remains as to whether the fault lies in the method (of bounding) itself, or in the fact that such methods were applied to small problems like those of Examples Alpha and Bravo. Regardless, a useful method of bounding was obtained which has as its basis, the following theorem.

Theorem 2.14. Let f be a nonnegative, integrable demand function defined on $[0,c]$, and let $[\alpha,\beta] \subseteq [0,c]$ be the service interval of some facility located at a median $y^* \varepsilon\ (\alpha,\beta)$ and having capacity/supply, $s = \int_{\alpha}^{\beta} f(x)dx$.

If g is a nonnegative, integrable function such that for some $\alpha'$, $\beta'$ satisfying $\alpha < \alpha' < y^* < \beta' < \beta$, we have

$$\text{i)} \quad \int_{\alpha'}^{y^*} g(x)dx = \int_{y^*}^{\beta'} g(x)dx = s/2,$$

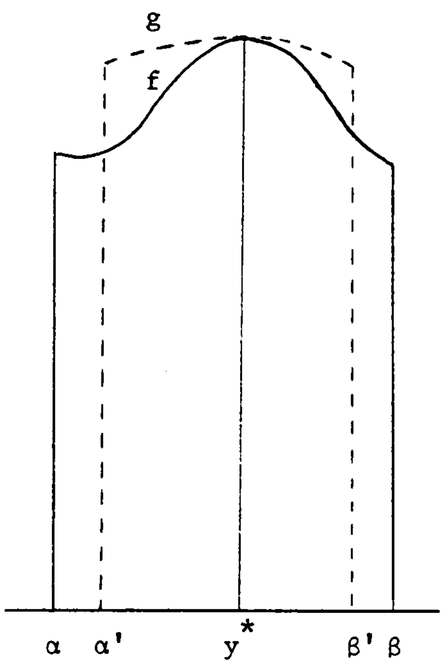
$$\text{ii)} \quad \int_{y}^{y^*} g(x)dx > \int_{y}^{y^*} f(x)dx, \text{ for all } \alpha' < y < y^*,$$

and $$\text{iii)} \quad \int_{y^*}^{y} g(x)dx > \int_{y^*}^{y} f(x)dx \text{ , for all } y^* < y < \beta',$$

then the transportation cost with respect to g of the service/supply provided to $[\alpha',\beta']$ by the facility located at $y^*$, is less than or equal to the cost with respect to f of the service/supply provided to $[\alpha,\beta]$, i.e.

$$\int_{\alpha'}^{\beta'} g(x)|x-y^*|dx < \int_{\alpha}^{\beta} f(x)|x-y^*|dx \quad . \tag{2.11}$$

(Note: the following schematic is given to help clarify the statement and proof of this theorem.)



Proof. In order to prove this theorem, it is sufficient to show that $\int_{\alpha'}^{y^*} g(x)(y^*-x)dx < \int_{\alpha}^{y^*} f(x)(y^*-x)dx$ and that $\int_{y_*}^{\beta'} g(x)(x-y^*)dx < \int_{y_*}^{\beta} f(x)(x-y^*)dx$.

Let us prove the second of these inequalities; a similar argument holds for the first inequality. Toward this end, define $F_*(x) = \int_{y_*}^{x} f(z)dz$ and $G_*(x) = \int_{y_*}^{x} g(z)dz$ and note from (i) and (iii), that by defining $g(x) \equiv 0$ on $(\beta',\beta]$, we have

$$s/2 > G_*(x) > F_*(x) \text{ for all } x\varepsilon[y^*,\beta]. \tag{2.12}$$

Hence, using (i) and integrating by parts, we get upon using (2.12) in the final step, that

$$\begin{aligned}
\int_{y_*}^{\beta} f(x)(x-y^*)dx - \int_{y_*}^{\beta'} g(x)(x-y^*)dx &= \int_{y_*}^{\beta} xf(x)dx - \int_{y_*}^{\beta'} xg(x)dx \\
&= \beta F_*(\beta) - \int_{y_*}^{\beta} F_*(x)dx - \beta' G_*(\beta') + \int_{y_*}^{\beta'} G_*(x)dx \\
&= (s/2)(\beta-\beta') + \int_{y_*}^{\beta'} [G_*(x)-F_*(x)]dx - \int_{\beta'}^{\beta} F_*(x)dx \\
&> (s/2)(\beta-\beta') - \int_{\beta'}^{\beta} F_*(x)dx > 0.
\end{aligned}$$

This completes the proof. $\square$

The following corollary is presented in the spirit of the intended use of Theorem 2.14.

Corollary 2.15. Let f be a nonnegative, integrable, unimodal demand function defined on $[0,c]$, and having a maximum at some $\alpha\varepsilon[0,c]$. Let $A_1$ and $A_2$ denote the total demands to the left and right of $\alpha$, respectively, i.e. $A_1 = \int_0^{\alpha} f(x)dx$ and $A_2 = \int_{\alpha}^{c} f(x)dx$. Pictorially, we might have something like that in Figure 2.16a.

"Squaring off" the demand over $[0,\alpha]$, we define

$$g = \begin{cases} f(\alpha), & \text{on } [\alpha - A_1/f(\alpha),\alpha] \\ f & \text{, on } [\alpha,c] \end{cases} .$$

Pictorially, we would have Figure 2.16b.

"Squaring off" the demand over $[\alpha,c]$, we define

$$h = \begin{cases} f & \text{, on } [0,\alpha] \\ f(\alpha), & \text{on } [\alpha,\alpha + A_2/f(\alpha)] \end{cases} .$$

Pictorally, we would have Figure 2.16c.

A lower bound on the optimal cost of serving/supplying the demand $A_1 + A_2$ with respect to f, is the larger of the optimal costs of doing the same, with respect to g and h.

<u>Proof</u>. Suppose that we know/have an optimal solution with respect to f. In particular, let us assume that it is of the type described in Theorem 2.5.

Consider the modified solution in g, in which those facility locations currently in $[\alpha,c]$ remain fixed, whereas those in $[0,\alpha]$ are optimally relocated with respect to g. Then by Theorem 2.14, the cost of the resulting solution in g is less than or equal to the cost of the optimal solution in f. Thus, the optimal solution with respect to g, must be less than or equal to the cost of the optimal soluton in f.

Similarly for h, and hence Corollary 2.15 is proven. $\square$

To illustrate how one might use Theorem 2.14/Corollary 2.15 to fathom partial orderings of an enumeration tree for the "p! problem", we direct the reader's attention to Example (Bravo) and in particular, to its enumeration tree shown in Figure 2.15.

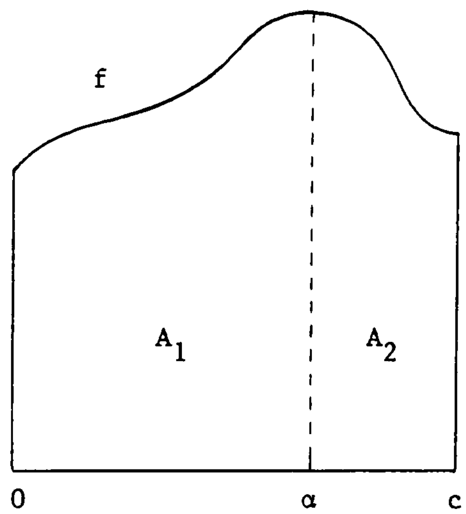



Figure 2.16a.

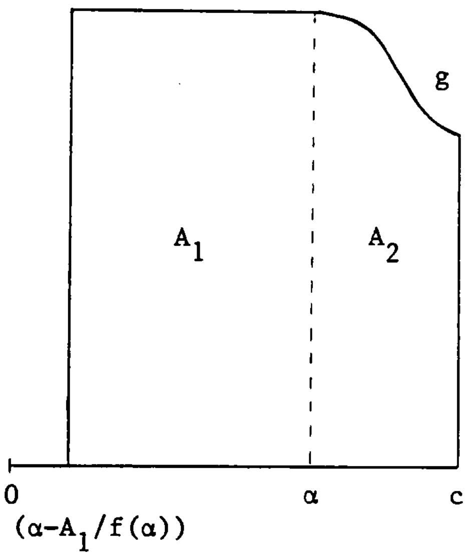



Figure 2.16b.

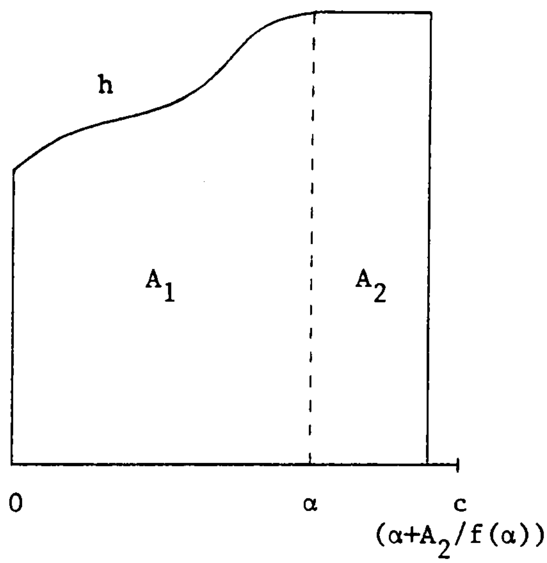



Figure 2.16c.

Figure 2.16. (See Corollary 2.15)

Again, noting that there are fourteen (14) orderings that need to be considered, i.e. their costs need to be computed, we will show how Theorem 2.14/Corollary 2.15 can be used to fathom nodes C,L and F. In doing so, we will "say" that the "equivalent" of three (3) cost computations are required. Noting that nodes C,L and F result in nine (9) completions/orderings, we remark that the fathoming of these nodes has saved us the equivalent of six (6) cost computations.

The basis for being able to fathom a node/partial ordering, is that a known lower bound on the costs of the completions of that node is greater than or equal to the cost of some incumbent solution/ordering. This being the case, suppose that for an incumbent solution, one selects the intuitively appealing ordering/permutation corresponding to node T. This ordering is such that the facilities have been located in an alternating fashion to the left and right of $\alpha = 10$, and has an associated cost of 51.79526. Subject to Theorem 2.13, one may attempt to improve upon this solution by interchanging the straddling facility (4) with one of its immediate neighbors. The interchange of facilities 4 and 2 violates Theorem 2.13, while the interchange of 4 and 5 satisfies the theorem and results in the ordering which corresponds to node S. This ordering has an associated cost of 51.09342 which is less than that of node T. Thus, let us use the ordering of node S as our incumbent solution. (Incidentally, this is an optimal solution.)

One final remark before fathoming nodes C, L and F, is that non-judicious attempts at fathoming can quickly "eat up" any savings in computation that may eventually result from successful fathoming. This is especially so for problems having a small number of facilities. If one attempts to fathom (via Theorem 2.14/Corollary 2.15) too high up in the tree, it is likely that Corollary 2.15 will square off too much demand and thus result in a lower bound that is not strong (large) enough to result in fathoming. With these thoughts in mind, let us proceed with the fathoming of nodes C, L and F.

First, node C.

At node C, we see that the assignments $s_1L$ and $s_2L$ have been made, and that they constitute fifteen (15) of the fifty (50) units of demand lying to the left of $\alpha = 10$.

"Squaring off" the remaining thirty-five (35) units of demand to the left of $\alpha$ via the function

$$g = \begin{cases} 10, & \text{on } [6.5,10] \\ f, & \text{on } [10,15] \end{cases},$$

and noting that g is non-increasing on $[6.5,15]$, we employ Theorem 2.8 and Corollary 2.6 to obtain an optimal solution to the problem of serving/supplying the demand with respect to g over the interval $[6.5,15]$. Per Corollary 2.15, the cost of this solution is a lower bound to the cost of serving/supplying the demand with respect to f over the interval $[\alpha_1=\sqrt{30},15]$. Figure 2.17 depicts the methodology involved in obtaining this lower bound. The cost of this bound is equal to 51.13777 (this includes the costs of $s_1$ and $s_2$ also) which is greater than that of our incumbent (i.e. 51.09342), and hence we can fathom node C.

Node L.

From Figure 2.15, we see that the assignments $s_1R$, $s_2L$ and $s_3R$ have been made.

"Squaring off" the remaining five (5) units of demand to the right

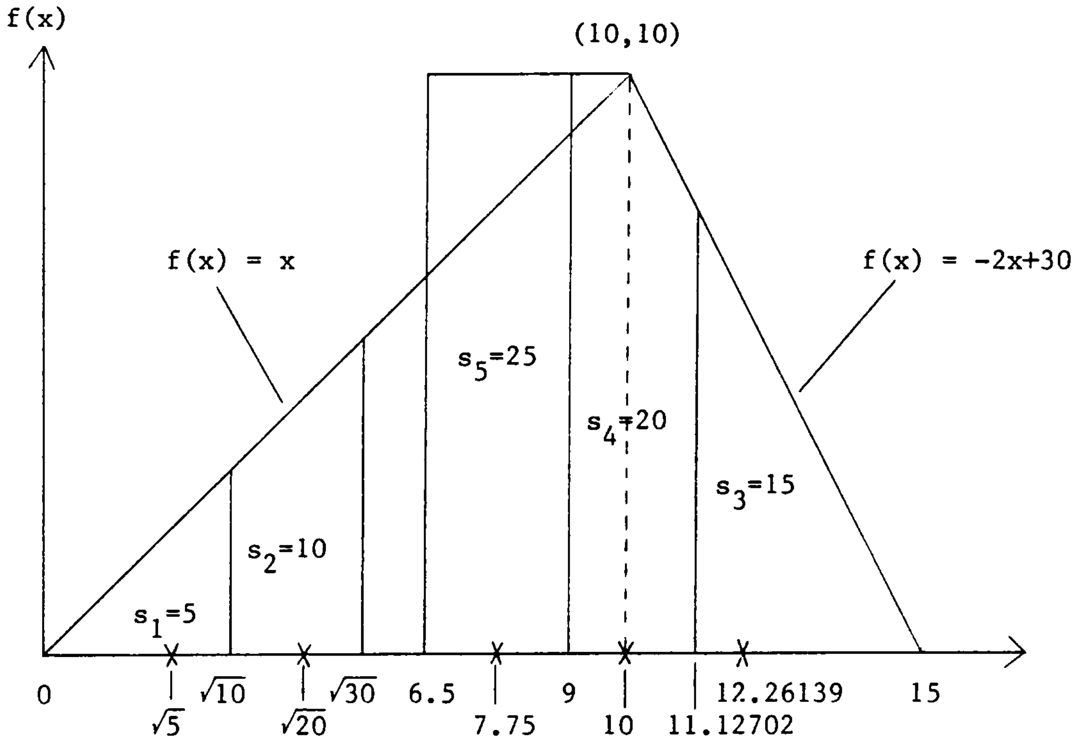



Figure 2.17. Fathoming Node C of Example Bravo

of $\alpha = 10$ via the function

$$h = \begin{cases} f, & \text{on } [0,10] \\ 10, & \text{on } [10,10.5] \end{cases},$$

and noting the monotonicity of h on [0,10.5], we employ Theorem 2.8 and Corollary 2.6 to obtain an optimal solution to the problem of serving/supplying the demand with respect to h over the interval [0,10.5]. Per Corollary 2.15, the cost of this solution is a lower bound to the costs of the completions of the partial ordering at node L. The cost of this bound is equal to 52.21267 which is greater than that of our incumbent, and hence we can fathom node L. For further clarification, see Figure 2.18.

Node F.

The reasoning is identical to that of nodes C and L. See Figure 2.19 for further detail.

This completes our analysis of the "p! problem" for the special case in which f satisfies either the conditions of Section 2.5 (i.e. nonnegative, integrable, symmetric, unimodal) or those of Section 2.6 (i.e. nonnegative, integrable, non-symmetric, unimodal). We conclude our analysis of Problem CP by offering some insight as to how one might "tackle" the "p! problem" for the case in which the demand function is simply nonnegative and integrable.

## 2.8 Treatment of Problem CP, When f is Simply Nonnegative and Integrable

In this section, we offer some suggestions/remarks as to how one might deal with the "p! problem", when the demand function of Problem CP is not as well behaved as in Sections 2.5 and 2.6. In particular, we present a theorem which may prove useful in fathoming the enumeration

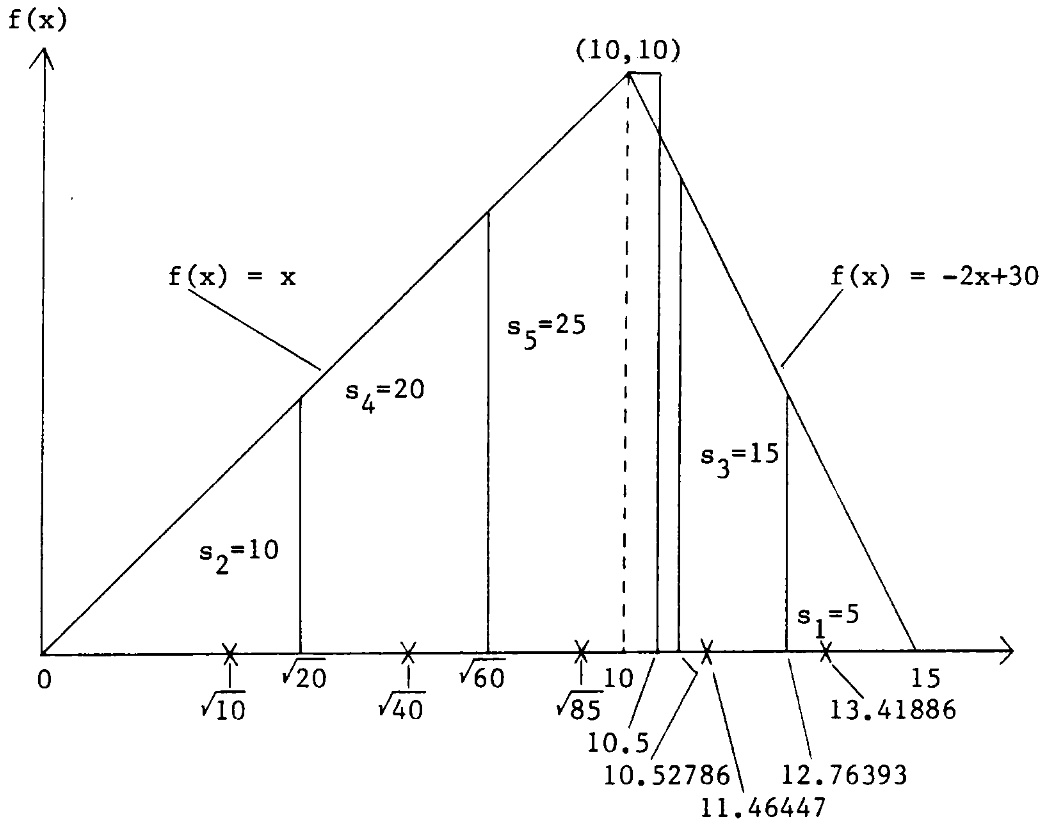



Figure 2.18. Fathoming Node L of Example Bravo

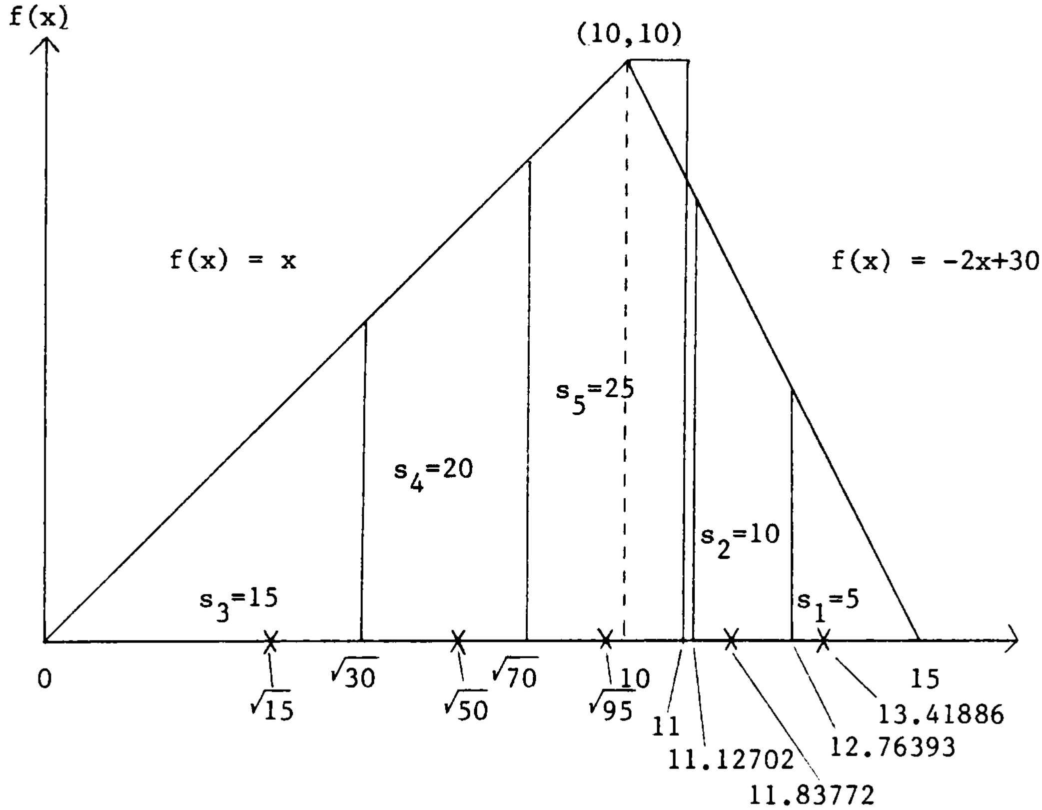



Figure 2.19. Fathoming Node F of Example Bravo

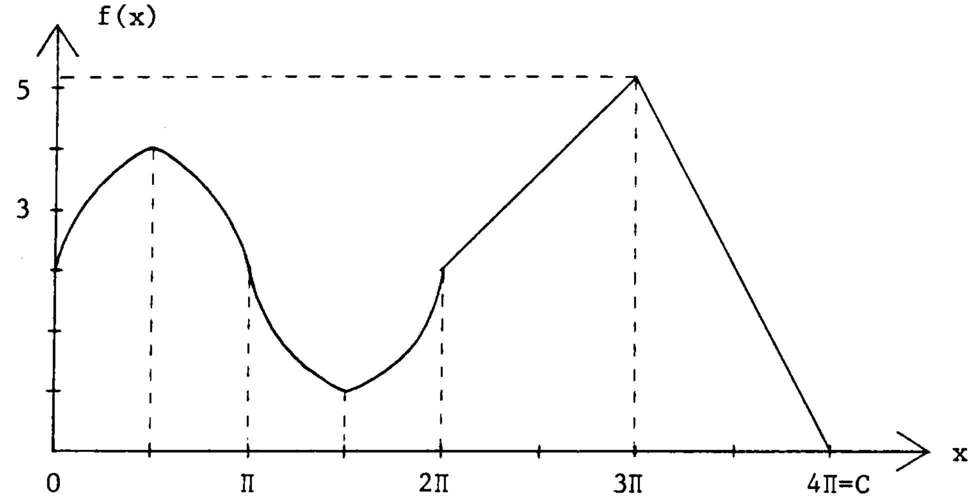
tree of the "p! problem" for the case in which f is simply nonnegative and integrable. Unlike the branching criteria used in Sections 2.5 and 2.6, construction of an enumeration tree for a problem having a more general demand function would require a more flexible branching policy, in that the nature of f and the nature of the fathoming criteria would be likely to dictate ones branching decisions. For instance, suppose that at some node of the enumeration tree, we find that f is monotone over a significant connected subset of the unserved portion of $[0,c]$. Then in view of Corollary 2.10, it would be advantageous to continue branching (if possible) so as to finally obtain a partial ordering having f monotone over the unserved portion of $[0,c]$, for then an optimal completion is known.

