Variable Sampling in Multiparameter Shewhart Charts.

by

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(ABSTRACT)

This dissertation deals with the use of Shewhart control charts, modified to have variable sampling intervals, to simultaneously monitor a set of parameters. Fixed sampling interval control charts are modified to utilize sampling intervals that vary depending on what is being observed from the data. Two problems are emphasized, namely, the simultaneous monitoring of the mean and the variance and the simultaneous monitoring of several means. For each problem, two basic strategies are investigated. One strategy uses separate control charts for each parameter. A second strategy uses a single statistic which combines the information in the entire sample and is sensitive to shifts in any of the parameters. Several variations on these two basic strategies are studied. Numerical studies investigate the optimal number of sampling intervals and the length of the sampling intervals to be used. Each procedure is compared to corresponding fixed interval procedures in terms of time and the number of samples taken to signal. The effect of correlation on multiple means charts is studied through simulation. For both problems, it is seen that the variable sampling interval approach is substantially more efficient than fixed interval procedures, no matter which strategy is used.

DEDICATION

This dissertation is dedicated to my parents, with love.

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Table of Contents

List of Illustrations

List of Tables

Chapter I

I. Introduction

The output of any repetitive process can be studied using a control chart. According to W.A. Shewhart (1931), the purpose of a control chart is threefold: firstly, to quantify the goal or standard that the process should attain, next, to help the process achieve that goal, and finally, to check whether that goal has been attained. Hence control are used in the specification of the product, as well as in production and inspection.

The theory of control charts is based on the breakdown of the variation in quality into two main components. The two components are

i) variation due to assignable causes or special causes

ii) variation due to chance.

If the observed variation in a process cannot be traced back to a single cause, then the variation is assumed to be chance variation. When the variation in the data is found to be due to chance alone, the process that generated this data is said to be "in control".

If there is a change in the process such that there is a deterioration in quality due to an assignable cause, then the change must be detected quickly, so that corrective action can be taken. In many cases, a change in the process can be represented by a change in a parameter or parameters of the distribution of the quality variable. For example, if the distribution of a variable of interest is approximately normal, then a process change may result in a change in the mean or variance or both. In this situation, both parameters will usually have to be monitored. When there are several quality variables of interest, then a shift in the process may change the parameters of the joint distribution of the variables. If the underlying distribution is multivariate normal, then the vector of means or the dispersion matrix or both may have to be monitored.

One of the most widely used control charts is the Shewhart control chart, named after its inventor, W.A. Shewhart (1931). The chart takes samples of fixed size at fixed sampling intervals and computes the appropriate statistics to determine whether the process is in control or not. The chart divides the range of the statistic into two distinct regions, the signal region and the in-control region. If the observed sample statistic is too far from a pre-set target value, (i.e if it falls in the signal region), the chart "signals" and the process is considered to have gone out of control. Otherwise, the process is allowed to continue to the next sample.

Statistical process control concentrates on monitoring the ongoing process with the help of control charts, rather than inspecting the product after the process is completed. Often, even if the time interval between samples is fixed, if the statistic is close to the out-of-control region, the process is viewed with suspicion and the next sample is taken earlier than scheduled. A control chart that is actually set up to vary the time interval between samples, depending on what has been observed by the previous sample, is referred to as a *variable sampling interval (VSI)* control chart. A *VSI* procedure is not to be confused with a *sampling inspection plan*. One difference is that the *VSI* procedure varies the sampling rate whereas a sampling inspection plan varies the sample size. Hence if the process is viewed as going off-target, on the basis of the most recent sample, a *VSI* control chart would sample earlier than a corresponding *fixed sampling interval (FSI)* chart. By the same token, if the process seems to be staying on-target, then the *VSI* chart would sample later than the *FSI* chart.

The properties of control charts that use variable sampling intervals were developed by Reynolds and Arnold (1987). Reynolds et al (1987) applied the *VSI* feature to two-sided $\overline{X}$ charts, and Amin et al (1988) applied it to cusum charts. Thus previous investigations of *VSI* control charts have looked in detail only at charts for monitoring one parameter. Typically, one parameter will not be sufficient to completely characterize the distribution of the quality variable or variables.

This dissertation is focussed on the simultaneous monitoring of several parameters. Although the problem of monitoring several parameters has frequently been treated by looking at each parameter separately, this is not necessarily the best procedure when a process change may involve more than one parameter. Chapter Two is a review of the literature that is pertinent to the dissertation and Chapter Three is an explanation of the *VSI* procedure including the notation and properties.

Two problems are studied in detail in this dissertation. One section of the dissertation deals with the case of monitoring the mean and the variance of one normally distributed variable while applying the *VSI* feature. This is developed in Chapter Four and the numerical results are tabulated. The results are amply illustrated with the help of plots and tables. FORTRAN programs were written to aid in the computations, some of which have been included in the Appendix. Chapter Five concentrates on a second problem, which is the monitoring of several means from a multivariate normal distribution using a *VSI* procedure. This was done for independent as well as correlated variables. Simulation studies were conducted to get results for the correlated case, since the problem was too difficult to tackle analytically.

Chapter II

II. Literature Review

This chapter reviews the development of the quality control literature with respect to VSI control charts. The most commonly used class of control charts is the class of Shewhart charts. A Shewhart chart takes samples of constant size at fixed intervals in time and its decision process is based on a statistic which is a function of the current sample only. Thus the chart determines whether to signal or to continue to the next sample, depending on the value of the plotted statistic.

