CLOSABILITY OF DIFFERENTIAL OPERATORS
AND SUBJORDAN OPERATORS

by

Thomas R. Fanney

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

APPROVED:

J. A. Ball, Chairman

K. B. Hannsgen J. E. Thomson

R. A. McCoy R. F. Olin

May, 1989

Blacksburg, Virginia
CLOSABILITY OF DIFFERENTIAL OPERATORS
AND SUBJORDAN OPERATORS

by

Thomas R. Fanney

J. A. Ball, Chairman, Mathematics

(ABSTRACT)

A (bounded linear) operator \( J \) on a Hilbert space is said to be jordan if \( J = S + N \)
where \( S = S^* \) and \( N^2 = 0 \). The operator \( T \) is subjordan if \( T \) is the restriction of a jordan
operator to an invariant subspace, and pure subjordan if no nonzero restriction of \( T \) to
an invariant subspace is jordan. The main operator theoretic result of the paper is that
a compact subset of the real line is the spectrum of some pure subjordan operator if and
only if it is the closure of its interior. The result depends on understanding when the
operator \( D = \theta + \frac{d}{dx} : L^2(\mu) \rightarrow L^2(\nu) \) is closable. Here \( \theta \) is an \( L^2(\nu) \) function, \( \mu \) and
\( \nu \) are two finite regular Borel measures with compact support on the real line, and the
domain of \( D \) is taken to be the polynomials. Approximation questions more general than
what is needed for the operator theory result are also discussed. Specifically, an explicit
characterization of the closure of the graph of \( D \) for a large class of \( (\theta, \mu, \nu) \) is obtained,
and the closure of the graph of \( D \) in other topologies is analyzed. More general results
concerning spectral synthesis in a certain class of Banach algebras and extensions to the
complex domain are also indicated.
Acknowledgements

For his unwavering support and example, and for his persistence in always asking "So, when are you going to be finished?", I would like to thank my father.

For her constant support and love, and for her insistence on never asking "So, when are you going to be finished?", I would like to thank my mother.

For his inspiration, guidance and saintly patience, I must especially thank – without him this would not have been possible, and with him I can finally give my father an answer.

Finally for her expertise and dedication, I would like to thank for typing this thesis.
Table of Contents

1. Introduction ..................................................... 1
2. The Closure of the Graph of a Differential Operator ................. 8
3. Spectra of Pure Subjordan Operators ................................ 56
4. Banach Algebras and Spectral Synthesis .............................. 82
5. Complex Jordan and Subjordan Operators ............................ 96
References ....................................................... 119
1. Introduction.

A (bounded, linear) operator $J$ on a Hilbert space $\mathcal{K}$ is said to be jordan (of order two) if $J = S + N$ where $S = S^*$, $SN = NS$ and $N^2 = 0$. We say that the operator $T$ on $\mathcal{H}$ is subjordan if there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a jordan operator $J$ on $\mathcal{K}$ such that $\mathcal{H}$ is invariant for $J$ and $T = J|\mathcal{H}$. The subjordan operator $T$ is said to be pure if there is no nonzero invariant subspace $\mathcal{H}_0$ for $T$ for which the restriction $T_0 = T|\mathcal{H}_0$ is jordan. We say that a compact subset $K$ of the real line $\mathbb{R}$ is regularly closed if and only if $K$ is the closure of its interior. We can now state one of the main results of this paper.

**Theorem A.** (See Theorem 3.17) A compact subset $K$ of the real line arises as the spectrum $\sigma(T)$ of a pure subjordan operator $T$ if and only if $K$ is regularly closed.

For the proof of Theorem A we use the model for subjordan operators given by Ball and Helton [BH] to deduce a model for pure subjordan operators. Characterizing these models requires characterizing when a differential operator of a certain class is closable. We actually obtain results on the closure of the graph of such a differential operator more general than what are required for the proof of Theorem A which should be of interest in their own right.

We now discuss the approximation question which comprises a large portion of the paper. We consider two finite compactly supported measures $\mu$ and $\nu$ on the real line and a complex valued function $\theta$ in $L^2(\nu)$. Define an operator $D$ from $L^2(\mu)$ into $L^2(\nu)$ with domain equal to the polynomials in $L^2(\mu)$ by

$$Dp = \theta p + \frac{d}{dx}p.$$

A main interest of this paper is the study of the closure of the graph of $D$

$$\mathcal{G} = \{p \oplus Dp : p \text{ a polynomial}\}$$

in $L^2(\mu) \oplus L^2(\nu)$. For a large class of $(\mu, \nu, \theta)$ we obtain a complete explicit description of the closure $\mathcal{G}^-$ of $\mathcal{G}$. As corollaries we are able to give complete characterizations of when
$D$ is closable (i.e. $\mathcal{G}^-$ contains no nonzero elements of the form $0 \oplus g$) as well as of when $\mathcal{G}$ is dense in $L^2(\mu) \oplus L^2(\nu)$.

To describe our results in detail, let $d\nu = w \, dx + \nu_{\text{sing}}$ be the Lebesgue decomposition of $\nu$ with respect to Lebesgue measure $dx$ on $\mathbb{R}$. When $w(x) = 0$ define $1/w(x) = \infty$. An interval $I$ is said to be an interval of local integrability for $1/w$ if $\int_a^b \frac{1}{w(x)} \, dx < \infty$ for every compact subinterval $[a, b] \subset I$; here none, either one or both endpoints of $I$ may belong to $I$. An interval $I$ (possibly open, closed or half open - half closed) is said to be a maximal interval of local integrability (MILI) for $1/w$ if $1/w$ is locally integrable over $I$ but no interval strictly containing $I$ has this property. The collection of all MILI's for $1/w$ forms a partitioning of a subset of the support $\text{spt} \nu$ of $\nu$. For $I$ an MILI, denote by $(\mathcal{G}|I)^-$ the closure of $\mathcal{G}$ in $L^2(\mu|I) \oplus L^2(\nu|I)$. Our first result shows how the characterization of $\mathcal{G}^-$ can be localized to MILI's.

**Theorem B.** (see Lemma 2.16 and Corollary 2.8) A given $h \oplus k \in L^2(\nu)$ is in $\mathcal{G}^-$ if and only if $(h \oplus k)|I$ is in $(\mathcal{G}|I)^-$ for each MILI $I$ for $1/w$. Moreover, $h \oplus k$ is in $\mathcal{G}^-$ if and only if $h \oplus k_a$ is in the closure of $\mathcal{G}$ in $L^2(\mu) \oplus L^2(w \, dx)$, where $k = k_a + k_s$ is the decomposition of $k \in L^2(\nu)$ with $k_a \in L^2(w \, dx)$ and $k_s \in L^2(\nu_{\text{sing}})$.

As a consequence of this localization result, characterizing $\mathcal{G}^-$ in $L^2(\mu) \oplus L^2(\nu)$ reduces to the case where $\nu = w \, dx$ and $\text{spt} \nu$ is a MILI for $1/w$. The analysis for this reduced case splits into eight cases summarized in Table 2.1. For five of the eight cases, we have a complete explicit characterization of $\mathcal{G}^-$; the remaining three remain open, although we have a natural conjecture for the form of $\mathcal{G}^-$ for these cases as well which we can verify for some simple examples. We also give examples to verify that all cases are not vacuous. We present here the simplest sample result.

**Theorem C.** (Type 1 in Table 2.1) Suppose $\nu = w \, dx$, $\text{spt} \nu$ is the closed interval $I = [a, b]$ and $\int_a^b \frac{1}{w(x)} \, dx < \infty$. Then a function $h \oplus k \in L^2(\mu) \oplus L^2(w \, dx)$ is in $\mathcal{G}^-$ if and only if there
exists a function \( h_1 \) such that

(i) \( h_1 \) is absolutely continuous on \( I \)

(ii) \( h = h_1 \mu - \text{a.e.} \)

(iii) \( e^{\Theta(x)}h_1(x) = h_1(a) + \int_a^x(e^{\Theta}k)(t)dt \)

where \( \Theta(x) = \int_a^x \theta(t)dt \).

For the general case the following partial result is sufficient for many purposes.

**Theorem D.** (see Corollary 2.18) Suppose \( \nu = wdx \) and \( \text{sppt}\nu \) is a MILI \( I \) for \( 1/w \).

Suppose that a function \( h \oplus k \in L^2(\mu) \oplus L^2(wdx) \) satisfies (i), (ii) and (iii) in Theorem B, where \( a \) is some point in \( I \), as well as

(iv) \( h \oplus k \) has compact support in \( I \).

Then \( h \oplus k \in G^- \).

Using these general results concerning \( G^- \) we are able to obtain complete answers to when \( D \) is closable and (the opposite extreme) when \( G \) is dense.

**Theorem E.** (see Theorem 2.24) The operator \( D = \theta + \frac{d}{dx} : L^2(\mu) \rightarrow L^2(\nu) \) is closable if and only if \( \nu \) is absolutely continuous (so \( d\nu = wdx \)), the union of the MILI's for \( 1/w \) is a carrier for \( wdx \), and the complement of \( \text{sppt}\mu \) contains no intervals in a MILI for \( 1/w \).

To give the characterization of when \( G \) is dense in \( L^2(\mu) \oplus L^2(\nu) \) we need a definition. If \( I \) is a MILI for \( 1/w \) with endpoints \( a \) and \( b \) (which may or may not be in \( I \)), we define an auxiliary interval \( J = J(I) \) to be an interval with the same endpoints as \( I \) such that

(i) \( a \in J \) if and only if \( \int_a^{a+\delta} e^{2Re(\theta(t))}w(t)^{-1}dt < \infty \) for some \( \delta > 0 \)

and

(ii) \( b \in J \) if and only if \( \int_{b-\delta}^b e^{2Re(\theta(t))}w(t)^{-1}dt < \infty \) for some \( \delta > 0 \).

Here \( \theta(t) = \int_{x_0}^t \theta(s)ds \) where \( x_0 \) is an arbitrary point of \( I \). It turns out that always \( J \supset I \). We can now state the result concerning the density of \( G \).

**Theorem F.** (see Theorem 2.31) \( G^- = L^2(\mu) \oplus L^2(\nu) \) (where \( d\nu = wdx + \nu_{\text{sing}} \)) if and
only if, for each MILI $I$ for $1/w$, either

(i) $\mu|I$ is a single point mass and $J = I$

or

(ii) $\mu|I = 0$ and $J \setminus I$ has at most one point.

Special cases of Theorem F were suggested by J. Agler (private communication).

Similar results hold for the closure of $\mathcal{G}$ in $L^p(\mu) \oplus L^p(\nu)$ where $1 \leq p < \infty$. In the case $1 < p < \infty$ one works with MILI's for $w^{1-q} \ (\text{where} \ \frac{1}{p} + \frac{1}{q} = 1)$ rather than for $w^{-1}$; if $p = 1$ one works with maximal intervals on which $1/w$ is locally essentially bounded.

We also consider the closure of $\mathcal{G}$ (for the special case $\theta = 0$) in the uniform norm of $C(E) \oplus C(E)$, where $E$ is a compact subset of $\mathbb{R}$ and $C(E)$ is the space of continuous functions on $E$. The result for this case is much simpler to state. Let the interior of $E$ be expressed as the countable disjoint union of open intervals $intE = \cup_i (a_i, b_i)$ and set $\hat{E} = \cup_i [a_i, b_i]$.

**Theorem G.** (see Theorems 2.33 and 2.40) Let $E$ be a compact subset of $\mathbb{R}$. The function $h \oplus k \in C(E)$ is in the closure of $\mathcal{G}$ in the norm of $C(E) \oplus C(E)$ if and only if

(i) $h|\hat{E}$ is absolutely continuous

and

(ii) $k|\hat{E} = (\frac{d}{dx})h|\hat{E}$.

From this result we read off that (for $\theta = 0$) $D$ is closable as an operator from $C(E)$ to $C(E)$ if and only if $E$ is the closure of its interior and that $\mathcal{G}$ is uniformly dense in $C(E) \oplus C(E)$ if and only if $E$ has empty interior.

The study of subjordan operators was initiated by Helton in [H1], [H2], [H3] and has continued in [BH], [Ag1], [Ag2]. Included in some of this work are some results concerning such generalizations as complex subjordan operators of higher order ($T$ is the restriction of $M + N$ to an invariant subspace where $M$ is normal, $M$ commutes with $N$ and $N^k = 0.$)
for some positive integer $k$) and commuting $n$-tuples of complex subjordan operators. Our
result Theorem A is an analogue for pure subjordan operators of the result of Clancey
and Putnam [CP] which characterizes the spectrum of a pure subnormal operator.

The closability of a differential operator as discussed above is closely connected with
the closability of a Dirichlet form as discussed by Fukushima in [F]; the motivation there
is the Beurling-Deny theory of Markovian semigroups. In particular our Theorem E has a
large intersection with Theorem 2.1.4 in [F] (see also [Ham]). There has also been interest
in the closure of the graph $G$ in the uniform topology (as in Theorem G) for $E$ a general
compact subset of the plane. Bishop [B1], [B2] and Agler [A2] have given conditions on $E$
for $G$ to be dense and Fixman and Rubel [FR] have given conditions for $D$ to be closable.
A complete characterization of $G^-$ as in Theorem G appears to be quite difficult for a
general $E \subset \mathbb{C}$. These authors also consider the closure of the more general manifold
$\{p \oplus p' \oplus \cdots \oplus p^{(n)} : p \text{ a polynomial (or rational)}\}$ in $C(E) \oplus \cdots \oplus C(E)$.

Our technique for all the different Banach space norms considered is a straightforward
duality analysis. We first obtain an explicit characterization of the annihilator $G^\perp$ of $G$
in the dual space. We then compute $G^-$ as the preannihilator $^\perp G^\perp$ of the annihilator. By
restricting to the real line setting, all the calculations can be carried out quite far in an
explicit form. This enables us to obtain an explicit characterization of the closure $G^-$ for
most cases.

From [B1], [B2], [FR] it appears that generalizations to planar domains will be quite
difficult. Nevertheless we conjecture the following qualitative generalization of Theorem
A to complex subjordan operators which is very much in the spirit of the local spectral
result of Clancey and Putnam for pure subnormal operators. Let us say that the operator
$T$ is complex subjordan if $T$ is the restriction of $M + N$ to an invariant subspace, where
$M$ is normal, $M$ commutes with $N$ and $N^2 = 0$. We say that $T$ is pure complex subjordan
if $T$ has no nonzero invariant subspace on which it is complex jordan. For $E$ a compact
subset of the plane, let \( R_2(E) \) be the closure of \( \{ r \oplus r' : r \text{ rational with no poles in } E \} \) in the uniform norm of \( C(E) \oplus C(E) \).

**Conjecture:** The compact subset \( K \) of the plane is the spectrum of some pure complex subjordan operator \( T \) if and only if \( R_2(K \cap \bar{D}) \neq C(K \cap \bar{D}) \oplus C(K \cap \bar{D}) \) whenever \( D \) is an open disk intersecting \( K \) in a nonempty set.

By using Theorem A combined with Theorem G, one can easily check that the conjecture is correct if \( K \subseteq \mathbb{R} \). Conceivably one might be able to prove the conjecture in general without the detailed knowledge of \( R_2(K) \) analogous to Theorem G.

We also mimic techniques used to model subjordans to do the same for complex subjordans in the case where the operator is cyclic. As before, the complex subjordan lives on the closure of a graph of a differential operator but here the closability of the latter operator is not a consequence of the construction in the pure case. Hence the precision of the real case is absent. We are able though to obtain a generalization of Bram's [Br] model for cyclic subnormals and use this to give conditions for when the complex subjordan operator is actually complex jordan or pure.

Furthermore, as an application to the complete characterization of the closure of the graph of \( D = \theta + d/dx \) given in Theorem C, we consider the Banach algebra \( A \) of \( L^2(\mu) \)-functions defined by

\[
\begin{bmatrix} I \\ D \end{bmatrix} A = \mathcal{G}^-
\]

where \( \| h \|_A^2 = \| h \|_{L^2(\mu)}^2 + \| Dh \|_{L^2(\mu)}^2 \) for \( h \in A \). We argue that the maximal ideal space of \( A \) is the support of the measure \( \mu \), here a closed interval in \( \mathbb{R} \). Moreover \( A \) is shown to be semi-simple and regular, and thus the question of spectral synthesis is addressed and answered affirmatively: Every closed set in the maximal ideal space of \( A \) is a set of spectral synthesis. This work expands on results given by Sarason [S] and Jorgensen [J].

Chapter 2 develops the needed duality machinery and proves the general results on
the closure of \( G \) in \( L^2(\mu) \oplus L^2(\nu) \) as listed in Theorems B-D and in Table 2.2. Chapter 2.a uses these results to characterize when \( D \) is closable (Theorem E) while Chapter 2.b characterizes when \( G \) is dense. Chapter 2.c indicates the modifications needed to compute \( G^- \) in the other norms which we consider. Chapter 3 sets down the model for pure subjordan operators and uses the results of Chapter 2 to prove Theorem A. The Banach algebra application is pursued in Chapter 4 and finally in Chapter 5, complex subjordan operators are the focus.
2. The Closure of the Graph of a Differential Operator.

In order to keep the notation more manageable, we investigate the closure of the graph $G$ of $D = \theta + d/dx$ only for the case $p = 2$. Later in this chapter we will indicate the modifications needed for general $p, 1 \leq p < \infty$. We also include results for closure of $G$ in the uniform topology in $C(E) \oplus C(E)$, where $C(E)$ is the space of continuous functions on the compact set $E$ in $\mathbb{R}$.

We begin by posing the problem and introducing the notation and terminology that will be used. Let $\mu$ and $\nu$ be two finite, positive measures both with support contained in $[0, 1]$. Let $m$ (or equivalently $dx$) represent Lebesgue measure on $[0, 1]$. Denote by $\nu = \nu_a + \nu_s$ the Lebesgue decomposition of $\nu$ with respect to $m$, where $\nu_a \ll m$ and $\nu_s \perp m$. (Note: throughout this paper the symbol "\ll" reads "is absolutely continuous with respect to", while "\perp" reads "is singular to" when referring to measures.) If we let $d\nu/dm = w \in L^1(dm)$, then $L^2(\nu)$ splits as $L^2(wdm) \oplus L^2(\nu_s)$, and for every $k \in L^2(\nu), k = k_a \oplus k_s$ where $k_a \in L^2(wdm)$ and $k_s \in L^2(\nu_s)$.

For $\theta \in L^2(\nu)$ define $D : L^2(\mu) \to L^2(\nu)$ via

$$Dp = \theta p + p'$$

for all polynomials $p$, and let

$$G = \{p \oplus Dp : p \text{ a polynomial}\}$$

denote the graph of $D$ in $L^2(\mu) \oplus L^2(\nu)$.

In the proofs of the following results a key observation is that

$$\int_{a}^{b} dm = \int_{a}^{b} \frac{1}{w} d\nu_a$$

since $wdm = d\nu_a$. Thus $1/w \in L^1(m|[a, b])$ if and only if $1/w \in L^2(\nu_a|[a, b])$. Now let $F \in L^2(\nu)$. Then, as mentioned above, $F$ can be written as $F_a \oplus F_s$ with $F_a \in L^2(wdm)$
and \( F_a \in L^2(\nu_a) \). Since \( \nu_a \perp m, |F|dm = |F_a|dm \) and it follows that

\[
\int_a^b |F|dm = \int_a^b |F| \frac{1}{w}d\nu_a \leq \|F\|_{L^2(\nu_a)} \frac{1}{w} \|L^2(\nu_a|[a,b])\].
\numbered{2.1}

Thus \( L^2(\nu_a|[a,b]) \) can be continuously embedded into \( L^1(m|[a,b]) \) when \( \int_a^b \frac{1}{w}dm < \infty \).

We will see that for \( h \oplus k \in \mathcal{G}^- \), if we let \( k = k_a \oplus k_s \) be the Lebesgue decomposition of \( k \), then \( k_s \) can be chosen independently of both \( h \) and \( k_a \). This fact and (2.1) will enable us to restrict our attention to intervals over which \( 1/w \) is integrable in a local sense. We therefore define the following terms and notation.

**Definition 2.2.** A property \( P \) is said to be local for an interval \( I \), written \( P_{loc}(I) \), if it holds for every compact subinterval \([c,d] \subseteq I\). This will be applied specifically to \( P : f \) is Lebesgue integrable over \( I \), and \( P : f \) is absolutely continuous over \( I \). We write these as \( f \in L^1_{loc}(I) \) and \( f \in AC_{loc}(I) \), respectively.

Thus \( f \in AC_{loc}(I) \) implies that

\[
f(y) - f(x) = \int_x^y f'(t)dt
\]

for all \( x, y \in I \).

**Definition 2.3.** An interval \( I \) will be termed a maximal interval of local integrability, (MILI) for a function \( f \), if \( f \) is \( L^1_{loc}(I) \), and if \( J \) is any other interval such that \( J \supset I \) with \( f \in L^1_{loc}(J) \), then \( J = I \).

The thrust of our analysis of the norm closure of \( \mathcal{G} \) relies on partitioning the support of \( \nu_a \) into MILI's for \( 1/w \), and directing our attention to one such interval. Towards this, let \( I \) be a MILI for \( 1/w \) and for some point \( t_0 \in I \), let

\[
\Theta(x) = \int_{t_0}^x \theta(t)dt.
\numbered{2.4}

Then, by observation (2.1), \( \Theta \in AC_{loc}(I) \). Later we shall see that \( \Theta \) plays a vital role in the determination of \( \mathcal{G}^- \) in certain circumstances.
Definition 2.4. For a point \( p \in \mathbb{R} \) and for a function \( f \geq 0 \), we say \( p \) is right integrable for \( f \), written \( p \) is \( RI(f) \), if for sufficiently small \( \delta > 0 \),
\[
\int_{p}^{p+\delta} f \, dm < \infty.
\]
Similarly, we say \( p \) is left integrable for \( f \), written \( p \) is \( LI(f) \), if for sufficiently small \( \delta > 0 \),
\[
\int_{p-\delta}^{p} f \, dm < \infty.
\]
If \( p \) is not right integrable for \( f \), we write \( p \) is \( NRI(f) \). Similarly we define the notation \( NLI(f) \).

Now suppose \( I \), with endpoints \( a \) and \( b \), is a MILI for \( 1/w \). We shall see that the characterization of \( G^- \) depends on whether or not the function \( 1/w \) and \( e^{2Re\Theta}(1/w) \) are right integrable at \( a \), and similarly left integrable at \( b \). Thus for each MILI \( I \) for \( 1/w \), we define a corresponding interval \( J \), with the same endpoints \( a \) and \( b \) as \( I \), but where \( a \in J \) if \( a \) is \( RI(e^{2Re\Theta}1/w) \), and \( b \in J \) if \( b \) is \( LI(e^{2Re\Theta}1/w) \).

As noted where \( 1/w \) is integrable, \( \Theta \) is absolutely continuous by (2.1), and hence \( e^{2Re\Theta} \) is bounded. That is, integrability of \( 1/w \) implies integrability of \( e^{2Re\Theta}1/w \). Thus \( I \) is always a subset of \( J \). From this and the symmetry of some cases, we have six qualitatively different types of intervals \( I \) which are MILI's for \( 1/w \):

- \((T.1)\): \( I = [a,b] = J \)
- \((T.2)\): \( I = (a,b) = J \)
- \((T.3)\): \( I = [a,b) = J \)
- \((T.4)\): \( I = (a,b), J = [a,b] \)
- \((T.5)\): \( I = [a,b), J = [a,b] \)
- \((T.6)\): \( I = (a,b), J = [a,b) \)

When \( I \) is of the first three types, the problem is totally solved. The next two types are handled under an additional assumption, and the last type remains an open question.
A table of examples of all these cases, with or without the additional assumption, will be provided after the presentation and proofs of the results.

Definition 2.5. Let \( \mathcal{G} \) denote the submanifold of \( L^2(\mu) \oplus L^2(\nu) \) consisting of all \( h \oplus k \) such that \( h \) and \( k \) satisfy the following: if \( I \) is an interval with endpoints \( a \) and \( b \) which is a MILI for \( 1/w \), and \( J \) is the corresponding interval determined by the integrability of \( e^{2\Re \Theta} 1/w \), then

there exists a function \( h_1 \in AC_{loc}(I) \) so that \( h_1 = h \mu- \text{a.e.} \), and

\[
e^{\Theta}(x)h_1(x) - e^{\Theta}(y)h_1(y) = \int_y^x (e^{\Theta}k)(t)dt; \tag{2.6.1}
\]

if \( a \in J \setminus I \), then for \( m- \text{a.e.} \, x \in J \),

\[
(e^{\Theta}h_1)(x) = \int_a^x (e^{\Theta}k)(t)dt, \tag{2.6.2}
\]

and similarly, if \( b \in J \setminus I \), then for \( m- \text{a.e.} \, x \in J \),

\[
-(e^{\Theta}h_1)(x) = \int_x^b (e^{\Theta}k)(t)dt. \tag{2.6.3}
\]

(We should note that 2.6.1 makes sense because \( e^{\Theta}k \in L^1_{loc}(J) \); to see this, let \( C \) be a compact subinterval of \( J \). Then since \( e^{\Theta}/w \in L^2(wdm|J) \) and \( k \in L^2(wdm|J) \), we have

\[
\int_C e^{\Theta}kdx = \int_C (e^{\Theta}/w)kwdx = \langle e^{\Theta}/w, k \rangle_{L^2(wdm|C)} < \infty.
\]

The conjecture is that \( \mathcal{G} = \mathcal{G}^- \), and our first theorem gives the inclusion in one direction: \( \mathcal{G}^- \subset \mathcal{G} \). A few points should be made about the definition of \( \mathcal{G} \). First of all, if \( k = k_a + k_s \), notice that no restriction has been placed on \( k_s \). Secondly, for the absolutely continuous extension \( h_1 \) of \( h \) chosen, \( k_a = \theta h_1 + h_1' \) m- a.e. on \( I \). Also given \( h \) there may be more than one absolutely continuous extension of \( h \) satisfying (2.6.2) of Definition 2.6 for some \( k \). Thus for a given \( h \) there may not be a unique \( k_a \) so that \( h \oplus k \) is in \( \mathcal{G} \).

For instance, if \( \mu \) consists of exactly one point mass in \( J \), (2.6.2) places no restriction on the pair \( h \oplus k \). Similarly, if \( \mu \) consists of two point masses in \( J \), say \( x_0 \) and \( y_0 \), then
the only restriction is that
\[ e^{\Theta(x_0)} h_1(x_0) - e^{\Theta(y_0)} h_1(y_0) = \int_{y_0}^{x_0} (e^{\Theta}) \kappa(t) dt \]

for an absolutely continuous extension \( h_1 \) of \( h \). On the other hand, if \( \mu = m \), (2.6.2) is a very stiff requirement. We also note that since \( L^2(\mu) \) consists of equivalence classes, to show a pair \( h \oplus k \in \hat{G} \) is in \( G^- \), we may use the absolutely continuous extension \( h_1 \) of \( h \) in place of \( h \), since \( h_1 = h \mu \) a.e.

Much of our analysis is based on considering \( G^- \) as a double orthogonal complement \((G^\perp)^\perp\) of \( G \). Towards this we characterize \( G^\perp \) in the following lemma. For convenience, we assume that \( \mu \) and \( \nu \) have support in the unit interval \([0,1]\).

**Lemma 2.7.** A pair \( f \oplus g \) in \( L^2(\mu) \oplus L^2(\nu) \) belongs to \( G^\perp \) if and only if the following conditions are satisfied:

\[
gd\nu = g\nu_a = gwdm
\]

\[
\int_{[0,1]} f(x) d\mu(x) + \int_{[0,1]} \theta(x) g(x) w(x) dx = 0
\]

\[
g(x) w(x) = -\left( \int_{(x,1]} f(t) d\mu(t) + \int_{(x,1]} \theta(t) g(t) w(t) dt \right).
\]

**Proof:** Let \( f \oplus g = G^\perp \). Then for all polynomials \( p \),

\[
0 = \int_{[0,1]} p(x) f(x) d\mu(x) + \int_{[0,1]} (\theta(x) p(x) + p'(x)) g(x) d\nu(x)
\]

\[
= p(0) [\int_{[0,1]} f(x) d\mu(x) + \int_{[0,1]} \theta(x) g(x) d\nu(x)] + \int_{[0,1]} (\int_{[0,1]} p'(t) dt) f(x) d\mu(x)
\]

\[
+ \int_{[0,1]} \theta(x) (\int_{[0,1]} p'(t) dt) g(x) d\nu(x) + \int_{[0,1]} p'(x) g(x) d\nu(x)
\]

\[
= p(0) [\int_{[0,1]} f(x) d\mu(x) + \int_{[0,1]} \theta(x) g(x) d\nu(x)]
\]

\[
+ \int_{[0,1]} p'(x) [\int_{(x,1]} f(t) d\mu(t) + \int_{(x,1]} \theta(t) g(t) d\nu(t)] dx + g(x) d\nu(x).
\]

We justify the change in order of integration by applying Fubini's theorem to the measures \( d\mu(t) \times d\mu(x) \) and \( d\nu(t) \times d\mu(x) \); with integrands \( \chi_{\{x \leq t\}}(x,t)p'(x)f(t) \) and \( \chi_{\{x \leq t\}}(x,t)p'(x)\theta(t)g(t) \).
respectively; here \( \chi_S(x,t) \) denotes the characteristic function of the set \( S \): \( \chi_S(s,t) = 1 \) if \( (s,t) \in S \) and 0 otherwise. Now using \( p = 1 \), we get that

\[
0 = \int_{[0,1]} f(x) d\mu(x) + \int_{[0,1]} \overline{\theta(x)} g(x) d\nu(x). \tag{2.7.4}
\]

Next, considering all polynomials \( p' \) with \( p(0) = 0 \), a dense set, we get for \( m \)-a.e. \( x \in [0,1] \)

\[
g(x) d\nu(x) = -\left( \int_{(x,1]} f(t) d\mu(t) + \int_{[x,1]} \overline{\theta(t)} g(t) d\nu(t) \right) dx. \tag{2.7.5}
\]

Since the measure on the right is absolutely continuous with respect to \( m \), we must have \( g d\nu \) is also. Thus, if \( d\nu(x) = w(x) dx + d\nu_*(x) \), then \( g(x) d\nu_*(x) = 0 \). This proves (2.7.1) of the lemma. Applying the result to equations (2.7.4) and (2.7.5) yields that \( f \oplus g \) in \( G^\perp \) must satisfy parts (2.7.2) and (2.7.3).

To complete the proof, simply reverse the argument. \( \square \)

Before we proceed to prove \( G^- \subset \mathcal{G} \), we give two corollaries which exhibit the ramifications of (2.7.1).

**Corollary 2.8.** For \( h \oplus k \in L^2(\mu) \oplus L^2(\nu) \), let \( k = k_\alpha \oplus k_* \) be the Lebesgue decomposition of \( k \) with respect to \( \nu_\alpha \oplus \nu_* \). Then \( h \oplus k \in G^- \) if and only if, for all \( f \oplus g \in G^\perp \),

\[
0 = \int_{[0,1]} f(x) \overline{h(x)} d\mu(x) + \int_{[0,1]} g(x) \overline{k_\alpha(x)} w(x) dx.
\]

**Proof:** Since \( (G^\perp)^\perp = G^- \), \( h \oplus k \in G^- \) if and only if for each \( f \oplus g \in G^\perp \)

\[
0 = \int_{[0,1]} f \overline{h} d\mu + \int_{[0,1]} g \overline{k} d\nu.
\]

But, from (2.7.1), the second integral is equal to

\[
\int_{[0,1]} g \overline{k_\alpha} wdm.
\]

The results follow.
**Corollary 2.9.** If \( \nu \neq 0 \), then \( D \) is not closable.

**Proof:** If \( \nu \neq 0 \), then there is a set \( A \) so that \( \nu([0,1] \setminus A) = \nu(A) = m(A) = 0 \). It follows that \( \chi_A(x) \) is a non-zero element of \( L^2(\nu) \) and from Corollary 2.8, \( 0 \oplus \chi_A \in G^\perp \). Thus \( D \) is not closable. \( \square \)

Notice we have shown that, as was the case for \( \hat{G} \), belonging to \( G^- \) places no restriction on \( k_* \). For this reason we see the closure depends only on the properties of \( \nu \) (that is, the integrability of \( 1/w \)), and is independent of \( \nu \). Thus, without loss of generality, we may assume \( d\nu = wdm \) at times in the following proofs.

**Theorem 2.10.** \( G^- \subset \hat{G} \).

**Proof:** We need to show \( h \oplus k \in G^- \) satisfies (2.6.1)-(2.6.3) of Definition 2.6. Let \( I \), with endpoints \( a \) and \( b \), be a MILI for \( 1/w \). If \( \Theta \) is defined by (2.4), then \( \Theta \) is \( AC \) in \( I \). Let \( \{p_n\} \) be a sequence of polynomials such that \( \{p_n \oplus Dp_n\} \) converges to \( h \oplus k \) in \( L^2(\mu) \oplus L^2(\nu) \). By choosing a subsequence if necessary we may assume that for \( \mu-\text{a.e.} \; x \) in \( I \)

\[
\lim_{n} p_n(x) = h(x)
\]

Since for \( \nu-\text{a.e.} \; x \) in \( I \),

\[
\{e^{\Theta}p_n\}' = e^{\Theta}(Dp_n)
\]

we have for all \( x, y \in I \),

\[
e^{\Theta(y)p_n(y)} - e^{\Theta(x)p_n(x)} = \int_{[x,y]} e^{\Theta(t)(Dp_n)(t)} dt.
\] (2.10.1)

Now, \( \lim_{n \to \infty} Dp_n = k \), so by (2.1), \( Dp_n \) approaches \( k \) in \( L^1_{\text{loc}}(I) \). Thus, if \( [c,d] \subset I \), we have for all \( x, y \in [c,d] \)

\[
\lim_{n \to \infty} \int_{[x,y]} e^{\Theta(t)(Dp_n)(t)} dt = \int_{[x,y]} e^{\Theta(t)k(t)} dt,
\]
since $e^\Theta$ is locally bounded in $I$. But the left hand side of (2.10.1) converges $\mu-$ a.e. to $e^{\Theta(y)}h(y) - e^{\Theta(x)}h(x)$. Thus, for $\mu-$ a.e. $x,y \in [c,d]$, 

$$e^{\Theta(t)}h(y) - e^{\Theta(x)}h(x) = \int_{[x,y]} e^{\Theta(x)}k(t)dt.$$ 

But, since $e^\Theta$ has bounded inverse and is $AC_{loc}(I)$, we have there is an $AC_{loc}(I)$ extension $h_1$ of $h$, so that $h = h_1 \mu-$ a.e. It follows that if $h \oplus k \in \mathcal{G}^-$, then $h \oplus k$ satisfies (2.6.2) of Definition 2.6. Moreover, since 

$$\int_{[x,y]} |e^{\Theta(t)}k(t)|dt = \int_{[x,y]} |e^{\Theta(t)}k(t)|1/w(t)dt \leq \|k\|_{L^2(\nu)}\|e^{2\text{Re}\Theta}1/w\|_{L^1(m)}^{1/2},$$

we see that $e^\Theta k$ is actually in $L^1_{loc}(J)$. Thus $h$ and $k$ satisfy property (2.6.1) of $\hat{\mathcal{G}}$.

For property (2.6.3), notice that if $a$ is $NRI(1/w)$ but $\int_{[a,a+\delta]} e^{2\text{Re}\Theta} \frac{1}{w}dm < \infty$, it must be the case that $\lim_{\delta \to 0} e^{2\text{Re}\Theta(a+\delta)} = 0$. Thus, for every $n, \lim_{x \to a} e^{\Theta(x)}p_n(x) = 0$. But then 

$$e^{\Theta(x)}p_n(x) = \int_{[a,x]} (e^{\Theta}p_n)'(t)dt.$$ 

Now the right hand side converges to 

$$\int_{[a,x]} (e^{\Theta}k)(t)dt,$$

while the left hand side converges for $\mu-$ a.e. $x$ in $I$ to $(e^{\Theta}h)(x)$.

Similarly, if $b \in J \setminus I$, we have 

$$(e^{\Theta}h)(x) = -\int_{[x,b]} (e^{\Theta}k)(t)dt. \quad \square$$

Addressing the question of when $\hat{\mathcal{G}} \subset \mathcal{G}^-$, we first prove five lemmas based on the characterization of $\mathcal{G}^\perp$ reached in Lemma 2.7. These results will enable us to localize our
concentration to one MILI for $1/w$. We begin with two lemmas which determine the set where all elements of $G^\perp$ vanish identically. If we consider the manifold consisting of all $h \oplus k$ that are zero off this set, we see that this manifold must be contained in $G^-$ since it is perpendicular to $G^\perp$.