Corollary 2.10 clearly states that f need only be nonegative and integrable. It is therefore fitting that it receive due mention in this section. The utility of this corollary in reducing the number of orderings/permutations to be considered, follows by noting that we need not consider (i.e. we may fathom) those orderings in which Corollary 2.10 is violated over some monotone section of some general f. Similarly, one need not consider any ordering which violates any of Theorems 2.11, 2.12, or 2.13 on some connected subset of $[0,c]$, over which f satisfies the requirements of the respective theorem. Thus, we see the usefulness of the results obtained for the special cases of Sections 2.5 and 2.6, in dealing with any Problem CP having a more general demand function. Admittedly, the procedure being described for the more general problem is presently less algorithmic than that described for the special cases. Also it is obvious that the effectiveness of such procedures are very much dependent on the nature of the demand function itself, and to some extent, on the ability of the problem solver.

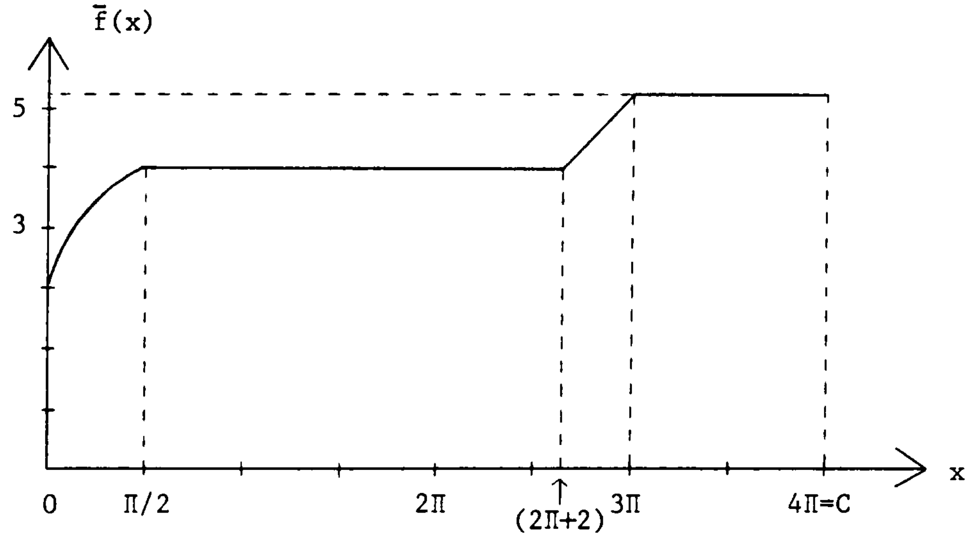
Continuing along the same line, we remark that the lower bounding result of Theorem 2.14 can be used to (easily) obtain an underestimate of the cost of a given solution to Problem CP. The value of this lies in the fact that if the underestimate exceeds some imcumbent value, then one may avoid the (more difficult) computation of the actual cost of the solution. Specifically, suppose we have a solution to Problem CP, and instead of evaluating the objective function value, we do the following. For each interval $[\alpha,\beta] \subseteq [0,c]$ served by some facility located at the median $y^*$ as in Theorem 2.14, we construct a function $g(\cdot)$ and determine $\alpha'$ and $\beta'$ such that (i) holds and that $g(x) = H_1 \equiv \sup \{f(x): \alpha < x < y^*\}$ on $[\alpha',y^*]$ and that $g(x) = H_2 \equiv \sup \{f(x): y^* < x < \beta\}$ on $(y^*,\beta']$. Then the conditions of Theorem 2.14 are satisfied, and so (2.11) holds. More importantly, the evaluation of the left-hand-side of (2.11) for each such service subinterval $[\alpha,\beta]$ is an easy task. In fact, it equals $(1/2)[H_1(y^*-\alpha')^2 + H_2(\beta'-y^*)^2]$. Hence, summing such quantities over all p subintervals of $[0,c]$, we obtain an underestimate of the actual cost of the ordering, which we would then compare to some incumbent value in the hope of avoiding the actual computation of the cost of the ordering.

We conclude our analysis of Problem CP by presenting a generalization of Corollary 2.15 which can be used to obtain a lower bound to the optimal value of Problem CP. Consider Problem CP with some nonnegative, integrable, demand function $f:[0,c] \to R$, and construct an associated nonnegative, integrable, demand function $\hat{f}(\cdot)$ defined on some $[0,c']$, $c' < c$, as follows. Note that since only the accumulated area under $f(\cdot)$ is of concern to us, we can assume without loss of generality that $f(\cdot)$ is lower semicontinuous. First, construct a nondecreasing envelope $\bar{f}(\cdot)$ for $f(\cdot)$. That is, let $\bar{f}(\cdot)$ be a nondecreasing function which is the pointwise infimum of all nondecreasing, continuous functions which exceed $f(\cdot)$ everywhere. (A similar result holds for a nonincreasing envelope construction.) Note that over all subsets of $[0,c]$ of positive measure where $\bar{f}(\cdot)$ exceeds $f(\cdot)$, the function $\bar{f}(\cdot)$ is a constant. Over each subinterval where $\bar{f}(\cdot)$ is a constant, shrink the length of this subinterval so that the area under $\bar{f}(\cdot)$ over the shrunken subinterval equals the area under $f(\cdot)$ over this original subinterval. Upon shrinking the lengths of such subintervals individually, leaving the length and function definitions in other subintervals unaltered, the function $\bar{f}(\cdot)$ transforms into the desired nondecreasing, nonnegative, function $\hat{f}(\cdot)$ defined on some resulting interval $[0,c']$, say, where $c' < c$. Note that the total area under $\hat{f}(\cdot)$ on $[0,c']$ equals the total area under $f(\cdot)$ on $[0,c]$. We provide Figure 2.20 to aid in explaining the construction of $\hat{f}(\cdot)$ and also refer the reader to Figure 2.16, where $\hat{f}_1 \equiv h$ and $\hat{f}_2 \equiv g$ are respectively obtained by "squaring off" the regions to the right and left of $\alpha$. We now present the main result of this section.

Theorem 2.16. Given $f:[0,c] \to R$ of Problem CP, let $\hat{f}:[0,c'] \to R$ be constructed as above. Then the PFL solution of Theorem 2.8 which solves Problem CP with f replaced by $\hat{f}$, gives an objective function value with respect to $\hat{f}$ which is a lower bound for the optimal value to Problem CP with respect to $f(\cdot)$.

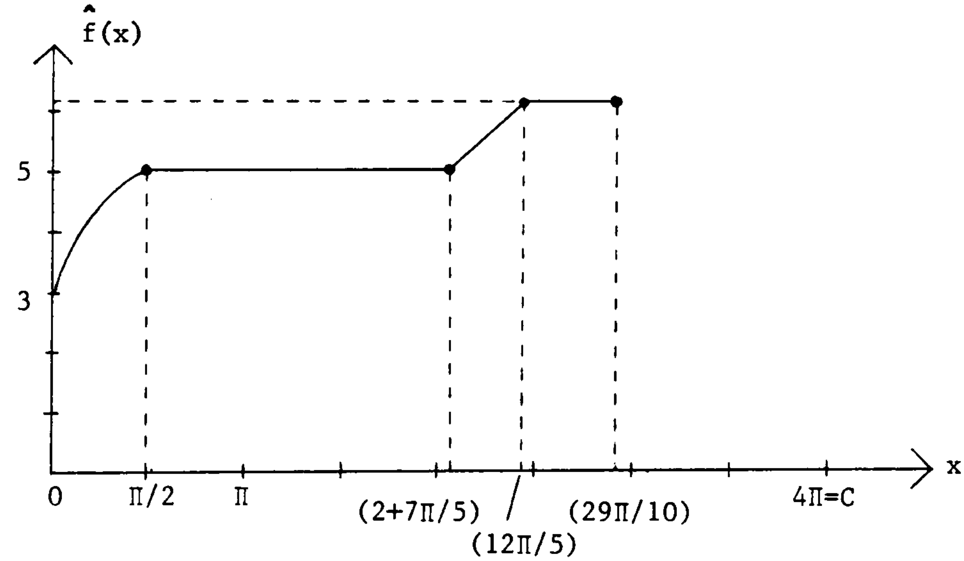



$$f(x) = \begin{cases} 2\sin x + 3, & 0 < x < 2\Pi \\ x + 3 - 2\Pi, & 2\Pi < x < 3\Pi \\ -((3+\Pi)/\Pi)x + 12 + 4\Pi, & 3\Pi < x < 4\Pi \end{cases}$$




$$\bar{f}(x) = \begin{cases} 2\sin x + 3, & 0 < x < \Pi/2 \\ 5, & \Pi/2 < x < 2\Pi+2 \\ x + 3 - 2\Pi, & 2\Pi+2 < x < 3\Pi \\ 3 + \Pi, & 3\Pi < x < 4\Pi \end{cases}$$

<u>Figure 2.20.</u> Bounding Partial Orderings




$$\hat{f}(x) = \begin{cases} 2\sin x + 3 , & 0 < x < \Pi/2 \\ 5 , & \Pi/2 < x < 2 + 7\Pi/5 \\ x - (7\Pi/5) + 3, & 2 + 7\Pi/5 < x < 12\Pi/5 \\ \Pi + 3 , & 12\Pi/5 < x < 29\Pi/10 \end{cases}$$

Figure 2.20. (continued)

Proof. Consider an optimal solution to Problem CP with respect to the demand function $f(\cdot)$, and let $\nu_1 = \sum_{i=1}^{p} \nu_{1i}$ be the optimal objective function value, where $\nu_{1i}$ is the contribution to $\nu_1$ due to facility $i=1, \ldots, p$, and let $\sigma$ be the corresponding optimal permutation of facilities. For this same $\sigma$, let $Y^*(\sigma)$ be the facility locations for Problem CP with respect to the demand function $\hat{f}(\cdot)$ and denote $\nu_2 \equiv \nu[Y^*(\sigma)] \equiv \sum_{i=1}^{p} \nu_{2i}$, where $\nu_{2i}$ is the contribution to $\nu_2$ due to facility $i$. In order to prove the result, it is sufficient to show that $\nu_{2i} < \nu_{1i}$ for each $i=1, \ldots, p$.

Hence, consider any $i \varepsilon \{1, \ldots, p\}$ and let $[\alpha,\beta] \subseteq [0,c]$ be the interval that facility $i$ serves from a median location $y^* \varepsilon (\alpha,\beta)$, at the stated cost $\nu_{1i}$. Now, let $\bar{f}(\cdot)$ be defined from $f(\cdot)$ as above, and consider $\bar{f}(\cdot)$ on the interval $[\alpha,\beta]$. Again, as before, on either side of $y^*$, shrink the lengths of each subinterval of $[\alpha,\beta]$ of positive measure on which $\bar{f}(\cdot)$ is a constant, so that the area under $\bar{f}(\cdot)$ over the shrunk subinterval is the same as the area under $f(\cdot)$ over the original subinterval. Packing the resulting function about $y^*$ gives a function $g(\cdot)$ defined on an interval $[\alpha',\beta']$ which clearly satisfies the conditions of Theorem 2.14, and hence, the cost $\nu_{1i}'$ associated with serving $g(\cdot)$ on $[\alpha',\beta']$ from $y^*$, satisfies $\nu_{1i}' < \nu_{1i}$. But by construction, $g(\cdot)$ on $[\alpha',\beta']$ is precisely the segment of $\hat{f}(\cdot)$ served from the same median location by facility $i$ in the solution $Y^*(\sigma)$. Hence, $\nu_{2i} = \nu_{1i}'$, and the proof is complete. $\square$

It is felt that Theorem 2.16, Corollary 2.10 and the results of Sections 2.5 and 2.6 provide one with a fair amount of ammunition with which to attack Problem CP in the case of general f. The objective of all of these results is to lessen the severity of the "p! problem", that is, to reduce the number of solutions for which one needs to evaluate the objective function of Problem CP. As is usually the case with combinatoral problems such as the "p! problem", further research is warranted for this general f case.

## 2.9 A Sequential One Facility Per Period Location-Allocation Problem

As a final case of analysis in this chapter, we consider a dynamic, sequential location-allocation problem, as in Scott [1971], Minieka [1980], and Cavalier and Sherali [1983b]. Here, a certain planning horizon with T periods is specified, and a single facility per period is required to be located. The individual period durations are assumed to be long enough so as to justify the location of a facility based on only the current period's information, in view of the unavailability or the unreliability of any future period data. Hence, a typical period's problem is as follows. Some (p-1) facilities having positive supplies $s_1, \ldots, s_{p-1}$ are known to be located at points $y_1, \ldots, y_{p-1}$ respectively, where $\zeta_0 \equiv 0 < y_1 = \zeta_1 < y_2 = \zeta_2 < \ldots < y_{p-1} = \zeta_{p-1} < c \equiv \zeta_p$, say. Additionally, for the period under question, an estimated demand distribution function $f(\cdot)$ is specified such that $\Delta \equiv \int_0^c f(x)dx - \sum_{i=1}^{p-1} s_i > 0$. The problem is to locate a $p^{th}$ facility having supply $s_p = \Delta$ at some point $y_p$ in $[0,c]$ so as to minimize the $p^{th}$ period's total cost, i.e., so as to solve Problem CP with the variables $y_1, \ldots, y_{p-1}$ fixed at the specified values $\zeta_1, \ldots, \zeta_{p-1}$ respectively. In light of Theorem 2.4, the problem can be restated as that of determining $y_p \varepsilon [0,c]$ so that the

PFL solution to Problem $CP(\zeta_1, \ldots, \zeta_{p-1}, y_p)$ is a least cost/optimal solution to the problem of supplying/serving the demand $\int_0^c f(x)dx$. Call this Problem CP(p).

We begin our analysis of Problem CP(p) by presenting a lemma which establishes the fact that one would not want to locate the additional facility so as to be coincident with one of the (p-1) existing facilities.

<u>Lemma 2.17</u>. Let $y_p^*$ be the location of the $p^{th}$ facility of any optimal solution to Problem CP(p). Then $y_p^* \notin \{\zeta_1, \ldots, \zeta_{p-1}\}$.

<u>Proof</u>. On the contrary, assume that $y_p^* = \zeta_i$ for some $i \in \{1, \ldots, p-1\}$. Thus, from Theorem 2.4 we have that the PFL solution to Problem $CP(\zeta_1, \ldots, \zeta_{p-1}, y_p^*)$ solves Problem CP(p). However, we will show that there exists a point $\hat{y}_p \neq \zeta_i$ in the interval $[0,c]$, for which the objective value of the PFL solution to Problem $CP(\zeta_1, \ldots, \zeta_{p-1}, \hat{y}_p)$, is strictly less than that of Problem $CP(\zeta_1, \ldots, \zeta_{p-1}, y_p^*)$, thus contradicting that $y_p^* = \zeta_i$ is the location of the $p^{th}$ facility for some optimal solution to Problem CP(p).

The PFL solution to Problem $CP(\zeta_1, \ldots, \zeta_{p-1}, y_p^*)$ is such that facilities i and p behave as a single facility having capacity $(s_i + s_p)$ and jointly serve some interval $[\alpha, \beta]$, say.

Define $\theta_L$ and $\theta_R$ in $(\alpha, \beta)$ according to

$$\theta_L = \max\{y: \int_\alpha^y f(x)dx = s_p/2\}, \quad \theta_R = \min\{y: \int_y^\beta f(x)dx = s_p/2\} \qquad (2.13)$$

and note that $\theta_L < \theta_R$ since $s_i > 0$. Now suppose that $\zeta_i > \theta_L$. Then, let facility p serve the subinterval $[\alpha, \gamma]$ and facility i serve the subinterval $[\gamma, \beta]$ where $\gamma$ is determined subject to their respective capacities. But this means that with the allocations fixed as such, a strict improvement in objective value results by moving $y_p$ leftwards to point $\hat{y}_p = \theta_L$ which is the rightmost median location with respect to the demand on $[\alpha,\gamma]$, thereby contradicting the optimality of $y_p^*$. Hence, we must have $\zeta_i < \theta_L < \theta_R$. However, in this case, again a strict improvement in objective value results by letting $\hat{y}_p = \theta_R$, and this completes the proof. □

(Note that the min and max operations in (2.13) and in (2.14) and (2.15) below, serve to accommodate the possibility that $f(\cdot)$ may be zero over some subintervals of positive measure.)

In light of Lemma 2.17, the problem of interest is one of determining in which of the (open) intervals $(0,\zeta_1)$, $(\zeta_1,\zeta_2)$, ..., $(\zeta_{p-2},\zeta_{p-1})$, $(\zeta_{p-1},c)$, to locate the $p^{th}$ facility. Thus, there are p permutations $\sigma_1$, ..., $\sigma_p$ that need to be considered, where permutation $\sigma_j$ corresponds to facility p being located in the $j^{th}$ position, i.e. $y_p \varepsilon (\zeta_{j-1},\zeta_j)$, and where Theorem 2.4 is used to obtain an optimal PFL allocation solution <u>independent</u> of the actual value of $y_p$. Thus, for a given permutation $\sigma_j$, the interval $[\alpha_j,\beta_j]$ served by facility p is known, and so it remains to determine an optimal value of $y_p$ in the interval $(\zeta_{j-1},\zeta_j)$. However, as we will now show, an optimal value of $y_p$ may not exist within $(\zeta_{j-1},\zeta_j)$, in which case we would discard permutation $\sigma_j$ in our search for a solution to Problem CP(p). Thus, for the moment, let us ignore the result of Lemma 2.17 and solve the problems

$$\text{CP}(p,\sigma_j): \quad \text{minimize } \left\{ \int_{\alpha_j}^{\beta_j} f(x)|x-y_p|dx : \zeta_{j-1} < y_p < \zeta_j \right\} \text{ for each}$$

$j = 1, \ldots, p,$ where

$$\alpha_j = \min \left\{ y > 0: \int_0^y f(x)dx = \sum_{i=1}^{j-1} s_i \right\} ,$$

$$\beta_j = \max \left\{ y < c: \int_y^c f(x)dx = \sum_{i=j}^{p-1} s_i \right\} , \tag{2.14}$$

and where undefined sums are zeroes.

Theorem 2.18. For each $j = 1, \ldots, p$, let $P_j = [\delta_j, \Delta_j]$ be the median interval (in $(\alpha_j, \beta_j)$) defined by

$$\delta_j = \min \left\{ y > \alpha_j: \int_{\alpha_j}^{y} f(x)dx = s_p/2 \right\}$$

and

$$\Delta_j = \max \left\{ y < \beta_j: \int_y^{\beta_j} f(x)dx = s_p/2 \right\} , \tag{2.15}$$

where $\alpha_j$, $\beta_j$, $j=1, \ldots, p$ are given by (2.14). If (a) $\Delta_j < \zeta_{j-1}$ or if $\delta_j > \zeta_j$, then $y_p = \zeta_{j-1}$ or $y_p = \zeta_j$ are respectively unique optimal solutions to $CP(p,\sigma_j)$. Otherwise, (b) any of the points in $P_j \cap [\zeta_{j-1}, \zeta_j]$ solve $CP(p,\sigma_j)$.

Proof. Note that if $\beta_j < \zeta_{j-1}$ or if $\alpha_j > \zeta_j$, then clearly condition (a) holds. Otherwise, $(\alpha_j, \beta_j) \cap [\zeta_{j-1}, \zeta_j] \neq \emptyset$, and since we must clearly have $y_p \varepsilon (\alpha_j, \beta_j) \cap [\zeta_{j-1}, \zeta_j]$ in this case, we can write $CP(p,\sigma_j)$ as

$$\text{minimize} \left\{ \int_{\alpha_j}^{y_p} f(x)(y_p - x)dx + \int_{y_p}^{\beta_j} f(x)(x - y_p)dx: \zeta_{j-1} < y_p < \zeta_j \right\}. \tag{2.16}$$

Noting that the Karush-Kuhn-Tucker (KKT) conditions are both necessary and sufficient for (2.16), (see Appendix A, Lemma 2.18*, for convexity of the objective function of (2.16)), we get $I_1 - I_2 - \mu_1 + \mu_2 = 0$, where $I_1 = \int_{\alpha_j}^{y_p} f(x)dx$, $I_2 = \int_{y_p}^{\beta_j} f(x)dx$, and where $\mu_1$ and $\mu_2$ are nonnegative, complementary slack Lagrange multipliers associated with the lower and upper bounding constraints for $y_p$ respectively. If a KKT solution has $y_p = \zeta_{j-1}$, then we must have $I_1 > I_2$. Similarly, if $y_p = \zeta_j$, then we must have $I_1 < I_2$, and if $\zeta_{j-1} < y_p < \zeta_j$, then we must have $I_1 = I_2 = s_p/2$. Now if $\Delta_j < \zeta_{j-1}$, then we must have $I_1 > I_2$ whenever $y_p > \zeta_{j-1}$, in which case $y_p = \zeta_{j-1}$ must hold at optimality. Similarly, $\delta_j > \zeta_j$ implies $y_p = \zeta_j$. Otherwise, any point in $P_j \cap [\zeta_{j-1}, \zeta_j]$ satisfies the KKT conditions with $I_1 = I_2 = s_p/2$, and $\mu_1 = \mu_2 = 0$ and this completes the proof. $\square$

Thus we see from Theorem 2.18, an optimal location for a facility constrained to the interval $[\zeta_{j-1}, \zeta_j]$ and serving some $[\alpha_j, \beta_j]$ is at a median location of $[\alpha_j, \beta_j]$, if such lies within $[\zeta_{j-1}, \zeta_j]$. Otherwise, it will be at whichever endpoint of $[\zeta_{j-1}, \zeta_j]$ lies nearest to a median location.

Now combining the result of Theorem 2.18 with that of Lemma 2.17, we obtain the following theorem which prescribes a reduced set of candidate optimal solutions to Problem CP(p).

Theorem 2.19. For each j=1, ..., p, define $S_j = \{y\}$ for some $y \varepsilon P_j \equiv [\delta_j, \Delta_j]$ in case $\zeta_{j-1} < \delta_j < \Delta_j < \zeta_j$, and $S_j = \phi$ otherwise. Then, an optimal solution to CP(p) lies in the set $S \equiv \bigcup_{j=1}^{p} S_j$.

