CONTRIBUTIONS TO
MODEL FOLLOWING CONTROL THEORY

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(ABSTRACT)

A standard form for linear and nonlinear perfect model following control problems is introduced, and the associated control laws developed. The error dynamics of such systems are analyzed with respect to stability of the error. The effects on the error dynamics of measurement errors and parameter variations are also analyzed, and it is seen that the perfect model following control problem is reduced to that of an error regulator.

The linear problem is analyzed to show that virtually all common problems are equivalent to standard form problems through similarity transformations. In the standard form, simple expressions for the control law and error dynamics are used to solve the problem.

The linear problem is also analyzed with respect to problems of different order model and plant systems, resulting in augmented system equations. These augmented systems are chosen so that the original dynamics are retained, and so that the higher order problem is in the standard form. The standard form problem is then solved as before.

Imperfect model following control problems are analyzed, with three associated results. First, a new test for perfect model following is developed. Pairs of models and plants that fail this or other tests are imperfect model following control problems. Second, the effect of using perfect model following control laws on such problems is determined to
be equivalent to the addition of a forcing function on the error regulator problem. Third, a new approach to the solution of imperfect model following control problems is shown. This approach seeks to find models that simultaneously satisfy the criteria for perfect model following while retaining the desired characteristics of the intended model.

The methods developed in this analysis are applied to problems that illustrate all the principles addressed. The final example is a detailed application to a nonlinear simulation of the F-18 airplane involving control of all degrees of freedom over a large range of angles of attack.
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INTRODUCTION

Model following control attempts to synthesize control laws that cause the states of a given physical system (the plant) to have responses to initial conditions and to control inputs that are the same, or nearly the same, as those of some conceptual model. For plants and models whose dynamics are described by linear, first order systems of differential equations, the implication is that the plant's complete eigenstructure will closely match that of the model. Model following control differs from pole placement methods in that model following considers all of the model's responses, including those due to control inputs. Seen this way, model following control may be considered to be a transfer function assignment method.

An intuitive understanding of model following control may be gained by considering a conventional airplane (the plant) employing this form of control to simulate the Dutch roll characteristics of some other airplane (the model). This may be desired purely for the sake of simulation, or perhaps to correct some flying qualities deficiency in the airplane. In any event, for a given pilot control input (stick and rudder pedal), the model following control law will drive the plant's control surfaces (ailerons and rudder) in an attempt to make the airplane roll and yaw in the same manner as that of the model with the same control input. Since the pilot is largely unaware of these additional control surface motions, he will perceive that the observed dynamic behavior is an aerodynamic response to his control input, and that the airplane actually handles like the one being modeled.
Model following control is divided into two major areas, called implicit and explicit model following. Implicit model following is a dynamics matching problem. It is normally accomplished by determining the plant state feedback gains that yield plant dynamics similar to those of the model. No attempt is made to make the plant follow the model’s state trajectory, and actual computation of the model’s states is not required. Note that in this form of model following the plant is dynamically the same as the model, and will respond to control inputs and to external disturbances the same as the model.

Explicit model following attempts to match both the dynamics and the trajectory of the plant and the model. It is thus a dynamic servo controller, wherein the plant is slaved to the model’s trajectory. The model states are calculated and fed forward for reference in matching trajectories, while the plant states are measured or estimated and are fed back to stabilize the system. In aircraft applications, this is often described as “flying the model.” The pilot's control inputs are applied to a computational representation of the model, and from this the model state trajectories are determined. Properly designed explicit model followers have stable error dynamics, such that (ideally) the difference between the plant and model states is continuously eliminated. Note that the model sees only the control inputs: it does not respond to external disturbances. In the event of an external disturbance to the plant, this control law will tend to return the plant to current, undisturbed, state of the model.

Both implicit and explicit model following control laws have been developed using linear optimal control methods. In the formulation for implicit model following, a quadratic performance measure consisting of plant and model state rates is minimized, while in explicit model following the performance measure also includes the plant and model states. Additionally, explicit model following control laws have been developed so that the
dynamics of the errors between the plant and model states may be arbitrarily assigned by the designer.

Aside from linear optimal control, several other design methods have been introduced. Two important ones work directly with the plant and model state space representations, but are only valid in cases where the plant and the model can be matched perfectly. Most other methods have failed to receive widespread acceptance, in many cases because they were targeted at specific problems, or because they offered no significant advantages over previous methods. While linear optimal control is frequently attacked because it provides no insights into the problem, its use in determining model following control laws still dominates the literature.

The primary use of model following control has been in the field of in-flight simulators, wherein an airplane is made to have the same flight characteristics as some other, the former being the plant and the latter being the model. Other applications of model following control have appeared; they mainly arise in design problems where the desired behavior of the plant, such as transient responses or handling qualities, has been determined through simulator study or through the identification of some other system that has the desired behavior. In those cases the desired responses are completely contained in the mathematical description of the model. Model following control may be applied to any control design problem if the design objectives can be embodied in the description of a model.

The term "perfect" is used to describe model following if the controlled plant dynamics are exactly the same as those of the model. Not all models can be perfectly followed by a given plant. Perfect model following tests have been developed for linear systems, and these criteria are the same for both implicit and explicit following. For
arbitrary models and plants, the model will generally not satisfy those criteria. Because one seldom has a model that can be perfectly followed by a given plant, the development of perfect model following control laws has received less attention than the more usual cases.

This dissertation returns attention to perfect, explicit model following control. A new formulation of the control law will be presented, and this formulation will be shown to offer insights into the mechanisms involved in effecting the desired control. In particular, the feedback and feedforward gains will be in a form that singles out the contributions to the matching of plant dynamics to those of the model, and the contributions to controlling the error between plant and model state trajectories. Other analytical results of this formulation will then be presented, and these results will be shown to provide means for dealing with several of the problems existing in the present design of model following controllers. The tool used for analysis is a standard form of plant and model equations of motion that assure that perfect following is possible, into which existing problems may be cast and analyzed.

Modern methods of control deal almost exclusively with linear systems, and model following control is not an exception. Many real problems in control are inadequately described by such linearized equations, and often it is in the region of nonlinearities that some form of automatic control is needed most. The model following control laws developed in this dissertation permit certain kinds of nonlinearities to be present. It will be shown that the plant and model equations of motion may be completely nonlinear in the states, but must satisfy certain requirements that are met if they are linear in the state rate terms and in the plant controls. Linearization may be required in the selection of gains for error dynamics, but not in the actual control laws themselves.
Model following control frequently deals with requirements wherein the plant and the model are of different order. In the present treatment, the plant and model equations of motion are augmented in ways that place them in the standard form while preserving their original dynamics. This will lead to an argument that the original problem was incompletely specified, and that the dynamics of the unmatched states of the model, plant, or both must be selected in order to complete the specification.

Another common problem involves the question of what to do when the model cannot be perfectly followed by the plant. When perfect model following is not possible, one must decide how the performance of the system is to be compromised. Frequently, perfect model following control laws are used for these problems without a clear understanding of the effects of doing so. This dissertation answers that question by analyzing the error dynamics involved. Alternatively, control laws are found that make the plant follow the desired model as well as possible. To follow a model "as well as possible" is a tailor-made problem for linear optimal control, and its use dominates the literature. There is an alternative approach that has not been previously investigated, namely the possibility of determining some other model that is similar to the desired model, and that satisfies the criteria for perfect following. The difference between the two approaches is that in the first, the quality of the model following is compromised at the expense of retaining a perfect model. In the second, the quality of the model being followed is compromised while retaining perfect following. This latter approach is one of the subjects of this dissertation, and it will be treated by finding candidate models (those that may be put in the standard form) that match the desired model's eigenstructure as nearly as possible.

The question of the robustness of the model following system will be addressed. The performance of model following control schemes in the presence of variations in plant
parameters or measurement error is usually determined by simulation and testing. While the
design methods that are normally used have a certain measure of inherent robustness, the
effect of errors and variations is never placed squarely on the table for dissection. The
formulation of the control laws for perfect explicit model following introduced in this
dissertation contains explicit expressions for the dynamics of the errors between the plant
and the model, and includes the effects of variations in plant parameters. The form of these
equations will be shown to permit an analysis of the effects of plant variations or modeling
inaccuracies while still in the design stage. The feedback and feedforward gains used in the
selection of error dynamics are directly available for use in the minimization of such effects
while maintaining stability of the errors.

The contributions to model following control theory presented in this dissertation
are as follows:

• A formulation of perfect model following control laws that proceeds from general,
  nonlinear equations of motion.

• A formulation of expressions for the error dynamics of perfect model following
  systems for the most general nonlinear cases.

• Analysis of the effects of measurement errors and plant parameter uncertainties on
  these error dynamics.

• The introduction of a standard form for linear model following control problems,
  and the proof that virtually all previously considered problems may be cast as
  standard form problems.
• Conditions for the assignment of arbitrary error dynamics of linear model following control problems when posed in the standard form.

• Analysis of the problem of different order plants and models.

• Formulation of a procedure for dealing with linear model following control problems in which perfect model following is not possible, including an analysis of the effects of applying perfect model following control laws to the solution of these problems.
In the early 1960s, Cornell Aeronautical Laboratory, Inc. (CAL) began design of a variable stability control system for a B-26 airplane. This airplane was to be capable of in-flight modification of its dynamic behavior, and was to be used extensively in determining flying qualities requirements for piloted airplanes. As implemented, this control system operated independently of the airplane's original system, and was engaged only for the duration of a test or demonstration. It operated by driving the airplane's control surfaces in such a way that it simulated some dynamic response of choosing. Different response characteristics could be selected in flight by varying the settings of several potentiometers, thus varying the gains in the control system.

The design of this airplane and the several that followed it gave impetus to the study of model following control. Many of the names most directly associated with model following control were (or still are) CAL employees, for example, Rynaski, Motyka, Tyler, Lebacqz, Asseo, and Buethe. NASA Ames appears to have had a long standing interest in the subject, as is evinced by the work of Meyer, Erzberger, and later, Lebacqz. Not all of the major contributions were made at CAL and Ames, however, the contributions of Chan and Kawahata being notable exceptions.

The earliest work that specifically addresses model following control as it is thought of today is due to Tyler [1]. He considered the problem of the synthesis of implicit and explicit model following control laws through the application of optimal control theory. Tyler makes reference to earlier work by R.E. Kalman, whose well-known studies
included a form of implicit model following control. Tyler's paper has served as a standard for most of the subsequent work done in the field; it is easily the most often cited reference, and his example using the lateral/directional equations of motion of a B-26 airplane has been borrowed frequently for illustration and comparison.

Tyler considered the usual linearized equations of motion of the plant and model, except the model description did not contain an external control input. In light of subsequent developments, this formulation implies that the model has the same input matrix as the plant. Performance indices for the implicit and for explicit model following were formed. For implicit model following, the performance index was intended to minimize the difference between the actual plant state rates and those of a plant having the model's system matrix instead of the original.

For explicit model following, the order of the plant and of the model was assumed to be the same, and an error vector was defined as the difference between model and plant states. The performance index was designed to minimize this error, with the inclusion of weights on the control effort. In both cases Tyler proceeded in the usual manner, deriving the matrix Riccati equations and solving them for the steady-state condition. Constant gain feedback and (for explicit following) feedforward matrices were thereby determined.

Tyler also addressed the case of explicit model following control with different order plants and models, but not in much detail. For models of lower order than the plant, he does point out that decisions have to be made as to which plant states should correspond to which model states. It is inferred that the remaining plant states are left to behave in a "natural" fashion.
Tyler provided several examples of applications of his results. It should be noted that at the time his paper was written, there were no criteria available for determining whether perfect model following was attainable. It is observed that all of his second order examples satisfy these criteria, and that his higher order examples nearly do. The term "nearly" in this sense means that a modal decomposition of the models actually used shows a similarity (in the relative magnitudes and phases of its eigenvectors) to that of models that can be followed perfectly. It is perhaps for this reason that Tyler was able to select weighting matrices in his performance indices that yielded good dynamics matching in response to initial conditions and to control inputs.

Soon after Tyler's paper, Rynaski and Whitbeck published a CAL Technical Report \[2\] that developed Tyler's results in greater detail and carried the analysis into the frequency domain. Through example, a model was introduced that had a control input matrix, but only the system matrix is used in the analysis.

Asseo \[3,4\] extended the optimal control analysis to model following with arbitrary model input. He selected explicit model following for his analysis because it is less sensitive to plant variations due to the dominance of the model poles in the closed loop system. The plant and model used by Asseo was of the usual form, but in his performance index he took an augmented state vector consisting of the plant states, model states, and plant controls. The performance index involved the norm of this vector, and included a weighted inner product of the augmented vector and its time derivative. The reason for this last term was to eliminate the dependence of the optimal control law on the model input description.

In these papers, Asseo established two conditions for perfect model following with plants and models of the same order. The first essentially required that the plant have an
independent controller for each plant degree of freedom, in which case it can follow any model of the same order. The second condition addressed a more general case wherein the number of linearly independent plant controls was greater than or equal to those of the model. His characterization of models that can be perfectly followed in this case is similar to that used in this dissertation.

Asseo also addressed the problem of reducing the sensitivity of the model following system to variations in plant parameters. This was accomplished by the design of a type-one perfect model following system. Such a system is shown to ensure zero steady state error to a step input to the model at any operating condition while providing perfect dynamics matching at the nominal operating condition. It should be noted that this result applies only to cases where perfect model following is possible, which is not generally the case.

Erzberger [5,6] addressed the question of when a given model can be followed perfectly by a given plant. This was done using state-space arguments. In doing so, he introduced the idea of "solving" the plant equations for the control. Since the plant control matrix is not generally invertible, Erzberger used the pseudoinverse. His results are based on the case of models without arbitrary control inputs. Chan [7] extended these results to models with arbitrary control inputs and to models of lower order than the plant. In the derivation of these conditions, an expression for the control vector that results in perfect model following was first assumed. The argument was that if perfect model following is attainable, then that same control vector is the one that makes it happen. These results are important because they were the first to offer a test to see if perfect following is possible. If it is, then the control law that accomplishes it is immediately available. This control law consisted of feedforward gains on the model states and external inputs, and feedback gains
on the plant states. These gains are not necessarily the same as would be attained through linear optimal control, since no penalty is placed on the use of control. The formulation offered no guarantees on the stability of the errors in the system or the sensitivity of the system to inaccuracies in plant modeling. It is, however, easy to apply and is used quite often, even in cases where perfect following is not attainable. In those cases, the performance of the system is often verified by simulation, and the results are deemed satisfactory if the plant follows the model reasonably well under a given set of initial conditions and control inputs. The term "reasonably well" is loosely construed to mean that the plant and model responses are near in phase and magnitude for all time considered.

Most previous works had assumed that all the plant states were available for feedback. Winsor and Roy [8] extended the application of optimal control to model following problems by considering partial state feedback. To do this, they employed an analysis tool known as specific optimal control. Partial state feedback is accomplished by constraining the gains on the unavailable states to be zero. The results were shown to have relative insensitivity to inaccuracies in plant modeling. The treatment in this paper considers only explicit model following. While the results presented are general, the design example offered considers only the case where perfect following is attainable.

Most analyses of model following control were by now solidly entrenched in time domain approaches, either through linear optimal control or the simple state space characterizations of Erzberger and Chan, Wolovich [9], and Wang and Desoer [10], considered the problem of perfect implicit model following from a frequency domain (transfer matrix) approach. Both provided algorithms for the construction of plant feedback matrices to provide model matching as specified by system closed loop transfer matrices. A completely different approach is taken by Moore and Silverman [11] using a linear system
input-output structure algorithm. Morse [12] provided extensions to the questions of solvability and system structure. While these and many other related works are of undoubted theoretical importance, they did not stir much interest among practical designers, and none of these papers have been cited in application oriented works.

In the early 1970s, CAL (now Calspan, Inc.), continued studies on variable stability airplanes such as the USAF Total In-Flight Simulator (TIFS) [13], and a proposal for a YT-2B in-flight simulator [14]. In these studies different, algebraically simple expressions for perfect explicit model following control laws were developed. Linear optimal control theory was not used at all. The reason that perfect following was assumed is probably because the designs included additional controllers not normally found on conventional airplanes, such as direct side force generators. With the inclusion of such controllers, the class of models that can be followed perfectly is greatly enlarged. Seen another way, models which could not be followed perfectly could be followed better than without the additional controllers. One such control law that evolved is known as response feedback, and requires feedback of the plant states and feedforward of the external inputs. By considering a different formulation of the performance index, Motyka derived what he called the open-loop implementation of the response feedback control law, requiring feedforward of model states and state rates (but not plant feedback). Neither of these control laws addressed the error dynamics of the system.

Because the relationships used in deriving these control laws are purely algebraic, Motyka [15] was able to relate them to nonlinear equations of motion for the plant and model. The controls appropriate to nonlinear problems depended on the the manner in which the controls appear in the equations. If the plant equations are linear in the control, then closed form expressions for the control laws were obtained. It should be noted at this
point that the tests for perfect model following no longer apply, since they are valid only for linear systems.

In the design of the TIFS model following system, Motyka, Rynaski, and Reynolds appear to have abandoned the weighting of error and control, and described a control law that was obtained by solving the plant equations for the plant control, then simply substituting into this expression the model states and state rates instead of those of the plant. The same approach was taken in determining the control law for the proposed YT-2B in flight simulator. This formulation left the plant state feedback completely free for use in sensitivity minimization. The means used to solve for the plant control was the pseudoinverse, and for matrices with fewer columns than rows ("tall" matrices, like most conventional airplane control matrices), the Penrose left pseudoinverse was used. The use of the pseudoinverse is justified in many ways, primarily based on the notion that in the absence of other considerations, the least squares approach is best. Other generalized inverses have been used, for example, one based on the controllable companion form of the plant equations [16].

Sensitivity minimization through the plant feedback gains was later addressed in more detail by Motyka [17]. The method used was a classical analysis based on closed-loop frequency domain characteristics. The basic approach was to make the bandwidth of a particular plant closed loop mode much wider than that of the model, and was shown through examples.

The primary means of determining model following control laws were thus established: linear optimal control, which was successfully applied to all forms of the problem; and two direct state space methods, strictly applicable only to perfect model following. Much of the rest of the work done in the field expanded on these results, or
offered applications of the methods. Buethe and Lebacqz [18] considered optimal control solutions using the state rates in addition to the states for feedback, and showed that this removed the requirements for the feedback of perturbation signals. Rynaski [19] used the direct form of model following control in considering adaptive multivariable model following. Armstrong [20] investigated the digital implementation of explicit model following through the discrete equivalent of linear optimal control, and applied the results to a typical fighter airplane using an unstable model. Kaufman and Berry [21] also considered digital implementation in their analysis of the requirements for an adaptive model following flight control system. Their design method was also the discrete equivalent of linear optimal control. Also using linear optimal control for analysis, Kreindler and Rothschild [22] contributed to the debate over explicit vs. implicit model following, making a strong case for implicit following on theoretical grounds but failing to show clear cut advantages in their examples.

