GEOMETRICALLY NONLINEAR ANALYSIS OF
COMPOSITE LAMINATES USING A REFINED SHEAR
DEFORMATION SHELL THEORY

by

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ABSTRACT

The theory is based on an assumed displacement field, in which the surface displacements are expanded in powers of the thickness coordinate up to the third order. The theory allows parabolic description of the transverse shear stresses, and therefore the shear correction factors of the usual shear deformation theory are not required in the present theory. The theory accounts for small strains but moderately large displacements (i.e., von Karman strains). Exact solutions for certain cross-ply shells and finite-element models of the theory are also developed. The finite-element model is based on independent approximations of the displacements and bending moments (i.e., mixed formulation), and therefore only C⁰-approximations are required. Further, the mixed variational formulations developed herein suggest that the bending moments can be interpolated using discontinuous approximations (across interelement boundaries). The finite element is used to analyze cross-ply and angle-ply laminated shells for bending, vibration and transient response. Numerical results are presented to show the effects of boundary conditions, lamination scheme (i.e., bending-stretching coupling and material anisotropy) shear deformation and geometric nonlinearity on deflections and frequencies. Many of the numerical results presented here for laminated shells should serve as references for future investigations.
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CHAPTER 1

INTRODUCTION

1.1 Background

The analyses of composite laminates in the past have been based on one of the following two classes of theories:

(i) Three-dimensional elasticity theory
(ii) Laminated plate theories

In three-dimensional elasticity theory each layer is treated as an elastic continuum with possibly distinct material properties in adjacent layers. Thus the number of governing differential equations will be $3N$, where $N$ is the number of layers in the laminate. At the interface of two layers the continuity of displacements and stresses give additional relations. Solution of the equations becomes intractable as the number of layers becomes large.

In a 'laminated plate theory', it is assumed that the laminate is in a state of plane stress, the individual laminae are elastic, and there is perfect bonding between layers. The laminate properties (i.e. stiffnesses) are obtained by integrating the lamina properties through the thickness. Thus, laminate plate theories are equivalent single-layer theories. In the 'classical laminate plate theory' (CLPT), which is an extension of the classical plate theory (CPT) to laminated plates, the transverse stress components are ignored. The classical laminate plate theory is adequate for many engineering problems. However, laminated plates made of advanced filamentary composite materials, whose elastic to shear modulus ratios are very large, are susceptible to thickness effects because their effective transverse shear moduli are
significantly smaller than the effective elastic moduli along fiber directions. These high ratios of elastic to shear modulus render the classical laminated plate theory inadequate for the analysis of thick composite plates.

The first, stress-based, shear deformation plate theory is due to Reissner [1-3]. The theory is based on a linear distribution of the inplane normal and shear stresses through the thickness,

\[
\sigma_1 = \frac{M_1}{(h^2/6)} \frac{z}{(h/2)}, \quad \sigma_2 = \frac{M_2}{(h^2/6)} \frac{z}{(h/2)}, \quad \sigma_6 = \frac{M_6}{(h^2/6)} \frac{z}{(h/2)} \quad (1.1)
\]

where \((\sigma_1, \sigma_2)\) and \(\sigma_6\) are the normal and shear stresses, \((M_1, M_2)\) and \(M_6\) are the associated bending moments (which are functions of the inplane coordinates \(x\) and \(y\)), \(z\) is the thickness coordinate and \(h\) is the total thickness of the plate. The distribution of the transverse normal and shear stresses \((\sigma_3, \sigma_4\) and \(\sigma_5\)\) is determined from the equilibrium equations of the three-dimensional elasticity theory. The differential equations and the boundary conditions of the theory were obtained using Castigliano's theorem of least work.

The origin of displacement-based theories is apparently attributed to Basset [4]. Basset assumed that the displacement components in a shell can be expanded in a series of powers of the thickness coordinate \(\zeta\). For example, the displacement component \(u_1\) along the \(\xi_1\) coordinate is written in the form

\[
u_1(\xi_1, \xi_2, \zeta) = u_1^0(\xi_1, \xi_2) + \sum \zeta^n u_1^{(n)}(\xi_1, \xi_2) \quad (1.2a)
\]

where \(\xi_1\) and \(\xi_2\) are the curvilinear coordinates in the middle surface of the shell, and \(u_1^{(n)}\) have the meaning
Basset's work did not receive as much attention as it deserves. In a 1949 NACA technical note, Hildebrand, Reissner and Thomas [5] presented (also see Hencky [6]) a displacement-based shear deformation theory for shells, which obviously can be specialized to flat plates. They assumed the following displacement field,

\[ u_1(\xi_1, \xi_2, \zeta) = u(\xi_1, \xi_2) + \zeta \phi_x(\xi_1, \xi_2), \]

\[ u_2(\xi_1, \xi_2, \zeta) = v(\xi_1, \xi_2) + \zeta \phi_y(\xi_1, \xi_2), \]

\[ u_3(\xi_1, \xi_2, \zeta) = w(\xi_1, \xi_2) \] (1.3)

The differential equations of the theory are then derived using the principle of minimum total potential energy. This gives five differential equations in the five displacement functions, \( u, v, w, \phi_x, \) and \( \phi_y. \)

The shear deformation theory based on the displacement field in Eq. (1.3) for plates is often referred to as the Mindlin's plate theory. Mindlin [7] presented a complete theory for isotropic plates based on the displacement field (1.3) taken from Hencky [6]. We shall refer to the shear deformation theory based on the displacement field (1.3) as the first-order shear deformation theory.

A generalization of the first-order shear deformation plate theory for homogeneous isotropic plates to arbitrarily laminated anisotropic plates is due to Yang, Norris, and Stavsky [8] and Whitney and Pagano [9]. Extensions of the classical von Karman plate theory [10] to shear
deformation theories can be found in the works of Reissner \([11,12]\) Medwadowski \([13]\), Schmidt \([14]\) and Reddy \([15]\).

Higher-order, displacement-based, shear deformation theories have been investigated by Librescu \([16]\) and Lo, Christensen and Wu \([17]\). In these higher-order theories, with each additional power of the thickness coordinate an additional dependent unknown is introduced into the theory. Levinson \([18]\) and Murthy \([19]\) presented third-order theories that assume transverse inextensibility. The nine displacement functions were reduced to five by requiring that the transverse shear stresses vanish on the bounding planes of the plate. However, both authors used the equilibrium equations of the first-order theory in their analysis. Stating in other words, the higher-order terms of the displacement field are accounted for in the calculation of the strains but not in the governing differential equations or in the boundary conditions. These theories can be shown (see Librescu and Reddy \([20]\)) to be the same as those described by Reissner \([1-3]\). Recently, Reddy \([21,22]\) developed a new third-order plate theory, which is extended in the present study to laminated shells.

1.2 Review of the Literature on Shells

Shell structures are abundant in structures around us on the earth and in space. Use of shell structures dates back to ancient Rome, where the roofs of the Pantheon can be classified today as thick shells. Shell structures have been built by experience and intuition. No logical and scientific study has been conducted on the design of shells until the eighteenth century.
The earliest need for design criterion for shell structures probably came with the development of the steam engine and its attendant accessories. However, it was not until 1888, by Love, that the first general theory was presented. Subsequent theoretical efforts have been directed towards improvements of Love's formulation and the solutions of the associated differential equations.

An ideal theory of shells require a realistic modeling of the actual geometry and material properties, and an appropriate description of the deformation. Even if we can take care of these requirements and set up the governing equations, analytical solutions to most shell problems are nevertheless limited in scope, and in general do not apply to arbitrary shapes, load distributions, and boundary conditions. Consequently, numerical approximation methods must be used to predict the actual behavior.

Many of the classical theories were developed originally for thin elastic shells, and are based on the Love-Kirchhoff assumptions (or the first approximation theory): (1) plane sections normal to the undeformed middle surface remain plane and normal to the deformed middle surface, (2) the normal stresses perpendicular to the middle surface can be neglected in the stress-strain relations, and (3) the transverse displacement is independent of the thickness coordinate. The first assumption leads to the neglect of the transverse shear strains. Surveys of various classical shell theories can be found in the works of Naghdi [23] and Bert [24,25]. These theories, known as the Love's first approximation theories (see Love [26]) are expected to yield sufficiently accurate results when (i) the lateral dimension-to-
thickness ratio \((a/h)\) is large; (ii) the dynamic excitations are within the low-frequency range; (iii) the material anisotropy is not severe. However, application of such theories to layered anisotropic composite shells could lead to as much as 30% or more errors in deflections, stresses, and frequencies.

As pointed out by Koiter [27], refinements to Love's first approximation theory of thin elastic shells are meaningless, unless the effects of transverse shear and normal stresses are taken into account in the refined theory. The transverse normal stress is, in general, of order \(h/R\) (thickness to radius ratio) times the bending stresses, whereas the transverse shear stresses, obtained from equilibrium conditions, are of order \(h/a\) (thickness to the length of long side of the panel) times the bending stresses. Therefore, for \(a/R < 10\), the transverse normal stress is negligible compared to the transverse shear stresses.

Ambartsumyan [28,29] was considered to be the first one to analyze laminates that incorporated the bending-stretching coupling. The laminates that Ambartsumyan analyzed were now known as laminated orthotropic shells because the individual orthotropic layers were oriented such that the principal axes of material symmetry coincided with the principal coordinates of the shell reference surface. In 1962, Dong, Pister and Taylor [30] formulated a theory of thin shells laminated of anisotropic shells. Cheng and Ho [31] presented an analysis of laminated anisotropic cylindrical shells using Flugge's shell theory [32]. A first approximation theory for the unsymmetric deformation of nonhomogeneous, anisotropic, elastic cylindrical shells
was derived by Widera and his colleagues [33,34] by means of asymptotic integration of the elasticity equation. For a homogeneous, isotropic material, the theory reduced to the Donnell's equation. An exposition of various shell theories can be found in the article by Bert [35].

The effect of transverse shear deformation and transverse isotropy as well as thermal expansion through the shell thickness were considered by Gulati and Essenber [36] and Zukas and Vinson [37]. Dong and Tso [38] presented a theory applicable to layered, orthotropic cylindrical shells. Whitney and Sun [39] developed a higher order shear deformation theory. This theory is based on a displacement field in which the displacements in the surface of the shell are expanded as linear functions of the thickness coordinate and the transverse displacement is expanded as a quadratic function of the thickness coordinate. Recently, Reddy [40] presented a shear deformation version of the Sanders shell theory for laminated composite shells. Such theories account for constant transverse shear stresses through thickness, and therefore require a correction to the transverse shear stiffness.

As far as the finite element of shells is concerned, the early works can be attributed to those of Dong [41], Dong and Selna [42], Wilson and Parsons [43], and Schmit and Monforton [44]. The studies in [41-44] are confined to the analysis of orthotropic shells of revolution. Other finite element analyses of laminated anisotropic composite shells include the works of Panda and Natarajan [45], Shivakumar and Krishna Murty [46], Rao [47], Seide and Chang [48], Venkatesh and Rao [49], Reddy and his colleagues [50-53], and Noor and his colleagues [54-56].
1.3 Present Study

While the three-dimensional theories give more accurate results than the lamination (classical or (see Seide [57], Srinivas et al. [58,59] and Pagano, et al. [60-61]) shear deformation) theories, they are intractable. For example, the 'local' theory of Pagano [62] results in a mathematical model consisting of 23N partial differential equations in the laminate's midplane coordinates and 7N edge boundary conditions, where N is the number of layers in the laminate. The computational costs, especially for geometrically nonlinear problems or transient analysis using the finite element method, preclude the use of such a theory. As demonstrated by Reddy [21,22] and his colleagues [63-65], the refined plate theory provides improved global response estimates for deflections, vibration frequencies and buckling loads for laminated composite plates. The present study, motivated by the above findings, deals with the extention of the third-order plate theory of Reddy [21,22] to laminated composite shells. The theory also accounts for the von Karman strains. The resulting theory contains, as special cases, the classical and first-order theories of plates and shells. Mixed variational formulations and associated finite-element models are developed in this dissertation. The significant and novel contributions of the research conducted for this dissertation are:

1. The formulation of a new third-order theory of laminated shells that accounts for a parabolic distribution of the transverse shear stresses and the von Karman strains.
2. The derivation of the exact solutions of the new theory for certain simply supported laminated composite shells.

3. The development of a mixed variational principle for the new theory of shells, which yields as special cases those of the classical (e.g., Love-Kirchhoff) and the first order theory.

4. The development of a mixed, C^0-finite-element and its application to the bending, vibration and transient analysis of laminated composite shells.

The theoretical formulation of the new shell theory is presented in Chapter 2, and the exact solutions for certain shells are presented in Chapter 3. Chapter 4 contains the development of a mixed variational principle, and the associated finite element model is described in Chapter 5. Chapter 6 contains the discussion of numerical results of the bending, vibration and transient analysis of laminated composite plates and shells. Comparison of the present results with the experimental results for certain unidirectional and bidirectional composites show excellent agreement. Many additional numerical results of laminated composite shells are also included, which should serve as references for future investigations. A summary, conclusions and recommendations of the present study are presented in Chapter 7.
CHAPTER 2

FORMULATION OF THE NEW THEORY

2.1 Kinematics

Let \((\xi_1, \xi_2, \varsigma)\) denote the orthogonal curvilinear coordinates (or shell coordinates) such that the \(\xi_1\)- and \(\xi_2\)- curves are lines of curvature on the midsurface \(\varsigma=0\), and \(\xi\)- curves are straight lines perpendicular to the surface \(\varsigma=0\). For cylindrical and spherical shells the lines of principal curvature coincide with the coordinate lines. The values of the principal radii of curvature of the middle surface are denoted by \(R_1\) and \(R_2\).

The position vector of a point on the middle surface is denoted by \(\mathbf{r}\), and the position of a point at distance \(\varsigma\) from the middle surface is denoted by \(\mathbf{R}\). The distance \(ds\) between points \((\xi_1, \xi_2, 0)\) and \((\xi_1+d\xi_1, \xi_2+d\xi_2, 0)\) is determined by (see Fig. 2.1)

\[
(ds)^2 = dr \cdot dr
\]

\[
= a_1^2(d\xi_1)^2 + a_2^2(d\xi_2)^2
\]

(2.1)

where \(dr = g_1d\xi_1 + g_2d\xi_2\), the vectors \(g_1\) and \(g_2\) (\(g_j = \frac{\partial \mathbf{r}}{\partial \xi_j}\)) are tangent to the \(\xi_1\) and \(\xi_2\) coordinate lines, and \(a_1\) and \(a_2\) are the surface metrics

\[
a_1^2 = g_1 \cdot g_1, \quad a_2^2 = g_2 \cdot g_2.
\]

(2.2)

The distance \(dS\) between points \((\xi_1, \xi_2, \varsigma)\) and \((\xi_1+d\xi_1, \xi_2+d\xi_2, \varsigma+d\varsigma)\) is given by

\[
(dS)^2 = d\mathbf{r} \cdot d\mathbf{r}
\]

\[
= L_1^2(d\xi_1)^2 + L_2^2(d\xi_2)^2 + L_3^2(d\varsigma)^2
\]

(2.3)
Figure 2.1  Geometry and stress resultants of a shell
where \( \frac{dR}{dz} = \frac{\partial R}{\partial \xi_1} d\xi_1 + \frac{\partial R}{\partial \xi_2} d\xi_2 + \frac{\partial R}{\partial \zeta} d\zeta \), and \( L_1, L_2 \) and \( L_3 \) are the Lame' coefficients

\[
L_1 = \alpha_1 \left( 1 + \frac{\xi}{R_1} \right), \quad L_2 = \alpha_2 \left( 1 + \frac{\xi}{R_2} \right), \quad L_3 = 1
\]

(2.4)

It should be noted that the vectors \( \mathbf{G}_1 \) and \( \mathbf{G}_2 \) are parallel to the vectors \( \mathbf{g}_1 \) and \( \mathbf{g}_2 \), respectively.

