

ECONOMIC EXPANSIBLE-CONTRACTIBLE SEQUENTIAL
FACTORIAL DESIGNS FOR EXPLORATORY EXPERIMENTS.

by

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(ABSTRACT)

Sequential experimentation, especially for factorial treatment structures, becomes important when one or more of the following conditions exist: observations become available quickly, observations are costly to obtain, experimental results need to be evaluated quickly, adjustments in experimental set-up may be desirable, a quick screening of the importance of various factors is important. The designs discussed in this study are suitable for these situations. Two approaches to sequential factorial experimentation are considered: one-run-at-a-time (ORAT) plans and one-block-at-a-time (OBAT) plans.

For 2^n experiments, saturated non-orthogonal 2^n_v fractions to be carried out as ORAT plans are reported. In such ORAT plans, only one factor level is changed between any two successive runs. Such plans are useful and economical for situations in which it is costly to change simultaneously more than one factor level at a given time. The estimable effects and the alias structure after each run have been provided. Formulas for the estimates of main-

effects and two-factor interactions have been derived. Such formulas can be used for assessing the significance of their estimates.

For 3^m and $2^n 3^m$ experiments, Webb's (1965) saturated non-orthogonal expandible-contractible $\langle 0, 1, 2 \rangle - 2_V^n$ designs have been generalized and new saturated non-orthogonal expandible-contractible 3_V^m and $2^n 3_V^m$ designs have been reported. Based on these 2_V^n , 3_V^m and $2^n 3_V^m$ designs, we have reported new OBAT 2_V^n , 3_V^m and $2^n 3_V^m$ plans which will eventually lead to the estimation of all main-effects and all two-factor interactions. The OBAT 2^n , 3^m and $2^n 3^m$ plans have been constructed according to two strategies: Strategy I OBAT plans are carried out in blocks of very small sizes, i.e. 2 and 3, and factor effects are estimated one at a time whereas Strategy II OBAT plans involve larger block sizes where factors are assumed to fall into disjoint sets and each block investigates the effects of the factors of a particular set. Strategy I OBAT plans are appropriate when severe time trends in the response may be present. Formulas for estimates of main-effects and two-factor interactions at the various stages of strategy I OBAT 2^n , 3^m and $2^n 3^m$ plans are reported.

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TABLE OF CONTENTS

	Page
ABSTRACT	ii
ACKNOWLEDGEMENTS	iv
Chapter	
I. INTRODUCTION	1
1.1 Background	1
1.2 Scope of the Present Study	4
II. MULTI-FACTOR EXPERIMENTS	8
2.1 One Factor at a Time (OFAT) Versus Factorial Designs	9
2.2 Historical Development of Factorial and Fractional Factorial Designs	13
2.3 Fractional Factorial Designs	17
2.3.1 Confirmatory Versus Exploratory Factorial Experiments	17
2.3.2 Optimality of Sequential Factorial Designs for Exploratory Experiments	17
2.3.3 Sequential Factorial Designs as Factor- screening Techniques for Industrial Experiments	
2.4 One-Run-at-a-Time (ORAT) Plans	19
2.5 One-Block-at-a-Time (OBAT) Plans	23
III. STRICT ORAT PLANS FOR THE 2^n FACTORIAL EXPERIMENTS	25
3.1 Literature Review of ORAT Plans	25
3.2 Parametrization of the 2^n Factorial Experiment	34
3.3 Treatment Combinations for the ORAT 2^n Plans	35

TABLE OF CONTENTS (contd.)

Chapter	Page
3.4 Strict ORAT 2_V^n Plans	57
3.5 An Example: Strict ORAT 2_V^6 Plan	74
IV. OBAT PLANS FOR THE 2^n FACTORIAL EXPERIMENTS	84
4.1 Literature Review of OBAT 2^n Plans	84
4.2 Treatment Combinations for OBAT 2^n Plans	96
4.3 OBAT 2^n Plans for Strategy I	101
4.4 OBAT 2^n Plans for Strategy II	112
4.5 An Example: OBAT Plan for the 2^6 Experiment	118
V. OBAT PLANS FOR THE 3^m FACTORIAL EXPERIMENTS	133
5.1 Literature Review of OBAT 3^m Plans	133
5.2 Parametrization of the 3^m Factorial Experiment	138
5.3 Economic Expansible-Contractible Permutation-Invariant 3_V^m Factorial Designs	140
5.4 OBAT 3^m Plans for Strategy I	154
5.5 OBAT 3^m Plans for Strategy II	182
5.6 An Example: OBAT Plan for the 3^4 Experiment	187
VI. OBAT PLANS FOR THE $2^n 3^m$ FACTORIAL EXPERIMENTS	195
6.1 Literature Review of OBAT $2^n 3^m$ Plans	195
6.2 Parametrization of the $2^n 3^m$ Factorial Experiment	198
6.3 Economic Expansible-Contractible Permutation-Invariant $2^n 3_V^m$ Factorial Design	201
6.4 OBAT $2^n 3^m$ Plans for Strategy I (order one)	229

TABLE OF CONTENTS (contd.)

Chapter	Page
6.5 OBAT $2^{n_3^m}$ Plans for Strategy I (order two)	257
6.6 OBAT $2^{n_3^m}$ Plans for Strategy II	268
VII. SUMMARY AND AREAS FOR FUTURE RESEARCH	276
7.1 Summary of Results	276
7.2 Areas for Future Research	286
APPENDIX A	288
APPENDIX B	290
REFERENCES	297
VITA	301

LIST OF TABLES

Table	Page
1. A Display of the Parameters in the Model (3.1) - One Factor Emphasized	39
2. A Display of the Parameters in the Model (3.1) - Two Factors Emphasized	45
3. Estimable Functions and Their Estimates from the 12-run Strict ORAT 2 ⁶ Plan Consisting of Sets (1) and (2) of (3.62)	77
4. The Frequency for which the First Four Factors of the Fraction F_m in (5.2) Occur Together One at a Time, Two at a Time, Three at a Time and Four at a Time	293

I. INTRODUCTION

1.1 Background

In many investigations it is required to study the effect of several factors on a particular response or process. This can be achieved by investigating one factor at a time while the other factors are held at a constant level or, more efficiently, by running a full factorial experiment in which all possible factor level combinations are utilized.

The difference between these two approaches is that in the first approach only so called simple main effects can be estimated each from only a subset of the available responses, whereas in the second approach all main effects and, most importantly, interactions can be estimated, each from all available responses.

The following are some situations where factorial experimentation is useful

1. In exploratory studies in which a relatively large number of factors are included in the experiment and interest is either in determining quickly the effect of each factor or in screening these factors in order to identify the most influential factors to the response.

2. In response surface exploration where an extensive study of a single factor or a few factors must be carried out to determine which factor levels produce an optimum response.

The special classes of 2^n , 3^m and $2^n 3^m$ factorial experiments are widely used in research and exploratory work. However, even in these classes, orthogonal as well as non-orthogonal factorial designs become uneconomically large when the number of factors is large. Alternatively

then, many exploratory factorial experiments and costly factorial experiments may be carried out sequentially such that factor effects can be studied one at a time, two at a time or a group at a time. Such sequential factorial experimentation allows the experimenter to modify his experiment after each stage by adding new factors, discarding factors which turn out to have insignificant effects, or terminating his experiment as soon as conclusive results are obtained. Hence, each additional stage in the experiment leads to the estimation of new effects and/or the increase of precision for already estimable effects.

Factorial designs suitable for sequential experimentation will be called sequential factorial designs (SFD). These SFD can be classified into two categories:

1. One-run-at-a-time (ORAT) designs or plans.
2. One-block-at-a-time (OBAT) designs or plans.

ORAT plans are factorial designs in which treatment combinations are applied to the experimental material one at a time and where evaluation takes place after each run. OBAT plans are factorial designs in which a set or a block of treatment combinations are applied to the experimental material one at a time and evaluation takes place after each block has been observed.

ORAT plans are useful for situations in which frequent or sudden interruptions in the experiment are expected. On the other hand, OBAT plans are useful for situations in which severe time trends are present in the responses of the various treatment combinations. Clearly, more parameters or effects are estimated after each block in OBAT plans than after each run in ORAT plans. Also, the order in which ORAT or OBAT

plans are carried out determines the order in which factor effects become estimable.

Although SFD have received considerable attention in the literature, a number of problems have not been addressed satisfactorily or not at all. Some of these problems are:

1. The emphasis in SFD that are ORAT plans has been on constructing plans that make as few factor level changes between successive runs as possible and plans that are robust (orthogonal) to time trends. However, the aspect of constructing ORAT plans according to which factor effects can be estimated after a particular run and how such effects can be estimated has received little attention. Daniel (1973), who initiated the idea of carrying out factorial experiments one run at a time, reported only two specific ORAT plans, namely 2^3_v and 2^4_v ORAT plans, in which he listed the effects that can be estimated after each run.

2. Most SFD that are OBAT plans have involved blocks of relatively large sizes. Only few results have been reported for using OBAT plans with blocks of size 2 for the 2^n experiment and blocks of size 3 for the 3^m experiment.

3. Most SFD have been developed to study the effect of a group of factors at each stage of the experiment. Designs utilized for this purpose are orthogonal and non-orthogonal resolution IV designs. Very few SFD have been reported with emphasis on the examination of only one factor effect at each stage or with emphasis on adding new factors to experiments which are in progress. Among these few SFD are Webb's (1965) 2^n_v designs.

Webb (1965), arguing that orthogonal 2^n_v designs are not economically

expansible for new factors since each additional factor doubles the number of runs in the design and arguing that these designs are not robust to premature or sudden termination of the experiment since most information on factor effects will be lost (each factor effect is estimated from all runs in the design), Webb reported saturated non-orthogonal expansible-contractible 2_V^n designs which allow new factors to be added with few new additional runs and which are robust to premature termination. Webb, however, did not derive explicit expressions for the estimates of main-effects and two-factor interactions that can be used for establishing the significance of their estimates nor did he consider running his designs sequentially as ORAT and OBAT plans.

These designs with good practical qualities which allow factors to be added to the experiment after it has started and which are robust to sudden interruptions or termination of the experiment, are potentially useful for sequentially carrying out factorial experiments in which factors can be studied one at a time, two at a time or a group at a time. Also an attempt to generalize Webb's (1965) saturated non-orthogonal expansible-contractible 2_V^n designs to 3_V^m and $2^n 3_V^m$ proves to be worthwhile.

1.2 Scope of the Present Study

The main concern in this dissertation is threefold:

1. constructing ORAT 2_V^n plans that deal with the problem (1) in Section 1.1.
2. generalizing Webb's (1965) saturated non-orthogonal expansible-

contractible 2_V^n designs to 3_V^m and $2^n 3_V^m$ designs.

3. constructing OBAT 2_V^n plans based on Webb's (1965) 2_V^n designs and OBAT 3_V^m and $2^n 3_V^m$ plans based on the 3_V^m and $2^n 3_V^m$ designs in 2 above. These OBAT plans address the problems in (2) and (3) in Section 1.1.

The construction of these designs will be based entirely on estimability consideration. After each step of the ORAT and OBAT plans, estimable factor effects will be provided. In addition, explicit expressions (formulas) for the estimates of main-effects and two-factor interactions will be derived. Such expressions indicate which treatment combinations are involved in the estimation of each factor effect.

In Chapter 2, we compare two approaches for the analysis of multi-factor experiments: the one-factor-at-a-time (OFAT) and the factorial. We also give a brief account of the historical development of factorial and fractional factorial designs. The subclass of sequential factorial designs will be discussed in detail together with an illustration of the appropriateness of these designs for exploratory industrial experiments. Attention was also given to the optimal properties of the two categories of sequential factorial designs, the ORAT and the OBAT designs.

In Chapter 3, we discuss ORAT plans for n factors at 2 levels each. In particular, we develop and characterize designs where only one factor level is changed between two successive runs. The implications with respect to estimability of main effects and two-factor interactions, after any run, are presented together with estimates of these effects.

OBAT designs for n factors each at 2 levels are discussed in Chapter 4. These plans, based on Webb's (1965) results, are developed for two

different strategies. Strategy I OBAT plans make use of blocks of size 2 evaluating one factor at a time. The plans for Strategy II are arranged in larger blocks. They are based on a grouping of factors into disjoint sets such that each block can be used to evaluate the effects of the factors in that set. For both types of designs, we provide at each step of the experiment, explicit expressions for estimable functions involving main-effects and/or two-factor interactions as well as estimates for these functions.

The idea of saturated non-orthogonal expandible-contractible advocated by Webb (1965) for 2^n experiments will be extended to 3^m experiments in Chapter 5 and $2^n 3^m$ experiments in Chapter 6.

In most situations of exploratory experimentation, factors with two and/or three levels are being used. Our aim here is to address these situations in the context of sequential experimentation. It is our goal to develop useful plans that

(i) provide readily estimates of main-effects and two-factor interactions,

(ii) are economical in the sense of using as few experimental units as possible,

(iii) allow the researcher to extract information after each step of experimentation,

(iv) provide opportunities to interrupt the investigation without loss of information on important factors,

(v) enable the investigator to easily and economically expand the experiment, and

(vi) eliminate systematic noise, such as time trends, from estimates.

In all this, the problem of estimability is of overriding concern.

Although the question of precision of estimates is not ignored, no attempt has been made to achieve necessarily design optimality.

II. MULTI-FACTOR EXPERIMENTS

In this chapter we shall give a brief description of the different types of multi-factor experimentation. In Section 2.1 we contrast the one-factor-at-a-time (OFAT) approach and the factorial approach by which the joint effect of two or more factors on the response can be described. In Section 2.2 a brief account is given of the historical development of factorial and fractional factorial designs. The usefulness of confounding techniques - which make use of the negligibility of high order interactions among the factors - in blocking factorial experiments and in constructing fractional factorial designs is indicated. The importance of fractional factorial designs comes from the fact that information on effects of major concern, namely, main effects and low order interactions can be obtained from a fraction of the full factorial experiment. The class of fractional factorial designs includes the regular or orthogonal designs, irregular or non-orthogonal designs, balanced and partially balanced designs, proportional frequency designs, augmented factorial designs and sequential factorial designs. Both of the last two categories of designs involve running the experiment in stages. Augmented designs are fractional factorial designs augmented by few additional runs where augmentation is done after the original fraction has been analyzed. On the other hand, sequential factorial designs are generally carried out in more than two stages where each stage is constructed according to the conclusions from preceding stages.

In Section 2.3 a comparison is made between confirmatory and exploratory multi-factor experiments. Confirmatory experiments involve

generally large factorial designs with fixed number of runs while exploratory experiments involve sequences of relatively smaller factorial designs hence the number of runs is not fixed and it is dependent on when sequential experimentation is stopped. Another comparison is made between two different design optimality criteria, one is related to the statistical characteristics of the experiment and the other is related to its physical characteristics. Statistical optimality criteria pertain to the precision of estimates, for example, orthogonality or D-optimality while the physical criteria pertain to the cost and time needed to perform the experiment. Confirmatory experiments are, in general, statistically optimal but not physically optimal while exploratory experiments are physically optimal but not statistically optimal. It was also shown that sequential factorial designs are quite useful and economical for screening factors in exploratory industrial type experiments where a large number of potential factors are included in the experiment. Sequential factorial designs are classified into two main categories, the one run at a time (ORAT) and the one block of runs at a time (OBAT). Properties of the ORAT and the OBAT plans were discussed in Sections 2.4 and 2.5.

2.1 One Factor at a Time (OFAT) Versus Factorial Designs

Many experiments involve the study of the effect of one or more factors (independent variables) on a particular response (dependent variable). For such multi-factor experiments, the OFAT approach was often used by experimenters until R. A. Fisher introduced the more efficient

approach, namely the factorial approach.

In the OFAT approach, factor effects are estimated one at a time by changing the levels of one factor while holding all other factors each at its lowest level. On the other hand, in factorial experiments all levels of a given factor are combined with all levels of every other factor and hence responses to the various treatment combinations allows the experimenter to investigate the joint effects of two or more factors on the response. Each factor's effect is estimated as the change in the response produced by a change in the levels of that particular factor. The other factors do not necessarily have to be at their lowest levels as in the OFAT approach.

The differences between the OFAT and factorial approaches can be seen in the following two-factor experiment where each factor has two levels. We denote the factors by A and B and their respective levels by a_0 and a_1 and b_0 and b_1 . Treatment combinations between the two factor levels are represented by $a_i b_j$, or just ij , where $i, j = 0, 1$ and the corresponding response by y_{ij} . Using the OFAT approach, information on both factors can be obtained by varying the levels of one factor keeping the other factor fixed or constant at its low level. The estimates of the two factor effects are

$$\begin{aligned}\hat{A} &= y_{10} - y_{00} \\ \hat{B} &= y_{01} - y_{00}\end{aligned}\tag{2.1}$$

and $\text{Var } \hat{A} = \text{Var } \hat{B} = 2\sigma^2$.

The changes or estimates in (2.1) can be used to make inferences about the effects of factor A and B. However, it is not clear whether these changes are real factor effects or just random variations in the response. Therefore, unless there is a prior estimate of the random variations in the response (experimental error) that can be used to judge the significance of the estimates in (2.1), the entire experiment or part of it must be replicated. Suppose 2 runs are made at each one of the 3 treatment combinations 00, 10 and 10.

Letting $y_{ij}^{(k)}$ denote the response to the treatment combination $a_i b_j$ of the k^{th} replicate where $i, j = 0, 1$ and $k = 1, 2$, we note that each one of the 3 differences

$$(y_{00}^{(2)} - y_{00}^{(1)}), (y_{10}^{(2)} - y_{10}^{(1)}) \text{ and } (y_{01}^{(2)} - y_{01}^{(1)})$$

is an estimate of the experimental error. Also, each factor's effect is now more precisely estimable since each factor is estimated as an average of two differences. That is,

$$\begin{aligned}\hat{A} &= \frac{1}{2}[(y_{10}^{(1)} - y_{00}^{(1)}) + (y_{10}^{(2)} - y_{00}^{(2)})] \\ \hat{B} &= \frac{1}{2}[(y_{01}^{(1)} - y_{00}^{(1)}) + (y_{01}^{(2)} - y_{00}^{(2)})]\end{aligned}\quad (2.2)$$

with $\text{Var } \hat{A} = \text{Var } \hat{B} = \sigma^2$ and $\text{Cov}(\hat{A}, \hat{B}) = \sigma^2$.

The following two deficiencies in the OFAT approach for analyzing multi-factor experiments are observed.

1. The OFAT approach does not allow the examination of the entire factor space of the experiment. In the example above, no response is

obtained at the treatment combination $a_1 b_1$ which involves changing both factor levels simultaneously. In general, the OFAT approach for an experiment involving n 2-level factors requires only the $(n+1)$ treatment combinations

00---0

10---0 and all its permutations.

That is, $2^n - (n+1)$ treatment combinations are left out.

2. The OFAT conclusions may be erroneous if the factors do not act independently on the response, i.e. they interact. In the example above, if y_{10} and y_{01} turn out to be "better" than y_{00} , one using the OFAT approach may conclude that y_{11} is even "much better". However, such conclusion is not necessarily true if the two factors interact.

These were some of the deficiencies that have led to the development of the factorial approach (see Fisher 1953). The superiority of the factorial approach over the OFAT can be seen in the two-factor experiment above where only one run is needed at each one of the four treatment combinations $a_0 b_0$, $a_1 b_0$, $a_0 b_1$ and $a_1 b_1$. That is, with only 4 runs involving the entire factor space of the two 2-level factors A and B, factor effects are estimable with the same precision as that of the replicated six-run OFAT plan. Hence, in 2^2 factorial experiment each factor's effect is estimable from two differences and

$$\hat{A} = \frac{1}{2}[(y_{10} - y_{00}) + (y_{11} - y_{01})]$$

$$\hat{B} = \frac{1}{2}[(y_{01} - y_{00}) + (y_{11} - y_{10})]$$

with $\text{Var } \hat{A} = \text{Var } \hat{B} = \sigma^2$ and $\text{Cov}(\hat{A}, \hat{B}) = 0$. From the factorial approach, information on the interaction AB between factors A and B can also be obtained where

$$\hat{AB} = \frac{1}{2}[(y_{11} - y_{01}) - (y_{10} - y_{00})]$$

with $\text{Var } \hat{AB} = \sigma^2$ and $\text{Cov}(\hat{AB}, \hat{A}) = \text{Cov}(\hat{AB}, \hat{B}) = 0$.

2.2 Historical Development of Factorial and Fractional Factorial Designs

As indicated in Section 2.1 that the factorial approach is the most efficient technique for the analysis of multi-factor experiments, yielding information on factor effects as well as their interactions. Factors may all have the same number of levels in which case the experiment is called symmetrical otherwise it is called asymmetrical. The special classes of 2^n and 3^m factorial experiments are widely used in research and exploratory work.

Effects in factorial experiments are classified as main effects, low-order interactions and high-order interactions. High order interactions are, generally, not easy to interpret and quite often they do not reflect real effects, i.e, they are negligible. Furthermore, as the number of factors in the experiment becomes large the number of treatment combinations in the factorial experiment becomes even larger in which case there may not be enough homogeneous experimental material to run the full factorial experiment in one whole block. In such cases, confounding techniques developed by R. A. Fisher [Kempthorne (1952)] allow the experimenter to run the full factorial experiment in small-

size blocks of homogeneous experimental material. The treatment combinations in each block are determined by the block size and the confounding scheme used to confound negligible interactions with blocks. The confounding scheme is chosen such that, if at all possible, main effects and low-order interactions remain estimable at the expense of high-order interactions.

There are also situations in which it is uneconomical or even impossible to run the entire factorial experiment, especially when the number of factors in the experiment is large. Confounding techniques can also be used to construct fractional factorials containing a subset of the treatment combinations of the full factorial experiment from which information on main effects as well as low order interactions can be obtained. These fractions are constructed such that non-negligible effects are confounded with negligible high order interactions.

Since information on main effects and low-order interactions can be obtained from a fraction of all the treatment combinations, researchers' main emphasis has been on developing technique for constructing fractional factorial designs. The theories of groups, Galois fields and projective geometry proved quite useful in the construction of balanced and partially balanced fractional factorial designs from which main effects and interaction effects up to a specified order are estimable.

Box and Hunter (1961) introduced the concept of resolution to classify fractional factorial designs according to their confounding structure and the capability of estimations. Resolutions III, IV and V designs received considerable attention from researchers.

Irregular or non-orthogonal fractional factorial designs, especially $\frac{3}{2^m}$ fractions of the 2^n experiments, were also introduced in the 1960's (see Addelman [1963]). They are more economical than regular or orthogonal factorial designs as they require a smaller number of runs for the estimation of the same number of factorial effects. However, this reduction in the number of runs is achieved at the cost of precision of parameter estimates and correlation among these estimates.

In 1962, Addelman introduced proportional frequency fractional factorial designs. In these designs any two factor levels occur together a proportional number of times. These designs provide considerable reduction in the number of runs required in a fraction of an asymmetrical factorial experiment. Addelman (1963) reviewed most construction techniques of orthogonal and non-orthogonal factorial designs.

Another class of factorial designs called augmented designs can be found in the recent literature on factorial experiments. These designs are actually regular or irregular fractional factorial designs augmented by a small number of additional runs. Augmentation may be done for a variety of reasons some of which are

1. To improve the estimates of already estimable parameters. For example, orthogonalizing a non-orthogonal design.
2. To break chains of aliases among confounded factorial effects so that non-estimable effects in the original fraction become estimable in the augmented design.
3. To provide some degrees of freedom for estimating the experimental error by replicating a few runs in the original fraction. This

replication is usually done when all available degrees of freedom in the original fraction are used for estimating factorial effects.

4. To expand or complete an interrupted experiment by adding more factors and hence more runs to the interrupted factorial design.

Factorial experiments that are carried out in a sequence of small fractions were first introduced in the 1960's (see Daniel [1962]) and they are still of major concern to researchers of industrial experiments. In sequential factorial designs, each succeeding fraction is constructed according to conclusions drawn from preceding fractions. These designs are mainly used in exploratory factorial experiments where a large number of potential factors are included but only a small subset of these factors are expected to have major influence on the response.

In his review on the development of factorial experiments, Addelman (1972) concluded that all contributions to the design of factorial experiments, since 1965, have involved fractional factorial designs. These designs are divided into three categories:

1. Fractions of asymmetrical factorial experiments.
2. Irregular or non-orthogonal fractional factorial designs.
3. Sequences of fractional factorial plans.

In addition to Addelman (1972), Steinberg and Hunter (1984) have reviewed recent work in the area of experimental design and have also indicated some important areas for future research one of which is sequential factorial experimentation.

The main objective of this dissertation is the construction of new sequential factorial designs for 2^n , 3^m and $2^n \cdot 3^m$ exploratory factorial experiments.

2.3 Fractional Factorial Designs

2.3.1 Confirmatory Versus Exploratory Factorial Experiments

Confirmatory experiments are experiments that are carried out either to discover something about a particular process or to compare the effects of several conditions (factors) on some phenomenon (response). On the other hand, exploratory experiments are experiments often carried out at the early stages of an investigation where a large number of potential factors are included. The main goal of the experiment is screening factors such that a subset of factors that are most influential for a particular response is identified.

Another difference between the two types of experiments is that confirmatory experiments involve a fixed number of runs (trials) while exploratory experiments are generally carried out sequentially hence allowing the experimenter to choose the number of runs according to certain accuracy and cost constraints.

2.3.2 Optimality of Sequential Factorial Designs for Exploratory Experiments

An experimental design is a plan as to how the experiment should be performed and since there may be more than one possible design for a given experimental situation each of which yields different information about the phenomenon under investigation, several optimality criteria were developed as tools for judging designs, i.e. which design is better or more efficient than another with respect to specific experimental goals.

These optimality criteria pertain to either the physical or the statistical characteristics of the experiment. Some of the physical criteria or constraints are time, effort and cost needed to conduct the experiment while the statistical criteria relate to the amount of information and precision of parameter estimates, such as orthogonality, balance or partial balance and D-optimality.

Confirmatory experiments are usually statistically optimal but generally not optimal in the physical sense since, for example, an orthogonal or a balanced factorial design may involve a large number of runs, a situation which the experimenter may not be able to afford. On the other hand, exploratory experiments are usually optimal in the physical sense, i.e. economical but not generally statistically optimal.

2.3.3 Sequential Factorial Designs as Factor-Screening Techniques for Industrial Experiments

Due to the fact that runs or trials in industrial type experiments are relatively more expensive than agricultural trials and also since industrial trials can be carried out or observed more quickly than agricultural trials, sequential factorial designs prove to be quite useful and economical for exploratory industrial experiments. Sequential factorial designs allow the experimenter to perform his experiment in a sequence of small fractions, where succeeding fractions can be constructed according to the results of the preceding ones. With such flexibility the experimenter has the freedom, once a fraction is completed, to terminate the experiment or to add new factors or to delete factors that turn out to be not influential.

More economy in the number of runs of sequential factorial designs can also be achieved since the assumption of negligibility of high-order interactions allows the experimenter to investigate many factors in each fraction in the sequence. Hence, with a sequence involving a small number of saturated fractions, the influential factors may be identified. However, since all degrees of freedom available are used for estimating factor effects, a prior estimate of the experimental error must be available in order to judge the significance of parameter estimates.

It is worth mentioning that designs other than sequential factorial designs have been used for factor screening such as supersaturated designs and group-screening designs. A comparison among several factor-screening designs is given by Kleijnen (1975). Also Srivastava (1975) characterized factor-screening designs as being a subclass of search designs. Search designs are designs that search for non-negligible effects in a factorial experiment.

Sequential factorial designs for exploratory experiments may be classified into two main categories:

1. One run at a time (ORAT) plans.
2. One block at a time (OBAT) plans.

Each one of these categories of designs is optimal with respect to some physical experimental criterion. These optimal properties will be discussed in the following two sections.

2.4 One Run at a Time (ORAT) Plans

ORAT plans are factorial designs in which treatment combinations

are applied to the experimental material one at a time. Between any two successive runs one factor level or more may be changed. However, by varying only one factor level between successive runs while holding the other factors constant, not necessarily at their lowest levels as in the OFAT plans, the experimenter can estimate one factor effect after each run. Furthermore, the run order determined by the ORAT plan determines the order in which factor effects become estimable.

ORAT plans are economical as screening designs for exploratory experiments especially ORAT 2^n plans for which only 2 runs are needed to assess any one factor effect, if factors do not interact. For example, under the assumption of no interactions, main effects of the three two-level factors A, B and C can be estimated from the ORAT plan (1), a, ab and abc with responses $y(1)$, $y(a)$, $y(ab)$ and $y(abc)$ respectively. The estimates are $\hat{A} = y(a) - y(1)$, $\hat{B} = y(ab) - y(a)$ and $\hat{C} = y(abc) - y(ab)$ with $\text{Var } \hat{A} = \text{Var } \hat{B} = \text{Var } \hat{C} = 2\sigma^2$. However, if two-factor interactions are non-negligible, the ORAT plan (1), ab, abc, bc, c and ac can be used to estimate the 6 factorial effects A, B, C, AB, AC and BC.

Of course, other ORAT plans for the 2^3 factorial experiment can be used, for example, the ORAT plan (1), a, b, c with responses $y(1)$, $y(a)$, $y(b)$ and $y(c)$ from which $\hat{A} = y(a) - y(1)$, $\hat{B} = y(b) - y(1)$ and $\hat{C} = y(c) - y(1)$. However, the ORAT plan (1), a, ab, abc varies only one factor level between successive runs while the ORAT plan (1), a, b, c changes more than one factor level between some runs. The ORAT plans which vary only one factor level between successive runs are quite useful and economical for situations in which it is costly to change more than one fac-

tor level between runs. ORAT plans which vary one factor level between runs are called strict ORAT plans by Daniel (1973). Strict ORAT 2^n plans are studied thoroughly in Chapter 3.

As factor-screening designs, estimates of factor effects in ORAT plans are not as precise as those in well-designed and well-randomized factorial designs. For example, in the ORAT (1), a, b, c we saw that each factor effect is estimable from only 2 runs while in the $\frac{1}{2}$ replicate of the 2^3 factorial experiment whose defining contrast is $I = ABC$ with runs a, b, c and abc, each factor effect is estimable from 4 runs, e.g. $\hat{A} = \frac{1}{2}[y(abc) - y(c) + y(a) - y(b)]$ and hence $\text{Var } \hat{A} = \text{Var } \hat{B} = \text{Var } \hat{C} = \sigma^2$. However, this is not a major disadvantage since the main interest in ORAT plans is not in describing factor effects precisely rather interest is in factor screening.

ORAT plans are saturated, i.e. the number of runs in the plan is equal to the number of parameters to be estimated. Such economical situation requires that a prior estimate of the experimental error be available in order to judge the significance of parameter estimates or some runs must be replicated from which an estimate of the error variance can be obtained. If, however, no such prior estimate is available or runs are expensive and cannot be replicated, ORAT plans can still be used for screening purposes. The researcher may use Daniel's (1959) graphical normal probability plots to judge the significance of factor effects in ORAT plans.

Normal probability plot is a graph of the cumulative distribution function of the normal distribution $N(0, \sigma^2)$ where the ordinate is scaled such that the graph is a straight line. With the assumption

that the error is distributed as $N(0, \sigma^2)$, responses as well as contrasts estimating single-degree-of-freedom factorial effects are also normally distributed. By arranging the effect estimates in an ascending order of magnitude and then plotting the j^{th} of these ordered estimates against $P_j = \frac{j-1/2}{N-1}$ where N is the number of runs in the plan, negligible effects will fall along the straight line while significant effects will be further from the line.

Another characteristic of ORAT plans is that if the experiment is stopped (interrupted) after any one run, information can still be obtained about the factor effects whose levels have been varied but no information can be obtained about factors kept at their lowest level. That is, ORAT plans are fully interruptible or contractible. However, this is not the case in fixed-size factorial designs where information about factor effects may be lost if the experiment is interrupted prematurely since, for example, in orthogonal p^{n-m} factorial designs each factorial effect is estimable from all runs in the design. ORAT plans are also expandable, i.e. new factors and hence new runs can be added to the ORAT plans after any run.

Since ORAT plans are run sequentially according to one particular order, i.e. there is no opportunity to randomize the runs order, responses to treatment combinations may drift with time hence biasing estimates of factor effects. However, if a substantial part of the uncontrolled variation in the responses is this time trend and if this trend follows a polynomial functional form, estimates of factor effects can be made orthogonal to the time effect and hence unbiased estimates of both factorial effects and time effects can be obtained. If the

time trend cannot be assumed of polynomial functional form, one block at a time (OBAT) plans can be used instead where blocking is expected to reduce or eliminate the time effect on the response.

ORAT plans will be discussed more fully in Chapter 3.

2.5 One Block at a Time (OBAT) Plans

In situations where factorial experiments are carried out sequentially, OBAT plans are regarded as the alternative to ORAT plans if responses to the treatment combinations drift with time and the time effect is of unknown functional form or if blocking is necessary for other reasons. That is, with sequences of small fractions or blocks the effect of time and other extraneous sources on the responses may greatly be reduced or eliminated and hence unbiased estimates of effects of interest can be obtained.

Unlike ORAT plans, each block in the OBAT plans may introduce more than one factor to the plan and hence more than one parameter or factorial effect is estimable after the completion of any one block. Each factor, after its first introduction to the OBAT plan, may be changed to any level in the succeeding blocks of the plan. The block order determined by the OBAT plan determines the order in which factor effects become estimable.

In contrast to the ORAT plans, OBAT plans can only be interrupted or discontinued after the completion of a block otherwise loss of information results on some factor effects. Once a block is completed the experimenter can make changes in his experiment such as adding new

factors or discarding non-influential factors from the experiment.

Similar to ORAT plans, OBAT plans are generally saturated and hence a prior estimate of the experimental error must be available or some runs be replicated or, when normality assumptions holds, Daniel's (1959) normal probability plots may be used to judge the significance of parameter estimates.

OBAT plans are dicussed in detail in Chapters 4, 5 and 6.

III. STRICT ORAT PLANS FOR THE 2^n FACTORIAL EXPERIMENTS

In this chapter ORAT non-orthogonal designs for the 2^n factorial experiment are developed in which any two successive runs differ only in the level of one factor. A literature review on ORAT plans is presented in Section 3.1. The set of treatment combinations from which the strict ORAT plans can be constructed is shown, in Section 3.3, to be a saturated $2^n/V$ fraction. From this fraction, all main effects and all two-factor interactions are estimable with the same precision. This fraction is partitioned into n disjoint sets and such sets are considered as stages for the strict ORAT plans in Section 3.4. At the completion of any one stage, estimable functions as well as their unbiased estimates are reported. An application of the strict ORAT 2^n plans for the 2^6 factorial experiment is provided in Section 3.5.

3.1 Literature Review of ORAT Plans

ORAT plans are factorial designs in which treatment combinations are carried out in a sequential order one after the other. Contributions to the literature of ORAT plans will be arranged in chronological order.

Daniel (1959) showed that the ORAT main effect plan for n 2-level factors given by the $(n+1)$ runs $(1), a_1, a_2, \dots, a_n$ where a_i represents the high level of factor A_i , $i = 1, 2, \dots, n$ can be augmented by additional runs to form a half-replicate of the full 2^n factorial

experiment plus the additional run (1). The run (1) is the run in which all factors are at their low level. This augmentation is done to improve precision of estimates.

For an illustration, we observe that each one of the main effects of the four 2-level factors A, B, C and D in the ORAT plan (1), a, b, c, d is estimable from only 2 runs, i.e. $\hat{A} = y(a) - y(1)$, $\hat{B} = y(b) - y(1)$, $\hat{C} = y(c) - y(1)$ and $\hat{D} = y(d) - y(1)$ with $\text{var } \hat{A} = \text{var } \hat{B} = \text{var } \hat{C} = \text{var } \hat{D} = 2\sigma^2$ where $y(1)$, $y(a)$, $y(b)$, $y(c)$ and $y(d)$ are the responses of the runs (1), a, b, c, d, respectively. When this ORAT plan is augmented by the 4 runs abc, abd, acd and bcd, the resulting set of 9 runs is the half-replicate of the 2^4 experiment whose defining contrast is $I = -ABCD$ plus the additional run (1). In this 8-run plan, each main effect is now estimable from 8 runs, e.g. $A = \frac{1}{4}[(a + ab + abd + acd) - (b - c - d - bcd)]$ with $\text{var } \hat{A} = \text{var } \hat{B} = \text{var } \hat{C} = \text{var } \hat{D} = \frac{1}{2}\sigma^2$, hence quadrupling the precision of estimates of factor effects.

Daniel pointed out that the resulting half-replicates from this augmentation scheme are embedded in each other, i.e. half-replicates of 2^n experiments are embedded in half-replicates of 2^{n+1} experiments. The ORAT plan (1) a, b, c for the 2^3 factorial with factors A, B and C can be augmented by the run abc where the resulting half-replicate is the one whose defining contrast is $I = ABC$ from which each main effect is estimable from 4 runs. This half-replicate of the 2^3 is embedded in the half-replicate of the 2^4 with $I = -ABCD$. However, this augmentation is not economical for 6 or more factors as this would require a large number of runs to be added to the original ORAT plan.

Daniel and Wilcoxon (1966) argued that since ORAT plans are carried out sequentially, responses to treatment combinations may drift with time hence biasing effect estimates. They reported ORAT 2^{p-q} factorial plans that are robust (orthogonal) to linear and quadratic time effects. Their plans involve as many as 14 two-level factors and 32 treatment combinations for $p - q = 2, 3, 4$ and 5.

Webb (1965) reported a catalog of non-orthogonal permutation-invariant designs for 2^n factorial experiments. This class of designs is characterized by a set of k integers m_1, m_2, \dots, m_k where $m_1 < m_2 < \dots < m_k < n$. Each $\langle m_1, m_2, \dots, m_k \rangle$ -design contains $\sum_{i=1}^k \binom{n}{m_i}$ treatment combinations. For a given m_i , the $\binom{n}{m_i}$ treatment combinations have exactly m_i factors at their high level and $(n - m_i)$ factors at their low level. For example, the $\langle 0, 1, 2 \rangle$ design for the 2^3 factorial is the design with $1 + 3 + \frac{3(3-1)}{2} = 7$ runs: 000, 100, 010, 001, 110, 101 and 011. Webb observed that the $\langle m_1, m_2, \dots, m_k \rangle$ -design series are embedded in the $\langle m_1, m_2, \dots, m_k \rangle$ -designs for the 2^{n+1} factorial. For an illustration, we observe that the $\langle 0, 1, 2 \rangle$ -design for the 2^2 experiment consisting of the 4 runs 00, 10, 01 and 11 is embedded in the $\langle 0, 1, 2 \rangle$ -seven-run design for the 2^3 experiment. This inclusion property makes these designs expandible, i.e. new factors can be added to the experiment after its initiation. These designs are also interruptible or contractible, i.e. robust to premature termination of the experiment in that information on some factors that have had their levels changed can still be obtained.

Referring to Daniel's (1959) augmentation of ORAT plans in nested orthogonal fractions for improving precision of parameter estimates, we notice that such augmentation is uneconomical as the number of additional runs needed is relatively large. To resolve this problem, Webb indicated that the $\langle m_1, \dots, m_k \rangle$ -design series for 2^n experiments are a compromise between two "extreme" expansible - contractible designs, the orthogonal 2^{p-q} designs and the ORAT designs, since

1. An orthogonal design which provides estimates with small variance are uneconomically expansible as the addition of one factor to the design doubles the number of runs. Also when orthogonal designs are interrupted prematurely, almost all information about factor effects will be lost.

2. ORAT plans which provide less precise estimates than orthogonal designs are economically expansible as the addition of one factor to the plan requires few additional runs in order that its effect be estimable. ORAT plans are fully contractible as the experiment can be interrupted after any one run.

Webb suggested that the saturated resolution $V \langle 0, 1, 2 \rangle$ - design series for 2^n experiments be conducted one run at a time where factors are arranged in descending order of importance for their first introduction to the plan. He also reported for the $\langle m_1, m_2, \dots, m_k \rangle$ - designs, the entries of the $X'X$ matrix as functions of n and m_1, m_2, \dots, m_k . The fraction $\langle 0, 1, 2 \rangle$ for the 2^n experiment will be used in Chapter IV for strategy-I-and-II OBAT 2^n plans.

Webb (1968) reported a class of main effect saturated plans for

n 2-level factors in $(n+1)$ runs where factors are assumed to be ordered such that the i^{th} factor appears at its high level for the first time in the $(i+1)^{\text{st}}$ run. In the runs after the $(i+1)^{\text{st}}$, the i^{th} factor may appear at any of its two levels. The first run is the run 00--0 where all factors are at their low level. The class of ORAT 2_{III}^n plans consists of the ORAT plans in which the k^{th} run - the run at which the $(k-1)^{\text{st}}$ factor appears at its high level for the first time - differs from some previous run in only the level of the $(k-1)^{\text{st}}$ factor. Two "extreme" ORAT plans belong to this class:

(i) the ORAT 2_{III}^n plan in which each factor returns to its low level after its first appearance at the high level, i.e. the ORAT plan (1), a_1, a_2, \dots, a_n .

(ii) the ORAT 2_{III}^n plan in which each factor stays at its high level after its first introduction at the high level, i.e. the ORAT plan (1) $a_1, a_1a_2, a_1a_2a_3, \dots, a_1a_2a_3 \dots a_n$.

For this class of ORAT saturated 2_{III}^n plans, Webb derived a lower bound on the variance of main effect estimates. The bound is $\frac{1}{2}\sigma^2$. Webb also derived a lower bound for the variance of effect estimates in 2_{V}^n ORAT plans. The bound is $\frac{1}{4}\sigma^2$ for the variance of main effect estimates for interacting factors and their two-factor interactions while for factors which do not interact the lower bound on the variance of their main effect estimates is $\frac{1}{2}\sigma^2$. For an illustration, Webb considered the ORAT saturated 2_{V}^4 plan (1), a, b, c, dc, ac, d for the 4 two-level factors A, B, C and D where factors A and C and factors C and D interact, it turns out that $\text{var } \hat{A} = \text{var } \hat{C} = \text{var } \hat{D} = \text{var } \hat{AC} = \text{var } \hat{CD} = \frac{1}{4}\sigma^2$ and $\text{var } \hat{B} = \frac{1}{2}\sigma^2$. Hence, the inclusion of two-factor

interactions improves the precision of main effect estimates.

In his 2_{III}^n and 2_V^n ORAT plans, Webb did not consider the problem of deriving formulas for effect estimates. However, such problem is discussed in this chapter.

Draper and Stoneman (1968b), using computer search techniques, examined the $8! = 40,320$ possible run orders for the ORAT full 2^3 factorial experiment and reported 11 orders which make a minimum number of factor level changes between successive runs that also have minimum linear time effect on the response. They also reported run orders with minimum factor level changes between runs for the ORAT eight-run plans 2^{4-1} , 2^{5-2} , 2^{6-3} and 2^{7-4} . Similar results for 2^4 and 2^5 main effect plans were reported by Dickinson (1974).

Tiahrt and Weeks (1970) proposed a restricted randomization technique for 2^n factorial experiments for $n = 2, 3, 4, 5$ and 6 , when at most a subset of the factors can have their levels changed between any two runs. As an illustration, we consider the 2^3 full factorial experiment when at most two factor level changes can be made between any two runs. Three plans are possible one of which is the plan (1), a, ab, b, abc, c and ac. The assignment of these treatment combinations to experimental unit can be done using the author's suggested randomization procedure.

Daniel (1973) divided ORAT plans for multi-factor experiments into four categories as follows:

1. Strict ORAT plans: plans in which only one factor level is changed between any two successive runs. Such plans are useful and economical in situations where it is costly or impossible to vary more than one factor level between runs. Strict ORAT plans for 2^n experiments

are discussed in this chapter.

2. Standard ORAT plans: plans which vary factors from some standard condition. As an example, the ORAT 2_{III}^n plan (1), a_1, a_2, \dots, a_n where the standard condition is the run (1).

3. Nested ORAT plans: plans where only a subset of the factor levels can be changed between successive runs. See Tiahrt and Weeks (1970).

4. Free ORAT plans: plans that make each new run with any combination of factor levels.

Assuming 3-and more-factor interactions negligible, Daniel argues that ORAT main effect plans lack full randomization of run order, since ORAT plans are performed in only one particular order. As a consequence, main effect estimates may be biased by two-factor interactions and/or by time trends. Such biases can be removed by augmenting the initial main effect plan with additional runs to free main effects from aliasing with two-factor interactions and by making time effects and factorial effects orthogonal to each other when the time trend follows a polynomial functional form. For an illustration, Daniel considered the ORAT 2_V^3 plan

$$(1)_1, a_2, ab_3, abc_4, bc_5, c_6, (1)_7, a_8, ac_9$$

where subscripts indicate the run order, from which all three main effects and all three two-factor interactions are estimable. The runs (1) and a have been replicated in order to estimate the linear and quadratic time effects when the time trend is a second-degree polynomial. Each one of the differences $D_1 = (1)_7 - (1)_1$ and $D_2 = a_8 - a_2$ represents the linear time effect T_L and hence $T_L = \frac{1}{2}(D_1 + D_2)$, and

$T_Q = \frac{1}{2}(D_1 - D_2)$ represents the quadratic time effect.

Daniel did not consider ORAT plans for general number of factors nor did he consider the problem of generating formulas for parameter estimates in ORAT plans.

Cotter (1979) reported an economical permutation - invariant factor screening design for the 2^n factorial experiment that can be carried out one run at a time (ORAT) or one block at a time (OBAT). The design consists of the $(2n+2)$ runs

- a. the run (00---0) and its foldover (111---1),
- b. the n runs $\pi(10---0)$ and their n foldovers $\pi(011---1)$,

where $\pi()$ denotes all the permutations of the treatment combination inside the bracket. Cotter called this design a systematic fractional replicate factor-screening design for factorial experiments with interactions and derived formulas from the model

$$E(y) = \pi + \sum_{i=1}^n A_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^n A_i A_j x_i x_j$$

$$+ \dots + (A_1 A_2 \dots A_n)(x_1 \cdot x_2 \cdot \dots \cdot x_n)$$

that can be used as approximate measures for the significance of factor effects.

An outline of how formulas from the linear model above can be derived that can be used to screen factors is given below.

Letting

y_0 denote the response of the run 00---0

y_r denote the response of the run $00\text{---}\xrightarrow{r}010\text{---}0$

y_{n+r} denote the response of the run $11\text{---}\xrightarrow{r}1011\text{---}1$

y_{2n+1} denote the response of the run 11---1

and

$S_0(r)$ = the sum of all odd interactions involving factor A_r

$\overline{S_0(r)}$ = the sum of all odd interactions not involving factor A_r

$S_e(r)$ = the sum of all even interactions involving factor A_r

$\overline{S_e(r)}$ = the sum of all even interactions not involving factor A_r .

One finds that

$$E(y_0) = \mu - S_0(r) + S_e(r)$$

$$E(y_r) = \mu + S_0(r) - S_e(r) - \overline{S_0(r)} + \overline{S_e(r)}$$

$$E(y_{n+r}) = \mu - S_0(r) - S_e(r) + \overline{S_0(r)} + \overline{S_e(r)}$$

$$E(y_{2n+1}) = \mu + S_e(r) + S_0(r) .$$

and

$$S_0(r) = \frac{1}{4}[(Ey_{2n+1} - Ey_{n+r}) + (Ey_r - Ey_0)]$$

$$S_e(r) = \frac{1}{4}[(Ey_{2n+1} - Ey_{n+r}) - (Ey_r - Ey_0)]$$

$$r = 1, 2, \dots, n .$$

Hence

$$C_0(r) = \frac{1}{4}[(y_{2n+1} - y_{n+r}) + (y_r - y_0)]$$

$$C_e(r) = \frac{1}{4}[(y_{2n+1} - y_{n+r}) - (y_r - y_0)]$$

are unbiased estimates of $S_0(r)$ and $S_e(r)$, respectively.

For screening factors, Cotter suggests that factor effects be approximately ordered for their importance by the quantity

$$M(r) = |C_0(r)| + |C_e(r)|, \quad r = 1, 2, \dots, n$$

saying that it is unlikely that both $C_0(r)$ and $C_e(r)$ have small values and hence all factors can be detected for significance.

As an application, Cotter considered the 2^4 experiment arguing that

1. if only one or at most 2 factors are significant, then 3 and 4 factor interactions may be zero. Hence $C_0(r)$ is an estimate of the main effect of factor A_r .

2. if just two factors A_r and A_s have even order effects then both $C_e(r)$ and $C_e(s)$ estimate the interaction $A_r A_s$.

3.2 Parametrization of the 2^n Factorial Experiment

In the 2^n factorial experiment, there are n 2-level factors denoted by A_1, A_2, \dots, A_n where factor A_i has levels a_i^0 and a_i^1 , or for short, 0 and 1. Treatment combinations among factor levels are represented by $a_1^{i_1} a_2^{i_2} \dots a_n^{i_n}$ where $i_j = 0, 1$ and $j = 1, 2, \dots, n$. The response for treatment combination $a_1^{i_1} a_2^{i_2} \dots a_n^{i_n}$ is denoted by

$y(a_1^{i_1} a_2^{i_2} \dots a_n^{i_n})$. Assuming three-factor and higher order interactions negligible, the linear model relating expected responses $Ey(a_1^{i_1} a_2^{i_2} \dots a_n^{i_n})$ and the non-negligible factorial effects is

$$\begin{aligned} a_1^{i_1} a_2^{i_2} \dots a_n^{i_n} &= Ey(a_1^{i_1} a_2^{i_2} \dots a_n^{i_n}) \\ &= \mu + \sum_{i=1}^n A_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^n A_{ij} x_i x_j \end{aligned} \quad (3.1)$$

where

$$x_k = \begin{cases} -1 & \text{if } a_k^{i_k} = a_k^0, \\ 1 & \text{if } a_k^{i_k} = a_k^1, \end{cases} \quad k = 1, 2, \dots, n.$$

The model (3.1) contains $p(n) = 1 + n + \frac{n(n-1)}{2}$ parameters representing the overall mean, main effects and 2-factor interactions.

3.3 Treatment Combinations for the ORAT 2^n Plans

The strict ORAT plans for the 2^n factorial experiment that will be developed in this chapter make use of only the subset or fraction f_n consisting of the $N = 1 + n + (n-1) + (n-2) + (n-3) + \dots + 4 + 3 + 2 + 1 = 1 + n + \frac{n(n-1)}{2}$ treatment combinations

```

0000---000000
1000---000000
1100---000000
1110---000000
  ⋮
1111---111000
1111---111100
1111---111110
1111---111111

```

(1)

```

0111---111111
0011---111111
0001---111111
  ⋮
0000---000111
0000---000011
0000---000001

```

(2)

```

1000---000001
1100---000001
1110---000001
  ⋮
1111---111001
1111---111101

```

(3)

```

0111---111101
0011---111101
0001---111101
  ⋮
0000---110101
0000---010101
0000---000101

```

(4)

```

1000---00010101
1100---00010101
1110---00010101
  ⋮
1111---10010101
1111---11110101

```

(5)

$$\begin{array}{l}
 0111\text{---}11010101 \\
 0011\text{---}11010101 \\
 0001\text{---}11010101 \\
 \vdots \\
 0000\text{---}00010101 \\
 \vdots \\
 \vdots
 \end{array}
 \left. \vphantom{\begin{array}{l} 0111\text{---}11010101 \\ 0011\text{---}11010101 \\ 0001\text{---}11010101 \\ \vdots \\ 0000\text{---}00010101 \\ \vdots \\ \vdots \end{array}} \right\} (6)$$

the last three sets are

$$\begin{array}{l}
 01110101\text{---}0101 \\
 00110101\text{---}0101 \\
 00010101\text{---}0101
 \end{array}
 \left. \vphantom{\begin{array}{l} 01110101\text{---}0101 \\ 00110101\text{---}0101 \\ 00010101\text{---}0101 \end{array}} \right\} (n-2)$$

$$\begin{array}{l}
 10010101\text{---}0101 \\
 11010101\text{---}0101
 \end{array}
 \left. \vphantom{\begin{array}{l} 10010101\text{---}0101 \\ 11010101\text{---}0101 \end{array}} \right\} (n-1)$$

$$010101\text{---}0101 \left. \vphantom{010101\text{---}0101} \right\} (n) \quad \text{if } n \text{ is even}$$

or

$$\begin{array}{l}
 100010101\text{---}0101 \\
 110010101\text{---}0101 \\
 111010101\text{---}0101
 \end{array}
 \left. \vphantom{\begin{array}{l} 100010101\text{---}0101 \\ 110010101\text{---}0101 \\ 111010101\text{---}0101 \end{array}} \right\} (n-2)$$

$$\begin{array}{l}
 011010101\text{---}0101 \\
 001010101\text{---}0101
 \end{array}
 \left. \vphantom{\begin{array}{l} 011010101\text{---}0101 \\ 001010101\text{---}0101 \end{array}} \right\} (n-1)$$

$$1010101\text{---}0101 \left. \vphantom{1010101\text{---}0101} \right\} (n) \quad \text{if } n \text{ is odd} \quad (3.2)$$

There are n sets in (3.2). Sets (1), (2), (3), ..., (n) contain $(n+1)$, $(n-1)$, $(n-2)$, ..., 3, 2, 1 treatment combinations, respectively. The first n treatment combinations in (3.2) can be written as

$$\begin{array}{c}
 11\text{---}1100\text{---}0 \\
 \xrightarrow{\quad r \quad}
 \end{array}
 \quad (3.3)$$

where $r = 0, 1, 2, \dots, (n-1)$. The second n treatment combinations are the foldovers of those in (3.3), i.e.,

$$\begin{array}{c} 00\text{---}0011\text{---}1 \\ \xrightarrow{r} \end{array} \quad (3.4)$$

Therefore, the first $n + n = 2n$ treatment combinations in (3.2) represent a 2_{IV}^n fraction, i.e. main effects are estimable while two-factor interactions are aliased with each other. This can be shown as follows:

Displaying the parameters of the linear model (3.1) as in Table 1, the treatment combinations in (3.3) and (3.4) can be expressed as

$$\begin{aligned} \begin{array}{c} 11\text{---}1100\text{---}0 \\ \xrightarrow{r} \end{array} &= \mu + \sum_{i=1}^{r-1} A_i + A_r - \sum_{i=r+1}^n A_i \\ &+ \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j + \sum_{i=1}^{r-1} A_i A_r \\ &- \sum_{i=1}^{r-1} \sum_{j=r+1}^n A_i A_j - \sum_{j=r+1}^n A_r A_j \\ &+ \sum_{i=r+1}^{n-1} \sum_{j=i+1}^n A_i A_j \end{array} \quad (3.5) \\ \begin{array}{c} 00\text{---}00111\text{---}1 \\ \xrightarrow{r} \end{array} &= \mu - \sum_{i=1}^{r-1} A_i - A_r + \sum_{i=r+1}^n A_i \\ &+ \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j + \sum_{i=1}^{r-1} A_i A_r \end{array}$$

TABLE 1. A Display of the Parameters in the Model (3.1) - One Factor Emphasized.

μ	A_1	A_2	A_3	A_4	-----	A_{r-1}	A_r	A_{r+1}	-----	A_{n-1}	A_n
	$A_1 A_2$	$A_1 A_3$			-----	$A_1 A_{r-1}$	$A_1 A_r$	$A_1 A_{r+1}$	-----	$A_1 A_{n-1}$	$A_1 A_n$
		$A_2 A_3$			-----	$A_2 A_{r-1}$	$A_2 A_r$	$A_2 A_{r+1}$	-----	$A_2 A_{n-1}$	$A_2 A_n$
							\vdots				
						$A_{r-1} A_r$	$A_{r-1} A_{r+1}$			$A_{r-1} A_{n-1}$	$A_{r-1} A_n$
								$A_r A_{r+1}$	-----	$A_r A_{n-1}$	$A_r A_n$
								$A_{r+1} A_{r+2}$	-----	$A_{r+1} A_{n-1}$	$A_{r+1} A_n$
											\vdots
											\vdots
											\vdots
										$A_{n-2} A_{n-1}$	$A_{n-2} A_n$
											$A_{n-1} A_n$

$$\begin{aligned}
& - \sum_{i=1}^{r-1} \sum_{j=r+1}^n A_i A_j - \sum_{j=r+1}^n A_r A_j \\
& + \sum_{i=r+1}^{n-1} \sum_{j=i+1}^n A_i A_j
\end{aligned} \tag{3.6}$$

From (3.5) we find

$$\begin{aligned}
& \xrightarrow[r]{(11\text{---}1100\text{---}0)} - \xrightarrow[r]{(11\text{---}1000\text{---}0)} = 2 \cdot (A_r + \sum_{i=1}^{r-1} A_i A_r - \sum_{j=r+1}^n A_r A_j)
\end{aligned} \tag{3.7}$$

and from (3.6)

$$\begin{aligned}
& \xrightarrow[r]{(00\text{---}0111\text{---}1)} - \xrightarrow[r]{(00\text{---}0011\text{---}1)} = 2 \cdot (A_r - \sum_{i=1}^{r-1} A_i A_r + \sum_{j=r+1}^n A_r A_j)
\end{aligned} \tag{3.8}$$

Hence

$$\begin{aligned}
& \frac{1}{2} [\xrightarrow[r]{(11\text{---}1100\text{---}0)} - \xrightarrow[r]{(11\text{---}1000\text{---}0)} + \xrightarrow[r]{(00\text{---}0111\text{---}1)} - \xrightarrow[r]{(00\text{---}0011\text{---}1)}] = A_r
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
& \frac{1}{2} \cdot \{ [\xrightarrow[r]{(00\text{---}0111\text{---}1)} - \xrightarrow[r]{(00\text{---}0011\text{---}1)}] - [\xrightarrow[r]{(11\text{---}1100\text{---}0)} - \xrightarrow[r]{(11\text{---}1000\text{---}0)}] \} \\
& = \sum_{j=r+1}^n A_r A_j - \sum_{i=1}^{r-1} A_i A_r, (r = 1, 2, \dots, n)
\end{aligned} \tag{3.10}$$

However, to prove that the entire fraction f_n in (3.2) is of resolution V , we regroup the treatment combinations such that the last treatment combination in each set is moved to the set following it. Therefore, the fraction f_n in (3.2) is now

000000----	000000	}	(1')
100000----	000000		
110000----	000000		
111000----	000000		
⋮			
111111----	100000		
111111----	110000		
111111----	111000		
111111----	111100		
111111----	111110		

111111----	111111	}	(2')
011111----	111111		
001111----	111111		
000111----	111111		
⋮			
000000----	011111		
000000----	001111		
000000----	000111		
000000----	000011		

000000----	000001	}	(3')
100000----	000001		
110000----	000001		
111000----	000001		
⋮			
111111----	100001		
111111----	110001		
111111----	111001		

$$\begin{array}{l}
 111111\text{----}111101 \\
 011111\text{----}111101 \\
 001111\text{----}111101 \\
 \vdots \\
 000000\text{----}011101 \\
 000000\text{----}001101
 \end{array}
 \left. \vphantom{\begin{array}{l} 111111 \\ 011111 \\ 001111 \\ \vdots \\ 000000 \\ 000000 \end{array}} \right\} (4')$$

$$\begin{array}{l}
 000000\text{----}000101 \\
 100000\text{----}000101 \\
 110000\text{----}000101 \\
 111000\text{----}000101 \\
 \vdots \\
 111111\text{----}100101
 \end{array}
 \left. \vphantom{\begin{array}{l} 000000 \\ 100000 \\ 110000 \\ 111000 \\ \vdots \\ 111111 \end{array}} \right\} (5')$$

$$\begin{array}{l}
 111111\text{----}110101 \\
 011111\text{----}110101 \\
 001111\text{----}110101 \\
 \vdots \\
 000000\text{----}0110101
 \end{array}
 \left. \vphantom{\begin{array}{l} 111111 \\ 011111 \\ 001111 \\ \vdots \\ 000000 \end{array}} \right\} (6')$$

$$\vdots$$

if n is even:

$$\begin{array}{l}
 1111010101\text{----}010101 \\
 0111010101\text{----}010101 \\
 0011010101\text{----}010101 \\
 0001010101\text{----}010101 \\
 1001010101\text{----}010101 \\
 1101010101\text{----}010101 \\
 0101010101\text{----}010101
 \end{array}
 \left. \vphantom{\begin{array}{l} 1111010101 \\ 0111010101 \\ 0011010101 \\ 0001010101 \\ 1001010101 \\ 1101010101 \\ 0101010101 \end{array}} \right\} \begin{array}{l} \\ ((n-2)') \\ \\ ((n-1)') \\ ((n)') \end{array}$$

or if n is odd:

$$\begin{array}{l}
 000010101\text{----}0101 \\
 100010101\text{----}0101 \\
 110010101\text{----}0101 \\
 111010101\text{----}0101 \\
 011010101\text{----}0101 \\
 001010101\text{----}0101 \\
 101010101\text{----}0101
 \end{array}
 \left. \vphantom{\begin{array}{l} 000010101 \\ 100010101 \\ 110010101 \\ 111010101 \\ 011010101 \\ 001010101 \\ 101010101 \end{array}} \right\} \begin{array}{l} \\ ((n-2)') \\ \\ ((n-1)') \\ ((n)') \end{array}$$

(3.11)

The number of treatment combinations in the successive sets of (3.11) are, respectively,

$$n, n-1, n-2, n-3, n-4, \dots, 5, 4, 3, 2 .$$

That is, the fraction f_n in (3.11) contains

$$\begin{aligned} N &= n + (n-1) + (n-2) + (n-3) + \dots + 3 + 3 + 1 = n + (n-1) + (n-2) \\ &\quad + \dots + 3 + 2 \\ &= 1 + n + (n-1) + (n-2) + (n-3) + \dots + 3 + 2 + 1 \\ &= 1 + n + \frac{n(n-1)}{2} \end{aligned}$$

treatment combinations, where $n = 1, 2, 3, \dots$. We also note that all treatment combinations in

- a. set (1') have their last digit 0
- b. set (2') have their last two digits 11
- c. set (3') have their last three digits 001
- d. set (4') have their last four digits 1101
- e. set (5') have their last five digits 00101
- ⋮
- ⋮
- ⋮
- etc.

Furthermore, each treatment combination in the odd numbered sets can be written in the general form

$$11\dots 1a_r 00\dots 0a_s \underbrace{010101\dots 0101}_{n-s} \quad (3.12)$$

$\xrightarrow{\quad r \quad}$
 $\xrightarrow{\quad s \quad}$

where $a_r, a_s = 0, 1, (r = 1, 2, 3, \dots, s-1), r < s \leq n$ and $(n-s)$ is an even number. On the other hand, each treatment combination in the even numbered sets can be written in the general form

$$000\dots 00a_r 111\dots 111a_s \underbrace{1010101\dots 0101}_{n-s} \quad (3.13)$$

$\xrightarrow{\quad r \quad}$
 $\xrightarrow{\quad s \quad}$

where $a_r, a_s = 0, 1, (r = 1, 2, 3, \dots, s-1), r < s \leq n-1$ and $(n-s)$ is an odd number.

Displaying the parameters of the linear model (3.1) as in Table 2 we notice that

$$\begin{aligned}
 & \underbrace{a_1^{i_1} a_2^{i_2} \dots a_{r-1}^{i_{r-1}} a_r^{i_r} a_{r+1}^{i_{r+1}} \dots a_{s-1}^{i_{s-1}} a_s^{i_s} a_{s+1}^{i_{s+1}} \dots a_{n-1}^{i_{n-1}} a_n^{i_n}}_{\substack{r \quad s \quad n-s}} = \\
 & \mu + \sum_{i=1}^{r-1} A_i x_i + A_r x_r + \sum_{i=r+1}^{s-1} A_i x_i + A_s x_s + \sum_{i=s+1}^n A_i x_i \\
 & + \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j x_i x_j + \sum_{i=1}^{r-1} A_i A_r x_i x_r + \sum_{i=1}^{r-1} \sum_{j=r+1}^{s-1} A_i A_j x_i x_j + \sum_{j=r+1}^{s-1} A_r A_j x_r x_j \\
 & + \sum_{i=1}^{r-1} A_i A_s x_i x_s + A_r A_s x_r x_s + \sum_{i=r+1}^{s-1} A_i A_s x_i x_s
 \end{aligned}$$

TABLE 2. A Display of the Parameters in Model (3.1) - Two Factors Emphasized.