Proof. For each j=1, ..., p, let $y_{pj}$ be an optimal solution to CP(p,$\sigma_j$). Then, clearly the best of the solutions $y_{p1}, \ldots, y_{pp}$ solves CP(p). Now, consider some $j \varepsilon \{1, \ldots, p\}$. If the case (a) of Theorem 2.18 holds, then by Lemma 2.17, $y_{pj}$ may be disregarded since it is strictly suboptimal. If case (b) of Theorem 2.18 holds, and if either $\zeta_{j-1} \varepsilon P_j$ or if $\zeta_j \varepsilon P_j$, then we may pick $y_{pj}$ to be $\zeta_{j-1}$ or $\zeta_j$ respectively, and this would again be strictly suboptimal for CP(p). Hence, we need only consider an optimal solution for CP(p) when $\zeta_{j-1} < \sigma_j \leq \Delta_j < \zeta_j$, and this completes the proof. $\square$

An algorithm to solve the sequential $p^{th}$ period problem is now evident.

<u>Algorithm for CP(p)</u>. Compute $\alpha_j$ and $\beta_j$ (recursively) from (2.14) and hence determine $\delta_j$ and $\Delta_j$ as in (2.15). Find the set S of Theorem 2.19 and note from the proof of this theorem that $S \neq \phi$. Select the best solution from S as an optimal solution to CP(p).

<u>Illustrative Example</u>. Consider a situation in which p = 4, with three facilities already located on $[0,c] \equiv [0,15]$ at locations $y_1 = \zeta_1 = 3$, $y_2 = \zeta_2 = 8$, and $y_3 = \zeta_3 = 13$, and with respective capacities $s_1 = 8$, $s_2 = 4$ and $s_3 = 6$. Furthermore, assume that the demand distribution function $f(\cdot)$ has been estimated as a step function given by $f(x) = 2$ for $0 \leq x < 5$, $f(x) = 1$ for $5 \leq x < 10$ and $f(x) = 3$ for $10 \leq x \leq 15$. Note that total (expected) demand is 30 units, and so, $s_p \equiv s_4 = 30 - 18 = 12$ units. From (2.14), we recursively compute $\alpha_1 = 0$, $\alpha_2 = 4$, $\alpha_3 = 7$ and $\alpha_4 = 11$, and we compute $\beta_4 = 15$, $\beta_3 = 13$, $\beta_2 = 35/3$ and $\beta_1 = 7$. Consequently, from (2.15), we obtain, $\delta_1 = \Delta_1 = 3$, $\delta_2 = \Delta_2 = 9$, $\delta_3 = \Delta_3 = 11$ and $\delta_4 = \Delta_4 = 13$. Note that when $f(\cdot)$ is positive (except on a set of measure zero), $\delta_j = \Delta_j$ for each $j=1, \ldots, p$ and we only need to compute either the $\alpha_j$ or the $\beta_j$ values. Finally, from Theorem 2.19, we determine that $S_1 = \phi$, $S_2 = \phi$, $S_3 = \{11\}$, and $S_4 = \phi$ so that $S = \{11\}$ is a singleton, and hence in this case, we can conclude that $y_4 = 11$ solves CP(4). Note that in the worst case, if $3 < \zeta_1 < 9$, $9 < \zeta_2 < 11$, and $11 < \zeta_3 < 13$, then we would have obtained $S = \{3,9,11,13\}$, and we would have had to evaluate the objective function of $CP(p,\sigma_j)$ for each of the p = 4 locations in S.

# CHAPTER III

# A CAPACITATED, BALANCED, LOCATION-ALLOCATION PROBLEM ON A TREE GRAPH HAVING BOTH NODAL AND LINK DEMANDS

## 3.1 Introduction

In this chapter, we consider another special case of Problem GAMNLAP. Specifically, we examine the problem of locating an absolute 2-median on an undirected tree network having both nodal and link demands, and for which the facilities have known finite capacities whose sum is equal to the total demand on the tree. Hence, the descriptors, "capacitated" and "balanced", used in the title of this chapter. We will refer to the problem considered herein, as the capacitated 2-median tree problem, where it is to be understood that the problem is balanced and that the tree is as described above.

Recall that Hakimi's [1964,1965] vertex optimality result rendered insignificant any effort to distinguish between the absolute p-median and the p-median problems. Such is not the case however, for minisum problems involving networks having link demands, and thus to be exact, all reference to the problem considered herein, as well as to the research of Chiu [1982], Cavalier and Sherali [1983a], and Batta, Brandeau, and Chiu [1983], should make use of the descriptor "absolute". However, for the sake of convenience, we have chosen not to use it in discussing the capacitated 2-median tree problem or in referencing any of the above papers, but wish to make it quite clear that the facilities are not restricted to the nodes of the tree in any of this work/research.

As was mentioned briefly in Chapter I, the papers by Chiu [1982] and Cavalier and Sherali [1983a], examine the 1-median problem involving the location of an uncapacitated facility on a tree network having both nodal and link demands. Both of these papers present the same Goldman-like [1971] algorithm for locating an uncapacitated 1-median on such a tree, and (both) do so, by using the following optimality criteria:

"It is easy to show that a point (on the tree) is optimal if and only if it is an optimal (constrained) location on all links containing that point. In other words, if an interior point on a link has exactly half the total demand weight on either side of it, then it is an optimal location on T, and if a node in T is such that if the subtrees obtained by disconnecting this node have demand weights no more than half the total demand, then this node is an optimal location in T." - Cavalier and Sherali [1983a].

Of course, the uncapacitated and capacitated 1-median tree problems are actually one and the same problem, and thus in a spirit of completeness, we can state that the capacitated 1-median tree problem has been solved (by the above authors).

In addition to the above papers, Batta, Brandeau, and Chiu [1983], and once again, Cavalier and Sherali [1983a], consider the uncapacitated 2-median tree problem, for such a tree as ours. Cavalier and Sherali's approach to solving this problem involves the solution of constrained problems which result from requiring the two facilities to lie on the unique path joining each pair of end nodes. They provide reduction type theorems to reduce the number of such pairs that need to be considered, thus making their approach more practical. Batta, Brandeau and Chiu's approach to the same problem, is to develop a locate-allocate type of algorithm. That is, "rather than trying to simultaneously find both the correct locations and allocations", their algorithm "works by sequential steps of location and allocation". Our approach to the capacitated 2-median tree problem differs from those above, in that we identify a finite set of points (on the tree) in which a 2-median is known to exist. The following paragraph outlines our approach.

This chapter proceeds as follows. We begin our analysis by extending an observation made by Cavalier and Sherali [1983a] which allows us to use the results of our chain graph analysis (of Chapter II) to solve a (path) constrained version of the capacitated 2-median tree problem. This capability is then exploited in order to develop necessary optimality criteria (Theorem 3.3) for the problem of interest. The optimality criteria are in turn, used to develop an algorithm which has as its purpose, the identification of a reduced (finite) set of candidate optimal solution pairs (on the tree) on which to locate the two facilities. Thereafter, the remaining analysis is directed towards reducing the cardinality of this set of pairs of points, and efficiently comparing their relative costs in order to determine an optimal pair.

## 3.2 Formulation of the Capacitated 2-median Tree Problem

In keeping with the mathematical notation and statement of Problem GAMNLAP presented in Chapter I, the capacitated 2-median tree problem can be formally stated as follows:

Given an n-vertex, undirected tree, $T \equiv T(N,A)$, whose vertices $v_1$, ..., $v_n \varepsilon N$ have associated nonnegative (expected) demands/weights $h_1$, ..., $h_n$, and on whose arcs/links $\ell \varepsilon A$ are defined nonnegative, integrable demand functions $f_\ell(\cdot)$, and given two (2) facilities having known finite capacities/supplies $s_1,s_2$ such that $s_1+s_2$ equals the total demand on T, determine points $y_1, y_2 \varepsilon T$, nodal allocations $\omega_{ik}$, $i=1,2$, $k=1, \ldots, n$, and nonnegative, integrable allocation functions $\Phi_{i\ell}: \ell \to R$, $i=1,2$, $\ell \varepsilon A$, which solve

$$\text{C2MTP}: \quad \underset{\substack{y_1, y_2 \varepsilon T \\ \omega, \Phi(\cdot) \geq 0}}{\text{minimize}} \left( \sum_{i=1}^{2} \sum_{k=1}^{n} \omega_{ik} d(y_i, v_k) + \sum_{i=1}^{2} \sum_{\ell \varepsilon A} \int_{\ell} \Phi_{i\ell}(x) d(x, y_i) dx \right) \quad (3.1)$$

$$\text{subject to} \quad \sum_{k=1}^{n} \omega_{ik} + \sum_{\ell \varepsilon A} \int_{\ell} \Phi_{i\ell}(x) dx = s_i \text{ for each } i=1,2 \quad (3.2)$$

$$\sum_{i=1}^{2} \omega_{ik} = h_k \text{ for each } k=1, \ldots, n \quad (3.3)$$

$$\sum_{i=1}^{2} \Phi_{i\ell}(x) = f_{\ell}(x) \text{ for each } x \varepsilon \ell, \text{ and } \ell \varepsilon A . \quad (3.4)$$

The reader is referred to Chapter I for a review of the parameters, decision variables, etc., of Problem C2MTP. We would mention however, that the orientation (with respect to integration) on any $\ell \varepsilon A$, is determined by that of $f_{\ell}(\cdot)$ in that the $\Phi_{i\ell}(\cdot)$ are defined so as to have the same orientation as $f_{\ell}(\cdot)$, for all $i=1,2$ and all $\ell \varepsilon A$.

Recalling the PFL solution to the chain problem of Chapter II, it should come as no surprise to the reader, that our analysis of the capacitated 2-median tree problem will result in an optimal solution for which no link contains a subset having positive measure over which the demand is jointly supplied. Rather, our optimal solution will be such that only nodal demands can be jointly supplied. (Recall that we are assuming without loss of generality, that all discrete demands are confined to the nodes of T.)

We begin our analysis, by establishing optimality criteria for the capacitated 2-median tree problem, which will be seen to be very similar to the 1-median optimality criteria used by Chiu, Cavalier and Sherali in their 1-median Goldman-like algorithms.

## 3.3 The 2-median Optimality Criteria

In anticipation of the analysis to follow, we remind the reader that Corollary 2.6 is as equally/readily applicable to a chain graph having discrete nodal demands, as to a chain graph like that of Chapter II, where it was assumed that all discrete nodal demands had been continuously spread over some $\varepsilon$-length links. Thus, for example, if one were tasked to locate two facilities having capacities $s_1=30$, $s_2=25$ on a chain graph having nodal and (cumulative) link demands as shown below in Figure 3.1, and such that $y_1$ is to the left of $y_2$ say, then in accordance with Corollary 2.6, $y_1$ would be located so as to coincide with node $v_2$ and the corresponding facility would serve/supply all of the demand from left to right, up to and including six (6) units of the demand at node $v_3$. Similarly, $y_2$ would be that point which is located 2 1/2 demand units to the left of node $v_4$, and the corresponding facility would supply the remainder of the demand on the graph.

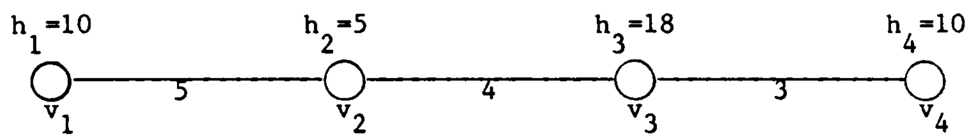
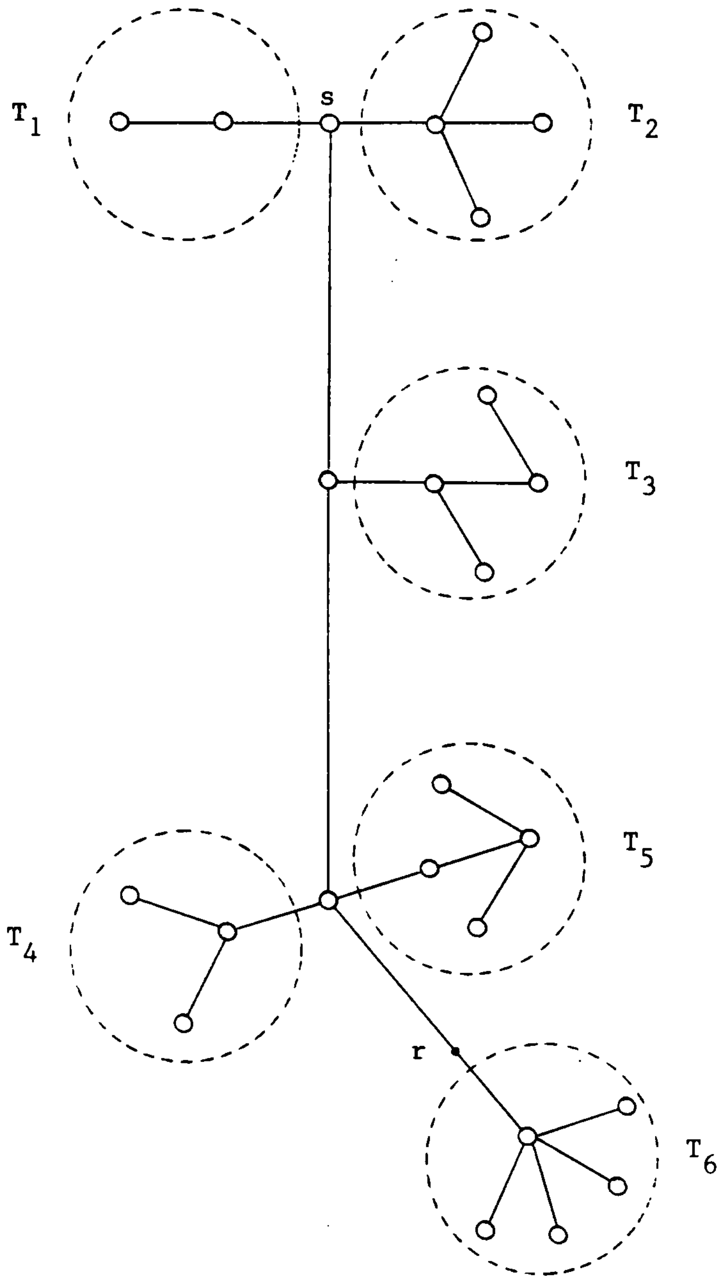



Figure 3.1. Chain Graph With Nodal and Link Demands

Before proceeding, we present some useful notation and definitions, the latter of which are those of Cavalier and Sherali [1983a]. Regarding the paths of T, and letting $r,s\varepsilon T$, we let P[r,s] denote the unique path joining r and s, including points r and s; whereas, P(r,s] say, denotes the path joining r and s, but does not include point r. (Similar interpretations apply to P[r,s) and P(r,s)). Cavalier and Sherali [1983a] provide us with the following two definitions. Let r,s be two points in T, and for some $t \varepsilon P[r,s]$, let $T_t$ be a subtree (if one exists) which results when T is disconnected at t and which is such that it shares no links with P[r,s]. Then $T_t$ will be said to be rooted (at t) on P[r,s]. Secondly, suppose $T_t$ is a subtree of T which is rooted at t on P[r,s]. Then the operation of deleting $T_t$ and adding the total demand/weight associated with $T_t$ to the demand at point t, will be referred to as collapsing $T_t$ into t. We note from the last sentence, that $T_t$ does not include the root node t. Rather, it is to be understood that an artificial node having an associated demand/weight of zero is added to $T_t$ in place of t. Figure 3.2 illustrates the first of these definitions, by depicting six subtrees rooted on P[r,s]. Notice that all but $T_6$ are rooted at nodes of T, whereas the root node of $T_6$ is point r, an interior point of some link of T. (Note: According to the textbook definition of a subtree, $T_6$ is not really a subtree of T since its root node is not an actual node of T. However, we feel that the intent of our presentation is clear, and will continue to refer to such subsets as (rooted) subtrees.)

The following development and resulting lemma, extend a similar result found in Cavalier and Sherali [1983a], which this author considers to be the very basis of their 1 and 2-median tree algorithms. In particular, we will see that we need not concern ourselves as to how two capacitated facilities located on P[r,s] say, would serve any subtree T' rooted on P[r,s], but rather we only need to know the amounts




(Note: Point r is not a node of T, but rather an interior point of some link of T.)

Figure 3.2. Rooted Subtrees

of supply provided T' by each facility, and the distance from each facility to the root node of T'. Thus, with respect to the problem of optimally locating two capacitated facities on P[r,s] to serve/supply the entire demand on T, we would not need to know the cost of serving T' per se, and could collapse T' into its root node and aggregate the total demand of T' with that of the root node. Referring once again to Figure 3.2, if one were interested in locating two capacitated facilities on P[r,s] so as to optimally serve/supply the demand on T, we will see that one could collapse all six subtrees into P[r,s], with each of their total demands being aggregated with that of their respective root node, and thus have a capacitated 2-median chain problem, for which one knows how to obtain an optimal solution via the analysis of Chapter II. (Note: we will refer to the problem in which the 2-median is to be located on some path P[r,s] onto which all of T has been collapsed, as the "2-median (location-allocation) problem collapsed onto the path P[r,s]", or as "Problem C2MTP collapsed onto the path P[r,s]").

More formally, let $T' \equiv T'(N'A')$ be any tree gotten by taking some connected proper subset of T and adding end nodes having associated demands of zero wherever necessary. Let p be any point in T-T', and let q be the unique point in T' which is closest to p (or any point on the path between p and that unique closest point). Then the cost of any feasible service/supply provided T' from a facility located at p, can be written as

$$C(p,T') = \sum_{v_j \in N'} \omega_j(p) d(p,v_j) + \sum_{\ell \in A'} \int_\ell \Phi_{p\ell}(x) d(p,x) dx , \qquad (3.5)$$

where $\omega_j(p)$ is the amount of supply provided to node $v_j$ in T' $(\omega_j(p)<h_j)$ by the facility at p, and where $\Phi_{p\ell}$ is a nonnegative, integrable

allocation function which describes the service/supply provided link $\ell \varepsilon A'$ from the facility at p.

Then applying the triangle inequality property as it applies to a tree, equation (3.5) can be continued so that we have,

$$\begin{aligned} C(p,T') &= \sum_{v_j \varepsilon N'} \omega_j(p)(d(p,q)+d(q,v_j)) + \sum_{\ell \varepsilon A'} \int_\ell \Phi_{p\ell}(x)(d(p,q)+d(q,x))dx \\ &= \Big( \sum_{v_j \varepsilon N'} \omega_j(p)d(q,v_j) + \sum_{\ell \varepsilon A'} \int_\ell \Phi_{p\ell}(x)d(q,x)dx \Big) + d(p,q)W(p) \\ &= C(q,T') + d(p,q)W(p) \ , \end{aligned} \tag{3.6}$$

where $C(q,T')$ is the cost of providing T' with the same (as that which the facility at p is to provide) service/supply but from point q instead, and $W(p)$ is the total amount of supply to be provided T' by the facility at p.

The following lemma and its proof are immediately obvious consequences of the above remarks/analysis, and hence the latter has been omitted.

Lemma 3.1. Let T' be a subtree of T obtained by disconnecting T at some point $q \varepsilon T$, and let the total demand on T' be jointly supplied by two facilities located at $y_1, y_2 \varepsilon$ T-T'. Then the cost of supplying the total demand on T' is given by

$$C(y_1,y_2,T') = C(q,T') + d(y_1,q)W_1 + d(y_2,q)W_2 \ , \tag{3.7}$$

where $C(q,T')$ is the cost of supplying the total demand on T' from q, and where $W_1, W_2$ are the amounts of supply provided T' from each of the facilities at $y_1, y_2$, respectively. (Note: $W_1 + W_2$ = Total Demand on T'.) □

Of course, in the event that q is an actual node of T and has an associated positive demand, we could easily modify equation (3.7) so as to include the cost of supplying/serving the demand at q. This would only require that $W_1$ and $W_2$ be adjusted to account for any such demand, since the cost of serving any demand at q from q is zero. However, given the way we have chosen to define T', we have that $q \notin T'$, and so the cost of serving any positive demand at q is not presently reflected in equation (3.7).

As a result of Lemma 3.1, if one were to consider the problem of optimally locating two capacitated facilities on some P[r,s] from which the entire demand on T is to be supplied, then for <u>any</u> subtree T' rooted at $q \in P[r,s]$, we can collapse T' into q, and equivalently solve the resulting problem. (That is, we can equivalently solve the 2-median location-allocation problem collapsed onto the path P[r,s]). This is easily seen, by noting that for any $y_1, y_2 \in P[r,s]$, equation (3.7) will always contain the value C(q,T'), and hence only the second and third terms of equation (3.7) need be considered in optimally locating $y_1, y_2 \in$ P[r,s]. Thus, as was mentioned earlier, with respect to optimally locating $y_1, y_2 \in P[r,s]$, we need not concern ourselves with how the facilities serve T' (or even with the value of C(q,T')), but only with their distances from q and the amounts of supply $W_1, W_2$, provided T' by each facility.

Lemma 3.1 is analagous to a result used by Cavalier and Sherali (in their 2-median tree algorithm) to solve constrained problems involving the location of two uncapacitated facilities on paths of the type $P[e_i, e_j]$ where $e_i, e_j$ are end nodes of T. Since their facilities are uncapacitated, the total demand of any subtree rooted on

$P[e_i,e_j]$ gets served/supplied by whichever facility is closest to the root node. As such, they are able to collapse the subtree into $P[e_i,e_j]$ via an equation like (3.7), but for which exactly one of $W_1,W_2$ is zero. In light of Lemma 3.1, it seems likely that we could approach/solve the capacitated 2-median tree problem in much the same way as Cavalier and Sherali solve the uncapacitated 2-median tree problem. In fact, we will see later that a result by Mirchandani and Oudjit [1980] concerning the relationship between a 1 and 2-median, and used by Cavalier and Sherali to lessen the computational burden of their algorithm, is still valid (and useful) for the capacitated 2-median tree problem. However, the reduction theorems used by Cavalier and Sherali are not valid for Problem C2MTP and thus we can not pursue a Cavalier-Sherali-like solution to this problem, but rather we will continue towards our objective (for this section) of establishing necessary optimality criteria for a 2-median of T, which will then be used (in a later section) to prescribe a reduced set of candidate optimal solution pairs (of points) in T.

The following lemma will be used in conjunction with Lemma 3.1, to assist us in obtaining the main result of this section.

Lemma 3.2. Suppose that $y_1^*,y_2^* \varepsilon T$ are the locations of an optimal solution to the capacitated 2-median tree problem (i.e. to Problem C2MTP). Then these same locations are optimal to the 2-median problem collapsed onto any path $P[r,s] \supseteq P[y_1^*,y_2^*]$, where $r,s \in T$. Furthermore, any alternative optimal solution to this additionally constrained problem solves Problem C2MTP.

<u>Proof</u>. The proof is immediate, since constraining the facilities to lie on any subset of T cannot result in a lesser cost of serving the demand on T, and since any solution to the constrained problem is feasible to Problem C2MTP. □

Regarding Lemmas 3.1 and 3.2, two remarks seem quite appropriate at this time. First, we would remark that Lemma 3.2 is equivalent to the very thinking of Cavalier and Sherali, when in their 2-median tree algorithm, they solve constrained (path) problems and then choose the one giving the least cost. Secondly, we note that solving constrained problems such as those of Lemma 3.2, is made easier thanks to Lemma 3.1 and the analysis of Chapter II.

We are now in a position to present the main result of this section and do so via the following theorem.