### 2.2 Displacement Field

The shell under consideration is composed of a finite number of orthotropic layers of uniform thickness (see Fig. 2.2). Let \( N \) denote the number of layers in the shell, and \( \zeta_k \) and \( \zeta_{k-1} \) be the top and bottom \( \zeta \)-coordinates of the \( k \)-th layer. Before we proceed, a set of simplifying assumptions that provide a reasonable description of the behaviors are as follows:

1. thickness to radius and other dimensions of shell are small.
2. transverse normal stress is negligible
3. strains are small, yet displacement can be moderately large compared to thickness

Following the procedure similar to that presented in [21] for flat plates, we begin with the following displacement field:

\[
\mathbf{U}(\xi_1, \xi_2, \zeta, t) = (1 + \frac{\xi}{R_1})u + \zeta \phi_1 + \zeta^2 \psi_1 + \zeta^3 \theta_1
\]

\[
\mathbf{V}(\xi_1, \xi_2, \zeta, t) = (1 + \frac{\xi}{R_2})v + \zeta \phi_2 + \zeta^2 \psi_2 + \zeta^3 \theta_2
\]

\[
\mathbf{W}(\xi_1, \xi_2, \zeta, t) = w
\]

(2.5)

where \( t \) is time, \((\mathbf{U}, \mathbf{V}, \mathbf{W})\) are the displacements along the \((\xi_1, \xi_2, \zeta)\)
Figure 2.2 Geometry of a laminated doubly-curved shell
coordinates, \( (u,v,w) \) are the displacements of a point on the middle surface and \( \phi_1 \) and \( \phi_2 \) are the rotations at \( \zeta = 0 \) of normals to the midsurface with respect to the \( \xi_2 \) and \( \xi_1 \) -axes, respectively. All of \( (u,v,w,\phi_1,\phi_2,\psi_1,\psi_2,\theta_1,\theta_2) \) are functions of \( \xi_1 \) and \( \xi_2 \) only. The particular choice of the displacement field in Eq. (2.5) is dictated by the desire to represent the transverse shear strains by quadratic functions of the thickness coordinate, \( \zeta \), and by the requirement that the transverse normal strain be zero. The kinematics of deformation of a transverse normal in various theories is shown in Fig. 2.3.

The functions \( \psi_i \) and \( \theta_i \) will be determined using the condition that the transverse shear stresses, \( \sigma_{13} = \sigma_5 \) and \( \sigma_{23} = \sigma_4 \) vanish on the top and bottom surfaces of the shell:

\[
\sigma_5(\xi_1,\xi_2, \pm \frac{h}{2}, t) = 0 \quad \sigma_4(\xi_1,\xi_2, \pm \frac{h}{2}, t) = 0 \quad (2.6)
\]

These conditions are equivalent to, for shells laminated of orthotropic layers, the requirement that the corresponding strains be zero on these surfaces. The transverse shear strains of a shell with two principal radii of curvature are given by

\[
\epsilon_5 = \frac{3u}{3\zeta} + \frac{1}{\alpha_1} \frac{3w}{3\xi_1} - \frac{u}{R_1}
\]

\[
= \frac{u}{R_1} + \phi_1 + 2\zeta \psi_1 + 3\zeta^2 \theta_1 + \frac{1}{\alpha_1} \frac{3w}{3\xi_1} - \frac{u}{R_1}
\]

\[
\epsilon_4 = \frac{3v}{3\zeta} + \frac{1}{\alpha_2} \frac{3w}{3\xi_2} - \frac{v}{R_2}
\]

\[
= \frac{v}{R_2} + \phi_2 + 2\zeta \psi_2 + 3\zeta^2 \theta_2 + \frac{1}{\alpha_2} \frac{3w}{3\xi_2} - \frac{v}{R_2}
\quad (2.7)
\]

Setting \( \epsilon_5(\xi_1,\xi_2, \pm \frac{h}{2}, t) \) and \( \epsilon_4(\xi_1,\xi_2, \pm \frac{h}{2}, t) \) to zero, we obtain
Figure 2.3 Assumed deformation patterns of the transverse normals in various displacement-based theories.
Substituting Eq. (2.8) into Eq. (2.5), we obtain

\[
\bar{u} = (1 + \frac{\zeta}{R_1})u + \zeta \phi_1 + \zeta^3 \frac{4}{3h^2} \left[ -\phi_1 - \frac{1}{a_1} \frac{aw}{\xi_1} \right]
\]

\[
\bar{v} = (1 + \frac{\zeta}{R_2})v + \zeta \phi_2 + \zeta^3 \frac{4}{3h^2} \left[ -\phi_2 - \frac{1}{a_2} \frac{aw}{\xi_2} \right]
\]

(2.9)

This displacement field is used to compute the strains and stresses, and then the equations of motion are obtained using the dynamic analog of the principle of virtual work.

2.3 Strain-Displacement Relations

Substituting Eq. (2.9) into the strain-displacement relations referred to an orthogonal curvilinear coordinate system, we obtain

\[
\varepsilon_1 = \varepsilon^0_1 + \zeta \kappa^0_1 + \zeta^2 \kappa^2_1
\]

\[
\varepsilon_2 = \varepsilon^0_2 + \zeta \kappa^0_2 + \zeta^2 \kappa^2_2
\]

\[
\varepsilon_4 = \varepsilon^0_4 + \zeta \kappa^1_4
\]

\[
\varepsilon_5 = \varepsilon^0_5 + \zeta \kappa^1_5
\]

\[
\varepsilon_6 = \varepsilon^0_6 + \zeta \kappa^0_6 + \zeta^2 \kappa^2_6
\]

(2.10)

where
Here \( x_i \) denote the cartesian coordinates \( (dx_i = a_i d\xi_i, i = 1,2) \).

2.4 Constitutive Relations

The stress-strain relations for the \( k \)-th lamina in the lamina coordinates are given by

\[
\begin{align*}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_6 \\
\sigma_4 \\
\sigma_5
\end{bmatrix} &=
\begin{bmatrix}
\overline{Q}_{11}^{(k)} & \overline{Q}_{12}^{(k)} & 0 & 0 & 0 \\
\overline{Q}_{21}^{(k)} & \overline{Q}_{22}^{(k)} & 0 & 0 & 0 \\
\overline{Q}_{66}^{(k)} & 0 & 0 & 0 & 0 \\
\overline{Q}_{44}^{(k)} & \overline{Q}_{45}^{(k)} & 0 & 0 & 0 \\
\overline{Q}_{55}^{(k)} & \overline{Q}_{55}^{(k)} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_6 \\
\varepsilon_4 \\
\varepsilon_5
\end{bmatrix}
\end{align*}
\]

(2.12)

where \( \overline{Q}_{ij}^{(k)} \) are the plane stress reduced stiffnesses of the \( k \)-th lamina.
in the lamina coordinate system. The coefficients $\overline{Q}_{ij}$ can be expressed in terms of the engineering constants of a lamina:

$$
\overline{Q}_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} \quad \overline{Q}_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}
$$

$$
\overline{Q}_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}
$$

$$
\overline{Q}_{44} = G_{23} \quad \overline{Q}_{55} = G_{13} \quad \overline{Q}_{66} = G_{12}
$$

To determine the laminate constitutive equations, Eq. (2.12) should be transformed to the laminate coordinates. We obtain

$$
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_6 (k)
\end{pmatrix} = 
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{12} & Q_{22} & Q_{26} \\
Q_{16} & Q_{26} & Q_{66} (k)
\end{bmatrix}
\begin{pmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_6 (k)
\end{pmatrix}
= 
\begin{bmatrix}
Q_{44} & Q_{45} \\
Q_{45} & Q_{55} (k)
\end{bmatrix}
\begin{pmatrix}
\epsilon_4 \\
\epsilon_5 (k)
\end{pmatrix}
$$

where

$$
Q_{11} = \overline{Q}_{11} \cos^4 \theta + 2(\overline{Q}_{12} + 2\overline{Q}_{66}) \sin^2 \theta \cos^2 \theta + \overline{Q}_{22} \sin^4 \theta
$$

$$
Q_{12} = (\overline{Q}_{11} + \overline{Q}_{22} - 4\overline{Q}_{66}) \sin^2 \theta \cos^2 \theta + \overline{Q}_{12} (\sin^4 \theta + \cos^4 \theta)
$$

$$
Q_{22} = \overline{Q}_{11} \sin^4 \theta + 2(\overline{Q}_{12} + 2\overline{Q}_{66}) \sin^2 \theta \cos^2 \theta + \overline{Q}_{22} \cos^4 \theta
$$

$$
Q_{16} = (\overline{Q}_{11} - \overline{Q}_{12} - 2\overline{Q}_{66}) \sin \theta \cos^3 \theta + (\overline{Q}_{12} - \overline{Q}_{22} + 2\overline{Q}_{66}) \sin^3 \theta \cos \theta
$$

$$
Q_{26} = (\overline{Q}_{11} - \overline{Q}_{12} - 2\overline{Q}_{66}) \sin^3 \theta \cos \theta + (\overline{Q}_{12} - \overline{Q}_{22} + 2\overline{Q}_{66}) \sin \theta \cos^3 \theta
$$

$$
Q_{66} = (\overline{Q}_{11} + \overline{Q}_{22} - 2\overline{Q}_{12} - 2\overline{Q}_{66}) \sin^2 \theta \cos^2 \theta + \overline{Q}_{66} (\sin^4 \theta + \cos^4 \theta)
$$

$$
Q_{44} = \overline{Q}_{44} \cos^2 \theta + \overline{Q}_{55} \sin^2 \theta
$$

$$
Q_{45} = (\overline{Q}_{55} - \overline{Q}_{44}) \cos \theta \sin \theta
$$
\[ Q_{55} = \overline{Q}_{55} \cos^2 \theta + \overline{Q}_{44} \sin^2 \theta \]  

(2.15)

2.5 Equations of Motion

The dynamic version of the principle of virtual work for the present case yields

\[
0 = \int_{-h/2}^{h/2} \sum_{k} \left[ f(k) \left[ \sigma_1 \delta \varepsilon_1(k) + \sigma_2 \delta \varepsilon_2(k) + \sigma_5 \delta \varepsilon_5(k) + \sigma_4 \delta \varepsilon_4(k) + \sigma_6 \delta \varepsilon_6(k) \right] \right] dx_1 \, dx_2 \, d\zeta - \int_{h/2}^{-h/2} \left\{ \int_{\Omega} \left[ (\ddot{u})^2 + (\ddot{v})^2 + (\ddot{w})^2 \right] dx_1 \, dx_2 \right\} d\zeta | dt \\

= \int_{-h/2}^{h/2} \left\{ \int_{\Omega} \left[ (N_1 \delta \varepsilon_1^0 \right] + M_1 \delta \kappa_1^0 + P_1 \delta \kappa_1^2 + N_2 \delta \varepsilon_2^0 + M_2 \delta \kappa_2^0 \\
+ P_2 \delta \kappa_2^2 + N_6 \delta \varepsilon_6^0 + M_6 \delta \kappa_6^0 + P_6 \delta \kappa_6^2 + Q_2 \delta \varepsilon_4 + K_2 \delta \kappa_4^1 + Q_1 \delta \varepsilon_5 + K_1 \delta \kappa_5^1 - \sigma \delta w \right] \\
+ \left[ (I_{11} \ddot{u} + I_{21} \ddot{v} - I_{31} \frac{\partial^2 w}{\partial x_1^2}) \delta u + (I_{11} \ddot{v} + I_{21} \ddot{v} - I_{31} \frac{\partial^2 w}{\partial x_2^2}) \delta v \\
+ (I_{13} \frac{\partial^2 u}{\partial x_1^2} + I_{53} \frac{\partial^2 u}{\partial x_2^2} + I_{73} \frac{\partial^2 v}{\partial x_2^2} + I_{91} \frac{\partial^2 v}{\partial x_1^2}) \delta w \right] \right\} dt \\
\]

(2.16)

where \( q \) is the distributed transverse load, \( N_i, M_i \) etc. are the resultants,

\[
(n_i, M_i, P_i) = \int_{\zeta_{k-1}}^{\zeta_k} \sigma_i(k) \left( 1, \zeta, \zeta^3 \right) d\zeta \quad (i = 1, 2, 6) \\

(Q_1, K_1) = \int_{\zeta_{k-1}}^{\zeta_k} \sigma_5(k) \left( 1, \zeta^2 \right) d\zeta \]
\[ (Q_2, K_2) = \frac{\pi}{\kappa} \int \kappa \sigma_0(1, \kappa^2) d\kappa \] (2.17)

The inertias \( \overline{I}_i \) and \( \overline{I}'_i \) \((i = 1, 2, 3, 4, 5)\) are defined by the equations,

\[
\overline{I}_1 = I_1 + \frac{2}{R_1} I_2 ,
\]

\[
\overline{I}'_1 = I_1 + \frac{2}{R_2} I_2
\]

\[
\overline{I}_2 = I_2 + \frac{1}{R_1} I_3 - \frac{4}{3h^2} I_4 - \frac{4}{3h^2 R_1} I_5 ,
\]

\[
\overline{I}'_2 = I_2 + \frac{1}{R_2} I_3 - \frac{4}{3h^2} I_4 - \frac{4}{3h^2 R_2} I_5
\]

\[
\overline{I}_3 = \frac{4}{3h^2} I_4 + \frac{4}{3h^2 R_1} I_5 ,
\]

\[
\overline{I}'_3 = \frac{4}{3h^2} I_4 + \frac{4}{3h^2 R_2} I_5
\]

\[
\overline{I}_4 = I_3 - \frac{8}{3h^2} I_5 + \frac{16}{9h^4} I_7
\]

\[
\overline{I}'_4 = I_3 - \frac{8}{3h^2} I_5 + \frac{16}{9h^4} I_7
\]

\[
\overline{I}_5 = \frac{4}{3h^2} I_5 - \frac{16}{9h^4} I_7
\]

\[
\overline{I}'_5 = \frac{4}{3h^2} I_5 - \frac{16}{9h^4} I_7
\] (2.18a)

and
\[(I_1, I_2, I_3, I_4, I_5, I_7) = \sum_{k=1}^{n} \rho(k) (1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^6) d\zeta \quad (2.18b)\]

The governing equations of motion can be derived from Eq. (2.16) by integrating the displacement gradients by parts and setting the coefficients of \(\delta u, \delta v, \delta w\) and \(\delta \phi_i\) \((i = 1, 2)\) to zero separately:

\[
\delta u: \quad \frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = -I_1 \ddot{u} + I_2 \ddot{\phi}_1 - I_3 \dot{\phi}_2
\]

\[
\delta v: \quad \frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} = -I_1 \ddot{v} + I_2 \ddot{\phi}_2 - I_3 \dot{\phi}_1
\]

\[
\delta w: \quad \frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} - \frac{4}{h^2} (\frac{\partial K_1}{\partial x_1} + \frac{\partial K_2}{\partial x_2}) + \frac{4}{3h^2} (\frac{\partial^2 p_1}{\partial x_1^2} + \frac{\partial^2 p_2}{\partial x_2^2}) + 2 \frac{\partial^2 p_6}{\partial x_1 \partial x_2} - \frac{N_1}{R_1} - \frac{N_2}{R_2} + N(w) = I_3 \frac{\partial u}{\partial x_1} + I_5 \frac{\partial \phi_1}{\partial x_1} + I_1 \ddot{w}
\]

\[
- \frac{16I_7}{9h^4} (\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2}) - q
\]

\[
\delta \phi_1: \quad \frac{\partial M_1}{\partial x_1} + \frac{\partial M_6}{\partial x_2} - Q_1 + \frac{4}{h^2} K_1 - \frac{4}{3h^2} (\frac{\partial^2 p_1}{\partial x_1^2} + \frac{\partial^2 p_6}{\partial x_2^2}) = I_2 \ddot{u} + I_4 \ddot{\phi}_1 - I_5 \frac{\partial w}{\partial x_1}
\]

\[
\delta \phi_2: \quad \frac{\partial M_6}{\partial x_1} + \frac{\partial M_2}{\partial x_2} - Q_2 + \frac{4}{h^2} K_2 - \frac{4}{3h^2} (\frac{\partial^2 p_6}{\partial x_1^2} + \frac{\partial^2 p_2}{\partial x_2^2}) = I_2 \ddot{v} + I_4 \ddot{\phi}_2 - I_5 \frac{\partial w}{\partial x_2}
\]

where

\[
N(w) = \frac{3}{3x_1} (N_1 \frac{3w}{3x_1} + N_6 \frac{3w}{3x_2}) + \frac{3}{3x_2} (N_6 \frac{3w}{3x_1} + N_2 \frac{3w}{3x_2}) \quad (2.20)
\]

The essential (i.e., geometric) and natural boundary conditions of the theory are given by:
Specify:

\[ u_n, u_{ns}, w, \frac{aw}{an}, \frac{aw}{as}, \phi_n, \phi_{ns} \text{ (essential)} \]

\[ N_n, N_{ns}, Q_n, P_n, P_s, M_n, M_{ns} \text{ (natural)} \]  
(2.21)

where

\[ N_n = n_x^2 N_1 + n_y^2 N_2 + 2n_x n_y N_6 \]
\[ N_{ns} = (N_2 - N_1) n_x n_y + N_6 (n_x^2 - n_y^2) \]
\[ P_n = n_x^2 P_1 + n_y^2 P_2 + 2n_x n_y P_6 \]
\[ P_{ns} = (P_2 - P_1) n_x n_y + P_6 (n_x^2 - n_y^2) \]
\[ M_n = n_x^2 M_1 + n_y^2 M_2 + 2n_x n_y M_6 \]
\[ M_{ns} = (M_2 - M_1) n_x n_y + M_6 (n_x^2 - n_y^2) \]
\[ Q_n = N_n \frac{aw}{an} + N_{ns} \frac{aw}{as} + \frac{4}{3h^2} \left( \frac{\alpha p_n}{an} + \frac{\alpha p_{ns}}{as} \right) + Q_n - \frac{4}{h^2} K_n \]
\[ Q_n = n_x Q_1 + n_y Q_2 \]
\[ K_n = n_x K_1 + n_y K_2 \]
\[ P_n = -\frac{4}{3h^2} P_n \]
\[ P_{ns} = -\frac{4}{3h^2} P_{ns} \]
\[ M_n = M_n - \frac{4}{3h^2} P_n \]
\[ M_{ns} = M_{ns} - \frac{4}{3h^2} P_{ns} \]  
(2.22)
and \( n_x \) and \( n_y \) are the direction cosines of the unit normal on the boundary of the laminate.