Control charts can be based on qualitative or quantitative characteristics. Examples of the former are p-charts and c-charts. P-charts measure the variation in the fraction defective of the process output. The size of the sample taken may remain constant or vary. If they vary, the control limits at each sampling point will also vary. C-charts measure the variation in the number of defects per unit. The number of defects per unit are assumed to follow a Poisson distribution and the control limits are set up accordingly.

Often, control charts based on variables are more economical than control charts based on attributes. Also, assignable causes are more easily found using control charts based on variables. Examples of control charts based on variables are $\overline{X}$ and R-charts. When the parameter of interest is the process mean μ, the Shewhart $\overline{X}$ chart has control limits set at $\mu_0 \pm 3\sigma/\sqrt{n}$, where μ_0 is the target value for the mean and σ is the standard error. These charts are easy to implement and are good at detecting large shifts from the target value. When the parameter in question is the process variance σ^2, R-charts or S^2 -charts can be used. R-charts use the sample range as the control statistic and are relatively simple to explain and to use. S^2 charts use the sample variance as the control statistic and are more efficient. Both of the charts are usually designed under the assumption that the process quality characteristic has a normal distribution.

Page (1955) incorporated warning lines within the control limits in Shewhart control charts. He suggested that action should be taken if r out of the last N points fell between the warning lines and control limits. He showed that this made the chart more efficient in detecting small shifts, since it used information from the last few samples instead of using only the last sample. Mosteller (1941) gave a mathematical basis for Shewhart's (1941) discussion of runs rules. In order to make the Shewhart chart more sensitive while still using a small sample size, it was suggested that an additional signal be produced if a given number of consecutive sample means fell above (or below) the median. Runs of different kinds and lengths were theoretically analyzed by Wolfowitz (1943). Weiler (1953) suggested that production be stopped if a specific number of consecutive means went over the control limits. Page (1962) showed that control charts with warning lines are better at detecting large shifts as compared to control charts using Moore's (1958) runs rules.

While using a Shewhart chart to monitor a single mean, Ghosh et al (1981) pointed out that if the variance is estimated, then the t-statistics based on that estimate will no longer be independent from one sample to the next. Further they proved that using a small sample size to estimate the variance would cause the ARL as well as the variance of the run length of the chart to increase. When the mean and the variance need to be monitored simultaneously, Reynolds and Ghosh (1981) discussed several alternative designs for control charts. This was done for the univariate case first and then extended to the multivariate case as well.

When monitoring several means, a T^2 chart (or a χ^2 chart, if the dispersion matrix is known) can be used. Hotelling (1947) introduced the T^2 statistic to be used under normal assumptions when the dispersion matrix is unknown. He partitioned the T^2 statistic into two components, namely the sum of squares of the deviations of the observations from the sample mean and the sum of squares of the deviations of the sample mean to the true mean. The variation about the sample mean was monitored using a chi-squared chart, since its asymptotic distribution is chi-squared.

Jackson (1956) plotted the control ellipses for the case where two means were being monitored. Jackson and Morris (1957) recommended using principal components along with the T^2 chart if the dispersion matrix is almost singular. Jackson (1959) extended the results of the bivariate case and developed a method of analyzing and controlling a multivariate process. The residuals associated with the principal components were used to monitor the unexplained variation. Alternative procedures for the residual analysis were studied by Jackson and Mudholkar (1979).

In multicharacteristic quality control, Ghare and Torgerson (1968) pointed out that if separate control charts are used at a fixed level α , the probability of rejecting when the process is in control is especially inaccurate if the quality variables are related. They also found that as the number of quality characteristics increase, the probability of falsely declaring the process to be out of control increases if separate charts are used.

Page (1954) designed cumulative sum control charts. These charts plot the cumulative sum of the deviations of the observed values from the target value. However the exact distribution of the run length could not be found. Page (1954) and Ewan and Kemp (1960) proved that if the ARL is large , the run length distribution of a one sided cusum chart is approximately geometric. Ewan and Kemp (1960) further developed the cusum chart by designing the V-mask, which is a V-shaped region used to detect significant changes in the process average.

Reynolds (1975) used a Brownian motion process to approximate the cusum process. Brook and Evans (1972) used a Markov chain to compute the ARL for a one-sided cusum. This method could be used for discrete distributions and gave very good approximations for continuous distributions. A similar approach was used by Woodall (1984) to determine the ARL for a two-sided cusum chart.

Lucas and Crosier (1982a) added an extra feature to cusum charts, namely, a fast initial response (FIR). This increased the sensitivity of the cusum charts at start-up or after a restart following an out-of-control signal. Lucas (1982) combined the strong points of the Shewhart and cusum charts by adding Shewhart limits to a cusum chart. This scheme would signal if the most recent sample was outside the Shewhart control limits or if the cusum chart signalled. Thus such a procedure would not only detect small to moderate shifts quickly, but would also be quick to detect large shifts. This scheme, however, was not robust to outliers. Lucas and Crosier (1982b) generated a more robust cusum procedure which was relatively resistant to outliers.

Rather than sampling at fixed time intervals it seems more reasonable to allow the time between samples to vary depending on what has been observed by the sample just taken. Variable sampling rates have been used previously in the context of acceptance sampling. In acceptance sampling, a sample of items from a lot is inspected after production and the lot is either rejected or accepted, depending on the observed quality. The primary objective in acceptance sampling is to reject a defective product rather than to directly control the production process.