**Lemma 2.11.** Let $f \oplus g \in G^\perp$. If $x$ is NRI($1/w$), then

$$\lim_{\delta \to 0} \left( \int_{[x+\delta,1]} f(t) d\mu(t) + \int_{[x+\delta,1]} \overline{\theta(t)} g(t) w(t) dt \right) = \int_{[x,1]} f(t) d\mu(t) + \int_{[x,1]} \overline{\theta(t)} g(t) w(t) dt = 0.$$

Similarly, if $x$ is NLI($1/w$), then

$$\lim_{\delta \to 0} \left( \int_{[x-\delta,1]} f(t) d\mu(t) + \int_{[x-\delta,1]} \overline{\theta(t)} g(t) w(t) dt \right) = \int_{[x,1]} f(t) d\mu(t) + \int_{[x,1]} \overline{\theta(t)} g(t) w(t) dt = 0.$$

**Proof:** Since the measure

$$\lambda(A) = \int_A f d\mu + \int_A \overline{\theta} g w dm$$

is countably additive,

$$\lim_{\delta \to 0} \left( \int_{[x+\delta,1]} f d\mu + \int_{[x+\delta,1]} \overline{\theta} g w dm \right) = \int_{[x,1]} f d\mu + \int_{[x,1]} \overline{\theta} g w dm,$$

for all $x$. Now assume $x$ is NRI($1/w$) and suppose the limit above is nonzero. Then there is an $\epsilon > 0$ and an interval $I$ with left end point $x$, so that for all $y$ in $I$,

$$\left| \int_{[y,1]} f d\mu + \int_{[y,1]} \overline{\theta} g w dm \right| > \epsilon.$$

But then from (2.7.3)

$$\|g\|_{L^2(w)}^2 = \int_{[0,1]} |g(y) w(y)|^2 \frac{1}{w(y)} dy = \int_{[0,1]} \left| \int_{[y,1]} f(t) d\mu(t) + \int_{[y,1]} \overline{\theta(t)} g(t) w(t) dt \right|^2 \frac{1}{w(y)} dy$$

$$\geq \int_{I} \left| \int_{[y,1]} f d\mu + \int_{[y,1]} \overline{\theta} g w dm \right|^2 \frac{1}{w(y)} dy > \epsilon^2 \int_I \frac{1}{w(y)} dy = \infty,$$

a contradiction. A similar argument applies if $x$ is NLI($1/w$). □
We next define a set $K$ by

$$K = \{x \in [0,1] : \text{x is both NRI}(1/w) \text{ and NLI}(1/w)\}. \tag{2.12.1}$$

As the next lemma will point out, $K$ is precisely the set where elements of $\mathcal{G}^\perp$ vanish identically. We note two other properties. First, $K$ is the complement of the union of all MILI's for $1/w$. Secondly, for each $x \in K$

$$\int_{[x,1]} f d\mu + \int_{[x,1]} \bar{g} d\mu = 0,$$  \tag{2.12.2}

by Lemma 2.11.

**Lemma 2.13.** If $f \oplus g \in \mathcal{G}^\perp$, then $f = 0$ $\mu -$ a.e. and $g = 0$ $\nu -$ a.e. on $K$.

**Proof:** Let $\lambda_1$ and $\lambda_2$ denote the measures defined by

$$\lambda_1(A) = \int_A f d\mu, \quad \lambda_2(A) = \int_A \bar{g} d\mu$$

for all subsets $A$ of $[0,1]$. Then $\lambda = \lambda_1 + \lambda_2$ where $\lambda$ was defined in the proof of Lemma 2.11. By (2.7.3) of Lemma 2.7 and Lemma 2.11, for $m-$ a.e. $x \in K$,

$$g(x)w(x) = -\lambda([x,1]) = 0.$$ 

That is, $gw = 0$, $m-$ a.e. on $K$, and hence $g = 0$, $\nu -$ a.e. on $K$. Moreover, this implies $\lambda_2(A) = 0$ for all subsets $A$ of $K$.

To finish the proof we need only show $\lambda_1(A) = 0$ for all $A \subseteq K$ as this implies $f = 0$, $\mu -$ a.e. on $K$. But $\lambda_2$ is zero on $K$ and $\lambda_1 = \lambda - \lambda_2$ so it suffices to show $\lambda(A) = 0$ for every subset $A$ of $K$.

Since $\{x\} = [x,1] \setminus (x,1)$, by additivity of $\lambda$ and (2.12.2), we have $\lambda(\{x\}) = 0$ for all $x \in K$. That is, $\lambda$ can have no point mass in $K$. Now let $A \subseteq K$. By regularity of the total variation measure $|\lambda|$, we may assume that $A$ is closed. Let $\epsilon > 0$ be given and choose an open set $U$ containing $A$ so that $|\lambda|(U \setminus A) < \epsilon$. Write

$$U = \bigcup_n (a_n, b_n),$$
as a disjoint union of open intervals. We construct a new open set $U'$ as follows: Let

$$Z' = \{ n \in \mathbb{Z} : (a_n, b_n) \cap A \neq \emptyset \},$$

$$a_n' = \inf \{ x : x \in (a_n, b_n) \cap A, n \in Z' \},$$

$$b_n' = \sup \{ x : x \in (a_n, b_n) \cap A, n \in Z' \},$$

and let

$$U' = \bigcup_{n \in \mathbb{Z}} (a_n', b_n').$$

Then notice that since $A$ is closed $a_n', b_n' \in A$ for all $n$, and $A \subset U'$ modulo these endpoints $a_n'$ and $b_n'$. But as these endpoints are a countable set and since $\lambda$ has no point masses in $K$, this set has $\lambda$-measure zero. Moreover, since each $a_n'$ and $b_n' \in K$, by additivity and (2.12.1) we have

$$\lambda((a_n', b_n')) = \lambda((a_n', 1]) - \lambda([b_n', 1]) = 0.$$

Thus $\lambda(U') = 0$. Hence

$$|\lambda(A)| = |\lambda(U') - \lambda(A)|$$

$$= |\lambda(U' \setminus A)|$$

$$\leq |\lambda|(U' \setminus A)$$

$$\leq |\lambda|(U \setminus A) < \epsilon.$$

So by the arbitrariness of $\epsilon$, we conclude that $\lambda(A) = 0$. This completes the proof. \qed

The next result, a corollary to Lemma 2.13, does not in itself contribute to the analysis of when $\mathcal{G}^- = \hat{\mathcal{G}}$, but does introduce the question of the closability of $D$. The answer to this question will be given at the end of this chapter.

**Corollary 2.14.** If $\nu << m$, and $\nu(K) > 0$, then $\mathcal{G}^-$ contains non-zero elements of the form $0 \oplus k$, and hence $D$ is not closable.
Suppose there is a set \( A \subset K \) so that \( \nu(A) > 0 \). Let \( k = \chi_A \), a nonzero element of \( L^2(\nu) \). By Lemma 2.13, if \( f \oplus g \in \mathcal{G}^\perp \), then
\[
\int_{[0,1]} g k d\nu = \int_{[0,1]\setminus K} g k d\nu = 0.
\]
Thus \( 0 \oplus k \in (\mathcal{G}^\perp)^\perp = \mathcal{G}^- \).

We next investigate the complement of the set \( K \) given above, namely the set of points where elements of \( \mathcal{G}^\perp \) can be nonzero, and thus where the elements of \( \mathcal{G}^- \) are constrained. We do this by localizing to intervals \( I \) which are MILI's for \( 1/w \).

**Lemma 2.15.** If \( I \) is a MILI for \( 1/w \), then for \( f \oplus g \in \mathcal{G}^\perp \),
\[
\int_I f(t) d\mu(t) + \int_I \overline{\theta(t)} g(t) w(t) dt = 0.
\]

**Proof:** Let \( I \) have endpoints \( a \) and \( b \). Then \( I \) has four forms depending on whether or not \( a \) and \( b \) belong to \( I \). Suppose \( a \notin I \), that is, \( a \) is NRI(1/w). Then for small \( \delta > 0 \),
\[
\int_a^{a+\delta} 1/w dm = \infty.
\]
If \( a \in I \), then by the maximality of \( I \), it must be the case that for small \( \delta > 0 \),
\[
\int_a^{a-\delta} 1/w dm = \infty.
\]
By Lemma 2.11, in the first case
\[
\lim_{\delta \to 0} \left( \int_{(a+\delta,1]} f(t) d\mu(t) + \int_{(a+\delta,1]} \overline{\theta(t)} g(t) w(t) dt \right) = \int_{(a,1]} f(t) d\mu(t) + \int_{(a,1]} \overline{\theta(t)} g(t) w(t) dt = 0,
\]
while in the second,
\[
\lim_{\delta \to 0} \left( \int_{(a-\delta,1]} f(t) d\mu(t) + \int_{(a-\delta,1]} \overline{\theta(t)} g(t) w(t) dt \right) = \int_{(a,1]} f(t) d\mu(t) + \int_{(a,1]} \overline{\theta(t)} g(t) w(t) dt = 0.
\]
Similar results hold for the endpoint \( b \). Thus, if \( b \in I \), then
\[
\int_{(b,1]} f(t) d\mu(t) + \int_{(b,1]} \overline{\theta(t)} g(t) w(t) dt = 0,
\]
and if $b \notin I$

$$\int_{[b,1]} f(t) d\mu(t) + \int_{[b,1]} \tilde{\theta}(t) g(t) w(t) dt = 0.$$  

In all cases, $I$ can be obtained as the difference of the first interval and the second. Thus

$$\int_I f(t) d\mu(t) + \int_I \tilde{\theta}(t) g(t) w(t) dt = 0$$

as claimed. \(\Box\)

**Lemma 2.16.** Suppose that for each $I$ which is a MILI for $1/w$ and for each $f \oplus g \in \mathcal{G}^\perp$, the element $h \oplus k$ of $L^2(\mu) \oplus L^2(\nu)$ satisfies

$$\int_I f h d\mu + \int_I g k d\nu = 0.$$

Then $h \oplus k \in \mathcal{G}^\perp$.

**Proof:** The observation here is that if $\mathcal{M} = \cup\{I : I$ is a MILI for $1/w\}$, then $[0,1] \setminus \mathcal{M} = K$. Since, by Lemma 2.13, $f \oplus g \in \mathcal{G}^\perp$ implies $f \oplus g = 0$ on $K$, we have

$$\int_{[0,1]} f h d\mu + \int_{[0,1]} g k d\nu = \int_\mathcal{M} f h d\mu + \int_\mathcal{M} g k d\nu.$$  

Thus by countable additivity, our assumption implies that this term is zero as needed. \(\Box\)

**Lemma 2.17.** Let $I$ be a MILI for $1/w$ and suppose that $h \oplus k \in L^2(\mu) \oplus L^2(\nu)$ satisfies

$$(h \oplus k)(x) = 0 \text{ for } x \in [0,1] \setminus I \quad (2.17.1)$$

there exists a function $h_1 \in AC_{loc}(I)$ with $h_1 = h \mu$ a.e. for $h_1$ as in (2.6.1),

$$k_a = \theta h_1 + h_1' \quad \nu_a - \text{ a.e. on } I \quad (2.17.3)$$

and for some sequences $\{c_n\}$ and $\{d_n\}$ such that $\{c_n\}$ approaches $a$ from the right and $\{d_n\}$ approaches $b$ from the left and for $h_1$ as in (2.17.2)

$$\lim_{n \to \infty} \left[ h_1(c_n) \left( \int_{[c_n,1]} f d\mu + \int_{[c_n,1]} \tilde{\theta} g d\nu \right) - h_1(d_n) \left( \int_{[d_n,1]} f d\mu + \int_{[d_n,1]} \tilde{\theta} g d\nu \right) \right] = 0 \quad (2.17.4)$$
for all $f \oplus g \in \mathcal{G}^\perp$.

Then $h \oplus k \in \mathcal{G}^\perp$.

**Proof:** It suffices to show that $h \oplus k$ is orthogonal to $\mathcal{G}^\perp$. Since $(h \oplus k)(x) = 0$ for $x \not\in I$, by Lemma 2.7 this amounts to showing

$$0 = \int_I f\bar{h}d\mu + \int_I g\bar{k}wdx$$

for all $f \oplus g$ in $\mathcal{G}^\perp$. But also by Lemma 2.7 and the assumptions on $h \oplus k$, the second integral becomes

$$\int_I g\bar{k}wdx = -\int_I \left\{ \int_{[t,t]} f(t)d\mu(t) + \int_{[t,t]} \overline{\theta(t)}g(t)w(t)dt \right\}k(x)dx.$$

Now suppose $[c,d]$ is a compact subinterval of $I$. Then $1/w \in L^1(dx|[c,d])$. Noting that $I$ can be written as the increasing union of such intervals, we have

$$\int_{[c,d]} g(x)\overline{k(x)}w(x)dx = -\int_{[c,d]} \left\{ \int_{[x,t]} f(t)d\mu(t) + \int_{[x,t]} \overline{\theta(t)}g(t)w(t)dt \right\}\overline{k(x)}dx$$

$$= -\int_{[c,d]} \int_{[0,1]} \chi_{\{x<t\}}(x,t)f(t)k(x)d\mu(t)dx - \int_{[c,d]} \int_{[0,1]} \chi_{\{x<t\}}(x,t)\overline{\theta(t)}g(t)w(t)\overline{k(x)}dt dx.$$

Now $f(t) \in L^2(\mu) \subset L^1(\mu)$, and $k \in L^1(m|[c,d])$, so the integrand in the first integral is in $L^1(\mu(t) \times m(x))[0,1] \times [c,d])$. Thus we may apply Fubini’s theorem to get this first integral equal to

$$-\int_{[0,1]} \int_{[c,d]} \chi_{\{x<t\}}(x,t)f(t)\overline{k(x)}dx d\mu(t) =$$

$$-\int_{[c,d]} f(t) \left( \int_{[c,t]} \overline{k(x)}dx \right) d\mu(t) - \int_{[d,1]} f(t) \left( \int_{[c,d]} \overline{k(x)}dx \right) d\mu(t).$$

In the second integral, $\overline{\theta(t)}g(t) \in L^2(wdt) \subset L^1(wdt)$. Thus $\overline{\theta(t)}g(t)w(t) \subset L^1(dt)$. Moreover, $k$ is $L^1_{loc}(I)$, so $k$ is $L^1(m|[c,d])$. Therefore the integrand is $L^1(m \times m)[0,1] \times [c,d])$, so using Fubini’s theorem again we get the second integral is equal to

$$-\int_{[0,1]} \int_{[c,d]} \chi_{\{x<t\}}(x,t)\overline{\theta(t)}g(t)w(t)\overline{k(x)}dx dt =$$
\[- \int_{[c,d]} \overline{\theta(t)} g(t) w(t) \left( \int_{[c,t]} k(x) dx \right) \, dt - \int_{(d,1]} \overline{\theta(t)} g(t) w(t) \left( \int_{[c,d]} \bar{k} dx \right) \, dt.\]

But for some absolutely continuous extension \( h_1 \) of \( h \),

\[ \int_{[c,d]} \overline{k(x)} dx = h_1(t) - \bar{h}_1(c) + \int_{[c,t]} (\partial h_1)(x) dx. \]

Using this, and \( h_1 = h \mu - \text{a.e.} \), we get

\[
\int_{[c,d]} g(t) \overline{k(t)} w(t) \, dt = - \int_{[c,d]} f(t) \overline{h_1(t)} d\mu(t) - \int_{[c,d]} \overline{\theta(t)} g(t) \overline{h_1(t)} w(t) \, dt \\
+ \overline{h_1(c)} \left[ \int_{[c,d]} f(t) d\mu(t) + \int_{[c,d]} \overline{\theta(t)} g(t) w(t) \, dt \right] - \int_{[c,d]} f(t) \left( \int_{[c,d]} \overline{\theta(x)} h_1(x) dx \right) d\mu(t) \\
- \int_{[c,d]} \overline{\theta(t)} g(t) w(t) \left( \int_{[c,d]} \overline{\theta(x)} h_1(x) dx \right) dt - \overline{h_1(d)} \left[ \int_{(d,1]} f(t) d\mu(t) + \int_{(d,1]} \overline{\theta(t)} g(t) w(t) \, dt \right] \\
+ \overline{h_1(c)} \left[ \int_{(d,1]} f(t) d\mu(t) + \int_{(d,1]} \overline{\theta(t)} g(t) w(t) \, dt \right] - \int_{(d,1]} f(t) \left( \int_{[c,d]} \overline{\theta(x)} h_1(x) dx \right) d\mu(t) \\
- \int_{(d,1]} \overline{\theta(t)} g(t) w(t) \left( \int_{[c,d]} \overline{\theta(x)} h_1(x) dx \right) dt. \]

Thus

\[
\int_I f \bar{h} d\mu + \int_I g \bar{k} d\nu = \int_{I \setminus [c,d]} f \overline{h_1} d\mu + \int_{[c,d]} f \overline{h_1} d\mu + \int_{I \setminus [c,d]} g \bar{k} \, dx - \int_{[c,d]} f \overline{h_1} d\mu - \int_{[c,d]} \bar{g} \overline{h_1} \, dx \\
+ \bar{h}_1(c) \left[ \int_{[c,d]} f d\mu + \int_{[c,d]} \bar{g} \, dx \right] - \int_{[c,d]} f(t) \left( \int_{[c,t]} \overline{\theta(x)} h_1(x) dx \right) d\mu(t) \\
- \int_{[c,d]} \overline{\theta(t)} g(t) w(t) \left( \int_{[c,t]} \overline{\theta(x)} h_1(x) dx \right) dt - \bar{h}_1(d) \left[ \int_{(d,1]} f d\mu + \int_{(d,1]} \bar{g} \, dx \right] \\
+ \bar{h}_1(c) \left[ \int_{(d,1]} f d\mu + \int_{(d,1]} \bar{g} \, dx \right] - \left( \int_{[c,d]} \overline{\theta h_1} \, dx \right) \left( \int_{(d,1]} f d\mu + \int_{(d,1]} \bar{g} \, dx \right) \\
= \int_{I \setminus [c,d]} f \overline{h_1} d\mu + \int_{I \setminus [c,d]} g \bar{k} \, dx + \bar{h}_1(c) \left[ \int_{[c,d]} f d\mu + \int_{[c,d]} \bar{g} \, dx \right] \\
- \bar{h}_1(d) \left[ \int_{(d,1]} f d\mu + \int_{(d,1]} \bar{g} \, dx \right] - \int_{[c,d]} \overline{\theta(t)} g(t) w(t) \left( \int_{[c,t]} \overline{\theta(x)} h_1(x) dx \right) dt - \left( \int_{[c,d]} \overline{\theta h_1} \, dx \right) \left( \int_{(d,1]} f d\mu + \int_{(d,1]} \bar{g} \, dx \right). \tag{2.17.5} \]
Now, carrying out another interchange of the order of integration, we get

\[- \int_{[c,d]} \int_{[c,d]} \chi_{\{x < t\}}(x, t) f(t)(\bar{\theta}_1)(x) dx d\mu(t) - \int_{[c,d]} \int_{[c,d]} \chi_{\{x < t\}}(x, t) \bar{\theta}(t) g(t) w(t)(\bar{\theta}_1)(x) dx dt \]

\[= - \int_{[c,d]} \left( \int_{[x,d]} f(t) d\mu(t) \right) (\bar{\theta}_1)(x) dx - \int_{[c,d]} \left( \int_{[x,d]} (\bar{\theta} w(t)) dt \right) (\bar{\theta}_1)(x) dx. \quad (2.17.6)\]

But

\[\int_{[x,d]} f(t) d\mu(t) + \int_{[x,d]} (\bar{\theta} w(t)) dt = \left[ \int_{[x,1]} f(t) d\mu(t) + \int_{[x,1]} (\bar{\theta} w(t)) dt \right] - \left[ \int_{[d,1]} f(t) d\mu(t) + \int_{[d,1]} (\bar{\theta} w(t)) dt \right] = -(gw)(x) - \int_{[d,1]} f(t) d\mu(t) - \int_{[d,1]} (\bar{\theta} w(t)) dt.\]

Thus (2.17.6) becomes

\[- \int_{[c,d]} f(t) \left( \int_{[c,t]} (\bar{\theta}_1)(x) dx \right) d\mu(t) - \int_{[c,d]} (\bar{\theta} w(t)) \left( \int_{[c,d]} (\bar{\theta}_1)(x) dx \right) dt = \int_{[c,d]} (gw)(x)(\bar{\theta}_1)(x) dx + \left[ \int_{[d,1]} f(t) d\mu(t) + \int_{[d,1]} (\bar{\theta} w(t)) dx \right] \left[ \int_{[c,d]} (\bar{\theta}_1)(x) dx \right].\]

Substituting this expression into (2.17.5) results in

\[\int_I \tilde{f} \tilde{h} d\mu + \int_I g \tilde{k} d\nu = \int_{I \setminus [c,d]} \tilde{f} \tilde{h} d\mu + \int_{I \setminus [c,d]} g \tilde{k} d\nu + \frac{1}{\bar{\theta}_1(c)} \left( \int_{[c,1]} f(t) d\mu(t) + \int_{[c,1]} (\bar{\theta} w(t)) dt \right) - \frac{1}{\bar{\theta}_1(d)} \left( \int_{[d,1]} f(t) d\mu(t) + \int_{[d,1]} (\bar{\theta} w(t)) dt \right).\]

Finally, notice that as \( c \) approaches \( a \) from the right and as \( d \) approaches \( b \) from the left, both \( \int_{I \setminus [c,d]} \tilde{f} \tilde{h} d\mu \) and \( \int_{I \setminus [c,d]} g \tilde{k} d\nu \) converge to zero by hypothesis (2.17.4). This completes the proof. \( \square \)
Corollary 2.18. Suppose $h\otimes k \in \mathcal{G}$ and for each MILI $I$ for $1/w$, $(h\otimes k)|I$ has compact support in $I$. Then $h \otimes k$ is in $\mathcal{G}^-$.

Proof: The result follows immediately from Lemmas 2.16 and 2.17. □

We now turn to the consideration of cases (T.1) through (T.5). The first three types rely on groundwork laid in the preceding lemmas. The analysis of (T.4) and (T.5) rely on Lemmas 2.11 through 2.16, an additional assumption, and on a transformation, by the function $e^\Theta$, to another direct sum space

$$L^2(e^{-2Re\Theta}d\mu|I) \oplus L^2(e^{-2Re\Theta}wdz|I).$$

Type (T.6) remains open.

Theorem 2.19. (T.1). Let $I$ be the MILI for $1/w$ which is of type (T.1). If $h \otimes k \in \mathcal{G}$ and is zero on $[0,1] \setminus I$, then $h \otimes k \in \mathcal{G}^-$.

Proof: By assumption $I = [a,b]$ is compact, and by maximality $a$ is NLI($1/w$) and $b$ is NRI($1/w$). Thus by Lemma 2.11 both

$$\left(\int_{[a,1]} fd\mu + \int_{[a,1]} \bar{\vartheta}gwdt\right) = \left(\int_{(b,1]} fd\mu + \int_{(b,1]} \bar{\vartheta}gwdt\right) = 0. \quad (2.19.1)$$

Since, we may take $[c,d] = [a,b]$ in the proof of Lemma 2.17 and since by assumption there is an AC($I$) extension $\bar{h}_1$ of $h$, it suffices only to show

$$\bar{h}_1(a) \left(\int_{[a,1]} fd\mu + \int_{[a,1]} \bar{\vartheta}gwdt\right) - \bar{h}_1(b) \left(\int_{(b,1]} fd\mu + \int_{(b,1]} \bar{\vartheta}gwdt\right) = 0.$$

But this follows immediately from (2.19.1). □

Theorem 2.20. (T.2). Let $I$ be a MILI for $1/w$ which is of type (T.2). If $h \otimes k \in \mathcal{G}$ and is zero on $[0,1] \setminus I$, then $h \otimes k \in \mathcal{G}^-$.

Proof: Here we have $I = (a,b)$. By Lemma 2.17, we need only show there are sequences $\{c_n\}$ and $\{d_n\}$, so that $c_n$ approaches $a$ from the right and $d_n$ approaches $b$ from the left,
so that
\[
\lim_{n \to \infty} \left[ \bar{h}_1(c_n) \left( \int_{[c_n,1]} f \, d\mu + \int_{[c_n,1]} \bar{g} \, dt \right) - \bar{h}_1(d_n) \left( \int_{(d_n,1]} f \, d\mu + \int_{(d_n,1]} \bar{g} \, dt \right) \right] = 0
\]
for all \( f \oplus g \in \mathcal{G} \). Suppose there is no sequence \( \{d_n\} \) approaching \( b \) from the left, so that
\[
\lim \left[ \bar{h}_1(d_n) \left( \int_{(d_n,1]} f \, d\mu + \int_{(d_n,1]} \bar{g} \, dt \right) \right] = 0.
\]
Then
\[
\liminf_{d \to b} \left| \bar{h}_1(d) \left( \int_{(d,1]} f \, d\mu + \int_{(d,1]} \bar{g} \, dt \right) \right| \neq 0.
\]
That is, there is an \( \epsilon > 0 \), and \( \delta > 0 \), so that
\[
\left| \bar{h}_1(x) \left( \int_{(x,1]} f \, d\mu + \int_{(x,1]} \bar{g} \, dt \right) \right| > \epsilon \quad (2.20.1)
\]
for all \( x \), with \( b - \delta < x < b \). Let \( t_0 \in I \), and define
\[
\Theta(x) = \int_{t_0}^x \theta(t) \, dt.
\]
Now, squaring both sides of the inequality (2.20.1) and multiplying by \( e^{2 \Re \Theta(x)} \) results in
\[
\left| e^{\Theta(x)} \bar{h}_1(x) \left( \int_{(x,1]} f \, d\mu + \int_{(x,1]} \bar{g} \, dt \right) \right|^2 > \epsilon^2 e^{2 \Re \Theta(x)}.
\]
But \( (e^{\Theta} \bar{h}_1)(x) = \int_{[t_0,x]} e^{\Theta(t)} k(t) \, dt + e^{\Theta(t_0)} \bar{h}_1(t_0) \).

Using this we get,
\[
\left| \left( \int_{[t_0,x]} e^{\Theta(t)} k(t) \, dt \right) \left( \int_{[x,1]} f \, d\mu + \int_{[x,1]} \bar{g} \, dt \right) + e^{\Theta(t_0)} \bar{h}_1(t_0) \left( \int_{[x,1]} f \, d\mu + \int_{[x,1]} \bar{g} \, dt \right) \right|^2 > \epsilon^2 e^{2 \Re \Theta(x)}.
\]
However, \( b \) is NLI \((1/w)\), so by Lemma 2.11,
\[
\lim_{x \to b} \left( \int_{[x,1]} f \, d\mu + \int_{[x,1]} \bar{g} \, dt \right) = \left( \int_{[b,1]} f \, d\mu + \int_{[b,1]} \bar{g} \, dt \right) = 0.
\]
So it suffices to only consider the situation where
\[
\left| \left( \int_{[t_0,x]} e^\Theta(t) k(t) \, dt \right) \left( \int_{[x,1]} f \, d\mu + \int_{[x,1]} \tilde{g} \, w \, dt \right) \right|^2 > e^{2e^{2Re\Theta(x)}},
\]
or
\[
\left| \int_{[x,1]} f \, d\mu + \int_{[x,1]} \tilde{g} \, w \, dt \right|^2 > e^{2e^{2Re\Theta(x)}} \left| \int_{[t_0,x]} e^\Theta(t) k(t) \, dt \right|^2. \tag{2.20.2}
\]

Now
\[
\left| \int_{[t_0,x]} e^\Theta(t) k(t) \, dt \right|^2 \leq \left[ \int_{[t_0,x]} e^{Re\Theta(t)} |k(t)| \frac{1}{w(t)} \, d\nu(t) \right]^2.
\]
\[
\leq \|k\|_{L^2(v)}^2 \int_{[t_0,x]} e^{2Re\Theta(t)} \left( \frac{1}{w(t)} \right)^2 \, d\nu(t)
\]
\[
\leq \|k\|_{L^2(v)}^2 \int_{[t_0,x]} e^{2Re\Theta(t)} \frac{1}{w(t)} \, dt.
\]

Thus, (2.20.2) may be replaced by the inequality
\[
|gw(x)|^2 \geq e^2 \|k\|^2 e^{2Re\Theta(x)} \left\{ \int_{[t_0,x]} e^{2Re\Theta(t)} \frac{1}{w(t)} \, dt \right\}^{-1}. \tag{2.20.3}
\]

Let \( b - \delta < t_1 < T < b \), and \( t_1 \neq t_0 \). Then
\[
\int_{[0,1]} |gw(x)|^2 \frac{1}{w(x)} \, dx = \|g\|_{L^2(v)}^2 \geq \int_{[t_1,T]} |g|^2 \, d\nu.
\]
So, integrating both sides of (2.20.3) with respect to \( 1/w(x) \, dx \) from \( t_1 \) to \( y \), where \( b - \delta < y < b \), we get
\[
\|g\|_{L^2(v)}^2 \geq \left( \frac{\epsilon}{\|k\|} \right)^2 \int_{[t_1,y]} e^{2Re\Theta(x)} \frac{1}{w(x)} \left( \int_{[t_0,x]} e^{2Re\Theta(t)} \frac{1}{w(t)} \, dt \right)^{-1} \, dx
\]
\[
= \left( \frac{\epsilon}{\|k\|} \right)^2 \ell_n \left( \int_{[t_0,x]} e^{2Re\Theta(t)} \frac{1}{w(t)} \, dt \right) \bigg|_{t_1}^{y}
\]
\[
= \left( \frac{\epsilon}{\|k\|} \right)^2 \left[ \ell_n \int_{[t_0,y]} e^{2Re\Theta(t)} \frac{1}{w(t)} \, dt - \ell_n \left( \int_{[t_0,t_1]} e^{2Re\Theta(t)} \frac{1}{w(t)} \, dt \right) \right].
\]

Finally, letting \( y \) tend to \( b \), and using the assumption that \( b \) is NLI \( (e^{2Re\Theta}1/w) \), we get the last expression goes to infinity, a contradiction. We conclude that there is a sequence \( \{d_n\}, d_n \) tending to \( b \) from the left such that
\[
\lim_{n \to \infty} \frac{1}{h_1(d_n)} \left( \int_{(d_n,1]} f \, d\mu + \int_{(d_n,1]} \tilde{g} \, w \, dt \right) = 0.
\]
Since \( a \) is \( NRI(1/w) \) we have

\[
\lim_{x \to a} \left( \int_{[x,1]} f \, d\mu + \int_{[a,1]} \bar{\theta} g \, dt \right) = \int_{(a,1]} f \, d\mu + \int_{(a,1]} \bar{\theta} g \, dt = 0.
\]

In addition we have that \( a \) is \( NRI(e^{2\Re\Theta}1/w) \), so an argument similar to the one above gives that there must be a sequence \( \{c_n\} \) approaching \( a \) from the right, so that

\[
\lim_{n \to \infty} h_1(c_n) \left( \int_{[c_n,1]} f \, d\mu + \int_{[c_n,1]} \bar{\theta} g \, dt \right) = 0. \quad \Box
\]

**Theorem 2.21.** (T.3). Let \( I \) be a MILI for \( 1/w \) which is of type (T.3). If \( h \oplus k \in \mathcal{G} \) and is zero on \([0,1] \setminus I\), then \( h \oplus k \in \mathcal{G}^- \).

**Proof:** In this case, \( I = [a,b) \), so we may take \( c_n = a \), for all \( n \). By maximality of \( I \), \( a \) is \( NLI(1/w) \), so by Lemma 2.11, for all \( f \oplus g \in \mathcal{G}^- \),

\[
\int_{[a,1]} f \, d\mu + \int_{[a,1]} \bar{\theta} g \, dt = 0.
\]

Moreover, \( h_1 \) is \( AC_{loc}[a,b) \), so \( h_1(a) \) is defined and finite. By Lemma 2.17, it suffices to show

\[
\lim_{n \to \infty} h_1(d_n) \left( \int_{[d_n,1]} f \, d\mu + \int_{[d_n,1]} \bar{\theta} g \, dt \right) = 0
\]

for some sequence \( \{d_n\} \) approaching \( b \) from the left. But since the endpoint \( b \) is as in Theorem 2.20, the same argument shows that such a sequence exists. \( \Box \)

In the consideration of cases where \( I \) is of the type (T.4) and (T.5), notice that

\[
\int_{[a,b]} e^{2\Re\Theta}1/w \, dt < \infty
\]

in both cases. From this we see that for any polynomial \( p \), \( (e^\Theta p)' \) is \( L^1[a,b] \), since

\[
\int_{[a,b]} |(e^\Theta p)'| \, dx = \int_{[a,b]} e^{\Re\Theta} |\Theta p + p'| \frac{1}{w} \, d\nu_a \leq \left( \int_{[a,b]} e^{2\Re\Theta} \frac{1}{w} \, dt \right)^{1/2} \left( \int_{[a,b]} |\Theta p + p'|^2 \, d\nu_a \right)^{1/2} < \infty.
\]
Thus $e^\Theta p \in AC[a,b]$ and

$$(e^\Theta p)(x) - (e^\Theta p)(y) = \int_y^x (e^\Theta p)'(t)dt$$

for all $x$ in $[a,b]$. To utilize this we consider the direct sum space

$$L^2(e^{-2Re\Theta}d\mu|I) \oplus L^2(e^{-2Re\Theta}wdx|I)$$

and the submanifold $G_\Theta = \{(e^\Theta p) \oplus (e^\Theta p)' : p \text{ a polynomial}\}$ Now notice that $h \oplus k \rightarrow e^\Theta h \oplus e^\Theta k$ is a transformation of $L^2(\mu) \oplus L^2(\nu_a)$ to $L^2(e^{-2Re\Theta}d\mu) \oplus L^2(e^{-2Re\Theta}wdx)$ when localized to $I$, and that $(e^\Theta p)' = e^\Theta(\theta p + p')$. Therefore $h \oplus k \in L^2(d\mu|I) \oplus L^2(wdx|I)$ belongs to $G^-$ if and only if $(e^\Theta h) \oplus (e^\Theta k)$ belongs to $G^-$. We show that, under the assumption that $(e^{-2Re\Theta}d\mu)|I$ is a finite measure, $h \oplus k \in G$ implies $(e^\Theta h) \oplus (e^\Theta k) \in G^-$. 

**Theorem 2.22. (T.4).** Let $I$ be a MILI for $1/w$ which is of type (T.4). Suppose the measure $e^{-2Re\Theta}d\mu|I$ is finite. If $h \oplus k \in G$ and is zero on $[0,1] \setminus I$, then $h \oplus k \in G^-$. 

**Proof:** Here we have $I = (a,b)$ but $J = [a,b]$. We begin by investigating the orthogonal complement of $G_\Theta$ in $L^2(e^{-2Re\Theta}d\mu) \oplus L^2(e^{-2Re\Theta}wdx)$. So let $f \oplus g \in G_\Theta^\perp$. Then for all polynomials $p$,

$$0 = \int_{[a,b]} (e^\Theta p)(x)f(x)e^{-2Re\Theta(x)}d\mu(x) + \int_{[a,b]} (e^\Theta p)'(x)g(x)e^{-2Re\Theta(x)}w(x)dx. \quad (2.22.1)$$

Now note that $(e^\Theta p)'(x) = \int_{(a,x]}(e^\Theta p)'(t)dt + (e^\Theta p)(a)$ and that by assumption $a \in J$ but $a \notin I$ so that $(e^\Theta p)(a) = 0$ (see the proof of Th. 2.10). Thus (2.22.1) becomes

$$0 = \int_{[a,b]} \left( \int_{(a,x]} (e^\Theta p)'(t)dt \right)f(x)e^{-2Re\Theta(x)}d\mu(x) + \int_{[a,b]} (e^\Theta p)'(x)g(x)e^{-2Re\Theta(x)}w(x)dx. \quad (2.22.2)$$

The hypothesis that $(e^{-2Re\Theta}d\mu)|I$ is a finite measure implies that $L^2((e^{-2Re\Theta}d\mu)|I) \subset$
Thus, we may apply Fubini's theorem to the measure \([e^{-2Re\Theta(x)}d\mu(x)]I\times[(dt)|I]\) and the integrand \(\chi_{\{t<z\}}\{t,x\}(e^\Theta p)'(t)f(x)\) in the first integral of (2.22.2), and get

\[
0 = \int_{[a,b]} (e^\Theta p)'(x) \left[ \left( \int_{[x,b]} \overline{f(t)} e^{-2Re\Theta(t)}d\mu(t) \right) e^{2Re\Theta(x)} \frac{1}{w(x)} + g(x) \right] e^{-2Re\Theta(x)} w(x)dx.
\]

(2.22.3)

If \(H(x)\) is the expression in the brackets in (2.22.3), then \(H(x)\) is in \(L^2(e^{-2Re\Theta(x)}w(x)dx)\), and satisfies, for every polynomial \(p\),

\[
0 = \int_{[a,b]} (e^\Theta p)'(x)H(x)e^{-2Re\Theta(x)}w(x)dx.
\]

Thus,

\[
0 = \int_{[a,b]} e^{\Theta(x)}(\theta(x)p(x) + p'(x))H(x)e^{-2Re\Theta(x)}w(x)dx
\]

\[
= \int_{[a,b]} e^{-\Theta(x)}\theta(x) \left( \int_{[a,x]} p'(t)dt \right) H(x)w(x)dx
\]

\[
+ \int_{[a,b]} e^{-\Theta(x)}p'(x)H(x)w(x)dx + p(a) \int_{[a,b]} e^{-\Theta(x)}\theta(x)H(x)w(x)dx.
\]

(2.22.4)

Now, considering polynomials \(p'\) with \(p(a) = 0\) and interchanging the order of integration in the first integral of (2.22.4), we get

\[
0 = \int_{[a,b]} p'(x)\left[ \int_{[x,b]} e^{-\Theta(t)}\theta(t)H(t)w(t)dt + e^{-\Theta(x)}H(x)w(x) \right]dx.
\]

The interchange is justified since \(H \in L^2(e^{-2Re\Theta}wdx)\) and \(e^{\Theta \theta} \in L^2(e^{-2Re\Theta}wdx)\) imply that \(e^{-\Theta \theta}Hw\) is integrable by the Schwarz inequality in the \(L^2(e^{-2Re\Theta}wdx)\) inner product.