The resultants can be expressed in terms of the strain components using Eqs. (2.10) and (2.12) in Eq. (2.14). We get

\[
N_i = A_{ij} \varepsilon_j^0 + B_{ij} \kappa_j^0 + E_{ij} \kappa_j^2
\]

\[
M_i = B_{ij} \varepsilon_j^0 + D_{ij} \kappa_j^0 + F_{ij} \kappa_j^2
\]

\[
P_i = E_{ij} \varepsilon_j^0 + F_{ij} \kappa_j^0 + H_{ij} \kappa_j^2
\]

\[
Q_2 = A_{4j} \varepsilon_j^0 + D_{4j} \kappa_j^1
\]

\[
Q_1 = A_{5j} \varepsilon_j^0 + D_{5j} \kappa_j^1
\]

\[
K_2 = D_{4j} \varepsilon_j^0 + F_{4j} \kappa_j^1
\]

\[
K_1 = D_{5j} \varepsilon_j^0 + F_{5j} \kappa_j^1
\]

where \( A_{ij}, B_{ij}, \) etc. are the plate stiffnesses,

\[
(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij})
\]

\[
= \sum_{k=1}^{n} \int_{\zeta_{k-1}}^{\zeta_k} Q_{ij}^{(k)} (1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5) d\zeta
\]

for \( i, j = 1, 2, 4, 5, 6. \)
3.1 Introduction

Exact solutions of the partial differential equations (2.19) on arbitrary domains and for general conditions is not possible. However, for simply supported shells whose projection in the \( x_1x_2 \)-plane is a rectangle, the linear version of Eq. (2.19) can be solved exactly, provided the lamination scheme is of antisymmetric cross-ply \([0^\circ/90^\circ/0^\circ/90^\circ...]\) or symmetric cross-ply \([0^\circ/90^\circ...]\) type. The Navier solution exists if the following stiffness are zero [21]:

\[
A_{i6} = B_{i6} = D_{i6} = E_{i6} = F_{i6} = H_{i6} = 0 \quad , \quad (i = 1,2) \quad (3.1)
\]

\[
A_{45} = D_{45} = F_{45} = 0
\]

The boundary conditions are assumed to be of the form,

\[
u(x_1,0) = u(x_1,b) = v(0,x_2) = v(a,x_2) = 0
\]

\[
w(x_1,0) = w(x_1,b) = w(0,x_2) = w(a,x_2) = 0
\]

\[
\frac{\partial w}{\partial x_1}(x_1,0) = \frac{\partial w}{\partial x_1}(x_1,b) = \frac{\partial w}{\partial x_2}(0,x_2) = \frac{\partial w}{\partial x_2}(a,x_2) = 0
\]

\[
N_2(x_1,0) = N_2(x_1,b) = N_1(0,x_2) = N_1(a,x_2) = 0
\]

\[
M_2(x_1,0) = M_2(x_1,b) = M_1(0,x_2) = M_1(a,x_2) = 0 \quad (3.2)
\]

\[
P_2(x_1,0) = P_2(x_1,b) = P_1(0,x_2) = P_1(a,x_2) = 0
\]

\[
\phi_1(x_1,0) = \phi_1(x_1,b) = \phi_2(0,x_2) = \phi_2(a,x_2) = 0
\]

where \( a \) and \( b \) denote the lengths along the \( x_1 \)- and \( x_2 \)-directions, respectively (see Fig. 3.1).
Figure 3.1 The geometry and the coordinate system for a projection of a shell element.
3.2 The Navier Solution

Following the Navier solution procedure (see Reddy [21]), we assume the following solution form that satisfies the boundary conditions in Eq. (3.2):

\[
U = \sum_{m,n=1}^{\infty} U_{mn} f_1(x_1,x_2)
\]

\[
v = \sum_{m,n=1}^{\infty} V_{mn} f_2(x_1,x_2)
\]

\[
w = \sum_{m,n=1}^{\infty} W_{mn} f_3(x_1,x_2)
\]

\[
\phi_1 = \sum_{m,n=1}^{\infty} \phi_{mn} f_1(x_1,x_2)
\]

\[
\phi_2 = \sum_{m,n=1}^{\infty} \phi_{mn} f_2(x_1,x_2)
\]

(3.3)

where

\[
f_1(x_1,x_2) = \cos \alpha x_1 \sin \beta x_2, \quad f_2(x_1,x_2) = \sin \alpha x_1 \cos \beta x_2
\]

\[
f_3(x_1,x_2) = \sin \alpha x_1 \sin \beta x_2, \quad \alpha = \frac{m\pi}{a}, \quad \beta = \frac{n\pi}{b}
\]

(3.4)

Substituting Eq. (3.3) into Eq. (2.19), we obtain

\[
[M] \begin{pmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{pmatrix} + [C] \begin{pmatrix} U_{mn} \\ V_{mn} \\ \phi_{mn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \phi_{mn} \end{pmatrix}, \text{ for any } m,n
\]

(3.5)

where \(Q_{mn}\) are the coefficients in the double Fourier expansion of the transverse load,
\[ q(x_1, x_2) = \sum_{m,n=1}^{\infty} Q_{mn} f_3(x_1, x_2) \]  

(3.6)

and the coefficients \( M_{ij} \) and \( C_{ij} \) \((i,j = 1,2,\ldots,5)\) are given in Appendix A.

Equation (3.5) can be solved for \( U_{mn}, V_{mn}, \) etc., for each \( m \) and \( n \), and then the solution is given by Eq. (3.3). The series in Eq. (3.3) are evaluated using a finite number of terms in the series. For free vibration analysis, Eq. (24) can be expressed as an eigenvalue equation,

\[
([C] - \omega^2[M])\{\Delta\} = \{0\}
\]

(3.7)

where \( \{\Delta\} = \{U_{mn}, V_{mn}, W_{mn}, \phi_{mn}^1, \phi_{mn}^2\}^T \), and \( \omega \) is the frequency of natural vibration. For static bending analysis, Eq. (3.5) becomes

\[
[C]\{\Delta\} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

(3.8)
CHAPTER 4
MIXED VARIATIONAL PRINCIPLES

4.1 Introduction

The variational formulations used for developing plate and shell bending elements can be classified into three major categories: (i) formulations based on the principle of virtual displacements (or the principle of total potential energy), (ii) formulations based on the principle of virtual forces (or the principle of complementary energy) and (iii) formulations based on mixed variational principles (Hu-Washizu-Reissner's principles). The finite-element models based on these formulations are called, respectively, the displacement models, equilibrium models and mixed models.

In the principle of virtual displacements, one assumes that the kinematic relations (i.e., strain-displacement relations and geometry boundary conditions) are satisfied exactly (i.e., point-wise) by the displacement field, and the equilibrium equations and force boundary conditions are derived as the Euler equations. No a-priori assumption concerning the constitutive behavior (i.e., stress-strain relations) is necessary in using the principle. The principle of (the minimum) total potential energy is a special case of the principle of virtual displacements applied to solid bodies that are characterized by the strain energy function \( U \) such that

\[
\frac{\partial U(\varepsilon_{ij})}{\partial \varepsilon_{ij}} = \sigma_{ij}
\]

where \( \varepsilon_{ij} \) and \( \sigma_{ij} \) are the components of strain and stress tensors, respectively. When the principle of total potential energy is used to
develop a finite-element model of a shell theory, the kinematic relations are satisfied point-wise but the equilibrium equations are met only in an integral (or variational) sense. The principle of complementary potential energy, a special case of the principle of virtual forces, can be used to develop equilibrium models that satisfy the equilibrium equations point-wise but meet the strain compatibility in a variational sense.

There are a number of mixed variational principles in elasticity (see [66-73]). The phrase 'mixed' is used to imply the fact that both the displacement (or primal) variables and (some of the) force (or dual) variables are given equal importance in the variational formulations. The associated finite-element models use independent approximations of dependent variables appearing in the variational formulation. There are two kinds of mixed models: (i) models in which both primal and dual variables are interpolated independently in the interior and on the boundary of an element, (ii) models in which the variables are interpolated inside the element and their values on the boundary are interpolated by the boundary values of the interpolation. The first kind are often termed hybrid models, and the second kind are known simply as the mixed models. In the present study the second kind (i.e., mixed model) will be discussed.

4.2 Mixed Variational Principles

To develop a mixed variational statement of the third-order laminate theory, we rewrite Eq. (2.23) in matrix notation as follows:

\[
\{N\} = [A^*]\{\varepsilon^0\} + [B^*]\{M\} + [E^*]\{P\}
\]
\[-\{\kappa^S\} = [B^*]^T\{\varepsilon^O\} - [D^*]\{M\} - [F^*]\{P\} \quad (4.1)\]

\[-\{\kappa\} = [E^*]^T\{\varepsilon^O\} - [F^*]^T\{M\} - [H^*]\{P\}\]

where

\[\{\kappa\} = -c_1(\{\kappa^S\} + \{\kappa^C\}), \quad \{\kappa^S\} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x} \\ \frac{\partial \phi_2}{\partial y} \\ \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \end{bmatrix}\]

\[\{\kappa^C\} = \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}, \quad \{\varepsilon^O\} = \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{w}{R_1} + \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{w}{R_2} + \frac{1}{2} \left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{bmatrix}\]

\[c_1 = \frac{4}{3h^2} \text{ and } c_2 = \frac{4}{h^2}\]

\[[A^*] = [A] - \left\{\begin{bmatrix} [B] \\ [E] \end{bmatrix}^T \begin{bmatrix} [D] & [F] \\ [F] & [H] \end{bmatrix}^{-1} \begin{bmatrix} [B] \\ [E] \end{bmatrix}\right\} \quad \text{(symmetric)}\]

\[[B^*] = [B][D^*] + [E][F^*] \quad \text{(not symmetric)}\]

\[[E^*] = [B][F^*] + [E][H^*] \quad \text{(not symmetric)}\]

\[\begin{bmatrix} [D^*] & [F^*] \\ [F^*] & [H^*] \end{bmatrix} = \begin{bmatrix} [D] & [F] \end{bmatrix}^{-1} \quad \text{(symmetric)}\]

Note that in this part of the dissertation we are using the notation \(x = x_1\) and \(y = x_2\).

Now we proceed to develop a mixed variational statement of the third-order laminate theory. In the present mixed formulation, we treat
the generalized displacements \((u, v, w, \phi_1, \phi_2)\) and generalized moments \((M_1, M_2, M_6, P_1, P_2, P_6)\) as the dependent variables.

The mixed functional associated with the static version of Eqs. (2.19), (4.1) is given by \(\Pi = (u, v, w, \phi_1, \phi_2, M_1, M_2, M_6, P_1, P_2, P_6)\),

\[
\Pi_{H1}(\Pi) = \int_{\Omega} \left[ \frac{1}{2} \{\epsilon^O\}^T[A^*]\{\epsilon^O\} + \{\epsilon^O\}^T([B^*][M] + [E^*][P]) \\
+ [P]^T[\kappa] - \frac{1}{2} [P]^T[H^*][P] \\
+ \frac{1}{2} \{\hat{\epsilon}^O\}^T[A]\{\hat{\epsilon}^O\} - qw d\xi d\eta \\
- \int_{\Gamma} (N_1 u_n + N_2 u_s + \hat{Q}_n w + \hat{M}_1 \phi_1 + \hat{M}_2 \phi_2) ds \right] (4.3)
\]

where

\[
Q_n = Q_1 n_x + Q_2 n_y \\
u_n = u_n x + v n_y , u_s = -u_n y + v n_x \\
\frac{\partial}{\partial x} = n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y} , \frac{\partial}{\partial s} = n_x \frac{\partial}{\partial y} - n_y \frac{\partial}{\partial x} \\
N_n = N_1 n_x^2 + 2N_6 n_x n_y + N_2 n_y^2 , N_s = (N_2 - N_1) n_x n_y + N_6 (n_x^2 - n_y^2) \\
(4.4)
\]

\[
\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} \phi_1 + \frac{\partial w}{\partial x} \\ \phi_2 + \frac{\partial w}{\partial y} \end{bmatrix} \\
[A] = [A] - 2c_2[O] + c_2^2[F] \quad (4.5)
\]
Note that the bending moments \((M_1, M_2, M_6)\) do not enter the set of essential boundary conditions, and the resultants \((P_1, P_2, P_6)\) do not enter the set of natural boundary conditions also.

The (mixed) variational principle corresponding to the functional in Eq. (4.3) can be stated as follows. Of all possible configurations a laminate can assume, the one that renders the functional \(\Pi_{H1}\) stationary also satisfies the equations of equilibrium (2.19) and the kinematic-constitutive equations [i.e. the last two equations of Eq. (4.3)].

We note that the variational statement in Eq. (4.3) contains the second-order derivatives of the transverse deflection (through \(\epsilon\)). To relax the continuity requirements on \(w\), we integrate the term \(T[M]\) by parts to trade the differentiation to \(P\). We obtain

\[
\Pi_{H2}(A) = \int_\Omega \left\{ \frac{1}{2} \left\{ \epsilon^0 \right\}^T[A^*][\epsilon^0] + \left\{ \epsilon^0 \right\}^T\left( [B^*][M] + [E^*][P] \right) \right. \\
- \left. \left\{ M \right\}^T[F^*][P] - \frac{1}{2} \left\{ M \right\}^T[D^*][M] \right. \\
+ \left( \kappa^S \right) T(\left\{ M \right\} - c_1[P]) - \frac{1}{2} \left\{ P \right\}^T[H^*][P] \right. \\
+ \frac{1}{2} \left\{ \epsilon^0 \right\}^T[A]\left[ \epsilon^0 \right] + c_1 [\left( \frac{\partial P_1}{\partial x} + \frac{\partial P_6}{\partial y} \right) \frac{\partial w}{\partial x} \\
+ \left( \frac{\partial P_1}{\partial y} + \frac{\partial P_2}{\partial y} \right) \frac{\partial w}{\partial y} ] - qw \right\} dx dy \\
- \int_\Gamma \left\{ N_n u_n + N_s u_s + Q_n w + \hat{M}_n \phi_n + \hat{M}_s \phi_s \right. \\
+ \left. c_1(\hat{\theta}_n \rho_n + \hat{\theta}_s \rho_s) \right\} ds \tag{4.6}
\]
\[ \theta_n = \frac{\partial w}{\partial n}, \quad \theta_s = \frac{\partial w}{\partial s} \]

We note that the stress resultants \( P_n \) and \( P_s \) (hence \( P_1, P_2 \) and \( P_6 \)) enter the set of essential boundary conditions in the case of the second mixed formulation of the higher-order theory. As special cases, the mixed variational statements for the classical and the first order theories can be obtained from Eq. (4.6) by setting appropriate terms to zero.
CHAPTER 5

FINITE-ELEMENT MODELS

5.1 Introduction

The mixed variational principles presented in Chapter 4 can be used to develop mixed finite-element models, which contain the bending moments as the primary variables along with the generalized displacements. Historically, at least with regard to bending of plates, the first mixed formulation is attributed to Herrmann [74,75] and Hellan [76].