μ	A_1	A_2	---	A_{r-2}	A_{r-1}	A_r	A_{r+1}	A_{r+2}	---	A_{s-2}	A_{s-1}	A_s	A_{s+1}	A_{s+2}	---	A_{n-1}	A_n	
	$A_1 A_2$	$A_1 A_3$	---	$A_1 A_{r-1}$	$A_1 A_r$	$A_1 A_{r+1}$	---	$A_1 A_{s-1}$	$A_1 A_s$	$A_1 A_{s+1}$	$A_1 A_{s+2}$	---	$A_1 A_{n-1}$	$A_1 A_n$				
		$A_2 A_3$	---	$A_2 A_{r-1}$	$A_2 A_r$	$A_2 A_{r+1}$	---	$A_2 A_{s-1}$	$A_2 A_s$	$A_2 A_{s+1}$	$A_2 A_{s+2}$	---	$A_2 A_{n-1}$	$A_2 A_n$				
					\vdots	\vdots		\vdots	\vdots	\vdots	\vdots		\vdots	\vdots				
					$A_{r-1} A_r$	$A_{r-1} A_{r+1}$	---	$A_{r-1} A_{s-1}$	$A_{r-1} A_s$	$A_{r-1} A_{s+1}$	$A_{r-1} A_{s+2}$	---	$A_{r-1} A_{n-1}$	$A_{r-1} A_n$				
						$A_r A_{r+1}$	---	$A_r A_{s-1}$	$A_r A_s$	$A_r A_{s+1}$	$A_r A_{s+2}$	---	$A_r A_{n-1}$	$A_r A_n$				
						$A_{r+1} A_{r+2}$	---	$A_{r+1} A_{s-1}$	$A_{r+1} A_s$	$A_{r+1} A_{s+1}$	---	$A_{r+1} A_{n-1}$	$A_{r+1} A_n$					
									\vdots									
									$A_{s-1} A_s$	$A_{s-1} A_{s+1}$	---	$A_{s-1} A_{n-1}$	$A_{s-1} A_n$					
										$A_s A_{s+1}$	---	$A_s A_{n-1}$	$A_s A_n$					
												$A_{s+1} A_{s+2}$	---	$A_{s+1} A_{n-1}$	$A_{s+1} A_n$			
																	\vdots	
																	$A_{n-2} A_{n-1}$	$A_{n-2} A_n$
																		$A_{n-1} A_n$

$$\begin{aligned}
& + \sum_{i=1}^{r-1} \sum_{j=s+1}^n A_i A_j x_i x_j + \sum_{j=s+1}^n A_r A_j x_r x_j + \sum_{j=s+1}^n A_s A_j x_s x_j \\
& + \sum_{i=r+1}^{s-2} \sum_{j=i+1}^{s-1} A_i A_j x_i x_j + \sum_{i=r+1}^{s-1} \sum_{j=s+1}^n A_i A_j x_i x_j + \sum_{i=s+1}^{n-1} \sum_{j=i+1}^n A_i A_j x_i x_j
\end{aligned}$$

where $x = \begin{cases} -1 & \text{for the low level of the factor} \\ 1 & \text{for the high level of the factor.} \end{cases}$

Hence, the treatment combinations in (3.12) and (3.13) can be expressed as

$$\begin{array}{c}
11\text{---}1a \quad 00\text{---}0a \quad 0101\text{---}0101 = \\
\begin{array}{ccc}
\begin{array}{c} \xrightarrow{r} \\ r \end{array} & & \begin{array}{c} \xrightarrow{n-s} \\ n-s \end{array} \\
\begin{array}{c} \xrightarrow{s} \\ s \end{array} & &
\end{array}
\end{array}$$

$$\mu + \sum_{i=1}^{r-1} A_i + A_r x_r - \sum_{i=r+1}^{s-1} A_i + A_s x_s + \sum_{i=1}^{n-s} (-1)^i A_{s+i}$$

$$+ \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j + \sum_{i=1}^{r-1} A_i A_r x_r - \sum_{i=1}^{r-1} \sum_{j=r+1}^{s-1} A_i A_j - \sum_{j=r+1}^{s-1} A_r A_j x_r$$

$$+ \sum_{i=1}^{r-1} A_i A_s x_s + A_r A_s x_r x_s - \sum_{i=r+1}^{s-1} A_i A_s x_s$$

$$+ \sum_{i=1}^{r-1} \sum_{j=1}^{n-s} (-1)^j A_i A_{s+j} + \sum_{j=1}^{n-s} (-1)^j A_r A_{s+j} x_r + \sum_{j=1}^{n-s} (-1)^j A_s A_{s+j} x_s$$

$$+ \sum_{i=r+1}^{s-2} \sum_{j=i+1}^{s-1} A_i A_j - \sum_{i=r+1}^{s-1} \sum_{j=1}^{n-s} (-1)^j A_i A_{s+j} + \sum_{i=1}^{n-s-1} \sum_{j=i+1}^{n-s} (-1)^i \cdot (-1)^j A_{s+i} A_{s+j}$$

(3.14)

$$\begin{aligned}
& + \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j + \sum_{i=1}^{r-1} A_i A_r - \sum_{i=1}^{r-1} \sum_{j=r+1}^{s-1} A_i A_j - \sum_{j=r+1}^{s-1} A_r A_j \\
& + \sum_{i=1}^{r-1} A_i A_s + A_r A_s - \sum_{i=r+1}^{s-1} A_i A_s \\
& + \sum_{i=1}^{r-1} \sum_{j=1}^{n-s} (-1)^j A_i A_{s+j} + \sum_{j=1}^{n-s} (-1)^j A_r A_{s+j} + \sum_{j=1}^{n-s} (-1)^j A_s A_{s+j} \\
& + \sum_{i=r+1}^{s-2} \sum_{j=i+1}^{s-1} A_i A_j - \sum_{i=r+1}^{s-1} \sum_{j=1}^{n-s} (-1)^j A_i A_{s+j} + \sum_{i=1}^{n-s-1} \sum_{j=i+1}^{n-s} (-1)^j (-1)^j A_{s+i} A_{s+j}
\end{aligned} \tag{3.16}$$

2) $11\text{---}1000\text{---}01010101\text{---}0101 =$

$$\begin{array}{c}
\overline{\hspace{1.5cm}} \\
\text{r} \\
\overline{\hspace{1.5cm}} \\
\text{s}
\end{array}$$

$$\begin{aligned}
\mu & + \sum_{i=1}^{r-1} A_i - A_r - \sum_{i=r+1}^{s-1} A_i + A_s + \sum_{i=1}^{n-s} (-1)^i A_{s+i} \\
& + \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j - \sum_{i=1}^{r-1} A_i A_r - \sum_{i=1}^{r-1} \sum_{j=r+1}^{s-1} A_i A_j + \sum_{j=r+1}^{s-1} A_r A_j \\
& + \sum_{i=1}^{r-1} A_i A_s - A_r A_s - \sum_{i=r+1}^{s-1} A_i A_s \\
& + \sum_{i=1}^{r-1} \sum_{j=1}^{n-s} (-1)^j A_i A_{s+j} - \sum_{j=1}^{n-s} (-1)^j A_r A_{s+j} + \sum_{j=1}^{n-s} (-1)^j A_s A_{s+j}
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=r+1}^{s-2} \sum_{j=i+1}^{s-1} A_i A_j - \sum_{i=r+1}^{s-1} \sum_{j=1}^{n-s} (-1)^j A_i A_{s+j} + \sum_{i=1}^{n-s-1} \sum_{j=i+1}^{n-s} (-1)^i (-1)^j A_{s+i} A_{s+j} \\
 & \hspace{25em} (3.17)
 \end{aligned}$$

3) 11-----100-----00010101-----010101 =

$$\begin{array}{c}
 \xrightarrow{\quad r \quad} \\
 \xrightarrow{\quad s \quad}
 \end{array}$$

$$\mu + \sum_{i=1}^{r-1} A_i + A_r - \sum_{i=r+1}^{s-1} A_i - A_s + \sum_{i=1}^{n-s} (-1)^i A_{s+i}$$

$$\begin{aligned}
 & + \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j + \sum_{i=1}^{r-1} A_i A_r - \sum_{i=1}^{r-1} \sum_{j=r+1}^{s-1} A_i A_j - \sum_{j=r+1}^{s-1} A_r A_j \\
 & - \sum_{i=1}^{r-1} A_i A_s - A_r A_s + \sum_{i=r+1}^{s-1} A_i A_s
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^{r-1} \sum_{j=1}^{n-s} (-1)^j A_i A_{s+j} + \sum_{j=1}^{n-s} (-1)^j A_r A_{s+j} - \sum_{j=1}^{n-s} (-1)^j A_s A_{s+j} \\
 & + \sum_{i=r+1}^{s-2} \sum_{j=i+1}^{s-1} A_i A_j - \sum_{i=r+1}^{s-1} \sum_{j=1}^{n-s} (-1)^j A_i A_{s+j} + \sum_{i=1}^{n-s-1} \sum_{j=i+1}^{n-s} (-1)^i \cdot (-1)^j \cdot
 \end{aligned}$$

$$\begin{aligned}
 & A_{s+i} A_{s+j} \\
 & \hspace{25em} (3.18)
 \end{aligned}$$

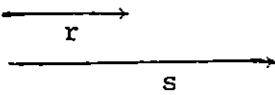
4) 11-----1000-----00010101-----0101 =

$$\begin{array}{c}
 \xrightarrow{\quad r \quad} \\
 \xrightarrow{\quad s \quad}
 \end{array}$$

$$\begin{aligned}
& \mu + \sum_{i=1}^{r-1} A_i - A_r - \sum_{i=r+1}^{s-1} A_i - A_s + \sum_{i=1}^{n-s} (-1)^i A_{s+i} \\
& + \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j - \sum_{i=1}^{r-1} A_i A_r - \sum_{i=1}^{r-1} \sum_{j=r+1}^{s-1} A_i A_j + \sum_{j=r+1}^{s-1} A_r A_j \\
& - \sum_{i=1}^{r-1} A_i A_s + A_r A_s + \sum_{i=r+1}^{s-1} A_i A_s \\
& + \sum_{i=1}^{r-1} \sum_{j=1}^{n-s} (-1)^j A_i A_{s+j} - \sum_{j=1}^{n-s} (-1)^j A_r A_{s+j} - \sum_{j=1}^{n-s} (-1)^j A_s A_{s+j} \\
& + \sum_{i=r+1}^{s-2} \sum_{j=i+1}^{s-1} A_i A_j - \sum_{i=r+1}^{s-1} \sum_{j=1}^{n-s} (-1)^j A_i A_{s+j} + \sum_{i=1}^{n-s-1} \sum_{j=i+1}^{n-s} (-1)^i \cdot (-1)^j \cdot \\
& \qquad \qquad \qquad A_{s+i} A_{s+j} \tag{3.19}
\end{aligned}$$

From (3.15), we have the following four results

$$1) 00\text{---}0111\text{---}111101\text{---}0101 =$$



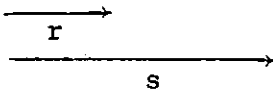
$$\mu - \sum_{i=1}^{r-1} A_i + A_r + \sum_{i=r+1}^{s-1} A_i + A_s + \sum_{i=1}^{n-s} (-1)^{i+1} A_{s+i}$$

$$+ \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j - \sum_{i=1}^{r-1} A_i A_r - \sum_{i=1}^{r-1} \sum_{j=r+1}^{s-1} A_i A_j + \sum_{j=r+1}^{s-1} A_r A_j$$

$$- \sum_{i=1}^{r-1} A_i A_s + A_r A_s + \sum_{i=r+1}^{s-1} A_i A_s$$

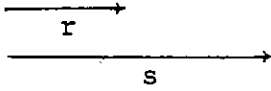
$$\begin{aligned}
& - \sum_{i=1}^{r-1} \sum_{j=1}^{n-s} (-1)^{j+1} A_i A_{s+j} + \sum_{j=1}^{n-s} (-1)^{j+1} A_r A_{s+j} + \sum_{j=1}^{n-s} (-1)^{j+1} A_s A_{s+j} \\
& + \sum_{i=r+1}^{s-2} \sum_{j=i+1}^{s-1} A_i A_j + \sum_{i=r+1}^{s-1} \sum_{j=1}^{n-s} (-1)^{j+1} A_i A_{s+j} + \sum_{i=1}^{n-s-1} \sum_{j=i+1}^{n-s} (-1)^{i+1} \cdot (-1)^{j+1} \cdot \\
& A_{s+i} A_{s+j} \tag{3.20}
\end{aligned}$$

2) 00---0011-----11110101-----0101 =



$$\begin{aligned}
\mu & - \sum_{i=1}^{r-1} A_i - A_r + \sum_{i=r+1}^{s-1} A_i + A_s + \sum_{i=1}^{n-s} (-1)^{i+1} A_{s+i} \\
& + \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j + \sum_{i=1}^{r-1} A_i A_r - \sum_{i=1}^{r-1} \sum_{j=r+1}^{s-1} A_i A_j - \sum_{j=r+1}^{s-1} A_r A_j \\
& - \sum_{i=1}^{r-1} A_i A_s - A_r A_s + \sum_{i=r+1}^{s-1} A_i A_s \\
& - \sum_{i=1}^{r-1} \sum_{j=1}^{n-s} (-1)^{j+1} A_i A_{s+j} - \sum_{j=1}^{n-s} (-1)^{j+1} A_r A_{s+j} + \sum_{j=1}^{n-s} (-1)^{j+1} A_s A_{s+j} \\
& + \sum_{i=r+1}^{s-2} \sum_{j=i+1}^{s-1} A_i A_j + \sum_{i=r+1}^{s-1} \sum_{j=1}^{n-s} (-1)^{j+1} A_i A_{s+j} + \sum_{i=1}^{n-s-1} \sum_{j=i+1}^{n-s} (-1)^{i+1} \cdot \\
& (-1)^{j+1} A_{s+i} A_{s+j} \tag{3.21}
\end{aligned}$$

$$3) 00\text{----}0111\text{----}11010101\text{----}0101 =$$



$$\mu - \sum_{i=1}^{r-1} A_i + A_r + \sum_{i=r+1}^{s-1} A_i - A_s + \sum_{i=1}^{n-s} (-1)^{i+1} A_{s+i}$$

$$+ \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j - \sum_{i=1}^{r-1} A_i A_r - \sum_{i=1}^{r-1} \sum_{j=r+1}^{s-1} A_i A_j + \sum_{j=r+1}^{s-1} A_r A_j$$

$$+ \sum_{i=1}^{r-1} A_i A_s - A_r A_s - \sum_{i=r+1}^{s-1} A_i A_s$$

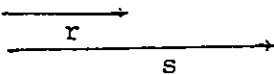
$$- \sum_{i=1}^{r-1} \sum_{j=1}^{n-s} (-1)^{j+1} A_i A_{s+j} + \sum_{j=1}^{n-s} (-1)^{j+1} A_r A_{s+j} - \sum_{j=1}^{n-s} (-1)^{j+1} A_s A_{s+j}$$

$$+ \sum_{i=r+1}^{s-2} \sum_{j=i+1}^{s-1} A_i A_j + \sum_{i=r+1}^{s-1} (-1)^{j+1} A_i A_{s+j} + \sum_{i=1}^{n-s-1} \sum_{j=i+1}^{n-s} (-1)^{i+1} \cdot (-1)^{j+1} \cdot$$

$$A_{s+i} A_{s+j}$$

(3.22)

$$4) 00\text{----}0011\text{----}11010101\text{----}0101 =$$



$$\mu - \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j + \sum_{i=1}^{r-1} A_i A_r - \sum_{i=1}^{r-1} \sum_{j=r+1}^{s-1} A_i A_j - \sum_{j=r+1}^{s-1} A_r A_j$$

$$+ \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} A_i A_j + \sum_{i=1}^{r-1} A_i A_r - \sum_{i=1}^{r-1} \sum_{j=r+1}^{s-1} A_i A_j - \sum_{j=r+1}^{s-1} A_r A_j$$

$$\begin{aligned}
& + \sum_{i=1}^{r-1} A_i A_s + A_r A_s - \sum_{i=r+1}^{s-1} A_i A_s \\
& - \sum_{i=1}^{r-1} \sum_{j=1}^{n-s} (-1)^{j+1} A_i A_{s+j} - \sum_{j=1}^{n-s} (-1)^{j+1} A_r A_{s+j} - \sum_{j=1}^{n-s} (-1)^{j+1} A_s A_{s+j} \\
& + \sum_{i=r+1}^{s-2} \sum_{j=i+1}^{s-1} A_i A_j + \sum_{i=r+1}^{s-1} \sum_{j=1}^{n-s} (-1)^{j+1} A_i A_{s+j} + \sum_{i=1}^{n-s-1} \sum_{j=i+1}^{n-s} (-1)^{i+1} \cdot \\
& \qquad \qquad \qquad (-1)^{j+1} A_{s+i} A_{s+j} \tag{3.23}
\end{aligned}$$

1) We also find from (3.16) and (3.17),

$$\begin{aligned}
& \begin{array}{c} 11\text{---}1100\text{---}010101\text{---}0101 \\ \xrightarrow{\quad r \quad} \\ \xrightarrow{\quad s \quad} \end{array} - \begin{array}{c} 11\text{---}1000\text{---}010101\text{---}0101 \\ \xrightarrow{\quad r \quad} \\ \xrightarrow{\quad s \quad} \end{array} = \\
& 2A_r + 2 \sum_{i=1}^{r-1} A_i A_r - 2 \sum_{j=r+1}^{s-1} A_j A_r + 2A_r A_s + 2 \sum_{j=1}^{n-s} (-1)^j A_r A_{s+j} . \tag{3.24}
\end{aligned}$$

2) From (3.18) and (3.19),

$$\begin{aligned}
& \begin{array}{c} 11\text{---}1100\text{---}00010101\text{---}0101 \\ \xrightarrow{\quad r \quad} \\ \xrightarrow{\quad s \quad} \end{array} - \begin{array}{c} 11\text{---}1000\text{---}00010101\text{---}0101 \\ \xrightarrow{\quad r \quad} \\ \xrightarrow{\quad s \quad} \end{array} = \\
& 2A_r + 2 \sum_{i=1}^{r-1} A_i A_r - 2 \sum_{j=r+1}^{s-1} A_r A_j - 2A_r A_s + 2 \sum_{j=1}^{n-s} (-1)^j A_r A_{s+j} . \tag{3.25}
\end{aligned}$$

3) From (3.20) and (3.21),

$$\begin{array}{c} 00\text{---}0111\text{---}11110101\text{---}0101 \\ \xrightarrow{r} \\ \xrightarrow{s} \end{array} - \begin{array}{c} 00\text{---}00111\text{---}11110101\text{---}0101 \\ \xrightarrow{r} \\ \xrightarrow{s} \end{array} =$$

$$2A_r - 2 \sum_{i=1}^{r-1} A_i A_r + 2 \sum_{j=r+1}^{s-1} A_r A_j + 2A_r A_s + 2 \sum_{j=1}^{n-s} (-1)^{j+1} A_r A_{s+j} \quad (3.26)$$

4) From (3.22) and (3.23),

$$\begin{array}{c} 00\text{---}0111\text{---}11010101\text{---}0101 \\ \xrightarrow{r} \\ \xrightarrow{s} \end{array} - \begin{array}{c} 00\text{---}0011\text{---}11010101\text{---}0101 \\ \xrightarrow{r} \\ \xrightarrow{s} \end{array} =$$

$$2A_r - 2 \sum_{i=1}^{r-1} A_i A_r + \sum_{j=r+1}^{s-1} A_r A_j - 2A_r A_s + 2 \sum_{j=1}^{n-s} (-1)^{j+1} A_r A_{s+j} \quad (3.27)$$

From (3.24) and (3.25), it follows that

$$\begin{aligned} & \left[\begin{array}{c} (11\text{---}1100\text{---}010101\text{---}0101 \\ \xrightarrow{r} \\ \xrightarrow{s} \end{array} \underbrace{\hspace{10em}}_{(n-s)} - \begin{array}{c} 11\text{---}1000\text{---}010101\text{---}0101 \\ \xrightarrow{r} \\ \xrightarrow{s} \end{array} \underbrace{\hspace{10em}}_{(n-s)} \right] \\ & - \left[\begin{array}{c} (11\text{---}1100\text{---}000101\text{---}0101 \\ \xrightarrow{r} \\ \xrightarrow{s} \end{array} \underbrace{\hspace{10em}}_{(n-s)} - \begin{array}{c} 11\text{---}1000\text{---}000101\text{---}0101 \\ \xrightarrow{r} \\ \xrightarrow{s} \end{array} \underbrace{\hspace{10em}}_{(n-s)} \right] \end{aligned}$$

$$= 4A_r A_s, \text{ where } (n-s) \text{ is an even number.} \quad (3.28)$$

From (3.26) and (3.27), it follows that

$$\begin{aligned}
 & [(00 \text{---} 0111 \text{---} 11110101 \text{---} 0101 - 00 \text{---} 0011 \text{---} 11110101 \text{---} 0101) \\
 & \quad \xrightarrow[r]{s} \quad \underbrace{\hspace{10em}}_{(n-s)} \quad \xrightarrow[r]{s} \quad \underbrace{\hspace{10em}}_{(n-s)} \\
 & - (00 \text{---} 0111 \text{---} 11010101 \text{---} 0101 - 00 \text{---} 0011 \text{---} 11010101 \text{---} 0101)] \\
 & \quad \xrightarrow[r]{s} \quad \underbrace{\hspace{10em}}_{(n-s)} \quad \xrightarrow[r]{s} \quad \underbrace{\hspace{10em}}_{(n-s)}
 \end{aligned}$$

$$= 4A_{r_s}, \text{ where } (n-s) \text{ is an odd number and } r = 1, 2, \dots, (s-1). \quad (3.29)$$

Also from (3.24) and (3.27),

$$\begin{aligned}
 & (11 \text{---} 1100 \text{---} 010101 \text{---} 0101 - 11 \text{---} 1000 \text{---} 010101 \text{---} 0101) \\
 & \quad \xrightarrow[r]{s} \quad \underbrace{\hspace{10em}}_{(n-s)} \quad \xrightarrow[r]{s} \quad \underbrace{\hspace{10em}}_{(n-s)} \\
 & + (00 \text{---} 0111 \text{---} 11010101 \text{---} 0101 - 0 \text{---} 0011 \text{---} 11010101 \text{---} 0101) \\
 & \quad \xrightarrow[r]{s} \quad \underbrace{\hspace{10em}}_{(n-s)} \quad \xrightarrow[r]{s} \quad \underbrace{\hspace{10em}}_{(n-s)}
 \end{aligned}$$

$$= 4A_r, \quad (r = 1, 2, \dots, n), \quad (3.30)$$

and from (3.25) and (3.26),

Therefore, the quantities (3.30), (3.31) and (3.9) are identical hence the fraction f_n in (3.11) is a 2_V^n plan.

3.4 Strict ORAT 2_V^n Plans

Strict ORAT plans are ORAT plans where only one factor level may be changed between successive runs. Such plans are quite useful in situations where it is costly to change more than one factor level between runs. Strict ORAT plans are generally based on the assumption that factors are ordered for their potential importance to the response where most influential factors are investigated first (see Section 2.4). The strict ORAT 2_V^n plans that will be developed in this chapter are based on the fraction f_n in (3.2). This fraction has the following properties:

1. In Set (1), we notice that

a) the r^{th} 2-level factor A_r is first introduced at its high level at the $(r+1)^{\text{st}}$ run

$$\frac{11\text{---}10000\text{---}0}{r}$$

$$r = 0, 1, 2, \dots, n$$

b) the r^{th} factor A_r stays at its high level after the $(r+1)^{\text{st}}$ run until the last factor A_n is introduced for the first time at its high level in the run 11----1 .

2. In set (2) whose runs are of the form $00\text{---}00111\text{---}1$ and

$$\frac{\text{---}}{r}$$

$r = 1, 2, \dots, (n-1)$, factors A_1, A_2, \dots, A_{n-1} are lowered each to its low level one at a time in the same order they were raised each to

its high level in set (1). Lowering the level of factor A_n would result in a replication of the run 00---0.

3. In set (3) whose runs are of the form $11\text{---}100\text{---}01$ and $r =$
 $\xrightarrow{\quad r \quad}$
 1, 2, ..., (n-2), factors A_1, A_2, \dots, A_{n-2} are successively raised each to its high level.

4. In the remaining sets, factor levels are raised (lowered) in the order determined by set (1) as long as the resulting run is not a replicate of a previously used one.

For the strict ORAT 2^n plan based on the fraction f_n in (3.2), we consider the sets partitioning f_n as stages after each of which we report the estimable functions as well as their unbiased estimates.

Stage A:

Strict ORAT 2^n plan in (n+1) runs, the runs being from set (1) and are of the form

$$\frac{111\text{---}1100\text{---}0}{\xrightarrow{\quad r \quad}} \} (1) \quad (3.32)$$

where $r = 0, 1, 2, \dots, n$. We found in (3.7), for the difference between any two successive runs, that

$$A_r + \sum_{i=1}^{r-1} A_i A_r - \sum_{j=r+1}^n A_r A_j = \frac{1}{2} \left[\frac{111\text{---}1100\text{---}0}{\xrightarrow{\quad r \quad}} - \frac{11\text{---}100\text{---}0}{\xrightarrow{\quad r \quad}} \right]$$

i.e.

$$A_1 - A_1A_2 - A_1A_3 - \dots - A_1A_r - A_1A_{r+1} - \dots - A_1A_n$$

$$= \frac{1}{2}[(100\text{---}0) - (00\text{---}0)]$$

$$A_2 + A_1A_2 - A_2A_3 - \dots - A_2A_r - A_1A_{r+1} - \dots - A_2A_n$$

$$= \frac{1}{2}[(110\text{---}0) - (100\text{---}0)]$$

⋮

$$A_r + A_1A_r + A_2A_r + \dots + A_{r-1}A_r - A_rA_{r+1} - \dots - A_rA_n$$

$$= \frac{1}{2}[(\underbrace{11\text{---}110\text{---}0}_r) - (\underbrace{11\text{---}100\text{---}0}_r)]$$

⋮

$$A_n + A_1A_n + A_2A_n + \dots + A_{r-1}A_n + A_rA_n + \dots + A_{n-1}A_n$$

$$= \frac{1}{2}[(111\text{---}11) - (111\text{---}10)] \tag{3.33}$$

Hence, unbiased estimates of the functions above are, respectively

$$\frac{1}{2}(y_{100\text{---}0} - y_{00\text{---}0})$$

$$\frac{1}{2}(y_{110\text{---}0} - y_{10\text{---}0})$$

⋮

$$\frac{1}{2}(y_{\underbrace{11\text{---}110\text{---}0}_r} - y_{\underbrace{11\text{---}100\text{---}0}_r})$$

$$\begin{aligned} & \vdots \\ & \frac{1}{2}(y_{11\dots 11} - y_{11\dots 10}) \end{aligned} \quad (3.34)$$

all with variance $\frac{1}{2}\sigma^2$.

It is worth mentioning here that the strict $(n+1)$ -run ORAT plan in (3.32) belongs to Webb's (1968) class of $(n+1)$ -run ORAT main effect plans. The class consists of ORAT plans in which the k^{th} run - the run at which the $(k-1)^{\text{st}}$ factor is first introduced at its high level - differ from a preceding run in only the level of the $(k-1)^{\text{st}}$ factor. In (3.32) any two successive runs, the r^{th} and the $(r+1)^{\text{st}}$, differ only in the level of the r^{th} factor A_r , $r = 1, 2, \dots, n$. Clearly, if interactions are negligible, then (3.32) is a main effect $(n+1)$ -run plan.

In the presence of two-factor interactions, the estimates in (3.34) cannot be used to judge the significance of main effects unless the alias chains in (3.33) are broken by adding more runs to the ORAT $(n+1)$ -run plan in (3.32). Therefore, we consider stage B in which set (2) is added to set (1).

Stage B:

Strict ORAT 2_{IV}^n plans in $1 + n + (n-1) = 2n$ runs from sets (1) and (2). The runs are

$$\begin{aligned} & \frac{11\dots 1100\dots 0}{\xrightarrow{r}} \{ (1), r = 0, 1, 2, \dots, n \\ & \frac{00\dots 0111\dots 1}{\xrightarrow{r}} \{ (2), r = 1, 2, \dots, n-1 \} \end{aligned} \quad (3.35)$$

We have seen in Section 3.3 that the fraction consisting of the $2n$ runs in (3.35) is of resolution IV. The estimable functions after $2n$ runs are given in (3.9) and (3.10). Letting

$$\Sigma_{r,1,n} = \sum_{j=r+1}^n A_r A_j - \sum_{i=1}^{r-1} A_i A_r \quad (3.36)$$

the unbiased estimates of the functions in (3.9) and (3.10) are

$$\begin{aligned} 1. \quad \hat{A}_r &= \frac{1}{4} \left[\left(y_{\overbrace{11\dots110\dots0}^r} - y_{\overbrace{11\dots100\dots0}^r} \right) + \left(y_{\overbrace{0\dots0111\dots1}^r} \right. \right. \\ &\quad \left. \left. - y_{\overbrace{0\dots0011\dots1}^r} \right) \right] \\ 2. \quad \hat{\Sigma}_{r,1,n} &= \frac{1}{4} \left[\left(y_{\overbrace{0\dots0111\dots1}^r} - y_{\overbrace{0\dots0011\dots1}^r} \right) - \left(y_{\overbrace{11\dots110\dots0}^r} \right. \right. \\ &\quad \left. \left. - y_{\overbrace{11\dots100\dots0}^r} \right) \right] \end{aligned} \quad (3.37)$$

all with variance $\frac{1}{4}\sigma^2$, where $r = 1, 2, \dots, n$. From (3.37) we notice that each factor's main effect is estimated as an average of two contrasts: one contrast involving the difference between two successive runs from set (1) of the fraction (3.2) and the other involving the difference between two successive runs from set (2).

For the 2_{IV}^n fraction in (3.35), the alias structure for two-factor interactions given by (3.36) is explicitly

$$\begin{aligned}
\Sigma_{1,n} &= A_1 A_2 + A_1 A_3 + \dots + A_1 A_r + A_1 A_{r+1} + \dots + A_1 A_n \\
\Sigma_{2,n} &= - A_1 A_2 + A_2 A_3 + \dots + A_2 A_r + A_2 A_{r+1} + \dots + A_2 A_n \\
&\vdots \\
\Sigma_{r,n} &= - A_1 A_r - A_2 A_r - \dots - A_{r-1} A_r + A_r A_{r+1} + \dots + A_r A_n \\
&\vdots \\
\Sigma_{(n-1),n} &= - A_1 A_n - A_2 A_n - \dots - A_r A_n - \dots - A_{n-1} A_n .
\end{aligned} \tag{3.38}$$

Therefore, while a decision on the importance of main effects can be made after $2n$ runs, no decision can be made about the importance of any one of the two-factor interactions $A_i A_j$, $i < j$ unless additional runs are added to the ORAT plan in (3.35). However, as we shall show that the successive sets (3), (4), ... etc. of the fraction f_n in (3.2) will break the alias chains in (3.38) in a particular order which can be summarized as follows:

1. In the strict ORAT plan

$$\begin{aligned}
&\frac{11\text{---}1100\text{---}0}{r}, (r = 0, 1, 2, \dots, n) \\
&\frac{00\text{---}0111\text{---}1}{r}, (r = 1, 2, \dots, (n-1)) \\
&\frac{11\text{---}1100\text{---}01}{r}, (r = 1, 2, \dots, (n-2))
\end{aligned} \tag{3.39}$$

from sets (1), (2), and (3), all interactions involving factor A_n , i.e. $A_i A_n$ for $i = 1, 2, \dots, n-1$ are estimable.

2. In the strict ORAT plan

$$\begin{aligned}
 & \frac{11\text{---}1100\text{---}0}{r}, (r = 0, 1, 2, \dots, n) \\
 & \frac{00\text{---}0111\text{---}1}{r}, (r = 1, 2, \dots, n-1) \\
 & \frac{11\text{---}100\text{---}01}{r}, (r = 1, 2, \dots, n-2) \\
 & \frac{00\text{---}0111\text{---}101}{r}, (r = 1, 2, \dots, n-3)
 \end{aligned} \tag{3.40}$$

from sets (1), (2), (3) and (4), all interactions involving factors A_n and A_{n-1} , i.e.

$$A_i A_n \text{ for } i = 1, 2, \dots, n-1$$

$$A_j A_{n-1} \text{ for } j = 1, 2, \dots, n-2$$

are estimable.

3. In a similar fashion, runs in the remaining sets when added to the ORAT plan in (3.40) will make interactions involving factors $A_{n-2}, A_{n-3}, \dots, A_2, A_1$ successively estimable.

For the sequential estimation of two-factor interactions we consider the following two stages and then the general stage (t).

Stage One: The estimation of $A_r A_n$, $r = 1, 2, \dots, (n-1)$ from the strict ORAT 2^n plan consisting of sets (1), (2) and (3) in (3.2).

Referring to the ORAT plan in (3.39) and since all estimates are obtained as averages of differences between successive runs, we display

(3.39) as

$$\begin{aligned}
 & \left. \begin{array}{l} \xrightarrow[r]{11\text{---}1a_r00\text{---}0} \\ \xrightarrow[r]{00\text{---}0a_r11\text{---}1} \end{array} \right\} (1), (a_r = 0, 1 \text{ and } r = 0, 1, 2, \dots, n) \\
 & \left. \begin{array}{l} \xrightarrow[r]{00\text{---}0a_r11\text{---}1} \\ \xrightarrow[r]{11\text{---}1a_r00\text{---}01} \end{array} \right\} (2), (a_r = 1, 0 \text{ and } r = 1, 2, \dots, (n-1)) \\
 & \left. \begin{array}{l} \xrightarrow[r]{11\text{---}1a_r00\text{---}01} \\ \xrightarrow[r]{11\text{---}1a_r00\text{---}01} \end{array} \right\} (3), (a_r = 0, 1 \text{ and } r = 1, 2, \dots, (n-2))
 \end{aligned} \tag{3.41}$$

or more explicitly

$$\begin{aligned}
 & \left. \begin{array}{l} \xrightarrow[r]{11\text{---}1000\text{---}0} \\ \xrightarrow[r]{11\text{---}1100\text{---}0} \end{array} \right\} (1) \quad r = 0, 1, 2, \dots, n \\
 & \left. \begin{array}{l} \xrightarrow[r]{00\text{---}0111\text{---}1} \\ \xrightarrow[r]{00\text{---}0011\text{---}1} \end{array} \right\} (2) \quad r = 1, 2, \dots, n-1 \\
 & \left. \begin{array}{l} \xrightarrow[r]{11\text{---}1000\text{---}01} \\ \xrightarrow[r]{11\text{---}1100\text{---}01} \end{array} \right\} (3) \quad r = 1, 2, \dots, (n-2)
 \end{aligned} \tag{3.42}$$

The estimable functions after the $(n+1) + (n-1) + (n-2) = 3n - 2$ runs in sets (1), (2) and (3) are

1. A_r , $r = 1, 2, \dots, n$
2. $A_r A_n$, $r = 1, 2, \dots, (n-1)$
3. $\sum_r 1, n-1 = \sum_{i=r+1}^{n-1} A_i A_r - \sum_{i=1}^{r-1} A_i A_r$, $r = 1, 2, \dots, n-2$ (3.43)

The unbiased estimates of A_r , ($r = 1, 2, \dots, n$) are given by (1) in (3.37). The unbiased estimates of the remaining functions in (3.43) are

$$\begin{aligned}
 1. \quad \hat{A}_r A_n &= \frac{1}{2} \left[\underbrace{y_{11 \dots 1100 \dots 01}}_r \xrightarrow{\quad} \underbrace{\quad}_n - \underbrace{y_{11 \dots 1000 \dots 01}}_r \xrightarrow{\quad} \underbrace{\quad}_n \right. \\
 &\quad \left. - \left(\underbrace{y_{11 \dots 1100 \dots 0}}_r \xrightarrow{\quad} \underbrace{\quad}_n - \underbrace{y_{11 \dots 1000 \dots 0}}_r \xrightarrow{\quad} \underbrace{\quad}_n \right) \right] \\
 2. \quad \hat{z}_{r, n-1} &= \frac{1}{2} \left[\underbrace{y_{00 \dots 0111 \dots 11}}_r \xrightarrow{\quad} \underbrace{\quad}_{n-1} - \underbrace{y_{00 \dots 0011 \dots 11}}_r \xrightarrow{\quad} \underbrace{\quad}_{n-1} \right. \\
 &\quad \left. - \left(\underbrace{y_{11 \dots 1100 \dots 01}}_r \xrightarrow{\quad} \underbrace{\quad}_{n-1} - \underbrace{y_{11 \dots 100 \dots 01}}_r \xrightarrow{\quad} \underbrace{\quad}_{n-1} \right) \right] \tag{3.44}
 \end{aligned}$$

all with variance $\frac{1}{4}\sigma^2$.

Referring to (3.9), it follows that $E\hat{A}_r = A_r$ and setting $s = n-1$ in (3.26) we find that

$$\underbrace{00 \dots 0111 \dots 11}_r \xrightarrow{\quad} \underbrace{\quad}_{n-1} - \underbrace{00 \dots 0011 \dots 11}_r \xrightarrow{\quad} \underbrace{\quad}_{n-1} = 2(A_r + \sum_{i=1}^{r-1} A_i A_r + \sum_{i=r+1}^{n-2} A_r A_i + A_r A_{n-1} + A_r A_n) \tag{3.45}$$

From the difference between (3.45) and (3.24) for $s = n$, unbiasedness of (2) in (3.44) follows.

Stage Two: The estimation of $A_r A_{n-1}$, ($r = 1, 2, \dots, (n-2)$) from the strict ORAT 2^n plan consisting of sets (1), (2), (3) and (4) in (3.2).

Referring to the alias between structure $\Sigma_{r, n-1}$ in (3.43) we note that it can be written explicitly as

$$\begin{aligned}
 \Sigma_{1, n-1} &= A_1 A_2 + A_1 A_3 + \dots + A_1 A_r + A_1 A_{r+1} + \dots + A_1 A_{n-1} \\
 \Sigma_{2, n-1} &= -A_1 A_2 + A_2 A_3 + \dots + A_2 A_r + A_2 A_{r+1} + \dots + A_2 A_{n-1} \\
 &\vdots \\
 \Sigma_{r, n-1} &= -A_1 A_r - A_2 A_r - \dots - A_{r-1} A_r + A_r A_{r+1} + \dots + A_r A_{n-1} \\
 &\vdots \\
 \Sigma_{n-2, n-1} &= -A_1 A_{n-1} - A_2 A_{n-1} - \dots - A_{r-1} A_{n-1} - A_r A_{n-1} - \dots + A_{n-2} A_{n-1}
 \end{aligned} \tag{3.46}$$

For stage two we have the strict ORAT plan in (3.40) which we write as

$$\begin{aligned}
 &\left. \begin{array}{l} 11\text{---}1a_r 00\text{---}0 \end{array} \right\} (1), (a_r = 0, 1 \text{ and } r = 0, 1, 2, \dots, n) \\
 &\left. \begin{array}{l} 00\text{---}0a_r 11\text{---}1 \end{array} \right\} (2), (a_r = 1, 0 \text{ and } r = 1, 2, \dots, n-1) \\
 &\left. \begin{array}{l} 11\text{---}1a_r 00\text{---}01 \end{array} \right\} (3), (a_r = 0, 1 \text{ and } r = 1, 2, \dots, n-2) \\
 &\left. \begin{array}{l} 00\text{---}0a_r 11\text{---}101 \end{array} \right\} (4), (a_r = 1, 0 \text{ and } r = 1, 2, \dots, (n-3)) \tag{3.47}
 \end{aligned}$$

The estimable functions after the $(n+1) + (n-1) + (n-2) + (n-3) = 3n - 5$ runs of sets (1), (2), (3) and (4) are

1. A_r , ($r = 1, 2, \dots, n$)
2. $A_r A_n$, ($r = 1, 2, \dots, n-1$)
3. $A_r A_{n-1}$, ($r = 1, 2, \dots, n-2$)
4. $\Sigma_{r, n-2} = \sum_{i=r+1}^{n-2} A_r A_i - \sum_{i=1}^{r-1} A_i A_r$, ($r = 1, 2, \dots, n-3$) (3.48)

The estimates of \hat{A}_r and $\hat{A}_r A_n$ are given by (1) in (3.37) and (1) in (3.44) whereas the estimates of $A_r A_{n-1}$ and $\Sigma_{r, n-2}$ are

1.
$$\hat{A}_r A_{n-1} = \frac{1}{2} \left[\begin{array}{c} y_{00\text{---}0111\text{---}11} \\ \xrightarrow{r} \\ \xrightarrow{n-1} \end{array} - \begin{array}{c} y_{00\text{---}0011\text{---}11} \\ \xrightarrow{r} \\ \xrightarrow{n-1} \end{array} \right] \\ - \left(\begin{array}{c} y_{00\text{---}0111\text{---}101} \\ \xrightarrow{r} \\ \xrightarrow{n-1} \end{array} - \begin{array}{c} y_{00\text{---}0011\text{---}101} \\ \xrightarrow{r} \\ \xrightarrow{n-1} \end{array} \right) \Big]$$
2.
$$\hat{\Sigma}_{r, n-2} = \frac{1}{2} \left[\begin{array}{c} y_{00\text{---}011\text{---}1101} \\ \xrightarrow{r} \\ \xrightarrow{n-2} \end{array} - \begin{array}{c} y_{00\text{---}011\text{---}1101} \\ \xrightarrow{r} \\ \xrightarrow{n-2} \end{array} \right] \\ - \left(\begin{array}{c} y_{11\text{---}1100\text{---}001} \\ \xrightarrow{r} \\ \xrightarrow{n-2} \end{array} - \begin{array}{c} y_{11\text{---}100\text{---}001} \\ \xrightarrow{r} \\ \xrightarrow{n-2} \end{array} \right) \Big]$$
 (3.49)

all with variance $\frac{1}{2}\sigma^2$.

The unbiasedness of \hat{A}_r and $\hat{A}_r A_n$ was proven in stage one. The unbiasedness of $\hat{A}_r A_{n-1}$ follows from (3.29) setting $s = n - 1$ and replacing treatment combinations by their corresponding response. The

unbiasedness of $\hat{\Sigma}_{r, n-2}$ can be shown as follows:

Referring to (3.26) and setting $s = n-3$, i.e., $n - s = 3$, we find for any two successive runs in set (4) that

$$\begin{aligned} \underbrace{00\text{---}0111\text{---}11101}_{\substack{\xrightarrow{r} \\ \xrightarrow{n-2}}} - \underbrace{00\text{---}0011\text{---}11101}_{\substack{\xrightarrow{r} \\ \xrightarrow{n-2}}} &= 2(A_r - \sum_{i=1}^{r-1} A_i A_r + \sum_{j=r+1}^{n-4} A_r A_j \\ &+ A_r A_{n-3} + A_r A_{n-2} - A_r A_{n-1} + A_r A_n) \end{aligned} \quad (3.50)$$

and from (3.25), setting $s = n - 2$, i.e., $n - s = 2$, we find for any two successive runs in set (3) that

$$\begin{aligned} \underbrace{11\text{---}1100\text{---}001}_{\substack{\xrightarrow{r} \\ \xrightarrow{n-2}}} - \underbrace{11\text{---}1000\text{---}001}_{\substack{\xrightarrow{r} \\ \xrightarrow{n-2}}} &= 2(A_r + \sum_{i=1}^{r-1} A_i A_r - \sum_{j=r+1}^{n-3} A_r A_j \\ &- A_r A_{n-2} - A_r A_{n-1} + A_r A_n) \end{aligned} \quad (3.51)$$

From the difference between (3.50) and (3.51), unbiasedness of $\hat{\Sigma}_{r, n-2}$ follows.

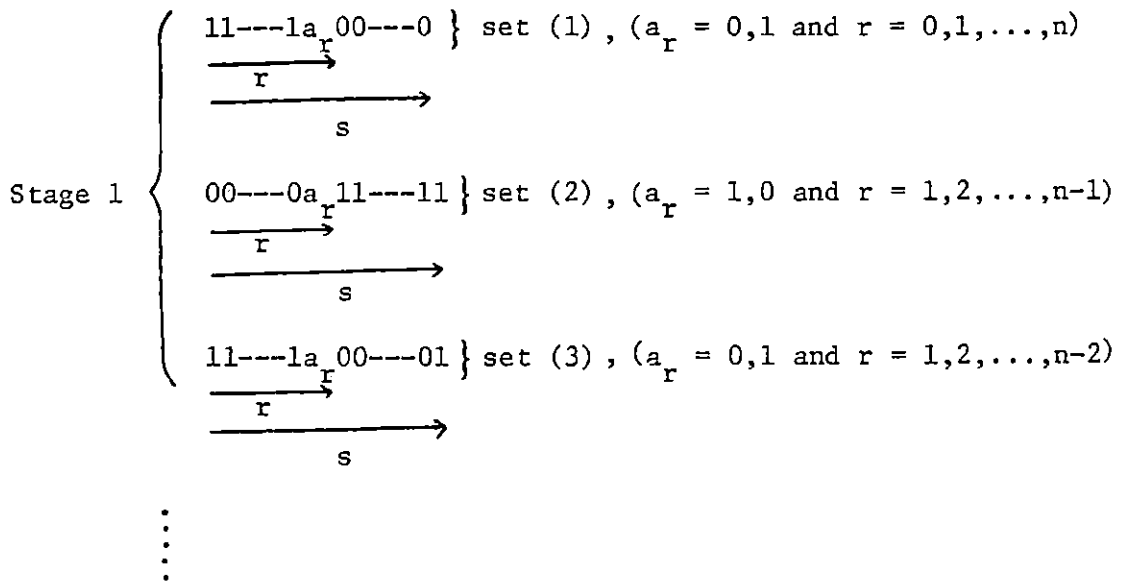
From the results in stages one and two we arrive at the following general result for strict ORAT 2^n plans.

Stage (t): The estimation of $A_r A_s$, ($s = n, n-1, \dots, 3, 2$), ($r = 1, 2, 3, 4, \dots, (s-1)$) and ($t = (n-s) + 1$).

From stages one and two we note that three sets from the fraction f_n in (3.2) are needed for the estimation of the two-factor interactions $A_r A_s$, ($s = n, n-1, \dots, 3, 2$) involving the s^{th} factor A_s as well as the alias chain of two-factor interactions $\Sigma_{r, s-1}$ in (3.36) involving factor

A_{s-1} . These three sets are sets $(n-s) + 1 = t$, $(n-s) + 2 = t + 1$ and $(n-s) + 3 = t + 2$ containing, respectively, s , $s-1$, $s-2$ treatment combinations. Sets t and $(t+2)$ contain treatment combinations differing only in the level of the s^{th} factor. In set $(t+1)$, factor A_s occurs at its low level if $(n-s)$ is odd or at its high level if $(n-s)$ is even. The interaction effects $A_r A_s$ are estimated as averages of two differences, one difference involves two successive runs from set t and the other difference involves two successive runs from set $(t+2)$ while the alias chain $\sum_r A_{1,s-1}$ is estimated as the average of two differences, one difference involves two successive runs from set $(t+1)$ and one difference involves two successive runs from set $(t+2)$.

For stage t , we have the following strict ORAT 2^n plan



$$4. A_r A_{n-2}, (r = 1, 2, \dots, n-3)$$

$$5. A_r A_{n-3}, (r = 1, 2, \dots, n-4)$$

⋮
⋮
⋮

$$(t+1). A_r A_{n-s}, (r = 1, 2, \dots, s-1)$$

$$(t+2). \sum_{r=1}^{s-1} A_r A_j - \sum_{i=1}^{r-1} A_i A_r, (r = 1, 2, \dots, s-2) \quad (3.53)$$

The unbiased estimates of A_r , ($r = 1, 2, \dots, n$) are given by (1) in (3.9). The unbiased estimates of the two-factor interactions $A_r A_{n-k}$, for $(n-k)$ even, are of the form

$$\begin{aligned} \hat{A}_r A_{n-k} = \frac{1}{4} & \left[\left(y_{\overbrace{11 \dots 1100 \dots 010101 \dots 01}^{r \quad n-k}} - y_{\overbrace{11 \dots 1000 \dots 010101 \dots 0101}^{r \quad n-k}} \right) \right. \\ & \left. - \left(y_{\overbrace{11 \dots 1100 \dots 000101 \dots 0101}^{r \quad n-k}} - y_{\overbrace{11 \dots 1000 \dots 000101 \dots 0101}^{r \quad n-k}} \right) \right] \end{aligned} \quad (3.54)$$

while the unbiased estimates of the two-factor interactions $A_r A_{n-k}$, for $(n-k)$ odd, are of the form

$$\hat{A}_r A_{n-k} = \frac{1}{4} \left[\left(y_{\overbrace{00 \dots 0111 \dots 1110101 \dots 0101}^{r \quad n-k}} - y_{\overbrace{00 \dots 0011 \dots 1110101 \dots 0101}^{r \quad n-k}} \right) \right]$$

$$- (y_{00\text{---}0111\text{---}1010101\text{---}0101} - y_{00\text{---}0011\text{---}1010101\text{---}0101}) \quad (3.55)$$

$\xrightarrow[r]{k}$
 $\xrightarrow[r]{k}$

$\xrightarrow[r]{n-k}$
 $\xrightarrow[r]{n-k}$

As the interaction effects $A_r A_s$, the unbiased estimate of the function

$\hat{\Sigma}_{r, s-1}$ depends on whether $(n-s)$ is even or odd. Therefore,

a) for $(n-s)$ even,

$$\hat{\Sigma}_{r, s-1} = \frac{1}{4} [(y_{00\text{---}0111\text{---}11010101\text{---}0101} - y_{00\text{---}0011\text{---}110101\text{---}0101}) - (y_{11\text{---}1100\text{---}010101\text{---}0101} - y_{11\text{---}1000\text{---}010101\text{---}0101})] \quad (3.56)$$

$\xrightarrow[r]{s-1}$
 $\xrightarrow[r]{s-1}$

$\xrightarrow[r]{(n-s)+1}$
 $\xrightarrow[r]{(n-s)+1}$

b) for $(n-s)$ odd,

$$\hat{\Sigma}_{r, s-1} = \frac{1}{4} (y_{00\text{---}0111\text{---}1110101\text{---}0101} - y_{00\text{---}0011\text{---}1110101\text{---}0101}) - (y_{11\text{---}1100\text{---}0010101\text{---}0101} - y_{11\text{---}1000\text{---}0010101\text{---}0101}) \quad (3.57)$$

$\xrightarrow[r]{s-1}$
 $\xrightarrow[r]{s-1}$

$\xrightarrow[r]{(n-s)+1}$
 $\xrightarrow[r]{(n-s)+1}$

The unbiasedness of the estimates in (3.54) and (3.55) follows by referring to (3.28) and (3.29). The unbiasedness of the estimates in

(3.56) and (3.57) can be shown as follows:

a. unbiasedness of $\hat{\Sigma}_{r, s-1}$ for $(n-s)$ even.

Replacing s by $(s-1)$ in (3.26) we find that

$$\begin{aligned}
 & \begin{array}{c} \overbrace{00\text{---}0111\text{---}110101\text{---}0101}^r \\ \underbrace{\hspace{10em}}_{s-1} \end{array} - \begin{array}{c} \overbrace{00\text{---}0011\text{---}110101\text{---}0101}^r \\ \underbrace{\hspace{10em}}_{s-1} \end{array} = 2(A_r - \sum_{i=1}^{r-1} A_i A_r) \\
 & \quad + \sum_{j=r+1}^{s-2} A_r A_j + A_r A_{s-1} + \sum_{j=1}^{(n-s)+1} (-1)^{j+1} A_r A_{(s-1)+j} \\
 & = 2 \left(A_r - \sum_{i=1}^{r-1} A_i A_r + \sum_{j=r+1}^{s-1} A_r A_j + \sum_{j=1}^{n-s} (-1)^j A_r A_{s+j} + A_r A_s \right) \quad (3.58)
 \end{aligned}$$

Also replacing s by $(s-2)$ in (3.25) we find that

$$\begin{aligned}
 & \begin{array}{c} \overbrace{11\text{---}1100\text{---}0010101\text{---}0101}^r \\ \underbrace{\hspace{10em}}_{s-2} \end{array} - \begin{array}{c} \overbrace{11\text{---}1000\text{---}0010101\text{---}0101}^r \\ \underbrace{\hspace{10em}}_{s-2} \end{array} \\
 & = 2(A_r + \sum_{i=1}^{r-1} A_i A_r - \sum_{j=r+1}^{s-3} A_r A_j - A_r A_{s-2} + \sum_{j=1}^{(n-s)+2} (-1)^j A_r A_{(s-2)+j}) \\
 & = 2(A_r + \sum_{i=1}^{r-1} A_i A_r - \sum_{j=r+1}^{s-1} A_r A_j + A_r A_s + \sum_{j=1}^{n-s} (-1)^j A_r A_{s+j}) \quad (3.59)
 \end{aligned}$$

From the difference between (3.58) and (3.59), unbiasedness follows.

b. unbiasedness of $\hat{\Sigma}_{r, s-1}$ for $(n-s)$ odd.

Referring to (3.27) we find that

$$\begin{aligned}
 & \begin{array}{ccc}
 00\text{---}0111\text{---}1110101\text{---}0101 & - & 00\text{---}0011\text{---}1110101\text{---}0101 \\
 \xrightarrow[r]{s-1} & & \xrightarrow[r]{s-1} \\
 & \underbrace{\hspace{10em}}_{(n-s)+1} & \underbrace{\hspace{10em}}_{(n-s)+1}
 \end{array} \\
 & = 2(A_r - \sum_{i=1}^{r-1} A_i A_r + \sum_{j=r+1}^{s-1} A_r A_j + A_r A_s + \sum_{j=1}^{n-s} (-1)^{j+1} A_r A_{s+j}) \tag{3.60}
 \end{aligned}$$

Replacing s by $(s-1)$ in (3.25) we find that

$$\begin{aligned}
 & \begin{array}{ccc}
 11\text{---}1100\text{---}0010101\text{---}0101 & - & 11\text{---}1000\text{---}0010101\text{---}0101 \\
 \xrightarrow[r]{s-1} & & \xrightarrow[r]{s-1} \\
 & \underbrace{\hspace{10em}}_{(n-s)+1} & \underbrace{\hspace{10em}}_{(n-s)+1}
 \end{array} \\
 & = 2(A_r + \sum_{i=1}^{r-1} A_i A_r - \sum_{j=r+1}^{s-2} A_r A_j - A_r A_{s-1} + \sum_{j=1}^{(n-s)+1} (-1)^j A_r A_{(s-1)+j}) \\
 & = 2(A_r + \sum_{i=1}^{r-1} A_i A_r - \sum_{j=r+1}^{s-1} A_r A_j + \sum_{j=1}^{n-s} (-1)^{j+1} A_r A_{s+j}) \tag{3.61}
 \end{aligned}$$

From the difference between (3.60) and (3.61), unbiasedness follows.

3.5 An Example: Strict ORAT 2^6_V Plan

The six factors of the 2^6 factorial experiment are A_1, A_2, A_3, A_4, A_5 and A_6 . The treatment combinations needed for the strict ORAT 2^6_V plans are obtained from (3.2) by setting $n = 6$. These treatment combinations are the 21 runs

$$\begin{aligned}
1. & A_1 - A_1A_2 - A_1A_3 - A_1A_4 - A_1A_5 - A_1A_6 \\
2. & A_2 + A_1A_2 - A_2A_3 - A_2A_4 - A_2A_5 - A_2A_6 \\
3. & A_3 + A_1A_3 + A_2A_3 - A_3A_4 - A_3A_5 - A_3A_6 \\
4. & A_4 + A_1A_4 + A_2A_4 + A_3A_4 - A_4A_5 - A_4A_6 \\
5. & A_5 + A_1A_5 + A_2A_5 + A_3A_5 + A_4A_5 - A_4A_6 \\
6. & A_6 + A_1A_6 + A_2A_6 + A_3A_6 + A_4A_6 + A_5A_6
\end{aligned} \tag{3.64}$$

Their respective unbiased estimates are

$$\begin{aligned}
1. & \frac{1}{2}[y(100000) - y(000000)] \\
2. & \frac{1}{2}[y(110000) - y(100000)] \\
3. & \frac{1}{2}[y(111000) - y(110000)] \\
4. & \frac{1}{2}[y(111100) - y(111000)] \\
5. & \frac{1}{2}[y(111110) - y(111100)] \\
6. & \frac{1}{2}[y(111111) - y(111110)]
\end{aligned} \tag{3.65}$$

all with variance $\frac{1}{2}\sigma^2$. If all interactions are negligible, then set (1) in (3.62) is a 2_{III}^6 plan from which main effects are estimable. However, if interactions are not negligible then more runs must be added to the 7 runs in stage A.

Stage B: The estimation of main effects A_r , ($r = 1, 2, \dots, 6$) from the strict ORAT 2^6 plan consisting of sets (1) and (2) in (3.62). After the completion of the first 12 runs in (3.62) the estimable functions and their unbiased estimates are in Table 3.

All estimates in Table (3) have variance $\frac{1}{4}\sigma^2$.

Stage One: The strict ORAT 2^6 plan consisting of sets (1), (2)

Table 3. Estimable Functions and Their Estimates from the 12-run
 Strict ORAT 2^6 Plan Consisting from Sets (1) and (2) of (3.62).

	Estimable functions	Estimate
1	A_1	$\frac{1}{4}[y(100000) - y(000000) + y(111111) - y(011111)]$
2	A_2	$\frac{1}{4}[y(110000) - y(100000) + y(011111) - y(001111)]$
3	A_3	$\frac{1}{4}[y(111000) - y(110000) + y(001111) - y(000111)]$
4	A_4	$\frac{1}{4}[y(111100) - y(111000) + y(000111) - y(000011)]$
5	A_5	$\frac{1}{4}[y(111110) - y(111100) + y(000011) - y(000001)]$
6	A_6	$\frac{1}{4}[y(111111) - y(111110) + y(000001) - y(000000)]$
7	$A_1A_2 + A_1A_3 + A_1A_4 + A_1A_5 + A_1A_6$	$\frac{1}{4}[y(111111) - y(011111) - y(100000) + y(000000)]$
8	$-A_1A_2 + A_2A_3 + A_2A_4 + A_2A_5 + A_2A_6$	$\frac{1}{4}[y(011111) - y(001111) - y(110000) + y(100000)]$
9	$-A_1A_3 - A_2A_3 + A_3A_4 + A_3A_5 + A_3A_6$	$\frac{1}{4}[y(001111) - y(000111) - y(111000) + y(110000)]$
10	$-A_1A_4 - A_2A_4 - A_3A_4 + A_4A_5 + A_4A_6$	$\frac{1}{4}[y(000111) - y(000011) - y(111100) + y(111000)]$
11	$-A_1A_5 - A_2A_5 - A_3A_5 - A_4A_5 + A_5A_6$	$\frac{1}{4}[y(000011) - y(000001) - y(111110) + y(111100)]$

and (3) of (3.62) for the estimation of $A_r A_6$, ($r = 1, 2, 3, 4, 5, 6$)

After the completion of the first 16 runs in (3.62) the estimable functions are

1. $A_1 ; A_2 ; A_3 ; A_4 ; A_5 ; A_6$
2. $A_1 A_2 + A_1 A_3 + A_1 A_4 + A_1 A_5 = \Sigma_{1,5}$
3. $- A_1 A_2 + A_2 A_3 + A_2 A_4 + A_2 A_5 = \Sigma_{2,5}$
4. $- A_1 A_3 - A_2 A_3 + A_3 A_4 + A_3 A_5 = \Sigma_{3,5}$
5. $- A_1 A_4 - A_2 A_4 - A_3 A_4 + A_4 A_5 = \Sigma_{4,5}$
6. $A_1 A_6 ; A_2 A_6 ; A_3 A_6 ; A_4 A_6 ; A_5 A_6$. (3.66)

The unbiased estimates of main effects A_r , $r = 1, 2, \dots, 6$ are given in Table 3. The unbiased estimates of the remaining functions in (3.66) are

1. $\hat{\Sigma}_{1,5} = \frac{1}{4} [(y_{111111} - y_{011111}) - (y_{100001} - y_{000001})]$
2. $\hat{\Sigma}_{2,5} = \frac{1}{4} [(y_{011111} - y_{001111}) - (y_{110001} - y_{100001})]$
3. $\hat{\Sigma}_{3,5} = \frac{1}{4} [(y_{001111} - y_{000111}) - (y_{111001} - y_{110001})]$
4. $\hat{\Sigma}_{4,5} = \frac{1}{4} [(y_{000111} - y_{000011}) - (y_{111101} - y_{111001})]$ (3.67)
5. $\hat{A}_1 A_6 = \frac{1}{4} [(y_{100001} - y_{000001}) - (y_{100000} - y_{000000})]$
6. $\hat{A}_2 A_6 = \frac{1}{4} [(y_{110001} - y_{100001}) - (y_{110000} - y_{100000})]$

$$\begin{aligned}
7. \quad \hat{A}_3 A_6 &= \frac{1}{4}[(y_{111001} - y_{110001}) - (y_{111000} - y_{110000})] \\
8. \quad \hat{A}_4 A_6 &= \frac{1}{4}[(y_{111101} - y_{111001}) - (y_{111100} - y_{111000})] \\
9. \quad \hat{A}_5 A_6 &= \frac{1}{4}[(y_{111111} - y_{111101}) - (y_{111110} - y_{111100})] \quad (3.68)
\end{aligned}$$

all with variance $\frac{1}{4}\sigma^2$.

Stage Two: The estimation of $A_r A_5$, ($r = 1, 2, 3, 4$) from the strict ORAT 2^6 plan consisting of sets (1), (2), (3) and (4) in (3.62).

After the completion of the first 19 runs in (3.62) the estimable functions are

$$\begin{aligned}
1. \quad A_r, & (r = 1, 2, \dots, 6) \\
2. \quad A_r A_6, & (r = 1, 2, \dots, 5) \\
3. \quad A_r A_5, & (r = 1, 2, 3, 4) \\
4. \quad \Sigma_{1,4} &= A_1 A_2 + A_1 A_3 + A_1 A_4 \\
&\Sigma_{2,4} = -A_1 A_2 + A_2 A_3 + A_2 A_4 \\
&\Sigma_{3,4} = -A_1 A_3 - A_2 A_3 + A_3 A_4 \quad (3.69)
\end{aligned}$$

Unbiased estimates of A_r , ($r = 1, 2, \dots, 6$) are given in Table 3 and the unbiased estimates of $A_r A_6$, ($r = 1, 2, \dots, 5$) are given by (5)-(9) in (3.68). The unbiased estimates of the remaining functions in (3.69) are

$$\begin{aligned}
1. \quad \hat{A}_1 A_5 &= \frac{1}{4}[(y_{111111} - y_{011111}) - (y_{111101} - y_{011101})] \\
2. \quad \hat{A}_2 A_5 &= \frac{1}{4}[(y_{011111} - y_{001111}) - (y_{011101} - y_{001101})] \\
3. \quad \hat{A}_3 A_5 &= \frac{1}{4}[(y_{001111} - y_{000111}) - (y_{001101} - y_{000101})]
\end{aligned}$$

$$4. \hat{A}_4 A_5 = \frac{1}{4} [(y_{000111} - y_{000011}) - (y_{000101} - y_{000001})] \quad (3.70)$$

$$5. \hat{\Sigma}_{1,4} = \frac{1}{4} [(y_{111101} - y_{011101}) - (y_{100001} - y_{000001})]$$

$$\hat{\Sigma}_{2,4} = \frac{1}{4} [(y_{011101} - y_{001101}) - (y_{110001} - y_{100001})]$$

$$\hat{\Sigma}_{3,4} = \frac{1}{4} [(y_{001101} - y_{000101}) - (y_{111001} - y_{110001})] \quad (3.71)$$

all with variance $\frac{1}{4}\sigma^2$.

Stage Three: The estimation of $A_r A_4$, ($r = 1, 2, 3$) from the strict ORAT 2^6 plan consisting of sets (1), (2), (3), (4) and (5) in (3.62).

After the completion of the first 19 runs in (3.62) the estimable functions are

$$1. A_r, (r = 1, 2, \dots, 6)$$

$$2. A_r A_6, (r = 1, 2, \dots, 5)$$

$$3. A_r A_5, (r = 1, 2, 3, 4)$$

$$4. A_r A_4, (r = 1, 2, 3)$$

$$5. \Sigma_{1,3} = A_1 A_2 + A_1 A_3$$

$$\Sigma_{2,3} = -A_1 A_2 + A_2 A_3. \quad (3.72)$$

The unbiased estimates of A_r , ($r = 1, 2, \dots, 6$) are given in Table 3.

The unbiased estimates of $A_r A_6$, ($r = 1, 2, \dots, 5$) and $A_s A_5$, ($s = 1, 2, 3, 4$) are given in (3.68) and (3.70). The unbiased estimates of the remaining functions in (3.72) are

$$\begin{aligned}
1. \quad \hat{A}_1 A_4 &= \frac{1}{4}[(y_{100101} - y_{000101}) - (y_{100001} - y_{000001})] \\
2. \quad \hat{A}_2 A_4 &= \frac{1}{4}[(y_{110101} - y_{100101}) - (y_{110001} - y_{100001})] \\
3. \quad \hat{A}_3 A_4 &= \frac{1}{4}[(y_{111101} - y_{110101}) - (y_{111001} - y_{110001})] \quad (3.73)
\end{aligned}$$

$$\begin{aligned}
4. \quad \hat{\Sigma}_{1,3} &= \frac{1}{4}[(y_{111101} - y_{011101}) - (y_{100101} - y_{000101})] \\
\hat{\Sigma}_{2,3} &= \frac{1}{4}[(y_{011101} - y_{001101}) - (y_{110101} - y_{100101})] \quad (3.74)
\end{aligned}$$

all with variance $\frac{1}{4}\sigma^2$.

Stage Four: The estimation of $A_r A_3$, ($r = 1, 2$) from the strict ORAT 2^6 plan consisting of sets (1), (2), (3), (4) and (5) in (3.62).

After the completion of all the runs in (3.62), all the parameters in (3.63) become estimable. The unbiased estimates of A_r , ($r = 1, 2, \dots, 6$) and $A_r A_s$, for ($s = r+1, r+2, \dots, 6$) and ($r = 4, 5, \text{ and } 6$), are given in the preceding stages while the unbiased estimates of the effects $A_1 A_3$, $A_2 A_3$ and $A_1 A_2$ are

$$\begin{aligned}
1. \quad \hat{A}_1 A_3 &= \frac{1}{4}[(y_{111101} - y_{011101}) - (y_{110101} - y_{010101})] \\
2. \quad \hat{A}_2 A_3 &= \frac{1}{4}[(y_{011101} - y_{001101}) - (y_{010101} - y_{000101})] \\
3. \quad \hat{A}_1 A_2 &= \frac{1}{4}[(y_{110101} - y_{010101}) - (y_{100101} - y_{000101})] \quad (3.75)
\end{aligned}$$

all with variance $\frac{1}{4}\sigma^2$.

II. Time Effect Non-negligible.

Assuming time effect follows a second degree polynomial, the linear model (3.1) is now

$$E y(a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5} a_6^{i_6})_h = \mu + \sum_{i=1}^n A_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^n A_i A_j x_i x_j + T_L \cdot t_h + T_Q \cdot t_h^2 \quad (3.76)$$

where

1. T_L and T_Q represent time linear and quadratic effects.
2. $h = 1, 2, \dots, N$ representing the run order, N is the total

number of runs in the ORAT plan,

3. t_h and t_h^2 are the coefficients of the orthogonal polynomial contrasts T_L and T_Q for N runs.

For $N = 22$ runs, the orthogonal polynomial coefficients t_h and t_h^2

are

$$\begin{array}{l} \text{linear:} \quad -21 \quad -19 \quad -17 \quad -15 \quad -13 \quad -11 \quad -9 \quad -7 \quad -5 \quad -3 \quad -1 \quad 1 \quad 3 \\ \quad \quad \quad 5 \quad 7 \quad 9 \quad 11 \quad 13 \quad 15 \quad 17 \quad 19 \quad 21 \\ \text{quadratic:} \quad -35 \quad -25 \quad -16 \quad -8 \quad -1 \quad 5 \quad 10 \quad 14 \quad 17 \quad 19 \quad 20 \quad -20 \quad -19 \\ \quad \quad \quad -17 \quad -14 \quad -10 \quad -5 \quad 1 \quad 8 \quad 16 \quad 25 \quad 35 \end{array} \quad (3.77)$$

From the linear model (3.76) and after the completion of all 22 runs in (3.62), the estimable functions become

$$\begin{array}{ll} A_1 & + T_Q \quad A_1 A_5 \\ A_2 & + 2T_Q \quad A_2 A_5 \quad + T_Q \\ A_3 & + T_Q \quad A_3 A_5 \quad + T_Q \\ A_4 & + 2T_Q \quad A_4 A_5 - 4T_L - 7T_Q \\ A_5 & + 11T_Q \quad A_1 A_4 \quad + 2T_Q \\ A_6 + 6T_L + 5T_Q & A_2 A_4 \quad + 2T_Q \\ \underline{A_1 A_6 - 2T_Q} & \underline{A_3 A_4 - 3T_L - 10T_Q} \end{array}$$

$$\begin{array}{rcl}
 A_2A_6 & - & 2T_Q \quad A_1A_3 \quad + \quad T_Q \\
 A_3A_6 & - & T_Q \quad A_2A_3 - 2T_L - 8T_Q \\
 A_4A_6 & - & T_Q \quad A_1A_2 - T_L - 4T_Q \\
 \underline{A_5A_6 - 5T_L + 4T_Q} & &
 \end{array} \tag{3.78}$$

Since in (3.78) factorial effects are aliased with linear and quadratic time effects, two runs must be replicated to dealias factorial effects from the time effects T_L and T_Q . Any two runs can be replicated, however, since the last run in (3.75) is 010101 and to preserve the strictness of the ORAT plan we replicate the runs 000101 and 000001. Letting $(a_1 a_2 a_3 a_4 a_5 a_6)_{i_h}$ denote the h^{th} run in the strict ORAT 2^6 plan

$$\left\{ \begin{array}{l}
 (000000)_1, (100000)_2, (110000)_3, (111000)_4, (111100)_5, (111110)_6, (111111)_7, \\
 (011111)_8, (001111)_9, (000111)_{10}, (000011)_{11}, (000001)_{12}, \\
 (100001)_{13}, (110001)_{14}, (111001)_{15}, (111101)_{16}, \\
 (011101)_{17}, (001101)_{18}, (000101)_{19}, \\
 (100101)_{20}, (110101)_{21}, \\
 (010101)_{22}, (000101)_{23}, (000001)_{24}
 \end{array} \right. \tag{3.79}$$

and using the model (3.76), we find that all $22 + 2 = 24$ parameters become estimable after 24 runs. The linear and quadratic time effects are of the form

$$\hat{T}_L = c_1(y(000101)_{23} - y(000101)_{19}) + c_2(y(000001)_{24} - y(000001)_{12})$$

$$\hat{T}_Q = c_1(y(000101)_{23} - y(000101)_{19}) - c_2(y(000001)_{24} - y(000001)_{12}) .$$

c_1 and c_2 are real numbers.

IV. OBAT PLANS FOR THE 2^n FACTORIAL EXPERIMENTS

In this chapter OBAT designs are developed for exploratory factorial experiments involving n 2-level factors. These designs can be used to estimate factor effects according to strategies I and II (see Section 7.1). A literature review of OBAT 2^n plans is provided in Section 4.1. The treatment combinations for the OBAT 2^n plans reported in this work are those used in generating Webb's (1965) expansible-contractible permutation-invariant $\langle 0, 1, 2 \rangle - 2^n$ designs (see Section 3.1). This $\langle 0, 1, 2 \rangle - 2^n$ design series is shown, in Section 4.2, to be $2^n/V$ fractional factorial designs and formulas, as functions of n , for estimates of main effects and two-factor interactions were provided. Strategy I OBAT 2^n plans are developed in Section 4.3 whereas strategy II OBAT 2^n plans are developed in Section 4.4. An application of OBAT 2^n plans based on strategies I and II to the 2^6 experiment is given in Section 4.5.

4.1 Literature Review of OBAT 2^n Plans

OBAT plans are factorial plans that are carried out in two or more stages (blocks). Contributions to the literature on OBAT 2^n plans are presented in chronological order. Included in this review are factor-screening 2^n designs as well as augmented factorial 2^n plans.

Dykstra (1959) presented a catalog of partially duplicated $2^n/V$ fractional factorial plans in two blocks (fractions) from two different defining contrasts, for $n \leq 11$ factors. The second 2^{n-k} fraction duplicates some of the runs in the first $2^n/V$ fraction. This augmentation with

partial replication is done to provide an unbiased estimate of the experimental error and to improve precision of already estimable factorial effects, whereas non-estimable effects in the first fraction remain non-estimable in the combined fraction. Similar work was done by Patel (1963) using irregular fractions for the first fraction.

Box and Hunter (1961) showed how two regular fractional factorial plans from the same family of defining contrasts for the 2^n factorial experiment can be combined to break chains of aliases among factorial effects. For example, the two defining contrasts $I = ABC = ACD = BD$ and $I = -ABC = ACD = -BD$ are from the same family generating two fractions from the 2^4 experiment with factors A, B, C and D. They proved that if the two defining contrasts differ from each other by switching only the sign of one factor then the effect of the factor whose sign is switched and all two-factor interactions involving this factor are estimable in the combined fraction. They also proved that if the two defining contrasts are such that one is obtainable from the other by switching the signs of all factors then all factors' main effects are estimable in the combined fraction. Three-factor and higher order interactions are assumed negligible.

Daniel (1962) who is mainly responsible for initiating the idea of running exploratory industrial type experiments in a sequence of fractional factorial plans, showed how resolution IV plans (2^{4-1} , 2^{7-3} , 2^{8-4} and 2^{16-11}) can be augmented - after they have been analyzed - by few additional runs so that information about some two-factor interactions can be obtained. In 2_{IV}^{n-m} fractions, two-factor interactions

are aliased with each other where each alias set contains at least two two-factor interactions. In order to break alias chains by additional runs Daniel assumes the following:

1. A small number of the factors in 2_{IV}^{n-m} plans are influential and hence some but not all two-factor interactions are non-negligible.
2. Small contrasts, i.e. non-significant, indicate that individual effects estimated by the contrasts can be dropped from the model.
3. Precision of estimates is not important, rather breaking alias chains with as few additional runs as possible. For an illustration Daniel considered the half-replicate 2_{IV}^{4-1} whose defining contrast is $I = ABCD$ and whose runs are: (1), ab, ac, ad, bc, bd, cd, abcd. The fraction 2_{IV}^{4-1} provides unbiased estimates for A, B, C, D, $AB + CD$, $AC + BD$ and $AD + BC$. Then, if it turns out that \hat{A} , \hat{C} and $AC \hat{+} BD$ are "large" and the other estimates are "small", one may conclude that $AC \hat{+} BD$ is large due to AC since \hat{B} and \hat{D} are insignificant. On the other hand, if \hat{A} , \hat{C} and $AB \hat{+} CD$ are "large" and the other estimates are "small", it may not be clear which of AB or CD is contributing to $AB + CD$ most. Therefore, the original fraction 2_{IV}^{4-1} must be augmented by additional runs in order to resolve this ambiguity. The additional runs are selected from the fraction 2_{IV}^{4-1} whose defining contrast is $I = -ABCD$. The following situations may occur:

1. No block effects. One additional run resolves the ambiguity in $AB \hat{+} CD$, say run a, and based on Daniel's assumptions we find that

$$a = \mu + A - C - AB + CD$$

or

$$AB - CD = \mu + A - C - a$$

and

$$AB \hat{=} CD = \mu + \hat{A} - \hat{C} - y(a)$$

where $y(a)$ is the response of the run a . Since $AB \hat{+} CD$ is available from the original fraction, it follows that

$$\hat{A}B = \frac{1}{2}[(AB \hat{+} CD) + (AB \hat{-} CD)]$$

$$\hat{C}D = \frac{1}{2}[(AB \hat{+} CD) - (AB \hat{-} CD)]$$

2. Blockeffects exist: A block of two runs is needed from which both the ambiguity in $AB \hat{+} CD$ can be resolved and the block effect can be eliminated. Selecting the block with runs a and abd, we note that

$$a = \mu + A - C - AB + CD + \text{an effect due to block 2.}$$

$$abd = \mu + A - C + AB - CD + \text{an effect due to block 2.}$$

Then

$$(AB - CD) = \frac{1}{2}(abd - a)$$

and

$$AB \hat{-} CD = \frac{1}{2}(y(abd) - y(a))$$

Since $AB \hat{+} CD$ and $AB \hat{-} CD$ are now available then AB and CD become available too.