<u>Theorem 3.3</u>. Suppose that $y_1^*, y_2^* \varepsilon T$ are the locations of an optimal solution to the capacitated 2-median tree problem, and that $y_1^* \neq y_2^*$. (Note that if $y_1^* = y_2^*$, then the two facilities and their supplies constitute a 1-median, and thus the theorem follows from the 1-median optimality conditions/criteria of Chiu [1982] and Cavalier and Sherali [1983a].) Disconnect T at $y_1^*$ and let $T_1, \ldots, T_m$ be the resulting subtrees, with $y_2^* \varepsilon T_m$, say. In keeping with our earlier development, let us assume that $T_i$ does not include $y_1^*$, but rather that an artificial node having an associated demand of zero has been added in its place, for i=1, ..., m. Then letting $\omega(\cdot)$ denote a total weight function, we have that

$$\omega(T_i) < s_1/2 \quad \text{for each } i=1, \ldots, m-1 , \tag{3.8}$$

and (more importantly)

$$\sum_{i=1}^{m-1} \omega(T_i) + \omega(y_1^*) > s_1/2 \ . \tag{3.9}$$

(A symmetric statement holds with respect to $y_2^*$.)

Proof. Given any subtree T' of T, define E[T'] to be the set of ends of T'.

If m=1, let $e_1 \equiv y_1^*$ and $e_2 \varepsilon$ E[T] such that $y_2^* \varepsilon P[e_1,e_2]$. If m>1, pick any $i \varepsilon \{1, \ldots, m-1\}$ and let $e_1 \varepsilon E[T] \cap T_i$ and $e_2 \varepsilon E[T] \cap T_m$ such that $P[e_1,e_2] \supseteq P[y_1^*,y_2^*]$. (In either case, $e_2$ may be $y_2^*$.) Consider the 2-median location-allocation problem collapsed onto the path $P[e_1,e_2]$. By Lemma 3.2, $y_1^*,y_2^*$ remain optimal. But $\omega(T_i)$ equals the total weight on the collapsed path $P[e_1,y_1^*)$, i.e. not including the collapsed weight at $y_1^*$, and $(\sum_{i=1}^{m-1} \omega(T_i) + \omega(y_1^*))$ equals the total weight on the collapsed path $P[e_1,y_1^*]$, i.e. including the collapsed weight at $y_1^*$. Hence, from the optimality conditions for the chain graph (see Chapter II), we obtain 3.8 and 3.9 and this completes the proof. □

As promised, Theorem 3.3 provides us with necessary optimality criteria/conditions for a solution to the capacitated 2-median tree problem, i.e. to Problem C2MTP. This theorem will prove itself to be quite useful in our approach to solving this problem.

## 3.4 Subproblems Inherent to Problem C2MTP and a Useful Reduction Theorem

As somewhat of an aside to the current direction of our analysis, but still pertinent (and informative) to the problem of interest, we take a moment to discuss the location and allocation subproblems inherent to Problem C2MTP. We will begin with the location subproblem, which can be described as follows.

Suppose that we are given nonnegative values $\omega_{ij}$, i=1,2, j=1, ..., n, and nonnegative, integrable functions $\Phi_{i\ell}:\ell \to R$, i=1,2, $\ell \varepsilon A$, which satisfy conditions (3.2), (3.3), and (3.4) of Problem C2MTP. Such values and functions are said to constitute a feasible allocation to Problem C2MTP. The corresponding location subproblem becomes that of determining points $y_1^*, y_2^* \varepsilon$ T which minimize the objective function given in (3.1). It is clear that for a given and fixed (feasible) allocation, the objective function can be separated into two functions, one in each of $y_1$ and $y_2$, whose minima (each) occur at points which are 1-medians with respect to their associated allocations. Thus, if one were given an allocation feasible to (3.2), (3.3), and (3.4) for each of two capacitated facilities, and one were asked to locate the facilities on T so as to minimize the objective function in (3.1), one would need to solve two independent 1-median tree problems, for which one would use the 1-median Goldman-like algorithms of Chiu, Cavalier and Sherali. We remark that this is so, regardless (assuming feasibility) of how a facility's supply is allocated. Even if entire nodes and/or links receive no supply from a facility, that facility's location is determined by applying the 1-median algorithm to the tree, where the allocation of the other facility is simply ignored while doing so. So we see that solving the location subproblem is a relatively simple task, thanks to the work of researchers such as Goldman, Chiu, Cavalier and Sherali.

Turning now to the allocation subproblem, and letting $y_1, y_2 \varepsilon$ T be given and fixed locations of two capacitated facilities having corresponding supplies $s_1, s_2$ (where $s_1 + s_2$ = total demand on T), the problem of interest is to determine values for the allocation variables

$\omega_{ij}$, and the allocation functions $\Phi_{i\ell}$, $i=1,2$, $j=1, \ldots, n$ and $\ell \varepsilon A$, so that $y=(y_1,y_2)$, the $\omega_{ij}$, and the $\Phi_{i\ell}$ satisfy (3.2), (3.3), (3.4), and minimize the objective function in (3.1). We do so quite easily, by collapsing T into $P[y_1,y_2]$ and then determining the optimal PFL allocation for the resulting chain graph. We remind the reader however, that the actual allocation on any subtree which gets collapsed into $P[y_1,y_2]$ has been (up to now) unknown, unimportant (i.e. with respect to the value of the objective function), and is clearly not unique. That is, if some subtree T' rooted on $P[y_1,y_2]$ has a total weight/demand of $\omega(T')$, and the facilities at $y_1,y_2$ supply $W_1,W_2$, respectively, of the demand on T' such that $\omega(T') = W_1 + W_2$, it simply does not matter (due to Lemma 3.1) how these facilities allocate their respective supply to T'. Thus, there is (or could be) a certain amount of flexibility in choosing the elements of the optimal PFL allocation solution to the allocation subproblem.

So we see that the location and allocation subproblems are easily solved, a fact that we will soon appreciate, when confronted by a reduced set of candidate optimal solution points from which to choose $y_1^*$ and $y_2^*$.

Before concluding this section, we present a theorem which extends a result by Mirchandani and Oudjit [1980] describing a useful relationship between the 1 and 2-medians of a graph. We would remark however, that their line of proof is not valid in the current situation.

<u>Theorem 3.4</u>. Suppose that $T = (N,A)$, $s_1,s_2$ and $\omega(\cdot)$ are as defined above. Let $\bar{y}\varepsilon T$ be the location of a 1-median, and $y_1^*,y_2^*\varepsilon T$, the locations of a 2-median (i.e. of an optimal solution to the capacitated 2-median

tree problem). Then we have that $\bar{y} \in P[y_1^*,y_2^*]$ .

<u>Proof</u>. If either of $y_1^*,y_2^*$ coincides with $\bar{y}$, then the theorem is trivially true. Hence, assume that this is not the case. Now, disconnect T at $\bar{y}$. If $y_1^*$ and $y_2^*$ lie in different resulting subtrees, then again the theorem holds. Hence, assume that $y_1^*$ and $y_2^*$ both belong to the same subtree $T_m$, say, and note that $\omega(T_m) < (s_1+s_2)/2$ by the definition of $\bar{y}$. (Recall that $\bar{y} \notin T_m$, but rather that $\bar{y}$ has been replaced by an artifical node of weight zero. Consequently, $\omega(T_m)$ does not include $\omega(\bar{y})$). Now consider two cases.

First, suppose that $y_2^* \in P[y_1^*,\bar{y}]$. (It may be that $y_1^*$ and $y_2^*$ coincide.) Select $e_1,e_2 \in E[T]$ such that $P[e_1,e_2] \supseteq P[y_1^*,\bar{y}]$ with $e_1 \in T_m$, and consider the 2-median location-allocation problem collapsed onto the path $P[e_1,e_2]$. Accordingly, $T-T_m$ gets collapsed onto $P[\bar{y},e_2] \subseteq P[e_1,e_2]$, and by Lemma 3.2, $y_1^*,y_2^*$ are optimal to Problem C2MTP collapsed onto $P[e_1,e_2]$. However, $\omega(T-T_m) > (s_1+s_2)/2$ and so the PFL solution results in the facility at $y_2^*$ serving strictly more than $s_2/2$ on one side of itself, a contradiction. Similarly, a contradiction results if $y_1^* \in P[y_2^*,\bar{y}]$.

Hence, suppose finally that $y_1^* \neq y_2^*$, $y_1^*,y_2^* \in T_m$, neither coincide with $\bar{y}$ and that the paths $P[y_1^*,\bar{y}]$ and $P[y_2^*,\bar{y}]$ have some $q \in T_m$ as the first intersection point, where q does not coincide with any of $y_1^*,y_2^*$ or $\bar{y}$. Now, select $e_1,e_2 \in E[T] \cap T_m$ such that $P[e_1,e_2] \supseteq P[y_1^*,y_2^*]$ and consider the 2-median location-allocation problem collapsed onto the path $P[e_1,e_2]$. By Theorem 3.3, the total weight on the collapsed path $P[e_1,y_1^*]$, i.e. including the collapsed weight at $y_1^*$, is $> s_1/2$.

Similarly, the total weight collapsed onto $P[e_2,y_2^*]$ is $\geq s_2/2$. If either of these inequalities was to hold as a strict inequality, we would have that $\omega(T_m) > (s_1+s_2)/2$, a contradiction to the fact that $\omega(T_m) \leq (s_1+s_2)/2$. Hence, both must hold as equalities. Furthermore, the remainder of the total demand on T, i.e. $(s_1+s_2)/2$, must be concentrated at q in the collapsed problem, or else we would again have $\omega(T_m) > (s_1+s_2)/2$. But this means that the pair $(y_1^*,q)$ is an alternative optimal solution to the collapsed 2-median problem on $P[e_1,e_2]$, and hence by Lemma 3.2, is optimal to Problem C2MTP as well. Now, applying the first case, since $q \in P[y_1^*,\bar{y}]$, we obtain a contradiction and the proof is complete. □

Note that this theorem says that for <u>any</u> 1-median $\bar{y}$, and <u>any</u> 2-median $y_1^*,y_2^*$, the relationship $\bar{y} \in P[y_1^*,y_2^*]$ must hold.

Note also (from the second paragraph of the above proof), that if there exists a 2-median $(y_1^*,y_2^*)$ such that $y_1^* = y_2^*$, then its location must coincide with that of the <u>unique</u> 1-median of T.

We will of course use the above theorem for the same purpose as do Mirchandani and Oudjit [1980], that being to reduce the set of candidate optimal solution points/pairs.

## 3.5 Obtaining a Reduced Set of Candidate Optimal Solutions

In this section, we present an algorithm/procedure which utilizes the optimality criteria of Theorem 3.3 (in particular (3.8)) to construct two trees, denoted by $T(s_1/2)$ and $T(s_2/2)$, which are contained entirely within T and whose ends constitute candidate optimal solution points for $y_1^*$ and $y_2^*$, respectively. More generally, the algorithm defines a mapping of $[0,(s_1+s_2)/2]$ into the set of all connected subsets of T, but it is $T(s_1/2)$ and $T(s_2/2)$ that are of most interest to us. We would remark that the mapping need not be one-to-one. (Note: most likely, $T(s_1/2)$ and $T(s_2/2)$ will not be actual subtrees of T, for their ends are not all likely to be ends of T but rather artificial ends created by collapsing portions of T into interior points of T.)

To facilitate our presentation of this algorithm, we introduce the following notation involving the total weight function $\omega(\cdot)$. Specifically, for any link $\ell = (r,s)\varepsilon A$, $\omega(r,s]$ will denote the sum of the weight/demand on $\ell$ and at s, i.e. $(\int_\ell f_\ell(x)dx + h_s)$. Similarly, we would define the weights, $\omega[r,s)$, $\omega[r,s]$, and $\omega(r,s)$, where the latter is equal to $\int_\ell f_\ell(x)dx$.

Algorithm $T(\theta)$. Consider the capacitated 2-median tree problem, and without loss of generality, assume that $s_1 < s_2$. Then for any constant $0 < \theta < (s_1+s_2)/2$, we construct a tree, $T(\theta) \subseteq T$, as follows:

Initialization Let T be the tree of Problem C2MTP and having no nodes flagged.

Step 1 Pick an end e of the current tree which is not flagged. If all ends are flagged, stop; the current tree is $T(\theta)$. Otherwise, proceed to Step 2.

Step 2 Let (v,e) be the link incident at e. If $\omega(v,e] < \theta$, collapse (v,e) into v, add $\omega(v,e]$ to the weight of v, and return to Step 1. Otherwise, i.e. $\omega(v,e]>\theta$, go to Step 3.

Step 3 If it exists, let $\bar{v}\ \varepsilon$ int(v,e) be the point closest to v such that $\omega[\bar{v},e] = \theta$. In this case, collapse $(\bar{v},e)$ into $\bar{v}$, denote $\bar{v}$ as being a node (of the resulting tree) whose weight is equal to $\omega[\bar{v},e]$, flag $\bar{v}$, and go to Step 1. If no such point exists, then flag e and go to Step 1.

Remarks Concerning Algoritm $T(\theta)$

R1) Without loss of generality, we assume that the tree T of Problem C2MTP is such that there does not exist an end e for which $\omega[\bar{v},e] = 0$ for some $\bar{v} \varepsilon \text{int}(v,e)$, where $(v,e)$ is the link incident at e. As a result, we have that $T = T(0)$.

R2) Algorithm $T(\theta)$ is such that every end of $T(\theta)$ has either a total weight $> \theta$ or has a total weight $= \theta$ and has a continuum of demand in some immediately adjacent neighborhood of itself. That is, in the case of the latter, if e is an end of $T(\theta)$ with $\omega(e) = \theta$, and if $(v,e)$ is the link (in $T(\theta)$) incident at e, then there exists some $\bar{v} \varepsilon \text{int}(v,e)$ for which the demand function is positive on $(\bar{v},e)$.

R3) In light of the second remark, $T((s_1+s_2)/2)$ is a single point, and is in fact, a 1-median of T.

R4) If $0 < \theta_1 < \theta_2 < (s_1+s_2)/2$, then $T(\theta_2)$can be determined via the above algorithm by first determining $T(\theta_1)$ and then initializing the algorithm with this tree, whence, the resulting $T(\theta_2)$ would be a subset of the $T(\theta_1)$ obtained above.

Recall that we mentioned in the beginning of this section that Algorithm $T(\theta)$ defines a mapping of $[0,(s_1+s_2)/2]$ into the set of all connected subsets of T. To be totally correct, we would remark that the mapping is a point-to-set mapping in that $T((s_1+s_2)/2)$ need not always be uniquely determined. However, the following theorem proves that $T(\theta)$ is uniquely determined for any $\theta \varepsilon [0,(s_1+s_2)/2)$.

Theorem 3.5 Let $0 < \theta < (s_1+s_2)/2$, and let $T(\theta)$ be obtained via the foregoing algorithm. Then, $T(\theta)$ is uniquely determined by this algorithm and contains all of the 1-medians of T. In particular, if $T(\theta)$ is a single node, then it is the unique 1-median of T.

Proof. We consider two cases.

Case (i) Suppose that $T(\theta)$ has at least two ends, and let e be any such end. Note that $T(\theta)$ has a link (v,e) incident at e. Disconnect T at e and let $T_1, \ldots, T_m$ be the subtrees (in T) so obtained, with $v \in T_m$, say. (Note that m=1 is possible.) Observe that it must be the case that $\omega(T_i) < \theta$ for each i=1, ..., m-1, and $\sum_{i=1}^{m-1} \omega(T_i) + \omega(e) > \theta$, with remark R2 above holding in addition. Consequently, e must be an end of any $T(\theta)$ resulting from the algorithm, and since e was arbitrarily selected, $T(\theta)$ must be unique in this case. Furthermore, since $\omega(T_i) < \theta < (s_1+s_2)/2$ for each i=1, ..., m-1, no 1-median of T could lie in any such $T_i - \{e\}$. Since this is true for each end of $T(\theta)$, we can conclude that all 1-medians of T must lie in $T(\theta)$.

Case (ii) Next, suppose that $T(\theta)$ is a single node, which is some point $v \in T$, say. Disconnect T at v and let $T_1, \ldots, T_m$ be the resulting subtrees. It must be the case that $\omega(T_i) < \theta < (s_1+s_2)/2$ for each i=1, ..., m, and so v is a 1-median of T. By this same inequality and the median property, no 1-median of T could possibly lie in $T_i - \{v\}$ for any i=1, ..., m, and so v must be the unique 1-median of T. This completes the proof. $\square$

A Numerical Example

As an illustration of Algorithm $T(\theta)$, consider the tree T shown in

Figure 3.3a whose nodes have been labeled A through I. The numbers appearing in parentheses denote nodal demands/weights, while those adjacent to the links, denote the cumulative demands/weights on the corresponding links. We will assume that all link demand functions are positive on their respective links.

Letting $\theta = 42$, say, application of Algorithm T(42) begins with the initialization step in which T is as shown in Figure 3.3a, and none of the six ends are flagged.

Proceeding to Step 1 and arbitrarily choosing node A say, we then proceed to Step 2 and collapse (H,A) into H, augment the demand at H by 15, and return to Step 1. The same procedure is performed for each of nodes B,C, ..., F, at which point in time, we find ourselves at Step 1 with the current tree being that of Figure 3.3b.

Noting that the weights of G,H and I have changed but that neither G or H is flagged, we arbitrarily choose end H and then proceed to Step 2. The "otherwise" of Step 2 applies, and we therefore move immediately to Step 3 where we flag H (via an asterisk) and return to Step 1.

At this point, we are at Step 1 with end G being the only end which is not flagged. Thus, we select G and proceed to Step 2, at which time we are immediately sent to Step 3, where we determine and flag $\bar{v} \equiv G'$, and then return to Step 1 with the current tree being that of Figure 3.3c. Noting that both G' and H are flagged, we stop with T(42) being uniquely determined (recall Theorem 3.5) as the tree in Figure 3.3c.

The usefulness of the above algorithm can best be seen in the following theorem, which can be accurately described as the main result of this section.

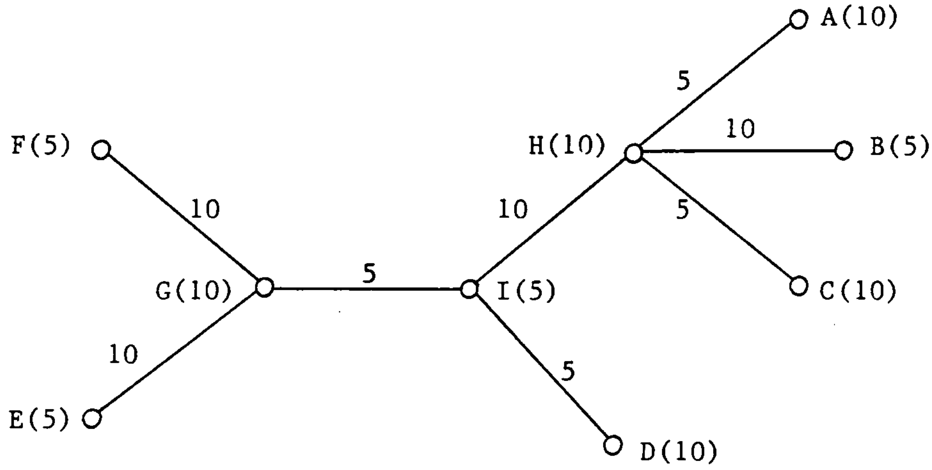



Figure 3.3a

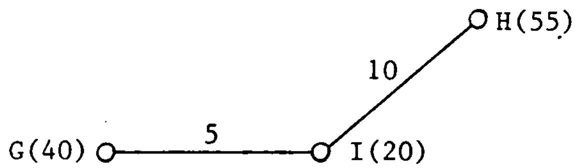



Figure 3.3b

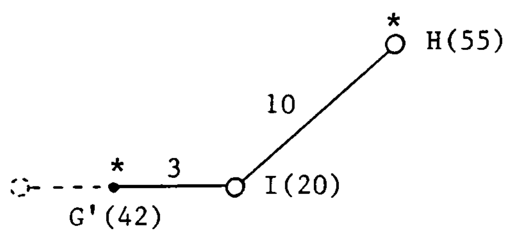



Figure 3.3c

Figure 3.3. Example of Algorithm T(θ)

Theorem 3.6. Let T, T($\theta$), $s_1 < s_2$, $y_1^*$ and $y_2^*$ all be as defined above (and note that $T(s_2/2) \subseteq T(s_1/2)$). Then, there exists an optimal 2-median $(y_1^*, y_2^*)$ such that $y_1^*, y_2^*$ are ends of $T(s_1/2)$, $T(s_2/2)$, respectively. Similarly, let $y_1$ be located at some end of $T(s_1/2)$. Then there exists an optimal location for $y_2$ (with $y_1$ fixed) among the ends of $T(s_2/2)$, and vice versa.

Proof. Consider an optimal solution $(y_1^*, y_2^*)$ to Problem C2MTP.

If $y_1^* = y_2^*$, then by Theorem 3.4, they must coincide with the unique 1-median location $\bar{y}$ on T. Now, let $T_1, \ldots, T_m$ be the subtrees of T obtained by disconnecting T at $\bar{y}$. Then, $\omega(T_i) < s_1/2$ for each i=1, ..., m, since if not, say, $\omega(T_1) > s_1/2$, then by picking an end $e_1 \varepsilon E[T] \cap T_1$ and considering the 2-median problem collapsed onto the path $P[e_1, \bar{y}]$, the solution $y_1^* = y_2^* = \bar{y}$ is suboptimal, and so by Lemma 3.2, leads to a contradiction. Similarly, $\omega(T_i) < s_2/2$ and $\omega(T_i) < (s_1+s_2)/2$ for each i=1, ..., m. Therefore, $y_1^* = y_2^* = \bar{y} \equiv T(s_1/2) \equiv T(s_2/2) \equiv T((s_1+s_2)/2)$, and the theorem holds.

Now suppose that $y_1^* \neq y_2^*$, and consider Problem C2MTP collapsed onto path $P[y_1^*, y_2^*]$. Then for this collapsed problem, we have from (3.9) of Theorem 3.3 that $\omega(y_1^*) \geq s_1/2$ and $\omega(y_2^*) \geq s_2/2$. If $\omega(y_1^*) > s_1/2$, let $v_1 = y_1^*$, and if $\omega(y_1^*) = s_1/2$, define $v_1 \varepsilon P[y_1^*, y_2^*]$ to be the point closest to $y_2^*$ for which $\omega(P[y_1^*, v_1]) = s_1/2$. Similarly, if $\omega(y_2^*) > s_2/2$, let $v_2 = y_2^*$, and if $\omega(y_2^*) = s_2/2$, let $v_2$ be the point closest to $y_1^*$ for which $\omega(P[v_2, y_2^*]) = s_2/2$. Clearly, $v_2 \varepsilon P[v_1, y_2^*]$. Moreover, $(v_1, v_2)$ is an alternative optimal solution to the collapsed path problem, and so by Lemma 3.2, to Problem C2MTP also. If $v_1 = v_2$, then the case is as the preceeding one. Otherwise, by Theorem 3.3 and Theorem 3.5, $v_1$ and $v_2$ are respectively ends of $T(s_1/2)$ and $T(s_2/2)$ by construction, and the first part of the proof is complete.

