A STUDY OF SOME FUNDAMENTAL EQUATIONS FOR THE DEFORMATION OF A VARIABLE THICKNESS PLATE

BY

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Thesis submitted to the Graduate Faculty of the
Virginia Polytechnic Institute
in candidacy for the degree of
DOCTOR OF PHILOSOPHY
in
Engineering Mechanics

1961

Blacksburg, Virginia

Grateful acknowledgement is made to Dr. D. M. Frederick for his inspiration and guidance during the preparation of this work and to my wife for her encouragement and patience through the years.

NOMENCLATURE

Θ_i = Dimensionless coordinate system

t = Thickness of plate

L = Characteristic length

λ = Non-dimensional parameter, t/L

$\bar{g}_i$ = covariant base vectors of the plate

$\bar{a}_\alpha$ = Covariant base vectors of the middle surface

g_{ij}, g^{ij} = Metric tensors of the plate

$a_{\alpha\beta}$, $a^{\alpha\beta}$ = Metric tensors of the middle surface

g = Determinant of the components g_{ij}

τ^{ij} = Contravariant stress tensor

$\bar{v}$ = Displacement vector in middle surface

v_α = Components of displacement vector

$n^{\alpha\beta}$ = Stress resultant

q^α = Shearing force

$m^{\alpha\rho}$ = Stress couple

p = Surface load

p^α = Surface traction

s^α = Surface couple

$\bar{t}$ = Stress vector

$\bar{t}_i$ = Stress vector for each coordinate surface

γ_{ij} = Covariant strain tensor

w = Deflection of middle surface in Θ_3 direction

$\bar{w}$ = Displacement vector with respect to middle surface

w_α = Components of $\bar{w}$

η = Poisson's ratio

μ = One of Lame's constants

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CHAPTER I

1. Introduction

The purpose of this dissertation is to develop a general approach to the problem of variable thickness plates. The notation and tensor approach of Green and Zerna (34) is used. In this way, the work is independent of the coordinate system until we are ready to apply the theory.

Most of the work done in the field of variable thickness plates has been done in the last twenty-five years. Olsson (see (1), (2), (4)) did much to lay the foundation for an approach to this problem. This approach is now what is called the classical one. The theory that was used for constant thickness plates was extended to allow the flexural rigidity to vary but not enough to invalidate the assumptions made for constant thickness plates. Federhofer (see (3), (6)) and Egger (see (5), (6)) also used this approach and made additional progress.

Reissner (see (7), (8), (15)) developed theory which includes the effect of shear deformation and normal stress or pressure. This theory was first applied only to constant thickness plates but recently Essenburg and Naghdi (see (32)) have extended it to varying thickness plates. Reissner's theory was used in other fields also, such as vibrations and elastic foundations. It allows three boundary conditions to be satisfied on each edge instead of two as in classical theory. Reissner (see (7)) proves a theorem that if we minimize a certain integral, we can derive simultaneously by variational principles, the stress-strain relations and the equations of equilibrium.

Considerable work has been done by Conway (see (10), (11), (20), (21), (24), (25), (28), (30)) but all of it has been done in the classical vein in the sense that Reissner's refined theory is not used. Others having done work in the same direction are Favre (see (13), (17)) and Contri (see (19), (23)).

Most of the work done in connection with variable thickness plates has concerned deflections and stresses; however, some work has been done in buckling by Chadaya (see (16)), Mansfield (see (27)), and Klein (see (28)). Critical values are found by approximate methods or by computer.

The approach of Essenburg and Naghdi (see (32)), as mentioned previously, seems to be a needed approach in that it compares the effects of transverse shear deformation, normal stress, and variation in thickness, with the usual terms found in the classical approach. It was found that the effects of these were small and that for practical purposes, the so called classical approach yielded very good results. Their technique consists of choosing stress-displacement relations which satisfy the equations of equilibrium and the boundary conditions and then using these relations in connection with Reisner's variational theorem in order to obtain suitable stress-strain relations.

In this dissertation, we shall show that we can start with the general stress-strain relations of elasticity and then by use of the strain-displacement relations, obtain the stress-displacement relations. We shall also derive the equations of equilibrium and other relations between the stress resultants, stress couples, and loads. This approach is almost the reverse of that of Essenburg and Naghdi. We will obtain the terms of their stress-displacement relations and in addition a term which involves the in-plane forces thus enabling us to consider the buckling of a variable thickness plate. Very little has been done in this field because of its inherent difficulties. The tensor approach to be used helps to keep the problem general. Much of the work done to date has been done by starting with a particular shaped plate and a particular type of thickness variation, mostly linear.

2. Base Vectors and the Metric Tensor

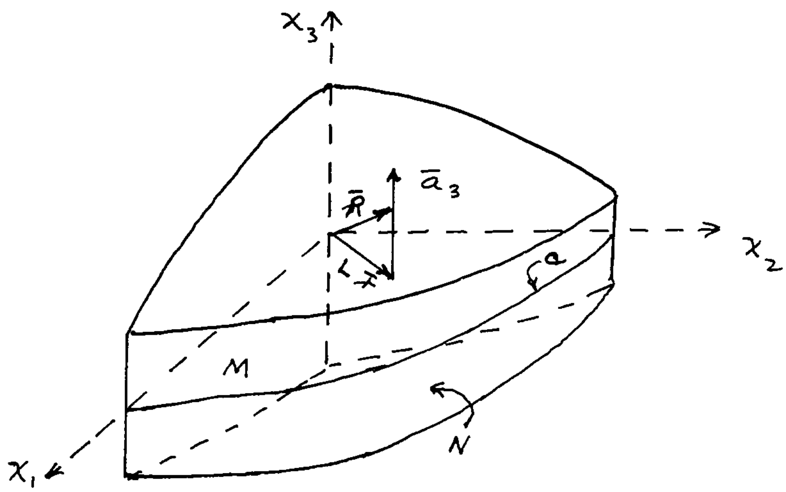

Figure 1. Coordinate System for a Plate of Variable Thickness

Consider a plane surface M and let L $\bar{r}$ denote the position vector of any point on M referred to some origin O. Then on M, we have

$$\bar{r} = \bar{r}\,(x_1, x_2) \quad , \tag{1.2.1}$$

where the bar over the letter denotes a vector. $\bar{a}_3$ is a unit normal vector on M and is not a function of the curvilinear coordinates x_1 and x_2. $x_3 = 0$ is the equation of M.

We restrict x_3 to lie in the interval $(-\frac{1}{2}t \leq x_3 \leq \frac{1}{2}t)$ where $t = t(x_1, x_2)$ is the thickness of the plate. Then $x_3 = \pm \frac{1}{2}t$ defines two more surfaces which may be taken as two boundaries of a plate S which includes the plane surface M as a middle surface. Let C be the curve which is the intersection of the edge boundary N and the middle surface M. The equation of the curve C is expressible as a function of x_1 and x_2.

The position vector of any point in the plate may be defined by

$$\bar{R} = L\,\bar{r} + x_3\,\bar{a}_3 \quad . \tag{1.2.2}$$

Let us replace the curvilinear coordinates x_i by non-dimensional coordinates θ_i, where

$$\theta_1 = \frac{x_1}{L_1}, \qquad \theta_2 = \frac{x_2}{L_2}, \qquad \theta_3 = \frac{x_3}{t} \tag{1.2.3}$$

and L_1 and L_2 are chosen so that θ_1 and θ_2 have ranges of unit length. Then θ_3 has the range $(-\frac{1}{2} \leq \theta_3 \leq \frac{1}{2})$.

Let us define a non-dimensional parameter

$$\lambda = t/L \tag{1.2.4}$$

so that we can write

$$\bar{R} = \bar{R}(\theta_1, \theta_2, \theta_3) = L\;\bar{r}(\theta_1, \theta_2) + \lambda\,\theta_3\,\bar{a}_3 \quad . \tag{1.2.5}$$

We have followed up to this point the approach to the study of shells used in (34). λ would be a constant for a plate of constant thickness. In this study, let λ be a function of θ_1 and θ_2 in order that we may study the case of a varying thickness plate.

Consider now the base vector $\bar{g}_\alpha$ and $\bar{g}_3$ which are defined by

$$\bar{g}_\alpha = \bar{R}_{,\alpha} \qquad \text{and} \qquad \bar{g}_3 = \bar{R}_{,3} \tag{1.2.6}$$

where the comma denotes partial differentiation. In all following work, Latin indices are to range over 1, 2, and 3 and Greek indices over only 1 and 2. Then we have

$$\begin{aligned} \bar{g}_\alpha &= L\left[\bar{r}_{,\alpha} + \lambda_{,\alpha}\,\theta_3\,\bar{a}_3\right] = L\left[\bar{a}_\alpha + \lambda_{,\alpha}\,\theta_3\,\bar{a}_3\right] \\ \bar{g}_3 &= L\,\lambda\,\bar{a}_3 \end{aligned} \tag{1.2.7}$$

if we define $\bar{a}_\alpha$ as

$$\bar{a}_\alpha = \bar{r}_{,\alpha} \quad . \tag{1.2.8}$$

We may now work out the components of the metric tensor by using (1.2.7) and

$$\begin{aligned} g_{\alpha\beta} &= \bar{g}_\alpha \cdot \bar{g}_\beta \\ g_{\alpha 3} &= \bar{g}_\alpha \cdot \bar{g}_3 \\ g_{33} &= \bar{g}_3 \cdot \bar{g}_3 \quad . \end{aligned} \tag{1.2.9}$$

We obtain

$$g_{\alpha\beta} = L^2 \left[a_{\alpha\beta} + \lambda_{,\alpha} \lambda_{,\beta} \theta_3^2 \right]$$

$$g_{\alpha 3} = L^2 \lambda \lambda_{,\alpha} \theta_3 \qquad (1.2.10)$$

$$g_{33} = L^2 \lambda^2$$

where $a_{\alpha\beta} = \bar{a}_\alpha \cdot \bar{a}_\beta$.

From (1.2.10) we can develop expressions for the contravariant components $g^{\alpha\beta}$, $g^{\alpha 3}$, g^{33} and for $g = \left| g_{ij} \right|$. Forming the determinant g, we have

$$g = \begin{vmatrix} L^2(a_{11} + \lambda_{,1} \lambda_{,1} \theta_3^2) & L^2(a_{12} + \lambda_{,1} \lambda_{,2} \theta_3^2) & L^2 \lambda \lambda_{,1} \theta_3 \\ L^2(a_{21} + \lambda_{,2} \lambda_{,1} \theta_3^2) & L^2(a_{22} + \lambda_{,2} \lambda_{,2} \ _3^2) & L^2 \lambda \lambda_{,2} \theta_3 \\ L^2 \lambda \lambda_{,1} \theta_3 & L^2 \lambda \lambda_{,2} \ _3 & L^2 \lambda^2 \end{vmatrix}$$

(1.2.11)

Expanding the determinant, we find that due to symmetry, many of the terms vanish and we get

$$g = \lambda^2 L^6 (a_{11} a_{22} - a_{12}^2) \qquad (1.2.12)$$

and if we let

$$a = a_{11} a_{22} - a_{12}^2 \qquad (1.2.13)$$

we have

$$g = \lambda^2 L^6 a$$

which can be written in the form

$$\frac{\sqrt{g}}{\sqrt{a}} = \lambda L^3 \quad . \tag{1.2.14}$$

If we now use

$$g^{ij} = \frac{D^{ij}}{g} \tag{1.2.15}$$

where D^{ij} is the cofactor of g_{ij} in g, we get

$$g\, g^{11} = L^4 \lambda^2 a_{22}$$

$$g\, g^{22} = L^4 \lambda^2 a_{11} \tag{1.2.16}$$

$$g\, g^{12} = -L^4 \lambda^2 a_{12}$$

$$g\, g^{13} = L^4(\lambda \lambda_{,2} a_{21} \theta_3 - \lambda \lambda_{,1} a_{22} \theta_3)$$

$$g\, g^{23} = -L^4(\lambda \lambda_{,2} a_{11} \theta_3 - \lambda \lambda_{,1} a_{12} \theta_3) \tag{1.2.17}$$

$$g\, g^{33} = L^4(a_{11}a_{22} + \lambda_{,2} \lambda_{,2} a_{11} \theta_3^2 + \lambda_{,1} \lambda_{,1} a_{22} \theta_3^2 - a_{12}^2$$

$$- 2 \lambda_{,1} \lambda_{,2} a_{12} \theta_3^2) \quad . \tag{1.2.18}$$

From the relationship $a_{\alpha\beta}\, a^{\beta\gamma} = \delta_\alpha^\gamma$, we show that

$$a^{11} = a_{22}/a$$

$$a^{12} = a^{21} = -a_{12}/a \tag{1.2.19}$$

$$a^{22} = a_{11}/a \quad .$$

By using (1.2.19), we find that (1.2.16), (1.2.17) and (1.2.18) can be written as

$$g^{\alpha\beta} = \frac{a^{\alpha\beta}}{L^2} \tag{1.2.20}$$

$$g^{\alpha 3} = \frac{-\lambda_{,\beta}\, a^{\alpha\beta}\, \theta_3}{\lambda L^2} \tag{1.2.21}$$

$$g^{33} = \frac{1 + \lambda_{,\alpha}\lambda_{,\beta}\, a^{\alpha\beta}\, \theta_3^2}{\lambda^2 L^2} \; . \tag{1.2.22}$$

It will be convenient to work out here the value of $\sqrt{g^{33}}$ evaluated when $\theta_3 = \frac{1}{2}$ and $\theta_3 = -\frac{1}{2}$. Since g^{33} is an even function in θ_3, we see that

$$\sqrt{g^{33}}\Big]^{\theta_3 = \frac{1}{2}} = \sqrt{g^{33}}\Big]_{\theta_3 = -\frac{1}{2}} = \frac{\sqrt{1 + \frac{1}{4}\lambda_{,\alpha}\,\lambda_{,\beta}\,a^{\alpha\beta}}}{\lambda L} \tag{1.2.23}$$

and if we let

$$\Theta = \sqrt{1 + \tfrac{1}{4}\,\lambda_{,\alpha}\,\lambda_{,\beta}\,a^{\alpha\beta}} \tag{1.2.24}$$

then we can write

$$\sqrt{g^{33}}\Big]^{\theta_3 = \frac{1}{2}} = \sqrt{g^{33}}\Big]_{\theta_3 = -\frac{1}{2}} = \frac{\Theta}{\lambda L} \; . \tag{1.2.25}$$

The triple scalar product $[\,\bar{g}_\alpha \;\bar{g}_\beta \;\bar{g}_3\,]$ may be evaluated by (1.2.7) and we find that

$$\sqrt{g/a} = \lambda L^3 \tag{1.2.26}$$

where

$$g = |g_{ij}| \quad \text{and} \quad a = |a_{\alpha\beta}| \; . \tag{1.2.27}$$

3. Surface Tensors

The stress vector on θ_i = constant may be represented by

$$\bar{t}_i = \frac{\tau^{ik}}{\sqrt{g^{ii}}} \bar{g}_k \qquad (1.3.1)$$

where τ^{ik} is the stress tensor. Then from (1.2.7) we have

$$\bar{t}_i = \frac{L}{\sqrt{g^{ii}}} \left[\tau^{i\rho} \bar{a}_\rho + (\tau^{i\alpha} \lambda_{,\alpha} \theta_3 + \tau^{i3} \lambda) \bar{a}_3 \right] . \qquad (1.3.2)$$

The force on an element in the surface θ_1 = constant is $\bar{t}_1 \, ds_1$ and $ds_1 = \sqrt{gg^{11}} \, d\theta^2 \, d\theta^3$. Then the force is $\bar{t}_1 \sqrt{gg^{11}} \, d\theta^2 \, d\theta^3 = \bar{T}_1 \, d\theta^2 \, d\theta^3$ where

$$\bar{T}_i = \bar{t}_i \sqrt{gg^{ii}} . \qquad (1.3.3)$$

The length of the corresponding line element of the middle surface M is

$$L \sqrt{a_{22}} \, d\theta^2 = L \sqrt{aa^{11}} \, d\theta^2 . \qquad (1.3.4)$$

We can now get a physical stress resultant $\bar{n}_1$ and a physical stress couple L $\bar{m}_1$ measured per unit length in M by integrating over θ_3.

$$\bar{n}_1 = \frac{1}{L \quad aa^{11}} \int_{-1/2}^{1/2} \bar{T}_1 \, d\theta^3 \qquad (1.3.5)$$

and

$$L\,\bar{m}_1 = \frac{1}{L\,aa^{11}} \int_{-1/2}^{1/2} (\bar{a}_3 \times \bar{T}_1)\, x_3 \, d\theta^3 . \tag{1.3.6}$$

We may do the same for θ_2 = constant and then write

$$\bar{n}_\alpha = \frac{\bar{N}_\alpha}{\sqrt{aa^{\alpha\alpha}}} \tag{1.3.7}$$

and

$$\bar{m}_\alpha = \frac{\bar{M}_\alpha}{\sqrt{aa^{\alpha\alpha}}} \tag{1.3.8}$$

where

$$\bar{N}_\alpha = 1/L \int_{-1/2}^{1/2} \bar{T}_\alpha \; d\,\theta^3 \tag{1.3.9}$$

and

$$\bar{M}_\alpha = \lambda/L \int_{-1/2}^{1/2} (\bar{a}_3 \times \bar{T}_\alpha)\;\; \theta_3 \, d\,\theta^3 . \tag{1.3.10}$$

If we put (1.3.2) into (1.3.3), we have

$$\bar{T}_i = L\sqrt{g}\,\left[\tau^{i\rho}\,\bar{a}_\rho + (\tau^{i\alpha}\,\lambda_{,\alpha}\;\;\theta_3 + \tau^{i3}\,\lambda)\;\bar{a}_3 \right] . \tag{1.3.11}$$

Then putting (1.3.11) into (1.3.7) and (1.3.9) we get

$$\sqrt{a^{\alpha\alpha}}\;\bar{n}_\alpha = 1/L \int_{-1/2}^{1/2} L\sqrt{\frac{g}{a}}\left[\tau^{\alpha\rho}\,\bar{a}_\rho + (\tau^{\alpha\rho}\,\lambda_{,\rho}\;\theta_3 + \tau^{\alpha 3}\lambda)a_3\right] d\theta^3. \tag{1.3.12}$$

Now if we use (1.2.11) and define $n^{\alpha\rho}$, q^{α}, and f^{α} by the following expressions,

$$n^{\alpha\rho} = \int_{-1/2}^{1/2} L^3 \ \tau^{\alpha\rho} \ d\,\theta^3 \tag{1.3.13}$$

$$q^{\alpha} = \lambda \int_{-1/2}^{1/2} \lambda \ L^3 \ \tau^{\alpha 3} \ d\,\theta^3 \tag{1.3.14}$$

$$f^{\alpha} = \lambda_{,\rho} \int_{-1/2}^{1/2} \lambda \, L^3 \ \tau^{\alpha\rho} \ \theta_3 \ d\,\theta^3 \tag{1.3.15}$$

then we can write (1.3.12) as

$$\sqrt{a^{\alpha\alpha}} \ \bar{n}_{\alpha} = n^{\alpha\rho} \ \bar{a}_{\rho} + q^{\alpha} \ \bar{a}_3 + f^{\alpha} \ \bar{a}_3 \ . \tag{1.3.16}$$

By using (1.3.8) and (1.3.10) in the same manner, we get

$$\sqrt{a^{\alpha\alpha}} \ \bar{m}_{\alpha} = \frac{\lambda}{L} \int_{-1/2}^{1/2} L\sqrt{\frac{g}{a}} \ \tau^{\alpha\rho} \ \theta_3 \ \bar{a}_3 \times \bar{a}_{\rho} \ d\,\theta_3 = m^{\alpha\rho} \ \bar{a}_3 \times \bar{a}_{\rho} \tag{1.3.17}$$

if we let

$$m^{\alpha\rho} = \lambda \int_{-1/2}^{1/2} \lambda \, L^3 \ \tau^{\alpha\rho} \ \theta_3 \ d\,\theta^3 \ . \tag{1.3.18}$$

The quantities defined by (1.3.13), (1.3.14), (1.3.15), and (1.3.18) are surface tensors and are called stress resultants, shearing forces, and stress couples.

We will need the components of $\bar{n}_{\alpha}$ and $L\,\bar{m}_{\alpha}$ when they are expressed in terms of unit base vectors. Now $\bar{a}_3$ is already a unit vector parallel to the x_3 axis but $\bar{a}_\alpha$ was defined as $\bar{a}_\alpha = \bar{r}_{,\alpha}$ so that we can write

$$\bar{n}_\alpha = n_{(\alpha 1)} \frac{\bar{a}_1}{\sqrt{a_{11}}} + n_{(\alpha 2)} \frac{\bar{a}_2}{\sqrt{a_{22}}} + (q_{(\alpha)} + f_{(\alpha)})\, \bar{a}_3 \tag{1.3.19}$$

$$L\, \bar{m}_\alpha = m_{(\alpha 1)} \frac{\bar{a}^2}{\sqrt{a^{22}}} + m_{(\alpha 2)} \frac{\bar{a}^1}{\sqrt{a^{11}}} \tag{1.3.20}$$

where parentheses have been used around the subscripts to indicate that the quantities involved are nottensors but are the physical stress resultants, physical shearing forces, and physical stress couples.