Addelman (1962) considered the situation when the experimenter

feels that one or more additional 2-level factors should have been included in a factorial experiment which has just been performed. Addelman described a procedure for augmenting 2^n factorial designs by additional regular fractions such that main effects of all the factors and two-factor interactions involving only the original factors are estimable in the combined, generally irregular, fraction. An uneconomical consequence of the augmentation procedure is that if one additional factor can be introduced to the original plan by adding $2^{(n+1)-k}$ treatment combinations, for some k , then t additional factors can only be accommodated with the addition of $t2^{(n+1)-k}$ treatment combinations. The $t \cdot 2^{(n+1)-k}$ treatment combinations can be introduced to the original design either simultaneously or one block at a time where blocks are of size $2^{(n+1)-k}$. Block effects as well as three-factor and higher order interactions were assumed negligible. For an illustration, Addelman considered expanding the full 2^3 factorial experiment with factors A, B and C by one, two and three additional 2-level factors. Denoting the two levels of each factor by 0 and 1, we consider the following:

1. The addition of one factor, say D.

The original 2^3 design may then be regarded as the half replicate of the 2^4 design defined by $I = D_0$, where the subscript indicates the level at which factor D occurs in each treatment combination. Adding the quarter replicate of the 2^4 design defined by $I = D_1 = ABC_0 = ABCD_1$ to the original $\frac{1}{2}$ replicate, the resulting augmented plan is a $\frac{3}{4}$ replicate of the 2^4 experiment. Each of the four treatment combinations of the fraction $I = D_1 = ABC_0 = ABCD_1$ must have

- a. factor D occurring at its high level.
- b. the sum of the levels of factors A, B, C modulo 2 is 0.
- d. the sum of the levels of factors A, B, C and D modulo 2 is 1.

Moreover, the original $\frac{1}{2}$ replicate can be regarded as the union of the two $\frac{1}{4}$ replicates defined by $I = D_0 = ABC_0 = ABCD_0$ and $I = D_0 = ABC_1 = ABCD_1$. Therefore, the combined $3/4$ replicates of the 2^4 experiment which contains 12 runs may be represented in tabular form as

Defining Contrast	Fractional Replicate		
	1	2	3
I			
D	0	0	1
ABC	0	1	0
ABCD	0	1	1

From this $3/4$ replicate estimates for A, B, C, D, AB, AC, BC, AD, BD and CD can be obtained.

2. The addition of two factors, say D and E.

The original 2^3 design may then be regarded as the $\frac{1}{4}$ replicate of the 2^5 design defined by $I = D_0 = E_0 = DE_0$. Augmenting this $\frac{1}{4}$ replicate by the two $\frac{1}{4}$ replicates defined by

$$I = D_0 = E_1 = DE_1$$

and

$$I = D_1 = E_0 = DE_1,$$

the resulting plan is a $3/4$ replicate of the 2^5 design containing 24

runs, from which all 5 main effects and all two-factor interactions, except DE, can be estimated. This augmented 3/4 replicate can be presented in tabular form as

Defining Contrast	Fractional Replicate		
	1	2	3
I			
D	0	0	1
E	0	1	0
DE	0	1	1

However, further reduction in the number of runs may be achieved by adding just two 1/8 replicates of the 2^5 design to the original 2/8 replicate. The augmented design is then a 4/8 replicate of the 2^5 experiment containing 16 runs from which all 5 main effects and all two-factor interactions, except DE, can be estimated. This augmented 4/8 replicate is displayed in the following tabular form containing a sequence of four 1/8 replicates of the 2^5 design.

Defining Contrast	Fractional Replicates			
	1	2	3	4
I				
D	0	0	1	0
ABC	0	1	0	1
ABCD	0	1	1	1
E	0	0	0	1
DE	0	0	1	1
ABCE	0	1	0	0
ABCDE	0	1	1	0

For three or more additional factors, a similar argument can be used but two-factor interactions, involving the additional factors remain non-estimable in the augmented design. Furthermore, expanding 2^n factorial experiments using the augmentation by regular fractions of the 2^n designs is not as economical as Webb's (1965) expansible-contractible permutation-invariant saturated 2^n designs which are described in Section 3.1.

Hunter (1964) derived what he called a predictor-corrector equation for improving the precision of least squares estimates obtained from 2^{n-k} fractional factorial plans in the light of additional runs. The original fraction's least squares parameter estimates are the elements of the vector $\hat{\beta} = (X'X)^{-1}X'Y_1$, and from the linear model

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X \\ Z \end{bmatrix} \beta + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$$

for the combined fraction, Hunter derived an updating least squares formula for $\hat{\beta}$. However, Hunter did not consider the problem of breaking alias chains as Daniel (1962) did, i.e. estimable effects in the original fraction become more precisely estimable in the combined fraction but non-estimable effects remain non-estimable.

John (1966) generalized Daniel's (1962) and Hunter's (1964) results by describing a procedure for augmenting the 2^{n-1} fraction by additional new runs not overlapping with the runs in the original fraction in order to both improve the precision of some already estimable effects and break chains of aliased effects. Since the half-replicate 2^{n-1} is

defined by the defining contrast $I = P$ where P is generally a negligible high order interaction, non-negligible factorial effects will be aliased in pairs. Consequently, each alias set of the augmenting 2^{n-k} fraction will contain one or more pairs of the aliased effects. John proved that in a given alias set of the 2^{n-k} fraction if only one pair contains two non-negligible effects then that chain can be broken, otherwise it cannot.

Holms (1967) reported sequences of regular 2^{n-m} fractional factorial plans from the same family of defining contrasts for $n \leq 8$ factors. Each sequence allows, eventually, the estimation of the mean, all main effects and all two factor interactions. Parameter estimates after each stage of the sequence are orthogonal since the number of stages (blocks) in each sequence is a power of 2, i.e. 2^λ , $\lambda = 0, 1, 2, 3$. However, such sequences are generally uneconomical as the number of runs in the sequences is much larger than the number of parameters to be estimated. Assuming three-factor and higher order interactions negligible, Holms reported the estimable functions after each stage in the sequence.

Webb (1968) argues that ORAT plans are of practical importance as they provide a means for minimizing the impact of a sudden unexpected termination of the experiment after any run. Situations may arise, however, where the experimenter is reasonably sure that a block of runs can be completed before it is likely that the experiment must be discontinued hence reducing time trend effects from which ORAT plans suffer. This leads to an improvement of the design. Therefore, the experiment is then run as a sequence of blocks. Each block is of size $k < n$ and

introduces k new factors, where factors are assumed to be ordered for their first introduction at the high level. Similar to ORAT plans, factors after their first introduction at the high level may be changed to any level in the subsequent blocks. Assuming that all interactions are negligible, Webb proved that for blocks of size k a lower bound for the variance of each main effect estimate is $\sigma^2/(k+1)$ where each factor appears at its low level in the blocks following the block in which that factor first appears at its high level. The bound $\sigma^2/(k+1)$ and is achieved by the OBAT 2_{III}^n plans with $k = 0 \pmod{4}$.

Daniel (1973) attempting to reduce the effect of time trends in ORAT plans, reported an OBAT plan for the special case involving four 2-level factors in 12 blocks of size 2 from which the overall mean, all four main effects, all six two-factor interactions and the 12 block effects are eventually estimable. Also arguing that different block orders result in different orders in which the factorial effects become estimable, Daniel recommended the following two block ordering strategies:

1. Running the 12 blocks one at a time such that all four main effects are estimable first and all six two-factor interactions are estimable second.

2. Ordering the factors and then running the 12 blocks one at a time such that the main effect of the first factor and all two-factor interactions involving this factor are estimable first. Then the main effect of the second factor and all two-factor interactions involving this factor are estimable second ... etc.

Pajak and Addelman (1975), dropping the requirement of orthogonality of factorial effect estimates, reported more economical sequences of regular 2^{n-m} factorial plans than Holms' (1967) requiring a minimum number of blocks - not necessarily powers of 2 as those of Holms' (1967) - such that all main effects and all two-factor interactions are eventually estimable, for $n = 3, 4, \dots, 17$ 2-level factors. The sequences are of two types.

1. A type α sequence: which is a sequence of regular 2_{III}^{n-m} fractions from the same family of defining contrasts.

2. A type β sequence: which is a sequence of regular fractions of the 2^n experiment such that the first fraction is a 2_{III}^{n-m} fraction and the remaining fractions are either all 2_{II}^{n-m} or all 2_I^{n-m} fractions.

The regular fractions in the sequences above having come from the same family of defining contrasts (i.e. all fractions have the same alias structure), each additional fraction helps to break or reduce the size of the alias sets, and hence non-estimable effects in the first fraction of the sequence become sequentially estimable. The different orders in which the various fractions in the sequence can be introduced results in different orders in which non-estimable effects in the first fraction become estimable. Furthermore, the minimum number of blocks or fractions needed for a sequence to be full (i.e. all main effects and all two-factor interactions become estimable) is determined by the size of the largest alias set in the first fraction.

Although type α and type β sequences involve a smaller number of runs than required by Holms' (1967) for the estimation of all main effects and all two-factor interactions, more economy on the number of runs

can still be achieved using sequences of irregular fractions of the 2^n experiment. Pajak and Adelman reported the following two types of sequences.

3. A type γ sequence: which is a sequence in which the first stage of fraction is an irregular main effect $(n + 1)$ -run plan. The second stage is the set of treatment combinations which when added to the first stage results in a 2_{III}^{n-m} plan, i.e. the second block contains $2^{n-m} - (n + 1)$ treatment combinations. The remaining stages are either all 2_{II}^{n-m} or all 2_I^{n-m} fractions.

4. A type δ sequence: which is a sequence of irregular main effect $(n + 1)$ -run plans.

Unlike sequences of types α and β whose alias structure can be determined from the defining contrasts, the alias structure after each stage of the sequences γ and δ as well as the minimum number of blocks needed for the sequences to be full can only be determined by matrix computer algorithms.

For each one of the four types of sequences, Pajak and Adelman reported

1. the number of estimable effects after each stage.
2. the average variance of main effect estimates.
3. the average variance of two-factor interaction estimates. They also compared the efficiency of their sequences with Webb's (1965) resolution V permutation-invariant expansible-contractible saturated $\langle 0, 2, n-1 \rangle$ designs. However, they did not consider deriving formulas for parameter estimates.

Cotter (1979) suggested that the ORAT, factor-screening design described in Section 3.1 can be used as an OBAT plan with blocks of any

size as long as half the number of blocks are the foldover of the other half. The importance of the effect A_r ($r = 1, 2, \dots, n$) as discussed in Section 3.1, can approximately be judged from only two blocks, say i and j' , where

$$C_0(r) = \frac{1}{2}[(y_{2n+1}^{(i)} - y_{n+r}^{(i)}) + (y_r^{(j)} - y_0^{(j)})]$$

and

$$C_e(r) = \frac{1}{2}[(y_{2n+1}^{(i)} - y_{n+r}^{(i)}) - (y_r^{(j)} - y_0^{(j)})]$$

However, Cotter did not consider the problem of reporting what effects become estimable after a particular block.

4.2 Treatment Combinations for OBAT 2^n Plans

Referring to the linear model (3.1) relating expected responses and factorial effects in the 2^n factorial experiments, the subset of treatment combinations that will be used in developing the OBAT 2^n plans is the fraction F_n consisting of the $N = 1 + n + \frac{n(n-1)}{2}$ runs of the form

$$00\text{-----}0 a_r^{i_r} 00\text{-----}0 a_s^{i_s} 00\text{-----}0 \quad (4.1)$$

where $(a_r^{i_r}, a_s^{i_s} = 0, 1)$, ($s = r + 1, r + 2, \dots, n$) and ($r = 1, 2, \dots, (n-1)$). The runs in (4.1) will be denoted by $(a_r^{i_r} a_s^{i_s})_{rs}$, or for short by $(a_r a_s)_{rs}$, and the corresponding response by $y(a_r a_s)_{rs}$. The linear model in (3.1) based on the fraction F_n in (4.1) can now be written as

$$\begin{aligned}
(a_{rs} a_{rs})_{rs} &= E y(a_{rs} a_{rs})_{rs} \\
&= \mu + A_r x_r + A_s x_s - \sum_{i \neq r,s} A_i + A_r A_s x_r x_s \\
&\quad - \left(\sum_{i \neq r,s} A_r A_i \right) x_r - \left(\sum_{i \neq r,s} A_s A_i \right) x_s + \sum_{i < j} A_i A_j \quad (4.2)
\end{aligned}$$

The fraction F_n possesses the following characteristics:

1. It is permutation-invariant since the runs in (4.1) are actually: the one run 00----0, the n runs $\pi(100----0)$ and the $\frac{n(n-1)}{2}$ runs $\pi(110----0)$ where $\pi(\)$ denotes all the permutations of the treatment combination inside the bracket. The information matrix $X'X$ is also invariant to permutations of factors and hence all factor effects are estimated with the same precision.

2. It is saturated, hence economical, since the number of runs N in (4.1) is the same as the number of parameters in model (4.2).

3. It is irregular, i.e. non-orthogonal since it is not of the form of a 2^{n-k} fraction.

4. It is economically expansible since an additional 2-level factor A_{n+1} requires " $F_{n+1} - F_n$ " = $1 + n$ runs, namely the run $(1)_{n+1}$ and the n runs $(11)_{s(n+1)}$, ($s = 1, 2, \dots, n$). This is unlike orthogonal 2^{n-k} fractions, where an additional factor doubles the number of runs in the design.

5. It is a resolution V fraction and hence all main effects and all two-factor interactions are estimable. A proof of this assertion will not be presented.

Letting

$$A_{(r,s)} = \sum_{i \neq r,s} A_i$$

$$A_r A_{(r,s)} = \sum_{i \neq r,s} A_r A_i$$

$$A_s A_{(r,s)} = \sum_{i \neq r,s} A_s A_i$$

$$(AA)_{(r,s)} = \sum_{\substack{i < j \\ i, j \neq r, s}} A_i A_j$$

where $\sum_{i \neq r,s}$ contains $(n-2)$ terms and $\sum_{\substack{i < j \\ i, j \neq r, s}}$ contains $\frac{n(n-1)}{2} - 1 - (n-2) -$

$(n-2) = \frac{(n-2)(n-3)}{2}$ terms, the model in (4.2) can then be written as

$$(a_{rs})_{rs} = \mu + A_r x_r + A_s x_s - A_{(r,s)} + A_r A_s x_r x_s - A_r A_{(r,s)} x_r - A_s A_{(r,s)} x_s + (AA)_{(r,s)} \quad (4.3)$$

For the treatment combinations in (4.1) which are

$$(0)$$

$$(1)_r, \quad r = 1, 2, \dots, n$$

$$(11)_{rs}, \quad s = r+1, r+2, \dots, n \text{ and } r = 1, 2, \dots, (n-1) \quad (4.4)$$

the model (4.3) can be written explicitly as

$$(0) = \mu - A_r - A_s - A_{(r,s)} + A_r A_s + A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)_{(r,s)}$$

$$(1)_r = \mu + A_r - A_s - A_{(r,s)} - A_r A_s - A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)_{(r,s)}$$

$$(1)_s = \mu - A_r + A_s - A_{(r,s)} - A_r A_s + A_r A_{(r,s)} - A_s A_{(r,s)} + (AA)_{(r,s)}$$

$$(11)_{rs} = + A_r + A_s - A_{(r,s)} + A_r A_s - A_r A_{(r,s)} - A_s A_{(r,s)} + (AA)_{(r,s)} \quad (4.5)$$

From (4.5), we find

$$\begin{aligned} (1)_r - (0) &= 2(A_r - A_r A_s - A_r A_{(r,s)}) \\ &= 2(A_r - \sum_{s \neq r} A_r A_s) \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} (11)_{rs} - (1)_s &= 2(A_r + A_r A_s - A_r A_{(r,s)}) \\ &= 2(A_r + A_r A_s - \sum_{i \neq r,s} A_r A_i) \end{aligned} \quad (4.7)$$

From (4.7) it follows that

$$\sum_{s \neq r} ((11)_{rs} - (1)_s) = 2[(n-1)A_r + \sum_{s \neq r} A_r A_s - \sum_{s \neq r} A_r A_{(r,s)}] \quad (4.8)$$

But

$$\begin{aligned} \sum_{s \neq r} A_r A_{(r,s)} &= \sum_{s \neq r} [\sum_{i \neq r,s} A_r A_i] \\ &= [(A_3 A_r + A_4 A_r + A_5 A_r + \dots + A_{r-1} A_r + \dots + A_r A_{n-1} + A_r A_n) \\ &\quad + (A_2 A_r + A_4 A_r + A_5 A_r + \dots + A_{r-1} A_r + \dots + A_r A_{n-1} + A_r A_n) \\ &\quad \vdots \\ &\quad + (A_2 A_r + A_3 A_r + A_4 A_r + \dots + A_{r-1} A_r + \dots + A_r A_{n-1})] \end{aligned}$$

$$\begin{aligned}
&= (n-2)[A_1A_r + A_2A_r + \dots + A_{r-1}A_r + A_rA_{r+1} + \dots + A_rA_{n-1} + A_rA_n] \\
&= (n-2)[A_rA_s + \sum_{i \neq r, s} A_rA_i] = (n-2) \sum_{i \neq r} A_rA_i \quad (4.9)
\end{aligned}$$

From (4.8) and (4.9), we then find that

$$\sum_{s \neq r} ((11)_{rs} - (1)_s) = 2(n-1)A_r - 2(n-3) \sum_{s \neq r} A_rA_s \quad (4.10)$$

Finally we obtain from (4.6), (4.7) and (4.10) the following formulas

$$\begin{aligned}
1) \quad A_rA_s &= \frac{1}{4} [((11)_{rs} - (1)_s) - ((1)_r - (0))] \\
2) \quad A_r &= \frac{1}{4} \sum_{s \neq r} ((11)_{rs} - (1)_s) - \frac{1}{4}(n-3)((1)_r - (0))
\end{aligned}$$

$$\text{where } s = r+1, r+2, \dots, n \text{ and } r = 1, 2, \dots, (n-1). \quad (4.11)$$

Therefore, all main effects A_r , ($r = 1, 2, \dots, n$) and all two-factor interactions A_rA_s , ($r < s$) are estimable, i.e. the fraction F_n in (4.1) is of resolution V. The unbiased estimates of main effects and two-factor interactions are obtained by using the linear model

$$y(a_r a_s)_{rs} = (a_r a_s)_{rs} + \epsilon_{rs}$$

where $E(\epsilon_{rs}) = 0$ and $\text{var}(\epsilon_{rs}) = \sigma^2$ as

$$\begin{aligned}
1) \quad \hat{A}_rA_s &= \frac{1}{4} [(y(11)_{rs} - y(1)_s) - (y(1)_r - y(0))] \\
2) \quad \hat{A}_r &= \frac{1}{4} \cdot \sum_{s \neq r} (y(11)_{rs} - y(1)_s) - \frac{1}{4}(n-3)(y(1)_r - y(0)) \quad (4.12)
\end{aligned}$$

with $\text{var}(\hat{A}_{rs}) = \frac{1}{4}\sigma^2$ and

$$\text{var } \hat{A}_r = \frac{1}{16} [(n-1)(2\sigma^2) + (n-3)^2(2\sigma^2)] = \frac{1}{8} (n^2 - 5n + 8)\sigma^2$$

$$n = 1, 2, 3, \dots \quad (4.13)$$

It is seen from (4.13) that main effects are estimable with higher precision than two-factor interactions. Such property of the saturated fraction F_n makes F_n appropriate for screening both main effects as well as two-factor interactions with more emphasis on main effects.

It is also worth mentioning that the fraction F_n is Webb's (1965) expansible-contractible permutation-invariant $\langle 0, 1, 2 \rangle$ -design for the 2^n experiment. Webb suggested that this fraction be carried out as an ORAT plan although, unlike the fraction f_n in (3.2), more than one factor level is changed between successive runs. However, we are suggesting the fraction F_n for the two strategies of the OBAT 2^n plans that will be developed in the remaining sections of this chapter. Webb reported the entries of the information matrix $X'X$ of the fraction F_n as functions of n (see Appendix A) but he did not consider the problem of deriving formulas for parameter estimates as functions of n .

4.3 OBAT 2^n Plans for Strategy I

In strategy I, factor effects are investigated one at a time in blocks of small size, namely of size 2, according to Daniel's (1973) second recommended block ordering (see Section 4.1). The n factors A_1, A_2, \dots, A_n of the 2^n experiment are assumed ordered and hence

blocks are formed such that the i^{th} factor A_i as well as two-factor interactions involving factor A_i become estimable before the j^{th} factor A_j and two-factor interactions involving A_j , for $j > i$ and $(i, j = 1, 2, \dots, n)$. The treatment combinations in all blocks of the OBAT plan are taken from the 2^n fraction F_n in (4.1). The linear model (4.2) is now changed to incorporate block effects (BE) as well. Hence, for an observation in block h we have

$$y(a_{r,s})_{rs,h} = a_r a_s + (BE)_h + \epsilon_{rs,h} \quad (4.14)$$

where $E(\epsilon_{rs,h}) = 0$, $\text{var}(\epsilon_{rs,h}) = \sigma^2$, and $\sum_{h=1}^b (BE)_h = 0$, b is the total number of blocks in the plan.

For strategy I we consider the following two cases

Case (1): 2^n_{III} plans in a sequence of blocks of size 2.

When all interactions are negligible, model (4.14) becomes

$$y(a_{r,s})_{rs,h} = \mu + A_r x_r + A_s x_s - A_{(r,s)} + (BE)_h + \epsilon_{rs,h} \quad (4.15)$$

Since there are n 2-level factors, n blocks of size 2 are needed in order that all n main effects A_1, A_2, \dots, A_n become estimable. The two treatment combinations in block h , where $h = 1, 2, \dots, n$ will be of the form $(1)_{r,h}$ and $(0)_{,h}$, for some r , where $r = 1, 2, \dots, n$. The assignment of the treatment combinations to the n blocks, their being $n!$ possible orderings, determines the order in which main effects become estimable. Since the treatment combinations in (4.1) are permutation-invariant, we assume without loss of generality, that

block r contains $(1)_{r,r}$, $(0)_{r,r}$, ($r = 1, 2, \dots, n$). (4.16)

In the blocking scheme (4.16), each factor's main effect is estimable from only one block since from (4.15) and (4.16), $((1)_{r,r} - (0)_{r,r}) = 2A_r$ and hence

$$\hat{A}_r = \frac{1}{2}(y(1)_{r,r} - y(0)_{r,r}), \quad r = 1, 2, \dots, n \quad (4.17)$$

with $\text{var } \hat{A}_r = \frac{1}{2}\sigma^2$.

It can also be seen from (4.16) that the "control" treatment (0) is replicated n times hence allowing the estimation of block effects $(BE)_h$, $h = 1, 2, \dots, n$. Block effects are estimated as contrasts among the responses $y(0)_{,h}$, i.e.

$$(BE)_h = \frac{1}{n} [n y(0)_{,h} - \sum_{h' \neq h}^n y(0)_{,h'}] \quad (4.18)$$

where $\sum_{h=1}^n (BE)_h = 0$. Obviously, additional factors can be added to the OBAT plan in (4.16) sequentially and their effects can be estimated as in (4.17). That is, the OBAT 2_{III}^n plan in (4.16) is expansible.

Case (2): 2_{IV}^n plans in a sequence of blocks of size 2.

When two-factor interactions are non-negligible, i.e. model (4.2), the contrasts in (4.17) do not estimate main effects rather they estimate Q_{rs} where

$$Q_{rs} = A_r - \sum_{s \neq r} A_r A_s = A_r - \sum_{i=1}^{r-1} A_i A_r - \sum_{i=r+1}^n A_r A_i \quad (4.19)$$

Therefore, to free any one factor, say A_r , from being aliased with the

(n-1) two-factor interactions $A_r A_s$, $s \neq r$, (n-1) more blocks of size 2 will be added to the OBAT plan in (4.16).

Assuming that factors are ordered such that factor A_1 is to be freed from aliasing first, factor A_2 second, ... , factor A_n last, we consider stages one, two and then the general stage (r).

Stage One: The estimation of A_1 and $A_1 A_s$, $s = 2, 3, \dots, n$.

From the block structure

$$\begin{aligned}
 \text{block 1:} & \quad (1)_{1,1} \quad (0)_{,1} \\
 \text{block } s: & \quad (1)_{s,s} \quad (0)_{,s} \\
 \text{block } (n+s-1): & \quad (11)_{1s,n+s-1} \quad (0)_{,n+s-1} \\
 & \quad s = 2, 3, \dots, n \quad (4.20)
 \end{aligned}$$

we can show (see below) that the estimable functions after $1 + (n-1) + (n-1) = 2n - 1$ blocks in (4.20) are

$$\begin{aligned}
 1. & \quad A_1 \\
 2. & \quad Q_{2s} = A_s - \sum_{i \neq 1, s}^n A_s A_i = A_s - \sum_{i=2}^{s-1} A_i A_s - \sum_{i=s+1}^n A_s A_i \\
 3. & \quad A_1 A_s \quad s = 2, 3, \dots, n. \quad (4.21)
 \end{aligned}$$

Letting $h = n - s + 1$, the unbiased estimates of the functions in (4.21) are

$$\begin{aligned}
 1. & \quad \frac{1}{4} \sum_{s=2}^n [(y(11)_{1s,h} - y(0)_{,h}) - (y(1)_{s,s} - y(0)_{,s})] \\
 & \quad - \frac{1}{4}(n-3)(y(1)_{1,1} - y(0)_{,1})
 \end{aligned}$$

$$2. \quad \frac{1}{4}[(y(11)_{1s,h} - y(0)_{,h}) - (y(1)_{1,1} - y(0)_{,1}) + (y(1)_{s,s} - y(0)_{,s})]$$

$$3. \quad \frac{1}{4}[(y(11)_{1s,h} - y(0)_{,h}) - (y(1)_{1,1} - y(0)_{,1}) - (y(1)_{s,s} - y(0)_{,s})] \quad (4.22)$$

The variances of the estimates in (4.22) are $\frac{1}{8}(n^2 - 4n + 7)\sigma^2$, $\frac{3}{8}\sigma^2$, $\frac{3}{8}\sigma^2$, respectively.

To prove the unbiasedness of the estimates in (4.22), we note that each estimate involves two or more differences of the responses in the same block so that block effects cancel out in their expected values.

Therefore

$$\begin{aligned} 1. \quad E(\hat{A}_1) &= \frac{1}{4} \sum_{s=2}^n [((11)_{1s} - (0)) - ((1)_s - (0))] \\ &\quad - \frac{1}{4}(n-3)((1)_1 - (0)) \\ &= \frac{1}{4} \sum_{s=2}^n ((11)_{1s} - (1)_s) - \frac{1}{4}(n-3)((1)_1 - (0)) \end{aligned}$$

and from (4.11), it follows then that $E\hat{A}_1 = A_1$.

$$2. \quad E(\hat{Q}_{2s}) = \frac{1}{4}[((11)_{1s} - (1)_1) + ((1)_s - (0))] .$$

$$\begin{aligned} 3. \quad E(\hat{A}_1 A_s) &= \frac{1}{4}[((11)_{1s} - (0)) - ((1)_1 - (0)) \\ &\quad - ((1)_s - (0))] \\ &= \frac{1}{4}[(11)_{1s} - (1)_1 - (1)_s + (0)] \end{aligned}$$

and from (4.11), we conclude that $E(\hat{A}_1 A_s) = A_1 A_s$, ($s = 2, 3, \dots, n$).

For $E(\hat{Q}_{2s})$ and from (4.5), we find that

$$(11)_{1s} - (1)_1 = 2(A_s + A_1 A_s - A_s A_{(1,s)})$$

$$(1)_s - (0) = 2(A_s - A_1 A_s - A_s A_{(1,s)})$$

and

$$((11)_{1s} - (1)_1) + ((1)_s - (0)) = 4(A_s - A_s A_{(1-s)}) = 4 \cdot Q_{2s}$$

Hence, unbiasedness of \hat{Q}_{2s} follows.

Stage two: The estimation of A_2 and $A_2 A_s, (s = 3, 4, \dots, n)$.

Referring to (4.21), we note that each factor $A_s, (s = 2, 3, \dots, n)$ is aliased with $(n-2)$ two-factor interactions $A_s A_{s'}, (s' = s+1, s+2, \dots, n)$. Therefore, to free A_2 from aliasing, $(n-2)$ more blocks of size 2 must be added to the $(2n-1)$ blocks in (4.20). Therefore, the block structure for stage two is

$$\begin{array}{lll} \text{block 1:} & (1)_{1,1} & (0)_{,1} \\ \text{block 2:} & (1)_{2,2} & (0)_{,2} \\ \text{block } s: & (1)_{s,s} & (0)_{,s}, (s = 3, 4, \dots, n) \\ \text{block } (n+s-1): & (11)_{1s, n+s-1} & (0)_{, n+s-1}, (s = 3, 4, \dots, n) \\ \text{block } (2n-1+s-2): & (11)_{2s, 2n-1+s-2} & (0)_{, 2n-1+s-2}, (s = 3, 4, \dots, n) \end{array} \quad (4.23)$$

The estimable functions after the completion of the $2n-1+n-2 = 3n-3$ blocks in (4.23) are

1. A_1
2. $Q_{3s} = A_s - \sum_{i=3}^{s-1} A_i A_s - \sum_{i=s+1}^n A_s A_i, (s = 3, 4, \dots, n)$

3. $A_1 A_{s'} , (s' = 2, 3, \dots, n)$
4. A_2
5. $A_2 A_s , (s = 3, 4, \dots, n).$ (4.24)

The estimates of $A_1, A_1 A_s$ are as given in (4.22). Letting $h = n-s+1$ and $k = 2n-1+s-2$, the unbiased estimates of the functions in 2, 4, and 5 in (4.24) are

2. $\frac{1}{2}[(y(11)_{1s,h} - y(0)_{,h}) - (y(1)_{1,1} - y(0)_{,1})]$
 $+ [(y(11)_{2s,k} - y(0)_{,k}) - (y(1)_{2,2} - y(0)_{,2})]$
4. $\frac{1}{2}[(y(11)_{12,n+1} - y(0)_{,n+1}) - (y(1)_{1,1} - y(0)_{,1})]$
 $+ \frac{1}{2} \sum_{s=3}^n [(y(11)_{2s,k} - y(0)_{,k}) - (y(1)_{s,s} - y(0)_{,s})]$
 $- \frac{1}{2}(n-3)(y(1)_{2,2} - y(0)_{,2})$
5. $\frac{1}{2}[(y(11)_{2s,k} - y(0)_{,k}) - (y(1)_{2,2} - y(0)_{,2}) - (y(1)_{s,s} - y(0)_{,s})]$ (4.25)

The unbiasedness of the estimates \hat{A}_2 and $\hat{A}_2 A_s$ in (4.25) can be shown in a similar argument as that used in proving the unbiasedness of \hat{A}_1 and $\hat{A}_1 A_s$ in (4.22). To show the unbiasedness of the estimate (2) in (4.25) we refer to (4.5) and find that

$$\begin{aligned}
 (11)_{rs} - (1)_r &= 2(A_s + A_r A_s - A_s A_{(r,s)}) \\
 &= 2(A_s + A_r A_s - \sum_{i \neq r,s} A_s A_i) \quad (4.26)
 \end{aligned}$$

from which

estimates and setting $r = 2$ in (4.29), we find for the estimate (2) in (4.25), that

$$\begin{aligned} E(\hat{Q}_{3s}) &= \frac{1}{4}\{[(11)_{1s} - (0)] - [(1)_1 - (0)]\} + \{[(11)_{2s} - (0)] \\ &\quad - [(1)_2 - (0)]\} \\ &= \frac{1}{4}\{[(11)_{1s} - (1)_1] + [(11)_{2s} - (1)_2]\} \end{aligned}$$

hence unbiasedness follows.

After stage two and referring to (4.24), we note that main effects A_r , ($r \neq 1, 2$) are each aliased with $(n-3)$ two-factor interactions $A_r A_s$, $r < s$. In order to free factors A_3, A_4, \dots, A_n from being aliased with two-factor interactions according to strategy I, more blocks must be added to the blocks in (4.23). Hence for stage r , $(n-r)$ blocks are needed to free A_r from $A_r A_s$ for $(s = r+1, r+2, \dots, n)$ and $(r = 1, 2, \dots, n)$. Each of these blocks is of the form

$$\text{block } h_{r(s-r)}: (11)_{rs, h_{r(s-r)}}^{(0)}, h_{r(s-r)}$$

where $[h_{r(s-r)} = n + (n-1) + (n-2) + \dots + (n-r-1) + (s-r)]$ and $s = r+1, r+2, \dots, n$. Therefore, we have the following general result for Strategy I OBAT 2^n plans.

Stage r : The estimation of A_r and $A_r A_s$, ($s = r+1, r+2, \dots, n$) and $(r = 1, 2, \dots, n)$.

For stage r , the block structure is

$$\begin{array}{l}
 \left. \begin{array}{l}
 \text{block 1: } (1)_{1,1} \quad (0)_{,1} \\
 \text{block 2: } (1)_{2,2} \quad (0)_{,2} \\
 \vdots \\
 \text{block } r: (1)_{r,r} \quad (0)_{,r} \\
 \text{block } s: (1)_{s,s} \quad (0)_{,s} \quad , (s = r+1, r+2, \dots, n)
 \end{array} \right\} \\
 \text{n blocks}
 \end{array}$$

$$\begin{array}{l}
 \left. \begin{array}{l}
 \text{block } h_{11}: (11)_{12, h_{11}} \quad (0)_{, h_{11}} \quad , h_{11} = n+1 \\
 \text{block } h_{12}: (11)_{13, h_{12}} \quad (0)_{, h_{12}} \quad , h_{12} = n+2 \\
 \vdots \\
 \text{block } h_{1(n-1)}: (11)_{1n, h_{1(n-1)}} \quad (0)_{, h_{1(n-1)}} \quad , h_{1(n-1)} = n+n-1 = 2n-1
 \end{array} \right\} \\
 \text{Stage 1} \\
 \text{(n-1) blocks}
 \end{array}$$

$$\begin{array}{l}
 \left. \begin{array}{l}
 \text{block } h_{21}: (11)_{23, h_{21}} \quad (0)_{, h_{21}} \quad , h_{21} = 2n-1+1 \\
 \text{block } h_{22}: (11)_{24, h_{22}} \quad (0)_{, h_{22}} \quad , h_{22} = 2n-1+2 \\
 \vdots \\
 \text{block } h_{2(n-2)}: (11)_{2n, h_{2(n-2)}} \quad (0)_{, h_{2(n-2)}} \quad , h_{2(n-2)} = 2n-1+n-2 \\
 \vdots
 \end{array} \right\} \\
 \text{Stage 2} \\
 \text{(n-2) blocks}
 \end{array}$$

$$\begin{array}{l}
 \left. \begin{array}{l}
 \text{block } h_{r1}: (11)_{r(r+1), h_{r1}} \quad (0)_{, h_{r1}} \quad , h_{r1} = n + \sum_{i=1}^{r-1} (n-i) + 1 \\
 \text{block } h_{r2}: (11)_{r(r+2), h_{r2}} \quad (0)_{, h_{r2}} \quad , h_{r2} = n + \sum_{i=1}^{r-1} (n-i) + 2 \\
 \vdots \\
 \text{block } h_{r(n-r)}: (11)_{rn, h_{r(n-r)}} \quad (0)_{, h_{r(n-r)}} \quad , h_{r(n-r)} = n + \sum_{i=1}^{r-1} (n-i) + (n-r)
 \end{array} \right\} \\
 \text{Stage } r \\
 \text{(n-r) blocks}
 \end{array}$$

The estimable functions after $n + \sum_{i=1}^r (n-i)$ blocks are

1. A_j , ($j = 1, 2, \dots, r$) and ($r = 1, 2, \dots, n$).
2. $Q_{(r+1)s} = A_s - \sum_{i=r+1}^{s-1} A_i A_s - \sum_{i=s+1}^n A_s A_i$, ($s = r+1, r+2, \dots, n$).
3. $A_j A_s$, ($j = 1, 2, \dots, r$) and ($s = j+1, j+2, \dots, n$). (4.31)

The unbiased estimates of the functions in (4.31) are

1. $\hat{A}_j = \frac{1}{4} \sum_{i=1}^{j-1} [(y(11)_{ij, h_{i(j-1)}} - y(0)_{, h_{i(j-1)}}) - (y(1)_{i,i} - y(0)_{, i})]$
 $+ \frac{1}{4} \sum_{s=j+1}^n [(y(11)_{js, h_{j(s-j)}} - y(0)_{, h_{j(s-j)}}) - (y(1)_{s,s} - y(0)_{, s})]$
 $- \frac{1}{4}(n-3)(y(1)_{j,j} - y(0)_{, j})$, ($j = 1, 2, \dots, r$).
2. $\hat{Q}_{rs} = \frac{1}{4} \sum_{i=1}^r [(y(11)_{is, h_{i(s-i)}} - y(0)_{, h_{i(s-i)}}) - (y(1)_{i,i} - y(0)_{, i})]$
 $- \frac{1}{4}(r-2)(y(1)_{s,s} - y(0)_{, s})$, ($s = r+1, r+2, \dots, n$).
3. $\hat{A}_j A_s = \frac{1}{4} [(y(11)_{js, h_{j(s-j)}} - y(0)_{, h_{j(s-j)}})$
 $- (y(1)_{j,j} - y(0)_{, j}) - (y(1)_{s,s} - y(0)_{, s})]$
 $h_{j(s-j)} = n + \sum_{\ell=1}^j (n-\ell)$, ($j = 1, 2, \dots, r$) and ($s = j+1, j+2, \dots, n$). (4.32)

To establish the unbiasedness of the estimates in (4.32) we recall that block effects cancel out in the expected values of the estimates hence

$$\begin{aligned}
1. \quad E\hat{A}_j &= \frac{1}{2} \sum_{i=1}^{j-1} [((11)_{ij} - (0)) - ((1)_i - (0))] \\
&\quad + \frac{1}{2} \sum_{s=j+1}^n [((11)_{js} - (0)) - ((1)_s - (0))] \\
&\quad - \frac{1}{2}(n-3)((1)_j - (0)) \\
&= \frac{1}{2} \sum_{i=1}^{j-1} ((11)_{ij} - (1)_i) + \frac{1}{2} \sum_{s=j+1}^n ((11)_{js} - (1)_s) \\
&\quad - \frac{1}{2}(n-3)((1)_j - (0)) \\
&= \frac{1}{2} \sum_{s \neq j} ((11)_{js} - (1)_s) - \frac{1}{2}(n-3)((1)_j - (0))
\end{aligned}$$

referring to (4.12) it follows that $E\hat{A}_j = A_j$.

$$\begin{aligned}
2. \quad E(\hat{Q}_{rs}) &= \frac{1}{2} \sum_{i=1}^r [((11)_{is} - (0)) - ((1)_i - (0))] - \frac{1}{2}(r-2)((1)_s - (0)) \\
&= \frac{1}{2} \sum_{i=1}^r ((11)_{is} - (1)_i) - \frac{1}{2}(r-2)((1)_s - (0))
\end{aligned}$$

referring to (4.29) it follows that $E\hat{Q}_{rs} = Q_{rs}$.

3. Referring to 3 in (4.22) unbiasedness of $\hat{A}_j A_s$ follows.

4.4 OBAT 2^n Plans for Strategy II

We consider here the three cases mentioned in Section 7.1, namely case (1): All n 2-level factors A_1, A_2, \dots, A_n are considered as one set of potentially interacting factors.

The 2^n fraction F_n in (4.1) can then be used to estimate n main

effects and $\frac{n(n-1)}{2}$ two-factor interactions. Parameter estimates are given in (4.12).

Case (2): The n 2-level factors A_1, A_2, \dots, A_n are partitioned into g disjoint sets where set h contains n_h factors and $\sum_{h=1}^g n_h = n$. Factors in each set may interact with each other but not with factors from another set.

The experiment will then be carried out in a sequence of g blocks such that the h^{th} block, ($h = 1, 2, \dots, g$) is the 2^{n_h} fraction F_{n_h} of (4.1) with $N_h = 1 + n_h + \frac{n_h(n_h-1)}{2}$ treatment combinations. Each treatment combination among the two levels of the n_h factors will be written as $(00\dots 0a_r 00\dots 0a_s 00\dots 0)_h$, or just as

$$(a_r a_s)_{rs,h} \tag{4.33}$$

where $(a_r, a_s = 0 \text{ or } 1), (r < s), (r, s = 1, 2, \dots, n_h)$ and $(h = 1, 2, \dots, g)$.

The sequence of g blocks in (4.33) can be written explicitly as

$$\begin{array}{llll} \text{block 1:} & (0)_{,1} & (1)_{r,1} & (11)_{rs,1} \\ \text{block 2:} & (0)_{,2} & (1)_{r,2} & (11)_{rs,2} \\ & \vdots & & \\ \text{block } g: & (0)_{,g} & (1)_{r,g} & (11)_{rs,g} \end{array} \tag{4.34}$$

We now have the following linear model for Strategy II OBAT 2^n plans.

$$y(a_r a_s)_{rs,h} = (a_r a_s)_{rs,h} + \epsilon_{rs,h} \tag{4.35}$$

Denoting the $(N_h - 1)$ factorial effects of the h^{th} 2^{n_h} fraction in

(4.33) by $(A_r)_h$ and $(A_r A_s)_h$ and referring to (4.12), their unbiased estimates are

$$(\hat{A}_r)_h = \frac{1}{2} \sum_{s \neq r} (y(11)_{rs,h} - y(1)_{s,h}) - \frac{1}{2}(n_h - 1)(y(1)_{r,h} - y(0)_{,h})$$

$$(\hat{A}_{rs})_h = \frac{1}{2} [(y(11)_{rs,h} - y(1)_{s,h}) - (y(1)_{r,h} - y(0)_{,h})] \quad (4.36)$$

where $(r = 1, 2, \dots, n_h)$, $(s = r+1, r+2, \dots, n)$ and $(h = 1, 2, \dots, g)$.

Case (3): The n 2-level factors A_1, A_2, \dots, A_n are partitioned into $g_1 + g_2 + 1$ disjoint sets where

a. factors in each set of the g_1 sets interact with each other but not with factors from another set. That is, these g_1 sets are the g sets in case (2).

b. in each one of the g_2 sets, some but not all factors interact with all factors in that particular set. However, we will assume that only two factors in each one of the g_2 sets interact with each other as well as with the remaining factors in the set. These two factors are considered to be the first and second factors in each one of the g_2 sets.

c. factors in the last set do not interact with each other.

We let n_{ij} represent the number of factors in the j^{th} set of the i^{th} group where $(j = 1, 2, 3, \dots, g_i)$ and $(i = 1, 2, 3)$ such that

$\sum_{j=1}^{g_i} n_{ij} = n_i$. is the total number of factors in the i^{th} group and

$\sum_{i=1}^3 \sum_{j=1}^{g_i} n_{ij} = n$. This partitioning of the n factors A_1, A_2, \dots, A_n can be displayed as

Group 1

	1	2	-----	g_1
	$(A_1)_1$	$(A_1)_2$		$(A_1)_{g_1}$
	$(A_2)_1$	$(A_2)_2$	---	$(A_2)_{g_1}$
	\vdots	\vdots		\vdots
	$(A_{n_{11}})_1$	$(A_{n_{12}})_2$		$(A_{n_1 g_1})_{g_1}$
Total num. of factors	n_{11}	n_{12}	---	$n_1 g_1$

Group 2

Group 3

	1	2	-----	g_2	1
	$(A_1)_{g_1+1}$	$(A_1)_{g_1+2}$		$(A_1)_{g_1+g_2}$	$(A_1)_{g_1+g_2+1}$
	$(A_2)_{g_1+1}$	$(A_2)_{g_1+2}$	---	$(A_2)_{g_1+g_2}$	$(A_2)_{g_1+g_2+1}$
	\vdots	\vdots		\vdots	\vdots
	$(A_{n_{21}})_{g_1+1}$	$(A_{n_{22}})_{g_1+2}$		$(A_{n_2 g_2})_{g_1+g_2}$	$(A_{n_{31}})_{g_1+g_2+1}$
Tot. num. of factors	n_{21}	n_{22}	---	$n_2 g_2$	n_{31}

The experiment will then be carried out in a sequence of $g_1 + g_2 + 1$ blocks such that

a. each one of the g_1 blocks in the $2_{v}^{n_{1j}}$ fraction $F_{n_{1j}}$ in (4.1) with $N_{n_{1j}} = 1 + n_{1j} + \frac{1}{2} n_{1j} (n_{1j} - 1)$ treatment combinations. Each treatment combination among the n_{1j} factor levels is of the form

$$(\dot{a}_r \dot{a}_s)_{rs,j} \quad (4.37)$$

where $(\dot{a}_r, \dot{a}_s = 0, 1), (r < s), (r, s = 1, 2, \dots, n_{1j})$ and $(j = 1, 2, \dots, g_1)$.

The contents of the g_1 blocks are similar to those in (4.34).

b. with the assumption that each set of the g_2 sets contains two interacting factors, each one of the g_2 blocks, say block j , is a $2_V^{n_{2j}}$ fraction containing $N_{2j} = 1 + n_{2j} + (n_{2j} - 1) + (n_{2j} - 2) = 3n_{2j} - 2$ treatment combinations. The N_{2j} treatment combinations among the two levels of the n_{2j} factors can be written explicitly as

$$\begin{aligned} & (0)_{,g_1+j} \\ & (1)_{r,g_1+j}, \quad r = 1, 2, \dots, n_{2j} \\ & (11)_{1s,g_1+j}, \quad s = 2, 3, \dots, n_{2j} \\ & (11)_{2s,g_1+j}, \quad s = 3, 4, \dots, n_{2j} \end{aligned} \quad (4.38)$$

where $j = 1, 2, \dots, g_2$.

c. the last set of non-interacting factors is a $2_{III}^{n_{31}}$ fraction containing $N_{31} = 1 + n_{31}$ treatment combinations. The N_{31} treatment combinations among the two levels of the n_{31} factors can be written explicitly as

$$\begin{aligned} & (0)_{,g_1+g_2+1} \\ & (1)_{r,g_1+g_2+1}, \quad r = 1, 2, \dots, n_{31} \end{aligned} \quad (4.39)$$

Referring to model (4.35) and denoting the factorial effects of the fractions $2_V^{n_{1j}}$, $2_V^{n_{2j}}$ and $2_{III}^{n_{31}}$, respectively, by

a. $(A_r)_j$ and $(A_r A_s)_j$

where $(r = 1, 2, \dots, (n_{1j}-1)), (s = r+1, r+2, \dots, n_{1j})$ and $(j = 1, 2, \dots, g_1)$.

b. $(A_r)_{g_1+j}$, $(A_1 A_s)_{g_1+j}$ and $(A_2 A_{s'})_{g_1+j}$

where $(r = 1, 2, \dots, (n_{2j}-1)), (s = 2, 3, \dots, n_{2j}), (s' = 3, 4, \dots, n_{2j})$ and $(j = 1, 2, \dots, g_2)$.

c. $(A_r)_{g_1+g_2+1}$

where $r = 1, 2, \dots, n_{31}$.

(4.40)

The unbiased estimates of the parameters in (4.40) are

a. $(\hat{A}_r)_j = \frac{1}{4} \sum_{s \neq r} (y(11)_{rs,j} - y(1)_{s,j}) - \frac{1}{4}(n_{1j}-3)(y(1)_{r,j} - y(0)_{,j})$

$$(\hat{A}_r A_s)_j = \frac{1}{4}[y(11)_{rs,j} - y(1)_{r,j} - y(1)_{s,j} + y(0)_{,j}] \quad (4.41)$$

with $\text{var}(\hat{A}_r)_j = \frac{1}{8} (n_{1j}^2 - 5n_{1j} + 8)\sigma^2$ and $\text{var}(\hat{A}_r A_s)_j = \frac{1}{4}\sigma^2$.

b. Letting $i = g_1 + j$, we find that

$$(\hat{A}_1)_i = \frac{1}{4} \sum_{s \neq 1}^{n_{2j}} (y(11)_{1s,i} - y(1)_{s,i}) - \frac{1}{4}(n_{2j} - 3)(y(1)_{1,i} - y(0)_{,i})$$

$$(\hat{A}_2)_i = \frac{1}{4}[(y(11)_{12,i} - y(1)_{1,i}) + \sum_{s=3}^{n_{2j}} (y(11)_{2s,i} - y(1)_{s,i})]$$

$$- \frac{1}{4}(n_{2j} - 3)(y(1)_{2,i} - y(0)_{,i})$$

$$(\hat{A}_r A_s)_i = \frac{1}{4}[(y(11)_{rs,i} - y(1)_{r,i} - y(1)_{s,i} + y(0)_{,i})]$$

($r = 1, 2$), ($s = r+1, r+2, \dots, n_{2j}$) and referring to (2) in (4.32),

$$(\hat{A}_s)_i = \frac{1}{2}[(y(11)_{1s,i} - y(1)_{1,i}) + (y(11)_{2s,i} - y(1)_{2,i})] \quad (4.42)$$

$$s = 3, 4, \dots, n_{2j}$$

with respective variances

$$\frac{1}{8}(n_{2j}^2 - 5n_{2j} + 8)\sigma^2, \frac{1}{8}(n_{2j}^2 - 5n_{2j} + 8)\sigma^2, \frac{1}{2}\sigma^2, \frac{1}{2}\sigma^2.$$

$$c. \quad (\hat{A}_r)_{g_1+g_2+1} = \frac{1}{2}[y(1)_{r,g_1+g_2+1} - y(0)_{g_1+g_2+1}] \quad (4.43)$$

with $\text{var}(\hat{A}_r)_{g_1+g_2+1} = \frac{1}{2}\sigma^2$.

4.5 An Example: OBAT Plans for the 2^6 Experiment

Referring to the $1 + n + \frac{n(n-1)}{2}$ treatment combinations of the 2_V^n fraction in (4.1), we have for the 2_V^6 experiment the following 22 runs

000000	100000	010000	001000	000100	000010	000001
		110000	101000	100100	100010	100001
			011000	010100	010010	010001
				001100	001010	001001
					000110	000101
						000011

which can be written using the $(a_r a_s)_{rs}$ notation as

$$\begin{array}{ccccccc}
(0) & (1)_1 & (1)_2 & (1)_3 & (1)_4 & (1)_5 & (1)_6 \\
& & (11)_{12} & (11)_{13} & (11)_{14} & (11)_{15} & (11)_{16} \\
& & & (11)_{23} & (11)_{24} & (11)_{25} & (11)_{26} \\
& & & & (11)_{34} & (11)_{35} & (11)_{36} \\
& & & & & (11)_{45} & (11)_{46} \\
& & & & & & (11)_{56}
\end{array} \quad (4.44)$$

Also referring to the linear model in (4.2), we have for the 2^6_V experiment the following effects

$$\begin{array}{ccccccc}
\mu & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\
& & A_1A_2 & A_1A_3 & A_1A_4 & A_1A_5 & A_1A_6 \\
& & & A_2A_3 & A_2A_4 & A_2A_5 & A_2A_6 \\
& & & & A_3A_4 & A_3A_5 & A_3A_6 \\
& & & & & A_4A_5 & A_4A_6 \\
& & & & & & A_5A_6
\end{array} \quad (4.45)$$

a. Strategy I OBAT 2^6 plans:

Assuming factors are ordered for their importance as A_1, A_2, \dots, A_6 , we consider

Case (1): The sequential estimation of all 6 main effects in 6 blocks of size 2 when all interactions are negligible, i.e. model (4.15) where $b = 6$.

The 6 blocks are

block order	block content	
1	(0) _{,1} (1) _{1,1}	
2	(0) _{,2} (1) _{2,2}	
3	(0) _{,3} (1) _{3,3}	
4	(0) _{,4} (1) _{4,4}	
5	(0) _{,5} (1) _{5,5}	
6	(0) _{,6} (1) _{6,6}	(4.46)

From the linear model (4.15) we have

$$y(a_r)_{r,r} = \mu + A_r x_r - \sum_{i \neq r} A_i + (BE)_{r,r} + \varepsilon_{r,r}$$

where $a_r = 0, 1$, $\sum_{r=1}^6 (BE)_{r,r} = 0$, $E(\varepsilon_{r,r}) = 0$ and $\text{var}(\varepsilon_{r,r}) = \sigma^2$. We find that each main effect A_r is estimable from the 2 runs in block r and its unbiased estimate is

$$\hat{A}_r = \frac{1}{2}(y(1)_{r,r} - y(0)_{r,r}), \quad (r = 1, 2, \dots, 6) \quad (4.47)$$

with $\text{var} \hat{A}_r = \frac{1}{2}\sigma^2$.

Case (2): The sequential estimation of main effects and two-factor interactions in blocks of size 2 such that A_r and $A_r A_s$ for $(r = 1, 2, \dots, 6)$ and $(s = r+1, r+2, \dots, 6)$ are estimable before $A_{r'}$ and $A_{r'} A_{s'}$ for $(r' > r)$ and $(s' = r' + 1, r' + 2, \dots, 6)$ from the model (4.14).

We then have the following stages

Stage One: The estimation of A_1 and all $A_1 A_s$, $(s = 2, 3, \dots, 6)$.

From any one block in (4.46) and using the model (4.14) we find that

$$E\frac{1}{2}(y(1)_{s,s} - y(0)_{s,s}) = \frac{1}{2}((1)_{s,s} - (0)_{s,s})$$

$$= A_s - \sum_{i=1}^{s-1} A_i A_s - \sum_{i=s+1}^n A_s A_i$$

and referring to (4.19)

$$= Q_{1s}, \quad (s = 1, 2, \dots, 6) \quad (4.48)$$

That is, the alias structure in (4.48) is

$$\begin{aligned} A_1 - A_1 A_2 - A_1 A_3 - A_1 A_4 - A_1 A_5 - A_1 A_6 &= Q_{11} \\ A_2 - A_1 A_2 - A_2 A_3 - A_2 A_4 - A_2 A_5 - A_2 A_6 &= Q_{12} \\ A_3 - A_1 A_3 - A_2 A_3 - A_3 A_4 - A_3 A_5 - A_3 A_6 &= Q_{13} \\ A_4 - A_1 A_4 - A_2 A_4 - A_3 A_4 - A_4 A_5 - A_4 A_6 &= Q_{14} \\ A_5 - A_1 A_5 - A_2 A_5 - A_3 A_5 - A_4 A_5 - A_5 A_6 &= Q_{15} \\ A_6 - A_1 A_6 - A_2 A_6 - A_3 A_6 - A_4 A_6 - A_5 A_6 &= Q_{16} \end{aligned} \quad (4.49)$$

Since A_1 in (4.49) is aliased with 5 two-factor interactions, 5 more blocks are needed to free A_1 from aliasing. Therefore, a total of 11 blocks are needed for stage one where

<u>block order</u>	<u>block content</u>	
1	(1) _{1,1}	(0) ₁
2	(1) _{2,2}	(0) ₂
⋮	⋮	⋮
6	(1) _{6,6}	(0) _{6,6}
7	(0) ₇	(11) _{12,7}
8	(0) ₈	(11) _{13,8}
⋮	⋮	⋮
⋮	⋮	⋮
11	(0) ₁₁	(11) _{16,11}

$$(4.50)$$

Referring to the model in (4.14) where $b = 11$, the estimable functions after the completion of the 11 blocks are

1. A_1
2. $A_2 - A_2A_3 - A_2A_4 - A_2A_5 - A_2A_6 = Q_{22}$
 $A_3 - A_2A_3 - A_3A_4 - A_3A_5 - A_3A_6 = Q_{23}$
 $A_4 - A_2A_4 - A_3A_4 - A_4A_5 - A_4A_6 = Q_{24}$
 $A_5 - A_2A_5 - A_3A_5 - A_4A_5 - A_5A_6 = Q_{25}$
 $A_6 - A_2A_6 - A_3A_6 - A_4A_6 - A_5A_6 = Q_{26}$
3. $A_1A_2 ; A_1A_3 ; A_1A_4 ; A_1A_5 ; A_1A_6 .$ (4.51)

The unbiased estimates of the functions in (4.51) are

1. $\hat{A}_1 = \frac{1}{4} \sum_{s=2}^6 [(y(11)_{1s,s+5} - y(0)_{,s+5}) - (y(1)_{s,s} - y(0)_{,s})]$
 $- \frac{3}{4}(y(1)_{1,1} - y(0)_{,1})$
2. $\hat{Q}_{2s} = \frac{1}{4}[y(11)_{1s,s+5} - y(0)_{,s+5}) - (y(1)_{1,1} - y(0)_{,1})]$
 $+ (y(1)_{s,s} - y(0)_{,s})] , (s = 2, 3, \dots, 6)$
3. $\hat{A}_1A_s = \frac{1}{4}[(y(11)_{1s,s+5} - y(0)_{,s+5}) - (y(1)_{1,1} - y(0)_{,1})]$
 $- (y(1)_{s,s} - y(0)_{,s})] \quad (s = 2, 3, \dots, 6) .$ (4.52)

Stage Two: The estimation of A_2 and A_2A_s , ($s = 3, 4, 5, 6$).

In (4.51), A_2 is aliased with four two-factor interactions hence four more blocks of size 2 must be added to the blocks in (4.50) to free A_2 from aliasing. The block structure for stage two is

$$\begin{aligned}
\text{block } r: & \quad (1)_{r,r} & (0)_{,r}, & \quad (r = 1, 2, \dots, 6) \\
\text{block}(s+5): & \quad (11)_{1s,s+5} & (0)_{,s+5}, & \quad (s = 2, 3, 4, 5, 6) \\
\text{block } 12: & \quad (11)_{23,12} & (0)_{,12} & \\
\text{block } 13: & \quad (11)_{24,13} & (0)_{,13} & \\
& \quad \vdots & & \\
\text{block } 15: & \quad (11)_{26,15} & (0)_{,15} & \quad (4.53)
\end{aligned}$$

The estimable functions after 15 blocks are

$$\begin{aligned}
1. & \quad A_1 ; A_2 \\
2. & \quad A_3 - A_3A_4 - A_3A_5 - A_3A_6 = Q_{33} \\
& \quad A_4 - A_3A_4 - A_4A_5 - A_4A_6 = Q_{34} \\
& \quad A_5 - A_3A_5 - A_4A_5 - A_5A_6 = Q_{35} \\
& \quad A_6 - A_3A_6 - A_4A_6 - A_5A_6 = Q_{36} \\
3. & \quad A_1A_2 ; A_1A_3 ; A_1A_4 ; A_1A_5 ; A_1A_6 \\
4. & \quad A_2A_3 ; A_2A_4 ; A_2A_5 ; A_2A_6 \quad (4.54)
\end{aligned}$$

The unbiased estimates of the functions involving A_1 are given in (4.52).

The unbiased estimate of the remaining functions in (4.54) are

$$\begin{aligned}
1. \quad \hat{A}_2 &= \frac{1}{4}[(y(11)_{12,7} - y(0)_{,7}) - (y(1)_{1,1} - y(0)_{,1})] \\
& \quad + \frac{1}{2} \sum_{s=3}^6 (y(11)_{2s,s+9} - y(0)_{,s+9}) - (y(1)_{s,s} - y(0)_{,s})] \\
& \quad - \frac{3}{4}(y(1)_{2,2} - y(0)_{,2})
\end{aligned}$$

$$\begin{aligned}
2. \quad \hat{Q}_{3s} &= \frac{1}{4}[(y(11)_{1s,s+5} - y(0)_{,s+5}) - (y(1)_{1,1} - y(0)_{,1})] \\
&\quad + \frac{1}{4}[(y(11)_{2s,s+9} - y(0)_{,s+9}) - (y(1)_{2,2} - y(0)_{,2})] \\
4. \quad \hat{A}_2 A_s &= \frac{1}{4}[(y(11)_{2s,s+9} - y(0)_{,s+9}) - (y(1)_{2,2} - y(0)_{,2}) \\
&\quad - (y(1)_{s,s} - y(0)_{,s})], \quad (s = 3, 4, 5, 6) \quad (4.55)
\end{aligned}$$

Stage Three: The estimation of A_3 and $A_3 A_s$, ($s = 4, 5, 6$)

In (4.54), A_3 is aliased with three two-factor interactions hence three more blocks of size 2 must be added to the blocks in (4.53) to free A_3 from aliasing. The block structure for stage three consists of the blocks 1 - 15 in (4.53) and the blocks

$$\begin{aligned}
\text{block 16:} & \quad (11)_{34,16} \quad (0)_{,16} \\
\text{block 17:} & \quad (11)_{35,17} \quad (0)_{,17} \\
\text{block 18:} & \quad (11)_{36,18} \quad (0)_{,18} \quad (4.56)
\end{aligned}$$

The estimable functions after 18 blocks are

$$\begin{aligned}
1. \quad & A_1 ; A_2 ; A_3 \\
2. \quad & A_4 - A_4 A_5 - A_4 A_6 = Q_{44} \\
& A_5 - A_4 A_5 - A_5 A_6 = Q_{45} \\
& A_6 - A_4 A_6 - A_5 A_6 = Q_{46} \\
3. \quad & A_1 A_2 ; A_1 A_3 ; A_1 A_4 ; A_1 A_5 ; A_1 A_6 \\
4. \quad & A_2 A_3 ; A_2 A_4 ; A_2 A_5 ; A_2 A_6 \\
5. \quad & A_3 A_4 ; A_3 A_5 ; A_3 A_6 \quad (4.57)
\end{aligned}$$

The unbiased estimates of the functions involving A_1 and A_2 are as before. The unbiased estimates of the remaining functions in (4.57) are

$$\begin{aligned}
 1. \quad \hat{A}_3 &= \frac{1}{4}[(y(11)_{13,8} - y(0)_{,8}) - (y(1)_{1,1} - y(0)_{,1})] \\
 &\quad + \frac{1}{4}[(y(11)_{23,10} - y(0)_{,10}) - (y(2)_{2,2} - y(0)_{,2})] \\
 &\quad + \frac{1}{4} \sum_{s=4}^6 [(y(11)_{3s,s+12} - y(0)_{,s+12}) - (y(1)_{s,s} - y(0)_{,s})] \\
 &\quad - \frac{3}{4} (y(1)_{3,3} - y(0)_{,3}) \\
 2. \quad \hat{Q}_{4s} &= \frac{1}{4}[(y(11)_{1s,s+5} - y(0)_{,s+5}) - (y(1)_{1,1} - y(0)_{,1})] \\
 &\quad + \frac{1}{4}[(y(11)_{2s,s+9} - y(0)_{,s+9}) - (y(1)_{2,2} - y(0)_{,2})] \\
 &\quad + \frac{1}{4}[(y(11)_{3s,s+12} - y(0)_{,s+12}) - (y(1)_{3,3} - y(0)_{,3})] \\
 &\quad - \frac{1}{4}(y(1)_{s,s} - y(0)_{,s}), \quad (s = 4, 5, 6) \\
 \hat{A}_3 A_s &= \frac{1}{4}[(y(11)_{3s,s+12} - y(0)_{,s+12}) - (y(1)_{3,3} - y(0)_{,3}) \\
 &\quad - (y(1)_{s,s} - y(0)_{,s})], \quad (s = 4, 5, 6) \tag{4.58}
 \end{aligned}$$

Stage Four: The estimation of A_4 and $A_4 A_s$, ($s = 5, 6$).

In (4.57), A_4 is aliased with two two-factor interactions hence two more blocks of size 2 must be added to the blocks in (4.56) to free A_4 from aliasing. The block structure for stage four consists of the blocks 1-18 in (4.56) and the blocks

$$\begin{aligned}
 \text{block 19: } & (11)_{45,19} \quad (0)_{,19} \\
 \text{block 20: } & (11)_{46,20} \quad (0)_{,20} \tag{4.59}
 \end{aligned}$$

The estimable functions after 20 blocks are

1. $A_1 ; A_2 ; A_3 ; A_4$
2. $A_5 - A_5 A_6 = Q_{55}$
 $A_6 - A_5 A_6 = A_{56}$
3. $A_1 A_2 ; A_1 A_3 ; A_1 A_4 ; A_1 A_5 ; A_1 A_6$
4. $A_2 A_3 ; A_2 A_4 ; A_2 A_5 ; A_2 A_6$
5. $A_3 A_4 ; A_3 A_5 ; A_3 A_6$
6. $A_4 A_5 ; A_4 A_6$ (4.60)

The unbiased estimates of the functions involving A_1 , A_2 and A_3 are as before. The unbiased estimates of the remaining functions in (4.60) are

1. $\hat{A}_4 = \frac{1}{4}[(y(11)_{14,9} - y(0)_{,9}) - (y(1)_{1,1} - y(0)_{,1})]$
 $+ \frac{1}{4}[(y(11)_{24,13} - y(0)_{,13}) - (y(1)_{2,2} - y(0)_{,2})]$
 $+ \frac{1}{4}[(y(11)_{34,16} - y(0)_{,16}) - (y(1)_{3,3} - y(0)_{,3})]$
 $+ \frac{1}{4} \sum_{s=5}^6 [(y(11)_{4s,s+14} - y(0)_{,s+14}) - (y(1)_{s,s} - y(0)_{,s})]$
 $- \frac{3}{4} (y(1)_{4,4} - y(0)_{,4})$
2. $\hat{Q}_{5s} = \frac{1}{4}[(y(11)_{1s,s+5} - y(0)_{,s+5}) - (y(1)_{1,1} - y(0)_{,1})]$
 $+ \frac{1}{4}[(y(11)_{2s,s+9} - y(0)_{,s+9}) - (y(1)_{2,2} - y(0)_{,2})]$

$$\begin{aligned}
& + \frac{1}{4}[(y(11)_{3s,s+12} - y(0)_{,s+12}) - (y(1)_{3,3} - y(0)_{,3})] \\
& + \frac{1}{4}[(y(11)_{4s,s+14} - y(0)_{,s+14}) - (y(1)_{4,4} - y(0)_{,4})] \\
& - \frac{2}{4}(y(1)_{s,s} - y(0)_{,s}), \quad (s = 5, 6)
\end{aligned}$$

$$\begin{aligned}
\hat{A}_{4,s} = & \frac{1}{4}[(y(11)_{4s,s+14} - y(0)_{,s+14}) - (y(1)_{4,4} - y(0)_{,4}) \\
& - (y(1)_{s,s} - y(0)_{,s})], \quad (s = 5, 6)
\end{aligned} \tag{4.61}$$

Stage Five: The estimation of A_5 , A_5A_6 and A_6 .

In (4.60), each of A_5 and A_6 is aliased with A_5A_6 hence one more block of size 2 must be added to the blocks in (4.59). The block structure for stage five consists of the blocks 1-20 in (4.59) and the block

$$\text{block 21: } (11)_{56,21} \quad (0)_{,21} \tag{4.62}$$

The estimable functions after 21 blocks are all the parameters in (4.45).

The unbiased estimates of the functions involving A_1 , A_2 , A_3 and A_4 are as before. The unbiased estimates of A_5 , A_5A_6 and A_6 are

$$\begin{aligned}
\hat{A}_5 = & \frac{1}{4}[(y(11)_{15,10} - y(0)_{,10}) - (y(1)_{1,1} - y(0)_{,1})] \\
& + \frac{1}{4}[(y(11)_{25,14} - y(0)_{,14}) - (y(1)_{2,2} - y(0)_{,2})] \\
& + \frac{1}{4}[(y(11)_{35,17} - y(0)_{,17}) - (y(1)_{3,3} - y(0)_{,3})] \\
& + \frac{1}{4}[(y(11)_{45,19} - y(0)_{,19}) - (y(1)_{4,4} - y(0)_{,4})] \\
& + \frac{1}{4}[(y(11)_{56,21} - y(0)_{,21}) - (y(1)_{6,6} - y(0)_{,6})]
\end{aligned}$$

$$\begin{aligned}
& - \frac{3}{4}(y(1)_{5,5} - y(0)_{,5}) \\
\hat{A}_5 A_6 &= \frac{1}{4}[(y(11)_{56,21} - y(0)_{,21}) - (y(1)_{5,5} - y(0)_{,5}) \\
& \quad - (y(1)_{6,6} - y(0)_{,6})] \\
\hat{A}_6 &= \frac{1}{4}[(y(11)_{16,11} - y(0)_{,11}) - (y(1)_{1,1} - y(0)_{,1})] \\
& \quad + \frac{1}{4}[(y(11)_{26,15} - y(0)_{,15}) - (y(1)_{2,2} - y(0)_{,2})] \\
& \quad + \frac{1}{4}[(y(11)_{36,18} - y(0)_{,18}) - (y(1)_{3,3} - y(0)_{,3})] \\
& \quad + \frac{1}{4}[(y(11)_{46,20} - y(0)_{,20}) - (y(1)_{4,4} - y(0)_{,4})] \\
& \quad + \frac{1}{4}[(y(11)_{56,21} - y(0)_{,21}) - (y(1)_{5,5} - y(0)_{,5})] \\
& \quad - \frac{3}{4}(y(1)_{6,6} - y(0)_{,6})
\end{aligned}$$

b. Strategy II OBAT 2^6 plans

Case (1): All six factors A_1, A_2, \dots, A_6 interact with each other hence the 21-run 2^6_v fraction in (4.44) is needed to assess the significance of the 21 parameters in (4.45). The unbiased estimates of these parameters are given in (4.12).