The work of Lebacqz and Govindaraj [23] was concerned with model following using unstable models. The implicit model following optimal regulator procedure results in stable closed loop systems, whether the model is stable or not; Lebacqz and Govindaraj made a coordinate transformation along the real axis to circumvent this. And finally, Meyer [24] introduced an algorithmic approach to generating model following control laws that has application to a class of nonlinear plants that can be transformed into canonic forms involving bundles of integrators of a certain form.

Some of the more important results concerned the error dynamics of explicit model following control. Kudva and Gourishankar [25] considered a stability problem occurring in certain instances of different order plants and models. They assumed perfect following and used Erzberger's control law formulation to analyze the root of the instability. In doing so,
they claimed that perfect model following is contingent on the stability of the closed-loop system. Kawahata [26] described a method for arbitrarily assigning the stability characteristics of the error. To do this, he proposed a different design method that considers not only the errors of the system but their derivatives.

A more general method appears in the paper by Anderson and Schmidt [27]. The equations of state are taken as before, except output matrices are defined for the plant and the model. By first assuming that the error dynamics were completely assignable, and then proceeding in a manner analogous to Erzberger, the control law and conditions for perfect model following were derived.

A largely unanswered question in model following control lies in the selection of the model. As mentioned above, models are selected because they have desirable responses to initial conditions and to control inputs. What is not always clear is what it is about the model that makes its responses desirable. In simulation applications, models are taken from the process being simulated and the answer is immediate. In designing control systems for vehicles to be operated by humans, "desirable" must be taken to mean that the model has good handling qualities. In airplane applications, good flying qualities are sometimes expressed in terms of several parameters related to the airplane dynamics, mostly taken from frequency domain analysis. It is well known, however, that an airplane that satisfies the letter of flying qualities requirements may still have poor flying qualities in some flight regime while performing some task. Nonetheless, such a characterization may be useful. Goddard and Gleason [28] recently reported their attempts to find the optimal longitudinal plant matrix. This was done by assuming ideal values for a number of flying qualities related parameters, such as short period damping, and finding a system matrix that simultaneously satisfied all of these requirements.
Several studies of applications of model following control to aircraft design have been reported. Only those that introduced fundamentally different ideas in model following have been addressed in this literature search. These reports will not be discussed here, although some will be referenced in the text as a source of examples. The rest are relegated to the bibliography, to which the interested reader is referred.
NOMENCLATURE

A  System matrix
B  Control input matrix
β  Sideslip angle
Δ  Matrix of differences between model and plant matrices: \([A_m-A_p : B_m-B_p]\)
Δ  Small variation
δ  Control input
e  Error vector
φ  Euler Bank angle
G  Matrix relating rows of a matrix to a basis for its row space
η  State vector dimension, different order plants and models
i  Current
I  Moment of inertia
J  Polar moment of inertia
Ke  Error feedback gain matrix
Kp  Motor constant
Λ  Diagonal matrix of Eigenvalues
L  Length
μ  Control vector dimension, different order plants and models
m  Dimension of control vector; Mass
M Modal matrix
N Dimension of augmented system (different order problems)
n Dimension of state vector
p Body axis roll rate
θ Euler pitch angle; Control stick deflection
q State vector in modal space; Body axis pitch rate
qi Euler parameter, i = 0 ... 3
r Body axis yaw rate
T Transformation matrix, or Torque
u Model control vector; X-axis velocity
up Plant control vector
v Y-axis velocity
w Z-axis velocity
W Weighting matrix.
w Dimension of common states (different order problems)
ω Frequency
x State vector
y Transformed state vector
+ve Positive
Ø Null set

Superscripts
aero Due to airframe aerodynamics
* Conjugate operation (matrix conjugate transpose)
* (When applied to control vector) Model following control
\begin{itemize}
\item $+ \quad$ Pseudo inverse
\item $^{-1} \quad$ Matrix inverse
\item $T \quad$ Matrix transpose
\item $n \quad$ Right generalized inverse
\item $w \quad$ Left generalized inverse
\end{itemize}

**Subscripts**

\begin{itemize}
\item $A \quad$ Applied
\item $a, A \quad$ aileron
\item $c \quad$ Candidate model
\item $cm \quad$ Control model
\item $F \quad$ Static friction
\item $HT \quad$ Horizontal tail
\item $L \quad$ Left
\item $LEF \quad$ Leading edge flap
\item $m \quad$ Desired model
\item $P \quad$ Peak, max
\item $p \quad$ Plant
\item $R \quad$ Right
\item $R \quad$ Ripple
\item $R, r \quad$ Rudder
\item $\text{ref} \quad$ Evaluated at reference conditions
\item $T \quad$ Thrust, throttle
\item $TF \quad$ Trailing edge flap
\item $V \quad$ Viscous damping
\end{itemize}
xx  X-axis
yy  Y-axis
zz  Z-axis
xz  Cross product, X- and Z-axes
z   Dummy variable

Diacritical Marks

·   (Dot) Time derivative

~   (Tilde) Difference between actual and measured, estimated, or assumed value

^   (Circumflex) Measured, estimated, or assumed value; Transformed variables

VECTOR AND MATRIX CONVENTIONS

Vectors and matrices are presented in plain text, and are usually differentiated from scalar values by context. Where confusion might arise, matrices are enclosed in brackets. Individual elements of vectors are indicated by a single subscript, and of a matrix, by a double subscript. Single subscripts on vectors are also used to indicate the system referred to (plant, model, etc.), but only the subscripts listed above are used for this purpose. A single subscript on a matrix indicates that it is square and of the dimension indicated. Other matrix dimensions are indicated by subscripts m×n, where m and n are the row and column dimensions. Partitions of a vector are indicated by a single superscript, and of a matrix, by a either single or double superscript, depending on how it is partitioned. The only instance in which a superscripted numeral does not indicate a partition is the squaring of certain scalar quantities.
We will consider plants and models that are described by differential equations of motion of the forms

\[ f_p(\dot{x}_p, x_p, u, u_p) = 0 \quad (4-1) \]

for the plant, and for the model,

\[ f_m(\dot{x}_m, x_m, u) = 0 \quad (4-2) \]

In these equations, \( x_p \) and \( x_m \) are the plant and model state vectors of dimension \( n \), where it is assumed that \( x_p \) and \( x_m \) represent the same variables on a one-to-one basis. The functions \( f_p \) and \( f_m \) are vector valued of dimension \( n \). The vector \( u_p \) is the control applied to the plant, and \( u \) is an external control applied to the model and possibly to the plant.

We further assume that each of these equations may be separated into three distinct groupings that correspond to a partitioning of the respective state vectors as follows:

- A type I grouping consists of those relationships that are kinematic in nature in that they do not explicitly depend on any control:

\[ f_p^1(\dot{x}_p^1, x_p) = 0 \quad (4-3) \]

\[ f_m^1(\dot{x}_m^1, x_m) = 0 \quad (4-4) \]
Here, \( \dim(f^1_p) = \dim(f^1_m) \), and \( x^1_p \) and \( x^1_m \) represent the same variables on a one-to-one basis.

- A type 2 grouping consists of plant equations that are driven by the model controls only. There are corresponding model equations that are of the same dimension, that dynamically describe the same state variables, and that are driven by the same control inputs:

\[
f^2_p(x^2_p, x_p, u) = 0 \quad (4-5)
\]

\[
f^2_m(x^2_m, x_m, u) = 0 \quad (4-6)
\]

- A type 3 grouping consists of plant equations that are driven by the plant controls only. There are corresponding model equations that are of the same dimension, that dynamically describe the same state variables, and that are driven by the same control inputs:

\[
f^3_p(x^3_p, x_p, u_p) = 0 \quad (4-7)
\]

\[
f^3_m(x^3_m, x_m, u) = 0 \quad (4-8)
\]

With respect to these three types of equations, we assume that at every point of interest the given functions vanish, that the partial derivatives of the given functions with respect to each of their arguments are continuous, and further that the Jacobians

\[
\frac{\partial f^1_1(x^1, x)}{\partial x^1} \neq 0 \quad (4-9)
\]

\[
\frac{\partial f^2(\chi^2, x, u)}{\partial x^2} \neq 0 \quad (4-10)
\]
\[
\frac{\partial f^3(x^3, x, u)}{\partial x^3} \neq 0 \tag{4-11}
\]

for both plant and model. Note that unsubscripted state or control vectors are intended to signify either the plant or the model, as appropriate.

With respect to the plant controls, we require that they be independent of one another, and that the dimension of \( f_p^3 \) be the same as the dimension of \( u_p \). We assume that

\[
\frac{\partial f_p^3(x_p^3, x_p, u_p)}{\partial u_p} \neq 0 \tag{4-12}
\]

The implicit function theorem assures that there are unique functions \( \phi^{1,2,3} \) for plant and model, and \( g_p \) for the plant, such that

\[
\begin{align*}
\dot{x}_p^1 &= \phi_p^1(x_p) \tag{4-13a} \\
\dot{x}_m^1 &= \phi_m^1(x_m) \tag{4-13b} \\
\dot{x}_p^2 &= \phi_p^2(x_p, u) \tag{4-14a} \\
\dot{x}_m^2 &= \phi_m^2(x_m, u) \tag{4-14b} \\
\dot{x}_p^3 &= \phi_p^3(x_p, u_p) \tag{4-15a} \\
\dot{x}_m^3 &= \phi_m^3(x_m, u) \tag{4-15b} \\
u_p &= g_p(x_p^3, x_p) \tag{4-16}
\end{align*}
\]

Equation (4-16) will form the basis for the development of perfect, explicit model following control laws. A distinction is made between the existence of the function \( g_p \) (guaranteed by the implicit function theorem), and the availability of a closed form expression for it. Closed form expressions at all points of interest will be required in the implementation of the model following control laws that will be developed.
The development of these control laws will depend on our ability to find controls $u_p$ that, for fixed $x_p$, allow us to specify $\dot{x}_p^3$. In (4-16) we substitute a dummy variable for the plant state rates, viz

$$u_p = g_p(z, x_p)$$

and examine the relationship between $\dot{x}_p^3$ and $z$. Using (4-17) in (4-15),

$$\dot{x}_p^3 = \phi_p^3(x_p^3, g_p(z, x_p))$$

(4-18)

For fixed $x_p$,

$$\dot{x}_p^3 = \phi_p^3(\cdot, g_p(z, \cdot)) = \Phi(z)$$

(4-19)

Referring to the Jacobian matrices,

$$\left[ \frac{\partial \Phi(z)}{\partial z} \right] = \left[ \frac{\partial \phi_p^3(x_p^3, u_p)}{\partial u_p} \right] \left[ \frac{\partial g_p(z, x_p)}{\partial z} \right]$$

(4-20)

where

$$\left[ \frac{\partial \phi_p^3}{\partial u_p} \right] = - \left[ \frac{\partial f_p^3}{\partial x_p^3} \right]^{-1} \left[ \frac{\partial f_p^3}{\partial u_p} \right]$$

(4-21)

$$\left[ \frac{\partial g_p}{\partial z} \right] = - \left[ \frac{\partial f_p^3}{\partial u_p} \right]^{-1} \left[ \frac{\partial f_p^3}{\partial x_p^3} \right]$$

(4-22)

so that

$$\left[ \frac{\partial \Phi}{\partial z} \right] = \left[ \frac{\partial f_p^3}{\partial x_p^3} \right]^{-1} \left[ \frac{\partial f_p^3}{\partial u_p} \right]$$

(4-23)
We conclude that if \( \partial f^3_p / \partial x^3_p \) is independent of \( x^3_p \), then

\[
\left[ \frac{\partial \Phi}{\partial z} \right] = I \tag{4-24}
\]

and hence,

\[
u_p = g_p(z, x_p) \Rightarrow \dot{x}_p^3 = z \tag{4-25}
\]

Hereafter we assume that \( f^3_p \) is linear in \( x^3_p \), so that this conclusion is valid. We likewise assume that \( f^3_m \) is linear in \( \dot{x}_m^3 \).

We will speak of particular equations of motion being the same for both plant and model, which will be taken to mean that the equations are functionally the same with respect to their arguments. When such is the case, we will drop the subscript (p or m):

\[
f^i_p(\cdot, \cdot, \cdot) = f^i_m(\cdot, \cdot, \cdot) \Rightarrow f^i(\cdot, \cdot, \cdot) \tag{4-26}
\]

With respect to the plant equations of motion, we will consider only plants that are completely controllable with respect to the plant control, \( u_p \).
PERFECT MODEL FOLLOWING CONTROL

This section considers the general perfect model following control problem, and introduces a standard form of the plant and model equations of motion. Problems in this standard form are analyzed with respect to the development of the perfect model following control law, the assignment of error dynamics for the system, and the effects of measurement errors and plant parameter uncertainties.

STANDARD FORM

Consider a plant and model whose equations of motion are separable into the three types defined above, whose type 1 equations are the same \( f^1_m(\cdot, \cdot) = f^1_p(\cdot, \cdot) \equiv f^1(\cdot, \cdot) \), and whose type 2 equations are the same \( f^2_m(\cdot, \cdot, \cdot) = f^2_p(\cdot, \cdot, \cdot) \equiv f^2(\cdot, \cdot, \cdot) \):

\[
\begin{align*}
    f^1(x^1_p, x_p) &= 0 \\
    f^2(x^2_p, x_p, u) &= 0 \\
    f^3_p(x^3_p, x_p, u_p) &= 0 \\
    f^1(x^1_m, x_m) &= 0 \\
    f^2(x^2_m, x_m, u) &= 0 \\
    f^3_m(x^3_m, x_m, u) &= 0
\end{align*}
\]

This form of plant and model equations, subject to the hypotheses stated in the preliminary section, is defined as the Standard Form for perfect model following control.
CONTROL LAW AND ERROR DYNAMICS

Dynamics Matching

Dynamics matching is taken to mean that $\dot{x}_p(t) = \dot{x}_m(t)$. With the assumption of complete controllability, the evolution of $x^1_p$ is determined through the control of $x^2_p$ and $x^3_p$ according to the equations of motion described by $f^1(\cdot, \cdot) = 0$. Likewise, the evolution of $x^2_p$ is determined through the control of $x^3_p$ according to the equations of motion described by $f^2(\cdot, \cdot, u) = 0$. Because the type 1 and type 2 equations are the same for both plant and model, we expect that if $x_p = x_m$ initially, then we can cause $\dot{x}_p = \dot{x}_m$ by selecting $u_p(t)$ such that $x^2_p(t) = \dot{x}^3_m(t)$. By assuming that $f^3_p$ is linear in $\dot{x}^3_p$, this is accomplished by substituting the model state rates for the corresponding plant state rates in the control equation (see (4-23)):

$$u_p = g_p(\dot{x}^3_m, x_p) \quad (5-3)$$

Now consider the error between model and plant,

$$e = 
\begin{bmatrix}
e^1 \\
e^2 \\
e^3
\end{bmatrix} =
\begin{bmatrix}
x^1_m - x^1_p \\
x^2_m - x^2_p \\
x^3_m - x^3_p
\end{bmatrix} \quad (5-4)$$

and the error rate,

$$\dot{e} = 
\begin{bmatrix}
\dot{e}^1 \\
\dot{e}^2 \\
\dot{e}^3
\end{bmatrix} =
\begin{bmatrix}
\dot{x}^1_m - \dot{x}^1_p \\
\dot{x}^2_m - \dot{x}^2_p \\
\dot{x}^3_m - \dot{x}^3_p
\end{bmatrix} \quad (5-5)$$
where

\begin{align}
\dot{x}_m^1 - \dot{x}_p^1 &= \phi^1(x_m) - \phi^1(x_p) \\
\dot{x}_m^2 - \dot{x}_p^2 &= \phi^2(x_m, u) - \phi^2(x_p, u) \\
\dot{x}_m^3 - \dot{x}_p^3 &= 0
\end{align} \quad (5-6)

Note that if the plant and model states are initially aligned, then \( e = 0 \) and

\[ \dot{e} \big|_{e=0} = 0 \quad (5-7) \]

so that the expected result of dynamically matching plant and model is attained.

**Assignment of Error Dynamics**

The error dynamics may be arbitrarily assigned by modifying the model equations. This modified model is referred to as the control model, denoted by the subscript \( cm \), and is defined by

\begin{align}
f^1(\dot{x}_{cm}^1, x_m) &= 0 \\
f^2(\dot{x}_{cm}^2, x_m, u) &= 0 \\
f^3(\dot{x}_{cm}^3, x_m, u) + f_{cm}(x_m, x_p) &= 0
\end{align} \quad (5-8)

The only difference between the control model and the original model is therefore the addition of the function \( f_{cm}(x_m, x_p) \) to the control model equations of motion. This function has the appearance of a full state feedback controller, where the feedback states are those of both the model and the plant.

By previous assumptions regarding the linearity of the model equations in the state rate variables, (5-8) implies that
The function $f_{cm}(x_m, x_p)$ is typically (but not necessarily) a function of the error, and at any rate is taken such that it vanishes whenever the error is zero. That is,

$$f_{cm}(x_m, x_m) \equiv 0 \quad (5-10)$$

so that

$$\dot{x}_{cm}^3 \big|_{e=0} = \dot{x}_m^3 \quad (5-11)$$

The control model still fits the standard form, and the control law

$$u_p^* = g_p(x_{cm}^3, x_p) \quad (5-12)$$

causes the plant to dynamically follow the control model. We will therefore select the function $f_{cm}(x_m, x_p)$ such that, if there is an error between the model and plant trajectories, then the control model will dynamically drive the plant states along a path that tends toward the model states. The effect of the function $f_{cm}(x_m, x_p)$ is most clearly seen through the error dynamics. Thus, with respect to the control model,

$$\dot{x}_{cm}^1 - \dot{x}_p^1 = \dot{x}_m^1 - \dot{x}_p^1 = \phi^1(x_m) - \phi^1(x_p)$$
$$\dot{x}_{cm}^2 - \dot{x}_p^2 = \dot{x}_m^2 - \dot{x}_p^2 = \phi^2(x_m, u) - \phi^2(x_p, u)$$
$$\dot{x}_{cm}^3 - \dot{x}_p^3 = \dot{x}_m^3 + f_{cm}(x_m, x_p) - \dot{x}_p^3 = 0 \quad (5-13)$$
so that

\[
\begin{align*}
\dot{e}^1 &= \phi^1(x_m) - \phi^1(x_p) \\
\dot{e}^2 &= \phi^2(x_m, u) - \phi^2(x_p, u) \\
\dot{e}^3 &= -f_{cm}(x_m, x_p)
\end{align*}
\] (5–14)

We may now arbitrarily select \( f_{cm}(x_m, x_p) \) (subject to \( f_{cm}(x_m, x_m) = 0 \)) such that the error dynamics are stable. One means of accomplishing this is through selection of an appropriate Lyapunov function, and will be demonstrated later for the nonlinear equations of motion of a rigid body.