But the class of the polynomials \(p'\) with \(p(a) = 0\) is dense in \(L^2(dx|I)\), so we conclude

\[
\int_{[x,b]} e^{-\Theta(t)}\theta(t)H(t)w(t)dt = -e^{-\Theta(x)}H(x)w(x)
\]

(2.22.5)

for \(m - a.e. \ x \in I\). Denoting the lefthand side of (2.22.5) by \(Y\), we see from (2.22.5) that \(Y\) satisfies the differential equation

\[Y' - \theta Y = 0,\]
which has the general solution $Y = ce^\Theta(x)$ (where $c$ is a constant dependent on $H$). But by definition of $Y$ and from (2.22.5),

$$Y = -H(x)e^{-\Theta(x)}w(x).$$

Thus $H(x) = ce^{2\text{Re}\Theta(x)/w(x)}$. In summary, for $f \oplus g \in \mathcal{G}_\Theta^1$, for $m$ a.e. $x \in I$,

$$\int_{[a,b]} \left\{ f(t)e^{-2\text{Re}\Theta(t)}d\mu(t) + g(x)e^{-2\text{Re}\Theta(x)}w(x) \right\} = \left[ c e^{2\text{Re}\Theta(x)} \frac{1}{w(x)} \right][e^{-2\text{Re}\Theta(x)}w(x)] = c$$

(2.22.6)

where $c$ is a constant dependent on the pair $f \oplus g$.

Now, let $h \oplus k \in \hat{\mathcal{G}}$. We want to show $e^{\Theta}h \oplus e^{\Theta}k$ is in $(\mathcal{G}_\Theta)$. That is, for every $f \oplus g \in \mathcal{G}_\Theta^1$,

$$0 = \int_{[a,b]} f(x)(e^{\Theta}h(x))e^{-2\text{Re}\Theta(x)}d\mu(x) + \int_{[a,b]} g(x)(e^{\Theta}k(x))e^{-2\text{Re}\Theta(x)}w(x)dx.$$

By the definition of $\hat{\mathcal{G}}$ in this case, $e^{\Theta}k = (e^{\Theta}h_1)'$. Also $e^{\Theta}h_1$ is $AC(J)$ with $(e^{\Theta}h_1)(a) = 0$, $e^{\Theta}k \in L^1(m|[a,b])$ and

$$e^{\Theta}h_1(x) = \int_{[a,x]} e^{\Theta}k(t)dt.$$

Thus, for $f \oplus g \in (\mathcal{G}_\Theta)^1$,

$$\int_{[a,b]} f(x)(e^{\Theta}h(x))e^{-2\text{Re}\Theta(x)}d\mu(x) + \int_{[a,b]} g(x)(e^{\Theta}k(x))e^{-2\text{Re}\Theta(x)}w(x)dx$$

$$= \int_{[a,b]} f(x) \left( \int_{[a,x]} e^{\Theta}k(t)dt \right) e^{-2\text{Re}\Theta(x)}d\mu(x) + \int_{[a,b]} g(x)(e^{\Theta}k(x))e^{-2\text{Re}\Theta(x)}w(x)dx$$

$$= \int_{[a,b]} (e^{\Theta}k(x)) [\int_{[a,x]} f(t)e^{-2\text{Re}\Theta(t)}d\mu(t) + g(x)e^{-2\text{Re}\Theta(x)}w(x)]dx.$$  

Finally, applying (2.22.6), we see that the last expression becomes

$$\tilde{c} \int_{[a,b]} (e^{\Theta}k)(t)dt = \tilde{c}[(e^{\Theta}h_1)(b) - (e^{\Theta}h_1)(a)] = 0. \quad \square$$
Theorem 2.23 (T.5). Let $I$ be a MILI for $\frac{1}{w}$ which is of type (T.5) and suppose that the measure $(e^{-2Re(\Theta)}d\mu)|I$ is finite. If $h \oplus k \in G$ and is zero on $[0,1] \setminus I$, then $h \oplus k \in G^c$.

Proof: Here we have $I = [a, b)$ and $J = [a, b]$. This proof follows the proof of the previous theorem with some modifications. First of all, note that $(e^\Theta p)(a)$ is well defined but not necessarily zero. In this case, we can take the normalization

$$\Theta(x) = \int_{[a,x]} \theta(t)dt,$$

so we do have $(e^\Theta p)(a) = p(a)$. Thus for $f \oplus g \in (G^c)^1$, (2.22.3) becomes, for all polynomials $p$,

$$0 = p(a) \int_{[a,b]} f(x)e^{-2Re(\Theta(x))}d\mu(x)$$

$$+ \int_{[a,b]} (e^\Theta p)'(x)[(\int_{[x,b]} f(t)e^{-2Re(\Theta(t))}d\mu(t))e^{2Re(\Theta(x))}\frac{1}{w(x)} + g(x))e^{-2Re(\Theta(x))}w(x)dx.$$  

Now, considering polynomials $p'$ with $p(a) = 0$, again a dense set in $L^1(dx|I)$, we get as in the proof of Theorem 2.22

$$\int_{[x,b]} f(t)e^{-2Re(\Theta(t))}d\mu(t) + g(x)e^{-2Re(\Theta(x))}w(x) = c$$  

for $m - a.e. x \in I$ and $f \oplus g \in (G^c)^1$, where the constant $c$ depends on $f \oplus g$.

But considering the polynomial $p \equiv 1$, (2.23.1) yields

$$0 = \int_{[a,b]} f(x)e^{-2Re(\Theta(x))}d\mu(x)$$

$$+ \int_{[a,b]} \theta(x)e^{\Theta(x)}[(f(t)e^{-2Re(\Theta(t))}d\mu(t))e^{2Re(\Theta(x))}\frac{1}{w(x)}$$

$$+ g(x)e^{-2Re(\Theta(x))}w(x)dx.$$ 

So, from (2.22.6)

$$\int_{[a,b]} f(x)e^{-2Re(\Theta(x))}d\mu(x) = -c \int_{[a,b]} \theta(x)e^{\Theta(x)}dx = -c(e^{\Theta(b)} - e^{\Theta(a)}) = c$$  

(2.23.2)
since $e^{\Theta(a)} = 1$, by our normalization and $e^{\Theta(b)} = 0$, because $b \in J$ but $b \notin I$. Now let $h \oplus k \in \hat{G}$. Then we must show for each $f \oplus g \in (G_\Theta)^\perp$,

$$0 = \int_{[a,b]} f(x)(e^{\Theta h})(x)e^{-2\text{Re}\Theta(x)}d\mu(x) + \int_{[a,b]} g(x)(e^{\Theta k})(x)e^{-2\text{Re}\Theta(x)}w(x)dx. \quad (2.23.3)$$

As before, we use the identity

$$(e^{\Theta h_1})(x) = \int_{[a,x]} (e^{\Theta k})(t)dt + (e^{\Theta h_1})(a) = \int_{[a,x]} (e^{\Theta k})(t) + h_1(a),$$

interchange the order of integration, and use (2.23.2), (2.22.6), to see that (2.23.3) reduces to showing

$$0 = \overline{h_1(a)} \int_{[a,b]} f(x)e^{-2\text{Re}\Theta(x)}d\mu(x) = \overline{c} \int_{[a,b]} (e^{\Theta \overline{k}})(x)dx.$$ 

But the right hand side becomes

$$\overline{ch_1(a)} + \overline{c}((e^{\Theta \overline{h}_1})(b)) - (e^{\Theta \overline{h}_1})(a) = 0,$$

as needed. $\Box$

The following table summarizes our results. Most of the table is self-explanatory. The fifth, sixth and seventh column give specific choices of $\mu$, $w$, and $\theta$ which satisfy all the conditions required by the first, second, and fourth columns of the same row. This shows that the special cases corresponding to each of the eight rows are all nonvacuous. Note that in all cases $G^- \subset \hat{G}$ by Theorem 2.10. In five of eight cases, as indicated in the last column we have characterized $G^-$ as exactly equal to $\hat{G}$. In the table the interval $I$ with endpoints 0 and 1 is assumed to be an MILI for $\frac{1}{w}$, and $J$ is the associated interval with the same endpoints as in Definition 2.5.
<table>
<thead>
<tr>
<th>Type</th>
<th>I</th>
<th>J</th>
<th>Added assumption</th>
<th>$\mu$</th>
<th>$w$</th>
<th>$\theta$</th>
<th>$e^{2Re\Theta \frac{1}{w}}$</th>
<th>$\hat{G} = G$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[0,1]</td>
<td>[0,1]</td>
<td>None</td>
<td>arbitrary</td>
<td>1</td>
<td>1</td>
<td>$e^{2x}$</td>
<td>yes</td>
</tr>
<tr>
<td>2</td>
<td>(0,1)</td>
<td>(0,1)</td>
<td>None</td>
<td>arbitrary</td>
<td>$x(1-x)$</td>
<td>1</td>
<td>$e^{2x}$</td>
<td>yes</td>
</tr>
<tr>
<td>3</td>
<td>[0,1]</td>
<td>[0,1]</td>
<td>None</td>
<td>arbitrary</td>
<td>$(1-x)$</td>
<td>1</td>
<td>$e^{2x}$</td>
<td>yes</td>
</tr>
<tr>
<td>4.a</td>
<td>(0,1)</td>
<td>[0,1]</td>
<td>the measure $e^{-2Re\Theta d\mu}$ is finite</td>
<td>$wdx$</td>
<td>$x^2(1-x)^2$</td>
<td>$\frac{1-2x}{\pi(1-x)}$</td>
<td>1</td>
<td>yes</td>
</tr>
<tr>
<td>4.b</td>
<td>(0,1)</td>
<td>[0,1]</td>
<td>the measure $e^{-2Re\Theta d\mu}$ is infinite</td>
<td>$dx$</td>
<td>$x^2(1-x)^2$</td>
<td>$\frac{1-2x}{\pi(1-x)}$</td>
<td>1</td>
<td>open</td>
</tr>
<tr>
<td>5.a</td>
<td>[0,1]</td>
<td>[0,1]</td>
<td>the measure $e^{-2Re\Theta d\mu}$ is finite</td>
<td>$wdx$</td>
<td>$(1-x)^2$</td>
<td>$\frac{-1}{1-x}$</td>
<td>1</td>
<td>yes</td>
</tr>
<tr>
<td>5.b</td>
<td>[0,1]</td>
<td>[0,1]</td>
<td>the measure $e^{-2Re\Theta d\mu}$ is infinite</td>
<td>$dx$</td>
<td>$(1-x)^2$</td>
<td>$\frac{-1}{1-x}$</td>
<td>1</td>
<td>open</td>
</tr>
<tr>
<td>6</td>
<td>(0,1)</td>
<td>(0,1)</td>
<td>None</td>
<td>arbitrary</td>
<td>$x(1-x)^2$</td>
<td>$\frac{-1}{1-x}$</td>
<td>$\frac{1}{x}$</td>
<td>open</td>
</tr>
</tbody>
</table>
As an illustration of the general ideas developed so far, we settle two questions: 1) when is $D$ a closable operator, and 2) when is $D$ the "most unclosable", i.e. when is $\mathcal{G}$ dense in $L^2(\mu) \oplus L^2(\nu)$. The answers are more subtle than one might at first expect.

2a. Closability of $D$

The following theorem gives a complete characterization (in terms of the data $\theta$, $\mu$, $\nu$) of when $D = \theta + \frac{d}{dx} : L^2(\mu) \to L^2(\nu)$ is closable.

**Theorem 2.24.** $D = \theta + \frac{d}{dx} : L^2(\mu) \to L^2(\nu)$ is closable if and only if $\nu << \mu$, $\nu(K) = 0$ (where $K$ is given by (2.11.1)) and the complement of $\text{supp} \mu$ contains no intervals in a MILI for $\frac{1}{w}$.

**Proof:** Suppose $\nu << \mu$, $\nu(K) = 0$ and the complement of $\text{supp} \mu$ contains no intervals in a MILI for $\frac{1}{w}$. We want to show that if $0 \otimes k \in \mathcal{G}^-$, then $k = 0 \nu - a.e..$ Since $\nu(K) = 0$, it suffices to show $k|I = 0$ for each MILI for $\frac{1}{w}$. Since $\mathcal{G}^- \subset \hat{\mathcal{G}}$ (Theorem 2.10), we know $e^{\Theta} k \in L^1_{\text{loc}}(J)$ and there is a $h_1 \in AC_{\text{loc}}(I)$ with $h_1 = 0 \mu - a.e.$ so that

$$e^{\Theta(x)} h_1(x) - e^{\Theta(y)} h_1(y) = \int_y^x (e^{\Theta} k)(t) dt.$$ 

Now $\text{supp} \mu \cap I$ is dense in $I$ by assumption, so $h_1 \equiv 0$ on $I$ (since $h_1$ is continuous). Thus

$$\int_y^x (e^{\Theta} k)(t) dt = 0$$

for all $x, y \in I$. It follows that $(e^{\Theta} k)(t) = 0 \mu - a.e.$ on $I$ and hence so is $k(t)$. Finally since we assume $\nu << \mu$, we have $k = 0 \nu - a.e.$ on $I$. We conclude that $D$ is closable.

Conversely, if $\nu$ is not absolutely continuous with respect to $\mu$, we have $D$ is not closable by Corollary 2.9. If $\nu << \mu$ and $\nu(K) > 0$ consider the non-zero element of $L^2(\mu) \oplus L^2(\nu)$, $0 \otimes \chi_K$. This element is orthogonal to $\mathcal{G}^\perp$ by Lemma 2.13, and hence is in $\mathcal{G}^-$, so $D$ is not closable in this case. Finally, suppose $\nu << \mu$, $\nu(K) = 0$ but there is an interval $A = (a', b')$ in $I \setminus \text{supp} \mu$ where $I$ with endpoints $a, b$ is some MILI for $1/w$. 


Choose \(a'' , b''\) with \(a' < a'' < b'' < b'.\) Set

\[ k(t) = e^{-\Phi(t)} [X_{[a'', \xi]}(t) - X_{[\xi, b'']}](t) \]

where \(\xi = a'' + \frac{1}{2}(b'' - a'').\) Then

\[ \int_{a''}^{b''} e^{\Phi(t)} k(t) dt = 0 \]

and \(\text{supp} \ k \subset [a'', b''].\) If we define \(h_1\) on \([a, b]\) by

\[ h_1(x) = e^{-\Phi(x)} \int_{a''}^{x} e^{\Phi(t)} k(t) dt \]

then \(h_1\) is absolutely continuously on \([a, b]\) with support contained in \([a'', b'']\). Thus in particular, \(h_1 = 0\) \(\mu - \text{a.e.}\) and \(0 \oplus k\) has compact support inside \(I\). By Corollary 2.18 we see that \(0 \oplus k \in \mathcal{G}^-\) and hence \(D\) is not closable in this case either. \(\square\)

We will need the following Corollary of Theorem 2.24 for our operator theory application in Chapter 3.

**Corollary 2.24.1.** Suppose \(D = \theta + \frac{d}{dx} : L^2(\mu) \to L^2(\nu)\) is closable. Then the support of \(\nu\) is a regularly closed subset of \(\mathbb{R}\) (i.e. \(\text{supp} \nu\) is the closure of its interior).

**Proof:** By Theorem 2.24, if \(D\) is closable then \(\nu \ll \mu\) and \(\nu(K) = 0\), i.e. \(\nu\) is carried by \(U = \cup \{\text{int } I : I \text{ is a MILI for } \frac{1}{w}\}\). Thus \(\text{supp} \nu\) is the closure of the open set \(U\). Hence \(\text{supp} \nu \supset (\text{int } \text{supp} \nu)^- \supset U^- = \text{supp} \nu\) and thus \(\text{supp} \nu\) is regularly closed. \(\square\)

### 2b. Density of \(G\)

We finally consider the most degenerate case, that is when \(\mathcal{G}^- = L^2(\mu) \oplus L^2(\nu)\). Before we proceed, though, we give another localization theorem about elements in \(\mathcal{G}^-\).

**Lemma 2.25.** Suppose \(I\) is an MILI for \(1/w\) and \(f \oplus g \in L^2(\mu) \oplus L^2(\nu)\) has support in \(I\). Then \(f \oplus g \in \mathcal{G}^\perp\) if and only if

\[ gd\nu = gd\nu_a = gwdm \text{ on } I \quad (2.25.1) \]
\[
\int_I f \, d\mu + \int_I \bar{g} \, d\omega \, dm = 0 \quad (2.25.2)
\]

For \( m \)-a.e. \( x \) in \( I \),
\[
g(x)w(x) = -\int_{(x,1] \cap I} f(t) \, d\mu(t) - \int_{(x,1] \cap I} \bar{\theta}(t) g(t) w(t) \, dt.
\]

**Proof:** It is easy to check that if \( f \oplus g \) has support in \( I \) and satisfies (2.25.1)-(2.25.3) above, then \( f \oplus g \) satisfies (2.7.1)-(2.7.3) in Lemma 2.7 and hence is in \( G^\perp \). Conversely, suppose that \( f \oplus g \) has support in \( I \) and is in \( G^\perp \). Then condition (2.25.1) is an immediate consequence of (2.7.1). To discuss condition (2.25.2), suppose \( a < b \) are the endpoints of \( I \). If \( a \in I \), by maximality of \( I \) necessarily \( a \) is a \( NLI(1/w) \) so by Lemma 2.11
\[
\int_{[a,1]} f \, d\mu + \int_{[a,1]} \bar{g} \, d\omega \, dm = 0. \quad (2.25.4)
\]

On the other hand, if \( a \notin I \), then again by maximality of \( I \) a is \( NRI(1/w) \), so again by Lemma 2.11
\[
\int_{[a,1]} f \, d\mu + \int_{[a,1]} \bar{g} \, d\omega \, dm = 0. \quad (2.25.5)
\]

Similarly, if \( b \in I \), then \( b \) is \( NRI(1/w) \) so
\[
\int_{(b,1]} f \, d\mu + \int_{(b,1]} \bar{g} \, d\omega \, dm = 0 \quad (2.25.6)
\]
while if \( b \notin I \), then \( b \) is \( NLI(1/w) \) so
\[
\int_{[b,1]} f \, d\mu + \int_{[b,1]} \bar{g} \, d\omega \, dm = 0. \quad (2.25.7)
\]

When we appropriately subtract (2.25.6) or (2.25.7) from (2.25.4) or (2.25.5), we get precisely (2.25.2). Finally, (2.25.3) follows upon combining (2.25.6) or (2.25.7) with (2.7.3).

\[\square\]

To characterize the conditions for which \( G^- = L^2(\mu) \oplus L^2(\nu) \), we proceed by a series of lemmas. We begin by showing \( \mu|I \) must be zero or a single point mass for each MILI for \( \frac{1}{w} \). Throughout the discussion we utilize the local formulation given by Lemma 2.25 and the equivalence that \( G^- = L^2(\mu) \oplus L^2(\nu) \) if and only if \( G^\perp = (0) \).
**Lemma 2.26.** If for some MILI $I$ for $\frac{1}{w}$, $\mu|I$ is not zero or a single point mass, then $\mathcal{G}^+ \neq (0)$ (and thus $\mathcal{G}^- \neq L^2(\mu) \oplus L^2(\nu)$).

**Proof:** Let $I$ be a MILI for $\frac{1}{w}$ with endpoints $a$ and $b$, $a < b$, and suppose $\mu|I$ is non-zero and not a single point mass. For simplicity we first illustrate the argument for the special case where $\theta = 0$ $\nu$-a.e. on $I$. Since $\mu|I$ is more than a single point mass, there is a set $S \subset \text{sppt}(\mu|I)$, so that $0 < \mu(S) < \mu(\text{sppt}(\mu|I))$. Let

$$f(x) = \begin{cases} \frac{-\mu(I \setminus S)}{\mu(S)} \chi_S(x) + \chi_{I \setminus S}(x), & x \in I \\ 0, & x \notin I. \end{cases}$$

Then $f \in L^2(\mu)$ and is nonzero. Set

$$g(x) = \begin{cases} -\frac{1}{w}(x) \int_{[x,b]} f(t) d\mu(t), & w(x) \neq 0 \\ 0, & \text{elsewhere}. \end{cases}$$

Then $g$ is in $L^2(\nu)$ and satisfies (2.25.1) and (2.25.3) of Lemma 2.25. Finally we see (2.25.2) is satisfied, since

$$\int_{[a,b]} f(t) d\mu(t) = \frac{-\mu(I \setminus S)}{\mu(S)} \int_{[a,b]} \chi_S d\mu + \int_{[a,b]} \chi_{I \setminus S} d\mu = \frac{-\mu(I \setminus S)}{\mu(S)} \mu(S) + \mu(I \setminus S) = 0.$$

Now we consider the case $\theta(x) \neq 0$ on a set of positive $\nu$-measure in $I$. Let $[\alpha, \beta]$ be a compact subinterval of $I$, so that $\mu|[\alpha, \beta]$ is more than a single point mass. Since $1/w \in L^1_{\text{loc}}(I)$, we have $1/w \in L^1[\alpha, \beta]$, and thus by observation (2.1), $\theta \in L^1[\alpha, \beta]$. Further if $\Theta(x) \equiv -\int_{[x,\beta]} \theta(t) dt$ for $x \in [\alpha, \beta]$, then $\Theta \in AC[\alpha, \beta]$ and $\Theta' = \theta$ $m-a.e.$ on $[\alpha, \beta]$.

Consider the equation ($\theta$ times equation (2.25.3) of Lemma 2.25) for $m-a.e. x \in [\alpha, \beta]$,

$$\overline{\theta(x)}g(x)w(x) = -\overline{\theta(x)} \int_{[x,\beta]} f(t) d\mu(t) - \overline{\theta(x)} \int_{[x,\beta]} \theta(t) g(t) w(t) dt. \quad (2.26.1)$$

If we set $y = \int_{[x,\beta]} \theta(t) g(t) w(t) dt$ for $x \in [\alpha, \beta]$, then we notice that $y' = -\overline{\theta(x)} g(x)w(x)$ $m-a.e.$ on $[\alpha, \beta]$. Thus (2.26.1) becomes the initial value problem on $[\alpha, \beta]$

$$\begin{cases} y' - \overline{\theta(x)} \int_{[x,\beta]} f(t) d\mu(t) \\ y(\beta) = 0. \end{cases} \quad (2.26.2)$$
Now (2.26.2) has absolutely continuous solution on \([\alpha, \beta]\)

\[
y = e^{\Theta(x)} \int_{[\alpha, \beta]} \overline{\theta(t)} e^{-\Theta(t)} \left( \int_{(t, \beta]} f(s) \, d\mu(s) \right) \, dt.
\] (2.26.3)

Choosing an \(f \in L^2(\mu)\) with \(f = 0\) off \([\alpha, \beta]\), let \(y\) be defined by (2.26.3) and set for \(\nu_\alpha\)-a.e. \(x\) in \(I\),

\[
g(x)w(x) = \begin{cases} -\int_{[\alpha, \beta]} f(t) \, d\mu(t) - y(x), & x \in [\alpha, \beta] \\ 0, & \text{otherwise} \end{cases}
\] (2.26.4)

Then for such an \(f\) and \(g\), (2.25.1) is satisfied. The verification of (2.25.3) requires

\[
- \int_{[\alpha, \beta]} f(t) \, d\mu(t) - y(x) = - \int_{[\alpha, \beta]} f(t) \, d\mu(t) - \int_{[\alpha, \beta]} \overline{\theta(t)} \left[ - \int_{(t, \beta]} f(s) \, d\mu(s) - y(t) \right] \, dt
\]

or more simply

\[
y(x) = - \int_{[\alpha, \beta]} \overline{\theta(t)} \left[ \int_{(t, \beta]} f(s) \, d\mu(s) + y(t) \right] \, dt.
\]

To verify this note that both sides are 0 at \(\beta\) so it suffices to check that the derivatives of both sides agree:

\[
y'(x) = \overline{\theta(x)} \int_{[\alpha, \beta]} f(s) \, d\mu(s) + \overline{\theta(x)} y(x).
\]

This in turn follows from the defining differential equation (2.26.2) for \(y\), and hence (2.25.3) holds for \(x \in [\alpha, \beta]\). Moreover since \(f \in L^1(\mu|[\alpha, \beta])\) and \(y \in C[\alpha, \beta]\), it is clear that

\[
m = \max \{|g(x)w(x)| : x \in [\alpha, \beta]\} < \infty.
\]

Thus

\[
\int_{[\alpha, \beta]} |g|^2 \, w \, dx = \int_{[\alpha, \beta]} |g| \, |w|^2 \frac{1}{w} \, dx \leq m^2 \int_{[\alpha, \beta]} 1/\nu \, dx < \infty
\]

and \(g \in L^2(\nu)\).

Our proof is then complete if we can show there is a nonzero \(f \in L^2(\mu)\), \(f = 0\) off \([\alpha, \beta]\), so that with \(f\), and \(g\) defined by (2.26.4), (2.25.2) is satisfied. That is,

\[
\int_{[\alpha, \beta]} \overline{\theta(t)} g(t) w(t) \, dt = - \int_{[\alpha, \beta]} f(t) \, d\mu(t).
\]
But since $f$ is assumed to be zero off $[\alpha, \beta]$, and $gw$, as defined, is also, this reduces to

$$\int_{[\alpha, \beta]} \overline{\theta(t)} g(t) w(t) dt = - \int_{[\alpha, \beta]} f(t) d\mu(t).$$

This then will verify (2.25.2) for $x < \alpha$ as well. Moreover, with $gw$ defined by (2.26.4), we have

$$y(x) = \int_{[\alpha, \beta]} \overline{\theta(t)} g(t) w(t) dt.$$

Thus, showing (2.25.2) is reduced to finding a nonzero $f \in L^2(\mu)$, zero off $[\alpha, \beta]$ and for which

$$y(\alpha) = - \int_{[\alpha, \beta]} f(t) d\mu(t). \quad (2.26.5)$$

or, from (2.26.3)

$$y(\alpha) = e^{\Theta(\alpha)} \int_{[\alpha, \beta]} \overline{\theta(t)} e^{-\Theta(t)} \left( \int_{[\alpha, \beta]} f(s) d\mu(s) \right) dt = - \int_{[\alpha, \beta]} f(t) d\mu(t),$$

or

$$\int_{[\alpha, \beta]} \overline{\theta(t)} e^{-\Theta(t)} \left( \int_{[\alpha, \beta]} f(s) d\mu(s) \right) dt = - e^{-\Theta(\alpha)} \int_{[\alpha, \beta]} f(t) d\mu(s).$$

Now, interchanging the order of integration on the left gives

$$\int_{[\alpha, \beta]} f(s) \left[ \int_{[\alpha, \beta]} \overline{\theta(t)} e^{-\Theta(t)} dt \right] d\mu(s) = - e^{-\Theta(\alpha)} \int_{[\alpha, \beta]} f(t) d\mu(s),$$

or

$$\int_{[\alpha, \beta]} f(s) \left[ \int_{[\alpha, \beta]} \overline{\theta(t)} e^{-\Theta(t)} dt \right] d\mu(s) + e^{-\Theta(\alpha)} d\mu(s) = 0.$$

But notice

$$\int_{[\alpha, \beta]} \overline{\theta(t)} e^{-\Theta(t)} dt = - \int_{[\alpha, \beta]} \left( e^{-\Theta(t)} \right)' dt = e^{-\Theta(\alpha)} - e^{-\Theta(s)}.$$

So let $F(s) = 2e^{-\Theta(\alpha)} - e^{-\Theta(s)}$, noting that $F \in AC[\alpha, \beta] \subset L^2(\mu|\alpha, \beta])$, and defining the continuous linear functional $L_F : L^2(\mu|\alpha, \beta]) \to \mathbb{C}$, by

$$L_F(f) = \int_{[\alpha, \beta]} f \overline{F} d\mu, \quad f \in L^2(\mu|\alpha, \beta]).$$
Then (2.26.5) is satisfied if there exists a nonzero $f$ in $\ker(L_F)$. But since $\mu|[\alpha, \beta]$ is more than a single point mass, $\text{Dim}(L^2[\alpha, \beta]) \geq 2$, and therefore $\ker(L_F)$ is nontrivial.

Choose a nonzero $\tilde{f} \in \ker(L_F)$, and let

$$f(x) = \begin{cases} \tilde{f}(x), & x \in [\alpha, \beta] \\ 0, & \text{elsewhere.} \end{cases}$$

Define $y$, using this $f \in L^2(\mu)$, via (2.26.3), and $g$ by (2.26.4), and we get $f \oplus g \in L^2(\mu) \oplus L^2(\nu)$, is nonzero, zero off $I$, and on $I$ satisfies Lemma 2.25. Thus $f \oplus g \in G^\perp$, so $G^\perp$ contains a nontrivial element. □

**Lemma 2.27.** Suppose $\theta = 0 \nu$ – a.e. on a MILI $I$ for $\frac{1}{\nu}$. Then, if $\mu|I$ is zero or a single point mass, $G^-|I = (L^2(\mu) \oplus L^2(\nu))|I$.

**Proof:** First suppose $\mu|I = 0$. Then $L^2(\mu)|I = (0)$. Let $a < b$ be the endpoints of $I$. Suppose $g \in L^2(\nu)$ is such that $0 \oplus g \in G^\perp$. Then for $m$ – a.e. $x$ in $I$,

$$g(x)w(x) = -\int_{[x,b]} \overline{\theta(t)}g(t)w(t)dt = 0,$$

by (2.25.3). Thus $gw = 0 \nu$ – a.e. But by (2.25.1), this implies $g = 0 \nu$ – a.e. Thus $G^\perp|I = 0$. Now suppose $\mu|I$ is a single point mass at $x_0 \in I$. Let $f \oplus g \in G^\perp$. Then (2.25.2) gives

$$-f(x_0)\mu(\{x_0\}) = \int_{[a,b]} \overline{\theta(t)}g(t)w(t)dt = 0.$$

Thus $f = 0 \mu$ – a.e. on $I$. It therefore follows from (2.25.3) that for $m$ – a.e. $x \in I$

$$g(x)w(x) = -\int_{[x,b]} \overline{\theta(t)}g(t)w(t)dt = 0.$$

Thus, with (2.25.1), this gives $g = 0 \nu$ – a.e. on $I$. We conclude $G^\perp|I = 0$. □

**Proposition 2.28.** Suppose $\theta = 0 \nu$ – a.e. on $\cup\{I : I$ is a MILI for $\frac{1}{\nu}\}$. Then $G^- = L^2(\mu) \oplus L^2(\nu)$ if and only if $\mu|I$ is no more than a single point mass for each MILI $I$ for $\frac{1}{\nu}$.

**Proof:** Combine Lemmas 2.26 and 2.27. □
It is left to consider when \( \mu \) is at most a point mass on a MILI \( I \) for \( \frac{1}{w} \) for a general \( \theta \). We begin with a result for when \( \mu|I = 0 \).

**Proposition 2.29.** Let \( I \) be a MILI for \( 1/w \) such that \( \mu|I = 0 \). Then \( G^{-1}|I = (L^2(\mu) \oplus L^2(\nu))|I \) if and only if \( J \setminus I \) is at most one point, where \( J \) is the interval associated with \( I \) as in Definition 2.6.

**Proof:** If \( I \) has endpoints \( a \) and \( b \) then by Lemma 2.25 \( 0 \oplus g \in G^\perp \) implies

\[
\int_{[a,b]} \overline{\partial} g w dt = 0 \tag{2.29.1}
\]

and for \( m \)-a.e. \( x \in I \)

\[
g(x)w(x) = -\int_{[x,b]} \overline{\partial} g w dt. \tag{2.29.2}
\]

Let \( Y(x) = \int_{[x,b]} \overline{\partial} g w dm \in AC[a,b] \). Then \( Y' = -\overline{\partial} g w \) \( m \)-a.e. on \( I \). Define \( \overline{\Theta}(x) = \int_{x_0}^{x} \overline{\theta}(t) dt \) for some \( x_0 \in I \). Then \( \overline{\Theta} \in AC_{loc}(I) \) since \( \theta \in L^1_{loc}(I) \). Consider the equation, from (2.29.2),

\[
\overline{\Theta}(x)g(x)w(x) = -\overline{\Theta}(x) \int_{[x,b]} \overline{\partial} g w dt,
\]

or

\[
Y' - \overline{\partial} Y = 0,
\]

with \( Y(a) = Y(b) = 0 \) (from (2.29.1) and the definition of \( Y \)). If we solve this first order equation on compact intervals containing \( x_0 \), we get \( Y(x) = de^{\overline{\Theta}(x)} \) for some \( d \). If \( I \) contains the endpoint \( a \), \( \overline{\theta} \) is integrable on \([a,x_0]\) and the boundary condition \( Y(a) = 0 \) together with the uniqueness theorem for differential equations (see e.g. Problem \#1f, p. 97 of [CL]) forces \( d = 0 \). By a similar argument, \( d = 0 \) if the endpoint \( b \) is in \( I \). Thus if \( I \) contains either endpoint \( a \) or \( b \), then \( g = 0 \nu_a - a.e. \) on \( I \).

If \( I = (a,b) \) and \( J = [a,b) \), then \( \int_{b-a} e^{2Re\Theta} \frac{1}{w} dm = \infty \). But since

\[
\int |g|^2 w dm = \int |gw|^2 \frac{1}{w} dm = \int |Y|^2 \frac{1}{w} dm = |d|^2 \int e^{2Re\Theta} \frac{1}{w} dm
\]
and $g$ by assumption is in $L^2(\nu)$, we must have $d = 0$. A similar argument shows $d$ must be zero if $I = (a, b)$ and $J = (a, b]$.

Conversely if $I = (a, b)$ and $J = [a, b]$, then $\Theta$ is such that while $\int_a^{a+\delta} \frac{1}{w} dx = \infty$, we have $\int_a^{a+\delta} e^{2Re\Theta \frac{1}{w}} dx < \infty$. Thus $\lim_{x \to a^+} \Theta(x) = -\infty$. Similarly, $\lim_{x \to b^-} \Theta(x) = -\infty$. Therefore $Y = de^{\Theta(x)}$ satisfies $Y' - \bar{\Theta} Y = 0$ and $Y(a) = Y(b) = 0$. Hence we can define $g$, nonzero in $L^2(\nu)$, via

$$g(x) = \begin{cases} -\frac{1}{w(x)} Y(x) & \text{if } x \in I, \ w(x) \neq 0 \\ 0, & \text{elsewhere} \end{cases}$$

and get $0 \oplus g \in \mathcal{G}^\perp$. Thus $\mathcal{G}^\perp \neq (0)$. □

The last case to consider is that where $\mu$ consists of a point mass on $I$.

**Proposition 2.30.** Let $I$ be a MILI for $\frac{1}{w}$ such that $\mu|I$ is a single point mass. Then $\mathcal{G}^-|I = (L^2(\mu) \oplus L^2(\nu))|I$ if and only if $J = I$.

**Proof:** Let $I$ be a MILI for $\frac{1}{w}$ with endpoints $a$ and $b$, such that $\mu|I$ is a single point mass at $x = x_0$. Define $\Theta(x) = \int_{x_0}^x \Theta(t) dt$. Then $\Theta(x) \in AC_{loc}(I)$ and $\Theta(x_0) = 0$. Consider (2.25.3)

$$g(x)w(x) = -\int_{(x,b)} \bar{\Theta(t)} g(t) w(t) dt - \int_{(x,b)} f(t) d\mu(t) = -\int_{(x,b)} \bar{\Theta} gw dm - f(x_0) \mu(\{x_0\}) \chi_{[a,x_0]}(x).$$

Multiplying this equation by $\theta(x)$ and considering $Y = \int_{[x,b]} \bar{\Theta} gw dm \in AC[a,b]$, we get

$$Y' - \bar{\Theta} Y = f(x_0) \mu(\{x_0\}) \chi_{[a,x_0]}(x) \quad (2.30.1)$$

with boundary conditions $Y(a) = f(x_0) \mu(\{x_0\})$ from (2.25.2), and $Y(b) = 0$ by definition of $Y$. The idea is that $(f \oplus g)|I \in \mathcal{G}^\perp|I$ if and only if there is a $Y \in AC[a,b]$ satisfying (2.30.1) with the boundary conditions, and with

$$g(x)w(x) = -Y(x) - f(x_0) \mu(\{x_0\}) \chi_{[a,x_0]}(x)$$

for $m$-a.e. $x \in I$. 