Since the bending moments do not enter the set of essential boundary conditions, they are not required to be continuous across interelement boundaries. Thus, one can develop mixed models with discontinuous (between elements) bilinear or higher-order approximation of the bending moments. The present section deals with the development of a mixed model based on $\Pi_{H^2}$.

5.2 Finite-Element Discretization

Let $\Omega$, the domain (i.e., the midplane) of the laminate, be represented by a collection of finite elements

$$\Omega = \bigcup_{e=1}^{E} \Omega^e, \quad \Omega^e \cap \Omega^f = \text{empty for } e \neq f$$

(5.1)

where $\Omega = \Omega \cup \Gamma$ is the closure of the open domain $\Omega$ and $\Gamma$ is its boundary. Over a typical element $\Omega^e$, each of the variables $u, v, w,$ etc. are interpolated by expressions of the form

$$u = \sum_{j=1}^{N} u_j \psi_j, \quad v = \sum_{j=1}^{N} v_j \psi_j, \quad w = \sum_{j=1}^{N} w_j \psi_j$$

(5.2)
etc., where \( \{\psi_j\} \) denote the set of admissible interpolation functions (see Reddy and Putcha [64]). Although the same set of interpolation functions is used, for simplicity, to interpolate each of the dependent variables, in general, different sets of interpolation can be used for \((u,v), w, (\phi_x, \phi_y), (M_1, M_2, M_6)\) and \((P_1, P_2, P_6)\). From an examination of the variational statement in Eq. (4.6), it is clear that the linear, quadratic, etc., interpolation functions of the Lagrange type are admissible. The resulting element is called a \( C^0 \)-element, because no derivative of the dependent variable is required to be continuous across interelement boundaries.

To develop the finite-element equations of a typical element, we first compute the strains, rotations, etc. in terms of the interpolation functions \( \psi_j \) and nodal values \( u_j, v_j \), etc. We have

\[
\{\varepsilon^0\} = \{\varepsilon^{0L}\} + \{\varepsilon^{0N}\} \tag{5.3}
\]

\[
\{\varepsilon^{0L}\} = \begin{pmatrix}
du{\partial u}{\partial x} + \frac{w}{R_1} \\
du{\partial u}{\partial y} + \frac{w}{R_2} \\
du{\partial v}{\partial x} + \frac{v}{R_2}
\end{pmatrix} = \begin{bmatrix}
\{\psi_i, x\} & \{0\} \\
\{0\} & \{\psi_i, y\} \\
\{\psi_i, y\} & \{\psi_i, x\}
\end{bmatrix} \begin{bmatrix}
\{u\} \\
\{v\} \\
\{0\}
\end{bmatrix} + \begin{bmatrix}
\frac{1}{R_1} \{\psi_i\} \\
\frac{1}{R_2} \{\psi_i\}
\end{bmatrix} \begin{bmatrix}
w
\end{bmatrix}
\]

\[
\equiv [H^L]\{\Delta^1\} + [H^O]\{\Delta^2\} \tag{5.4}
\]
\[
\{ \varepsilon^{ON} \} = \frac{1}{2} \left( \begin{array}{c} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_i}{\partial y} \\ \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_i}{\partial y} \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} \{ f_x \psi_i, x \} \\ \{ f_y \psi_i, y \} \end{array} \right) \quad \{ w \} \\
\left( \begin{array}{c} f_x \psi_i + f_y \psi_i \end{array} \right) \equiv \frac{1}{2} [H^N] \{ \Delta^2 \} (5.5)
\]

\[
\{ \delta \varepsilon^{ON} \} = [H^N] \{ \delta \Delta^2 \}
\]

\[
\phi_1 = \begin{bmatrix} \{ \psi_i \} & \{ 0 \} \\ \{ 0 \} & \{ \psi_i \} \end{bmatrix} \quad \phi_2 = \begin{bmatrix} \{ \phi_1 \} \\ \{ \psi_i \} \end{bmatrix} \equiv [H^2] \{ \Delta^3 \} (5.6)
\]

\[
\kappa^S = \begin{bmatrix} \frac{\partial \phi_1}{\partial x} \\ \frac{\partial \phi_2}{\partial y} \\ \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \end{bmatrix} \equiv [H^L] \{ \Delta^3 \} (5.7)
\]

\[
\{ \theta \} = \begin{bmatrix} \frac{\partial \psi_i}{\partial x} \\ \frac{\partial \psi_i}{\partial y} \end{bmatrix} = \begin{bmatrix} \{ \psi_i, x \} \\ \{ \psi_i, y \} \end{bmatrix} \equiv [H^1] \{ \Delta^2 \} (5.8)
\]

\[
\{ \varepsilon^0 \} = \{ \theta \} + \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = [H^1] [H^2] \begin{bmatrix} \{ \Delta^2 \} \\ \{ \Delta^3 \} \end{bmatrix} (5.9)
\]

\[
\{ M \} = \begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} \{ \psi_i \} & \{ 0 \} & \{ 0 \} \\ \{ 0 \} & \{ \psi_i \} & \{ 0 \} \\ \{ 0 \} & \{ 0 \} & \{ \psi_i \} \end{bmatrix} \equiv [H^3] \{ \Delta^4 \} (5.10)
\]
Here, \( \frac{\partial w}{\partial x} = f_x \), \( \frac{\partial w}{\partial y} = f_y \) and

\[
\{\psi_i\} = \{\psi_1 \ \psi_2 \ \vdots \ \psi_N\}, \ \{\psi_{i,x}\} = \{\psi_{1,x} \ \psi_{2,x} \ \vdots \ \psi_{N,x}\}, \ etc. \quad (5.14)
\]

and \( \{u\}, \ \{v\}, \ etc. \) denote the columns of the nodal values of \( u, \ v, \ etc. \), respectively.

5.3 Finite-Element Model

The finite-element model for the refined shear deformation theory is derived from the variational statement in Eq. (4.6) over an element. The first variation of the functional in (4.6) for a typical element \( \Omega^e \) is given by

\[
0 = \int_{\Omega^e} \left[ \left( [\delta \varepsilon^0]^T [\varepsilon^0] + [B^*][M] + [E^*][P] \right) \right] \\
+ \delta M^T \left( \left\{ \begin{array}{c} \kappa^S \\ \end{array} \right\} + [B^*]^T[\varepsilon^0] - [D^*][M] - [F^*][P] \right),
\]
\[ + \{\delta P\}^T(- c_1\{\kappa^S\} + [E^*]\{\varepsilon^O\} - [F^*]\{M\} - [H^*]\{P\}) \]

\[ + \{\delta e^O\}^T\{A\}\{e^O\} + \{\delta \kappa^S\}^T\{(M) - c_1\{P\}\} \]

\[ + c_1\left(\frac{\partial \delta P_1}{\partial x} + \frac{\partial \delta P_6}{\partial y}\right) + (\frac{\partial \delta P_6}{\partial x} + \frac{\partial \delta P_2}{\partial y}) + (\frac{\partial \delta P_1}{\partial x} + \frac{\partial \delta P_6}{\partial y}) \]

\[ + (\frac{\partial \delta P_6}{\partial x} + \frac{\partial \delta P_2}{\partial y}) \right) \delta w] \text{dxdy} \]

\[ - \int \left( N_n \delta u_n + N_s \delta u_s + Q_n \delta w + M_n \delta \phi_n + M_s \delta \phi_s \right) \]

\[ + c_1 \theta_n \delta P_n + c_1 \theta_s \delta P_s \text{ds} \quad (5.15) \]

Substitution of Eqs. (5.2)-(5.14) into the variational statement in Eq. (5.15), we obtain

\[
\begin{pmatrix} [K_{11}] & [K_{12}] & \ldots & [K_{15}] \\ [K_{21}] & [K_{22}] & \ldots & [K_{25}] \\ \vdots & \vdots & \ddots & \vdots \\ [K_{51}] & [K_{52}] & \ldots & [K_{55}] \end{pmatrix} \begin{pmatrix} \{\Delta^1\} \\ \{\Delta^2\} \\ \vdots \\ \{\Delta^5\} \end{pmatrix} = \begin{pmatrix} \{F^1\} \\ \{F^2\} \\ \vdots \\ \{F^5\} \end{pmatrix} \quad (5.16) \]

where

\[ [K_{11}] = \int_{\Omega^e} [H^L]^T[A^*][H^L]dA, \quad \{F^1\} = \int_{\Gamma^e} [H^2]^T N_n \text{ds} \]

\[ [K_{12}] = \frac{1}{2} \int_{\Omega^e} [H^L]^T[A^*]([H^N] + 2[H^O])dA \]

\[ [K_{21}] = \int_{\Omega^e} ([H^N] + H^O)^T[A^*][H^L]dA \]

\[ [K_{14}] = \int_{\Omega^e} [H^L]^T[\delta^*][H^3]dA = [K^4_{14}]^T \]
\[ [K^{15}] = \int_{\Omega^e} [H^1]^T [E^*] [H^3] dA = [K^{51}]^T \]

\[ [K^{22}] = \int_{\Omega^e} (H^0) + (H^N)^T [A^*] ([H^0] + \frac{1}{2} [H^N]) dA + \int_{\Omega^e} [H^1]^T \hat{A} [H^1] dA \]

\[ [K^{24}] = \int_{\Omega^e} (([H^0] + [H^N])^T [B^*] [H^3]) dA \]

\[ [K^{24}] = \int_{\Omega^e} (\frac{1}{2} [H^3]^T [B^*]^T ([H^N] + 2[H^0])) dA \]

\[ [K^{25}] = \int_{\Omega^e} ([H^0] + [H^N])^T [E^*] [H^3] + c_1 [H^1] T [H^1] T dA \]

\[ [K^{52}] = \int_{\Omega^e} ([H^3]^T [E^*] ([H^0] + \frac{1}{2} [H^N]) + c_1 [H^1] [H^1]) dA \]

\[ \{F^2\} = \int_{\Omega^e} \{\psi_1\}^T q dA + \int_{\rho^e} \{\psi_1\}^T Q_n ds \]

\[ [K^{33}] = \int_{\Omega^e} [H^2]^T \hat{A} [H^2] dA, \quad [K^{23}] = \int_{\Omega^e} [H^1]^T \hat{A} [H^2] dA = [K^{32}]^T \]

\[ [K^{34}] = \int_{\Omega^e} [H^1]^T [H^3] dA = [K^{43}]^T \]

\[ [K^{35}] = - \int_{\Omega^e} c_1 [H^1] T [H^3] dA = [K^{53}]^T \]

\[ \{F^3\} = \int_{\rho^e} [H^2]^T [R]^T \{ \begin{bmatrix} n_x \\ n_y \end{bmatrix} \} ds \]

\[ [R] = \begin{bmatrix} n_x & n_y \\ -n_y & n_x \end{bmatrix} \]

\[ [K^{44}] = - \int_{\Omega^e} [H^3]^T [D^*] [H^3] dA, \]

\[ [K^{45}] = - \int_{\Omega^e} [H^3]^T [F^*] [H^3] dA = [K^{54}]^T \]
Clearly, the element stiffness matrix is not symmetric. For the linear case (i.e. \([H^N] = [0]\)), the element stiffness matrix is symmetric. The finite-element models for the classical and first-order theories can be obtained as special cases from Eq. (5.16). An explicit form of the coefficients of the stiffness matrix in Eq. (5.16) is given in Appendix B.

### 5.4 Solution Procedure

The assembled finite-element equations are of the form

\[
[K^D(\Delta)]\{\Delta\} = \{F\}
\]  

(5.18)

where \([K^D]\) is the assembled stiffness matrix, unsymmetric in general, and \(\{\Delta\}\) is the global solution vector. Because the stiffness matrix is a nonlinear function of the unknown solution, Eq. (5.18) should be solved iteratively. Two iterative methods of analysis are quite commonly used in the finite-element analysis of nonlinear problems: the Picard type direction iteration and the Newton-Raphson iteration methods. Here the Newton-Raphson method is used to solve the nonlinear equations.
Suppose that the solution is required for a load of \( P \). The load is divided into a sequence of load steps \( \Delta P_1, \Delta P_2, \ldots, \Delta P_n \) such that \( P = \sum_{i=1}^{n} \Delta P_i \). At any load step, Eq. (5.18) is solved iteratively to obtain the solution. At the end of the \( r \)-th iteration the solution for the next iteration is obtained solving the following equation
\[
[K^T(\Delta r)]\{\delta \Delta r^{+1}\} = -\{R\} - [\hat{K}^D(\Delta r)]\{\Delta r\} + \{F\}
\]
(5.19)
for the increment of the solution \( \{\delta \Delta r^{+1}\} \), and the total solution is computed from
\[
\{\Delta r^{+1}\} = \{\Delta r\} + \{\delta \Delta r^{+1}\}
\]
(5.20)
where \( [K^T] \) is the tangent stiffness matrix,
\[
[K^T] = [K]^D + \frac{\partial [K]^D}{\partial \{\Delta\}} \{\Delta\}
\]
(5.21)
A geometrical explanation of the Newton-Raphson iteration is given in Fig. 5.1.

For the finite-element models developed here, the tangent stiffness matrix is symmetric. For example, the tangent stiffness matrix for the model in Eq. (5.16) for plates \( (1/R_1 = 1/R_2 = 0) \) is given by
\[
[K^T] = \begin{bmatrix}
[K^{11T}] \\
[K^{21T}] & [K^{22T}] & \text{symm.} \\
\vdots \\
[K^{51T}] & \cdots & \cdots & \cdots & [K^{55T}]
\end{bmatrix}
\]
(5.22)
where
\[
[K^{11T}] = [K^{11}], \quad [K^{21T}] = [K^{21}],
\]
\[
[K^{22T}] = [\hat{K}^{22}] + \int_a^e \{[H]^T[A^]*[H]^1\}dA\{\Delta^1\}
\]
Figure 5.1 Convergence of the Newton-Raphson Iteration
For all coupled (i.e. bending-stretching coupling) problems, the mixed models of classical, first-order and third-order theories have six, eight and eleven degrees of freedom per node, respectively. If the bending moments are eliminated at the element level, the element degrees of freedom can be reduced by 3 (see Fig. 5.2).

The finite-element model (5.16) for the dynamic case is of the form,

\[ [K]\{\Delta\} + [M]\ddot{\{\Delta\}} = \{F\} \quad (5.24) \]

where \([M]\) is the mass matrix (see Appendix B). For free vibration, Eq. (5.24) can be written as
Figure 5.2 The displacement and mixed finite elements for the third-order shear deformation theory
\[ [K][\Delta] = \lambda [M][\Delta] \]  

(5.25)

where \( \lambda \) is the square of the frequency. For transient analysis, Eq. (5.24) can be reduced, after using the temporal approximation (see Reddy [77]), to the form

\[ [\hat{K}][\Delta]_{n+1} = [\hat{F}]_{n,n+1} \]  

(5.26)

where \( \{\Delta\}_{n+1} \) denotes the solution at time \( t = (n+1)\Delta t \), \( \Delta t \) being the time step. In the present study the Newmark beta method is used. In this case \([\hat{K}]\) and \([\hat{F}]\) are given by (see Reddy [77])

\[ [\hat{K}] = [K] + a_0[M] \]

\[ [\hat{F}] = [F]_{n+1} + [M](a_0[\Delta]_n + a_1[\dot{\Delta}]_n + a_2[\ddot{\Delta}]_n) \]

\( a_0 = \frac{1}{(\beta \Delta t)^2} \), \( a_1 = a_0 \Delta t \), \( a_2 = \frac{1}{2\beta} - 1 \)  

(5.27)

Once the solution \( \{\Delta\} \) is known at time \( t = (n + 1)\Delta t \), the first and second derivatives (velocity and accelerations) of \( \{\Delta\} \) at \( t = (n + 1)\Delta t \) can be computed from

\[ \{\dot{\Delta}\}_{n+1} = a_0(\{\Delta\}_{n+1} - \{\Delta\}_n) - a_1[\dot{\Delta}]_n - a_2[\ddot{\Delta}]_n \]

\[ \{\ddot{\Delta}\}_{n+1} = \{\dot{\Delta}\}_n + a_3[\ddot{\Delta}]_n + a_4[\ddot{\Delta}]_{n+1} \]  

(5.28)

where \( a_3 = (1 - \alpha)\Delta t \) and \( a_4 = \alpha \Delta t \).