If we compare (1.3.19) with (1.3.16) and (1.3.20) with (1.3.17), we see that

$$n_{(\alpha\beta)} = n^{\alpha\beta} \sqrt{a_{\beta\beta} / a^{\alpha\alpha}} \tag{1.3.21}$$

$$q_{(\alpha)} = q^\alpha / \sqrt{a^{\alpha\alpha}} \tag{1.3.22}$$

$$f_{(\alpha)} = f^\alpha / \sqrt{a^{\alpha\alpha}} \tag{1.3.23}$$

$$\begin{aligned} m_{(11)} &= L\, m^{11} \sqrt{a_{11}/a^{11}} \qquad & m_{(12)} &= -L\, m^{12} \sqrt{a} \\ m_{(21)} &= L\, m^{21} \sqrt{a} & m_{(22)} &= -L\, m^{22} \sqrt{a_{22}/a^{22}} \quad . \end{aligned} \tag{1.3.24}$$

From (1.3.7) and (1.3.16) we can write

$$\bar{N}_\alpha = N^{\alpha\rho}\, \bar{a}_\rho + Q^\alpha\, \bar{a}_3 + F^\alpha\, \bar{a}_3 \tag{1.3.25}$$

if we let

$$N^{\alpha\beta} = \sqrt{a}\; n^{\alpha\beta} \quad \text{and} \quad Q^{\alpha} = \sqrt{a}\; q^{\alpha} \quad \text{and} \quad F^{\alpha} = \sqrt{a}\; f^{\alpha}\,. \tag{1.3.26}$$

From (1.3.17) and (1.3.8), we have

$$\bar{M}_{\alpha} = \sqrt{a}\; \bar{m}_{\alpha} \sqrt{a^{\alpha\alpha}} = \sqrt{a}\; m^{\alpha\beta} \;(\bar{a}_3 \times \bar{a}_{\beta}) \tag{1.3.27}$$

and thus if we let

$$M^{\alpha\beta} = \sqrt{a}\; m^{\alpha\beta} \tag{1.3.28}$$

we have

$$\bar{M}_{\alpha} = M^{\alpha\beta} \;(\bar{a}_3 \times \bar{a}_{\beta})\,. \tag{1.3.29}$$

4. Loads

Consider an element of the middle surface M of the plate which is bounded by the curves θ_1 = constant, θ_2 = constant, $\theta_1 + d\theta^1$ = constant and $\theta_2 + d\theta^2$ = constant. The area of middle surface of the element of volume is $L^2 \sqrt{a}\, d\theta^1\, d\theta^2$.

We replace the surface forces by a resultant force $\bar{p}$, measured per unit area of the middle surface and a couple $\bar{s}$ about the point (θ_1, θ_2) of M.

The boundary conditions on the faces of the plate require that the forces per unit area acting on these faces should be

$$\left[\bar{t}_3\right]_{\theta_3 = \frac{1}{2}} \quad , \quad \left[-\bar{t}_3\right]_{\theta_3 = -\frac{1}{2}}$$

where $\bar{t}_3$ is as given by (1.3.2).

We now replace the surface forces by a force

$$L\,\bar{p} = L\left[\bar{t}_3\right]_{\theta_3 = +\frac{1}{2}} - L\left[\bar{t}_3\right]_{\theta_3 = -\frac{1}{2}} = L\left[\bar{t}_3\right]_{-\frac{1}{2}}^{\frac{1}{2}} . \qquad (1.4.1)$$

and a couple

$$L\,\bar{s} = \lambda L \left[(\bar{a}_3 \times \bar{t}_3)\,\theta_3\right]_{-\frac{1}{2}}^{\frac{1}{2}} \qquad (1.4.2)$$

where $\bar{p}$ and $\bar{s}$ are measured per unit area of the middle surface M.
From (1.4.1) and (1.3.2), we have

$$L\,\bar{p} = \left[\frac{L^2}{g^{33}}\right]^{\frac{1}{2}} \left\{ \tau^{3\rho}\right]^{\frac{1}{2}} \bar{a}_\rho + \left(\lambda_{,\alpha}\tau^{3\alpha}\,\theta_3\right]^{\frac{1}{2}} + \lambda\tau^{33}\Big]^{\frac{1}{2}}\Big)\bar{a}_3\Big\}$$

$$(1.4.3)$$

$$- \left[\frac{L^2}{g^{33}}\right]_{-\frac{1}{2}} \left\{ \tau^{3\rho}\right]_{-\frac{1}{2}} \bar{a}_\rho + \left(\lambda_{,\alpha}\,\tau^{3\alpha}\,\theta_3\right]_{-\frac{1}{2}} + \lambda\tau^{33}\Big]_{-\frac{1}{2}}\Big)\bar{a}_3\Big\} .$$

If we now use (1.2.25), we have

$$L\,\bar{p} = \frac{\lambda L^3}{\Theta}\left\{\tau^{3\alpha}\right]_{-\frac{1}{2}}^{\frac{1}{2}} \bar{a}_\alpha + \left(\lambda_{,\alpha}\tau^{3\alpha}\,\theta_3\right]_{-\frac{1}{2}}^{\frac{1}{2}} + \lambda\tau^{33}\Big]_{-\frac{1}{2}}^{\frac{1}{2}}\Big)\bar{a}_3\Big\} .$$

$$(1.4.4)$$

If we now define p^α and p and s^α as

$$p^\alpha = \frac{\lambda L^3}{\Theta}\left[\tau^{3\alpha}\right]_{-\frac{1}{2}}^{\frac{1}{2}} \qquad (1.4.5)$$

$$p = \frac{\lambda^2 L^3}{\Theta}\left[\tau^{33}\right]_{-\frac{1}{2}}^{\frac{1}{2}} \qquad (1.4.6)$$

$$s^{\alpha} = \frac{\lambda L^3}{\oplus} \begin{bmatrix} \tau^{3\alpha} & \theta_3 \end{bmatrix}_{-\frac{1}{2}}^{\frac{1}{2}} \tag{1.4.7}$$

then we may write (1.4.4) as

$$L\,\bar{p} = p^{\alpha}\ \bar{a}_{\alpha} + \left(\frac{\lambda_{,\alpha}}{\lambda} s^{\alpha} + p \right) \bar{a}_3 \ . \tag{1.4.8}$$

In terms of unit base vectors and physical components, we shall write

$$\bar{p} = p_{(\alpha)} \frac{\bar{a}}{\sqrt{a_{\alpha\alpha}}} + p_{(3)}\ \bar{a}_3 \ . \tag{1.4.9}$$

Comparing (1.4.8) and (1.4.9), we see that

$$p^{\alpha} = \frac{L\ p_{(\alpha)}}{\sqrt{a_{\alpha\alpha}}} \quad \text{and} \quad p + \lambda_{,\alpha}\ s^{\alpha} = L\ p_{(3)} \ . \tag{1.4.10}$$

Then from (1.4.2) and (1.3.2), we have

$$L\,\bar{s} = \left[\frac{\lambda^2\ L^3}{\oplus} \tau^{3\rho} \quad \theta_3\ (\bar{a}_3 \times \bar{a}_{\rho}) \right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

which can be written by use of (1.47) as

$$L\,\bar{s} = s^{\alpha}\ (\bar{a}_3 \times \bar{a}_{\alpha}) = s^{\alpha}\ \epsilon_{\alpha\rho}\ \ \bar{a}^{\rho} \ . \tag{1.4.11}$$

In terms of unit base vectors and physical components, we could write

$$\bar{s} = s_{(2)} \frac{\bar{a}^1}{\sqrt{a^{11}}} + s_{(1)} \frac{\bar{a}^2}{\sqrt{a^{22}}} \tag{1.4.12}$$

so that from (1.4.11) and (1.4.12), we conclude

$$L\, s_{(1)} = s^1 \sqrt{aa^{22}} = s^1 \sqrt{a_{11}}$$

$$L\, s_{(2)} = -s^2 \sqrt{aa^{11}} = -s^2 \sqrt{a_{22}} \quad . \tag{1.4.13}$$

It is convenient to introduce the following notation

$$P^{\alpha} = p^{\alpha} \sqrt{a} \quad \text{and} \quad P = (p + \lambda_{,\alpha}\, s^{\alpha}) \sqrt{a} \tag{1.4.14}$$

and

$$S^{\alpha} = s^{\alpha} \sqrt{a} \tag{1.4.15}$$

so that

$$\bar{P} = L\, \bar{p} \sqrt{a} = P^{\alpha}\, \bar{a}_{\alpha} + P\, \bar{a}_3 \quad . \tag{1.4.16}$$

5. <u>Equations of Equilibrium</u>.

If we neglect the body forces, we can write the equations of equilibrium as

$$\bar{T}_{i,i} = 0 \quad . \tag{1.5.1}$$

We can express these equations in terms of the stress resultants and stress couples if we integrate with respect to x_3 over the thickness of the plate. We have $x_3 = \lambda L\, \theta_3$ so that (1.5.1) becomes

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{T}_{i,i}\, \lambda\, L\, d\,\theta^3 = 0 \quad . \tag{1.5.2}$$

Simplifying and integrating we get

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{T}_{\alpha,\alpha}\, d\,\theta^3 + \Big[\bar{T}_3 \Big]_{-\frac{1}{2}}^{\frac{1}{2}} = 0 \tag{1.5.3}$$

Using (1.3.9) and the fact that

$$\bar{P} = L \left[\bar{T}_3 \right]_{-1/2}^{1/2} \qquad (1.5.4)$$

we have

$$\bar{N}_{\alpha,\alpha} + \bar{P} = 0 \ . \qquad (1.5.5)$$

From (1.3.25) and (1.4.14), we have (1.5.5) in the form

$$N^{\alpha\rho}{}_{,\alpha}\, \bar{a}_\rho + N^{\alpha\rho}\, \bar{a}_{\rho,\alpha} + Q^{\alpha}{}_{,\alpha}\, \bar{a}_3 + F^{\alpha}{}_{,\alpha}\, \bar{a}_3 + P^{\rho}\, \bar{a}_\rho + P\, \bar{a}_3 = 0. \qquad (1.5.6)$$

Simplifying and equating coefficients of $\bar{a}_i$, we get

$$N^{\alpha\beta}{}_{,\alpha} + \left\{ {\beta \atop \rho \ \alpha} \right\} N^{\alpha\rho} + P^{\beta} = 0 \qquad (1.5.7)$$

and

$$Q^{\alpha}{}_{,\alpha} + F^{\alpha}{}_{,\alpha} + P = 0 \ . \qquad (1.5.8)$$

Using (1.3.26) and (1.4.14), we can write (1.5.8) as

$$q^{\alpha}\big|_{\alpha} + f^{\alpha}\big|_{\alpha} + \circledast p + \lambda_{,\alpha} \circledast s^{\alpha} = 0 \qquad (1.5.9)$$

and in the same manner, (1.5.7) becomes

$$n^{\alpha\beta}\big|_{\alpha} + \circledast\, p^{\beta} = 0 \ . \qquad (1.5.10)$$

If we take the vector product of $x_3\, \bar{a}_3$ with both sides of (1.5.1) and integrate with respect to x_3, we get

$$\int_{-t/2}^{t/2} \left[\bar{a}_3 \times \bar{T}_{i,i} \right] x_3\, dx_3 = 0 \ . \qquad (1.5.11)$$

In terms of the variable θ_3, (1.5.11) becomes

$$\int_{-1/2}^{1/2} \left[\bar{a}_3 \times \bar{T}_{\alpha,\alpha} \right] \quad \theta_3 \, d\,\theta^3 + \int_{-1/2}^{1/2} \left[\bar{a}_3 \times \bar{T}_{3,3} \right] \theta_3 \, d\,\theta^3 = 0 \quad . \tag{1.5.12}$$

Integrating the last term by parts, we have

$$\int_{-1/2}^{1/2} \left[\bar{a}_3 \times \bar{T}_{\alpha,\alpha} \right] \quad \theta_3 \, d\,\theta^3 - \int_{-1/2}^{1/2} \left[\bar{a}_3 \times T_3 \right] d\,\theta^3 + \left[\theta_3 (\bar{a}_3 \times T_3) \right]_{-1/2}^{1/2} = 0. \tag{1.5.13}$$

From (1.3.10), we obtain

$$\bar{M}_{\alpha,\alpha} = \frac{\lambda_{,\alpha}}{L} \int_{-1/2}^{1/2} (\bar{a}_3 \times \bar{T}_\alpha) \, \theta_3 \, d\,\theta^3 + \frac{\lambda}{L} \int_{-1/2}^{1/2} (\bar{a}_3 \times \bar{T}_{\alpha,\alpha}) \, \theta_3 \, d\theta^3 \, . \tag{1.5.14}$$

Solving for the last integral, we have

$$\int_{-1/2}^{1/2} (\bar{a}_3 \times \bar{T}_{\alpha,\alpha}) \, \theta_3 \, d\,\theta^3 = L/\lambda \; \bar{M}_{\alpha,\alpha} - \frac{\lambda_{,\alpha}}{\lambda} \int_{-1/2}^{1/2} (\bar{a}_3 \times \bar{T} \;) \, \theta_3 \, d\,\theta^3. \tag{1.5.15}$$

Since from (1.3.27), we have

$$\bar{M}_\alpha = \sqrt{a} \quad m^{\alpha\rho} \; (\bar{a}_3 \times \bar{a}_\rho) \tag{1.3.27}$$

we can find $\bar{M}_{\alpha,\alpha}$ to be

$$\bar{M}_{\alpha,\alpha} = m^{\alpha\rho}{}_{,\alpha} \sqrt{a} \, (\bar{a}_3 \times \bar{a}_\rho) + m^{\alpha\rho} \sqrt{a} \begin{Bmatrix} \beta \\ \beta \;\; \alpha \end{Bmatrix} (\bar{a}_3 \times a_\rho) + m^{\alpha\rho} \sqrt{a} \begin{Bmatrix} \beta \\ \rho \;\; \alpha \end{Bmatrix} (\bar{a}_3 \times \bar{a}_\beta) \; . \tag{1.5.16}$$

From (1.3.11), we find that

$$\bar{a}_3 \times \bar{T}_\alpha = L \sqrt{g}\, \tau^{3\rho} (\bar{a}_3 \times \bar{a}_\rho)$$

and thus by (1.3.18),

$$\int_{-1/2}^{1/2} (\bar{a}_3 \times \bar{T}_\alpha)\, \theta_3\, d\,\theta_3 = L \sqrt{g}\, (\bar{a}_3 \times \bar{a}_\rho) \int_{-1/2}^{1/2} \tau^{3\rho}\, \theta_3\, d\,\theta^3$$

$$= \frac{L}{\lambda} \sqrt{a}\, (\bar{a}_3 \times \bar{a}_\rho)\, m^{\alpha\rho} \,. \tag{1.5.17}$$

If we put (1.5.16) and (1.5.17) into (1.5.15), we have

$$\int_{-1/2}^{1/2} (\bar{a}_3 \times \bar{T}_{\alpha,\alpha})\, \theta_3\, d\,\theta^3 = \frac{L\sqrt{a}}{\lambda} m^{\alpha\rho}{}_{,\alpha} (\bar{a}_3 \times \bar{a}_\rho) + \frac{L}{\lambda} \sqrt{a}\, m^{\alpha\rho} \left\{ {\beta \atop \beta\ \alpha} \right\} (\bar{a}_3 \times \bar{a}_\rho)$$

$$+ \frac{L}{\lambda} \sqrt{a}\, m^{\alpha\rho} \left\{ {\beta \atop \rho\ \alpha} \right\} (\bar{a}_3 \times \bar{a}_\beta)$$

$$- \frac{L\sqrt{a}}{\lambda} \frac{\lambda_{,\alpha}}{\lambda} m^{\alpha\rho} (\bar{a}_3 \times \bar{a}_\rho) \,. \tag{1.5.18}$$

We now have an expression for the first term of (1.5.13). If we simplify the last two terms of (1.5.13), we get from (1.3.11) and (1.3.14)

$$\int_{-1/2}^{1/2} (\bar{a}_3 \times \bar{T}_3)\, d\,\theta^3 = \frac{L\sqrt{a}}{\lambda} (\bar{a}_3 \times \bar{a}_\rho)\, q^\rho \tag{1.5.19}$$

and from (1.3.11) and (1.4.7)

$$\Big[\theta_3 (\bar{a}_3 \times \bar{T}_3) \Big]_{-1/2}^{1/2} = \frac{\Phi\, L\sqrt{a}}{\lambda} (\bar{a}_3 \times \bar{a}_\rho)\, s^\rho \,. \tag{1.5.20}$$

If we now put (1.5.18), (1.5.19), and (1.5.20) back into (1.5.13), we get

$$\frac{L}{\lambda}\sqrt{a}\;m^{\alpha\rho}{}_{,\alpha}(\bar{a}_3 \times \bar{a}_\rho) + \frac{L}{\lambda}\sqrt{a}\;m^{\alpha\rho}\left\{{}_{\beta}{}^{\beta}{}_{\alpha}\right\}(\bar{a}_3 \times \bar{a}_\rho)$$

$$+\frac{L}{\lambda}\sqrt{a}\;m^{\alpha\rho}\left\{{}_{\rho}{}^{\beta}{}_{\alpha}\right\}(\bar{a}_3 \times \bar{a}_\beta) - \frac{L\sqrt{a}}{\lambda}\frac{\lambda_{,\alpha}}{\lambda}m^{\alpha\rho}(\bar{a}_3 \times \bar{a}_\rho)$$

$$-\frac{L\sqrt{a}}{\lambda}(\bar{a}_3 \times \bar{a}_\rho)\;q^\rho + \frac{\oplus L\sqrt{a}}{\lambda}(\bar{a}_3 \times \bar{a}_\rho)\;s^\rho = 0.$$

Equating coefficients and simplifying, we get

$$m^{\alpha\rho}\Big|_\alpha - \frac{\lambda_{,\alpha}}{\lambda}m^{\alpha\rho} - q^\rho + \oplus s^\rho = 0 \quad . \tag{1.5.21}$$

From (1.3.15) and (1.3.18), it is obvious that

$$f^\rho = \frac{\lambda_{,\alpha}}{\lambda}\;m^{\alpha\rho} \tag{1.5.22}$$

so that (1.5.21) can be written as

$$m^{\alpha\rho}\Big|_\alpha - f^\rho - q^\rho + \oplus s^\rho = 0 \quad . \tag{1.5.23}$$

Equation (1.5.23) along with (1.5.9) and (1.5.10) give us five equations of equilibrium in terms of the stress resultants, stress couples and loads. Note that for a constant thickness plate, we have $f^\rho = 0$ and the equations are those given in (34). We note also that they differ from those obtained in (32) in the terms involving f^ρ, since their equilibrium equation does not contain f^ρ at all.

CHAPTER II

1. Stress-Strain Relations

It is shown in Green and Zerna (34) that in general for a homogeneous isotropic material, we have

$$\tau^{ik} = \mu\left\{g^{ir}\, g^{ks} + g^{is}\, g^{ir} + \frac{1}{1-2\eta} g^{ik}\, g^{rs}\right\}\gamma_{rs} \tag{2.1.1}$$

where μ and η are elastic constants. The contravariant components of the metric tensor are as given in (1.2.20), (1.2.21), and (1.2.22).

From (2.1.1), we find that

$$\begin{aligned}\tau^{\alpha\beta} &= \mu\left\{g^{\alpha\omega}\, g^{\beta\rho} + g^{\alpha\rho}\, g^{\beta\omega} + \frac{2\eta}{1-2\eta} g^{\alpha\beta} g^{\omega\rho}\right\}\gamma_{\omega\rho}\\ &+ 2\mu\left\{g^{\alpha3}\, g^{\beta\rho} + g^{\alpha\rho}\, g^{\beta3} + \frac{2\eta}{1-2\eta} g^{\alpha\beta}\, g^{3\rho}\right\}\gamma_{3\rho}\\ &+ \mu\left\{2\, g^{\alpha3}\, g^{\beta3} + \frac{2\eta}{1-2\eta} g^{\alpha\beta}\, g^{33}\right\}\gamma_{33}\end{aligned} \tag{2.1.2}$$

$$\begin{aligned}\tau^{\alpha3} &= \mu\left\{g^{\alpha\beta}\, g^{3\rho} + g^{\alpha\rho}\, g^{3\beta} + \frac{2\eta}{1-2\eta} g^{\alpha3}\, g^{\beta\rho}\right\}\gamma_{\beta\rho}\\ &+ 2\mu\left\{g^{\alpha3}\, g^{3\rho} + g^{\alpha\rho}\, g^{33} + \frac{2\eta}{1-2\eta} g^{\alpha3}\, g^{3\rho}\right\}\gamma_{3\rho}\\ &+ \mu\left\{2\, g^{\alpha3}\, g^{33} + \frac{2\eta}{1-2\eta} g^{\alpha3}\, g^{33}\right\}\gamma_{33}\end{aligned} \tag{2.1.3}$$

$$\begin{aligned}\tau^{33} &= \mu\left\{g^{3\rho}\, g^{3\beta} + g^{3\beta}\, g^{3\rho} + \frac{2\eta}{1-2\eta} g^{33}\, g^{\rho\beta}\right\}\gamma_{\rho\beta}\\ &+ 2\mu\left\{g^{33}\, g^{3\beta} + g^{3\beta}\, g^{33} + \frac{2\eta}{1-2\eta} g^{33}\, g^{3\beta}\right\}\gamma_{3\beta}\\ &+ \mu\left\{g^{33}\, g^{33} + g^{33}\, g^{33} + \frac{2\eta}{1-2\eta} g^{33}\, g^{33}\right\}\gamma_{33}.\end{aligned} \tag{2.1.4}$$