Continuous sampling inspection plans inspect all or a fraction of the items being produced depending on the observed quality. Such a plan was first introduced by Dodge in 1943 and is referred to as the CSP-1 plan. As with acceptance sampling, the primary objective was to remove defective items. Variations on the CSP-1 plan were studied by Wald and Wolfowitz (1945). Their plan was more efficient when there was a sequence of poor quality. Lieberman and Solomon (1955) designed multilevel sampling inspection plans in which there were several levels of partial inspection and switching could be done only to adjacent levels. This was found to be better than the previous two-level plans since the switches were not as abrupt. This plan was further developed by Derman, Littauer and Solomon (1957). They proposed using "tightened" multilevel plans which made it possible to switch to high levels of inspection faster.

The idea of sampling from a process at varying time intervals was investigated by Arnold (1970) where the problem was formulated in a Markovian structure. He derived a formula for the expected sample size and evaluated several sampling schemes based on that criterion. An economically optimal sampling plan based on Arnold's model was obtained in a dissertation by Crigler (1973). The results were extended by Crigler and Arnold (1979 and 1986). Smeach and Jernigan (1977) derived a formula for the variance of the expected sample size using Arnold's model. They also developed approximate formulae for the expected sample size and its variance which were simplifications of the exact formulae. Hui & Jensen (1980) extended this problem to the multivariate case. Their model included the time required for process adjustments when the process was out of control. They used Markov processes to find the expected numbers of inspections and adjustments along with their variances. None of these papers studied the properties of the variable sampling interval feature.

Intuition would suggest, and studies have confirmed, that the sampling interval should be long if the sample statistic is close to target and short if the statistic is not close to target. It has been shown (see, for example, Reynolds et al (1987)), that variable interval procedures can lead to detection of process changes much earlier than the usual fixed interval procedures, while maintaining the same average sampling rate when the process is in control. Reynolds et al (1987) considered the properties of a $\overline{X}$ chart with the variable sampling plan and showed that it was more efficient than the corresponding $FSI\,\overline{X}$ chart. Reynolds & Arnold (1987) developed expressions for the average time required for a VSI Shewhart chart to signal and the average number of samples taken before signalling. They showed that the optimal VSI control chart uses only the shortest possible interval and the longest possible interval from the entire range of sampling intervals. Extensions to cusum charts and other similar work are given in Amin et al (1987).

Chapter III

III. Notation And Theory

Suppose that the quality variables of interest are represented by $\underline{X} = (X_1, X_2, ..., X_m)'$ and the distribution family for $\underline{X}$ is indexed by a vector of parameters $\underline{\theta} = (\theta_1, \theta_2, ..., \theta_r)$. (In the situation where a single quality variable is of interest, $\underline{X}$ will be replaced by the scalar X). Since the process changes over time, $\underline{\theta}$ is a function of time, but the above notation is used for simplicity. Let $\underline{\theta}_0$ represent the target value for $\underline{\theta}$. Any process change which produces a deterioration in quality is assumed to be reflected by a change in $\underline{\theta}$.

Assume that the process is monitored by taking a random sample of size n at each sampling point. Let the sample for the jth variable at the ith sampling point be represented by $\underline{X}_j^{(i)} = (X_1^{(i)}, X_2^{(i)}, ..., X_n^{(i)})$. The entire sample is represented by the matrix $X^{(i)}_{m \times n}$, which will be denoted by X from now on, for simplicity of notation. If the chart signals after a sample is observed, then either the process is stopped or it continues onto the next sample while an assignable cause is sought. It is also assumed that a Shewhart-type chart is being used, so that the decision made after a sample is taken depends only on the present sample and not on any previous samples. The *VSI* feature can be used with other types of charts but this study was restricted to Shewhart charts because of their simplicity and widespread use.

A good control chart should have a low probability of signalling when the process is in control and yet be able to detect a shift quickly. Traditionally, the number of samples taken before the chart signals is called the run length and the expected number of samples is called the *average run length (ARL)*. For a FSI chart, the time taken for a chart to signal can be obtained by simply multiplying the run length by the fixed time interval between samples. When the process is on target, the ARL should be large so that the frequency of false alarms is low and when the process changes, the ARL should be small so that the change will be detected quickly.

A Shewhart control chart for monitoring a single parameter is equivalent to applying a sequence of hypothesis tests. At each sample, the null hypothesis that the parameter is equal to its target value is tested. If the null hypothesis is accepted, the chart continues onto the next sample but if the null hypothesis is rejected, the chart signals. For example, monitoring the mean of the process using a FSI chart is analogous to testing a simple hypothesis about the mean of the population. The Neymann-Pearson Lemma establishes the existence of an optimal test for simple null and alternative hypotheses. This optimality can be extended to obtain uniformly most powerful tests for a general one-sided problem if the distribution of the observations has a monotone likelihood ratio. A test which has the highest power at a specified alternative among tests of the same size corresponds to the control chart which has the smallest ARL at the specified shift among charts with the same ARL when on target. For example, the optimal Shewhart control charts for detecting one-sided changes in the process mean corresponds to the uniformly most powerful test for a one-sided test on the mean. Similarly, the optimal procedure, within a restricted class of two-sided $\overline{X}$ charts, corresponds to a uniformly most powerful unbiased test for a two-sided test on the mean.

The problem of monitoring a vector $\underline{\theta}$ using a Shewhart control procedure is also analogous to applying a sequence of hypothesis tests. If a process change can produce a simultaneous change of unknown composition in the components of $\underline{\theta}$ then a uniformly most powerful test will usually not exist and thus an optimal control procedure will usually not exist. In practice, separate control charts are usually used for each component of $\underline{\theta}$. Although this procedure has the advantage of simplicity and ease of interpretation, it may not be particularly effective at detecting certain types of process changes and its properties may be difficult to evaluate when the statistics used in the separate charts are correlated.