Now all solutions to the differential equation (2.30.1) (with boundary conditions ignored for the moment) have the form

\[ Y(x) = -ce^{\Theta(x)} \int_{x_0}^{x} e^{-\Theta(t)} \theta(t) \chi_{[x_0, x]}(t) dt + de^{\Theta(x)} \]

where \( c = -f(x_0)\mu(\{x_0\}) \). To fit the boundary conditions we must choose \( d \) so that \( Y(a) = c \) and \( Y(b) = 0 \). Using that \( (e^{-\Theta(t)})' = -\theta(t)e^{-\Theta(t)} \) and \( e^{\Theta(x_0)} = 1 \), we then have

\[ Y(x) = ce^{\Theta(x)} \int_{x_0}^{x} (e^{-\Theta(t)})' dt + de^{\Theta(x)} = ce^{\Theta(x)}(e^{-\Theta(x)} - e^{-\Theta(x_0)}) + de^{\Theta(x)} = (d-c)e^{\Theta(x)} + c, \]

for \( x < x_0 \). Further for \( x > x_0 \), \( Y(x) = de^{\Theta(x)} \), and \( Y(x_0) = d \) by either formula. Thus

\[ Y(x) = \begin{cases} (d-c)e^{\Theta(x)} + c & x \leq x_0 \\ de^{\Theta(x)} & x > x_0. \end{cases} \tag{2.30.2} \]

Notice that from (2.30.2)

i) \( Y(a) = c \) implies either \( d = c \) or \( e^{\Theta(a)} = 0 \), that is, \( d = c \) or

\[ \lim_{t \to a^+} \Theta(t) = -\infty \tag{2.30.3} \]

ii) \( Y(b) = 0 \) implies \( d = 0 \) or \( \lim_{t \to b^-} \Theta(t) = -\infty \).

Now by iii) of Lemma 2.25, \( g(x)w(x) = cX_{[a, x_0]}(x) - Y(x) \), or by (2.30.2)

\[ g(x)w(x) = \begin{cases} (c-d)e^{\Theta(x)} & x \leq x_0 \\ -de^{\Theta(x)} & x > x_0. \end{cases} \tag{2.30.4} \]

Considering that \( g \in L^2(\nu) \), we have

\[ \int_{[a, b]} |g|^2 w dx = \int |gw|^2 \frac{1}{w} dx = \int_{[a, x_0]} |d - c|e^{2Re\Theta} \frac{1}{w} dx + \int_{[x_0, b]} |d|e^{2Re\Theta} \frac{1}{w} dx. \]

Since, \( g \in L^2(\nu) \), we must have

i) \( d = c \) or \( \int_a^{a+\delta} e^{2Re\Theta} \frac{1}{w} dm < \infty (a \in J) \),

and \( \tag{2.30.5} \)

ii) \( d = 0 \), or \( \int e^{2Re\Theta} \frac{1}{w} dm < \infty (b \in J) \).
We finish by considering all cases for $I$ and $J$:

1) $I = [a, b] = J$ : we have $\Theta \in AC[a, b]$, thus both $e^{\Theta(a)}$ and $e^{\Theta(b)}$ are not zero. So by (2.30.3) $d = c$ and $d = 0$. Now $0 = c = -f(x_0)\mu(\{x_0\})$ implies $f = 0 \mu$ - a.e. on $I$, and $c = d = 0$ implies $g = 0 \nu_a$ - a.e. by (2.30.4). Thus $G^\perp|I = (0)$.

2) $I = (a, b) = J$ : Here $a, b \notin J$, so $\int_a^{a+\delta} e^{2Re\Theta} \frac{1}{w} \, dm = \infty$ so by (2.30.5) we have $c = d = 0$. Thus, as above, $G^\perp|I = (0)$.

3) $I = (a, b) = J$ : Since $b \in I$, $\theta$ is left integrable at $b$, so $e^{\Theta(b)} \neq 0$. So $d = 0$ by (2.30.3). But $a \notin J$ implies $\int_a^{a+\delta} e^{2Re\Theta} \frac{1}{w} \, dm = \infty$, so by (2.30.5) $c = d = 0$. Thus $G^\perp|I = (0)$.

4) $I = [a, b] = J$ : Here $e^{\Theta(a)} \neq 0$, so $d = c$ by (2.30.3). But $b \notin J$, so $d = 0$ by (2.30.5). Thus $G^\perp|I = (0)$.

5) $I = (a, b), J = [a, b]$ : Since $a \notin J$, $d = 0$ by (2.30.5), but there is no restriction on $c$. So we can let $f(x) \equiv -c/\mu(\{x_0\}) \neq 0$ on $I$, and define $g \nu_a$ - a.e. on $I$ by (2.30.4). Since $\int_a^{a+\delta} e^{2Re\Theta} \frac{1}{w} \, dm < \infty$ but $\int_a^{a+\delta} \frac{1}{w} \, dm = \infty$, we must have $\lim_{x \to a^+} e^{\Theta(x)} = 0$, that is $Y(a) = c$. Of course $Y(b) = 0$ since $d = 0$. We conclude that with $f$ and $g$ so defined $f \oplus g \in G^\perp|I$ by Lemma 2.25. Hence $G^\perp \neq (0)$.

6) $I = (a, b), J = (a, b]$ : Here $a \notin J$ implies $d = c$, but $d$ can be chosen freely. So by the same argument as above, a nonzero $f \oplus g \in G^\perp|I$ can be constructed.

7) $I = (a, b), J = [a, b]$ : Here both $c$ and $d$ are unrestricted, so $G^\perp|I \neq (0)$.

8) $I = [a, b], J = [a, b]$ : Since $a \in I$, $e^{\Theta(a)} \neq 0$, thus $d = c$. But $d$ can be anything. So $G^\perp|I \neq (0)$.

9) $I = (a, b), J = [a, b]$ : Here $e^{\Theta(b)} \neq 0$, thus $d = 0$, but $c$ is not restricted. Thus $G^\perp|I \neq (0)$.

This completes the proof of Proposition 2.30. □

We finally summarize results 2.26 through 2.30 in the following:
Theorem 2.31. \( G^- = L^2(\mu) \oplus L^2(\nu) \) if and only if for each MILI \( I \) for \( \frac{1}{w} \) either

\[
\mu|I \text{ is a point mass and } J = I, \tag{2.31.1}
\]
or

\[
\mu|I = 0 \text{ and } J \setminus I \text{ is at most one point.} \tag{2.31.2}
\]

We finish this discussion by showing that \( G^- = \hat{\mathcal{G}} \) in the case where \( \mu|I = \delta_{x_0} \), a point mass at \( x = x_0 \) in \( I \), and thus substantiating our conjecture that \( G^- = \hat{\mathcal{G}} \) for this case.

Proposition 2.32. If \( \mu|I \) consists of a single point mass at \( x = x_0 \) in \( I \) for a MILI \( I \) for \( 1/w \), then \( \hat{\mathcal{G}}|I = G^-|I \).

Proof: Without loss of generality we assume that \( \mu|I \) equals the unit point mass measure \( \delta_{x_0} \) supported on \( \{x_0\} \subset I \). For this simple \( \mu \) the measure \( e^{-2Re\Theta}d\mu \) is certainly finite. Thus, by the results summarized in Table 2.1, we already know \( \hat{\mathcal{G}}|I = G^-|I \) when \((I, J)\) is of types 1-5. There remains only Type 6 where \( I \) is open and the set \( J \setminus I \) consists of one point. We proceed by showing \( \hat{\mathcal{G}}^\perp|I = G^\perp|I \) in this case as well.

First of all, from Definition 2.6 for \( \hat{\mathcal{G}} \), \( h \oplus k \in \hat{\mathcal{G}}|I \) where \( I \) is a MILI for \( \frac{1}{w} \) with endpoints \( a \) and \( b \), means \( h = h(x_0) \mu - a.e. \) on \( I \) and for some \( h_1 \in AC_{loc}(I) \), \( h_1(x_0) = h(x_0) \),

\[
e^{\Theta(x)}h_1(x) - e^{\Theta(y)}h_1(y) = \int_y^x (e^{\Theta k})(t)dt
\]

for \( m-a.e. \) \( x \in I \), where we let \( \Theta(x) = \int_{x_0}^x \theta(t)dt \). Letting \( y = x_0 \), and noting \( e^{\Theta(x_0)} = 1 \), we get

\[
e^{\Theta(x)}h_1(x) = \int_{x_0}^x (e^{\Theta k})(t)dt + h(x_0) \tag{2.32.1}
\]

for \( m-a.e. \in I \). Now if \( a \in J \setminus I \), by (2.32.1) and (2.6.2), we get

\[
\int_{x_0}^x (e^{\Theta k})(t)dt + h(x_0) = (e^{\Theta}h_1)(x) = \int_a^x (e^{\Theta k})(t)dt = \int_a^{x_0} (e^{\Theta k})(t)dt + \int_{x_0}^x (e^{\Theta k})(t)dt.
\]
Thus, if $a \in J \setminus I$,
\[
\int_a^{x_0} (e^{\Theta} k)(t) = h(x_0).
\] (2.32.2)

Similarly, if $b \in J \setminus I$, by (2.32.1) and (2.6.2) of Def. 2.6,
\[
\int_x^b (e^{\Theta} k)(t) dt + h(x_0) = (e^{\Theta} h_1)(x) = -\int_x^b (e^{\Theta} k)(t) dt
\]
\[
= \int_{x_0}^x (e^{\Theta} k)(t) dt - \int_{x_0}^b (e^{\Theta} k)(t) dt.
\]

That is, if $b \in J \setminus I$,
\[
\int_{x_0}^b (e^{\Theta} k)(t) dt = -h(x_0).
\] (2.32.3)

We now consider explicitly the two cases where $(J \setminus I)$ consists of one point.

1) $I = (a, b)$, $J = [a, b)$: By (2.32.2) we have
\[
\mathcal{G}^\perp|I = \{c(-1 \oplus -e^{\Theta} \chi_{[a, x_0]}1/w) : c \in \mathbb{C}\}
\]
which agrees with $\mathcal{G}^\perp$ by case (5) of the proof of Theorem 2.30, via (2.30.4).

2) $I = (a, b)$, $J = (a, b\,]$: By (2.31.3),
\[
\mathcal{G}^\perp|I = \{c[(-1 \oplus -e^{\Theta} \chi_{[x_0, b]}1/w) : c \in \mathbb{C}\}
\]
which, from (2.30.4), agrees with $\mathcal{G}^\perp$ by the case (6) of Theorem 2.30. □

2c. Closure of the graph in other topologies

In this subchapter we study the closure of $\mathcal{G}$ in various other topologies. We begin
with the norm topology of $L^p(\mu) \oplus L^p(\nu)$ where $1 \leq p < \infty$. In this case we require
$\theta \in L^p(\nu)$. As mentioned earlier, with certain modifications of the statements and duality
relationships, all the results given carry over to $p$ with $1 \leq p < \infty$.

Consider first $1 < p < \infty$. Let $q$ represent the conjugate index for $p$ defined by
$1/p + 1/q = 1$. Recall for $p = 2$ the important observation (2.1) that $F \in L^2(\nu|[a, b])$
implies $F \in L^1(m|[a,b])$ whenever $\int_{[a,b]} 1/w \, dm < \infty$. Correspondingly, for the case of a more general $p$,

$$
\int_{[a,b]} |F| \, dm = \int_{a}^{b} |F| \frac{1}{w} \, d\nu \leq \|F\|_{L^p(v|[a,b])} \left( \int_{[a,b]} \left( \frac{1}{w} \right)^q \, d\nu \right)^{1/q}
$$

by Hölder's inequality. Thus if $w^{1-q} \in L^1(m|[a,b])$ then $F \in L^p(\nu)$ implies that $F \in L^1(m|[a,b])$. Therefore the localization occurs at intervals $I$, which are MILI's for $w^{1-q}$.

Now the other observation, for $p = 2$, was that $e^{\Theta}p$ is absolutely continuous and $(e^{\Theta}p)' = e^{\Theta}(\Theta p + p')$, whenever $e^{2Re\Theta} \frac{1}{w}$ is integrable. Since

$$
\int_{[a,b]} |(e^{\Theta}p)'| \, dm = \int_{[a,b]} e^{Re\Theta} |\Theta p + p'| \frac{1}{w} \, d\nu \\
\leq \left( \int_{[a,b]} e^{Re\Theta} \frac{1}{w^q} \, d\nu \right)^{1/q} \|\Theta p + p'\|_{L^p(\nu)} \\
= \left( \int_{[a,b]} e^{Re\Theta} w^{1-q} \, dm \right)^{1/q} \|\Theta p + p'\|_{L^p(\nu)}
$$

we see that the corresponding requirement is that $e^{qRe\Theta} w^{1-q} \in L^1(w|[a,b])$. Thus for an interval $I$ which is a MILI for $w^{1-q}$, the analogous interval $J$ is defined with respect to the integrability of $e^{qRe\Theta} w^{1-q}$ at the endpoints of $I$. As before, there are six cases to consider, depending on whether $I$ and $J$ include their endpoints.

Recall, for $p = 2$, we conjectured that $\hat{\mathcal{G}} = \mathcal{G}^-$ and that in most cases, this was in fact true. For general $p$, $1 \leq p < \infty$, $\hat{\mathcal{G}}$ is defined as in Def. 2.6, but with the modifications mentioned above. As before, $\mathcal{G}^- \subset \hat{\mathcal{G}}$; the proof is completely analogous. To argue the other inclusion we considered $\mathcal{G}^-$ as $(\mathcal{G}^\perp)^\perp$. To make sense of this for $p \neq 2$, we simply interpret this in the context of $L^p - L^q$ duality. For $\mathcal{G} \subset L^p(\mu) \oplus L^p(\nu)$ define the annihilator of $\mathcal{G}$, $\mathcal{G}^\perp \subseteq L^q(\mu) \mp L^q(\nu)$, via

$$
\mathcal{G}^\perp = \{ f \oplus g \in L^q(\mu) \oplus L^q(\nu) : \int f p d\mu + \int g p d\nu = 0 \} \text{ for all polynomials } p\}.
$$
Similarly, the second annihilator of $\mathcal{G}$, $\perp(\mathcal{G}^\perp)$, is defined by

$$\perp(\mathcal{G}^\perp) = \{h \oplus k \in L^p(\mu) \oplus L^p(\nu) : \int fh d\mu + \int gk d\nu = 0 \text{ for all } f \oplus g \in \mathcal{G}^\perp\}.$$ 

Since a standard result in functional analysis is that $\perp(\mathcal{G}^\perp)$ is the norm closure of $\mathcal{G}$, we see that arguments similar to those for $p = 2$ will give analogous results for all $p, 1 < p < \infty$.

For $p = 1$, the observation corresponding to (2.1) is that if $\frac{1}{w}$ is essentially bounded on $[a, b]$ then $F \in L^1(\nu|[a, b])$ implies $F \in L^1(m|[a, b])$. Thus the localization is to maximal intervals $I$ on which $\frac{1}{w} \in L^\infty(I)$. The intervals $J$ are then defined according to whether or not $e^{\Re \theta} \frac{1}{w}$ remains essentially bounded at the endpoints of $I$. Letting $q = \infty$, $\mathcal{G}^\perp$ and $\perp\mathcal{G}^\perp$ are defined as in the cases $1 < p < \infty$. Using this and the above modifications, the results and proofs for $p = 1$ follow those given.

We finish this chapter by considering the closure of the graph of $\theta + \frac{d}{dx}$ acting on the space of continuous functions on a compact subset of $[0, 1]$ into itself. By analogy $\theta$ would then be assumed to be continuous. But since multiplication by $\theta$ then is bounded, we may simplify our discussion, without loss of generality, to the consideration of simply $D = \frac{d}{dx}$.

So let $E$ be a compact subset of the interval $[0, 1]$, and let $C(E)$ denote the space of continuous functions on $E$ with the uniform norm. Consider the densely defined operator $D : C(E) \rightarrow C(E)$ given by $Dp = p'$ for all polynomials $p$. Let $\mathcal{G}$ represent the graph of $D$, that is

$$\mathcal{G} = \{p \oplus p' : p \text{ a polynomial}\}.$$ 

We wish to characterize the closure of $\mathcal{G}, \mathcal{G}^-$, in the direct sum topology of uniform convergence in each component.

The analysis relies on the topological properties of $E$. Denote by $\text{int } E$ the interior of $E$. Write $\text{int } E$ as a countable disjoint union of intervals

$$\text{int } E = \bigcup_i (a_i, b_i).$$
Let \( \hat{E} \) be given by

\[ \hat{E} = \bigcup_i [a_i, b_i]. \]

Notice first that the expression for \( \hat{E} \) is also a disjoint union; indeed, if \([a_i, b_i] \cup [a_j, b_j] \neq \emptyset\) where \(a_i < a_j\), then \((a_i, b_j) \subset \text{int } E\), a contradiction. Secondly \( E \setminus \hat{E} \) contains no intervals since it is a subset of the nowhere dense set \( E \setminus (\text{Int } E)\). Now let \( \hat{G} \) denote the closed submanifold of \( C(E) \oplus C(E) \) defined by

\[ \hat{G} = \{(h \oplus k) \in C(E) \oplus C(E) : h \text{ is } AC(\hat{E}) \text{ and } k|\hat{E} = h'|\hat{E}\}. \]

Our result is that \( G^\bot = \hat{G} \). We begin by proving \( G^\bot \subset \hat{G} \) which is straightforward. The other inclusion relies on a characterization of the annihilator of \( G \), \( G^\bot \), where the duality relationship is that of \( C(E) \) with \( M(E) \), the Banach space of all regular Borel measures supported in \( E \).

**Theorem 2.33.** \( G^\bot \subseteq \hat{G} \).

**Proof:** Let \((h \oplus k) \in G^\bot\). Let \( \{p_n\} \) be a sequence of polynomials such that \( p_n \to h \) and \( p_n' \to k \) uniformly on \( E \). If \( \hat{E} = \bigcup_i [a_i, b_i] \), we need to show for each \( i \), \( h \) is \( AC([a_i, b_i]) \) and \( h'|_{[a_i, b_i]} = k \). Fix one such \( i \). We have, for all \( x, y \in [a_i, b_i] \)

\[ p_n(x) - p_n(y) = \int_{[y, x]} p_n'(t) \, dt. \tag{2.33.1} \]

Furthermore for each \( x, y \in [a_i, b_i] \)

\[ p_n(x) - p_n(y) \to h(x) - h(y). \]

Finally, since \( p_n' \to k \) uniformly, we have

\[ \lim_{n \to \infty} \int_{[y, x]} p_n'(t) \, dt = \int_{[y, x]} k(t) \, dt \]

for all \( x, y \in [a_i, b_i] \). The theorem then follows by taking the limit as \( n \) goes to infinity in (2.33.1). \( \square \)

The proof of the other inclusion involves establishing some lemmas which characterize \( G^\bot \) and localizing our concerns to each \([a_i, b_i]\).
LEMMA 2.34. For two regular Borel measures $m_1$ and $m_2$ supported on $E$, $(m_1 \oplus m_2) \in \mathcal{G}^\perp$ if and only if

$$\int_{[0,1]} dm_1 = 0 \quad (2.34.1)$$

the measure $\left[ \int_{(x,1]} dm_1(t) \right] dx$ is supported in $E$, \quad (2.34.2)

and

$$dm_2(t) = - \left[ \int_{(x,1]} dm_1(t) \right] dx. \quad (2.34.3)$$

PROOF: We first note that (2.34.3) states that the Radon-Nikodym derivative of $m_2$ with respect to $m$ (Lebesgue measure) is $- \left[ \int_{(x,1]} dm_1(t) \right]$. That is

$$dm_2/dm = - \left[ \int_{(x,1]} dm_1(t) \right].$$

Now suppose $(m_1 \oplus m_2) \in \mathcal{G}^\perp$. Then for all polynomials $p$,

$$0 = \int_{[0,1]} p(x)dm_1(x) + \int_{[0,1]} p'(x)dm_2(x)$$

$$= p(0) \int_{[0,1]} dm_1(x) + \int_{[0,1]} \left( \int_{[0,x]} p'(t)dt \right) dm_1(x) + \int_{[0,1]} p'(x)dm_2(x)$$

$$= p(0) \int_{[0,1]} dm_1(t) + \int_{[0,1]} p'(x) \left( \int_{[0,1]} dm_1(t) \right) dx + \int_{[0,1]} p'(x)dm_2(x). \quad (2.34.4)$$

Letting $p \equiv 1$, in (2.34.4), we get $m_1$ must satisfy (2.34.1). Next, by considering polynomials $p'$ with $p(0) = 0$ (a dense set in $C([0,1])$ by the Stone-Weierstrass theorem), it follows from (2.34.4) that $m_1$ and $m_2$ must satisfy (2.34.2) and (2.34.3).

To prove the other direction simply reverse the argument. \square

The next key idea is that the intervals $[a_i, b_i]$ which comprise $\tilde{E}$ play a role similar to that of the MILI's for $\frac{1}{w}$ in the $L^p$ cases. That is, the elements in $\mathcal{G}^\perp$ must be zero off their union.
LEMMA 2.35. If \((m_1 \oplus m_2) \in G^1\), then \(m_1\) has no point masses in \(E \setminus \hat{E}\).

PROOF: As noted above \(E \setminus \hat{E}\) contains no intervals. Thus any open interval must intersect the complement of \(E \setminus \hat{E}\). Let \(x_0 \in E \setminus \hat{E}\). By (2.34.2), for \(m - a.e. \ x \in [0,1] \setminus E\)

\[
m_1([x,1]) = \int_{[x,1]} dm_1(t) = 0. \tag{2.35.1}
\]

Thus, by additivity, \(m_1((x,y]) = 0\) for \(m - a.e. \ x,y \in [0,1] \setminus E\). Now choose a sequence of positive numbers \(\{\xi_n\}\) so that \(\xi_n \searrow 0\). Let \(I_n = (x_0 - \xi_n, x_0 + \xi_n)\). We wish to show that, for each \(n\), there are points \(x_n\) and \(y_n\) belonging to \(I_n\), with \(x_n < x_0 < y_n\), so that

\[
m_1((x_n,y_n]) = 0.
\]

To do this, from (2.35.1), it suffices to show each \(I_n\) intersects the complement of \(E\) in a set of positive Lebesgue measure on each side of \(x_0\). We first show that \(I_n\) must intersect the complement of \(E\) on each side of \(x_0\). Indeed, if say \(I_n\) did not intersect \([0,1] \setminus E\) to the left of \(x_0\), then it would follow that \((x_0 - \xi_n, x_0] \subset E\). But this would contradict \(x_0 \notin \hat{E}\). Next note that both \([0,1] \setminus E\) and \(I_n\) are open. Thus their non-empty intersection, which from above contains an open interval on each side of \(x_0\), must have positive Lebesgue measure.

Thus we have shown that there are sequences \(\{x_n\}, \{y_n\}\) such that

\[
\bigcap_n (x_n, y_n] = \{x_0\}
\]

and

\[
m_1(x_n, y_n] = 0
\]

for each \(n\). Finally, since \(m_1\) is regular, we have

\[
\lim_{n \to \infty} m_1((x_n,y_n]) = m_1(\{x_0\})
\]

and the lemma follows. ∎
LEMMA 2.36. Let \( A = \{ x \in E \setminus \hat{E} : \int_{[x, 1]} d m_1(t) = 0 \} \). Then \( m_1(A) = 0 \).

PROOF: We use in our arguments the total variation of \( m_1, |m_1| \), which is regular if \( m_1 \) is regular. So without loss of generality we may assume \( A \) is closed. Let \( \epsilon > 0 \) be given and choose an open set \( U \supset A \) so that

\[ |m_1|(U \setminus A) < \epsilon \]

Write \( U \) as a disjoint union of open intervals.

\[ U = \bigcup_n (\alpha_n, \beta_n) \]

We construct a new open set \( U_1 \supset A \) as follows. Delete each interval \( (\alpha, \beta) \) for which \( (\alpha, \beta) \cap A = \emptyset \) from the collection \( \{(\alpha_n, \beta_n)\} \). Let

\[ \alpha'_n = \inf \{ x : x \in (\alpha_n, \beta_n) \cap A \} \]

\[ \beta'_n = \sup \{ x : x \in (\alpha_n, \beta_n) \cap A \} \]

Then for each \( n \) not deleted, let

\[ U_1 = \bigcup_n (\alpha'_n, \beta'_n) \]

Then \( A \subset U_1 \), modulo the endpoints \( \alpha'_n \) and \( \beta'_n \) which are in \( A \), since \( A \) is closed. But the collection of all endpoints is a countable set in \( E \setminus \hat{E} \), and so by Lemma 1.4 has \( m_1 \)-measure zero. Furthermore

\[ m_1((\alpha'_n, \beta'_n)) = \int_{(\alpha'_n, 1]} d m_1(t) - \int_{(\beta'_n, 1]} d m_1(t) = 0 \]

by assumption. Thus \( m_2(U_1) = 0 \). Finally, then

\[ |m_1(A)| = |m_1(U_1) - m_1(A)| \]

\[ = |m_1(U_1 \setminus A)| = |m_1(U_1) - m_1(A)| \]

\[ = |m_1(U_1 \setminus A)| \]

\[ \leq |m_1|(U_1 \setminus A) \leq |m_1|(U \setminus A) < \epsilon. \]

So by the arbitrariness of \( \epsilon \), Lemma 2.36 follows. \( \square \)
**Lemma 2.37.** For \((m_1 \oplus m_2) \in \mathcal{G}^\perp, |m_1|(E \setminus \hat{E}) = 0.\)

**Proof:** Suppose there is an \(A \subset E \setminus \hat{E}\) with \(m_1(A) \neq 0.\) Then by Lemma 2.35, there exists \(x_0 \in E \setminus \hat{E}\) so that

\[
\int_{(x_0, 1]} dm_1(t) \neq 0.
\]

But since \(m_1\) has no point masses in \(E \setminus \hat{E},\) the function \(\int_{[x, 1]} dm_1(x)\) is continuous in \(E \setminus \hat{E}.\)

Thus there is an interval \(I\) containing \(x_0\) and \(\xi > 0,\) so that

\[
\left| \int_{(x, 1]} dm_1(t) \right| > \xi
\]

for all \(x \in I \cap (E \setminus \hat{E}).\) Since \(I\) must intersect \([0, 1] \setminus E\) and \(m_1\) is supported on \(E,\) this contradicts (2.34.2). \(\Box\)

**Corollary 2.38.** For \((m_1 \oplus m_2) \in \mathcal{G}^\perp, |m_2|(E \setminus \hat{E}) = 0.\)

**Proof:** Let \(B \subset E \setminus \hat{E}.\) Then if \(A\) is defined as in Lemma 2.36,

\[
B = (B \cap A) \cup (B \cap [(E \setminus \hat{E}) \setminus A]) = B_1 \cup B_2
\]

Now, by (2.34.3), for any set \(C\)

\[
m_2(C) = \int_C dm_2(t) = -\int_C \int_{(x, 1]} dm_1(t) dx.
\]

So \(m_2(B_1) = 0\) by definition of \(A.\) So suppose \(m_2(B_2) \neq 0.\) Then there must be an \(x_0 \in E \setminus \hat{E}\) so that \(\int_{(x_0, 1]} dm_1(t) \neq 0.\) But this leads to a contradiction as in the proof of Lemma 2.37. \(\Box\)

Recall that \([a_i, b_i](i = 1, 2, \ldots)\) are the maximal intervals in \(\hat{E}.\)

**Lemma 2.39.** For each \(i,\)

\[
0 = \int_{[a_i, 1]} dm_1(t) = \int_{(b_i, 1]} dm_1(t).
\]
PROOF: By regularity of $m$, for $x_n \not\to a_i$.

$$\lim_{n \to \infty} m_1((x_n, 1]) = m_1([a_i, 1]).$$ \hfill (2.39.1)

Let $I$ be an open interval containing $a_i$. Then $I$ must intersect $[0, 1] \setminus E$, an open set of positive Lebesgue measure. But since (2.34.2) states

$$\int_{(x, 1]} dm(t) = 0$$

for $m$-a.e. $x \in [0, 1] \setminus E$, there is a sequence $\{x_n\}$ converging up to $a_i$ so that, for each $n$,

$$m_1((x_n, 1]) = \int_{(x_n, 1]} dm_1(t) = 0.$$ 

Using this sequence in (2.39.1), we get

$$\int_{[a, 1]} dm(t) = 0.$$ 

A similar argument shows $\int_{[b, 1]} dm_1(t) = 0$. \hfill \Box

THEOREM 2.40. $\hat{G} \subseteq G^-$. 

PROOF: Let $(h \oplus k) \in \hat{G}$. We show $(h \oplus k) \in G^-$ by showing

$$\int_{[0, 1]} h(x) dm_1(x) + \int_{[0, 1]} k(x) dm_2(x) = 0$$

for all $(m_1 \oplus m_2) \in G^\perp$. But using Lemmas 2.37 and 2.38, it suffices to show for each $i$ and for each $(m_1 \oplus m_2) \in G^\perp$

$$\int_{[a, b]} h(x) dm_1(x) + \int_{[a, b]} k(x) dm_2(x) = 0.$$ 

So fix $i$, and let $(m_1 \oplus m_2) \in G^\perp$. Using (2.34.3) we have

$$\int_{[a, b]} k(x) dm_2(x) = -\int_{[a, b]} k(x) \left( \int_{(x, 1]} dm_1(t) \right) dx$$

$$= -\int_{[a, b]} \left( \int_{[a, x]} k(t) dt \right) dm_1(x) - \int_{[b, 1]} \left( \int_{[a, b]} k(t) dt \right) dm_1(x)$$
by interchanging the order of integration. But by assumption \( h(y) - h(x) = \int_{[x,y]} k(t) dt \) for all \( x, y \in [a_i, b_i] \). Thus,

\[
\int_{[a_i, b_i]} k(x) dm_2(x) = -\int_{[a_i, b_i]} h(x) dm_1(x) + h(\alpha_i) \int_{[a_i, b_i]} dm_1(x) \\
- h(b_i) \int_{[a_i, b_i]} dm_1(x) + h(\alpha_i) \int_{[a_i, b_i]} dm_1(x)
\]

\[
= -\int_{[a_i, b_i]} h(x) dm_1(x) + h(\alpha_i) \int_{[a_i, b_i]} dm_1(x) - h(b_i) \int_{[a_i, b_i]} dm_1(x)
\]

So

\[
\int_{[a_i, b_i]} h(x) dm_1(x) + \int_{[a_i, b_i]} k(x) dm_2(x) = h(\alpha_i) \int_{[a_i, b_i]} dm_1(x) - h(b_i) \int_{[a_i, b_i]} dm_1(x) = 0
\]

by Lemma 2.39. □

We close this chapter by answering the questions when is \( D \) closable and when is \( \mathcal{G}^- \) all of \( C(E) \oplus C(E) \).

**Theorem 2.41.** \( D \) is closable, if and only if \( E = (\hat{E})^- \) (that is, \( E \) is the closure of its interior).

**Proof:** First suppose \((\hat{E})^- = E\). Let \((0 \oplus g) \in \mathcal{G}^-\). Then by Theorem 2.33, \( g \equiv 0 \) on \( \hat{E} \). Thus, by continuity \( g \equiv 0 \) on \( (\hat{E})^- \).

Conversely, suppose \((\hat{E})^- \neq E\). Let \( x_0 \in E \setminus (\hat{E})^- \). By Urysohn’s Lemma, there is a function \( g \), continuous on \( E \), so that \( g(x_0) = 1 \) and \( g(x) = 0 \) for all \( x \in (\hat{E})^- \). Now \((0 \oplus g) \in \tilde{G} \subset \mathcal{G}^-, \) by Theorem 2.40, but is non-zero. Thus \( D \) is not closable. □

**Theorem 2.42.** \( \mathcal{G}^- = C(E) \oplus C(E) \) if and only if \( \hat{E} = \phi \) (that is, \( E \) has empty interior).

**Proof:** If \( \hat{E} \neq \phi \), we have for \( h(x) \equiv 1 \) on \( E \), \((h \oplus h)\) is in \( C(E) \oplus C(E) \) but not in \( \mathcal{G}^- \). Conversely, if \( \hat{E} = \phi \), then there are no requirements on an element of \( C(E) \oplus C(E) \) to be in \( \tilde{G} \subset \mathcal{G}^- \). Thus \( \mathcal{G}^- = C(E) \oplus C(E) \). □

3a. Introduction

In this chapter we address the question we originally set out to answer: can a definitive characterization be made of those compact subsets of the real line which arise as spectra of pure subjordan operators?

Towards this we introduce the players involved.

Definition 3.1. Let $\mathcal{K}$ be a Hilbert space, and let $S$ and $N$ be bounded operators on $\mathcal{K}$ satisfying

\begin{align*}
S &= S^* \quad (3.1.1) \\
N^2 &= 0 \quad (3.1.2) \\
SN &= NS \quad (3.1.3)
\end{align*}

The operator $J = S + N$ is called a jordan operator on $\mathcal{K}$. (Note: $J$ is sometimes referred to as a real jordan operator since the spectrum of $J$ is a subset of the real line. In fact, it can be shown that $\sigma(J) = \sigma(S)$ (see Proposition 3.7). Complex jordan operators are ones for which $S$ is only required to be normal. These will be mentioned in the last chapter. The reader will also find in depth discussions of complex jordans (and subjordans) in [A2], [BH], [Bu].)

Definition 3.2. Let $J$ be a jordan operator on $\mathcal{K}$ and suppose $\mathcal{H}$ is a subspace of $\mathcal{K}$ invariant for $J$. We call the operator $T$ on $\mathcal{H}$ defined by $T \equiv J|\mathcal{H}$ a subjordan operator. Further, if it is the case that the smallest subspace of $\mathcal{K}$ containing $\mathcal{H}$ on which both $S$ and $N$ are invariant is itself $\mathcal{K}$, then $J$ is said to be a minimal jordan extension of $T$. Finally, a subjordan operator $T$ on $\mathcal{H}$ is called pure, if for all invariant subspaces $\mathcal{H}_0$ for $T$ with $\mathcal{H}_0 \neq \mathcal{H}$ and so that $T|\mathcal{H}_0$ is jordan, then it must be that $\mathcal{H}_0 = (0)$.

The following two propositions give equivalent descriptions of purity.

56
Proposition 3.3. $T$ is a pure subJordan operator on $\mathcal{H}$ if and only if there are no nonzero invariant subspaces for $T$ on which $T$ is self-adjoint.

Proof: Let $\mathcal{H}_0 \subset \mathcal{H}$ with $T(\mathcal{H}_0) \subset \mathcal{H}_0$. First, if $T|\mathcal{H}_0$ is self-adjoint then it is trivially Jordan on $\mathcal{H}_0$. Conversely, let $T|\mathcal{H}_0 = S_0 + N_0$ be Jordan. That is $S_0^* = S_0$, $N_0^2 = 0$, and $S_0N_0 = N_0S_0$. Assuming $N_0 \neq 0$ (otherwise $T|\mathcal{H}_0$ is self-adjoint and we are done), let $\mathcal{H}_1 = \text{Ran } N_0 \neq 0$ and consider $T|\mathcal{H}_1$. Let $h_0 \in \mathcal{H}_0$ and set $h_1 = Nh_0 \in \mathcal{H}_1$.

Then

$$Th_1 = (S_0 + N_0)Nh_0 = S_0N_0h_0 + N_0^2h_0 = S_0h_1$$

and

$$S_0h_1 = S_0N_0h_0 = N_0(S_0h_0) \subset \mathcal{H}_1.$$ 

So we see $\mathcal{H}_1$ is a nonzero invariant subspace for $T$ on which it is self-adjoint. □

The second equivalent notion of purity actually gives that the only manner in which $T = J|\mathcal{H} = S_0 + N_0$ can be Jordan itself is for $S_0 = S|\mathcal{H}$ and $N_0 = N|\mathcal{H}$. The proof relies on machinery exposited in [BH], which we presently set forth, and the ensuing technical lemma.

Definition 3.4. The symbol expansion of an operator $T$ is the function

$$Q_T(s) = e^{-isT^*}e^{isT} = \sum_{k=0}^{\infty} B_k s^k$$

for some sequence of operators $\{B_k\}_{k=1}^{\infty}$. An operator $T$ is said to be coadjoint (of order 2) if there are operators $B_1$ and $B_2$ so that

$$Q_T(s) = I + B_1s + B_2s^2.$$ 

An operator $A$ is said to be a nilpart for a coadjoint operator $T$, if

$$A^*A \leq B_2 \quad (3.4.1)$$
\[ A - A^* = T - T^* \] \hspace{1cm} (3.4.2)

and

\[ T^*(A + A^*) = (A + A^*)T. \] \hspace{1cm} (3.4.3)

As a point of reference, nilparts are the basic building blocks used by Ball and Helton in constructing Jordan extensions (again, see [BH]). In [H3], Helton proved that an operator is Jordan if and only if both it and its adjoint are co-adjoint. Using that we present this technical result.

**Lemma 3.5.** Let \( J = S + N \) be a Jordan operator on \( K \). Then the nilpart for \( J \) is unique and equal to \( N \).