If we now let

$$B^{\alpha\beta\rho\omega} = \mu(g^{\alpha\omega} g^{\beta\rho} + g^{\alpha\rho} g^{\beta\omega} + \frac{2\eta}{1-2\eta} g^{\alpha\beta} g^{\rho\omega}) \tag{2.1.5}$$

$$B^{\alpha\beta\rho 3} = 2\mu(g^{\alpha 3} g^{\beta\rho} + g^{\alpha\rho} g^{\beta 3} + \frac{2\eta}{1-2\eta} g^{\alpha\beta} g^{3\rho}) \tag{2.1.6}$$

$$B^{\alpha\beta 3} = \mu(2 g^{\alpha 3} g^{\beta 3} + \frac{2\eta}{1-2\eta} g^{\alpha\beta} g^{33}) \tag{2.1.7}$$

$$A^{\alpha\beta\rho 3} = \mu(g^{\alpha\beta} g^{3\rho} + g^{\alpha\rho} g^{3\beta} + \frac{2\eta}{1-2\eta} g^{\alpha 3} g^{\beta\rho}) \tag{2.1.8}$$

$$A^{\alpha\rho 3} = 2\mu(g^{\alpha 3} g^{3\rho} + g^{\alpha\rho} g^{33} + \frac{2\eta}{1-2\eta} g^{\alpha 3} g^{3\rho}) \tag{2.1.9}$$

$$A^{\alpha} = \mu(2 g^{\alpha 3} g^{33} + \frac{2\eta}{1-2\eta} g^{\alpha 3} g^{33}) \quad , \tag{2.1.10}$$

$$C = \frac{2\mu(1-\eta)}{1-2\eta} g^{33} g^{33} \tag{2.1.11}$$

then we can write

$$\tau^{\alpha\beta} = B^{\alpha\beta\rho\omega} \gamma_{\omega\rho} + B^{\alpha\beta\rho 3} \gamma_{3\rho} + B^{\alpha\beta 3} \gamma_{33} \tag{2.1.12}$$

$$\tau^{\alpha 3} = A^{\alpha\omega\rho 3} \gamma_{\omega\rho} + A^{\alpha\rho 3} \gamma_{3\rho} + A^{\alpha} \gamma_{33} \tag{2.1.13}$$

$$\tau^{33} = B^{\omega\rho 3} \gamma_{\omega\rho} + 2 A^{\rho} \gamma_{3\rho} + C \gamma_{33} \quad . \tag{2.1.14}$$

We may now use (2.1.14) to eliminate γ_{33} from (2.1.12) and (2.1.13) and we get

$$\tau^{\alpha\beta} = \left(B^{\alpha\beta\rho\omega} - \frac{B^{\alpha\beta 3}\,B^{\omega\rho 3}}{C}\right)\gamma_{\omega\rho} + \left(B^{\alpha\beta\rho 3} - \frac{2\,A^{\alpha\beta 3}\,A^{\rho}}{C}\right)\gamma_{3\rho}$$

$$+ \frac{B^{\alpha\beta 3}}{C}\tau^{33} \tag{2.1.15}$$

$$\tau^{\alpha 3} = \left(A^{\alpha\omega\rho 3} - \frac{A^{\alpha}\,B^{\omega\rho 3}}{C}\right)\gamma_{\omega\rho} + \left(A^{\alpha\rho 3} - \frac{2\,A^{\alpha}\,A^{\rho}}{C}\right)\gamma_{3\rho} + \frac{A^{\alpha}}{C}\tau^{33} . \tag{2.1.16}$$

Expressions (2.1.5) through (2.1.11) may be found in terms of the surface metric by using (1.2.20), (1.2.21), and (1.2.22). We get

$$B^{\alpha\beta\rho\omega} = \frac{\mu}{L^4}\left(a^{\alpha\omega}\,a^{\beta\rho} + a^{\alpha\rho}\,a^{\beta\omega} + \frac{2\eta}{1-2\eta}\,a^{\alpha\beta}\,a^{\rho\omega}\right) \tag{2.1.17}$$

$$B^{\alpha\beta\rho 3} = -\frac{2\,\lambda_{,\omega}}{\lambda}\,B^{\alpha\beta\rho\omega}\,\theta_3 \tag{2.1.18}$$

$$B^{\alpha\beta 3} = \frac{2\mu}{L^4}\left[\frac{\lambda_{,\rho}\,\lambda_{,\omega}}{\lambda^2}\,a^{\alpha\rho}\,a^{\beta\omega}\,\theta_3^2 + \frac{\eta\,a^{\alpha\beta}}{1-2\eta}\left(\frac{1+\lambda_{,\rho}\,\lambda_{,\omega}\,a^{\rho\omega}\,\theta_3^2}{\lambda^2}\right)\right] \tag{2.1.19}$$

$$A^{\alpha\beta\rho 3} = \frac{-\lambda_{,\omega}\,\mu}{\lambda\,L^4}\left(a^{\alpha\beta}\,a^{\rho\omega} + a^{\alpha\rho}\,a^{\beta\omega} + \frac{2\eta}{1-2\eta}\,a^{\beta\rho}\,a^{\alpha\omega}\right)\theta_3 \tag{2.1.20}$$

$$A^{\alpha\rho 3} = \frac{2\mu}{L^4}\left[\frac{\lambda_{,\beta}\,\lambda_{,\omega}\,a^{\alpha\beta}}{\lambda^2\,(1-2\eta)}\,a^{\rho\omega}\,\theta_3^2 + a^{\alpha\rho}\left(\frac{1+\lambda_{,\beta}\,\lambda_{,\omega}\,a^{\beta\omega}\,\theta_3^2}{\lambda^2}\right)\right] \tag{2.1.21}$$

$$A^{\alpha} = -\frac{2\,\mu\,\lambda_{,\omega}\,(1-\eta)\,g^{33}\,a^{\alpha\omega}\,\theta_3}{\lambda\,L^2\,(1-2\eta)} \tag{2.1.22}$$

where g^{33} is as given in (1.2.22).

We now have the coefficients of the strains in (2.1.15) and (2.1.16) in terms of λ, the surface metric $a^{\alpha\beta}$, and θ_3. If we could now find the strains as a function of θ_3 then we would have the stresses as functions of θ_3.

2. Strain-Displacement Relations

The covariant strain tensor γ_{ik} is expressed in terms of the displacement vector $\bar{V}$ in (34) as the following

$$\gamma_{ik} = \tfrac{1}{2} (\bar{g}_i \cdot \bar{V}_{,k} + \bar{g}_k \cdot \bar{V}_{,i}) \quad . \tag{2.2.1}$$

Let us now assume that the displacement vector $\bar{V}$ is represented approximately by

$$\bar{V} = L\,\bar{v} + x_3 \bar{w} = L\,(\bar{v} + \lambda\,\theta_3\,\bar{w}) \tag{2.2.2}$$

where

$$\bar{v} = v_\alpha\,\bar{a}^\alpha + w\,\bar{a}_3 \tag{2.2.3}$$

and

$$\bar{w} = w_\alpha\,\bar{a}^\alpha \quad . \tag{2.2.4}$$

$\bar{v}$ is the displacement of the middle surface and λ, v_α, w, and w_α are functions of θ_1 and θ_2 only, i.e. they are independent of θ_3. We can write $\bar{V}$ as

$$\bar{V} = L\left[(v_\alpha + \lambda\,\theta_3\,w_\alpha)\,\bar{a}^\alpha + w\,\bar{a}_3 \right] \quad . \tag{2.2.5}$$

Using (1.2.7) and (2.2.5), we may now express the strains in terms of displacements. From (2.2.5), we obtain

$$\bar{V}_{,\alpha} = L\Big[v_\beta \bar{a}^\beta_{,\alpha} + v_{\beta,\alpha} \bar{a}^\beta + \lambda \theta_3 w_\beta \bar{a}^\beta_{,\alpha} + \lambda \theta_3 w_{\beta,\alpha} \bar{a}^\beta + \lambda_{,\alpha} \theta_3 w_\beta \bar{a}^\beta + w_{,\alpha} \bar{a}_3 \Big] \tag{2.2.6}$$

Since $\bar{a}^\beta_{,\alpha} = - \left\{ {}_\alpha{}^\beta{}_\rho \right\} \bar{a}^\rho$, we have

$$v_\beta \bar{a}^\beta_{,\alpha} + v_{\rho,\alpha} \bar{a}^\rho = - \left\{ {}_\alpha{}^\beta{}_\rho \right\} v_\beta \bar{a}^\rho + v_{\rho,\alpha} \bar{a}^\rho = v_\rho|_\alpha \bar{a}^\rho$$

so that we can write (2.2.6) as

$$\bar{V}_{,\alpha} = L\Big[v_\rho|_\alpha \bar{a}^\rho + \lambda \theta_3 w_\rho|_\alpha \bar{a}^\rho + \lambda_{,\alpha} \theta_3 w_\rho \bar{a}^\rho + w_{,\alpha} \bar{a}_3 \Big]$$

or

$$\bar{V}_{,\alpha} = L\Big[(v_\rho|_\alpha + \lambda \theta_3 w_\rho|_\alpha + \lambda_{,\alpha} \theta_3 w_\rho) \bar{a}^\rho + w_{,\alpha} \bar{a}_3 \Big] . \tag{2.2.7}$$

From (2.2.1), we have

$$\gamma_{\alpha 3} = ½ (\bar{g}_\alpha \cdot \bar{V}_{,3} + \bar{g}_3 \cdot \bar{V}_{,\alpha}) . \tag{2.2.8}$$

Now

$$\bar{V}_{,3} = L\Big[\lambda w_\alpha \bar{a}^\alpha + (v_\alpha + \lambda \theta_3 w_\alpha) \bar{a}^\alpha_{,3} \Big] = L \lambda w_\alpha \bar{a}^\alpha . \tag{2.2.9}$$

Then

$$2 \gamma_{\alpha 3} = L^2 \Big[(\bar{a}_\alpha + \lambda_{,\alpha} \theta_3 \bar{a}_3) \cdot (\lambda w_\beta \bar{a}^\beta) \Big] + L^2 \lambda w_{,\alpha} \bar{a}_3 \cdot \bar{a}_3$$

$$2\gamma_{\alpha 3} = L^2\lambda\, w_\beta\, a^{\beta\rho}\, \bar{a}_\alpha \cdot \bar{a}_\rho + L^2\, \lambda\, w_{,\alpha}$$

$$2\gamma_{\alpha 3} = L^2\lambda\left[w_\beta\ a^{\beta\rho}\ a_{\alpha\rho} + w_{,\alpha}\right] = L^2\lambda\left[w_\beta\ \delta_\alpha^\beta + w_{,\alpha}\right]$$

$$\gamma_{\alpha 3} = \frac{L^2\lambda}{2}\,(w_\alpha + w_{,\alpha}) \quad . \tag{2.2.10}$$

From (2.2.1),

$$\gamma_{\alpha\beta} = \tfrac{1}{2}\,(\bar{g}_\alpha \cdot \bar{V}_{,\beta} + \bar{g}_\beta \cdot \bar{V}_{,\alpha}) \quad . \tag{2.2.11}$$

From (1.2.7) and (2.2.7),

$$2\gamma_{\alpha\beta} = L^2\left[(\bar{a}_\alpha + \lambda_{,\alpha}\theta_3\bar{a}_3)\cdot\left\{(v_\rho|_\beta + \lambda\theta_3 w_\rho|_\beta + \lambda_{,\beta}\theta_3 w_\rho)\bar{a}^\rho + w_{,\beta}\bar{a}_3\right\}\right]$$

$$+ L^2\left[(\bar{a}_\beta + \lambda_{,\beta}\theta_3\bar{a}_3)\cdot\left\{(v_\rho|_\alpha + \lambda\theta_3 w_\rho|_\alpha + \lambda_{,\alpha}\theta_3 w_\rho)\bar{a}^\rho + w_{,\alpha}\bar{a}_3\right\}\right] .$$

Simplifying, we get

$$\gamma_{\alpha\beta} = \frac{L^2}{2}\left[(v_\alpha|_\beta + v_\beta|_\alpha) + \theta_3\left\{\lambda(w_\alpha|_\beta + w_\beta|_\alpha) + \lambda_{,\beta}(w_\alpha + w_{,\alpha})\right.\right.$$

$$\left.\left. + \lambda_{,\alpha}(w_\beta + w_{,\beta})\right\}\right] \quad .$$

Let

$$H_{\alpha\beta} = \lambda(w_\alpha|_\beta + w_\beta|_\alpha) + \lambda_{,\beta}(w_\alpha + w_{,\alpha}) + \lambda_{,\alpha}(w_\beta + w_{,\beta})\,; \tag{2.2.12}$$

then

$$\gamma_{\alpha\beta} = \frac{L^2}{2}\left[v_\alpha|_\beta + v_\beta|_\alpha + \theta_3\, H_{\alpha\beta}\right] \quad . \tag{2.2.13}$$

It may be observed here that if γ_{33} is found by using (2.2.1), (2.2.2), (2.2.3) and (2.2.4), then $\gamma_{33} = 0$. However, we shall not use this result since it is based on an approximation. We have seen that we could eliminate γ_{33} between (2.1.12), (2.1.13), and (2.1.14) and thus avoid the issue of using an approximation.

3. Stress-Displacement Relations

When we solved for γ_{33} in (2.1.14) and substituted in (2.1.12), we obtained

$$\tau^{\alpha\beta} = (B^{\alpha\beta\rho\omega} - \frac{B^{\alpha\beta 3}\, B^{\omega\rho 3}}{C})\gamma_{\omega\rho} + (B^{\alpha\beta\rho 3} - 2\,\frac{B^{\alpha\beta 3}\, A^{\rho}}{C})\gamma_{3\rho} + \frac{B^{\alpha\beta 3}}{C}\tau^{33} \,. \tag{2.3.1}$$

We now use (2.1.17), (2.1.18), (2.1.19), (2.1.22), and (2.1.1') and letting

$$J^{\alpha\beta} = \frac{\lambda_{,\sigma}\,\lambda_{,\omega}}{\lambda^2}\, a^{\alpha\sigma}\, a^{\beta\omega} \tag{2.3.2}$$

$$\psi = \lambda_{,\sigma}\,\lambda_{,\omega}\, a^{\sigma\omega} \tag{2.3.3}$$

so that

$$g^{33} = \frac{1 + \psi\,\theta_3^2}{\lambda^2 L^2} \tag{2.3.4}$$

we can begin to express $\tau^{\alpha\beta}$ as a function of θ_3 and displacements.

If we neglect terms involving θ_3^4 and higher powers of θ_3, we find that

$$B^{\alpha\beta\rho\omega} - \frac{B^{\alpha\beta 3}\, B^{\omega\rho 3}}{C} = \overset{\prime}{B}{}^{\alpha\beta\rho\omega} - \frac{2\mu\lambda^2\eta}{L^4(1-\eta)}\, K^{\alpha\beta\rho\omega}\, \theta_3^2 \qquad (2.3.5)$$

where

$$\overset{\prime}{B}{}^{\alpha\beta\rho\omega} = \frac{\mu}{L^4}\left(a^{\alpha\omega}\, a^{\beta\rho} + a^{\alpha\rho}\, a^{\beta\omega} + \frac{2\eta}{1-\eta}\, a^{\alpha\beta}\, a^{\omega\rho}\right) \qquad (2.3.6)$$

and

$$K^{\alpha\beta\rho\omega} = J^{\alpha\beta}\, a^{\omega\rho} + J^{\omega\rho}\, a^{\alpha\beta} \quad . \qquad (2.3.7)$$

We also get

$$\begin{aligned} B^{\alpha\beta\rho 3} - 2\,\frac{B^{\alpha\beta 3}\, A^{\rho}}{C} = & - \frac{2\mu\lambda_{,\omega}}{\lambda L^4}\left(a^{\alpha\omega}\, a^{\beta\rho} + a^{\alpha\rho}\, a^{\beta\omega}\right)\theta_3 \\ & + \frac{4\mu\lambda\lambda_{,\omega}}{L^4}\, a^{\rho\omega}\, J^{\alpha\beta}\, \theta_3^3 \end{aligned} \qquad (2.3.8)$$

and

$$\frac{B^{\alpha\beta 3}}{C} = \frac{(1-2\eta)\, J^{\alpha\beta}\, \lambda^4}{1-\eta}\, \theta_3^2 + \frac{\eta\,\lambda^2\, a^{\alpha\beta}}{1-\eta}\left(1-\psi\,\theta_3^2\right) . \qquad (2.3.9)$$

We now substitute (2.3.5), (2.3.8), (2.3.9), (2.2.10) and (2.2.13) into (2.3.1) and get

$$\tau^{\alpha\beta} = \frac{L^2}{2} B'^{\alpha\beta\rho\omega} \left[v_{\omega}|_{\rho} + v_{\rho}|_{\omega} + \theta_3 H_{\omega\rho} \right]$$

$$- \frac{\mu \lambda^2 \eta}{L^2(1-\eta)} K^{\alpha\beta\rho\omega} \theta_3^2 \left[v_{\omega}|_{\rho} + v_{\rho}|_{\omega} + \theta_3 H_{\omega\rho} \right]$$

$$- \frac{\mu \lambda_{,\omega}}{L^2} (a^{\alpha\omega} a^{\beta\rho} + a^{\alpha\rho} a^{\beta\omega})(w_{\rho} + w_{,\rho}) \theta_3 \tag{2.3.10}$$

$$+ \frac{2 \mu \lambda^2 \lambda_{,\omega}}{L^2} a^{\rho\omega} J^{\alpha\beta} (w_{\rho} + w_{,\rho}) \theta_3^3$$

$$+ \left[\frac{(1-2\eta) J^{\alpha\beta} \lambda^4 \theta_3^2}{1-\eta} + \frac{\eta \lambda^2 a^{\alpha\beta}}{1-\eta} (1-\psi \theta_3^2) \right] \tau^{33} .$$

Let us assume that we can write

$$\tau^{33} = A + B \theta_3 + C \theta_3^2 + D \theta_3^3 \tag{2.3.11}$$

and placing this in (2.3.10) and neglecting terms involving powers of θ_3 higher than the third, we can rewrite (2.3.10) as

$$\tau^{\alpha\beta} = \frac{L^2}{2} B'^{\alpha\beta\rho\omega} (v_{\omega}|_{\rho} + v_{\rho}|_{\omega}) + \frac{A \eta \lambda^2 a^{\alpha\beta}}{1-\eta}$$

$$+ \left[\frac{L^2}{2} B'^{\alpha\beta\rho\omega} H_{\rho\omega} - \frac{\mu \lambda_{,\omega}}{L^2} (a^{\alpha\omega} a^{\beta\rho} + a^{\alpha\rho} a^{\beta\omega})(w_{\rho} + w_{,\rho}) + \frac{B \eta \lambda^2 a^{\alpha\beta}}{1-\eta} \right] \theta_3$$

$$+ \left[\frac{\eta \lambda^2 a^{\alpha\beta} C}{1-\eta} + \frac{(1-2\eta) J^{\alpha\beta} A \lambda^4}{1-\eta} - \frac{\eta \lambda^2 a^{\alpha\beta} \psi A}{1-\eta} - \frac{\mu \lambda^2 \eta}{L^2(1-\eta)} K^{\alpha\beta\rho\omega} (v_{\omega}|_{\rho} + v_{\rho}|_{\omega}) \right] \theta_3^2$$

$$+ \left[\frac{(1-2\eta) J^{\alpha\beta} \lambda^4 B}{1-\eta} - \frac{\eta \lambda^2 a^{\alpha\beta} \psi B}{1-\eta} + \frac{\eta \lambda^2 a^{\alpha\beta} D}{1-\eta} \right. \tag{2.3.12}$$

$$\left. + \frac{2 \mu \lambda^2 \lambda_{,\omega}}{L^2} a^{\rho\omega} J^{\alpha\beta} (w_{\rho} + w_{,\rho}) - \frac{\mu \lambda^2 \eta}{L^2(1-\eta)} K^{\alpha\beta\rho\omega} H_{\rho\omega} \right] \theta_3^3 .$$

Then we can write

$$\tau^{\alpha\beta} = A^{\alpha\beta} + B^{\alpha\beta}\,\theta_3 + C^{\alpha\beta}\,\theta_3^2 + D^{\alpha\beta}\,\theta_3^3 \quad . \tag{2.3.13}$$

If we now use the definition of $n^{\alpha\beta}$ from (1.3.13), we get

$$\frac{n^{\alpha\beta}}{\lambda L^3} = A^{\alpha\beta} + \frac{C^{\alpha\beta}}{12} \tag{2.3.14}$$

and from (1.3.18), we get

$$\frac{m^{\alpha\beta}}{\lambda^2 L^3} = \frac{B^{\alpha\beta}}{12} + \frac{D^{\alpha\beta}}{80} \tag{2.3.15}$$

Then we can write $\tau^{\alpha\beta}$ as

$$\tau^{\alpha\beta} = \frac{n^{\alpha\beta}}{\lambda L^3} + \frac{12m^{\alpha\beta}}{\lambda^2 L^3}\,\theta_3 - \frac{C^{\alpha\beta}}{12}\,(1-12\theta_3^2) - \frac{3D^{\alpha\beta}}{20}(\theta_3 - \frac{20}{3}\theta_3^3) \;, \tag{2.3.16}$$

where

$C^{\alpha\beta}$ = coefficient of θ_3^2 in (2.3.12)

and (2.3.17)

$D^{\alpha\beta}$ = coefficient of θ_3^3 in (2.3.12) .

Then we have

$$\frac{n^{\alpha\beta}}{\lambda L^3} = \frac{L^2}{2}\,B'^{\alpha\beta\rho\omega}\,(v_\omega|_\rho + v_\rho|_\omega) + \frac{A\,\eta\,\lambda^2\,a^{\alpha\beta}}{1-\eta} + \frac{C^{\alpha\beta}}{12} \tag{2.3.18}$$

and

$$\frac{12m^{\alpha\beta}}{\lambda^2 L^3} = \frac{L^2}{2} B'^{\alpha\beta\rho\omega} H_{\rho\omega} + \frac{B\eta\lambda^2 a^{\alpha\beta}}{1-\eta} + \frac{3}{20} D^{\alpha\beta}$$
$$- \frac{\mu\lambda_{,\omega}}{L^2} (a^{\alpha\omega} a^{\beta\rho} + a^{\alpha\rho} a^{\beta\omega})(w_\rho + w_{,\rho}) \quad . \tag{2.3.19}$$

Simplifying (2.3.18) and (2.3.19), we get

$$n^{\alpha\beta} = \mu\lambda L\left[v^\alpha|^\beta + v^\beta|^\alpha + \frac{2\eta a^{\alpha\beta}}{1-\eta} v^\rho|_\rho\right] + \frac{A\eta\lambda^3 L^3 a^{\alpha\beta}}{1-\eta} + \frac{\lambda L^3 C^{\alpha\beta}}{12} \tag{2.3.20}$$

and

$$m^{\alpha\beta} = \frac{\mu\lambda^2 L}{12}\left[\lambda(w^\alpha|^\beta + w^\beta|^\alpha) + \frac{2\lambda\eta}{1-\eta} a^{\alpha\beta} w^\rho|_\rho + \frac{2\eta a^{\alpha\beta}}{1-\eta}\lambda|_\rho (w^\rho + w|^\rho)\right.$$
$$+ \frac{\eta\lambda^4 L^3 a^{\alpha\beta}}{1-\eta}\left(\frac{B}{12} + \frac{D}{80}\right) + \frac{\lambda^4 L}{80(1-\eta)}\left[J^{\alpha\beta} B\lambda^2 L^2 (1-2\eta)\right. \tag{2.3.21}$$
$$\left.\left. -\eta B L^2 a^{\alpha\beta}\psi + 2\mu\lambda_{,\omega} a^{\rho\omega} J^{\alpha\beta}(1-\eta)(w_\rho + w_{,\rho}) + \mu\eta K^{\alpha\beta\rho\omega} H_{\rho\omega}\right]\right] .$$

Let us now obtain an expression for $\tau^{\alpha 3}$ in terms of displacements just as we have done for $\tau^{\alpha\beta}$. Using (2.1.13) and (2.1.14), we can write $\tau^{\alpha 3}$ as

$$\tau^{\alpha 3} = \left(A^{\alpha\omega\rho 3} - \frac{A^\alpha B^{\omega\rho 3}}{C}\right)\gamma_{\omega\rho} + \left(A^{\alpha\rho 3} - \frac{2A^\alpha A^\rho}{C}\right)\gamma_{3\rho} + \frac{A^\alpha}{C}\tau^{33} \ . \tag{2.3.22}$$

We now simplify this by using (2.1.20), (2.1.19), (2.1.21), (2.1.22), and (2.1.11).