In order to prove the "similarly" part of the theorem, we proceed in much the same way as we did for the first part of the theorem. Specifically, let $y_1$ be fixed at some end of $T(s_1/2)$. Let us denote Problem C2MTP so constrained, as Problem C2MTP($y_1$), and let $y_2^*$ be any optimal solution to Problem C2MTP($y_1$). Clearly, a result like that of Lemma 3.2 can be shown to hold for Problem C2MTP($y_1$). Specifically, $y_2^*$ remains optimal to Problem C2MTP($y_1$) collapsed onto any path containing $y_1$ and $y_2^*$, and any alternative optimal solution to this collapsed problem would also solve Problem C2MTP($y_1$). It is this result that is used in the remainder of this proof.

If $y_1 = y_2^*$, and $T_1, \ldots, T_m$ are the subtrees obtained by disconnecting T at $y_2^*$, then $\omega(T_i) \leq s_2/2$ for each $i=1, \ldots, m$, since otherwise, if $\omega(T_i) > s_2/2$, then by picking $e \in E[T] \cap T_i$ and examining Problem C2MTP($y_1$) collapsed onto the path $P[y_2^*,e]$, an improved solution would result. But $\omega(T_i) \leq s_2/2$ for all $i=1, \ldots, m$, implies that $y_2^* \equiv T(s_2/2)$ and the result holds.

Hence, suppose that $y_1 \neq y_2^*$ and consider Problem C2MTP($y_1$) collapsed onto $P[y_1,y_2^*]$. Clearly, we must have that $\omega(y_2^*) \geq s_2/2$ or else $y_2^*$ would not be optimal to Problem C2MTP($y_1$). Then, by letting $v = y_2^*$ if $\omega(y_2^*) > s_2/2$, and otherwise defining it to be that point closest to $y_1$ for which $\omega(P[v,y_2^*]) = s_2/2$, we see that v is an alternative optimal solution to Problem C2MTP($y_1$) collapsed onto $P[y_1,y_2^*]$, and hence to Problem C2MTP($y_1$). Clearly then, v is an end of $T(s_2/2)$ by construction, and the proof is complete. □

Corollary 3.7. $y_1^* = y_2^*$ is an optimal (2-median) solution to Problem C2MTP iff $T(s_1/2) = T(s_2/2) = T((s_1+s_2)/2)$.

Proof. The necessary part of the theorem follows from the proof of Theorem 3.6. The sufficiency part follows from Remark R3 and Theorem 3.6. □

Note: We caution the reader to be careful so as not to think that all 2-medians must necessarily lie within the ends of $T(s_1/2)$, $T(s_2/2)$. Recall that Algorithm $T(\theta)$ would collapse $(v,e)$ into $v$, should $e$ be an end of $T$ having a weight of $\theta$ and should $\omega(v,e) = 0$. Thus, if $\theta = s_1/2$ and $y_1^* = v$, say, where $y_1^*$ is an element of some 2-median $(y_1^*, y_2^*)$, then any point in the interior of $(v,e)$ could be used in place of $y_1^*$. Algorithm $T(s_1/2)$ would pass by/ignore such points.

The above theorem goes a long way towards reducing the number of candidate optimal solution points/pairs that need to be considered. However, depending on the number of ends of the two trees involved, the problem that remains may still be quite difficult. It is this very problem to which we will devote the remainder of this chapter.

## 3.6 Methods of Determining Optimal Locations for $y_1, y_2$ Amongst the Ends of $T(s_1/2)$ and $T(s_2/2)$, Respectively

Before proceeding with this section, we would remark that just as we did in Theorem 3.6, we will assume that $s_1 < s_2$ throughout the remainder of this chapter.

Now letting $E[T(s_1/2)]$ and $E[T(s_2/2)]$ denote the set of ends of each of the respective trees, and noting that the elements of these sets must have weights greater than or equal to $s_1/2$ and $s_2/2$ respectively, we can determine a least upper bound for $|E[T(s_2/2)]|$, where the bars are used to denote cardnality. We do so , via the following lemma.

<u>Lemma 3.8</u>. In accordance with the above definitions, theorems, lemmas, and algorithm, it follows that $|E[T(s_2/2)]| \leq 3$.

<u>Proof</u>. Suppose that $|E[T(s_2/2)]| \geq 4$. Then letting ND denote the total demand on the nodes of $T(s_2/2)$, and letting D denote the total demand on T, we have

$$ND \geq 4(s_2/2) = 2s_2 \begin{cases} = D \text{ if } s_1 = s_2 \,. & (3.10a) \\ > D \text{ if } s_1 < s_2 \,. & (3.10b) \end{cases}$$

Clearly (3.10b) is impossible, and (3.10a) would imply that ND = D = $2s_2$, so that $T(s_2/2)$ has 4 ends with each end having a weight of $s_2/2$, whereas the remainder of the tree has no demand upon it. Such can not be the case however, for Algorithm $T(s_2/2)$ would not produce such a tree, but rather would continue to collapse such a tree into a single point. Therefore, we have that $|E[T(s_2/2)]| \leq 3$ and the proof is complete. $\square$

Now, defining a tripod to be any tree having exactly three (3) ends, we present the following useful corollary to the above lemma.

<u>Corollary 3.9</u>. If $T(s_2/2)$ is a tripod, then $T(s_1/2) \supseteq T(s_2/2)$ is also a tripod.

<u>Proof</u>. If $s_1 = s_2$, the proof follows immediately, since $T(s_1/2) = T(s_2/2)$ from Theorem 3.5. Thus, let us suppose that $s_1 < s_2$ <u>and</u> that $T(s_2/2)$ is a proper subset of $T(s_1/2)$, so that $\Gamma = T(s_1/2) - T(s_2/2) \neq \emptyset$. (Note: $s_1 < s_2$ does not imply that $T(s_2/2)$ is a proper subset of $T(s_1/2)$. It is

possible to have $s_1 < s_2$ and $T(s_1/2) = T(s_2/2)$). Recall that the point-to-set mapping defined by Algorithm $T(\theta)$ need not be 1-1.)

Now suppose that e is an end of $T(s_1/2)$ which is also an end of some tree (in $\Gamma$) rooted at an interior point of $T(s_2/2)$. (Figure 3.4a depicts such a situation where $T(s_2/2)$ is the tripod connecting the ends labelled a,b,c, and where the tree rooted at $\alpha$ lies in $\Gamma$.) Then the combined sum of the weights/demands at e and the three (3) ends of $T(s_2/2)$ is $\geq 3(s_2/2) + (s_1/2) = s_2 + s_2/2 + s_1/2 > s_2 + s_1 = D$, since $s_1 < s_2$. Clearly this is impossible, and so we now know that all of $\Gamma$ must be rooted on the ends of $T(s_2/2)$. Thus, it follows that each and every end of $T(s_1/2)$ must either be coincident with an end of $T(s_2/2)$, or it must be an end of some tree (in $\Gamma$) rooted at an end of $T(s_2/2)$.

Finally, suppose that $e_1, \ldots, e_m$ are all of the ends of all of the trees (in $\Gamma$) rooted at some end e of $T(s_2/2)$. (See Figure 3.4b.) Then the combined sum of the weights at $e_1, \ldots, e_m$ and at the two ends of $T(s_2/2)$ different from e is $\geq m(s_1/2) + 2(s_2/2) = m(s_1/2) + s_2$. Clearly then, $m \leq 2$. Note however, that $m=2$ would imply that there is no demand on the links of $T(s_2/2)$, and that the ends of $T(s_2/2)$ other than e have weights equal to $s_2/2$. This contradicts Algorithm $T(s_2/2)$. Thus, it must be the case that $m=1$, and since this is true for any of the three ends of $T(s_2/2)$, we can conclude that $T(s_1/2)$ has exactly three ends and is therefore a tripod. This completes the proof. $\square$

A result such as that of Corollary 3.9 is very nice indeed. Unfortunately, such a result does not apply to the case where $|E[T(s_2/2)]| = 1$ or 2, as is shown by the following examples.

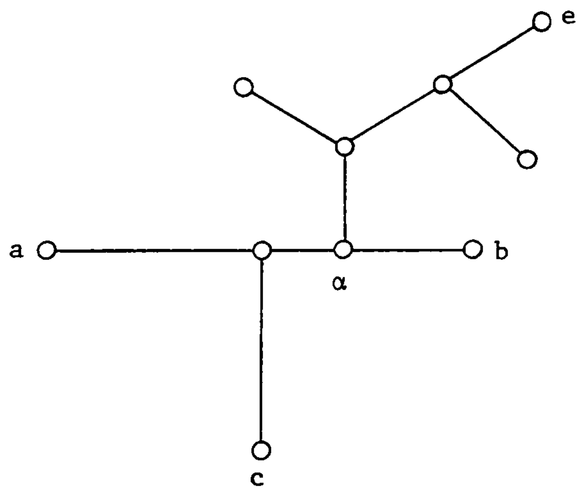



Figure 3.4a. (See Corollary 3.9)

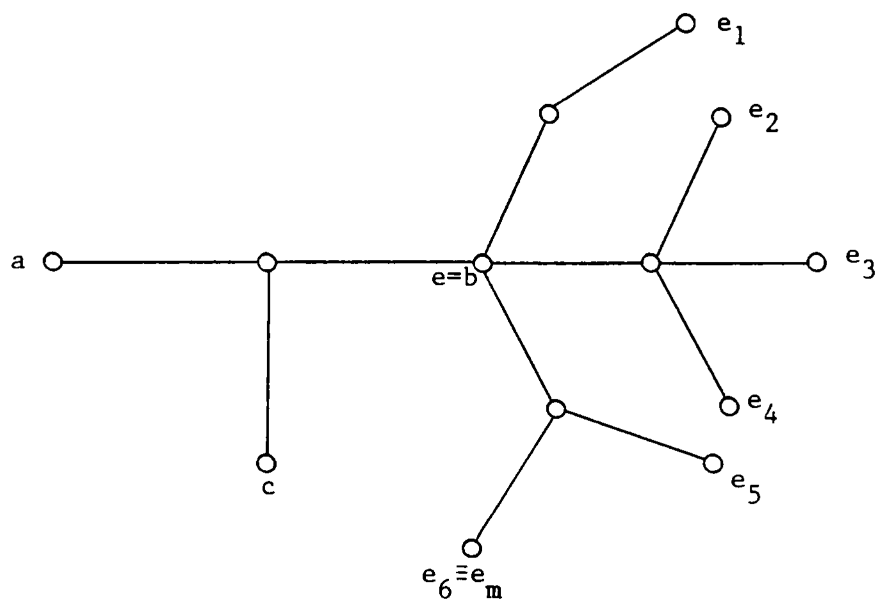
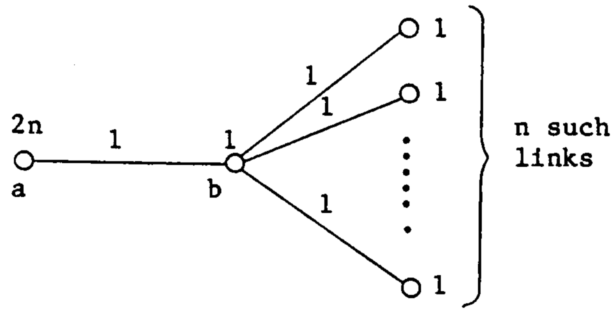



Figure 3.4b. (See Corollary 3.9)

Example 3.1. (A case where $|E[T(s_2/2)]| = 2$ and $|E[T(s_1/2)]| = n+1$, for any $n = 1,2, \ldots$ )

Let T be the following tree, where the numbers appearing on the links denote the total cumulative weights on the links and are due to uniform density functions say, and where D, $s_1$ and $s_2$ are as given:

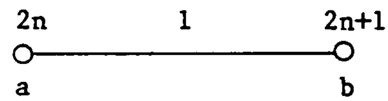



$D = 4n + 2 =$ Total Demand

$s_1 = 2$

$s_2 = 4n$

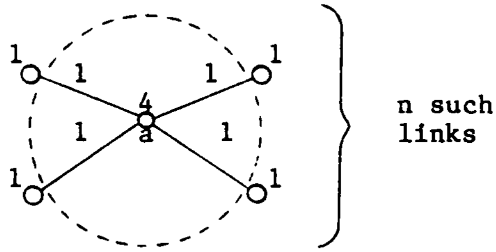
Then according to Algorithm T(θ), $T(s_1/2) = T$ and $T(s_2/2)$ is the subtree,




Note: Admittedly, this is not a very interesting example, in that the decision as to where to locate the facilities is relatively simple due to the n links having identical demand distributions (and assuming all link lengths are equal). However, the example does illustrate the issue at hand.

Example 3.2. (A case where $|E[T(s_2/2)]| = 1$ and $|E[T(s_1/2)]| = n$, for any $n = 1,2, \ldots$ )

Let T be a tree having n links all of which are incident at a common node having a weight of four (4), and such that each link contains an end of T. Such a tree would look like the following, with D, $s_1$ and $s_2$ as given:




$D = 4 + 2n$ = Total Demand

$s_1 = 2$

$s_2 = 2 + 2n$

Then according to Algorithm $T(\theta)$, $T(s_1/2) = T$ and $T(s_2/2)$ is the single point "a", having a weight of D. This concludes the example.

Continuing in our effort to determine $y_1^* \varepsilon\ E[T(s_1/2)]$ and $y_2^* \varepsilon\ E[T(s_2/2)]$ (thanks to Theorem 3.6), we next present a "catch-all" type of lemma which consists of properties involving $T(s_1/2)$, $T(s_2/2)$, and $\bar{y}$, any 1-median of T.

<u>Lemma 3.10.</u>

(a) $\bar{y}\ \varepsilon\ T(s_2/2) \subseteq T(s_1/2)$ for any 1-median $\bar{y}$.

(b) If $\bar{y}\ \varepsilon\ E[T(s_1/2)]$, then $\bar{y}\ \varepsilon\ E[T(s_2/2)]$.

(c) If $\bar{y}\ \varepsilon\ E[T(s_2/2)]$, then $|E[T(s_2/2)]| < 2$.

(d) If $|E[T(s_2/2)]| = 3$, then T has a unique 1-median $\bar{y}$, and it is located at the first point of intersection of the paths from any two (2) ends to the third end of $T(s_2/2)$. We denote this point as q.

<u>Proof.</u>

(a) By Theorem 3.5, we know that $T(s_1/2)$ and $T(s_2/2)$ are uniquely determined and that both contain all 1-medians of T. Also, since $s_1 < s_2$, we have that $T(s_2/2) \subseteq T(s_1/2)$ by construction. Thus, we have that $\bar{y}\ \varepsilon\ T(s_2/2) \subseteq T(s_1/2)$ for any 1-median $\bar{y}$.

(b) This follows directly from (a).

(c) We know from Lemma 3.8 that $|E[T(s_2/2)]| < 3$. Suppose for the moment, that $T(s_2/2)$ has three (3) ends, and that some 1-median $\bar{y}$ is coincident with one of them. Then, letting ND denote the total demand on the nodes of $T(s_2/2)$, we have

$$ND \geq 2(s_2/2) + (s_1+s_2)/2 \quad \left.\right\} \begin{array}{lll} > D & \text{if} & s_1 < s_2 . \quad (3.11a) \\ = D & \text{if} & s_1 = s_2 . \quad (3.11b) \end{array}$$

Since (3.11a) is physically impossible, and since (3.11b) implies that the links of $T(s_2/2)$ have no demand upon them (which is also impossible due to Algorithm $T(s_2/2)$), we must conclude that $|E[T(s_2/2)]| \leq 2$.

(d) If $|E[T(s_2/2)]| = 3$, then $T(s_2/2)$ is a tripod, whose ends we will label as A, B and C. Let $\bar{y}$ be any 1-median of T. By parts (a) and (c), we know that $\bar{y}$ lies in the interior of $T(s_2/2)$, i.e. $\bar{y} \varepsilon T(s_2/2) - E[T(s_2/2)]$. Now, without loss of generality, let us assume that $\bar{y} \varepsilon \text{int}(A,q)$, say. Then, collapse both (q,B) and (q,C) into point q, and consider the 1-median problem on (A,q). Due to the fact that the demand/weight at each of B and C is $\geq s_2/2$, and also due to remark R2, we have that the collapsed weight at point q is strictly greater than $(s_1 + s_2)/2$. Clearly then, $\bar{y} \varepsilon \text{int}(A,q)$ does not satisfy the necessary conditions of a 1-median, and hence it must be the case that $\bar{y}$ is uniquely located at point q. This completes the proof of Lemma 3.10. □

(Note: all future reference to the above lemma will be according to Lemma 3.10x, where $x \varepsilon \{a,b,c,d\}$.)

The utility of the above lemma is best seen and most appreciated, when the lemma is used in conjunction with a property/result like that of Theorem 3.4, in which a physical relationship between the 1 and 2-medians of a tree is established. Given what we now know regarding the locations of $y_1^*$, $y_2^*$ (i.e. Theorem 3.6), the value/worth of results such as Theorem 3.4 and Lemma 3.10 becomes quite apparent.

Finally, we present a theorem which brings together many of the above results and which hints of an algorithm to be used in determining $y_1^* \varepsilon E[T(s_1/2)]$, $y_2^* \varepsilon E[T(s_2/2)]$. We feel that this theorem allows for a timely, necessary and expeditious transition from our current "ivory-towered" type of analysis of Problem C2MTP, to one in which we are forced to "get our hands dirty" so to speak; that is, one in which we actually examine specific cases of Problem C2MTP in the hope of discovering (computationally) efficient methods of determining $y_1^*$, $y_2^*$ for such cases.

<u>Theorem 3.11</u>. Let $\bar{y}$ be any (fixed) 1-median of T. (Note that if $|E[T(s_2/2)]| = 1$ or 3, then $\bar{y}$ is in fact the unique 1-median of T.) Choose any end of $T(s_1/2)$ and denote it by $e^*$, say. Then suppose that $y_1^*$ is located (fixed) at $e^*$, and determine the set $\Lambda(e^*) \equiv \{e \varepsilon E[T(s_2/2)]: \bar{y} \varepsilon P[e^*, e], \bar{y} \neq e\}$.

Regarding the choice of $y_2^* \varepsilon E[T(s_2/2)]$ (recall Theorem 3.6), the following holds:

If $\Lambda(e^*) = \emptyset$, then $y_2^* = \bar{y}$ is optimal.

If $\Lambda(e^*) \neq \emptyset$, then one of the ends in $\Lambda(e^*)$ is the best location for $y_2^*$.

Additionally, $|\Lambda(e^*)| < 2$, and so if $|\Lambda(e^*)| = 1$, the location of $y_2^*$ is known immediately.

<u>Proof</u>. That $|\Lambda(e^*)| < 2$ follows immediately by first noting that $\Lambda(e^*) \subseteq E[T(s_2/2)]$ and then calling upon Lemma 3.8, Corollary 3.9 and

Lemma 3.10d, in that order.

Now, $\Lambda(e^*) = \emptyset$ implies that $|E[T(s_2/2)]| = 1$ or 2, and if it is 1, then $T(s_2/2)$ is a single point and is in fact equal to the unique 1-median of T (see Theorem 3.5). Thus, $\bar{y} = T(s_2/2)$ and so by Theorem 3.6, we have that $y_2^* = \bar{y}$ and the theorem holds. In the case where $|E[T(s_2/2)]| = 2$, $T(s_2/2)$ is a chain graph, and so $\Lambda(e^*) = \emptyset$ implies that $\bar{y} \in T(s_2/2)$ is one of the two ends of the chain. More importantly, $y_2^*$ must be located coincident to $\bar{y}$ since locating it at the other end of $T(s_2/2)$ would not satisfy Theorem 3.4 (since $\Lambda(e^*) = \emptyset$). This completes the "$\Lambda(e^*) = \emptyset$" part of the theorem.

If $\Lambda(e^*) \neq \emptyset$, then two cases arise.

(i) First, suppose that $e^* = y_1^* = \bar{y}$. Then by Lemma 3.10 b and c and our assumption that $\Lambda(e^*) \neq \emptyset$, we have that $|E[T(s_2/2)]| = 2$, so that $T(s_2/2)$ is a chain graph of which $\bar{y}$ is one of the two ends, and point v say, is the other (end). More importantly, v is both the optimal location of $y_2^*$ (otherwise, more than $s_2/2$ units of demand would lie to one side of $y_2^*$) and the sole element of $\Lambda(e^*)$.

(ii) Now suppose that $e^* = y_1^* \neq \bar{y}$. If $\bar{y} \notin E[T(s_2/2)]$, the theorem follows from Theorems 3.4 and 3.6.

If $\bar{y} \in E[T(s_2/2)]$, then for any $e \in \Lambda(e^*) \neq \emptyset$, we have a path $P[e,y_1^*]$ in $T(s_1/2)$ which can be decomposed into $P[e,\bar{y}] \cup P[\bar{y},y_1^*]$, where $P[e,\bar{y}] \subseteq T(s_2/2)$ and $P(\bar{y},y_1^*]$ is contained in $T(s_1/2) - T(s_2/2)$. Now clearly $y_2^*$ could not be located at $\bar{y}$, for then the (collapsed) demand on $[e,\bar{y})$ would exceed $s_2/2$ in violation of Theorem 3.3. (Recall our second remark concerning Algorithm $T(\theta)$.) Thus, it follows that $y_2^*$ will be located coincident to some element of $\Lambda(e^*)$. This completes the proof. □

Having reduced Problem C2MTP to that of using Theorem 3.11 to assist us in determining $y_1^*$, $y_2^*$ in $E[T(s_1/2)]$ and $E[T(s_2/2)]$, respectively, we are now in need of computationally efficient methods of determining the cost of any given pair of locations, $y_1 \varepsilon E[T(s_1/2)]$, $y_2 \varepsilon E[T(s_2/2)]$. As such, the following section constitutes what might be descriptively referred to as a "post-localization" analysis of Problem C2MTP.

## 3.7 Comparing the Relative Costs of Candidate Solutions

In organizing this section, we have decided to follow the lead of Lemma 3.8 and to partition the section into three cases, according to the cardinality of $E[T(s_2/2)]$. We begin with the case in which $T(s_2/2)$ has exactly three ends.

### Case 1: $|E[T(s_2/2)]| = 3$

Recall that from Corollary 3.9 and Lemma 3.10d, we have that $T(s_1/2) \supseteq T(s_2/2)$ is also a tripod, i.e. $|E[T(s_1/2)]| = 3$, and that T has a unique 1-median, $\bar{y} \varepsilon T(s_2/2)$, which is located at the first point of intersection of the paths from any two ends of $T(s_2/2)$ to the third end of $T(s_2/2)$. Consequently, the situation of Case 1 can be pictorally represented as shown in Figure 3.5a, where $\{z_1, z_2, z_3\} \equiv E[T(s_2/2)]$ and $\{e_1, e_2, e_3\} \equiv E[T(s_1/2)]$. Note of course, that unlike the situation depicted in Figure 3.5a, it is possible for any of the pairs $(e_i, z_i)$, $i=1,2,3$, to be such that $e_i = z_i$. Similarly, $T(s_1/2)$ and $T(s_2/2)$ could have many more discrete points of positive demand than as shown in the above figure. However, since the following analysis (of Case 1) applies in any case, one can without loss of generality, assume the situation to be that of Figure 3.5a.