**Proof:** Let \( B_1 \) and \( B_2 \) be operators so that

\[ Q_J(s) = I + sB_1 + s^2B_2, \]

and suppose \( A \) is a nilpart for \( J \). By Lemma 11.5 in [BH], \( B_2 = N^*N \) and since \( J = S + N \) is Jordan, \( J - J^* = N - N^* \). Therefore 3.4.1 and 3.4.2 become

\[ A^*A \leq N^*N \] \hspace{1cm} (3.5.1)

and

\[ A - A^* = N - N^*. \] \hspace{1cm} (3.5.2)

Now write \( \mathcal{K} = (\text{Ran}N)^\perp \oplus (\text{Ran}N) \) and for some \( \Gamma : (\text{Ran}N)^\perp \to \text{Ran}N \) we have

\[ N = \begin{bmatrix} 0 & 0 \\ \Gamma & 0 \end{bmatrix}. \]

By (3.5.1), we have \( \text{Ran}A^* \subseteq \text{Ran}N^* \subseteq \ker N^* = (\text{Ran}N)^\perp \). Thus, if with respect to \( \mathcal{K} = (\text{Ran}N)^\perp \oplus \text{Ran}N \),

\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \]
it follows that

\[ A^* = \begin{bmatrix} A_{11}^* & A_{21}^* \\ 0 & 0 \end{bmatrix}. \]

Hence \( A_{12} = A_{22} = 0 \). Using (3.5.2), we have

\[ \begin{bmatrix} A_{11} - A_{11}^* & -A_{11}^* \\ A_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\Gamma^* \\ \Gamma & 0 \end{bmatrix}. \]

Thus \( A_{11} = A_{11}^* \) and \( A_{21} = \Gamma \). Returning to (3.5.1) and computing products yields

\[ \begin{bmatrix} A_{11}^2 + \Gamma^* \Gamma & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} \Gamma^* \Gamma & 0 \\ 0 & 0 \end{bmatrix}, \]

or \( A_{11}^2 + \Gamma^* \Gamma \leq \Gamma^* \Gamma \). Therefore \( A_{11} = A_{12} = A_{22} = 0 \) and \( A_{21} = \Gamma \) from which we conclude \( A = N \). \( \Box \)

**Proposition 3.6.** Let \( J = S + N \) be a jordan operator on \( \mathcal{K} \). Let \( \mathcal{H} \leq \mathcal{K} \) be invariant for \( J \) with \( T = J|\mathcal{H} \). Then \( T \) is jordan if and only if \( \mathcal{H} \) is invariant for both \( S \) and \( N \).

**Proof:** Let \( J = S + N \) on \( \mathcal{K} \), let \( \mathcal{H} \leq \mathcal{K} \) be invariant for \( J \), let \( T = J|\mathcal{H} = S_0 + N_0 \) and assume both \( J \) and \( T \) are jordan. Further let

\[ Q_J(s) = 1 + B_1 s + B_2 s^2 \]

and

\[ Q_T(s) = 1 + B_{10} s + B_{20} s^2 \]

be the symbol expansions for \( J \) and \( T \). We claim \( B_{20} = P_{\mathcal{H}} B_2|\mathcal{H} \). To see this consider

\[ P_{\mathcal{H}} Q_J(s)|\mathcal{H} = P_{\mathcal{H}} e^{-isJ} e^{isJ}|\mathcal{H} \]

\[ = P_{\mathcal{H}} e^{isT^*} P_{\mathcal{H}} e^{isT}|\mathcal{H} \]

\[ = e^{isT^*} e^{isT} = Q_T(s). \]

Now by Lemma 1.5 of [BH], we have \( B_2 = N^* N \) and \( B_{20} = N_0^* N_0 \). Thus

\[ N_0^* N_0 = P_{\mathcal{H}} N^* N|\mathcal{H}. \]
But from Theorem 1.1 of [BH], we have \( P_{\mathcal{H}} N|\mathcal{H} \) is a nilpart for \( T \), which, by applying Lemma 3.5, gives
\[
(P_{\mathcal{H}} N|\mathcal{H})^* P_{\mathcal{H}} N|\mathcal{H} = P_{\mathcal{H}} N^* N|\mathcal{H},
\]
or
\[
P_{\mathcal{H}} N^* P_{\mathcal{H}} N|\mathcal{H} = P_{\mathcal{H}} N^* N|\mathcal{H}.
\]
This gives
\[
P_{\mathcal{H}} N^* (I - P_{\mathcal{H}}) N|\mathcal{H} = 0,
\]
or
\[
(I - P_{\mathcal{H}}) N|\mathcal{H} = 0.
\]
Hence \( \mathcal{H} \) is invariant for \( N \) and hence also invariant for \( S \).

Conversely, if \( \mathcal{H} \) is invariant for both \( S \) and \( N \), then \( T = (S|\mathcal{H}) + (N|\mathcal{H}) \), where \( (S|\mathcal{H})^* = S|\mathcal{H}, (N|\mathcal{H})^2 = 0 \) and \( (S|\mathcal{H})(N|\mathcal{H}) = SN|\mathcal{H} = NS|\mathcal{H} = (N|\mathcal{H})(S|\mathcal{H}). \) Hence \( T \) is jordan on \( \mathcal{H} \). \( \square \)

We complete this introductory section with a result which gives, for \( J \) jordan, \( \sigma(J) = \sigma(S) \). In fact this proposition is used in Chapter 5 in the more general context of complex jordan operators.

**Proposition 3.7.** Let \( A \) and \( N \) be bounded operators on a Hilbert space \( \mathcal{K} \) so that \( AN = NA \) and \( N^2 = 0 \). Then \( \sigma(A + N) = \sigma(A) \).

**Proof:** Since \( A \) and \( N \) commute, we have
\[
[(\lambda I - A)^{-1} N]^2 = (\lambda I - A)^{-2} N^2 = 0
\]
for all \( \lambda \) in the resolvent set of \( A \).

Thus \((\lambda I - A)^{-1} N \) is nilpotent from which it follows that \( I - (\lambda I - A)^{-1} N \) is invertible. So considering the formal equation
\[
(\lambda I - (A + N))^{-1} = (\lambda I - A)^{-1} [I - (\lambda I - A)^{-1} N]^{-1},
\]
we see $\lambda \in \rho(A + N)$ if and only if $\lambda \in \rho(A)$. Hence $\sigma(A + N) = \sigma(A)$. □

3b. The Model

We now introduce a concrete construction of a subjordan operator $T = T(\mu, \nu, \theta)$ which we will later show serves as a model for general pure subjordan operators. Earlier models for co-adjoint (or 2-symmetric) operators ([H1], [H2], [H4], [A]) were more complicated and involved the theory of distributions. The model introduced in [BH] assumes that the operator is already known to be subjordan; the former models were built to yield properties under which a jordan extension existed. The model from [BH] uses only measure theory while avoiding distributions. Under the latter perspective, our contribution here is to specialize it to characterize pure subjordan operators.

Definition 3.8. Let $\mu$ and $\nu$ be two real measures whose supports are compact and contained in some finite closed real interval, satisfying the following:

$$\text{sppt}\mu = \text{sppt}\nu \equiv E$$

(3.8.1)

$$\nu \ll \mu$$

(3.8.2)

The inclusion map $\Gamma : L^2(\mu) \rightarrow L^2(\nu)$ is continuous

(3.8.3)

$\nu \ll m$, where $m$ as before denotes Lebesgue measure (restricted to $E$)

(3.8.4)

$\nu$ is carried by the union of all MILI's for $1/w$, where $w = dv/dm \in L^1(m)$. (3.8.5)

Notationally, let $\mathcal{K}_1 = L^2(\mu), \mathcal{K}_2 = L^2(\nu), \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2, S_i$ be multiplication by $x$ on $\mathcal{K}_i (i = 1,2)$. Let $\Gamma : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ be the operator defined by $\Gamma f = f$ for all $f \in \mathcal{K}_1$, and finally let $J = S + N : \mathcal{K} \rightarrow \mathcal{K}$, where

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ \Gamma & 0 \end{bmatrix}$$
is thought of as acting on an element $f$ of $\mathcal{K}$ written as $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ with $f_i \in \mathcal{K}_i (i = 1, 2)$. Further, let $\theta \in L^2(\nu)$, and define the unbounded operator $D : L^2(\mu) \to L^2(\nu)$, densely on all polynomials $p$ by

$$Dp = \theta p + p'.$$

We let $\mathcal{H}$ denote the subspace of $\mathcal{K}$ obtained by closing $\mathcal{G} \equiv \left\{ \begin{bmatrix} p \\ Dp \end{bmatrix} : p \text{ a polynomial} \right\}$, the graph of $D$. Finally, let $T = T(\mu, \nu, \theta) \equiv J|\mathcal{H}$.

Later we will show that given a pure subjordan operator (under an additional cyclic property) it is unitarily equivalent to a $T(\mu, \nu, \theta)$ for certain choices of $\mu, \nu$, and $\theta$. The following theorem states that a given $T = T(\mu, \nu, \theta)$ is subjordan, and with an additional hypothesis (which we think is probably unnecessary) is in fact, pure.

**Theorem 3.9.** The operator $T = T(\mu, \nu, \theta)$, as defined by Def. 3.8, with $\mu$ and $\nu$ satisfying 3.8.1 through 3.8.5, is subjordan on $\mathcal{H}$ with minimal jordan extension $J$. Moreover if the following is also satisfied

**The measure $\mu$ is carried by the union of the MILI's for $1/w$, \hspace{1cm} (3.9.1)**

then $T = T(\mu, \nu, \theta)$ is pure.

**Proof:** We first show $J$ is jordan. Clearly, $S = S_1 \oplus S_2$ is self-adjoint and $N$ is nilpotent. Thus $J$ is jordan, if $SN = NS$. Now

$$SN = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \Gamma & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ S_2 \Gamma & 0 \end{bmatrix}$$

and

$$NS = \begin{bmatrix} 0 & 0 \\ \Gamma & 0 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \Gamma S_1 & 0 \end{bmatrix}.$$

Since by (3.8.1), $\text{sppt} \mu = \text{sppt} \nu$ for all $f \in \mathcal{K}_1$,

$$S_1 \Gamma(f(x)) = S_2(f(x)) = xf(x)$$
and
\[ \Gamma S_1(f(x)) = \Gamma(xf(x)) = xf(x), \]
it follows that \( J \) is jordan on \( \mathcal{K} \).

Next we note that \( D \) is closable. This follows from Theorem 2.24, since \( \nu << m \) by (3.8.4), \( \nu(K) = 0 \) by (3.8.5), and the complement of \( \text{sppt}\mu \) contains no intervals in a MILI for \( 1/w \) by (3.8.1).

To see \( H = G' \) is invariant for \( J \), consider \( p \) a polynomial and denoting by \( J \) the operator of multiplication by \( \begin{bmatrix} x & 0 \\ 1 & x \end{bmatrix} \), we get
\[
\begin{bmatrix} x & 0 \\ 1 & x \end{bmatrix} \begin{bmatrix} p \\ Dp \end{bmatrix} = \begin{bmatrix} xp \\ p + xDp \end{bmatrix} = \begin{bmatrix} xp \\ p + x\theta p + xp' \end{bmatrix} = \begin{bmatrix} (xp) \\ D(xp) \end{bmatrix} \in \mathcal{G}
\]
Thus \( J(\mathcal{H}) \subseteq \mathcal{H} \), since \( \mathcal{H} = \mathcal{G}^- \) and \( D \) is closable.

Now let \( \mathcal{K} \) be a subspace so that \( \mathcal{H} \subseteq \mathcal{K} \subseteq \mathcal{K} \) and which is invariant for both \( S \) and \( N \). Then
\[
N \begin{bmatrix} p \\ Dp \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ Dp \end{bmatrix} = \begin{bmatrix} 0 \\ p \end{bmatrix} \in \mathcal{K}.
\]
For all polynomials \( p \), thus, \( [N(\mathcal{H})]^- = [N(\mathcal{G}^-)]^- = \left\{ \begin{bmatrix} 0 \\ p \end{bmatrix} : p \text{ a polynomial} \right\}^- = \left[ L^2(\nu) \right] \subseteq \mathcal{K} \). Moreover since \( Dp \in L^2(\nu) \) for all polynomials, \( \begin{bmatrix} 0 \\ Dp \end{bmatrix} \in \mathcal{K} \) for all polynomials. Finally since \( \mathcal{H} \subseteq \mathcal{K} \), we have \( \begin{bmatrix} p \\ Dp \end{bmatrix} \in \mathcal{K} \) for all polynomials, and it follows that
\[
\begin{bmatrix} p \\ Dp \end{bmatrix} - \begin{bmatrix} 0 \\ Dp \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix} \in \mathcal{K}.
\]
Taking the closure over all polynomials \( p \), we have \( \left[ L^2(\mu) \right] \subseteq \mathcal{K} \). Thus
\[
\mathcal{K} \supseteq \left[ L^2(\mu) \right] \oplus \left[ 0 \\ L^2(\nu) \right] = \mathcal{K}
\]
which proves that \( J \) is a minimal jordan extension of \( T \). (Note: Minimal jordan extensions may not be unique but are unitarily equivalent under certain conditions. See [BH]).
It is left to show that if in addition $\mu$ satisfies 3.9.1, then $T$ is in fact pure. To do this we show every nonzero invariant subspace for $T$ contains an invariant subspace on which $T$ is not self-adjoint. Since the restriction of a self-adjoint operator to an invariant subspace is also self-adjoint, it follows that $T$ has no invariant subspaces on which it is self-adjoint, hence by Prop. 3.3, $T$ is pure.

Let $\mathcal{H}_0$ be a nonzero invariant subspace for $T$. Since $\mathcal{H}_0 \subseteq \mathcal{H} = \begin{bmatrix} I \\ D \end{bmatrix} \text{Dom} D$, there is a $h_0 \in \text{Dom} D \subseteq L^2(\mu)$, $h_0 \neq 0$, so that $\begin{bmatrix} h_0 \\ Dh_0 \end{bmatrix} \in \mathcal{H}_0$. Using the definition of $\hat{\mathcal{G}}$ (Def. 2.6) and the fact that $\hat{\mathcal{G}} \subseteq \mathcal{G}^{-}$ (Th. 2.10), we may assume $h_0 \in AC_{loc}(I)$ for each MILI $I$ for $1/w$. Moreover, by 3.9.1, $\mu$ is carried by the union of the MILI's for $1/w$, and hence not identically zero on some MILI.

Consider the cyclic subspace $\hat{\mathcal{H}}_0$ of $\mathcal{H}_0$ generated by $\begin{bmatrix} h_0 \\ Dh_0 \end{bmatrix}$. Precisely, let

$$\hat{\mathcal{H}}_0 = [\text{span}\{T^n \begin{bmatrix} h_0 \\ Dh_0 \end{bmatrix} : n = 0, 1, 2, \ldots\}]^{-}.$$ 

By an inductive argument one can show in general that

$$T^n \begin{bmatrix} h \\ Dh \end{bmatrix} = \begin{bmatrix} x^{n-1}h \\ D(x^{n-1}h) \end{bmatrix},$$

for $h \in \text{Dom} D$, $n = 1, 2, \ldots$; hence $\hat{\mathcal{H}}_0$ can also be identified as

$$\hat{\mathcal{H}}_0 = \left\{ \begin{bmatrix} p h_0 \\ D(p h_0) \end{bmatrix} : p \text{ a polynomial} \right\}^{-}.$$

Now let $d\hat{\mu} = |h_0|^2 d\mu$, $d\hat{\nu} = |h_0|^2 d\nu$, and $\hat{\theta} = \theta + h'_0/h_0$, and consider the transformation

$$U : \begin{bmatrix} L^2(d\mu|\text{supp} h_0) \\ L^2(d\nu|\text{supp} h_0) \end{bmatrix} \to \begin{bmatrix} L^2(d\hat{\mu}) \\ L^2(d\hat{\nu}) \end{bmatrix}$$

defined by

$$U \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} h/h_0 \\ k/h_0 \end{bmatrix}.$$ 

Since for the polynomials $p$,

$$\left\| \begin{bmatrix} p h_0 \\ D(p h_0) \end{bmatrix} \right\|^2_\mathcal{G} = \|ph_0\|^2_{L^2(\mu)} + \|\theta ph_0 + ph'_0 + p'h_0\|^2_{L^2(\nu)} = \|p\|^2_{L^2(\hat{\mu})} + \|\hat{\theta} p + p'\|^2_{L^2(\hat{\nu})},$$
(where \( \| \cdot \|_G \) denotes the graph norm on \( H \)) we see \( U \) is a unitary map from \( \mathcal{H}_0 \) to \( \mathcal{H}_1 \), where \( \mathcal{H}_1 \) is the closure of the graph of \( \hat{D} : \text{Dom} \hat{D} \subseteq L^2(\hat{\mu}) \rightarrow L^2(\hat{\nu}) \), defined densely on the polynomials as

\[
\hat{D}p = \hat{\theta}p + p'.
\]

Noting that \( \hat{\theta} \in L^2(\hat{\nu}) \), since \( \|\hat{\theta}\|_{L^2(\hat{\nu})} = \|Dh_0\|_{L^2(\nu)}^2 \), and letting \( \hat{T} : \left[ \frac{I}{\hat{D}} \right] \text{Dom} \hat{D} \rightarrow \mathcal{H}_1 \) be defined densely on all polynomials \( p \) by

\[
\hat{T} \left[ \begin{array}{c} p \\ \hat{D}p \end{array} \right] = T \left[ \begin{array}{c} ph_0 \\ D(ph_0) \end{array} \right],
\]

we see that \( T \) is unitarily equivalent to \( \hat{T} \).

We next argue that \( \hat{T} = T(\hat{\mu}, \hat{\nu}, \hat{\theta}) \) is an operator of the type defined in 3.8, i.e. \( \hat{\mu} \) and \( \hat{\nu} \) satisfy 3.8.1 through 3.8.5. First note that

\[
\text{sppt} \hat{\mu} = (\text{sppt} |h_0|^2) \cap E = (\text{sppt} |h_0|^2) \cap \text{sppt} \nu = \text{sppt} \hat{\nu}.
\]

Further we have

\[
d\hat{\nu} = |h_0|^2d\nu = |h_0|^2w\mu d\mu,
\]

and

\[
d\hat{\nu} = |h_0|^2d\nu = |h_0|^2(d\nu/d\mu)d\mu = (d\nu/d\mu)d\hat{\mu}.
\]

Thus \( \hat{\mu} \) and \( \hat{\nu} \) satisfy 3.8.1-3.8.4.

Let \( G = \cup \{\text{Int } I : I \text{ a MILI for } 1/w\} \). By 3.9.1, the previously noted fact that \( h_0 \) is \( \mu \)-a.e. locally absolutely continuous, and assuming, since \( h_0 \in L^2(\nu) \), \( h_0 = 0 \) when \( w = 0 \), we may conclude \( |h_0|^2 \) is carried by the open subset \( \hat{G} \) of \( G \) where

\[
\hat{G} = E \setminus \{x : h_0(x) = 0\}.
\]

Thus \( d\hat{\nu} = |h_0|^2d\nu \) is carried by \( \hat{G} \) which is a countable union of open intervals contained in \( G \). Furthermore, \((|h_0|^2w)^{-1}\) is locally integrable on each of these intervals and not
integrable on $E \setminus \tilde{G}$. Hence the union of the MILI's for $(|h_0|^2 \omega)^{-1}$ must be contained in $(\tilde{G})^-$. We conclude $d\tilde{\nu} = |h_0|^2 \omega d\mu$ is carried by the union of the MILI's for $(|h_0|^2 \omega)^{-1}$. Thus $\tilde{\nu}$ satisfies 3.8.5.

We conclude $\hat{T} = T(\mu, \nu, \theta)$ is subjordan on $\hat{H}_1$. The proof is completed by the following.

**Proposition 3.10.** If $T = T(\mu, \nu, \theta)$ is defined as in 3.8, then $T$ is not self-adjoint.

**Proof:** Let $p$ and $q$ be polynomials. We compare graph inner products

$$< T \begin{bmatrix} p \\ Dp \end{bmatrix}, \begin{bmatrix} q \\ Dq \end{bmatrix} >_\sigma$$

and

$$< \begin{bmatrix} p \\ Dp \end{bmatrix}, T \begin{bmatrix} q \\ Dq \end{bmatrix} >_\sigma,$$

where $T$ is multiplication by \begin{bmatrix} x & 0 \\ 1 & x \end{bmatrix}. Expanding (3.10.1) gives

$$\int xp\tilde{q}d\mu + \int [(xp)' + \theta xp][(\tilde{q} + \theta \tilde{q})]d\nu,$$

while (3.10.2) yields

$$\int xp\tilde{q}d\mu + \int [p' + \theta p][(x\tilde{q})' + x\tilde{q}]d\nu.$$

Canceling the first term of each and then considering just the integrands of the second terms, we have

$$xp'\tilde{q}' + p\tilde{q}' + xp'\tilde{\theta}q + p\tilde{\theta}q + \theta xp\tilde{q} + |\theta|^2 xp\tilde{q}$$

and

$$xp'\tilde{q}' + p'\tilde{q} + p'x\tilde{\theta}q + x\theta p\tilde{q}' + \theta p\tilde{q} + |\theta|^2 xp\tilde{q}.$$

Dropping like terms, we see that self-adjointness of $T$ requires

$$pq' + p\tilde{\theta}q = p'\tilde{q} + \theta p\tilde{q} \quad (3.10.3)$$
for all polynomials $p$ and $q$. Letting $p = x, q = 1$, (3.10.3) yields

$$x \bar{\theta} = 1 + \theta x$$

or

$$x \left( \frac{1}{2} \text{Im}(\theta) \right) = i,$$

which is a contradiction. Thus $T = T(\mu, \nu, \theta)$ is not self-adjoint. □

3c. The Cyclic Case

In the following we show that an arbitrary subjordan operator $T$ with cyclic vector is unitarily equivalent to an operator of the form presented in Definition 3.8. This is done through a series of lemmas. The results of Chapter 2 are then cited to conclude that $\sigma(T)$ must be regularly closed.

**Lemma 3.11.** Let $J$ be a Jordan operator on $\mathbb{K}$. Let $\mathcal{H} \subseteq \mathbb{K}$ be invariant for $J$ and let $T \in J|\mathcal{H}$. Then $J$ can be represented on $\mathbb{K} = \mathbb{K}_1 \oplus \mathbb{K}_2$ by

$$J = \begin{bmatrix} S_1 & 0 \\ \Gamma & S_2 \end{bmatrix}$$

where

$S_i : \mathbb{K}_i \to \mathbb{K}_i$ is self-adjoint $(i = 1, 2)$. \hfill (3.11.1)

$$\Gamma S_1 = S_2 \Gamma$$ \hfill (3.11.2)

$\text{Ran}\Gamma$ is dense in $\mathbb{K}_2$, \hfill (3.11.3)

and

There is a closed operator $D : \mathbb{K}_1 \to \mathbb{K}_2$ \hfill (3.11.4)

so that $\mathcal{H} = \begin{bmatrix} I_D \\ D \end{bmatrix} \text{Dom} \ D + \begin{bmatrix} 0 \\ \mathcal{N} \end{bmatrix}$, $S_1(\text{Dom} \ D) \subseteq \text{Dom} \ D, \mathcal{N} \subseteq (\text{Ran} D)^\perp, S_2 \mathcal{N} \subseteq \mathcal{N}$ and $DS_1 = S_2 D + \Gamma$. 

Proof: Let \( J = S + N \) where \( S = S^* \), \( N^2 = 0 \) and \( SN = NS \). Since \( S \) is self-adjoint, it can be represented as multiplication by \( x \) on a direct integral space (see [Cl])

\[
\mathcal{K} = \int \oplus \mathcal{K}(x) d\alpha(x)
\]

for some measure \( \alpha \). Further since \( SN = NS \), \( N \) can be represented as a diagonal operator of multiplication by the measurable operator function \( N(x) \) on \( \mathcal{K} \). Now for each \( x \), put \( \mathcal{K}(x) = \mathcal{K}_1(x) \oplus \mathcal{K}_2(x) \), where \( \mathcal{K} = [\text{Ran}(N(x))]' \) and \( \mathcal{K}_2(x) = \text{Ran}N(x) \). Then we can write \( N(x) \) as

\[
N(x) = \begin{bmatrix}
0 & 0 \\
\Gamma(x) & 0
\end{bmatrix}
\]

for some operator \( \Gamma(x) : \mathcal{K}_1(x) \rightarrow \mathcal{K}_2(x) \) which has dense range for each \( x \) (3.11.3). With respect to \( \mathcal{K}(x) = \mathcal{K}_1(x) \oplus \mathcal{K}_2(x) \), write

\[
S(x) = \begin{bmatrix}
S_1(x) & 0 \\
0 & S_2(x)
\end{bmatrix}
\]

where for \( i = 1, 2 \), \( S_i(x) \) is multiplication by \( x \) on \( \mathcal{K}_i(x) \) and \( S_i(x)^* = S_i(x) \). Then

\[
J = \begin{bmatrix}
S_1 & 0 \\
\Gamma & S_2
\end{bmatrix}
\]

on \( \mathcal{K}_1 \oplus \mathcal{K}_2 = (\int \oplus \mathcal{K}_1(x) d\alpha(x)) \oplus (\int \oplus \mathcal{K}_2(x) d\alpha(x)) \). Further, since \( SN = NS \), a quick calculation shows \( \Gamma(x)S_1(x) = S_2(x)\Gamma(x) \), for each \( x \).

It is left to prove 3.11.4. Towards this, define a subspace, \( \text{Dom} \, D \), of \( \mathcal{K} \) via

\[
\text{Dom} \, D = \{ h \in \mathcal{K}_1 : \exists k \in \mathcal{K}_2 \text{ with } \begin{bmatrix} h \\ k \end{bmatrix} \in \mathcal{H} \}.
\]

Let \( D \) be the linear operator on \( \text{Dom} \, D \) defined by: for \( h \in \text{Dom} \, D \), let \( Dh = k \) where \( k \) is the element of minimal norm in the closed subspace \( \{ \ell \in \mathcal{K}_2 : \begin{bmatrix} h \\ \ell \end{bmatrix} \in \mathcal{H} \} \).

We first show \( D \) is closed. Let \( \{ f_n \} \) be a sequence of functions in \( \text{Dom} \, D \) so that \( f_n \rightarrow f \) and \( Df_n \rightarrow g \). Now, \( \begin{bmatrix} f_n \\ Df_n \end{bmatrix} \in \mathcal{H} \) for all \( n \), and since \( \mathcal{H} \) is closed and \( \begin{bmatrix} f_n \\ Df_n \end{bmatrix} \rightarrow \begin{bmatrix} f \\ g \end{bmatrix} , \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{H} \). Thus \( f \in \text{Dom} \, D \). Let

\[
\hat{g}_n = D(f - f_n) \text{ and } g_n = Df_n.
\]
Then for each \( n \),

\[
\|\hat{g}_n\| = \inf \{ \|k\| \mid \begin{pmatrix} f - f_n \\ g - g_n \end{pmatrix} \in \mathcal{H} \}.
\]

Further, since \( \begin{pmatrix} f - f_n \\ g - g_n \end{pmatrix} \in \mathcal{H} \), we have

\[
\|\hat{g}_n\| \leq \|g - g_n\|
\]

for all \( n \). But \( \lim_{n \to \infty} \|g - g_n\| = 0 \). So

\[
\|Df - g\| \leq \|Df - Df_n\| + \|Df_n - g\|
= \|\hat{g}_n\| + \|g_n - g\| \to 0.
\]

Thus \( Df = g \).

Let \( h \in \text{Dom}D \) and choose \( k \in \mathcal{K}_2 \) so that \( \begin{pmatrix} h \\ k \end{pmatrix} \in \mathcal{H} \). Thus, by the invariance of \( \mathcal{H} \) for \( J \), we get

\[
\begin{pmatrix} S_1 & 0 \\ \Gamma & S_2 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} S_1 h \\ \Gamma h + S_2 k \end{pmatrix} \in \mathcal{H}.
\]

So \( S_1 h \in \text{Dom}D \) and we conclude that \( \text{Dom}D \) is invariant for \( S_1 \). Further, using \( \begin{pmatrix} S_1 h \\ DS_1 h \end{pmatrix} \) in the previous calculation gives \( DS_1 = \Gamma + S_2 D \).

Finally, defining \( \mathcal{N} \) to be the subspace of \( \mathcal{K}_2 \) given by

\[
\mathcal{N} = \{ k \in \mathcal{K}_2 : \begin{pmatrix} 0 \\ k \end{pmatrix} \in \mathcal{H} \}.
\]

Then obviously \( \mathcal{N} \subseteq (\text{Ran}D)^\perp \) and

\[
\mathcal{H} = \begin{bmatrix} f \\ D \end{bmatrix} \text{Dom}D + \begin{bmatrix} 0 \\ \mathcal{N} \end{bmatrix}.
\]

Moreover, if \( k \in \mathcal{N} \), then \( \begin{pmatrix} 0 \\ k \end{pmatrix} \in \mathcal{H} \) and

\[
J \begin{pmatrix} 0 \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ S_2 k \end{pmatrix} \in \mathcal{H}.
\]

Thus \( S_2 \mathcal{N} \subseteq \mathcal{N} \), and the proof is complete. □
LEMMA 3.12. Let $T$ be sub-Jordan on $\mathcal{H}$ with cyclic vector. Let $J$ be a minimal Jordan extension of $T$ on $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ satisfying 3.11.1, 3.11.2, and 3.11.3. Then there are compactly supported real measures $\mu$ and $\nu$, with

$$\mathcal{K}_1 = L^2(\mu), \mathcal{K}_2 = L^2(\nu)$$

(3.12.1)

and

$$S_i \text{ is multiplication by } x \text{ on } \mathcal{K}_i \ (i = 1, 2).$$

(3.12.2)

PROOF: By the proof of Lemma 3.11, for $i = 1, 2$

$$\mathcal{K}_i = (\int \oplus \mathcal{K}_i(x) d\alpha(x)).$$

Thus, to show 3.12.1 and 3.12.2, we argue that the fiber of $\mathcal{K}_i$ over $x$, for each $x$, is one-dimensional. Let $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ be a cyclic vector for $T$. We look for an invariant subspace for both $S$ and $N$ (and thus for $J$) which contains

$$\mathcal{H} = \left\{ p(T) \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} : p \text{ a polynomial} \right\}^-.$$

Using the easily verified identity

$$T^n = \begin{bmatrix} x & 0 \\ \Gamma(x) & x \end{bmatrix}^n = \begin{bmatrix} x^n & 0 \\ n\Gamma(x)x^{n-1} & x^n \end{bmatrix}$$

we see that $\mathcal{H}$ is also given by

$$= \left\{ p'(x)\Gamma(x)\xi_1(x) + p(x)\xi_2(x) \right\} : p \text{ a polynomial}.$$

Let $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ where

$$\mathcal{K}_1(x) = (\vee \{\xi_1(x)\})^-$$

and

$$\mathcal{K}_2(x) = (\vee \{\Gamma(x)\xi_1(x), \xi_2(x)\})^-.$$
Notice that \( \mathcal{H} \subseteq \mathcal{K} \) and since, if \( \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{K} \) then
\[
N \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ \Gamma(x)f(x) \end{bmatrix} \in \mathcal{K}
\]
and
\[
S \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} xf(x) \\ xg(x) \end{bmatrix} \in \mathcal{K},
\]
we have \( \mathcal{K} \) is invariant for both \( S \) and \( N \).

Hence, by minimality of \( J, \mathcal{K} \subseteq \mathcal{K} \), and in particular \( \mathcal{K}_1 \subseteq \mathcal{K}_1 \) which has a one dimensional fiber. Thus \( \mathcal{K}_1 \) also has a one dimensional fiber. To see that \( \mathcal{K}_2 \) has a one dimensional fiber, note that \( \mathcal{K}_2 = \text{Ran}N \) and
\[
N \begin{bmatrix} \xi_1(x) \\ \xi_2(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \Gamma(x)\xi_1(x) \end{bmatrix}.
\]
That is, the fiber of \( \text{Ran}N \) over \( x \) is spanned by \( \begin{bmatrix} 0 \\ \Gamma(x)\xi_1(x) \end{bmatrix} \) and therefore is one-dimensional.

We conclude that there exists real measures \( \mu \) and \( \nu \) so that up to unitary equivalence \( \mathcal{K}_1 = L^2(\mu) \) and \( \mathcal{K}_2 = L^2(\nu) \), and that \( S_i \) is multiplication by \( x \) on \( \mathcal{K}_i \) \((i = 1, 2)\). \( \square \)

**Lemma 3.13.** Under the assumptions of Lemma 3.12, the following hold:

There is a function \( \Gamma(x) \in L^2(\nu) \) so that for all \( h \in L^2(\mu), (\Gamma h)(x) = \Gamma(x)h(x) \) \( (3.12.1) \)

\[
\nu << \mu
\]
(3.12.2)

\[
\Gamma(x) \neq 0 \quad \nu - \text{a.e.}
\]
(3.12.3)

**Proof:** By 3.11.2, 3.12.1, and 3.12.2, the operator \( \Gamma : L^2(\mu) \rightarrow L^2(\nu) \) intertwines multiplication by \( x \) on \( L^2(\mu) \) and \( L^2(\nu) \). Citing Abrahamse \([A]\), we thus have there exists a function \( \Gamma(x) \) so that for all \( h \in L^2(\mu), (\Gamma h)(x) = \Gamma(x)h(x) \) which satisfies \( \mu \)-a.e. on the
set $E = \{x|(d\mu/d\nu)(x) > 0\}$ (where $d\mu/d\nu$ is the Radon-Nikodym derivative of $\mu$ with respect to $\nu$)

$$\Gamma(x) \leq c(d\mu/d\nu)^{1/2}, \text{ (some } c > 0) \quad (3.12.4)$$

and

$$\Gamma(x) = 0, \quad (x \notin E). \quad (3.12.5)$$

To show $\nu << \mu$, we argue by contradiction as follows.

Let $\nu = \nu_a + \nu_s$ where $\nu_a << \mu$ and $\nu_s \perp \mu$. Suppose there is a set $A$ so that $\nu_s(A) > 0$ while $\nu_a(A) = \mu(A) = 0$. Consider the function $\chi_A(x)$, a non-zero element of $L^2(\nu)$. Pick $h \in \text{Ran} \Gamma$. Then $h(x) = \Gamma(x)g(x)$ $\nu$- a.e., for some $g \in L^2(\mu)$. Since $\mu(A) = 0$, we may assume $g = 0$ on $A$. Then

$$<h, \chi_A >_{\mathcal{K}_2} = \int \Gamma(x)g(x)\overline{\chi_A(x)}d\nu$$
$$= \int_A \Gamma(x)g(x)d\nu = 0.$$

Thus $\chi_A$ is orthogonal to $\text{Ran} \Gamma$, but by 3.11.3, $\text{Ran} \Gamma$ is dense in $\mathcal{K}_2$, a contradiction. We conclude $\nu_s = 0$, implying $\nu << \mu$.

Now note that since $\nu << \mu$, the set

$$E = \{x|(d\mu/d\nu)(x) > 0\}$$
$$= \{x|(d\nu/d\mu)(x) > 0\}$$

is a carrier for $\nu$. Thus, by (3.13.4)

$$\int |\Gamma|^2d\nu = \int_E |\Gamma|^2d\nu \leq \int_E c|d\mu/d\nu|d\nu$$
$$= \int_E cd\mu < \infty,$$

and it follows that $\Gamma(x) \in L^2(\nu)$.

Finally, using again that $E$ is a carrier for $\nu$ and (3.13.4), we have

$$\frac{1}{\Gamma(x)} \geq \frac{1}{c(d\nu/d\mu)^{1/2}} > 0$$

$\nu$-a.e., and 3.13.3 follows. }
**Lemma 3.14.** If in addition to the assumptions of Lemma 3.12, $T$ is pure subjordan, then

$$\mathcal{H} = \begin{bmatrix} I \\ D \end{bmatrix} \text{Dom} D$$

(3.14.1)

The operator $D$ can be represented as $\theta + d/dx$ where $\theta \in L^2(\nu)$, and

$$\mathcal{H} = \{ \begin{bmatrix} p \\ Dp \end{bmatrix} | p \text{ a polynomial} \}^-$$

(3.14.2)

**Proof:** By 3.11.4

$$\mathcal{H} = \begin{bmatrix} I \\ D \end{bmatrix} \text{Dom} D + \begin{bmatrix} 0 \\ \mathcal{N} \end{bmatrix}$$

where

$$\begin{bmatrix} 0 \\ \mathcal{N} \end{bmatrix} = \mathcal{H} \cap \begin{bmatrix} 0 \\ \mathcal{K}_2 \end{bmatrix}.$$  

Thus $\begin{bmatrix} 0 \\ \mathcal{N} \end{bmatrix}$ is the intersection of subspaces invariant for $J$, and hence is itself invariant for $J$. Moreover, since $S_2\mathcal{N} \subseteq \mathcal{N}$ and $J \begin{bmatrix} 0 \\ \mathcal{N} \end{bmatrix} = \begin{bmatrix} 0 \\ S_2\mathcal{N} \end{bmatrix}$, it follows from Proposition 3.3, that $T$ is not pure. We conclude $\mathcal{N} = 0$, and 3.14.1 follows.