Assuming again that we can neglect terms involving powers of θ_3 higher than the third, we get

$$A^{\alpha\omega\rho 3} - \frac{A^{\alpha} B^{\omega\rho 3}}{C} = - \frac{\mu \lambda_{,\beta}}{\lambda L^4} (a^{\alpha\omega} a^{\beta\rho} + a^{\alpha\rho} a^{\beta\omega}) \theta_3 + \frac{\mu \lambda_{,\beta}}{\lambda L^4} (2 J^{\omega\rho} a^{\alpha\beta} \lambda^2) \theta_3^3 \tag{2.3.23}$$

$$A^{\alpha\rho 3} - \frac{2 A^{\alpha} A^{\rho}}{C} = \frac{2\mu}{L^4} \left[- J^{\alpha\rho} \theta_3^2 + \frac{a^{\alpha\rho}}{\lambda^2} (1 + \psi \theta_3^2) \right] \tag{2.3.24}$$

$$\frac{A^{\alpha}}{C} = - \lambda \lambda_{,\beta} a^{\alpha\beta} \theta_3 \quad . \tag{2.3.25}$$

If we now substitute (2.3.23), (2.3.24), (2.3.25), (2.2.10), (2.2.13), and (2.3.11) into (2.3.22), we get

$$\begin{aligned} \tau^{\alpha 3} = {} & \frac{\mu a^{\alpha\rho}}{\lambda L^2} (w_{\rho} + w_{,\rho}) - \left[\frac{\mu \lambda_{,\beta}}{\lambda L^2} (v^{\alpha}|^{\beta} + v^{\beta}|^{\alpha}) - \lambda \lambda_{,\beta} a^{\alpha\beta} A \right] \theta_3 \\ & + \left[\frac{\mu \lambda_{,\beta} J^{\omega\rho} a^{\alpha\beta} \lambda}{L^2} (v_{\omega}|_{\rho} + v_{\rho}|_{\omega}) - \frac{\mu \lambda_{,\beta}}{2\lambda L^2} (a^{\alpha\omega} a^{\rho\beta} + a^{\alpha\rho} a^{\beta\omega}) H_{\omega\rho} \right. \\ & \left. - \frac{\lambda \mu}{L^2} J^{\alpha\rho} (w_{\rho} + w_{,\rho}) + \frac{\mu a^{\alpha\rho} \psi}{\lambda L^2} (w_{\rho} + w_{,\rho}) - \lambda \lambda_{,\beta} a^{\alpha\beta} B \right] \theta_3^2 \end{aligned} \tag{2.3.26}$$

We can now write (2.3.26) as

$$\tau^{\alpha 3} = A^{\alpha} + B^{\alpha} \theta_3 + C^{\alpha} \theta_3^2 \tag{2.3.27}$$

where this A^{α} is not the same as defined by (2.1.10).

Let us now place (2.3.27) in (2.3.26) and multiply both sides of (2.3.26) by $1 - 4\theta_3^2$ and integrate from $\theta_3 = -½$ to $\theta_3 = +½$. We find that

$$A^{\alpha} = \frac{3\,q^{\alpha}}{2\lambda^2 L^3} - \frac{\oplus\, s^{\alpha}}{2\lambda^2 L^3} \tag{2.3.28}$$

$$C^{\alpha} = \frac{6}{\lambda^2 L^3}\,(-q^{\alpha} + \oplus\, s^{\alpha}) \tag{2.3.29}$$

and also that

$$\begin{aligned}\frac{2}{3}A^{\alpha} + \frac{C^{\alpha}}{30} &= \frac{4\,q^{\alpha}}{5\lambda^2 L^3} - \frac{2\,\oplus\, s^{\alpha}}{15\lambda^2 L^3}\\ &= \frac{2\mu a^{\alpha\rho}}{3\lambda L^2}(w_{\rho} + w_{,\rho}) + \frac{\lambda\mu a^{\alpha\rho}\psi}{30\lambda^2 L^2}(w_{\rho} + w_{,\rho})\\ &\quad - \frac{\lambda\mu J^{\alpha\rho}}{30 L^2}(w_{\rho} + w_{,\rho}) - \lambda\lambda_{,\beta}\; a^{\alpha\beta} B\\ &\quad + \frac{\mu\lambda_{,\beta} J^{\omega\rho} a^{\alpha\beta}}{30 L^2}(v_{\omega}|_{\rho} + v_{\rho}|_{\omega}) - \frac{\mu\lambda_{,\beta}}{60\lambda L^2}(a^{\alpha\omega}a^{\rho\beta} + a^{\alpha\rho}a^{\beta\omega}) H_{\omega\rho}.\end{aligned} \tag{2.3.30}$$

Using (2.2.12), (2.3.20), (2.3.21), we can write (2.3.30) as

$$\begin{aligned}\frac{4\,q^{\alpha}}{5\lambda^2 L^3} - \frac{2\,\oplus\, s^{\alpha}}{15\lambda^2 L^3} &= \frac{2\mu a^{\alpha\rho}}{3\lambda L^2}(w_{\rho} + w_{,\rho}) + \frac{\lambda\,\mu\, a^{\alpha\rho}\psi}{30\,\lambda^2 L^2}(w_{\rho} + w_{,\rho})\\ &\quad - \frac{\mu\lambda J^{\alpha\rho}}{30 L^2}(w_{\rho} + w_{,\rho}) - \frac{\lambda\lambda_{,\beta}\, a^{\alpha\beta}\, B}{30}\\ &+ \frac{\mu\lambda_{,\beta}\lambda_{,\omega}\lambda_{,\rho}\; a^{\alpha\beta}}{30\, L^2\lambda^2}\left(\frac{n^{\omega\rho}}{\mu\lambda L} - \frac{2\eta a^{\omega\rho} v^{\sigma}|_{\sigma}}{1-\eta} - \frac{A\,\eta\,\lambda^2 L^2 a^{\omega\rho}}{\mu\,(1-\eta)}\right)\\ &- \frac{\mu\lambda_{,\beta}}{60\lambda L^2}\left(\frac{24 m^{\alpha\beta}}{\mu\lambda^2 L} - \frac{4\lambda\eta a^{\alpha\beta} w^{\rho}|_{\rho}}{1-\eta} - \frac{4\eta a^{\alpha\beta}\lambda|_{\rho}(w^{\rho} + w|^{\rho})}{1-\eta}\right)\\ &- \frac{24B\eta a^{\alpha\beta}}{\mu L(1-\eta)} + 2\lambda|^{\beta}(w^{\alpha} + w|^{\alpha}) + 2\,\lambda|^{\alpha}(w^{\beta} + w|^{\beta})\quad .\end{aligned} \tag{2.3.31}$$

After we have determined A and B, we shall return to (2.3.31) and simplify it considerably. It will yield a relationship between w_ρ and $w_{,\rho}$ which will be necessary in simplifying some of the equations which we already have.

Let us now find $\tau^{\alpha 3}$ from $\tau^{\alpha\beta}$ by using the equations of equilibrium on stresses,

$$\tau^{\alpha\beta}\big|_{\beta} + \tau^{\alpha 3}{}_{,3} = 0 \quad . \qquad (2.3.32)$$

From (2.3.13) and (2.3.16), if $\tau^{\alpha\beta}$ is assumed linear in θ_3, we have

$$\tau^{\alpha\beta} = \frac{n^{\alpha\beta}}{\lambda L^3} + \frac{12\ m^{\alpha\beta}}{\lambda^2\ L^3}\quad \theta_3 \qquad (2.3.33)$$

so that

$$\tau^{\alpha\beta}\big|_{\beta} = \frac{1}{\lambda^2 L^3}\left[\lambda\ n^{\alpha\beta}\big|_{\beta} - \lambda\big|_{\beta}\ n^{\alpha\beta} + 12\ \theta_3\ m^{\alpha\beta}\big|_{\beta} - 24\ \frac{\lambda|_{\beta}}{\lambda}\ m^{\alpha\beta}\ \theta_3\right] .$$

We have already shown that

$$\tau^{\alpha 3} = A^{\alpha} + B^{\alpha}\theta_3 + C^{\alpha}\ \theta_3^2$$

so that

$$\tau^{\alpha 3}{}_{,3} = B^{\alpha} + 2C^{\alpha}\ \theta_3 \quad .$$

From (1.5.23), we have also

$$m^{\alpha\beta}\big|_{\beta} = + f^{\alpha} + q^{\alpha} - \oplus\ s^{\alpha} \quad .$$

By equating coefficients, we have

$$B^\alpha = \frac{\lambda|_\beta n^{\alpha\beta} - \lambda n^{\alpha\beta}|_\beta}{\lambda^2 L^3} \tag{2.3.34}$$

$$C^\alpha = \frac{6\,(f^\alpha - q^\alpha + \oplus s^\alpha)}{\lambda^2 L^3} \quad . \tag{2.3.35}$$

From the definition of q^α (1.3.14), we have

$$q^\alpha = \lambda^2 L^3 \int_{-1/2}^{1/2} \tau^{\alpha 3}\, d\theta^3 = \lambda^2 L^3 \int_{-1/2}^{1/2} (A^\alpha + B^\alpha \theta_3 + C^\alpha \theta_3^2)\, d\theta^3$$

or

$$q^\alpha = \lambda^2 L^3 \left(A^\alpha + \frac{C^\alpha}{12}\right)$$

so that

$$A^\alpha = \frac{q^\alpha}{\lambda^2 L^3} - \frac{C^\alpha}{12} = \frac{3q^\alpha - f^\alpha - \oplus s^\alpha}{2\lambda^2 L^3} \quad . \tag{2.3.36}$$

Thus we can write $\tau^{\alpha 3}$ as

$$\tau^{\alpha 3} = \frac{3q^\alpha}{2\lambda^2 L^3}(1 - 4\theta_3^2) - \frac{f^\alpha}{2\lambda^2 L^3}(1 - 12\,\theta_3^2)$$
$$- \frac{\oplus s^\alpha}{2\lambda^2 L^3}(1 - 12\,\theta_3^2) + \frac{\lambda|_\beta n^{\alpha\beta} - \lambda n^{\alpha\beta}|_\beta}{\lambda^2 L^3}\theta_3 \quad .$$

We can now find τ^{33} from $\tau^{\alpha 3}$ by using the equilibrium equation

$$\tau^{\alpha 3}|_\alpha + \tau^{33}{}_{,3} = 0 \quad . \tag{2.3.38}$$

From (2.3.37), we have

$$2L^3 \tau^{3\alpha}\big|_\alpha = \frac{3\lambda^2 q^\alpha|_\alpha - 6\lambda\lambda|_\alpha q^\alpha}{\lambda^4}(1 - 4\theta_3^2)$$

$$- \frac{\lambda^2 f^\alpha|_\alpha - 2\lambda\lambda|_\alpha f^\alpha}{\lambda^4}(1 - 12\theta_3^2)$$

$$- \frac{\lambda^2 \oplus s^\alpha|_\alpha - 2\lambda\lambda|_\alpha \oplus s^\alpha}{\lambda^4}(1 - 12\theta_3^2) \qquad (2.3.39)$$

$$+ 2\left[\frac{\lambda^2(\lambda|_\beta n^{\alpha\beta}|_\alpha + \lambda|_{\beta\alpha} n^{\alpha\beta}) - 2\lambda\lambda|_\alpha \lambda|_\beta n^{\alpha\beta}}{\lambda^4}\right]\theta_3$$

$$- 2\left[\frac{\lambda^2(\lambda n^{\alpha\beta}|_{\beta\alpha} + \lambda|_\alpha n^{\alpha\beta}|_\beta) - 2\lambda^2\lambda|_\alpha n^{\alpha\beta}|_\beta}{\lambda^4}\right]\theta_3 .$$

Now we assumed that

$$\tau^{33} = A + B\theta_3 + C\theta_3^2 + D\theta_3^3 \qquad (2.3.11)$$

so that

$$\tau^{33}{}_{,3} = B + 2C\theta_3 + 3D\theta_3^2 .$$

By equating coefficients, we have

$$2\lambda^2 L^3 C = \lambda n^{\alpha\beta}|_{\beta\alpha} - 2\lambda|_\alpha n^{\alpha\beta}|_\beta - \lambda|_{\beta\alpha} n^{\alpha\beta} + 2\frac{\lambda|_\alpha \lambda|_\beta}{\lambda} n^{\alpha\beta} \qquad (2.3.40)$$

$$-2\lambda^3 L^3 B = 3\lambda q^\alpha|_\alpha - 6\lambda|_\alpha q^\alpha - \lambda f^\alpha|_\alpha + 2\lambda|_\alpha f^\alpha - \lambda\oplus s^\alpha|_\alpha + 2\lambda|_\alpha \oplus s^\alpha \qquad (2.3.41)$$

$$6\lambda^3 L^3 D = 12\lambda q^\alpha|_\alpha - 24\lambda|_\alpha q^\alpha - 12\lambda f^\alpha|_\alpha + 24\lambda|_\alpha f^\alpha - 12\lambda\oplus s^\alpha|_\alpha + 24\lambda|_\alpha \oplus s^\alpha . \qquad (2.3.42)$$

From (1.4.6), we have that

$$\oplus\ p = \lambda^2 L^3 \left[\tau^{33}\right]_{-1/2}^{1/2} \quad .$$

Then from (2.3.11),

$$\frac{\oplus\ p}{\lambda^2 L^3} = B + \frac{D}{4} \quad .$$

From (2.3.41) and (2.3.42), we get

$$B + \frac{D}{4} = \frac{-\lambda\ q^{\alpha}|_{\alpha} + 2\ \lambda|_{\alpha}\ q^{\alpha}}{\lambda^3 L^3}$$

so that

$$\lambda\ q^{\alpha}|_{\alpha} - 2\ \lambda|_{\alpha}\ q^{\alpha} = -\lambda\ \oplus\ p \quad . \tag{2.3.43}$$

Now let

$$\oplus\ p^{*} = \lambda^2 L^3 \left(\left[\tau^{33}\right]^{\theta_3 = 1/2} + \left[\tau^{33}\right]_{\theta_3 = -1/2}\right) \tag{2.3.44}$$

so that by (2.3.11), we have

$$\frac{\oplus\ p^{*}}{\lambda^2 L^3} = 2A + \frac{C}{2}$$

or

$$A = \frac{\oplus\ p^{*}}{2\lambda^2 L^3} - \frac{C}{4} \quad . \tag{2.3.45}$$

Then using (2.3.11), (2.3.45), (2.3.40), (2.3.41), and (2.3.42), we have after simplifying,

$$\tau^{33} = \frac{\oplus p^{*}}{2\lambda^2 L^3} -$$

$$- \frac{1}{8\lambda^2 L^3}\left[\lambda\, n^{\alpha\beta}\big|_{\beta\alpha} - 2\,\lambda\big|_{\alpha} n^{\alpha\beta}\big|_{\beta} - \lambda\big|_{\beta\alpha} n^{\alpha\beta} + \frac{2\,\lambda|_{\alpha}\,\lambda|_{\beta}}{\lambda} n^{\alpha\beta}\right](1-4\theta_3^2)$$

$$+ \frac{3\oplus p}{2\lambda^2 L^3}(\theta_3 - \tfrac{4}{3}\theta_3^3) + \frac{f^{\alpha}|_{\alpha}}{2\lambda^2 L^3}(\theta_3 - 4\,\theta_3^3) \qquad (2.3.46)$$

$$- \frac{\lambda|_{\alpha} f^{\alpha}}{\lambda^3 L^3}(\theta_3 - 4\,\theta_3^3) + \frac{\oplus s^{\alpha}|_{\alpha}}{2\lambda^2 L^3}(\theta_3 - 4\,\theta_3^3)$$

$$- \frac{\lambda|_{\alpha} \oplus s^{\alpha}}{\lambda^3 L^3}(\theta_3 - 4\,\theta_3^3) \quad .$$

We now have in (2.3.33), (2.3.37), and (2.3.46) expressions for $\tau^{\alpha\beta}$, $\tau^{\alpha 3}$ and τ^{33} in terms of θ_3 and the surface tensors q^{α}, s^{α}, f^{α}, $n^{\alpha\beta}$, $m^{\alpha\beta}$ and the variable parameter λ.

$\tau^{\alpha\beta}$ and $\tau^{\alpha 3}$ as given in (2.3.33) and (2.3.37) agree with Reissner's theory for a constant thickness plate with no in-plane forces.

$\tau^{\alpha\beta}$ and $\tau^{\alpha 3}$ as given in (2.3.33) and (2.3.37) almost agree with Essenburg and Naghdi (32) for variable thickness plates with no in-plane forces and with $s^{\alpha} = 0$, i.e. with a symmetrical load.

τ^{33} as given by (2.3.46) agrees with Reissner's theory for a constant thickness plate but does not quite agree with the form obtained by Essenburg and Naghdi (34). If, in (2.3.46), we have no in-plane forces and if $s^{\alpha} = 0$, then we have a term involving f^{α} that is not used in (34) and the coefficient of f^{α} does not quite agree with (34).

It is easily verified that (2.3.33), (2.3.37), and (2.3.46) satisfy exactly the equations of equilibrium on the stresses. Since we have found A and B in the expression for τ^{33}, we may now return to equation (2.3.31) and simplify it. If we substitute (2.3.45) and (2.3.41) into (2.3.31) and then multiply both sides by $\frac{3\lambda^2 L^3}{2}$, we get

$$
\begin{aligned}
&6/5q^{\alpha} - 1/5 \oplus s^{\alpha} + 3/5f^{\alpha} - \frac{\lambda_{,\beta}\lambda_{,\omega}\lambda_{,\rho}\, a^{\alpha\beta} n^{\omega\rho}}{20\,\lambda} = \mu\,\lambda\, L\, a^{\alpha\rho}(w_{\rho} + w_{,\rho}) \\
&+ \frac{\mu L \lambda\, \lambda_{,\sigma}\lambda_{,\omega}\, a^{\alpha\rho} a^{\sigma\omega}}{20}(w_{\rho}+w_{,\rho}) - \frac{\mu\lambda L}{20}\lambda_{,\sigma}\lambda_{,\omega}\, a^{\alpha\sigma} a^{\rho\omega}(w_{\rho}+w_{,\rho}) \\
&+ \frac{\lambda_{,\beta}\, a^{\alpha\beta}}{40}\Big[-3\lambda \oplus p - \lambda f^{\alpha}|_{\alpha} + 2\,\lambda|_{\alpha} f^{\alpha} - \lambda \oplus s^{\alpha}|_{\alpha} + 2\,\lambda|_{\alpha} \oplus s^{\alpha} \Big] \\
&+ \frac{\mu L\, \lambda_{,\beta}\lambda_{,\omega}\lambda_{,\rho}\, a^{\alpha\beta}}{20}\left[- \frac{2\eta\, a^{\omega\rho}\, v^{\sigma}|_{\sigma}}{1-\eta} - \frac{\eta\lambda^2 L^2 a^{\omega\rho}}{\mu\,(1-\eta)}\left(\frac{\oplus p^{*}}{2\lambda^2 L^3} - \frac{C}{4}\right)\right] \qquad (2.3.47) \\
&+ \frac{\mu\lambda\lambda_{,\beta} L}{40}\Bigg[\frac{4\eta\, a^{\alpha\beta}\, w^{\rho}|_{\rho}}{1-\eta} + 4\eta\, a^{\alpha\beta}\, \lambda|_{\rho}\, (w^{\rho} + w|^{\rho}) \\
&\qquad - 2\,\lambda|^{\beta}(w^{\alpha} + w|^{\alpha}) - 2\,\lambda|^{\alpha}(w^{\beta} + w|^{\beta}) \\
&\qquad + \frac{24\,\eta\, a^{\alpha\beta}}{\mu L\,(1-\eta)}\left\{ \frac{-3\lambda\oplus p - \lambda f^{\alpha}|_{\alpha} + 2\lambda|_{\alpha} f^{\alpha} - \lambda\oplus s^{\alpha}|_{\alpha} + 2\lambda|_{\alpha}\oplus s^{\alpha}}{2\lambda^3 L^3}\right\}\Bigg] .
\end{aligned}
$$

For a constant thickness plate, (2.3.47) agrees with Reissner's theory. If we assume that $\lambda << 1$ and that $\lambda_{,\alpha} << 1$ and thus neglect terms of higher order involving these factors, then (2.3.47) becomes

$$6/5q^{\alpha} - 1/5 \oplus s^{\alpha} + 3/5f^{\alpha} = \mu \lambda L a^{\alpha\rho} (w_{\rho} + w_{,\rho}) \quad . \tag{2.3.46}$$

This agrees with (32) if $s^{\alpha} = 0$ except for the sign of the term involving f^{α}.