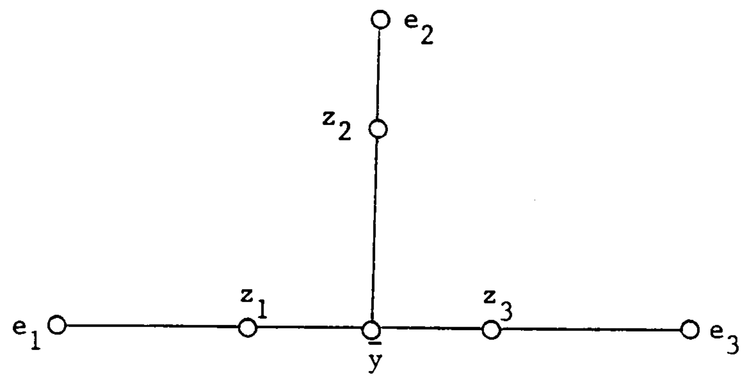



Figure 3.5a. Case 1

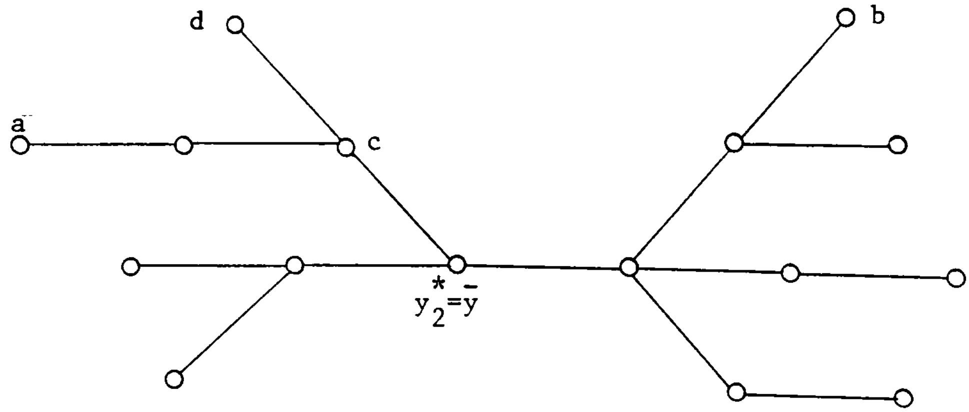



Figure 3.5b. Case 2

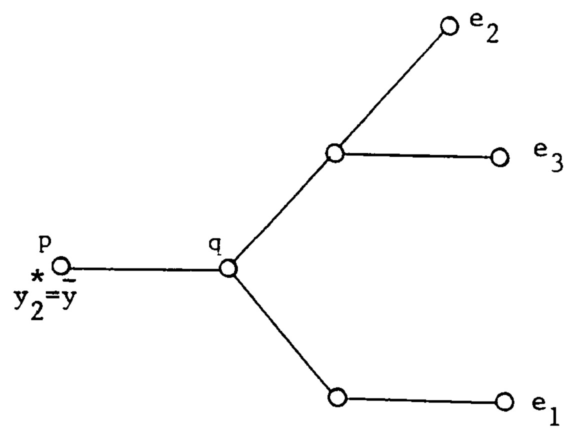



Figure 3.5c. Case 2

Now we know from Theorem 3.4, that $(y_1^*, y_2^*) = (e_i, z_j)$, $i,j\varepsilon\{1,2,3\}$, must be such that $i \neq j$. Therefore, an exhaustive determination of $y_1^* \varepsilon\ E[T(s_1/2)]$, $y_2^* \varepsilon\ E[T(s_2/2)]$ would require one to compute and compare the relative costs of only six (6) candidate solution pairs, $(y_1, y_2) = (e_i, z_j)$, $i \neq j$, $i,j\varepsilon\{1,2,3\}$. Even so, it would be to ones advantage to have a computationally efficient method of determining such costs. As such, consider the following two remarks.

<u>Remark 1</u>: If $y_1^* = e_i$ for some $i\varepsilon\{1,2,3\}$, then the corresponding facility must serve more than $P[e_i, \bar{y}]$, since $\omega(z_j^+) + \omega(z_k^+) > s_2$ for $j,k \neq i$, $j,k\varepsilon\{1,2,3\}$ and where $\omega(z_j^+) \equiv \omega(z_j)$ plus the weight/demand on some $\varepsilon$-segment of the link incident at $z_j$ in $T(s_2/2)$ (recall remark R2 in section 3.5). Similarly, if $y_2^* = z_j$ for some $j\varepsilon\{1,2,3\}$, then the corresponding facility must serve more than $P[e_j, \bar{y}]$.

We would ask the reader to note, that in making Remark 1, we have utilized Lemma 3.1 so as to be able to restrict our attention (with respect to cost computation) to $T(s_1/2)$, since the costs of serving any subtrees (of T) rooted on $T(s_1/2)$ remain the same regardless of where we finally locate $y_1 \varepsilon\ E[T(s_1/2)]$, $y_2 \varepsilon\ E[T(s_2/2)]$. Note also, that Remark 1 makes use of the fact (from Chapter II) that if $(y_1^*, y_2^*) = (e_i, z_j)$ for some $i \neq j$, then the PFL allocation is optimal to the chain graph problem obtained by collapsing $T(s_1/2)$ onto $P[e_i, e_j]$. Thus , what Remark 1 is telling us, is that if $(y_1^*, y_2^*) = (e_i, z_j)$, $i \neq j$, then the facility located at $e_i$ serves all of $P[e_i, \bar{y})$, the facility located at $z_j$ serves all of $P[e_j, \bar{y})$, and that $P[\bar{y}, e_k]$, $k \neq i,j$, is jointly supplied/served by the facilities at $e_i$ and $z_j$.

The following general remark, used in conjunction with Remark 1, will enable us to efficiently compute the cost of any candidate solution pair, $(y_1,y_2) = (e_i,z_j)$, $i \neq j$, $i,j \in \{1,2,3\}$.

Remark 2: Consider any path $P[r,s]$ in T and let $C_r,C_s$ be the costs of serving/supplying the entire demand on $P[r,s]$ from the points r and s, respectively. Then, letting $\ell \equiv d(r,s)$ and $W \equiv \omega(P[r,s])$, we have that $C_r + C_s = \ell W$.

With respect to the issue at hand, the value of Remark 2 lies in the fact that if one of $C_r,C_s$ has been computed, then the other is easily obtained. Note that Remark 2 follows easily from the observation that the cost of serving a demand $\Delta$ on an incremental segment a distance x from r is $x\Delta$ if served from r and $(\ell-x)\Delta$ if served from s, which totals $\ell\Delta$.

The remainder of our analysis of Case 1 defines various partial costs that are to be used in computing the costs of candidate solutions, $(y_1,y_2) = (e_i,z_j)$, $i \neq j$, $i,j \in \{1,2,3\}$. It is to be assumed throughout our analysis of this case, that Remark 2 is used to facilitate the computation of these partial costs. Specifically, for each of $j = 1,2,3$, let

$$\delta_j^1 \equiv \text{cost of serving } P(z_j,\bar{y}) \text{ from } z_j, \text{ and}$$

$$\delta_j^2 \equiv \text{cost of serving } P(z_j,e_j) \text{ from } z_j .$$

Then by using $\delta_j^1,\delta_j^2$, $j=1,2,3$, and only appropriate weights and distances (i.e. no further integrations are required), one can compute the following:

$$\alpha_i \equiv [\text{cost of serving } P[e_i,\bar{y}) \text{ from } e_i] + d(e_i,\bar{y})[s_1-\omega(P[e_i,\bar{y}))],\ i=1,2,3$$

$$\beta_j \equiv [\text{cost of serving } P[e_j,\bar{y}) \text{ from } z_j] + d(z_j,\bar{y})[s_2-\omega(P[e_j,\bar{y}))],\ j=1,2,3$$

$$\gamma_k \equiv \text{cost of serving } P(\bar{y},e_k] \text{ from } \bar{y},\ k=1,2,3$$

Clearly then, for any candidate solution $(y_1, y_2) = (e_i, z_j)$, $i \neq j$, the total cost relevant for comparative purposes is given by

$$C_{ij} = \alpha_i + \beta_j + \gamma_k \text{ , where } k \neq i,j, \text{ and } i,j,k \in \{1,2,3\} .$$

To see this, we would again refer the reader to Lemma 3.1 and remark that $P(\bar{y}, e_k]$ is a subtree (in $T(s_1/2)$) rooted on $P[e_i, e_j]$ at point $\bar{y}$.

Finally, we note that since $\{C_{ij} : i \neq j,\ i,j \in \{1,2,3\}\}$ has so few elements, one could simply compute all $C_{ij}$ and then pick the smallest, thereby obtaining an optimal solution (i.e. a 2-median) to Problem C2MTP. On the other hand, a more elegant solution procedure would be to construct the 3x3 matrix

$$\Pi \equiv \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \text{ , and then solve the following}$$

assignment problem:

$$\text{minimize } z(x) = \sum_{i=1}^{3} \sum_{j=1}^{3} \pi_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^{3} x_{ij} = 1 \text{ for } i=1,2,3$$

$$\sum_{i=1}^{3} x_{ij} = 1 \text{ for } j=1,2,3$$

$$\text{all } x_{ij} \geq 0 .$$

Any basic feasible solution to this linear system will necessarily have $x_{ij} = 0$ or $1$ for all $i,j$, and so there will exist only one allocation from each row and column. Therefore, the $\pi_{ij}$ corresponding to the nonzero variables of any optimal solution to the above assignment problem will determine the minimum $C_{ij}$ value and thus identify the location of $y_1^* \epsilon\ E[T(s_1/2)]$, $y_2^* \epsilon\ E[T(s_2/2)]$.

This concludes our analysis of Case 1.

Case 2: $|E[T(s_2/2)]| = 1$

In this case, $T(s_2/2)$ is a single point and is therefore equal to the unique 1-median of T (see Theorem 3.5). Thus we have that $y_2^* = \bar{y}$, and so if $T(s_1/2)$ is a single node, we are done. It is unfortunate however, that unlike the situation in Case 1, $T(s_1/2)$ can now have any number of ends (see Example 3.2), and so as one might expect, the task of determining $y_1^* \varepsilon E[T(s_1/2)]$ will (usually) require a greater effort for this case than it did for Case 1. Nevertheless, we will present a very simple iterative procedure by which one can determine $y_1^* \varepsilon E[T(s_1/2)]$, but first, we must consider a problem for which an efficient solution methodology is crucial to our procedure for determining $y_1^*$.

Reduction Algorithm R(r,z,s)

Let $r,z,s \varepsilon T$ be such that $z \varepsilon P(r,s)$. Consider the chain problem, denoted by R(r,z,s), which results from collapsing all of T onto P[r,s], fixing $y_2$ at point z, and then having to determine which of points r and s is the better location for $y_1$.

We know from Chapter II, that the PFL allocation is optimal to Problem R(r,z,s) for each of $(y_1,y_2) = (r,z)$ and $(y_1,y_2) = (s,z)$, and so we can compute the cost corresponding to each of these solutions and then simply choose the one having least cost. Obviously, the dominant factor with respect to the efficiency of any such cost computation is the number of integrations required to determine the cost of a solution. In particular, it would seem that those integrals involving a distance component are the keys to the success or failure (with respect to computational efficiency) of any method of cost computation, and so we will proceed under the assumption that only such integrations are of any importance with respect to the issue of efficiency, and that distance and weight computations are negligible with respect to the same. Consequently, and without loss of generality, we will assume that Problem R(r,z,s) is such that all discrete nodal demands have been continuously spread across $\varepsilon$-length links so that P[r,s] contains no discrete points of positive demand. This assumption will serve to expedite our discussion of Problem R(r,z,s).

Any method of computing the costs of $(y_1,y_2) = (r,z)$ and $(y_1,y_2) = (s,z)$ will necessarily require that one first determine the service intervals defined by the corresponding PFL allocations. As such, one must determine the breakpoints $\alpha,\beta \in P[r,s]$ such that $[\omega(P[r,\alpha]) > s_1$ and $\omega(P[\alpha,s]) > s_2]$ and $[\omega(P[r,\beta]) > s_2$ and $\omega(P[\beta,s]) > s_1]$. Note that $\alpha$ will always be less than or equal to $\beta$ since $s_1 < s_2$, but that the relationship between $z,\alpha$ and $\beta$ can vary from problem to problem. For the sake of discussion/illustration, let us assume that we are dealing with a situation in which $r < \alpha < z < \beta < s$.

Now, direct computation of the costs corresponding to the solutions $(y_1,y_2) = (r,z)$ and $(y_1,y_2) = (s,z)$ requires the evaluation of six (6) integrals, and is given by

$$C(r,z) \equiv \int_r^{\alpha} f(x)d(x,r)dx + \int_{\alpha}^{z} f(x)d(x,z)dx + \int_z^{s} f(x)d(x,z)dx$$

$$C(s,z) \equiv \int_r^{z} f(x)d(x,z)dx + \int_z^{\beta} f(x)d(x,z)dx + \int_{\beta}^{s} f(x)d(x,s)dx \ ,$$

where $f(\cdot)$ is the demand function defined on P[r,s].

We propose a procedure, denoted by Algorithm R(r,z,s), whereby one can compare the values of C(r,z) and C(s,z) by using only two (2) integrals. Specifically, we utilize Remark 2 (see Case 1), the integrals $\Pi_1 \equiv \int_r^\alpha f(x)d(x,r)dx$, $\Pi_2 \equiv \int_\beta^s f(x)d(x,\beta)dx$, and some appropriate weights and distances to compute the difference, $C(r,z) - C(s,z)$. We proceed as follows (recall that we are assuming $r < \alpha < z < \beta < s$ and that $P[r,s]$ has no discrete points of positive demand):

$$C(r,z) \equiv \int_r^\alpha f(x)d(x,r)dx + \int_\alpha^z f(x)d(x,z)dx + \int_z^\beta f(x)d(x,z)dx + \int_\beta^s f(x)d(x,z)dx$$

$$C(s,z) \equiv \int_r^\alpha f(x)d(x,z)dx + \int_\alpha^z f(x)d(x,z)dx + \int_z^\beta f(x)d(x,z)dx + \int_\beta^s f(x)d(x,s)dx$$

$$\therefore\ C(r,z)-C(s,z) = \int_r^\alpha f(x)d(x,r)dx + \int_\beta^s f(x)d(x,z)dx - \int_r^\alpha f(x)d(x,z)dx - \int_\beta^s f(x)d(x,s)dx \qquad (3.12)$$

Now,

$$\int_\beta^s f(x)d(x,z)dx = \int_\beta^s f(x)[d(z,\beta)+d(x,\beta)]dx = d(z,\beta)\omega(P[\beta,s]) + \int_\beta^s f(x)d(x,\beta)dx = d(z,\beta)\omega(P[\beta,s]) + \Pi_2\ , \qquad (3.13)$$

$$\begin{aligned}\int_r^\alpha f(x)d(x,z)dx &= \int_r^\alpha f(x)[d(x,\alpha)+d(\alpha,z)]dx = \int_r^\alpha f(x)d(x,\alpha)dx+d(\alpha,z)\omega(P[r,\alpha]) \\ &= d(r,\alpha)\omega(P[r,\alpha]) - \int_r^\alpha f(x)d(x,r)dx + d(\alpha,z)\omega(P[r,\alpha]) \\ &= d(r,z)\omega(P[r,\alpha]) - \Pi_1\ , \qquad (3.14)\end{aligned}$$

$$\begin{aligned}\int_\beta^s f(x)d(x,s)dx &= d(\beta,s)\omega(P[\beta,s]) - \int_\beta^s f(x)d(x,\beta)dx \\ &= d(\beta,s)\omega(P[\beta,s]) - \Pi_2. \qquad (3.15)\end{aligned}$$

Then, substituting (3.13),(3.14) and (3.15) into (3.12) gives us,

$$C(r,z)-C(s,z) = 2(\Pi_1+\Pi_2) - \omega(P[r,\alpha])d(r,z) + \omega(P[\beta,s])[d(z,\beta)-d(\beta,s)]. \qquad (3.16)$$

Now, the very essence of Algorithm R(r,z,s) is embodied in equation (3.16), wherein one must simply compute the values of its "ingredients" and thereby determine the relationship between C(r,z) and C(s,z). Such a task would lend itself readily to computer implementation, and hence our decision to refer to such as an algorithm.

Finally, we would remark that an equation like that of (3.16) could be derived for any given relationship involving $r,z,s,\alpha,\beta$ and for any demand distribution on P[r,s]. Consequently, we will assume that Algorithm R(r,z,s) is such that it can solve any Problem R(r,z,s). This concludes our discussion of Problem R(r,z,s) and so we now return to the original task of determining $y_1^* \varepsilon$ $E[T(s_1/2)]$ for Case 2.

Our procedure for determining $y_1^* \varepsilon$ $E[T(s_1/2)]$ simply compares two ends (of $T(s_1/2)$) at a time, and then eliminates the one having a greater associated cost. This process is continued until $y_1^* \varepsilon$ $E[T(s_1/2)]$ is determined. Specifically, let $T_1, \ldots, T_m$ be the subtrees of $T(s_1/2)$ obtained by disconnecting T at $y_2^* = \bar{y}$, and for the moment, let us assume that $m > 1$ so that $y_2^* = \bar{y}$ is an interior point of $T(s_1/2)$. The situation could then be something like that of Figure 3.5b, in which $m = 3$ and $|E[T(s_1/2)]| = 8$. We would mention that in general, $|E[T(s_1/2)]| < 2(1+s_2/s_1)$. To see this, let $\rho = |E[T(s_1/2)]|$, ND $\equiv$ the total weight of the ends of $T(s_1/2)$, and note that $\rho(s_1/2) \leq \mathrm{ND} < s_1 + s_2$, where the first inequality follows from Theorem 3.3 and the second, from the construction methodology of Algorithm $T(s_1/2)$.

Now, proceed as follows. Choose any ends (of $T(s_1/2)$) $e_i, e_j$ such that they belong to different elements of $\{T_1, \ldots, T_m\}$. Apply Algorithm $R(e_i, y_2^*, e_j)$ to Problem $R(e_i, y_2^*, e_j)$ and thereby eliminate one of $e_i, e_j$ from further consideration. In so doing, one would then collapse the eliminated end and a portion of $P[e_i, e_j]$ into its respective subtree ($T_i$ or $T_j$) so as to obtain a reduced subtree having one less end. For example, suppose that one selects the ends denoted by "a" and "b" in Figure 3.5b, and that the solution to Problem $R(a, y_2^*, b)$ is such that end "a" gets eliminated. One would then collapse $P[a,c]$ into point "c" and repeat the process by considering Problem $R(d, y_2^*, b)$, say. Continuing as such, one will arrive at a point in which only one subtree rooted at $y_2^*$ remains, such as is depicted in Figure 3.5c, say. (Note that the remainder of our discussion handles the $m = 1$ case.) Note that fewer than $(1+s_2/s_1)$ ends of $T(s_1/2)$ will survive this process since any subtree of $T(s_1/2)$ rooted at $y_2^*$ must have a total demand $< (s_1 + s_2)/2$. Now, for comparative purposes among survivors, one can collapse $P[p,q]$ into point q and continue the process just described by solving Problem ($e_1, y_2^*$ at q, $e_2$ or $e_3$), etc.. Thus, by collapsing and using Algorithm $R(\cdot,\cdot,\cdot)$ in this manner, one will eventually obtain $y_1^*$. This concludes our analysis of Case 2.

<u>Case 3: $|E[T(s_2/2)]| = 2$</u>

In this case, $T(s_2/2)$ is a chain graph having ends $z_1, z_2$, say. We consider two subcases.

<u>Subcase 3a</u> Suppose that one of the ends of $T(s_2/2)$ is a 1-median of T. Clearly, $z_1$ and $z_2$ can not both be 1-median locations of T, and so without loss of generality, let us suppose that $z_1 = \bar{y}$.

Now, $T(s_1/2)$ can not have any subtrees rooted on $P(z_1, z_2)$, and so $T(s_1/2)$ might look something like that of Figure 3.6a. To show that this is true, one can argue by contradiction that if there was a subtree

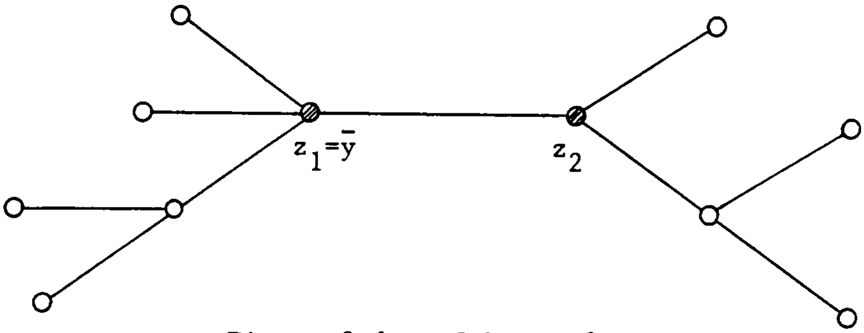



Figure 3.6a. Subcase 3a

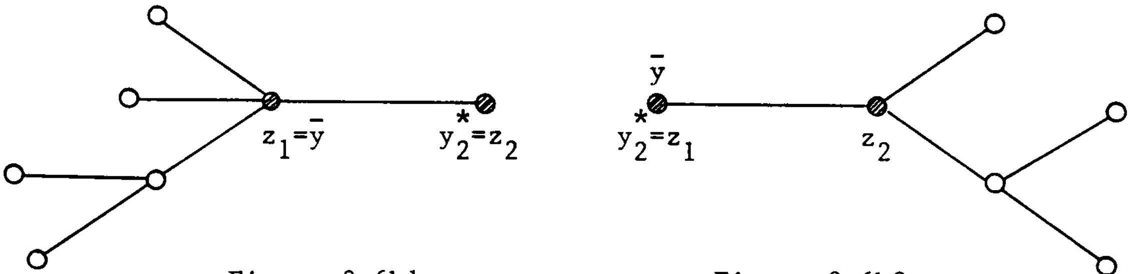



Figure 3.6b1 Figure 3.6b2

Subcase 3a

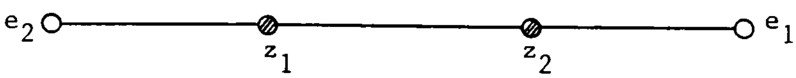



Figure 3.6c. Subcase 3a

rooted on $P(z_1,z_2)$, then $z_1 = \bar{y}$ would imply that $\omega(z_2) = z_2/2$ and $\omega(P(z_1,z_2)) = 0$, in which case, Algorithm $T(s_2/2)$ would result in $T(s_2/2) \equiv z_1$, thereby contradicting that $|E[T(s_2/2)]| = 2$.