Alternatively, we may linearize the error dynamic equations and evaluate them at the zero error condition for some fixed values of \( x_m \) and \( u \) to yield expressions for small perturbations in the error, \( \Delta e \):

\[
\begin{align*}
\Delta \dot{e}^1 &= \left[ \frac{\partial \phi^1}{\partial x_m} \right]_{\text{ref}} \Delta e \\
\Delta \dot{e}^2 &= \left[ \frac{\partial \phi^2}{\partial x_m} \right]_{\text{ref}} \Delta e \\
\Delta \dot{e}^3 &= \left[ \frac{\partial f_{cm}}{\partial x_m} \right]_{\text{ref}} \Delta e
\end{align*}
\] (5–15)

Now if the various partial derivatives that appear in these equations are constant, then the problem is one of selecting the gains (by selection of \( f_{cm} \), through the matrix \( \partial f_{cm}/\partial x_m \)) as a regulator problem. Otherwise, the solution is valid only in the vicinity of the selected operating point. It will be shown in the section on linear model following control that the assumption of complete plant controllability assures that the dynamics of the error are arbitrarily assignable.
ERRORS AND UNCERTAINTIES

Control and State Measurement Error

The control law developed for perfect model following requires knowledge of the model states, state rates, the externally applied control, and the plant states. In the implementation of this control law, complete knowledge of the model is assumed known, either through real time computation or through previously stored tables. The externally applied control and the plant states are usually measured or estimated, however, and may be inaccurately known. The effect of such measurement errors is analyzed through their effect on the error dynamics.

The measured or estimated values of the control and the states are denoted by variables with a circumflex (^) mark, and the differences between these and the actual values are denoted with tilde (~) marks (e.g., \( \hat{x}_p = x_p + \tilde{x}_p \)). The control law is being calculated using the measured or estimated values,

\[
\begin{align*}
    u_p^* &= g_p(\dot{x}_p^3, \hat{x}_p) \\
         &= g_p(\dot{x}_p^3, (x_p + \tilde{x}_p)) \\
\end{align*}
\]  

(5-16)

The plant is responding to this control using the actual values of the states and the erroneous values of the external control:

\[
\begin{align*}
    \dot{x}_p^1 &= \phi^1(x_p) \\
    \dot{x}_p^2 &= \phi^2(x_p, \hat{u}) \\
    \dot{x}_p^3 &= \phi^3_p(x_p, u_p^*(\dot{x}_p^3, \hat{x}_p)) \\
\end{align*}
\]  

(5-17)
Now consider series expansions of the plant equations:

\[
\begin{align*}
\dot{x}_p^1 &= \phi^1(x_p) \\
\dot{x}_p^2 &= \phi^2(x_p, u) + \left[ \frac{\partial \phi^2(x_p, u)}{\partial u} \right] \bar{u} + \ldots \\
\dot{x}_p^3 &= \phi^3_p(x_p, u_p^*(\hat{x}_{cm}^3, x_p)) + \left[ \frac{\partial \phi^3_p(x_p, u_p^*(\hat{x}_{cm}^3, x_p))}{\partial u_p(\hat{x}_{cm}^3, x_p)} \right] \dot{x}_p + \ldots (5-18)
\end{align*}
\]

By assuming that the measurement errors are small and retaining only the terms shown, the error dynamics may be approximated by

\[
\begin{align*}
\dot{e}^1 &= \phi^1(x_m) - \phi^1(x_p) \\
\dot{e}^2 &= \phi^2(x_m, u) - \phi^2(x_p, u) - \left[ \frac{\partial \phi^2}{\partial u} \right] \bar{u} \\
\dot{e}^3 &= -f_{cm}(x_m, x_p) - \left[ \frac{\partial \phi^3}{\partial u_p} \right] \left[ \frac{\partial u_p^*}{\partial x_p} \right] \bar{x}_p (5-19)
\end{align*}
\]
or, for the linearized error dynamics,

\[
\begin{align*}
\Delta \dot{e}^1 &= \left[ \frac{\partial \phi^1}{\partial x_m} \right]_{ref} \Delta e \\
\Delta \dot{e}^2 &= \left[ \frac{\partial \phi^2}{\partial x_m} \right]_{ref} \Delta e - \left[ \frac{\partial \phi^2}{\partial u} \right] \bar{u} \\
\Delta \dot{e}^3 &= -\left[ \frac{\partial f_{cm}}{\partial x_m} \right]_{ref} \Delta e - \left[ \frac{\partial \phi^3}{\partial u_p} \right] \left[ \frac{\partial u_p^*}{\partial x_p} \right] \bar{x}_p (5-20)
\end{align*}
\]

**Plant Parameter Uncertainties**

We now assume that the control and state measurement errors are negligible, and examine the effects of uncertainties in plant parameters alone. Within the plant and model
equations of motion, the function \( f^1(\dot{x}, x) \) is assumed to be known exactly for this analysis. In most cases it represents kinematic relationships, and is not subject to uncertainties. The function \( f^2(\dot{x}, x, u) \) is most often an artificiality introduced when the plant and model are of different order, as will be shown, and arises from the model equations of motion, which are assumed known. The function \( f^2(\dot{x}, x, u) \) may also arise when certain of the plant controls are not available for synthesis, in which case the function is not known exactly. Such problems have never appeared in the literature, however, and will not be considered here. In any case, the effect of variations in the parameters of either of these functions may be analyzed in a manner similar to that which follows.

In problems involving aircraft dynamics, the function \( f^3_p(\dot{x}_p, x_p, u_p) \) typically contains force and moment relationships that are functions of the state variables, and are almost never known with certainty. This is reflected in the control law through uncertainties in the parameters in the function \( g_p(x_{cm}, x_p) \). For given values of \( x_{cm} \) and \( x_p \), consider the assumed value of the function to be in error by some difference such that \( \hat{g}_p(x_{cm}, x_p) = g_p(x_{cm}, x_p) + \tilde{g}_p(x_{cm}, x_p) \), equivalent to \( u_p^* = u_p + \tilde{u}_p^* \). The equation for \( \dot{x}_p^3 \) may then be written as

\[
\dot{x}_p^3 = \phi_p^3(x_p, u_p^* + \tilde{u}_p^*)
\]

(5–21)

Expanding this equation as before,

\[
\dot{x}_p^3 \approx \phi_p^3(x_p, u_p^*) + \left[ \frac{\partial \phi_p^3(x_p, u_p^*)}{\partial u_p^*} \right] \tilde{u}_p^*
\]

(5–22)

Equivalently,

\[
\dot{x}_p^3 = \phi_p^3(x_p, u_p^*) + \left[ \frac{\partial \phi_p^3(x_p, u_p^*)}{\partial u_p^*} \right] \tilde{g}_p(x_{cm}, x_p)
\]

(5–23)
The error dynamics are modified from the nominal case as follows:

\[
\dot{e}^3 = -f_{cm}(x_m, x_p) - \left[ \frac{\partial \phi_p^3(x_p^*, u_p^*)}{\partial u_p} \right] \hat{g}_p(x_m, x_p) \tag{5-24}
\]

\[
\Delta \dot{e}^3 \approx \left[ \frac{\partial f_{cm}}{\partial x_m} \right]_{ref} \Delta e - \left[ \frac{\partial \phi_p^3(x_p^*, u_p^*)}{\partial u_p} \right] \tilde{g}_p(x_m, x_p) \tag{5-25}
\]

**Remarks**

As was the case with no measurement error, \(f_{cm}(x_m, x_p)\) may now be selected to stabilize the error, except now consideration must be made of the additional terms arising from the measurement error, parameter uncertainty, or both. It should be noted that these additional terms are functions of the plant states. This dependency may be removed for analysis of the error dynamics by the substitution \(x_p = x_m - e\).

At best, the additional terms will result in non-zero steady state errors under steady state operating conditions, but at worst will affect the stability of the error dynamics. Given estimates of the expected measurement errors, the feedback gains may be selected to stabilize the error dynamics and to minimize some norm of the error. This, however, becomes an \(H_\infty\) control problem, and is well beyond the scope of this dissertation. A simpler approach is to consider only the linearized versions of the error dynamics, and to apply optimal regulator theory to the nominal case to obtain feedback gains. Following this, estimates of the errors and uncertainties are added to the closed loop solution and their effects evaluated by simulation.
LINEAR MODEL FOLLOWING CONTROL

PERFECT MODEL FOLLOWING

In this section, the perfect model following control problem for linear systems is discussed. The standard form for the plant and model equations of motion in linear form is introduced, and it is shown that all linear perfect model following control problems are equivalent to this standard form through similarity transformations on the plant and model state vectors. The control law for systems in the standard form, and the assignment of dynamics for the model following control error, are developed.

The equations of motion of the plant and model are assumed to be provided in the forms

\[
\dot{x}_p = A_p x_p + B_p u_p \quad (6-1)
\]

\[
\dot{x}_m = A_m x_m + B_m u \quad (6-2)
\]

Standard Form

The usual formulation of the linear model following control problem does not include the external control \( u \) in the plant equations of motion, and treats only those cases where there are fewer plant controllers than degrees of freedom. To make a valid comparison, we will consider only equations that are separable into types 1 and 3, and later treat the more general linear case by example. We will, however, keep the superscript
notation consistent with the standard form introduced previously. The linear equivalent to the standard form will be taken as:

$$\dot{x}_p = \begin{bmatrix} \dot{x}_p^1 \\ \dot{x}_p^2 \\ \dot{x}_p^3 \end{bmatrix} = \begin{bmatrix} A_1^1 & \cdots & A_1^3 \\ A_2^1 & \cdots & A_2^3 \\ A_3^1 & \cdots & A_3^3 \end{bmatrix} \begin{bmatrix} x_p^1 \\ x_p^2 \\ x_p^3 \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix} u_p$$

(6-3)

$$x_p^1 : (n-m)x1 \quad x_p^3 : mx1$$

$$A_1^1 : (n-m)xn \quad A_3^3 : mxn$$

$$\dot{x}_m = \begin{bmatrix} \dot{x}_m^1 \\ \dot{x}_m^2 \\ \dot{x}_m^3 \end{bmatrix} = \begin{bmatrix} A_1^1 & \cdots & A_1^3 \\ A_2^1 & \cdots & A_2^3 \\ A_3^1 & \cdots & A_3^3 \end{bmatrix} \begin{bmatrix} x_m^1 \\ x_m^2 \\ x_m^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B_m^3 \end{bmatrix} u$$

(6-4)

$$x_m^1 : (n-m)x1 \quad x_m^3 : mx1$$

$$A_1^1 : (n-m)xn \quad A_3^3 : mxn$$

$$B_m^3 : mxm$$

The following characteristics of these equations are emphasized:

(1) The plant and the model are of the same order (n);

(2) For a plant with m controls, there are exactly m plant equations that depend on these controls, and these controls appear through the identity matrix for those m equations.

(3) For the n-m plant equations that do not directly depend on the controls, there are n-m model equations in which the controls do not appear. These n-m equations are identical with the corresponding plant equations (the sub-matrix $A_1^1$ is the same for both plant and model).

(4) The remaining m equations are arbitrary, including $B_m^3$. The externally applied control vector $u$ is of any order less than or equal to m.
By comparison with the general definition of the standard form, we have

\[ f^1(x_p^1, x_p) = x_p^1 - A^1 x_p = 0 \]
\[ f^2(x_p^2, x_p, u) = \emptyset \]
\[ f_p^2(x_p^3, x_p, u_p) = x_p^3 - A_p^3 x_p - u_p = 0 \quad (6-5) \]
\[ f^1(x_m^1, x_m) = x_m^1 - A^1 x_m = 0 \]
\[ f^2(x_m^2, x_m, u) = \emptyset \]
\[ f_m^2(x_m^3, x_m, u) = x_m^3 - A_m^3 x_m - B_m^3 u = 0 \quad (6-6) \]

Systems in this standard form satisfy the usual criteria for perfect model following given by Erzberger, Chan, et al.\[5'7\] Additionally, all systems that satisfy the criteria for perfect model following may be put in the form of (6-5) and (6-6) by simultaneous similarity transformations on the plant and model equations. These assertions are proven as follows:

For the case of full state feedback, criteria for perfect model following may be written as:

\[ [I - B_p B_p^+] B_m = [0] \quad (6-7) \]
\[ [I - B_p B_p^+] [A_m - A_p] = [0] \quad (6-8) \]

where \( B_m, A_m, \) and \( A_p \) are, in general, not in the form given by (6-5) and (6-6). The superscript \( + \) indicates the pseudoinverse of the matrix\(^*\). In the form given by equation

\[^*\text{Notation and definitions for generalized inverses follow Boullion and Odell.} [29] \text{ B}^W \text{ is the left generalized inverse, while B}^+ \text{ is the (unique) pseudoinverse. B}^W \text{satisfies (BB}^W\text{B)=B, (B}^W\text{BB}^W\text{)=B}^W, and} \]
(6-3), \( B_p^+ = B_p^T = [0 : 1] \), and \( [I - B_p B_p^+] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). The proof that (6-5) and (6-6) imply (6-7) and (6-8) is immediate, since the submatrix \( A^1 \) is the same for plant and model. To prove that (6-7) and (6-8) imply the existence of a similarity transformation that results in (6-5) and (6-6), we first consider the class \( L \) of all left generalized inverses of \( B_p \),

\[
\]

and note that \( B_p^+ \in L \). We partition \( B_p \) as

\[
B_p = \begin{bmatrix} B_p^1 \\ B_p^2 \end{bmatrix} \quad (6-10)
\]

and require, without loss of generality (since \( B_p \) is assumed to be of full rank), that \( \text{rank}(B_p^2) = m \). This implies that \( \exists \ G \ni B_p^1 = GB_p^2 \) or

\[
B_p = \begin{bmatrix} GB_p^2 \\ B_p^2 \end{bmatrix} \quad (6-11)
\]

We take a general form of the left inverse as \(^{[30]}\)

\[
B_p^w = [P : B_p^{2^{-1}} - PG] \quad (6-12)
\]

where \( P \) is arbitrary. Using this form,

\[
[I - B_p B_p^w] = \begin{bmatrix} I - GB_p^2 P & -[I - GB_p^2 P]G \\ -B_p^2 P & B_p^2 PG \end{bmatrix} \quad (6-13)
\]

\((B_p^w B)^* = (B_p B)^*\). \( B_p^+ \) additionally satisfies \((BB^+)^* = (BB^+)\). The use of the Penrose left pseudo-inverse, which appears frequently in analyses of this sort, is not necessarily the inverse of choice.
We also need to partition the other matrices to conform with left multiplication by (6-13):

\[
B_m = \begin{bmatrix} B^1_m \\ B^2_m \end{bmatrix}, \quad A_m = \begin{bmatrix} A^1_m \\ A^2_m \end{bmatrix}, \quad A_p = \begin{bmatrix} A^1_p \\ A^2_p \end{bmatrix}
\]

(6-14)

Note that the submatrices in these expressions are not the same as those in the standard form equations, since we have assumed that the given equations are not in the standard form.

Using (6-13) and performing the operations indicated by (6-7) and (6-8), it is found that the conditions for perfect model following hold if and only if

\[
B^1_p - GB^2_p = [0]
\]

(6-15a)

\[
[A^1_m - A^1_p] - G[A^2_m - A^2_p] = [0]
\]

(6-15b)

For convenience, introduce the matrix \( \Delta \) such that

\[
\Delta = \begin{bmatrix} \Delta^1 \\ \Delta^2 \end{bmatrix} = \begin{bmatrix} A^1_m - A^1_p & B^1_m - B^1_p \\ A^2_m - A^2_p & B^2_m - B^2_p \end{bmatrix}
\]

(6-16)

The conditions for perfect model following are that

\[
\Delta^1 = GA^2
\]

(6-17)

The rest of the proof is constructive. Consider the similarity transformation represented by

\[
T = \begin{bmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{bmatrix} = \begin{bmatrix} T^{11} & -T^{11}G \\ P & B^2_p - PG \end{bmatrix}
\]

(6-18)
where $T^{11}$ and $P$ are arbitrary, except that $T$ must be nonsingular. Now note that if a plant and model satisfy (6-7) and (6-8), then (6-17) must be satisfied, and that (6-18) is the required similarity transformation.

As a consequence of this proof, virtually every linear perfect model following control problem that appears in the literature may be put in the standard form by the given transformation. Thus, the analysis of the equations in the standard form, including the development of the control law and error dynamics, is valid for all such problems.