Case (2): Assuming that the six factors A_1, A_2, \dots, A_6 are partitioned into two non-interacting (disjoint) sets: one containing four factors and the other two factors. That is,

set 1 includes 4 interacting factors: $(A_1)_1, (A_2)_1, (A_3)_1$

and $(A_4)_1$

set 2 includes 2 interacting factors: $(A_1)_2$ and $(A_2)_2$

The six factor effects can then be investigated in a sequence of two blocks where

a. block one is a 2^4_v fraction whose runs are

$$\begin{array}{ccccc} (0000)_1 & (1000)_1 & (0100)_1 & (0010)_1 & (0001)_1 \\ (1100)_1 & (1010)_1 & (1001)_1 & (0110)_1 & (0011)_1 \end{array}$$

written as

$$\begin{array}{cccccc} (0)_{,1} & (1)_{1,1} & (1)_{2,1} & (1)_{3,1} & (1)_{4,1} & \\ (11)_{12,1} & (11)_{13,1} & (11)_{14,1} & (11)_{23,1} & (11)_{24,1} & (11)_{34,1} \end{array} \quad (4.63)$$

and whose parameters are

$$\begin{array}{ccccccc} (\mu)_1 & (A_1)_1 & (A_2)_1 & (A_3)_1 & (A_4)_1 & & \\ & (A_1A_2)_1 & (A_1A_3)_1 & (A_1A_4)_1 & (A_2A_3)_1 & (A_2A_4)_1 & (A_3A_4)_1 \end{array} \quad (4.64)$$

b. block two is a 2^2_v fraction whose runs are

$$(00)_2 \quad (10)_2 \quad (01)_2 \quad (11)_2$$

written as

$$(0)_{,2} \quad (1)_{1,2} \quad (1)_{2,2} \quad (11)_{12,2} \quad (4.65)$$

and whose parameters are

$$(\mu)_2 \quad (A_1)_2 \quad (A_2)_2 \quad (A_1A_2)_2 \quad (4.66)$$

After the completion of the first block, the parameters in (4.64) become estimable. Their unbiased estimates are given in (4.36) where $(h = 1)$, $(r = 1, 2, 3, 4)$ and $(s = r+1, r+2, \dots, 4)$.

After the completion of the second block, the parameters in (4.66) become estimable. Their unbiased estimates are also given in (4.36) where $(h = 2)$, $(r = 1, 2)$ and $(s = r+1)$.

Case (3); Assuming that the six factors A_1, A_2, \dots, A_6 are partitioned into two disjoint sets:

1. set 1 includes four factors $(A_1)_1, (A_2)_1, (A_3)_1,$ and $(A_4)_1$ such that factors $(A_1)_1$ and $(A_2)_1$ interact and each interacts with factors $(A_3)_1$ and $(A_4)_1$. Factors $(A_3)_1$ and $(A_4)_1$ do not interact with each other.
2. set 2 includes two non-interacting factors: $(A_1)_2$ and $(A_2)_2$.

The six factor effects can then be studied in a sequence of two blocks where

a. block one is a 2^4_v whose runs are

$(0000)_1$	$(1000)_1$	$(0100)_1$	$(0010)_1$	$(0001)_1$
$(1100)_1$	$(1010)_1$	$(1001)_1$	$(0110)_1$	$(0101)_1$

written as

$(0)_{,1}$	$(1)_{1,1}$	$(1)_{2,1}$	$(1)_{3,1}$	$(1)_{4,1}$	
$(11)_{12,1}$	$(11)_{13,1}$	$(11)_{14,1}$	$(11)_{23,1}$	$(11)_{24,1}$	(4.67)

and whose parameters are

$$\begin{aligned}
(\mu)_1 & \quad (A_1)_1 & (A_2)_1 & (A_3)_1 & (A_4)_1 \\
& (A_1A_2)_1 & (A_1A_3)_1 & (A_1A_4)_1 & (A_2A_3)_1 & (A_2A_4)_1 & (4.68)
\end{aligned}$$

b. block two is a 2_{III}^2 fraction whose runs are

$$(00)_2 \quad (10)_2 \quad (01)_2$$

written as

$$(0)_{,2} \quad (1)_{1,2} \quad (1)_{2,2} \quad (4.69)$$

and whose parameters are

$$(\mu)_2, (A_1)_2, (A_2)_2. \quad (4.70)$$

After the completion of the first block, the parameters in (4.68) become estimable. Referring to (4.42), the unbiased estimates of these parameters are

$$(\hat{A}_1)_1 = \frac{1}{4} \sum_{s=2}^4 (y(11)_{1s,1} - y(1)_{s,1}) - \frac{1}{4}(4-3)(y(1)_{1,1} - y(0)_{,1})$$

$$\begin{aligned}
(\hat{A}_2)_1 &= \frac{1}{4}(y(11)_{12,1} - y(1)_{1,1}) + \frac{1}{4} \sum_{s=3}^4 (y(11)_{2s,1} - y(1)_{s,1}) \\
&\rightarrow \frac{1}{4}(y(1)_{2,1} - y(0)_{,1})
\end{aligned}$$

$$(\hat{A}_{r s})_1 = \frac{1}{4}[y(11)_{rs,1} - y(1)_{r,1} - y(1)_{s,1} + y(0)_{,1}]$$

($s = r+1, r+2, \dots, n_{2j}$) and ($r = 1, 2$). For ($s = 3, 4$), we have

$$(\hat{A}_s)_1 = \frac{1}{4}[(y(11)_{1s,1} - y(1)_{1,1}) + (y(11)_{2s,1} - y(1)_{2,1})].$$

After the completion of the second block, the parameters in (4.70)

become estimable. Their unbiased estimates are given in (4.43) where $(g_1 = 0)$, $(g_2 = 1)$ and $r = 1, 2$.

V. OBAT 3^m PLANS FOR THE 3^m FACTORIAL EXPERIMENTS

This chapter extends the results of Chapter IV involving OBAT plans for exploratory 2^n experiments to the 3^m experiments. After a brief literature review (Section 5.1) and a model statement for the 3^m experiment (Section 5.2), an economic expandible-contractible 3^m_V factorial design is developed in Section 5.3. This 3^m_V design will be used to construct the OBAT 3^m plans we report in this dissertation. OBAT 3^m plans for strategy I are discussed in Section 5.4 and OBAT 3^m plans for strategy II are discussed in Section 5.5. An example of the OBAT 3^m plans is given in Section 5.6.

5.1 Literature Review of OBAT 3^m Plans

Hoke (1974) and Campbell and Carr (1977) studied the 3^m experiment from two different perspectives. Hoke considered the problem of sequential fitting of quadratic response surfaces in stages whereas Campbell and Carr considered the problem of screening factors using 3^m_{IV} irregular fractions.

a) Attempting to construct more economical factorial designs for fitting the quadratic response surface

$$E(y) = \beta_0 + \sum_{i=1}^m \beta_i x_i + \sum_{i=1}^m \beta_{ii} x_i^2 + \sum_{i=1}^{m-1} \sum_{j=i+1}^m \beta_{ij} x_i x_j \quad (M)$$

with $p(m) = 1 + m + m + \frac{m(m-1)}{2} = (m+1)(m+2)/2$ parameters and with $x = -1, 0, 1$ representing the levels of each of the m 3-level factors, Hoke (1974) argues that

1. Regular or orthogonal 3^{m-k} fractional factorial plans are not economical as they involve a large number of runs relative to the number of parameters in the quadratic response surface model.

2. Composite designs which are augmented regular 2^{m-k} fractions and which are often used to fit quadratic response surfaces are not generally saturated hence they are, in some situations uneconomical.

Hoke's work is based on an idea by Fry (1961) who partitioned the 3^m factor space into $(m+1)$ concentric hyperspheres

$$S_r = \{(x_1, x_2, \dots, x_m) : \sum_{i=1}^m x_i^2 = r^2, (x_i = -1, 0, 1) \text{ and } r = 0, 1, 2, \dots, m\}$$

where (x_1, x_2, \dots, x_m) represent points in the m -dimensional space as well as the treatment combinations of the 3^m factorial experiment. The $(m+1)$ hyperspheres $S_r, (r = 0, 1, 2, \dots, m)$ correspond to the $(m+1)$ terms in the binomial expansion

$$3^m = (1+2)^m = [1 + 2m + \dots + \binom{m}{r} 2^r + \dots + 2^m] .$$

Hence the hypersphere S_r contains $\binom{m}{r} 2^r$ treatment combinations. The points (x_1, x_2, \dots, x_m) on the r^{th} hypersphere have r coordinates non-zero and $(m-r)$ coordinates zero. The center point is the point $(0, 0, \dots, 0)$. Hoke (1974) partitioned each hypersphere S_r even further into partially balanced irregular fractions of the 3^m experiment.

Letting $(x_i = 0, 1, 2)$ represent the 3 levels of each factor, the center point of the factor space is now $(1, 1, \dots, 1)$ and the r^{th} hypersphere S_r is the set which contains the treatment combinations (x_1, x_2, \dots, x_m) having r digits not equal to 1. Hoke partitioned

every hypersphere S_r into $(r+1)$ disjoint permutation-invariant sets

$$S_r(j) = \pi(\underbrace{22\dots 2}_j \underbrace{11\dots 1}_{m-r} \underbrace{00\dots 0}_{r-j}), \quad (j = 0, 1, 2, \dots, r) \text{ and } (r = 0, 1, 2, \dots, m).$$

where each set $S_r(j)$ contains all the treatment combinations (x_1, x_2, \dots, x_m) having j factors at level 2, $(m-r)$ factors at level 1 and $(r-j)$ factors at level 0 and where $\pi(\cdot)$ is used to represent all permutations of the treatment combination inside the bracket. Each set $S_r(j)$ contains $\frac{m!}{j!(m-r)!(r-j)!}$ treatment combinations and the hypersphere S_r is the union of the disjoint sets $S_r(j)$, i.e.

$$S_r = S_r(0) \cup S_r(1) \cup \dots \cup S_r(r) = \bigcup_{j=0}^r S_r(j)$$

with

$$\sum_{j=0}^r \frac{m!}{j!(m-r)!(r-j)!} = \binom{m}{r} 2^r$$

treatment combinations.

Hoke's (1974) economical procedure for fitting the quadratic response surface (M) is sequential in its nature and is carried out in three nested and saturated stages where

Stage 1. fits the first order model, which includes β_0 and β_i ($i = 1, 2, \dots, m$), by a saturated irregular fraction of the 3^m factorial design.

Stage 2. fits the main effect model, which includes β_0 , β_i and β_i^2 ($i = 1, 2, \dots, m$), by a saturated irregular fraction containing the fraction in stage 1.

Stage 3. fits the model with all $p(m) = (m+1)(m+2)/2$ parameters by a saturated irregular fraction containing the fraction in stage 2.

The irregular fractions in the three stages are actually combinations or unions of several $S_r(j)$ sets. Hoke actually constructed the irregular saturated nested fractions for the three stages above by combining $S_r(j)$ sets from the following sets

A: (00---0), (11---1), (22---2)

B: $S_1(0)$, $S_1(1)$, $S_{m-1}(0)$, $S_{m-1}(m-1)$, $S_m(1)$, $S_m(m-1)$

C: $S_2(0)$, $S_2(2)$, $S_{m-2}(0)$, $S_{m-2}(m-2)$, $S_m(2)$, $S_m(m-2)$.

For stages (1), (2) and (3) the following saturated designs can be used, respectively,

(00---0) \cup $S_m(m-1)$ with $1 + m$ runs

(00---0) \cup $S_m(m-1) \cup S_{m-1}(0)$ with $1 + m + m = 1 + 2m$ runs

(00---0) \cup $S_m(m-1) \cup S_{m-1}(0) \cup S_m(2)$ with $1 + m + m + \frac{m(m-1)}{2}$ runs

However, since the designs for each stage are not unique, Hoke derived a formula for the trace of $(X'X)^{-1}$ and reported, using computer search techniques designs for stages (1), (2) and (3) that minimize trace $(X'X)^{-1}$ within the class of saturated permutation-invariant 3^m designs. Hoke reported formulas for parameter estimates, as functions of m , for the designs in the three stages and he reported the alias structure for the designs in stages (1) and (2). Furthermore, Hoke used the measure of efficiency

$$100 \times \frac{N_1}{N_2} \times \frac{\text{trace}(X'X)_1^{-1}}{\text{trace}(X'X)_2^{-1}}$$

for design 1 relative to design 2 and concluded that the irregular permutation-invariant designs

$$S_0(0) \cup S_m(m-1) \cup S_{m-1}(0) \cup S_m(2) \cup S_1(1)$$

containing

$$1 + m + m + \frac{m(m-1)}{2} + m = \frac{(m+1)(m+2)}{2} + m$$

treatment combinations ($m = 1, 2, \dots$) are more economical and more efficient than Box and Behnken's (1960) irregular permutation-invariant designs $3^3//15$, $3^4//27$, $3^5//46$, $3^6//54$, $3^7//62$ and $3^9//30$ (the symbol $3^m//N$ represents 3^m designs containing N runs).

It is worth mentioning here that Mitchel and Bayne (1978), using computer algorithm, reported several D-optimal partially balanced fractions for both the quadratic response surface and the 3^m factorial effect model involving up to 5 3-level factors. Efficiency comparisons with Hoke's (1974) 3^m designs and other 3^m designs in the literature were made.

b) Campbell and Carr (1977) reported an irregular resolution IV 3^m factorial design in $3(2m+1)$ runs that can be used for screening a group of m 3-level factors in exploratory factorial experiments based on the linear model (5.1). An additional 3-level factor requires $3[2(m+1) + 1] - 3(2m+1) = 6$ more runs. The treatment combinations of the 3_{IV}^m design is the union of the nine sets

$$(000\text{---}0), (111\text{---}1), (22\text{---}2)$$

$$\pi(111\text{---}10), \pi(111\text{---}12)$$

$$\pi(000\text{---}02), \pi(222\text{---}20)$$

$$\pi(000\text{---}01), \pi(222\text{---}21)$$

or in Hoke's (1974) $S_r(j)$ notation the nine sets are

$$S_m(0), S_0(0), S_m(m)$$

$$S_1(0), S_1(1)$$

$$S_m(1), S_m(m-1)$$

$$S_{m-1}(1), S_{m-1}(m-1)$$

containing 1, 1, 1, m, m, m, m, m, m runs, respectively.

The authors reported the entries of the matrices $X'X$ and $(X'X)^{-1}$ as functions of m . They also reported the alias structure among the two-factor interactions: linear by linear, linear by quadratic, quadratic by linear and quadratic by quadratic. However, the authors did not derive general formulas for parameter estimates as functions of m nor did they indicate how to break the alias chains involving two-factor interactions in the light of additional runs. They also did not consider the problem of blocking their 3_{IV}^m designs.

5.2 Parametrization of the 3^m Factorial Experiment

The 3^m experiment contains m three-level factors which we denote by B_t ($t = 1, 2, \dots, m$) with levels b_t^0, b_t^1 and b_t^2 . Treatment combinations are denoted by $b_1^{i_1} b_2^{i_2} \dots b_m^{i_m}$ where ($i_j = 0, 1, 2$) and ($j = 1, 2, \dots, m$) and the corresponding response by $y(b_1^{i_1} b_2^{i_2} \dots b_m^{i_m})$. Using single degree of freedom contrasts to describe factor effects and

interactions and denoting the linear and quadratic effects of factor B_t by B_t and B_t^2 , respectively, the linear model relating expected responses and factorial effects up to two-factor interactions is

$$\begin{aligned} b_1^{i_1} b_2^{i_2} \dots b_m^{i_m} &= E y(b_1^{i_1} b_2^{i_2} \dots b_m^{i_m}) \\ &= \mu + \sum_{k=1}^m B_k y_k + \sum_{k=1}^m B_k^2 y_k' + \sum_{k=1}^{m-1} \sum_{\ell=k+1}^m B_k B_\ell y_k y_\ell \\ &\quad + \sum_{k=1}^{m-1} \sum_{\ell=k+1}^m B_k B_\ell^2 y_k y_\ell' + \sum_{k=1}^{m-1} \sum_{\ell=k+1}^m B_k^2 B_\ell y_k' y_\ell \\ &\quad + \sum_{k=1}^{m-1} \sum_{\ell=k+1}^m B_k^2 B_\ell^2 y_k' y_\ell', \end{aligned}$$

where

$$y_j = \begin{cases} -1 & \text{if } b_j^{i_j} = b_j^0 \\ 0 & \text{if } b_j^{i_j} = b_j^1 \\ 1 & \text{if } b_j^{i_j} = b_j^2 \end{cases} \quad \text{and} \quad y_j' = \begin{cases} 1 & \text{if } b_j^{i_j} = b_j^0 \\ -2 & \text{if } b_j^{i_j} = b_j^1 \\ 1 & \text{if } b_j^{i_j} = b_j^2 \end{cases} \quad (5.1)$$

Model (5.1) contains $p(m) = 1 + m + m + 4 \cdot \frac{m(m-1)}{2} = 1 + 2m^2$

parameters representing the overall mean, m linear effects, m quadratic effects, $\frac{m(m-1)}{2}$ two-factor interactions $B_t B_u$, $\frac{m(m-1)}{2}$ two-factor interactions $B_t B_u^2$, $\frac{m(m-1)}{2}$ two-factor interactions $B_t^2 B_u$ and $\frac{m(m-1)}{2}$ two-factor interactions $B_t^2 B_u^2$ for $(u = t+1, t+2, \dots, m)$ and $(t = 1, 2, \dots, (m-1))$. For quantitative factors model (5.1) without the last three terms is referred to as a quadratic response surface [see Hoke (1974)].

5.3 Economic Expansible-Contractible 3_{IV}^m Designs for OBAT 3^m Plans

Parallel to Webb's (1965) expansible-contractible $\langle 0, 1, 2 \rangle$ - designs (see Section 4.1) for the 2^n factorial experiment where $\langle 0, 1, 2 \rangle$ - designs for the 2^n experiment are embedded in $\langle 0, 1, 2 \rangle$ - designs for the 2^{n+1} experiment, we report here economic expansible-contractible $\langle 0, 1, 2 \rangle$ - designs for 3_V^m factorial experiments where $\langle 0, 1, 2 \rangle$ - designs for the 3^m experiment are nested in $\langle 0, 1, 2 \rangle$ - designs for the 3^{m+1} experiment. The treatment combinations generating these 3_V^m designs is the fraction F_m consisting of the $N = 1 + m + m + 4 \cdot \frac{m(m-1)}{2} = 1 + 2m^2$ runs of the form

$$00\text{---}0b_t^{i_t}00\text{---}0b_u^{i_u}00\text{---}0 \quad (5.2)$$

where $(b_t^{i_t}, b_u^{i_u} = 0, 1, 2), (u = t+1, t+2, \dots, m)$ and $(t = 1, 2, \dots, m)$. These $1 + 2m^2$ runs will be used for developing the OBAT 3^m later in this chapter. The runs in (5.2) will be denoted by $(b_t^{i_t} b_u^{i_u})_{tu}$, or for short by $(b_t b_u)_{tu}$ and the corresponding response by $y(b_t b_u)_{tu}$. The linear model in (5.1) based on the fraction F_m in (5.2) can now be written as

$$\begin{aligned} (b_t b_u)_{tu} &= E y(b_t b_u)_{tu} \\ &= \mu + B_t \cdot y_t + B_u \cdot y_u - \sum_{k \neq t, u} B_k + B_t^2 y_t + B_u^2 y_u + \sum_{k \neq t, u} B_k^2 \\ &\quad + B_t B_u \cdot y_t y_u - \left(\sum_{k \neq t, u} B_t B_k \right) \cdot y_t - \left(\sum_{k \neq t, u} B_u B_k \right) \cdot y_u \end{aligned}$$

$$\begin{aligned}
& + \sum_{k < \ell} \sum_{k, \ell \neq t, u} B_k B_\ell + B_t B_u^2 \cdot y_t y'_u + \left(\sum_{k \neq t, u} B_t B_k^2 \right) \cdot y_t \\
& + \left(\sum_{k \neq t, u} B_u B_k^2 \right) \cdot y_u - \sum_{k < \ell} \sum_{k, \ell \neq t, u} B_k B_\ell^2 + B_t^2 B_u \cdot y'_t y'_u \\
& - \left(\sum_{k \neq t, u} B_t^2 B_k \right) \cdot y'_t - \left(\sum_{k \neq t, u} B_u^2 B_k \right) \cdot y'_u - \sum_{k < \ell} \sum_{k, \ell \neq t, u} B_k^2 B_\ell \\
& + B_t^2 B_u^2 \cdot y'_t y'_u + \left(\sum_{k \neq t, u} B_t^2 B_k^2 \right) \cdot y'_t + \left(\sum_{k \neq t, u} B_u^2 B_k^2 \right) \cdot y'_u \\
& + \sum_{k < \ell} \sum_{k, \ell \neq t, u} B_k^2 B_\ell^2 \tag{5.3}
\end{aligned}$$

The fraction F_m in (5.2) possesses the following characteristics:

1. It is permutation-invariant since F_m can be written as the union of disjoint permutation invariant sets. Referring to Hoke (1974) (see Section 5.1), each set is of the form

$$S_r(j) = \pi(\underbrace{22 \dots 2}_j \underbrace{11 \dots 1}_{m-r} \underbrace{00 \dots 0}_{r-j}) \tag{5.4}$$

Therefore

$$\begin{aligned}
F_m &= \pi(00 \dots 0) \cup \pi(100 \dots 0) \cup \pi(200 \dots 0) \cup \pi(110 \dots 0) \\
&\cup \pi(120 \dots 0) \cup \pi(220 \dots 0) \\
&= S_m(0) \cup S_{m-1}(0) \cup S_m(1) \cup S_{m-2}(0) \cup S_{m-1}(1) \cup S_m(2) \tag{5.5}
\end{aligned}$$

where sets $S_m(0)$, $S_{m-1}(0)$, $S_m(1)$, $S_{m-2}(0)$, $S_{m-1}(1)$ and $S_m(2)$ contain 1, m, m, $\frac{m(m-1)}{2}$, $m(m-1)$, $\frac{m(m-1)}{2}$ runs, respectively. The information matrix $X'X$ is also invariant to permutations of factors hence all factor effects are estimated with the same precision.

2. It is saturated hence economical since the number of runs N in (5.2) is the same as the number of parameters in model (5.3).

3. It is irregular, i.e. non-orthogonal since it is not of the form of a 3^{m-q} fraction.

4. It is economically expansible since an additional 3-level factor B_{m+1} requires " $F_{m+1} - F_m$ " = $1 + 2(m+1)^2 - 1 - 2m^2 = 2 + 4m$ runs, namely

a. the two runs $(1)_{m+1} = \frac{00---01}{m+1}$ and $(2)_{m+1} = \frac{00---02}{m+1}$

b. the m runs $(11)_{t(m+1)}$, $t = 1, 2, \dots, m$

c. the m runs $(12)_{t(m+1)}$, $t = 1, 2, \dots, m$

d. the m runs $(21)_{t(m+1)}$, $t = 1, 2, \dots, m$

e. the m runs $(22)_{t(m+1)}$, $t = 1, 2, \dots, m$

which are used for the estimation of

a. the linear and quadratic effects $B_{(m+1)}$ and $B_{(m+1)}^2$.

b. the m linear by linear interactions $B_t B_{(m+1)}$.

c. the m linear by quadratic interactions $B_t B_{(m+1)}^2$.

d. the m quadratic by linear interactions $B_t^2 B_{(m+1)}$.

e. the m quadratic by quadratic interactions $B_t^2 B_{(m+1)}^2$.

This economical characteristic of the fraction F_m is not shared by orthogonal 3^{m-q} fractions as an additional factor triples the number of runs in the design.

5. Fractions in (5.2) are nested with each other. To see this, we note that for 3^2 experiment F_2 consists of the 9 runs

00, 10, 01, 20, 02, 11, 12, 21, 22

and for the 3^3 experiment F_3 consists of the 19 runs

000, 100, 010, 001, 200, 020, 002, 110, 101, 011,
120, 102, 012, 210, 201, 021, 220, 202, 022 .

The fraction F_2 can also be considered as a subset of F_3 in which the third factor is at its low level hence F_2 can be written as

000, 100, 010, 200, 020, 110, 120, 210, 220

6. The information matrix $X'X$ corresponding to the fraction F_m contains 22 distinct entries, i.e. independent of m . These entries which are functions of m are listed in Appendix B.

7. It is a resolution V fraction of the 3^m experiment hence all main effects and all two-factor interactions are estimable. A proof of this assertion will now be presented.

We let

$$L(t,u) = \sum_{k \neq t,u} B_k$$

$$Q(t,u) = \sum_{k \neq t,u} B_k^2$$

$$\begin{aligned}
L_t L(t,u) &= \sum_{k \neq t,u} B_t B_k, & L_u L(t,u) &= \sum_{k \neq t,u} B_u B_k \\
L_t Q(t,u) &= \sum_{k \neq t,u} B_t B_k^2, & L_u Q(t,u) &= \sum_{k \neq t,u} B_u B_k^2 \\
Q_t L(t,u) &= \sum_{k \neq t,u} B_t^2 B_k, & Q_u L(t,u) &= \sum_{k \neq t,u} B_u^2 B_k \\
Q_t Q(t,u) &= \sum_{k \neq t,u} B_t^2 B_k^2, & Q_u Q(t,u) &= \sum_{k \neq t,u} B_u^2 B_k^2
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
(LL)(t,u) &= \sum_{\substack{k < \ell \\ k, \ell \neq t, u}} B_k B_\ell, & (QL)(t,u) &= \sum_{\substack{k < \ell \\ k, \ell \neq t, u}} B_k^2 B_\ell \\
(LQ)(t,u) &= \sum_{\substack{k < \ell \\ k, \ell \neq t, u}} B_k B_\ell^2, & (QQ)(t,u) &= \sum_{\substack{k < \ell \\ k, \ell \neq t, u}} B_k^2 B_\ell^2
\end{aligned} \tag{5.7}$$

We also let

$$\begin{aligned}
L_t L(t) &= \sum_{u \neq t} B_t B_u, & Q_t L(t) &= \sum_{u \neq t} B_t^2 B_u \\
L_t Q(t) &= \sum_{u \neq t} B_t B_u^2, & Q_t Q(t) &= \sum_{u \neq t} B_t^2 B_u^2
\end{aligned} \tag{5.8}$$

We note here that

$\sum_{k \neq t, u}$ contains $(m-2)$ terms

$\sum_{\substack{k < \ell \\ k, \ell \neq t, u}}$ contains $\frac{m(m-1)}{2} - 1 - (m-2) - (m-2) = \frac{(m-2)(m-3)}{2}$ terms

and

$\sum_{u \neq t}$ contains (m-1) terms .

With this notation, model (5.3) can now be written as

$$\begin{aligned}
 (b_{t \ b_u}^{i_t \ i_u})_{tu} &= \mu + B_t \cdot y_t + B_u \cdot y_u - L_{(t,u)} + B_t^2 \cdot y_t' + B_u^2 \cdot y_u' + Q_{(t,u)} \\
 &+ B_t B_u \cdot y_t \cdot y_u - L_t L_{(t,u)} \cdot y_t - L_u L_{(t,u)} \cdot y_u + (LL)_{(t,u)} \\
 &+ B_t B_u^2 \cdot y_t \cdot y_u' + L_t Q_{(t,u)} \cdot y_t + L_u L_{(t,u)} \cdot y_u - (LQ)_{(t,u)} \\
 &+ B_t^2 B_u \cdot y_t' \cdot y_u - Q_t L_{(t,u)} \cdot y_t' - Q_u L_{(t,u)} \cdot y_u' - (QL)_{(t,u)} \\
 &+ B_t^2 B_u^2 \cdot y_t' \cdot y_u' + Q_t Q_{(t,u)} \cdot y_t' + Q_u Q_{(t,u)} \cdot y_u' + (QQ)_{(t,u)}
 \end{aligned} \tag{5.9}$$

For the treatment combinations $(b_{t \ b_u}^{i_t \ i_u})_{tu}$ in (5.2), the model (5.9) can be written explicitly as

$$\begin{aligned}
 (0) &= \mu - B_t - B_u - L_{(t,u)} + B_t^2 + B_u^2 + Q_{(t,u)} \\
 &+ B_t B_u + L_t L_{(t,u)} + L_u L_{(t,u)} + (LL)_{(t,u)} \\
 &- B_t B_u^2 - L_t Q_{(t,u)} - L_u Q_{(t,u)} - (LQ)_{(t,u)} \\
 &- B_t^2 B_u - Q_t L_{(t,u)} - Q_u L_{(t,u)} - (QL)_{(t,u)} \\
 &+ B_t^2 B_u^2 + Q_t Q_{(t,u)} + Q_u Q_{(t,u)} + (QQ)_{(t,u)} \\
 (1)_t &= \mu - B_u - L_{(t,u)} - 2B_t^2 + B_u^2 + Q_{(t,u)} \\
 &+ L_u L_{(t,u)} + (LL)_{(t,u)} - L_u Q_{(t,u)} - (LQ)_{(t,u)}
 \end{aligned}$$

$$\begin{aligned}
 & + 2B_t^2 B_u + 2Q_t L(t,u) - Q_u L(t,u) - (QL)(t,u) \\
 & - 2B_t^2 B_u^2 - 2Q_t Q(t,u) + Q_u Q(t,u) + (QQ)(t,u)
 \end{aligned}$$

$$(1)_u = \mu - B_t - L(t,u) + B_t^2 - 2B_u^2 + Q(t,u)$$

$$+ L_t L(t,u) + (LL)(t,u) - L_t Q(t,u) - (LQ)(t,u)$$

$$+ 2B_t B_u^2 - Q_t L(t,u) + 2Q_u L(t,u) - (QL)(t,u)$$

$$- 2B_t^2 B_u^2 + Q_t Q(t,u) - 2Q_u Q(t,u) + (QQ)(t,u)$$

$$(2)_t = \mu + B_t - B_u - L(t,u) + B_t^2 + B_u^2 + Q(t,u)$$

$$- B_t B_u - L_t L(t,u) + L_u L(t,u) + (LL)(t,u)$$

$$+ B_t B_u^2 + L_t Q(t,u) - L_u Q(t,u) - (LQ)(t,u)$$

$$- B_t^2 B_u - Q_t L(t,u) - Q_u L(t,u) - (QL)(t,u)$$

$$+ B_t^2 B_u^2 + Q_t Q(t,u) + Q_u Q(t,u) + (QQ)(t,u)$$

$$(2)_u = \mu - B_t + B_u - L(t,u) + B_t^2 + B_u^2 + Q(t,u)$$

$$- B_t B_u + L_t L(t,u) - L_u L(t,u) + (LL)(t,u)$$

$$- B_t B_u^2 - L_t Q(t,u) + L_u Q(t,u) - (LQ)(t,u)$$

$$+ B_t^2 B_u - Q_t L(t,u) - Q_u L(t,u) - (QL)(t,u)$$

$$+ B_t^2 B_u^2 + Q_t Q(t,u) + Q_u Q(t,u) + (QQ)(t,u)$$

$$\begin{aligned}
(11) \quad {}_{tu} = & \mu - L_{(t,u)} - 2B_t^2 - 2B_u^2 + Q_{(t,u)} \\
& + (LL)_{(t,u)} - (LQ)_{(t,u)} + 2Q_t L_{(t,u)} + 2Q_u L_{(t,u)} - (QL)_{(t,u)} \\
& + 4B_t^2 B_u^2 - 2Q_t Q_{(t,u)} - 2Q_u Q_{(t,u)} + (QQ)_{(t,u)}
\end{aligned}$$

$$\begin{aligned}
(12) \quad {}_{tu} = & \mu + B_u - L_{(t,u)} - 2B_t^2 + B_u^2 + Q_{(t,u)} \\
& - L_u L_{(t,u)} + (LL)_{(t,u)} + L_u Q_{(t,u)} - (LQ)_{(t,u)} \\
& - 2B_t^2 B_u + 2Q_t L_{(t,u)} - Q_u L_{(t,u)} - (QL)_{(t,u)} \\
& - 2B_t^2 B_u^2 - 2Q_t Q_{(t,u)} + Q_u Q_{(t,u)} + (QQ)_{(t,u)}
\end{aligned}$$

$$\begin{aligned}
(21) \quad {}_{tu} = & \mu + B_t - L_{(t,u)} + B_t^2 - 2B_u^2 + Q_{(t,u)} \\
& - L_t L_{(t,u)} + (LL)_{(t,u)} - Q_t L_{(t,u)} + 2Q_u L_{(t,u)} - (QL)_{(t,u)} \\
& - 2B_t^2 B_u + L_t Q_{(t,u)} - (LQ)_{(t,u)} \\
& - 2B_t^2 B_u^2 + Q_t Q_{(t,u)} - 2Q_u Q_{(t,u)} + (QQ)_{(t,u)}
\end{aligned}$$

$$\begin{aligned}
(22) \quad {}_{tu} = & \mu + B_t + B_u - L_{(t,u)} + B_t^2 + B_u^2 + Q_{(t,u)} \\
& + B_t B_u - L_t L_{(t,u)} - L_u L_{(t,u)} + (LL)_{(t,u)} \\
& + B_t B_u^2 + L_t Q_{(t,u)} + L_u Q_{(t,u)} - (LQ)_{(t,u)} \\
& + B_t^2 B_u - Q_t L_{(t,u)} - Q_u L_{(t,u)} - (QL)_{(t,u)} \\
& + B_t^2 B_u^2 + Q_t Q_{(t,u)} + Q_u Q_{(t,u)} + (QQ)_{(t,u)}
\end{aligned} \tag{5.10}$$

From (5.10) we find the following

$$\begin{aligned}
 \text{a) } (2)_t - (0) &= 2[B_t - B_t B_u - L_t L(t,u) + B_t B_u^2 + L_t Q(t,u)] \\
 &= 2(B_t - L_t L(t) + L_t Q(t)) \quad (5.11)
 \end{aligned}$$

$$\begin{aligned}
 (22)_{tu} - (2)_u &= 2[B_t + B_t B_u - L_t L(t,u) + B_t B_u^2 + L_t Q(t,u)] \\
 &= 2(B_t + B_t B_u - L_t L(t,u) + L_t Q(t)) \quad (5.12)
 \end{aligned}$$

$$(21)_{tu} - (1)_u = 2(B_t - L_t L(t,u) - 2B_t B_u^2 + L_t Q(t,u)) \quad (5.13)$$

$$\begin{aligned}
 \text{b) } (0) - 2 \cdot (1)_t + (2)_t &= 6[B_t^2 - B_t^2 B_u - Q_t L(t,u) + B_t^2 B_u^2 + Q_t Q(t,u)] \\
 &= 6(B_t^2 - Q_t L(t) + Q_t Q(t)) \quad (5.14)
 \end{aligned}$$

$$\begin{aligned}
 (2)_u - 2 \cdot (12)_{tu} + (22)_{tu} &= 6[B_t^2 + B_t^2 B_u - Q_t L(t,u) + B_t^2 B_u^2 + Q_t Q(t,u)] \\
 &= 6(B_t + B_t B_u - Q_t L(t,u) + Q_t Q(t)) \quad (5.15)
 \end{aligned}$$

$$(1)_u - 2 \cdot (11)_{tu} - (21)_{tu} = 6(B_t^2 - Q_t L(t,u) - 2B_t^2 B_u^2 + Q_t Q(t,u))$$

$$\text{where } (u = t+1, t+2, \dots, m) \text{ and } (t = 1, 2, \dots, m) \dots \quad (5.16)$$

From (5.11), (5.12), and (5.13), we will show that the linear effect B_t is estimable from F_m .

1) From (5.12),

$$\begin{aligned}
 \sum_{u \neq t} ((22)_{tu} - (2)_u) &= 2 \sum_{u \neq t} [B_t + B_t B_u - L_t L(t,u) + B_t B_u^2 + L_t Q(t,u)] \\
 &= 2(m-1)B_t + 2 \cdot \sum_{u \neq t} B_t B_u + 2 \cdot \sum_{u \neq t} L_t L(t,u) \\
 &\quad + 2 \cdot \sum_{u \neq t} B_t B_u^2 + 2 \cdot \sum_{u \neq t} L_t Q(t,u) \quad (5.17)
 \end{aligned}$$

Referring to (4.9), we find that

$$\sum_{u \neq t} L_t L(t, u) = \sum_{u \neq t} \left[\sum_{k \neq t, u} B_t B_k \right] = (m-2) \sum_{u \neq t} B_t B_u = (m-2) L_t L(t) \quad (5.18)$$

and

$$\sum_{u \neq t} L_t Q(t, u) = \sum_{u \neq t} \left[\sum_{k \neq t, u} B_t B_k \right] = (m-2) \sum_{u \neq t} B_t B_u^2 = (m-2) L_t Q(t) \quad (5.19)$$

From (5.17), (5.18) and (5.19), it follows that

$$\begin{aligned} \sum_{u \neq t} ((22)_{tu} - (2)_u) &= 2[(m-1)B_t + L_t L(t) - (m-2)L_t L(t) + L_t Q(t) \\ &\quad + (m-2)L_t Q(t)] \\ &= 2[(m-1)B_t - (m-3)L_t L(t) + (m-1)L_t Q(t)] \quad (5.20) \end{aligned}$$

2) From (5.13),

$$\begin{aligned} \sum_{u \neq t} ((21)_{tu} - (1)_u) &= 2 \cdot \sum_{u \neq t} [B_t - L_t L(t, u) - 2B_t B_u^2 + L_t Q(t, u)] \\ &= 2(m-1)B_t - 2 \cdot \sum_{u \neq t} L_t L(t, u) - 4 \sum_{u \neq t} B_t B_u^2 \\ &\quad + 2 \cdot \sum_{u \neq t} L_t Q(t, u) \quad (5.21) \end{aligned}$$

From (5.18), (5.19) and (5.21), it follows that

$$\begin{aligned} \sum_{u \neq t} ((21)_{tu} - (1)_u) &= 2(m-1)B_t - 2(m-2)L_t L(t) - 4L_t Q(t) + 2(m-2)L_t Q(t) \\ &= 2[(m-1)B_t - (m-2)L_t L(t) + (m-4)L_t Q(t)] \quad (5.22) \end{aligned}$$

From (5.20) and (5.22), we find that

$$\begin{aligned} \sum_{u \neq t} [(22)_{tu} - (2)_u] + [(21)_{tu} - (1)_u] \\ = 4(m-1)B_t - 2(2m-5)L_t L(t) + 2(2m-5)L_t Q(t) \end{aligned} \quad (5.23)$$

From (5.11) and (5.23), it follows that

$$\frac{1}{6} \sum_{u \neq t} [(22)_{tu} - (2)_u] + [(21)_u - (1)_u] - \frac{1}{6}(2m-5)((2)_t - (0)) = B_t \quad (5.24)$$

which establishes estimability of the linear effect B_t .

For the quadratic effect B_t^2 , we find from (5.15),

$$\begin{aligned} \sum_{u \neq t} ((2)_u - 2 \cdot (12)_{tu} + (22)_{tu}) = 6(m-1)B_t^2 + 6(m-3)Q_t L(t) \\ + 6(m-1)Q_t Q(t) \end{aligned} \quad (5.25)$$

and from (5.16),

$$\begin{aligned} \sum_{u \neq t} ((1)_u - 2 \cdot (11)_{tu} + (21)_{tu}) = 6(m-1)B_t^2 - 6(m-2)Q_t L(t) \\ + 6(m-4)Q_t Q(t) \end{aligned} \quad (5.26)$$

From (5.25) and (5.26),

$$\begin{aligned} \sum_{u \neq t} [((2)_u - 2 \cdot (12)_{tu} + (22)_{tu}) + ((1)_u - 2 \cdot (11)_{tu} + (21)_{tu})] \\ = 12(m-1)B_t^2 - 6(2m-5)Q_t L(t) + 6(2m-5)Q_t Q(t) \end{aligned} \quad (5.27)$$

and from (5.14) and (5.27) it follows that

$$\frac{1}{18} \sum_{u \neq t} [((2)_u - 2 \cdot (12)_{tu} + (22)_{tu}) + ((1)_u - 2 \cdot (11)_{tu} + (21)_{tu})]$$

$$-\frac{1}{18} (2m-5) ((0) - 2 \cdot (1)_t + (2)_t) = B_t^2 \quad (5.28)$$

For the interaction effects we find that

1. From (5.11) and (5.12),

$$\frac{1}{2} [((22)_{tu} - (2)_u) - ((2)_t - (0))] = B_t B_u \quad (5.29)$$

2. From (5.14) and (5.15),

$$\begin{aligned} \frac{1}{12} [((2)_u - 2 \cdot (12)_{tu} + (22)_{tu}) - ((0) - 2 \cdot (1)_t \\ + (2)_t)] = B_t^2 B_u \end{aligned} \quad (5.30)$$

also

$$\begin{aligned} \frac{1}{12} [((2)_t - 2 \cdot (21)_{tu} + (22)_{tu}) - ((0) - 2 \cdot (1)_u \\ + (2)_u)] = B_t B_u^2 \end{aligned} \quad (5.31)$$

3. From (5.14), (5.15) and (5.16),

$$\begin{aligned} \frac{1}{36} [((0) - 2 \cdot (1)_t + (2)_t) - 2((1)_u - 2 \cdot (11)_{tu} + (21)_{tu}) \\ + ((2)_u - 2 \cdot (12)_{tu} + (22)_{tu})] = B_t^2 B_u^2 \end{aligned} \quad (5.32)$$

Therefore, all parameters of the linear model (5.1) are estimable hence the fraction F_m in (5.2) is of resolution V. The unbiased estimates of the parameters of the linear model

$$y(b_t b_u)_{tu} = (b_t b_u)_{tu} + \epsilon_{tu} \quad (5.33)$$

where $E(\epsilon_{tu}) = 0$ and $\text{var}(\epsilon_{tu}) = \sigma^2$ are obtained by replacing $(b_t b_u)_{tu}$

in (5.24), (5.28), (5.29), (5.30), (5.31) and (5.32) by its response $y(b_t b_u)_{tu}$. We also find that

$$\begin{aligned} \text{var } \hat{B}_t &= \frac{1}{36} [(m-1)(4\sigma^2)] + \frac{1}{36} (2m-5)^2 (2\sigma^2) = \frac{1}{18} (4m^2 - 18m + 23)\sigma^2 \\ \text{var } \hat{B}_t^2 &= \frac{1}{324} [(m-1)(12\sigma^2)] + \frac{1}{324} (2m-5)^2 (6\sigma^2) = \frac{1}{54} (4m^2 - 18m + 23)\sigma^2 \\ \text{var } \hat{B}_t B_u &= \frac{1}{4}\sigma^2 \\ \text{var } \hat{B}_t^2 B_u &= \frac{1}{12} \sigma^2 \\ \text{var } \hat{B}_t B_u^2 &= \frac{1}{12} \sigma^2 \\ \text{var } \hat{B}_t^2 B_u^2 &= \frac{1}{36} \sigma^2 \end{aligned} \tag{5.34}$$

It can be seen from (5.34) that the linear and quadratic effects of each factor are estimated with higher precision than the two-factor interactions. Such property of the saturated fraction F_m makes it appropriate for screening m 3-level factors for their main effects as well as their two-factor interactions with more emphasis on main effects.

It is worth mentioning here that, as Webb (1965) suggested that the $\langle 0, 1, 2 \rangle$ - designs for the 2^n experiment be carried out as ORAT 2^n plans where factors are ordered in descending order of importance, the $\langle 0, 1, 2 \rangle$ - designs for the 3^m experiments in (5.2) can also be carried out as ORAT 3^m plans. As ORAT plans, in both $\langle 0, 1, 2 \rangle$ - designs for 2^n and 3^m experiments, more than one factor level is changed between successive runs. Assuming that the m factors of the 3^m experiment are ordered as B_1, B_2, \dots, B_m , we suggest that the $(1 + 2m^2)$ runs be in the order

1. (0)
2. $(1)_t \rightarrow (2)_t$ for $t = 1, 2, 3, \dots, m$.
3. $(11)_{1u} \rightarrow (12)_{1u} \rightarrow (21)_{1u} \rightarrow (22)_{1u}$ for $u = 2, 3, \dots, m$.
4. $(11)_{2u} \rightarrow (12)_{2u} \rightarrow (21)_{2u} \rightarrow (22)_{2u}$ for $u = 3, 4, \dots, m$.
- ⋮
- $(t+2)$. $(11)_{tu} \rightarrow (12)_{tu} \rightarrow (21)_{tu} \rightarrow (22)_{tu}$ for $u = t+1, t+2, \dots, m$.
- ⋮
- $(m+1)$. $(11)_{(m-1)m} \rightarrow (12)_{(m-1)m} \rightarrow (21)_{(m-1)m} \rightarrow (22)_{(m-1)m}$ (5.35)

where the arrow \rightarrow indicates "followed by". For the ORAT 3^m plan in (5.35) we outline, but do not prove, some important results. We find that

a. after the completion of the $1 + (m+m) + 4(m-1) = 6m - 3$ runs in steps 1, 2, and 3, all factorial effects involving the first factor become estimable, namely the effects $B_1, B_1^2, B_1B_u, B_1^2B_u, B_1B_u^2$ and $B_1^2B_u^2$ for $u = 2, 3, \dots, m$.

b. after the completion of the $6m - 3 + 4(m-2) = 10m - 5$ runs in steps 1, 2, 3 and 4, all factorial effects involving the first and second factor become estimable.

c. in general, after the completion of the $1 + (m+m) + 4 \cdot \sum_{i=1}^t (m-i)$ runs in steps 1, 2, \dots , $(t+2)$, all factorial effects involving the first, the second, \dots , the t^{th} factor become estimable ($t = 1, 2, \dots, m$).

Parameter estimates are as given in (5.24), (5.28), (5.29), (5.30),

(5.31) and (5.32). These estimates can be used to judge the "significance" of their corresponding parameters.

We now consider the fraction F_m in (5.2) for developing strategy I and II OBAT 3^m plans.

5.4 OBAT 3^m Plans for Strategy I.

In strategy I, factor effects are investigated one at a time in blocks of small size, namely size 3, according to a particular order where the m factors B_1, B_2, \dots, B_m of the 3^m experiment are assumed ordered and hence blocks are formed such that the t^{th} factor as well as two-factor interactions involving this factor become estimable before the u^{th} factor and two-factor interactions involving it for ($u = t+1, t+2, \dots, m$) and ($t = 1, 2, \dots, m$). The treatment combinations in all blocks of the OBAT plan are taken from the 3^m_V fraction in (5.2). The linear model (5.9) is now changed to incorporate block effects (BE) as well. Hence, for an observation in block h we have

$$y(b_t b_u)_{tu,h} = (b_t b_u)_{tu} + (BE)_h + \epsilon_{tu,h} \quad (5.36)$$

where $E(\epsilon_{tu,h}) = 0$, $\text{var}(\epsilon_{tu,h}) = \sigma^2$ and $\sum_{n=1}^b (BE)_h = 0$, b is the total number of blocks in the plan.

For strategy I OBAT 3^m plans, we consider the following two cases.

Case (1): 3^m_{III} plans in a sequence of m blocks of size 3.

When all interactions are negligible, model (5.36) becomes

$$\begin{aligned}
 y(b_{t,u} b_u)_{tu,h} &= \mu + B_t \cdot y_t + B_u \cdot y_u - L_{(t,u)} + B_t^2 \cdot y'_t + B_u^2 \cdot y'_u \\
 &+ Q_{(t,u)} + (BE)_h + \epsilon_{tu,h}
 \end{aligned} \tag{5.37}$$

Since there are m 3-level factors, m blocks of size 3 are needed in order that all m linear effects B_1, B_2, \dots, B_m and all m quadratic effects $B_1^2, B_2^2, \dots, B_m^2$ become estimable. The three treatment combinations in block h , for $(h = 1, 2, \dots, m)$ will be of the form $(0)_{t,h}$, $(1)_{t,h}$ and $(2)_{t,h}$, for some t where $(t = 1, 2, \dots, m)$. However, since the treatment combinations in (5.2) are permutation-invariant, we assume without loss of generality that

$$\text{block } t \text{ contains } (0)_{t,t}, (1)_{t,t} \text{ and } (2)_{t,t} \text{ for } (t = 1, 2, \dots, m). \tag{5.38}$$

In the blocking scheme (5.38), each factor's linear and quadratic effects are estimable from only one block since from (5.11), (5.14) and (5.37),

$$\begin{aligned}
 (2)_{t,t} - (0)_{t,t} &= 2 \cdot B_t \\
 (0)_{t,t} - 2 \cdot (1)_{t,t} + (2)_{t,t} &= 6 \cdot B_t^2
 \end{aligned}$$

hence

$$\hat{B}_t = \frac{1}{2}(y(2)_{t,t} - y(0)_{t,t})$$

and

$$\hat{B}_t^2 = \frac{1}{6} (y(0)_{t,t} - 2 \cdot y(1)_{t,t} + y(2)_{t,t}) \tag{5.39}$$

with $\text{var } \hat{B}_t = \frac{1}{2}\sigma^2$ and $\text{var } \hat{B}_t^2 = \frac{1}{6}\sigma^2$. Obviously, additional factors

can be added to the OBAT 3^m plan in (5.38) and their effects can be estimated as in (5.39).

Case (2): 3^m_V Plans in a Sequence of Blocks of Size 3.

When two-factor interactions are non-negligible, i.e. model (5.36) holds the contrasts in (5.39) do not estimate the linear and quadratic effects of the t^{th} factor rather they estimate, respectively

$$\begin{aligned} 1. \quad B_t - \sum_{u \neq t} B_t B_u + \sum_{u \neq t} B_t B_u^2 &= B_t - L_t L(t) + L_t Q(t) \\ 2. \quad B_t^2 - \sum_{u \neq t} B_t^2 B_u + \sum_{u \neq t} B_t^2 B_u^2 &= B_t^2 - Q_t L(t) + Q_t Q(t) \end{aligned} \quad (5.40)$$

where ($t = 1, 2, \dots, m$).

The functions in (5.40) will be denoted by

$$\begin{aligned} 1. \quad S_{tu} &= B_t - \left(\sum_{u=1}^{t-1} B_t B_u + \sum_{u=t+1}^m B_t B_u \right) + \left(\sum_{u=1}^{t-1} B_t B_u^2 + \sum_{u=t+1}^m B_t B_u^2 \right) \\ 2. \quad S_{tu}^2 &= B_t^2 - \left(\sum_{u=1}^{t-1} B_t^2 B_u + \sum_{u=t+1}^m B_t^2 B_u \right) + \left(\sum_{u=1}^{t-1} B_t^2 B_u^2 + \sum_{u=t+1}^m B_t^2 B_u^2 \right) \end{aligned} \quad (u = t+1, t+2, \dots, m) \text{ and } (t = 1, 2, \dots, m). \quad (5.41)$$

In (5.41), the linear effect B_t is aliased with the $(m-1)$ interaction effects $B_t B_u$, $u \neq t$ and with the $(m-1)$ interaction effects $B_t B_u^2$, $u \neq t$. Therefore, to free B_t and B_t^2 from aliasing more blocks must be added to the OBAT plan in (5.38). Assuming factors are ordered such that factor B_1 is to be freed from aliasing first, factor B_2 second, \dots , factor B_m last, in stages, such that in stage t the following functions become estimable B_t , B_t^2 , $B_t B_u$, $B_t B_u^2$, $B_t^2 B_u$ and $B_t^2 B_u^2$ ($u = t+1, t+2, \dots, m$ and $t = 1, 2, \dots, m$). For purposes of illustration we describe

stages 1 and 2 first and then discuss the general stage t .

Stage One: The estimation of $B_1, B_1B_u, B_1B_u^2, B_1^2, B_1^2B_u$ and $B_1^2B_u^2$, ($u = 2, 3, 4, \dots, m$).

Stage one will be carried out in two steps:

a) step 1: The estimation of the $(m-1)$ interaction effects B_1B_u ($u = 2, 3, \dots, m$) by adding $(m-1)$ blocks to (5.38) where

$$\text{block}(m+u): (0)_{,m+u-1}, (11)_{1u,m+u-1}, (22)_{1u,m+u-1}, (u = 2, 3, \dots, m). \quad (5.42)$$

b) step 2: The estimation of the remaining effects involving the first factor, i.e. the effects $B_1, B_1B_u^2, B_1^2, B_1^2B_u$ and $B_1^2B_u^2$, ($u = 2, 3, \dots, m$) by adding $(m-1)$ more blocks to the $m + m-1 = 2m-1$ blocks in (5.42) where

$$\text{block}(2m-1+u-1): (0)_{,2m-1+u-1}, (12)_{1u,2m-1+u-1}, (21)_{1u,2m-1+u-1}, \\ (u = 2, 3, \dots, m).$$

We notice here that step 2 is not needed if the interactions $B_1B_u^2, B_1^2B_u$ and $B_1^2B_u^2$, ($u = 2, 3, \dots, m$) are negligible. That is, from step 1 the effects B_1, B_1^2 and B_1B_u become estimable.

a) For step 1 we have the $m + (m-1) = 2m - 1$ blocks

$$\text{block } t: (0)_{,t}, (1)_{t,t}, (2)_{t,t}, (t = 1, 2, \dots, m). \\ \text{block } (m+u-1): (0)_{,m+u-1}, (11)_{1u,m+u-1}, (22)_{1u,m+u-1}, (u = 2, 3, \dots, m). \quad (5.43)$$

Estimable functions after the completion of the $(2m - 1)$ blocks in (5.43) are

1. $B_1 + \sum_{u=2}^m B_1 B_u^2$
 2. The $(m-1)$ interaction effects $B_1 B_u$, $(u = 2, 3, \dots, m)$.
 3. The $(m-1)$ alias chains $B_1 B_u^2 + B_1^2 B_u - 3B_1^2 B_u^2$, $(u = 2, 3, \dots, m)$.
 4. $(B_1^2 + B_1 B_u^2 - 2B_1^2 B_u^2) + (\sum_{v \neq 1, u} B_1 B_v^2 - 2 \cdot \sum_{v=1, u} B_1^2 B_v^2)$
 $= B_1^2 + \sum_{u=2}^m B_1 B_u^2 - 2 \cdot \sum_{u=2}^m B_1^2 B_u^2$.
 5. $(B_u - B_1 B_u^2 + 3B_1^2 B_u^2) - \sum_{v \neq 1, u} B_u B_v + \sum_{v \neq 1, u} B_u B_v^2$, $(u = 2, 3, \dots, m)$.
 6. $(B_u^2 - B_1 B_u^2 + B_1^2 B_u^2) - \sum_{v \neq 1, u} B_u^2 B_v + \sum_{v \neq 1, u} B_u^2 B_v^2$, $(u = 2, 3, \dots, m)$.
- (5.44)

Letting $h = m + u - 1$, the unbiased least squares estimates of the functions in (5.44) are, respectively,

1. $\frac{1}{4} \cdot \sum_{u=2}^m [(y(22)_{1u,h} - y(0)_{,h}) - (y(2)_{u,u} - y(0)_{,u})]$
 $- \frac{1}{4(m-3)} (y(2)_{1,1} - y(0)_{,1})$
2. $\hat{B}_1 B_u = \frac{1}{4} [(y(22)_{1u,h} - y(0)_{,h}) - (y(2)_{1,1} - y(0)_{,1})]$
 $- (y(2)_{u,u} - y(0)_{,u})]$
3. $\frac{1}{12} [(3y(0)_{,h} - 4y(11)_{1u,h} + y(22)_{1u,h}) - (3y(0)_{,1} - 4y(1)_{1,1})]$
 $+ y(2)_{1,1} - (3y(0)_{,u} - 4y(1)_{u,u} + y(2)_{u,u})]$

$$4. \frac{1}{12} \sum_{u=2}^m [(3y(0)_{,h} - 4y(11)_{1u,h} + y(22)_{1u,h}) - (3y(0)_{,u} - 4y(1)_{u,u} + y(2)_{u,u})] - \frac{1}{12}(c_1 y(0)_{,1} - (c_1 + c_2)y(1)_{1,1} + c_2 y(2)_{1,1})$$

where $c_1 = 3m - 5$ and $c_2 = m - 3$.

$$5. \frac{1}{6} [(2y(11)_{1u,h} + y(22)_{1u,h} - 3y(0)_{,h}) - (2y(1)_{1,1} + y(2)_{1,1} - 3y(0)_{,1})] + \frac{1}{3} (y(2)_{u,u} - y(1)_{,u})$$

$$6. \frac{1}{6} (y(0)_{,u} - 2y(1)_{u,u} + y(2)_{u,u}) \quad (5.45)$$

Variances of the estimates in (5.45) are

$$1. \frac{1}{8} (m^2 - 4m + 7)\sigma^2$$

$$2. \frac{3}{8} \sigma^2$$

$$4. \frac{13}{24} \sigma^2$$

$$5. \frac{7}{9} \sigma^2$$

$$6. \frac{1}{6} \sigma^2 .$$

We show below that the estimates in (5.45) are unbiased.

All estimates in (5.45) involve comparisons between responses in the same block hence block effects cancel out. Therefore,

1. The expected value of the estimate (1) in (5.45) is

$$\begin{aligned} & \frac{1}{4} \sum_{u=2}^m [((22)_{1u} - (0)) - ((2)_u - (0))] - \frac{1}{4}(m-3)((2)_1 - (0)) \\ & = \frac{1}{4} \sum_{u=2}^m ((22)_{1u} - (2)_u) - \frac{1}{4}(m-3)((2)_1 - (0)) \end{aligned}$$

Referring to (5.20) and (5.11)

$$\begin{aligned}
 &= \frac{1}{2}[2(m-1)B_1 - 2(m-3)L_1L_{(1)} + 2(m-1)L_1Q_{(1)}] \\
 &\quad - \frac{1}{2}(m-3)[2B_1 - 2L_1L_{(1)} + 2L_1Q_{(1)}] \\
 &= B_1 + L_1Q_{(1)} = B_1 + \sum_{u=2}^m B_1 B_u^2 .
 \end{aligned}$$

2. The expected value of the estimate (2) in (5.45) is

$$\begin{aligned}
 &\frac{1}{2}[(22)_{1u} - (0)] - ((2)_1 - (0)) - ((2)_u - (0)) \\
 &= \frac{1}{2}[(22)_{1u} - (2)_u] - ((2)_1 - (0))
 \end{aligned}$$

referring to (5.12) and (5.11)

$$= B_1 B_u .$$

3. The expected value of the estimate (3) in (5.45) is

$$\begin{aligned}
 &\frac{1}{2}[(3(0) - 4(11)_{1u} + (22)_{1u}) - (3(0) - 4(1)_1 + (2)_1) \\
 &\quad - (3(0) - 4(1)_u + (2)_u)] \\
 &= \frac{1}{2}[(22)_{1u} - (2)_u] - 4((11)_{1u} - (1)_u) - (3(0) - 4(1)_1 + (2)_1)
 \end{aligned} \tag{5.46}$$

From (5.10) we find that

$$\begin{aligned}
 ((11)_{1u} - (1)_u) &= B_1 - 3B_1^2 - L_1L_{(1,u)} + L_1Q_{(1,u)} - 2B_1B_u^2 + 3Q_1L_{(1,u)} \\
 &\quad + 6B_1^2B_u^2 - 3Q_1Q_{(1,u)}
 \end{aligned} \tag{5.47}$$

$$\begin{aligned}
(3(0) - 4(1)_1 + (2)_1) &= -2B_1 + 12B_1^2 + 2B_1B_u + 2L_1L_{(1,u)} - 2B_1B_u^2 - 2L_1Q_{(1,u)} \\
&\quad - 12B_1B_u^2 - 12Q_1L_{(1,u)} + 12B_1^2B_u^2 + 12Q_1Q_{(1,u)}
\end{aligned} \tag{5.48}$$

From (5.12), (5.46), (5.47) and (5.48) unbiasedness follows.

4. The expected value of the estimate (4) in (5.45) is

$$\begin{aligned}
\frac{1}{12} \sum_{u=2}^m [((22)_{1u} - (2)_u) - 4((11)_{1u} - (1)_u)] \\
- \frac{1}{12} [(3m-5)(0) - 4(m-2)(1)_1 - (m-3)(2)_1]
\end{aligned} \tag{5.49}$$

and from (5.12) and (5.47),

$$\begin{aligned}
\frac{1}{12} \sum_{u=2}^m [((22)_{1u} - (2)_u) - 4((11)_{1u} - (1)_u)] &= \frac{1}{12} \sum_{u=2}^m [-2B_1 + 12B_1^2 + 2B_1B_u \\
&\quad + 2L_1L_{(1,u)} + 10B_1B_u^2 - 2L_1Q_{(1,u)} - 12Q_1L_{(1,u)} - 24B_1^2B_u + 12Q_1Q_{(1,u)}]
\end{aligned}$$

referring to (4.9)

$$\begin{aligned}
&= \frac{1}{12} [-2(m-1)B_1 + 12(m-1)B_1^2 + 2L_1L_{(1)} + 2(m-2)L_1L_{(1)} \\
&\quad + 10L_1Q_{(1)} - 2(m-2)L_1Q_{(1)} - 12(m-2)Q_1L_{(1)} \\
&\quad - 24Q_1Q_{(1)} + 12(m-2)Q_1Q_{(1)}] \\
&= \frac{1}{12} [-2(m-1)B_1 + 12(m-1)B_1^2 + 2(m-1)L_1L_{(1)} - 2(m-7)L_1Q_{(1)} \\
&\quad - 12(m-2)Q_1L_{(1)} + 12(m-4)Q_1Q_{(1)}]
\end{aligned} \tag{5.50}$$

From (5.11) we find that

$$\begin{aligned}
((3m-5)(0) - 4(m-2)(1)_1 - (2)_1) &= -2(m-2)B_1 + 12(m-2)B_1^2 \\
&+ 2(m-2)L_1L_{(1)} - 2(m-2)L_1Q_{(1)} \\
&- 12(m-2)Q_1L_{(1)} + 12(m-2)Q_1Q_{(1)}
\end{aligned} \tag{5.51}$$

From (5.50) and (5.51) unbiasedness of the estimate in (4) follows.

5. The expected value of the estimate (5) in (5.45) is

$$\frac{1}{6} [2((11)_{1u} - (1)_1) + ((22)_{1u} - (2)_u)] + \frac{1}{3} ((2)_u - (1)_u)$$

from (5.10),

$$\begin{aligned}
(2)_u - (1)_u &= B_u + 3B_u^2 - B_t B_u - L_u L_{(t,u)} - 3B_t B_u^2 + L_u Q_{(t,u)} + B_t^2 B_u \\
&- 3Q_u L_{(t,u)} + 3B_t^2 B_u^2 + 3Q_u Q_{(t,u)}
\end{aligned} \tag{5.52}$$

from (5.47), (5.52) and (5.12), unbiasedness follows too.

6. Unbiasedness of the estimate (6) in (5.45) follows by referring to (5.39) and (5.40).

b) For step 2 we have the $m + (m-1) + (m-1) = (3m-2)$ blocks

block t: $(0)_{,t} \quad (1)_{t,t} \quad (2)_{t,t}, \quad (t = 1, 2, \dots, m).$

block h: $(0)_{,h} \quad (11)_{1u,h} \quad (22)_{1u,h}, \quad h = m+u-1 \text{ and } u = 2, 3, \dots, m.$

block k: $(0)_{,k} \quad (12)_{1u,k} \quad (21)_{1u,k}, \quad k = 2m-1+u-1 \text{ and } u = 2, 3, \dots, m.$

(5.53)

Estimable functions after the completion of the $(3m-2)$ blocks in (5.53) are

1. B_1

2. B_1^2

3. a.
$$S_{2u} = B_u - \left(\sum_{v=3}^{u-1} B_u B_v + \sum_{v=u+1}^m B_u B_v \right) + \left(\sum_{v=3}^{u-1} B_u B_v^2 + \sum_{v=u+1}^m B_u B_v^2 \right)$$

$$= B_u - L_u L(1,u) + L_u Q(1,u), (u = 2, 3, \dots, m).$$

b.
$$S_{2u}^2 = B_u^2 - \left(\sum_{v=3}^{u-1} B_u^2 B_v + \sum_{v=u+1}^m B_u^2 B_v \right) + \left(\sum_{v=3}^{u-1} B_u^2 B_v^2 + \sum_{v=u+1}^m B_u^2 B_v^2 \right)$$

$$= B_u^2 - Q_u L(1,u) + Q_u Q(1,u), (u = 2, 3, \dots, m).$$

4. The $4(m-1)$ interaction effects

a. $B_1 B_u, u = 2, 3, \dots, m.$

b. $B_1 B_u^2, u = 2, 3, \dots, m.$

c. $B_1^2 B_u, u = 2, 3, \dots, m.$

d. $B_1^2 B_u^2, u = 2, 3, \dots, m.$

(5.54)

The unbiased estimates of the functions in (5.54) are

1.
$$\hat{B}_1 = \frac{1}{6} \sum_{u=2}^m [(y(22)_{1u,h} - y(0)_{,h}) + (y(21)_{1u,k} - y(0)_{,k}) - (y(1)_{u,u} + y(2)_{u,u} - 2y(0)_{,u})] - \frac{1}{6}(2m-5)(y(2)_{1,1} - y(0)_{,1})$$

2.
$$\hat{B}_1^2 = \frac{1}{18} \sum_{u=2}^m [(y(0)_{,h} - 2y(11)_{1u,h} + y(22)_{1u,h}) + (y(0)_{,k} - 2y(12)_{1u,k} + y(21)_{1u,k}) + (y(1)_{u,u} + y(2)_{u,u} - 2y(0)_{,u})] - \frac{1}{18}(2m-5)(y(0)_{,1} - 2y(1)_{1,1} + y(2)_{1,1})$$

$$\begin{aligned}
3. \quad a. \quad \hat{S}_{2u} &= \frac{1}{6} [(y(22)_{1u,h} - y(0)_{,h}) + (y(12)_{1u,k} - y(0)_{,k})] \\
&\quad + \frac{1}{6} [(y(2)_{u,u} - y(0)_{,u}) - (y(1)_{1,1} + y(2)_{1,1} - 2y(0)_{,1})] \\
b. \quad \hat{S}_{2u}^2 &= \frac{1}{18} [(y(0)_{,h} - 2y(11)_{1u,h} + y(22)_{1u,h}) + (y(0)_{,k} \\
&\quad - 2y(21)_{1u,k} + y(12)_{1u,k})] + \frac{1}{18} [(y(0)_{,u} - 2y(1)_{u,u} \\
&\quad + y(2)_{u,u}) + (y(1)_{1,1} + y(2)_{1,1} - 2y(0)_{,1})] \\
4. \quad a. \quad \hat{B}_1 B_u &\text{ is the same as that given by (2) in (5.45).} \\
b. \quad \hat{B}_1 B_u^2 &= \frac{1}{12} [(y(2)_{1,1} - y(0)_{,1}) - (y(0)_{,u} - 2y(1)_{u,u} + y(2)_{u,u}) \\
&\quad + (y(22)_{1u,h} - y(0)_{,h}) - 2(y(21)_{1u,k} - y(0)_{,k})] \\
c. \quad \hat{B}_1^2 B_u &= \frac{1}{12} [(y(22)_{1u,h} - y(0)_{,h}) + (y(2)_{u,u} - y(0)_{,u}) \\
&\quad - 2(y(12)_{1u,k} - y(0)_{,k}) - (y(0)_{,1} - 2y(1)_{1,1} + y(2)_{1,1})] \\
d. \quad \hat{B}_1^2 B_u^2 &= \frac{1}{36} [(y(0)_{,1} - 2y(1)_{1,1} + y(2)_{1,1}) + (y(0)_{,u} - 2y(1)_{u,u} \\
&\quad + y(2)_{u,u}) + (4y(11)_{1u,h} + y(22)_{1u,h} - 5y(0)_{,h}) \\
&\quad - 2(y(12)_{1u,k} + y(21)_{1u,k} - 2y(0)_{,k})] \quad (5.55)
\end{aligned}$$

Variances of the estimates in (5.55) are, respectively

$$\begin{aligned}
1. \quad \text{var } \hat{B}_1 &= \frac{1}{36}(m-1)[(2\sigma^2) + (2\sigma^2) + (1+1+4)\sigma^2] \\
&\quad + \frac{1}{36}(2m-5)^2(2\sigma^2) \\
&= \frac{1}{18}(4m^2 - 15m + 20)\sigma^2
\end{aligned}$$

$$\begin{aligned}
 2. \quad \text{var } \hat{B}_1^2 &= \left(\frac{1}{18}\right)^2 (m-1) [3(1+4+1)^2] \\
 &\quad + \left(\frac{1}{18}\right)^2 (2m-5)^2 (1+4+1) \sigma^2 \\
 &= \frac{1}{54} (4m^2 - 17m + 22) \sigma^2
 \end{aligned}$$

$$3. \quad \text{a.} \quad \text{var } \hat{S}_{2u} = \frac{1}{3} \sigma^2$$

$$\text{b.} \quad \text{var } \hat{S}_{2u}^2 = \frac{2}{27} \sigma^2$$

$$4. \quad \text{a.} \quad \text{var } \hat{B}_{1u} = \frac{3}{8} \sigma^2$$

$$\text{b.} \quad \text{var } \hat{B}_{1u}^2 = \frac{2}{27} \sigma^2$$

$$\text{c.} \quad \text{var } \hat{B}_{1u}^2 = \frac{2}{27} \sigma^2$$

$$\text{d.} \quad \text{var } \hat{B}_{1u}^{2^2} = \frac{39}{648} \sigma^2$$

Recalling that block effects cancel out in the expected values of the estimates in (5.55), their unbiasedness can be shown as follows.

$$\begin{aligned}
 1. \quad \hat{EB}_1 &= \frac{1}{6} \sum_{u=2}^m [((22)_{1u} - (2)_u) + ((21)_{1u} - (1)_u)] \\
 &\quad - \frac{1}{6} (2m-5)((2)_1 - (0))
 \end{aligned}$$

and referring to (5.24) it follows that $\hat{EB}_1 = B_1$.

$$\begin{aligned}
 2. \quad \hat{EB}_1^2 &= \frac{1}{18} \sum_{u=2}^m [((2)_u - 2(12)_{1u} + (22)_{1u}) + ((1)_u - 2(11)_{1u} \\
 &\quad + (21)_{1u})] - \frac{1}{18} (2m-5)((0) - 2(1)_1 + (2)_1)
 \end{aligned}$$

and referring to (5.28) it follows that $\hat{EB}_1^2 = B_1^2$.

$$3. \quad a. \quad \widehat{ES}_{2u} = \frac{1}{6} [((22)_{1u} - (2)_1) + ((12)_{1u} - (1)_1) + ((2)_u - (0))]]$$

From (5.10) we find that

$$(22)_{1u} - (2)_1 = 2B_u + 2B_1B_u - 2L_uL(1,u) + 2L_uQ(1,u) + 2B_1^2B_u \quad (5.56)$$

$$(12)_{1u} - (1)_1 = 2B_u - 2L_uL(1,u) + 2L_uQ(1,u) - 4B_1^2B_u \quad (5.57)$$

$$(2)_u - (0) = 2B_u - 2B_1B_u - 2L_uL(1,u) + 2L_uQ(1,u) + 2B_1^2B_u \quad (5.58)$$

Adding (5.56), (5.57) and (5.58), unbiasedness of \widehat{S}_{2u} follows

$$b. \quad \widehat{ES}_{2u}^2 = \frac{1}{18} [((1)_1 - 2(11)_{1u} + (12)_{1u}) + ((2)_1 - 2(21)_{1u} + (22)_{1u}) \\ + ((0) - 2(1)_u + (2)_u)]$$

From (5.10) we find that

$$(1)_1 - 2(11)_{1u} + (12)_{1u} = 6B_u^2 - 6Q_uL(1,u) - 12B_1^2B_u + 6Q_uQ(1,u) \quad (5.59)$$

$$(2)_1 - 2(21)_{1u} + (22)_{1u} = 6B_u^2 + 6B_tB_u^2 - 6Q_uL(1,u) + 6B_t^2B_u^2 + 6Q_uQ(1,u) \quad (5.60)$$

$$(0) - 2(1)_u + (2)_u = 6B_u^2 - 6B_1B_u^2 - 6Q_uL(1,u) + 6B_1^2B_u^2 + 6Q_uQ(1,u) \quad (5.61)$$

Adding (5.59), (5.60) and (5.61), unbiasedness of \widehat{S}_{2u}^2 follows.