3.1 Background

The decision process of the *VSI* control chart is assumed to be represented by two functions of the current sample $\mathbf{X}$, a signal function $s(\mathbf{x})$ and a sampling interval function $d(\mathbf{x})$. For each $\mathbf{x}$,

$$s(\mathbf{x}) = \begin{cases} 1 & \text{if the chart signals} \\ 0 & \text{otherwise} \end{cases} \tag{3.1.1}$$

and

$$d(\mathbf{x}) = \text{sampling interval to be used before the next sample when } \mathbf{X} = \mathbf{x}$$

Thus the signal function determines when the chart signals and the sampling interval function determines the length of the next sampling interval or the time to wait until taking the next sample. The set of possible sampling intervals will be restricted to a finite set $A = \{d_1, d_2, ..., d_k\}$, such that

$$0 < \ell_1 \le d_i \le \ell_2 \ . \tag{3.1.2}$$

Here ℓ_1 denotes the shortest time at which the next sample can be taken while assuring independence between samples, and ℓ_2 denotes the longest time that is reasonable to allow the process to run before observing the next sample. An example of a Shewhart $\overline{X}$ chart using the *VSI* procedure is shown in Figure 3.1.

For a *VSI* control chart the time required to signal is not a constant multiple of the *ARL*. Thus, following Reynolds et al. (1988), separate measures for the time to signal and the number of samples to signal are developed. The number of samples to signal is defined to be the number of

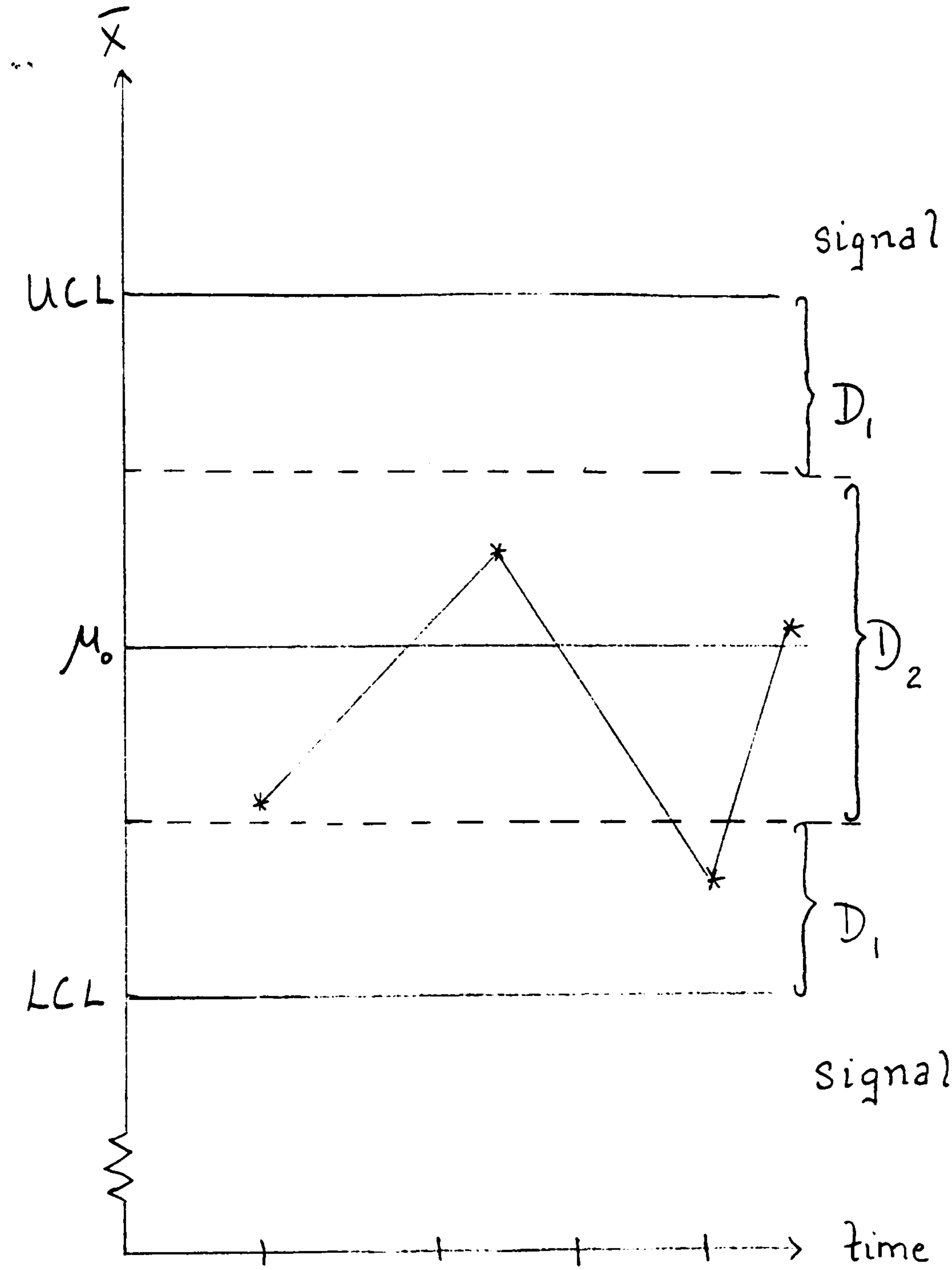


Figure 3. 1. Shewhart X-chart using the VSI procedure

samples taken from time zero to the time the chart signals. The *average number of samples to signal (ANSS)* is the expected value of the number of samples to signal. Although the $ANSS$ has the same definition as the ARL, the ARL was not used because it tends to be associated with time as well as number of samples in FSI charts. The time to signal is defined to be the length of time from the start of the process to the time when the chart signals and the *average time to signal (ATS)* is the expected value of the time to signal. Let

$$\begin{aligned} N &= \text{ number of samples to signal} \\ R_i &= \text{length of sampling interval used before the ith sample is taken, given} \\ &\quad \text{that the chart does not signal at the previous sample} \\ T &= \text{ time to signal} \\ D_i &= \text{ region in which the sampling interval } d_i \text{ is used} \end{aligned} \quad . \qquad (3.1.3)$$