**Control Law and Error Dynamics**

From the standard form, the control law for perfect model following control is:

$$u_p^* = \dot{x}_{cm}^3 - A_p^3 x_p$$  \hspace{1cm} (6-19)

For our control model, we anticipate a linear regulator problem and take

$$x_{cm} = \begin{bmatrix} x_{cm}^1 \\
\quad x_{cm}^3 
\end{bmatrix} \equiv \begin{bmatrix} \dot{x}_m^1 \\
\quad \dot{x}_m^3 
\end{bmatrix} + \begin{bmatrix} 0 \\
K_e 
\end{bmatrix} e$$  \hspace{1cm} (6-20)

where $K_e$ is a constant matrix of appropriate dimensions and, using the general notation,

$$f_{cm}(x_m, x_p) = K_e (x_m - x_p)$$  \hspace{1cm} (6-21)

Now consider the error between model and plant,

$$e = \begin{bmatrix} e^1 \\
\quad e^3 
\end{bmatrix} \equiv \begin{bmatrix} x_m^1 - x_p^1 \\
\quad x_m^3 - x_p^3 
\end{bmatrix}$$  \hspace{1cm} (6-22)

and the error rate,
\[
\begin{bmatrix}
\dot{x}_m^1 - \dot{x}_p^1 \\
\dot{x}_m^3 - \dot{x}_p^3 \\
\end{bmatrix} = \begin{bmatrix}
A^1 x_m - A^1 x_p \\
A^3 x_m - [A_p^3 x_p + u_p] \\
\end{bmatrix} = \begin{bmatrix}
A^1 \begin{bmatrix} x_m - x_p \end{bmatrix} \\
\dot{x}_m^3 - [A_p^3 x_p + \dot{x}_m - A_p^2 x_p] \\
\end{bmatrix} = \begin{bmatrix}
A^1 \\
-K_e \\
\end{bmatrix} e
\]

(6-23)

**Conditions for Arbitrary Error Dynamics**

We now consider under what conditions the matrix $K_e$ may be selected in order that the error dynamics may be arbitrarily assigned. An equivalent expression for the error dynamics takes the form of a full state feedback regulator problem:

\[
\dot{e} = \begin{bmatrix} A^{11} & A^{12} \\ 0 & 0 \end{bmatrix} e + \begin{bmatrix} 0 \\ I \end{bmatrix} u_e
\]

(6-24)

where $A^{11}$ and $A^{12}$ are appropriately partitioned submatrices of $A^1$, and

\[
u_e = -K_e e
\]

(6-25)

The poles of the error dynamics may be arbitrarily placed if

\[
\text{Rank} \begin{bmatrix} 0 : A^{12} : A^{11} A^{12} : \cdots : A^{11^{n-2}} A^{12} \\ I : 0 : 0 : \cdots : 0 \end{bmatrix} = n
\]

(6-26)

$A_p$ has been partitioned as

\[
A_p = \begin{bmatrix}
A^{11} & A^{12} \\
A_p^{31} & A_p^{32} \\
\end{bmatrix}
\]

(6-27)

where the partitions superscript 21 and 22 are null. The plant is completely controllable if
It follows that the errors are completely controllable if and only if the plant is. All plants considered in this dissertation have been assumed to be completely controllable.

**DIFFERENT ORDER PLANTS AND MODELS**

Linear perfect model following control problems are frequently stated with plant and model equations of motion of different order. Since our standard form requires that they be of the same order, this section will analyze the different order problem and show how it may be handled by the standard form. In so doing, we will show that such problems may be incompletely specified, and that all are equivalent to larger order problems where the order of plant and model is the same.

**Problem Formulation**

Consider a plant and model described by the usual linear differential equations of motion,

\[
\dot{x}_p = A_p x_p + B_p u_p \quad (6-29)
\]

\[
\dot{x}_m = A_m x_m + B_m u \quad (6-30)
\]

The plant is of dimension \(n\) with \(m\) controls, and the model of dimension \(\eta\) with \(\mu\) controls. Denote by \(w\) the number of states common to the plant and the model. Then \((n-w)\) is the number of plant states not present in the model equations, and \((\eta-w)\) is the number of model states not present in the plant equations.
The perfect model following control objective is to find the control law that makes the w common states of the plant follow the corresponding w states of the model. To accomplish this, we augment the plant and model equations as necessary to make both have the same states, namely the w common ones plus the (n-w) and (n1-w) states not originally common to the plant and model. The augmentation of these states will be performed so that the original equations are uncoupled from the added ones; i.e., so that the original plant and model dynamics are retained. The dynamics of the augmented states will be selected to satisfy the necessary and sufficient conditions for perfect model following, to ensure the stability of the augmented model, and to ensure the controllability of the augmented plant.

Augmented Equations of Motion

Assume that the plant equations have been arranged so that the first (n-w) states are those not common to the model, and that the last w states are those that are common. Also assume that the model equations have been arranged so that the first w states are the common ones, and that the last (n1-w) states are those not common to the plant. Denote by N the dimension of the augmented systems, so that N=w+(n-w)+(n1-w) = (n+n1-w). The augmented plant (subscript ap) and augmented model (subscript am) are then formed as follows:

\[
A_{ap} = \begin{bmatrix} A_{ap1}^1 \\ A_{ap1}^2 \\ A_{ap2}^1 \\ A_{ap2}^2 \end{bmatrix} = \begin{bmatrix} A_p : [0] \\ A_{ap}^2 \end{bmatrix} \quad (6-31)
\]

\[
B_{ap} = \begin{bmatrix} B_{ap1}^1 \\ B_{ap1}^2 \\ B_{ap2}^1 \\ B_{ap2}^2 \end{bmatrix} = \begin{bmatrix} B_p : [0] \\ B_{ap}^2 \end{bmatrix} \quad (6-32)
\]
\[
\begin{align*}
\mathbf{x}_{ap} &= \begin{bmatrix} x_{ap}^1 \\ x_{ap}^2 \\
\end{bmatrix} = \begin{bmatrix} x_p^1 \\ x_p^2 \\
\end{bmatrix} = \begin{bmatrix} x_m^1 \\ x_m^2 \\
\end{bmatrix} \\
(6-33) \\
\mathbf{u}_{ap} &= \begin{bmatrix} u_{ap}^1 \\ u_{ap}^2 \\
\end{bmatrix} = \begin{bmatrix} u_p^1 \\ u_p^2 \\
\end{bmatrix} \\
(6-34) \\
\mathbf{A}_{am} &= \begin{bmatrix} A_{am}^1 \\ A_{am}^2 \\
\end{bmatrix} = \begin{bmatrix} A_{am}^1 \\ [0] : [0] : A_m \\
\end{bmatrix} \\
(6-35) \\
\mathbf{B}_{am} &= \begin{bmatrix} B_{am}^1 \\ B_{am}^2 \\
\end{bmatrix} = \begin{bmatrix} B_{am}^1 \\ [0] : [0] : B_m \\
\end{bmatrix} \\
(6-36) \\
\mathbf{x}_{am} &= \begin{bmatrix} x_{am}^1 \\ x_{am}^2 \\
\end{bmatrix} = \begin{bmatrix} x_{am}^1 \\ x_m^1 \\
\end{bmatrix} = \begin{bmatrix} x_m^1 \\ x_m^2 \\
\end{bmatrix} \\
(6-37) \\
\mathbf{u}_{am} &= \begin{bmatrix} u_{am}^1 \\ u_{am}^2 \\
\end{bmatrix} = \begin{bmatrix} u_{am}^1 \\ u \\
\end{bmatrix} \\
(6-38)
\end{align*}
\]

The following notes apply to these equations:

1. The matrices \( \mathbf{A}_p, \mathbf{B}_p, \mathbf{A}_m, \) and \( \mathbf{B}_m, \) and the vectors \( \mathbf{x}_p, \mathbf{x}_m, \mathbf{u}_p, \) and \( \mathbf{u}, \) are the same as in the original plant and model equations.

2. The submatrices of zeros added to \( \mathbf{A}_{ap} \) and \( \mathbf{A}_{am} \) are of dimensions such that both matrices have \( N \) columns.

3. The submatrices \( \mathbf{A}_{ap}^2 \) and \( \mathbf{A}_{am}^1 \) have \( N \) columns and sufficient rows to make the matrices \( \mathbf{A}_{ap} \) and \( \mathbf{A}_{am} \) square. That is, \( \mathbf{A}_{ap}^2 \) has dimension \((\eta - w) \times N\), and \( \mathbf{A}_{am}^1 \) has dimension \((n - w) \times N\).
(4) The vectors $x_{ap}$ and $x_{am}$ represent the same states, while the vectors $x_{p}^2$ and $x_{m}^1$ represent the $w$ common states.

(5) The submatrices of zeros added to $B_{ap}$ and $B_{am}$ are of dimensions such that both matrices have the same number of columns, denoted $p$.

(6) The submatrices $B_{ap}^2$ and $B_{am}^1$ have $(n-w)$ and $(n-w)$ columns, respectively, and sufficient rows to give $B_{ap}$ and $B_{am}$ $N$ rows each.

**Matrix Element Selection**

If $A_{ap}^2$, $B_{ap}^2$, $A_{am}^1$, and $B_{am}^1$ can be selected such that the augmented systems satisfy the criteria for perfect model following, then the augmented plant can be made to follow the augmented model perfectly. In particular, the common states of the plant and model will exhibit perfect model following, as specified in the design objective. If, in addition, $A_{ap}^2$ and $B_{ap}^2$ are selected so that the augmented plant is completely controllable, then the error dynamics of the model following system may be arbitrarily assigned. Note that if the common states of the plant and model are driven by any of the external controllers, type 2 equations will be introduced, and some or all of the elements of $A_{ap}^2$ and $B_{ap}^2$ must be selected accordingly.

Some thought must go in to the selection of $A_{am}^1$ and $B_{am}^1$. The point is that all the states of the augmented plant will be following those of the augmented model. This means that the $(n-w)$ plant states not originally common to the plant and the model will be following their counterparts in the augmented model, and these dynamics will be determined by the selection of $A_{am}^1$ and $B_{am}^1$. In many practical cases, especially where plant controls do not appear in these relationships, the criteria for perfect model following will
dictate that $A_{am}^1$ and $B_{am}^1$ be identical to the corresponding parts of $A_{ap}$ and $B_{ap}$. This is tantamount to requiring that the (n-w) plant states which are not common behave the same in the model as in the plant; i.e., that the original plant dynamics are retained for these states. If the selection of these matrices is in fact arbitrary, then the designer must decide how these states should behave, and this is equivalent to specifying a larger order model.

**Imperfect Model Following**

To this point, we have considered only problems that satisfied the criteria for perfect model following. Design objectives, however, arise from considerations unrelated to whether or not the model to be followed satisfies these criteria. Considering the stringency of these criteria, arbitrarily selected models will in general not satisfy them. The problem of what to do if a given model cannot be perfectly followed is called imperfect model following, and there are generally three alternatives available.

The first alternative is to apply perfect model following control laws and see how well they do. Erzberger [5] suggested that, if the entries in his tests (the right hand sides of (6-7) and (6-8)) are small in comparison to the entries in the system matrices, then perfect model following control laws might be expected to produce satisfactory results. He did not defend this idea; there was no quantitative measure of the degradation in model following performance that will result from this approach, and only extensive simulation could tell whether the degradation was acceptable or not.

The second alternative is the application of linear optimal control theory, and this is the means most often used. This method, however, does not directly reveal what compromises have been caused by the plant's inability to perfectly follow the given model. An optimal control solution to an imperfect model following problem could loosely be
thought of as causing the plant to perfectly follow some model (not the original), but an understanding of the structure of that model is buried in the selection of weighting matrices in the problem formulation.

The third alternative is to give up on the original model, and use instead one that can be followed perfectly. This new model must somehow embody certain qualities of the original model. This raises the question of why a particular model was selected for control implementation in the first place. Here we will define these qualities through modal analysis, and assume that everything desirable about a particular model is described by its stability characteristics, mode shapes, and control response characteristics.

A class of candidate models that can be followed perfectly by the given plant will be defined. A candidate model that most closely matches the dynamics of the desired model is then determined through constrained parameter optimization. The result is perfect model following of a model that has an eigenstructure that resembles that of the desired model.

Effects of Imperfect Models

Now assume that the problem is given such that the plant equations are in the standard form, either naturally or through transformations. Also assume that the model has failed the tests for perfect model following. The result is that its type 1 equations will be functionally different from those of the plant,

\[ A_m^1 \neq A_p^1 \]  \hspace{1cm} (6-39)

or, with reference to the previously defined matrix \( A^1 \):

\[
\begin{align*}
A_p^1 &= A^1 \\
A_m^1 &= A^1 + \delta A^1
\end{align*}
\]  \hspace{1cm} (6-40)
with the obvious definition for $\delta A^1$. If we now attempt to follow this model perfectly, the dynamics of $e^1$ are modified:

\[
\dot{e}^1 = A_m^1x_m - A^1x_p \\
= [A^1 + \delta A^1]x_m - A^1x_p \\
= A^1e + \delta A^1x_m
\]  

(6-41)

The effect of this difference alone on the total error dynamics (6-23), after selection of a control model, is

\[
\dot{e} = \begin{bmatrix} A^1 \\ -K_e \end{bmatrix}e + \begin{bmatrix} \delta A^1 \\ 0 \end{bmatrix}x_m
\]

(6-42)

Thus, if $x_m$ is considered to be an exogenous input, then the error regulator problem now has an additional forcing function. If $\delta A^1$ is constant, then for steady values of $x_m$ the error seeks a generally non-zero equilibrium, but the stability of the error about its equilibrium is not affected. For dynamically varying $\delta A^1(t)$ and $x_m(t)$, the problem requires further analysis, and needs knowledge or approximations of the dynamic nature of the forcing function.

Erzberger's intuitive notion regarding small entries in the Erzberger tests are supported by this analysis, since "almost" satisfying these tests is loosely equivalent to saying that the elements of $\delta A^1$ are small. If they are in fact small, and if the error dynamics are significantly faster than those of $x_m$, then the actual errors as a percentage of $x_m$ will be small as well.
Candidate Model Definition

It is now assumed that the desired model can not be followed perfectly by the given plant. Consider as alternatives all models that can be perfectly followed by the given plant. These candidates must satisfy (6-17). With $\Delta_c = [A_c - A_p : B_c - B_p]$, $\Delta_c = [\Delta_c^1 \Delta_c^2]^T$ (subscript c is introduced to denote a candidate model), the requirement is that

$$[\Delta_c^1 - GA_c^2] = 0$$

Equation (6-43) represents $(n-m)(n+m)$ algebraic equations for the $n\cdot(n+m)$ undetermined parameters of the candidate system matrices, $A_c$ and $B_c$. We wish to select from all candidate models the one that best approximates the dynamics of the desired model.

Candidate Model Selection

The dynamic responses of two systems may be compared through their respective modal decompositions. Associated with the desired model and the candidate model are their modal matrices, $M_m$ and $M_c$, whose columns are the eigenvectors or generalized eigenvectors of the respective system.

With the transformation

$$x = Mq$$

the system dynamics are given by

$$\dot{q} = M^{-1}AMq + M^{-1}Bu$$
$$= \Lambda q + M^{-1}Bu$$
where $\Lambda$ is (for distinct eigenvalues) the diagonal matrix of system eigenvalues. If we now select the coefficients of $A_c$ and $B_c$ so that equations (6-44) and (6-45) for the candidate model are as nearly as possible in some sense like those for the desired model, and such that their modal matrices are similar as well, then the dynamic characteristics of the candidate will be the "best" match possible. This problem can be cast in the form of a parameter optimization problem, where the cost to be minimized is

$$
C = \| \Lambda_m - \Lambda_c \| W_1 + \| M_m - M_c \| W_2 + \| M_m^{-1} B_m - M_c^{-1} B_c \| W_3
$$

(6-46)

$W_1$, $W_2$, and $W_3$ are weighting matrices. They are included so that particular system eigenvalues, modes, or control response characteristics may be more faithfully reproduced at the expense of others (for example, fast modes at the expense of slow ones).

The selection of candidate models is therefore to minimize the cost given by (6-46) by selecting the elements of $A_c$ and $B_c$, subject to the constraints given by (6-43).
APPLICATIONS

Four examples of applications of the preceding results will be presented.

The first deals with the design of a control system intended to provide realistic force and motion cues in the control stick of an airplane simulator. The design features fully nonlinear equations of motion for both the plant and model, control and state measurement errors, and variations in plant parameters.

The second design problem is an attitude command system for an unstable F-104 airplane. This is an example of linear perfect model following control, and fully illustrates the methods applicable to problems of different order plants and models. The problem addresses the effects of a shift in center of gravity of the airplane, and further illustrates the means for dealing with variations in plant parameters.

The methods related to imperfect model following control are demonstrated in the third example. Here, a variation on a classical linear model following control problem is addressed, in which perfect following is unattainable. The example details the necessary transformations involved in posing the parameter optimization problem, and presents its solution.

The final example involves the all axis aerodynamic control of an F-18 airplane over a wide range of operating conditions. Nonlinear six degree of freedom rigid body equations of motion are used in the analysis. In this example, nonlinear error feedback gains are derived using Lyapunov stability criteria, which results in a controller that does not depend on the linearization of the dynamics about fixed operating conditions.
FLIGHT SIMULATOR CONTROL LOADER

Perfect explicit model following control laws for a flight simulator control loader problem are developed. The control loader is digitally controlled and uses a direct drive DC motor (torque motor) to simulate airplane control stick forces and displacements. The control laws deal directly with the nonlinearities present in the modeled control system and in the motor, and reduce the problem to one of a second order linear regulator involving the error. The linear error dynamics for the nonlinear problem are determined by conventional pole placement methods. In this example, the simulated control system features breakout forces, nonlinear friction, and hysteresis in the spring forces.

PROBLEM DESCRIPTION

The design problem is that of a digitally controlled flight simulator control loader system. The flight simulator control stick is driven by an electric torque motor, which opposes or assists the pilot’s applied forces on the stick in a way that makes the stick forces and motions simulate that of a real flight control system. The simulated system contains various nonlinearities, including freeplay about the trim position, breakout forces, hysteresis, and a nonlinear variation of friction with displacement.

Figure 7-1 is a schematic of the system in the longitudinal axis. With modifications, the following discussion applies to lateral and directional control loaders as well.
**Equations of Motion**

The plant equations of motion are

\[ \dot{J}_p = m_p g L_p \sin \theta_p + T_A - K_v \dot{\theta}_p \pm T_F + K_p i_p \]  

\[ (7-1) \]

In (7-1), the sign of the static friction term \( T_F \) is taken to oppose the direction of motion, and is zero when \( \dot{\theta} = 0 \). The last term in (7-1) reflects the assumption that the motor develops torque instantaneously with changes in current. Expressed in terms of state variables, this equation is in the standard form for perfect model following.

With \( x_p^1 = \theta_p \), \( u_p = \frac{K_p}{J} i_p \):

\[ \begin{align*}
\dot{x}_p^1 &= x_p^3 \\
\dot{x}_p^2 &= \emptyset \\
\dot{x}_p^3 &= f_p(x_p) + u_p
\end{align*} \]

\[ (7-2) \]
where

\[ f_p = \frac{1}{j}[m_p g L_p \sin x_p^1 + T_A - K_v x_p^2 \pm T_F] \]  (7-3)

The control system being simulated is described by a similar second order equation of the form

\[ \dot{\theta}_m = \phi_m^3(\theta_m, \dot{\theta}_m, T_A) \]  (7-4)

This equation represents the breakout, friction, viscous damping, etc. present in the control system being simulated.