4. a. unbiasedness of \widehat{B}_1B_u was established in step 1.

$$b. \quad E\widehat{B}_1B_u^2 = \frac{1}{12} [((22)_{1u} - (2)_u) - 2((21)_{1u} - (1)_u) + ((2)_1 - (0))]]$$

$$= \frac{1}{12} [((2)_1 - 2(21)_{1u} + (22)_{2u}) - ((0) - 2(1)_u + (2)_u)]$$

From (5.11), (5.12) and (5.13), unbiasedness follows.

c. unbiasedness of $B_1^2 B_u$ can be shown in a similar way as the unbiasedness of $B_1^2 B_u^2$ in b.

$$\begin{aligned} \text{d. } EB_1^2 B_u^2 &= \frac{1}{36} [((0) - 2(1)_1 + (2)_1) - 2((1)_u - 2(11)_{1u} + (21)_{1u}) \\ &\quad + ((2)_u - 2(12)_{1u} + (22)_{1u})] \end{aligned}$$

referring to (5.32) it follows that $EB_1^2 B_u^2 = B_1^2 B_u^2$.

Stage Two: The estimation of B_2 , $B_2 B_u$, $B_2 B_u^2$, B_2^2 , $B_2^2 B_u$ and $B_2^2 B_u^2$, for $(u = 3, 4, \dots, m)$.

Referring to (3) in (5.54), we notice that

a. the linear effect B_2 is still aliased with the $(m-2)$ interaction effects $B_2 B_u$, $(u = 3, 4, \dots, m)$ and the $(m-2)$ interactions $B_2 B_u^2$.

b. the quadratic effect B_2^2 is still aliased with the $(m-2)$ interactions $B_2^2 B_u$, $(u = 3, 4, \dots, m)$ and the $(m-2)$ interactions $B_2^2 B_u^2$.

Therefore, to free both B_2 and B_2^2 from aliasing, $2(m-2)$ blocks of size 3 must be added to the $(3m-2)$ blocks in (5.53). However, the $2(m-2)$ blocks will be added in two steps as we did in stage one where

Step 1 involves the estimation of the $(m-1)$ interactions $B_2 B_u$, $(u = 3, 4, \dots, m)$ by adding the $(m-2)$ blocks

$$\text{block } [(3m-2)+(u-2)] : (0)_{,3m+u-4}, (11)_{2u,3m+u-4}, (22)_{2u,3m+u-4},$$

$$u = 3, 4, \dots, m. \quad (5.62)$$

and

Step 2 involves the estimation of the remaining effects involving the second factor, i.e. B_2 , B_2^2 , $B_2 B_u^2$, $B_2^2 B_u$ and $B_2^2 B_u^2$, by adding the (m-2) blocks

$$\text{block}(4m-3+u-2): \quad (0)_{,4m+u-5}, \quad (12)_{2u,4m+u-5}, \quad (21)_{2u,4m+u-5},$$

$$u = 3, 4, \dots, m. \quad (5.63)$$

along with the (m-2) blocks in step 1 above to the blocks in (5.53).

For stage two, we find the following

a) After the completion of the (4m-4) blocks in step 1 of stage two, the estimable functions are

1. B_1 ; B_1^2
2. $B_1 B_u$; $B_1 B_u^2$; $B_1^2 B_u$; $B_1^2 B_u^2$.
3. $B_2 + \sum_{u=3}^m B_2 B_u^2$, ($u = 3, 4, \dots, m$).
4. The (m-1) linear by linear interactions $B_2 B_u$, $u = 3, 4, \dots, m$.
5. The (m-1) alias chains $B_2 B_u^2 + B_2^2 B_u - 3B_2^2 B_u^2$, $u = 3, 4, \dots, m$.
6. $(B_2^2 + B_2 B_u^2 - 2B_2^2 B_u^2) + \sum_{u=3}^m B_2 B_u^2 - 2 \cdot \sum_{u=3}^m B_2^2 B_u^2$.
7. a. $(B_u - B_2 B_u^2 + 3B_2^2 B_u^2) - \left[\sum_{v=4}^{u-1} B_u B_v + \sum_{v=u+1}^m B_u B_v \right]$
 $+ \left[\sum_{v=4}^{u-1} B_u B_v^2 + \sum_{v=u+1}^m B_u B_v^2 \right]$, ($u = 3, 4, \dots, m$).

$$\begin{aligned}
 \text{b. } & (B_u^2 - B_2 B_u^2 + B_2^2 B_u^2) - \left[\sum_{v=4}^{u-1} B_u^2 B_v^2 + \sum_{v=u+1}^m B_u^2 B_v^2 \right] \\
 & + \left[\sum_{v=4}^{u-1} B_u^2 B_v^2 + \sum_{v=u+1}^m B_u^2 B_v^2 \right] \quad , \quad (u = 3, 4, \dots, m) \quad (5.64)
 \end{aligned}$$

The unbiased estimates of the functions (1) and (2) in (5.64) are given by (1), (2) and (4) in (5.55). Letting $i = 3m - 2 + u - 2$, for $(u = 3, 4, \dots, m)$, the unbiased estimate of the remaining functions in (5.64) are, respectively

$$\begin{aligned}
 3. \quad & \frac{1}{4} \sum_{u=3}^m [(y(22)_{2u,i} - y(0)_{,i}) - (y(2)_{u,u} - y(0)_{,u})] \\
 & + \frac{1}{6} [(y(22)_{12,h} - y(0)_{,h}) + (y(12)_{12,k} - y(0)_{,k}) \\
 & - (y(1)_{1,1} + y(2)_{1,1} - 2y(0)_{,1})] - c_1 (y(2)_{2,2} - y(0)_{,2})
 \end{aligned}$$

where $h = m + 2 - 1 = m + 1$ and $k = (2m-1) + (2-1) = 2m$. The scalar c_1 is also a function of m .

$$\begin{aligned}
 4. \quad & \hat{B}_2 B_u = \frac{1}{4} [(y(22)_{2u,i} - y(0)_{,i}) - (y(2)_{2,2} - y(0)_{,2}) \\
 & - (y(2)_{u,u} - y(0)_{,u})]
 \end{aligned}$$

$$\begin{aligned}
 5. \quad & = \frac{1}{12} [(3y(0)_{,i} - 4y(11)_{2u,i} + y(22)_{2u,i}) - (3y(0)_{,2} \\
 & - 4y(1)_{2,2} + y(2)_{2,2}) - (3y(0)_{,u} - 4y(1)_{u,u} + y(2)_{u,u})]
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & \frac{1}{12} \sum_{u=3}^m [(3y(0)_{,i} - 4y(11)_{2u,i} + y(22)_{2u,i}) - (3y(0)_{,u} - 4y(1)_{u,u} \\
 & + y(2)_{u,u})] + \frac{1}{18} [(y(0)_{,h} - 2y(11)_{12,h} + y(22)_{12,h})
 \end{aligned}$$

$$\begin{aligned}
& + (y(0)_{,k} - 2y(21)_{12,k} + y(12)_{12,k}) \\
& + (y(1)_{1,1} + y(2)_{1,1} - 2y(0)_{,1})] \\
& + (c_2 y(0)_{,2} - (c_2 + c_3) y(1)_{2,2} + c_3 y(2)_{2,2})
\end{aligned}$$

where $h = m + 1$ and $k = 2m$. The scalars c_2 and c_3 are functions of m .

$$\begin{aligned}
7. \quad a. \quad & \frac{1}{6} [(2y(11)_{2u,i} + y(22)_{2u,i} - 3y(0)_{,i}) - (2y(1)_{2,2} + y(2)_{2,2} \\
& - 3y(0)_{,2})] + \frac{1}{6} [(y(22)_{1u,h} - y(0)_{,h}) + (y(12)_{1u,k} \\
& - y(0)_{,k}) - (y(1)_{,1} + y(2)_{1,1} - 2y(0)_{,1})] \\
& - \frac{1}{3} (y(1)_{u,u} - y(0)_{,u})
\end{aligned}$$

$$\begin{aligned}
b. \quad & \frac{1}{18} [y(0)_{,h} - 2y(11)_{1u,h} + y(22)_{1u,h}) + (y(0)_{,k} - 2y(21)_{1u,k} \\
& + y(12)_{1u,k}) + (y(1)_{1,1} + y(2)_{1,1} - 2y(0)_{,1})] \\
& + \frac{1}{18} (y(0)_{,u} - 2y(1)_{u,u} + y(2)_{u,u}) \quad (5.65)
\end{aligned}$$

where $(h = m + u - 1)$, $(k = 2m - 1 + u - 1)$ and $(u = 3, 4, \dots, m)$. The scalars c_1 in (3) and c_2 and c_3 in (6) above can be obtained by equating the expected values of the estimates in (3) and (6) with their corresponding functions (3) and (6) in (5.64). The unbiasedness of the estimates (3) - (7) can be established in a manner similar to that in step 1 of stage one.

b) After the completion of the $4m - 4 + m - 2 = 5m - 6$ blocks in step 2 of stage two, the estimable functions are

$$\begin{aligned}
3. \quad \hat{B}_2^2 &= \frac{1}{18} [(y(0)_{,h} - 2y(11)_{12,h} + y(22)_{12,h}) + (y(0)_{,k} \\
&\quad - 2y(21)_{12,k} + y(12)_{12,k}) + (2y(0)_{,1} - y(1)_{1,1} - y(2)_{1,1})] \\
&\quad + \frac{1}{18} \sum_{u=3}^m [(y(0)_{,j} - 2y(12)_{2u,j} + y(21)_{2u,j}) \\
&\quad + (y(1)_{u,u} + y(2)_{u,u} - 2y(0)_{,u}) \\
&\quad + (y(0)_{,i} - 2y(11)_{2u,i} + y(22)_{2u,i})] \\
&\quad - \frac{1}{6} (2m-5)(y(0)_{,2} - 2y(1)_{2,2} + y(2)_{2,2})
\end{aligned}$$

$$\begin{aligned}
4. \quad a. \quad \hat{B}_2^2 B_u &= \frac{1}{4} [(y(22)_{2u,i} - y(0)_{,i}) - (y(2)_{2,2} - y(0)_{,2}) \\
&\quad - (y(2)_{u,u} - y(0)_{,u})]
\end{aligned}$$

$$\begin{aligned}
b. \quad \hat{B}_2^2 B_u^2 &= \frac{1}{12} [y(2)_{2,2} - y(0)_{,1}) - (y(0)_{,u} - 2y(1)_{u,u} + y(2)_{u,u}) \\
&\quad + (y(22)_{2u,i} - y(0)_{,i}) - 2(y(21)_{2u,j} - y(0)_{,j})]
\end{aligned}$$

$$\begin{aligned}
c. \quad \hat{B}_2^2 B_u &= \frac{1}{12} [(y(22)_{2u,i} - y(0)_{,i}) + (y(2)_{u,u} - y(0)_{,u}) \\
&\quad - 2(y(12)_{2u,j} - y(0)_{,j}) - (y(0)_{,2} - 2y(1)_{2,2} + y(2)_{2,2})]
\end{aligned}$$

$$\begin{aligned}
d. \quad \hat{B}_2^2 B_u^2 &= \frac{1}{36} [(y(0)_{,2} - 2y(1)_{2,2} + y(2)_{2,2}) + (y(0)_{,u} - 2y(1)_{u,u} \\
&\quad + y(2)_{u,u}) + (4y(11)_{2u,i} + y(22)_{2u,i} - 5y(0)_{,i}) \\
&\quad - 2(y(12)_{2u,j} + y(21)_{2u,j} - 2y(0)_{,j})]
\end{aligned}$$

$$\begin{aligned}
5. \quad a. \quad \hat{S}_{u3} &= \frac{1}{6} [(y(22)_{1u,h} - y(0)_{,h}) + (y(12)_{1u,k} - y(0)_{,k}) \\
&\quad - (y(1)_{1,1} + y(2)_{1,1} - 2y(0)_{,1})] \\
&\quad + \frac{1}{6} [(y(22)_{2u,i} - y(0)_{,i}) + (y(12)_{2u,j} - y(0)_{,j}) \\
&\quad - (y(1)_{2,2} + y(2)_{2,2} - 2y(0)_{,2})] \\
&\quad - \frac{1}{6}(y(2)_{u,u} - y(0)_{,u}) \\
b. \quad \hat{S}_{u3}^2 &= \frac{1}{18} [(y(0)_{,h} - 2y(11)_{1u,h} + y(22)_{1u,h}) + (y(0)_{,k} \\
&\quad - 2y(21)_{1u,k} + y(12)_{1u,k}) + (y(1)_{1,1} + y(2)_{1,1} \\
&\quad - 2y(0)_{,1})] \\
&\quad + \frac{1}{18} [(y(0)_{,i} - 2y(11)_{2u,i} + y(22)_{2u,i}) + (y(0)_{,j} \\
&\quad - 2y(21)_{2u,j} + y(12)_{2u,j}) + (y(1)_{2,2} + y(2)_{2,2} \\
&\quad - 2y(0)_{,2})] \\
&\quad - \frac{1}{18} (y(0)_{,u} - 2y(1)_{u,u} + y(2)_{u,u}) \quad (5.67)
\end{aligned}$$

The unbiasedness of the estimates in (5.67) can be established in a manner similar to that in step 2 of stage one.

With factors ordered as B_1, B_2, \dots, B_m and from stages one and two, we arrive at the following general result.

Stage (t): The estimation of $B_t, B_t^2, B_t B_u, B_t B_u^2, B_t^2 B_u$ and $B_t^2 B_u^2$ for $(u = t+1, t+2, \dots, m)$ and $(t = 1, 2, \dots, m)$.

After the completion of stage $(t-1)$ we notice the following

a. the linear effect B_t is aliased with the $(m-t)$ interaction effects $B_t B_u$, $(u = t+1, t+2, \dots, m)$ and the $(m-t)$ interactions $B_t B_u^2$.

b. the quadratic effect B_t^2 is aliased with the $(m-t)$ interactions $B_t^2 B_u$ and the $(m-t)$ interactions $B_t^2 B_u^2$, $(u = t+1, t+2, \dots, m)$.

Therefore, to free both B_t and B_t^2 from aliasing, $2(m-t)$ blocks of size 3 must be added to the

$$m + 2(m-1) + 2(m-2) + \dots + 2(m-t+1)$$

blocks in stage $(t-1)$, $(t = 1, 2, \dots, m)$. Letting

$$h_{t(u-t)} = m + 2(m-1) + 2(m-2) + \dots + 2(m-t+1) + (u-t)$$

and

$$k_{t(u-t)} = m + 2(m-1) + 2(m-2) + \dots + 2(m-t+1) + (m-t) + (u-t)$$

and conducting stage (t) in two steps, we have

Step 1. Involves the estimation of the $(m-t)$ interactions, $B_t B_u$, $(u = t+1, t+2, \dots, m)$ by adding the $(m-t)$ blocks

$$\text{block } h_{t(u-t)}: \quad (0)_{,h_{t(u-t)}}, \quad (11)_{tu,h_{t(u-t)}}, \quad (22)_{tu,h_{t(u-1)}} \quad (5.68)$$

to the $m + 2(m-1) + \dots + 2(m-t+1)$ blocks in stage $(t-1)$.

Step 2. Involves the estimation of the remaining effects involving the t^{th} factor, i.e. B_t , B_t^2 , $B_t B_u^2$, $B_t^2 B_u$ and $B_t^2 B_u^2$ by adding the blocks

$$\text{block } k_{t(u-t)}: (0)_{,k_{t(u-t)}}, (12)_{tu,k_{t(u-t)}}, (21)_{tu,k_{t(u-t)}} \quad (5.69)$$

to the $m + 2(m-1) + \dots + 2(m-t+1) + (m-t)$ blocks in step 1. We note here that Step 2 is not needed if the linear by quadratic, quadratic by linear and quadratic by quadratic interactions are negligible and hence the effects B_t , B_t^2 and $B_t B_u$ become estimable after the completion of step 1.

For stage t , the block structure is

$$\begin{array}{l}
 \text{Stage one} \\
 m+2(m-1) \text{ blocks} \\
 \left\{ \begin{array}{l}
 \text{block 1: } (0)_{,1} \quad (1)_{1,1} \quad (2)_{1,1} \\
 \text{block 2: } (0)_{,2} \quad (1)_{2,2} \quad (2)_{2,2} \\
 \vdots \\
 \text{block } m: (0)_{,m} \quad (1)_{m,m} \quad (2)_{m,m} \\
 \\
 \text{block } h_{11}: (0)_{,h_{11}} \quad (11)_{12,h_{11}} \quad (22)_{12,h_{11}}, \\
 \qquad \qquad \qquad h_{11} = m+1 \\
 \text{block } h_{12}: (0)_{,h_{12}} \quad (11)_{13,h_{12}} \quad (22)_{13,h_{12}}, \\
 \qquad \qquad \qquad h_{12} = m+2 \\
 \vdots \\
 \text{block } h_{1(m-1)}: (0)_{,h_{1(m-1)}} \quad (11)_{1m,h_{1(m-1)}} \quad (22)_{1m,h_{1(m-1)}}, \\
 \qquad \qquad \qquad h_{1(m-1)} = m+m-1 \\
 \\
 \text{block } k_{11}: (0)_{,k_{11}} \quad (12)_{12,k_{11}} \quad (21)_{12,k_{11}}, \\
 \qquad \qquad \qquad k_{11} = (2m-1)+1 \\
 \text{block } k_{12}: (0)_{,k_{12}} \quad (12)_{13,k_{12}} \quad (21)_{13,k_{12}}, \\
 \qquad \qquad \qquad k_{12} = (2m-1)+2 \\
 \vdots
 \end{array} \right.
 \end{array}$$

$$\left(\text{block } k_{1(m-1)} : (0)_{,k_{1(m-1)}} \quad (12)_{1m,k_{1(m-1)}} \quad (21)_{1m,k_{1(m-1)}} \right),$$

$$k_{1(m-1)} = 3m-2$$

⋮

$$\left\{ \begin{array}{l} \text{block } h_{t1} : (0)_{,h_{t1}} \quad (11)_{t(t+1),h_{t1}} \quad (22)_{t(t+1),h_{t1}}, \\ h_{t1} = m + 2 \sum_{i=1}^{t-1} (m-i) + 1 \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{block } h_{t2} : (0)_{,h_{t2}} \quad (11)_{t(t+2),h_{t2}} \quad (22)_{t(t+2),h_{t2}}, \\ h_{t2} = m + 2 \sum_{i=1}^{t-1} (m-i) + 2 \end{array} \right.$$

⋮

$$\left\{ \begin{array}{l} \text{block } h_{t(m-t)} : (0)_{,h_{t(m-t)}} \quad (11)_{tm,h_{t(m-t)}} \quad (22)_{tm,h_{t(m-t)}}, \\ h_{t(m-t)} = m + 2 \sum_{i=1}^{t-1} (m-i) + (m-t) \end{array} \right.$$

Stage t
2(m-t) blocks

$$\left\{ \begin{array}{l} \text{block } k_{t1} : (0)_{,k_{t1}} \quad (12)_{t(t+1),k_{t1}} \quad (21)_{t(t+1),k_{t1}}, \\ k_{t1} = m + 2 \sum_{i=1}^{t-1} (m-i) + (m-t) + 1 \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{block } k_{t2} : (0)_{,k_{t2}} \quad (12)_{t(t+2),k_{t2}} \quad (21)_{t(t+2),k_{t2}}, \\ k_{t2} = m + 2 \sum_{i=1}^{t-1} (m-i) + (m-t) + 2 \end{array} \right.$$

⋮

$$\left\{ \begin{array}{l} \text{block } k_{t(m-t)} : (0)_{,k_{t(m-t)}} \quad (12)_{tm,k_{t(m-t)}} \quad (21)_{tm,k_{t(m-t)}}, \\ k_{t(m-t)} = m + 2 \sum_{i=1}^{t-1} (m-i) + (m-t) + (m-t) \end{array} \right.$$

(5.70)

After the completion of all the blocks (5.69) for Steps 1 and 2 of stage t , we notice that the estimable functions are

$$\begin{aligned}
 1. & B_1 ; B_1^2 ; B_1 B_u ; B_1 B_u^2 ; B_1^2 B_u ; B_1^2 B_u^2 , (u = 2, 3, \dots, m). \\
 2. & B_2 ; B_2^2 ; B_2 B_u ; B_2 B_u^2 ; B_2^2 B_u ; B_2^2 B_u^2 , (u = 3, 4, \dots, m). \\
 & \vdots \\
 t. & a. B_t ; B_t^2 ; B_t B_u ; B_t B_u^2 ; B_t^2 B_u ; B_t^2 B_u^2 , (u = t+1, t+2, \dots, m). \\
 & b. S_{(t+1)u} = B_u - \left(\sum_{v=t+1}^{u-1} B_u B_v + \sum_{v=u+1}^m B_u B_v \right) + \left(\sum_{v=t+1}^{u-1} B_u B_v^2 + \sum_{v=u+1}^m B_u B_v^2 \right) \\
 & c. S_{(t+1)u}^2 = B_u^2 - \left(\sum_{v=t+1}^{u-1} B_u^2 B_v + \sum_{v=u+1}^m B_u^2 B_v \right) + \left(\sum_{v=t+1}^{u-1} B_u^2 B_v^2 + \sum_{v=u+1}^m B_u^2 B_v^2 \right)
 \end{aligned}
 \tag{5.71}$$

The unbiased estimates of the functions in (5.71) are

$$\begin{aligned}
 1. \hat{B}_j &= \frac{1}{6} \sum_{i=1}^{j-1} [(y(22)_{ij, h_i(j-i)} - y(0)_{, h_i(j-i)}) + (y(12)_{ij, k_i(j-i)} \\
 & - y(0)_{, k_i(j-i)}) - (y(1)_{i,i} + y(2)_{i,i} - 2y(0)_{, i})] \\
 & + \frac{1}{6} \sum_{u=j+1}^m [(y(22)_{ju, h_j(u-j)} - y(0)_{, h_j(u-j)}) + (y(21)_{ju, k_j(u-j)} \\
 & - y(0)_{, k_j(u-j)}) - (y(1)_{u,u} + y(2)_{u,u} - 2y(0)_{, u})] \\
 & - \frac{1}{6} (2m-5)(y(2)_{j,j} - y(0)_{, j}) , j = 1, 2, \dots, t . \\
 2. \hat{B}_j^2 &= \frac{1}{18} \sum_{i=1}^{j-1} [(y(0)_{, h_i(j-i)} - 2y(11)_{ij, h_i(j-i)} + y(22)_{ij, h_i(j-i)}) \\
 & + (y(0)_{, k_i(j-i)} - 2y(21)_{ij, k_i(j-i)} + y(12)_{ij, k_i(j-i)}) \\
 & - 2(y(0)_{, i} + y(1)_{i,i} + y(2)_{i,i})]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{18} \sum_{u=j+1}^m [(y(0)_{,h_j(u-j)} - 2y(11)_{j^u,h_j(u-j)} \\
& + y(22)_{j^u,h_j(u-j)}) + (y(0)_{,k_j(u-j)} - 2y(12)_{j^u,k_j(u-j)} \\
& + y(21)_{j^u,k_j(u-j)}) - (2y(0)_{,u} + y(1)_{u,u} + y(2)_{u,u})] \\
& - \frac{1}{6} (2m-5)(y(0)_{,j} - 2y(1)_{j,j} + y(2)_{j,j}), \quad j = 1, 2, \dots, t.
\end{aligned}$$

$$\begin{aligned}
3. \quad \hat{B}_j^2 B_u &= \frac{1}{4} [(y(22)_{j^u,h_j(u-j)} - y(0)_{,h_j(u-j)}) - (y(2)_{j,j} - y(0)_{,j}) \\
& - (y(2)_{u,u} - y(0)_{,u})], \quad j = 1, 2, \dots, t \text{ and } u = j+1, j+2, \dots, m.
\end{aligned}$$

$$\begin{aligned}
4. \quad \hat{B}_j^2 B_u^2 &= \frac{1}{12} [(y(22)_{j^u,h_j(u-j)} - y(0)_{,h_j(u-j)}) - 2(y(21)_{j^u,k_j(u-j)} \\
& - y(0)_{,k_j(u-j)}) - (y(0)_{,u} - 2y(1)_{u,u} + y(2)_{u,u}) \\
& + (y(2)_{j,j} - y(0)_{,j})], \quad j = 1, 2, \dots, t \text{ and } u = j+1, j+2, \dots, m.
\end{aligned}$$

$$\begin{aligned}
5. \quad \hat{B}_j^2 \hat{B}_u &= \frac{1}{12} [(y(22)_{j^u,h_j(u-j)} - y(0)_{,h_j(u-j)}) - 2(y(12)_{j^u,k_j(u-j)} \\
& - y(0)_{,k_j(u-j)}) - (y(0)_{,j} - 2y(1)_{j,j} + y(2)_{j,j}) \\
& + (y(2)_{u,u} - y(0)_{,u})]
\end{aligned}$$

$$\begin{aligned}
6. \quad \hat{B}_j^2 \hat{B}_u^2 &= \frac{1}{36} [(4y(11)_{j^u,h_j(u-j)} + y(22)_{j^u,h_j(u-j)} - 5y(0)_{,h_j(u-j)}) \\
& - 2(y(12)_{j^u,k_j(u-j)} + y(21)_{j^u,k_j(u-j)} - 2y(0)_{,k_j(u-j)}) \\
& + (y(0)_{,j} - 2y(1)_{j,j} + y(2)_{j,j}) + (y(0)_{,u} + 2y(1)_{u,u} \\
& + y(2)_{u,u})], \quad j = 1, 2, \dots, t \text{ and } u = j+1, j+2, \dots, m.
\end{aligned}$$

$$\begin{aligned}
7. \hat{S}_{(t+1)u} &= \frac{1}{6} \sum_{j=1}^t [(y(22)_{ju, h_j(u-j)} - y(0)_{, h_j(u-j)}) \\
&\quad + (y(12)_{ju, k_j(u-j)} - y(0)_{, k_j(u-j)}) \\
&\quad - (y(1)_{j,j} + y(2)_{j,j} - 2y(0)_{, j})] \\
&\quad - \frac{1}{6} (2t-3)(y(2)_{u,u} - y(0)_{, u}) \\
8. \hat{S}_{(t+1)u}^2 &= \frac{1}{18} \sum_{j=1}^t [(y(0)_{, h_j(u-j)} - 2y(11)_{ju, h_j(u-j)} \\
&\quad + y(22)_{ju, h_j(u-j)}) + (y(0)_{, k_j(u-j)} - 2y(21)_{ju, k_j(u-j)} \\
&\quad + y(12)_{ju, k_j(u-j)}) + (y(1)_{j,j} + y(2)_{j,j} + 2y(0)_{, j})] \\
&\quad - \frac{1}{18} (2t-3)(y(0)_{, u} - 2y(1)_{u,u} + y(2)_{u,u}) \quad (5.72)
\end{aligned}$$

Clearly, the OBAT 3^m plan in (5.70) can be expanded by additional 3-level factors, their effects can be estimated as given by (5.72). Also recalling that block effects cancel out from the expected values of the estimates in (5.72) and referring to (5.24), (5.28), (5.29), (5.30), (5.31) and (5.32) the unbiasedness of \hat{B}_j , \hat{B}_j^2 , $B_j \hat{B}_u$, $B_j \hat{B}_u^2$, $B_j^2 \hat{B}_u$ and $B_j^2 \hat{B}_u^2$ follows immediately. The unbiasedness of $\hat{S}_{(t+1)u}$ and $\hat{S}_{(t+1)u}^2$ can be shown as follows:

a. for $\hat{S}_{(t+1)u}$ we notice that

$$\begin{aligned}
E\hat{S}_{(t+1)u} &= \frac{1}{6} \sum_{j=1}^t [((22)_{ju} - (2)_j) + ((12)_{ju} - (1)_j)] \\
&\quad - \frac{1}{6} (2t-3)((1)_u - (0))
\end{aligned}$$

but from (5.12) and (5.13),

$$(22)_{ju} - (2)_j = 2(B_u + B_j B_u - L_u L(j,u) + B_u B_j^2 + L_u Q(j,u))$$

$$(12)_{ju} - (1)_j = 2(B_u - L_u L(j,u) - 2B_u B_j^2 + L_u Q(j,u))$$

and

$$[(22)_{ju} - (2)_j] + [(12)_{ju} - (1)_j] = 4B_u + 2B_j B_u - 4L_u L(j,u) - 2B_j^2 B_u + 4L_u Q(j,u)$$

hence

$$\begin{aligned} & \sum_{j=1}^t [((22)_{ju} - (2)_j) + ((12)_{ju} - (1)_j)] \\ &= 4tB_u + (2B_1 B_u - 4B_2 B_u - 4B_3 B_u - \dots - 4B_t B_u - 4B_{t+1} B_u \\ & \quad - 4B_{t+2} B_u - \dots - 4B_{u-1} B_u - \dots - 4B_u B_u) \\ & \quad + (-4B_1 B_u + 2B_2 B_u - 4B_3 B_u - \dots - 4B_t B_u - 4B_{t+1} B_u - 4B_{t+2} B_u \\ & \quad - \dots - 4B_u B_u) + \dots + (-4B_1 B_u - 4B_2 B_u - \dots + 2B_t B_u \\ & \quad - 4B_{t+1} B_u - \dots - 4B_u B_u) - 2 \sum_{j=1}^t B_j^2 B_u + 4 \sum_{j=1}^t L_u Q(j,u) \\ &= 4tB_u - 2(2t-3) \sum_{j=1}^t B_j B_u - 4t \left(\sum_{v=t+1}^{u-1} B_u B_v + \sum_{v=u+1}^m B_u B_v \right) \\ & \quad + 2(2t-3) \sum_{j=1}^t B_j^2 B_u + 4t \left(\sum_{v=t+1}^{u-1} B_u B_v^2 + \sum_{v=u+1}^m B_u B_v^2 \right) \end{aligned} \quad (5.73)$$

and from (5.11) we find that

$$\begin{aligned}
(2)_u - (0) &= 2(B_u - B_j B_u - L_u L(j,u) + B_j^2 B_u + L_u Q(j,u)) \\
&= 2B_u - 2 \sum_{j=1}^t B_j B_u - 2 \left(\sum_{v=t+1}^{u-1} B_u B_v^2 + \sum_{v=u+1}^m B_u B_v^2 \right) \\
&\quad + 2 \sum_{j=1}^t B_j^2 B_u - 2 \left(\sum_{v=t+1}^{u-1} B_u B_v^2 + \sum_{v=u+1}^m B_u B_v^2 \right) \quad (5.74)
\end{aligned}$$

From the difference between (5.73) and (5.74) unbiasedness of $\hat{S}_{(t+1)u}$ follows.

b. for $S_{(t+1)u}^2$ we notice that

$$\begin{aligned}
\hat{E}S_{(t+1)u}^2 &= \frac{1}{18} \sum_{j=1}^t [((2)_j - 2(21)_{ju} + (22)_{ju}) + ((1)_j - 2(11)_{ju} \\
&\quad + (12)_{ju})] - \frac{1}{18} (2t-3)((0) - 2(1)_u + (2)_u)
\end{aligned}$$

but from (5.15) and (5.16),

$$\begin{aligned}
((2)_j - (21)_{ju} + (22)_{ju}) &= 6(B_u^2 + B_u^2 B_j - Q_u L(j,u) + B_j^2 B_u^2 \\
&\quad + Q_u Q(j,u))
\end{aligned}$$

$$((1)_j - 2(11)_{ju} + (12)_{ju}) = 6(B_u^2 - Q_u L(j,u) - 2B_j^2 B_u^2 + Q_u Q(j,u))$$

and

$$\begin{aligned}
&((2)_j - (21)_{ju} + (22)_{ju}) + ((1)_j - 2(11)_{ju} + (12)_{ju}) = \\
&12B_u + 6B_j B_u^2 - 12Q_u L(j,u) - 6B_j^2 B_u^2 + 12Q_u Q(j,u)
\end{aligned}$$

hence

$$\begin{aligned}
& \sum_{j=1}^t [((2)_j - (21)_{ju} + (22)_{ju}) + ((1)_j - 2(11)_{ju} + (12)_{ju})] = \\
& 12tB_u^2 - 6(2t-3) \sum_{j=1}^t B_j B_u^2 - 12t \left(\sum_{v=t+1}^{u-1} B_u^2 B_v + \sum_{v=u+1}^m B_u^2 B_v \right) \\
& + 6(2t-3) \sum_{j=1}^t B_j^2 B_u^2 + 12t \left(\sum_{v=t+1}^{u-1} B_u^2 B_v^2 + \sum_{v=u+1}^m B_u^2 B_v^2 \right) \quad (5.75)
\end{aligned}$$

and from (5.14) we find that

$$\begin{aligned}
(0) - 2(1)_u + (2)_u &= 6(B_u - B_j B_u^2 - Q_u L(j,u) + B_j^2 B_u^2 + Q_u Q(j,u)) \\
&= 6B_u - 6 \sum_{j=1}^t B_j B_u^2 - 6 \left(\sum_{v=t+1}^{u-1} B_u^2 B_v + \sum_{v=u+1}^m B_u^2 B_v \right) \\
&\quad + 6 \sum_{j=1}^t B_j^2 B_u^2 + 6 \left(\sum_{v=t+1}^{u-1} B_u^2 B_v^2 + \sum_{v=u+1}^m B_u^2 B_v^2 \right) \quad (5.76)
\end{aligned}$$

From the difference between (5.75) and (5.76), unbiasedness of $\hat{S}_{(t+1)u}^2$ follows.

5.5 OBAT 3^m Plans for Strategy II

We consider here the three cases mentioned in Chapter 1, namely

Case (1): All m 3-level factors B_1, B_2, \dots, B_m are considered as one set of potentially interacting factors.

For this case, the 3^m fraction F_m in (5.2) can then be used to estimate the linear and quadratic effects as well as the $4 \cdot \frac{m(m-1)}{2}$ two-factor interactions of the m 3-level factors. Parameter estimates are given in (5.24), (5.28), (5.29), (5.30), (5.31) and (5.32).

Case (2): The m 3-level factors B_1, B_2, \dots, B_m are partitioned into g disjoint sets where set h contains m_h factors and $\sum_{h=1}^g m_h = m$.

Factors in each set may interact with each other but not with factors from another set.

For case (2), the experiment will then be carried out in a sequence of g blocks such that the h^{th} block is the $3_{V_h}^{m_h}$ fraction F_{m_h} of (5.2) with $N_h = 1 + 2m_h + 4 \cdot \frac{m_h(1-m_h)}{2} = 1 + 2m_h^2$ treatment combinations ($h = 1, 2, \dots, g$). Each treatment combination among the three levels of the m_h factors in the fraction F_{m_h} will be written as $(00\text{---}0b_t00\text{---}0b_u00\text{---}0)_h$, or just as

$$(b_t b_u)_{tu,h} \tag{5.77}$$

where $(b_t, b_u = 0, 1, 2), (u = t+1, t+-, \dots, m_h), (t = 1, 2, \dots, m_h)$ and $(h = 1, 2, \dots, g)$.

The sequence of g blocks in (5.77) can be written explicitly as

$$\begin{array}{l} \text{block 1: } (0)_{,1} \quad (1)_{t,1} \quad (2)_{t,1} \quad (11)_{tu,1} \quad (12)_{tu,1} \quad (21)_{tu,1} \quad (22)_{tu,1} \\ \text{block 2: } (0)_{,2} \quad (1)_{t,2} \quad (2)_{t,2} \quad (11)_{tu,2} \quad (12)_{tu,2} \quad (21)_{tu,2} \quad (22)_{tu,2} \\ \vdots \\ \text{block } g: \quad (0)_{,g} \quad (1)_{t,g} \quad (2)_{t,g} \quad (11)_{tu,g} \quad (12)_{tu,g} \quad (21)_{tu,g} \quad (22)_{tu,g} \end{array} \tag{5.78}$$

We now have the following linear model for Strategy II OBAT 3^m plans

$$y(b_t b_u)_{tu,h} = (b_t b_u)_{tu,h} + \epsilon_{tu,h} \tag{5.79}$$

Denoting the $(N_h - 1)$ factorial effects of the h^{th} $2_{V_h}^{m_h}$ fraction by

$$(B_t)_h, (B_t^2)_h, (B_t B_u)_h, (B_t B_u^2)_h, (B_t^2 B_u)_h \text{ and } (B_t^2 B_u^2)_h, \quad (5.80)$$

and referring to (5.24), (5.28), (5.29), (5.30), (5.31) and (5.32), the unbiased estimates of the parameters in (5.80) are

$$\begin{aligned} 1. \quad (\hat{B}_t)_h &= \frac{1}{6} \sum_{u \neq t} [(y(22)_{tu,h} - y(2)_{u,h}) + (y(21)_{tu,h} - y(1)_{u,h})] \\ &\quad - \frac{1}{6} (2m_h - 5) (y(2)_{t,h} - y(0)_{,h}) \\ 2. \quad (\hat{B}_t^2)_h &= \frac{1}{18} \sum_{u \neq t} [(y(2)_{u,h} - 2 \cdot y(12)_{tu,h} + y(22)_{tu,h}) \\ &\quad + (y(1)_{u,h} - 2y(11)_{tu,h} + y(21)_{tu,h})] \\ &\quad - \frac{1}{18} (2m_h - 5) (y(0)_{,h} - 2y(1)_{t,h} + y(2)_{t,h}) \\ 3. \quad (\hat{B}_t B_u)_h &= \frac{1}{4} [(y(22)_{tu,h} - y(2)_{u,h}) - (y(2)_{t,h} - y(0)_{,h})] \\ 4. \quad (\hat{B}_t^2 B_u)_h &= \frac{1}{12} [(y(2)_{u,h} - 2y(12)_{tu,h} + y(22)_{tu,h}) \\ &\quad - (y(0)_{,h} - 2y(1)_{t,h} + y(2)_{t,h})] \\ 5. \quad (\hat{B}_t B_u^2)_h &= \frac{1}{12} [(y(2)_{t,h} - 2y(21)_{tu,h} + y(22)_{tu,h}) \\ &\quad - (y(0)_{,h} - 2y(1)_{u,h} + y(2)_{u,h})] \\ 6. \quad (\hat{B}_t^2 B_u^2)_h &= \frac{1}{36} [(y(0)_{,h} - 2y(1)_{t,h} + y(2)_{t,h}) \\ &\quad - 2(y(1)_{u,h} - 2y(11)_{tu,h} + y(21)_{tu,h}) \\ &\quad + (y(2)_{u,h} - 2y(12)_{tu,h} + y(22)_{tu,h})] \end{aligned} \quad (5.81)$$

Case (3): The m 3-level factors B_1, B_2, \dots, B_m are partitioned into three groups containing altogether $(g_1 + g_2 + 1)$ disjoint sets such that

a. group one contains g_1 sets.

Factors in each set of the g_1 sets interact with each other but not with factors from another set. That is, these g_1 sets are actually the g sets in case (2).

b. group two contains g_2 sets.

In each one of the g_2 sets, some but not all factors interact with all factors in that particular set. The remaining factors in each set do not interact with each other.

However, we will assume that only two factors in each one of the g_2 sets interact with each other and with the remaining factors in the set. These two factors are considered to be the first and the second, i.e. $(B_1)_h$ and $(B_2)_h$ ($h = g_1 + 1, g_1 + 2, \dots, g_1 + g_2$).

c. group three consists of only one set.

Factors in this set do not interact with each other.

Let m_{ij} represent the number of factors in the j^{th} set of the i^{th} group where ($j = 1, 2, 3, \dots, g_i$) and ($i = 1, 2, 3$) such that $\sum_{j=1}^{g_i} m_{ij} = m_i$. m_i is the total number of factors in the i^{th} group and $\sum_{i=1}^3 \sum_{j=1}^{g_i} m_{ij} = m$.

For case (3), the 3^m experiment will then be carried out in a sequence of $(g_1 + g_2 + 1)$ blocks such that

a. each one of the g_1 blocks is the $3_{v_1}^{m_{1h}}$ fraction in (5.2) with $N_{1h} = 1 + 2m_{1h}^2$ treatment combinations. Each treatment combination among the m_{1h} factor levels is of the form

$$(b_t b_u)_{tu,h} \quad (5.82)$$

where $(b_t, b_u = 0, 1, 2), (t = 1, 2, \dots, m_{1j}), (u = t+1, t+2, \dots, m_{1j})$ and $(h = 1, 2, \dots, g_1)$. The contents of the g_1 blocks are similar to those in (5.78) with $g_1 = g$.

b. Each one of the g_2 blocks is a $3_V^{m_{2h}}$ fraction and with the assumption that each set of the g_2 sets contains only two interacting factors, the fraction $3_V^{m_{2h}}$ contains

$$N_{2h} = (1 + 2m_{2h}) + 4(m_{2h} - 1) + 4(m_{2h} - 2)$$

treatment combinations. These N_{2j} treatment combinations among the 3 levels of the m_{2j} factors can be written explicitly as

$$\begin{aligned} & (0)_{,g_1+j} \quad (1)_{t,g_1+j} \quad (2)_{t,g_1+j} \quad , \quad t = 1, 2, \dots, m_{2j} \\ & (11)_{1u,g_1+j} \quad (12)_{1u,g_1+j} \quad (21)_{1u,g_1+j} \quad (22)_{1u,g_1+j} \\ & (11)_{2v,g_1+j} \quad (12)_{2v,g_1+j} \quad (21)_{2v,g_1+j} \quad (22)_{2v,g_1+j} \end{aligned} \quad (5.83)$$

where $(u = 2, 3, \dots, m_{2j}), (v = 3, 4, \dots, m_{2j})$ and $(j = 1, 2, \dots, g_2)$

c. The last set which contains non-interacting factors is a $3_V^{m_{31}}$ fraction containing $N_{31} = 1 + 2m_{31}$ treatment combinations. These treatment combinations can be written explicitly as

$$(0)_{g_1+g_2+1} \quad (1)_{t,g_1+g_2+1} \quad (2)_{t,g_1+g_2+1} \quad (t = 1, 2, \dots, m_{31}) \quad (5.84)$$

Referring to the model (5.79) and denoting the factorial effects of the fractions $3_V^{m_{1h}}$, $3_V^{m_{2h}}$ and $3_{III}^{m_{31}}$, respectively, by

$$a. (B_t)_h, (B_t^2)_h, (B_t B_u)_h, (B_t^2 B_u)_h, (B_t B_u^2), (B_t^2 B_u^2)_h$$

where $(t = 1, 2, \dots, m_{1j}), (u = t+1, t+2, \dots, m_{1j})$ and $(h = 1, 2, \dots, g_1)$.

$$b. (B_t)_h, (B_t^2)_h, (B_1 B_u)_h, (B_1 B_u^2)_h, (B_1^2 B_u)_h, (B_1^2 B_u^2)_h, (B_2 B_v)_h, (B_2 B_v^2)_h, (B_2^2 B_v)_h, (B_2^2 B_v^2)_h$$

where $(t = 1, 2, \dots, m_{2j}), (u = 2, 3, \dots, m_{2j}), (v = 3, 4, \dots, m_{2j})$ and $(h = g_1+1, g_1+2, \dots, g_1+g_2)$.

$$c. (B_t)_h, (B_t^2)_h$$

where $(t = 1, 2, \dots, m_{3j})$ and $(h = g_1 + g_2 + 1)$. (5.84)

The unbiased estimates of the parameters in (a) above are similar to those in (5.81) with m_h replaced by m_{1h} . The unbiased estimates of the interaction effects in (b) in (5.85) are given by (3), (4), (5) and (6) in (5.81) setting $(t = 1, 2)$ and $(v = u)$. The unbiased estimates of $(B_t)_h$ and $(B_t^2)_h$ for $(t = 1, 2)$, in (b) in (5.85) are given by (1) and (2) in (5.81) setting $(t = 1, 2)$ and $(m_h = m_{2h})$ whereas the unbiased estimates of $(B_t)_h$ and $(B_t^2)_h$ for $(t = 3, 4, \dots, m_{2h})$ and $(h = g_1+1, g_1+2, \dots, g_1+g_2)$ are given by (7) and (8) in (5.72) setting $t = 2$. Finally, the unbiased estimates of the parameters in (c) in (5.85) are given by (5.39).

5.6 An Example: OBAT 3^4 Plans

Referring to the $1 + 2m + 4 \cdot \frac{m(m-1)}{2}$ treatment combinations of the 3^m_V fraction in (5.2), we have for the 3^4_V experiment the following 33 runs

0000	1000	0100	0010	0001	2000	0200	0020	0002
	1100	1010	1001	2200	2020	2002		
	1200	1020	1002	2100	2010	2001		
	0110	0101	0220	0202	0120	0102	0210	0201
	0011	0022	0012	0021				

which can be written using the $(b_{t'u})_{tu}$ notation as

$$\begin{aligned}
 (0) \quad & (1)_1 \quad (1)_2 \quad (1)_3 \quad (1)_4 \quad (2)_1 \quad (2)_2 \quad (2)_3 \quad (2)_4 \\
 & (11)_{12} \quad (11)_{13} \quad (11)_{14} \quad (22)_{12} \quad (22)_{13} \quad (22)_{14} \\
 & (12)_{12} \quad (12)_{13} \quad (12)_{14} \quad (21)_{12} \quad (21)_{13} \quad (21)_{14} \\
 & (11)_{23} \quad (11)_{24} \quad (22)_{23} \quad (22)_{24} \quad (12)_{23} \quad (12)_{24} \quad (21)_{23} \quad (21)_{24} \\
 & (11)_{34} \quad (22)_{34} \quad (12)_{34} \quad (21)_{34}
 \end{aligned} \tag{5.86}$$

Also referring to the linear model in (5.1), we have for the 3^4_V experiment the following 33 effects

$$\begin{array}{cccccccc}
 \mu & B_1 & B_2 & B_3 & B_4 & B_1^2 & B_2^2 & B_3^2 & B_4^2 \\
 & B_1 B_2 & B_1 B_3 & B_1 B_4 & & B_1^2 B_2^2 & B_1^2 B_3^2 & B_1^2 B_4^2 & \\
 & B_1^2 B_2 & B_1^2 B_3 & B_1^2 B_4 & & B_1 B_2^2 & B_1 B_3^2 & B_1 B_4^2 & \\
 & B_2 B_3 & B_2 B_4 & B_2^2 B_3^2 & B_2^2 B_4^2 & B_2^2 B_3 & B_2^2 B_4 & B_2 B_3^2 & B_2 B_4^2 \\
 & B_3 B_4 & B_3^2 B_4^2 & B_3^2 B_4 & B_3 B_4^2 & & & &
 \end{array} \tag{5.87}$$

OBAT 3^4_V Plan for Strategy I

For the linear model in (5.36) and with factors ordered for their importance as B_1, B_2, B_3 and B_4 , we consider the two cases:

a) Case (1): 3^4_{III} plan in a sequence of 4 blocks of size 3.

When all interactions are negligible, we have the following

model

$$y(b_t)_{t,h} = \mu + B_t \cdot y_t - \sum_{u \neq t} B_u + B_t^2 \cdot y_t' + \sum_{u \neq t} B_u^2 + (BE)_h + \varepsilon_{t,h} \quad (5.88)$$

where $(b_t = 0, 1, 2)$, $(t, u = 1, 2, 3, 4)$, $(h = 1, 2, 3, 4)$. The block structure for case (1) consists of the four blocks

$$\text{block } t: (0)_{,t} \quad (1)_{t,t} \quad (2)_{t,t}, \quad (t = 1, 2, 3, 4) \quad (5.89)$$

Blocks (5.89) make use of only the first 9 treatment combinations in (5.86). From (5.89), the t^{th} factor's linear and quadratic effects are estimable from block t where

$$\hat{B}_t = \frac{1}{2}(y(2)_{t,t} - y(0)_{,t})$$

and

$$\hat{B}_t^2 = \frac{1}{6} (y(0)_{,t} - 2y(1)_{t,t} + y(2)_{t,t}) \quad (5.90)$$

with $\text{var } \hat{B}_t = \frac{1}{2}\sigma^2$ and $\text{var } \hat{B}_t^2 = \frac{1}{6}\sigma^2$, $(t = 1, 2, 3, 4)$.

b) Case (2): 3^4_V plan in a sequence of blocks of size 3.

When two-factor interactions are non-negligible, we have the linear model (5.36) and the contrasts in (5.90) estimate now the functions in (5.40), $(t = 1, 2, 3, 4)$. These functions are explicitly

$$1. \quad B_1 - (B_1B_2 + B_1B_3 + B_1B_4) + (B_1B_2^2 + B_1B_3^2 + B_1B_4^2) = S_{11}$$

$$B_1^2 - (B_1^2B_2 + B_1^2B_3 + B_1^2B_4) + (B_1^2B_2^2 + B_1^2B_3^2 + B_1^2B_4^2) = S_{11}^2$$

$$2. \quad B_2 - (B_1B_2 + B_2B_3 + B_2B_4) + (B_1^2B_2 + B_2B_3^2 + B_2B_4^2) = S_{21}$$

$$\begin{aligned}
& B_2^2 - (B_1B_2^2 + B_2^2B_3 + B_2^2B_4) + (B_1^2B_2^2 + B_2^2B_3^2 + B_2^2B_4^2) = S_{21}^2 \\
3. \quad & B_3 - (B_1B_3 + B_2B_3 + B_3B_4) + (B_1^2B_3 + B_2^2B_3 + B_3^2B_4^2) = S_{31} \\
& B_3^2 - (B_1B_3^2 + B_2^2B_3^2 + B_3^2B_4^2) + (B_1^2B_3^2 + B_2^2B_3^2 + B_3^2B_4^2) = S_{31}^2 \\
4. \quad & B_4 - (B_1B_4 + B_2B_4 + B_3B_4) + (B_1^2B_4 + B_2^2B_4 + B_3^2B_4^2) = S_{41} \\
& B_4^2 - (B_1B_4^2 + B_2^2B_4^2 + B_3^2B_4^2) + (B_1^2B_4^2 + B_2^2B_4^2 + B_3^2B_4^2) = S_{41}^2 \quad (5.91)
\end{aligned}$$

To break the alias chains in (5.91), additional blocks of size 3 must be added to the four blocks in (5.89). We consider the following stages:

Stage One: The estimation of B_1 , B_1^2 , B_1B_u , $B_1B_u^2$, $B_1^2B_u$ and $B_1^2B_u^2$, ($u = 2, 3, 4$) in two steps such that step 1 involves the estimation of B_1B_u , ($u = 2, 3, 4$) and step 2 involves the estimation of the remaining effects of factor 1.

For step 1, the block structure is

$$\begin{aligned}
\text{block (t):} & \quad (0)_{,t} \quad (1)_{t,t} \quad (2)_{t,t}, \quad t = 1, 2, 3, 4 \\
\text{block (h):} & \quad (0)_{,h} \quad (11)_{lu,h} \quad (22)_{lu,h}, \quad u = 2, 3, 4 \text{ and } h = 4 + (u-1), \\
& \hspace{20em} (5.92)
\end{aligned}$$

The estimable functions after the 7 blocks in (5.92) are

1. $B_1 + (B_1B_2^2 + B_1B_3^2 + B_1B_4^2)$.
2. The three two-factor interactions B_1B_u , ($u = 2, 3, 4$).
3. The three alias chains $B_1B_u^2 + B_1^2B_u - 3B_1^2B_u^2$, ($u = 2, 3, 4$).

$$\begin{aligned}
4. \quad & B_1^2 + (B_1B_2^2 + B_1B_3^2 + B_1B_4^2) - 2(B_1^2B_2^2 + B_1^2B_3^2 + B_1^2B_4^2), \\
5. \quad & \text{a. } B_3 - (B_1B_3^2 - 3B_1^2B_3^2) - (B_2B_3 + B_3B_4) + (B_2^2B_3 + B_3^2B_4) \\
& \text{b. } B_2 - (B_1B_2^2 - 3B_1^2B_2^2) - (B_2B_3 + B_2B_4) + (B_2^2B_3 + B_2^2B_4) \\
& \text{c. } B_4 - (B_1B_4^2 - 3B_1^2B_4^2) - (B_2B_4 + B_3B_4) + (B_2^2B_4 + B_3^2B_4), \\
6. \quad & \text{d. } B_3^2 - (B_1B_3^2 + B_1^2B_3^2) - (B_2B_3^2 + B_3^2B_4) + (B_2^2B_3^2 + B_3^2B_4^2) \\
& \text{e. } B_2^2 - (B_1B_2^2 + B_1^2B_2^2) - (B_2^2B_3 + B_2^2B_4) + (B_2^2B_3^2 + B_2^2B_4^2) \\
& \text{f. } B_4^2 - (B_1B_4^2 + B_1^2B_4^2) - (B_2B_4^2 + B_3B_4^2) + (B_2^2B_4^2 + B_3^2B_4^2) \quad (5.93)
\end{aligned}$$

The unbiased estimates of the functions in (5.93) are given by (5.45) setting $h = 4 + (u-1) = u + 3$ and $m = 4$. Clearly, the effects B_1 , B_1^2 and B_1B_u , ($u = 2, 3, 4$) become estimable from step 1 if the interactions $B_1B_u^2$, $B_1^2B_u$ and $B_1^2B_u^2$ are negligible, otherwise we consider step 2 which has the block structure

$$\begin{aligned}
\text{block } (t): \quad & (0)_{,t} \quad (1)_{t,t} \quad (2)_{t,t} \\
\text{block } (u+3): \quad & (0)_{,u+3} \quad (11)_{1u,u+3} \quad (22)_{1u,u+3}, \quad u = 2,3,4 \\
\text{block } (u+6): \quad & (0)_{,u+6} \quad (12)_{1u,u+6} \quad (21)_{1u,u+6}, \quad u = 2,3,4 \quad (5.94)
\end{aligned}$$

The runs in the 10 blocks are the first 21 runs in (5.86). Estimable functions after the completion of the 10 blocks in (5.94) are

$$\begin{aligned}
1. \quad & B_1 \\
2. \quad & B_1^2 \\
3. \quad & \text{a. } B_2 - (B_2B_3 + B_2B_4) + (B_2^2B_3 + B_2^2B_4) = S_{23} \\
& \text{b. } B_2^2 - (B_2^2B_3 + B_2^2B_4) + (B_2^2B_3^2 + B_2^2B_4^2) = S_{23}^2 \\
& \text{c. } B_3 - (B_2B_3 + B_3B_4) + (B_2^2B_3 + B_3^2B_4) = S_{32} \\
& \text{d. } B_3^2 - (B_2B_3^2 + B_3^2B_4) + (B_2^2B_3^2 + B_3^2B_4^2) = S_{32}^2 \\
& \text{e. } B_4 - (B_2B_4 + B_3B_4) + (B_2^2B_4 + B_3^2B_4) = S_{42}
\end{aligned}$$

3. $B_2 + B_2B_3^2 + B_2B_4^2$.
4. The two interaction effects B_2B_3, B_2B_4 .
5. The two alias chains $B_2B_u^2 + B_2^2B_u - 3B_2^2B_u^2, (u = 3, 4)$.
6. $B_2^2 + (B_2B_3^2 + B_2B_4^2) - 2(B_2^2B_3^2 + B_3^2B_4^2)$.
7. a. $(B_3 - B_2B_3^2 + 3B_2^2B_3^2) - B_3B_4 + B_3B_4^2$
 b. $B_3^2 - (B_2B_3^2 + B_3^2B_4) + (B_2^2B_3^2 + B_3^2B_4^2)$
 c. $(B_4 - B_2B_4^2 + 3B_2^2B_4^2) - B_3B_4 + B_3^2B_4$
 d. $B_4^2 - (B_2B_4^2 + B_3^2B_4^2) + (B_2^2B_4^2 + B_3^2B_4^2)$ (5.97)

The unbiased estimates of the parameters in (1) and (2) in (5.97) were given in stage one whereas the unbiased estimates of the remaining functions in (5.97) are given by (5.65) setting ($h = u + 3, k = u + 6, \text{ for } u = 2, 3, 4$), ($i = u + 8 \text{ for } u = 3, 4$) and $m = 4$. The scalars c_1, c_2 and c_3 in (3) and (6) in (5.65) are $\frac{1}{3}, \frac{4}{9}$ and $\frac{1}{9}$, respectively. Clearly, the effects B_2, B_2^2 and $B_2B_u, u = 3, 4$ become estimable after step 1 of stage two if the interactions $B_2B_u^2, B_2^2B_u$ and $B_2^2B_u^2$ are negligible, otherwise we consider step 2 which has a block structure consisting of the 12 blocks in (5.96) plus the two blocks

$$\text{block } (u + 10): \quad (0)_{,u+10} \quad (12)_{2u,u+10} \quad (21)_{2u,u+10}, (u = 3, 4) \quad (5.98)$$

The estimable functions after the completion of the 14 blocks in (5.98) are

1. $B_1 ; B_1^2 ; B_1B_u ; B_1B_u^2 ; B_1^2B_u ; B_1^2B_u^2, (u = 2, 3, 4)$.
2. B_2 .
3. B_2^2 .

4. The eight interaction effects

$$\begin{aligned}
 \text{a. } & B_2B_3 ; B_2B_4 \\
 \text{b. } & B_2B_3^2 ; B_2B_4^2 \\
 \text{c. } & B_2^2B_3 ; B_2^2B_4 \\
 \text{d. } & B_2^2B_3^2 ; B_2^2B_4^2 .
 \end{aligned}
 \tag{5.99}$$

The unbiased estimates of parameters in (1) in (5.99) were given in stage one whereas the unbiased estimates of the remaining functions in (5.99) are given by (5.67) setting $(h = u + 3, k = u + 6$ for $u = 2, 3, 4)$, $(i = u + 8, j = u + 10$ for $u = 3, 4)$ and $m = 4$.

Stage Three: The estimation of $B_3, B_4, B_3B_4, B_3^2B_4, B_3^2B_4^2$ and $B_3^2B_4^2$ in two steps such that step 1 involves the estimation of B_3B_4 and step 2 involves the remaining effects involving factor 3. In fact, all the parameters in (5.87) become estimable after stage three and their estimates are given by (5.72),

VI. OBAT PLANS FOR THE $2^n 3^m$ FACTORIAL EXPERIMENTS

This chapter extends the results of chapter 4 on OBAT 2^n plans and the results of chapter 5 on OBAT 3^m plans for exploratory 2^n and 3^m experiments. After a brief literature review (Section 6.1) and a model statement for the $2^n 3^m$ experiment (Section 6.2), an economic expandible-contractible $2^n 3_V^m$ factorial design is developed in Section 6.3. The treatment combinations in these $2^n 3_V^m$ designs will be used to construct the OBAT $2^n 3^m$ plans we report in this dissertation. The $(n+m)$ factors of the $2^n 3^m$ experiment were assumed ordered for their importance to the response according to two orders

Order one: $B_1, B_2, \dots, B_m, A_1, A_2, \dots, A_n$.

Order two: $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_m$.

where B_k represents a 3-level factor ($k = 1, 2, \dots, m$) and A_i represents a 2-level factor ($i = 1, 2, \dots, n$). Strategy I OBAT $2^n 3^m$ plans for order one were developed in Section 6.4 whereas strategy I OBAT $2^n 3^m$ plans for order two were developed in Section 6.5. Strategy II OBAT $2^n 3^m$ plans are discussed in section 6.6.

6.1 Literature Review of OBAT $2^n 3^m$ Plans

The $2^n 3^m$ factorial experiment as well as orthogonal fractions thereof become uneconomically large as n and/or m become large. Therefore, research into non-orthogonal fractions of the $2^n 3^m$ experiment has been of particular importance.

Draper and Stoneman (1968) described a procedure for fitting

the quadratic response surface

$$E(y) = \beta_0 + \sum_{i=1}^n \beta_i x_i + \sum_{j=n+1}^{n+m} \beta_j y_j + \sum_{j=n+1}^{n+m} \beta_{jj} y_j^2 + \sum_{j=n+1}^{n+m-1} \sum_{k=j+1}^{n+m} \beta_{jk} y_j y_k$$

which contains $P = 1 + n + m + m + \frac{m(m-1)}{2} = 1 + n + 2m + \frac{m(m-1)}{2}$ parameters. The procedure is based on an idea by Fry (1961) who partitioned the $2^n 3^m$ factor space which contains the $2^n 3^m$ points $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)$ with $(x = -1, 1)$ and $(y = -1, 0, 1)$ into $(m+1)$ concentric hyperspheres

$$S_{n+r} = \{(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m) : (\sum_{i=1}^n x_i^2 + \sum_{j=1}^m y_j^2) = (n+r)^2, r = 0, 1, 2, \dots, m\}.$$

The $(m+1)$ hyperspheres S_{n+r} ($r = 0, 1, 2, \dots, m$) correspond to the $(m+1)$ terms in the binomial expansion

$$\begin{aligned} 2^n 3^m &= 2^n (1+2)^m = 2^n [1 + 2m + \dots + \binom{m}{r} 2^r + \dots + 2^m] \\ &= 2^n + 2^{n+1} \cdot m + \dots + \binom{m}{r} 2^{n+r} + \dots + 2^{n+m} \end{aligned}$$

Hence the hypersphere S_{n+r} contains $\binom{m}{r} 2^{n+r}$ treatment combinations.

The points $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)$ on the r^{th} hypersphere have the first n digits either (1 or -1), r of the next m digits are either (1 or -1) and the remaining $(m-r)$ digits are zero. Draper and Stoneman regarded each hypersphere S_{n+r} as the union of $\binom{m}{r}$ disjoint sets of 2^{n+r} full factorials. Regular fractions $2^{(n+r)-k}$, ($k = 0, 1, 2, \dots, n+r$) using the standard techniques based on defining contrasts can be generated, and by sequentially combining fractions from the

same or different hyperspheres until a non-singular design is obtained, the quadratic response surface above can be fit. The resulting non-singular $2^n 3^m$ design may not necessarily be orthogonal and generally it is not saturated.

Hoke (1974), attempting to construct even more economical non-orthogonal $2^n 3^m$ fractional factorial designs to fit quadratic response surfaces involving n 2-level factors with levels(0, 2) and m 3-level quantitative factors with levels(0, 1, 2) suggested the following partitioning of the $2^n \cdot 3^m$ factor space into permutation-invariant disjoint sets $S(i:j)$. Each hypersphere S_{n+k} is partitioned such that

$$S_{n+k} = \bigcup_{i=0}^n \left[\bigcup_{j=0}^k S(i:j) \right], \quad k = 1, 2, \dots, m$$

where

$$S(i:j) = \pi(\underbrace{22\dots 2}_i \underbrace{00\dots 0}_{n-i} \cdot \underbrace{22\dots 2}_j \underbrace{00\dots 0}_{k-j} \underbrace{11\dots 1}_{m-k})$$

and where $\pi(\cdot)$ denotes all permutations of the first n digits each combined with the separate permutations of the last m digits. The

number of treatment combinations in the sets $S(i:j)$ and S_{n+k} is $\frac{n!}{i!(n-i)!} \cdot \frac{m!}{j!(k-j)!(m-k)!}$ and $\sum_{i=0}^n \left[\sum_{j=0}^k \frac{n!}{i!(n-i)!} \cdot \frac{m!}{j!(k-j)!(m-k)!} \right] = \binom{m}{k} 2^{n+k}$, respectively. Clearly, the sets $S(i:j)$ are permutation-

invariant and the sets S_{n+k} are permutation-invariant too. Hoke (1974) suggested that by sequentially combining sets $S(i:j)$ from the same hypersphere and/or from different hyperspheres, economic permutation-invariant non-orthogonal $2^n 3^m$ fractions can be obtained from which factor effects can be estimated with the same precision. However,

Hoke did not report any $2^n 3^m$ designs constructed this way.

Extensive work on $2^n 3^m$ experiments was done by Margolin (1967, 1968, 1969a and 1969b). Margolin (1967) described a procedure for analyzing $2^n 3^m$ factorial and fractional factorial designs using an algorithm similar to that of Yates' for the analysis of 2^{n-m} factorial designs. Margolin (1968) reported a procedure for generating the alias matrix of Addelman's (1962) orthogonal main-effect plans when two-factor interactions are non-negligible.

Margolin (1969a) reported some orthogonal main effect $2^n 3^m$ plans that also permit non-orthogonal estimation of all two-factor interactions when three-factor and higher order interactions are negligible. Non-orthogonality among two-factor interactions allows saving in the number of runs in the design.

Margolin (1969b) derived a lower bound on the number of runs required for resolution IV 2^n and $2^n 3^m$ factorial plans. The bound for 2^n experiments is $2n$ and for $2^n 3^m$ experiments is $3(n + 2m - 1)$, $m > 0$ and $n \geq 0$.

6.2 Parametrization of the $2^n \cdot 3^m$ Factorial Experiment

The $2^n \cdot 3^m$ factorial experiment contains n two-level factors denoted by A_r ($r = 1, 2, \dots, n$) with levels a_r^0 and a_r^1 and m three-level factors denoted by B_t ($t = 1, 2, \dots, m$) with levels b_t^0, b_t^1 and b_t^2 . Treatment combinations are denoted by $(a_1^{i_1} a_2^{i_2} \dots a_n^{i_n} b_1^{k_1} b_2^{k_2} \dots b_m^{k_m})$ where $(i_r = 0, 1)$ for $(r = 1, 2, \dots, n)$ and $(k_t = 0, 1, 2)$ for $(t = 1, 2, \dots, m)$ and the corresponding response is

denoted by $y(a_1^{i_1} a_2^{i_2} \dots a_n^{i_n} \cdot b_1^{k_1} b_2^{k_2} \dots b_m^{k_m})$. Using single degree of freedom contrasts to describe factor effects and interactions for the three-level factors and denoting the linear and quadratic effects of factor B_t by B_t and B_t^2 , respectively, the linear model relating expected responses and factorial effects up to two-factor interactions is

$$\begin{aligned}
 (a_1^{i_1} a_2^{i_2} \dots a_n^{i_n} \cdot b_1^{k_1} b_2^{k_2} \dots b_m^{k_m}) &= E y(a_1^{i_1} a_2^{i_2} \dots a_n^{i_n} \cdot b_1^{k_1} b_2^{k_2} \dots b_m^{k_m}) \\
 &= \mu + \sum_{i=1}^n A_i x_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^n A_{ij} x_i x_j + \sum_{k=1}^m B_k y_k + \sum_{k=1}^m B_k^2 y_k' \\
 &\quad + \sum_{k=1}^{m-1} \sum_{\ell=k+1}^m B_k B_\ell y_k' y_\ell + \sum_{k=1}^{m-1} \sum_{\ell=k+1}^m B_k B_\ell^2 y_k y_\ell' + \sum_{k=1}^{m-1} \sum_{\ell=k+1}^m B_k^2 B_\ell y_k' y_\ell' \\
 &\quad + \sum_{k=1}^{m-1} \sum_{\ell=k+1}^m B_k^2 B_\ell^2 y_k' y_\ell' + \sum_{i=1}^n \sum_{k=1}^m A_i B_k x_i y_k + \sum_{i=1}^n \sum_{k=1}^m A_i B_k^2 x_i y_k'
 \end{aligned}$$

where

$$x_i = \begin{cases} -1 & \text{if } a_i^{i_r} = a_i^0 \\ 1 & \text{if } a_i^{i_r} = a_i^1 \end{cases}, \quad r = 1, 2, \dots, n$$

$$y_k = \begin{cases} -1 & \text{if } b_k^{k_t} = b_k^0 \\ 0 & \text{if } b_k^{k_t} = b_k^1 \\ 1 & \text{if } b_k^{k_t} = b_k^2 \end{cases}$$

$$y_k' = \begin{cases} -1 & \text{if } b_k^{k_t} = b_k^0 \\ -2 & \text{if } b_k^{k_t} = b_k^1 \\ 1 & \text{if } b_k^{k_t} = b_k^2 \end{cases}, \quad t = 1, 2, \dots, m.$$

(6.1)

Model (6.1) contains $p(n,m) = 1 + n + \frac{n(n-1)}{2} + m + m + 4 \cdot \frac{m(m-1)}{2} +$

$nm + nm$ parameters representing

1. the overall mean μ .
2. the n main effects of two-level factors, i.e. A_r , $r = 1, 2, \dots, n$.
3. the $\frac{n(n-1)}{2}$ two-factor interactions between two-level factors, i.e. $A_r A_s$, $s = r+1, r+2, \dots, n$ and $r = 1, 2, \dots, n-1$.
4. the m linear effects of three-level factors, i.e. B_t , $t = 1, 2, \dots, m$.
5. the m quadratic effects of three-level factors, i.e. B_t^2 , $t = 1, 2, \dots, m$.
6. a) the $\frac{m(m-1)}{2}$ linear by linear effects $B_t B_u$ between three-level factors.
 b) the $\frac{m(m-1)}{2}$ linear by quadratic effects $B_t B_u^2$ between three-level factors.
 c) the $\frac{m(m-1)}{2}$ quadratic by linear effects $B_t^2 B_u$ between three-level factors.
 d) the $\frac{m(m-1)}{2}$ quadratic by quadratic effects $B_t^2 B_u^2$ between three-level factors, where $u = t+1, t+2, \dots, m$ and $t = 1, 2, \dots, m-1$.
7. the mn two-factor interactions $A_r B_t$ between two-level factors A_r ($r = 1, 2, \dots, n$) and the linear effect B_t of a three-level factor ($t = 1, 2, \dots, m$).
8. the mn two-factor interactions $A_r B_t^2$ between two-level factors A_r ($r = 1, 2, \dots, n$) and the quadratic effect B_t^2 of a three-level factor ($t = 1, 2, \dots, m$).