From Ball and Helton [BH], $T$ is unitarily equivalent to $\begin{bmatrix} S_1 & 0 \\ \Gamma & S_2 \end{bmatrix}$ restricted to

$$\left\{ \begin{bmatrix} p(x) \\ Dp(x) \end{bmatrix} | p \text{ a polynomial} \right\}^-$$

where $\xi$ is a cyclic vector for $T$, and $D = \theta + \Gamma d/dx$ for some $\theta \in L^2(\nu)$. Define the operator

$$U : L^2(\nu) \rightarrow L^2(|\Gamma|^2 d\nu)$$

via

$$(Uf)(x) = (f/\Gamma)(x)$$

for all $f \in L^2(\nu)$. Then

$$\|Uf\|^2_{L^2(|\Gamma|^2 d\nu)} = \int |f/\Gamma|^2 |\Gamma|^2 d\nu$$

$$= \|f\|^2_{L^2(\nu)},$$
so $U$ is isometric. Moreover from 3.13.1, 3.13.3, and 3.13.4, $|\Gamma|^2 \in L^1(d\nu)$ and \textit{esssuppt} $\Gamma = \text{sppt} \nu$, we have $U$ is in fact unitary. Let

$$V : L^2(\mu) \oplus L^2(\nu) \to L^2(\mu) \oplus L^2(|\Gamma|^2d\nu)$$

be defined by

$$V \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} f \\ g/\Gamma \end{bmatrix}.$$ 

Then $V$ is unitary and if

$$\hat{J} = \begin{bmatrix} S_1 & 0 \\ I & S_2 \end{bmatrix} : L^2(\mu) \oplus L^2(|\Gamma|^2d\nu) \to L^2(\mu) \oplus L^2(|\Gamma|^2d\nu)$$

Then $VJ = \hat{J}V$. Hence $J \cong \hat{J}$, and we may without loss of generality assume $\Gamma(x) \equiv 1 \nu$-a.e. (Thus $\Gamma : L^2(\mu) \to L^2(\nu)$ can be thought of as the inclusion map which under the assumptions is continuous.)

We now argue that if $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ is a cyclic vector for $T$ then $\xi_1 \neq 0 \mu$-a.e. If this were not the case then there would be a set $A$ with $\mu(A) > 0$ and $\xi_1 \equiv 0$ on $A$. By considering

$$\hat{\mathcal{K}} = \chi_A \mathcal{K}_a \oplus \mathcal{K}_2$$

we see that $\hat{\mathcal{K}}$ is invariant for both $S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ but strictly smaller than $\mathcal{K}$. This contradicts our assumption that $J$ was a minimal extension.

Let $U : L^2(\mu) \to L^2(|\xi_1|^2d\mu)$ be defined by $Uf = \xi_1^{-1}f$. Then $U$ is unitary. Let $\hat{D} = UDU^*$, where $D = \theta + d/dx$ given in the model developed by Ball and Helton [BH]. Then

$$(Df)(x) = \theta(x)f(x) + \xi_1(x)(d/dx)f(x)$$

and it follows that

$$\mathcal{H} = \left\{ \begin{bmatrix} D \\ Dp \end{bmatrix} \mid p \text{ a polynomial} \right\}^{-}.$$ 

Finally, by the argument above we may take $\xi = 1$ and thus we have 3.4.2. \qed
Theorem 3.5. If $T$ is a pure sub-Jordan operator with cyclic vector, then there are real measures $\mu$ and $\nu$ satisfying 3.8.1 through 3.8.5 and a $\theta \in L^2(\nu)$ so that $T \cong T(\mu, \nu, \theta)$ as defined in Definition 3.8.

Proof: Lemmas 3.11 through 3.14 give all of Definition 3.8, but 3.8.1, 3.8.3, 3.8.4, and 3.8.5. Citing Theorem 2.24 gives 3.8.4 and 3.8.5. To see 3.8.3, we use the fact that $\Gamma = 1$ is a bounded operator from $L^2(\mu)$ to $L^2(\nu)$. Thus for some $m > 0$, for all $f \in L^2(\mu)$,

$$||f||_{L^2(\nu)}^2 \leq m||f||_{L^2(\mu)}^2.$$  

Letting $s = d\nu/d\mu$, we then get for $f \in L^2(\mu)$

$$||f||_{L^2(\nu)}^2 = \int |f(x)|^2 s(x) d\mu$$

$$= \int |f(x)||s(x)|^{1/2}^2 d\mu$$

$$\leq m||f||_{L^2(\mu)}^2.$$  

This implies that $M_s : L^2(\mu) \rightarrow L^2(\mu)$, where $M_s$ is multiplication by $s$ is bounded. So, in fact, $s \in L^\infty(\mu)$.

It is left to show 3.8.1, that is $sppt\mu = sppt\nu$. Now since $\nu << \mu, sppt\nu \subseteq sppt\mu$ so we need only show $sppt\mu \subseteq sppt\nu$. Suppose there is a set $A \subseteq (sppt\mu) \setminus (sppt\nu)$ with $\mu(A) > 0$. Let $A = X_A L^2(\mu)$. Then $A \subseteq ker D$ and it follows that the subspace $M = \begin{bmatrix} A \\ 0 \end{bmatrix}$ is contained in $H$. Moreover, since $S_A A \subseteq A$ and

$$T|M = \begin{bmatrix} S_A & 0 \\ 0 & 0 \end{bmatrix}$$

we have by Proposition 3.3, $T$ is not pure, a contradiction. We conclude $sppt\mu = sppt\nu$.  

□

Corollary 3.16. A set $E$ is the spectrum of a pure sub-Jordan operator with cyclic vector if and only if $E$ is regularly closed.

Proof: Suppose $T$ is a pure subjordan operator on $H$ with minimal Jordan extension $J$ on $K$. By Theorem 3.15, we have $T$ is unitarily equivalent to an operator of the form
T(μ, ν, θ) as in Definition 3.8 on $H = \left\{ \left[ \begin{array}{c} p \\ Dp \end{array} \right] | p \text{ a polynomial} \right\}$ where $D = \theta + d/dx : L^2(μ) \to L^2(ν)$. Let $E = \text{spptμ} = \text{spptν}$. By Corollary 2.25, since $D$ is closed, it follows that $E$ is a regularly closed compact set. Thus we need only argue that $\sigma(T) = E$.

Since $J$ is assumed minimal, by Lemma 6.7 of [BH], we know $\sigma(J) \subseteq \sigma(T)$. Moreover, by Theorem 3.8 of Agler [A2], we have $\sigma(T) \setminus \sigma(J)$ is either empty or a union of components of $\mathbb{C} \setminus \sigma(J)$. But since $\sigma(J) \subseteq \mathbb{R}$, $\mathbb{C} \setminus \sigma(J)$ has exactly one component, itself. Thus $\sigma(T) \setminus \sigma(J) = \mathbb{C} \setminus \sigma(J)$ or is empty. The former is impossible since $\sigma(T)$ is bounded, so we conclude $\sigma(J) = \sigma(T)$. (Note: This argument does not rely on $T$ being cyclic.)

It is left to show $\sigma(J) = E$. Since

$$J = \begin{bmatrix} x & 0 \\ 1 & x \end{bmatrix} : L^2(μ) \oplus L^2(ν) \to L^2(μ) \oplus L^2(ν),$$

we have $\sigma(J) = \sigma(S)$, where $S = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$. That is, $S = S_1 \oplus S_2$ with $S_1 = M_x$ on $L^2(μ)$ and $S_2 = M_x$ on $L^2(ν)$. Thus

$$\sigma(J) = \sigma(S) = \sigma(S_1) \cup \sigma(S_2) = \sigma(S_1) \cup \sigma(S_2) = (\text{spptμ}) \cup (\text{spptν}) = E \cup E = E.$$  

Now let $E$ be any regularly closed compact subset of $\mathbb{R}$. Let $μ = ν = m_E$, where $m_E$ is Lebesgue measure restricted to $E$. Then $μ$ and $ν$ satisfy 3.8.1 through 3.8.5 of Definition 3.8, and 3.9.1 of Theorem 3.9. Constructing the operator $T = T(μ, ν, θ)$ as in Definition 3.8, we have $T$ is pure subjordan by Theorem 3.5. Finally, by the argument above $\sigma(T) = E$. □

3d. The General Case

Here we prove the main theorem of this chapter. To do this we model the general (i.e. possibly non-cyclic) pure subjordan in a slightly different manner than in the proof of Lemma 3.11.
**THEOREM 3.17.** A compact subset $E$ of the real line is the spectrum of a pure subjordan operator if and only if $E$ is regularly closed.

**Proof:** One direction of the implication follows from Corollary 3.16. For the other direction, let $T$ be pure subjordan on $\mathcal{H}$ with minimal Jordan extension $J = S + N$ on $\mathcal{K}$. Let $\mathcal{K}_1 = (\text{Ran}N)^\perp, \mathcal{K}_2 = \text{Ran}N$, and decompose

$$J = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Gamma \end{bmatrix}$$

with respect to $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Then (via Theorem 9.1, Chapter II, [C]) we represent $S_1$ as $\oplus \sum M_x$ on $\oplus \sum L^2(\mu_j)$ with $\mu_{j+1} << \mu_j$ for all $j$ and $\sigma(S_1) = \text{sppt}\mu_1$. Similarly $S_2$ can be represented as $\oplus \sum M_x$ on $\oplus \sum L^2(\nu_i)$ with $\nu_{i+1} << \nu_i$ for all $i$ and $\sigma(S_2) = \text{sppt}\nu_1$. Finally, $\Gamma$ can be viewed as a matrix operator, $\Gamma = [\Gamma_{ij}]$, where

$$\Gamma_{ij} : L^2(\mu_j) \to L^2(\nu_i)$$

is a bounded multiplication operator for all $i$ and $j$, via Abrahamse [A].

The idea of the proof is find a vector $\xi$ so that $T$ restricted to the cyclic subspace generated by $\xi$ has spectrum the same as $T$ and then cite Corollary 3.16.

We begin by showing $\sigma(T) = \sigma(J) = \text{sppt}\mu_1$. That $\sigma(T) = \sigma(J)$ follows in exactly the same manner as in the proof of Corollary 3.16. To see that $\sigma(J) = \text{sppt}\mu_1$, it suffices to show $\nu_1 << \mu_1$, since this implies $\text{sppt}\nu_1 \subseteq \text{sppt}\mu_1$, and thus

$$\sigma(J) = \sigma(S) = \sigma(S_1) \cup \sigma(S_2)$$

$$= \text{sppt}\mu_1 \cup \text{sppt}\nu_1 = \text{sppt}\mu_1. \quad (3.17.1)$$

So suppose $\nu_1$ is not absolutely continuous with respect to $\mu_1$. Then, if we write $\nu_1 = \nu_a + \nu_s$ where $\nu_a << \mu_1$, and $\nu_s \perp \mu_1$, there is a set $A$ with $\nu_s(A) > 0$ and $\nu_a(A) = \mu_1(A) = 0$. Let $f(x) = \oplus \sum_i f_i(x)$, where

$$f_i(x) = \begin{cases} \chi_A(x) & i = 1 \\ 0 & i \neq 1 \end{cases}.$$
Then $f(x)$ is nonzero in $K_2$. Let $h(x) = (\Gamma g)(x)$ for some $g \in K_1$. Then $h \in \text{Ran} \Gamma$ which is dense in $K_2$ by definition. Further, since $\mu_j << \mu_1$ for all $j$, and $\mu_1(A) = 0$, we may assume $g(x) = \oplus \sum g_j(x)$ satisfies $g_j(x) = 0$ on $A$ for all $j$. It follows that

$$< h, f >_{K_2} = \sum_{i=1}^{\infty} \int (\sum_{j=1}^{\infty} \Gamma_{ij}(x)g_j(x))f_i(x)d\nu_i(x) = \int (\sum_{j=1}^{\infty} \Gamma_{1j}(x)g_j(x))f_1(x)d\nu_1(x) = \int_A (\sum_{j=1}^{\infty} \Gamma_{1j}(x)g_j(x))d\nu_1(x) = 0.$$  

Thus $f$ is orthogonal to $\text{Ran} \Gamma$, a contradiction. We next show that there is vector $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \in H$ so that if $f_1(x) = \oplus \sum_{j=1}^{\infty} \xi_{ij}$, then $\xi_{11} \neq 0$ $\mu_1$-a.e. Let $C$ be the nonempty linear submanifold of $L^2(\mu_1)$ defined by

$$C = \{f_1 \in L^2(\mu_1) : \text{there is a } g \in K_2, f_j \in L^2(\mu)(j > 1) \text{ so that } f = \oplus \sum_{j=1}^{\infty} f_j \in K_1, \begin{bmatrix} f \\ g \end{bmatrix} \in H \}.$$  

Define a norm, $\| \cdot \|_C$, on $C$ via

$$\|f_1\|_C = \inf \{\| \begin{bmatrix} f \\ g \end{bmatrix} \|_C : f_j \in L^2(\mu_j)(j \geq 2), g \in K_2$$

such that $f = \oplus \sum_{j=1}^{\infty} f_j \in K_1, g \in K_2$ and $\begin{bmatrix} f \\ g \end{bmatrix} \in H \}$.

Under this norm, $C$ is closed. We now cite a result of Chaumat [Ch] to get the existence of $\xi_{11}$ in $C \subseteq L^2(\mu_1)$, so that for all $f \in C$, $f d\mu_1 << \xi_{11}d\mu_1$.

Before proceeding to prove $\xi_{11} \neq 0 \mu_1$ - a.e., we first argue that for any set $A$ with $\mu_1(A) > 0$, there is a $\begin{bmatrix} f \\ g \end{bmatrix} \in H$ so that $f_1 \neq 0$ on $A$ where $f = \oplus \sum_{j} f_j$. If this were not

Chaumat's Lemma 3 states that if $C$ is a closed, nonempty convex subset of $L^1(\mu)$ for a finite measure $\mu$, then there exists a $f_c \in C$ so that for all $f \in C$, $f d\mu << f_c d\mu$. But he actually proves a more general result: $C$ need only be closed in some norm. Using this and noting that $L^1(\mu_1) \subseteq L^2(\mu_1)$, we have the result cited.
the case, that is, if for all \( \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{H}, f_1 \equiv 0 \) on \( A \), then letting \( B = (\text{sppt} \mu_1) \setminus A \),

\[
\hat{\mathcal{K}}_1 = \chi_B L^2(\mu_1) \oplus \sum_{j=2}^{\infty} L^2(\mu_j)
\]

and \( \hat{\mathcal{K}} = \hat{\mathcal{K}}_1 \oplus \mathcal{K}_2 \), we have \( \hat{\mathcal{K}} \) contains \( \mathcal{H} \), is strictly smaller than \( \mathcal{K} \), and is invariant for both \( S \) and \( N \) (since these are multiplication operators), and hence for \( J \); this contradicts minimality.

Now suppose \( \xi_{11} = 0 \) on some set \( A \) with \( \mu_1(A) > 0 \). By the above there is a \( \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{H} \) so that \( f_1 \neq 0 \) on \( A \). But then

\[
\int_A |f_1| d\mu_1 > 0, \quad \text{while} \quad \int_A \xi_{11} d\mu_1 = 0
\]

contradicting that \( f_1 d\mu << \xi_{11} d\mu_1 \). It follows that \( \xi_{11} \neq 0 \mu_1 - a.e. \), or equivalently \( \text{ess sppt} \xi_{11} = \text{sppt} \mu_1 \).

Consider the closed subspace

\[
\mathcal{M} = [\vee \{ T^n \xi \}]^-.\]

By definition, \( \mathcal{M} \) is an invariant cyclic subspace and for \( T \) and if we let \( \hat{T} = T|\mathcal{M} \), then \( \hat{T} \) is subjordan. In fact, \( \hat{T} \) is pure, for if there is an invariant subspace \( \mathcal{M}_0 \) of \( \mathcal{M} \) on which \( \hat{T} \) is self-adjoint, then \( \mathcal{M}_0 \) is an invariant subspace for \( T \) on which it is self-adjoint. Let \( \hat{J} = \hat{S} + \hat{N} \) be a minimal Jordan extension. We claim

\[
\sigma(\hat{J}) = \sigma(\hat{T}) = \text{sppt} \mu_1 = \sigma(T) = \sigma(J). \tag{3.17.2}
\]

Since \( T \) is pure subjordan with cyclic vector, by Corollary 3.16, the spectrum is necessarily regularly closed. Thus once (3.17.2) is established it follows that \( \sigma(T) \) is regularly closed and the proof is complete.

Now the first (and as already noted, the last) equality of (3.17.2) follows from a phenomenon previously mentioned: the spectrum of a (real) subjordan operator is equal
to the spectrum of any minimal (real) Jordan extension (see proof of Corollary 3.16). So it is left to show \( \sigma(\hat{T}) = \text{sppt}\mu_1 \). Let \( \lambda \in \mathbb{R} \setminus (\text{sppt}\mu_1) \). Since \( \mu_i < \mu_1, \nu_1 < \mu_1, \) and \( \nu_i < \nu_1 \), for all \( i, j \), it follows that

\[
\lambda \in [\mathbb{C} \setminus (\text{sppt}\mu_j)] \cap [\mathbb{C} \setminus (\text{sppt}\nu_i)]
\]

for all \( i, j \). Thus \( g(x) = (x - \lambda)^{-1} \) is continuous on

\[
\bigcup_{i,j} ((\text{sppt}\mu_j) \cup (\text{sppt}\nu_i)) \subseteq \text{sppt}\mu_1.
\]

By adjusting the definition of \( g(x) \) on a small neighborhood of \( \lambda \) if necessary, we may assume that there is a continuously differentiable function \( g_0 \) defined on the closed convex hull \([a, b]\) of \( \text{supp}\mu_1 \) such that \( g_0|\text{supp}\mu_1 = g|\text{supp}\mu_1 \). Since \( g_0' \) is continuous on \([a, b]\), there is a sequence of polynomials \( \{p_n'\} \) converging uniformly to \( g_0' \) on \([a, b]\). Define a sequence of polynomials \( p_n \) by

\[
p_n(x) = \int_a^x p_n'(t)dt + g_0(a).
\]

Since, for \( x \in [a, b] \)

\[
g_0(x) = \int_a^x g'(t)dt + g_0(a),
\]

it follows that, for all \( x \in [a, b] \),

\[
|p_n(x) - g(x)| = |\int_a^x (p_n'(t) - g'(t))dt|
\]

\[
\leq (b - a)\sup\{|p_n'(x) - g_0'(x)| : x \in [a, b]\}.
\]

Thus the sequence \( \{p_n\} \) and its derived sequence \( \{p_n'\} \) converge to \( g \) and \( g_0' \), respectively uniformly on \([a, b]\), and hence also on \( \text{sppt}\mu_1 \). Using this, we have

\[
\lim_{n \to \infty} p_n(\hat{T})\xi = \left[ \frac{g(x)\xi_1(x)}{\Gamma(x)g_0(x)\xi_1 + g(x)\xi_2(x)} \right] \in \mathcal{M}.
\]

Further, since \( p_n \to g \) and \( p'_n \to g'_0 \) uniformly on \( \text{supp}\mu_1 \), we have \( p_n(\hat{T}) \to g(\hat{T}) = (\hat{T} - \lambda)^{-1} \) strongly. Therefore, \( \lambda \in \rho(\hat{T}) \). We conclude that \( \sigma(\hat{T}) \subseteq \text{sppt}\mu_1 \).
Now let $\lambda_0 \in \rho(\hat{T})$. Then, since $\sigma(\hat{T}) = \sigma(\hat{J}) = \sigma(S_1)$ (see (3.17.1)), we have $\lambda_0 \in \rho(S_1)$. Choose an open set $G$ so that $\lambda_0 \in G \subseteq \rho(S_1)$. Then, for all $\lambda \in G$, $(x - \lambda)^{-1}$ is a bounded multiplication operator on $L^2(\mu_1)$. Thus $(x - \lambda)^{-1}\xi_{11}(x) \in L^2(\mu_1) \subseteq L^1(\mu_1)$, for all $\lambda \in G$. Hence $G \cap \text{esssupp} \xi_{11} = \emptyset$. In particular, $\lambda_0 \notin \text{esssupp} \xi_{11} = \text{supp} \mu_1$. Thus $\text{supp} \mu_1 \subseteq \sigma(\hat{T})$, which completes the proof. $\Box$

4a. Introduction

In this chapter we consider a class of Banach algebras of functions which arise as the domain of a closed operator $D = \theta + d/dx$ as defined in Chapter 2. We identify the maximal ideal space and argue that the problem of spectral synthesis can be solved in this setting. It turns out that closed ideals are determined by the characterization of the closure of the graph of the operator $D$ in cases where the closure is definitive (see Table 2.1). Special cases of this result have been given by Sarason [S] (with more of a operator theoretic point of view) and by Jorgensen [J].

Though we include some standard definitions we refer the reader to Larsen's book [L] (or the appropriate chapter of almost any functional analysis text) for most of the basic definitions and elementary properties of commutative Banach algebras. We begin by introducing the class of functions of interest here.

Definition 4.1. Let $\mu$ and $\nu$ be positive finite measures both with common support in the interval $[0,1]$. Let $\nu << m$ where $m$ is Lebesgue measure restricted to $[0,1]$, and let $d\nu/dm = w$. Further, suppose $w^{-1} \in L^1(m)$ and let $\theta \in L^2(\nu)$. Define a normed linear space of functions $(A, \| \cdot \|_A)$ by the following:

4.1.1. $h \in A$ if and only if $h \in AC[0, 1]$ (absolutely continuous functions on $[0, 1]$) and for some $k \in L^2(\nu)$,

$$h(x) = e^{-\Theta(x)}c + e^{-\Theta(x)} \int_0^x e^{\Theta(t)}k(t)dt$$

where

$$\Theta(x) = \int_0^x \theta(t)dt.$$  

4.1.2. For $h \in A$,

$$\|h\|^2_A = \|h\|_{L^2(\mu)} + \|k\|^2_{L^2(\nu)}.$$ 

The following proposition describes the connection between $(A, \| \cdot \|_A)$ and the closure of
the graph of the operator $D$ as investigated in Chapter 2.

**Proposition 4.2.** The space $A$ can be identified as follows: if $D = \theta + d/dx : L^2(\mu) \rightarrow L^2(\nu)$ (with $\mu, \nu$, and $\theta$ defined as above) and if

$$G = \left\{ \begin{bmatrix} p \\ Dp \end{bmatrix} : p \text{ a polynomial} \right\},$$

then

$$G^{-1}(L^2(\mu) \oplus L^2(\nu)) = \begin{bmatrix} I \\ D \end{bmatrix} A.$$

In particular, $A$ is closed under $\| \cdot \|_A$.

**Proof:** In the language of Chapter 2, there is only one MILI for $w^{-1}$, namely the closed interval $[0,1]$. Referring to Definition 2.6, we see that elements of $A$ are precisely the first components of $\hat{G}(\mu - a.e.)$, by letting $I = [a,b] = [0,1]$ and noting that $J = I$.

Furthermore, citing Theorems 2.10 and 2.19, we see that $\hat{G} = G^-$. The key observation here is that if $h \in AC[0,1]$ and $k \in L^2(\nu)$ where $h$ is defined as in 4.1.1, then

$$Dh = \theta h + h' = k$$

and thus

$$\|h\|^2_A = \|h\|^2_{L^2(\mu)} + \|Dh\|^2_{L^2(\nu)}. \square$$

The next result gives that not only is $(A, \| \cdot \|_A)$ a Banach space but actually a Banach algebra.

**Theorem 4.3.** $(A, \| \cdot \|_A)$ is a closed commutative algebra with identity and there exists a norm, equivalent to $\| \cdot \|_A$, under which $A$ is a Banach algebra.

**Proof:** First note that if $f \in A$, then $f' \in L^2(\nu)$. To see this, let $f \in A$. Then by Proposition 4.2, $\theta f + f' \in L^2(\nu)$. Furthermore, since

$$\|\theta f\|_{L^2(\nu)} \leq \|\theta\|_{L^2(\nu)} \|f\|_{L^2[0,1]}.$$
we see that $\theta f \in L^2(\nu)$. Therefore,

$$f' = \theta f + f' - (\theta f) \in L^2(\nu).$$

Now let $f, g \in A$. Since both $f, g \in AC[0, 1]$, it follows that $(fg)^2 \in AC[0, 1]$, and

$$\|fg\|_{L^2(\mu)}^2 = \int_0^1 |fg|^2 d\mu \leq \|(fg)^2\|_{L^\infty[0,1]} \mu([0,1]) < \infty.$$

Furthermore,

$$\|\theta(fg) + (fg)\|_{L^2(\mu)}^2 = \|\theta fg + f'g + g'f\|_{L^2(\mu)}^2$$

$$\leq \|fg\|_{L^\infty[0,1]}^2 \|\theta\|_{L^2(\mu)}^2$$

$$+ \|f\|_{L^\infty[0,1]}^2 \|f'\|_{L^2(\mu)}^2 + \|g\|_{L^\infty[0,1]}^2 \|g'\|_{L^2(\mu)}^2$$

which is finite by the argument above. Noting that pointwise multiplication of functions is commutative and that the finiteness of $\mu$ together with $\theta \in L^2(\nu)$ implies $1 \in A$, we conclude that $A$ is a commutative algebra with unit.

To argue that $A$ may be interpreted as a Banach algebra, we cite a theorem of Gelfand (Theorem 1.3.1 of [L]). This result asserts the existence of a norm, equivalent to $\|\cdot\|_A$, under which $A$ is a Banach algebra provided that $A$ is a Banach space for which multiplication from $A \times A$ to $A$ is separately continuous. For our purpose, since $A$ is commutative, it suffices to show that for any given $f \in A$, the map $g \to fg$ is continuous from $A$ to $A$. Moreover, by the closed graph theorem, we need only show that $\{fgg : g \in A\}$ is closed in $A \times A$.

So let $g_n \to g$ and $fg_n \to h$ in $A$. Then, since $A$ is closed and hence $h \in A$, to complete the proof it is left to argue that $fg = h$. Now since $g_n \to g$ in $A$ implies $g_n \to g$ in $L^1(\mu)$, there is a subsequence converging pointwise $\mu$-a.e. to $g$. Similarly, $fg_n \to h$ in $A$ tells us that there is a subsequence converging pointwise $\mu$-a.e. to $h$; without loss of generality, we may assume $g_n \to g$ and $fg_n \to h$ pointwise $\mu$-a.e. Thus for $\mu$-a.e. $x \in [0,1]$,

$$|f(x)g(x) - h(x)| \leq |f(x)g(x) - f(x)g_n(x)|$$
+ |f(x)g_n(x) - h(x)|
\leq ||f||_{L^\infty[0,1]}|g(x) - g_n(x)|
+ |f(x)g_n(x) - h(x)|

which approaches zero. Therefore \(fg = h\) in \(A\). \(\square\)

Here we should note some justifications for our preclusions. The assumption that \(w^{-1} \in L^1(m)\) is crucial for this gives that the only MILI for \(w^{-1}\) is closed. Without this we could only assume \(f \in AC_{loc}(I)\) and still maintain the connection with the closure of the graph of \(D\). But then \(f\) may not be uniformly bounded and \(A\) would lose its multiplicative structure. Also, if \(\nu\) is not absolutely continuous with respect to Lebesgue measure, or if the set where \(w^{-1}\) is not locally integrable has positive \(\nu\)-measure, or if the interior of the support of \(\mu\) contains an interval outside the support of \(wdx\), then by Theorem 2.24, \(D = \theta + d/dx\) is not closable. Hence \(A\) cannot have both an algebra and graph structure. Finally, the case where \(sppt\nu\) consists of more than one closed MILI for \(w^{-1}\) can be reduced to the case of a single MILI by considering direct sums. (For this, see Lemma 2.13 which gives that the closure of \(\mathcal{G}\) is independently determined on distinct MILI’s for \(w^{-1}\)).

Thus from a Banach algebra point of view and in keeping with the previous work of this paper, it seems natural to look at the case we are presently considering.

4b. The Maximal Ideal Space and Spectral Synthesis

Having established that \(A\) is a commutative Banach algebra with identity, it is natural to investigate its maximal ideal space. The standard technique for such an investigation is to consider the kernels of a certain class of linear functionals on \(A\). By the term \textit{continuous multiplicative linear functional on} \(A\), we mean a linear functional \(\tau : A \to \mathbb{C}\) which is multiplication preserving and continuous in the Banach algebra norm (or equivalently
\[ \Delta(A) = \{ \tau : \tau \text{ is a continuous multiplicative linear functional on } A \}. \]

The idea we use is that there is a one-to-one onto correspondence between \( \Delta(A) \) and the collection of maximal ideals of \( A \) determined by the mapping \( \tau \to \ker(\tau) \). With respect to this association, we call \( \Delta(A) \) the **maximal ideal space of \( A \)** and give it the relative weak-* topology. We can think of its elements as functionals or maximal ideals as needed in context, and we do so. To begin our analysis, we prove the following result.

**Lemma 4.4.** For every \( x_0 \in [0, 1] \), the map \( \delta_{x_0} : A \to \mathbb{C} \), defined by \( \delta_{x_0}(f) = f(x_0) \) belongs to \( \Delta(A) \).

**Proof:** Let \( x_0 \in [0, 1] \). Clearly \( \delta_{x_0} \) is a multiplicative linear homomorphism from \( A \) to \( \mathbb{C} \), so we need only show \( \delta_{x_0} \) is continuous. By the nature of the definition of \( f \in A \) (4.1.1), we begin with the special case \( x_0 = 0 \).

Let \( f_n \to f \) in \( A \). Then \( f_n \to f \) in \( L^2(\mu) \) and \( k_n = \theta f_n + f'_n \to \theta f + f' = k \) in \( L^2(\nu) \).

Since
\[
\int_0^1 |k_n - k|dm \leq \|k_n - k\|_{L^2(\nu)}w^{-1}\|\frac{1}{k}\|_{L^1(m)},
\]
we have \( k_n \to k \) in \( L^1(m) \). By 4.1.1, solving for \( c_n \), we get
\[
c_n = e^{\Theta(x)}f_n(x) - \int_0^x e^{\Theta(t)}k_n(t)dt. \tag{4.4.1}
\]

Furthermore, \( e^{\Theta} \in L^\infty[0, 1] \) implies that \( \mu \)-a.e.
\[
c_n \to e^{\Theta(x)}f(x) - \int_0^x e^{\Theta(t)}k(t)dt.
\]

Now, by definition
\[
f(x) = e^{-\Theta(x)}c + e^{-\Theta(x)}\int_0^x e^{\Theta(t)}k(t)dt
\]
and solving for \( c \), we get
\[
c = e^{\Theta(x)}f(x) - \int_0^x e^{\Theta(t)}k(t)dt. \tag{4.4.2}
\]
But, by letting \( x = 0 \) in (4.4.1) and (4.4.2), we have \( c = f(0) \) and \( c_n = f_n(0) \) for all \( n \). It follows that \( f(0) = \lim_{n \to 0} f(0) \), and hence \( \delta_{x_0} \) is continuous for \( x_0 = 0 \).

Now let \( x_0 \) be an arbitrary point in \([0,1]\). Using Theorem 1.18 [R], to show \( \delta_{x_0} \) is continuous, it suffices to show

\[
\ker \delta_{x_0} = \{ f \in A : \delta_{x_0}(f) = f(x_0) = 0 \}
\]

is closed. Let \( f_n \to f \) in \( A \) so that \( f_n(x_0) = 0 \) for all \( n \), and let \( k_n = \theta f_n + f'_n \) and \( k = \theta f + f' \). Then

\[
|f(x_0)| = |f_n(x_0) - f(x_0)| \\
= |e^{-\Theta(x_0)}[(f_n(0) - f(0)) \\
+ \int_0^{x_0} e^{\Theta(t)}(k_n(t) - k(t))dt]| \\
\leq e^{-\Theta(x_0)}|f_n(0) - f(0)| + ||e^{\Theta}||_{L^\infty[0,1]}||k_n - k||_{L^1(m)}. 
\]

By our preliminary arguments, we know \( f_n(0) \to f(0) \) and \( ||k - k_n||_{L^1(m)} \to 0 \). Thus \( f(x_0) = 0 \) and \( \ker(\delta_{x_0}) \) is closed. \( \square \)

The previous lemma gives a mapping from the interval \([0,1]\) into \( \Delta(A) \) which is clearly one-to-one since \( A \) contains the polynomials and hence separates points. The following result gives that this map is also onto.

**Theorem 4.5.** The maximal ideal space of \( A, \Delta(A) \), is topologically isomorphic to \([0,1]\) and the Gelfand transform \( \tilde{f} : A \to C(\Delta(A)) \) is the identity.

**Proof:** By Lemma 4.4 and the subsequent comment, to show there exists a bijection from \([0,1]\) to \( \Delta(A) \), we need only show that if \( \tau \in \Delta(A) \), then there is an \( x_0 \in [0,1] \) so that \( \tau(f) = f(x_0) \) for all \( f \in A \).

So let \( \tau \in \Delta(A) \). The function \( i(x) = x \) belongs to \( A \), so let \( x_0 = \tau(i) \). We first show \( x_0 \in [0,1] \). To do this we utilize the fact that if \( f \in A \) so that \( f(x) \neq 0 \) for all...
$x \in [0,1]$, then $f^{-1} \in A$; which we argue as follows. As noted, since $f \in L^2(\mu)$ we have that $f' \in L^2(\nu)$. Further, if $f$ is never zero, then $1/f \in AC[0,1]$ and there is an $\epsilon > 0$ so that $|f(x)| \geq \epsilon$ for all $x \in [0,1]$. Thus to prove $f^{-1} \in A$, it is left to show $\theta(f^{-1}) + (f^{-1})' = \theta f^{-1} - f'f^{-2} \in L^2(\nu)$. But

$$\|\theta f^{-1} - f'f^{-2}\|^2_{L^2(\nu)} \leq 1/\epsilon^2\|\theta\|^2_{L^2(\nu)} + 1/\epsilon^4\|f'\|^2_{L^2(\nu)},$$

which is finite.

Now for each $c \in \mathbb{C} \setminus [0,1], f(x) = x - c$ is not zero on $[0,1]$. Hence $f$ is invertible in $A$ and thus $\tau(f) \neq 0$. But $\tau(f) = \tau(i) - c = x_0 - c \neq 0$. We conclude $x_0 \in [0,1]$.

Next we show $\tau(f) = f(x_0)$, for all $f \in A$. By Proposition 4.2, the polynomials are dense in $A$. Moreover, since $\tau$ is multiplicative and linear, for any polynomial $p(x) = \sum_{k=0}^{m} a_k x^k$,

$$\tau(p) = \sum_{k=0}^{m} a_k \tau(i)^k = p(x_0).$$

Let $f \in A$, and let $\{p_n\}$ be a sequence of polynomials so that $p_n \rightarrow f$ in $A$. Then

$$|\tau(f) - f(x_0)| \leq |\tau(f) - \tau(p_n)| + |\tau(p_n) - f(x_0)|$$

$$= |\tau(f - p_n)| + |\delta_{x_0}(p_n - f)|.$$

Now $\tau(f - p_n) \rightarrow 0$ since $\tau$ is assumed to be continuous and $\delta_{x_0}(p_n - f) \rightarrow 0$ since $\delta_{x_0}$ is continuous by Lemma 4.4. It follows that $\tau = \delta_{x_0}$.

To see that the topologies coincide, first let $x_n \rightarrow x$ in $[0,1]$. Since every $f \in A$ is continuous, it then follows that $\delta_{x_n}(f) = f(x_n) \rightarrow f(x) = \delta_x$ for all $f \in A$. Thus $\delta_{x_n} \rightarrow \delta_x$ in $\Delta(A)$ with the relative weak-* topology. Secondly, if $\delta_{x_n} \rightarrow \delta_x$ in $\Delta(A)$, then $\delta_{x_n}(f) \rightarrow \delta_x(f)$ for all $f \in A$. But using $i(x) = x$, we get $x_n \rightarrow x$ in $[0,1]$.

Finally, the Gelfand transform is defined by the formula

$$\hat{f}(\delta_x) = \delta_x(f) = f(x)$$
and thus is the identity. □

Having characterized the maximal ideal space of \( A \), we proceed to lay the groundwork for the problem of spectral synthesis and the main result of this chapter. We begin by showing that \( A \) is semisimple and regular. By *semisimple* we mean that the radical of \( A \), \( \text{Rad}(A) \), contains only the zero function, where

\[
\text{Rad}(A) = \cap \{ \mathcal{M} : \mathcal{M} \text{ a maximal ideal of } A \}.
\]

The property of regularity is defined in terms of the hull-kernel topology on \( \Delta(A) \). Towards this, let \( E \) be a closed subset of \( \Delta(A) = [0, 1] \). The *kernel of \( E \)*, \( k(E) \), is defined to be

\[
k(E) = \{ f \in A : f(x) = 0 \text{ for all } x \in E \}.
\]

Let \( I \) be an ideal of \( A \). Then the *hull of \( I \)*, \( h(I) \), is defined as

\[
h(I) = \{ x \in [0, 1] : f(x) = 0 \text{ for all } f \in I \}.
\]

The *hull-kernel topology on \( \Delta(A) \)* is the topology generated by taking as the closure of a set \( E \subset \Delta(A) \), the set \( h(k(E)) \). A commutative Banach algebra is said to be *regular* if the hull-kernel topology is Hausdorff (or equivalently the hull-kernel and relative weak-* topologies on \( \Delta(A) \) coincide (Theorem 7.12 [L])).