Then $T = \sum_{i=1}^{N} R_i$ and by Wald's identity, $E(T) = E(N) \cdot E(R_i)$, assuming that the process parameter is constant. Thus,

$$ATS = ANSS \cdot E(R_i) \ .$$

The distribution of R_i is the conditional distribution of $d(\mathbf{x})$ given that the procedure does not signal. For simplicity it is assumed that the sampling interval used before the first sampling interval is determined by a random sampling interval whose distribution is the same as that of R_i. In practice, to make the process sensitive to start-up problems, it may be preferable to use the shortest interval ℓ_1 first. Since the distribution of N is geometric with parameter $p = P(s(\mathbf{X}) = 1)$ the $ANSS$ is therefore p^{-1}. Further as the $ANSS$ is simply the reciprocal of the signal probability of the procedure, it is independent of the sampling interval function. So if two charts have the same control limits and one uses the FSI procedure and the other uses the VSI procedure, they will both have the same $ANSS$ function. Hence the $ANSS$ is identical for procedures that differ only in their sampling interval functions.

As shown in Reynolds and Arnold (1987), the ATS can also be written as

$$ATS = \frac{ANSS}{(1-p)} \sum_{i=1}^{k} d_i P(D_i) \ . \tag{3.1.4}$$

on the set A , when the regions D_i are as defined in equation (3.1.3). Note that p and $P(D_i)$ are functions of $\underline{\theta}$. Equation (3.1.4) reduces to d/p when a fixed sampling interval d is used. Equation (3.1.3.) is valid for any $\underline{\theta}$ as long as $\underline{\theta}$ is constant. If $\underline{\theta} = \underline{\theta}_0$ then the ATS is the expected time to the first false alarm, but if $\underline{\theta} = \underline{\theta}_1$ then the ATS is the expected time to detect $\underline{\theta}_1$. If only two sampling intervals are used, specifying the sampling intervals and setting the ATS at a constant when the process is in control, fixes the probabilities of the sampling intervals. This determines the boundary values for the regions D_i for the individual charts. Alternatively, if the regions D_i are specified, they will determine the probabilities of the sampling intervals and hence the values of d_i.

3.2 Restricted Optimality

When a single parameter was being monitored using a VSI Shewhart chart Reynolds and Arnold (1987) proved the somewhat counterintuitive result that it is best to use only two sampling intervals spaced as far apart as possible. This result is fortunate since it keeps the complexity of the VSI chart at a reasonable level. It is of interest to determine whether this result extends to the case where several parameters are being monitored.

Consider the situation where a control procedure uses r individual charts to monitor the r components of $\underline{\theta}$ and the statistics used in the charts are independent. When separate charts are being used to monitor each parameter, the VSI feature cannot be added independently to each chart since the sampling intervals assigned by separate charts could be different. In such situations, a decision rule has to be set up to specify the sampling interval to be used. One possibility is to have the sampling interval depend on only one of the charts. For example, if a shift in a certain parameter is more critical or likely than shifts in any other parameter, then the sampling interval could depend only on the chart monitoring that parameter. If the sampling interval is to depend on all the charts then one could use the minimum or some other function of the sampling intervals, such as the average. Another possibility is to simply combine the information about all the parameters into a single statistic and use only one chart.

The following regions are defined:

$$\begin{aligned} \{S_i\} &= \{\underline{X}_i \mid i\text{th chart signals}\} \\ \{D_{1i}\} &= \{\underline{X}_i \mid i\text{th chart assigns the sampling interval } d_1\} \\ \{D_{2i}\} &= \{\underline{X}_i \mid i\text{th chart assigns the sampling interval } d_2\} \end{aligned} \quad . \tag{3.2.1}$$

The signal regions $\{S_i\}$ and the regions $\{D_{1i}\}$ and $\{D_{2i}\}$ are assumed to be fixed for each chart with d_1 and d_2 chosen to give a specified ATS when $\underline{\theta} = \underline{\theta}_0$. It is also assumed that each chart uses the same two sampling intervals d_1 and d_2 and that the control procedure uses the minimum of the sampling intervals assigned by the r individual charts. Thus D_2 is the cartesian product of the regions $D_{21}, D_{22}, \ldots, D_{2r}$. For this situation, it was possible to establish conditions under which it is best to space the two intervals as far apart as possible.

The ATS for a given $\underline{\theta}$ is

$$\frac{d_1 P(D_1 \mid \underline{\theta}) + d_2 P(D_2 \mid \underline{\theta})}{P(S \mid \underline{\theta})[1 - P(S \mid \underline{\theta})]} \quad . \tag{3.2.2}$$

If the sampling interval and signal regions $\{D_{1i}\},\{D_{2i}\}$ and $\{S_i\}$ are given and if the ATS at $\underline{\theta} = \underline{\theta}_0$ must equal a specified constant a_0, then d_1 and d_2 must satisfy

$$d_1 P(D_1 \mid \underline{\theta}_0) + d_2 P(D_2 \mid \underline{\theta}_0) = c_0 \quad . \tag{3.2.3}$$

where

$$c_0 = a_0 P(S|\underline{\theta}_0)[1 - P(S|\underline{\theta}_0)] \ .$$

It is assumed that a_0 satisfies

$$\frac{l_1}{P(S|\theta_0)} < a_0 < \frac{l_2}{P(S|\theta_0)} \ ,$$

so that there will exist a d_1 and d_2 satisfying both (3.1.2) and (3.2.3). If d_1 and d_2 satisfy (3.2.3) then d_2 must satisfy

$$d_2 = \frac{c_0}{P(D_2|\underline{\theta}_0)} - \frac{P(D_1|\underline{\theta}_0)}{P(D_2|\underline{\theta}_0)} d_1 \ . \tag{3.2.4}$$

The following theorem gives conditions under which the procedure that takes d_1 as small as possible (and the corresponding d_2 as large as possible) minimizes the ATS at a fixed $\underline{\theta} = \underline{\theta}_1$.