4. Stress Resultants and Stress Couples

Let us now simplify the expression for $m^{\alpha\beta}$ as was expressed in (2.3.21). It was found that

$$\begin{aligned} m^{\alpha\beta} = \frac{\mu\lambda^2 L}{12}\Big[\lambda(w^{\alpha}|^{\beta} + w^{\beta}|^{\alpha}) + \frac{2\lambda\eta}{1-\eta} a^{\alpha\beta} w^{\rho}|_{\rho} + \frac{2\eta a^{\alpha\beta}}{1-\eta}\lambda|_{\rho}(w^{\rho} + w|^{\rho})\Big] \\ + \frac{\eta\lambda^4 L^3 a^{\alpha\beta}}{1-\eta}\Big(\frac{B}{12} + \frac{D}{80}\Big) + \frac{\lambda^4 L}{30(1-\eta)}\Big[J^{\alpha\beta} B \lambda^2 L^2 (1-2\eta) \\ - \eta B L^2 a^{\alpha\beta}\psi + 2\mu\lambda_{,\omega} a^{\rho\omega} J^{\alpha\beta}(1-\eta)(w_{\rho} + w_{,\rho}) + \mu\eta K^{\alpha\beta\rho\omega} H_{\rho\omega}\Big] . \end{aligned} \tag{2.3.21}$$

Using (2.3.41), (2.3.42), and (2.3.43), we find that

$$\frac{B}{12} + \frac{D}{80} = \frac{1}{\lambda^3 L^3}\Big[\frac{\lambda \oplus p}{10} + \frac{\lambda f^{\alpha}|_{\alpha}}{60} - \frac{\lambda|_{\alpha} f^{\alpha}}{30} + \frac{\lambda \oplus s^{\alpha}|_{\alpha}}{60} - \frac{\lambda|_{\alpha} \oplus s^{\alpha}}{30}\Big] . \tag{2.4.1}$$

Then from (2.3.41), (2.3.2), and (2.3.3), we have also

$$J^{\alpha\beta}\lambda^2 L^2 B = \lambda_{,\sigma}\lambda_{,\omega} \frac{a^{\alpha\sigma} a^{\beta\omega} L^2}{2\lambda^3 L^3}\Big[3\lambda \oplus p + \lambda f^{\rho}|_{\rho} - 2\lambda|_{\rho} f^{\rho} + \lambda \oplus s^{\rho}|_{\rho} - 2\lambda|_{\rho} \oplus s^{\rho}\Big], \tag{2.4.2}$$

$$\psi \eta L^2 a^{\alpha\beta} B = \lambda_{,\sigma}\lambda_{,\omega} \frac{a^{\sigma\omega} a^{\alpha\beta} \eta L^2}{2\lambda^3 L^3}\Big[3\lambda \oplus p + \lambda f^{\rho}|_{\rho} - 2\lambda|_{\rho} f^{\rho} + \lambda \oplus s^{\rho}|_{\rho} - 2\lambda|_{\rho} \oplus s^{\rho}\Big] \tag{2.4.3}$$

$$2\mu\lambda_{,\omega} a^{\rho\omega} J^{\alpha\beta}(1-\eta)(w_{\rho} + w_{,\rho}) = 2\,\frac{\mu\lambda|_{\rho}\lambda|_{\sigma}\lambda|_{\omega}}{\lambda^2} a^{\alpha\sigma} a^{\beta\omega}(1-\eta)(w^{\rho} + w|^{\rho}) . \tag{2.4.4}$$

From (2.3.7) and (2.2.12), we have

$$K^{\alpha\beta\rho\omega} H_{\rho\omega} = (J^{\alpha\beta} a^{\omega\rho} + J^{\omega\rho} a^{\alpha\beta}) H_{\rho\omega}$$

$$K^{\alpha\beta\rho\omega} H_{\rho\omega} = \frac{2 \lambda_{,\sigma} \lambda_{,\omega}}{\lambda^2} a^{\alpha\sigma} a^{\beta\omega} \left[\lambda w^{\rho}|_{\rho} + \lambda|_{\rho} (w^{\rho} + w|^{\rho})\right]$$

$$+ \frac{\lambda_{,\sigma} \lambda_{,\omega}}{\lambda^2} a^{\alpha\beta} \left[\lambda (w^{\omega}|^{\sigma} + w^{\sigma}|^{\omega}) + \lambda|^{\sigma} (w^{\omega} + w|^{\omega}) + \lambda|^{\omega}(w^{\sigma} + w|^{\sigma})\right] . \qquad (2.4.5)$$

Then placing (2.4.1), (2.4.2), (2.4.3), (2.4.4), and (2.4.5) into (2.3.21), we have

$$m^{\alpha\beta} = \frac{\mu\lambda^3 L}{12} (w^{\alpha}|^{\beta} + w^{\beta}|^{\alpha}) + \frac{\mu\lambda^3 L\eta}{6(1-\eta)} a^{\alpha\beta} w^{\rho}|_{\rho}$$

$$+ \frac{\mu\lambda^2 L\eta}{6(1-\eta)} a^{\alpha\beta} \lambda|_{\rho}(w^{\rho} + w|^{\rho}) + \frac{\eta\lambda a^{\alpha\beta}}{1-\eta}\left(\frac{\lambda \oplus p}{10} + \frac{\lambda \oplus s^{\alpha}|_{\alpha}}{60}\right)$$

$$+ \frac{\eta\lambda a^{\alpha\beta}}{1-\eta}\left(\frac{\lambda f^{\rho}|_{\rho}}{60} - \frac{\lambda|_{\rho} f^{\rho}}{30} - \frac{\lambda|_{\rho} \oplus s^{\rho}}{30}\right) \qquad (2.4.6)$$

$$+ \frac{(1-2\eta)\lambda\lambda_{,\sigma}\lambda_{,\omega} a^{\alpha\sigma} a^{\beta\omega}}{160(1-\eta)}\Big(3\lambda \oplus p + \lambda f^{\rho}|_{\rho} - 2 \lambda|_{\rho} f^{\rho}$$

$$+ \lambda \oplus s^{\rho}|_{\rho} - 2 \lambda|_{\rho} \oplus s^{\rho}\Big)$$

$$+ \frac{\mu\lambda^2 L \lambda_{,\rho} \lambda_{,\sigma} \lambda_{,\omega}}{40} a^{\alpha\sigma} a^{\beta\omega} (w^{\rho} + w|^{\rho})$$

$$+ \frac{\lambda^2 L \mu \eta}{40(1-\eta)} \lambda_{,\sigma}\lambda_{,\omega} a^{\alpha\sigma} a^{\beta\omega} \left[\lambda w^{\rho}|_{\rho} + \lambda_{,\rho} (w^{\rho} + w|^{\rho})\right]$$

$$+ \frac{\lambda^2 L \mu \eta}{80(1-\eta)} \lambda_{,\sigma}\lambda_{,\omega} a^{\alpha\beta} \Big[\lambda(w^{\omega}|^{\sigma} + w^{\sigma}|^{\omega}) + \lambda|^{\sigma}(w^{\omega} + w|^{\omega})$$

$$+ \lambda|^{\omega}(w^{\sigma} + w|^{\sigma})\Big]$$

$$- \frac{\eta \lambda \lambda_{,\sigma} \lambda_{,\omega} a^{\sigma\omega} a^{\alpha\beta}}{160(1-\eta)} \left(3\lambda \circledast p + \lambda f^{\rho}|_{\rho} - 2\lambda|_{\rho} f^{\rho} + \lambda \circledast s^{\rho}|_{\rho} - 2\lambda|_{\rho} \circledast s^{\rho}\right)$$

(continued 2.4.6)

If we now use (2.3.48), (1.5.9), and if we let

$$h^{\alpha\sigma\beta\omega} = (1-2\eta)\, a^{\alpha\sigma} a^{\beta\omega} - \eta\, a^{\sigma\omega} a^{\alpha\beta} \tag{2.4.7}$$

then we can write $m^{\alpha\beta}$ in a more simplified form as follows:

$$\begin{aligned} m^{\alpha\beta} = & - \frac{\mu \lambda^3 L}{6}\Big[w|^{\alpha\beta} + \frac{\eta\, a^{\alpha\beta}}{1-\eta} w|^{\rho}_{\rho} \\ & + \frac{\lambda^2}{60}\left(6q^{\alpha}|^{\beta} - \circledast s^{\alpha}|^{\beta} + 3f^{\alpha}|^{\beta} + 6q^{\beta}|^{\alpha} - \circledast s^{\beta}|^{\alpha} + 3f^{\beta}|^{\alpha}\right) \\ & - \frac{\lambda \eta\, a^{\alpha\beta}}{60(1-\eta)}\Big[6\lambda \circledast p + \lambda \circledast s^{\rho}|_{\rho} + 3\lambda f^{\rho}|_{\rho} - 12\lambda|_{\rho} q^{\rho} \\ & + (4\lambda|_{\rho} + 12\lambda\lambda|_{\rho}) \circledast s^{\rho}\Big] \\ & + \frac{h^{\alpha\sigma\beta\omega}}{160(1-\eta)} \lambda \lambda_{,\sigma} \lambda_{,\omega} (3\lambda \circledast p + \lambda f^{\rho}|_{\rho} - 2\lambda|_{\rho} f^{\rho} + \lambda \circledast s^{\rho}|_{\rho} - 2\lambda|_{\rho} s^{\rho}) \Big] . \end{aligned} \tag{2.4.8}$$

Let us at this point restrict ourselves to problems in which $s^{\rho} = 0$ and letting

$$D = \frac{\mu \lambda^3 L}{6(1-\eta)} \quad , \tag{2.4.9}$$

we can by using (2.4.7) write $m^{\alpha\beta}$ as

$$
\begin{aligned}
m^{\alpha\beta} = -D\Big[&(1-\eta)\, w\big|^{\alpha\beta} + \eta\, a^{\alpha\beta}\, w\big|^{\rho}_{\rho} \\
&+ \frac{\lambda^2}{20}\left(2\, q^{\alpha}\big|^{\beta} + 2\, q^{\beta}\big|^{\alpha} + f^{\alpha}\big|^{\beta} + f^{\beta}\big|^{\alpha}\right) \\
&- \frac{\lambda^2 \eta\, a^{\alpha\beta}}{20(1-\eta)}\left(2 \circledast p + f^{\rho}\big|_{\rho} - 4\,\frac{\lambda|_{\rho}}{\lambda}\, q\right) \qquad (2.4.10) \\
&+ \frac{1-2\eta}{160(1-\eta)}\lambda^2\, \lambda\big|^{\alpha}\, \lambda\big|^{\beta}\left(3 \circledast p + f^{\rho}\big|_{\rho} - 2\,\frac{\lambda|_{\rho}}{\lambda}\, f^{\rho}\right) \\
&- \frac{\eta}{160(1-\eta)}\lambda^2\, \lambda\big|_{\sigma}\, \lambda\big|_{\omega}\, a^{\sigma\omega}\, a^{\alpha\beta}\left(3 \circledast p + f^{\rho}\big|_{\rho} - 2\,\frac{\lambda|_{\rho}}{\lambda}\, f^{\rho}\right).
\end{aligned}
$$

If λ is constant, this agrees with Reissner's theory.

In the usual approach to plate problems, a deflection equation for the middle surface is obtained and solved for certain loads and boundary conditions. The stresses and other desired information may then be found from the deflections. This approach has been tried for the present problem. Equation (2.4.10) was differentiated covariantly with respect to θ_α and $m^{\alpha\beta}\big|_{\alpha}$ was then substituted in (1.5.21) which is with $s^{\beta} = 0$,

$$m^{\alpha\beta}\big|_{\alpha} - f^{\beta} - q^{\beta} = 0.$$

After another differentiation of this equation with respect to θ_β, we arrive at the deflection equation. This procedure was carried out and the equation was found to be so involved and unwieldy that another approach seemed to be desired.

Let us now simplify $n^{\alpha\beta}$ as given in (2.3.20) in the same manner in which we simplified $m^{\alpha\beta}$.

From (2.3.20) we have

$$n^{\alpha\beta} = \mu\lambda L\left[v^{\alpha}|^{\beta} + v^{\beta}|^{\alpha} + \frac{2\eta a^{\alpha\beta}}{1-\eta} v^{\rho}|_{\rho}\right] + \frac{A\eta\lambda^3 L^3 a^{\alpha\beta}}{1-\eta} + \frac{\lambda L^3 C}{12} .$$

In using $\tau^{\alpha\beta}$ in the form (2.3.33) as against the first assumed form (2.3.13), it can be seen that we are assuming C to be approximately zero. When we compare (2.3.13) and (2.3.12), we see that this is implying that

$$C = \frac{\eta\lambda^2 a^{\alpha\beta} C}{1-\eta} + \frac{(1-2\eta) J^{\alpha\beta} A\lambda^4}{1-\eta} - \frac{\eta\lambda^2 a^{\alpha\beta}\psi A}{1-\eta}$$

$$- \frac{\mu\lambda^2\eta}{L^2(1-\eta)} K^{\alpha\beta\rho\omega}(v_{\omega}|_{\rho} + v_{\rho}|_{\omega}) = 0 .$$

Then using (2.3.2), (2.3.3), and (2.3.7), we have

$$\eta\lambda^2 a^{\alpha\beta} C + (1-2\eta)\lambda_{,\sigma}\lambda_{,\omega} a^{\alpha\sigma} a^{\beta\omega}\lambda^2 A - \eta\lambda_{,\sigma}\lambda_{,\omega} a^{\sigma\omega} a^{\alpha\beta}\lambda^2 A$$

$$- \frac{\mu\lambda^2\eta}{L^2}(v_{\Phi}|_{\rho} + v_{\rho}|_{\Phi})\left(\frac{\lambda_{,\sigma}\lambda_{,\omega}}{\lambda^2} a^{\alpha\sigma} a^{\beta\omega} a^{\Phi\rho} + \frac{\lambda_{,\sigma}\lambda_{,\omega}}{\lambda^2} a^{\Phi\sigma} a^{\rho\omega} a^{\alpha\beta}\right) = 0.$$

From (2.3.45), it is observed that A and C are of the same order of magnitude. Thus if we assume that $\lambda_{,\alpha}$ is small compared to λ then C is approximately zero, which means that by (2.3.45), we have

$$A = \frac{\Phi p^*}{2\lambda^2 L^3}$$

and thus

$$n^{\alpha\beta} = \mu\lambda L\left[v^{\alpha}|^{\beta} + v^{\beta}|^{\alpha} + \frac{2\eta a^{\alpha\beta}}{1-\eta} v^{\rho}|_{\rho}\right] + \frac{\eta\lambda a^{\alpha\beta}}{2(1-\eta)}\Phi p^* . \quad (2.4.11)$$

5. Fundamental Equations

We need now to examine the equations that we have derived relating the stress resultants, stress couples, shearing forces and loads. They will be collected here so that we may see how they fit together. The equilibrium equations, with $s^{\alpha} = 0$, are

$$q^{\alpha}\big|_{\alpha} + f^{\alpha}\big|_{\alpha} + \circledast p = 0 \tag{1.5.9}$$

$$n^{\alpha\beta}\big|_{\alpha} + \circledast p^{\beta} = 0 \tag{1.5.10}$$

$$m^{\alpha\beta}\big|_{\alpha} - f^{\beta} - q^{\beta} = 0. \tag{1.5.23}$$

The relationship between f^{β} and $m^{\alpha\beta}$ is

$$f^{\beta} = \frac{\lambda|_{\alpha}}{\lambda} m^{\alpha\beta}. \tag{1.5.22}$$

We have ten unknowns q^{α}, f^{α}, $n^{\alpha\beta}$, and $m^{\alpha\beta}$ related by the seven equations above. If we examine (2.4.10) and (2.4.11), we see that we have the three displacements v^{α} and w added to the ten unknowns above giving us thirteen unknowns but at the same time, we have six additional equations to add to the seven previous ones. Thus, we have thirteen unknowns in thirteen equations. Under suitable boundary conditions, this should lead to a solution of the problem.

Let us take a closer look at (2.4.10) and simplify it to some extent. We may, by grouping the terms differently, write (2.4.10) in the form

$$m^{\alpha\beta} = -D\left[(1-\eta)\, w\big|^{\alpha\beta} + \eta\, a^{\alpha\beta}\, w\big|^{\rho}_{\rho}\right]$$

$$+ \frac{\lambda^2}{20}\left(2q^{\alpha}\big|^{\beta} + 2q^{\beta}\big|^{\alpha}\right) + \frac{\lambda\,\lambda|_{\rho}\,\eta\, a^{\alpha\beta}}{5(1-\eta)} q^{\rho}$$

$$+ \frac{\lambda^2}{160(1-\eta)}\left[-16\eta a^{\alpha\beta} + 3\lambda|^{\alpha}\lambda|^{\beta}(1-2\eta) - 3\eta\lambda|_{\sigma}\lambda|_{\omega} a^{\sigma\omega} a^{\alpha\beta}\right] \circledast p$$

$$+ \frac{\lambda^2}{20}\left(f^{\alpha}\big|^{\beta} + f^{\beta}\big|^{\alpha}\right) \qquad (2.5.1)$$

$$+ \left(\frac{\lambda|_{\rho}}{\lambda}\right) \frac{\lambda^2}{160(1-\eta)}\left[-2(1-2\eta)\lambda|^{\alpha}\lambda|^{\beta} + 2\eta\lambda|_{\sigma}\lambda|_{\omega} a^{\sigma\omega} a^{\alpha\beta}\right] f^{\rho}$$

$$+ \frac{\lambda^2}{160(1-\eta)}\left[(1-2\eta)\lambda|^{\alpha}\lambda|^{\beta} - \eta\lambda|_{\sigma}\lambda|_{\omega} a^{\sigma\omega} a^{\alpha\beta} - 8\eta a^{\alpha\beta}\right] f^{\rho}\big|_{\rho}\,.$$

Now we let

$$P^{\alpha\beta} = \frac{\lambda^2}{160(1-\eta)}\left[-16\eta a^{\alpha\beta} + 3\lambda|^{\alpha}\lambda|^{\beta}(1-2\eta) - 3\eta\lambda|_{\sigma}\lambda|_{\omega} a^{\sigma\omega} a^{\alpha\beta}\right] \qquad (2.5.2)$$

$$Q^{\alpha\beta} = \frac{\lambda^2}{160(1-\eta)}\left[(1-2\eta)\lambda|^{\alpha}\lambda|^{\beta} - \eta\lambda|_{\sigma}\lambda|_{\omega} a^{\sigma\omega} a^{\alpha\beta} - 8\eta a^{\alpha\beta}\right] \qquad (2.5.3)$$

$$R^{\alpha\beta} = \frac{\lambda^2}{160(1-\eta)}\left[-2(1-2\eta)\lambda|^{\alpha}\lambda|^{\beta} + 2\eta\lambda|_{\sigma}\lambda|_{\omega} a^{\sigma\omega} a^{\alpha\beta}\right]. \qquad (2.5.4)$$

We note that each of these three tensors depend only on λ, η, and the surface metric $a^{\alpha\beta}$. We can now write (2.5.1) in a simpler form,

$$m^{\alpha\beta} = -D\left[(1-\eta)w|^{\alpha\beta} + \eta a^{\alpha\beta} w|^{\rho}_{\rho}\right]$$

$$+ \frac{\lambda^2}{10}(q^{\alpha}|^{\beta} + q^{\beta}|^{\alpha}) + \frac{\lambda\lambda|_{\rho}\eta a^{\alpha\beta}}{5(1-\eta)} q^{\rho} \tag{2.5.5}$$

$$+ \frac{\lambda^2}{20}(f^{\alpha}|^{\beta} + f^{\beta}|^{\alpha}) + P^{\alpha\beta} \circledast p + Q^{\alpha\beta} f^{\rho}|_{\rho} + \frac{\lambda|_{\rho}}{\lambda} R^{\alpha\beta} f^{\rho}.$$

We can now use (1.5.22) and (1.5.23) to eliminate q^{α} and f^{α} from the right hand side of (2.5.5). Since we have

$$f^{\beta} = \frac{\lambda|_{\alpha}}{\lambda} m^{\alpha\beta}, \tag{1.5.22}$$

then

$$f^{\beta}|^{\alpha} = \frac{\lambda|_{\rho}}{\lambda} m^{\rho\beta}|^{\alpha} + \frac{\lambda\lambda|^{\alpha}_{\rho} - \lambda|_{\rho}\lambda|^{\alpha}}{\lambda^2} m^{\rho\beta}. \tag{2.5.6}$$

From (1.5.23), we get

$$q^{\beta} = m^{\alpha\beta}|_{\alpha} - \frac{\lambda|_{\alpha}}{\lambda} m^{\alpha\beta}. \tag{2.5.7}$$