**Theorem 4.6.** \( A \) is semisimple and regular.

**Proof:** By Proposition 7.1.1 of [L], \( k[\Delta(A)] = \text{Rad}(A) \). Using this, we have

\[
k[\Delta(A)] = k[[0, 1]] = \{ f \in A : f(x) = 0 \forall x \in [0, 1] \} = \{0\}.
\]

Thus \( A \) is semisimple.

To see that \( A \) is regular we first use that it is sufficient to show for any closed set \( E \subset [0, 1] \), for any \( x_0 \notin E \), there exists a function \( f \in A \) so that \( f|_E \equiv 0 \), but \( f(x_0) \neq 0 \) (see Theorem 7.1.2 of [L]). So choose a continuously differentiable function \( f \) so that \( f|_E \equiv 0 \)
and \( f(x_0) = 1 \). We want to show \( f \in A \). Since \( f' \) is continuous and bounded on \([0,1]\), we can choose a sequence of polynomials \( \{p'_n\} \) so that \( p'_n \to f' \) uniformly on \([0,1]\). Let

\[
p_n(x) = \int_{x_0}^{x} p'_n(t)dt + 1
\]

and notice that

\[
f(x) = \int_{x_0}^{x} f'(t)dt + 1.
\]

Thus

\[
|p_n(x) - f(x)| = |\int_{x_0}^{x} (p'_n(t) - f'(t))dt| \\
\leq \int_{0}^{1} |p'_n(t) - f'(t)|dt \\
\leq \|p'_n - f'\|_{L^\infty[0,1]}.
\]

Therefore \( p_n \to f \) uniformly on \([0,1]\). It follows that \( p_n \to f \) in \( L^2(\mu) \) and \( \theta p_n + p'_n \to \theta f + f' \) in \( L^2(\nu) \). Hence \( p_n \to f \) in \( A \), and it follows that \( f \in A \). \( \square \)

We are now ready to present and prove the main result of this chapter. For a semisimple regular commutative Banach Algebra \( A \), a closed set \( E \subset \Delta(A) \) is said to be a set of spectral synthesis if \( I = k(E) \) is the only closed ideal in \( A \) such that \( h(I) = E \). The problem of spectral synthesis is the determination of exactly which closed sets in \( \Delta(A) \) are sets of spectral synthesis. We show all closed sets are of this type in this instance. The idea of the proof is as follows: if \( E \subset \Delta(A) = [0,1] \) is closed, if \( I = k(E) \) and if \( J \) is any closed ideal in \( A \) so that \( h(J) = E \), then \( J \subset I \). Furthermore, there is an ideal \( I_g \), generated by a particular function \( g \in J \), so that \( I_g^{\|\cdot\|_A} \) (the closure of \( I_g \) in \( A \)) = \( I \). But noting \( I_g \subset J \subset I \), we conclude \( J = I \). The characterization of the norm closure of \( I_g \) relies heavily on the techniques and results of Chapter 2.

**Theorem 4.7.** Every closed set in \( \Delta(A) \) is a set of spectral synthesis.
PROOF: Let $E \subseteq [0,1]$ be closed, let

$$I = k(E) = \{ f \in A : f(x) = 0 \text{ for all } x \in E \}$$

and let $J$ be any closed ideal in $A$ so that $h(J) = E$. Then if $f \in J$, we have $f|_E \equiv 0$. Hence $f \in I$ and it follows that $J \subseteq I$.

We claim there is a function $g \in J$ so that the zero set of $g$,

$$Z(g) = \{ x \in [0,1] : g(x) = 0 \},$$

is exactly $E$. Indeed, since

$$E = h(J) = \{ x \in [0,1] : f(x) = 0 \text{ for all } f \in J \},$$

for each $x \in [0,1] \setminus E$ there is a $f_x \in J$ and a neighborhood $U_x$ of $\{x\}$ so that $f_x(y) \neq 0$ for all $y \in U_x$. Without loss of generality, we assume $\|f_x\|_A \leq 1$. Since $[0,1] \setminus E$ is relatively open and hence a countable union of relatively open sets whose closures are compact, clearly we can extract a countable subcover $\{U_x, : i = 1, 2, 3, \ldots\}$ of $[0,1] \setminus E$ from $\{U_x : x \in [0,1] \setminus E\}$. Let

$$g(x) = \sum_{i=1}^{\infty} 2^{-i}|f_x|^2.$$ 

Then $g \in J$, since $J$ is a closed ideal, and is nonzero on $[0,1] \setminus E$. Therefore $Z(g) = E$.

Now let

$$I_g = \{ fg : f \in A \}.$$

Clearly $I_g$ is an ideal with $I_g \subseteq J \subseteq I$. We show $I_g^{-\|\cdot\|_A} = I$, from which we may conclude $J = I$ as needed.

To do this, we characterize the closure of $\{pg : p \text{ a polynomial}\} \subseteq I_g$ by equating it to the closure of a graph as in Chapter 2. Consider the unitary transformation

$$U : L^2(\mu|(sptg) \oplus L^2(\nu|sptg) \to L^2(|g|^2d\mu) \oplus L^2(|g|^2d\nu).$$
via $U(h \oplus k) = h g^{-1} \oplus k g^{-1}$. $U$ transforms our problem to one of characterizing the closure of

$$
\mathcal{G} = \{ p \oplus \bar{\theta} p + p' : p \text{ a polynomial} \}
$$

in $L^2(\mu) \oplus L^2(\nu)$, where $\bar{\theta} = \theta + g' g^{-1} \in L^2(\nu) = L^2(|g|^2 d\nu)$ and $d\bar{\mu} = |g|^2 d\mu$. This is easily seen by the computation: for $p$ a polynomial,

$$
\|pg\|_A^2 = \|pg\|_{L^2(\mu)}^2 + \|\theta pg + pg' + p'g\|_{L^2(\nu)}^2
$$

$$
= \int |p|^2 |g|^2 d\mu + \int |\theta pg + pg' + p'g|^2 d\nu
$$

$$
= \|p\|_{L^2(|g|^2 d\mu)}^2 + \|\|(\theta + g' g^{-1})p + p'|^2|g|^2 d\nu
$$

$$
= \|p\|_{L^2(\mu)}^2 + \|\theta p + p'|_{L^2(\nu)}^2.
$$

As before the analysis of the closure of $\mathcal{G}$ depends on investigating the nature of the endpoints of the MILI's for $\bar{w}^{-1} = (|g|^2 w)^{-1}$ and $e^{2Re \bar{\theta}} \bar{w}^{-1}$. Since $g$ is continuous, the set $[0,1] \setminus Z(g) = \{ x : g(x) \neq 0 \}$ is open relative to $[0,1]$. Thus we may write

$$
[0,1] \setminus Z(g) = \cup_{i=1}^{\infty} \tilde{I}_i \subseteq \text{sppt } g
$$

where each $\tilde{I}_i$ is an open interval possibly modulo the endpoints 0 and 1 and where the union is disjoint.

We restrict our attention to one such $\tilde{I}_i$ with endpoints $a$ and $b$. If $x$ is an interior point of $\tilde{I}_i$, then $g$ is bounded away from zero in a neighborhood of $x$. Throughout we have assumed $w^{-1} \in L^1(m)$; thus $w^{-1}$ is both right and left integrable at $x$. We conclude the same holds for $\bar{w}^{-1} = (|g|^2 w)^{-1}$. Next let $x_0$ be an interior point of $\tilde{I}_i$ and set

$$
\bar{\sigma}(x) = \int_{x_0}^{x} \bar{\theta}(t) dt, \quad x \in \tilde{I}_i.
$$

Then

$$
\bar{\sigma}(x) = \int_{x_0}^{x} (\theta + g' g^{-1})(t) dt
$$
where \( c_0 \) is a constant. Therefore, \( e^{\hat{\Theta}(x)} = c(e^\Theta g)(x) \) with \( c \) being a nonzero constant.

Looking at endpoints, for small \( \delta > 0 \), we have

\[
\int_a^{a+\delta} e^{2\text{Re}\Theta(x)} \tilde{w}^{-1}(x) dx = |c|^2 \int_a^{a+\delta} e^{2\text{Re}\Theta(x)} w^{-1}(x) dx < \infty,
\]

since \( w^{-1} \in L^1(m) \) implies \( e^{2\text{Re}\Theta} w^{-1} \in L^1(m) \). Similarly,

\[
\int_{b-\delta}^b e^{2\text{Re}\Theta(x)} \tilde{w}^{-1}(x) dx = |c|^2 \int_{b-\delta}^b e^{2\text{Re}\Theta(x)} w^{-1}(x) dx < \infty.
\]

Thus in the notation of Chapter 2, \( \tilde{J}_i = [a, b] \).

Now to determine the MILI's for \( \tilde{w}^{-1} \), suppose \( a \notin \tilde{I}_i \). Then \( g(a) = 0 \), and since \( g \in A \) we have for \( x \in \tilde{I}_i \),

\[
g(x) = e^{-\Theta(x)} \int_a^x e^{\Theta(t)}(\theta(t)g(t) + g'(t)) dt.
\]

By using \( d\nu = w dx \) and the Schwarz inequality we get

\[
|g(x)|^2 \leq e^{-2\text{Re}\Theta(x)} \|e^{\Theta(\theta g + g')}\|_{L^2(\nu)}^2 \int_a^x w^{-1}(t) dt.
\]

Thus

\[
\int_a^{a+\delta} \tilde{w}^{-1}(x) dx = \int_a^{a+\delta} |g(x)|^{-2} w^{-1}(x) dx
\]

\[
\geq \sup_{a \leq x \leq a+\delta} \left\{ e^{2\text{Re}\Theta(x)} \right\} \|e^{\Theta(\theta g + g')}\|_{L^2(\nu)}^{-2} \left( \int_a^{a+\delta} (w^{-1}(x) / \int_a^x w^{-1}(t) dt) dx \right)
\]

\[
= m \|e^{\Theta(\theta g + g')}\|_{L^2(\nu)}^{-2} [\ln \left\{ \int_a^x w^{-1}(t) dt \right\}]_{x=a+\delta}^{x=a}
\]

\[
= m \|e^{\Theta(\theta g + g')}\|_{L^2(\nu)}^{-2} [\ln(\int_a^{a+\delta} w^{-1}(t) dt - \ln(\int_a^a w^{-1}(t) dt))]
\]

\[
= \infty.
\]

In the same way, if \( b \in \tilde{I}_i \) then \( g(b) = 0 \), and for \( x \in \tilde{I}_i \),

\[
g(x) = -e^{-\Theta(x)} \int_b^x e^{\Theta(t)}(\theta(t)g(t) + g'(t)) dt,
\]
and it follows, arguing as above, that $\int_{0-\delta}^{b-\delta} \hat{w}^{-1} dm = \infty$. Therefore, if $\tilde{I}_i = (a, b)$, which is the case where $a \neq 0$ and $b \neq 1$, then the corresponding MILI for $\hat{w}^{-1}$ is $\tilde{I}_i = (a, b)$.

Considering $a = 0$ and $g(a) \neq 0$, we have, since $g$ is continuous and $w^{-1} \in L^1(m)$, for appropriate $\delta > 0$,

$$
\int_{0}^{\delta} \hat{w}^{-1}(x)dx = \int_{0}^{\delta} |g|^{-2} w^{-1}(x)dx < \infty.
$$

Similarly, if $b = 1$ and $g(b) \neq 0$, we have $\int_{\delta}^{1} \hat{w}^{-1}(x)dx < \infty$.

In summary, when restricting to a $\tilde{I}_i$ with endpoints $a$ and $b$, the corresponding MILI for $\hat{w}^{-1}$ is $\tilde{I}_i$ and has one of the following forms: $(a, 1], [0, b), (a, b)$, or $[0, 1]$ while $\tilde{J}_i = [a, b]$ in each case. Noting that

$$
\int_{0}^{1} e^{-2Re\Theta} d\hat{\mu} = \int_{0}^{1} e^{-2Re\Theta} |g|^{-2}(|g|^2 d\mu)
$$

$$
= \int_{0}^{1} e^{-2Re\Theta} d\mu
$$

$$
\leq M\mu([0, 1]) < \infty,
$$

we see we are in cases in which the analysis of Chapter 2 is complete (see cases 1, 4.a, 5.a of Table 2.1).

Consequently, the closure of $\tilde{G}$ is $L^2(\hat{\mu}) \oplus L^2(\hat{\nu})$ is exactly the space described by Definition 2.6. That is, $(\tilde{G})^{-L^2(\hat{\mu})\oplus L^2(\hat{\nu})}$ is the collection of all pairs $\tilde{h} \oplus \tilde{k} \in L^2(\hat{\mu}) \oplus L^2(\hat{\nu})$ so that, restricting to a particular $\tilde{I}_i$ with endpoints $a$ and $b$,

$$
e^{\Theta} h(x) = e^{\Theta} h(y) + \int_{y}^{x} (e^{\Theta} \tilde{k})(t)dt \quad (4.7.1)$$

and, if $a \in \tilde{J}_i \setminus \tilde{I}_i$, then

$$
e^{\Theta} h(x) = \int_{a}^{x} (e^{\Theta} k)(t)dt \quad (4.7.2)$$

and, if $b \in \tilde{J}_i \setminus \tilde{I}_i$, then

$$
e^{\Theta} h(x) = -\int_{x}^{b} (e^{\Theta} k)(t)dt. \quad (4.7.3)$$
Now (4.7.2) implies \((e^{\hat{\Theta}} h)(a) = 0\) when \(a \in \tilde{J}_i \setminus \tilde{I}_i\) which from above we have seen occurs when \(g(a) = 0\). In like manner, (4.7.3) gives that \((e^{\hat{\Theta}} h)(b) = 0\) when \(g(b) = 0\). Summarizing, for \(x \in \bigcup_i \tilde{I}_i\), \((e^{\hat{\Theta}} h)(x) = 0\) when \(g(x) = 0\).

In order to show \(I_g^{-\|\cdot\|} = k(E)\), it remains to prove, if \(h \in A\) and has the property that \(h(x) = 0\) when \(g(x) = 0\), then \(h \in I_g^{-\|\cdot\|}\). We accomplish this by showing that, for such an \(h\), we have

\[
U h \oplus \tilde{D}(U h) = \tilde{h} \oplus \tilde{k}
\]

satisfies (4.7.1), (4.7.2), and (4.7.3) and is thus in the closure of \(\tilde{G}\).

Since \(U h = h^{-1} g = \tilde{h}\) and \(e^{\hat{\Theta}} = ce^{\Theta} g (c \neq 0)\), it follows that \(e^{\Theta(x)} h(x) = c^{-1} e^{\hat{\Theta}(x)} \tilde{h}(x)\). So we need only demonstrate that \(e^{\hat{\Theta}} \tilde{h}\) satisfies (4.7.1), (4.7.2), and (4.7.3) given that \(h\) is zero when \(g\) is zero. First of all, for \(x, y \in \tilde{I}_i\)

\[
(e^{\hat{\Theta}} \tilde{h})(y) - (e^{\hat{\Theta}} \tilde{h})(x) = c[(e^{\Theta} h)(y) - (e^{\Theta} h)(x)]
\]

\[
= c \int_x^y (e^{\Theta} k)(t) dt \\
= c \int_x^y c^{-1} e^{\hat{\Theta}(t)} g^{-1}(t) k(t) dt \\
= \int_x^y (e^{\hat{\Theta}} \tilde{k})(t) dt.
\]

Moreover, (4.7.2) and (4.7.3) are satisfied if we can exhibit that \((e^{\hat{\Theta}} h)(a) = 0\) when \(g(a) = 0\), and similarly for \(b\). But, if \(g(a) = 0\) then \(h(a) = 0\) implying \(e^{\Theta(a)} h(a) = e^{\hat{\Theta}(a)} \tilde{h}(a) = 0\). In like manner, if \(g(b) = 0\), then \(e^{\hat{\Theta}(b)} \tilde{h}(b) = 0\). Therefore \(h \in k(E)\) results in \(\tilde{h} \oplus \tilde{k} \in \tilde{G} L^2(\tilde{I}) L^2(\tilde{I})\) which gives that \(h \in I_g^{-\|\cdot\|}\). This completes the proof of the theorem and Chapter 4. \(\square\)
5. Complex Jordan and Subjordan Operators.

We conclude with a brief examination of how some of the work done in Chapter 2 and Chapter 3 may be expanded into the complex plane. Though few results are actually proved, it is our purpose to pose some questions, point out some difficulties, relate our ideas to work previously done by others, and to formulate reasonable conjectures.

5a. Definitions and a formulation of purity

Definition 5.1. An operator $J$ on a Hilbert space $K$ is said to be complex jordan (of order 2), if $J = M + N$ where $M$ is normal, $MN = NM$, and $N^2 = 0$. An operator $T$ is said to be complex subjordan (of order 2) on $H$ if there is a complex jordan $J$ on $K \supset H$ so that $T = J|_H$.

Complex jordan operators have been studied in [A1], [A2], [B], and [BH]. In particular, Bunce ([B]) characterized all complex jordan operators as ones satisfying three algebraic equations in a way analogous to Helton's algebraic formulation for real jordan operators ($J$ and $J^*$ being coadjoint via Definition 3.4 [H3]). Complex subjordan operators have been considered in [A1], [A2], and [BH]. The primary aim of this discussion is to define a notion of pure complex subjordan which generalizes simultaneously the notions of purity for the case of subnormal (see [C]) and for real subjordan as in Chapter 3 of this paper. To serve these dual purposes we actually give three notions which we will belatedly prove to be equivalent.

Definition 5.2. Let $T$ be a complex subjordan operator on $H$ with complex jordan extension $J = M + N$ on $K$. Let $H_0 \subseteq H$ be a proper invariant subspace for $T$. Then we say

\[ T \text{ is pure, if } T|_{H_0} \text{ being complex jordan implies } H = \{0\}, \text{ or } \]

\[ T \text{ is pure, if whenever } H_0 \text{ is invariant for } M, M^*, \text{ and } N \text{ then } H_0 = \{0\}, \text{ or } \]

\[ (5.2.1) \]

\[ (5.2.2) \]
$T$ is pure, if $T|H_0$ being normal implies $H_0 = \{0\}$. \hfill (5.2.3)

We should note that in the real case (i.e. where $M = M^*$), the analogous meanings of purity coincide. To be exact, in 5.2.1, if the word "complex" is dropped, we have Definition 3.2, while Proposition 3.3 (replacing normal with self-adjoint) gives 5.2.3. Finally, again taking $M = M^*$, Proposition 3.6 is exactly 5.2.2. So we see Definition 5.2 is consistent with, in the natural way, Definition 3.2.

Moreover these three notions generalize the idea of purity for subnormal operators. A pure subnormal is the restriction of a normal to an invariant subspace so that it has no nonzero reducing subspace on which it is normal (see Definition 2.2, p. 127 [C]). Clearly, a subnormal operator $A$ can be thought of as complex subjordan; that is, the restriction of a complex jordan with nilpotent part $N = 0$. If in addition $A$ is a pure subnormal operator, then by 5.2.2 $A$ can also be regarded as pure complex subjordan. Conversely if $A$ is a pure complex subjordan operator with normal extension in the sense of 5.2.3, then $A$ is pure subnormal.

We now show the equivalence of 5.2.1, 5.2.2 and 5.2.3, beginning with the first and third.

**Proposition 5.3.** Let $T$ be complex subjordan on $K$. Then $T$ has a nonzero invariant subspace on which it is complex jordan if and only if $T$ has a nonzero invariant subspace on which it is normal.

**Proof:** If $TH \subseteq H$ with $H$ nonzero, $H \neq K$, and $T|H$ normal, then $T|H$ is complex jordan. Conversely, suppose $T$ is complex subjordan on $K$ and $H$ is a nonzero invariant subspace of $K$ so that $T|H = M_0 + N_0$ is complex jordan. Without loss of generality, we assume $N_0 \neq 0$. Let $H_0 = N_0H \neq (0)$. Considering $h \in H$, we see $M_0N_0h = N_0M_0h \in H_0$. Further, by the Fugledge-Putnam Theorem (Theorem 5.4 in [C]), $M_0^*N_0h = N_0M_0^*h \in H_0$. 

Thus $\mathcal{H}_0$ is reducing for $M_0$. Moreover

$$T|\mathcal{H}_0 = (M_0 + N_0)|\mathcal{H}_0 = M_0|\mathcal{H}_0.$$ 

Thus $\mathcal{H}_0$ is a nontrivial proper invariant subspace on which $T$ is normal. \(\square\)

In order to exhibit the equivalence of 5.2.1 and 5.2.2 we argue via spectral subspaces. Let $J = M + N$ be complex Jordan on $\mathcal{K}$. We define a spectral measure $E_J(\cdot)$ for $J$ to be simply $E_M(\cdot)$ the spectral measure for the normal operator $M$. The idea used here is a consequence of Proposition 3.7, that is, $\sigma(J) = \sigma(M)$. The following result gives that, with respect to $E_J(\cdot)$, $J$ is a spectral operator in the sense of Dunford and Schwartz (see [DS], pp 1930-1931).

**Proposition 5.4.** For $J = M + N$, a complex Jordan operator on $\mathcal{K}$, let $E_J(\cdot)$ be the spectral measure for the normal operator $M$. Then for all Borel subsets $\delta$ of the complex plane $\mathbb{C}$

$$E_J(\delta)J = JE_J(\delta), \text{ and}$$

$$\sigma(J|E_J(\delta)\mathcal{K}) \subset \overline{\delta}.$$ 

Furthermore,

for all $h, k \in \mathcal{K}$, the scalar valued measure $\langle E_J(\cdot)h, k \rangle$ is countably additive.

**Proof:** First of all 5.4.3 follows directly from the definition of a spectral measure for a normal operator (see [C]: §2 and §3).

Since $N$ commutes with $M$ and $M^*$ (via Fuglede-Putnam) we have, by the spectral theorem for normal operators (Theorem 3.1, p. 67, [C]) that both $M$ and $N$ commute with $E_J(\delta)$ for each Borel subset of the plane. Thus 5.6.1 follows.

Now let $\delta$ be a Borel subset of $\mathbb{C}$. Since $E_J(\cdot)$ is the spectral projection for $M$, and since normal operators are spectral operators, we know $\sigma(M|E_J(\delta)\mathcal{K}) \subset \overline{\delta}$. Further, by
above, the subspace $E_J(\delta)\mathcal{K}$ is invariant for both $M$ and $N$. Let $M_\delta = M|E_J(\delta)\mathcal{K}$ and $N_\delta = N|E_J(\delta)\mathcal{K}$. Then $M_\delta N_\delta = N_\delta M_\delta$ and $(N_\delta)^2 = 0$. So by Proposition 3.7

$$\sigma(\mathcal{J}|E_J(\delta)\mathcal{K}) = \sigma((M + N)|E_J(\delta)\mathcal{K})$$

$$= \sigma((M_\delta + N_\delta) = \sigma(M_\delta)$$

$$= \sigma(M|E_J(\delta)\mathcal{K}) \subseteq \bar{\delta},$$

which proves 5.4.2. □

Our argument giving the equivalence of 5.2.1 and 5.2.2 relies on a procedure for recovering spectral subspaces $E_J(\delta)\mathcal{K}$ for Borel subsets of the plane directly from $J$. We define for $k \in \mathcal{K}$, the set

$$\rho_J(k) = \{\lambda_0 \in \mathbb{C} : \text{there exists a function } \lambda \to k(\lambda) \in \mathcal{K},$$

\text{defined and analytic on a neighborhood of } \lambda_0, \text{ so that } (\lambda I - J)k(\lambda) \equiv k\}.$$  

By Theorem 2, p. 1933 of [DS], $J$ has the single valued extension property. Thus $k(\lambda)$ is unique and is referred to as the local resolvent of $k$. Further, by definition, $\rho_J(k)$ is open and contains the resolvent set $\rho(J)$ of $J$ for all $k$. Let $\sigma_J(k) = \mathbb{C} \setminus \rho_J(k)$. Then $\sigma_J(k)$ is compact and nonempty if $k \neq 0$. Given a Borel set $\delta \subseteq \mathbb{C}$, a spectral maximal subspace for $J$ is one of the form

$$\mathcal{K}_J(\delta) = \{k \in \mathcal{K} : \sigma_J(k) \subseteq \delta\}.$$  

By Theorem 4, p. 1934 of [DS],

$$\mathcal{K}_J(\delta) = E_J(\delta)\mathcal{K}.$$  

The proof of the following theorem, which gives that 5.2.1 and 5.2.2 are equivalent, follows from the two succeeding lemmas.
THEOREM 5.5. Let $J = M + N$ be a complex jordan operator on a Hilbert space $\mathcal{K}$. Suppose $\mathcal{H} \subseteq \mathcal{K}$ is invariant for $J$. Then $J_0 = J|_\mathcal{H}$ is itself jordan if and only if $\mathcal{H}$ is invariant for $M, M^*$, and $N$. Furthermore, if $J_0 = M_0 + N_0$, then $M_0 = M|_\mathcal{H}$ and $N_0 = N|_\mathcal{H}$.

LEMMA 5.6. With $J$ and $J_0$ defined and jordan on $\mathcal{K}$ and $\mathcal{H}$ respectively as in Theorem 5.5, for every Borel set $\delta \subseteq \mathbb{C}$

$$E_{J_0}(\delta)\mathcal{H} \subseteq E_J(\delta)\mathcal{H}$$

where $E_{J_0}(\cdot)$ and $E_J(\cdot)$ are the spectral measures for $J_0$ and $J$ respectively.

PROOF: (Note that spectral measures are unique via Corollary 9, p. 1935 of [DS]; thus the use of the definite article in the lemma is justified.) By the previously cited result from [DS], we need only show

$$\mathcal{K}_{J_0}(\delta) \subseteq \mathcal{K}_J(\delta).$$

So let $h \in \mathcal{H}$ and $\lambda_0 \in \mathbb{C} \setminus \delta$. Then there exists a vector-valued function $\lambda \rightarrow h(\lambda)$, defined and analytic on a neighborhood of $\lambda_0$, so that $(\lambda I - J_0)h(\lambda) \equiv h$. But since $\mathcal{H} \subseteq \mathcal{K}$ this says there is a $\mathcal{K}$-vector-valued function $\lambda \rightarrow h(\lambda)$, defined and analytic on a neighborhood of $\lambda_0$, so that $(\lambda I - J)h(\lambda) \equiv h$. That is, for $h \in \mathcal{H}$, $\sigma_{J_0}(h) \subseteq \delta$ implies $\sigma_J(h) \subseteq \delta$. Hence $\mathcal{K}_{J_0}(\delta) \subseteq \mathcal{K}_J(\delta)$.

LEMMA 5.7. For all $h \in \mathcal{H}$, for all Borel sets in the plane,

$$E_{J_0}(\delta)h = E_J(\delta)h.$$

PROOF: By the definition of the spectral projections for $J$ and $J_0$, we have, for a Borel set $\delta \subseteq \mathbb{C}$,

$$\mathcal{H} = E_{J_0}(\delta)\mathcal{H} \oplus E_{J_0}(\mathbb{C} \setminus \delta)\mathcal{H}$$

and

$$\mathcal{K} = E_J(\delta)\mathcal{K} \oplus E_J(\mathbb{C} \setminus \delta)\mathcal{K}.$$
Also from Lemma 5.6, $E_{J_0}(\delta)\mathcal{H} \subseteq E_J(\delta)\mathcal{H}$ and $E_{J_0}(\mathbb{C} \setminus \delta)\mathcal{H} \subseteq E_J(\mathbb{C} \setminus \delta)\mathcal{H}$. So if $h \in \mathcal{H}$, we have

\[
E_{J_0}(\delta)h \oplus E_{J_0}(\mathbb{C} \setminus \delta)h = E_J(\delta)h \oplus E_J(\mathbb{C} \setminus \delta)h
\]

which implies $E_{J_0}(\delta)h = E_J(\delta)h$. □

**Proof of Theorem 5.5:** From Lemma 5.7, $\mathcal{H}$ is invariant under $E_J(\cdot)$ for all Borel sets $\delta \subseteq \mathbb{C}$. Since $E_J(\cdot) = E_M(\cdot)$, we have $\mathcal{H}$ is invariant under all the spectral projections for $M$. In particular, $\mathcal{H}$ is invariant for $M = \int \lambda dE(\lambda)$, and via the Fugledge-Putnam theorem, invariant for $M^*$. Since $\mathcal{H}$ is invariant also under $J = M + N$, it follows immediately that $\mathcal{H}$ is invariant for $N$. This gives the implication in one direction. On the other hand, if we assume $\mathcal{H}$ is invariant for $M$, $M^*$, and $N$, it is easily seen that $J_0 = J|\mathcal{H} = M|\mathcal{H} + N|\mathcal{H}$ is jordan. □

---

**5b. The spectra of pure complex subjordan operators and a "Clancey-Putnam" type conjecture**

Having proposed and proved the equivalence of the three notions of purity which provide a definition simultaneously consistent with the real and the subnormal cases, we address the characterization of the spectra of pure complex subjordan operators. Our main objective is to suggest an answer that is consistent with the Clancey-Putnam Theorem [CP] for the case of pure subnormals and with the results of Chapter 3 for the real case. Finally we show the conjecture holds in two other particular cases. We begin by stating the Clancey-Putnam Theorem followed soon after by our conjecture for pure complex subjordans. For a compact subset $K$ of the plane, we denote by $R(K)$ the uniform closure of all rational functions with poles in $\mathbb{C} \setminus K$ thought of as a subspace of $C(K)$.

**Theorem 5.8.** (Clancey-Putnam [CP]) A compact set $E$ of the complex plane is the spec-
trum of some pure subnormal operator if and only if for every open disc $D$ with $D \cap E \neq \emptyset$, 

$$R(E \cap \bar{D}) \neq C(E \cap \bar{D}).$$

As we saw in Chapter 3, the canonical model for cyclic (real) Jordan operators is multiplication by

$$J = \begin{bmatrix} x & 0 \\ 1 & x \end{bmatrix}$$

on a direct sum of $L^2$-spaces supported on the real line. Here the pure subjordan $T$ with minimal jordan extension $J$ is $J$ restricted to the $L^2$-closure of $\{p \oplus p' : p \text{ a polynomial}\}$ where $T$ acts as multiplication by $x$, in that for polynomials $p$,

$$T \begin{bmatrix} p \\ p' \end{bmatrix} = \begin{bmatrix} xp \\ (xp)' \end{bmatrix}.$$ 

Thus when expanding into the plane we naturally have in mind a pure complex subjordan $T$ with cyclic vector as restriction of $\begin{bmatrix} z & 0 \\ 1 & z \end{bmatrix}$ to the closure of either $\{p \oplus p' : p \text{ a polynomial}\}$ or $\{r \oplus r' : r \text{ a rational function with poles off } \sigma(T)\}$ thought of as subspaces of the direct sum of $L^2$-spaces supported in the plane. Now in the real case, as we shall see, the uniform closures coincide, so to maintain the connection to the Clancey-Putnam Theorem, we make the following definition and conjecture.

**Definition 5.9.** Let $E$ be a compact subset of the complex plane. We define the subspace $R_2(E)$ of $C(E) \oplus C(E)$ to be the uniform closure in $C(E) \oplus C(E)$ of the manifold

$$\{r \oplus r' : r \text{ a rational function with no poles in } E\}.$$ 

**Conjecture 5.10.** A compact subset $E$ of the complex plane is the spectrum of some pure complex subjordan operator if and only if

$$R_2(E \cap \bar{D}) \neq C(E \cap \bar{D}) \oplus C(E \cap \bar{D}).$$
for all open discs $D$ so that $D \cap E \neq \emptyset$.

First we notice that in at least one direction this conjecture is consistent with Theorem 5.8. Indeed a compact set which is the spectrum of a pure subnormal operator in particular is the spectrum of a pure complex subjordan operator. By the result of Clancey-Putnam,

$$R(E \cap \tilde{D}) \neq C(E \cap \tilde{D})$$

for each open disc $D$ for which $D \cap E \neq \emptyset$. But it is easily seen directly that

$$R_2(E \cap \tilde{D}) = C(E \cap \tilde{D}) \oplus C(E \cap \tilde{D})$$

$$\Rightarrow R(E \cap \tilde{D}) = C(E \cap \tilde{D});$$

for if $r_n \oplus r_n' \to h \oplus k$, then in particular $r_n \to h$. Then by the contrapositive, $R(E \cap \tilde{D}) \neq C(E \cap \tilde{D})$ implies that $R_2(E \cap \tilde{D}) \neq C(E \cap \tilde{D}) \oplus C(E \cap \tilde{D})$. Thus the condition of Conjecture 5.10 holds for a subset $E$ which is the spectrum of a pure subnormal operator.

We next show that the conjecture holds in the case that $T$ is a pure (real) subjordan operator. We begin with a useful lemma.

**Lemma 5.11.** If $E$ is a compact subset of the real line, then $R_2(E)$ is equal to the uniform closure on $E$ of the manifold \{p \oplus p' : p a polynomial\}.

**Proof:** Since

$$\{p \oplus p' : p a polynomial\} \subseteq \{r \oplus r' : r \text{ rational with poles in } \mathbb{R} \setminus E\},$$

we need only show that given a rational function $r$ with poles off $E$, we can find a polynomial $p$ so that $p \oplus p'$ uniformly approximates $r \oplus r'$ on $E$.

So let $r$ be rational with poles in $\mathbb{R} \setminus E$. Since $r$, and hence $r'$, can only have a finite number of discontinuities, the poles of $r$ form a compact set in $\mathbb{R}$ disjoint from $E$. Thus, since by assumption $E$ is compact, we can cover $E$ by a finite set of closed finite intervals disjoint from the poles of $r$ and $r'$. Let $\hat{E}$ denote the union of these closed intervals. Then
\( \hat{E} \) is compact and if \( a = \min\{x : x \in \hat{E}\} \) and \( b = \max\{x : x \in \hat{E}\} \), then \( (b - a) < \infty \). Further both \( r \) and \( r' \) are continuous on \( \hat{E} \).

Now let \( f \) be a smooth continuous extension of \( r|\hat{E} \) to \([a,b]\). Then \( f' \) is defined on \([a,b]\) and \( r = f \) and \( r' = f' \) on \( \hat{E} \). Let \( \epsilon > 0 \) be given and choose a polynomial \( p' \) so that

\[
\sup_{x \in [a,b]} |p'(x) - f'(x)| < \frac{\epsilon}{2}(\max\{1, b - a\})^{-1}.
\]

Define the polynomial \( p \) via

\[
p(x) = \int_a^x p'(t)dt + f(a).
\]

Noticing that

\[
f(x) = \int_a^x f'(t)dt + f(a)
\]
we have

\[
\|p - r\|_{L^\infty(\hat{E})} + \|p' - r'\|_{L^\infty(\hat{E})}
= \|p - f\|_{L^\infty(\hat{E})} + \|p' - f'\|_{L^\infty(\hat{E})}
\leq \sup_{x \in [a,b]} \int_a^x |p'(t) - f'(t)|dt + \epsilon/2(\max\{1, b - a\})^{-1}
< (b - a)^{\frac{\epsilon}{2}}(\max\{1, b - a\})^{-1} + \epsilon/2(\max\{1, b - a\})^{-1}
< \epsilon. \qed
\]

**Theorem 5.12.** If \( E \) is a compact set of real numbers (considered as a subset of the complex plane), then \( E \) is the spectrum of a pure subjordan operator if and only if

\[
R_2(E \cap \bar{I}) \neq C(E \cap \bar{I}) \oplus C(E \cap \bar{I})
\]

for all open intervals \( I \) so that \( I \cap E \neq \phi \).

**Proof:** By Theorem 3.14, we know \( E \) is the spectrum of pure (real) subjordan operator if and only if \( E \) is regularly closed. So let \( T \) be a pure subjordan with \( \sigma(T) = E \). We need to show for all open intervals \( I \) with \( E \cap I \neq \phi \),

\[
R_2(E \cap \bar{I}) \neq C(E \cap \bar{I}) \oplus C(E \cap \bar{I}).
\]
Or, equivalently, by Lemma 5.10, we need to show the uniform closure, on $E \cap \bar{I}$, of

$$ \{p \oplus p' : p \text{ a polynomial}\} $$

is not all of $C(E \cap \bar{I}) \oplus C(E \cap \bar{I})$. But by Theorem 2.42 it suffices to show $\text{int}(E \cap \bar{I}) \neq \emptyset$ for all open intervals $I$ which have nonempty intersection with $E$. So let $I$ be such an interval. If $I \cap \text{int}(E) = \emptyset$, then $(\text{int}(E))^c \subset I^C$ since $I^C$ is closed. Thus $I \cap E$ is a subset of $E$ disjoint from the closure of the interior of $E$; this contradicts our assumption that $E$ is regularly closed. Thus $\text{int}(\bar{I} \cap E) \supset \text{int}(I \cap E) = I \cap \text{int}(E) \neq \emptyset$, and one direction is exhibited.