With \( x_m^1 \equiv \theta_m, u \equiv T_A \):

\[ \begin{align*}
\dot{x}_m^1 &= x_m^3 \\
\dot{x}_m^2 &= \emptyset \\
\dot{x}_m^3 &= \phi_m^3(x_m^1, x_m^3, u) \\
\end{align*} \]  (7-5)

The control law is given by (5-12),

\[ u_p^* = \dot{x}_m^3 + f_{cm}(x_m, x_p) - f_p(x_p) \]  (7-6)

and the error dynamics by (5-14), with

\[ \begin{align*}
\dot{e}_1 &= \phi^1(x_m) - \phi^1(x_p) = x_m^3 - x_p^3 = e^3 \\
\dot{e}_2 &= \emptyset \\
\dot{e}_3 &= -f_{cm}(x_m - x_p) \\
\end{align*} \]  (7-7)
If we now take
\[ f_{cm}(x_m, x_p) = k_1 e^1 + k_3 e^3 \equiv K_e e \] (7-8)
and drop references to \( e^2 \), the error dynamics may be put in convenient matrix form,
\[ \dot{e} = \begin{bmatrix} e^1 \\ e^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_3 \end{bmatrix} e \] (7-9)

The values of \( k_1 \) and \( k_3 \) may then be picked to place the poles of the error dynamics as desired. Note that linearization of the error dynamics equations was not required, and (7-9) is valid for any operating condition.

**Measurement errors**

Assume that the physical characteristics of the system are known and remain constant during operation. The measured or estimated values of the states and of applied torque are denoted by variables with a circumflex (^) mark, and the differences between these and the actual values are denoted with tilde (~) marks (e.g., \( \hat{x}_p = x_p + \tilde{x}_p \)). The control being applied to the plant is given by
\[ u^* = \dot{x}_m^3 + K_e (x_m^3 - \hat{x}_p) - f_p(\hat{x}_p) \] (7-10)

This leads to
\[ \dot{x}_p^3 = \dot{x}_m^3 - f_p(x_p) - f_p(\hat{x}_p) + K_e (x_m - \hat{x}_p) \] (7-11)
Assuming small errors in measurement and estimation, $f_p(\hat{x}_p)$ may be linearized about the exact values of the variables, resulting in:

$$x_p^3 = \hat{x}_m^3 + K_e e + \frac{1}{J} \hat{T}_A - K^* \ddot{x}_p \tag{7-12}$$

where

$$K^* = K_e - \begin{bmatrix} \frac{m_p g L_p \cos \theta_p}{J} & -K_V \end{bmatrix} \tag{7-13}$$

The error dynamics now are given by:

$$\dot{e} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} e + \begin{bmatrix} 0 \\ K^* \end{bmatrix} \ddot{x}_p - \begin{bmatrix} 0 \\ \frac{T_A}{J} \end{bmatrix} \tag{7-14}$$

The original error dynamics are thus modified by the addition of exogenous inputs of the form shown.

**Commutation Ripple**

Because commutation in the motor is done in discrete steps, there is a small variation in average torque during rotation of the armature. This ripple may be modeled as a variation in $K_p$ with $\theta_p$. Figure 7-2 (greatly exaggerated) shows the effect of ripple on torque for a given current input.
Ripple is specified as a percentage deviation ($T_R$) about the average torque, where the average is the arithmetic average of maximum and minimum torque for a given current input. Define $K_{PO}$ as the nominal value of $K_p$ and the ripple function $f_R(x_p^1) (= f_R(\theta_p))$ such that

$$K_p(x_p^1) = K_{PO}[1 + f_R(x_p^1)]$$  \hspace{1cm} (7-15)

Applying this to the error dynamics yields

$$e^3 = -K_e e - f_R(x_p^1)u_p^*$$  \hspace{1cm} (7-16)

where $u_p^*$ is determined using the nominal value of $K_p$.

Analysis of the effect of ripple on the error dynamics is complicated by the fact that $f_R(t)$ depends on the rate of stick deflection. The error feedback gains may be made large to minimize the ripple effect, but this is at the expense of noisy tracking.

Alternatively, if sufficiently accurate measurements of $\theta_p$ are available, the control may be calculated using an approximation to the actual value of $K_p$ instead of $K_{PO}$. This
will change the form of the forcing function in (7-16) and, if the approximation is good, will reduce its magnitude.

**SOLUTION**

The parameters of the torque motor considered for this application are as follows:

- \( T_p \) .............. 60 lb-ft
- \( K_P \) .............. 3.11 lb-ft/Amp
- \( T_f \) .............. 0.83 lb-ft
- \( K_v \) .............. 8.0 lb-ft-sec/rad
- \( T_R \) .............. 4 %
- \( \omega_R \) .............. 97 cycles/rev
- \( J_{motor} \) .............. 0.041 lb-ft-sec²
- Peak Current . .19.3 Amps

The control stick was taken as having a length of 1.5 feet and a mass of 0.2 slug. Commutation ripple was modeled as a rectified sine wave:

\[
f_R = T_R \left( 2 \left| \sin(\omega_R \theta_p) \right| - 1 \right)
\]  

(7-17)

This function was applied to the input to the motor. In order to simulate the fact that the ripple function is not known exactly, an approximation to (7-17) was used in calculating the control laws. This approximation consisted of a linear interpolation of a table of seven values of the function given by (7-17). The error dynamics resulting from this approximation are given by (7-16), where the forcing function involves the difference between the actual and approximated ripple functions:
\[ e^3 = -K_e e - [f_R(x_p^1)_{\text{Actual}} - f_R(x_p^1)_{\text{Approx}}] u_p^* \]  

(7-18)

For the simulation results shown, a constant measurement bias error of +0.02 radian was assumed for \( \theta_p \), and of 0.02 radian per second for the angle rate. A +2\% error in the measurement of applied torque was also assumed. The effect of this is to further modify the error dynamics, first by affecting the accuracy of the approximation given in (7-18), and second by the factors given in (7-14):

\[ e^3 = -K_e e - [f_R(x_p^1)_{\text{Actual}} - f_R(x_p^1)_{\text{Approx}}] u_p + \left( k_1 \frac{m_p g L \cos x_p^1}{J} \right) \ddot{x}_p^1 \]

\[ - \left( k_3 + \frac{k_V}{J} \right) \dot{x}_p^3 - \frac{T_A}{J} \]  

(7-19)

where \( \ddot{x}_p^1 = \ddot{x}_p^3 = 0.02 \), and \( \tilde{T}_A = 0.02 T_A \).

Equation (7-19) is difficult to analyze, primarily because of the motor ripple term. It is comprised of an unknown part, \( f_R(x_p^1)_{\text{Actual}} \), and an estimate based on possibly erroneous measurement, \( f_R(x_p^1)_{\text{Approx}} \). Increased gains will not generally reduce the effects of bias errors in measurements of angle and angle rates, since these terms are linear in \( k_1 \) and \( k_3 \). In this simulation, values of \( k_1 \) and \( k_3 \) were determined by trial and error.

**Model Characteristics.**

The control stick characteristics to be simulated were modeled as follows:

**Static friction:** a nonlinear function of stick position, varying from five ft-lbs at one radian forward displacement to four ft-lbs at one radian aft displacement, with values less than 1.5 ft-lbs within 0.5 radian of centered.

**Spring constant:** 30.0 ft-lbs per radian, with hysteresis of ± 0.05 radian.
Breakout: $\pm 2.0$ ft-lbs.

Viscous damping: $30.0$ ft-lb-sec per radian.

Dead space: $\pm 2.0$ ft-lbs about the centered position.

**RESULTS AND DISCUSSION**

The resulting system was simulated using fourth order Runge-Kutta integration with a fixed step size of 0.025 second. Initial conditions for the simulations were taken as zero. The control input was a series of ramp functions, as shown in figure 7-3. The acceleration response to this input of the modeled control stick is shown in figure 7-4. The highly nonlinear nature of the model is clearly shown by this response.

For gains of $k_1 = 1.0$, $k_3 = 2.0$, angle following errors on the order of 0.2 radian were seen. Gains were increased to $k_1 = 25.0$, $k_3 = 5.0$, with results as shown in figures 7-5 and 7-6. The angle tracking (figure 7-5) exhibited errors on the order 0.04 radian at worst, due primarily to the bias errors in measurement. The effect of commutator ripple was seen in angle rate tracking (figure 7-6) in two places, first at the initial control input, and later from 6.5 to 8.0 seconds after the external control input was removed. Although the rate tracking errors did not exceed 0.03 radian/second, they fluctuated rapidly, and are likely to be perceived by the pilot as a roughness, or grinding feeling, in the stick response.

The power supply requirements for this case are shown in figure 7-7. Since the motor acceleration is directly proportional to the current supplied, this figure further demonstrates the roughness in stick response expected as a result of the ripple effect. Whether this would be objectionable to the pilot depends on whether it is within the threshold of his perception of accelerations, for which data are not available. Also to be
noted from this figure is the requirement for the motor power supply to respond rapidly to current requirements. Data from this figure may be used in the design of the power supply with respect to its time constants.

Experimentation with this simulation indicated that ripple effects may be reduced by a combination of higher feedback gains, and by "dividing them out" using an approximation based on the measured angle. Higher gains ultimately lead to noise tracking, and higher peak current requirements. For the motor characteristics chosen, ripple will go through a complete cycle each 0.065 radian of stick travel. Unless the measured value of $\theta_p$ is near this value, any scheme to approximate the ripple function may be useless. The latter method requires a precision in measurement that is inversely proportional to the ripple frequency. This argues for selecting a motor with smaller values of ripple frequency, or for the use of gearing. Increased feedback gains are effective in eliminating both ripple and measurement errors. In the example presented above, satisfactory performance was attained without excessively high gains, and the current requirements remained well within the maximum allowed.
Figure 7-3

Control History
Figure 7-4

Angular Acceleration (Model)

Model Acceleration
Figure 7-5

Stick Position
Figure 7-6

Stick Rate
Figure 7-7

Current Requirements
Hirzinger [31] analyzed a flight path angle command design problem for an aerodynamically unstable F-104 airplane. He employed a second order model to be followed by the linearized fourth order longitudinal dynamics of the airplane, and used linear optimal control to derive the control laws. The problem included performance specifications on the response to step commands and to external disturbances, as well as a requirement for satisfactory performance in the presence of changes in the plant parameters due to center of gravity shifts. The problem is interesting because the model introduced a new state variable, namely $\frac{dy}{dt}$. This problem will be analyzed using the methods introduced in the section on different order plants and models.

**Problem Formulation**

The following is taken from Hirzinger’s paper, with changes in notation as required for consistency with the present treatment.

The plant in this problem is represented by the F-104 linearized longitudinal equations of motion, $\dot{x}_p = A_p x_p + B_p u_p$, with the following definitions:

\[
\begin{align*}
    x_{p1} &= \frac{Ax}{v_0} : \text{Incremental speed} \\
    x_{p2} &= \dot{\theta} : \text{Pitch rate} \\
    x_{p3} &= \theta : \text{Pitch angle} \\
    x_{p4} &= \gamma : \text{Flight path angle}
\end{align*}
\]  

(8-1)
\[ u_{p1} = \frac{\Delta s}{s_0} : \text{Incremental thrust} \]
\[ u_{p2} = \eta_e : \text{Elevator deflection} \]
\[ u_{p3} = \eta_f : \text{Flap deflection} \] (8–2)

The system matrices are given for a center of gravity location of 50% MAC. It should be noted that the aircraft is unstable at this condition, with a positive real system Eigenvalue at \( \lambda = 2.38 \).

\[
A_p = \begin{bmatrix}
-.00127 & 0.0 & -.00105 & .000334 \\
-.0249 & -1.134 & 13.22 & -13.22 \\
0 & 1 & 0 & 0 \\
5.606 & .00363 & 1.438 & -1.438
\end{bmatrix} \quad \text{(8–3)}
\]

\[
B_p = \begin{bmatrix}
.00603 & -.00018 & 0.0 \\
.5186 & 30.08 & .8875 \\
0.0 & 0.0 & 0.0 \\
-.00373 & .28 & .2872
\end{bmatrix} \quad \text{(8–4)}
\]

The model formulated by Hirzinger was intended to provide certain desired characteristics in response to flight path angle commands. It is given by a second order equation, \( \dot{x}_m = A_m x_m + B_m u \), where

\[
x_{m1} = \gamma \\
x_{m2} = \dot{\gamma} \quad \text{(8–5)}
\]

\[ u = \gamma_c \quad \text{(8–6)} \]

and where \( \gamma_c \) is the commanded flight path angle.

The system matrices were given as:

\[
A_m = \begin{bmatrix}
0 & 1 \\
-6.25 & -5.0
\end{bmatrix} \quad \text{(8–7)}
\]

\[
B_m = \begin{bmatrix}
0 \\
1
\end{bmatrix} \quad \text{(8–8)}
\]
Augmented Equations of Motion

The augmented states and controls are given by:

\[
\begin{align*}
    x_{ap1} &= x_{am1} = x_{p1} \\
    x_{ap2} &= x_{am2} = x_{p2} \\
    x_{ap3} &= x_{am3} = x_{p3} \\
    x_{ap4} &= x_{am4} = x_{p4} = x_{m4} \\
    x_{ap5} &= x_{am5} = x_{m5} \\
    u_{ap1} &= u_{am1} = u_{p1} \\
    u_{ap2} &= u_{am2} = u_{p2} \\
    u_{ap3} &= u_{am3} = u_{p3} \\
    u_{ap4} &= u_{am4} = u
\end{align*}
\]  

(8–9)

We note that the external control (u) has been introduced into the augmented plant equations, and expect a type 2 equation to result. The augmented system matrices are:

\[
A_{ap} = \begin{bmatrix} A_p & [0]_{4x1} \\ \tilde{A}_{ap} \end{bmatrix} \quad \text{(8–11)}
\]

\[
B_{ap} = \begin{bmatrix} B_p & [0]_{4x1} \\ \tilde{B}_{ap} \end{bmatrix} \quad \text{(8–12)}
\]

\[
A_{am} = \begin{bmatrix} A_{am} & [0]_{2x3} \\ [0]_{2x3} & A_m \end{bmatrix} \quad \text{(8–13)}
\]

\[
B_{am} = \begin{bmatrix} B_{am} & [0]_{2x3} \\ [0]_{2x3} & B_m \end{bmatrix} \quad \text{(8–14)}
\]
where the new pieces have the following dimensions:

\[
\begin{align*}
A_{ap}^2 & : 1 \times 5 \\ B_{ap}^2 & : 1 \times 4 \\ A_{am}^1 & : 3 \times 5 \\ B_{am}^1 & : 3 \times 4
\end{align*}
\] (8–15)

**Matrix element selection**

In order to satisfy the requirements placed on type 2 equations, we must select the augmented plant submatrices so that they are identical to the corresponding parts of the augmented model matrices, i.e.,

\[
\begin{align*}
A_{ap}^2 &= [0 \ 0 \ 0 \ -6.25 \ -5.0] \\
B_{ap}^2 &= [0 \ 0 \ 0 \ 1]
\end{align*}
\] (8–16) (8–17)

It remains to select the elements of the augmented model matrices. Since they are completely arbitrary, this is tantamount to completing the specification of the model, as discussed in the section on different order models and plants. In other words, we are being asked to decide how the other plant states are to behave while the flight path angle is being controlled in the desired manner.

In the absence of other information, it would seem reasonable to instill these model states with the same dynamics as the plant, but these plant dynamics are unstable.¹

¹This doesn't mean that perfect model following can't be accomplished, only that it doesn't make much sense to do it.
Alternatively, the arbitrary elements of the augmented model matrices could all be set to zero, which requires that the actual airspeed and pitch angle remain constant throughout flight path angle changes. This approach was tried with success, and in the simulation the flight path angle was varied primarily with flap changes, using power and elevator to maintain constant speed and pitch angle. This is not, however, what was intended by the performance specification. Hirzinger assumed that the pitch angle dynamics would be the same as the flight path angle dynamics, thus implicitly specifying a fourth order model. Based on this, the pitch angle dynamics of the augmented model were made identical to the flight path angle dynamics, while the speed dynamics were made to be constant. This results in the following augmented system matrices:

\[
A_{ap} = \begin{bmatrix}
-.00127 & 0 & -.00105 & .00033 & 0 \\
-.02488 & -1.134 & 13.22 & -13.22 & 0 \\
0 & 1 & 0 & 0 & 0 \\
5.606 & .00363 & 1.438 & -1.438 & 0 \\
0 & 0 & 0 & -6.25 & -5.0 \\
\end{bmatrix}
\]

\[
B_{ap} = \begin{bmatrix}
6.026 \cdot 10^{-3} & -1.802 \cdot 10^{-4} & 0 & 0 \\
0.5186 & 30.08 & 0.8875 & 0 \\
0 & 0 & 0 & 0 \\
-3.725 \cdot 10^{-3} & 0.28 & 0.2872 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
A_{am} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & -5.0 & -6.25 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -6.25 & -5.0 \\
\end{bmatrix}
\]

\[
B_{am} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Standard Form

Having completely specified the augmented model, we now proceed to put the equations in the standard form. A transformation that does this is given by $T = T_2 T_1$, where

$$T_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (8-22)$$

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1.023 \times 10^{-3} & 1.659 \times 10^2 & -3.161 \times 10^{-3} \\ 0 & 0 & 3.421 \times 10^{-2} & -3.010 & -0.1057 \\ 0 & 0 & -3.333 \times 10^{-2} & 5.085 & 3.585 \end{bmatrix} \quad (8-23)$$

Here, $T_1$ puts the zeros of $B_{ap}$ in the first row, the common control in the second row, and does a little esthetic shuffling of the remaining rows. The last three rows of $T_2$ constitute a generalized left inverse of the matrix $T_1 B_{ap}$. After this transformation, the systems are in the standard form. Denoting the transformed matrices and vectors with the circumflex accent, they are:

$$\hat{x}_{ap} = T x_{ap}$$
$$\hat{x}_{am} = T x_{am} \quad (8-24)$$

$$\hat{A}_{ap} = \begin{bmatrix} 0 & 0 & 0.519 & 30.08 & 0.887 \\ 0 & -5.0 & 2.33 \times 10^{-2} & -1.75 & -1.79 \\ -0.165 & 0 & -2.15 \times 10^{-3} & -2.24 \times 10^{-2} & 1.21 \times 10^{-2} \\ 0.303 & 0 & -2.27 \times 10^{-2} & -1.26 & -0.121 \\ 4.71 & 0 & 0.165 & 0.205 & -1.31 \end{bmatrix} \quad (8-25)$$

$$\hat{B}_{ap} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (8-26)$$
Control law

The three plant controls are taken directly from the augmented system equations according to equation (6-19) as applied to equations (8-25) - (8-28).

\[
\dot{u}_p = \frac{d}{dt} \begin{bmatrix} \hat{x}_{am3} \\ \hat{x}_{am4} \\ \hat{x}_{am5} \end{bmatrix} + K_e e - \begin{bmatrix} -0.166 & 0 & -2.16 \cdot 10^{-3} & -2.22 \cdot 10^{-2} & 1.23 \cdot 10^{-2} \\ 0.303 & 0 & -2.27 \cdot 10^{-2} & -1.26 & -0.121 \\ 4.71 & 0 & 0.165 & 0.206 & -1.31 \end{bmatrix} \begin{bmatrix} \hat{x}_{am3} \\ \hat{x}_{am4} \\ \hat{x}_{am5} \end{bmatrix}
\]  

(8-29)

Here, \( e \) is the error vector between the augmented and transformed states, and \( K_e \) is a 3x5 matrix whose elements will be selected next.