It follows from Theorem 3.4 that the current situation (as depicted in Figure 3.6a) can be decomposed into considering the two situations shown in Figure 3.6b. Note that with respect to Theorem 3.4, $y_2^*$ could have been located at either end of $T(s_2/2)$ in Figure 3.6b1, but that the optimality of the PFL allocation requires that $y_2^* = z_2$. One can now determine an optimal location for $y_1^*$ for each of Figures 3.6b1 and 2 via the $m = 1$ procedure of Case 2, and thereby obtain the two candidate solutions $(y_1^*,y_2^*) = (e_2,z_2)$ and $(y_1^*,y_2^*) = (e_1,z_1)$, as depicted in Figure 3.6c.

The task that remains is to compare the above two solutions, and as such, one could proceed in much the same way as was done for Case 2. Specifically, one would first determine the breakpoint corresponding to the PFL allocation of each solution, and then use these two points in conjunction with $e_2,z_1,z_2,e_1$ to decompose $P[e_2,e_1]$ into segments. Next, one would determine the cost of serving each such segment from its left endpoint say, and then use these costs/expressions to derive an expression/formulae for the difference in costs between the two solutions. As indicated in Case 2, such a formulae could easily be incorporated into a computer algorithm to be used as often as necessary.

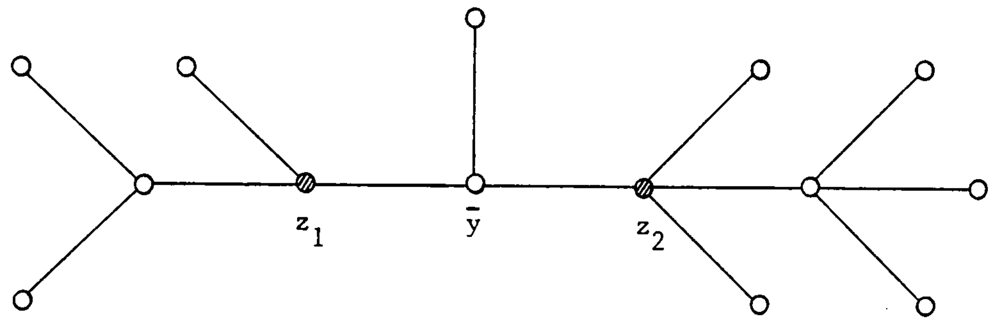
<u>Subcase 3b</u> Suppose that neither end of $T(s_2/2)$ is a 1-median of T, but recall (from Theorem 3.5) that $T(s_2/2)$ contains all 1-median locations of T. Now, unlike Subcase 3a in which all subtrees (of $T(s_1/2)$) rooted

on $P[z_1,z_2]$ must be rooted at $z_1$ or $z_2$, Subcase 3b is such that there can also exist exactly one subtree having exactly one end which is rooted at some 1-median $\bar{y} \in P(z_1,z_2)$. As such, $T(s_1/2)$ might look something like that of Figure 3.7a. A simple proof by contradiction can be used to establish the above remark concerning the geometry of $T(s_1/2)$.

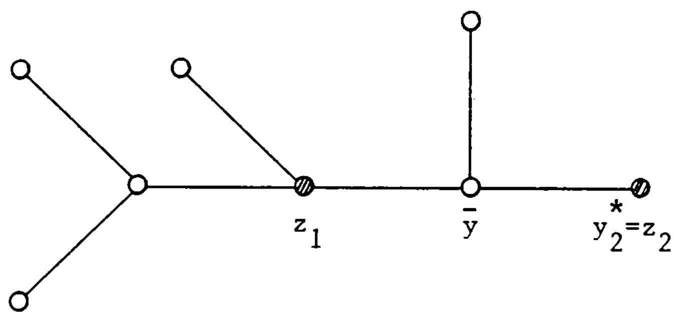
In the event that $T(s_1/2)$ has no subtree rooted on $P(z_1,z_2)$, one is again dealing with Subcase 3a, and so let us assume that the present situation is that of Figure 3.7a. As such, one can utilize Theorem 3.4 to decompose the present situation into those of Figures 3.7b and c, in which $y_2^*$ is fixed at the points $z_2$ and $z_1$, respectively.

Now by utilizing the m = 1 procedure of Case 2, one can obtain a pair of candidate solutions, $(y_1^*,y_2^*) = (e_1,z_1)$, $(y_1^*,y_2^*) = (e_2,z_2)$ which are themselves located according to one of Figures 3.8a,b,c, and d. The task that remains is to choose the least costly of these solutions. As such, one would proceed as in Subcase 3a, should the candidate solutions be located as in Figure 3.8a. On the other hand, if Figure 3.8b describes their locations, one would simply collapse $P[p,q]$ into q, locate $y_1^*$ at $q = \bar{y}$, and then solve Problem $R(z_1,y_1^* = \bar{y},z_2)$ via Algorithm $R(z_1,y_1^* = \bar{y},z_2)$.

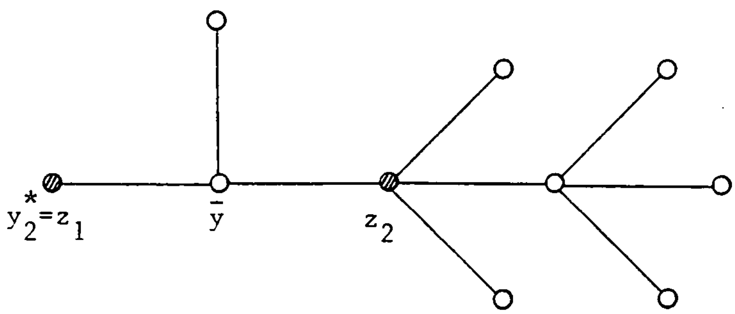
Finally, should $(y_1^*,y_2^*) = (e_1,z_1)$, $(y_1^*,y_2^*) = (e_2,z_2)$ be located according to Figure 3.8c (or Figure 3.8d), one could proceed in much the same way as in Subcase 3a. Specifically, one would determine the breakpoint corresponding to the PFL allocation of each solution, and then partition Figure 3.8c accordingly. Then utilizing the costs of serving the resulting segments of the figure, one could derive a




<u>Figure 3.7a</u>. Subcase 3b




<u>Figure 3.7b</u>. Subcase 3b




<u>Figure 3.7c</u>. Subcase 3b

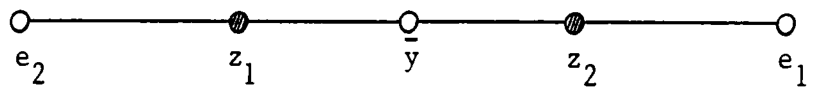



Figure 3.8a

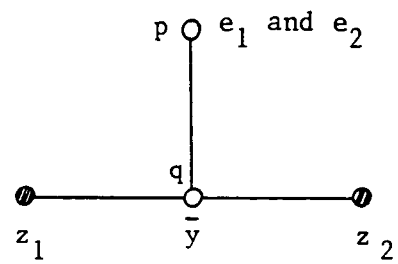



Figure 3.8b

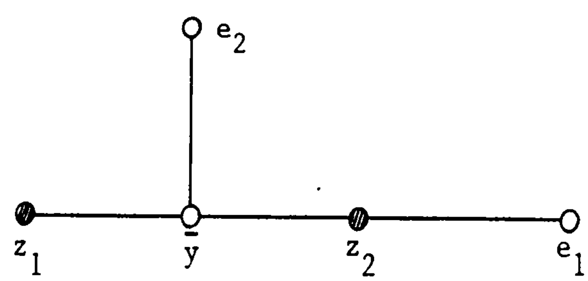



Figure 3.8c

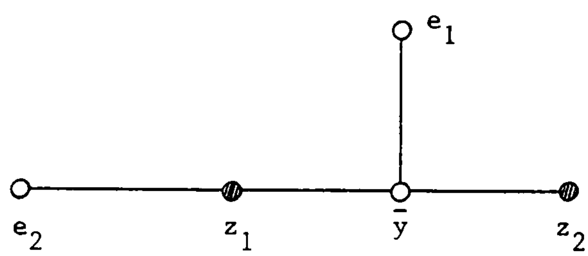



Figure 3.8d. Subcase 3b (Cont.)

formula/expression for the difference in costs between the two candidate solutions. This concludes our analysis of Case 3 and also our analysis of Problem C2MTP.

# CHAPTER IV

# CONCLUSIONS AND FUTURE RESEARCH

## 4.1 Conclusions

An exact solution procedure has been developed for an absolute p-median problem in which $p > 1$ capacitated facilities are to be located on a chain graph having both nodal and link demands. This procedure enables one to determine a global optimum to this nonconvex problem by exploiting the optimality of a particular type of location-allocation scheme for any given ordering (from left to right, on the chain) of the facilities. Specifically, the methodology of branch and bound was used to implicitly evaluate the minimum cost of every possible ordering of the p facilities.

The results of the chain graph problem were then used to develop an algorithm to solve a dynamic, sequential location-allocation problem in which a single facility per period is required to be located. Specifically, the algorithm prescribes a reduced set of candidate optimal solutions from which to choose the best.

Finally, an exact solution procedure was developed for an absolute 2-median problem, in which two capacitated facilities are to be located on a tree graph having both nodal and link demands. This procedure utilizes the results of our chain graph analysis, as well as a tree-reduction type of result, to develop an algorithm with which one can identify two subtrees, each of whose ends constitute a set of candidate optimal locations for one of the two elements of an absolute 2-median. Further localization results were provided to assist in reducing the number of candidate pairs (of ends) that need to be considered. In addition, a post-localization analysis was effected, wherefrom efficient methods of comparing the relative costs of candidate pairs resulted.

## 4.2 Future Research

As was mentioned previously, the formulation of location-allocation problems seems to be limited only by one's imagination, and consequently, it seems relatively easy to identify interesting problems/areas of future research. With respect to the problems analyzed herein, this author can think of three (3) immediate extensions/variants which appear to be both interesting and useful. A brief discussion of each of these problems is given below. Before proceeding however, we would remark that the intent of this section is not to provide the reader with rigorously formulated (future) research problems, nor is it to provide detailed (or even semi-detailed) solution procedures for any such problem. Rather, the purpose of this section is to simply suggest possible problems/areas of future research and to briefly discuss these problems with respect to possible solution methodologies and/or difficulties one might encounter in analyzing them. With this in mind, let us proceed.

### Unbalanced Versions of Problems CP and C2MTP

Recall that it was important in both Problem CP and Problem C2MTP, that the total demand ($\equiv$D) equal the total supply ($\equiv$S). The condition $S \geq D$ was necessary in order that the transportation approximation used to establish the optimality (to Problem CP(y)) of the PFL allocation be feasible (i.e. to have feasible solutions). The standard transportation problem is considered infeasible when $D > S$. Furthermore, if $S > D$, it is usually suggested that one amend the problem via a dummy destination having demand equal to the excess supply and such that the per unit transportation cost from each origin to the "dummy" be zero. In neither case, does the above conclusion/suggestion seem appropriate to Problem CP say, for one can certainly imagine a real-world situation in which $D > S$, and furthermore, how one might amend a dummy destination to Problem CP (in the case where $D < S$) is not at all apparent to this author.

For the case in which $D > S$, and addressing the chain graph problem in particular, one would most likely want to incorporate some sort of penalty into one's objective function, whereby one is penalized according to the distribution of that demand left unserved. As such, the problem would become one of determining locations and allocations which minimize the sum of the costs due to service and to the penalty. The PFL solution, for example, would result in all of the unserved demand lying in some interval $[\alpha,c] \subseteq (0,c]$, contrary to this author's intuition that it would be better to have a more uniform distribution of unserved demand. Metaphorically speaking, it may be better to "spread the dissatisfaction across all customers rather than to lay it on the shoulders of a few." Obviously, both the solution and solution procedure would be very much dependent on how one chose to define such a penalty. Finally, we would simply remark that a benchmark/starting point for any solution procedure to any such problem may exist in the solution obtained by replacing the demand function $f(\cdot)$ with $(f(\cdot) - (D-S)/c)$, solving the resulting balanced problem, and thus obtaining a uniform distribution of the unserved demand from which to compute the penalty cost.

For the case in which $S > D$, and again addressing the chain graph problem in particular, the difficulty that one is likely to encounter, should one attempt to effect a PFL type of allocation/solution, is best illustrated by noting the behavior of such an allocation in the case of uncapacitated facilities. Specifically, for such a case, a PFL type of allocation would be such that the facility located at $y_1$, where $0 < y_1 < \ldots < y_p < c$, would serve/supply all demand on the interval $[0, (y_1 + y_2)/2]$. Similarly, the facility at $y_2$ would serve/supply the interval $[(y_1 + y_2)/2, (y_2 + y_3)/2]$, etc.. Obviously, with $S > D$ and having capacitated facilities, a PFL allocation is likely to require that one decide whether to allow the allocation of a particular facility at $y_i$, say, to serve past the point $(y_i + y_{i+1})/2$, or whether to terminate its service at this point and allow the remaining capacity (of the $i^{th}$ facility) to go unused.

It is the above kind of complications/peculiarities which one is certain to encounter while analyzing unbalanced versions of Problems CP and C2MTP. Notwithstanding such complications, it is this author's opinion that exact solution procedures are possible for these problems.

Locating an Additional Facility on a Tree

As a counterpart to the "Sequential One Facility Per Period Location-Allocation Problem" on the chain graph, one might be interested in determining how to optimally locate a third facility on a tree such as that of Problem C2MTP. Unlike the chain graph problem, in which it was assumed that the locations of p-1 facilities were given and fixed for any $p > 1$, the "additional facility on a tree" problem that is being suggested here, involves locating only a third facility on a tree for which an absolute 2-median is given and fixed. Such is the case, since it is presently unknown how one would determine an absolute p-median on such a tree as ours for any $p > 3$. This in itself is a problem which awaits future research/solution, but is much too difficult to consider as a next step in extending the current analysis of this study, and would be more than likely to result in a heuristic rather than an exact solution procedure.

This author feels quite certain that an exact solution procedure can be developed for solving the "additional facility on a tree" problem. In particular, an approach like that of the following seems worthy of further/future investigation.

a) Given the locations $y_1^*, y_2^* \varepsilon T$ of an absolute 2-median (and assuming that $y_1^* \neq y_2^*$), first determine an optimal location $y_3$ for a third facility on the path $P[y_1^*, y_2^*]$. This can be done easily by collapsing all of T onto $P[y_1^*, y_2^*]$ and then determining the PFL allocation in accordance with the capacities of the three facilities.

Now let e be any end of T, and note that it must be the case that e is an end of some subtree $T_e$ rooted on $P[y_1^*, y_2^*]$. (Note: it may be that $T_e \equiv e = y_1^*$ or $y_2^*$.) We must consider two cases.

b) Suppose that $T_e$ is rooted at an end of $P[y_1^*, y_2^*]$, and without loss of generality, suppose that it is rooted at $y_2^*$. Then collapse all of T onto $P[y_1^*, e]$ and determine an optimal location for $y_3 \varepsilon P[y_2^*, e]$ via the PFL allocation.

c) Suppose that $T_e$ is rooted at some point $\alpha \varepsilon P(y_1^*, y_2^*)$. Then collapsing all of T onto the tripod formed by $\{y_1^*, y_2^*, e\}$, it is again a simple matter to determine an optimal $y_3 \varepsilon P[e, \alpha]$.

Having performed a), and b) or c) for every end of T, it seems plausible that any $y_3$ having minimum cost would in fact be $y_3^*$. Of course, further effort is warranted to reduce the work required by such a brute-force approach. Regardless, it does apear that such an approach lends itself to an exact solution procedure of this problem. We would add that if $y_1^* = y_2^*$, an argument similar to that above can be made by disconnecting T at $\bar{y} = y_1^* = y_2^*$ and then considering each end e of each subtree so obtained, with respect to locating $y_3 \varepsilon P[\bar{y},e]$.

Locating an Absolute 2-median on a General Network With a Continuum of Demand

A natural, but certainly not immediate, extension of Problem C2MTP is that of locating a capacitated absolute 2-median on a general network having both nodal and link demands, and for which it is again assumed that the total demand is equal to the total supply.

Exact, but not necessarily efficient, solution procedures exist for locating an absolute p-median ($p > 1$) on a general network having only nodal demands (see Hakimi [1965]). With respect to networks having both nodal and link demands, Chiu [1982] has provided us with both exact and heuristic solution procedures for locating an absolute 1-median on a general network. In this same paper, he develops an efficient (exact) algorithm for locating an absolute 1-median on a tree. Cavalier and Sherali [1983a] also provide an exact solution procedure for the absolute 1-median on a tree problem in which the tree has both nodal and link demands. In addition, the papers by Batta, Brandeau, and Chiu [1983], and again, Cavalier and Sherali [1983a], solve the uncapacitated absolute 2-median problem on such a tree. It remains for someone to solve the uncapacitated absolute 2-median problem on a general network having a continuum of demand. In fact, there remains a need for researchers to develop more efficient algorithms for solving some of the above "already solved" problems.

Returning now to capacitated, minisum, network-based location-allocation problems having a continuum of demand, this study represents the first of any such analysis (known by this author). In particular, we have developed an exact solution procedure for Problem C2MTP. In the next (and last) section of this study, we present a brief discussion of a possible heuristic for locating a capacitated absolute 2-median on a general network having both nodal and link demands, and for which the total demand is equal to the total supply. We make no remarks regarding the goodness of this heuristic, but rather present it in order to demonstrate one type of approach that one may wish to pursue in analyzing this problem.

### 4.3 Locating a Capacitated Absolute 2-median on a General Network Having a Continuum of Demand

The following constitutes a non-rigorous discussion/description of a "locate-allocate" type of heuristic for determining a capacitated absolute 2-median on a general network G having both nodal and link demands. The location phase of this heuristic simply requires that one solve two absolute 1-median problems on a general network having a continuum of demand (see Chiu [1982]). The allocation phase requires that one formulate and solve a transportation approximation to the original problem for given and fixed locations of the two facilities. Before actually presenting an algorithmic (step by step) description of this heuristic, we will briefly describe its allocation phase so that we can simply reference this phase at the proper point in time.

Allocation Phase

Suppose that one is given the locations $y_1, y_2 \varepsilon$ G of two capacitated facilities from which one wishes to optimally serve/supply the demand on G. One can formulate/construct an approximating transportation problem by first discretizing/partitioning all of G, and then by aggregating the partitioned demand so as to have a finite number of "destination" points distributed throughout G. In keeping with the usual terminology used in describing a transportation problem, points $y_1$ and $y_2$ would represent the "sources" of our problem, the decision variables would be the number of units of supply provided to the destination points from each of $y_1$ and $y_2$, and finally, the cost of supplying a single unit of demand from a facility ($y_1$ or $y_2$) to any destination point is simply the distance (on G) between the two points.

Now, assuming that we have ordered/numbered the destination/demand points, let $d_{1i}$ and $d_{2i}$ denote the shortest path distances between the $i^{th}$ destination/demand point and the facilities at $y_1$ and $y_2$, respectively. Such distances are easily obtained as follows. To determine the $d_{1i}$ distances, first construct the shortest path (spanning) tree from $y_1$ to all nodes of G. For every destination point i which lies on this tree, the value of $d_{1i}$ is easily obtained. For any link of G which is not a part of the shortest path tree, one can utilize the known distances from $y_1$ to the two ends of the link to determine a "shortest distance breakpoint" on the link, and thereby know the shortest path (and hence distance) between any point on the link and $y_1$. Similarly, one can obtain the $d_{2i}$ values.

Now, for each destination/demand point i, define $\Delta_i \equiv d_{1i} - d_{2i}$, and without loss of generality, reindex the $\Delta_i$ values so that they are ordered according to increasing value. As such, the first elements ($\Delta_i$) of this finite ordering correspond to destinations which are closer to $y_1$ than $y_2$ (assuming the existence of such a demand distribution on G), whereas the latter elements correspond to those destinations closer to $y_2$ than to $y_1$.

If one were to now allocate the supplies $s_1, s_2$ (corresponding to the facilities at $y_1, y_2$, respectively) to the destination/demand points in accordance with the increasing $\Delta_i$ sequence and beginning with $s_1$, the resulting allocation would be the same basic feasible solution that one would obtain by applying the Northwest Corner Rule to the transportation problem. More importantly, it is an optimal basic feasible solution, i.e. it is an optimal allocation. To see this, suppose there are n destination points and that the facility at $y_1$ exhausts its supply $s_1$ while serving the $k^{th}$ (i.e. corresponding to $\Delta_k$) destination/demand point. (Note, that in the event of non-degeneracy, $x_{2k} > 0$.) Now, letting $\mu_1, \mu_2$ and $v_1, \ldots, v_n$ denote the dual variables corresponding to the the sources (at $y_1, y_2$) and to the destinations (corresponding to ($\Delta_1, \ldots, \Delta_n$), respectively, one finds that they are of the following form:

$$\begin{aligned}
\mu_1 &= d_{1k} - d_{2k} + d_{2n} \\
\mu_2 &= d_{2n} \\
v_j &= d_{1j} - d_{1k} + d_{2k} - d_{2n} \text{ for } j=1, \ldots, k-1 \\
v_j &= d_{2j} - d_{2n} \quad \text{for } j=k, \ldots, n-1 \\
v_n &= 0 \quad .
\end{aligned}$$

As such, the reduced costs corresponding to the nonbasic variables (arcs) are of the form,

$$z_{1j} - c_{1j} \equiv z_{1j} - d_{1j} = \Delta_k - \Delta_j \text{ for } j=k+1, \ldots, n$$

$$z_{2j} - c_{2j} \equiv z_{2j} - d_{2j} = \Delta_j - \Delta_k \text{ for } j=1, \ldots, k-1.$$

Thus, by recalling the ordering that was imposed upon the $\Delta_j$'s, one can see immediately that dual feasibility exists, and hence the Northwest Corner Rule solution is optimal.

Now, in keeping with the (announced) non-rigorous format of this section, we would only remark that in the limit, as the mesh of the partitioning of G goes to zero, one would obtain an optimal allocation to the original allocation problem. This concludes our discussion of the allocation phase, and we are now prepared to present the following heuristic for locating a capacitated absolute 2-median on a general network having a continuum of demand.

Algorithm C2MGNP

STEP 1: locate an absolute 1-median $\bar{y}$ on G. (see Chiu [1982]).

STEP 2: determine a shortest path (spanning) tree $T \subseteq G$ from $\bar{y}$ to all the nodes of G. Determine the "shortest distance breakpoints" for all links not in T, and then "break" these links at these points to obtain a tree T'.

STEP 3: use the procedure contained herein (for solving Problem C2MTP) to locate a capacitated absolute 2-median $(y_1, y_2)$ on T'.

STEP 4: execute the Allocation Phase in order to determine optimal allocations on G for the facilities at $y_1, y_2$.

STEP 5: with respect to each of the allocations determined in the preceding step, locate absolute 1-medians $\bar{y}_1$ and $\bar{y}_2$ on G, for each facility.

STEP 6: If $y_1$ and $y_2$ are not themselves absolute 1-median locations corresponding to their assigned allocations, replace $(y_1, y_2)$ by $(\bar{y}_1, \bar{y}_2)$ and return to Step 4. Otherwise, stop.

Note that an improved solution results upon each loop through Steps 4-6. Although this heuristic is intuitively appealing, as was mentioned previously, we make no claims as to its goodness, but hope that it serves to stimulate the reader's interest.

This concludes our study of balanced, capacitated, location-allocation problems involving networks having a continuum of demand.