Now suppose $E$ is compact and not the spectrum of any pure (real) sub Jordain operator; or equivalently, suppose $E$ is not regularly closed. Write $E = E_1 \cup E_2$ where $E_1 = \text{int}(E)$ and $E_2 = E \setminus E_1 \neq \emptyset$. Notice $E_2$ is nowhere dense and compact. Moreover there must exist a component of $E_2$ which can be separated from $E_1$ by an open interval. Otherwise $E_2 \subseteq \partial E = \partial(\text{int}E)$ which implies $E$ is regularly closed, contradicting our assumption. Choose such an interval $I$ so that $I \cap E_1 = \emptyset$ but $I \cap E_2 \neq \emptyset$. Then $\bar{I} \cap E_2$ is nowhere dense and by Theorem 3.42 and Lemma 5.10,

$$ R_2(E \cap \bar{I}) = C(E \cap \bar{I}) \oplus C(E \cap \bar{I}). \square $$

The remainder of this chapter involves lending more credence to Conjecture 5.10 by exhibiting two cases for which it holds. As in the development in the real instance we rely on knowledge of when an arbitrary pair of functions $(h,k)$ can be approximated by one of the form $(f,f')$. Here we cite results by other authors instead of developing our own. Our first example concerns totally disconnected sets.

**Proposition 5.13.** Let $T$ be a complex sub Jordan operator with $E = \sigma(T)$ a totally disconnected set. Then $T$ is Jordan. In particular, $T$ is not pure.

**Proof:** This is an immediate consequence of Theorem 3.18 in [Ag2]. \square
According to E. Bishop [B2], if \( K \) is totally disconnected subset of the plane then the uniform closure on \( K \) of \( \{p \oplus p' : p \text{ a polynomial}\} \) is \( C(K) \oplus C(K) \). This coupled with the preceding proposition tells us that if \( T \) is complex subjordan with totally disconnected spectrum \( E \), then both \( T \) is not pure and

\[
R_2(\bar{D} \cap E) = C(\bar{D} \cap E) \oplus C(\bar{D} \cap E),
\]

which is consistent with our conjecture.

We next construct a pure complex subjordan operator \( T \) with \( \sigma(T) = E \) in a situation in which we know

\[
R_2(\bar{D} \cap E) \neq C(\bar{D} \cap E) \oplus C(\bar{D} \cap E)
\]

for all open discs \( D \) with \( D \cap E \neq \emptyset \).

To begin let \( E \) be any smooth simple curve with no critical points. That is, there is a one-to-one function \( f \in C[0, 1] \) so that \( f' \) is nonzero and continuous on \([0, 1] \), with \( E = f([0, 1]) \). Consequently, \( E \) is rectifiable. We turn to a paper by U. Fixman and L. A. Rubel [FR] to show that in this case

\[
R_2(\bar{D} \cap E) \neq C(\bar{D} \cap E) \oplus C(\bar{D} \cap E)
\]

for all open discs \( D \) which satisfy \( D \cap E \neq \emptyset \). In their paper they investigate sets \( K \) which have the property that the graph of the differential operator \( f \to f' \), with domain consisting of all functions \( f \) analytic on \( K \), is closed under uniform limits. They call such sets \( D \)-sets. Theorem 2.1 of this reference asserts that rectifiable curves are \( D \)-sets.

If \( E \) is defined as above then certainly for every open \( D \) with \( D \cap E \neq \emptyset \), \( \bar{D} \cap E \) is a rectifiable curve. Thus \( \bar{D} \cap E \) is a \( D \)-set and it follows that the uniform closure of

\[
\{f \oplus f' : f \text{ analytic on } \bar{D} \cap E\}
\]

cannot be all of \( C(\bar{D} \cap E) \oplus C(\bar{D} \cap E) \). To make this result fit our purposes we simply note that

\[
\{r \oplus r' : r \text{ a rational function with poles in } \mathbb{C} \setminus (\bar{D} \cap E)\}
\]
The following construction ties up this example and completes the section.

**Proposition 5.14.** Let $E$ be a smooth simple curve with no critical points. Then there is a pure complex subjordan operator $T$ so that $\sigma(T) = E$.

**Proof:** Let $f$ be a continuous complex-valued function defined on $[0,1]$ satisfying

1. $f$ is one-to-one,
2. $f' \in C[0,1]$, 
3. $f'(t) \neq 0$ for all $t \in [0,1]$, and
4. $f([0,1]) = E$.

Let $J_0$ be multiplication by \[
\begin{bmatrix}
  t & 0 \\
  1 & t 
\end{bmatrix}
\]
on $\mathcal{K} = L^2(m) \oplus L^2(m)$ where $m$ is Lebesgue measure on $[0,1]$. Let $T_0 = J_0|\mathcal{H}$ where $\mathcal{H}$ is the $\mathcal{K}$-closure of the manifold \{ $p \oplus p' : p$ a polynomial \}. Then by Theorem 3.9, $T_0$ is a pure (real) subjordan operator with minimal extension $J$.

As we have seen previously, we have a $C^1$-functional calculus available for use. That is, since both $f$ and $f'$ are continuous on $[0,1]$, then $f(T_0)$ is multiplication by \[
\begin{bmatrix}
  f(t) & 0 \\
  f'(t) & f(t) 
\end{bmatrix}
\]
restricted to $\mathcal{H}$. (See the latter part of the proof of Theorem 3.17.) If we let $J = M_f + N_f$ where $M_f$ is multiplication by \[
\begin{bmatrix}
  f(t) & 0 \\
  0 & f(t) 
\end{bmatrix}
\]
on $\mathcal{K}$ and $N_f$ is multiplication by \[
\begin{bmatrix}
  0 & 0 \\
  f'(t) & 0 
\end{bmatrix}
\]
on $\mathcal{K}$, then clearly $J$ is jordan and hence $T$ is subjordan. It is left to show $T$ is pure and $\sigma(T) = E$.

By citing the Stone-Weierstrass Theorem in the complex plane (see Theorem 7.3.8 of [Se]), the manifold

\[\{ p(f, \bar{f}) : p \text{ a polynomial in } z \text{ and } \bar{z} \}\]
is dense in $C[0,1]$. (Note: Here we use 5.14.1 to get that the manifold separates points.)

Now suppose $\mathcal{H}_0$ is a subspace of $\mathcal{H}$ which is invariant for $M_f$ and $M_f^*$. Then $\mathcal{H}_0$ is invariant for $p(M_f)$ and $p(M_f^*)$, for all polynomials $p$. Thus $\mathcal{H}_0$ is invariant for $p(M_f, M_f^*)$.
where \( p \) is any polynomial in \( z \) and \( \bar{z} \). By the result noted above, there is a sequence of polynomials \( \{ p_n \} \) in \( z \) and \( \bar{z} \) so that \( \{ p_n(f, \bar{f}) \} \) converges uniformly to \( g(t) = t \) on \([0, 1]\).

Thus
\[
p_n(M_f, M_{\bar{f}}) = \begin{bmatrix} p_n(f, \bar{f}) & 0 \\ 0 & p_n(f, \bar{f}) \end{bmatrix} \longrightarrow \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} = S,
\]
implies that \( \mathcal{H}_0 \) is invariant for \( S \).

Suppose further that \( \mathcal{H}_0 \) is invariant for \( N_f \). By 5.14.2 and 5.14.3 we have as before a sequence \( \{ p_n \} \) of polynomials in two variables so that \( \{ p_n(f, \bar{f}) \} \) converges uniformly to \( (f')^{-1} \) on \([0, 1]\). Since \( \mathcal{H}_0 \) is invariant for
\[
\begin{bmatrix} \frac{1}{f'(t)} p_n(f, \bar{f}) & 0 \\ 0 & p_n(f, \bar{f}) \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{1}{f'(t)} & 0 \\ 0 & 1 \end{bmatrix} = N,
\]
it follows that \( \mathcal{H}_0 \) is invariant for \( N \). Therefore \( \mathcal{H}_0 \subseteq \mathcal{H} \) being invariant for \( M_f, M_{\bar{f}} \) and \( N_f \) implies \( \mathcal{H}_0 \) is invariant for both \( S \) and \( N \). Since \( T_0 \) is pure this says \( \mathcal{H}_0 = \{0\} \). Hence \( T \) is also pure.

To show \( \sigma(T) = E \), we first show \( J = f(T_0) \) is the minimal jordan extension of \( T = f(T_0) \). Considering \( \mathcal{H} = \{ h \\ h' \} : h \in AC[0, 1] \}, we need to show that \( \mathcal{K} \) is the smallest space containing \( \mathcal{H} \) that is invariant for both \( M_f \) and \( N_f \). Notice that for all \( h \in AC[0, 1] \), for all polynomials \( p \)
\[
p(N_f) \begin{bmatrix} h \\ h' \end{bmatrix} = \begin{bmatrix} 0 \\ p(f')h \end{bmatrix} \in \mathcal{K}.
\]
But since functions of the form \( p(f')h \) are dense in \( L^2(m) \), this says \( \begin{bmatrix} 0 \\ L^2(m) \end{bmatrix} \subseteq \mathcal{K} \). In particular, for all polynomials \( p \),
\[
\begin{bmatrix} 0 \\ p(f')h + p(f)h' \end{bmatrix} \in \mathcal{K}.
\]
Thus
\[
p(J) \begin{bmatrix} h \\ h' \end{bmatrix} - \begin{bmatrix} 0 \\ p(f')h + p(f)h' \end{bmatrix} = \begin{bmatrix} p(f)h \\ 0 \end{bmatrix} \in \mathcal{K}.
for all polynomials \( p \) and all \( h \in AC[0,1] \). Since the manifold \( \{ p(f)h : p \text{ a polynomial}, h \in AC[0,1] \} \) is dense in \( L^2(m) \), we get \( \begin{bmatrix} L^2(m) \\ 0 \end{bmatrix} \subseteq \mathcal{K} \). Therefore \( \mathcal{K} = \begin{bmatrix} L^2(m) \\ L^2(m) \end{bmatrix} \).

By Agler's Theorem 3.8 [Ag2], \( \sigma(J) \subseteq \sigma(T) \) and \( \sigma(T) \setminus \sigma(J) \) is empty or a union of components of \( \mathbb{C} \setminus \sigma(J) \). Now if \( \sigma(J) \subseteq E \) is a non-selfintersecting arc, then \( \mathbb{C} \setminus \sigma(J) \) has precisely one component, itself. Thus \( \sigma(T) \setminus \sigma(J) = \mathbb{C} \setminus \sigma(J) \) or is empty. Since the former is absurd, we conclude \( \sigma(J) = \sigma(T) \), provided \( \sigma(J) \subseteq E \).

To see this is indeed the case, we show \( \mathbb{C} \setminus E \subseteq \rho(J) \). So let \( \lambda \in \mathbb{C} \setminus E \). Then both functions \( (\lambda - f(t))^{-1} \) and \( (\lambda - f(t))^{-2}(\lambda - f(t))' \) are defined and continuous on \([0,1]\). Thus multiplication by

\[
\begin{bmatrix}
(\lambda - f)^{-1} \\
-(\lambda - f)^{-2}(\lambda - f)' \\
0 \\
(\lambda - f)^{-1}
\end{bmatrix}
\]
is bounded. The following calculation shows this operator is in fact \( (\lambda I - f(J_0))^{-1} \):

\[
\begin{bmatrix}
\lambda - f \\
(\lambda - f)' \\
0 \\
(\lambda - f)
\end{bmatrix}
\begin{bmatrix}
(\lambda - f)^{-1} \\
-(\lambda - f)^{-2}(\lambda - f)' \\
0 \\
(\lambda - f)^{-1}
\end{bmatrix}
= \begin{bmatrix}
1 \\
(\lambda - f)'(\lambda - f)^{-1} - (\lambda - f)^{-1}(\lambda - f)' \\
0 \\
1
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}.
\]

In conclusion, we cite Proposition 3.7, to get that both \( \sigma(J_0) = \sigma(S) \) and \( \sigma(J) = \sigma(M_J) \). Obviously, \( \sigma(S) = [0,1] \), so by the spectral mapping Theorem, \( E = f([0,1]) = f(\sigma(S)) = \sigma(f(S)) = \sigma(M_J) = \sigma(J) = \sigma(T) \).

5c. A Model for Complex Subjordan Operators with Cyclic Vector

We conclude with the construction of a model for a cyclic complex subjordan operator \( T \) on \( \mathcal{H} \) similar to that of the real case presented in Chapter 3. The basic ideas used there are mimicked and some proofs follow exactly as before.

Recall we originally characterized \( \mathcal{H} \) as the closure of the graph of a closable differential operator \( D \) if \( T \) is pure. Here the purity of \( T \) does not imply the closability of \( D \) and
hence specific results using the structure of the closure of the graph cannot be utilized as before. Though less precise, an analysis of purity is still possible.

Definition 5.15. Let $T$ be a complex subjordan operator on $\mathcal{H}$ with complex jordan extension $J = M + N$ on $\mathcal{K}$ where $M$ is normal, $N^2 = 0$, and $MN = NM$. We say $J$ is a \textit{minimal complex jordan extension} if $\mathcal{K}$ is the smallest space containing $\mathcal{H}$ for which $M, M^*,$ and $N$ are invariant.

Theorem 5.16. Let $T$ be a complex subjordan operator on $\mathcal{H}$ with cyclic vector and minimal complex jordan extension $J$ on $\mathcal{K}$. Then there are finite positive measures $\mu$ and $\nu$ compactly supported in the complex plane with $\nu$ absolutely continuous with respect to $\mu$, and a function $\theta \in L^2(\nu)$ so that

\[
\mathcal{K} = L^2(\mu) \oplus L^2(\nu). \tag{5.16.1}
\]

$J$ is unitarily equivalent to multiplication by the matrix function $\begin{bmatrix} z & 0 \\ 1 & z \end{bmatrix}$ on $\mathcal{K}$, \tag{5.16.2}

and

$T$ is unitarily equivalent to $J$ restricted to the closure of the graph of

$D = \theta + d/dz$ defined on all polynomials with cyclic vector $\begin{bmatrix} 1 \\ \theta \end{bmatrix}$. \tag{5.16.3}

Conversely, an operator $T$ defined by (5.16.3) is a cyclic complex subjordan operator with minimal complex jordan extension $J$ defined by (5.16.2) on $\mathcal{K}$ as defined by (5.16.1), provided

the inclusion map $1 : L^2(\mu) \to L^2(\nu)$ is continuous. \tag{5.16.4}

Proof: Let $J = M + N$ where $M$ is normal, $N^2 = 0$, and $MN = NM$. Let $\mathcal{K}_1 = (\text{Ran} N)^\perp$ and $\mathcal{K}_2 = \text{Ran} N$. Letting $M_i = M|\mathcal{K}_i$, we see by Fuglede-Putnam that $\mathcal{K}_i$ is reducing for


$M_i$, hence $M_i$ is normal for $i = 1, 2$. Furthermore, for some operator $\Gamma : \mathcal{K}_1 \to \mathcal{K}_2$ with dense range we can write

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & 0 \\ \Gamma & 0 \end{bmatrix}$$

with respect to the decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Note also, since $MN = NM$, we have $\Gamma M_1 = M_2 \Gamma$.

Citing Theorem 9.1, p. 99 of [C], there exist two sequences of measures $\{\mu_i\}_{i=1}^{\infty}$ and $\{\nu_j\}_{j=1}^{\infty}$ satisfying $\mu_{i+1} \ll \mu_i$ and $\nu_{j+1} \ll \nu_j$ with $M_1$ represented as multiplication by $z$ on $\mathcal{K}_1 = \oplus \sum_{i=1}^{\infty} L^2(\mu_i)$ and $M_2$ as multiplication by $z$ on $\mathcal{K}_2 = \oplus \sum_{j=1}^{\infty} L^2(\nu_j)$. With respect to these decompositions of $\mathcal{K}_1$ and $\mathcal{K}_2$, $\Gamma$ can be viewed as the matrix operator $
abla = [\Gamma_{ij}]$ where $\Gamma_{ij} : L^2(\mu_i) \to L^2(\mu_j)$. If we designate, for $i, j = 1, 2, \ldots, M_1 = M_1|L^2(\mu_i)$ and $M_2 = M_2|L^2(\nu_j)$, it follows that $M_2 \Gamma_{ij} = \Gamma_{ij} M_1$. Therefore by Abrahamse [A], for each $i, j$ we get the existence of a function $\Gamma_{ij}(z)$ so that for all $h \in L^2(\mu_i)$,

$$(\Gamma_{ij} h)(z) = \Gamma_{ij}(z) h(z).$$

Let $\Gamma(z)$ be the matrix valued function with $ij$-coordinate $\Gamma_{ij}(z)$. Then for each $h(z) = h_1(z) \oplus h_2(z) \oplus \ldots$ in $\oplus \sum_{i=1}^{\infty} L^2(\mu_i)$

$$(\Gamma h)(z) = \oplus \sum \Gamma_{ij}(z) h_i(z).$$

We argue that $T$ being cyclic and $J$ being a minimal jordan extension of $T$ implies that without loss of generality there are compactly supported measures $\mu$ and $\nu$ so that $\mathcal{K}_1 = L^2(\mu)$ and $\mathcal{K}_2 = L^2(\nu)$. Let $\zeta_i = \zeta_{i1} \oplus \zeta_{i2} \oplus \cdots \in \mathcal{K}_i$ for $i = 1, 2$ be so that $\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$ is a cyclic vector for $T$. Then

$$\mathcal{H} = \{p(T)\zeta : p \text{ a polynomial}\}^\perp = \{ \begin{bmatrix} p(z) \zeta_1(z) \\ p'(z) \Gamma(z) \zeta_1(z) + p(z) \zeta_2(z) \end{bmatrix} : p \text{ a polynomial} \}^\perp.$$
Now, if we let

$$\hat{K}_1 = \{ p(z, \bar{z}) \zeta_1(z) : p \text{ a polynomial in two arguments} \},$$

$$\hat{K}_2 = \text{span}\{ p(z, \bar{z}) \Gamma(z) \zeta_1(z), p(z, \bar{z}) \zeta_2(z) : p \text{ a polynomial in two arguments} \},$$

and $\hat{K} = \hat{K}_1 \oplus \hat{K}_2$, we see $\mathcal{H} \subseteq \hat{K}$. Moreover, if \[
\begin{bmatrix}
 h \\
 k
\end{bmatrix} \in \hat{K},
\]

then

$$N \begin{bmatrix}
 h \\
 k
\end{bmatrix} = \begin{bmatrix}
 0 \\
 \Gamma(z) h(z)
\end{bmatrix} \in \hat{K},$$

$$M \begin{bmatrix}
 h \\
 k
\end{bmatrix} = \begin{bmatrix}
 z h(z) \\
 z k(z)
\end{bmatrix} \in \hat{K},$$

and

$$M^* \begin{bmatrix}
 h \\
 k
\end{bmatrix} = \begin{bmatrix}
 \bar{z} h(z) \\
 \bar{z} k(z)
\end{bmatrix} \in \hat{K}.$$

Thus $\hat{K}$ is invariant for $M, M^*$ and $N$. So by minimality of $J$, $\mathcal{K} \subseteq \hat{K}$. In particular $\mathcal{K}_1 \subseteq \hat{K}_1$. We conclude $M_1$ is star-cyclic. By Theorem 4.3, p. 14 of [C], there is a measure $\mu$ compactly supported in $\mathbb{C}$ so that $M_1$ is unitarily equivalent to multiplication by $z$ on $L^2(\mu)$.

Furthermore, since $\mathcal{K}_2 = \text{Ran}N$ and

$$N \zeta = \begin{bmatrix}
 0 \\
 \Gamma(z) \zeta_1(z)
\end{bmatrix},$$

we see $\mathcal{K}_2$ is spanned (over the polynomials) by $\begin{bmatrix}
 0 \\
 \Gamma(z) \zeta_1(z)
\end{bmatrix}$. Thus $M_2$ is star-cyclic and we have as above a measure $\nu$ compactly supported in $\mathbb{C}$ so that $M_2$ is unitarily equivalent to multiplication by $z$ on $L^2(\nu)$. This gives (5.16.1).

Noticing that now $\Gamma : L^2(\mu) \to L^2(\nu)$ with $M_2 \Gamma = \Gamma M_1$, it follows from [A] that there is a measurable function $\Gamma(z)$ satisfying, for some $c > 0$, $|\Gamma(z)| \leq c(d\mu/d\nu)^{1/2}$, $\Gamma(z) = 0$ on the set $\{ z | (d\mu/d\nu)(z) > 0 \}$, such that for every $h \in L^2(\mu)$, for $\nu$-a.e. $z$,

$$(\Gamma h)(z) = \Gamma(z) h(z).$$
Furthermore, by arguments identical to those of Lemma 3.13, it follows that \( \nu \) is absolutely continuous with respect to \( \mu \), \( \Gamma \in L^2(\nu) \) and \( \Gamma \neq 0 \) \( \nu \)-a.e. This last fact gives that the transformation

\[
U : L^2(\nu) \to L^2(|\Gamma|^2d\nu)
\]

via

\[
Uf = f/\Gamma
\]

is an isometry and hence the mapping

\[
V : L^2(\mu) \oplus L^2(\nu) \to L^2(\mu) \oplus L^2(|\Gamma|^2d\nu)
\]

defined by

\[
V \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} h \\ k/\Gamma \end{bmatrix}
\]

is unitary (see the proof of Lemma 3.14). Therefore, replacing \( d\nu \) by \( |\Gamma|^2d\nu \) and \( J \) by \( VJV^{-1} \) but retaining the prior notation, we have

\[
J = \begin{bmatrix} z & 0 \\ 1 & z \end{bmatrix}
\]
on \( L^2(\mu) \oplus L^2(\nu) \). Thus (5.16.2) holds.

Now consider the cyclic vector \( \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \) for \( T \). Suppose there is a set \( A \subseteq \text{sppt}\mu \) so that \( \mu(A) > 0 \) and \( \zeta_1 = 0 \) on \( A \). Then letting \( \tilde{\mathcal{H}} = \chi_A \cdot L^2(\mu) \oplus L^2(\nu) \), we see that \( \mathcal{H} \leq \tilde{\mathcal{H}} \), \( \tilde{\mathcal{K}} \) is a proper subspace of \( \mathcal{K} \), and \( \tilde{\mathcal{K}} \) is invariant for \( M, M^* \), and \( N \). This contradicts the minimality of the complex jordan extension \( J \) on \( \mathcal{K} \). So we may assume \( \zeta_1 \neq 0 \) \( \mu \)-a.e.

Note that \( \nu << \mu \) implies also \( \zeta_1 \neq 0 \) \( \nu \)-a.e. Consider the isometry

\[
\begin{bmatrix} h \\ k \end{bmatrix} \to \begin{bmatrix} \zeta_1^{-1}h \\ \zeta_1^{-1}k \end{bmatrix}
\]
from \( L^2(\mu) \oplus L^2(\nu) \to L^2(|\zeta_1|^2d\mu) \oplus L^2(|\zeta_1|^2d\nu) \). Using this we may take as our cyclic vector for \( T \) the vector \( \zeta = \begin{bmatrix} 1 \\ \theta \end{bmatrix} \) (where \( \theta = \zeta_1^{-1}\zeta_2 \in L^2(|\zeta_1|^2d\nu) \) which we rename \( L^2(d\nu) \)
just as we identify \( L^2(|\zeta_1|^2d\mu) \) with \( L^2(d\mu) \)).
Finally, noting that

\[ T^n \begin{bmatrix} 1 \\ \theta \end{bmatrix} = \begin{bmatrix} \theta(z)z^n + n z^{n-1} \end{bmatrix}, \]

we see that

\[ \mathcal{H} = \left\{ \begin{bmatrix} p \\ Dp \end{bmatrix} : p \text{ a polynomial} \right\}^{-} \]

where \( D = \theta + d/dz : L^2(\mu) \rightarrow L^2(\nu) \). This gives (5.16.3) and completes the proof in one direction.

Conversely, if \( \mu \) and \( \nu \) are positive compactly supported measures with \( \nu << \mu \), if \( \theta \in L^2(\nu) \) so that (5.16.1), (5.16.2), (5.16.3), and (5.16.4) hold then clearly \( T \) is complex subjordan on \( \mathcal{H} \) with cyclic vector \( \begin{bmatrix} 1 \\ \theta \end{bmatrix} \) and with complex jordan extension \( J \). We need only show \( J \) is minimal. Let \( \hat{\mathcal{K}} \) be a space so that \( \mathcal{H} \subseteq \hat{\mathcal{K}} \subseteq \mathcal{K} \) and which is invariant for \( M, M^* \), and \( N \). Then we have

\[ N \begin{bmatrix} 1 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \hat{\mathcal{K}}. \]

Therefore if \( p(\cdot, \cdot) \) is a polynomial in two arguments it follows that

\[ p(M, M^*) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p(z, \bar{z}) \\ 0 \\ p(z, \bar{z}) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ p(z, \bar{z}) \end{bmatrix} \in \hat{\mathcal{K}}. \]

Since \( p \) is arbitrary, we conclude that \( \begin{bmatrix} 0 \\ L^2(\nu) \end{bmatrix} \subseteq \hat{\mathcal{K}}. \) In particular for all polynomials \( p(\cdot) \) is one argument \( \begin{bmatrix} 0 \\ Dp \end{bmatrix} \in \hat{\mathcal{K}}. \) Thus

\[ \begin{bmatrix} p(z) \\ (Dp)(z) \end{bmatrix} - \begin{bmatrix} 0 \\ Dp \end{bmatrix} = \begin{bmatrix} p(z) \\ 0 \end{bmatrix} \in \hat{\mathcal{K}}. \]

Now let \( q(\cdot) \) be a polynomial in one argument. Then by the above

\[ q(M^*) \begin{bmatrix} p(z) \\ 0 \end{bmatrix} = \begin{bmatrix} q(\bar{z})p(z) \\ 0 \end{bmatrix} \in \hat{\mathcal{K}}. \]

Consider the collection

\[ \{ q(\bar{z})p(z) : q, p \text{ polynomials} \}. \]
By the Stone-Weierstrass theorem in the complex case (see [Se], Theorem 7.3.8, p. 155), this is a dense set in $L^2(\mu)$. Therefore, 
\[
\begin{bmatrix}
L^2(\mu) \\
0
\end{bmatrix} \subseteq \hat{\mathcal{K}}. \quad \text{We have shown } \mathcal{K} = \begin{bmatrix} L^2(\mu) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ L^2(\nu) \end{bmatrix} \subseteq \hat{\mathcal{K}}. \quad \text{Thus } J \text{ is minimal.}
\]

For measures $\mu$ and $\nu$ compactly supported in the complex plane with $\nu \ll \mu$ such that $h \in L^2(\mu)$ implies $h \in L^2(\nu)$ and for $\theta \in L^2(\nu)$, we denote by $T(\mu, \nu, \theta)$ the restriction of the operator $J(\mu, \nu, \theta) \equiv \begin{bmatrix} z & 0 \\ 1 & z \end{bmatrix}$ on $L^2(\mu) \oplus L^2(\nu)$, to the subspace
\[
P^2_2(\mu, \nu, \theta) \equiv \{ \begin{bmatrix} p \\ Dp \end{bmatrix} : p \text{ a polynomial, } D = \theta + d/dz \}.
\]

Here the closure is in $L^2(\mu) \oplus L^2(\nu)$. Theorem 5.16 states that all cyclic complex subjordans look like a $T(\mu, \nu, \theta)$ on $P^2_2(\mu, \nu, \theta)$ for some triple $(\mu, \nu, \theta)$. Also, it should be noted that by taking $\nu = 0$ we get Bram's characterization of cyclic subnormal operators (see [Br]).

The next two theorems describe exactly when $T(\mu, \nu, \theta)$ is actually complex jordan or the other extreme, pure.

**Theorem 5.17.** $T(\mu, \nu, \theta)$ is complex jordan if and only if $P^2_2(\mu, \nu, \theta) = L^2(\mu) \oplus L^2(\nu)$.

**Proof:** Obviously, by the preceding theorem, if $P^2_2(\mu, \nu, \theta) = L^2(\mu) \oplus L^2(\nu)$, then $T = T(\mu, \nu, \theta)$ is complex jordan. So assume $T$ is complex jordan on $P^2_2(\mu, \nu, \theta)$. Then by Theorem 5.5, $\mathcal{H} = P^2_2(\mu, \nu, \theta)$ is invariant for $M, M^*$, and $N$. Thus by the same argument used in the last part of the proof of Theorem 5.16, replacing $\hat{\mathcal{K}}$ with $\mathcal{H}$, we see $P^2_2(\mu, \nu, \theta) = L^2(\mu) \oplus L^2(\nu)$. □

**Theorem 5.18.** $T(\mu, \nu, \theta)$ is pure if and only if there is no nonzero positive measure $\alpha$, compactly supported in the complex plane, for which $P^2_2(\mu, \nu, \theta)$ contains the subspace $L^2(\alpha) \oplus (0)$ or $(0) \oplus L^2(\alpha)$.

**Proof:** By Theorem 5.5, $T = T(\mu, \nu, \theta)$ is not pure if and only if there is a nontrivial subspace $\mathcal{H}_0$ of $P^2_2(\mu, \nu, \theta)$ invariant for $M, M^*$, and $N$. So assume $\mathcal{H}_0$ is a nontrivial subspace of $P^2_2(\mu, \nu, \theta)$ invariant for $M, M^*$, and $N$. We consider three cases:
1. $\mathcal{H}_0 \not\subseteq \ker N$. Here we look at the non-zero subspace $\tilde{\mathcal{H}} = N\mathcal{H}_0$. First note that $N\tilde{\mathcal{H}} \neq (0)$. Secondly,

$$M\tilde{\mathcal{H}} = MN\mathcal{H}_0 = NM\mathcal{H}_0 \subseteq N\mathcal{H} = \tilde{\mathcal{H}}.$$ 

Finally, using Fuglede-Putnam, we have

$$M^*\tilde{\mathcal{H}} = M^*N\mathcal{H}_0 = NM^*\mathcal{H}_0 \subseteq N\mathcal{H} = \tilde{\mathcal{H}}.$$ 

Thus $\tilde{\mathcal{H}} = N\mathcal{H}_0 \subseteq (0) \oplus L^2(\nu)$ and $T|\tilde{\mathcal{H}}$ is normal. Thus there is a measure $\alpha$ so that $\tilde{\mathcal{H}} = 0 \oplus L^2(\alpha)$.

2. $\mathcal{H}_0 \subseteq \text{Ran} N = 0 \oplus L^2(\nu)$. Here as in the previous case, $T|\mathcal{H}_0$ is normal. Thus $\mathcal{H}_0 = 0 \oplus L^2(\alpha)$ for some measure $\alpha$.

3. $\mathcal{H}_0 \subseteq \text{Ker} N$ but $\mathcal{H}_0 \not\subseteq \text{Ran} N$. Then we consider the nonzero subspace

$$\tilde{\mathcal{H}} = \begin{bmatrix} \mathcal{H}_0 & (0) \end{bmatrix} \subseteq L^2(\mu) \oplus (0).$$

Let $\begin{bmatrix} h \\ 0 \end{bmatrix} \in \tilde{\mathcal{H}}$. Then there is a $k \in L^2(\nu)$ so that $\begin{bmatrix} h \\ k \end{bmatrix} \in \mathcal{H}_0$. Now $M \begin{bmatrix} h \\ k \end{bmatrix} \in \mathcal{H}_0$ implies $M \begin{bmatrix} h \\ 0 \end{bmatrix} \in \tilde{\mathcal{H}}$. Thus $\tilde{\mathcal{H}}$ is invariant for $M$. Similarly $\tilde{\mathcal{H}}$ is invariant for $M^*$ since $\mathcal{H}_0$ is invariant for $M^*$. Finally, noting that for $\begin{bmatrix} h \\ k \end{bmatrix} \in \mathcal{H}_0$, $N \begin{bmatrix} h \\ k \end{bmatrix} = 0$ implies $N \begin{bmatrix} h \\ 0 \end{bmatrix} = 0$ for $\begin{bmatrix} h \\ 0 \end{bmatrix} \in \tilde{\mathcal{H}}$. We conclude that $T|\tilde{\mathcal{H}}$ is normal and hence there is a measure $\alpha$ so that $\tilde{\mathcal{H}} = L^2(\alpha) \oplus 0$.

Conversely, suppose $P_2^2(\mu, \nu, \theta)$ contains a non-zero subspace of the form $\mathcal{H}_0 = L^2(\alpha) \oplus (0)$. If $\mathcal{H}_0 \subseteq \text{Ker} N$ then we are in Case 3, above, and it follows that $T|\mathcal{H}_0$ is normal. Thus by Proposition 5.4, $T$ is not pure. On the other hand, if $\mathcal{H}_0 \not\subseteq \text{Ker} N$, then $\tilde{\mathcal{H}} = N\mathcal{H}_0$ is nonzero and as seen in Case 1, $T|\tilde{\mathcal{H}}$ is normal. Hence $T$ is not pure.

Finally, if $P_2^2(\mu, \nu, \theta)$ contains a nonzero subspace of the form $\mathcal{H}_0 = (0) \oplus L^2(\alpha)$, then $\mathcal{H}_0 \subseteq \text{Ran} N$. Here we are in Case 2 and it follows that $T$ is not pure. 

The last two theorems suggest the following questions:
For what measures $\mu$ and $\nu$ compactly supported in the complex plane, for what functions $\theta \in L^2(\nu)$, do we have

(1) $P^2_1(\mu, \nu, \theta) = L^2(\mu) \oplus L^2(\nu)$

or

(2) $P^2_1(\mu, \nu, \theta)$ contains a nontrivial subspace of the form $L^2(\alpha) \oplus (0)$ or $(0) \oplus L^2(\alpha)$?

Both of these problems seem particularly difficult in general though complete answers are given in Chapters 2 and 3 when $\mu$ and $\nu$ are supported on the real line (see Theorem 2.31, the model construction and Theorem 3.17 of Chapter 3).

When $\nu$ is taken to be zero, then the questions above reduce to:

For what measure $\mu$, compactly supported in the plane does

$$(1) P^2(\mu) = L^2(\mu)$$

or

$$(2) P^2(\mu) \text{ splits into a direct sum with one summand an } L^2 - \text{ space.}$$

Here $P^2(\mu)$ denotes the closure of the polynomials as a linear manifold in $L^2(\mu)$. These latter questions have been addressed and answered in some cases by various authors. The reader may pursue work done regarding $(1)$ in [Bre1], [Bre2], [He], [Ho], [G], [M], [Tr1], [Tr2], [Tr3], and [Tr4]. In particular, a consequence of a version of Szegö’s theorem generalized by Kolmogoroff and Krein is that for $\mu$ supported on the unit circle, $P^2(\mu) = L^2(\mu)$ provided that

$$\int \log \left(\frac{d\mu}{dm}\right) \, dm = -\infty$$

where $dm$ is Lebesgue measure on the circle (see [Ho], [G]). T. T. Trent characterized when $P^2(\mu) \neq L^2(\mu)$ in general with the condition that there exist a finite measure $\nu$ singular with respect to $\mu$ so for some positive constant $c$,

$$\|\nu\|_{1,\mu} \leq c \|\nu\|_{2,\mu}$$
for all polynomials $p$.

Question (20) is addressed in [Kr1], [Kr2], and [KM] in the case where $\mu$ is supported on the closed unit disk $\bar{D}$ and $\mu = \nu + wdm$ where $\nu$ is carried by the open disc $D$, $m$ is Lebesgue measure on $\partial D$, and $w \in L^1(dm)$. T. Kriete shows that if $\text{supp} \nu \subseteq D$ and $\nu$ is circular symmetric then

$$P^2(\mu) = P^2(\nu) + L^2(wdm).$$

Here "circular symmetric" means that $d\nu = dm(r)d\theta$ for some Borel measure $m$ on $[0,1)$. The interplay between $m$ and $w$ is investigated with various results. For example if $dm(r) = G(r)rdr$, then the rate of decay of $G$ as $r \to 1$ is a determining factor for splitting, as is the logarithmic integrability of $w$. In particular, if for small $\delta > 0$

$$\int_{1-\delta}^1 \log \log \frac{1}{G(r)} dr = \infty$$

and if $w = 0$ on a set of positive Lebesgue measure in $\partial D$, then $P^2(\mu)$ splits. Conversely, if the above integral is finite and $\partial D$ contains a subarc for which $1/w$ is integrable, then $P^2(\mu)$ does not split—a condition similar to those examined in Chapter 2 of this paper. Further references concerning when $P^2(\mu)$ splits include [Bre3], [Ha], [Hr1], [Hr2], [K], [N], [V1], and [V2].
References


[Kr2] _____, Splitting and boundary behavior in certain $H^2(\mu)$ spaces, in *Linear


The vita has been removed from the scanned document