Substituting (2.5.6) and (2.5.7) in (2.5.5), we get

$$m^{\alpha\beta} = -D\left[(1-\eta)w|^{\alpha\beta} + \eta a^{\alpha\beta} w|^{\rho}_{\rho}\right]$$

$$+ \frac{\lambda^2}{10}\left[m^{\sigma\alpha}|^{\beta}_{\sigma} + m^{\sigma\beta}|^{\alpha}_{\sigma} - \frac{\lambda|_{\sigma}}{\lambda}(m^{\sigma\alpha}|^{\beta} + m^{\sigma\beta}|^{\alpha})\right.$$

$$\left. - \frac{\lambda\lambda|^{\beta}_{\sigma} - \lambda|_{\sigma}\lambda|^{\beta}}{\lambda^2} m^{\sigma\alpha} - \frac{\lambda\lambda|^{\alpha}_{\sigma} - \lambda|_{\sigma}\lambda|^{\alpha}}{\lambda^2} m^{\sigma\beta}\right]$$

$$+ \frac{\lambda^2\eta a^{\alpha\beta}\lambda|_{\rho}}{5(1-\eta)\lambda}\left(m^{\sigma\rho}|_{\sigma} - \frac{\lambda|_{\sigma}}{\lambda} m^{\sigma\rho}\right) \tag{2.5.8}$$

$$+ \frac{\lambda^2}{20}\left[\frac{\lambda|_{\rho}}{\lambda}(m^{\rho\beta}|^{\alpha} + m^{\rho\alpha}|^{\beta}) + \frac{\lambda\lambda|^{\alpha}_{\rho} - \lambda|_{\rho}\lambda|^{\alpha}}{\lambda^2} m^{\rho\beta}\right.$$

$$\left. + \frac{\lambda\lambda|^{\beta}_{\rho} - \lambda|_{\rho}\lambda|^{\beta}}{\lambda^2} m^{\rho\alpha}\right]$$

$$+ P^{\alpha\beta} \circledast p + Q^{\alpha\beta}\left[\frac{\lambda|_{\rho}}{\lambda} m^{\rho\sigma}|_{\sigma} + \frac{\lambda\lambda|_{\sigma\rho} - \lambda|_{\rho}\lambda|_{\sigma}}{\lambda^2} m^{\rho\sigma}\right]$$

$$+ \frac{\lambda|_{\rho}}{\lambda} R^{\alpha\beta} \frac{\lambda|_{\sigma}}{\lambda} m^{\rho\sigma}.$$

If we now let

$$S^{\alpha\beta} = \frac{32\eta a^{\alpha\beta}\lambda^2}{160(1-\eta)} , \tag{2.5.9}$$

and regroup terms, we can write (2.5.8) as

$$\begin{aligned} m^{\alpha\beta} &= -D\left[(1-\eta)w|^{\alpha\beta} + \eta a^{\alpha\beta} w|^{\rho}_{\rho}\right] \\ &+ \frac{\lambda^2}{10}\left[m^{\sigma\alpha}|_{\sigma}^{\ \beta} + m^{\sigma\beta}|_{\sigma}^{\ \alpha} - \frac{\lambda|_{\sigma}}{\lambda}(m^{\sigma\alpha}|^{\beta} + m^{\sigma\beta}|^{\alpha}) \right. \\ &\qquad \left. - \frac{\lambda\lambda|^{\beta}_{\sigma} - \lambda|_{\sigma}\lambda|^{\beta}}{\lambda^2} m^{\sigma\alpha} - \frac{\lambda\lambda|^{\alpha}_{\sigma} - \lambda|_{\sigma}\lambda|^{\alpha}}{\lambda^2} m^{\sigma\beta}\right] \\ &+ \left[-S^{\alpha\beta}\frac{\lambda|_{\sigma}\lambda|_{\rho}}{\lambda^2} + R^{\alpha\beta}\frac{\lambda|_{\sigma}\lambda|_{\rho}}{\lambda^2} + Q^{\alpha\beta}\left(\frac{\lambda\lambda|_{\sigma\rho} - \lambda|_{\rho}\lambda|_{\sigma}}{\lambda^2}\right)\right] m^{\rho\sigma} \\ &+ \left[S^{\alpha\beta}\frac{\lambda|_{\rho}}{\lambda} + Q^{\alpha\beta}\frac{\lambda|_{\rho}}{\lambda}\right] m^{\rho\sigma}|_{\sigma} + P^{\alpha\beta}\, \text{ⓟ}\, p. \end{aligned} \tag{2.5.10}$$

If we now differentiate (1.5.23) covariantly with respect to θ_{β}, we get

$$m^{\alpha\beta}|_{\alpha\beta} - f^{\beta}|_{\beta} - q^{\beta}|_{\beta} = 0$$

and then by using (1.5.9), we get

$$m^{\alpha\beta}|_{\alpha\beta} + \text{ⓟ}\, p = 0 . \tag{2.5.11}$$

In equations (2.5.10) and (2.5.11), we have a system of four simultaneous differential equations in four unknowns in $m^{\alpha\beta}$ and $w.^{\alpha\beta}$ It can be seen that it would be difficult indeed to get a general solution for this set of equations. We shall see that we may obtain a solution under certain conditions.

Let us now consider the remainder of the thirteen equations that we have, i.e.

$$n^{\alpha\beta}\big|_{\alpha} + \oplus p^{\beta} = 0 \qquad (1.5.10)$$

and

$$n^{\alpha\beta} = \mu\lambda L\left[v^{\alpha}\big|^{\beta} + v^{\beta}\big|^{\alpha} + \frac{2\eta a^{\alpha\beta}}{1-\eta} v^{\rho}\big|_{\rho}\right] + \frac{\eta\lambda a^{\alpha\beta}}{2(1-\eta)}\oplus p.^{*} \qquad (2.4.11)$$

We have here five equations in the five unknowns $n^{\alpha\beta}$ and v^{α}. We may differentiate (2.4.11) covariantly with respect to θ_{α} and substitute in (1.5.10) and get an equation involving only the unknown displacements v^{α}.

Suppose that in (2.5.10) and (2.5.11), we have λ as a linear function of θ_2 only, i.e. it is independent of θ_1. We shall see that many terms in (2.5.10) will vanish. We have

$$\begin{aligned}
m^{\alpha\beta} = &- D\left[(1-\eta)\, w\big|^{\alpha\beta} + \eta\, a^{\alpha\beta}\, w\big|^{\rho}_{\rho}\right] \\
&+ \frac{\lambda^{2}}{10}\Big[m^{\sigma\alpha}\big|_{\sigma}^{\ \beta} + m^{\sigma\beta}\big|_{\sigma}^{\ \alpha} - \frac{\lambda|_2}{\lambda}\left(m^{2\alpha}\big|^{\beta} + m^{2\beta}\big|^{\alpha}\right) \\
&- \frac{\lambda\lambda|_2^{\beta} - \lambda|_2\lambda|^{\beta}}{\lambda^{2}} m^{2\alpha} - \frac{\lambda\lambda|_2^{\alpha} - \lambda|_2\lambda|^{\alpha}}{\lambda^{2}} m^{2\beta}\Big] \qquad (2.5.12) \\
&+ \frac{\lambda|_2\lambda|_2}{\lambda^{2}}\left[R^{\alpha\beta} - S^{\alpha\beta} - Q^{\alpha\beta}\right] m^{22} \\
&+ \frac{\lambda|_2}{\lambda}\left[S^{\alpha\beta} + Q^{\alpha\beta}\right] m^{2\sigma}\big|_{\sigma} + P^{\alpha\beta}\oplus p.
\end{aligned}$$

Suppose that we now assume that the stresses are independent of θ_1. If we examine (2.3.33), (2.3.37), and (2.3.46) under this assumption, we see that $m^{\alpha\beta}$, $n^{\alpha\beta}$, q^{α}, and f^{α} are independent of θ_1. We can now write (2.5.12) as three equations.

$$m^{11} = -D\left[(1-\eta)\; w|^{11} + \eta\, a^{11}\, w|^{\rho}_{\rho}\right]$$

$$+ \frac{\lambda|_2\, \lambda|_2}{\lambda^2}\left[R^{11} - S^{11} - Q^{11}\right] m^{22}$$

$$+ \frac{\lambda|_2}{\lambda}\left[S^{11} + Q^{11}\right] m^{22}|_2 + P^{11} \circledast p.$$

If we examine (2.5.2), (2.5.3), (2.5.4), and (2.5.9) with the present assumptions, we have

$$m^{11} = -D\left[(1-\eta)\; w|^{11} + \eta\, a^{11}\, w|^{\rho}_{\rho}\right]$$

$$+ \frac{\lambda|_2\, \lambda|_2\, \eta}{160(1-\eta)}\left[\lambda|_2\; \lambda|_2\; a^{11}\, a^{22} - 40\, a^{11}\right] m^{22}$$

$$+ \frac{\lambda\, \lambda|_2\, \eta}{160(1-\eta)}\left[-\lambda|_2\; \lambda|_2\; a^{11}\, a^{22} + 24\, a^{11}\right] m^{22}|_2 \qquad (2.5.13)$$

$$+ \frac{\lambda^2\, \eta}{160(1-\eta)}\left[-3\, a^{11}\, a^{22} - 16\, a^{11}\right] \circledast p.$$

In the same manner, we get

$$m^{12} = -D\left[(1-\eta)\,w\Big|^{12} + \eta\ a^{11}\ w\Big|^{\rho}_{\rho}\right]$$

$$+\frac{\lambda^2}{10}\left[m^{21}\Big|^{22} - \frac{\lambda|_2}{\lambda}m^{21}\Big|^{2} + \frac{\lambda|_2\,\lambda|_2}{\lambda^2}m^{21}\right]$$

$$+\frac{a^{12}\,\eta\,\lambda|_2\,\lambda|_2}{160(1-\eta)}\left[3\,\lambda|_2\ \lambda|_2\ a^{22} - 24\right]m^{22} \qquad (2.5.14)$$

$$+\frac{a^{12}\,\eta\,\lambda\,\lambda|_2}{160(1-\eta)}\left[-\lambda|_2\ \lambda|_2\ a^{22} + 24\right]m^{22}\Big|_2$$

$$\frac{a^{12}\,\eta\,\lambda^2}{160(1-\eta)}\left[-3\,\lambda|_2\ \lambda|_2\ a^{22} - 16\right] \oplus p.$$

Then for m^{22}, we get

$$m^{22} = -D\left[(1-\eta)\,w\Big|^{22} + \eta\ a^{22}\ w\Big|^{\rho}_{\rho}\right]$$

$$+\frac{\lambda^2}{5}\left[m^{22}\Big|^{22} - \frac{\lambda|_2}{\lambda}m^{22}\Big|^{2} + \frac{\lambda|_2\,\lambda|^2}{\lambda^2}m^{22}\right]$$

$$+\frac{\lambda|_2\ \lambda|_2}{160(1-\eta)}\left[-3(1-2\eta)\,\lambda|^2\,\lambda|^2 + 3\eta\,\lambda|_2\ \lambda|_2\ a^{22}a^{22}\right.$$

$$\left. - 24\,\eta\,a^{22}\right]m^{22} \qquad (2.5.15)$$

$$+\frac{\lambda\ \lambda|_2}{160(1-\eta)}\left[+(1-2\eta)\,\lambda|^2\,\lambda|^2 - \eta\,\lambda|_2\ \lambda|_2\ a^{22}a^{22} + 24\eta a^{22}\right]m^{22}\Big|_2$$

$$\frac{\lambda^2}{160(1-\eta)}\left[-16\,\eta\,a^{22} + 3\lambda|^2\ \lambda|^2\ (1-2\eta) - 3\,\eta\lambda|_2\ \lambda|_2\ a^{22}a^{22}\right] \oplus p.$$

From (2.5.11), we get under the same assumptions,

$$m^{22}\Big|_{22} + \oplus\, p = 0. \qquad (2.5.16)$$

Now in equations (2.5.13), (2.5.14), (2.5.15) and (2.5.16), we have four unknowns in four equations. We could now specialize these equations to rectangular cartesian coordinates or cylindrical coordinates since these are the simpler of the coordinate systems.

Another assumption that we can make to simplify (2.5.13), (2.5.14), and (2.5.15) still further is that the derivatives of λ are small when compared to λ.

Even after the assumptions that we have made, we see that we must select a particular problem in a particular coordinate system to make further progress.

CHAPTER III

1. A Buckling Problem Solved by Classical Theory

Conway (29) found a Levy type solution for the deflection of a variable thickness rectangular plate by using the classical theory. This solution can be extended so that an approximate solution of the buckling of the plate may be found.

It is shown in (33) that for a plate subjected to a lateral load q and forces in the middle plane, the equation of equilibrium is

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\left(q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y}\right), \tag{3.1.1}$$

where

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) , \tag{3.1.2}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) , \tag{3.1.3}$$

$$M_{xy} = -M_{yx} = D\,(1-\nu)\frac{\partial^2 w}{\partial x \partial y} . \tag{3.1.4}$$

For a plate of variable thickness, D, the flexural rigidity, will be a function of x and y so that placing (3.1.2), (3.1.3) and (3.1.4) into (3.1.1), we get

$$D \nabla^4 w + 2 \frac{\partial D}{\partial x} \frac{\partial}{\partial x} (\nabla^2 w) + 2 \frac{\partial D}{\partial y} \frac{\partial}{\partial y} (\nabla^2 w) + \nabla^2 D \nabla^2 w$$

$$- (1-\nu) \quad \frac{\partial^2 D}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 D}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2} \frac{\partial^2 w}{\partial x^2} \quad =$$

$$q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \quad . \tag{3.1.5}$$

Consider a rectangular plate of length 2a in the x direction and b in the y direction such that

$$D = D_o \quad e^{cy} \tag{3.1.6}$$

where D_o and C are constants, and let the plate be loaded with a lateral load q = a constant and let

$$N_x = P_o \quad e^{cy}$$

$$N_y = 0 \tag{3.1.7}$$

$$N_{xy} = 0 \quad .$$

Placing (3.1.6) and (3.1.7) in (3.1.5), we have

$$\nabla^4 w + 2c \frac{\partial}{\partial y} (\nabla^2 w) + c^2 \frac{\partial^2 w}{\partial y^2} + c^2 \nu \frac{\partial^2 w}{\partial x^2} - \frac{P_o}{D_o} \frac{\partial^2 w}{\partial x^2} = \frac{q}{D_o} e^{-cy} \quad .$$

(3.1.8)

Assume a solution for the homogeneous equation of (3.1.8) in the form

$$w_1 = \sum_{n=1,3,5}^{\infty} F(y) \cos \gamma_n x \tag{3.1.9}$$

where

$$\gamma_n = \frac{n\pi}{2a} \quad . \tag{3.1.10}$$

Substituting (3.1.9) in (3.1.8) and simplifying, we get

$$F'''' + 2cF''' + (c^2 - 2\gamma_n^2)\, F'' - 2c\gamma_n^2\, F' + (\gamma_n^4 - c^2\nu\gamma_n^2 - \frac{P_o}{D_o}\,\gamma_n^2)F=0, \tag{3.1.11}$$

where the primes denote differentiation with respect to y. For solutions of the form

$$F = A\, e^{\lambda y} \quad , \tag{3.1.12}$$

we have the equation

$$\lambda^4 + 2c\lambda^3 + (c^2-2\gamma_n^2)\,\lambda^2 - 2c\gamma_n^2\lambda + (\gamma_n^4 - c^2\nu\gamma_n^2 - \frac{P_o}{D_o}\,\gamma_n^2) = 0. \tag{3.1.13}$$

If we write (3.1.13) in the form

$$(\lambda + \alpha + \beta_1)(\lambda + \alpha - \beta_1)(\lambda + \alpha + \beta_2)(\lambda + \alpha - \beta_2) = 0, \tag{3.1.14}$$

we can by expanding (3.1.14) and equating coefficients between (3.1.14) and (3.1.13), solve for α, β_1, and β_2. We obtain

$$\alpha = c/2 \tag{3.1.15}$$

$$\beta_1 = \tfrac{1}{2}\left[c^2 + 4\gamma_n^2 + 4\gamma_n\,(\nu c^2 + \frac{P_o}{D_o})^{1/2}\right]^{1/2} \tag{3.1.16}$$

$$\beta_2 = \tfrac{1}{2}\left[c^2 + 4\gamma_n^2 - 4\gamma_n\,(\nu c^2 + \frac{P_o}{D_o})^{1/2}\right]^{1/2} \quad . \tag{3.1.17}$$

We may then write

$$\lambda_{1,2} = -\alpha \pm \beta_1 \tag{3.1.18}$$

$$\lambda_{3,4} = -\alpha \pm \beta_2 \,. \tag{3.1.19}$$

Suppose that the plate to be considered is simply supported on the edges $x = \pm a$. Then if we expand unity in a Fourier series of $\cos \gamma_n x$, we can write the loading intensity as

$$q = \frac{4q}{\pi} \sum_{n=1,3}^{\infty} \frac{1}{n} (-1)^{\frac{n-1}{2}} \cos \gamma_n x \quad , \gamma_n = \frac{n\pi}{2a}. \tag{3.1.20}$$

Assume a particular solution of (3.1.8) in the form

$$w_2 = e^{-cy} \sum_{n=1,3}^{\infty} F_n \cos \gamma_n x \quad . \tag{3.1.21}$$

Substitute (3.1.21) in (3.1.8) and find that

$$F_n = \frac{4q}{\pi D_o} \frac{\frac{1}{n} (-1)^{\frac{n-1}{2}}}{(\gamma_n^4 - c^2 \nu \gamma_n^2 + \frac{P_o}{D_o} \gamma_n^2)} \quad . \tag{3.1.22}$$

Then the complete solution is

$$w = \frac{4q}{\pi D_o} \sum_{n=1,3}^{\infty} \left[c_1' e^{\lambda_1 y} + c_2' e^{\lambda_2 y} + c_3' e^{\lambda_3 y} + c_4' e^{\lambda_4 y} + \frac{(-1)^{\frac{n-1}{2}} e^{-cy}}{n(\gamma_n^4 - c^2 \nu \gamma_n^2 + \frac{P_o}{D_o} \gamma_n^2)} \right] \cos \gamma_n x \quad . \tag{3.1.23}$$

We can write (3.1.23) in the form

$$w = \sum_{n=1,3}^{\infty} \Bigg[c_1 e^{\lambda_1 y} + c_2 e^{\lambda_2 y} + c_3 e^{\lambda_3 y} + c_4 e^{\lambda_4 y}$$

$$+ \frac{4q}{\pi D_o} \frac{(-1)^{\frac{n-1}{2}} e^{-cy}}{n(\gamma_n^4 - c^2 \nu \gamma_n^2 + \frac{P_o}{D_o} \gamma_n^2)} \Bigg] \cos \gamma_n x \ . \tag{3.1.24}$$

Let q = 0 in (3.1.24) and consider the buckling problem. We then have

$$w = \sum_{n=1,3}^{\infty} \left[c_1 e^{\lambda_1 y} + c_2 e^{\lambda_2 y} + c_3 e^{\lambda_3 y} + c_4 e^{\lambda_4 y} \right] \cos \gamma_n x \ , \tag{3.1.25}$$

where

$$\lambda_1 = - \frac{c}{2} + \frac{1}{2} \left[c^2 + 4\gamma_n^2 + 4\gamma_n (\nu c^2 + \frac{P_o}{D_o})^{1/2} \right]^{1/2} = -\alpha + \beta_1$$

$$\lambda_2 = - \frac{c}{2} - \frac{1}{2} \left[c^2 + 4\gamma_n^2 + 4\gamma_n (\nu c^2 + \frac{P_o}{D_o})^{1/2} \right]^{1/2} = -\alpha - \beta_1$$

$$\lambda_3 = - \frac{c}{2} + \frac{1}{2} \left[c^2 + 4\gamma_n^2 - 4\gamma_n (\nu c^2 + \frac{P_o}{D_o})^{1/2} \right]^{1/2} \tag{3.1.26}$$

$$\lambda_4 = - \frac{c}{2} - \frac{1}{2} \left[c^2 + 4\gamma_n^2 - 4\gamma_n (\nu c^2 + \frac{P_o}{D_o})^{1/2} \right]^{1/2} \ .$$

In (3.1.26), we see that λ_1 and λ_2 will be real numbers and that λ_3 and λ_4 may be real or complex depending on the values of c, γ_n, P_o, D_o, and ν. In order to satisfy certain conditions on the boundaries y = o and y = b, we need for λ_3 and λ_4 to be complex. This will be the case if

$$4\,\gamma_n\,(\nu\, c^2 + \frac{P_o}{D_o})^{1/2} > c^2 + 4\gamma_n^2 \quad . \tag{3.1.27}$$

Let us consider this to be the case so that we can write

$$\lambda_3 = -\frac{c}{2} + i\beta_2 = -\alpha + i\beta_2$$

and (3.1.28)

$$\lambda_4 = -\frac{c}{2} - i\beta_2 = -\alpha - i\beta_2$$

where β_2 is now the real number

$$\beta_2 = 1/2\left[-c^2 - 4\gamma_n^2 + 4\gamma_n\,(\nu c^2 + \frac{P_o}{D_o})^{1/2}\right]^{1/2} \quad . \tag{3.1.29}$$

We can then write after some simplification

$$w = \sum_{n=1,3}^{\infty} e^{-\alpha y}\left[c_1 e^{\beta_1 y} + c_2 e^{-\beta_1 y} + c_5 \sin\beta_2 y + c_6 \cos\beta_2 y\right]\cos\gamma_n x, \tag{3.1.30}$$

or

$$w = \sum_{n=1,3}^{\infty} F(y)\cos\gamma_n x \tag{3.1.31}$$

where F(y) is now

$$F(y) = e^{-\alpha y}\left[c_1 e^{\beta_1 y} + c_2 e^{-\beta_1 y} + c_5 \sin \beta_2 y + c_6 \cos \beta_2 y\right]. \qquad (3.1.32)$$

Case I: Consider the case where y = 0 is simply supported and y = b is free. Then we have

$$w = 0, \quad \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{for} \quad y = 0 \qquad (3.1.33)$$

and

$$\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y} = 0 \quad \text{for } y = b. \qquad (3.1.34)$$

(3.1.33) implies that

$$F(y)\Big]_{y=0} = 0 \qquad (3.1.35)$$

and

$$F''(y)\Big]_{y=0} - \nu\,\gamma_n^2\, F(y)\Big]_{y=0} = 0. \qquad (3.1.36)$$

(3.1.34) implies that

$$F''(y)\Big]_{y=b} - \nu\,\gamma_n^2\, F(y)\Big]_{y=b} = 0 \qquad (3.1.37)$$

and

$$F'''(y)\Big]_{y=b} - (2-\nu)\,\gamma_n^2\, F'(y)\Big]_{y=b} = 0. \qquad (3.1.38)$$