The evaluation of the element matrices requires numerical integration. Reduced integration is used to evaluate the stiffness coefficients associated with the shear energy terms. More specifically, the 2 x 2 Gauss rule is used for shear terms and the standard 3 x 3 Gauss rule is used for the bending terms when the nine node quadratic isoparametric element is considered.
CHAPTER 6
SAMPLE APPLICATIONS

6.1 Introduction

A number of representative problems are analyzed using the higher-order theory developed in the present study. First few problems are included here to illustrate the accuracy of the present theory. These problems are solved using the closed-form solutions presented in Chapter 3. Then problems that do not allow closed-form solutions are solved by the mixed finite-element model described in Chapter 5.

The geometries of typical cylindrical and spherical shells are shown in Fig. 6.1. Of course, plates are derived as special cases from cylindrical or spherical shells.

6.2 Exact Solutions

It is well known that the series solution in Eq. (3.3) converges faster for uniform load than for a point load. For a sinusoidal distribution of the transverse load, the series reduces to a single term.

1. Four-layer, cross-ply (0/90/90/0) square laminate under sinusoidal load

This example is included here to demonstrate the relative accuracy of the present higher-order theory when compared to three-dimensional elasticity theory and the first-order theory. Square, cross-ply laminates under simply-supported boundary conditions [see Eq. (3.2)] and sinusoidal distribution of the transverse load are studied for deflections. The lamina properties are assumed to be
Figure 6.1 Geometry of a typical cylindrical and spherical shell
$E_1 = 25E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, $\nu_{12} = \nu_{13} = 0.25$

Figure 6.2 contains plots of the nondimensional center deflection [$\ddot{w} = wE_2h^3)/q_oa^4$] versus the side to thickness ratio (a/h) obtained using various theories. It is clear that the present third-order theory gives the closest solution to the three-dimensional elasticity solution [60] than either the first-order theory or the classical theory.

2. **An isotropic spherical shell under point load at the center.**

   The problem data is
   
   \[R_1 = R_2 = 96.0 \text{ in.}, \quad a = b = 32.0 \text{ in.}, \quad h = 0.1 \text{ in.},\]
   \[E_1 = E_2 = 10^7 \text{ psi}, \quad \nu = 0.3, \quad \text{intensity of load} = 100 \text{ lbs}.\]

   A comparison of the center transverse deflection of the present theory (HSDT) with that obtained using the first-order shear deformation theory (FSDT) and classical shell theory (CST) for various terms in the series is presented in Table 1. It should be noted that Vlasov [79] did not consider transverse shear strains in his study. The difference between the values predicted by HSDT and FSDT is not significant for the thin isotropic shell problem considered here.

3. **Cross-ply spherical shells under sinusoidal, uniform, and point loads.**

   The geometric and material parameters used here are the same as those used in Problem 1. Table 2-4 contain the nondimensionalized center deflection of various cross-ply shells under sinusoidal, uniform, and point loads, respectively. The difference between the solutions predicted by the higher-order theory and the first-order theory increases with the increasing value of $R/a$. Also, the difference is the
Figure 6.2 Nondimensionalized center deflection versus side to thickness ratio for four-layer, symmetric cross-ply [0 /90 /90 /0] square laminated plate.
Table 1. Center deflection (-w x 10^3) of a spherical shell under point load at the center

<table>
<thead>
<tr>
<th>h</th>
<th>Theory</th>
<th>N = 9</th>
<th>N = 49</th>
<th>N = 99</th>
<th>N = 149</th>
<th>N = 199</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>CST</td>
<td>-</td>
<td>39.591</td>
<td>39.647</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Vlasov [79]</td>
<td>-</td>
<td>39.560</td>
<td></td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.32</td>
<td>FSDT [40]</td>
<td>32.594</td>
<td>39.469</td>
<td>39.724</td>
<td>39.786</td>
<td>39.814</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>32.584</td>
<td>39.458</td>
<td>39.714</td>
<td>39.775</td>
<td>39.803</td>
</tr>
<tr>
<td>1.6</td>
<td>FSDT</td>
<td>3.664</td>
<td>3.9019</td>
<td>3.9194</td>
<td>3.9269</td>
<td>3.9318</td>
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<tr>
<td></td>
<td>HSDT</td>
<td>3.661</td>
<td>3.8986</td>
<td>3.9158</td>
<td>3.9230</td>
<td>3.9274</td>
</tr>
<tr>
<td>3.2</td>
<td>FSDT</td>
<td>0.1646</td>
<td>0.1713</td>
<td>0.1735</td>
<td>0.1748</td>
<td>0.1757</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>0.1637</td>
<td>0.1701</td>
<td>0.1715</td>
<td>0.1720</td>
<td>0.1722</td>
</tr>
<tr>
<td>6.4</td>
<td>FSDT</td>
<td>0.03485</td>
<td>0.03756</td>
<td>0.03864</td>
<td>0.03927</td>
<td>0.03972</td>
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<tr>
<td></td>
<td>HSDT</td>
<td>0.03450</td>
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<td></td>
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<td>0.007275</td>
<td>0.007337</td>
<td>0.007350</td>
<td>0.007354</td>
</tr>
</tbody>
</table>
Table 2. Nondimensionalized center deflections, \( \bar{w} = \left( -wh^3E_2/q_o a^4 \right) 10^3 \), of cross-ply laminated spherical shells under sinuso\-dally distributed load (\( a/b = 1, R_1 = R_2 = R, q_0 = 100 \))

<table>
<thead>
<tr>
<th>R/a</th>
<th>0°/90°</th>
<th>0°/90°/0°</th>
<th>0°/90°/90°/0°</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>a/h=100</td>
<td>a/h=10</td>
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<tr>
<td>5</td>
<td>FSDT</td>
<td>1.1948</td>
<td>11.429</td>
</tr>
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<td></td>
<td>HSDT</td>
<td>1.1937</td>
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<td>HSDT</td>
<td>3.5733</td>
<td>11.896</td>
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<td></td>
<td>HSDT</td>
<td>7.1236</td>
<td>12.094</td>
</tr>
<tr>
<td>100</td>
<td>FSDT</td>
<td>10.446</td>
<td>12.370</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10.651</td>
<td>12.161</td>
</tr>
</tbody>
</table>
Table 3. Nondimensionalized center deflections, \( \bar{w} = (-wEh^3/q_a^2) \times 10^3 \), of cross-ply laminated spherical shells under uniformly distributed load.

<table>
<thead>
<tr>
<th>R/a</th>
<th>0°/90°</th>
<th>0°/90°/0°</th>
<th>0°/90°/90°/0°</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>a/h=100</td>
<td>a/h=10</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>1.7519</td>
<td>17.566</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>5.5388</td>
<td>18.744</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>11.268</td>
<td>19.064</td>
</tr>
<tr>
<td>50</td>
<td>FSDT</td>
<td>15.714</td>
<td>19.452</td>
</tr>
</tbody>
</table>
Table 4. Nondimensionalized center deflection of cross-ply spherical shells under point load at the center (101 terms are used in the series)

\[ \bar{w} = - \left( \frac{wh^3E_2}{Pa^2} \right) \times 10^2, \ a/b = 1, \ a/h = 10 \]

<table>
<thead>
<tr>
<th>R/a</th>
<th>Theory</th>
<th>$0^\circ/90^\circ$</th>
<th>$0^\circ/90^\circ/0^\circ$</th>
<th>$0^\circ/90^\circ/90^\circ/0^\circ$</th>
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<tbody>
<tr>
<td>5</td>
<td>FSDT</td>
<td>7.1015</td>
<td>5.1410</td>
<td>4.9360</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
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<td>4.4340</td>
<td>4.3574</td>
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<td>7.3836</td>
<td>5.2273</td>
<td>5.0186</td>
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<td>HSDT</td>
<td>6.1913</td>
<td>4.5470</td>
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</tr>
<tr>
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<td>5.0472</td>
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<td>HSDT</td>
<td>6.2987</td>
<td>4.5865</td>
<td>4.5081</td>
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</table>
most for the symmetric laminates, especially for shallow shells and plates. Note that for point-loaded shells the difference between the solution predicted by the first-order theory and the higher-order theory is more significant, especially for antisymmetric cross-ply laminates.

4. **Natural vibration of cross-ply cylindrical shells**

Table 5 contains the nondimensionalized fundamental frequencies of cross-ply cylindrical shells for three lamination schemes: [0°/90°], [0°/90°/0°], and [0°/90°/90°/0°]. For thin antisymmetric cross-ply shells, the first-order theory underpredicts the natural frequencies when compared to the higher-order theory. However, for symmetric cross-ply shells, the trend reverses.

5. **Natural vibration of cross-ply spherical shells**

Table 6 contains nondimensionalized natural frequencies obtained using the first- and higher-order theories for various cross-ply spherical shells. Analogous to cylindrical shells, the first-order theory underpredicts fundamental natural frequencies of antisymmetric cross-ply shells; for symmetric thick shells and symmetric shallow thin shells the trend reverses.

6.3 Approximate (Finite-Element) Solutions

6.3.1 Bending Analysis

1. **Rectangular plate under uniformly distributed load and mixed boundary conditions.**

The geometry and boundary conditions for the problem are shown in Fig. 6.3. The plate is assumed to be made of steel (E = 30 x 10^6 and ν = 0.3). The problem was also solved by Timoshenko [78], Herrmann
Table 5 Nondimensionalized fundamental frequencies of cross-ply cylindrical shells

\[ \bar{\omega} = \omega \frac{a^2}{h} \sqrt{\rho/E_2} \]

<table>
<thead>
<tr>
<th>R/a</th>
<th>Theory</th>
<th>a/h=100</th>
<th>a/h=10</th>
<th>a/h=100</th>
<th>a/h=10</th>
<th>a/h=100</th>
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<td>15.5511.78</td>
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<td>20</td>
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<td>15.18412.26</td>
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<tr>
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<td>FSDT</td>
<td>9.7816</td>
<td>8.8951</td>
<td>15.244</td>
<td>12.163</td>
<td>15.1911.78</td>
<td>15.17011.78</td>
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<tr>
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<td>HSDT</td>
<td>9.783</td>
<td>8.9730</td>
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<td>15.1911.78</td>
<td>15.17011.78</td>
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<td>15.1911.78</td>
<td>15.17011.78</td>
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<td>15.19</td>
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<td>15.1911.78</td>
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</tr>
<tr>
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<td>8.8998</td>
<td>15.183</td>
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<td>15.18412.226</td>
<td>15.17011.78</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>9.6880</td>
<td>8.9760</td>
<td>15.170</td>
<td>11.790</td>
<td>15.17011.78</td>
<td>15.17011.78</td>
</tr>
</tbody>
</table>
Table 6. Nondimensionalized fundamental frequencies of cross-ply laminated spherical shells

\[
\bar{\omega} = \omega \frac{a^2}{h} \sqrt{h/\alpha E_2}
\]

<table>
<thead>
<tr>
<th>R/a</th>
<th>Theory</th>
<th>a/h=100</th>
<th>a/h=10</th>
<th>a/h=100</th>
<th>a/h=10</th>
<th>a/h=100</th>
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<td>15.426</td>
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<td>15.170</td>
<td>11.790</td>
<td>15.170</td>
<td>11.780</td>
</tr>
</tbody>
</table>
FIGURE 6.3 COMPARISON OF THE TRANSVERSE DEFLECTION OBTAINED BY VARIOUS INVESTIGATORS ($x_2 = 0.4$)
and Prato [80]. Figures 6.3-6.5 contain plots of the transverse displacement and bending moments obtained by various investigators (using the linear theory). The agreement between the present solution and others is very good.

2. Rectangular laminates under uniformly distributed load

Figure 6.6 contains load—deflection curves for a simply supported square unidirectional laminate (0°) under uniformly distributed load. The following geometric and material properties are used:

\[ a = b = 12 \text{ in.}, \quad h = 0.138 \text{ in.} \]
\[
E_1 = 3 \times 10^6 \text{psi}, \quad E_2 = 1.28 \times 10^6 \text{psi}, \quad G_{12} = G_{13} = G_{23} = 0.37 \times 10^6 \text{psi} \\
\nu = 0.32
\]

The experimental results and classical solutions are taken from the paper by Zaghloul and Kennedy [86]. The agreement between the present solution and the experimental solution is extremely good.

Figure 6.7 contains load—deflection curves for a clamped bidirectional laminate [0/90/90/0] under uniformly distributed load. The geometric parameters and layer properties are given by

\[ a = b = 12 \text{ in.}, \quad h = 0.096 \text{ in.} \]
\[
E_1 = 1.8282 \times 10^6 \text{ psi}, \quad E_2 = 1.8315 \times 10^6 \text{ psi}, \\
G_{12} = G_{13} = G_{23} = 3.125 \times 10^6 \text{ psi}, \quad \nu = 0.2395
\]

The experimental results and the classical laminate solutions are taken from [81]. The present solution is in good agreement with the experimental results, and the difference is attributed to possible errors in the simulation of the material properties and boundary conditions.
Figure 6.4 Comparison of the bending moment $M_1$ along the line $x_1 = 0.4$
FIGURE 6.5 COMPARISON OF $M_6$ ALONG THE LINE $x_2 = 0.4$
Figure 6.6 Center deflection versus load intensity for a simply supported square orthotropic plate under uniformly distributed transverse load. The following simply supported boundary conditions were used:

\[
\begin{align*}
    v &= w = \psi_y = M_x = P_x = 0 \text{ on side } x = a \\
    u &= w = \psi_x = M_y = P_y = 0 \text{ on side } y = b
\end{align*}
\]
Figure 6.7 Center deflection versus load intensity for a clamped (CC-1) square laminate [0 /90 /90 /0 ] under uniform transverse load (von Karman theory).

CC-1: \( u = v = w = \psi_x = \psi_y = 0 \) on all four clamped edges
3. **Orthotropic cylinder subjected to internal pressure.**

Consider a clamped orthotropic cylinder with the following geometric and material properties (see Fig. 6.8)

\[
R_1 = 20 \text{ in. }, R_2 = \infty, \\
E_1 = 2 \times 10^6 \text{ psi, } E_2 = 7.5 \times 10^6 \text{ psi} \\
\nu_{21} = 0.25, \ G_{12} = 1.25 \times 10^6 \text{ psi} \\
G_{13} = G_{23} = 0.625 \times 10^6 \text{ psi, } h = 1 \text{ in.} \\
a = 10 \text{ in.}, \ P = 6.41/\pi \text{ psi}
\]

This problem has an analytical solution (see [78]) for linear case, and Rao [47] used the finite-element method to solve the same problem. Both are based on the classical theory. The center deflections from [78] and [47] are 0.000367 in. and 0.000366 in., respectively. Chao and Reddy [52] obtained 0.0003764 in. and 0.0003739 in. using the finite-element model based on the first-order shear deformation theory and 3-D degenerate element, respectively. The current result is 0.0003761 in., which is closer to Chao and Reddy [52], as expected.

For the nonlinear analysis of the same problem, the present results are compared to those of Chang and Sawamipahakdi [81] and Chao and Reddy [52] in Fig. 6.9, which contains plots of the center deflection versus the load obtained by various investigators. The agreement between various results is very good.

4. **Nine-layer cross-ply spherical shell subjected to uniform loading.**

Consider a nine-layer \([0^\circ/90^\circ/0^\circ.../0^\circ]\) cross-ply laminated spherical shell with the following material and geometric data:

\[
R_1 = R_2 = 1000 \text{ in.}, \ a = b = 100 \text{ in.}
\]
RADIUS OF SHELL = 20"

FIGURE 6.3 GEOMETRY AND BOUNDARY CONDITIONS FOR THE OCTANT OF THE CLAMPED CYLINDRICAL SHELL
FIGURE 6.9 CENTER DEFLECTION VERSUS LOAD FOR THE CLAMPED CYLINDER WITH INTERNAL PRESSURE (ORTHOTROPIC)
The present results are compared with those obtained by Noor and Hartley [54] and Chao and Reddy [52] in Fig. 6.10. Noor and Hartley used a 13 degree-of-freedom element based on a shear deformation shell theory. The present results agree with both investigations.