Theorem If the regions $\{S_i\}$,$\{D_{1i}\}$ and $\{D_{2i}\}$ are fixed and the sampling intervals d_1 and d_2 satisfy (3.2.3), then the ATS at $\underline{\theta} = \underline{\theta}_1$ is a strictly increasing function of d_1 if and only if

$$P(D_1|\underline{\theta}_1) > P(D_2|\underline{\theta}_1) \frac{P(D_1|\underline{\theta}_0)}{P(D_2|\underline{\theta}_0)} \ . \tag{3.2.5}$$

A sufficient condition for (3.2.5) to hold is that

$$P(D_{1i}|\underline{\theta}_1) > P(D_{2i}|\underline{\theta}_1) \frac{P(D_1|\underline{\theta}_0)}{P(D_2|\underline{\theta}_0)} \ : \tag{3.2.6}$$

for some $i = 1, 2, ..., r$.

Proof If (3.2.3) and (3.2.4) hold then the ATS (3.1) at $\underline{\theta} = \underline{\theta}_1$ reduces to

$$d_1 \left[P(D_1|\underline{\theta}_1) - P(D_2|\underline{\theta}_1) \frac{P(D_1|\underline{\theta}_0)}{P(D_2|\underline{\theta}_0)} \right] b + \frac{c_0 P(D_2|\underline{\theta}_1)}{P(D_2|\underline{\theta}_0)} b \ .$$

where

$$b = \left[P(S|\underline{\theta}_1)(1 - P(S|\underline{\theta}_1))\right]^{-1} .$$

Since $b > 0$, it follows that the ATS at $\underline{\theta} = \underline{\theta}_1$ is increasing in d_1 if and only if (3.2.5) holds. Since D_1 represents the event that at least one chart uses d_1 and no chart signals, it follows that $P(D_1)$ is no less than the probability of the event that exactly one of the events $D_{11}, D_{12}, \ldots, D_{1r}$ occurs and there is no signal. Using this fact with the fact that $P(D_2) = \prod_{i=1}^{r} P(D_{2i})$ gives

$$\begin{aligned} P(D_1|\underline{\theta}_1) - P(D_2|\underline{\theta}_1)\frac{P(D_1|\underline{\theta}_0)}{P(D_2|\underline{\theta}_0)} &\geq \sum_{i=1}^{r} P(D_{1i}|\underline{\theta}_1)\prod_{j\neq i} P(D_{2j}|\underline{\theta}_1) - \prod_{i=1}^{r} P(D_{2i}|\underline{\theta}_1)\frac{P(D_1|\underline{\theta}_0)}{P(D_2|\underline{\theta}_0)} \\ &\geq P(D_{1i}|\underline{\theta}_1)\prod_{j\neq i} P(D_{2j}|\underline{\theta}_1) - \prod_{i=1}^{r} P(D_{2i}|\underline{\theta}_1)\frac{P(D_1|\underline{\theta}_0)}{P(D_2|\underline{\theta}_0)} \\ &= \left[P(D_{1i}|\underline{\theta}_1) - P(D_{2i}|\underline{\theta}_1)\frac{P(D_1|\underline{\theta}_0)}{P(D_2|\underline{\theta}_0)}\right]\prod_{j\neq i} P(D_{2j}|\underline{\theta}_1) \end{aligned}$$

Thus if (3.2.6) holds for i then (3.2.5) also holds and the proof is complete.

When two sampling intervals are used with $P(D_1|\underline{\theta}_0) = P(D_2|\underline{\theta}_0)$, this theorem implies that a sufficient condition for the widest possible spacing of d_1 and d_2 to be optimal is that at least one of the marginal charts has a greater probability of using the short sampling interval when there is a shift in any of the process parameters. This condition is not trivially satisfied but would generally hold true if a moderate to large shift occurred for any process parameter.

For example, consider the case where there are only two charts and the overall probability of assigning the long sampling interval is set at 0.5 when $\underline{\theta} = \underline{\theta}_0$. If the individual probabilities of each of the charts assigning the long sampling interval are equal, and hence approximately 0.7 each, then the probability of at least one of the charts assigning the short sampling interval has to increase from 0.3 to 0.6, for the condition to hold. If this condition is satisfied, then the optimal procedure is to keep the two sampling intervals as far apart as possible.

3.3 Adjusted Average Time to Signal

If the ATS is computed for some $\underline{\theta}_1 \neq \underline{\theta}_0$ then equation (3.1.4) assumes that $\underline{\theta} = \underline{\theta}_1$ from time zero, i.e. the process was out of control from the time that the chart was started. This gives an unfair advantage to the VSI chart, since it can sample earlier. Hence a more realistic model was considered in which the shift could occur after the process had been running and also that it could occur at random between samples. This model is important for evaluating a VSI procedure when ℓ_2 from equation (3.1.2) is large, since it accounts for the possibility of a shift occurring during a long sampling interval.