Placing (3.1.32) in (3.1.35), (3.1.36), (3.1.37), and (3.1.38), we get four homogeneous equations in the four unknowns c_1, c_2, c_5, and c_6. These equations are

$$c_1 + c_2 + c_6 = 0 \quad , \tag{3.1.39}$$

$$c_1\left[(\beta_1-\alpha)^2 - \nu\gamma_n^2\right] + c_2\left[(\beta_1+\alpha)^2 - \nu\gamma_n^2\right] - 2\,c_5\alpha\beta_2 + c_6\left[\alpha^2 - \beta_2^2 - \nu\gamma_n^2\right] = 0 \;, \tag{3.1.40}$$

$$\begin{aligned} c_1 e^{\beta_1 b}\left[(\beta_1-\alpha)^2 - \nu\gamma_n^2\right] &+ c_2 e^{-\beta_1 b}\left[(\beta_1+\alpha)^2 - \nu\gamma_n^2\right] \\ &+ c_5\left[(-\beta_2^2+\alpha^2-\nu\gamma_n^2)\sin\beta_2 b - 2\alpha\beta_2\cos\beta_2 b\right] \\ &+ c_6\left[2\alpha\beta_2\sin\beta_2 b + (-\beta_2^2+\alpha^2-\nu\gamma_n^2)\cos\beta_2 b\right] = 0 \;, \end{aligned} \tag{3.1.41}$$

$$\begin{aligned} c_1 e^{\beta_1 b}&(\beta_1-\alpha)\left[(\beta_1-\alpha)^2 - (2-\nu)\gamma_n^2\right] \\ &- c_2 e^{-\beta_1 b}(\beta_1+\alpha)\left[(\beta_1+\alpha)^2 - (2-\nu)\gamma_n^2\right] \\ &+ c_5\left\{\beta_2\left[3\alpha^2-\beta_2^2 - (2-\nu)\gamma_n^2\right]\cos\beta_2 b + \alpha\left[3\beta_2^2-\alpha^2 \right.\right. \\ &\left.\left.+ (2-\nu)\gamma_n^2\right]\sin\beta_2 b\right\} + c_6\left\{\alpha\left[3\beta_2^2-\alpha^2+(2-\nu)\gamma_n^2\right]\cos\beta_2 b \right. \\ &\left. + \beta_2\left[\beta_2^2 - 3\alpha^2 + (2-\nu)\gamma_n^2\right]\sin\beta_2 b\right\} = 0. \end{aligned} \tag{3.1.42}$$

If we make the following substitutions,

$$A = (\beta_1 - \alpha)^2 - \nu\,\gamma_n^2 \;, \qquad B = (\beta_1 + \alpha)^2 - \nu\,\gamma_n^2$$

$$D = \alpha^2 - \beta_2^2 - \nu\,\gamma_n^2$$

$$E = (\beta_1 - \alpha)^2 - (2 - \nu)\,\gamma_n^2 \;, \quad F = (\beta_1 + \alpha)^2 - (2 - \nu)\,\gamma_n^2 \tag{3.1.43}$$

$$G = 3\alpha^2 - \beta_2^2 - (2 - \nu)\,\gamma_n^2 \;, \quad H = 3\beta_2^2 - \alpha^2 + (2 - \nu)\,\gamma_n^2$$

Then we can write (3.1.39), (3.1.40), (3.1.41), and (3.1.42) as

$$c_1 + c_2 + c_6 = 0 \tag{3.1.39}$$

$$Ac_1 + Bc_2 - 2\alpha\beta_2\, c_5 + Dc_6 = 0 \tag{3.1.40}$$

$$Ae^{\beta_1 b}c_1 + Be^{-\beta_1 b}c_2 + (D\sin\beta_2 b - 2\alpha\beta_2\cos\beta_2 b)c_5 + (2\alpha\beta_2\sin\beta_2 b + D\cos\beta_2 b)c_6 = 0 \tag{3.1.41}$$

$$Ee^{\beta_1 b}(\beta_1 - \alpha)c_1 - F\,e^{-\beta_1 b}(\beta_1 + \alpha)c_2 + \left[\beta_2 G\cos\beta_2 b + \alpha H\sin\beta_2 b\right] c_5$$

$$+ \left[\alpha H\cos\beta_2 b - \beta_2 G\sin\beta_2 b\right] c_6 = 0 \,. \tag{3.1.42}$$

Now if a solution is to exist for this problem, we must satisfy the condition that the determinant of the coefficients of c_1, c_2, c_5, and c_6 must be zero. This determinant is

$$\begin{vmatrix} 1 & 1 & 0 & 1 \\ A & B & -2\alpha\beta_2 & D \\ A\,e^{\beta_1 b} & B\,e^{-\beta_1 b} & D\sin\beta_2 b - 2\alpha\beta_2\cos\beta_2 b & 2\alpha\beta_2\sin\beta_2 b + D\cos\beta_2 b \\ E\,e^{\beta_1 b}(\beta_1 - \alpha) & -F\,e^{-\beta_1 b}(\beta_1 + \alpha) & \beta_2 G\cos\beta_2 b + \alpha H\sin\beta_2 b & \alpha H\cos\beta_2 b - \beta_2 G\sin\beta_2 b \end{vmatrix} = 0 \tag{3.1.43}$$

If we expand this determinant and equate to zero, we have the characteristic equation for the solution of this buckling problem. Since $\alpha = \frac{c}{2}$, if we assume that c is small, then we may neglect powers of α higher than the first and we now make this assumption. The expansion of the above determinant may be written after some simplification in the form

$$(R\, e^{\beta_1 b} + S\, e^{-\beta_1 b}) \cos \beta_2 b + (T\, e^{\beta_1 b} + \quad e^{-\beta_1 b}) \sin \beta_1 b$$

$$- 4\alpha\beta_1\beta_2 DG - 2\alpha\beta_1\beta_2(AF + BE) = 0 \,, \tag{3.1.44}$$

where

$$R = A\, G\, \beta_2(B - D) + 2\alpha\beta_1\beta_2\, B\, E \tag{3.1.45}$$

$$S = B\, G\, \beta_2(D - A) + 2\alpha\beta_1\beta_2\, A\, F \tag{3.1.46}$$

$$T = \beta_1\, D\, E(D - B) + D\, E\, \alpha\, (B - D) + A\, H(B - D) + 2\alpha\, A\, G\beta_2^2 \tag{3.1.47}$$

$$U = F\, D\beta_1\, (D - A) + \alpha(D - A)\, (BH + FD) - 2\alpha\, BG\beta_2^2 \,. \tag{3.1.48}$$

Equation (3.1.44) is our characteristic equation for determining the critical values of the buckling load. If in (3.1.44) we let $\alpha = \frac{c}{2} = 0$, then we should get the characteristic equation for the buckling of a constant thickness plate. This is indeed the case and we have

$$\beta_2(\beta_1^2 - \nu\gamma_n^2) \tanh \beta_1 b = \beta_1(\beta_2^2 + \nu\gamma_n^2) \tan \beta_1 b \quad , \tag{3.1.49}$$

for $\alpha = \frac{c}{2} = 0$.

In (3.1.44), β_1 and β_2 are functions of the critical load P_o and thus R, S, T, U, A, B, D, E, F, and G may be expressed as functions of P_o and the dimensions and properties of the plate. We could thus use (3.1.44) to find the least critical value P_o which would cause buckling.

Case II: Consider now the case where the edge $y = 0$ is fixed instead of simply supported, while the edge $y = b$ is still free. We would then have for the boundary conditions on $y = 0$

$w = 0$ and $\frac{\partial w}{\partial y} = 0$. The only change in this case from Case I is that $\frac{\partial w}{\partial y} = 0$ at $y = 0$ instead of the condition

$$\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } y = 0.$$

Using (3.1.31) and (3.1.32) we see that $\frac{\partial w}{\partial y} = 0$ implies that

$$F'(y)\Big]_{y=0} = 0 \qquad \text{which yields}$$

$$c_1(\beta_1 - \alpha) - c_2(\beta_1 + \alpha) + c_5\beta_2 - c_6\alpha = 0 \,. \tag{3.1.50}$$

Now we set up the determinant formed from (3.1.50), (3.1.39), (3.1.41) and (3.1.42). There will be no change in A, B, D, E, F, G, and H. When this determinant has been expanded and equated to zero, we can write the characteristic equation in the form

$$\begin{aligned}(R\, e^{\beta_1 b} + T\, e^{-\beta_1 b}) \cos\beta_2 b + (S\, e^{\beta_1 b} + U\, e^{-\beta_1 b}) \sin\beta_2 b \\ + \beta_1\beta_2(2DG + AF + BE) + \alpha\beta_2(AF - BE) = 0 \,,\end{aligned} \tag{3.1.51}$$

where

$$R = -\beta_1\beta_2(DE + AG) + \alpha\beta_2(DE + AH) - 2\alpha\beta_1^2\beta_2 E \tag{3.1.52}$$

$$S = \beta_1^2\, DE - \beta_2^2 AG - \alpha\beta_1(DE + AH) - 2\alpha\beta_1\beta_2^2\, E \tag{3.1.53}$$

$$T = -\beta_1\beta_2(FD + BG) - \alpha\beta_2(DF + BH) + 2\alpha\beta_1^2\beta_2\, F \tag{3.1.54}$$

$$U = \beta_2^2\, BG - \beta_1^2\, FD - \alpha\beta_1(DF + BH) - 2\alpha\beta_1\beta_2^2\, F \quad . \tag{3.1.55}$$

Equation (3.1.51) for $\alpha = 0$ becomes

$$2(\beta_1^2 - \nu\gamma_n^2)\,(\beta_2^2 + \nu\gamma_n^2) + \Big[\,(\beta_1^2 - \nu\,\gamma_n^2)^2 + (\beta_2^2 + \nu\,\gamma_n^2)^2 \quad \cosh\beta_1 b\,\cos\beta_2 b =$$

$$\frac{1}{\beta_1\beta_2}\Big[\beta_1^2(\beta_2^2 + \nu\gamma_n^2) - \beta_2^2(\beta_1^2 - \nu\gamma_n^2) \quad \sinh\beta_1 b\,\sin\beta_2 b \quad , \tag{3.1.56}$$

which is the characteristic equation for a constant thickness plate with same boundary conditions.

Case III: Consider now the case where both edges are fixed at $y=0$ and $y=b$. The equations then become

$$c_1 + c_2 + c_6 = 0 \tag{3.1.57}$$

$$c_1(\beta_1-\alpha) - c_2(\beta_1+\alpha) + \beta_2 c_5 - \alpha c_6 = 0 \tag{3.1.58}$$

$$c_1 e^{\beta_1 b} + c_2 e^{-\beta_1 b} + c_5 \sin\beta_2 b + c_6 \cos\beta_2 b = 0 \tag{3.1.59}$$

$$c_1 e^{\beta_1 b}(\beta_1 - \alpha) - c_2 e^{-\beta_1 b}(\beta_1 + \alpha) + c_5(\beta_2 \cos\beta_2 b - \alpha \sin\beta_2 b)$$

$$- c_6(\beta_2 \sin\beta_2 b + \alpha\cos\beta_2 b) = 0 \quad . \qquad (3.1.60)$$

Expanding the determinantin the usual manner, we can write the characteristic equation for this case in the form

$$(R\, e^{\beta_1 b} + T\, e^{-\beta_1 b}) \cos\beta_2 b + (S\, e^{\beta_1 b} + U\, e^{-\beta_1 b}) \sin\beta_2 b$$
$$+ 4\beta_1\beta_2 = 0 \quad , \qquad (3.1.61)$$

where

$$R = - 2\beta_1\beta_2 \qquad (3.1.62)$$

$$S = \beta_1^2 - \beta_2^2 \qquad (3.1.63)$$

$$T = - 2\beta_1\beta_2 \qquad (3.1.64)$$

$$U = -\beta_1^2 + \beta_2^2 \quad . \qquad (3.1.65)$$

For $\alpha = 0$, equation (3.1.61) becomes

$$2 - 2\cos\beta_2 b \cosh\beta_1 b + \frac{\beta_1^2 - \beta_2^2}{\beta_1\beta_2} \sin\beta_2 b \sinh\beta_1 b = 0, \qquad (3.1.66)$$

which is the characteristic equation for a constant thickness plate with the same boundary conditions.

Thus, we have found characteristic equations for three cases of variable thickness plates provided the flexural rigidity and the load P_x do not vary too much from the uniform case,ie. provided that $\alpha = c/2$ is small enough such that we can neglect powers of α higher than the first.

2. Application of Theory Presented in Chapters I and II

We shall now see how the theory which has been developed can be applied to solve actual plate problems. Two problems will be solved, one in rectangular and the other in cylindrical coordinates. Both are problems for which the classical solution exists and we shall compare the solutions obtained by using the present theory with the classical solutions. Consider the plate shown in Figure 2 below.

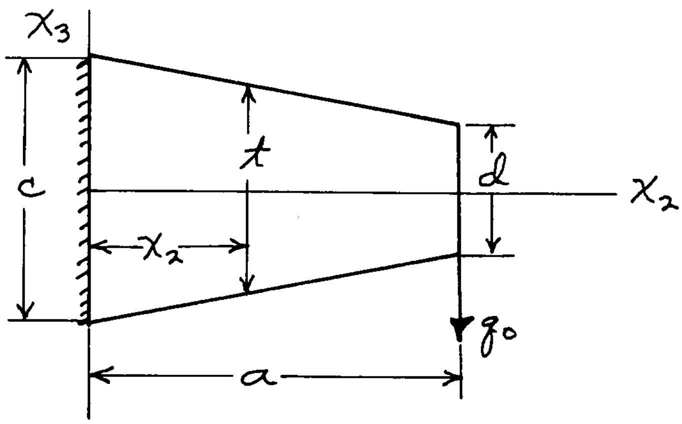

Figure 2. Rectangular Plate with Shear Load on One End

The plate to be considered is free on two edges $x_1 = \pm b/2$, clamped on $x_2 = 0$ and loaded with a uniform shear load q_0 on $x_2 = a$. The top and bottom surfaces are free of loads. Let the plate be long enough in the x_1 direction, say b units, such that we may assume that the stresses and displacements are functions of x_2 and x_3 but not of x_1. In our dimensionless coordinates, we have $\theta_2 = x_2/L$ and $\lambda = t/L$. Let the characteristic length L be a so that we have

$$\theta_2 = \frac{x_2}{L} = \frac{x_2}{a} \tag{3.2.1}$$

and

$$\lambda = \frac{t}{L} = \frac{c + (d - c)\theta_2}{a} \tag{3.2.2}$$

and if we let $C_1 = c/a$ and $C_2 = (d - c)/a$, then

$$\lambda = C_1 + C_2\theta_2. \tag{3.2.3}$$

The boundary conditions are such that we have on $x_2 = 0$

$$w = 0, \qquad w_{,2} = 0, \qquad m_{(22)} = -q_0 a \tag{3.2.4}$$

and on $x_2 = a$,

$$q_{(2)} = q_0, \qquad m_{(22)} = 0, \qquad m_{(12)} = 0 \tag{3.2.5}$$

and on $x_1 = \pm\, b/2$,

$$q_{(1)} = 0, \qquad m_{(11)} = 0, \qquad m_{(12)} = 0. \tag{3.2.6}$$

Under the assumption that the stresses are functions independent of θ_1, we have q_1, m_{11}, and m_{12} independent of θ_1 and since they are zero on the ends $x_1 = \pm\, b/2$ then they must be zero throughout the plate,

$$\therefore\ q_1 = m_{11} = m_{12} = 0. \tag{3.2.7}$$

From (2.4.9), we have that

$$D = \frac{\mu \lambda^3 L}{6(1 - \eta)} = \frac{\mu a(C_1 + C_2\,\theta_2)^3}{6(1 - \eta)}. \tag{3.2.8}$$

We may now apply (2.5.10) and (2.5.11) to the problem of finding deflections and stresses in the plate. Since we are using rectangular coordinates here, with $\varpi p = 0$, (2.5.11) becomes

$$m^{\alpha\beta}\big|_{\alpha\beta} = 0. \tag{3.2.9}$$

Under our assumptions, this is

$$m^{22}{}_{,22} = 0 \;. \tag{3.2.10}$$

Integrating (3.2.10) a couple of times and applying boundary conditions (3.2.5) and (3.2.4), we get

$$m^{22} = q_o(1 - \theta_2) \;. \tag{3.2.11}$$

If we now substitute (3.2.11) into (2.5.15), we observe that we can solve for $w_{,22}$ and get

$$\frac{\mu\lambda^3 a}{6(1-\eta)} w_{,22} = \left[-1 + \frac{\lambda_{,2}\lambda_{,2}}{5} + \frac{\lambda_{,2}\lambda_{,2}}{160(1-\eta)}\left\{-3(1-3\eta)\lambda_{,2}\lambda_{,2} - 24\eta\right\}\right] q_o(1-\theta_2)$$

(3.2.12)

$$+ \left[-\frac{\lambda_{,2}}{5} + \frac{\lambda_{,2}}{160(1-\eta)}\left\{(1-3\eta)\lambda_{,2}\lambda_{,2} + 24\eta\right\}\right](-\lambda q_o) .$$

Since $\lambda = C_1 + C_2\theta_2$, we have $\lambda_{,2} = C_2$ and (3.2.12) simplifies to

$$w_{,22} = \left(1 - \frac{C_2^2}{8}\right)\frac{4q_o}{\mu}\frac{(\theta_2 - 1)}{(C_1 + C_2\theta_2)^3} + \frac{C_2 q_o}{2\mu}\frac{1}{(C_1 + C_2\theta_2)^2} \;. \tag{3.2.13}$$

If we integrate (3.2.13) with respect to θ_2 and apply the boundary condition that $w_{,2} = 0$ when $\theta_2 = 0$, we get

$$w_{,2} = \frac{q_o}{\mu C_1^2 (C_1 + C_2\theta_2)^2}\left[C_1(-4\theta_2 + 2\theta_2^2) + C_1 C_2\left(\frac{+C_1}{2} + \frac{C_2}{2}\right)\theta_2 + C_2\left(-2 + \frac{C_1 C_2}{4} + \frac{C_2^2}{4}\right)\theta_2^2\right] .$$

(3.2.14)

Integrating (3.2.14) with respect to θ_2 and applying the condition that $w = 0$ when $\theta_2 = 0$, we get

$$w = \frac{q_o}{\mu C_2^3 (C_1 + C_2\theta_2)}\left[4C_2\theta_2 + \left(2\frac{C_2^2}{C_1} - 2\frac{C_2^3}{C_1^2} + \frac{C_2^4}{4C_1} + \frac{C_2^5}{4C_1^2}\right)\theta_2^2 - 4(C_1 + C_2\theta_2)\ln\frac{C_1 + C_2\theta_2}{C_1}\right] .$$

(3.2.15)

This gives us the deflection of the middle surface of the plate. If we examine our problem, we see that for a constant thickness plate, we have $C_2 = 0$. Thus if we let C_2 go to zero in (3.2.15), we should obtain the solution for the constant thickness plate. We note that direct substitution of $C_2 = 0$ yields the indeterminate form of $\frac{0}{0}$. We have to use L'Hospital's Rule three times before we get a finite limit and this limit is the desired one which is

$$w\Big]_{C_2 = 0} = \frac{q_o}{\mu a C_1^3}\left[\frac{2}{3}\theta_2^3 - 2\theta_2^2\right]. \qquad (3.2.16)$$

We would now like to obtain the classical solution to this problem. We use (3.1.5) where $N_x = N_y = N_{xy} = 0$ and $D = \frac{\mu a}{6(1-\eta)}(C_1 + C_2\theta_2)^3$. After simplifying, (3.1.5) becomes

$$(c + C_2 y)^3 \frac{d^4 w}{dy^4} + 6C_2(c + C_2 y)^2 \frac{d^3 w}{dy^3} + 6C_2^2(c + C_2 y)\frac{d^2 w}{dy^2} = 0. \qquad (3.2.17)$$

The solution of (3.2.17) is

$$w = A_1 + A_2(c + C_2 y) + \frac{A_3}{c + C_2 y} + A_4 \log(c + C_2 y),$$

and this in terms of θ_2 becomes

$$w = A_1 + A_2 a(C_1 + C_2\theta_2) + \frac{A_3}{a(C_1 + C_2\theta_2)} + A_4 \log a(C_1 + C_2\theta_2). \qquad (3.2.18)$$

If we apply the boundary conditions (3.2.4) and (3.2.5) to (3.2.18) and evaluate the constants A_1, A_2, A_3, and A_4, we get

$$w = \frac{q_o}{\mu}\left[\frac{+4C_2\theta_2 + \left(2\frac{C_2^2}{C_1} - 2\frac{C_2^3}{C_1^2}\right)\theta_2^2 - 4(C_1 + C_2\theta_2)\log\frac{C_1 + C_2\theta_2}{C_1}}{C_2^3(C_1 + C_2\theta_2)}\right]. \qquad (3.2.19)$$

If we compare (3.2.19) with (3.2.15) we see that the only difference is that two terms in the coefficient of θ_2^2 in our solution do not appear in the classical solution. These two terms represent a refinement of the classical solution.

In order to compare the solutions, let us plot the deflection at $\theta_2 = 1$ as a function of $C_1 = c/a$. Now we have that $C_2 = \frac{d - c}{a} = d/a - c/a = d/a - C_1$. Since $d < c$, C_2 is negative. Let $d/a = 1/20$ so that $C_2 = 1/20 - C_1$. We shall now plot $(w/w_c)_{\theta_2 = 1}$ against c/a, where w_c represents the classical deflection.

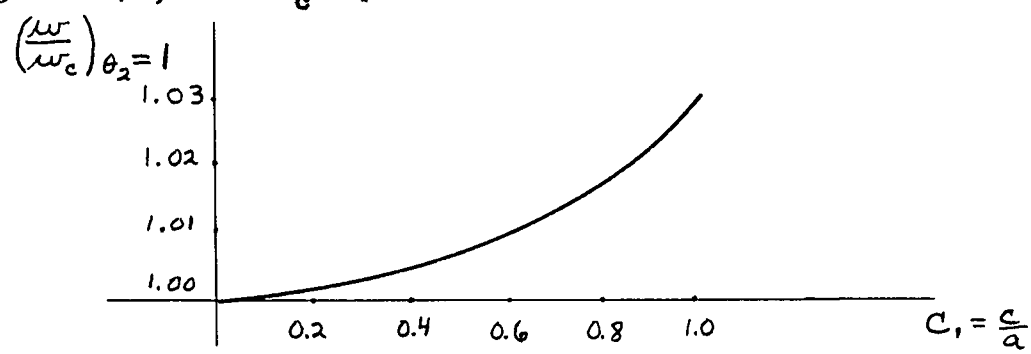

Figure 3. The Relationship Between w and w_c.