6.3.2 Vibration Analysis

Equation (5.24) can be expressed in the alternative form as

\[
\begin{bmatrix}
[K_{11}] & [K_{12}] \\
[K_{21}] & [K_{22}]
\end{bmatrix}
\begin{bmatrix}
\Delta_1 \\
\Delta_2
\end{bmatrix}
+
\begin{bmatrix}
[M_{11}] & [M_{12}] \\
[M_{21}] & [M_{22}]
\end{bmatrix}
\begin{bmatrix}
\ddot{\Delta}_1 \\
\ddot{\Delta}_2
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\tag{6.1}
\]

where

\[
\begin{align*}
\{\Delta_1\}^T &= \{u\}^T\{v\}^T\{w\}^T\{\phi_1\}^T\{\phi_2\}^T \\
\{\Delta_2\}^T &= \{M_1\}^T\{M_2\}^T\{M_6\}^T\{P_1\}^T\{P_2\}^T\{P_6\}^T
\end{align*}
\tag{6.2}
\]

For free vibration analysis, we wish to eliminate \(\{\Delta_2\}\) as follows. From Eq. (6.1) we have

\[
[K_{11}]{\Delta_1} + [K_{12}]{\Delta_2} = -[M_{11}]\ddot{\Delta}_1 
\tag{6.3}
\]

\[
[K_{21}]{\Delta_1} + [K_{22}]{\Delta_2} = 0 
\tag{6.4}
\]

From (6.4) we have,

\[
{\Delta_2} = -[K_{22}]^{-1}[K_{21}]{\Delta_1} 
\tag{6.5}
\]

Substituting Eq. (6.5) into Eq. (6.3), we obtain,

\[
([K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}])\{\Delta_1\} = -[M_{11}]\ddot{\Delta}_1 
\tag{6.6}
\]

For the free vibration case, Eq. (6.6) reduces to

\[
([\mathcal{K}] - \omega^2[M])\{\Delta_1\} = 0 
\tag{6.7}
\]
Figure 6.10 Nonlinear bending of a nine-layer cross-ply spherical shell [0/90/0/90/.../0]
where $\omega$ is the natural frequencies of the system.

1. **Natural frequencies of a two-layer [0°/90°] laminated plate.**

   The geometric and material properties used are
   
   \[ a = b = 100 \text{ in.}, \quad h = 0.1 \text{ in.} \]
   \[ E_1 = 40 \times 10^6 \text{ psi}, \quad E_2 = 10^6 \text{ psi} \]
   \[ G_{12} = G_{13} = G_{23} = 0.5 \times 10^6 \text{ psi}, \quad \nu_{12} = 0.25 \]
   \[ \rho = 1 \]

   The boundary conditions are shown in Fig. 6.11, which also contains the plots of the ratios $\omega_{NL}/\omega_L$ versus $w_0/h$. Here $\omega_{NL}$ and $\omega_L$ denote the nonlinear and linear natural frequencies, and $w_0$ is the normalized center deflection of the first node. The results are compared with those of Chia and Prabhakara [82], and Reddy [83]. The present results are slightly higher compared to those obtained by the classical and first-order shear deformation theories.

2. **Natural frequencies of two-layer [45°/-45°] angle-ply square plate.**

   The material and geometric parameters used are
   
   \[ E_1 = 10 \times 10^6 \text{ psi}, \quad E_2 = 10^6 \text{ psi} \]
   \[ G_{12} = G_{13} = G_{23} = 0.3333 \times 10^6 \text{ psi}, \quad \nu_{12} = 0.3 \]
   \[ a = b = 100 \text{ in.}, \quad h = 0.1 \text{ in.} \]

   The boundary conditions used are shown in Fig. 6.12, which also contains plots of $\omega_{NL}/\omega_L$ versus $w_0/h$. For this case no results are available in the literature for comparison.
FIGURE 6.11  FUNDAMENTAL FREQUENCIES OF A TWO-LAYER CROSS-PLY 
(0/90) SQUARE LAMINATE UNDER SIMPLY SUPPORTED BOUNDARY CONDITIONS
FIGURE 6.12 RATIO OF NONLINEAR TO LINEAR FREQUENCY VERSUS AMPLITUDE TO THICKNESS RATIO FOR TWO-LAYER ANGLE-PLY SQUARE PALTE (45/-45).
3. **Natural vibration of two-layer \([0^\circ/90^\circ]\) cross-ply plates.**

Consider a two-layer cross-ply plate with the following geometric and material properties:

\[
E_1 = 7.07 \times 10^6 \text{ psi}, \quad E_2 = 3.58 \times 10^6 \text{ psi} \\
G_{12} = G_{23} = G_{13} = 1.41 \times 10^6 \text{ psi}, \quad v_{12} = 0.3 \\
a/h = 1000, \quad a = 100 \text{ in.}
\]

The results of \(\omega_{\text{NL}}/\omega_L\) versus \(w_0/h\) are shown in Fig. 6.13. Compared to the results of Reddy [83] and Chandra and Raju [84], the results of the present study are in general a little higher.

6.3.3 **Transient Analysis**

The selection of a time step is a critical problem in transient analysis. We may expect better results if smaller time steps are used. In practice, one wishes to take as large a time step as possible to save computational expense. In the present study, the selection of time step follows that given by Tsui and Tong [85]. Tsui and Tong presented an estimate formula of the time step for conditionally stable integration schemes:

\[
(\Delta t)^2 \leq \frac{(\Delta x)^2 [p(1 - v^2)/E_1]}{2 + (1 - v) \frac{\pi}{12} [1 + 1.5 \left(\frac{\Delta x}{h}\right)^2]}
\]

where \(\Delta x\) is the minimum distance between the element node points.

1. **Two layer cross-ply \([0^\circ/90^\circ]\) laminated, clamped cylinder under internal pressure load.**

The geometric parameters and boundary conditions are the same as those shown in Fig. 6.8. The material properties used are

\[
E_1 = 7.5 \times 10^6 \text{ psi}, \quad E_2 = 2.0 \times 10^6 \text{ psi}
\]
BC1 boundary conditions:

- \( u = w = \phi_1 = 0 \); \( M_2 = P_2 = 0 \) along \( x_2 = b/2 \)
- \( v = w = \phi_2 = 0 \); \( M_1 = P_1 = 0 \) along \( x_1 = a/2 \)
- \( u = \phi_1 = 0 \); \( M_6 = P_6 = 0 \) along \( x_1 = 0 \)
- \( v = \phi_2 = 0 \); \( M_6 = P_6 = 0 \) along \( x_2 = 0 \)

**Figure 6.13** Ratio of the nonlinear to linear frequency versus the amplitude to thickness ratio of a two-layer cross-ply (0/90) square laminate
\[
G_{12} = G_{13} = G_{23} = 1.25 \times 10^6 \text{ psi}, \quad r = 0.25
\]

\[
\rho = 1 \text{ lb-sec}^2/\text{in}^4, \quad q = 5000 \text{ psi}
\]

Figure 6.14 contains the plot of the center deflection versus time of the current solutions and compare to those by Reddy and Chandrashekhara [53]. With the time step \( \Delta t = 0.005 \), the current solutions match very well with those results by Reddy and Chandrashekhara with a smaller step \( \Delta t = 0.001 \). In other words, a lot of computation time has been saved by the current formulation to get the same accuracy.

2. Two-layer angle-ply \([-45^\circ/45^\circ]\) laminated, clamped cylinder with internal pressure load.

The geometric and material properties are the same as those used in Problem 1 of this section. The boundary conditions are also the same except that the moment resultants \( M_i, P_i \) are not specified. A 2 x 2 mesh of nine-node element was used to simulate the transient response of the cylinder. Only one octant was used. Figure 6.15 contains the plot of the transverse deflection versus time for \( q = 5000 \text{ psi} \) and \( \Delta t = 0.005 \) and compare to the solutions by Reddy and Chandrashekhara [53].
Figure 6.14  NONLINEAR TRANSIENT RESPONSE OF A TWO-LAYER CROSS-PLY (0/90) CYLINDRICAL SHELL UNDER INTERNAL PRESSURE
FIGURE 6.15 NONLINEAR TRANSIENT RESPONSE OF A ANGLE-PLY 
(-45/45) CYLINDRICAL SHELL UNDER INTERNAL 
PRESSURE
7.1 Summary and Conclusions

The present study dealt with the following major topics:

(i) The development of a third-order shear deformation theory of laminated composite doubly-curved shells. The theory accounts for (a) the parabolic variation of the transverse shear strains, and (b) the von Karman strains. It does not require the use of the shear correction coefficients.

(ii) The development of the closed-form solutions (for the linear theory) for the simply supported cross-ply laminates. These solutions can be used for a check in the numerical analysis of shells.

(iii) The construction of a mixed variational principle for the third-order theory that includes the classical theory and the first order theory as special cases.

(iv) The development and application of the finite-element model of the third-order theory for laminated composite shells, accounting for the geometric nonlinearity in the sense of von Karman.

The increased accuracy of the present third-order theory over the classical or first-order shear deformation theory is demonstrated via examples that have either the three-dimensional elasticity solution or experimental results. Many of the other results on bending, vibration and transient analysis included here can serve as references for future investigations.
7.2 Some Comments on Mixed Models

The displacement model of the classical laminate theory requires the use of $C^1$ elements, which are algebraically complex and computationally expensive. The $C^0$ mixed elements are algebraically simple and allows the direct computation of the bending moments at the nodes. The inclusion of the bending moments as the nodal degrees of freedom not only results in increased accuracy of the average stress over that determined from the displacement model but it allows us to determine stresses at the nodes. This feature is quite attractive in contact problems and singular problems in general. It should be noted that the bending moments are not required to be continuous across interelement boundaries, as was shown in Chapter 4. The mixed models based on the shear deformation theories also have the same advantages, except that the displacement model of the first-order shear deformation theory is also a $C^0$ element. In general, the formulative and programming efforts are less with the mixed elements.

7.3 Recommendations

The theory presented here can be extended to a more general theory; for example, the development of the theory in general curvilinear coordinates, and for more general shells (than the doubly-curved shells considered here). Extension of the present theory to include nonlinear material models is awaiting. Of course, the inclusion of thermal loads and damping in the present theory is straightforward.

The displacement models require $C^1$-approximation of the transverse deflection, $C^0$-approximation of other displacement variables. Such a
displacement finite-element model was not developed in the present study, and interested researchers could pursue the development.
REFERENCES


APPENDIX A

COEFFICIENTS OF THE NAVIER SOLUTION

\[ C(1,1) = -A_{11}a^2 - A_{66}b^2 \]

\[ C(1,2) = -A_{12}a^2 - A_{66}b^2 \]

\[ C(1,3) = A_{11} \frac{a}{R_1} + A_{12} \frac{a}{R_2} + \frac{4a}{3h^2} \left( E_{11}a^2 + E_{12}b^2 + 2E_{66}b^2 \right) \]

\[ C(1,4) = a^2 \left( -B_{11} + \frac{4E_{11}}{3h^2} \right) + b^2 \left( -B_{66} + \frac{4E_{66}}{3h^2} \right) \]

\[ C(1,5) = ab \left( -B_{12} - B_{66} + \frac{4E_{12}}{3h^2} + \frac{4E_{66}}{3h^2} \right) \]

\[ C(2,2) = -a^2 A_{66} - b^2 A_{22} \]

\[ C(2,3) = b \left( \frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} \right) + \frac{ab}{2} \left( \frac{4E_{12}}{3h^2} + \frac{8E_{66}}{3h^2} \right) + \frac{ab}{3h^2} E_{22} \]

\[ C(2,4) = \frac{ab}{2} \left( -B_{12} - B_{66} + \frac{4E_{12}}{3h^2} + \frac{4E_{66}}{3h^2} \right) \]

\[ C(2,5) = a^2 \left( -B_{66} + \frac{4E_{66}}{3h^2} \right) + b^2 \left( -B_{22} + \frac{4E_{22}}{3h^2} \right) \]

\[ C(3,3) = a^4 \left( -\frac{16H_{11}}{9h^4} \right) + b^4 \left( -\frac{16H_{22}}{9h^4} \right) - \frac{32ab^2}{9h^4} \left( H_{12} + 2H_{66} \right) \]

\[ + a^2 \left( -A_{55} + \frac{80}{h^2} - \frac{16F_{55}}{h^4} - \frac{8E_{11}}{3h^2 R_1} - \frac{8E_{12}}{3h^2 R_2} \right) \]

\[ + b^2 \left( -A_{44} + \frac{80}{h^2} - \frac{16F_{44}}{h^4} - \frac{8E_{12}}{3h^2 R_1} + \frac{8E_{22}}{3h^2 R_2} \right) \]
\[
C(3,4) = \frac{4a^3}{3h^2} (F_{11} - \frac{4H_{11}}{3h^2}) + \frac{4a^2}{3h^2} (F_{12} + 2F_{66}) - \frac{16a^2}{9h^4} (H_{12} + 2H_{66}) + \alpha \left[ \frac{B_{11}}{R_1} + \frac{B_{12}}{R_2} - \frac{4}{3h^2} \left( \frac{E_{11}}{R_1} + \frac{E_{12}}{R_2} \right) \right] - A_{55}^a
+ \frac{8aD_{55}}{h^2} - \frac{16aF_{55}}{h^4}
\]

\[
C(3,5) = \frac{4a^3}{3h^2} (F_{22} - \frac{4H_{22}}{3h^2}) + \frac{4a^2}{3h^2} (F_{12} + 2F_{66}) - \frac{16a^2}{9h^4} (H_{12} + 2H_{66}) + \beta \left[ \frac{B_{12}}{R_1} + \frac{B_{22}}{R_2} - \frac{4}{3h^2} \left( \frac{E_{12}}{R_1} + \frac{E_{22}}{R_2} \right) \right] - A_{44}^\beta
+ \frac{8aD_{44}}{h^2} - \frac{16aF_{44}}{h^4}
\]

\[
C(4,4) = \frac{8a^2}{3h^2} F_{11} + \frac{8a^2}{3h^2} F_{66} - \frac{16}{9h^4} (H_{11} \alpha^2 + H_{66} \beta^2) - D_{11} \alpha^2 - D_{66} \beta^2
- A_{55} + \frac{8D_{55}}{h^2} - \frac{16F_{55}}{h^4}
\]

\[
C(4,5) = \alpha \beta [-D_{12} - D_{66} + \frac{8}{3h^2} (F_{12} + F_{66}) - \frac{16}{9h^4} (H_{12} + H_{66})]
\]

\[
C(5,5) = \alpha^2 (-D_{66} + \frac{8F_{66}}{3h^2} - \frac{16H_{66}}{9h^4}) + \beta^2 (-D_{22} + \frac{8F_{22}}{3h^2} - \frac{16H_{22}}{9h^4}) - A_{44}
+ \frac{8D_{44}}{h^2} - \frac{16F_{44}}{h^4}
\]

\[
M(1,1) = - (I_1 + \frac{2I_2}{R_1})
\]
\[ M(1,2) = 0 \]

\[ M(1,3) = \left( -\frac{4}{3h^2} I_4 + \frac{4}{3h^2 R_1} I_5 \right) \alpha \]

\[ M(1,4) = -\left( I_2 + \frac{I_3}{R_1} - \frac{4}{3h^2} I_4 - \frac{4}{3h^2 R_1} I_5 \right) \]

\[ M(1,5) = 0 \]

\[ M(2,2) = -(I_1 + \frac{2I_2}{R_2}) \]

\[ M(2,3) = \left( -\frac{4}{3h^2} I_4 + \frac{4}{3h^2 R_2} I_5 \right) \beta \]

\[ M(2,4) = 0 \]

\[ M(2,5) = -(I_2 + \frac{I_3}{R_2} - \frac{4}{3h^2} I_4 - \frac{4}{3h^2 R_2} I_5) \]

\[ M(3,3) = -\frac{16}{9h^4} I_7 \alpha^2 - \frac{8}{3h^2} I_5 + \frac{16}{9h^4} I_7 \beta^2 \]