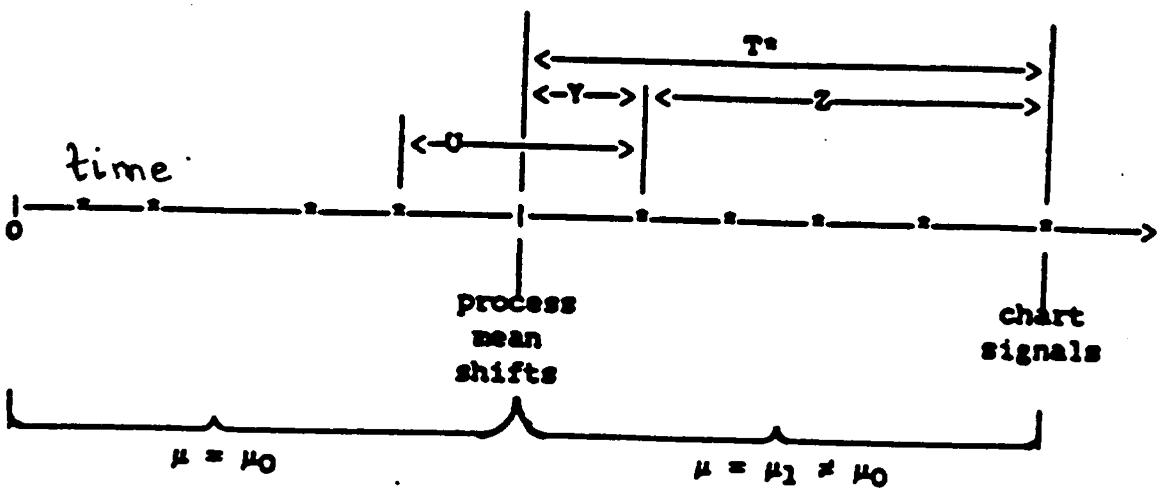
If a shift occurs between two samples, it is assumed that the point of occurrence is uniformly distributed over the interval and that the probability of its occurrence is proportional to the length of the interval and the frequency with which an interval of that length occurs when the process is in control. Under such a model, the relevant statistics are the length of time and the number of samples taken, from the time that the process goes out of control till the time that the chart signals. Let

$$\begin{aligned} Y &= \text{ time from the process shift until the next sample} \\ Z &= \text{ time from the next sample after the shift until a signal} \\ T^* &= \text{ time from the shift until a signal} \end{aligned} .$$

The relationship between T^*, Y and Z can be seen from Figure 3.2. Clearly, $T^* = Y + Z$ where Z is distributed like $\sum_1^{N-1} d(\underline{X}_i \mid \underline{\theta}_1)$ and the distribution of $d(\underline{X}_i \mid \underline{\theta}_1)$ is the conditional distribution of $d(\underline{X})$, given no signal when $\underline{\theta} = \underline{\theta}_1$. Reynolds and Arnold (1987) calculated the ATS using such a model. It is referred to as the *adjusted ATS* and denoted by ATS^*. Hence

$$\begin{aligned} ATS^* = E(T^* \mid \theta) &= E(Y \mid \underline{\theta}_0) + E(Z \mid \underline{\theta}_1) \\ &= E(Y \mid \underline{\theta}_0) + E(N-1)E[d(\underline{X}_i \mid \underline{\theta}_1)] \end{aligned} .$$

* = points where samples are taken



Time required to detect a shift in the process mean when shift occurs at a random time.

Figure 3. 2. Development of the Adjusted ATS

The distribution of N is still geometric with parameter p . On the set A , the adjusted ATS is simply

$$ATS^{*} = \frac{\sum_{l=1}^{k}\{d_l^2 p_{0l}\}}{2\sum_{l=1}^{k}\{d_l p_{0l}\}} + \frac{\sum_{l=1}^{k}\{d_l p_l\}}{P(s = 1 \mid \underline{\theta})} .$$

where $p_{0l} = P[d(\underline{X}) = d_l, s(\underline{X}) = 0 \mid \underline{\theta}_0]$ and $p_l = P[d(\underline{X}) = d_l, s = 0 \mid \underline{\theta}]$. For a FSI chart, the adjusted ATS reduces to $ANSS - d/2$.

Chapter IV

IV. Monitoring The Normal Mean And Variance

Consider the situation where one quality variable is of interest and the distribution of this variable is normal with mean μ and variance σ^2. Assume that the objective is to monitor both of these parameters so that $\underline{\theta} = (\mu, \sigma^2)$ and $\underline{\theta}_0 = (\mu_0, \sigma_0^2)$.

The problem of monitoring both μ and σ^2 is analogous to the problem of simultaneously testing the mean and the variance. The null hypothesis would be $H_0{:}\mu = \mu_0, \sigma^2 = \sigma_0^2$ and the alternative hypothesis would be $H_1{:}\mu \neq \mu_0$ and/or $\sigma^2 \neq \sigma_0^2$ or $H_1{:}\mu \neq \mu_0$ and/or $\sigma^2 > \sigma_0^2$. In this dissertation, the focus is on the second alternative hypothesis. Although this may seem like a relatively simple hypothesis testing problem involving a normal distribution, no uniformly optimal test exists and there seems to be no test that is well known and widely used.

There are several ways of monitoring the mean and variance simultaneously. In practice, the typical approach is to use separate charts for each parameter and then to signal if either chart signals. Although S^2 is the optimal statistic for testing a hypothesis about the variance of a normal distribution, R-charts are frequently used in practice to control the variance. In this study, S^2 -charts were used because the S^2-chart is more efficient than the R-chart and because the distribution of S^2 is much simpler. Primary interest is usually in detecting only an increase in the variance and so only one-sided charts were considered.