6.3 Economic Expansible-Contractible $2^n \cdot 3^m$ Designs for OBAT $2^n 3^m$ Plans

In this section we extend Webb's (1965) expansible-contractible $\langle 0, 1, 2 \rangle$ - designs (see Section 4.1) for the 2^n factorial experiment and the expansible-contractible $\langle 0, 1, 2 \rangle$ - designs for the 3^m experiment (see Section 5.3) to the $2^n 3^m$ experiment in the form of economic expansible-contractible $\langle 0, 1, 2 \rangle$ - $2^n 3^m$ designs. Such designs reduce to Webb's (1965) $\langle 0, 1, 2 \rangle$ - 2^n designs when $m = 0$ and reduce to the $\langle 0, 1, 2 \rangle$ - 3^m design in (5.2) when $n = 0$. The set of treatment combinations generating these $\langle 0, 1, 2 \rangle$ - $2^n 3^m$ designs is the fraction $F_{n,m}$ consisting of the $N = 1 + n + \frac{n(n-1)}{2} + 2m + \frac{4m(m-1)}{2} + 2nm$ runs of the form

$$00\cdots 0a_r^{i_r} 00\cdots 0a_s^{i_s} 00\cdots 0 \cdot 00\cdots 0b_t^{k_t} 00\cdots 0b_u^{k_u} 00\cdots 0 \quad (6.2)$$

where

$$(a_r^{i_r}, a_s^{i_s} = 0, 1), (s = r+1, r+2, \dots, n \text{ and } r = 1, 2, \dots, n) .$$

$$(b_t^{k_t}, b_u^{k_u} = 0, 1, 2) \text{ and } (u = t+1, t+2, \dots, m \text{ and } t = 1, 2, \dots, m) .$$

These N runs will be used for developing the OBAT $2^n 3^m$ plans later in this chapter. The runs in (6.2) will be denoted by $(a_r^{i_r} a_s^{i_s} \cdot b_t^{k_t} b_u^{k_u})_{rs \cdot tu}$, or for short $(a_r a_s \cdot b_t b_u)_{rs \cdot tu}$ and the corresponding response by $y(a_r a_s \cdot b_t b_u)_{rs \cdot tu}$. The linear model in (6.1) for the fraction $F_{n,m}$ can be written as

$$\begin{aligned}
(a_r a_s \cdot b_t b_u)_{rs \cdot tu} &= \mu + A_r x_r + A_s x_s - \sum_{i \neq r, s} A_i \\
&+ A_r A_s x_r x_s - \left(\sum_{j=r, s} A_r A_j \right) x_r - \left(\sum_{j \neq r, s} A_s A_j \right) x_s + \sum_{i < j} \sum_{i, j \neq r, s} A_i A_j \\
&+ B_t y_t + B_u y_u - \sum_{k \neq t, u} B_k + B_t^2 y_t' + B_u^2 y_u' + \sum_{k \neq t, u} B_k^2 \\
&+ B_t B_u y_t y_u - \left(\sum_{k \neq t, u} B_t B_k \right) y_t - \left(\sum_{k \neq t, u} B_u B_k \right) y_u + \sum_{k < \ell} \sum_{k, \ell \neq t, u} B_k B_\ell \\
&+ B_t B_u^2 y_t y_u' + \left(\sum_{k \neq t, u} B_t B_k^2 \right) y_t + \left(\sum_{k \neq t, u} B_u B_k^2 \right) y_u - \sum_{k < \ell} \sum_{k, \ell \neq t, u} B_k B_\ell^2 \\
&+ B_t^2 B_u y_t y_u' - \left(\sum_{k \neq t, u} B_t^2 B_k \right) y_t' - \left(\sum_{k \neq t, u} B_u^2 B_k \right) y_u' - \sum_{k < \ell} \sum_{k, \ell \neq t, u} B_k^2 B_\ell \\
&+ B_t^2 B_u^2 y_t y_u' + \left(\sum_{k \neq t, u} B_t^2 B_k^2 \right) y_t' + \left(\sum_{k \neq t, u} B_u^2 B_k^2 \right) y_u' + \sum_{k < \ell} \sum_{k, \ell \neq t, u} B_k^2 B_\ell^2 \\
&+ \left[\{A_r B_t x_r y_t + A_r B_u x_r y_u - \left(\sum_{k \neq t, u} A_r B_k \right) x_r\} + \{A_s B_t x_s y_t + A_s B_u x_s y_u \right. \\
&\quad \left. - \left(\sum_{k \neq t, u} A_s B_k \right) x_s\} + \left\{ - \left(\sum_{i \neq r, s} A_i B_t \right) y_t - \left(\sum_{i \neq r, s} A_i B_u \right) y_u \right. \right. \\
&\quad \left. \left. + \sum_{i \neq r, s} \sum_{k \neq t, u} A_i B_k \right\} \right] \\
&+ \left[\{A_r B_t^2 x_r y_t' + A_r B_u^2 x_r y_u' + \left(\sum_{k \neq t, u} A_r B_k^2 \right) x_r\} + \{A_s B_t^2 x_s y_t' + A_s B_u^2 x_s y_u' \right. \\
&\quad \left. + \left(\sum_{k \neq t, u} A_s B_k^2 \right) x_s\} + \left\{ - \left(\sum_{i \neq r, s} A_i B_t^2 \right) y_t' - \left(\sum_{i \neq r, s} A_i B_u^2 \right) y_u' \right. \right. \\
&\quad \left. \left. - \sum_{i \neq r, s} \sum_{k \neq t, u} A_i B_k^2 \right\} \right] \tag{6.3}
\end{aligned}$$

The fraction $F_{n,m}$ in (6.2) possesses the following characteristics:

1. It is permutation-invariant since the treatment combinations in (6.2) can also be written as

$$\pi(00\cdots 0a_r 00\cdots 0a_s 00\cdots 0 \cdot 00\cdots 0b_t 00\cdots 0b_u 00\cdots 0)$$

or for short as

$$\pi(a_r a_s \cdot b_t b_u)_{rs \cdot tu} \quad (6.4)$$

From (6.4), we notice that $F_{n,m}$ is the union of the permutation invariant sets

(0)

$$\pi(1\cdot)_{r\cdot}, \quad r = 1, 2, \dots, n.$$

$$\pi(11\cdot)_{rs\cdot}, \quad s = r+1, r+2, \dots, n \text{ and } r = 1, 2, \dots, n-1.$$

$$\pi(\cdot 1)_{\cdot t} \text{ and } \pi(\cdot 2)_{\cdot t}, \quad t = 1, 2, \dots, m.$$

$$\pi(\cdot 11)_{\cdot tu}, \pi(\cdot 12)_{\cdot tu}, \pi(\cdot 21)_{\cdot tu} \text{ and } \pi(\cdot 22)_{\cdot tu},$$

$$u = t+1, t+2, \dots, m \text{ and } t = 1, 2, \dots, m-1.$$

$$\pi(1\cdot 1)_{r\cdot t} \text{ and } \pi(1\cdot 2)_{r\cdot t}, \quad r = 1, 2, \dots, n \text{ and } t = 1, 2, \dots, m.$$

(6.5)

Therefore, $F_{n,m}$ is permutation-invariant. The information matrix $X'X$ for $F_{n,m}$ is also invariant to permutations of factors hence factor effects are estimated with the same precision.

2. It is saturated hence economical since the number of runs N in $F_{n,m}$ is the same as the number of parameters in model (6.3).

3. It is economically expansible since

a) an additional 3-level factor B_{m+1} requires " $F_{n,m-1} - F_{n,m}$ "

= 2 + 4m + 2n runs, namely

the two runs $(\cdot 1)_{\cdot(m+1)}$ and $(\cdot 2)_{\cdot(m+1)}$

the 4m runs $(\cdot 11)_{\cdot t(m+1)}$, $(\cdot 12)_{\cdot t(m+1)}$, $(\cdot 21)_{\cdot t(m+1)}$ and

$(\cdot 22)_{\cdot t(m+1)}$, ($t = 1, 2, \dots, m$)

the 2n runs $(1 \cdot 1)_{r \cdot (m+1)}$ and $(1 \cdot 2)_{r \cdot (m+1)}$, $r = 1, 2, \dots, n$

which can be used for estimating the effects

$B_{(m+1)}$, $B_{(m+1)}^2$, $B_t B_{(m+1)}$, $B_t B_{(m+1)}^2$, $B_t^2 B_{(m+1)}$, $B_t^2 B_{(m+1)}^2$,

$A_r B_{(m+1)}$ and $A_r B_{(m+1)}^2$, ($t = 1, 2, \dots, n$) and ($r = 1, 2, \dots, n$).

b) an additional 2-level factor A_{n+1} requires " $F_{n+1,m} - F_{n,m}$ "
= 1 + n + 2m runs, namely

the run $(1 \cdot)_{(n+1)}$.

the n runs $(11 \cdot)_{r(n+1)}$, $r = 1, 2, \dots, n$.

the 2m runs $(1 \cdot 1)_{(n+1) \cdot t}$ and $(1 \cdot 2)_{(n+1) \cdot t}$, $t = 1, 2, \dots, m$

which are needed for estimating the effects

A_{n+1} , $A_r A_{n+1}$, $A_{n+1} B_t$, $A_{n+1} B_t^2$, ($r=1,2,\dots,n$) and ($t=1,2,\dots,m$).

This property of the non-orthogonal fraction $F_{n,m}$ of being economical is not shared by orthogonal $2^n 3^m$ fractions where an additional factor requires a large number of runs relative to the number of additional parameters to be estimated.

4. Fractions in (6.2) are nested with each other. This can be seen as follows: For the 2 · 3 experiment, $F_{1,1}$ consists of the 6 runs

00, 01, 02, 10, 11 and 12 and for the $2^2 \cdot 3$ experiment, $F_{2,1}$ consists of the ten runs 000, 100, 010, 110, 001, 002, 101, 102, 011 and 012.

The fraction $F_{1,1}$ can also be considered as a subset of F_3 in which the additional two level factor is at its low level hence $F_{1,1}$ can be written as 000, 001, 002, 010, 001 and 012, i.e. $F_{1,1}$ is nested in $F_{2,1}$.

5. It is a resolution V fraction of the $2^n 3^m$ experiment hence all main effects and all two-factor interactions are estimable.

A proof of this assertion will now be presented. We let

$$\begin{aligned} \sum_{k \neq t, u} A_r B_k &= A_r L(t, u), \quad \sum_{k \neq t, u} A_s B_k = A_s L(t, u) \\ \sum_{i \neq r, s} A_i B_t &= A_{(r, s)} L_t, \quad \sum_{i \neq r, s} A_i B_u = A_{(r, s)} L_u, \quad \sum_{i \neq r, s} \sum_{k \neq t, u} A_i B_k \\ &= (AL)_{(rstu)} \\ \sum_{k \neq t, u} A_r B_k^2 &= A_r Q(t, u), \quad \sum_{k \neq t, u} A_s B_k^2 = A_s Q(t, u) \\ \sum_{i \neq r, s} A_i B_t^2 &= A_{(r, s)} Q_t, \quad \sum_{i \neq r, s} A_i B_u^2 = A_{(r, s)} Q_u, \quad \sum_{i \neq r, s} \sum_{k \neq t, u} A_i B_k^2 \\ &= (AQ)_{(rstu)} \end{aligned} \quad (6.6)$$

where $\sum_{i \neq r, s} \sum_{k \neq t, u}$ contains $(n-2)(m-2)$ terms. With the notation given in (4.5), (5.8) and (6.6), the linear model (6.3) can now be written as

$$\begin{aligned} (a_{r s} a_{t u} \cdot b_{r s} b_{t u})_{rs \cdot tu} &= \mu + A_r x_r + A_s x_s - A_{(r, s)} + A_r A_s x_r x_s - A_r A_{(r, s)} x_r \\ &\quad - A_s A_{(r, s)} x_s + (AA)_{(r, s)} + B_t y_t + B_u y_u - L(t, u) \\ &\quad + B_t^2 y_t^2 + B_u^2 y_u^2 + Q(t, u) \\ &\quad + B_t B_u y_t y_u - L_t L(t, u) y_t - L_u L(t, u) y_u + (LL)_{(t, u)} \end{aligned}$$

$$\begin{aligned}
& + B_t B_u^2 y_t' y_u' + L_t Q(t,u) y_t' + L_u Q(t,u) y_u' - (LQ)(t,u) \\
& + B_t^2 B_u y_t' y_u' - Q_t L(t,u) y_t' - Q_u L(t,u) y_u' - (QL)(t,u) \\
& + B_t^2 B_u^2 y_t' y_u' + Q_t Q(t,u) y_t' + Q_u Q(t,u) y_u' + (QQ)(t,u) \\
& + A_r B_t x_r y_t + A_r B_u x_r y_u - A_r L(t,u) x_r + A_s B_t x_s y_t + A_s B_u x_s y_u - A_s L(t,u) x_s \\
& - A_{(r,s)} L_t y_t - A_{(r,s)} L_u y_u + (AL)(rstu) \\
& + A_r B_t^2 x_r y_t' + A_r B_u^2 x_r y_u' + A_r Q(t,u) + A_s B_t^2 x_s y_t' + A_s B_u^2 x_s y_u' + A_s Q(t,u) x_s \\
& - A_{(r,s)} Q_t y_t' - A_{(r,s)} Q_u y_u' - (AQ)(rstu) \tag{6.7}
\end{aligned}$$

For the various treatment combinations $(a_r a_s \cdot b_t b_u)_{rs \cdot tu}$ in (6.5), the model (6.7) can be written more explicitly as

$$\begin{aligned}
(0) & = \mu - A_r - A_s - A_{(r,s)} + A_r A_s + A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)_{(r,s)} \\
& - B_t - B_u - L(t,u) + B_t^2 + B_u^2 + Q(t,u) \\
& + B_t B_u + L_t L(t,u) + L_u L(t,u) + (LL)(t,u) \\
& - B_t B_u^2 - L_t Q(t,u) - L_u Q(t,u) - (LQ)(t,u) \\
& - B_t^2 B_u - Q_t L(t,u) - Q_u L(t,u) - (QL)(t,u) \\
& + B_t^2 B_u^2 + Q_t Q(t,u) + Q_u Q(t,u) + (QQ)(t,u) \\
& + A_r B_t + A_r B_u + A_s B_t + A_s B_u \\
& + A_{(r,s)} L_t + A_{(r,s)} L_u + A_r L(t,u) + A_s L(t,u) + (AL)(rstu)
\end{aligned}$$

$$\begin{aligned}
& - A_r B_t^2 - A_r B_u^2 - A_s B_t^2 - A_s B_u^2 \\
& - A_{(r,s)} Q_t - A_{(r,s)} Q_u - A_r Q_{(t,u)} - A_s Q_{(t,u)} - (AQ)_{(rstu)} \\
(1\cdot)_r = & \mu + A_r - A_s - A_{(r,s)} - A_r A_s - A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)_{(r,s)} \\
& - B_t - B_u - L_{(t,u)} + B_t^2 + B_u^2 + Q_{(t,u)} \\
& + B_t B_u + L_t L_{(t,u)} + L_u L_{(t,u)} + (LL)_{(t,u)} \\
& - B_t B_u^2 - L_t Q_{(t,u)} - L_u Q_{(t,u)} - (LQ)_{(t,u)} \\
& - B_t^2 B_u - Q_t L_{(t,u)} - Q_u L_{(t,u)} - (QL)_{(t,u)} \\
& + B_t^2 B_u^2 + Q_t Q_{(t,u)} + Q_u Q_{(t,u)} + (QQ)_{(t,u)} \\
& - A_r B_t - A_r B_u + A_s B_t + A_s B_u \\
& + A_{(r,s)} L_t + A_{(r,s)} L_u - A_r L_{(t,u)} + A_s L_{(t,u)} + (AL)_{(rstu)} \\
& + A_r B_t^2 + A_r B_u^2 - A_s B_t^2 - A_s B_u^2 \\
& - A_{(r,s)} Q_t - A_{(r,s)} Q_u + A_r Q_{(t,u)} - A_s Q_{(t,u)} - (AQ)_{(rstu)}
\end{aligned}$$

$$\begin{aligned}
(1\cdot)_s = & \mu - A_r + A_s - A_{(r,s)} - A_r A_s + A_r A_{(r,s)} - A_s A_{(r,s)} + (AA)_{(r,s)} \\
& - B_t - B_u - L_{(t,u)} + B_t^2 + B_u^2 + Q_{(t,u)} \\
& + B_t B_u + L_t L_{(t,u)} + L_u L_{(t,u)} + (LL)_{(t,u)} \\
& - B_t B_u^2 - L_t Q_{(t,u)} - L_u Q_{(t,u)} - (LQ)_{(t,u)} \\
& - B_t^2 B_u - Q_t L_{(t,u)} - Q_u L_{(t,u)} - (QL)_{(t,u)}
\end{aligned}$$

$$\begin{aligned}
& + B_t^2 B_u^2 + Q_t Q(t,u) + Q_u Q(t,u) + (QQ)(t,u) \\
& + A_r B_t + A_r B_u - A_s B_t - A_s B_u \\
& + A_{(r,s)} L_t + A_{(r,s)} L_u + A_r L(t,u) - A_s L(t,u) + (AL)(rstu) \\
& - A_r B_t^2 - A_r B_u^2 + A_s B_t^2 + A_s B_u^2 \\
& - A_{(r,s)} Q_t - A_{(r,s)} Q_u - A_r Q(t,u) + A_s Q(t,u) - (AQ)(rstu)
\end{aligned}$$

$$\begin{aligned}
(11.)_{rs} &= \mu + A_r + A_s - A_{(r,s)} + A_r A_s - A_r A_{(r,s)} - A_s A_{(r,s)} + (AA)(r,s) \\
& - B_t - B_u - L(t,u) + B_t^2 + B_u^2 + Q(t,u) \\
& + B_t B_u + L_t L(t,u) + L_u L(t,u) + (LL)(t,u) \\
& - B_t B_u^2 - L_t Q(t,u) - L_u Q(t,u) - (LQ)(t,u) \\
& - B_t^2 B_u - Q_t L(t,u) - Q_u L(t,u) - (QL)(t,u) \\
& + B_t^2 B_u^2 + Q_t Q(t,u) + Q_u Q(t,u) + (QQ)(t,u) \\
& - A_r B_t - A_r B_u - A_s B_t - A_s B_u \\
& + A_{(r,s)} L_t + A_{(r,s)} L_u - A_r L(t,u) - A_s L(t,u) + (AL)(rstu) \\
& + A_r B_t^2 + A_r B_u^2 + A_s B_t^2 + A_s B_u^2 \\
& - A_{(r,s)} Q_t - A_{(r,s)} Q_u + A_r Q(t,u) + A_s Q(t,u) - (AQ)(rstu)
\end{aligned}$$

$$\begin{aligned}
(1.)_t &= \mu - A_r - A_s - A_{(r,s)} + A_r A_s + A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)(r,s) \\
& - B_u - L(t,u) - 2B_t^2 + B_u^2 + Q(t,u)
\end{aligned}$$

$$\begin{aligned}
& + L_u L(t,u) + (LL)(t,u) - L_u Q(t,u) - (LQ)(t,u) \\
& + 2B_t^2 B_u + 2Q_t L(t,u) - Q_u L(t,u) - (QL)(t,u) \\
& - 2B_t^2 B_u^2 - 2Q_t Q(t,u) + Q_u Q(t,u) + (QQ)(t,u) \\
& + A_r B_u + A_s B_u + A_{(r,s)} L_u + A_r L(t,u) + A_s L(t,u) + (AL)(rstu) \\
& + 2A_r B_t^2 - A_r B_u^2 + 2A_s B_t^2 - A_s B_u^2 \\
& + 2A_{(r,s)} Q_t - A_{(r,s)} Q_u - A_r Q(t,u) - A_s Q(t,u) - (AQ)(rstu)
\end{aligned}$$

$$\begin{aligned}
(\cdot 1) \cdot u & = \mu - A_r - A_s - A_{(r,s)} + A_r A_s + A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)(r,s) \\
& - B_t - L(t,u) + B_t^2 - 2B_u^2 + Q(t,u) + L_t L(t,u) + (LL)(t,u) \\
& + 2B_t B_u^2 - L_t Q(t,u) - (LQ)(t,u) - Q_t L(t,u) + 2Q_u L(t,u) - (QL)(t,u) \\
& - 2B_t^2 B_u^2 + Q_t Q(t,u) - 2Q_u Q(t,u) + (QQ)(t,u) \\
& + A_r B_t + A_s B_t + A_{(r,s)} L_t + A_r L(t,u) + A_s L(t,u) + (AL)(rstu) \\
& - A_r B_t^2 + 2A_r B_u^2 - A_s B_t^2 + 2A_s B_u^2 \\
& - A_{(r,s)} Q_t + 2A_{(r,s)} Q_u - A_r Q(t,u) - A_s Q(t,u) - (AQ)(rstu)
\end{aligned}$$

$$\begin{aligned}
(\cdot 2) \cdot t & = \mu - A_r - A_s - A_{(r,s)} + A_r A_s + A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)(r,s) \\
& + B_t - B_u - L(t,u) + B_t^2 + B_u^2 + Q(t,u) \\
& - B_t B_u - L_t L(t,u) + L_u L(t,u) + (LL)(t,u) \\
& + B_t B_u^2 + L_t Q(t,u) - L_u Q(t,u) - (LQ)(t,u)
\end{aligned}$$

$$\begin{aligned}
& - B_t^2 B_u - Q_t L(t,u) - Q_u L(t,u) - (QL)(t,u) \\
& + B_t^2 B_u^2 + Q_t Q(t,u) + Q_u Q(t,u) + (QQ)(t,u) \\
& - A_r B_t + A_r B_u - A_s B_t + A_s B_u \\
& - A_{(r,s)} L_t + A_{(r,s)} L_u + A_r L(t,u) + A_s L(t,u) + (AL)(rstu) \\
& - A_r B_t^2 - A_r B_u^2 - A_s B_t^2 - A_s B_u^2 \\
& - A_{(r,s)} Q_t - A_{(r,s)} Q_u - A_r Q(t,u) - A_s Q(t,u) - (AQ)(rstu)
\end{aligned}$$

$$\begin{aligned}
(\cdot 2) \cdot_u = & \mu - A_r - A_s - A_{(r,s)} + A_r A_s + A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)(r,s) \\
& - B_t + B_u - L(t,u) + B_t^2 + B_u^2 + Q(t,u) \\
& - B_t B_u + L_t L(t,u) - L_u L(t,u) + (LL)(t,u) \\
& - B_t B_u^2 - L_t Q(t,u) + L_u Q(t,u) - (LQ)(t,u) \\
& + B_t^2 B_u - Q_t L(t,u) - Q_u L(t,u) - (QL)(t,u) \\
& + B_t^2 B_u^2 + Q_t Q(t,u) + Q_u Q(t,u) + (QQ)(t,u) \\
& + A_r B_t - A_r B_u + A_s B_t - A_s B_u \\
& + A_{(r,s)} L_t - A_{(r,s)} L_u + A_r L(t,u) + A_s L(t,u) + (AL)(rstu) \\
& - A_r B_t^2 - A_r B_u^2 - A_s B_t^2 - A_s B_u^2 \\
& - A_{(r,s)} Q_t - A_{(r,s)} Q_u - A_r Q(t,u) - A_s Q(t,u) - (AQ)(rstu)
\end{aligned}$$

$$\begin{aligned}
(\cdot 11) \cdot_{tu} &= \mu - A_r - A_s - A_{(r,s)} + A_r A_s + A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)_{(r,s)} \\
&\quad - L_{(t,u)} - 2B_t^2 - 2B_u^2 + Q_{(t,u)} + (LL)_{(t,u)} - (LQ)_{(t,u)} \\
&\quad + 2Q_t L_{(t,u)} + 2Q_u L_{(t,u)} - (QL)_{(t,u)} \\
&\quad + 4B_t^2 B_u^2 - 2Q_t Q_{(t,u)} - 2Q_u Q_{(t,u)} + (QQ)_{(t,u)} + A_r L_{(t,u)} \\
&\quad + A_s L_{(t,u)} + (AL)_{(rstu)} \\
&\quad + 2A_r B_t^2 + 2A_r B_u^2 + 2A_s B_t^2 + 2A_s B_u^2 \\
&\quad + 2A_{(r,s)} Q_t + 2A_{(r,s)} Q_u - A_r Q_{(t,u)} - A_s Q_{(t,u)} - (AQ)_{(rstu)}
\end{aligned}$$

$$\begin{aligned}
(\cdot 12) \cdot_{tu} &= \mu - A_r - A_s - A_{(r,s)} + A_r A_s + A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)_{(r,s)} \\
&\quad + B_u - L_{(t,u)} - 2B_t^2 + B_u^2 + Q_{(t,u)} - L_u L_{(t,u)} + (LL)_{(t,u)} \\
&\quad + L_u Q_{(t,u)} - (LQ)_{(t,u)} \\
&\quad - 2B_t^2 B_u + 2Q_t L_{(t,u)} - Q_u L_{(t,u)} - (QL)_{(t,u)} \\
&\quad - 2B_t^2 B_u^2 - 2Q_t Q_{(t,u)} + Q_u Q_{(t,u)} + (QQ)_{(t,u)} \\
&\quad - A_r B_u - A_s B_u - A_{(r,s)} L_u + A_r L_{(t,u)} + A_s L_{(t,u)} + (AL)_{(rstu)} \\
&\quad + 2A_r B_t^2 - A_r B_u^2 + 2A_s B_t^2 - A_s B_u^2 \\
&\quad + 2A_{(r,s)} Q_t - A_{(r,s)} Q_u - A_r Q_{(t,u)} - A_s Q_{(t,u)} - (AQ)_{(rstu)}
\end{aligned}$$

$$\begin{aligned}
(\cdot 21) \cdot_{tu} &= \mu - A_r - A_s - A_{(r,s)} + A_r A_s + A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)_{(r,s)} \\
&\quad + B_t - L_{(t,u)} + B_t^2 - 2B_u^2 + Q_{(t,u)} - L_t L_{(t,u)} + (LL)_{(t,u)}
\end{aligned}$$

$$\begin{aligned}
& - 2B_t B_u^2 + L_t Q(t,u) - (LQ)(t,u) - Q_t L(t,u) + 2Q_u L(t,u) \\
& \quad - (QL)(t,u) \\
& - 2B_t^2 B_u^2 + Q_t Q(t,u) - 2Q_u Q(t,u) + (QQ)(t,u) \\
& - A_r B_t - A_s B_t - A_{(r,s)} L_t + A_r L(t,u) + A_s L(t,u) + (AL)(rstu) \\
& - A_r B_t^2 + 2A_r B_u^2 - A_s B_t^2 + 2A_s B_u^2 \\
& - A_{(r,s)} Q_t + 2A_{(r,s)} Q_u - A_r Q(t,u) - A_s Q(t,u) - (AQ)(rstu) \\
(22) \cdot_{tu} = & \mu - A_r - A_s - A_{(r,s)} + A_r A_s + A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)(r,s) \\
& + B_t + B_u - L(t,u) + B_t^2 + B_u^2 + Q(t,u) \\
& + B_t B_u - L_t L(t,u) - L_u L(t,u) + (LL)(t,u) \\
& + B_t^2 B_u^2 + L_t Q(t,u) + L_u Q(t,u) - (LQ)(t,u) \\
& + B_t^2 B_u - Q_t L(t,u) - Q_u L(t,u) - (QL)(t,u) \\
& + B_t^2 B_u^2 + Q_t Q(t,u) + Q_s Q(t,u) + (QQ)(t,u) \\
& - A_r B_t - A_r B_u - A_s B_t - A_s B_u \\
& - A_{(r,s)} L_t - A_{(r,s)} L_u + A_r L(t,u) + A_s L(t,u) + (AL)(rstu) \\
& - A_r B_t^2 - A_r B_u^2 - A_s B_t^2 - A_s B_u^2 \\
& - A_{(r,s)} Q_t - A_{(r,s)} Q_u - A_r Q(t,u) - A_s Q(t,u) - (AQ)(rstu)
\end{aligned}$$

$$\begin{aligned}
(1.1)_{r.t} &= \mu + A_r - A_s - A_{(r,s)} - A_r A_s - A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)_{(r,s)} \\
&\quad - B_u - L_{(t,u)} - 2B_t^2 + B_u^2 + Q_{(t,u)} + L_u L_{(t,u)} + (LL)_{(t,u)} \\
&\quad - L_u Q_{(t,u)} - (LQ)_{(t,u)} \\
&\quad + 2B_t^2 B_u + 2Q_t L_{(t,u)} - Q_u L_{(t,u)} - (QL)_{(t,u)} \\
&\quad - 2B_t^2 B_u^2 - 2Q_t Q_{(t,u)} + Q_u Q_{(t,u)} + (QQ)_{(t,u)} \\
&\quad - A_r B_u + A_s B_u + A_{(r,s)} L_u - A_r L_{(t,u)} + A_s L_{(t,u)} + (AL)_{(rstu)} \\
&\quad - 2A_r B_t^2 + A_r B_u^2 + 2A_s B_t^2 - A_s B_u^2 \\
&\quad + 2A_{(r,s)} Q_t - A_{(r,s)} Q_u + A_r Q_{(t,u)} - A_s Q_{(t,u)} - (AQ)_{(rstu)}
\end{aligned}$$

$$\begin{aligned}
(1.2)_{r.t} &= \mu + A_r - A_s - A_{(r,s)} - A_r A_s - A_r A_{(r,s)} + A_s A_{(r,s)} + (AA)_{(r,s)} \\
&\quad + B_t - B_u - L_{(t,u)} + B_t^2 + B_u^2 + Q_{(t,u)} \\
&\quad - B_t B_u - L_t L_{(t,u)} + L_u L_{(t,u)} + (LL)_{(t,u)} \\
&\quad + B_t B_u^2 + L_t Q_{(t,u)} - L_u Q_{(t,u)} - (LQ)_{(t,u)} \\
&\quad - B_t^2 B_u - Q_t L_{(t,u)} - Q_u L_{(t,u)} - (QL)_{(t,u)} \\
&\quad + B_t^2 B_u^2 + Q_t Q_{(t,u)} + Q_u Q_{(t,u)} + (QQ)_{(t,u)} \\
&\quad + A_r B_t - A_r B_u - A_s B_t + A_s B_u \\
&\quad - A_{(r,s)} L_t + A_{(r,s)} L_u - A_r L_{(t,u)} + A_s L_{(t,u)} + (AL)_{(rstu)} \\
&\quad + A_r B_t^2 + A_r B_u^2 - A_s B_t^2 - A_s B_u^2
\end{aligned}$$

$$- A_{(r,s)} Q_t - A_{(r,s)} Q_u + A_r Q(t,u) - A_s Q(t,u) - (AQ)_{(rstu)} \quad (6.8)$$

From (6.8) we find the following differences

a) for the main effect A_r

$$(1.2)_{r.t} - (.2)_{.t} = 2[A_r - A_r A_s - A_r A_{(r,s)} + A_r B_t - A_r B_u - A_r L(t,u) + A_r B_t^2 + A_r B_u^2 + A_r Q(t,u)] \quad (6.9)$$

$$(1.1)_{r.t} - (.1)_{.t} = 2[A_r - A_r A_s - A_r A_{(r,s)} - A_r B_u - A_r L(t,u) - 2A_r B_t^2 + A_r B_u^2 + A_r Q(t,u)] \quad (6.10)$$

$$(11.)_{rs.} - (1.)_{s.} = 2[A_r + A_r A_s - A_r A_{(r,s)} - A_r B_t - A_r B_u - A_r L(t,u) + A_r B_t^2 + A_r B_u^2 + A_r Q(t,u)] \quad (6.11)$$

$$(1.)_{r.} - (0) = 2[A_r - A_r A_s - A_r A_{(r,s)} - A_r B_t - A_r B_u - A_r L(t,u) + A_r B_t^2 + A_r B_u^2 + A_r Q(t,u)] \quad (6.12)$$

b) for the linear effect B_t ,

$$(.22)_{.tu} - (.2)_{.u} = 2[B_t + B_t B_u - L_t L(t,u) + B_t B_u^2 + L_t Q(t,u) - A_r B_t - A_s B_t - A_{(r,s)} L_t] \quad (6.13)$$

$$(.21)_{.tu} - (.1)_{.u} = 2[B_t - L_t L(t,u) - 2B_t B_u^2 + L_t Q(t,u) - A_r B_t - A_s B_t - A_{(r,s)} L_t] \quad (6.14)$$

$$(1\cdot2)_{r\cdot t} - (1\cdot)_{r\cdot} = 2[B_t^2 - B_t^2 B_u - L_t L(t,u) + B_t^2 B_u^2 + L_t Q(t,u) + A_r B_t^2 - A_s B_t^2 - A_{(r,s)} L_t] \quad (6.15)$$

$$(\cdot2)_{\cdot t} - (0) = 2[B_t^2 - B_t^2 B_u - L_t L(t,u) + B_t^2 B_u^2 + L_t Q(t,u) - A_r B_t^2 - A_s B_t^2 - A_{(r,s)} L_t] \quad (6.16)$$

c) for the quadratic effect B_t^2 ,

$$(\cdot1)_{\cdot u} - 2(\cdot11)_{\cdot tu} + (\cdot21)_{\cdot tu} = 6[B_t^2 - Q_t L(t,u) - 2B_t^2 B_u^2 + Q_t Q(t,u) - A_r B_t^2 - A_s B_t^2 - A_{(r,s)} Q_t] \quad (6.17)$$

$$(\cdot2)_{\cdot u} - 2(\cdot12)_{\cdot tu} + (\cdot22)_{\cdot tu} = 6[B_t^2 + B_t^2 B_u - Q_t L(t,u) + B_t^2 B_u^2 + Q_t Q(t,u) - A_r B_t^2 - A_s B_t^2 - A_{(r,s)} Q_t] \quad (6.18)$$

$$(1\cdot)_{r\cdot} - 2(1\cdot1)_{r\cdot t} + (1\cdot2)_{r\cdot t} = 6[B_t^2 - B_t^2 B_u - Q_t L(t,u) + B_t^2 B_u^2 + Q_t Q(t,u) + A_r B_t^2 - A_s B_t^2 - A_{(r,s)} Q_t] \quad (6.19)$$

$$(0) - 2(\cdot1)_{\cdot t} + (\cdot2)_{\cdot t} = 6[B_t^2 - B_t^2 B_u - Q_t L(t,u) + B_t^2 B_u^2 + Q_t Q(t,u) - A_r B_t^2 - A_s B_t^2 - A_{(r,s)} Q_t] \quad (6.20)$$

Using (6.9)-(6.12) we shall show that the main effect A_r of the r^{th} 2-level factor ($r = 1, 2, \dots, n$) is estimable from $F_{n,m}$. We find from (6.9) and (6.12),

$$\begin{aligned} & [((1\cdot2)_{r\cdot t} - (\cdot2)_{\cdot t}) + ((1\cdot1)_{r\cdot t} - (\cdot1)_{\cdot t})] \\ & = 4A_r - 4A_r A_s - 4A_r A_{(r,s)} + 2A_r B_t^2 - 4A_r B_u^2 - 4A_r L(t,u) \end{aligned}$$

$$- 2A_r B_t^2 + 4A_r B_u^2 + 4A_r Q(t,u) \quad (6.21)$$

$$\begin{aligned} & \sum_{t=1}^m [((1 \cdot 2)_{r \cdot t} - (\cdot 2)_{\cdot t}) + ((1 \cdot 1)_{r \cdot t} - (\cdot 1)_{\cdot t})] \\ &= 4mA_r - 4m \sum_{s \neq r} A_r A_s + 2 \sum_{t=1}^m A_r B_t - 4 \sum_{t=1}^m (A_r B_u + A_r L(t,u)) \\ & \quad - 2 \sum_{t=1}^m A_r B_t^2 + 4 \sum_{t=1}^m (A_r B_u^2 + A_r Q(t,u)) \end{aligned} \quad (6.22)$$

Referring to (4.9), we find

$$\sum_{t=1}^m (A_r B_u + A_r L(t,u)) = (m-2) \sum_{t=1}^m A_r B_t \quad (6.23)$$

and

$$\sum_{t=1}^m (A_r B_u^2 + A_r Q(t,u)) = (m-2) \sum_{t=1}^m A_r B_t^2 \quad (6.24)$$

From (6.22), (6.23) and (6.24) we find

$$\begin{aligned} & \sum_{t=1}^m [((1 \cdot 2)_{r \cdot t} - (\cdot 2)_{\cdot t}) + ((1 \cdot 1)_{r \cdot t} - (\cdot 1)_{\cdot t})] = 4mA_r - 4m \sum_{s \neq r} A_r A_s \\ & \quad - 2(2m-3) \sum_{t=1}^m A_r B_t + 2(2m-3) \sum_{t=1}^m A_r B_t^2 \end{aligned} \quad (6.25)$$

From (6.11),

$$\begin{aligned} \sum_{s \neq r} ((11 \cdot)_{rs \cdot} - (1 \cdot)_{s \cdot}) &= 2(n-1)A_r + 2 \sum_{s \neq r} A_r A_s - 2 \sum_{s \neq r} A_r A(r,s) \\ & \quad - 2(n-1)[A_r B_t - A_r B_u - A_r L(t,u)] \\ & \quad + 2(n-1)[A_r B_t^2 + A_r B_u^2 + A_r Q(t,u)] \end{aligned} \quad (6.26)$$

Referring to (4.9),

$$\sum_{s \neq r} A_r A_s(r,s) = (n-2) \sum_{s \neq r} A_r A_s \quad (6.27)$$

From (6.26) and (6.27),

$$\begin{aligned} \sum_{s \neq r} ((11 \cdot)_{rs} - (1 \cdot)_{s \cdot}) &= 2(n-1)A_r - 2(n-3) \sum_{s \neq r} A_r A_s \\ &\quad - 2(n-1) \sum_{r=1}^n A_r B_t + 2(n-1) \sum_{r=1}^n A_r B_t^2 \end{aligned} \quad (6.28)$$

From (6.25) and (6.28),

$$\begin{aligned} 2 \cdot \sum_{t=1}^m [((1 \cdot 2)_{r \cdot t} - (\cdot 2)_{\cdot t}) + ((1 \cdot 1)_{r \cdot t} - (\cdot 1)_{\cdot t})] \\ + 3 \cdot \sum_{s \neq r} ((11 \cdot)_{rs} - (1 \cdot)_{s \cdot}), \\ = 2(4m + 3n - 3)A_r - 2(4m + 3n - 9) \left[\sum_{s \neq r} A_r A_s + \sum_{t=1}^m A_r B_t \right. \\ \left. - \sum_{t=1}^m A_r B_t^2 \right] \end{aligned} \quad (6.28)$$

From (6.12),

$$\begin{aligned} (4m + 3n - 9)((1 \cdot)_{r \cdot} - (0)) &= 2(4m + 3n - 9)A_r - 2(4m + 3n - 9) \\ &\cdot \left[\sum_{s \neq r} A_r A_s + \sum_{t=1}^m A_r B_t - \sum_{t=1}^m A_r B_t^2 \right] \end{aligned} \quad (6.29)$$

From the difference between (6.28) and (6.29), it follows that

$$\begin{aligned}
A_r = & \frac{1}{6} \sum_{t=1}^m [((1 \cdot 2)_{r \cdot t} - (\cdot 2)_{\cdot t}) + ((1 \cdot 1)_{r \cdot t} - (\cdot 1)_{\cdot t})] \\
& + \frac{1}{2} \sum_{s \neq r} ((11 \cdot)_{rs \cdot} - (1 \cdot)_{s \cdot}) - \frac{1}{12} (4m+3n-9)((1 \cdot)_{r \cdot} - (0)) \\
& , (r = 1, 2, \dots, n) . \quad (6.30)
\end{aligned}$$

Hence, main effects of 2-level factors are estimable from $F_{n,m}$.

Using (6.13)-(6.16) we shall now show that the linear effect B_t of the t^{th} 3-level factor ($t = 1, 2, \dots, m$) is estimable from $F_{n,m}$.

We find from (6.13) and (6.14),

$$\begin{aligned}
((\cdot 22)_{\cdot tu} - (\cdot 2)_{\cdot u}) + ((\cdot 21)_{\cdot tu} - (\cdot 1)_{\cdot u}) &= 4B_t + 2B_t B_u \\
- 4L_t L(t,u) - 2B_t B_u^2 + 4L_t Q(t,u) - 4(A_r B_t - A_s B_t - A_{(r,s)} L_t) & \quad (6.31)
\end{aligned}$$

$$\begin{aligned}
\sum_{u \neq t} [((\cdot 22)_{\cdot tu} - (\cdot 2)_{\cdot u}) + ((\cdot 21)_{\cdot tu} - (\cdot 1)_{\cdot u})] &= 4(m-1)B_t \\
+ 2 \sum_{u \neq t} B_t B_u - 4 \sum_{u \neq t} L_t L(t,u) - 2 \sum_{u \neq t} B_t B_u^2 + 4 \sum_{u \neq t} L_t Q(t,u) & \\
- 4(m-1) \sum_{r=1}^n A_r B_t & \quad (6.32)
\end{aligned}$$

Referring to (4.9), we find

$$\begin{aligned}
\sum_{u \neq t} [((\cdot 22)_{\cdot tu} - (\cdot 2)_{\cdot u}) + ((\cdot 21)_{\cdot tu} - (\cdot 1)_{\cdot u})] &= 4(m-1)B_t \\
+ 2 \sum_{u \neq t} B_t B_u - 4(m-2) \sum_{u \neq t} B_t B_u - 2 \sum_{u \neq t} B_t B_u^2 + 4(m-2) \sum_{u \neq t} B_t B_u^2 & \\
- 4(m-1) \sum_{r=1}^n A_r B_t &
\end{aligned}$$

or

$$= 4(m-1)B_t - 2(2m-5) \sum_{u \neq t} B_t B_u + 2(2m-5) \sum_{u \neq t} B_t B_u^2 - 4(m-1) \sum_{r=1}^n A_r B_t \quad (6.33)$$

From (6.15),

$$\begin{aligned} \sum_{r=1}^n ((1 \cdot 2)_{r \cdot t} - (1 \cdot)_{r \cdot}) &= 2nB_t - 2n(B_t B_u + L_t L(t, u)) + 2n(B_t B_u^2 \\ &+ L_t Q(t, u)) + 2 \sum_{r=1}^n A_r B_t - 2 \sum_{r=1}^n (A_s B_t + A_{(r, s)} B_t) \end{aligned} \quad (6.34)$$

Referring to (4.9),

$$\sum_{r=1}^n (A_s B_t + A_{r, s} B_t) = \sum_{r=1}^n \left[\sum_{s \neq r} A_s B_t \right] = (n-2) \sum_{r=1}^n A_r B_t \quad (6.35)$$

From (6.34) and (6.35),

$$\begin{aligned} \sum_{r=1}^n ((1 \cdot 2)_{r \cdot t} - (1 \cdot)_{r \cdot}) &= 2nB_t - 2n \sum_{u \neq t} B_t B_u + 2n \sum_{u \neq t} B_t B_u^2 \\ &- 2(n-2) \sum_{r=1}^n A_r B_t \end{aligned} \quad (6.36)$$

From (6.33) and (6.36)

$$\begin{aligned} 2 \cdot \sum_{u \neq t} [((\cdot 22)_{\cdot tu} - (\cdot 2)_{\cdot u}) + ((\cdot 21)_{\cdot tu} - (\cdot 1)_{\cdot u})] \\ + 3 \cdot \sum_{r=1}^n ((1 \cdot 1)_{r \cdot t} - (1 \cdot)_{r \cdot}) \\ = 2(4m+3n-4)B_t - 2(4m+3n-10) \left[\sum_{u \neq t} B_t B_u - \sum_{u \neq t} B_t B_u^2 + \sum_{r=1}^n A_r B_t \right] \end{aligned} \quad (6.37)$$

From (6.16),

$$\begin{aligned}
(4m+3n-10)((\cdot 2)_{\cdot t} - (0)) &= 2(4m+3n-10)B_t - 2(4m+3n-10) \\
&\cdot [(B_t B_u + L_t L(t,u) - (B_t B_u^2 + L_t Q(t,u)) + (A_r B_t + A_s B_t + A_{(r,s)} B_t))] \\
&= 2(4m+3n-10)B_t - 2(4m+3n-10) \left[\sum_{u \neq t} B_t B_u - \sum_{u \neq t} B_t B_u^2 + \sum_{r=1}^n A_r B_t \right]
\end{aligned} \tag{6.38}$$

From the difference between (6.37) and (6.38), it follows that

$$\begin{aligned}
B_t &= \frac{1}{6} \sum_{u \neq t} [((\cdot 22)_{\cdot tu} - (\cdot 2)_{\cdot u}) + ((\cdot 21)_{\cdot tu} - (\cdot 1)_{\cdot u})] \\
&\quad + \frac{1}{4} \sum_{r=1}^n ((1 \cdot 2)_{r \cdot t} - (1 \cdot)_{r \cdot}) - \frac{1}{12} (4m+3n-10)((\cdot 2)_{\cdot t} - (0)) \\
&\quad , (t = 1, 2, \dots, m). \tag{6.39}
\end{aligned}$$

Hence, linear effects of 3-level factors are estimable from $F_{n,m}$.

Using (6.17)-(6.20) we shall show next that the quadratic effect B_t^2 of the t^{th} 3-level factor ($t = 1, 2, \dots, m$) is estimable from $F_{n,m}$. We find from (6.17) and (6.18)

$$\begin{aligned}
& [((\cdot 1)_{\cdot u} - 2(\cdot 11)_{\cdot tu} + (\cdot 21)_{\cdot tu}) + ((\cdot 2)_{\cdot u} - 2(\cdot 12)_{\cdot tu} + (\cdot 22)_{\cdot tu})] \\
&= 12B_t^2 + 6B_t^2 B_u - 12Q_t L(t,u) - 6B_t^2 B_u^2 + 12Q_t Q(t,u) \\
&\quad - 12(A_r B_t^2 + A_s B_t^2 + A_{(r,s)} Q_t) \tag{6.40}
\end{aligned}$$

$$\begin{aligned}
& \sum_{u \neq t} [((\cdot 1)_{\cdot u} - 2(\cdot 11)_{\cdot tu} + (\cdot 21)_{\cdot tu}) + ((\cdot 2)_{\cdot u} - 2(\cdot 12)_{\cdot tu} + (\cdot 22)_{\cdot tu})] \\
&= 12(m-1)B_t^2 + 6 \sum_{u \neq t} B_t^2 B_u - 12 \sum_{u \neq t} Q_t L(t,u) - 6 \sum_{u \neq t} B_t^2 B_u^2 \\
&\quad + 12 \sum_{u \neq t} Q_t Q(t,u) - 12(m-1) \sum_{r=1}^n A_r B_t^2 \tag{6.41}
\end{aligned}$$

Referring to (4.9), we find

$$\sum_{u \neq t} Q_t L(t, u) = (m-2) \sum_{u \neq t} B_t^2 B_u \quad \text{and} \quad \sum_{u \neq t} Q_t Q(t, u) = (m-2) \sum_{u \neq t} B_t^2 B_u^2 \quad (6.42)$$

From (6.41) and (6.42),

$$\begin{aligned} & \sum_{u \neq t} [((\cdot 1)_{\cdot u} - 2(\cdot 11)_{\cdot tu} + (\cdot 21)_{\cdot tu}) + ((\cdot 2)_{\cdot u} - 2(\cdot 12)_{\cdot tu} + (\cdot 22)_{\cdot tu})] \\ &= 12(m-1)B_t^2 - 6(2m-5) \sum_{u \neq t} B_t^2 B_u + 6(2m-5) \sum_{u \neq t} B_t^2 B_u^2 - 12(m-1) \sum_{r=1}^n A_r B_t^2 \end{aligned} \quad (6.43)$$

From (6.19),

$$\begin{aligned} & \sum_{r=1}^n ((1\cdot)_{r\cdot} - 2(1\cdot 1)_{r\cdot t} + (1\cdot 2)_{r\cdot t}) = 6n - 6n(B_t^2 B_u + Q_t L(t, u)) \\ & + 6n(B_t^2 B_u^2 + Q_t Q(t, u)) + 6 \sum_{r=1}^n A_r B_t^2 - 6 \sum_{r=1}^n (A_s B_t^2 + A_{(r,s)} B_t^2) \end{aligned} \quad (6.44)$$

Referring to (4.9),

$$\sum_{r=1}^n (A_s B_t^2 + A_{(r,s)} B_t^2) = (n-2) \sum_{r=1}^n A_r B_t^2 \quad (6.45)$$

From (6.44) and (6.45),

$$\begin{aligned} & \sum_{r=1}^n ((1\cdot)_{r\cdot} - 2(1\cdot 1)_{r\cdot t} + (1\cdot 2)_{r\cdot t}) = 6nB_t^2 - 6n \sum_{u \neq t} B_t^2 B_u \\ & + 6n \sum_{u \neq t} B_t^2 B_u^2 - 6(n-1) \sum_{r=1}^n A_r B_t^2 \end{aligned} \quad (6.46)$$

From (6.43) and (6.46)

$$\begin{aligned}
& 2 \sum_{u \neq t} [((\cdot 1)_{\cdot u} - 2(\cdot 11)_{\cdot tu} + (\cdot 21)_{\cdot tu}) + ((\cdot 2)_{\cdot u} - 2(\cdot 12)_{\cdot tu} \\
& \quad + (\cdot 22)_{\cdot tu})] + 3 \cdot \sum_{r=1}^n ((1\cdot)_r - 2(1\cdot)_{r\cdot t} + (1\cdot 2)_{r\cdot t}) \\
& = 6(4m+3n-4)B_t^2 - 6(4m+3n-10) \left[\sum_{u \neq t} B_t^2 B_u^2 - \sum_{u \neq t} B_t^2 B_u^2 + \sum_{r=1}^n A_r B_t^2 \right]
\end{aligned} \tag{6.47}$$

From (6.20),

$$\begin{aligned}
(4m+3n-10)((0) - 2(\cdot 1)_{\cdot t} + (\cdot 2)_{\cdot t}) &= 6(4m+3n-10)B_t^2 - 6(4m+3n-10) \\
&\cdot [(B_t^2 B_u^2 + Q_t L(t,u)) - (B_t^2 B_u^2 + Q_t Q(t,u)) + (A_r B_t^2 + A_s B_t^2 \\
& + A_{(r,s)} B_t^2)] \\
&= 6(4m+3n-10) - 6(4m+3n-10) \left[\sum_{u \neq t} B_t^2 B_u^2 - \sum_{u \neq t} B_t^2 B_u^2 + \sum_{r=1}^n A_r B_t^2 \right]
\end{aligned} \tag{6.48}$$

From the difference between (6.47) and (6.48) it follows that

$$\begin{aligned}
B_t^2 &= \frac{1}{18} \sum_{u \neq t} [((\cdot 1)_{\cdot u} - 2(\cdot 11)_{\cdot tu} + (\cdot 21)_{\cdot tu}) + ((\cdot 2)_{\cdot u} - 2(\cdot 12)_{\cdot tu} \\
& \quad + (\cdot 22)_{\cdot tu})] + \frac{1}{12} \sum_{r=1}^n ((1\cdot)_r - 2(1\cdot)_{r\cdot t} + (1\cdot 2)_{r\cdot t}) \\
& \quad - \frac{1}{36} (4m+3n-10)((0) - 2(\cdot 1)_{\cdot t} + (\cdot 2)_{\cdot t})
\end{aligned} \tag{6.49}$$

Hence, quadratic effects of 3-level factors are estimable from $F_{n,m}$.

To prove that two-factor interactions are also estimable from the fraction $F_{n,m}$ we find

a) from the difference between (6.11) and (6.12),

$$A_r A_s = \frac{1}{2} [((11\cdot)_{rs} - (1\cdot)_s) - ((1\cdot)_r - (0))] \tag{6.50}$$

b) from the difference between (6.15) and (6.16),

$$A_r B_t = \frac{1}{2} [((1.2)_{r,t} - (1.)_{r.}) - ((.2)_{.t} - (0))] \quad (6.51)$$

c) from the difference between (6.19) and (6.20),

$$A_r B_t^2 = \frac{1}{12} [((1.)_{r.} - 2(1.1)_{r,t} + (1.2)_{r,t}) - ((0) - 2(.1)_{.t} + (.2)_{.t})] \quad (6.52)$$

d) from the difference between (6.13) and (6.16),

$$B_t B_u = \frac{1}{2} [((.22)_{.tu} - (.2)_{.u}) - ((.2)_{.t} - (0))] \quad (6.53)$$

e) from the difference between (6.18) and (6.20),

$$B_t^2 B_u = \frac{1}{12} [((.2)_{.u} - 2(.12)_{.tu} + (.22)_{.tu}) - ((0) - 2(.1)_{.t} + (.2)_{.t})] \quad (6.54)$$

f) from (6.18), we find

$$\begin{aligned} (.2)_{.t} - 2(.21)_{.tu} + (.22)_{.tu} &= 6[B_u^2 + B_u^2 B_t - Q_u L_{(t,u)} + B_t^2 B_u^2 \\ &+ Q_u Q_{(t,u)} - A_r B_u^2 - A_s B_u^2 - A_{(r,s)} Q_u] \end{aligned} \quad (6.55)$$

and from (6.20),

$$\begin{aligned} (0) - 2(.1)_{.u} + (.2)_{.u} &= 6[B_u^2 - B_u^2 B_t - Q_u L_{(t,u)} + B_t^2 B_u^2 + Q_u Q_{(t,u)} \\ &- A_r B_u^2 - A_s B_u^2 - A_{(r,s)} Q_u] \end{aligned} \quad (6.56)$$

From the difference between (6.55) and (6.56) it follows that

$$B_t B_u^2 = \frac{1}{12} [((.2)_{.t} - 2(.21)_{.tu} + (.22)_{.tu}) - ((0) - 2(.1)_{.u} + (.2)_{.u})] \quad (6.57)$$

g) from (6.17), (6.18) and (6.20), we find

$$B_{tu}^2 = \frac{1}{36} [((0) - 2(\cdot 1)_{\cdot t} + (\cdot 2)_{\cdot t}) - 2((\cdot 1)_{\cdot u} - 2(\cdot 11)_{\cdot tu} + (\cdot 21)_{\cdot tu}) + ((\cdot 2)_{\cdot u} - 2(\cdot 12)_{\cdot tu} + (\cdot 22)_{\cdot tu})] \quad (6.58)$$

Therefore, all parameters of the model (6.1) are estimable hence the fraction $F_{n,m}$ in (6.2) is of resolution V. The unbiased estimates of the parameters of the linear model

$$y(a_r a_s \cdot b_t b_u)_{rs \cdot tu} = (a_r a_s \cdot b_t b_u)_{rs \cdot tu} + \varepsilon_{rs \cdot tu} \quad (6.59)$$

where $E(\varepsilon_{rs \cdot tu}) = 0$ and $\text{var}(\varepsilon_{rs \cdot tu}) = \sigma^2$ are obtained by replacing $(a_r a_s \cdot b_t b_u)_{rs \cdot tu}$ in (6.30), (6.39), (6.49), (6.50), (6.51), (6.52), (6.53), (6.54), (6.57) and (6.58) by its response $y(a_r a_s \cdot b_t b_u)_{rs \cdot tu}$.

We also find that

$$\begin{aligned} \text{var } \hat{A}_r &= \left[\frac{1}{36} (2m+2m) + \frac{1}{16} (2n-2) + \frac{2}{144} (4m+3n-9)^2 \right] \sigma^2 \\ \text{var } \hat{B}_t &= \left[\frac{1}{36} (2(m-1)+2(m-1)) + \frac{1}{16} (2n) + \frac{2}{144} (4m+3n-10) \right] \sigma^2 \\ \text{var } \hat{B}_t^2 &= \left[\frac{1}{(18)^2} (6(m-1)+6(m-1)) + \frac{1}{(12)^2} (6n) + \frac{6}{(36)^2} (4m+3n-10)^2 \right] \sigma^2 \\ \text{var } \hat{A}_r \hat{A}_s &= \text{var } \hat{A}_r \hat{B}_t = \text{var } \hat{B}_t \hat{B}_u = \frac{1}{4} \sigma^2 \\ \text{var } \hat{A}_r \hat{B}_t^2 &= \text{var } \hat{B}_t \hat{B}_u^2 = \text{var } \hat{B}_t^2 \hat{B}_u = \frac{1}{12} \sigma^2 \\ \text{var } \hat{B}_t^2 \hat{B}_u^2 &= \frac{1}{36} \sigma^2 \end{aligned} \quad (6.60)$$

It can be seen from (6.50) that the main effect A_r and the linear and quadratic effects B_t and B_t^2 are estimated with higher precision than two-factor interactions. Such property of the fraction $F_{n,m}$ makes

it appropriate for screening n two-level factors and m three-level factors as well as their two-factor interactions with more emphasis on main effects.

Considering only the following two orders:

Order one: $B_1, B_2, \dots, B_m, A_1, A_2, \dots, A_n$.

Order two: $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_m$.

Expressing the order of importance of the $(n+m)$ factors of the $2^n 3^m$ experiment to the response, the fraction $F_{n,m}$ can be carried out sequentially as an ORAT or an OBAT $2^n 3^m$ plan. As an ORAT $2^n 3^m$ plan we suggest that the N runs in $F_{n,m}$ be carried out

a) for order one:

1. (0)

2. $(\cdot 1)_{\cdot t} \rightarrow (\cdot 2)_{\cdot t} \rightarrow (1 \cdot)_{r \cdot}$ $t = 1, 2, \dots, m$ and $r = 1, 2, \dots, n$.

3. $(\cdot 11)_{\cdot 1u} \rightarrow (\cdot 22)_{\cdot 1u} \rightarrow (\cdot 12)_{\cdot 1u} \rightarrow (\cdot 21)_{\cdot 1u} \rightarrow (1 \cdot 1)_{r \cdot 1} \rightarrow (1 \cdot 2)_{r \cdot 1}$

for $u = 2, 3, \dots, m$ and $r = 1, 2, \dots, n$.

4. $(\cdot 11)_{\cdot 2u} \rightarrow (\cdot 22)_{\cdot 2u} \rightarrow (\cdot 12)_{\cdot 2u} \rightarrow (\cdot 21)_{\cdot 2u} \rightarrow (1 \cdot 1)_{r \cdot 2} \rightarrow (1 \cdot 2)_{r \cdot 2}$

for $u = 3, 4, \dots, m$ and $r = 1, 2, \dots, n$.

⋮

$(t+2)$. $(\cdot 11)_{\cdot tu} \rightarrow (\cdot 22)_{\cdot tu} \rightarrow (\cdot 12)_{\cdot tu} \rightarrow (\cdot 21)_{\cdot tu} \rightarrow (1 \cdot 1)_{r \cdot t} \rightarrow (1 \cdot 2)_{r \cdot 2}$

for $u = t+1, t+2, \dots, m$ and $r = 1, 2, \dots, n$.

$(m+1)$. $(\cdot 11)_{\cdot (m-1)m} \rightarrow (\cdot 22)_{\cdot (m-1)m} \rightarrow (\cdot 12)_{\cdot (m-1)m} \rightarrow (\cdot 21)_{\cdot (m-1)m} \rightarrow$

$(1 \cdot 1)_{r \cdot (m-1)} \rightarrow (1 \cdot 2)_{r \cdot (m-1)} \rightarrow (1 \cdot 1)_{r \cdot m} \rightarrow (1 \cdot 2)_{r \cdot m}$

$$\left\{ \begin{array}{l} (m+2) \cdot (11 \cdot)_{1s}, \quad s = 2, 3, \dots, n. \\ (m+3) \cdot (11 \cdot)_{.2s}, \quad s = 3, 4, \dots, n. \\ \vdots \\ (m+1+r) \cdot (11 \cdot)_{.rs}, \quad s = r+1, r+2, \dots, n. \\ \vdots \\ (m+1+n-1) \cdot (11 \cdot)_{(n-1)n} \end{array} \right. \quad (6.61)$$

where the arrow \rightarrow indicates "followed by".

For the ORAT $2^n 3^m$ plan in (6.61) we outline but do not prove some important results. We find that

a. After the completion of the $1 + m + m + n + 4(m-1) + 2n$ runs in steps 1, 2 and 3, all factorial effects involving factor B_1 become estimable, namely

$$B_1, B_1^2, B_1 B_u, B_1^2 B_u, B_1 B_u^2, B_1^2 B_u^2, A_r B_1 \text{ and } A_r B_1^2$$

for $u = 2, 3, \dots, m$ and $r = 1, 2, \dots, n$.

b. After the completion of the $[1 + 2m + n + 4(m-1) + 2n + 4(m-2) + 2n]$ runs in steps 1, 2, 3 and 4, all factorial effects involving B_1 and B_2 become estimable.

c. In general, after the $[1 + 2m + n + 4 \sum_{i=1}^t (m-i) + 2tn]$ runs in steps 1, 2, \dots , $(t+2)$, all factorial effects involving B_1, B_2, \dots, B_t become estimable ($t = 1, 2, \dots, m$).

d. After the completion of the $[1 + 2m + n + 4 \sum_{i=1}^{m-1} (m-i) + 2mm + (n-1)]$ runs in steps 1, 2, \dots , $(m+1), (m+2)$, all factorial effects involving the B_1, B_2, \dots, B_m and A_1 become estimable, namely

$B_t, B_t^2, B_t B_u, B_t^2 B_u, B_t B_u^2, B_t^2 B_u^2, A_r B_t, A_r B_t^2, A_1$ and $A_1 A_s$ for $(t = 1, 2, \dots, m), (u = t+1, t+2, \dots, m), (r = 1, 2, \dots, n)$ and $(s = 2, 3, 4, \dots, n)$.

e. After the completion of the $[1 + 2m + n + 4 \sum_{i=1}^{m-1} (m-i) + 2mn + (n-1) + (n-2)]$ runs in steps 1, 2, ..., (m+1), (m+2) and (m+3), all factorial effects involving B_1, B_2, \dots, B_m and A_1, A_2 become estimable.

f. In general, after the completion of the $[1 + 2m + n + 4 \sum_{i=1}^{m-1} (m-i) + 2mn + \sum_{j=1}^r (n-j)]$ runs in steps 1, 2, ..., (m+1), (m+2), ..., (m+1+r), all factorial effects involving B_1, B_2, \dots, B_m and the A_1, A_2, \dots, A_r become estimable ($r = 1, 2, \dots, n$).

b) for order two:

1. (0)
2. $(1\cdot)_r \rightarrow (1\cdot)_t \rightarrow (2\cdot)_t, r = 1, 2, \dots, n$ and $t = 1, 2, \dots, m$.
3. $(11\cdot)_{1s} \rightarrow (1\cdot 1)_{1\cdot t} \rightarrow (1\cdot 2)_{1\cdot t}, s = 2, 3, \dots, m$.
4. $(11\cdot)_{2s} \rightarrow (1\cdot 1)_{2\cdot t} \rightarrow (1\cdot 2)_{2\cdot t}, s = 3, 4, \dots, m$.
- ⋮
- $(r+2)\cdot (11\cdot)_{rs} \rightarrow (1\cdot 1)_{r\cdot t} \rightarrow (1\cdot 2)_{r\cdot t}, s = r+1, r+2, \dots, n$.
- ⋮
- $(n+1)\cdot (11\cdot)_{(n-1)n} \rightarrow (1\cdot 1)_{(n-1)\cdot t} \rightarrow (1\cdot 1)_{(n-1)\cdot t} \rightarrow$
 $(1\cdot 1)_{n\cdot t} \rightarrow (1\cdot 2)_{n\cdot t}$
- $(n+2)\cdot (\cdot 11)_{\cdot 1u} \rightarrow (\cdot 22)_{\cdot 1u} \rightarrow (\cdot 12)_{\cdot 1u} \rightarrow (\cdot 21)_{\cdot 1u}, u = 2, 3, \dots, m$.
- $(n+3)\cdot (\cdot 11)_{\cdot 2u} \rightarrow (\cdot 22)_{\cdot 2u} \rightarrow (\cdot 12)_{\cdot 2u} \rightarrow (\cdot 21)_{\cdot 2u}, u = 3, 4, \dots, m$.
- ⋮

$$(n+1+m-1) \cdot (011) \cdot (m-1)_m \rightarrow (022) \cdot (m-1)_m \rightarrow (012) \cdot (m-1)_m \rightarrow (021) \cdot (m-1)_m \quad (6.62)$$

For the ORAT $2^n 3^m$ plan in (6.62) we also outline some important results. We find that

1. After the completion of the $[1 + n + m + m + (n-1) + 2m]$ runs in steps 1, 2 and 3, main effect A_1 and all interaction effects involving A_1 become estimable, namely $A_1 A_s$, $B_1 B_t$ and $B_1 B_t^2$ ($s = 2, 3, \dots, n$ and $t = 1, 2, \dots, m$).

2. After the completion of the $[1 + n + m + m + (n-1) + 2m + (n-2) + 2m]$ runs in steps 1, 2, 3 and 4, main effects A_1, A_2 and all interaction effects involving A_1 and A_2 become estimable, namely $A_1 A_2, A_1 A_s, A_2 A_s, A_1 B_t, A_1 B_t^2, A_2 B_t$ and $A_2 B_t^2$ ($s = 3, 4, \dots, n$ and $t = 1, 2, \dots, m$).

3. In general, after the $[1 + n + 2m + \sum_{i=1}^r (n-i) + 2mr]$ runs in steps 1, 2, $\dots, (r+2)$, main effects A_1, A_2, \dots, A_r and all interaction effects involving A_1, A_2, \dots, A_r become estimable.

4. After the completion of the $[1 + n + 2m + \sum_{i=1}^{n-1} (n-i) + 2mn + 4(m-1)]$ runs in steps 1, 2, $\dots, (n+1), (n+2)$, all factorial effects involving A_1, A_2, \dots, A_n and B_1 become estimable, namely $A_r, A_r A_s, A_r B_t, A_r B_t^2, B_1 B_u, B_1 B_u^2, B_1^2 B_u$ and $B_1^2 B_u^2$ [$(r, s = 1, 2, \dots, n, (r \neq s), t = 1, 2, \dots, m$ and $u = 2, 3, \dots, m)$].

5. After the completion of the $[1 + n + 2m + \sum_{i=1}^{n-1} (n-i) + 2mn + 4(m-1) + 4(m-2)]$ runs in steps 1, 2, $\dots, (n+1), (n+2), (n+3)$, all factorial effects involving $A_1, A_2, \dots, A_n, B_1$ and B_2 become estimable.

6. In general, after the completion of the $[1 + n + 2m + \sum_{i=1}^{n-1} (n-i) + 2mn + 4 \sum_{j=1}^t (m-j)]$ all factorial effects involving $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_t$ become estimable.

Parameter estimates are as given in (6.30), (6.39), (6.49), (6.50), (6.51), (6.52), (6.53), (6.54), (6.57) and (6.58). These estimates can be used to judge the "significance" of their corresponding parameters.

We now consider the fraction $F_{n,m}$ in (6.2) for developing strategy I and II OBAT $2^n 3^m$ plans.

6.4 OBAT $2^n 3^m$ Plans for Strategy I (Order One)

Assuming factors are ordered for their investigation as $B_1, B_2, \dots, B_m, A_1, A_2, \dots, A_n$, factor effects are investigated one at a time according to strategy I in blocks of small sizes, namely sizes 2 and 3 such that from a set of blocks the effects $B_t, B_t^2, B_t B_u, B_t B_u^2, B_t^2 B_u, B_t^2 B_u^2, A_r B_t$ and $A_r B_t^2$ ($u = t+1, t+2, \dots, m$ and $r = 1, 2, \dots, n$) become estimable before the effects $B_{t'}, B_{t'}^2, B_{t'} B_{u'}, B_{t'} B_{u'}^2, B_{t'}^2 B_{u'}, B_{t'}^2 B_{u'}^2, A_r B_{t'}$ and $A_r B_{t'}^2$ ($t' > t, u' = t' + 1, t' + 2, \dots, m$ and $r = 1, 2, \dots, n$) and before the effects A_r and $A_r A_s$ ($r = 1, 2, \dots, n$ and $s = r+1, s+2, \dots, n$). The treatment combinations in all blocks of the OBAT $2^n 3^m$ plan are taken from the $2^n 3^m$ fraction in (6.2).

For strategy I, the linear model in (6.3) is now changed to incorporate block effects (BE) as well. Hence, for an observation in block h we have

$$y(a_r a_s \cdot b_t b_u)_{rs \cdot tu, h} = (a_r a_s \cdot b_t b_u)_{rs \cdot tu} + (BE)_h + \epsilon_{rs \cdot tu, h} \quad (6.63)$$

where $E(\epsilon_{rs \cdot tu, h}) = 0$, $\text{var}(\epsilon_{rs \cdot tu, h}) = \sigma^2$ and $\sum_{h=1}^b (BE)_h = 0$ with b the total number of blocks in the plan.

Also for strategy I OBAT $2^n 3^m$ plans, we consider the following two case:

Case (1): $2^n 3^m_{III}$ plans in a sequence of m blocks of size 3 and n blocks of size 2.