This indicates that the additional terms obtained in our solution are very small in this problem and have little effect on the deflection of the middle surface of the plate.

As a second problem for checking our theory, we choose a circular ring plate of varying thickness. This problem will provide a better test in that we will be using curvilinear coordinates and thus will have a check on the terms involving the metric of the coordinate system.

Consider a plate as shown in Figure 4. It is clamped at the outer edges r = a and loaded by a uniform shear load q_o on r = b and a uniform load p_o on the upper surface.

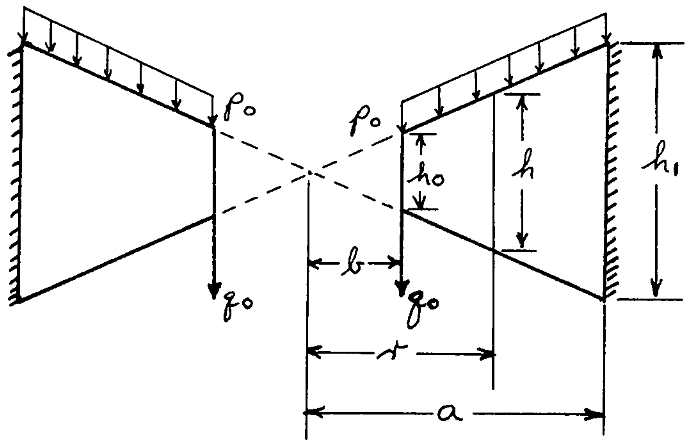

Figure 4. Circular Ring Plate of Variable Thickness

From the geometry of the figure, we have that $\lambda = h/L = h/a$ and $h/r = h_o/b$ so that

$$\lambda = \frac{h_o r}{ab} = \frac{h_o}{b}\left(\frac{r}{a}\right) = C\,\theta_1 \;, \tag{3.2.20}$$

if we let

$$C = \frac{h_o}{b} \quad \text{and} \quad \theta_1 = \frac{r}{a} \;. \tag{3.2.21}$$

We will let the subscript 1 refer to r and 2 to θ in cylindrical coordinates, i.e.,

$$\theta_1 = \frac{r}{a}\,, \quad \theta_2 = \theta, \quad \text{and } \theta_3 = \frac{z}{h} \;. \tag{3.2.22}$$

Due to the symmetry of the plate and its loading, we can say that the stresses and displacements are independent of θ_2. Thus we have $q_\theta = q_2 = 0$, $m_{r\theta} = m_{12} = 0$, but $m_{\theta\theta} = m_{22} \neq 0$.

From (2.4.9), we have that

$$D = \frac{\mu \lambda^3 L}{6(1 - \eta)} = \frac{\mu C^3 \theta_1^{\,3} a}{6(1 - \eta)} \;. \tag{3.2.23}$$

From (2.5.10), with λ independent of θ_2, we have

$$m^{\alpha\beta} = -D\,[\,(1-\eta)w|^{\alpha\beta} + \eta\, a^{\alpha\beta}\, w|^{\rho}_{\rho}\,]$$

$$+ \frac{\lambda^2}{10}\,[\,m^{\sigma\alpha}|^{\beta}_{\sigma} + m^{\sigma\beta}|^{\alpha}_{\sigma} - \frac{\lambda|_1}{\lambda}\,(m^{1\alpha}|^{\beta} + m^{1\beta}|^{\alpha})$$

$$- \frac{\lambda\,\lambda|^{\beta}_{1} - \lambda|_1\,\lambda|^{\beta}}{\lambda^2}\, m^{1\alpha} - \frac{\lambda\,\lambda|^{\alpha}_{1} - \lambda|_1\,\lambda|^{\alpha}}{\lambda^2}\, m^{1\beta}\,] \qquad (3.2.24)$$

$$+ [\,- S^{\alpha\beta}\,\frac{\lambda|_1\,\lambda|_1}{\lambda^2} + R^{\alpha\beta}\,\frac{\lambda|_1\,\lambda|_1}{\lambda^2} + Q^{\alpha\beta}\Big(\frac{\lambda\,\lambda|_{11} - \lambda|_1\,\lambda|_1}{\lambda^2}\Big)\,]\, m^{11}$$

$$+ [\,S^{\alpha\beta}\,\frac{\lambda|_1}{\lambda} + Q^{\alpha\beta}\,\frac{\lambda|_1}{\lambda}\,]\,m^{11}|_1 + P^{\alpha\beta}\;\theta p.$$

We now must lower the suffixes on the derivatives by using the metric tensor. This yields after some simplification,

$$m^{\alpha\beta} = -D[\,(1-\eta)a^{\alpha 1}\, a^{\beta 1}\, w|_{11} + \eta\, a^{\alpha\beta}\, a^{11}\, w|_{11}\,]$$

$$+ \frac{\lambda^2}{10}\,[\,a^{\beta 1}\, m^{1\alpha}|_{11} + a^{\alpha 1}\, m^{1\beta}|_{11} - \frac{\lambda|_1}{\lambda}\,(a^{\beta 1}\, m^{1\alpha}|_1 + a^{\alpha 1}\, m^{1\beta}|_1)$$

$$+ a^{\beta 1}\,\frac{\lambda_1\,\lambda_1}{\lambda^2}\, m^{1\alpha} + a^{\alpha 1}\,\frac{\lambda_1\,\lambda_1}{\lambda^2}\, m^{1\beta}\,] \qquad (3.2.25)$$

$$+ \frac{\lambda|_1\,\lambda|_1}{\lambda^2}(-S^{\alpha\beta} + R^{\alpha\beta} - Q^{\alpha\beta})\,m^{11}$$

$$+ \frac{\lambda|_1}{\lambda}\,(S^{\alpha\beta} + Q^{\alpha\beta})\,m^{11}|_1 + P^{\alpha\beta}\;\theta p.$$

Since $m^{12} = 0$ in this problem we will be interested in m^{11} and m^{22}. From (3.2.25), we get

$$
\begin{aligned}
m^{11} = & - D\, w\big|_{11} + \frac{\lambda^2}{5}\, m^{11}\big|_{11} - \frac{\lambda\, \lambda\big|_1}{5}\, m^{11}\big|_1 + \frac{\lambda\big|_1\, \lambda\big|_1}{5}\, m^{11} \\
& + \frac{\lambda\, \lambda\big|_1}{160(1-\eta)} \left[24\eta + (1 - 3\eta)\, \lambda\big|_1\, \lambda\big|_1 \right] m^{11}\big|_1 \\
& + \frac{\lambda\big|_1\, \lambda\big|_1}{160(1-\eta)} \left[-24\eta - 3(1 - 3\eta)\, \lambda\big|_1\, \lambda\big|_1 \right] m^{11} \\
& + \frac{\lambda^2}{160(1-\eta)} \left[-16\eta + 3(1 - 3\eta)\, \lambda\big|_1\, \lambda\big|_1 \right] \theta\, p.
\end{aligned}
\tag{3.2.26}
$$

Using (3.2.23), we can solve for $w\big|_{11}$ in terms of m^{11} and its derivatives and get

$$
\begin{aligned}
5\, \mu\, a\, c\, w\big|_{11} = & \frac{6(1-\eta)}{\theta_1}\, m^{11}\big|_{11} - \frac{6(1-\eta)}{\theta_1^{\,2}}\, m^{11}\big|_1 + \frac{6(1-\eta)}{C^2\, \theta_1^{\,3}}\, (C^2 - 5)\, m^{11} \\
& + \frac{3}{16\theta_1^{\,2}} \left[24\eta + (1 - 3\eta) C^2 \right] m^{11}\big|_1 \\
& + \frac{3}{16\theta_1^{\,3}} \left[-24\eta - 3(1 - 3\eta) C^2 \right] m^{11} \\
& + \frac{3}{16 C^2 \theta_1^{\,3}} \left[-16\eta + 3(1 - 3\eta) C^2 \right] \theta\, p.
\end{aligned}
\tag{3.2.27}
$$

Thus, we see that if we can obtain m^{11} as a function of θ_1 then we can find w as a function of θ_1.

Using expressions for covariant derivatives and the Christoffel symbols for cylindrical coordinates, we can show that

$$m^{11}\big|_{1} = m^{11}{}_{,1} \tag{3.2.28}$$

$$m^{11}\big|_{11} = m^{11}{}_{,11} + \frac{1}{\theta_1} m^{11}{}_{,1} \; . \tag{3.2.29}$$

From (2.5.11) we have

$$m^{\alpha\beta}\big|_{\alpha\beta} + \circledast\, p = 0, \tag{2.5.11}$$

which for our problem becomes

$$m^{11}{}_{,11} + \frac{1}{\theta_1} m^{11}{}_{,1} = - \circledast\, p. \tag{3.2.30}$$

Since m^{11} is a function of θ_1 only and not of θ_2, we can write (3.2.30) as

$$\frac{d^2 m^{11}}{d\,\theta_1{}^2} + \frac{1}{\theta_1}\frac{d\,m^{11}}{d\,\theta_1} = - \circledast\, p. \tag{3.2.31}$$

For boundary conditions on (3.2.31) we have that

$$m^{11} = 0 \quad \text{when} \quad \theta_1 = b/a \tag{3.2.32}$$

$$q^{1} = -q_o \ \text{when} \quad \theta_1 = b/a \; . \tag{3.2.33}$$

Solving (3.2.31) subject to (3.2.32) and (3.2.33) we get

$$m^{11} = \left(\frac{\circledast p}{2}\frac{b^2}{a^2} - q_o\,\frac{b}{a}\right)\log\frac{a\,\theta_1}{b} + \frac{\circledast p}{4}\left(\frac{b^2}{a^2} - \theta_1{}^2\right) \; . \tag{3.2.34}$$

We now substitute m^{11} from (3.2.24) into (3.2.27) and get an expression for $w\big|_{11}$.

Let us simplify the problem here to the case where $\eta = 1/3$ and $\Theta p = 0$. We get

$$5 \mu a C w_{,11} = \frac{5 q_o b}{2a} \left(\frac{1}{\theta_1^3}\right) + \frac{q_o b(40 - 5c^2)}{2aC^2} \left(\frac{1}{\theta_1^3} \log \frac{a \theta_1}{b}\right) . \qquad (3.2.35)$$

The boundary conditions on w in (3.2.35) are

$$w = 0 \quad \text{and} \quad w_{,1} = 0 \quad \text{when} \quad \theta_1 = 1 . \qquad (3.2.36)$$

Solving (3.2.35) subject to (3.2.36), and using the fact that

$$\mu = \frac{E}{2(1 + \eta)}$$

we get

$$w_{,1} = \frac{q_o b}{Ea^2} \left(\frac{32 - 6C^2}{3C^3}\right)\left(\frac{1}{\theta_1^2} - 1\right) - \frac{q_o b}{Ea^2}\left(\frac{16-2C^2}{3C^3}\right)\left(\frac{1}{\theta_1^2} \log \frac{a\theta_1}{b} - \log \frac{a}{b}\right).$$

(3.2.37)

For w, we get

$$w = \frac{q_o b}{Ea^2} \left(\frac{32 - 6C^2}{3C^3}\right)\left(2 - \frac{1}{\theta_1} - \theta_1\right) + \frac{q_o b}{Ea^2}\left(\frac{16-2C^2}{3C^3}\right)\left[\frac{1}{\theta_1}\left(\log \frac{a\theta_1}{b} + 1\right) - \left(\log \frac{a}{b} + 1\right)\right]$$

(3.2.38)

$$+ \frac{q_o b}{Ea^2}\left(\frac{16-2C^2}{3C^3}\right)\left[\theta_1 \log \frac{a}{b} - \log \frac{a}{b}\right] .$$

The particular problem that we have solved for the deflection w is solved by the classical approach in (33). The solution as obtained there is

$$\frac{dw}{dr} = A + \frac{B}{r^3} + \frac{Pb^3}{4\pi D_o r^2} , \qquad (3.2.39)$$

where $P = 2\pi b q_o$ (3.2.40)

and $$D_o = \frac{E h_o^2}{12(1-\eta^2)} = \frac{9 E h_o^3}{96}$$ (3.2.41)

since we have taken $\eta = 1/3$. Now we have that $\theta_1 = r/a$ or $r = a\theta_1$, so that

$$\frac{dw}{dr} = A + \frac{B}{a^3\theta_1^3} + \frac{q_o b}{Ea^3}\left(\frac{48}{9C^3\theta_1^2}\right).$$ (3.2.42)

We want to compare our solution with the classical solution so we take the numerical values

$$C = h_o/b = 1/2 \quad \text{and} \quad a/b = 5 .$$ (3.2.43)

We can obtain A and B by using the boundary conditions

$$\frac{dw}{d\theta_1} = 0 \quad \text{when } \theta_1 = 1$$ (3.2.44)

and

$$M_r = 0 \quad \text{when} \quad r = b, \quad \text{i.e. when } \theta_1 = b/a ,$$ (3.2.45)

where

$$M_r = D\left(\frac{d^2w}{dr^2} + \frac{1}{3r}\frac{dw}{dr}\right).$$ (3.2.46)

When A and B are found and (3.2.43) is used, we obtain

$$w_{,1} = \frac{q_o b}{Ea^2}\left[-37.2 - \frac{5.37}{\theta_1^3} + \frac{42.6}{\theta_1^2}\right] .$$ (3.2.47)

Integrating (3.2.47) and using the boundary condition that

$$w = 0 \quad \text{when } \theta_1 = 1$$ (3.2.48)

we have for the classical solution,

$$w_C = \frac{q_o b}{Ea^2}\left[-37.2\,\theta_1 + \frac{2.68}{\theta_1^2} - \frac{42.6}{\theta_1} + 77.1\right].$$ (3.2.49)

If we substitute the values given in (3.2.43) into (3.2.38), we get

$$w = \frac{q_o b}{Ea^2}\left[-14.8\,\theta_1 - \frac{40.0}{\theta_1} + \frac{41.3}{\theta_1}\log 5\theta_1 - 11.7\right], \qquad (3.2.50)$$

as the solution of our system of equations. Now we see that (3.2.49) and (3.2.50) are definitely not the same solution when we compare terms. Both solutions satisfy the same boundary conditions but they are not solutions of the same differential equation. Let us now plot the two solutions and see how they compare. The comparison is shown in Figure 5. It is noted that there is a definite correspondence in shape.

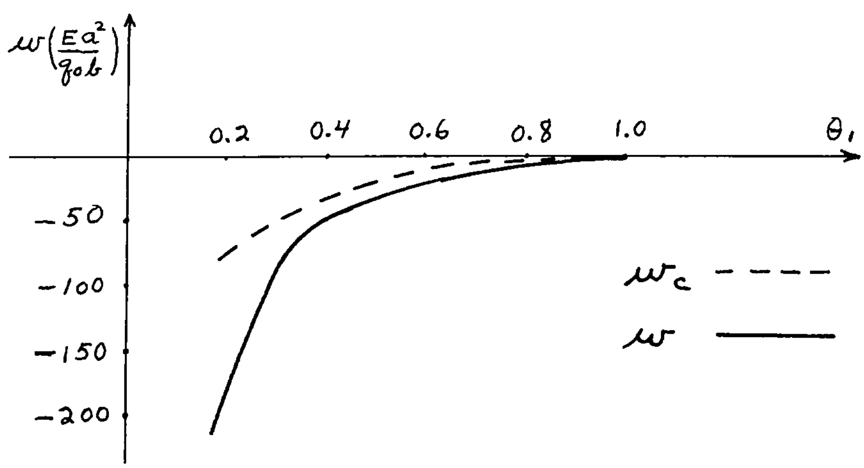

Figure 5. Comparison of w_c with w.

However, it is also noted that for $0.2 < \theta_1 < 0.4$, our solution departs rather rapidly from the classical solution while for the range $0.4 < \theta_1 < 1$, there is good agreement. Let us now plot the ratio w/w_c against θ_1.

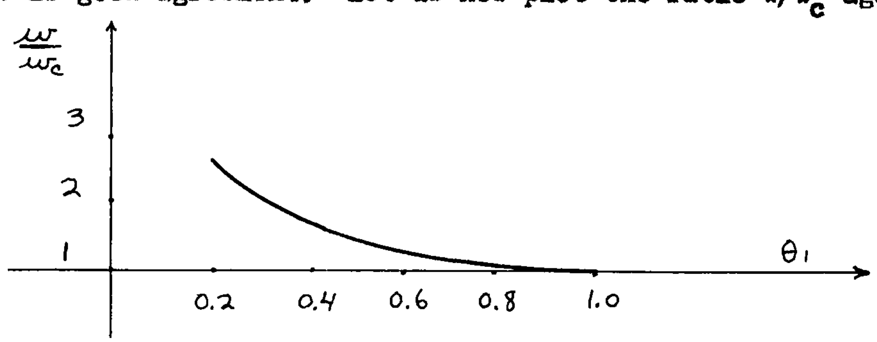

Figure 6. The Deflection Ratio Plot.

3. Summary of Results

The equations derived in this thesis compare very favorably with those derived by Essenburg and Naghdi (32) even though an entirely different approach is used. We started with general stress-strain relations for an isotropic body and made approximations and assumptions only when necessary. Some assumptions must be made in order to obtain a set of equations that are solvable.

Most of the assumptions that have been made are concerned with the order of magnitude of certain terms. In finding the stresses in terms of the displacements, we assumed that we could neglect terms in θ_3^4 and higher. Note that in (2.3.16), we have $\tau^{\alpha\beta}$ as cubic in θ_3 but then we assumed that we could neglect terms in $\tau^{\alpha\beta}$ of order higher than the first in θ_3. This corresponds to the assumption that was made in (32). We would have been more general if we had kept the terms through θ_3^3. Another approximation that we made was that $\lambda << 1$ and also that $\lambda_{,\alpha} << 1$ so that we could simplify (2.3.47) into the form (2.3.48) which corresponds to the form obtained in (32). Many terms were neglected as a result of this assumption but more investigation would be needed to determine the effect of these neglected terms.

When the solutions of the two problems are compared, we note a marked difference in the magnitude of the ratio w/w_c.

The reason for the difference is not obvious in view of the similar types of loading. The solution of the second problem would seem to indicate that the classical solution is not too good near the inner surface of the ring plate.

Some extensions of what has been done are now obvious. Since we now have the deflection of the middle surface and the moment couples, we could obtain the stresses in the plate. If in-plane forces existed on the boundaries, then we would have to use (1.5.10) and (2.4.11) in order to find $n^{\alpha\beta}$ and v^{α}. This could lead to a study of the buckling of a variable thickness plate, a field in which little has been done. In view of what can still be done here, a quotation by Melville is appropriate at this point. Melville writes, "For small erections may be finished by their first architects; grand ones, true ones, ever leave the copestone to posterity. God keep me from ever completing anything. This whole book is but a draught-nay, but a draught of a draught. Oh, Time, Strength, Cash and Patience!"

In (2.5.10), (2.5.11), (1.5.10), and (2.4.11), we have nine equations and nine unknowns and if they can be solved subject to a given set of boundary conditions, then we can obtain the stresses which will be consistent with the assumptions that have been made.

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ABSTRACT

The approach to the problem of a variable thickness plate used in this paper is different from the usual approach in that this paper starts with general stress-strain relations and a generalized form of the position vector as used by Green and Zerna in "Theoretical Elasticity".

They use

$$\bar{R} = L\left[\bar{r}(\theta_1, \theta_2) + \lambda\theta_3\, \bar{a}_3(\theta_1, \theta_2)\right]$$

where θ_1, θ_2, and θ_3 are curvilinear coordinates with θ_1 and θ_2 being the coordinates of the middle surface and $\lambda = t/L$ being a constant for a plate of constant thickness t.

This paper takes $\lambda = \lambda(\theta_1, \theta_2)$ as a function of θ_1 and θ_2 so that the variable thickness may be taken into account. General tensor notation is used so as to work independent of coordinate systems.

Making simplifying assumptions only when necessary, the equations of equilibrium and stress-strain relations are derived in terms of tensors connected with the middle surface as was done by Green and Zerna for a constant thickness plate. The additional terms obtained in these equations due to the variation in λ help us to evaluate the effects of the varying thickness.

Expressions for stress are developed and they include the effects of transverse shear deformation and normal stress as well as the variation in thickness. These expressions are very

much like those used by Essenburg and Naghdi in a paper presented at the Third U. S. National Congress of Applied Mechanics, June, 1958. However, they assumed the form for the stresses while the present paper arrived at their assumed forms with some additional terms after starting with general stress-strain relations.

Using the notation of Green and Zerna, a set of nine equations involving the nine unknowns, $m^{\alpha\beta}$, w, $n^{\alpha\beta}$, and v^{α} is derived and under appropriate boundary conditions, this set will yield a solution to the problem which will be better than the classical solution.

Two problems are solved and numerical results are obtained and compared with the classical solutions. One of the problems involves a rectangular plate clamped on one edge with a uniform shear load on the other. The other problem involves a circular ring plate clamped on the outer edge with a uniform shear load on the inner edge. A much better correlation for the deflection of the middle surface is obtained for the rectangular than for the circular ring plate. The deflection at the inner edge of the ring plate obtained by the theory of this paper is over twice that obtained in the classical solution of the same problem.

In the previously mentioned set of nine fundamental equations, we have the stress resultants $n^{\alpha\beta}$ and the deflections v^{α}. With appropriate boundary conditions, these equations could lead to a study of in-plane forces and buckling of variable thickness plates, a field in which not much progress has been made. This paper does not include any numerical work in this direction. It is felt, however, that one of the principal contributions of this paper to the literature is that the set of nine fundamental equations includes the stress resultants in $n^{\alpha\beta}$ thus enabling us to study the effect of in-plane forces as well as that of transverse shear deformation, normal stress, and surface tractions.