Selection of error dynamics

With the plant controls as given above, the error dynamics are

\[
\dot{e} = A_e e = \begin{bmatrix} 0 & 0 & 0.519 & 30.08 & 0.887 \\ 0 & -5.0 & 2.33 \cdot 10^{-2} & -1.75 & -1.79 \end{bmatrix} 
- K_e
\]

(8-30)
If the plant parameters do not vary, this is a simple pole placement problem, whereby the desired error dynamics may be arbitrarily selected. If the plant parameters vary, then the error dynamics are given by

\[
\dot{e} = A_e e - \tilde{A}_{ap} \hat{x}_{ap}
\]  

(8–31)

where \( \tilde{A}_{ap} \) represents the difference between the assumed and actual plant parameters (after the transformation to standard form). Alternatively, this equation may be written

\[
\dot{e} = [A_e + \tilde{A}_{ap}] e - \tilde{A}_{ap} \hat{x}_{am}
\]  

(8–32)

By treating \( \hat{x}_{am} \) as an exogenous control input, (8-32) can be viewed as a robust regulator problem.

Hirzinger’s problem required satisfactory performance of the system with a center of gravity shift from 0.5 to 0.6 MAC. This was approximated by a change in the plant system matrix given by

\[
\tilde{A}_p = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 10 & -10 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]  

(8–33)

which, when augmented and transformed, becomes

\[
\tilde{A}_{ap} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1.02 \cdot 10^{-2} & 0 & 0 & -2.86 \cdot 10^{-3} & -2.94 \cdot 10^{-3} \\
0.342 & 0 & 1.27 \cdot 10^{-3} & -9.58 \cdot 10^{-2} & -9.82 \cdot 10^{-2} \\
-0.333 & 0 & -1.24 \cdot 10^{-3} & 9.33 \cdot 10^{-2} & 9.57 \cdot 10^{-2} \\
\end{bmatrix}
\]  

(8–34)
The elements of $K_e$ were determined by treating (8-30), using (6-24), as a linear quadratic regulator problem with unity weighting on the states and controls. This resulted in the following control gains for the system defined by equation (8-30):

$$K_e = \begin{bmatrix} 1.73 \cdot 10^{-2} & 7.32 \cdot 10^{-4} & 1.0 & 0.118 & 1.93 \cdot 10^{-3} \\ 1.0 & -8.73 \cdot 10^{-3} & 0.118 & 7.82 & 0.199 \\ 2.70 \cdot 10^{-2} & -2.91 \cdot 10^{-2} & 1.93 \cdot 10^{-3} & 0.199 & 1.05 \end{bmatrix} \quad (8-35)$$

Applying these gains to the nominal system resulted in closed loop error dynamic eigenvalues of -1.0, -1.06, -4.98, and -3.92±j3.85. The effect of the change in parameters due to the CG shift was assessed by evaluating the poles of the system defined by (8-32), and (8-34). This caused the eigenvalues to shift to -0.89, -1.0, -4.98, and -4.0±j2.16. The largest effect is therefore in the natural frequency of the oscillatory roots, but overall stability has not been compromised. The remaining robustness consideration is the effect on the error of the exogenous control input ($\hat{x}_{em}$), which will be assessed through simulation.

**RESULTS AND DISCUSSION**

The system was simulated using the gains given by equation (8-35) and the control law given by equation (8-29). The input was a step flight path angle command of plus ten degrees, but note that there is a scaling of the response of the model and plant. Results of the simulation are shown in the figures that follow.

The first three figures are based on initial conditions of zero error. Figure 8-1 shows the responses of the model, and figure 8-2 of the aircraft at the 60% MAC condition. The response of the aircraft was identical to the responses computed for the model to within $10^{-5}$ for both center of gravity conditions, indicating that the selected error
dynamic gains are satisfactory from this consideration. Figure 8-3 displays the error between model and plant for the 60% MAC condition, further confirming this conclusion.

The remaining figure (8-4) is from the simulation of the same system (at 60% MAC) with initial errors present in the four states. The time to half amplitude of the deadbeat errors is roughly 0.5 second, which is somewhat faster than the slowest eigenvalue, and the fast oscillatory response has nearly the same damped natural frequency as the complex pair of error eigenvalues.
Figure 8-1

Model Responses
Figure 8-2

Aircraft Responses
Figure 8-3
Tracking Errors
Errors Following Disturbances

![Graph showing errors following disturbances with time in seconds on the x-axis and error magnitude on the y-axis. The graph includes lines for Theta, U, d(Theta)/dt, and Gamma, each with a different line style.](Figure 8-4)

Initial Condition Error Correction
IMPERFECT MODEL FOLLOWING CONTROL PROBLEM

This example is based on the linearized lateral equations of motion of the B-26 airplane. All system matrices, including those of the desired model, are taken from Tyler, Erzberger, et al. The desired model control matrix has been changed from that stated in the works cited, where it was equal to the plant control matrix. It is easily shown that the systems do not satisfy criteria for perfect model following.

PROBLEM FORMULATION

The states and controls of the system are \( x = [\phi \; \dot{\phi} \; \beta \; r]^T \), \( u = [\delta_r \; \delta_a]^T \). The \( A \) and \( B \) matrices are as follows:

\[
A_p = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & -2.93 & -4.75 & -0.78 \\
0.086 & 0 & -0.11 & -1.0 \\
0 & -0.042 & 2.59 & -0.39
\end{bmatrix}
\]
\quad (9-1)

\[
B_p = \begin{bmatrix}
0 & 0 \\
0 & -3.91 \\
0.035 & 0 \\
-2.53 & 0.31
\end{bmatrix}
\]  
\quad (9-2)

\[
A_m = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & -1.0 & -73.14 & 3.18 \\
0.086 & 0 & -0.11 & -1.0 \\
0.0086 & 0.086 & 8.95 & -0.49
\end{bmatrix}
\]
\quad (9-3)

\[
B_m = \begin{bmatrix}
0 & 0 \\
0 & -3.91 \\
0.175 & 0 \\
-2.53 & 0.31
\end{bmatrix}
\]  
\quad (9-4)
The differences between the desired model matrices and those of the plant introduce nonzero entries in the right hand sides of (6-7), (6-8), (6-15a) and (6-15b), ensuring that the desired model fails all the tests for perfect model following.

**Candidate Models**

We take $B_p^2 = \begin{bmatrix} 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix}$. For this selection, $G = \begin{bmatrix} 0 & 0 \\ -911.7 & -12.61 \end{bmatrix}$. From (6-39), candidate models are therefore those for which

\[
\begin{align*}
 & a_{c1j} - a_{p1j} = 0 \\
 & (a_{c2j} - a_{p2j}) - C_1(a_{c3j} - a_{p3j}) - C_2 (a_{c4j} - a_{p4j}) = 0 \\
 & b_{c1j} - b_{p1j} = 0 \\
 & (b_{c2j} - b_{p2j}) - C_1(b_{c3j} - b_{p3j}) - C_2 (b_{c4j} - b_{p4j}) = 0
\end{align*}
\]

(9-5)

where $j = 1...4$, $C_1 = -911.7$, $C_2 = -12.61$

**Solution**

The problem was cast as a parameter optimization problem as described in Section 6, “Imperfect Model Following”. It was solved using a general nonlinear programming problem solver with finite difference gradients. The variables were the 18 undetermined elements of the candidate system and control matrices, and the eight equality constraints were as defined by (9-5) and (9-6). The line searches used by the optimization procedure were sensitive to the cost associated with differences in the modal control matrices, and required that this portion of the cost be deweighted considerably. This yielded the following candidate system:
Table 9-1 presents the eigenvalues, eigenvectors, and the modal control matrices, for this solution and for those of the desired model.
Table 9-1  Eigenstructure Comparison

<table>
<thead>
<tr>
<th>Candidate Model</th>
<th>Desired Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalues</td>
<td></td>
</tr>
<tr>
<td>-1.023</td>
<td>-1.023</td>
</tr>
<tr>
<td>-0.2887 - j2.942</td>
<td>-0.2882 - j2.942</td>
</tr>
<tr>
<td>-0.2887 + j2.942</td>
<td>-0.2882 + j2.942</td>
</tr>
<tr>
<td>-0.2729x10⁻³</td>
<td>-0.2752x10⁻³</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Magnitude</th>
<th>Angle</th>
<th>Magnitude</th>
<th>Angle</th>
</tr>
</thead>
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<tr>
<td>1st E’vector</td>
<td></td>
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<td></td>
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<tr>
<td>-0.9772</td>
<td>0.0</td>
<td>-0.9772</td>
<td>0.0</td>
</tr>
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<td>1.0</td>
<td>0.0</td>
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<td>0.0</td>
<td>-0.3473x10⁻²</td>
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</tr>
<tr>
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<td>0.0</td>
<td>-0.8721x10⁻¹</td>
<td>0.0</td>
</tr>
<tr>
<td>2nd E’vector</td>
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<td></td>
</tr>
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<td>0.3383</td>
<td>95.6</td>
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</tr>
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<td>0.4262x10⁻¹</td>
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</table>

\[
M^{-1}_cB_c = \begin{bmatrix} -18.07 & -1.642 \\ 9.053-5.729 & -1.136+j0.429 \\ 9.053+j5.729 & -1.136-j0.429 \\ -20.92 & -1.391 \end{bmatrix} \quad \quad M^{-1}_mB_m = \begin{bmatrix} -18.04 & -1.645 \\ 9.018-5.796 & -1.133+j0.435 \\ 9.018+j5.796 & -1.133-j0.435 \\ -20.94 & -1.389 \end{bmatrix}
\]

The candidate model has eigenvalues, eigenvectors, and modal control matrices that are virtually the same as those of the desired model. The most notable differences occur in
the least significant components of the eigenvectors, and should have minimal effect on the similarity of the dynamic responses of the desired and candidate models.

Control Law Formulation

The transformation to standard form was taken as (cf equation (6-18))

\[
T = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1.0968 \times 10^{-3} & 1 & 1.3834 \times 10^{-2} \\
0 & -3.134 \times 10^{-2} & 0 & -0.3952 \\
1 & -0.2558 & 0 & 0
\end{bmatrix}
\]  

(9–9)

The states now are \( y_p \) and \( y_c \), where \( y = Tx \). In standard form, we have

\[
TA_p T^{-1} = \begin{bmatrix}
3.909 & 0 & 0 & -3.909 \\
0.383 & -7.938 \times 10^{-2} & 2.543 & -0.297 \\
0.369 & -0.875 & -0.482 & -0.369 \\
6.777 & 1.215 & -0.462 & -6.777
\end{bmatrix}
\]  

(9–10a)

\[
TB_p = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 1
\end{bmatrix}
\]  

(9–10b)

\[
TA_c T^{-1} = \begin{bmatrix}
3.909 & 0 & 0 & -3.909 \\
0.383 & -7.938 \times 10^{-2} & 2.543 & -0.297 \\
-2.229 \times 10^{-2} & -1.236 & -0.278 & 2.097 \times 10^{-2} \\
5.208 & 18.91 & 5.285 & -5.153
\end{bmatrix}
\]  

(9–11a)

\[
TB_c = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0.987 & 2.59 \times 10^{-4} \\
0.987 & 2.59 \times 10^{-4} & -1.036 \times 10^{-2} & 1.001
\end{bmatrix}
\]  

(9–11b)

Using the control law given by (6-19), the error feedback gains were obtained by treating (6-23) as a linear quadratic regulator problem with unity weighting. As a result, the error dynamics are
Here, the errors are between $y_c$ and $y_p$. This feedback selection results in error eigenvalues of $-1.59 \pm 1.23$ and $-34.7 \pm 33.4$. Finally, the control law is given by

$$u_p = \dot{y}_c^2 + \begin{bmatrix} 6.70 \times 10^{-2} & 0.925 & 2.388 & -5.704 \times 10^{-2} \\ -10.56 & -4.753 \times 10^{-2} & -5.704 \times 10^{-2} & 9.141 \\ 0.369 & -0.875 & -0.482 & -0.369 \\ 6.777 & 1.215 & -0.462 & -6.777 \end{bmatrix} (y_c - y_p)$$

(9-13)

**RESULTS AND DISCUSSION**

The systems were simulated to determine responses to initial conditions, and to unit step inputs in rudder and aileron with zero initial conditions. In all cases the plant followed the candidate model perfectly, as expected. The only remaining question is whether the candidate model behaves dynamically the same as the desired model.

Selected time histories of the responses of the plant (identical to those of the candidate model) and of the desired model are shown in figures 9-1 through 9-8. Not shown are the time histories of bank angle and roll rate responses to rudder and aileron inputs, in which there was no discernible difference between plant and desired model responses.

Figures 9-1 through 9-4 are the responses due to initial conditions alone, with no control inputs. The initial conditions for this simulation were selected to ensure that all system modes were equally excited. The effects of the small differences between desired
and candidate model eigenvectors are noticeable in all four time histories, but the responses are virtually identical.

Figures 9-5 and 9-6 show the sideslip and yaw rate responses of the plant and the desired model following a unit step rudder input at time zero. Figures 9-7 and 9-8 are the same, but for a unit step aileron input. In both cases, the differences in response to control inputs are most pronounced in the sideslip angle time histories. They are, however, dynamically similar in that the describing features of the excited modes (such as frequency and damping of the Dutch roll) have been preserved.

Finally, the error correcting attributes of the control law are demonstrated in figure 9-9. The initial conditions of the candidate model were taken as [0 0 0 0]T, and of the plant as [1 1 1 1]T. The systems were excited with step rudder and aileron inputs at time zero. Only the differences between plant and candidate model states are shown in this figure. The initial errors are effectively eliminated within three seconds.

In the interpretation of these results, it is emphasized that it was not the intent of the analysis to make the plant follow the desired model trajectory exactly. The fact that it very nearly does results from the fact that a candidate model was found whose Eigenstructure was very nearly the same as that desired. The primary conclusions to be drawn from these time histories are that (1) the control law corrects for errors in initial conditions and causes the plant to follow the candidate model perfectly, and (2) the candidate model has dynamic responses that are similar to those of the desired model.
Figure 9-1

Bank Angle Response to Initial Conditions
Figure 9-2

Roll Rate Response to Initial Conditions
Figure 9-3

Sideslip Response to Initial Conditions
Figure 9-4

Yaw Rate Response to Initial Conditions
Figure 9-5

Sideslip Response to Rudder Input
Figure 9-6

Yaw Rate Response to Rudder Input
Figure 9-7
Sideslip Response to Aileron Input
Figure 9-8

Yaw Rate Response to Aileron Input
Figure 9-9

Initial Condition Error Responses
This problem demonstrates the application of the methods presented previously to the control of an airplane over a wide range of flight conditions. The airplane to be controlled is represented by a nonlinear, six degree of freedom aerodynamic model of the F-18 obtained from NASA Langley. The model of the F-18 consisted of FORTRAN subroutines which utilized table look-up and interpolation to yield force and moment coefficient information. The forces and moments were functions of angle of attack; sideslip angle; Mach; body axis roll, pitch, and yaw rate; rate of change of angle of attack; and control positions. Because of possible confusion over two uses of the word “model,” the aerodynamic model will be referred to as if it were an actual airplane.

The objective of this application is to cause the airplane to have response characteristics over a wide range of angles of attack that are the same as the airplane has at a specified, fixed angle of attack. In formulating this objective, it is assumed that the airplane has desirable handling qualities at the specified angle of attack. The model to be followed therefore had these characteristics:

- The baseline model was the linearized version of the actual airplane at a particular reference flight condition which included the specified angle of attack. This also means that the physical characteristics of the model (mass, moments of inertia, etc.) are the same as those of the airplane.
As the model was flown away from the reference flight condition, all stability and control derivatives remained fixed as a function of angle of attack. The single exception was that the lift-curve slope of the model was nonlinear, based on a polynomial approximation to that of the airplane.

PROBLEM FORMULATION

In this problem, there are no type 2 equations of motion. To avoid repeated references to this, the notation has been modified somewhat in that all standard form superscript 2 equations are ignored.

States and Controls

The states used were \( x = [q_0 q_1 q_2 q_3 p q r u v w]^T \), defined as follows:

- \( q_0 \ldots q_3 \) Euler Parameters (Quaternions)
- \( p, q, r \) Body axis roll, pitch, and yaw rates
- \( u, v, w \) Body axis X, Y, and Z velocities

The state vector is partitioned as \( x^1 = [q_0 q_1 q_2 q_3]^T \) and \( x^3 = [p q r u v w]^T \).

Controls were \( u = [\delta_{HTL} \delta_{HTR} \delta_{TFL} \delta_{TFR} \delta_{AR} \delta_{LER} \delta_R \delta_T]^T \), where the \( \delta \) is taken to mean the deflection measured from zero displacement. These controls are defined (with positive sense and limits of travel) as follows:

- \( \delta_{HTL}, \delta_{HTR} \) Horizontal tail, Left and Right (+ve trailing edge down, +10.5°, -24.0°)
- \( \delta_{TFL}, \delta_{TFR} \) Trailing edge flap, Left and Right (+ve trailing edge down, +45.0°, -8.0°)
The left and right horizontal tail, trailing edge flap, and aileron were capable of independent motion and were treated as separate controllers. While left and right leading edge flap and rudder were also capable of independent motion, asymmetric leading edge flap generated no lateral-directional forces or moments, and asymmetric rudder generated no longitudinal forces or moments (a feature of the data used, not of the actual airplane). These controls were therefore limited to symmetric deflections and treated as single controllers.

The F-18 engine model was not employed. Instead, an “ideal” throttle was used as a controller. It was assumed that the thrust acted along the body fixed X-axis, and this control was applied as necessary to exactly balance the body fixed X-forces.