When all interactions are negligible, model (6.63) becomes

$$y(a_{rs} a_{tu})_{rs \cdot tu, h} = \mu + B_t y_t + B_u y_u - L_{(t,u)} + B_t^2 y_t' + B_u^2 y_u' + Q_{(t,u)} + A_r x_r + A_s x_s + (BE)_h + \epsilon_{rs \cdot tu, h} \quad (6.64)$$

Since the $2^n 3^m$ experiment contains m 3-level factors, m blocks of size 3 are needed in order that all m linear effects B_1, B_2, \dots, B_m and all m quadratic effects $B_1^2, B_2^2, \dots, B_m^2$ become estimable. Since the $2^n 3^m$ experiment also contains n 2-level factors, n blocks of size 2 are needed in order that the n main effects A_1, A_2, \dots, A_n become estimable. Furthermore, with the assumption that factors are ordered as $B_1, B_2, \dots, B_m, A_1, A_2, \dots, A_n$, the sequence of blocks for case (1) of strategy I is the following

$$\begin{aligned} \text{block } t: & \quad (0)_{,t} \quad (\cdot 1)_{\cdot t, t} \quad (\cdot 2)_{\cdot t, t}, \quad t=1, 2, \dots, m \\ \text{block } (m+r): & \quad (0)_{,m+r} \quad (1 \cdot)_{r \cdot, m+r}, \quad r=1, 2, \dots, n. \end{aligned} \quad (6.65)$$

In the blocking scheme (6.65), each effect is estimated from only one block since

a) from (6.16) and (6.64)

$$(\cdot 2)_{\cdot t, t} - (0)_{\cdot t} = 2B_t$$

b) from (6.20) and (6.64)

$$(0)_{\cdot t} - 2(\cdot 1)_{\cdot t, t} + (\cdot 2)_{\cdot t, t} = 6B_t^2$$

c) from (6.12) and (6.64)

$$(1\cdot)_{r\cdot, m+r} - (0)_{\cdot, m+r} = 2A_r$$

and hence

$$\begin{aligned} 1. \quad \hat{B}_t &= \frac{1}{2}(y(\cdot 2)_{\cdot t, t} - y(0)_{\cdot t}) \\ 2. \quad \hat{B}_t^2 &= \frac{1}{6}(y(0)_{\cdot t} - 2y(\cdot 1)_{\cdot t, t} + y(\cdot 2)_{\cdot t, t}) \\ 3. \quad \hat{A}_r &= \frac{1}{2}(y(1\cdot)_{r\cdot, m+r} - y(0)_{\cdot, m+r}) \end{aligned} \quad (6.66)$$

with $\text{var } \hat{B}_t = \text{var } \hat{A}_r = \frac{1}{2} \sigma^2$ and $\text{var } \hat{B}_t^2 = \frac{1}{6} \sigma^2$.

Obviously, the OBAT $2^n 3^m$ plan in (6.65) is expandible as additional 3-level and 2-level factors can be added to the plan and their effects can be estimated as in (6.66).

Case (2): $2^n 3^m$ plans in a sequence of blocks of sizes 2 and 3.

When two-factor interactions are non-negligible, i.e. if model (6.62) holds the contrasts in (6.66) do not estimate B_t , B_t^2 and A_r , respectively, rather they estimate

$$1. \quad B_t - \sum_{u \neq t} B_t B_u + \sum_{u \neq t} B_t B_u^2 - \sum_{r=1}^n A_r B_t$$

$$\begin{aligned}
 2. \quad B_t^2 &= \sum_{u \neq t} B_t^2 B_u + \sum_{u \neq t} B_t^2 B_u^2 - \sum_{r=1}^n A_r B_t^2 \\
 3. \quad A_r &= \sum_{t=1}^m A_r B_t + \sum_{t=1}^m A_r B_t^2 - \sum_{s \neq r} A_r A_s
 \end{aligned} \tag{6.67}$$

which can be written as

$$\begin{aligned}
 1. \quad S_{tu} &= B_t - \left(\sum_{u=1}^{t-1} B_t B_u + \sum_{u=t+1}^m B_t B_u \right) + \left(\sum_{u=1}^{t-1} B_t B_u^2 + \sum_{u=t+1}^m B_t B_u^2 \right) \\
 &\quad - \sum_{r=1}^n A_r B_t \\
 2. \quad S_{tu}^2 &= B_t^2 - \left(\sum_{u=1}^{t-1} B_t^2 B_u + \sum_{u=t+1}^m B_t^2 B_u \right) + \left(\sum_{u=1}^{t-1} B_t^2 B_u^2 + \sum_{u=t+1}^m B_t^2 B_u^2 \right) \\
 &\quad - \sum_{r=1}^n A_r B_t^2 \\
 3. \quad Q_{rs} &= A_r - \sum_{t=1}^m A_r B_t + \sum_{t=1}^m A_r B_t^2 - \left(\sum_{i=1}^{r-1} A_i A_r + \sum_{i=r+1}^n A_r A_i \right)
 \end{aligned} \tag{6.68}$$

It follows from (6.68) that

1. the linear effect B_t is aliased with the $(m-1)$ interactions $B_t B_u$ ($u \neq t$), with the $(m-1)$ interactions $B_t B_u^2$ ($u \neq t$) and with the n interactions $A_r B_t$.

2. the quadratic effect B_t^2 is aliased with the $(m-1)$ interactions $B_t^2 B_u$, with the $(m-1)$ interactions $B_t^2 B_u^2$ and with the n interactions $A_r B_t^2$.

3. the main effect A_r is aliased with the m interactions $A_r B_t$ ($t = 1, 2, \dots, m$), with the m interactions $A_r B_t^2$ and with the $(n-1)$ interactions $A_r A_s$ ($s \neq r$).

Therefore, to free B_t , B_t^2 and A_r from aliasing more blocks of sizes

2 and 3 must be added to the OBAT plan in (6.65). With factors ordered

such that B_1 is freed from aliasing first, B_2 second, \dots , B_m m^{th} , A_1 $(m+1)^{\text{st}}$, A_2 $(m+2)^{\text{nd}}$, \dots , A_n last, the experiment is then carried out in $(m + n - 1)$ stages where

a. for stage t ($t \leq m - 1$) the parameters B_t , B_t^2 , $B_t B_u$, $B_t^2 B_u$, $B_t B_u^2$, $B_t^2 B_u^2$, $A_r B_t$ and $A_r B_t^2$ become estimable ($u = t+1, t+2, \dots, m$ and $r = 1, 2, \dots, n$).

b. for stage m the effects B_m , B_m^2 , $A_r B_m$ and $A_r B_m^2$ ($r = 1, 2, \dots, m$) become estimable.

c. for stage $m + r$ ($r \leq n - 1$) the parameters A_r and $A_r A_s$ become estimable ($(s \neq r) = 1, 2, \dots, n$).

For purposes of illustration, we describe stage 1 first and then discuss the general stages t , m , $m + r$.

Stage One: The estimation of B_1 , B_1^2 , $B_1 B_u$, $B_1 B_u^2$, $B_1^2 B_u$, $B_1^2 B_u^2$, $A_r B_1$ and $A_r B_1^2$ ($u = 2, 3, \dots, m$ and $r = 1, 2, \dots, n$).

Stage one will be carried out in three steps such that

a) Step 1 involves the estimation of the $(m-1)$ interactions $B_1 B_u$ ($u = 2, 3, \dots, m$) by adding the $(m-1)$ blocks

$$\text{block } h_{1(u-1)}: \quad (0), h_{1(u-1)}, (\cdot 11) \cdot 1u, h_{1(u-1)}, (\cdot 22) \cdot 1u, h_{1(u-1)}$$

$$(h_{1(u-1)} = m + n + u - 1 \text{ and } u = 2, 3, \dots, m) \quad (6.69)$$

to the $(m + n)$ blocks in (6.65).

b) Step 2 involves the estimation of the $3(m-1)$ interactions $B_1 B_u^2$, $B_1^2 B_u$ and $B_1^2 B_u^2$ ($u = 2, 3, \dots, m$) by adding the $(m-1)$ blocks

block $k_{1(u-1)}$: $(0)_{,k_{1(u-1)}}$, $(\cdot 12)_{\cdot 1u, k_{1(u-1)}}$, $(\cdot 21)_{\cdot 1u, k_{1(u-1)}}$

$$(k_{1(u-1)} = m + n + (m-1) + (u-1) \text{ and } u = 2, 3, \dots, m) \quad (6.70)$$

to the $m + n + (m-1)$ blocks of step 1.

c) Step 3 involves the estimation of the remaining functions involving the first 3-level factor, namely B_1 , B_1^2 , $A_r B_1$ and $A_r B_1^2$ ($r = 1, 2, \dots, n$) by adding the n blocks

block l_{1r} : $(1\cdot)_{r\cdot, l_{1r}}$ $(1\cdot 1)_{r\cdot 1, l_{1r}}$ $(1\cdot 2)_{r\cdot 1, l_{1r}}$

$$(l_{1r} = m + n + (m-1) + (m-1) + r \text{ and } r = 1, 2, \dots, n) \quad (6.71)$$

to the $[m + n + (m-1) + (m-1)]$ blocks in step 2.

Step 2 is not needed if the two-factor interactions involving quadratic effects of 3-level factors are negligible hence the first 3-level factor effects (B_1 , B_1^2 , $B_1 B_u$, $A_r B_1$ and $A_r B_1^2$ for $u = 2, 3, \dots, m$ and $r = 1, 2, \dots, n$) become estimable in two steps, namely steps 1 and 3. We discuss below the three steps in more detail.

The estimable functions after the completion of the $m + n + (m-1)$ blocks in step 1 are

$$1. \quad B_1 + \sum_{u=2}^m B_1 B_u^2 - \sum_{r=1}^n A_r B_1$$

$$2. \quad \text{the } (m-1) \text{ interaction effects } B_1 B_u, \quad u = 2, 3, \dots, m.$$

$$3. \quad \text{the } (m-1) \text{ alias chains } B_1 B_u^2 + B_1^2 B_u - 3B_1^2 B_u^2, \quad u = 2, 3, \dots, m.$$

$$4. \quad (B_1^2 + B_1 B_u^2 - 2B_1^2 B_u^2) + \sum_{k \neq 1, u} B_1 B_k^2 - 2 \sum_{k \neq 1, u} B_1^2 B_k^2 - \sum_{r=1}^n A_r B_1^2$$

$$\begin{aligned}
5. \quad & (B_u - B_1 B_u^2 + 3B_1^2 B_u^2) - \sum_{v \neq 1, u} B_u B_v + \sum_{v \neq 1, u} B_u B_v^2 - \sum_{r=1}^n A_r B_t \\
6. \quad & (B_u^2 - B_1 B_u^2 + B_1^2 B_u^2) - \sum_{v \neq 1, u} B_u^2 B_v + \sum_{v \neq 1, u} B_u^2 B_v^2 - \sum_{r=1}^n A_r B_u^2 \\
7. \quad & A_r - \sum_{t=1}^m A_r B_t + \sum_{t=1}^m A_r B_t^2 - \sum_{s \neq r} A_r A_s \tag{6.72}
\end{aligned}$$

The unbiased least squares estimates of the functions in (6.72) are, respectively,

$$\begin{aligned}
1. \quad & \frac{1}{2} \sum_{u=2}^m [(y(\cdot 22)_{\cdot 1u, h_1(u-1)} - y(0)_{\cdot, h_1(u-1)}) - (y(\cdot 2)_{\cdot u, u} \\
& \quad - y(0)_{\cdot, u})] - \frac{1}{2}(m-3)(y(\cdot 2)_{\cdot 1, 1} - y(0)_{\cdot 1}) \\
2. \quad & \frac{1}{2} [(y(\cdot 22)_{\cdot 1u, h_1(u-1)} - y(0)_{\cdot, h_1(u-1)}) - (y(\cdot 2)_{\cdot 1, 1} - y(0)_{\cdot 1}) \\
& \quad - (y(\cdot 2)_{\cdot u, u} - y(0)_{\cdot, u})] \\
3. \quad & \frac{1}{12} [(3y(0)_{\cdot, h_1(u-1)} - 4y(\cdot 11)_{\cdot 1u, h_1(u-1)} + y(\cdot 22)_{\cdot 1u, h_1(u-1)}) \\
& \quad - (3y(0)_{\cdot 1} - 4y(\cdot 1)_{\cdot 1, 1} + y(\cdot 2)_{\cdot 1, 1}) \\
& \quad - (3y(0)_{\cdot, u} - 4y(\cdot 1)_{\cdot, u, u} + y(\cdot 2)_{\cdot, u, u})] \\
4. \quad & \frac{1}{12} \sum_{u=2}^m [(3y(0)_{\cdot, h_1(u-1)} - 4y(\cdot 11)_{\cdot 1u, h_1(u-1)} + y(\cdot 22)_{\cdot 1u, h_1(u-1)}) \\
& \quad - (3y(0)_{\cdot, u} - 4y(\cdot 1)_{\cdot, u, u} + y(\cdot 2)_{\cdot, u, u})] \\
& \quad - \frac{1}{12} (c_1 y(0)_{\cdot 1} - (c_1 + c_2) y(\cdot 1)_{\cdot 1, 1} + c_2 y(\cdot 2)_{\cdot 1, 1})
\end{aligned}$$

where $c_1 = 3m - 5$ and $c_2 = m - 3$.

$$\begin{aligned}
5. & \frac{1}{6} [(2y(\cdot 11)_{\cdot 1u, h_1(u-1)} + y(\cdot 22)_{\cdot 2u, h_1(u-1)} - 3y(0)_{\cdot, h_1(u-1)}) \\
& \quad - (2y(\cdot 1)_{\cdot 1, 1} + y(\cdot 2)_{\cdot 1, 1} - 3y(0)_{\cdot, 1})] + \frac{1}{3}(y(\cdot 2)_{\cdot u, u} - y(\cdot 1)_{\cdot u, u}) \\
6. & \frac{1}{6} (y(0)_{\cdot, u} - 2y(\cdot 1)_{\cdot u, u} + y(\cdot 2)_{\cdot u, u}) \\
7. & \frac{1}{2}(y(1\cdot)_{r, m+r} - y(0)_{\cdot, m+r}) \tag{6.73}
\end{aligned}$$

It can be seen from (6.73) that the estimates in (1), (2), ... , (6) are the same as those in (1), (2), ... , (6) in (5.45). For the unbiasedness of the estimates in (6.73), we note that estimates involve comparisons between responses in the same block hence block effects cancel out. Therefore

1. The expected value of the estimate (1) in (6.73) is

$$\frac{1}{2} \sum_{u=2}^m ((\cdot 22)_{\cdot 1u} - (\cdot 2)_{\cdot u}) - \frac{1}{2}(m-3)((\cdot 2)_{\cdot 1} - (0)) .$$

Summing (6.13) over u , we find

$$\begin{aligned}
\sum_{u=2}^m ((\cdot 22)_{\cdot 1u} - (\cdot 2)_{\cdot u}) &= 2[(m-1)B_t - (m-3) \sum_{u=2}^m B_t B_u \\
&+ (m-1) \sum_{u=2}^m B_t B_u^2 - (m-1) \sum_{r=1}^n A_r B_t] \tag{6.74}
\end{aligned}$$

and from (6.20) and (6.74) unbiasedness follows.

2. The expected value of the estimate (2) in (6.73) is

$$\frac{1}{4} [((\cdot 22)_{\cdot 1u} - (\cdot 2)_{\cdot u}) - ((\cdot 2)_{\cdot 1} - (0))] = B_1 B_u$$

3. The expected value of the estimate (3) in (6.73) is

$$\frac{1}{12} [((\cdot 22)_{\cdot 1u} - (\cdot 2)_{\cdot u}) - 4((\cdot 11)_{\cdot 1u} - (\cdot 1)_{\cdot u}) - 4(\cdot 1)_{\cdot 1} + (\cdot 2)_{\cdot 1}]$$

from (6.13), (5.47) and (5.48), unbiasedness follows.

4. The expected value of the estimate (4) in (6.73) is

$$\begin{aligned} \frac{1}{12} \sum_{u=2}^m [((\cdot 22)_{\cdot 1u} - (\cdot 2)_{\cdot u}) - 4((\cdot 11)_{\cdot 1u} - (\cdot 1)_{\cdot u})] \\ - \frac{1}{12} [(3m-5)(0) - 4(m-2)(\cdot 1)_{\cdot 1} + (m-3)(\cdot 2)_{\cdot 1}] \end{aligned}$$

from (5.50) and (5.51), unbiasedness follows.

5. The expected value of the estimate (5) in (6.73) is

$$\frac{1}{6} [2((\cdot 11)_{\cdot 1u} - (\cdot 1)_{\cdot 1}) + ((\cdot 22)_{\cdot 1u} - (\cdot 2)_{\cdot u})] + \frac{1}{3} ((\cdot 2)_{\cdot u} - (\cdot 1)_{\cdot u})$$

from (5.47), (5.52) and (6.13), unbiasedness follows.

6. The expected value of the estimate (6) in (6.73) follows from (6.20) setting $t = 1$.

7. The expected value of the estimate (7) in (6.73) follows from (6.12).

For step 2, the estimable functions after completion of the $m + n + (m-1) + (m-1)$ blocks are

$$1. B_1 - \sum_{r=1}^n A_r B_{r1}.$$

$$2. B_1^2 - \sum_{r=1}^n A_r B_{r1}^2.$$

$$3. a. S_{2u} = B_u - \left(\sum_{v=2}^{u-1} B_u B_{uv} + \sum_{v=u+1}^m B_u B_{uv} \right) + \left(\sum_{v=2}^{u-1} B_u B_{uv}^2 + \sum_{v=u+1}^m B_u B_{uv}^2 \right)$$

$$\begin{aligned}
 & - \sum_{r=1}^n A_r B_u . \\
 \text{b. } S_{2u}^2 & = B_u^2 - \left(\sum_{v=2}^{u-1} B_u B_v + \sum_{v=u+1}^m B_u B_v \right) + \left(\sum_{v=2}^{u-1} B_u^2 B_v^2 + \sum_{v=u+1}^m B_u^2 B_v^2 \right) \\
 & - \sum_{r=1}^n A_r B_u^2, \quad u = 2, 3, \dots, m .
 \end{aligned}$$

4. The $4(m-1)$ interaction effects,

$$\text{a. } B_1 B_u \quad \text{b. } B_1 B_u^2 \quad \text{c. } B_1^2 B_u \quad \text{and} \quad \text{d. } B_1^2 B_u^2 .$$

5. The function (7) in (6.72). (6.75)

The unbiased estimates of the functions in (6.75) are, respectively

$$\begin{aligned}
 1. \quad & \frac{1}{6} \sum_{u=2}^m [(y(\cdot 22)_{\cdot 1u, h_1(u-1)} - y(0)_{\cdot, h_1(u-1)}) \\
 & + (y(\cdot 21)_{\cdot 1u, k_1(u-1)} - y(0)_{\cdot, k_1(u-1)}) \\
 & - (y(\cdot 1)_{\cdot u, u} + y(\cdot 2)_{\cdot u, u} - 2y(0)_{\cdot, u})] \\
 & - \frac{1}{6} (2m-5)(y(\cdot 2)_{\cdot 1, 1} - y(0)_{\cdot, 1}) \\
 2. \quad & \frac{1}{18} \sum_{u=2}^m [(y(0)_{\cdot, h_1(u-1)} - 2y(\cdot 11)_{\cdot 1u, h_1(u-1)} + y(\cdot 22)_{\cdot 1u, h_1(u-1)}) \\
 & + (y(0)_{\cdot, k_1(u-1)} - 2y(\cdot 12)_{\cdot 1u, k_1(u-1)} + y(\cdot 21)_{\cdot 1u, k_1(u-1)}) \\
 & + (y(\cdot 1)_{\cdot u, u} + y(\cdot 2)_{\cdot u, u} - 2y(0)_{\cdot, u})] \\
 & - \frac{1}{18} (2m-5)(y(0)_{\cdot, 1} - 2y(\cdot 1)_{\cdot 1, 1} + y(\cdot 2)_{\cdot 1, 1}) \\
 3. \quad \text{a. } & \frac{1}{6} [(y(\cdot 22)_{\cdot 1u, h_1(u-1)} - y(0)_{\cdot, h_1(u-1)}) \\
 & + (y(\cdot 12)_{\cdot 1u, k_1(u-1)} - y(0)_{\cdot, k_1(u-1)})]
 \end{aligned}$$

$$+ (y(\cdot 2)_{,u,u} - y(0)_{,u}) - (y(\cdot 1)_{,1,1} + y(\cdot 2)_{,1,1} - 2y(0)_{,1})]$$

$$\begin{aligned} \text{b. } & \frac{1}{18} [(y(0)_{,h_1(u-1)} - 2y(\cdot 11)_{,1u,h_1(u-1)} + y(\cdot 22)_{,1u,h_1(u-1)}) \\ & + (y(0)_{,k_1(u-1)} - 2y(\cdot 21)_{,1u,k_1(u-1)} + y(\cdot 12)_{,1u,k_1(u-1)}) \\ & + (y(0)_{,u} - 2y(\cdot 1)_{,u,u} + y(\cdot 2)_{,u,u}) \\ & + (y(\cdot 1)_{,1,1} + y(\cdot 2)_{,1,1} - 2y(0)_{,1})] \end{aligned}$$

4. a. $B_1^{\wedge} B_u$ is the same as that given by (2) in (6.73).

$$\begin{aligned} \text{b. } & \frac{1}{12} [(y(\cdot 2)_{,1,1} - y(0)_{,1}) - (y(0)_{,u} - 2y(\cdot 1)_{,u,u} + y(\cdot 2)_{,u,u}) \\ & + (y(\cdot 22)_{,1u,h_1(u-1)} - y(0)_{,h_1(u-1)}) \\ & - 2(y(\cdot 21)_{,1u,k_1(u-1)} - y(0)_{,k_1(u-1)})] \end{aligned}$$

$$\begin{aligned} \text{c. } & \frac{1}{12} [(y(\cdot 22)_{,1u,h_1(u-1)} - y(0)_{,h_1(u-1)}) \\ & - 2(y(\cdot 12)_{,1u,k_1(u-1)} - y(0)_{,k_1(u-1)}) + (y(\cdot 2)_{,u,u} \\ & - y(0)_{,u}) - (y(0)_{,1} - 2y(\cdot 1)_{,1,1} + y(\cdot 2)_{,1,1})] \end{aligned}$$

$$\begin{aligned} \text{d. } & \frac{1}{36} [(y(0)_{,1} - 2y(\cdot 1)_{,1,1} + y(\cdot 2)_{,1,1}) \\ & + (y(0)_{,u} - 2y(\cdot 1)_{,u,u} + y(\cdot 2)_{,u,u}) \\ & + (4y(\cdot 11)_{,1u,h_1(u-1)} + y(\cdot 22)_{,1u,h_1(u-1)} - 5y(0)_{,h_1(u-1)}) \end{aligned}$$

$$\begin{aligned}
& - 2(y(\cdot 12)_{\cdot 1u, k_1(u-1)} + y(\cdot 21)_{\cdot 1u, k_1(u-1)} \\
& \quad - 2y(0)_{\cdot k_1(u-1)})]
\end{aligned}$$

5. The estimate is as given by (7) in (6.73). (6.76)

It can be seen from (6.76) that the estimates in (1), (2), (3) and (4) are the same as those in (1), (2), (3) and (4) in (5.55). The variances of the estimates in (6.76) are, respectively

1. $\frac{1}{36} [(m-1)(2 + 2 + 6) + 2(2m-5)^2] \sigma^2$.
2. $\frac{1}{(18)^2} [(m-1)(6 + 6) + 6(2m-5)^2] \sigma^2$.
3. a. $\frac{1}{3} \sigma^2$ and b. $\frac{2}{27} \sigma^2$.
4. a. $\frac{3}{8} \sigma^2$, b. $\frac{2}{27} \sigma^2$, c. $\frac{2}{27} \sigma^2$ and d. $\frac{39}{648} \sigma^2$.
5. $\frac{1}{2} \sigma^2$. (6.77)

For the unbiasedness of the estimates in (6.76), we recall block effects cancel out. Therefore

1. The expected value of the estimate (1) in (6.76) is

$$\frac{1}{6} \sum_{u=2}^m [((\cdot 22)_{\cdot 1u} - (\cdot 2)_{\cdot u}) + ((\cdot 21)_{\cdot 1u} - (\cdot 1)_{\cdot u})] - \frac{1}{6} (2m-5)((\cdot 2)_{\cdot 1} - (0))$$

Referring to (6.33) and (6.16), unbiasedness follows.

2. The expected value of the estimate (2) in (6.76) is

$$\frac{1}{18} \sum_{u=2}^m [((\cdot 2)_{\cdot u} - 2(\cdot 12)_{\cdot 1u} + (\cdot 22)_{\cdot 1u}) + ((\cdot 1)_{\cdot u} - 2(\cdot 11)_{\cdot 1u} + (\cdot 21)_{\cdot 1u})] \\ - \frac{1}{18} (2m-5)((0) - 2(\cdot 1)_{\cdot 1} + (\cdot 2)_{\cdot 1})$$

Referring to (6.43) and (6.20), unbiasedness follows.

3. a. The expected value of the estimate (3a) in (6.76) is

$$\frac{1}{6} [((\cdot 22)_{\cdot 1u} - (\cdot 2)_{\cdot 1}) + ((\cdot 12)_{\cdot 1u} - (\cdot 1)_{\cdot 1}) + ((\cdot 2)_{\cdot u} - (0))]$$

Referring to (5.56), (5.57) and (5.58), unbiasedness follows.

b. The expected value of the estimate (3b) in (6.76) is

$$\frac{1}{18} [((\cdot 1)_{\cdot 1} - 2(\cdot 11)_{\cdot 1u} + (\cdot 12)_{\cdot 1u}) + ((\cdot 2)_{\cdot 1} - 2(\cdot 21)_{\cdot 1u} + (\cdot 22)_{\cdot 1u}) \\ + ((0) - 2(\cdot 1)_{\cdot u} + (\cdot 2)_{\cdot u})]$$

Referring to (5.59), (5.60) and (5.61), unbiasedness follows.

4. The unbiasedness of the estimates (4a), (4b), (4c) and (4d) follows immediately once their expected values are taken.

5. The unbiasedness of the estimate 5 in (6.76) was established in step 1.

For step 3, the estimable functions after the completion of the $[m + n + (m-1) + (m-1) + n]$ blocks are

1. B_1 .
2. B_1^2 .

3. The $4(m-1)$ interactions

a. $B_1 B_u$ b. $B_1 B_u^2$ c. $B_1^2 B_u$ and d. $B_1^2 B_u^2$ ($u = 2, 3, \dots, m$).

4. The $2n$ interactions

a. $A_r B_1$ and b. $A_r B_1^2$, ($r = 1, 2, \dots, n$).

5. S_{2u} and S_{2u}^2 in (3) in (6.75).

$$6. A_r - \sum_{t=2}^m A_r B_t + \sum_{t=2}^m A_r B_t^2 - \sum_{s \neq r} A_r A_s. \quad (6.78)$$

The unbiased least squares estimates of the functions in (6.78) are, respectively

$$1. \frac{1}{6} \sum_{u=2}^m [(y(\cdot 22)_{\cdot 1u, h_{1(u-1)}} - y(0)_{\cdot, h_{1(u-1)}}) + (y(\cdot 21)_{\cdot 1u, k_{1(u-1)}} - y(0)_{\cdot, k_{1(u-1)}}) - (y(\cdot 1)_{\cdot u, u} + y(\cdot 2)_{\cdot u, u} - 2y(0)_{\cdot u})] \\ + \frac{1}{4} \sum_{r=1}^n (y(\cdot 2)_{r \cdot 1, \ell_{1r}} - y(1 \cdot)_{r \cdot, \ell_{1r}}) - \frac{1}{12} (4m+3n-10) \\ \cdot (y(\cdot 2)_{\cdot 1, 1} - y(0)_{\cdot 1})$$

$$2. \frac{1}{18} \sum_{u=2}^m [(y(0)_{\cdot, h_{1(u-1)}} - 2y(\cdot 11)_{\cdot 1u, h_{1(u-1)}} + y(\cdot 22)_{\cdot 1u, h_{1(u-1)}}) \\ + (y(0)_{\cdot, k_{1(u-1)}} - 2y(\cdot 12)_{\cdot 1u, k_{1(u-1)}} + y(\cdot 21)_{\cdot 1u, k_{1(u-1)}}) \\ + (y(\cdot 1)_{\cdot u, u} + y(\cdot 2)_{\cdot u, u} - 2y(0)_{\cdot u})] \\ + \frac{1}{12} \sum_{r=1}^n (y(1 \cdot)_{r \cdot, \ell_{1r}} - 2y(1 \cdot 1)_{r \cdot 1, \ell_{1r}} + y(1 \cdot 2)_{r \cdot 1, \ell_{1r}})$$

$$- \frac{1}{36} (4m+3n-10) (y(0)_{,1} - 2y(\cdot 1)_{\cdot 1,1} + y(\cdot 2)_{\cdot 1,1})$$

3. The estimates are given by (4) in (6.76).

$$4. \quad a. \quad \hat{A}_r B_1 = \frac{1}{4} [(y(1 \cdot 2)_{r \cdot 1, l_{1r}} + y(1 \cdot)_{r \cdot, l_{1r}}) - (y(\cdot 2)_{\cdot 1,1} - y(0)_{,1})]$$

$$b. \quad \hat{A}_r B_1^2 = \frac{1}{12} [(y(1 \cdot)_{r \cdot, l_{1r}} - 2y(1 \cdot 1)_{r \cdot 1, l_{1r}} + y(1 \cdot 2)_{r \cdot 1, l_{1r}}) - (y(0)_{,1} - 2y(\cdot 1)_{\cdot 1,1} + y(\cdot 2)_{\cdot 1,1})]$$

5. The estimates are given by (3a) and (3b) in (6.76).

$$6. \quad \frac{1}{6} [(y(1 \cdot 1)_{r \cdot 1, l_{1r}} + y(1 \cdot 2)_{r \cdot 1, l_{1r}} - 2y(1 \cdot)_{r \cdot, l_{1r}}) - (y(\cdot 1)_{\cdot 1,1} + y(\cdot 2)_{\cdot 1,1} - 2y(0)_{,1})] + \frac{1}{2} (y(1 \cdot)_{r \cdot, m+r} - y(0)_{, m+r}) \quad (6.79)$$

The variances of the estimates in (6.79) are, respectively

$$1. \quad [\frac{1}{36} (m-1)(2 + 2 + 6) + \frac{1}{16} (2n) + \frac{2}{144} (4m+3n-10)^2] \sigma^2$$

$$2. \quad [\frac{1}{(18)^2} (m-1)(6 + 6 + 6) + \frac{1}{144} (6n) + \frac{6}{(36)^2} (4m+3n-10)] \sigma^2$$

3. Given by (4) in (6.77)

$$4. \quad a. \quad \frac{1}{4} \sigma^2 \quad b. \quad \frac{1}{12} \sigma^2$$

5. given by (3) in (6.77)

$$6. \quad \frac{7}{12} \sigma^2 \quad (6.80)$$

It can be seen from (1), (2) and (3) in (6.80) and from the corresponding quantities in (6.60) that the parameters B_1 , B_1^2 , $B_1 B_u$, $B_1 B_u^2$, $B_1^2 B_u$, $B_1^2 B_u^2$, $A_r B_1$ and $A_r B_1^2$ are estimated with more precision from the blocked $2^n 3^m$ fraction $F_{n,m}$ than from the fraction $F_{n,m}$ itself.

Recalling that block effects cancel out in the expected values of the estimates in (6.79), the unbiasedness of (1), (2), (3) and (4) in (6.79) follows immediately. The unbiasedness of the estimates in (5) was shown in step 2. To prove the unbiasedness of the estimate (6) we note that its expected value is

$$\frac{1}{6} [((1.2)_{r.1} - (.2)_{.1}) + ((1.1)_{r.1} - (.1)_{.1}) + ((1.)_r - (0))]$$

Setting $t = 1$ in (6.25) and referring to (6.12), unbiasedness follows.

From the results in steps 1, 2, and 3 of stage one, we arrive at the following general result.

Stage t: The estimation of B_t , B_t^2 , $B_t B_u$, $B_t B_u^2$, $B_t^2 B_u$, $B_t^2 B_u^2$, $A_r B_t$ and $A_r B_t^2$ ($t < u$, $t = 1, 2, \dots, m$ and $r = 1, 2, \dots, n$).

This stage will be carried out in 3 steps such that

a) Step 1 leads to the estimation of the $(m-t)$ interactions $B_t B_u$ ($u = t+1, t+2, \dots, m$) with the addition of the $(m-t)$ blocks

$$\text{block } h_{t(u-t)}: (0)_{,h_{t(u-t)}}, (.11)_{.tu,h_{t(u-t)}}, (.22)_{.tu,h_{t(u-t)}} \\ (h_{t(u-t)} = m + n + 2 \sum_{i=1}^{t-1} (m-i) + n(t-1) + (u-t) \text{ and } t < u).$$

b) Step 2 leads to the estimation of the $3(m-t)$ interactions $B_t B_u^2$, $B_t^2 B_u$ and $B_t^2 B_u^2$ with the addition of the $(m-t)$ blocks

block $k_{t(u-t)}$: $(0)_{,k_{t(u-t)}}$, $(\cdot 12)_{\cdot tu, k_{t(u-t)}}$, $(\cdot 21)_{\cdot tu, k_{t(u-t)}}$

$$(k_{t(u-t)} = m + n + 2 \sum_{i=1}^{t-1} (m-i) + n(t-1) + (m-t) + (u-t)) .$$

c) Step 3 leads to the estimation of the $2n$ interactions $A_r B_t$ and $A_r B_t^2$ ($r = 1, 2, \dots, n$) with the addition of the n blocks

block l_{tr} : $(1\cdot)_{r\cdot, l_{tr}}$, $(1\cdot 1)_{r\cdot t, l_{tr}}$, $(1\cdot 2)_{r\cdot t, l_{tr}}$

$$(l_{tr} = m + n + 2 \cdot \sum_{i=1}^t (m-i) + n(t-1) + r) .$$

For stage t the block structure is

block t :	$(0)_{,t}$	$(\cdot 1)_{\cdot t, t}$	$(\cdot 2)_{\cdot t, t}$, $t=1, 2, \dots, m$.
block $(m+r)$:	$(0)_{,m+r}$	$(1\cdot)_{r\cdot, m+r}$, $r=1, 2, \dots, n$.
{	block $h_{1(u-1)}$:	$(0)_{,h_{1(u-1)}}$	$(\cdot 11)_{\cdot 1u, h_{1(u-1)}}$ $(\cdot 22)_{\cdot 1u, h_{1(u-1)}}$,
			$h_{1(u-1)} = m + n + u - 1$.
	block $k_{1(u-1)}$:	$(0)_{,k_{1(u-1)}}$	$(\cdot 12)_{\cdot 1u, k_{1(u-1)}}$ $(\cdot 21)_{\cdot 1u, k_{1(u-1)}}$,
			$k_{1(u-1)} = m + n + (m-1) + (u-1)$
block l_{1r} :	$(1\cdot)_{r\cdot, l_{1r}}$	$(1\cdot 1)_{r\cdot 1, l_{1r}}$	$(1\cdot 2)_{r\cdot 1, l_{1r}}$,
			$l_{1r} = m + n + 2(m-1) + r$.

⋮

$$\begin{array}{l}
 \text{Stage } t. \left\{ \begin{array}{l}
 \text{block } h_{t(u-t)}: (0)_{,h_{t(u-t)}} \quad (\cdot 11)_{\cdot tu, h_{t(u-t)}} \quad (\cdot 22)_{\cdot tu, h_{t(u-t)}} , \\
 \\
 h_{t(u-t)} = m + n + 2 \sum_{i=1}^{t-1} (m-i) + n(t-1) + (u-t) . \\
 \\
 \text{block } k_{t(u-t)}: (0)_{,k_{t(u-t)}} \quad (\cdot 12)_{\cdot tu, k_{t(u-t)}} \quad (\cdot 21)_{\cdot tu, k_{t(u-t)}} , \\
 \\
 k_{t(u-t)} = m + n + 2 \sum_{i=1}^{t-1} (m-i) + n(t-1) + (m-t) \\
 \\
 \quad \quad \quad + (u-t) . \\
 \\
 \text{block } \ell_{tr}: (1\cdot)_{r\cdot, \ell_{tr}} \quad (1\cdot 1)_{r\cdot t, \ell_{tr}} \quad (1\cdot 2)_{r\cdot t, \ell_{tr}} , \\
 \\
 \ell_{tr} = m + n + 2 \sum_{i=1}^t (m-i) + n(t-1) + r . \quad (6.81)
 \end{array} \right.
 \end{array}$$

After the completion of the $[m + n + 2 \cdot \sum_{i=1}^t (m-i) + nt]$ blocks in (6.81), the estimable functions are

$$1. \quad B_1 ; B_1^2 ; B_1 B_u ; B_1 B_u^2 ; B_1^2 B_u ; B_1^2 B_u^2 ; A_r B_1 ; A_r B_1^2 , \quad (u = 2, 3, \dots, m \text{ and } r = 1, 2, \dots, n) .$$

$$2. \quad B_2 ; B_2^2 ; B_2 B_u ; B_2 B_u^2 ; B_2^2 B_u ; B_2^2 B_u^2 ; A_r B_2 ; A_r B_2^2 , \quad (u = 3, 4, \dots, m \text{ and } r = 1, 2, \dots, n) .$$

⋮

$$t. \quad a. \quad B_t ; B_t^2 ; B_t B_u ; B_t B_u^2 ; B_t^2 B_u ; B_t^2 B_u^2 ; A_r B_t ; A_r B_t^2 , \quad (u = t+1, t+2, \dots, m \text{ and } r = 1, 2, \dots, n) .$$

$$\begin{aligned}
 b. \quad S_{(t+1)u} = & B_u - \left(\sum_{v=t+1}^{u-1} B_u B_v + \sum_{v=u+1}^m B_u B_v \right) + \left(\sum_{v=t+1}^{u-1} B_u B_v^2 \right. \\
 & \left. + \sum_{v=u+1}^m B_u B_v^2 \right) - \sum_{r=1}^n A_r B_u .
 \end{aligned}$$

$$c. S_{(t+1)u}^2 = B_u^2 - \left(\sum_{v=t+1}^{u-1} B_u^2 B_v^2 + \sum_{v=u+1}^m B_u^2 B_v^2 \right) + \left(\sum_{v=t+1}^{u-1} B_u^2 B_v^2 \right. \\ \left. + \sum_{v=u+1}^m B_u^2 B_v^2 \right) - \sum_{r=1}^n A_r B_u^2 .$$

$$d. Q_{rs} = A_r - \sum_{u=t+1}^m A_r B_u + \sum_{u=t+1}^m A_r B_u^2 - \left(\sum_{i=1}^{r-1} A_i A_r + \sum_{i=r+1}^n A_r A_i \right) \quad (6.82)$$

The unbiased estimates of the functions in (6.82) are

$$1. \hat{B}_j = \frac{1}{6} \sum_{i=1}^{j-1} \left[(y(\cdot 22)_{\cdot ij, h_{i(j-i)}} - y(0)_{, h_{i(j-i)}}) \right. \\ \left. + (y(\cdot 12)_{\cdot ij, k_{i(j-i)}} - y(0)_{, k_{i(j-i)}}) \right. \\ \left. - (y(\cdot 1)_{\cdot i, i} + y(\cdot 2)_{\cdot i, i} - 2y(0)_{, i}) \right] \\ + \frac{1}{6} \sum_{u=j+1}^m \left[(y(\cdot 22)_{\cdot ju, h_{j(u-j)}} - y(0)_{, h_{j(u-j)}}) \right. \\ \left. + (y(\cdot 12)_{\cdot ju, k_{j(u-j)}} - y(0)_{, k_{j(u-j)}}) \right. \\ \left. - (y(\cdot 1)_{\cdot u, u} + y(\cdot 2)_{\cdot u, u} - 2y(0)_{, u}) \right] \\ + \frac{1}{4} \sum_{r=1}^n (y(1 \cdot 2)_{r \cdot j, \ell_{jr}} - y(1 \cdot)_{r \cdot, \ell_{jr}}) \\ - \frac{1}{12} (4m+3n-10)(y(\cdot 2)_{\cdot j, j} - y(0)_{, j}), \quad j = 1, 2, \dots, t.$$

$$2. \hat{B}_j^2 = \frac{1}{18} \sum_{i=1}^{j-1} \left[(y(0)_{, h_{i(j-i)}} - 2y(\cdot 11)_{\cdot ij, h_{i(j-i)}} + y(\cdot 22)_{\cdot ij, h_{i(j-i)}}) \right. \\ \left. + (y(0)_{, k_{i(j-i)}} - 2y(\cdot 21)_{\cdot ij, k_{i(j-i)}} + y(\cdot 12)_{\cdot ij, k_{i(j-i)}}) \right. \\ \left. - (2y(0)_{, i} + y(\cdot 1)_{\cdot i, i} + y(\cdot 2)_{\cdot i, i}) \right]$$

$$\begin{aligned}
& + \frac{1}{18} \sum_{u=j+1}^m [(y(0)_{,h_{j(u-j)}} - 2y(\cdot 11)_{\cdot ju, h_{j(u-j)}} \\
& \quad + y(\cdot 22)_{\cdot ju, h_{j(u-j)}}) \\
& + (y(0)_{,k_{j(u-j)}} - 2y(\cdot 12)_{\cdot ju, k_{j(u-j)}} + y(\cdot 21)_{\cdot ju, k_{j(u-j)}}) \\
& - (2y(0)_{,u} + y(\cdot 1)_{\cdot u, u} + y(\cdot 2)_{\cdot u, u})] \\
& + \frac{1}{12} \sum_{r=1}^n (y(1\cdot)_{r, \ell_{jr}} - 2y(1\cdot 1)_{r, j, \ell_{jr}} + y(1\cdot 2)_{r, j, \ell_{jr}}) \\
& - \frac{1}{36} (4m+3n-10) (y(0)_{,j} - 2y(\cdot 1)_{\cdot j, j} + y(\cdot 2)_{\cdot j, j}) , \\
& \qquad \qquad \qquad j = 1, 2, \dots, t .
\end{aligned}$$

3. The unbiased estimate of $B_j B_u$ is given by (2) in (6.73) replacing $t = 1$ by j . The unbiased estimates of $B_j B_u^2$, $B_j^2 B_u$, $B_j^2 B_u^2$ are given by 4(a), 4(b), 4(c) and 4(d) in (6.76), respectively replacing $t = 1$ by j . The unbiased estimates $A_r B_j$ and $A_r B_j^2$ are given by 4(a) and 4(b) in (6.79), respectively replacing $t = 1$ by j .

$$\begin{aligned}
4. \quad a. \quad \hat{S}_{(t+1)u} &= \frac{1}{6} \sum_{j=1}^t [(y(\cdot 22)_{\cdot ju, h_{j(u-j)}} - y(0)_{,h_{j(u-j)}}) \\
& \quad + (y(\cdot 12)_{\cdot ju, k_{j(u-j)}} - y(0)_{,k_{j(u-j)}}) \\
& \quad - (y(\cdot 1)_{\cdot j, j} + y(\cdot 2)_{\cdot j, j} - 2y(0)_{,j})] \\
& \quad - \frac{1}{6} (2t-3) (y(\cdot 2)_{\cdot u, u} - y(0)_{,u}) .
\end{aligned}$$

$$\begin{aligned}
b. \quad \hat{S}_{(t+1)u}^2 &= \frac{1}{18} \sum_{j=1}^t [(y(0)_{,h_{j(u-j)}} - 2y(\cdot 11)_{\cdot ju, h_{j(u-j)}} \\
& \quad + y(\cdot 22)_{\cdot ju, h_{j(u-j)}})
\end{aligned}$$

$$\begin{aligned}
& + (y(0)_{,k_j(u-j)} - 2y(\cdot 21)_{\cdot j u, k_j(u-j)} \\
& \quad + y(\cdot 12)_{\cdot j u, k_j(u-j)}) \\
& + (y(\cdot 1)_{\cdot j, j} + y(\cdot 2)_{\cdot j, j} - 2y(0)_{\cdot j, j})] \\
& - \frac{1}{18} (2t-3)(y(0)_{\cdot u} - 2y(\cdot 1)_{\cdot u, u} + y(\cdot 2)_{\cdot u, u}) \\
5. \quad \hat{Q}_{rs} &= \frac{1}{6} \sum_{j=1}^t [(y(\cdot 1)_{r \cdot j, \ell_{jr}} + y(1 \cdot 2)_{r \cdot j, \ell_{jr}} - 2y(1 \cdot)_{r \cdot, \ell_{jr}}) \\
& \quad - (y(\cdot 1)_{\cdot j, j} + y(\cdot 2)_{\cdot j, j} - 2y(0)_{\cdot j, j})] \\
& \quad - \frac{1}{6} (2t-5)(y(1 \cdot)_{r \cdot, m+r} - y(0)_{\cdot, m+r}) \tag{6.83}
\end{aligned}$$

To establish the unbiasedness of the estimates in (6.83), we recall that block effects cancel out from the estimates. The unbiasedness of B_j , B_j^2 , $B_j B_u$, $B_j B_u^2$, $B_j^2 B_u$, $B_j^2 B_u^2$, $A_r B_j$ and $A_r B_j^2$ ($j = 1, 2, \dots, t$) follows immediately once expected values are taken and reference to (6.39), (6.49), (6.53), (6.57), (6.54), (6.58), (6.51) and (6.52) is made, respectively. Referring to (5.74) and (5.75) it follows that $ES_{(t+1)u}^{\hat{S}} = S_{(t+1)u}$ and referring to (5.76) and (5.77) it also follows that $ES_{(t+1)u}^{\hat{S}^2} = S_{(t+1)u}^2$. To prove the unbiasedness of \hat{Q}_{rs} , we note that

$$\begin{aligned}
E\hat{Q}_{rs} &= \frac{1}{6} \sum_{j=1}^t [((1 \cdot 2)_{r \cdot j} - (\cdot 2)_{\cdot j}) + ((1 \cdot 1)_{r \cdot j} - (\cdot 1)_{\cdot j}) \\
& \quad - 2((1 \cdot)_{r \cdot} - (0))] - \frac{1}{6} (2t-5)((1 \cdot)_{r \cdot} - (0)) \\
&= \frac{1}{6} \sum_{j=1}^t [((1 \cdot 2)_{r \cdot j} - (\cdot 2)_{\cdot j}) + ((1 \cdot 1)_{r \cdot j} - (\cdot 1)_{\cdot j})] \\
& \quad - \frac{1}{6} (2t-3)((1 \cdot)_{r \cdot} - (0)) \tag{6.84}
\end{aligned}$$

But from (6.21)

$$\begin{aligned}
 & \{((1 \cdot 2)_{r \cdot j} - (\cdot 2)_{\cdot j}) + ((1 \cdot 1)_{r \cdot j} - (\cdot 1)_{\cdot j})\} = 4A_r - 4 \cdot \sum_{s \neq r} A_r A_s + 2A_r B_j \\
 & \quad - 4 \sum_{u \neq j} A_r B_u - 2A_r B_j^2 + 4 \cdot \sum_{u \neq j} A_r B_u^2 \\
 & = 4A_r - 4 \cdot \sum_{s \neq r} A_r A_s + (-4A_r B_1 - 4A_r B_2 - \dots - 4A_r B_{j-1} + 2A_r B_j \\
 & \quad - 4A_r B_{j+1} - \dots - 4A_r B_m) + (4A_r B_1^2 + 4A_r B_2^2 + \dots + 4A_r B_{j-1}^2 \\
 & \quad - 2A_r B_j^2 + 4A_r B_{j+1}^2 + \dots + 4A_r B_m^2)
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{j=1}^t [((1 \cdot 2)_{r \cdot j} - (\cdot 2)_{\cdot j}) + ((1 \cdot 1)_{r \cdot j} - (\cdot 1)_{\cdot j})] & = 4tA_r - 4t \cdot \sum_{s \neq r} A_r A_s \\
 & \quad - 2(2t-3) \sum_{j=1}^t A_r B_j - 4t \sum_{j=t+1}^m A_r B_j + 2(2t-3) \sum_{j=1}^t A_r B_j^2 + 4t \sum_{j=t+1}^m A_r B_j^2
 \end{aligned} \tag{6.85}$$

From (6.12),

$$\begin{aligned}
 (2t-3)((1 \cdot)_{r \cdot} - (0)) & = (2t-3)[2A_r - 2 \sum_{s \neq r} A_r A_s - 2 \sum_{j=1}^t A_r B_j - 2 \sum_{j=t+1}^m A_r B_j \\
 & \quad + 2 \sum_{j=1}^t A_r B_j^2 + 2 \sum_{j=t+1}^m A_r B_j^2] .
 \end{aligned} \tag{6.86}$$

From the difference between (6.85) and (6.86) it follows that $\hat{E}Q_{rs} = Q_{rs}$.

Stage m: The estimation of B_m , B_m^2 , $A_r B_m$ and $A_r B_m^2$, $r = 1, 2, \dots, n$.

After stage (m-1), all interactions between two 3-level factors

become estimable but the linear and quadratic effects B_m and B_m^2 are still aliased with the interactions $A_r B_m$ and $A_r B_m^2$. Therefore, for stage m , the n blocks

$$\text{block } l_{mr}: (1\cdot)_{r\cdot, l_{mr}}, (1\cdot 1)_{r\cdot m, l_{mr}}, (1\cdot 2)_{r\cdot m, l_{mr}},$$

$$r = 1, 2, \dots, n. \quad (6.87)$$

must be added to the blocks in (6.81) in order that the effects B_m , B_m^2 , $A_r B_m$ and $A_r B_m^2$ be estimable.

After the completion of the $[m + n + 2 \sum_{i=1}^{m-1} (m-i) + nm]$ blocks in (6.81) and (6.87), the estimable functions are

1. a. B_t b. B_t^2 , $t = 1, 2, \dots, m$.
2. a. $B_t B_u$ b. $B_t B_u^2$ c. $B_t^2 B_u$ d. $B_t^2 B_u^2$, $u = t+1, t+2, \dots,$
 m and $t = 1, 2, \dots, m-1$.
3. a. $A_r B_t$ b. $A_r B_t^2$, $t = 1, 2, \dots, m$.
4. $A_r - \sum_{s \neq r} A_r A_s = A_r - \left(\sum_{s=1}^{r-1} A_r A_s + \sum_{s=r+1}^n A_r A_s \right)$. (6.88)

The unbiased estimates of (1), (2) and (3) are given by (1), (2) and (3) in (6.83). The unbiased estimate of (4) in (6.88) is given by (5) in (6.83) setting $t = m$.

Stage (m+1): The estimation of A_1 and $A_1 A_s$ ($s = 1, 2, \dots, n$).

Referring to (4) in (6.88), each main effect A_r ($r = 1, 2, \dots, n$) is aliased with the $(n-1)$ interactions $A_r A_s$ ($s \neq r$). Therefore, to free

A_1 from aliasing, the $(n-1)$ blocks

$$\text{block } q_{1(s-1)}: \quad (0)_{,q_{1(s-1)}} \quad (11 \cdot)_{1s \cdot, q_{1(s-1)}} \quad (6.89)$$

$(q_{1(s-1)})_{m-1} = m + n + 2 \sum_{i=1}^{m-1} (m-i) + nm + (s-1)$ must be added to the $(m + n + 2 \sum_{i=1}^{m-1} (m-i) + nm)$ blocks in stage m .

The estimable functions after the completion of the $[m + n + 2 \sum_{i=1}^{m-1} (m-i) + mn + (n-1)]$ blocks in stage $(m+1)$ are

1. (1), (2) and (3) in (6.88).

2. A_1 .

3. $A_1 A_s$, $(s = 2, 3, \dots, n)$.

$$4. \quad A_s - \left(\sum_{x=2}^{s-1} A_s A_x + \sum_{x=s+1}^n A_s A_x \right), \quad (s = 2, 3, \dots, n) \quad (6.90)$$

The unbiased estimates of (1) in (6.90) are given by (1), (2) and (3) in (6.83). The unbiased estimates of the remaining functions in (6.90) are, respectively

$$\begin{aligned} 1. \quad & \frac{1}{6} \sum_{t=1}^m [(y(1 \cdot 1)_{1 \cdot t, \ell_{t1}} + y(1 \cdot 2)_{1 \cdot t, \ell_{t1}} - 2y(1 \cdot)_{1 \cdot, \ell_{t1}}) \\ & - (y(\cdot 1)_{\cdot t, t} + y(\cdot 2)_{\cdot t, t} - 2y(0)_{, t})] \\ & + \frac{1}{4} \sum_{s=2}^n [(y(11 \cdot)_{1s \cdot, q_{1(s-1)}} - y(0)_{, q_{1(s-1)}}) \\ & - (y(1 \cdot)_{s \cdot, m+s} - y(0)_{, m+s})] \\ & - \frac{1}{12} (4m+3n-9) (y(1 \cdot)_{1 \cdot, m+1} - y(0)_{, m+1}) . \end{aligned}$$

$$\begin{aligned}
2. & \frac{1}{4} [(y(11\cdot))_{1s\cdot, q_{1(s-1)}} - y(0)_{, q_{1(s-1)}}] \\
& - (y(1\cdot))_{1\cdot, m+1} - y(0)_{, m+1} \\
& - (y(1\cdot))_{s\cdot, m+s} - y(0)_{, m+s}] \\
3. & \frac{1}{6} \sum_{t=1}^m [(y(1\cdot 1))_{s\cdot t, \ell_{ts}} + y(1\cdot 2)_{s\cdot t, \ell_{ts}} - 2y(1\cdot)_{s\cdot, \ell_{ts}}] \\
& - (y(\cdot 1))_{\cdot t, t} + y(\cdot 2)_{\cdot t, t} - 2y(0)_{, t}] \\
& + \frac{1}{4} [(y(11\cdot))_{1s\cdot, q_{1(s-1)}} - y(0)_{, q_{1(s-1)}}] \\
& - (y(1\cdot))_{1\cdot, m+1} - y(0)_{, m+1} \\
& + (y(1\cdot))_{s\cdot, m+s} - y(0)_{, m+s}] . \tag{6.91}
\end{aligned}$$

The unbiasedness of A_1 and $A_1 A_s$ follows immediately by taking their expected values and comparing them with (6.30) and (6.50). The remaining estimate in (6.91) can be shown to be unbiased as follows:

The expected value of the estimate (3) in (6.91) is

$$\begin{aligned}
& \frac{1}{6} \sum_{t=1}^m [((1\cdot 1))_{s\cdot t} - (\cdot 1)_{\cdot t} + ((1\cdot 2))_{s\cdot t} - (\cdot 2)_{\cdot t} - 2((1\cdot)_{s\cdot} - (0))] \\
& + \frac{1}{4} ((11\cdot))_{1s\cdot} - (1\cdot)_{1\cdot} + \frac{1}{4} ((1\cdot)_{s\cdot} - (0)) \\
& = \frac{1}{6} \sum_{t=1}^m [((1\cdot 1))_{s\cdot t} - (\cdot 1)_{\cdot t} + ((1\cdot 2))_{s\cdot t} - (\cdot 2)_{\cdot t}] \\
& - \frac{1}{12} (4m-3)((1\cdot)_{s\cdot} - (0)) + \frac{1}{4} ((11\cdot))_{1s\cdot} - (1\cdot)_{1\cdot} . \tag{6.92}
\end{aligned}$$

Referring to (6.25),

$$\begin{aligned} \sum_{t=1}^m [((1 \cdot 1)_{s \cdot t} - (1 \cdot 1)_{\cdot t}) + ((1 \cdot 2)_{s \cdot t} - (1 \cdot 2)_{\cdot t})] &= 4mA_s - 4mA_1A_s - 4mA_sA(1, s) \\ &- 2(2m-3) \sum_{t=1}^m A_s B_t + 2(2m-3) \sum_{t=1}^m A_s B_t^2. \end{aligned} \quad (6.93)$$

Referring to (6.11),

$$((11 \cdot 1)_{1s \cdot} - (1 \cdot 1)_{1 \cdot}) = 2[A_s + A_1A_s - A_sA(1, s) - \sum_{t=1}^m A_s B_t + \sum_{t=1}^m A_s B_t^2] \quad (6.94)$$

and referring to (6.12),

$$((1 \cdot 1)_{s \cdot} - (0)) = 2[A_s - A_1A_s - A_sA(1, s) - \sum_{t=1}^m A_s B_t + \sum_{t=1}^m A_s B_t^2] \quad (6.95)$$

From (6.93), (6.94) and (6.95), unbiasedness of the estimate (3) in (6.91) follows.

From the results in stage $(m+1)$, we can generalize to

Stage $(m+r)$: The estimation A_r and A_rA_s ($r < s$, $r = 1, 2, \dots, m$).

For this stage, the $(n-r)$ blocks

$$\text{block } q_{r(s-r)}: \quad (0)_{\cdot, q_{r(s-r)}} \quad (11 \cdot 1)_{rs \cdot, q_{r(s-r)}} \quad (6.96)$$

$[q_{r(s-r)} = m + n + 2 \sum_{i=1}^{m-1} (m-i) + nm + \sum_{j=1}^{r-1} (n-j) + (s-r)]$ are needed to free A_r from being aliased with A_rA_s ($s = r+1, r+2, \dots, n$).

The estimable functions after the completion of the blocks $[m + n + 2 \sum_{i=1}^{m-1} (m-i) + nm + \sum_{j=1}^r (n-j)]$ in this stage are

1. (1), (2) and (3) in (6.88).

$$2. A_1 ; A_1 A_s, \quad s = 2, 3, \dots, n.$$

$$3. A_2 ; A_2 A_s, \quad s = 3, 4, \dots, n.$$

⋮

$$(r+1). A_r ; A_r A_s, \quad s = r+1, r+2, \dots, n.$$

$$(r+2). A_s - \left(\sum_{x=r+1}^{s-1} A_s A_x + \sum_{x=s+1}^n A_s A_x \right), \quad s = r+1, r+2, \dots, n. \quad (6.97)$$

The unbiased estimates of (1) in (6.97) are given by (1), (2) and (3) in (6.83). The unbiased estimates of the remaining functions in (6.97) are

$$\begin{aligned} \hat{A}_k &= \frac{1}{6} \sum_{t=1}^m [(y(1 \cdot 1)_{k \cdot t, l_{tk}} + y(1 \cdot 2)_{k \cdot t, l_{tk}} - 2y(1 \cdot)_{k \cdot, l_{tk}}) \\ &= (y(\cdot 1)_{\cdot t, t} + y(\cdot 2)_{\cdot t, t} - 2y(0)_{\cdot t})] \\ &+ \frac{1}{4} \sum_{i=1}^{k-1} [(y(11 \cdot)_{ik \cdot, q_{i(k-i)}} - y(0)_{\cdot, q_{i(k-i)}}) \\ &- (y(1 \cdot)_{i \cdot, m+i} - y(0)_{\cdot, m+i})] \\ &+ \frac{1}{4} \sum_{s=k+1}^n [(y(11 \cdot)_{ks \cdot, q_{k(s-k)}} - y(0)_{\cdot, q_{k(s-k)}}) \\ &- (y(1 \cdot)_{s \cdot, m+s} - y(0)_{\cdot, m+s})] - \frac{1}{4} (n-3)(y(1 \cdot)_{k \cdot, m+k} - y(0)_{\cdot, m+k}) \\ &(k = 1, 2, \dots, r). \end{aligned}$$

$$\begin{aligned} \hat{A}_{k s} &= \frac{1}{4} [(y(11 \cdot)_{ks \cdot, q_{k(s-k)}} - y(0)_{\cdot, q_{k(s-k)}}) \\ &- (y(1 \cdot)_{k \cdot, m+k} - y(0)_{\cdot, m+k}) - (y(1 \cdot)_{s \cdot, m+s} - y(0)_{\cdot, m+s})] \end{aligned}$$

where $k = 1, 2, \dots, r$ and $s = k+1, k+2, \dots, n$.

$$\begin{aligned}
(r+2) \cdot \frac{1}{6} \sum_{t=1}^m & [(y(1 \cdot 1)_{s \cdot t, l_{ts}} + y(1 \cdot 2)_{s \cdot t, l_{ts}} - 2y(1 \cdot)_{s \cdot, l_{ts}}) \\
& - (y(\cdot 1)_{\cdot t, t} + y(\cdot 2)_{\cdot t, t} - 2y(0)_{\cdot t})] \\
& + \frac{1}{4} \sum_{k=1}^r [(y(11 \cdot)_{ks \cdot, q_{k(s-k)}} - y(0)_{\cdot, q_{k(s-k)}}) \\
& - (y(1 \cdot)_{k \cdot, m+k} - y(0)_{\cdot, m+k})] \\
& - c(y(1 \cdot)_{s \cdot, m+s} - y(0)_{\cdot, m+s}) \tag{6.98}
\end{aligned}$$

where c is a function of n . With $\text{var } \hat{A}_k = [\frac{m}{36} (6 + 6) + \frac{1}{16} (n-1)(2 + 2) + \frac{2}{16} (n-3)^2] \sigma^2$ and $\text{var } A_k \hat{A}_s = \frac{3}{8} \sigma^2$.

The constant c in (r+2) in (6.98) is obtained by equating the expected value of this estimate with the function (r + 2) in (6.97). Recalling that block effects cancel out, the unbiasedness of the estimates \hat{A}_k and $A_k \hat{A}_s$ can be shown as follows:

$$\begin{aligned}
E\hat{A}_k &= \frac{1}{6} \sum_{t=1}^m [((1 \cdot 1)_{k \cdot t} - (\cdot 1)_{\cdot t}) + ((1 \cdot 2)_{k \cdot t} - (\cdot 2)_{\cdot t}) \\
&\quad - 2((1 \cdot)_{k \cdot} - (0))] \\
&\quad + \frac{1}{4} \sum_{i=1}^k ((11 \cdot)_{ik \cdot} - (1 \cdot)_{i \cdot}) + \frac{1}{4} \sum_{s=k+1}^n ((11 \cdot)_{ks \cdot} - (1 \cdot)_{s \cdot}) \\
&\quad - \frac{3}{12} (n-3)((1 \cdot)_{k \cdot} - (0)) \\
&= \frac{1}{6} \sum_{t=1}^m [((1 \cdot 1)_{k \cdot t} - (\cdot 1)_{\cdot t}) + ((1 \cdot 2)_{k \cdot t} - (\cdot 2)_{\cdot t})] \\
&\quad + \frac{1}{4} \sum_{s \neq k} ((11 \cdot)_{ks \cdot} - (1 \cdot)_{s \cdot}) - \frac{1}{12} (4m+3n-9)((1 \cdot)_{k \cdot} - (0)) \\
&= A_k
\end{aligned}$$

and

$$EA_{k's}^{\wedge A} = \frac{1}{4} [((11\cdot)_{ks} - (1\cdot1)_{s\cdot}) - ((1\cdot)_{k\cdot} - (0))] = A_k A_s .$$

6.5 OBAT $2^n 3^m$ Plans for Strategy I (Order Two)

Assuming factors are ordered as $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_m$, factor effects are investigated one at a time, according to strategy I, in blocks of sizes 2 and 3 such that from a set of blocks the effects $A_r, A_r A_s, A_r B_t$ and $A_r B_t^2$ ($r < s, r = 1, 2, \dots, n$ and $t = 1, 2, \dots, m$) become estimable before the effects $A_{r'}, A_{r'} A_{s'}, A_{r'} B_t$ and $A_{r'} B_t^2$ ($r' > r, r' < s',$ and $t = 1, 2, \dots, m$).

Referring to (6.68), to free A_r, B_t and B_t^2 from aliasing more blocks of sizes 2 and 3 must be added to the OBAT plan

$$\begin{aligned} \text{block } r: & \quad (0)_{,r} \quad (1\cdot)_{,r} \quad , \quad r = 1, 2, \dots, n . \\ \text{block } (n+t): & \quad (0)_{,n+t} \quad (\cdot 1)_{\cdot t} \quad (\cdot 2)_{\cdot t} \quad , \quad t = 1, 2, \dots, m . \end{aligned} \tag{6.99}$$

with factors ordered such that A_1 is freed from aliasing first, A_2 second, \dots, A_n n^{th} , B_1 $(n+1)^{\text{st}}$, B_2 $(n+2)^{\text{nd}}$, \dots, B_m last, the experiment is then carried out in $(n + m - 1)$ stages where

- a. for stage r ($r \leq n - 1$), the parameters $A_r, A_r A_s, A_r B_t$ and $A_r B_t^2$ become estimable ($s = r+1, r+2, \dots, n$ and $t = 1, 2, \dots, m$).
- b. for stage n , the effects $A_n, A_n B_t$ and $A_n B_t^2$ ($t = 1, 2, \dots, m$) become estimable.
- c. for stage $n + t$ ($t \leq m - 1$), the parameters $B_t, B_t B_u, B_t B_u^2, B_t^2 B_u$ and $B_t^2 B_u^2$ ($u = t+1, t+2, \dots, m$) become estimable.

For an illustration, we describe stage 1 first and then discuss the general stages r, n and $n + t$.

Stage One: The estimation of A_1 , $A_1 A_s$, $A_1 B_t$ and $A_1 B_t^2$ ($s = 2, 3, \dots, n$ and $t = 1, 2, \dots, m$).

Stage one will be carried out in two steps such that

a) step 1 involves the estimation of the $(n-1)$ interactions $A_1 A_s$ ($s = 2, 3, \dots, n$) by adding the $(n-1)$ blocks

$$\text{block } q_{1(s-1)}: \quad (0)_{,q_{1(s-1)}} \quad (11 \cdot)_{1s, q_{1(s-1)}} \quad (6.100)$$

($q_{1(s-1)} = n + m + s - 1$ and $s = 2, 3, \dots, n$) to the $(n + m)$ blocks in (6.99).

b) Step 2 involves the estimation of A_1 , $A_1 B_t$ and $A_1 B_t^2$ ($t = 1, 2, \dots, m$) by adding the m blocks

$$\text{block } (\ell_{t1}): \quad (1 \cdot)_{1 \cdot, 1, \ell_{t1}} \quad (1 \cdot 1)_{1 \cdot t, \ell_{t1}} \quad (1 \cdot 2)_{1 \cdot t, \ell_{t1}} \quad (6.101)$$

($\ell_{t1} = n + m + (n-1) + t$ and $t = 1, 2, \dots, m$) to the $n + m + (n-1)$ blocks in step 1.

The estimable functions after the completion of the $[n + m + (n-1)]$ blocks in step 1 are

1. $A_1 - \sum_{t=1}^m A_1 B_t + \sum_{t=1}^m A_1 B_t^2$
2. $A_1 A_s$
3. $A_s - \sum_{\omega \neq 1, s} A_s A_\omega - \sum_{t=1}^m A_s B_t + \sum_{t=1}^m A_s B_t^2$, $s = 2, 3, \dots, n$.
4. (1) and (2) in (6.68) (6.102)

The unbiased estimates of the functions in (6.102) are, respectively

1. $\frac{1}{4} \sum_{s=2}^n [(y(11\cdot))_{1s\cdot, q_{1(s-1)}} - y(0)_{\cdot, q_{1(s-1)}}] - (y(1\cdot)_{s\cdot, s} - y(0)_{\cdot, s})] - \frac{1}{4} (n-3)(y(1\cdot)_{1\cdot, 1} - y(0)_{\cdot, 1})$
2. $\frac{1}{4} [(y(11\cdot))_{1s\cdot, q_{1(s-1)}} - y(0)_{\cdot, q_{1(s-1)}}] - (y(1\cdot)_{1\cdot, 1} - y(0)_{\cdot, 1}) - (y(1\cdot)_{s\cdot, s} - y(0)_{\cdot, s})]$
3. $\frac{1}{4} [(y(11\cdot))_{1s\cdot, q_{1(s-1)}} - y(0)_{\cdot, q_{1(s-1)}}] - (y(1\cdot)_{1\cdot, 1} - y(0)_{\cdot, 1})] + \frac{1}{4} (y(1\cdot)_{s\cdot, s} - y(0)_{\cdot, s})]$
4. $\frac{1}{2} (y(\cdot 2)_{\cdot t, n+t} - y(0)_{\cdot, n+t})$ and
 $\frac{1}{6} (y(0)_{\cdot, n+t} - 2y(\cdot 1)_{\cdot t, n+t} + y(\cdot 2)_{\cdot t, n+t})$ (6.103)

From (6.103), the estimates in (1), (2) and (3) are the same as the estimates of the functions (1), (3) and (2) in (4.21) while the estimates in (4) are the same as those in (1), (2) in (6.66).

For step 2, the estimable functions after the completion of the $[n + m + (n-1) + m]$ blocks are

1. A_1
2. $A_1 A_s$
3. (3) in (6.102)
4. (a) $A_1 B_t$ and (b) $A_1 B_t^2$
5. $B_t - \sum_{r=2}^n A_r B_t - \sum_{u \neq t} B_t B_u + \sum_{u \neq t} B_t B_u^2$

$$6. B_t^2 - \sum_{r=2}^n A_r B_t^2 - \sum_{u \neq t} B_t^2 B_u^2 + \sum_{u \neq t} B_t^2 B_u^2. \quad (6.104)$$

The unbiased estimates of the functions (1), (2), (3), (4) are given by (1) in (6.91), (2) in (6.91), (3) in (6.103) and (4) in (6.79), respectively. The unbiased estimates of the remaining functions in (6.104) are

$$4. \frac{1}{4} [(y(1 \cdot 2)_{1 \cdot t, \ell_{t1}} - y(1 \cdot)_{1 \cdot, \ell_{t1}}) + (y(\cdot 2)_{\cdot t, n+t} - y(0)_{\cdot, n+t})] .$$

$$5. \frac{1}{12} [(y(0)_{\cdot, \ell_{t1}} - 2y(1 \cdot 1)_{1 \cdot t, \ell_{t1}} + y(1 \cdot 2)_{1 \cdot t, \ell_{t1}}) + (y(0)_{\cdot, n+t} - 2y(\cdot 1)_{\cdot t, n+t} + y(\cdot 2)_{\cdot t, n+t})] . \quad (6.105)$$

Unbiasedness of the estimates (4) in (6.105) can be established from (6.15) and (6.16) while the unbiasedness of (5) in (6.105) can be established from (6.19) and (6.20).

We consider next the general stages r , n and $n+t$.