\[ M(3,4) = \left( -\frac{4}{3h^2} I_5 - \frac{16}{9h} I_7 \right) \alpha \]

\[ M(3,5) = \left( -\frac{4}{3h^2} I_5 - \frac{16}{9h} I_7 \right) \beta \]

\[ M(4,4) = -(I_3 - \frac{8}{3h^2} I_5 + \frac{16}{9h^4} I_7) \]

\[ M(4,5) = 0 \]

\[ M(5,5) = -(I_3 - \frac{8}{3h^2} I_5 + \frac{16}{9h^4} I_7) \]
## APPENDIX B

### STIFFNESS COEFFICIENTS FOR THE MIXED MODEL

The element equation (5.16) can be written in the form

\[
\begin{bmatrix}
K^{1} & K^{12} & \ldots & K^{1(11)}
\end{bmatrix}
\begin{bmatrix}
\Delta^{1}
\end{bmatrix}
= \begin{bmatrix}
F^{1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
K^{12} & K^{2} & \ldots & K^{2(11)}
\end{bmatrix}
\begin{bmatrix}
\Delta^{2}
\end{bmatrix}
= \begin{bmatrix}
F^{2}
\end{bmatrix}
\]

\[
\vdots
\]

\[
\begin{bmatrix}
K^{(11)1} & K^{(11)2} & \ldots & K^{(11)(11)}
\end{bmatrix}
\begin{bmatrix}
\Delta^{11}
\end{bmatrix}
= \begin{bmatrix}
F^{11}
\end{bmatrix}
\]

where \( \Delta^{1} = \{U\} \), \( \Delta^{2} = \{V\} \), etc., and

\[
K^{11} = A_{11} [S_{11}] + A_{13} [S_{12}] + A_{31} [S_{12}] + A_{33} [S_{22}]
\]

\[
K^{12} = A_{12} [S_{12}] + A_{13} [S_{11}] + A_{32} [S_{22}] + A_{33} [S_{21}]
\]

\[
K^{13} = \frac{A_{11}}{R_1} [S_{10}] + \frac{1}{2} A_{11} \left( \frac{\partial W}{\partial x_1} S_{11} \right) + \frac{A_{12}}{R_2} [S_{10}]
\]

\[
+ \frac{1}{2} A_{12} \left( \frac{\partial W}{\partial x_2} S_{12} \right) + \frac{1}{2} A_{13} \left( \frac{\partial W}{\partial x_1} S_{12} \right) + \frac{1}{2} A_{13} \left( \frac{\partial W}{\partial x_2} S_{11} \right)
\]

\[
+ \frac{A_{31}}{R_1} [S_{20}] + \frac{1}{2} A_{31} \left( \frac{\partial W}{\partial x_1} S_{21} \right) + \frac{A_{32}}{R_2} [S_{20}]
\]

\[
+ \frac{1}{2} A_{32} \left( \frac{\partial W}{\partial x_2} S_{22} \right) + \frac{1}{2} A_{33} \left( \frac{\partial W}{\partial x_1} S_{22} \right) + \frac{1}{2} A_{33} \left( \frac{\partial W}{\partial x_2} S_{21} \right)
\]

\[
K^{14} = K^{15} = 0
\]

\[
K^{16} = B_{11} [S_{10}] + B_{31} [S_{20}]
\]

\[
K^{17} = B_{12} [S_{10}] + B_{32} [S_{20}]
\]

\[
K^{18} = B_{13} [S_{10}] + B_{33} [S_{20}]
\]
\[ [K^{19}] = E_{11}[S_{10}] + E_{31}[S_{20}] \]

\[ [K^{1,10}] = E_{12}[S_{10}] + E_{32}[S_{20}] \]

\[ [K^{1,11}] = E_{13}[S_{10}] + E_{33}[S_{20}] \]

\[ [K^{21}] = A_{31}[S_{11}] + A_{33}[S_{12}] + A_{21}[S_{21}] + A_{23}[S_{22}] \]

\[ [K^{22}] = A_{32}[S_{12}] + A_{33}[S_{11}] + A_{22}[S_{22}] + A_{23}[S_{21}] \]

\[ [K^{23}] = \frac{A_{31}}{R_1}[S_{10}] + \frac{1}{2} A_{31} \left( \frac{\partial^2 w}{\partial x_1^2} S_{11} \right) + \frac{A_{32}}{R_2}[S_{10}] + \frac{1}{2} A_{32} \left( \frac{\partial^2 w}{\partial x_2^2} S_{11} \right) \]

\[ + \frac{A_{21}}{R_1}[S_{20}] + \frac{1}{2} A_{21} \left( \frac{\partial^2 w}{\partial x_1^2} S_{21} \right) + \frac{A_{22}}{R_2}[S_{20}] + \frac{1}{2} A_{22} \left( \frac{\partial^2 w}{\partial x_2^2} S_{21} \right) \]

\[ + \frac{1}{2} A_{23} \left( \frac{\partial^2 w}{\partial x_1^2} S_{22} \right) + \frac{1}{2} A_{23} \left( \frac{\partial^2 w}{\partial x_1^2} S_{22} \right) + \frac{1}{2} A_{23} \left( \frac{\partial^2 w}{\partial x_2^2} S_{21} \right) \]

\[ [K^{24}] = [K^{25}] = 0 \]

\[ [K^{26}] = B_{31}[S_{10}] + B_{21}[S_{20}] \]

\[ [K^{27}] = B_{32}[S_{10}] + B_{22}[S_{20}] \]

\[ [K^{28}] = B_{33}[S_{10}] + B_{23}[S_{20}] \]

\[ [K^{29}] = E_{31}[S_{10}] + E_{21}[S_{20}] \]

\[ [K^{2,10}] = E_{32}[S_{10}] + E_{22}[S_{20}] \]
\[
[K^2, 11] = E_{33}[S_{10}] + E_{23}[S_{20}]
\]

\[
[K^{31}] = \frac{A_{11}}{R_1} [S_{01}] + \frac{A_{13}}{R_1} [S_{02}] + A_{11} \frac{\partial w}{\partial x_1} S_{11}
\]
\[
+ A_{13} \frac{\partial w}{\partial x_1} S_{12} + \frac{A_{21}}{R_2} [S_{01}] + \frac{A_{23}}{R_2} [S_{02}] + A_{21} \frac{\partial w}{\partial x_2} S_{21}
\]
\[
+ A_{23} \frac{\partial w}{\partial x_2} S_{22} + A_{31} \frac{\partial w}{\partial x_1} S_{21} + A_{33} \frac{\partial w}{\partial x_1} S_{22}
\]
\[
+ A_{31} \frac{\partial w}{\partial x_2} S_{11} + A_{33} \frac{\partial w}{\partial x_2} S_{12}
\]

\[
[K^{32}] = \frac{A_{12}}{R_1} [S_{02}] + \frac{A_{13}}{R_1} [S_{01}] + A_{12} \frac{\partial w}{\partial x_1} S_{12} + A_{13} \frac{\partial w}{\partial x_1} S_{11}
\]
\[
+ \frac{A_{22}}{R_2} [S_{02}] + \frac{A_{23}}{R_2} [S_{01}] + A_{22} \frac{\partial w}{\partial x_2} S_{22} + A_{23} \frac{\partial w}{\partial x_2} S_{21}
\]
\[
+ A_{32} \frac{\partial w}{\partial x_1} S_{22} + A_{33} \frac{\partial w}{\partial x_1} S_{21} + A_{32} \frac{\partial w}{\partial x_2} S_{12}
\]
\[
+ A_{33} \frac{\partial w}{\partial x_2} S_{11}
\]

\[
[K^{33}] = \frac{A_{11}}{R_1^2} [S_{00}] + \frac{1}{2} \frac{A_{11}}{R_1} \frac{\partial w}{\partial x_1} S_{01} + \frac{A_{12}}{R_1 R_2} [S_{00}]
\]
\[
+ \frac{1}{2} \frac{A_{12}}{R_1} \frac{\partial w}{\partial x_2} S_{02} + \frac{1}{2} \frac{A_{13}}{R_1} \frac{\partial w}{\partial x_1} S_{02} + \frac{1}{2} \frac{A_{13}}{R_1} \frac{\partial w}{\partial x_2} S_{01}
\]
\[
+ \frac{A_{11}}{R_1} \frac{\partial w}{\partial x_1} S_{10} + \frac{1}{2} A_{11} \left( \frac{\partial w}{\partial x_1} \right)^2 S_{11} + \frac{A_{12}}{R_2} \frac{\partial w}{\partial x_2} S_{10}
\]
\[
+ \frac{1}{2} A_{12} \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{12} + \frac{1}{2} A_{13} \left( \frac{\partial w}{\partial x_1} \right)^2 S_{12}
\]
\[+ \frac{1}{2} A_{13} \left( \frac{aw}{ax_1} \frac{aw}{ax_2} S_{11} \right) + \frac{A_{21}}{R_1 R_2} [S_{00}] + \frac{1}{2} \frac{A_{21}}{R_2} \left( \frac{aw}{ax_1} S_{01} \right)\]

\[+ \frac{A_{22}}{R_2} [S_{00}] + \frac{1}{2} \frac{A_{22}}{R_2} \left( \frac{aw}{ax_2} S_{02} \right) + \frac{1}{2} \frac{A_{23}}{R_2} \left( \frac{aw}{ax_1} S_{02} \right)\]

\[+ \frac{1}{2} \frac{A_{23}}{R_2} \left( \frac{aw}{ax_2} S_{01} \right) + \frac{A_{21}}{R_1} \left( \frac{aw}{ax_2} S_{20} \right) + \frac{1}{2} \frac{A_{21}}{R_1} \left( \frac{aw}{ax_1} \frac{aw}{ax_2} S_{21} \right)\]

\[+ \frac{A_{22}}{R_2} \left( \frac{aw}{ax_2} S_{20} \right) + \frac{1}{2} \frac{A_{22}}{R_2} \left( \frac{aw}{ax_2} \right)^2 S_{22} + \frac{1}{2} \frac{A_{23}}{R_2} \left( \frac{aw}{ax_1} \frac{aw}{ax_2} S_{22} \right)\]

\[+ \frac{1}{2} \frac{A_{23}}{R_2} \left( \frac{aw}{ax_2} \right)^2 S_{21} \]

\[+ \frac{A_{31}}{R_1} \left( \frac{aw}{ax_1} S_{20} \right) + \frac{1}{2} \frac{A_{32}}{R_2} \left( \frac{aw}{ax_1} \frac{aw}{ax_2} S_{22} \right) + \frac{1}{2} \frac{A_{33}}{R_2} \left( \frac{aw}{ax_1} \frac{aw}{ax_2} S_{22} \right)\]

\[+ \frac{1}{2} \frac{A_{32}}{R_2} \left( \frac{aw}{ax_1} \frac{aw}{ax_2} S_{21} \right) + \frac{A_{31}}{R_1} \left( \frac{aw}{ax_2} S_{10} \right) + \frac{1}{2} \frac{A_{31}}{R_1} \left( \frac{aw}{ax_1} \frac{aw}{ax_2} S_{11} \right)\]

\[+ \frac{A_{12}}{R_2} \left( \frac{aw}{ax_2} S_{10} \right) + \frac{1}{2} \frac{A_{32}}{R_2} \left( \frac{aw}{ax_2} \right)^2 S_{12} + \frac{1}{2} \frac{A_{33}}{R_2} \left( \frac{aw}{ax_1} \frac{aw}{ax_2} S_{12} \right)\]

\[+ \frac{1}{2} \frac{A_{33}}{R_2} \left( \frac{aw}{ax_2} \right)^2 S_{11} + \bar{A}_{55} [S_{11}] + \bar{A}_{45} [S_{12} + S_{21}]\]

\[+ \bar{A}_{44} [S_{22}]\]

\[[K^{34}] = \bar{A}_{55} [S_{10}] + \bar{A}_{45} [S_{20}]\]

\[[K^{35}] = \bar{A}_{45} [S_{10}] + \bar{A}_{44} [S_{20}]\]

\[[K^{36}] = \frac{B_{11}}{R_1} [S_{00}] + B_{11} \frac{aw}{ax_1} S_{10} + \frac{B_{21}}{R_2} [S_{00}] + B_{21} \frac{aw}{ax_2} S_{20}\]
\[ \begin{align*}
&+ B_{31} \left( \frac{3w}{\alpha x_1} S_{20} \right) + B_{31} \left( \frac{3w}{\alpha x_2} S_{10} \right) \\
\left[ K^{37} \right] &= \frac{B_{12}}{R_1} [S_{00}] + B_{12} \left( \frac{3w}{\alpha x_1} S_{10} \right) + \frac{B_{22}}{R_2} [S_{00}] + B_{22} \left( \frac{3w}{\alpha x_2} S_{20} \right) \\
&+ B_{32} \left( \frac{3w}{\alpha x_1} S_{20} \right) + B_{32} \left( \frac{3w}{\alpha x_2} S_{10} \right) \\
\left[ K^{38} \right] &= \frac{B_{13}}{R_1} [S_{00}] + B_{13} \left( \frac{3w}{\alpha x_1} S_{10} \right) + \frac{B_{23}}{R_2} [S_{00}] + B_{23} \left( \frac{3w}{\alpha x_2} S_{20} \right) \\
&+ B_{33} \left( \frac{3w}{\alpha x_1} S_{20} \right) + B_{33} \left( \frac{3w}{\alpha x_2} S_{10} \right) \\
\left[ K^{39} \right] &= \frac{E_{11}}{R_1} [S_{00}] + E_{11} \left( \frac{3w}{\alpha x_1} S_{10} \right) + \frac{E_{21}}{R_2} [S_{00}] + E_{21} \left( \frac{3w}{\alpha x_2} S_{20} \right) \\
&+ E_{31} \left( \frac{3w}{\alpha x_1} S_{20} \right) + E_{31} \left( \frac{3w}{\alpha x_2} S_{10} \right) + CP[S_{11}] \\
\left[ K^{3,10} \right] &= \frac{E_{12}}{R_1} [S_{00}] + E_{12} \left( \frac{3w}{\alpha x_1} S_{10} \right) + \frac{E_{22}}{R_2} [S_{00}] + E_{22} \left( \frac{3w}{\alpha x_2} S_{20} \right) \\
&+ E_{32} \left( \frac{3w}{\alpha x_1} S_{20} \right) + E_{32} \left( \frac{3w}{\alpha x_2} S_{10} \right) + CP[S_{22}] \\
\left[ K^{3,11} \right] &= \frac{E_{13}}{R_1} [S_{00}] + E_{13} \left( \frac{3w}{\alpha x_1} S_{10} \right) + \frac{E_{23}}{R_2} [S_{00}] + E_{23} \left( \frac{3w}{\alpha x_2} S_{20} \right) \\
&+ E_{33} \left( \frac{3w}{\alpha x_1} S_{20} \right) + E_{33} \left( \frac{3w}{\alpha x_2} S_{10} \right) + CP[(S_{12} + S_{21})] \\
\left[ K^{41} \right] &= \left[ K^{42} \right] = 0 \\
\left[ K^{43} \right] &= \overline{A}_{55} [S_{01}] + \overline{A}_{45} [S_{02}] 
\end{align*} \]
\[ k^{44} = A_{55} [S_{00}] \]
\[ k^{45} = A_{45} [S_{00}] \]
\[ k^{46} = [S_{10}] \quad k^{47} = 0 \quad k^{48} = [S_{20}] \]
\[ k^{49} = CP [S_{01}] \quad k^{4,10} = 0 \quad k^{4,11} = CP [S_{02}] \]
\[ k^{51} = [k^{52}] = 0 \]
\[ k^{53} = A_{45} [S_{01}] + A_{44} [S_{02}] \]
\[ k^{54} = A_{45} [S_{00}] \]
\[ k^{55} = A_{44} [S_{00}] \]
\[ k^{56} = 0 \quad k^{57} = [S_{20}] \quad k^{58} = [S_{10}] \]
\[ k^{59} = 0 \quad k^{5,10} = CP [S_{02}] \quad k^{5,11} = CP [S_{01}] \]
\[ k^{61} = B_{11} [S_{01}] + B_{31} [S_{02}] \]
\[ k^{62} = B_{21} [S_{02}] + B_{31} [S_{01}] \]
\[ k^{63} = \frac{B_{11}}{R_1} [S_{00}] + \frac{1}{2} B_{11} \left( \frac{\partial^2}{\partial x_1^2} \right) S_{01} + \frac{B_{21}}{R_2} [S_{00}] + \frac{1}{2} B_{21} \left( \frac{\partial^2}{\partial x_2^2} \right) S_{02} \]
\[ + \frac{1}{2} B_{31} \left( \frac{\partial^2}{\partial x_1^2} \right) S_{02} + \frac{1}{2} B_{31} \left( \frac{\partial^2}{\partial x_2^2} \right) S_{01} \]
\[ k^{64} = [S_{01}] \quad k^{65} = 0 \]
\[ k^{66} = -D_{11} [S_{00}] \]
\[
\begin{align*}
K_{67}^{67} &= -D_{12}(S_{00}) \\
K_{68}^{68} &= -D_{13}(S_{00}) \\
K_{69}^{69} &= -F_{11}(S_{00}) \\
K_{6,10}^{6,10} &= -F_{12}(S_{00}) \\
K_{6,11}^{6,11} &= -F_{13}(S_{00}) \\
K_{71}^{71} &= B_{12}(S_{01}) + B_{32}(S_{02}) \\
K_{72}^{72} &= B_{22}(S_{02}) + B_{32}(S_{01}) \\
K_{73}^{73} &= \frac{B_{12}}{R_1} (S_{00}) + \frac{1}{2} B_{12} \left[ \frac{\partial w}{\partial x_1} \right] (S_{01}) + \frac{B_{22}}{R_2} (S_{00}) + \frac{1}{2} B_{22} \left[ \frac{\partial w}{\partial x_2} \right] (S_{02}) \\
&\quad + \frac{1}{2} B_{32} \left[ \frac{\partial w}{\partial x_1} \right] (S_{02}) + \frac{1}{2} B_{32} \left[ \frac{\partial w}{\partial x_2} \right] (S_{01}) \\
K_{74}^{74} &= 0 \\
K_{75}^{75} &= (S_{02}) \\
K_{76}^{76} &= -D_{21}(S_{00}) \\
K_{77}^{77} &= -D_{22}(S_{00}) \\
K_{78}^{78} &= -D_{23}(S_{00}) \\
K_{79}^{79} &= -F_{21}(S_{00}) \\
K_{7,10}^{7,10} &= -F_{22}(S_{00}) \\
K_{7,11}^{7,11} &= -F_{23}(S_{00})
\end{align*}
\]
\[ [K^{81}] = B_{13} [S_{01}] + B_{33} [S_{02}] \]