Equations of Motion

The equations of motion are as follows:

\[ 0 = \frac{1}{2} (-q_1 p - q_2 q - q_3 r) - \dot{q}_0 \]  \hspace{1cm} (10–1a)

\[ 0 = \frac{1}{2} (q_0 p - q_3 q + q_2 r) - \dot{q}_1 \]  \hspace{1cm} (10–1b)

\[ 0 = \frac{1}{2} (q_3 p + q_0 q - q_1 r) - \dot{q}_2 \]  \hspace{1cm} (10–1c)

\[ 0 = \frac{1}{2} (-q_2 p + q_1 q + q_0 r) - \dot{q}_3 \]  \hspace{1cm} (10–1d)
In simulating the control of the airplane, the Euler parameters were initialized using

\[ q_0 = \cos \frac{\psi}{2} \cos \frac{\theta}{2} \cos \frac{\phi}{2} + \sin \frac{\psi}{2} \sin \frac{\theta}{2} \sin \frac{\phi}{2} \]  
\[ q_1 = \cos \frac{\psi}{2} \cos \frac{\theta}{2} \sin \frac{\phi}{2} - \sin \frac{\psi}{2} \sin \frac{\theta}{2} \cos \frac{\phi}{2} \]  
\[ q_2 = \cos \frac{\psi}{2} \sin \frac{\theta}{2} \cos \frac{\phi}{2} + \sin \frac{\psi}{2} \cos \frac{\theta}{2} \sin \frac{\phi}{2} \]  
\[ q_3 = \sin \frac{\psi}{2} \cos \frac{\theta}{2} \cos \frac{\phi}{2} - \cos \frac{\psi}{2} \sin \frac{\theta}{2} \sin \frac{\phi}{2} \]  

where \( \psi, \theta, \) and \( \phi \) are the usual Euler angles.

Simulation integrations were performed using equations (10-1a) through (10-1j). In presenting the results of the simulation, Euler angles were used. The Euler angles of interest are related to the Euler parameters by the following:

\[ \phi = \tan^{-1} \left( \frac{2(q_2q_3 + q_0q_1)}{q_0^2 - q_1^2 - q_2^2 + q_3^2} \right) \]  
\[ \theta = \sin^{-1} \left[ 2(q_0q_1 - q_2q_3) \right] \]  

In the equations of motion, the forces and moments due to control deflections appear on the left hand side in anticipation of "solving" the equations for the control vector. The forces and moments in these equations were assumed to be available as linearized
functions of the states. Note that this is not a requirement of the model following control law, but rather is the form in which the forces and moments are usually provided. With the definitions

\[
F_p^{\text{control}} = \begin{bmatrix} L \\ M \\ N \\ X \\ Y \\ Z \end{bmatrix}_p^{\text{control}} \quad F_p^{\text{aero}} = \begin{bmatrix} L \\ M \\ N \\ X \\ Y \\ Z \end{bmatrix}_p^{\text{aero}}
\]  

(10–2)

and with state and control vectors as defined above, we have

\[
F_p^{\text{control}} = \left[ \frac{\partial F_p^{\text{control}}}{\partial u_p} \right]^{\text{ref}} u_p
\]  

(10–3)

\[
F_p^{\text{aero}} = F_p^{\text{ref}} + \left[ \frac{\partial F_p^{\text{aero}}}{\partial x_p} \right]^{\text{ref}} \Delta x_p
\]  

(10–4)

Here the superscript "ref" on the matrices of partial derivatives indicates that they are to be evaluated at reference conditions, \( u_p^{\text{ref}} = 0 \) and \( x_p^{\text{ref}} \), while \( F_p^{\text{ref}} \) is the vector of forces and moments at the reference conditions. \( \Delta x \) is the vector of perturbed states. In the simulation of this system, the values of \( F_p^{\text{ref}} \) and of the matrices of partial derivatives were approximated by polynomials that were functions of the angle of attack. For convenience, define

\[
A_p \equiv \begin{bmatrix} [0]_{4\times 10} \\ \cdots \\ \frac{\partial F_p^{\text{aero}}}{\partial x_p} \end{bmatrix} \equiv \begin{bmatrix} [0]_{4\times 10} \\ \cdots \\ A_3 \end{bmatrix}
\]  

(10–5)
Note that this is not the usual definition of \( A_p \). With these definitions, the equations of motion may be written as

\[
B_p^{ref} u_p = f(x_p, \dot{x}_p) - [F_p^{ref} + A_p^{ref} \Delta x_p] \tag{10-7}
\]

where

\[
f(x_p, \dot{x}_p) = \begin{bmatrix}
\frac{1}{2}(-q_1 p - q_2 q_3 r) - \dot{q}_0 \\
\frac{1}{2}(q_0 p - q_3 q + q_2 r) - \dot{q}_1 \\
\frac{1}{2}(q_3 p + q_0 q - q_1 r) - \dot{q}_2 \\
\frac{1}{2}(-q_2 p + q_1 q + q_0 r) - \dot{q}_3 \\
\end{bmatrix}
\]

Note that \( f \) is linear in the state rate terms, and may be written as

\[
f(x_p, \dot{x}_p) = f_1(x_p) + f_2(\dot{x}_p) \tag{10-9}
\]

where
\[
f_2(\cdot) \equiv D(\cdot) = \begin{bmatrix}
-I_4 & [0]_{4 \times 3} & [0]_{4 \times 3} \\
[1]_{3 \times 4} & 0 & -1_{xx} \\
0 & [1]_{3 \times 3} & 0 \\
-1_{xz} & 0 & I_{zz} \\
[0]_{3 \times 4} & [0]_{3 \times 3} & mI_3 \\
\end{bmatrix}(\cdot)
\] (10-10)

For later use, this matrix is partitioned as

\[
D = \begin{bmatrix}
-I_4 & [0]_{4 \times 6} \\
[0]_{6 \times 4} & D^{22} \\
\end{bmatrix} \quad (10-11)
\]

with the obvious definition for \(D^{22}\).

Relating these equations of motion to equations (5-1) and (5-2) (the standard form), equations (10-1a) through (10-1d) represent \(f^1(\dot{x}_p, x_p)\), \(f^2(\dot{x}_p, x_p, u)\) is null, and (10-1e) through (10-11) represent \(f^3(\dot{x}_p, x_p, u_p)\), where the functional dependency on \(u_p\) is given through (10-3). Since the model equations are kinematically the same as the airplane, a change of subscript in these relationships completes the comparison.

**Control Law**

The matrix \(B^2_p\) has dimensions 6x9, and upon evaluation is found to have full rank. It should be noted that the airplane does not have direct side force control, and this matrix has full rank only because several of the controllers effect side forces indirectly, and because these controllers are redundant in their primary effects. We now seek an expression for the control vector \(u_p\). The minimum norm solution was selected, yielding:

\[
u_p = \left[ [0]_{9 \times 4} : B^2_p \left[ B_p^2 B_p^2 \right]^{-1} \right]^{\text{ref}} \left[ f(x_p, \dot{x}_p) - [F_p^{\text{ref}} + A_p^{\text{ref}} \Delta x_p] \right] \quad (10-12)
\]
The control law for perfect model following is now given by replacing the plant state rates with those of a control model:

\[ u_p^* = \begin{bmatrix} 0 & B_p^n \end{bmatrix} \begin{bmatrix} f(x_p, \dot{x}_{cm}) - [F_p + A_p \Delta x_p] \end{bmatrix} \] (10-13)

where it is understood that the matrices are to be evaluated at reference conditions, and

\[ B_p^n \equiv B_p^{2T} [B_p^2 B_p^{2T}]^{-1} \] (10-14)

If we partition \( f \) consistent with \( A_p \) and \( B_p \),

\[ f = \begin{bmatrix} f_{1x1}^1 \\ f_{6x1}^3 \end{bmatrix} \] (10-15)

then

\[ u_p^* = B_p^n [f^3(x_p, \dot{x}_{cm}) - [F_p + A_p \Delta x_p]] \] (10-16)

Note that

\[ f_{2}^3(\dot{x}) = D^{22} \dot{x}^3 \] (10-17)

**Uncertainties**

In the present problem there are no measurement errors, since the aircraft response is being computed. Errors arise because the derivatives \( A_p \) and \( B_p \) are estimated by a polynomial curve fit. The effect of these errors is found by considering the actual and estimated values of these derivatives. In this analysis, the diacritical circumflex accent (^) indicates estimated values, the tilde (~) represents the difference between actual and
estimated values, and no diacritical mark is used for the actual values. Thus, the control law is based on estimated values:

\[ u^*_g = \hat{B}_p^n [ f^3(x_p, \dot{x}_{cm}) - [\hat{F}_p + \hat{A}_p^2 \Delta x_p] ] \tag{10-18} \]

Using the relationships

\[ \hat{B}_p^n = B_p^n - \tilde{B}_p^n \tag{10-19a} \]
\[ \hat{F}_p = F_p - \tilde{F}_p \tag{10-19b} \]
\[ \hat{A}_p = A_p - \tilde{A}_p \tag{10-19c} \]

the effect of these differences on the control forces and moments applied to the airplane may be written as

\[ B_p u^*_g = B_p \hat{B}_p^n [ f^3(x_p, \dot{x}_{cm}) - [\hat{F}_p + \hat{A}_p^2 \Delta x_p] ] \]
\[ = B_p (B_p^n - \tilde{B}_p^n) [ f^3(x_p, \dot{x}_{cm}) - [(F_p - \tilde{F}_p) + (A_p^2 - \tilde{A}_p^2) \Delta x_p] ] \]
\[ = (I-B_p \tilde{B}_p^n) [ [f^3(x_p, \dot{x}_{cm}) - F_p - A_p^2 \Delta x_p] + [\tilde{F}_p + \tilde{A}_p^2 \Delta x_p] ] \]
\[ = [f^3(x_p, \dot{x}_{cm}) - F_p - A_p^2 \Delta x_p] + [\tilde{F}_p + \tilde{A}_p^2 \Delta x_p] \]
\[ - B_p \tilde{B}_p^n [f^3(x_p, \dot{x}_{cm}) - F_p - A_p^2 \Delta x_p] - B_p \tilde{B}_p^n [\tilde{F}_p + \tilde{A}_p^2 \Delta x_p] \tag{10-20} \]

Equating (10-20) to the right hand side of (10-3), and using (10-17),

\[ D^{22} x_p^3 = D^{22} x_{cm}^3 + [\tilde{F}_p + \tilde{A}_p^2 \Delta x_p] - B_p \tilde{B}_p^n [f^3(x_p, \dot{x}_{cm}) - F_p - A_p^2 \Delta x_p] \]
\[ - B_p \tilde{B}_p^n [\tilde{F}_p + \tilde{A}_p^2 \Delta x_p] \tag{10-21} \]

where the matrix D is the same for the plant and model. Equation (10-21) describes the evolution of \( x_p^3 \) in the presence of uncertainties, and will next be used in the formulation of the error dynamics.
**Error Dynamics**

We now consider the more general form of the control model, viz:

\[
\dot{x}_{cm} = \dot{x}_m + f_{cm}(x_m, x_p)
\]  
\hspace{1cm} (10-22)

where \( f_{cm}(x_m, x_p) \) is any vector valued function of the states that vanishes identically when \( e=0 \) and, as usual,

\[
e \equiv [e_1 \ldots e_{10}]^T = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}
\]  
\hspace{1cm} (10-23a)

\[
e_1 = [e_1 \ldots e_4]^T
\]  
\hspace{1cm} (10-23b)

\[
e_2 = \emptyset
\]  
\hspace{1cm} (10-23c)

\[
e_3 = [e_5 \ldots e_{10}]^T
\]  
\hspace{1cm} (10-23d)

Equation (10-21) may then be written

\[
\dot{x}_{cm}^3 - \dot{x}_p^3 = \dot{x}_m^3 + f_{cm}^3(x_m, x_p) - \dot{x}_p^3
\]

\[
= -D_{22}^{-1} \left\{ [\tilde{F}_p + \tilde{A}_p^2 \Delta x_p] - B_p \tilde{B}^n_p \left[ f_3(x_p, \dot{x}_{cm}) - F_p - \tilde{A}_p^2 \Delta x_p \right] \\
- B_p \tilde{B}^n_p [\tilde{F}_p + \tilde{A}_p^2 \Delta x_p] \right\}
\]  
\hspace{1cm} (10-24)

(noting that \( f_{cm} \) has been partitioned consistent with the state variables), or

\[
e_3^3 = - f_{cm}^3(x_m, x_p) - D_{22}^{-1} \left\{ [\tilde{F}_p + \tilde{A}_p^2 \Delta x_p] \\
- B_p \tilde{B}^n_p \left[ f_3(x_p, \dot{x}_{cm}) - F_p - \tilde{A}_p^2 \Delta x_p \right] - B_p \tilde{B}^n_p [\tilde{F}_p + \tilde{A}_p^2 \Delta x_p] \right\}
\]

\[
= - f_{cm}^3(x_m, x_p) + U^3(x_p)
\]  
\hspace{1cm} (10-25)

where \( U(x_p) \equiv [U_1 \ldots U_{10}]^T \) represents the vector of uncertainties arising from differences in actual plant parameter values from those assumed, i.e., those terms in (10-25) with tilde
accents. \( U(x_p) \) is partitioned into \( U^1(x_p) \) and \( U^3(x_p) \), where the elements of \( U^1 = [U_1 \ldots U_4]^T \) are identically zero since the kinematic relationships were given by God and Euler, and are not subject to uncertainties.

The error dynamics are now given by

\[
\dot{e} = \begin{bmatrix}
\frac{1}{2}(-q_1 p - q_2 q - q_3 r)_m - \frac{1}{2}(-q_1 p - q_2 q - q_3 r)_p \\
\frac{1}{2}( q_0 p - q_3 q + q_2 r)_m - \frac{1}{2}( q_0 p - q_3 q + q_2 r)_p \\
\frac{1}{2}( q_3 p + q_0 q - q_1 r)_m - \frac{1}{2}( q_3 p + q_0 q - q_1 r)_p \\
\frac{1}{2}(-q_2 p + q_1 q + q_0 r)_m - \frac{1}{2}(-q_2 p + q_1 q + q_0 r)_p
\end{bmatrix} - \mathbf{U}(x_p) + \mathbf{U}(x_p)
\]  

Control Model

We seek control model functions to ensure stability of the errors \( e_1 \ldots e_{10} \). Consider a candidate Lyapunov function \( V = e^T \Gamma e \), with \( \Gamma \equiv \text{diag}(\gamma_1 \gamma_2 \ldots \gamma_{10}) \), \( \Gamma > 0 \). We then require control model functions such that \( \dot{V} = 2e^T \Gamma \dot{e} \leq 0 \ \forall \ e, e^T \Gamma \dot{e} = 0 \iff e = 0 \). In performing the indicated operation, a good deal of simplification results if we set \( \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_q, \gamma_q > 0 \):

\[
2e^T \Gamma \dot{e} = \left[ \frac{1}{2} \gamma_q (q_{p0} q_{m1} - q_{m0} q_{p1} - q_{p2} q_{m3} + q_{m2} q_{p3}) - \gamma_5 f_{cm}(x_m, x_p) + U_5(x_p) \right] e_5 \\
+ \left[ \frac{1}{2} \gamma_q (q_{p0} q_{m2} + q_{p1} q_{m3} - q_{m0} q_{p2} - q_{m1} q_{p3}) - \gamma_6 f_{cm}(x_m, x_p) + U_6(x_p) \right] e_6 \\
+ \left[ \frac{1}{2} \gamma_q (q_{p0} q_{m3} - q_{p1} q_{m2} + q_{m1} q_{p2} - q_{m0} q_{p3}) - \gamma_7 f_{cm}(x_m, x_p) + U_7(x_p) \right] e_7 \\
+ [U_8(x_p) - \gamma_8 f_{cm}] e_8 + [U_9(x_p) - \gamma_9 f_{cm}] e_9 + [U_{10}(x_p) - \gamma_{10} f_{cm}] e_{10}
\]  

(10–27)
Now take

\[ f_{cm}(x_m, x_p) = \frac{1}{2} \gamma_s (q_{p0}q_{m1} - q_{m0}q_{p1} - q_{p2}q_{m3} + q_{m2}q_{p3}) - \lambda_5 e_5 \]  
\hspace{1cm} (10-28a)

\[ f_{cm}(x_m, x_p) = \frac{1}{2} \gamma_s (q_{p0}q_{m2} + q_{p1}q_{m3} - q_{m0}q_{p2} - q_{m1}q_{p3}) - \lambda_6 e_6 \]  
\hspace{1cm} (10-28b)

\[ f_{cm}(x_m, x_p) = \frac{1}{2} \gamma_s (q_{p0}q_{m3} - q_{p1}q_{m2} + q_{m1}q_{p2} - q_{m0}q_{p3}) - \lambda_7 e_7 \]  
\hspace{1cm} (10-28c)

\[ f_{cm}(x_m, x_p) = -\lambda_8 e_8 \]  
\hspace{1cm} (10-28d)

\[ f_{cm}(x_m, x_p) = -\lambda_9 e_9 \]  
\hspace{1cm} (10-28e)

\[ f_{cm}(x_m, x_p) = -\lambda_{10} e_{10} \]  
\hspace{1cm} (10-28f)

By setting \( q_{p0} = q_{m0}, q_{p1} = q_{m1}, \) etc., it is seen that \( f_{cm}(x_m, x_p) \) vanishes identically when the error is zero, as required. Now set \( \gamma_s ... \gamma_{10} = 1, \) with the result:

\[ 2e^T \Gamma \dot{e} = \sum_{i=5}^{10} \lambda_i e_i^2 + \sum_{i=1}^{10} U_i(x_p) e_i = \sum_{i=5}^{10} [\lambda_i e_i^2 + U_i(x_p) e_i] \]  
\hspace{1cm} (10-29)

If each \( U_i(x_p) \) were constant, then for \( \gamma_i < 0, \) each term in the summation would have an unstable equilibrium at \( e_i = 0 \) and a stable one at \( e_i = -U_i/\lambda_i. \) Since \( U_i \) actually varies with \( x_p \) (and hence with \((x_m - e))\), the correct analysis should consider this functional dependency. However, given the complex nature of equation (10-25) and the unavailability of analytic expressions for the terms in it, one is forced to make simplifying assumptions about the nature of \( U_i(x_p). \) Here it is assumed that \( U_i(x_m - e) \) varies slowly with the error for fixed values of \( x_m, \) and it is treated as a constant. The validity of this assumption is verified through simulation, while values of \( \lambda_i \) are selected experimentally.
Simulation Details

A schematic of the simulation steps is shown in figure 10-1. The parallelism in the figure is conceptual, since the steps were performed sequentially. However, care was taken to separate the model and plant computations except for the generation of the plant control, and to ensure that the timing of the integrations and availability of information was consistent with the application to a real airplane control problem.