Stage (r): The estimation of A_r , $A_r A_s$, $A_r B_t$ and $A_r B_t^2$ ($r < s$, $r = 1, 2, \dots, n-1$ and $t = 1, 2, \dots, m$).

This stage is carried out in two steps.

a) Step 1 leads to the estimation of the $(n-r)$ interactions $A_r A_s$ ($s = r+1, r+2, \dots, m$) with the addition of the $(n-r)$ blocks

$$\text{block } q_{r(s-r)}: \quad (0)_{\cdot, q_{r(s-r)}} \quad (11 \cdot)_{rs \cdot, q_{r(s-r)}}$$

$$(q_{r(s-r)} = n + m + \sum_{i=1}^{r-1} (n-i) + m(r-1) + s-r \text{ and } r < s).$$

b) Step 2 leads to the estimation of the $2m$ interactions $A_r B_t$ and $A_r B_t^2$ with the addition of the n blocks

$$\text{block } \ell_{tr}: \quad (1\cdot)_{r\cdot, \ell_{tr}} \quad (1\cdot 1)_{r\cdot t, \ell_{tr}} \quad (1\cdot 2)_{r\cdot t, \ell_{tr}}$$

$$(\ell_{tr} = n + m + \sum_{i=1}^r (n-i) + m(r-1) + t, \quad t = 1, 2, \dots, m).$$

For stage r, the block structure is

$$\text{block } r: \quad (0)_{,r} \quad (1\cdot)_{r\cdot, r}$$

$$\text{block } (n+t): \quad (0)_{,n+t} \quad (1\cdot)_{\cdot t, n+t} \quad (1\cdot 2)_{\cdot t, n+t}$$

$$\text{Stage one} \left\{ \begin{array}{l} \text{block } q_{1(s-1)}: \quad (0)_{,q_{1(s-1)}} \quad (11\cdot)_{1s\cdot, q_{1(s-1)}} \\ \text{block } \ell_{t1}: \quad (1\cdot)_{1\cdot, \ell_{t1}} \quad (1\cdot 1)_{1\cdot t, \ell_{t1}} \quad (1\cdot 2)_{1\cdot t, \ell_{t1}} \\ \vdots \\ \vdots \end{array} \right.$$

$$\text{Stage } r \left\{ \begin{array}{l} \text{block } q_{r(s-r)}: \quad (0)_{,q_{r(s-r)}} \quad (11\cdot)_{rs\cdot, q_{r(s-r)}} \\ \text{block } \ell_{tr}: \quad (1\cdot)_{1\cdot, \ell_{tr}} \quad (1\cdot 1)_{1\cdot t, \ell_{tr}} \quad (1\cdot 2)_{1\cdot t, \ell_{tr}} \end{array} \right.$$

(6.106)

After the completion of the $n + m + \sum_{i=1}^r (n-i) + mr$ blocks in (6.106),

the estimable functions are

$$1. \quad A_1 ; A_1 A_s ; A_1 B_t ; A_1 B_t^2 \quad (s = 2, 3, \dots, n \text{ and } t = 1, 2, \dots, m).$$

$$2. \quad A_2 ; A_2 A_s ; A_2 B_t ; A_2 B_t^2 \quad (s = 3, 4, \dots, n \text{ and } t = 1, 2, \dots, m).$$

⋮

$$r. \quad a. \quad A_r ; A_r A_s ; A_r B_t ; A_r B_t^2 .$$

$$b. \quad A_s - \left(\sum_{\omega=r+1}^{s-1} A_s A_\omega + \sum_{\omega=s+1}^n A_s A_\omega \right) - \sum_{t=1}^m A_s B_t + \sum_{t=1}^m A_s B_t^2 .$$

$$c. \quad B_t - \sum_{s=r+1}^n A_s B_t - \sum_{u \neq t} B_t B_u + \sum_{u \neq t} B_t B_u^2 .$$

$$d. B_t^2 - \sum_{s=r+1}^n A_s B_t^2 - \sum_{u \neq t} B_t^2 B_u^2 + \sum_{u \neq t} B_t^2 B_u^2. \quad (6.107)$$

The unbiased estimates of the functions (1), (2), ..., r(a) were given in the preceding section. The unbiased estimates of the remaining functions in (6.107) are

$$\begin{aligned}
 b. & \frac{1}{2} \sum_{i=1}^r [(y(11\cdot)_{is\cdot, q_i(s-i)} - y(0)_{, q_i(s-i)}) \\
 & - (y(1\cdot)_{i\cdot, i} - y(0)_{, i})] \\
 & - \frac{1}{2}(r-2)(y(1\cdot)_{s\cdot, s} - y(0)_{, s}) \\
 c. & \frac{1}{2} \sum_{i=1}^r (y(1\cdot 2)_{i\cdot t, l_{ti}} - y(1\cdot)_{i\cdot, l_{ti}}) \\
 & - c_1 (y(\cdot 2)_{\cdot t, n+t} - y(0)_{, n+t}) \\
 d. & \frac{1}{12} \sum_{i=1}^r (y(1\cdot)_{i\cdot, l_{ti}} - 2y(1\cdot 1)_{i\cdot t, l_{ti}} + y(1\cdot 2)_{i\cdot t, l_{ti}}) \\
 & - c_2 (y(0)_{, n+t} - 2y(\cdot 1)_{\cdot t, n+t} + y(\cdot 2)_{, n+t}) \quad (6.108)
 \end{aligned}$$

The estimate (b) in (6.108) is the same as (2) in (4.32). The constants c_1 and c_2 in (c) and (d) in (6.108) can be obtained by equating the expected values of (c) and (d) by their corresponding functions in (6.107).

Stage n: The estimation of A_n , $A_n B_t$ and $A_n B_t^2$ ($t = 1, 2, \dots, m$).

After stage (n-1), all interactions between two-level factors become estimable but main effect A_n is still aliased with the interactions $A_n B_t$ and $A_n B_t^2$. Therefore, for stage n, the m blocks

$$\text{block } \ell_{tn}: \quad (1\cdot)_{n\cdot, \ell_{tn}} \quad (1\cdot 1)_{n\cdot t, \ell_{tn}} \quad (1\cdot 2)_{n\cdot t, \ell_{tn}},$$

$$t = 1, 2, \dots, m.$$

$$(\ell_{tn} = n + m + \sum_{i=1}^{n-1} (n-i) + (n-1)m + t) \quad (6.109)$$

must be added to the blocks in (6.106) and (6.109) in order that the effects A_n , $A_n B_t$ and $A_n B_t^2$ be estimable.

After the completion of the $[n + m + \sum_{i=1}^{n-1} (n-i) + nm]$ blocks in (6.106) and (6.109), the estimable functions are

1. A_r , $r = 1, 2, \dots, n$.
2. $A_r A_s$, $s = r+1, r+2, \dots, n$ and $r = 1, 2, \dots, n-1$.
3. $A_r B_t$ and $A_r B_t^2$ ($r = 1, 2, \dots, n$ and $t = 1, 2, \dots, m$).
4. $B_t - \sum_{u \neq t} B_t B_u + \sum_{u \neq t} B_t B_u^2$.
5. $B_t^2 - \sum_{u \neq t} B_t^2 B_u + \sum_{u \neq t} B_t^2 B_u^2$. (6.110)

The unbiased estimates of (1), (2) are given by \hat{A}_k and $\hat{A}_k \hat{A}_s$ in (6.98), respectively, whereas the estimates of (3) above are given by (4) in (6.79). The unbiased estimates of the remaining functions in (6.110) are, respectively

$$4. \quad \frac{1}{2} \sum_{r=1}^n (y(1\cdot 2)_{r\cdot t, \ell_{tr}} - y(1\cdot)_{r\cdot, \ell_{tr}}) - c_3 (y(\cdot 2)_{\cdot t, n+t} - y(0)_{\cdot, n+t}).$$

$$5. \frac{1}{12} \sum_{r=1}^n (y(0)_{,l_{tr}} - 2y(1.1)_{r.t,l_{tr}} + y(1.2)_{r.t,l_{tr}}) - c_4 (y(0)_{,n+t} - 2y(\cdot 1)_{.t,n+t} + y(\cdot 2)_{.t,n+t}) \quad (6.111)$$

The constants c_3 and c_4 in (6.111) can be obtained by equating the expected values of the estimates (4) and (5) by their corresponding functions in (6.110).

Stage (n+1): The estimation of B_1 , B_1^2 , $B_1 B_u$, $B_1 B_u^2$, $B_1^2 B_u$ and $B_1^2 B_u^2$ ($u = 2, 3, \dots, m$).

Referring to (4) and (5) in (6.110), the linear effect B_t is aliased with the $(m-1)$ interactions $B_t B_u$ and the $(m-1)$ interactions $B_t B_u^2$ whereas the quadratic effect B_t^2 is aliased with the $(m-1)$ interactions $B_t^2 B_u$ and the $(m-1)$ interactions $B_t^2 B_u^2$. Therefore, to free the first factor B_1 from aliasing, the $2(m-1)$ blocks

$$a. \text{ block } h_{1(u-1)}: \quad (0)_{,h_{1(u-1)}} \quad (\cdot 11)_{.1u,h_{1(u-1)}} \quad (\cdot 22)_{.1u,h_{1(u-1)}}, \\ u = 2, 3, \dots, m$$

$$(h_{1(u-1)}) = n + m + \sum_{i=1}^{n-1} (n-i) + nm + (u-1).$$

$$b. \text{ block } k_{1(u-1)}: \quad (0)_{,k_{1(u-1)}} \quad (\cdot 12)_{.1u,k_{1(u-1)}} \quad (\cdot 21)_{.1u,k_{1(u-1)}},$$

$$(k_{1(u-1)}) = h_{1(u-1)} + (m-1) \quad (6.112)$$

must be added to the blocks of stage n . However, we carry out stage $(n+1)$ in two steps where

a. Step 1 involves the estimation of $B_1 B_u$ ($u = 2, 3, \dots, m$) by adding the $(m-1)$ blocks (a) in (6.112) to the blocks of stage n .

b. Step 2 involves the estimation of $B_1, B_1^2, B_1^2 B_u, B_1 B_u^2$ and $B_1^2 B_u^2$ ($u = 2, 3, \dots, m$) by adding the blocks (b) in (6.112) to the blocks of step 1 of stage $(n+1)$.

The estimable functions after stage $(n+1)$ are

1. A_r , $r = 1, 2, \dots, n$.
2. $A_r A_s$, $s = r+1, r+2, \dots, n$ and $r = 1, 2, \dots, n-1$.
3. $A_r B_t$ and $A_r B_t^2$, $r = 1, 2, \dots, n$ and $t = 1, 2, \dots, m$.
4. $B_1, B_1^2, B_1 B_u, B_1^2 B_u, B_1 B_u^2, B_1^2 B_u^2$, $u = 2, 3, \dots, m$.
5. 3(a) and 3(b) in (5.54). (6.113)

The unbiased estimates of (1) and (2) are given by \hat{A}_k and $\hat{A}_k \hat{A}_s$ in (6.98) and the unbiased estimates of (3) are given by (4) in (6.79).

The unbiased estimates of (4) are given by (1), (2) and (3) in (6.83)

The unbiased estimates of (5) in (6.113) are, respectively

$$\begin{aligned}
 \text{a. } & \frac{1}{2} \sum_{r=1}^n (y(1 \cdot 2)_{r \cdot u, \ell_{ur}} - y(1 \cdot)_{r \cdot, \ell_{ur}}) \\
 & + \frac{1}{6} [(y(\cdot 22)_{\cdot 1u, h_{1(u-1)}} - y(0)_{\cdot, h_{1(u-1)}}) \\
 & + (y(\cdot 12)_{\cdot 1u, k_{1(u-1)}} - y(0)_{\cdot, k_{1(u-1)}}) \\
 & - (y(\cdot 1)_{\cdot 1, n+1} + y(\cdot 2)_{\cdot 1, n+1} - 2y(0)_{\cdot, n+1})] \\
 & - c_5 (y(\cdot 2)_{\cdot u, n+u} - y(0)_{\cdot, n+u}) .
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } & \frac{1}{12} \sum_{r=1}^n (y(1\cdot)_{r\cdot, \ell_{ur}} - 2y(1\cdot 1)_{r\cdot u, \ell_{ur}} + y(1\cdot 2)_{r\cdot u, \ell_{ur}}) \\
 & + \frac{1}{18} [(y(0)_{\cdot, h_{1(u-1)}} - 2y(\cdot 11)_{\cdot 1u, h_{1(u-1)}} + y(\cdot 22)_{\cdot 1u, h_{1(u-1)}} \\
 & + (y(\cdot 1)_{\cdot 1, n+1} + y(\cdot 2)_{\cdot 1, n+1} - 2y(0)_{\cdot, n+1})] \\
 & - c_6 (y(0)_{\cdot, n+u} - 2y(\cdot 1)_{\cdot u, n+u} + y(\cdot 2)_{\cdot u, n+u}) \quad (6.114)
 \end{aligned}$$

The constants c_5 and c_6 can be found by equating the expected values of the estimates in (6.114) with their corresponding functions in (5) in (6.113).

From the results in stage $(n+1)$, we can generalize to

Stage $(n+t)$: The estimation of B_t , B_t^2 , $B_t B_u$, $B_t^2 B_u$, $B_t B_u^2$ and $B_t^2 B_u^2$ ($t < u$ and $t = 1, 2, \dots, m$).

For this stage, $2(m-t)$ blocks

$$\text{a. block } h_{t(u-t)}: \quad (0)_{\cdot, h_{t(u-t)}} \quad (\cdot 11)_{\cdot tu, h_{t(u-t)}} \quad (\cdot 22)_{\cdot tu, h_{t(u-t)}}$$

$$u = t+1, t+2, \dots, m$$

$$(h_{t(u-t)}) = n + m + \sum_{i=1}^{n-1} (n-i) + mn + 2 \sum_{j=1}^{t-1} (m-j) + u - t,$$

$$\text{b. block } k_{t(u-t)}: \quad (0)_{\cdot, k_{t(u-t)}} \quad (\cdot 12)_{\cdot tu, k_{t(u-t)}} \quad (\cdot 21)_{\cdot tu, k_{t(u-t)}}$$

$$(k_{tu} = h_{t(u-t)} + (m-t)) \quad (6.115)$$

are needed to free the t^{th} factor B_t from aliasing. This stage is carried out in two steps where step 1 involves the estimation of the $(m-t)$ interactions $B_t B_u$ and step 2 leads to the estimation of B_t , B_t^2 , $B_t^2 B_u$, $B_t B_u^2$ and $B_t^2 B_u^2$.

The estimable functions after stage (n+t) are

1. A_r , $r = 1, 2, \dots, n$.
2. $A_r A_s$, $s = r+1, r+2, \dots, n$ and $r = 1, 2, \dots, n-1$.
3. $A_r B_t$ and $A_r B_t^2$, $r = 1, 2, \dots, n$ and $t = 1, 2, \dots, m$.
4. $B_1, B_1^2, B_1 B_u, B_1 B_u^2, B_1^2 B_u$ and $B_1^2 B_u^2$ ($u = 2, 3, \dots, m$).
5. $B_2, B_2^2, B_2 B_u, B_2 B_u^2, B_2^2 B_u$ and $B_2^2 B_u^2$ ($u = 3, 4, \dots, m$).
- \vdots
- (t+3). a. $B_t, B_t^2, B_t B_u, B_t B_u^2, B_t^2 B_u$ and $B_t^2 B_u^2$ ($u = t+1, t+2, \dots, m$).
- b. (b) in (5.71).
- c. (c) in (5.71). (6.116)

The unbiased estimates of (1), (2), \dots , (t+3)(a) were given earlier in this section. The unbiased estimates of (t+3)(b) and (t+3)(c) are, respectively

$$\begin{aligned}
 \text{b. } & \frac{1}{2} \sum_{r=1}^n (y(1 \cdot 2)_{r \cdot u, \ell_{ur}} - y(1 \cdot)_{r \cdot, \ell_{ur}}) \\
 & + \frac{1}{6} \sum_{j=1}^t [(y(\cdot 22)_{\cdot j u, h_j(u-j)} - y(0)_{\cdot, h_j(u-j)}) \\
 & + (y(\cdot 12)_{\cdot j u, k_j(u-j)} - y(0)_{\cdot, k_j(u-j)}) \\
 & - (y(\cdot 1)_{\cdot j, n+j} + y(\cdot 2)_{\cdot j, n+j} - 2y(0)_{\cdot, n+j})] \\
 & - c_7 (y(\cdot 2)_{\cdot u, n+u} - y(0)_{\cdot, n+u}) .
 \end{aligned}$$

$$\begin{aligned}
c. \quad & \frac{1}{12} \sum_{r=1}^n (y(1\cdot)_{r\cdot, \ell_{ur}} - 2y(1\cdot 1)_{r\cdot u, \ell_{ur}} + y(1\cdot 2)_{r\cdot u, \ell_{ur}}) \\
& + \frac{1}{18} \sum_{j=1}^t [(y(0)_{\cdot, h_j(u-j)} - 2y(\cdot 11)_{\cdot, ju, h_j(u-j)} \\
& \quad + y(\cdot 22)_{\cdot, ju, h_j(u-j)}) \\
& + (y(0)_{\cdot, k_j(u-j)} - 2y(\cdot 21)_{\cdot, ju, k_j(u-j)} + y(\cdot 12)_{\cdot, ju, k_j(u-j)}) \\
& + (y(\cdot 1)_{\cdot, j, n+j} + y(\cdot 2)_{\cdot, j, n+j} - 2y(0)_{\cdot, n+j})] \\
& - c_8(y(0)_{\cdot, n+u} - 2y(\cdot 1)_{\cdot, u, n+u} + y(\cdot 2)_{\cdot, u, n+u}) \quad (6.117)
\end{aligned}$$

The constants c_7 and c_8 are functions of t and can be found by equating the expected values of the estimates (b) and (c) in (6.117) with their corresponding functions in (6.116).

6.6 OBAT $2^n 3^m$ Plans for Strategy II

For strategy II we consider the three cases

Case (1): All m 3-level factors B_1, B_2, \dots, B_m and n 2-level factors are considered as one set of potentially interacting factors.

In this case, the $2^n \cdot 3^m$ fraction $F_{n,m}$ in (6.2) can then be used to assess the significance of the factorial effects $B_t, B_t^2, B_t B_u, B_t^2 B_u, B_t B_u^2, B_t^2 B_u^2, A_r B_t, A_r B_t^2, A_r$ and $A_r A_s$ ($r < s, t < u, r = 1, 2, \dots, n$ and $t = 1, 2, \dots, m$) whose estimate can be obtained from (6.39), (6.49), (6.53), (6.54), (6.57), (6.58), (6.51), (6.52), (6.30) and (6.50), respectively.

Case (2): The m 3-level factors B_1, B_2, \dots, B_m and the n two-level factors A_1, A_2, \dots, A_n are partitioned into g disjoint sets where set h contains m_h 3-level factors ($m_h \leq m$) and n_h 2-level factors ($n_h \leq n$) and $\sum_{h=1}^g (n_h + m_h) = n + m$. Factors in each set may interact with each other but not with factors from another set.

For case (2), the experiment will then be carried out in a sequence of g blocks such that the h^{th} block is the $2^{n_h} 3^{m_h}$ fraction F_{n_h, m_h} of (6.2) with

$$N_h = 1 + n_h + \frac{n_h(n_h-1)}{2} + 2m_h + \frac{4m_h(m_h-1)}{2} + 2n_h m_h$$

treatment combinations ($h = 1, 2, \dots, g$). Each treatment combination among the levels of the n_h and m_h factors in the fraction will be written as $(00\text{---}0a_r 00\text{---}0a_s 00\text{---}0 \cdot 00\text{---}0b_t 00\text{---}0b_u 00\text{---}0)_h$, or just as

$$(a_r a_s \cdot b_t b_u)_{rs \cdot tu, h} \quad (6.118)$$

where ($a_r, a_s = 0, 1$), ($b_t, b_u = 0, 1, 2$) and ($t < u, r < s, r = 1, 2, \dots, n_h$ and $t = 1, 2, \dots, m_h$).

The sequence of g blocks in (6.118) can be written explicitly as

$$\begin{array}{l} \text{block 1: } (0)_{,1} \quad (1\cdot)_{r\cdot,1} \quad (\cdot 1)_{\cdot t,1} \quad (\cdot 2)_{\cdot t,1} \quad (11\cdot)_{rs\cdot,1} \quad (\cdot 11)_{\cdot tu,1} \\ \quad (\cdot 12)_{\cdot tu,1} \quad (\cdot 21)_{\cdot tu,1} \quad (\cdot 22)_{\cdot tu,1} \quad (1\cdot 1)_{r\cdot t,1} \quad (1\cdot 2)_{r\cdot t,1} \\ \text{block 2: } (0)_{,2} \quad (1\cdot)_{r\cdot,2} \quad (\cdot 1)_{\cdot t,2} \quad (\cdot 2)_{\cdot t,2} \quad (11\cdot)_{rs\cdot,2} \quad (\cdot 11)_{\cdot tu,2} \\ \quad (\cdot 12)_{\cdot tu,2} \quad (\cdot 21)_{\cdot tu,2} \quad (\cdot 22)_{\cdot tu,2} \quad (1\cdot 1)_{r\cdot t,2} \quad (1\cdot 2)_{r\cdot t,2} \\ \vdots \\ \text{block } g: (0)_{,g} \quad (1\cdot)_{r\cdot,g} \quad (\cdot 1)_{\cdot t,g} \quad (\cdot 2)_{\cdot t,g} \quad (11\cdot)_{rs\cdot,g} \quad (\cdot 11)_{\cdot tu,g} \end{array}$$

$$\begin{array}{ccccccc}
 (\cdot 12)_{\cdot tu, g} & (\cdot 21)_{\cdot tu, g} & (\cdot 22)_{\cdot tu, g} & (1 \cdot 1)_{r \cdot t, g} & (1 \cdot 2)_{r \cdot t, g} & & \\
 & & & & & & (6.119)
 \end{array}$$

We now have the following linear model for Strategy II OBAT $2^n 3^m$ plans.

$$y(a_r a_s \cdot b_t b_u)_{rs \cdot tu, h} = (a_r a_s \cdot b_t b_u)_{rs \cdot tu, h} + \varepsilon_{tu, h} \quad (6.120)$$

Denoting the factorial effects of the h^{th} $2^{n_h} 3^{m_h}$ fraction by $(B_t)_h$, $(B_t^2)_h$, $(B_t B_u)_h$, $(B_t^2 B_u)_h$, $(B_t B_u^2)_h$, $(B_t^2 B_u^2)_h$, $(A_r B_t)_h$, $(A_r B_t^2)_h$, $(A_r)_h$ and $(A_r A_s)_h$ (6.121)

and referring to (6.39), (6.49), (6.53), (6.54), (6.57), (6.58), (6.51), (6.52), (6.30) and (6.50), the unbiased estimates of the parameters in (6.121) are, respectively

1.
$$\frac{1}{6} \sum_{u \neq t} [(y(\cdot 22)_{\cdot tu, h} - y(\cdot 2)_{\cdot u, h}) + (y(\cdot 21)_{\cdot tu, h} - y(\cdot 1)_{\cdot u, h})] + \frac{1}{4} \sum_{r=1}^{n_h} (y(1 \cdot 2)_{r \cdot t, h} - y(1 \cdot)_{r \cdot, h}) - \frac{1}{12} (4m_h + 3n_h - 10)(y(\cdot 2)_{\cdot t, h} - y(0)_{\cdot, h})$$
2.
$$\frac{1}{18} \sum_{u \neq t} [(y(1 \cdot)_{\cdot u, h} - 2y(\cdot 11)_{\cdot tu, h} + y(\cdot 21)_{\cdot tu, h}) + (y(\cdot 2)_{\cdot u, h} - 2y(\cdot 12)_{\cdot tu, h} + y(\cdot 22)_{\cdot tu, h})] + \frac{1}{12} \sum_{r=1}^{n_h} (y(1 \cdot)_{r \cdot, h} - 2y(1 \cdot 1)_{r \cdot t, h} + y(1 \cdot 2)_{r \cdot t, h}) - \frac{1}{36} (4m_h + 3n_h - 10)(y(0)_{\cdot, h} - 2y(\cdot 1)_{\cdot t, h} + y(\cdot 2)_{\cdot t, h})$$
3.
$$\frac{1}{4} [(y(\cdot 22)_{\cdot tu, h} - y(\cdot 2)_{\cdot u, h}) - (y(\cdot 2)_{\cdot t, h} - y(0)_{\cdot, h})]$$

4. $\frac{1}{12} [(y(\cdot 2)_{\cdot u, h} - 2y(\cdot 12)_{\cdot tu, h} + y(\cdot 22)_{\cdot tu, h}) - (y(0)_{, h} - 2y(\cdot 1)_{\cdot t, h} + y(\cdot 2)_{\cdot t, h})]$
5. $\frac{1}{12} [(y(\cdot 2)_{\cdot t} - 2y(\cdot 21)_{\cdot tu, h} + y(\cdot 22)_{\cdot tu, h}) - (y(0)_{, h} - 2y(\cdot 1)_{\cdot u, h} + y(\cdot 2)_{\cdot u, h})]$
6. $\frac{1}{36} [(y(0)_{, h} - 2y(\cdot 1)_{\cdot t, h} + y(\cdot 2)_{\cdot t, h}) - 2(y(\cdot 1)_{\cdot u, h} - 2y(\cdot 11)_{\cdot tu, h} + y(\cdot 21)_{\cdot tu, h}) + (y(\cdot 2)_{\cdot u, h} - 2y(\cdot 12)_{\cdot tu, h} + y(\cdot 22)_{\cdot tu, h})]$
7. $\frac{1}{4} [(y(1 \cdot 2)_{r \cdot t, h} - y(1 \cdot)_{r \cdot, h}) - (y(\cdot 2)_{\cdot t, h} - y(0)_{, h})]$
8. $\frac{1}{12} [(y(1 \cdot)_{r \cdot, h} - 2y(1 \cdot 1)_{r \cdot t, h} + y(1 \cdot 2)_{r \cdot t, h}) - (y(0)_{, h} - 2y(\cdot 1)_{\cdot t, h} + y(\cdot 2)_{\cdot t, h})]$
9. $\frac{1}{6} \sum_{t=1}^{m_h} [(y(1 \cdot 2)_{r \cdot t, h} - y(\cdot 2)_{\cdot t, h}) + (y(1 \cdot 1)_{r \cdot t, h} - y(\cdot 1)_{\cdot t, h})] + \frac{1}{4} \sum_{s \neq r} (y(11 \cdot)_{rs \cdot, h} - y(1 \cdot)_{s \cdot, h}) - \frac{1}{12} (4m_h + 3n_h - 9)(y(1 \cdot)_{r \cdot, h} - y(0)_{, h})]$
10. $\frac{1}{4} [(y(11 \cdot)_{rs \cdot, h} - y(1 \cdot)_{s \cdot, h}) - (y(1 \cdot)_{r \cdot, h} - y(0)_{, h})]$ (6.122)

Case (3): The m 3-level factors and the n 2-level factors A_1, A_2, \dots, A_n are partitioned into three groups containing altogether $(g_1 + g_2 + 1)$ disjoint sets such that

- a. group one contains g_1 sets. The 2-level and 3-level factors

in each one of the g_1 sets interact with each other but not with factors from another set. That is, these g_1 sets are actually the g sets of case (2).

b. Group two contains g_2 sets. In each one of the g_2 sets, it is assumed here that only one 3-level factors (say $(B_1)_h$) and one 2-level factor (say $(A_1)_h$) interact with each other and with the remaining 2-level and 3-level factors in the set.

c. Group three consists of only one set. The 2-level and 3-level factors in this set do not interact with each other.

Let n_{ij} and m_{ij} represent, respectively, the number of 2-level factors and 3-level factors in the j^{th} set of the i^{th} group where $j = 1, 2, \dots, g_i$ and $i = 1, 2, 3$ such that $\sum_{j=1}^{g_i} (m_{ij} + n_{ij}) = n_i + m_i$ is the total number of factors in the i^{th} group and $\sum_{i=1}^3 \sum_{j=1}^{g_i} (m_{ij} + n_{ij}) = n + m$. For case (3), the $2^{n_3} 3^m$ experiment will then be carried out in a sequence of $(g_1 + g_2 + 1)$ blocks such that

a) each one of the g_1 blocks is the $2^{n_{1h}} 3^{m_{1h}}_V$ fraction (6.2) with

$$N_{1h} = 1 + n_{1h} + \frac{n_{1h}(n_{1h}-1)}{2} + 2m_{1h} + \frac{4m_{1h}(m_{1h}-1)}{2} + 2n_{1h}m_{1h}$$

treatment combinations. Each treatment combination among the $(n_{1h} + m_{1h})$ factor levels is of the form

$$(a_r a_s \cdot b_t b_u)_{rs \cdot tu, h} \quad (6.123)$$

where $(a_r, a_s = 0, 1)$, $(b_t, b_u = 0, 1, 2)$, $(h = 1, 2, \dots, g_1)$ and $(t < u, r < s, r = 1, 2, \dots, n_{1h}$ and $t = 1, 2, \dots, m_{1h})$. The contents of the g_1 blocks are identical to those in (6.119) with $g_1 = g$.

b) Each one of the g_2 blocks is a $2^{n_{2h}} 3^{m_{2h}}_V$ fraction containing

$$N_{2h} = 1 + n_{2h} + 2m_{2h} + (n_{2h} - 1) + 2(m_{2h} - 1) + 2$$

treatment combinations. These N_{2h} treatment combinations among the $(n_{2h} = m_{2h})$ factor levels can be written explicitly as

$$\begin{aligned} & (0)_{,h} \quad (1\cdot)_{r\cdot,h} \quad (\cdot 1)_{\cdot t,h} \quad (\cdot 2)_{\cdot t,h} \\ & (11\cdot)_{1s\cdot,h} \\ & (\cdot 11)_{\cdot 1u,h} \quad (\cdot 12)_{\cdot 1u,h} \quad (\cdot 21)_{\cdot 1u,h} \quad (\cdot 22)_{\cdot 1u,h} \\ & (1\cdot 1)_{1\cdot 1,h} \quad (1\cdot 2)_{1\cdot 1,h} \end{aligned} \quad (6.124)$$

($s = 2, 3, \dots, n_{2h}$, $u = 2, 3, \dots, m_{2h}$ and $h = g_1 + 1, g_1 + 2, \dots, g_1 + g_2$).

c) the last set which contains non-interacting 2-level and 3-level factors is a $2^{n_{3h}} 3^{m_{3h}}_{III}$ fraction containing $N_{31} = 1 + n_{31} + 2m_{31}$ treatment combinations. These treatment combinations among the $(n_{31} + m_{31})$ factor levels can be written explicitly as

$$(0)_{,h} \quad (1\cdot)_{r\cdot,h} \quad (\cdot 1)_{\cdot t,h} \quad (\cdot 2)_{\cdot t,h} \quad (6.125)$$

($r = 1, 2, \dots, n_{31}$, $t = 1, 2, \dots, m_{31}$ and $h = g_1 + g_2 + 1$).

Referring to the model (6.120) and denoting the factorial effects of the fractions $2^{n_{1h}} 3^{m_{1h}}_V$, $2^{n_{2h}} 3^{m_{2h}}_V$ and $2^{n_{31}} 3^{m_{31}}_{III}$, respectively, by

a. $(B_t)_n$, $(B_t^2)_h$, $(B_t B_u)_h$, $(B_t^2 B_u)_h$, $(B_t B_u^2)_h$, $(B_t^2 B_u^2)_h$, $(A_r B_t)_h$, $(A_r B_t^2)_h$, $(A_r)_h$, $(A_r A_s)_h$ ($r < s$, $t < u$, $r = 1, 2, \dots, n_{1h}$, $t = 1, 2, \dots, m_{1h}$ and $h = 1, 2, \dots, g_1$).

b. 1. $(A_1)_h$
2. $(B_1)_h$, $(B_1^2)_h$

3. $(A_{1s}A_s)_h$, $s = 2, 3, \dots, n_{2h}$
4. $(B_{1u}B_u)_h$, $(B_{1u}^2B_u)$, $(B_{1u}B_u^2)$, $(B_{1u}^2B_u^2)$, $u = 2, 3, \dots, m_{2h}$
5. $(A_{1l}B_l)_h$, $(A_{1l}B_l^2)_h$
6. $(A_r)_h$, $(B_t)_h$, $(B_t^2)_h$ ($r = 2, 3, \dots, n_{2h}$, $t = 2, 3, \dots, m_{2h}$ and $h = g_1 + 1, g_1 + 2, \dots, g_1 + g_2$)
- c. $(A_r)_h$, $(B_t)_h$, $(B_t^2)_h$ ($r = 1, 2, \dots, n_{31}$, $t = 1, 2, \dots, m_{31}$ and $h = g_1 + g_2 + 1$).

(6.126)

The unbiased estimates of the parameters (a) in (6.126) are similar to those in (6.122) with n_h and m_h replaced by n_{1h} and m_{1h} , respectively. The unbiased estimates of the parameters (b) in (6.126) can respectively be obtained from (6.122) as follows.

1. Setting $n_h = n_{2h}$, $m_h = m_{2h}$ and $r = 1$ in (9).
2. Setting $n_h = n_{2h}$, $m_h = m_{2h}$ and $t = 1$ in (1) and (2).
3. Setting $r = 1$ in (10).
4. Setting $t = 1$ in (3), (4), (5) and (6).
5. Setting $r = 1$ and $t = 1$ in (7) and (8).
6. $(\hat{A}_r)_h = \frac{1}{4} (y(11\cdot)_{1r\cdot,h} - y(1\cdot)_{1\cdot,h})$

$$\begin{aligned}
 &+ \frac{1}{6} [(y(1\cdot2)_{r\cdot1,h} - y(\cdot2)_{\cdot1,h}) \\
 &\quad + (y(1\cdot1)_{r\cdot1,h} - y(\cdot1)_{\cdot1,h})] \\
 &- \frac{1}{12} [y(1\cdot)_{r\cdot,h} - y(0)_{\cdot,h}]
 \end{aligned}$$

where $r = 2, 3, \dots, n_{2h}$ and $h = g_1 + 1, g_1 + 2, \dots, g_1 + g_2$.

$$(\hat{B}_{t,h}) = \frac{1}{6} [(y(\cdot 22)_{\cdot 1t,h} - y(\cdot 2)_{\cdot 1,h}) + (y(\cdot 12)_{\cdot 1t,h} - y(\cdot 1)_{\cdot 1,h})] \\ + \frac{1}{4} (y(1 \cdot 2)_{1 \cdot t,h} - y(1 \cdot)_{1 \cdot ,h}) - \frac{1}{12} (y(\cdot 2)_{\cdot t,h} - y(0)_{\cdot ,h})$$

$$(\hat{B}_{t,h}^2) = \frac{1}{18} [(y(\cdot 1)_{\cdot 1,h} - 2y(\cdot 11)_{\cdot 1t,h} + y(\cdot 12)_{\cdot 1t,h}) \\ + (y(\cdot 2)_{\cdot 1,h} - 2y(\cdot 21)_{\cdot 1t,h} + y(\cdot 22)_{\cdot 1t,h})] \\ + \frac{1}{12} (y(1 \cdot)_{1 \cdot ,h} - 2y(1 \cdot 1)_{1 \cdot t,h} + y(1 \cdot 2)_{1 \cdot t,h}) \\ - \frac{1}{36} (y(0)_{\cdot ,h} - 2y(\cdot 1)_{\cdot t,h} + y(\cdot 2)_{\cdot t,h})$$

where $t = 2, 3, \dots, m_{2h}$ and $h = g_1 + 1, g_1 + 2, \dots, g_1 + g_2$. Finally, the unbiased estimates of the parameters (c) in (6.126) are given by (6.66).

VII. SUMMARY AND FUTURE RESEARCH

7.1 Summary of Results

In this dissertation, we have reported new ORAT 2_V^n plans and new OBAT 2_V^n , 3_V^m and $2^n 3_V^m$ plans for sequentially carried out factorial experiments. Formulas for parameter estimates of the ORAT and OBAT plans have also been reported. Such formulas can be used for establishing the significance of the parameter estimates. ORAT plans were discussed in Chapter 3, OBAT 2^n plans in Chapter 4, OBAT 3^m plans in Chapter 5 and OBAT $2^n 3^m$ plans in Chapter 6.

In Chapter 3, we have reported a saturated non-orthogonal 2_V^n fraction f_n , for general n , in which only one factor level is changed between any two successive treatment combinations and we considered using this 2_V^n design as an ORAT plan. Such ORAT plans are called by Daniel (1973) strict ORAT plans. The $N = 1 + n + \frac{n(n-1)}{2}$ treatment combinations of the fraction f_n are given in (3.2) and are generated as follows: the n factors A_1, A_2, \dots, A_n are assumed to be ordered such that the i^{th} factor A_i appears at its high level for the first time at the $(i+1)^{\text{st}}$ run $(\underbrace{11\dots 1}_{i}00\dots 0)$, $i = 1, 2, \dots, n$. The first run in f_n is the run $(00\dots 0)$ where all factors are at their low level. After the $(n+1)^{\text{st}}$ run $(111\dots 1)$, factor levels are now lowered each to its low level in the same order they were raised to their high level until the $(2n)^{\text{th}}$ run $(00\dots 01)$. Then the process of raising factor levels each to its high level or lowering them each to its low level continues until no factor level changes between successive runs is possible. No run is replicated

more than once. The N runs above are partitioned into n disjoint sets (see (3.2)). Such sets are also considered stages of the experiment. For each stage, the estimable effects and their unbiased estimates have been reported. The main-effects and two-factor interactions become estimable from the fraction f_n in the following order: A_1, A_2, \dots, A_n ; $A_1A_n, A_2A_n, \dots, A_{n-1}A_n$; $A_1A_{n-1}, A_2A_{n-1}, \dots, A_{n-2}A_{n-1}$; \dots ; A_1A_3, A_2A_3 ; A_1A_2 . All these parameters are estimated with variance $\frac{1}{4}\sigma^2$.

The results we found concerning strict ORAT 2_V^n plans generalize those of Webb (1968). Webb proved that estimates of all main-effects and two-factor interactions all have variance $\frac{1}{4}\sigma^2$ but he did not report the actual runs nor did he consider deriving formulas for the estimates of these parameters. He also did not consider the problems of determining which effects become estimable after a particular run. Our results also generalize those of Daniel (1973) who only considered specific examples of ORAT plans, namely 2^3 and 2^4 designs. Comparing our 2_V^n ORAT plans with Cotter's (1979) 2^n ORAT plans, we notice that Cotter did not consider the problem of how many factor level changes are made between runs and the formulas that he derived for judging the significance of parameter estimates are just approximate and not exact. Cotter's formulas for judging the significance of factor effects have been derived without the assumption that high-order interactions are negligible. One of the new results found in this study for 2^n factorial experiments is that the alias structure for resolution IV fold-over 2^n designs in $2n$ runs as well as formulas for the estimates of main effects have been reported.

In Chapter 4, we have reported exact formulas for the estimates of main-effects and two-factor interactions using Webb's (1965) saturated

non-orthogonal expansible-contractible $\langle 0, 1, 2 \rangle - 2_V^n$ designs. Webb's 2_V^n designs consist of the following $1 + n + \frac{n(n-1)}{2}$ treatment combinations

000----0

← n →

$\pi(100----0)$

$\pi(110----0)$

where $\pi()$ denotes all permutations of the treatment combination inside the bracket. We have also reported new OBAT 2_V^n plans based on Webb's 2_V^n designs in blocks of size 2, which eventually lead to the estimation of all main-effects and two-factor interactions. The blocks are of the form

$$\text{a) } 00----0, \underbrace{00---010----0}_r$$

$$\text{b) } 00----0, \underbrace{00---010---010----0}_r$$

$$\underbrace{\hspace{10em}}_s$$

where $r < s$ and $r, s = 1, 2, \dots, n$.

In Chapter 5, we have generalized Webb's (1965) 2_V^n designs and reported new saturated non-orthogonal expansible-contractible permutation-invariant 3_V^m designs and we have reported exact formulas for parameter estimates. The parameters are the linear and quadratic effects of the factors and two-factor interactions involving the linear and quadratic effect of each factor. The treatment combinations in these 3_V^m designs are the following $1 + m + m + \frac{m(m-1)}{2} + m(m-1) + \frac{m(m-1)}{2}$ runs

00----0
 $\longleftrightarrow m \longrightarrow$

$\pi(1000----0)$

$\pi(2000----0)$

$\pi(1100----0)$

$\pi(1200----0)$

$\pi(2200----0)$

We have also reported OBAT 3^m plans based on these expansible-contractible 3^m_V designs which eventually lead to the estimation of all the parameters in the 3^m_V design. The blocks are of size 3 and are of the form

$$\begin{array}{l}
 \text{a) } 0000\text{---}0 \quad \begin{array}{c} 00\text{---}0100\text{---}0 \\ \xrightarrow{t} \end{array} \quad \begin{array}{c} 00\text{---}0200\text{---}0 \\ \xrightarrow{t} \end{array} \\
 \\
 \text{b) } 0000\text{---}0 \quad \begin{array}{c} 00\text{---}0100\text{---}0100\text{---}0 \\ \xrightarrow{t} \\ \xrightarrow{u} \end{array} \quad \begin{array}{c} 00\text{---}0200\text{---}0200\text{---}0 \\ \xrightarrow{t} \\ \xrightarrow{u} \end{array} \\
 \\
 \text{c) } 0000\text{---}0 \quad \begin{array}{c} 00\text{---}0100\text{---}0200\text{---}0 \\ \xrightarrow{t} \\ \xrightarrow{u} \end{array} \quad \begin{array}{c} 00\text{---}0200\text{---}0100\text{---}0 \\ \xrightarrow{t} \\ \xrightarrow{u} \end{array}
 \end{array}$$

where $t < u$ and $t, u = 1, 2, \dots, m$.

In Chapter 6, we have generalized Webb's (1965) 2^n_V designs and the 3^m_V designs of Chapter 5 and reported new saturated non-orthogonal expansible-contractible permutation-invariant $2^n 3^m_V$ designs and we have reported exact formulas for parameter estimates. The treatment combinations in these $2^n 3^m_V$ designs are the following $1 + n + \frac{n(n-1)}{2} + m + m + \frac{m(m-1)}{2} +$

$$m(m-1) + \frac{m(m-1)}{2} + nm + nm$$

$$\begin{array}{c} 0000\text{---}0 \cdot 0000\text{---}0 \\ \longleftarrow n \longrightarrow \cdot \longleftarrow m \longrightarrow \end{array}$$

$$\pi(1000\text{---}0 \cdot 0000\text{---}0)$$

$$\pi(1100\text{---}0 \cdot 0000\text{---}0)$$

$$\pi(0000\text{---}0 \cdot 1000\text{---}0)$$

$$\pi(0000\text{---}0 \cdot 2000\text{---}0)$$

$$\pi(0000\text{---}0 \cdot 1100\text{---}0)$$

$$\pi(0000\text{---}0 \cdot 1200\text{---}0)$$

$$\pi(0000\text{---}0 \cdot 2200\text{---}0)$$

$$\pi(1000\text{---}0 \cdot 1000\text{---}0)$$

$$\pi(1000\text{---}0 \cdot 2000\text{---}0)$$

where $\pi(\cdot)$ denotes all permutations of the first n digits combined with the separate permutations of the second m digits. We have also reported OBAT $2^n 3^m$ plan based on these expansible-contractible $2^n 3^m$ designs in blocks of sizes 2 and 3. The blocks of size 2 are of the form

$$\text{a) } 00\text{---}0 \cdot 00\text{---}0 \quad \begin{array}{c} 00\text{---}0100\text{---}0 \cdot 00\text{---}0 \\ \xrightarrow{\quad r \quad} \end{array}$$

$$\text{b) } 00\text{---}0 \cdot 00\text{---}0 \quad \begin{array}{c} 00\text{---}0100\text{---}0100\text{---}0 \cdot 00\text{---}0 \\ \xrightarrow{\quad r \quad} \\ \xrightarrow{\quad s \quad} \end{array}$$

where $r < s$ and $r, s = 1, 2, \dots, n$. The blocks of size 3 are of the form

$$c) \begin{array}{ccc} 00\text{---}0 \cdot 00\text{---}0 & 00\text{---}0 \cdot \begin{array}{c} 00\text{---}010\text{---}0 \\ \xrightarrow{t} \end{array} & 00\text{---}0 \cdot \begin{array}{c} 00\text{---}020\text{---}0 \\ \xrightarrow{t} \end{array} \end{array}$$

$$d) \begin{array}{ccc} 00\text{---}0 \cdot 00\text{---}0 & 00\text{---}0 \cdot \begin{array}{c} 00\text{---}0100\text{---}010\text{---}0 \\ \xrightarrow{t} \\ \xrightarrow{u} \end{array} \end{array}$$

$$\begin{array}{ccc} 00\text{---}0 \cdot 00\text{---}0200\text{---}020\text{---}0 \\ \xrightarrow{t} \\ \xrightarrow{u} \end{array}$$

$$e) \begin{array}{ccc} 0\text{---}0 \cdot 00\text{---}0 & 00\text{---}0 \cdot \begin{array}{c} 00\text{---}0100\text{---}020\text{---}0 \\ \xrightarrow{t} \\ \xrightarrow{u} \end{array} \end{array}$$

$$\begin{array}{ccc} 00\text{---}0 \cdot 00\text{---}0200\text{---}010\text{---}0 \\ \xrightarrow{t} \\ \xrightarrow{u} \end{array}$$

$$f) \begin{array}{ccc} 00\text{---}010\text{---}0 \cdot 00\text{---}0 & \begin{array}{c} 00\text{---}010\text{---}0 \\ \xrightarrow{r} \end{array} \cdot \begin{array}{c} 00\text{---}010\text{---}0 \\ \xrightarrow{t} \end{array} \end{array}$$

$$\begin{array}{ccc} 00\text{---}010\text{---}0 \cdot 00\text{---}020\text{---}0 \\ \xrightarrow{r} \quad \xrightarrow{t} \end{array}$$

where $t < u$ and $(t, u = 1, 2, \dots, m$ and $r = 1, 2, \dots, n)$.

Emphasizing the idea of investigating factors one-at-a-time in blocks of small sizes to reduce the effect of time trends on the response, the OBAT 2_V^n , 3_V^m and $2^n 3_V^m$ plans in blocks of sizes 2, 3 and 2 and 3, respectively are carried out according to the following strategy recommended by Daniel (1973), but since we will be considering more than one blocking strategy we shall refer to this strategy as Strategy I.

Strategy I assumes that factors are ordered for their investigation and hence blocks are ordered accordingly. The blocks are then grouped such that the first group of blocks (stage 1) provides estimates for the main effect of factor F_1 and all two-factor interactions involving F_1 , the second group of blocks (stage 2) provides estimates for the main-effect of factor F_2 and all two-factor interactions involving F_2 (except the two-factor interactions involving F_1 and F_2 which have been estimated in stage 1), the third group of blocks (stage 3) provides estimates for the main-effect of factor F_3 and all two-factor interactions involving F_3 (except two-factor interactions involving F_1 and F_3 and two-factor interactions involving F_2 and F_3), until the last stage in which the main-effects of the last two factors and their two-factor interactions are investigated.

From the alias structure of Strategy I OBAT 2_{V}^{n} , 3_{V}^{m} and $2^{\text{n}}3_{\text{V}}^{\text{m}}$ plans in the various stages, we note that in stage i ($i = 1, 2, \dots$), main-effect of factor F_i is aliased with two-factor interactions involving F_i and F_j ($j > i$). By adding the blocks of stage i to the blocks of stage $(i-1)$ one block at a time, the two-factor interactions involving F_i become sequentially estimable in the following order: those involving F_i and F_{i+1} first then those involving F_i and F_{i+2} second, etc. Hence after the addition of the last block in stage i , the main-effect F_i is free from aliasing, i.e. estimable. It follows then that the alias chains at stage i are of larger size than those at stage j ($j > i$) hence the number of blocks in stage i is larger than the number of blocks in stage j . Formulas for the estimates of main-effects and two-factor interactions in strategy I OBAT 2_{V}^{n} , 3_{V}^{m} and $2^{\text{n}}3_{\text{V}}^{\text{m}}$ plans have also been provided.

In contrast to strategy I OBAT 2_V^n , 3_V^m and $2^n 3_V^m$ plans, we have suggested Strategy II OBAT 2_V^n , 3_V^m and $2^n 3_V^m$ plans in which the effects of a group of factors are investigated in a block where each block leads to the estimation of the main-effects and two-factor interactions involving the factors in a particular group. For this strategy, we have considered the following cases regarding the availability of prior information about the nature of two-factor interactions:

Case (1): All factors in the experiment interact with each other in which case the strategy II OBAT plan contains only one block and this block is a saturated non-orthogonal expansible-contractible resolution V design.

Case (2): The factors in the experiment can be grouped into g disjoint groups where all factors in a group interact with each other, in which case the strategy II OBAT plan is considered as a sequence of g blocks where each block is a saturated non-orthogonal expansible-contractible resolution V design.

Case (3): The factors can be grouped into three groups of $(g_1 + g_2 + 1)$ disjoint sets where

a. In each of the g_1 sets of the first group, all factors interact with each other.

b. In each of the g_2 sets of the second group, some factors interact with each other as well as with all other factors in that particular set.

c. All factors in the third group do not interact with each other.

Therefore, the entire experiment for case (3) is carried out in a sequence of $(g_1 + g_2 + 1)$ blocks such that

1. Each one of the g_1 blocks is a saturated non-orthogonal expandable-contractible resolution V design.

2. Each one of the g_2 blocks is a saturated non-orthogonal expandable-contractible design providing information on all main-effects and all non-negligible two-factor interactions for the factors in that particular block.

3. The last block in the sequence is a saturated expandable-contractible resolution III design.

Formulas for the estimates in the various blocks of Strategy II OBAT 2^n , 3^m and $2^n 3^m$ plans have been provided.

Relating the results of this dissertation with those available in the literature we find

a) for strategy I OBAT plans:

(i) Our OBAT 2_V^n plans are for general n thereby generalizing Daniel's (1973) specific example for the 2^4 factorial experiment in 12 blocks of size 2.

(ii) Our OBAT 2_V^n plans generalize also Pajak and Addelman's (1975) OBAT 2_V^n plans ($n \leq 17$). Our designs use smaller block sizes than theirs.

(iii) Although Cotter (1979) indicated that his 2^n designs can be carried out in blocks, he did not actually report the block structure nor did he consider the problem of determining what effects become estimable after a particular block. In our OBAT 2_V^n plans, we give more explicit recommendations and results.

(iv) Our OBAT 3_V^m plans differ from Hoke's (1974) OBAT 3_V^m plans.

Each one of Hoke's OBAT plans involve only three blocks (or non-orthogonal

3^m fractions) and each fraction is contained in the succeeding fraction. The smallest of these three fractions contain $(m+1)$ treatment combinations. In Hoke's OBAT 3^m plans, two-factor interactions involving quadratic effects were considered negligible whereas in our OBAT 3^m plans these two-factor interactions were considered non-negligible and formulas for their estimates have been provided. Hoke did not also consider the problem of expanding factorial designs by additional factors but we did.

(v) The results of Strategy I OBAT $2^n 3_V^m$ plans are all new. These results reduce to the results of Chapter 4 for 2^n experiments if $m = 0$ and to the results of Chapter 5 for 3^m experiments if $n = 0$.

(vi) Although Daniel (1962) and Pajak and Addelman (1975) considered the problem of breaking alias chains for their limited 2^n designs and Hoke (1974) provided a computer matrix algorithm for this problem for his 3^m designs, no general explicit solutions were provided. For all cases considered, we derive analytically solutions to this problem and give instructions for step-by-step augmentation as well as explicit formulas for estimates of effects and interactions.

(vii) Although we have used blocks of sizes 2 and 3 in our OBAT plans, blocks of larger sizes based on our 2_V^n , 3_V^m and $2^n 3_V^m$ designs can also be used, for example, blocks from which all effects involving a particular factor can be estimated all at once. Our 2_V^n , 3_V^m and $2^n 3_V^m$ designs can also be carried out in blocks of different sizes.

b) for Strategy II OBAT plans:

(i) For case (1), our OBAT 2_V^n plans parallel Cotter's (1979). Cotter's designs are for the estimation of factor's main-effects where

approximate formulas for assessing significance were given whereas the formulas we have reported for the estimates of main effects and two-factor interactions are exact. Of course, our plans include more runs than Cotter's.

(ii) For case (1), our OBAT 3_{V}^{m} plans parallel Campbell and Carr (1977) 3_{IV}^{m} designs. They did not provide formulas for the estimates of the linear and quadratic effects of each factor but we did provide formulas for estimates of main-effects and two-factor interactions.

(iii) For case (2) and (3) our OBAT 2^{n} , 3^{m} and $2^{\text{n}}3^{\text{m}}$ plans provide some new results.

c) for saturated non-orthogonal expansible-contractible designs:

The expansible-contractible 3_{V}^{m} and $2^{\text{n}}3_{\text{V}}^{\text{m}}$ we have reported in Chapters 5 and 6, respectively, are all new.

7.2 Areas for Future Research

The following extensions to the work of this dissertation are possible.

1. The construction of strict ORAT 3_{V}^{m} and $2^{\text{n}}3_{\text{V}}^{\text{m}}$ plans along the lines of the strict ORAT 2_{V}^{n} plans for Chapter 3. The possibility of blocking such designs can also be studied.

2. The derivation of formulas for parameter estimates of Campbell and Carr's (1977) 3_{IV}^{m} designs. Such formulas are needed if these designs are used for factor-screening.

3. The investigation of optimality properties of the 2_{V}^{n} and 3_{V}^{m} fractions of Chapters 4 and 5 whose entries of the $X'X$ matrix are listed

in Appendices A and B. Hoke's (1975) approach might be helpful in this regard.

4. The construction of resolution IV $2^n 3^m$ designs corresponding to Campbell and Carr's (1977) 3_{IV}^m designs and the derivation of formulas for parameter estimates.

5. The generation of the entries of the information matrix $X'X$ of the $2^n 3_y^m$ fraction of Chapter 6 and the study of its optimality properties.

6. The possibility of programming all the designs we have reported in this dissertation so that they are easily available for practical applications.

APPENDIX A

This appendix contains the entries of the information matrix $X'X$ of Webb's (1965) resolution $V \langle m_1, m_2, \dots, m_k \rangle$ - designs for the 2^n experiments (see Section 3.1). Each entry of $X'X$ is the inner product of two columns of design matrix X in (3.1) relating expected responses to the overall mean μ , the n main effects A_r ($r = 1, 2, \dots, n$) and the $\frac{n(n-1)}{2}$ two-factor interactions $A_r A_s$ ($r < s$ and $r, s = 1, 2, \dots, n$). The entry in the i^{th} row and j^{th} column of $X'X$ is denoted by (P_i, P_j) where P_i and P_j represent any one of the $1 + n + \frac{n(n-1)}{2}$ effects above. Webb (1965) found out that the matrix $X'X$ of the $\langle m_1, m_2, \dots, m_k \rangle$ - designs contains only five distinct entries and they are

$$1. \quad (\mu, \mu) = (A_i, A_i) = (A_i A_j, A_i A_j) = \sum_{i=1}^k \binom{n}{m_i}$$

$$(i = 1, 2, \dots, n \text{ and } i \neq j)$$

$$2. \quad (\mu, A_i) = (A_i, A_i A_j) = \sum_{i=1}^k \left[\binom{n-1}{m_i-1} - \binom{n-1}{m_i} \right]$$

$$(i = 1, 2, \dots, n \text{ and } i \neq j)$$

$$3. \quad (\mu, A_i A_j) = (A_i, A_j) = (A_i A_j, A_i A_k) = (A_i A_j, A_l A_j)$$

$$= \sum_{i=1}^k \left[\binom{n-2}{m_i-2} - 2 \binom{n-2}{m_i-1} + \binom{n-2}{m_i} \right]$$

$$(i \neq j, j \neq k \text{ and } l \neq i)$$

$$4. \quad (A_i, A_j A_k) = \sum_{i=1}^k \left[\binom{n-3}{m_i-3} - 3 \binom{n-3}{m_i-2} + 3 \binom{n-3}{m_i-1} - \binom{n-3}{m_i} \right]$$

$$(i \neq j \neq k)$$

$$5. (A_i A_j, A_k A_l) = \sum_{i=1}^k \left[\binom{n-4}{m_i-4} - 4 \binom{n-4}{m_i-3} + 6 \binom{n-4}{m_i-2} - 4 \binom{n-4}{m_i-1} + \binom{n-4}{m_i} \right]$$

(i ≠ j ≠ k ≠ l)

(where $\binom{p}{q}$ is defined to be zero if $q < 0$ or $p < q$.)

For the $\langle 0, 1, 2 \rangle - 2^n$ designs which we have studied in Chapter 4, the entries of the $X'X$ matrix can be obtained by plugging in $m_1 = 0$, $m_2 = 1$, $m_3 = 2$ and $k = 3$ in the above formulas.

APPENDIX B

This appendix contains the entries of the information matrix $X'X$ of the saturated expandable-contractible 3^m_v design in (5.2). Each entry of $X'X$ is the inner product of two columns of the design matrix X in (5.1) relating expected responses to the overall mean μ , the $2m$ linear and quadratic effects B_t and B_t^2 ($t = 1, 2, \dots, m$) and the $\frac{4m(m-1)}{2}$ two-factor interactions $B_t B_u$, $B_t^2 B_u$, $B_t B_u^2$ and $B_t^2 B_u^2$ ($t < u$ and $t, u = 1, 2, \dots, m$). The entry in the i^{th} row and the j^{th} column of $X'X$ is denoted by (P_i, P_j) where P_i and P_j represent any one of the $1 + 2m + \frac{4m(m-1)}{2}$ effects above.

In generating the entries of the $X'X$ matrix, we make use of Srivastava and Anderson's (1969) frequency operator f . This operator operates on the symbols $b_t^{i_t}$ of the treatment combinations $b_1^{i_1} b_2^{i_2} \dots b_m^{i_m}$ which are regarded as ordinary products of $b_t^{i_t}$ symbols ($t = 1, 2, \dots, m$). The quantity $f(b_t^{i_t})$ represents the number of times the $(b_t^{i_t})^{\text{th}}$ level of t^{th} factor B_t occurs in the fraction F_m ($i_t = 0, 1, \text{ and } 2$). The quantity $f(b_t^{i_t} b_u^{i_u})$ represents the number of times the $(b_t^{i_t})^{\text{th}}$ level of factor B_t and the $(b_u^{i_u})^{\text{th}}$ level of factor B_u occur together in the fraction F_m ($i_t, i_u = 0, 1, 2$). The quantity $f(b_1^{i_1} b_2^{i_2} \dots b_m^{i_m})$ represents the number of times the levels $b_1^{i_1}, b_2^{i_2}, \dots, b_m^{i_m}$ of the m factors B_1, B_2, \dots, B_m occur together in the fraction F_m ($i_j = 0, 1, 2$ and $j = 1, 2, \dots, m$). Furthermore, since each treatment combination does not include more than one level of a certain factor we must have

1. $b_t^{i_t} b_t^{i'_t} = \begin{cases} b_t^{i_t} & \text{if } i_t = i'_t \\ \phi & \text{if } i_t \neq i'_t \end{cases}$
2. $(b_t^0 + b_t^1 + b_t^2) b_t^{i_t} = b_t^0 b_t^{i_t} + b_t^1 b_t^{i_t} + b_t^2 b_t^{i_t} = b_t^{i_t}$
3. $b_t^{i_t} + \phi = b_t^{i_t}$

That is, $I = b_t^0 + b_t^1 + b_t^2$ and ϕ are the multiplicative and the additive identities for the symbols $b_t^{i_t}$, respectively.

Since the fraction F_m in (5.2) contains $N = 1 + 2m^2$ treatment combinations, we must also have for any one factor that

$$f(b_t^0 + b_t^1 + b_t^2) = N = 1 + 2m^2$$

and

$$f(\phi) = \text{zero} .$$

Using the frequency operator f above, Srivastava and Anderson (1969) concluded that the entries of the $X'X$ matrix, (P_i, P_j) , for resolution $V 3^m$ fractions can be generated from the formula

$$(P_i, P_j) = f \prod_{k=1}^4 [(c_0(i_k) b_{i_k}^0 + c_1(i_k) b_{i_k}^1 + c_2(i_k) b_{i_k}^2)]$$

where c_0 , c_1 and c_2 are the coefficients of the orthogonal polynomial contrasts

$$\begin{array}{lll} c_0(0) = 1 & c_0(1) = -1 & c_0(2) = 1 \\ c_1(0) = 1 & c_1(1) = 0 & c_1(2) = -2 \\ c_2(0) = 1 & c_2(1) = 1 & c_2(2) = 1 \end{array}$$

The index k ranges from 1 to 4 depending on whether the inner product (P_i, P_j) involves 1 up to 4 factorial effects. For an illustration of the formula we note that

$$\begin{aligned}
 \text{a. } (B_t^2, B_u B_w) &= f[(b_t^0 - 2b_t^1 + b_t^2)(b_u^2 - b_u^0)(b_w^2 - b_w^0)] \\
 &= f(b_t^0 - 2b_t^1 + b_t^2)(b_u^2 b_w^2 - b_u^2 b_w^0 - b_u^0 b_w^2 + b_u^0 b_w^0) \\
 &= f(b_t^0 b_u^2 b_w^2) - f(b_t^0 b_u^2 b_w^0) - f(b_t^0 b_u^0 b_w^2) + f(b_t^0 b_u^0 b_w^0) \\
 &\quad - 2f(b_t^1 b_u^2 b_w^2) + 2f(b_t^1 b_u^2 b_w^0) + 2f(b_t^1 b_u^0 b_w^2) - 2f(b_t^1 b_u^0 b_w^0) \\
 &\quad + f(b_t^2 b_u^2 b_w^2) - f(b_t^2 b_u^2 b_w^0) - f(b_t^2 b_u^0 b_w^2) + f(b_t^2 b_u^0 b_w^0) \\
 \text{b. } (\mu, B_t) &= f(b_t^2 - b_t^0) = f(b_t^2) - f(b_t^0) \\
 \text{c. } (B_t^2, B_t B_w) &= f[(b_t^0 - 2b_t^1 + b_t^2)(b_t^2 - b_t^0)(b_w^2 - b_w^0)] \\
 &= f(b_t^0 b_t^2 + b_t^0 b_t^0 - 2b_t^1 b_t^2 + 2b_t^1 b_t^0 + b_t^2 b_t^2 - b_t^2 b_t^0)(b_w^2 - b_w^0) \\
 &= f(\phi + b_t^0 - \phi + \phi + b_t^2 - \phi)(b_w^2 - b_w^0) \\
 &= f(b_t^2 + b_t^0)(b_w^2 - b_w^0) \\
 &= f(b_t^2 b_w^2) - f(b_t^2 b_w^0) + f(b_t^0 b_w^2) - f(b_t^0 b_w^0)
 \end{aligned}$$

where $t \neq u \neq w$.

Table (4) gives the frequency for which 1 up to 4 factors occur together in the fraction F_m in (5.2). Due to the permutation invariance of the fraction F_m and without loss of generality we consider the first four factors in constructing this frequency table where $\pi(\)$

TABLE 4. The Frequency for which the first four factors of the fraction F_m in (5.2) occur together one at a time, two at a time, three at a time and four at a time.

No. of factors	Factor level combinations	Frequency
1	0	$f(0) = 1 + 2m^2 - 2(2m-1) = 2m^2 - 4m + 3$
	1	$f(1) = 1 + (m-1) + (m-1) = 2m - 1$
	2	$f(2) = 1 + (m-1) + (m-1) = 2m - 1$
2	10 and 01	$f(10) = f(01) = 1 + 2[(m-1)-1] = 2m - 3$
	11	$f(11) = 1$
	12 and 21	$f(12) = f(21) = 1$
	00	$f(00) = 1 + 2(m-2)$
	20 and 02	$f(20) = 1 + 2[(m-1)-1] = 2m - 3$
	22	$f(22) = 1$
3	$\pi(100)$	$f(100) = f(010) = f(001) = 1 + 2[(m-1)-2] = 2m-5$
	$\pi(110)$	$f(110) = f(101) = f(011) = 1$
	$\pi(120)$	$f(120) = f(102) = f(012) = f(210) = f(201) = f(021) = 1$
	000	$f(000) = 1 + 2(m-3) + 4\left[\frac{m(m-1)}{2} - (m-1) - (m-2) - (m-3)\right] = 2m^2 - 12m + 19$
	$\pi(200)$	$f(200) = f(020) = f(002) = 2m - 5$
	$\pi(220)$	$f(220) = f(202) = f(022) = 1$
4	$\pi(1000)$	$f(1000) = 1 + 2[(m-1)-3] = 2m - 7$
	$\pi(1100)$	$f(1100) = 1$
	$\pi(1200)$	$f(1200) = 1$
	0000	$f(0000) = 1 + 2(m-4) + 4\left[\frac{m(m-1)}{2} - (m-1) - (m-2) - (m-3) - (m-4)\right] = 2m^2 - 16m + 33$
	$\pi(2000)$	$f(2000) = 2m - 7$
	$\pi(2200)$	$f(2200) = 1$

represents all permutations of the treatment combination inside the bracket. From Srivastava and Anderson's formula and from Table (4) we find that the entries of the $X'X$ matrix for the saturated resolution $V 3^m$ fraction in (5.2) are

$$1. (\mu, \mu) = 1 + 2m^2$$

$$2. (\mu, B_t) = (B_t, B_t^2) = -2m^2 + 6m - 4$$

$$3. (\mu, B_t^2) = -(\mu, B_t)$$

$$\begin{aligned} 4. (\mu, B_t B_u) &= (\mu, B_t^2 B_u^2) = (B_t, B_u) = (B_t, B_t^2 B_u) = (B_t^2, B_u^2) \\ &= (B_t^2, B_t B_u) = (B_t B_u, B_t^2 B_u^2) = (B_t B_u^2, B_t^2 B_u) \\ &= 2m^2 - 12m + 16 \end{aligned}$$

$$\begin{aligned} 5. (\mu, B_t B_u^2) &= (\mu, B_t^2 B_u) = (B_t, B_u^2) = (B_t, B_t^2 B_u^2) = (B_t^2, B_t B_u^2) \\ &= -(\mu, B_t B_u) \end{aligned}$$

$$6. (B_t, B_t) = 2m^2 - 2m + 2$$

$$\begin{aligned} 7. (B_t, B_t B_u) &= (B_t B_u, B_t B_u^2) = (B_t B_u, B_t^2 B_u) \\ &= -2m^2 + 8m - 8 \end{aligned}$$

$$8. (B_t, B_t B_u^2) = -(B_t, B_t B_u)$$

$$\begin{aligned} 9. (B_t, B_u B_w) &= (B_t, B_u^2 B_w^2) = (B_t^2, B_u B_w) = (B_t B_u, B_w B_u^2) \\ &= (B_t B_u^2, B_t^2 B_w^2) = (B_t^2, B_u B_w^2) = (B_t^2, B_u^2 B_w) \end{aligned}$$

$$\begin{aligned}
 &= (B_t B_u, B_w^2 B_u) = (B_t^2 B_u, B_w^2 B_u^2) \\
 &= -2m^2 + 18m - 37
 \end{aligned}$$

$$\begin{aligned}
 10. \quad (B_t, B_u B_w^2) &= (B_t, B_u^2 B_w) = (B_t^2, B_u^2 B_w^2) = (B_t^2 B_u, B_t^2 B_w) \\
 &= (B_t B_u^2, B_w^2 B_u) = (B_t B_u, B_w^2 B_u^2) = (B_t B_u, B_t^2 B_w^2) \\
 &= (B_t B_u^2, B_t^2 B_w) \\
 &= -(B_t, B_u B_w)
 \end{aligned}$$

$$11. \quad (B_t^2, B_t^2) = 2m^2 + 6m - 2$$

$$\begin{aligned}
 12. \quad (B_t^2, B_t^2 B_u) &= (B_t B_u^2, B_t^2 B_u) = (B_t^2 B_u, B_t^2 B_u^2) \\
 &= -2m^2 + 8
 \end{aligned}$$

$$13. \quad (B_t^2, B_t^2 B_u^2) = -(B_t^2, B_t^2 B_u)$$

$$14. \quad (B_t B_u, B_t B_u) = 2m^2 - 4m + 4$$

$$\begin{aligned}
 15. \quad (B_t B_u, B_w B_u) &= (B_t B_u, B_t B_w) = (B_t B_u^2, B_t B_w^2) = (B_t^2 B_u, B_w^2 B_u) \\
 &= 2m^2 - 14m + 23
 \end{aligned}$$

$$16. \quad (B_t B_u, B_t B_w^2) = -(B_t B_u, B_t B_w)$$

$$\begin{aligned}
 17. \quad (B_t B_u, B_v B_w) &= (B_t B_u, B_v^2 B_w) = (B_t B_u, B_v^2 B_w^2) = (B_t^2 B_u, B_v^2 B_w) \\
 &= (B_t B_u^2, B_v^2 B_w) = (B_t B_u^2, B_v^2 B_w) = (B_t^2 B_u, B_v^2 B_w^2) \\
 &= 2m^2 - 24m + 67
 \end{aligned}$$

$$18. \quad (B_t B_u, B_v B_w^2) = (B_t B_u, B_t^2 B_w) = (B_t B_u^2, B_v^2 B_w^2) = (B_t^2 B_u, B_v^2 B_w^2) \\ = -(B_t B_u, B_v B_w)$$

$$19. \quad (B_t B_u^2, B_t B_u^2) = (B_t^2 B_u, B_t^2 B_u) = 2m^2 + 4m - 4$$

$$20. \quad (B_t B_u^2, B_w B_u^2) = 2m^2 - 6m - 5$$

$$21. \quad (B_t B_u^2, B_w^2 B_u^2) = (B_t^2 B_u, B_t^2 B_w^2) = (B_t^2 B_u^2, B_w^2 B_u^2) \\ = (B_t^2 B_u^2, B_t^2 B_w^2) \\ = -(B_t B_u^2, B_w B_u^2)$$

$$22. \quad (B_t^2 B_u^2, B_t^2 B_u^2) = 2m^2 + 12m + 4$$

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