\[ [K^{82}] = B_{23} [S_{02}] + B_{33} [S_{01}] \]

\[ [K^{83}] = \frac{B_{13}}{R_1} [S_{00}] + \frac{1}{2} B_{13} \left( \frac{\partial w}{\partial x_1} \right) S_{01} + \frac{B_{23}}{R_2} [S_{00}] + \frac{1}{2} B_{23} \left( \frac{\partial w}{\partial x_2} \right) S_{02} \]

\[ + \frac{1}{2} B_{33} \left( \frac{\partial w}{\partial x_1} \right) S_{02} + \frac{1}{2} B_{33} \left( \frac{\partial w}{\partial x_2} \right) S_{01} \]

\[ [K^{84}] = [S_{02}] \]

\[ [K^{85}] = [S_{01}] \]

\[ [K^{86}] = -D_{31} [S_{00}] \]

\[ [K^{87}] = -D_{32} [S_{00}] \]

\[ [K^{88}] = -D_{33} [S_{00}] \]

\[ [K^{89}] = -F_{31} [S_{00}] \]

\[ [K^{8,10}] = -F_{32} [S_{00}] \]

\[ [K^{8,11}] = -F_{33} [S_{00}] \]

\[ [K^{91}] = E_{11} [S_{01}] + E_{31} [S_{02}] \]

\[ [K^{92}] = E_{21} [S_{02}] + E_{31} [S_{01}] \]

\[ [K^{93}] = \frac{E_{11}}{R_1} [S_{00}] + \frac{1}{2} E_{11} \left( \frac{\partial w}{\partial x_1} \right) S_{01} + \frac{E_{21}}{R_2} [S_{00}] + \frac{1}{2} E_{21} \left( \frac{\partial w}{\partial x_2} \right) S_{02} \]
\[ + \frac{1}{2} E_{31}[\frac{\partial w}{\partial x_1} S_{02}] + \frac{1}{2} E_{31}[\frac{\partial w}{\partial x_2} S_{01}] + CP[S_{11}] \]

\[ [K^{94}] = CP[S_{10}] \quad [K^{95}] = 0 \]

\[ [K^{96}] = - F_{11}[S_{00}] \]

\[ [K^{97}] = - F_{21}[S_{00}] \]

\[ [K^{98}] = - F_{31}[S_{00}] \]

\[ [K^{99}] = - H_{11}[S_{00}] \]

\[ [K^{9,10}] = - H_{12}[S_{00}] \]

\[ [K^{9,11}] = - H_{13}[S_{00}] \]

\[ [K^{10,1}] = E_{12}[S_{01}] + E_{32}[S_{02}] \]

\[ [K^{10,2}] = E_{22}[S_{02}] + E_{32}[S_{01}] \]

\[ [K^{10,3}] = \frac{E_{12}}{R_1} [S_{00}] + \frac{1}{2} E_{12}[\frac{\partial w}{\partial x_1} S_{01}] + \frac{E_{22}}{R_2} [S_{00}] + \frac{1}{2} E_{22}[\frac{\partial w}{\partial x_2} S_{02}] + \frac{1}{2} E_{32}[\frac{\partial w}{\partial x_1} S_{02}] + \frac{1}{2} E_{32}[\frac{\partial w}{\partial x_2} S_{01}] + CP[S_{22}] \]

\[ [K^{10,4}] = 0 \quad [K^{10,5}] = CP[S_{20}] \]

\[ [K^{10,6}] = - F_{12}[S_{00}] \]

\[ [K^{10,7}] = - F_{22}[S_{00}] \]
\([K^{10,8}] = - F_{32}[S_{00}]\)
\([K^{10,9}] = - H_{21}[S_{00}]\)
\([K^{10,10}] = - H_{22}[S_{00}]\)
\([K^{10,11}] = - H_{23}[S_{00}]\)
\([K^{11,1}] = E_{13}[S_{01}] + E_{33}[S_{02}]\)
\([K^{11,2}] = E_{23}[S_{02}] + E_{33}[S_{01}]\)
\([K^{11,3}] = \frac{E_{13}}{R_1} [S_{00}] + \frac{1}{2} E_{13} \left( \frac{\partial w}{\partial x_1} S_{01} \right) + \frac{E_{23}}{R_2} [S_{00}] + \frac{1}{2} E_{23} \left( \frac{\partial w}{\partial x_2} S_{02} \right) + \frac{1}{2} E_{33} \left( \frac{\partial w}{\partial x_2} S_{01} \right) + CP[(S_{12} + S_{21})]\)
\([K^{11,4}] = CP[S_{20}]\)
\([K^{11,5}] = CP[S_{10}]\)
\([K^{11,6}] = - F_{13}[S_{00}]\)
\([K^{11,7}] = - F_{23}[S_{00}]\)
\([K^{11,8}] = - F_{33}[S_{00}]\)
\([K^{11,9}] = - H_{31}[S_{00}]\)
\([K^{11,10}] = - H_{32}[S_{00}]\)
\([K^{11,11}] = - H_{33}[S_{00}]\)
Mass Matrix $[M]$ for the Mixed Model

$$
[M] = 
\begin{bmatrix}
[M_{11}] & [M_{12}] & \cdots & [M_{1,11}] \\
[M_{21}] & [M_{22}] & \cdots & [M_{2,11}] \\
\cdots & \cdots & \cdots & \cdots \\
[M_{11,11}] & [M_{11,2}] & \cdots & [M_{11,11}] \\
\end{bmatrix}
$$

$$
[M_{11}] = I_1[S_{00}]
$$

$$
[M_{13}] = -\frac{4}{3h^2} I_4[S_{01}] \quad [M_{31}] = -\frac{4}{3h^2} I_4[S_{10}]
$$

$$
[M_{14}] = I_2[S_{00}] - \frac{4}{3h^2} I_4[S_{00}] \quad [M_{41}] = I_2[S_{00}] - \frac{4}{3h^2} I_4[S_{00}]
$$

$$
[M_{22}] = I_1[S_{00}]
$$

$$
[M_{23}] = -\frac{4}{3h^2} I_4[S_{02}] \quad [M_{32}] = -\frac{4}{3h^2} I_4[S_{20}]
$$

$$
[M_{25}] = (I_2 - \frac{4}{3h^2} I_4)[S_{00}] \quad [M_{52}] = [M_{25}]
$$

$$
[M_{33}] = I_1[S_{00}] + I_7\left(\frac{4}{3h^2}\right)^2[S_{11} + S_{22}]
$$

$$
[M_{34}] = (-\frac{4}{3h^2} I_5 + \left(\frac{4}{3h^2}\right)^2 I_7)[S_{10}]
$$

$$
[M_{43}] = (-\frac{4}{3h^2} I_5 + \left(\frac{4}{3h^2}\right)^2 I_7)[S_{01}]
$$

$$
[M_{35}] = (-\frac{4}{3h^2} I_5 + \left(\frac{4}{3h^2}\right)^2 I_7)[S_{20}]
$$

$$
[M_{53}] = (-\frac{4}{3h^2} I_5 + \left(\frac{4}{3h^2}\right)^2 I_7)[S_{02}]
$$

$$
[M_{44}] = (I_3 - \frac{8}{3h^2} I_5 + \left(\frac{4}{3h^2}\right)^2 I_7)[S_{00}]
$$

$$
[M_{55}] = [M_{44}]
$$

all others are zero, and $I_i$ are defined in Eq. (2.18b)
\[
\bar{A}_{44} = A_{44} - 2E_{44} \frac{4}{h^2} + G_{44} \left(\frac{4}{h^2}\right)^2
\]

\[
\bar{A}_{45} = A_{45} - 2E_{45} \frac{4}{h^2} + G_{45} \left(\frac{4}{h^2}\right)^2
\]

\[
\bar{A}_{55} = A_{55} - 2E_{55} \frac{4}{h^2} + G_{55} \left(\frac{4}{h^2}\right)^2
\]

\[
CP = 3 \cdot \frac{4}{h^2}
\]

\[
[S_{IJ}] = \int_{\Omega} \left[ \frac{\partial \psi_i}{\partial x_I} \frac{\partial \psi_j}{\partial x_J} \right] dx_1 dx_2
\]

with \( \frac{\partial \psi_i}{\partial x_0} = \psi_i \)

\[
\left[ \frac{\partial \psi_i}{\partial x_2} S_{IJ} \right] = \int_{\Omega} \left[ \frac{\partial \psi_i}{\partial x_2} \frac{\partial \psi_j}{\partial x_I} \frac{\partial \psi_j}{\partial x_J} \right] dx_1 dx_2
\]

For the tangent stiffness matrix, the coefficients are given by

\[
\left[ (K_T)^{13} \right] = 2[K^{13}] = [K^{31}]
\]

\[
\left[ (K_T)^{23} \right] = 2[K^{23}] = [K^{32}]
\]

\[
\left[ (K_T)^{33} \right] = [K^{33}]
\]

\[
\begin{align*}
+ A_{11} \left[ \frac{\partial u}{\partial x_1} S_{11} \right] + A_{21} \left[ \frac{\partial u}{\partial x_2} S_{22} \right] + A_{31} \left[ \frac{\partial u}{\partial x_1} \left( S_{21} + S_{12} \right) \right] \\
+ A_{13} \left[ \frac{\partial u}{\partial x_2} S_{11} \right] + A_{23} \left[ \frac{\partial u}{\partial x_2} S_{22} \right] + A_{33} \left[ \frac{\partial u}{\partial x_2} \left( S_{21} + S_{12} \right) \right]
\end{align*}
\]
+ \frac{1}{2} A_{12}\left[\frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{11}\right] + \left(\frac{\partial w}{\partial x_2}\right)^2 S_{11}

+ \frac{1}{2} A_{13}\left[\frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{12}\right] + \left(\frac{\partial w}{\partial x_2}\right)^2 S_{11}

+ \frac{1}{2} A_{21}\left[\frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{21}\right] + \left(\frac{\partial w}{\partial x_1}\right)^2 S_{22}

+ \frac{1}{2} A_{23}\left[\frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{22}\right] + \left(\frac{\partial w}{\partial x_2}\right)^2 S_{21}

+ \frac{1}{2} A_{31}\left[\frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{11}\right] + \left(\frac{\partial w}{\partial x_1}\right)^2 S_{12}

+ \frac{1}{2} A_{33}\left[\frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{22}\right] + \left(\frac{\partial w}{\partial x_2}\right)^2 S_{12}

+ \frac{1}{2} A_{32}\left[\frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{22}\right] + \left(\frac{\partial w}{\partial x_2}\right)^2 S_{21}

+ \frac{1}{2} A_{11}\left[\frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{21}\right] + A_{13}\left(\frac{\partial w}{\partial x_1}\right)^2 S_{11}

+ \frac{1}{2R_1} A_{11}\left(\frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_1} S_{01}\right) + A_{12}\left(\frac{\partial w}{\partial x_2} \frac{\partial w}{\partial x_2} S_{02}\right)
\[ + A_{13} \left( \frac{\partial w}{\partial x_2} S_{01} \right) + \frac{\partial w}{\partial x_1} S_{02} \right) \]

\[ + \frac{1}{2R_2} \left[ A_{21} \left( \frac{\partial w}{\partial x_1} S_{01} \right) + A_{22} \left( \frac{\partial w}{\partial x_2} S_{02} \right) + A_{23} \left( \frac{\partial w}{\partial x_2} S_{01} \right) \right. \]

\[ + \left( \frac{\partial w}{\partial x_1} S_{02} \right) \right] + \frac{1}{R_1} \left[ A_{11} [\text{WS}_{11}] + A_{21} [\text{WS}_{22}] \right] \]

\[ + \frac{1}{R_2} \left[ A_{12} [\text{WS}_{11}] + A_{22} [\text{WS}_{22}] \right] + \frac{1}{R_1} \left[ A_{31} [W(S_{12} + S_{21})] \right] \]

\[ + \frac{1}{R_2} \left[ A_{32} [(\text{WS}_{12} + S_{21})] \right] + A_{22} \left( \frac{\partial w}{\partial x_2} \right)^2 S_{22} \]

\[ + A_{23} \left( \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{22} \right) + A_{32} \left( \frac{\partial w}{\partial x_2} \right)^2 S_{12} \right) + A_{33} \left( \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{12} \right) \]

\[ + A_{31} \left( \frac{\partial w}{\partial x_1} \right)^2 S_{21} \right) + A_{33} \left( \frac{\partial w}{\partial x_1} \frac{\partial w}{\partial x_2} S_{21} \right) \]

\[ + M_1 [B_{11} [S_{11}] + B_{21} [S_{22}] + B_{31} [(S_{12} + S_{21})]] \]

\[ + M_2 [B_{12} [S_{11}] + B_{22} [S_{22}] + B_{32} [(S_{12} + S_{21})]] \]

\[ + M_6 [B_{13} [S_{11}] + B_{23} [S_{22}] + B_{33} [(S_{12} + S_{21})]] \]

\[ + P_1 [E_{11} [S_{11}] + E_{21} [S_{22}] + E_{31} [(S_{21} + S_{12})]] \]

\[ + P_2 [E_{12} [S_{11}] + E_{22} [S_{22}] + E_{32} [(S_{21} + S_{12})]] \]

\[ + P_6 [E_{13} [S_{11}] + E_{23} [S_{22}] + E_{33} [(S_{21} + S_{12})]] \]

\[ [(K_T)^{63}] = [K^{36}] \]
$$[ (K_T)^{73} ] = [K^{37}]$$

$$[ (K_T)^{83} ] = [K^{38}]$$

$$[ (K_T)^{93} ] = [K^{39}]$$

$$[ (K_T)^{10,3} ] = [K^{3,10}]$$

$$[ (K_T)^{11,3} ] = [K^{3,11}]$$

All others are same as those in [K].
The vita has been removed from the scanned document