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# APPENDIX A

## THEOREMS, COROLLARIES AND LEMMAS OF CHAPTER II

### 2.2 Formulation of a p-median Location-Allocation Problem on a Chain

<u>CP</u>: minimize $\sum_{i=1}^{p} \int_0^c \Phi_i(x)|x-y_i|dx$

subject to

$$\sum_{i=1}^{p} \Phi_i(x) = f(x) \text{ for all } x\varepsilon[0,c]$$

$$\int_0^c \Phi_i(x)dx = s_i \text{ for } i=1, \ldots, p$$

$$\Phi_i(x) \geq 0 \text{ for all } x\varepsilon[0,c],\ i=1, \ldots, p$$

$$0 \leq y_i \leq c \text{ for } i=1, \ldots, p.$$ (See page 38)

### 2.3 Characterizations of the Individual Location and Allocation Solution

<u>Lemma 2.1</u>. Let $y_1^*, \ldots, y_p^*, \Phi_1^*(\cdot), \ldots, \Phi_p^*(\cdot)$ represent an optimal solution to Problem CP. Then, there exists a reindexing of facilities such that $0 \leq y_1^* \leq \ldots \leq y_p^* \leq c$. (See page 41)

<u>Theorem 2.2</u>. Problems CP($\Delta, y_1, \ldots, y_p$) and CPT($\Delta, y_1, \ldots, y_p$) are equivalent in the following sense. There exists a one-to-one correspondence between the sets of feasible solutions to Problems CP($\Delta, y_1, \ldots, y_p$) and CPT($\Delta, y_1, \ldots, y_p$), and more importantly, corresponding solutions have equal objective function values. It follows then, that any optimal solution to one problem will correspond to an optimal solution of the other. (See page 45)

<u>Theorem 2.3</u>. The Northwest Corner Rule solution $x^*$ is optimal to Problem CPT($\Delta, y_1, \ldots, y_p$) for any $p \geq 2$. (See page 50)

Theorem 2.4. There exists an optimal solution $\Phi^* = (\Phi_1^*, \ldots, \Phi_p^*)$ to Problem CP($y_1, \ldots, y_p$) for which the allocation functions $\Phi_i^*(\cdot)$, i=1, ..., p, result in the facilities at $0 < y_1 < \ldots < y_p < c$ serving the intervals $I_1 = [0,\alpha_1]$, $I_2 = [\alpha_1,\alpha_2]$, ..., $I_p = [\alpha_{p-1},c]$, respectively, for some $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{p-1} < c$. (See page 51)

Theorem 2.5. There exists an optimal solution to Problem CP, for which

(a) each facility i serves an interval $[\alpha_i,\beta_i]$, with its location $y_i \varepsilon [\alpha_i,\beta_i]$, and such that

(b) $\int_{\alpha_i}^{y_i} f(x)dx = \int_{y_i}^{\beta_i} f(x)dx = s_i/2$ . (See page 54)

(Note that such $y_i$ are generally referred to as "median" locations.)

Corollary 2.6. Given any ordering of the p facilities, their optimal locations are known readily, i.e. simply mark off the intervals to be served (from left to right) according to their capacities, and then place the facilities at their respective median locations. (See p. 54)

## 2.4 A Useful and Insightful Special Case

Lemma 2.7. Consider a facility i having supply $s_i$ which serves an interval $I_i \equiv [\alpha_{i-1},\alpha_i]$, and suppose that this facility is located at a median location $y_i$ with respect to $I_i$, i.e. $y_i$ satisfies

$$\int_{\alpha_{i-1}}^{y_i} f(x)dx = \int_{y_i}^{\alpha_i} f(x)dx = s_i/2. \quad \text{Then,}$$

$$\int_{I_i} f(x)|x-y_i|dx = A_{id} + A_{iu} \text{ , where}$$

$$A_{id} \equiv \int_{\alpha_{i-1}}^{y_i} f(x)(y_i-x)dx = y_i(s_i/2) - \int_{F(\alpha_{i-1})}^{F(y_i)} F^{-1}(y)dy \quad \text{and}$$

$$A_{iu} \equiv \int_{y_i}^{\alpha_i} f(x)(x-y_i)dx = \int_{F(y_i)}^{F(\alpha_i)} F^{-1}(y)dy - y_i(s_i/2).$$

(Note that $F^{-1}$ exists since F is strictly increasing under our current assumptions on f.) (See page 56)

Theorem 2.8. Given that the demand function f is nonnegative, integrable and nondecreasing, that ordering in which the p facilities appear from left to right (in $[0,c]$) in nondecreasing order of capacity, is optimal to Problem CP. That is, the associated $(y,\Phi)$ of Corollary 2.6 is an optimal solution to Problem CP.
(Note: our proof of Th. 2.8 requires the convexity of F(x), and hence the following lemma.) (See page 58)

Lemma 2.9. Given that f is a nonnegative, integrable, nondecreasing function defined on $[0,c]$, the continuous function $F(x) = \int_0^x f(t)dt$, $x\varepsilon[0,c]$, is a convex function. (See page 59)

Corollary 2.10. Let f be any nonnegative, integrable, demand function for Problem CP. Then in determining an optimal solution to Problem CP, it is sufficient to restrict one's attention to those orderings/ permutations of the p facilities which give solutions (per Corollary 2.6) satisfying the following condition: sets of facilities which use their entire supply to serve a connected subset of $[0,c]$ over which f is monotone nondecreasing (nonincreasing), must be arranged from left to right in nondecreasing (nonincreasing) order of capacity. (See p. 66)

## 2.5 Analysis of the Symmetric, Unimodal Demand Distribution Case

Theorem 2.11. Suppose that the demand function f for Problem CP is nonnegative, integrable, unimodal and symmetric on $[0,c]$. Then, in determining an optimal solution to Problem CP, it is sufficient to restrict attention to those orderings/permutations of the p facilities which give solutions (per Corollary 2.6) satisfying the following condition in addition to that of Corollary 2.10: if a facility M straddles, then it has maximum capacity. (See page 69)

Theorem 2.12. Given a demand function f satisfying the requirements of Theorem 2.11, and $p = 3$ facilities having capacities $s_1 < s_2 < s_3$, the alternating ordering is optimal. For $p = 4$, such an ordering need not be optimal. (See page 84)

## 2.6. Analysis of the Non-Symmetric, Unimodal Demand Distribution Case

Theorem 2.13. Consider Problem CP and suppose that the demand function f is nonnegative, integrable and unimodal on $[0,c]$. (Recall that being unimodal means f is nondecreasing on $[0,\alpha]$ and nonincreasing on $[\alpha,c]$, for some $\alpha\varepsilon[0,c]$.) Then in determining an optimal solution to Problem CP, it is sufficient to restrict attention to those orderings/permutations of the p facilities which give solutions (per Corollary 2.6) satisfying the following condition in addition to that of Corollary 2.10: if facility M straddles, then $s_M$ must be a maximum with respect to the capacities of those facilities located on at least one side of it. (See page 92)

## 2.7. Using Lower Bounds to Fathom Partial Orderings

Theorem 2.14. Let f be a nonnegative, integrable demand function defined on $[0,c]$, and let $[\alpha,\beta]$ $[0,c]$ be the service interval of some facility located at a median $y^* \varepsilon (\alpha,\beta)$ and having capacity/supply,

$$s = \int_{\alpha}^{\beta} f(x)dx.$$

If g is a nonnegative, integrable function such that for some $\alpha'$, $\beta'$ satisfying $\alpha < \alpha' < y^* < \beta' < \beta$, we have

$$\text{i)} \quad \int_{\alpha'}^{y^*} g(x)dx = \int_{y^*}^{\beta'} g(x)dx = s/2,$$

$$\text{ii)} \quad \int_{y}^{y^*} g(x)dx \geq \int_{y}^{y^*} f(x)dx, \text{ for all } \alpha' < y < y^*,$$

and
$$\text{iii)} \quad \int_{y^*}^{y} g(x)dx \geq \int_{y^*}^{y} f(x)dx \text{ , for all } y^* < y < \beta',$$

then the transportation cost with respect to g of the service/supply provided to $[\alpha',\beta']$ by the facility located at $y^*$, is less than or equal to the cost with respect to f of the service/supply provided to $[\alpha,\beta]$, i.e.

$$\int_{\alpha'}^{\beta'} g(x)|x-y^*|dx \leq \int_{\alpha}^{\beta} f(x)|x-y^*|dx \quad .$$

(Note: the following schematic is given to help clarify the statement and proof of this theorem.) (See page 97)

<u>Corollary 2.15</u>. Let f be a nonnegative, integrable, unimodal demand function defined on $[0,c]$, and having a maximum at some $\alpha\varepsilon[0,c]$. Let $A_1$ and $A_2$ denote the total demands to the left and right of $\alpha$, respectively, i.e. $A_1 = \int_0^{\alpha} f(x)dx$ and $A_2 = \int_{\alpha}^{c} f(x)dx$. Pictorially, we might have something like that in Figure 2.16a.

"Squaring off" the demand over $[0,\alpha]$, we define

$$g = \begin{cases} f(\alpha), & \text{on } [\alpha - A_1/f(\alpha),\alpha] \\ f \quad , & \text{on } [\alpha,c] \ . \end{cases}$$

Pictorially, we would have Figure 2.16b.

"Squaring off" the demand over $[\alpha,c]$, we define

$$h = \begin{cases} f & \text{, on } [0,\alpha] \\ f(\alpha), & \text{on } [\alpha,\alpha + A_2/f(\alpha)] \end{cases}.$$

Pictorally, we would have Figure 2.16c.

A lower bound on the optimal cost of serving/supplying the demand $A_1 + A_2$ with respect to f, is the larger of the optimal costs of doing the same, with respect to g and h. (See page 99)

## 2.8 Treatment of Problem CP, When f is Simply Nonnegative and Integrable

Theorem 2.16. Given $f:[0,c] \to R$ of Problem CP, let $\hat{f}:[0,c'] \to R$ be constructed as above. Then the PFL solution of Theorem 2.8 which solves Problem CP with f replaced by $\hat{f}$, gives an objective function value with respect to $\hat{f}$ which is a lower bound for the optimal value to Problem CP with respect to $f(\cdot)$. (See page 110)

## 2.9 A Sequential One Facility Per Period Location-Allocation Problem

Lemma 2.17. Let $y_p^*$ be the location of the $p^{th}$ facility of any optimal solution to Problem CP(p). Then $y_p^* \varepsilon \{\zeta_1, \ldots, \zeta_{p-1}\}$. (See page 115)

Theorem 2.18. For each j = 1, ..., p, let $P_j = [\delta_j, \Delta_j]$ be the median interval (in $(\alpha_j, \beta_j)$) defined by

$$\delta_j = \min \{y > \alpha_j : \int_{\alpha_j}^{y} f(x)dx = s_p/2\}$$

and

$$\Delta_j = \max \{y < \beta_j : \int_{y}^{\beta_j} f(x)dx = s_p/2\} ,$$

where $\alpha_j$, $\beta_j$, j=1, ..., p are given by (2.14). If (a) $\Delta_j < \zeta_{j-1}$ or if $\delta_j > \zeta_j$, then $y_p = \zeta_{j-1}$ or $y_p = \zeta_j$ are respectively unique optimal solutions to CP(p,$\sigma_j$). Otherwise, (b) any of the points in $P_j \quad [\zeta_{j-1}, \zeta_j]$ solve CP(p,$\sigma_j$). (See page 117)

Lemma 2.18*. Let f be an integrable, nonnegative function defined on $[\alpha,\beta]$ $[0,c]$ . The function $\Gamma(x_0) = \int_\alpha^\beta f(x)|x-x_0|dx$ is convex on $[0,c]$. (See page 117) (See end of Appendix for proof.)

Theorem 2.19. For each $j=1, \ldots, p$, define $S_j = \{y\}$ for some $y \varepsilon P_j \equiv [\delta_j, \Delta_j]$ in case $\zeta_{j-1} < \delta_j < \Delta_j < \zeta_j$, and $S_j = \phi$ otherwise. Then, an optimal solution to CP(p) lies in the set $S \equiv \bigcup_{j=1}^{p} S_j$. (See page 118)

Algorithm for CP(p). Compute $\alpha_j$ and $\beta_j$ (recursively) from (2.14) and hence determine $\delta_j$ and $\Delta_j$ as in (2.15). Find the set S of Theorem 2.19 and note from the proof of this theorem that $S \neq \phi$. Select the best solution from S as an optimal solution to CP(p). (See page 119)

Proof of Lemma 2.18*: Let $x_1, x_2$ be any two points in $[0,c]$ and $\lambda \varepsilon (0,1)$. Let $\bar{x} = \lambda x_1 + (1-\lambda)x_2$.

Then $\lambda\Gamma(x_1) + (1-\lambda)\Gamma(x_2) - \Gamma(\bar{x})$

$$= (1-\lambda)(\Gamma(x_2)-\Gamma(\bar{x})) - \lambda(\Gamma(\bar{x})-\Gamma(x_1))$$

$$= (1-\lambda)\Big[\int_\alpha^\beta f(x)|x-x_2|dx - \int_\alpha^\beta f(x)|x-\bar{x}|dx\Big]$$

$$- \lambda\Big[\int_\alpha^\beta f(x)|x-\bar{x}|dx - \int_\alpha^\beta f(x)|x-x_1|dx\Big]$$

$$= \lambda\int_\alpha^\beta f(x)|x-x_1|dx + (1-\lambda)\int_\alpha^\beta f(x)|x-x_2|dx - \int_\alpha^\beta f(x)|x-\bar{x}|dx$$

$$= \int_\alpha^\beta f(x)(\lambda|x-x_1| + (1-\lambda)|x-x_2|)dx - \int_\alpha^\beta f(x)|x-\bar{x}|dx$$

$$= \int_\alpha^\beta f(x)\big|\lambda|x-x_1| + (1-\lambda)|x-x_2|\big|dx - \int_\alpha^\beta f(x)|x-\bar{x}|dx$$

since $(\lambda|x-x_1| + (1-\lambda)|x-x_2|) \geq 0$ ,

$$> \int_{\alpha}^{\beta} f(x)|\lambda(x-x_1) + (\lambda-1)(x_2-x)|dx - \int_{\alpha}^{\beta} f(x)|x-\bar{x}|dx$$

since $f > 0$ and $|\lambda|x-x_1|+(1-\lambda)|x-x_2|| > |\lambda(x-x_1) + (\lambda-1)(x_2-x)|$,

$$= \int_{\alpha}^{\beta} f(x)|-(\lambda x_1+(1-\lambda)x_2) + x|dx - \int_{\alpha}^{\beta} f(x)|x-\bar{x}|dx$$

$$= \int_{\alpha}^{\beta} f(x)|-\bar{x}+x|dx - \int_{\alpha}^{\beta} f(x)|x-\bar{x}|dx = 0 \quad . \quad \square$$

## APPENDIX B

## THEOREMS, COROLLARIES AND LEMMAS OF CHAPTER III

### 3.2 Formulation of the Capacitated 2-median Tree Problem

$$\text{C2MTP}: \quad \underset{\substack{y_1, y_2 \in T \\ \omega, \Phi(\cdot) \geq 0}}{\text{minimize}} \quad \left( \sum_{i=1}^{2} \sum_{k=1}^{n} \omega_{ik} d(y_i, v_k) + \sum_{i=1}^{2} \sum_{\ell \in A} \int_{\ell} \Phi_{i\ell}(x) d(x, y_i) dx \right)$$

subject to

$$\sum_{k=1}^{n} \omega_{ik} + \sum_{\ell \in A} \int_{\ell} \Phi_{i\ell}(x) dx = s_i \quad \text{for each } i=1,2$$

$$\sum_{i=1}^{2} \omega_{ik} = h_k \quad \text{for each } k=1, \ldots, n$$

$$\sum_{i=1}^{2} \Phi_{i\ell}(x) = f_{\ell}(x) \quad \text{for each } x \in \ell, \text{ and } \ell \in A .$$

(See page 123)

### 3.3 The 2-median Optimality Criteria

Lemma 3.1. Let T' be a subtree of T obtained by disconnecting T at some point $q \varepsilon T$, and let the total demand on T' be jointly supplied by two facilities located at $y_1, y_2 \varepsilon$ T−T'. Then the cost of supplying the total demand on T' is given by

$$C(y_1, y_2, T') = C(q, T') + d(y_1, q)W_1 + d(y_2, q)W_2 ,$$

where C(q,T') is the cost of supplying the total demand on T' from q, and where $W_1, W_2$ are the amounts of supply provided T' from each of the facilities at $y_1, y_2$, respectively. (Note: $W_1 + W_2$ = Total Demand on T'.) (See page 128)

Lemma 3.2. Suppose that $y_1^*, y_2^* \varepsilon T$ are the locations of an optimal solution to the capacitated 2-median tree problem (i.e. to Problem C2MTP). Then these same locations are optimal to the 2-median problem collapsed onto any path P[r,s] $P[y_1^*, y_2^*]$, where r,s ε T. Furthermore, any alternative optimal solution to this additionally constrained problem solves Problem C2MTP. (See page 130)

Theorem 3.3. Suppose that $y_1^*, y_2^* \varepsilon T$ are the locations of an optimal solution to the capacitated 2-median tree problem, and that $y_1^* \neq y_2^*$. (Note that if $y_1^* = y_2^*$, then the two facilities and their supplies constitute a 1-median, and thus the theorem follows from the 1-median optimality conditions/criteria of Chiu [1982] and Cavalier and Sherali [1983a].) Disconnect T at $y_1^*$ and let $T_1, \ldots, T_m$ be the resulting subtrees, with $y_2^* \varepsilon T_m$, say. In keeping with our earlier development, let us assume that $T_i$ does not include $y_1^*$, but rather that an artificial node having an associated demand of zero has been added in its place, for $i=1, \ldots, m$. Then letting $\omega(\cdot)$ denote a total weight function, we have that

$$\omega(T_i) < s_1/2 \quad \text{for each } i=1, \ldots, m-1 ,$$

and (more importantly)

$$\sum_{i=1}^{m-1} \omega(T_i) + \omega(y_1^*) > s_1/2 .$$

(A symmetric statement holds with respect to $y_2^*$.) (See page 131)

3.4 Subproblems Inherent to Problem C2MTP and a Useful Reduction Theorem

Theorem 3.4. Suppose that $T = (N,A)$, $s_1, s_2$ and $\omega(\cdot)$ are as defined above. Let $\bar{y} \varepsilon T$ be the location of a 1-median, and $y_1^*, y_2^* \varepsilon T$, the locations of a 2-median (i.e. of an optimal solution to the capacitated 2-median tree problem). Then we have that $\bar{y} \ \varepsilon \ P[y_1^*, y_2^*]$. (See page 134)

3.5 Obtaining a Reduced Set of Candidate Optimal Solutions

Algorithm T(θ). Consider the capacitated 2-median tree problem, and without loss of generality, assume that $s_1 < s_2$. Then for any constant

$0 < \theta < (s_1+s_2)/2$, we construct a tree, $T(\theta)$  T, as follows:

Initialization Let T be the tree of Problem C2MTP and having no nodes flagged.

Step 1 Pick an end e of the current tree which is not flagged. If all ends are flagged, stop; the current tree is $T(\theta)$. Otherwise, proceed to Step 2.

Step 2 Let (v,e) be the link incident at e. If $\omega(v,e] < \theta$, collapse (v,e) into v, add $\omega(v,e]$ to the weight of v, and return to Step 1. Otherwise, i.e. $\omega(v,e] > \theta$, go to Step 3.

Step 3 Let $\bar{v} \in \text{int}(v,e)$ be the point closest to v such that $\omega[\bar{v},e] = \theta$. In this case, collapse $(\bar{v},e)$ into $\bar{v}$, denote $\bar{v}$ as being a node (of the resulting tree) whose weight is equal to $\omega[\bar{v},e]$, flag $\bar{v}$, and go to Step 1. If no such point exists, then flag e and go to Step 1. (See p. 137)

Theorem 3.5 Let $0 < \theta < (s_1+s_2)/2$, and let $T(\theta)$ be obtained via the foregoing algorithm. Then, $T(\theta)$ is uniquely determined by this algorithm and contains all of the 1-medians of T. In particular, if $T(\theta)$ is a single node, then it is the unique 1-median of T. (See p. 139)

Theorem 3.6. Let T, $T(\theta)$, $s_1 < s_2$, $y_1^*$ and $y_2^*$ all be as defined above (and note that $T(s_2/2)$  $T(s_1/2)$). Then, there exists an optimal 2-median $(y_1^*,y_2^*)$ such that $y_1^*,y_2^*$ are ends of $T(s_1/2)$, $T(s_2/2)$, respectively. Similarly, let $y_1$ be located at some end of $T(s_1/2)$. Then there exists an optimal location for $y_2$ (with $y_1$ fixed) among the ends of $T(s_2/2)$, and vice versa. (See page 142)

Corollary 3.7. $y_1^* = y_2^*$ is an optimal (2-median) solution to Problem C2MTP iff $T(s_1/2) = T(s_2/2) = T((s_1+s_2)/2)$. (See page 144)

## 3.6 Methods of Determining Optimal Locations for $y_1, y_2$ Amongst the Ends of $T(s_1/2)$ and $T(s_2/2)$, Respectively

Lemma 3.8. In accordance with the above definitions, theorems, lemmas, and algorithm, it follows that $|E[T(s_2/2)]| < 3$. (See page 145)

Corollary 3.9. If $T(s_2/2)$ is a tripod, then $T(s_1/2) \quad T(s_2/2)$ is also a tripod. (See page 145)

Lemma 3.10.

(a) $\bar{y} \in T(s_2/2) \quad T(s_1/2)$ for any 1-median $\bar{y}$.

(b) If $\bar{y} \in E[T(s_1/2)]$, then $\bar{y} \in E[T(s_2/2)]$.

(c) If $\bar{y} \in E[T(S_2/2)]$, then $|E[T(s_2/2)]| < 2$.

(d) If $|E[T(s_2/2)]| = 3$, then T has a unique 1-median $\bar{y}$, and it is located at the first point of intersection of the paths from any two (2) ends to the third end of $T(s_2/2)$. We denote this point as q. (See p. 149)

Theorem 3.11. Let $\bar{y}$ be any (fixed) 1-median of T. (Note that if $|E[T(s_2/2)]| = 1$ or 3, then $\bar{y}$ is in fact the unique 1-median of T.) Choose any end of $T(s_1/2)$ and denote it by $e^*$, say. Then suppose that $y_1^*$ is located (fixed) at $e^*$, and determine the set $\Lambda(e^*) \equiv \{e \in E[T(s_2/2)]: \bar{y} \in P[e^*, e], \bar{y} \neq e\}$.

Regarding the choice of $y_2^* \in E[T(s_2/2)]$ (recall Theorem 3.6), the following holds:

If $\Lambda(e^*) = 0$, then $y_2^* = \bar{y}$ is optimal.

If $\Lambda(e^*) \neq 0$, then one of the ends in $\Lambda(e^*)$ is the best location for $y_2^*$.

Additionally, $|\Lambda(e^*)| \leq 2$, and so if $|\Lambda(e^*)| = 1$, the location of $y_2^*$ is known immediately. (See page 151)

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