With reference to the figure, details of the simulation are as follows:

1. Initialization. The baseline model was established in steady level flight at 10,000 feet and Mach 0.5. Balancing the airplane at this flight condition required an angle of attack of 0.077 radian. The airplane subroutines were called to evaluate the matrices $A_m$ and $B_m$, as defined in equations (10-5) and (10-6), with appropriate change of subscript. The derivatives were calculated by perturbing each of the appropriate parameters ($\alpha_m$, $\beta_m$, $M_m$, $p_m$, $q_m$, $r_m$, and control positions of the model) about the reference conditions, and dividing the resulting change in the force or moment by the amount the parameter was perturbed. That is,

$$\frac{\partial F_m^{\text{aero}}}{\partial x_m} \approx \frac{\Delta F_m^{\text{aero}}}{\Delta x_m}$$

$$\frac{\partial F_m^{\text{control}}}{\partial u} \approx \frac{\Delta F_m^{\text{control}}}{\Delta u}$$

The resulting values for $A_m$ and $B_m$, and for the model reference flight conditions, were then held fixed for the remainder of the simulation. All model forces and moments were calculated using the linear relationships $A_m x_m$ and $B_m u$ except, as previously noted,
the lift curve slope. The model lift curve slope was exactly the same as that approximated for the airplane. Reference values of the model control were determined by applying the model following control law to the model itself, with state rates explicitly set to zero. This yielded control positions for unaccelerated flight.

(2) The control \((u)\) applied to the model was provided as a simple function of time. Following one second of flight with the reference model controls applied, combinations of horizontal tail and thrust control were applied to cause the model to fly to an angle of attack near 20°. This trajectory, with no other controls applied, served as the baseline for simulating short period and Dutch roll responses. In subsequent simulations, the baseline trajectory was flown, and small horizontal tail and rudder inputs were applied at 20° angle of attack to excite the short period and Dutch roll responses of the model.

(3) The plant and model equations of motion were integrated using a fourth order Runge-Kutta scheme with step size of 0.025 second. Forces and moments for the model were determined as described above, while those for the airplane were obtained through calls to the F-18 subroutines.

(4) \(A_p, B_p,\) and \(B_p^N\) were determined by evaluating second and third order polynomial approximations (as functions of plant angle of attack, \(\alpha_p\)) to the individual entries in these matrices. These polynomials were generated by using curve fits to the actual data over a range of \(-10° \leq \alpha \leq 35°\). These matrices were reevaluated at increments of \(\alpha_p\) that were varied to determine the effect on the simulation.

(5) The control law was as described above. The \(\lambda_i\) of the control model functions \(f_{cm}(x_m, x_p)\), and the \(\gamma_q\) that appears in the Lyapunov function, were established by systematically varying the values of these parameters while observing the error and control
activity during the simulations. For purposes of comparison of short period and Dutch roll responses, simulations were performed in which the model following plant control was "turned off" and the airplane flown open loop, using the same additional controls that were applied to the model to excite these modes.

(6) The plant controls were limited to the values given above by simple if-then statements.

Control Saturation

The effects of control saturation on model following has not been analyzed in this work, but deserves mention because of the usually deleterious effect it had. A convenient way of looking at the model following control law as presented here is that we seek the control vector, \( u_p \), that yields a certain vector of forces and moments. These forces and moments are the differences between those that the control model is experiencing, and those that are provided by the airplane aerodynamics with no controls applied. That is,

\[
B_p u_p^* = F_{cm}^{Aero} + F_{cm}^{Control} - F_p^{Aero} \equiv F^{Req} \tag{10-31}
\]

Now, any errors between the model and airplane will (loosely) be seen in this formulation in the term \( F_{cm}^{Control} \),

\[
F_{cm}^{Control} = F_{m}^{Control} + F_{Error} \tag{10-32}
\]

The last term in this equation represents the forces and moments that result from the error feedback implicit in \( f_{cm}(x_m, x_p) \).
The consequence of plant control limiting (saturation) is that (10-31) will not be satisfied,

\[ B_p u_p^{\text{Sat}} = F^{\text{Sat}} = F^{\text{Req}} - \Delta F \]  \hspace{1cm} (10-33)

The superscript "sat" refers to cases wherein any or all plant controls are saturated. The forces and moments being generated by the saturated controls are deficient by an amount \( \Delta F \). This deficiency will generally lead to an increase in error between model and plant, followed by demands for more control to eliminate this error. This increased demand for control typically cannot be satisfied either, and all other requirements remaining constant, it continues to grow during the period of saturation roughly as the integral of the error.

If, after a period of saturation, the required forces and moments are such that they may be satisfied by (10-31), the system will recover according to the error dynamics selected. Since these requirements grow as the integral of the error, only brief periods of saturation can normally be handled. In the results that follow, an example of this phenomenon is found.

The plant control used to satisfy (10-31) depends on the choice of a right inverse for \( B_p \). In general, there are infinitely many choices, the minimum norm solution used here being the one that minimizes total control energy. Given more controllers than degrees of freedom, there is always redundancy in controller effects, and other solutions to (10-31) may be available that do not saturate the controls. Each of these other solutions may be viewed as a different weighting on the controls, such weighting being selected to reduce the requirement on a control that is near saturation in favor of one that is not. The difficulty here is that it is not known a priori which controls will saturate first.
It is conceivable that different inverses to $B_p$ could be calculated and stored for later use, each being constructed with the assumption that certain combinations of controls are saturated, and considering the force or moment contribution of those controls to be givens. This is impractical, since there are at least eight controllers, and for each there are two limits that must be considered, giving on the order of $16!$ combinations. If the rudder, throttle, and leading edge flaps are treated realistically, and if thrust vectoring controls are then added, the number becomes $26!$. And finally, if certain dynamic pressure limitations on maximum control deflections are considered, and if these inverses be approximated as functions of sideslip angle, Mach, etc., in addition to angle of attack, then the problem is truly formidable.

Real time calculation of the inverses is only slightly better. Given an approximation to $B_p$, the effects of the saturated controls may be removed by taking the force or moment contribution of those controls to be givens, and attempting to solve the problem with a reduced number of controllers. This requires the inversion of a $6 \times 6$ matrix at each reevaluation of the control law, which is bad enough. Worse is the fact that the new solution may saturate yet other controllers, requiring repeated reevaluations. Worst is that with the removal of controllers from consideration, the matrix inverse may not even exist.

Other schemes for dealing with the problem of saturated controls were considered, ranging from a constrained parameter optimization solution (which is computationally intensive), to simply declaring that certain controls should be dedicated to the generation of select forces or moments (which sidesteps the problem).

In the results that follow, the problem was left unsolved, and control saturation was avoided. This affected the results in two ways. First, the problem was limited to flight below 20° angle of attack. Lateral-directional control requirements above this angle of
attack inevitably led to saturation with disastrous consequences. Second, all sideforce requirements were removed in the computation of the control law. As mentioned earlier, there are no direct sideforce generators on the airplane, these forces being produced only as secondary effects. As a consequence, small sideforce requirements demanded very large asymmetric control deflections, usually to the point of saturation. The removal of this requirement did not noticeably affect the quality of the model following, presumably because the model and the airplane had similar sideforce response characteristics.

Results and Discussion

Results are presented for a baseline flight from reference conditions to approximately 20° angle of attack. Error feedback gains used in these simulations were as follows:

- Euler parameter errors ($\gamma_q$) .............30.0
- Roll rate error ($\lambda_5$) ..................30.0
- Pitch rate error ($\lambda_6$) ................. 5.0
- Yaw rate error ($\lambda_7$) .................. 5.0
- X-Velocity error ($\lambda_8$) ............... 1.0
- Y-Velocity error ($\lambda_9$) ...............10.0
- Z-Velocity error ($\lambda_{10}$) ............ 5.0

Baseline Case

Figure 10-2 shows the model and airplane control histories for the baseline case. The top graph summarizes the model controls, while the remaining four are the airplane controls generated by the model following control law. The strategy for flying the model to the desired angle of attack was essentially a nose up horizontal tail command accompanied
by a sharp decrease in the idealized throttle (actually, more akin to application of speed brakes, since the control is negative).

The initial conditions for these simulations were at the reference flight conditions. The airplane controls required at this time were roughly the same as those applied to the model, since the model was trimmed at that condition using the model following control laws, and because the model is simply a linearized version of the airplane at the reference conditions.

The airplane horizontal tail activity roughly parallels that of the model, except for its greater magnitude at higher angles of attack. This represents not only a difference in control power at higher angles of attack, but also the requirement to match the model’s pitch rate. Because the airplane dynamics become progressively more different from those of the model as angle of attack increases, its pitching moment for a given flight condition is different as well, and must be modified through the application of all its controls. This is shown in the graphs of aileron and flap histories. Also notable is the fact that the airplane drag characteristics become substantially different, requiring more (less negative) throttle as angle of attack is increased.

The sharp fluctuations in airplane control activity, particularly noticeable in the aileron and leading edge flap time histories, are due primarily to the updates performed on the assumed values of plant parameters ($A_p$, $B_p$, and $B_{p}^{n}$) used in the control law computations. In the simulations shown, these parameters were updated at one degree increments of airplane angle of attack, the updates being performed as the angle of attack was 60% of the way to the next integer value in order to prevent chattering. The fluctuations are observed to occur immediately following updates.
Figure 10-3 shows the actual responses of the model and airplane in pitch, pitch rate, and angle of attack. On the scale shown, the model and airplane responses appear virtually identical. Figure 10-4 displays the actual error between model and airplane responses. In all three cases, the magnitude of the errors is small, and represents instantaneously the effect of differences between actual and assumed airplane parameters. These are roughly proportional to the equilibria in the error dynamics at $e_i = (-U_i/\lambda_i)$ mentioned above; see (10-29). While the plant states are varying, the various $U_i$ are dynamic variables that presumably vary on a much slower time scale than the error dynamics. This presumption is borne out by these simulations, which display stable error dynamics. The magnitude of any of these errors could be reduced by increasing the appropriate error feedback gains, but higher gains ultimately led to excessive control requirements and saturation.

**Short period response**

The short period response of the model was excited by a $+2^\circ$ symmetric horizontal tail input (superimposed on the baseline model control) of 0.1 second duration at $t=36.0$ seconds. Figure 10-5 shows the airplane controls required to match the model's responses. The differing dynamical characteristics of the model and airplane at this angle of attack are clearly indicated by the fact that a simple horizontal tail input alone does not produce the same short period response. The coordinated application of all other longitudinal controls is required as well. The initial responses caused a saturation in trailing edge flaps, seen from 36.1 to 36.3 seconds. Since this saturation was of short duration, it did not adversely effect the quality of model following.

Noticeable on these time histories are clearer views of the effects of plant parameter updates on the control law. Updates were performed at 36.1, 37.0, and 40.0 seconds. The
effect of the first is lost in the initial response, but the other two updates resulted in
discernible fluctuations in the trailing edge flaps, ailerons, and throttle. These controls
quickly returned to values that they appeared to be tracking toward in the first place, which
suggests that the effects are due more to transients caused by discontinuities in the updates
than to actual differences in parameters. This in turn suggests that the updates did not have
to be performed as frequently as they were. However, this update frequency was selected
to give good model following over a wide range of angles of attack, and was left fixed for
all simulations.

Figure 10-6 shows the model and airplane short period responses. Two plant
responses are shown, one for the open loop case mentioned above ("O.L.Plant"), and the
other for the model following response ("MF Plant"). The open loop airplane response
indicates that the airplane differs from the model primarily in that the horizontal tail is
significantly less effective, and the short period frequency is nearly twice as fast. Short
period damping is not clear from these graphs, but appears to be similar for model and
open loop plant. Model following responses show no degradation in performance over the
baseline case, and no unexpected modes of the error dynamics have been excited.

The errors between model and airplane are shown in figure 10-7. The angle error
graphs appear benign, while the pitch rate error displays some rapid fluctuations following
the model control input. This is not particularly surprising, since the input was a pulse in
the horizontal tail. Following this initial activity, the pitch rate error subsides to more docile
behavior, as hoped for.
Dutch roll response

The model Dutch roll response was excited in a method similar to that described for the short period. The rudder control used was one cycle of a sinusoid of 0.5° amplitude, as shown in the first graph of figure 10-8. The frequency of this input was selected to be approximately the same as the damped frequency of the model response. The airplane control activity required to match the model's responses is shown in the remaining graphs. The dynamical differences between model and airplane are again demonstrated, since relatively large deflections of all lateral-directional controllers was required in addition to sustained rudder inputs of the same frequency.

Figure 10-9 more clearly demonstrates these dynamical differences. The airplane open loop Dutch roll characteristics differ from those of the model through an increase in rudder effectiveness and a response frequency that is roughly half. The model following responses show generally good results in that the right magnitude, frequency, and damping are displayed by the airplane. However, the actual trajectory following appears degraded, particularly in the roll and yaw rate terms. This represents further the effects of uncertainties in plant parameters. Although this maneuver was flown at nearly constant angle of attack, the parameters are also functions of sideslip angle, which was obviously not constant. The parameter approximations used did not incorporate considerations for varying sideslip angles, and is assumed to account for the phenomenon seen here.

Figure 10-10 shows the errors between model and airplane during the Dutch roll. No unstable tendencies are observed, indicating that the dynamics resulting from the uncertainties are not destabilizing.
Initialize Variables

New Model States

Get Model Control

Time

$u_m$

Integrate Model Equations

(2)

Get Plant Parameters

$\alpha_p$

Parameters

$\Sigma$

$e$

Evaluate Control Law

$\dot{x}_p$

$\dot{x}_m$

Control Limits

$u_p$

Integrate Plant Equations

(3)

New Plant States

Figure 10-1

Simulation Schematic
Figure 10-2

Model & Airplane Controls, 5° to 20° AOA
Figure 10-3

Model & Airplane States, 5° to 20° AOA
Figure 10-4

Errors, 5° to 20° AOA
Figure 10-5

Airplane Control History, Short Period Response
Figure 10-6

Model & Airplane States, Short Period Response
Figure 10-7

Errors, Short Period Response
Figure 10-8

Model & Airplane Controls, Dutch Roll Response
Figure 10-9

Model & Airplane States, Dutch Roll Response
Figure 10-10

Errors, Dutch Roll Response
CONCLUSIONS

Within the scope of the examples presented, the concepts of model following control introduced in this dissertation have been shown to have practical application. The methods have been successfully applied to both linear and nonlinear problems. Frequently encountered cases of different order plant and model, and of imperfect model following, have been solved with the methods developed here. Within three of these problems, various real world considerations have been present in the form of state measurement errors and plant parameter variations.

With respect to the control loader problem, model following control may be successfully applied to a design problem that uses a direct drive DC motor. Its implementation requires the measurement of stick position and rate, and of the applied torque. On-line calculations are required to determine the moments developed by the motor in response to the measured parameters, and to integrate the equations of motion of the control stick being simulated. The effects of motor ripple may be reduced by increasing the error feedback gains, by approximating the value of the ripple through accurate position measurement, or by a combination of both. Proportional bias errors in torque measurements may be compensated for by increased error feedback gains, whereas constant bias errors in position and rate measurements will degrade the performance of the system for any selection of gains. Model following control was shown to be relatively insensitive to nonlinearities in the equations of motion of the control system being simulated. In the example presented, the modeled control system incorporated several
nonlinearities often present in conventional flight control systems, and these were faithfully reproduced in the simulation results.

In the application to a control configured vehicle, it is noted that the simulation results were not particularly better or worse than those attained by Hirzinger. The primary difference lies in the method used to derive the feedforward and feedback gains for the implementation. Linear optimal control design requires a fair amount of juggling of the weighting matrices used in the performance index, accompanied by repeated simulations to assess the performance. The approach presented here is essentially a “one-pass” method, wherein all the characteristics of the system are determined without experimentation. Simulation is then used to verify the expected results. It was shown that the effect on model following of the center of gravity shift could be completely analyzed, and explicit formulations for the effect on the error dynamics obtained.

In considering a problem wherein perfect model following was unattainable, it was shown that the notion of selecting a candidate model based on comparisons of eigenstructure is valid. The procedure involved a difficult constrained parameter optimization problem, and is clearly not suited to real time implementation. However, the method has application to point design problems in which the problem has to be solved only once, or in certain trajectory following problems in which the model states and state rates are precalculated and stored.

Through the F-18 example it was shown that nonlinear model following control may be successfully applied to the control of an airplane as its flight condition is varied. The major limitation of the model following control method is the inability of the control laws to deal with the problem of control saturation. It was shown that nonlinear error feedback gains may be determined from considerations of Lyapunov stability criteria, and
that do not require linearization of the dynamical equations. These gains were selected to provide stable error dynamics for the variations in plant parameters that occurred in this problem.
SUMMARY

Two key concepts in perfect model following control have been introduced: a standard form for the plant and model equations of motion, and the associated control and error dynamics. From these ideas, several previously unanswered model following control questions have been analyzed and answered.

The standard form for the plant equations is seen to be a natural separation of the equations of motion into those that are uncontrolled or purely kinematic, those that are driven by external controllers, and those through which the controls available are to be applied. Models whose equations of motion differ from those of the plant only in the third type may be perfectly followed, and the control law for accomplishing this is easily obtained. This control law carries with it the concept of a control model, which completely specifies the response of the system to errors between model and plant. The nature of the control model is determined through analysis of the dynamics of these errors.

For linear systems, an equivalent standard form is defined. Every commonly encountered linear perfect model following control problem may be put in this standard form by means of a similarity transformation. In the standard form the control law is given, and the designer's attention is focused on the error dynamics. The error dynamics based on the standard form allow the analysis of several real world control implementation problems, namely measurement errors, plant parameter variations, and the effects of imperfect model following with perfect model following control laws. These analyses all result in a similar form, in which the error dynamics are viewed as a regulator problem. In all cases the error regulator problem is found to have external forcing functions, driven by the model states.
In some cases, the effect is to vary the parameters of the error regulator dynamics, determined by the variations in plant parameters.

Problems involving different order models and plants are equivalent to larger order problems in which the plant and model share the same states. The augmentation of the plant and model equations of motion results in a standard form problem, and forces the designer to completely specify the problem.

Perfect model following control solutions to imperfect model following control problems may be obtained by selecting from a class of candidate models one that has dynamic characteristics that are similar to those of the intended model. This requires that the desirable characteristics of the intended model be explicitly addressed to provide a basis for comparison. This solution offers all of the advantages of the perfect model following control law formulation and its associated error dynamics analysis.
LITERATURE CITED


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