Although using separate charts seems to be reasonable, it is not optimal for detecting certain types of process changes such as those involving a change in both μ and σ^2. As an alternative to using two separate charts, Reynolds and Ghosh (1981) proposed using a single statistic based on the squared standardized deviations of the observations from the target value μ_0. If X is the variable being monitored then $E((X-\mu_0)^2) = \sigma^2 + (\mu - \mu_0)^2$ and large values of $(X-\mu_0)^2$ will tend to be obtained when μ changes from μ_0 or σ^2 increases (or both). If the squared deviations from target, $(X_{ij} - \mu_0)^2$, are standardized and summed, the statistic

$$C_i = \sum_{j=1}^{n} [(X_{ij} - \mu_0)/\sigma_0]^2 no(4.1)$$

is obtained. When $\mu = \mu_0$ and $\sigma = \sigma_0$ the distribution of C_i is chi-squared with n degrees of freedom $(\chi^2_{(n)})$. This sum statistic thus includes information about both μ and σ^2 . It is uniformly most powerful for detecting an increase in σ^2 when μ remains at μ_0 and in general it is good for detecting process changes involving a significant change in the variance. The procedure also has the practical advantage of requiring that only one chart be plotted.

The statistic C_i can be interpreted as the sum of the statistics used by the $\overline{X}$- and S^2-charts. The two-sided $\overline{X}$-chart signals for large values of $|\overline{X}_i - \mu_0|$ or equivalently for large values of U_i where

$$U_i = n(\overline{X}_i - \mu_0)^2/\sigma_0^2 \; .$$

The S^2-chart signals for large values of S^2 or equivalently for large values of V_i where

$$V_i = (n-1)S_i^2/\sigma_0^2 \; .$$

Then C_i combines the information in U_i and V_i through the sum

$$C_i = U_i + V_i \ .$$

Note that U_i and V_i are independent and when $\mu = \mu_0$ and $\sigma = \sigma_0$, U_i is $\chi^2_{(1)}$ and V_i is $\chi^2_{(n-1)}$. When C_i is used and a sample gives a signal it may be necessary to examine both U_i and V_i as a diagnostic aid in finding the assignable cause.

4.1 Applying the VSI Feature

As stated in section 3.2 the *VSI* feature cannot be applied independently to the $\overline{X}$ and S^2 charts. There are several possible decision rules that could be used. If shifts in μ are more critical or likely than shifts in σ, the sampling interval could depend only on the $\overline{X}$ chart. If the sampling interval is to depend on both charts then the minimum of the two sampling intervals assigned by separate $\overline{X}$ and S^2 charts could be used. Another possibility is to combine the information about μ and σ^2 using the sum statistic defined in equation (4.1) and use only one chart.

In this study, the following four basic strategies are compared. To simplify notation, the subscripts x and s correspond to $\overline{X}$ charts and S^2 charts, respectively.

i) FSI_{Sep}: separate *FSI* charts for the mean and variance.

This procedure uses separate $\overline{X}$ and S^2 charts and signals if either the $\overline{X}$ or S^2 chart signals. Let the signal probabilities of the $\overline{X}$ and S^2 charts be denoted by α_x and α_s when $\mu = \mu_0$ and $\sigma = \sigma_0$. Since $\overline{X}$ and S^2 are independent under the assumption of normality, the overall signal probability is $\alpha = 1 - (1 - \alpha_x)(1 - \alpha_s)$. The control limits for $\overline{X}$ were set at

$$\{\mu_0 - z_{1-\alpha_x/2}\sigma_0/\sqrt{n}\, , \mu_0 + z_{1-\alpha_x/s}\sigma_0/\sqrt{n}\} \ .$$

where z_α is the α quantile of the standard normal distribution. Similarly, $(n-1)s^2/\sigma_0^2$ has a $\chi^2_{(n-1)}$ distribution. Since only an upper limit was required for the S^2 chart, the limits for S^2 were set at $\{0, \chi^2_{(n-1),1-\alpha_2}\sigma_0^2/(n-1)\}$ where $\chi^2_{(n-1),\alpha}$ denotes the α quantile of the $\chi^2_{(n-1)}$ distribution. Once the control limits are set, the signal probabilities for different parameter values can be computed.

ii) FSI_C: a single FSI chart using the sum statistic

Recall that the sum statistic

$$C_i = \sum_{j=1}^{n} [(X_{ij} - \mu_0)/\sigma_0]^2$$

has a $\chi^2_{(n)}$ distribution. Following the same arguments as above, the control limits were set at $\{0, \chi^2_{(n),1-\alpha}\}$.

iii) VSI_{Sep}: separate VSI charts for the mean and variance

The signal function used is the same as for the FSI_{Sep} procedure. The sampling interval used is the minimum of the two sampling intervals assigned by the $\overline{X}$ and S^2 charts, respectively, when $\underline{X} = \underline{x}$. Let D_{ix} and D_{is} denote marginal regions which are defined as follows:

$$D_{ix} = \{\overline{x} \mid d_X(\overline{x}) = d_i\}$$
$$D_{is} = \{s^2 \mid d_S(s^2) = d_i\} \,.$$

These regions are illustrated in Figure 4.1.

To illustrate for two sampling intervals, let $S_x = \{\overline{x} \mid \overline{X}\text{chart signals}\}$ and $S_s = \{s^2 \mid S^2\text{chart signals}\}$. If D_i denotes the joint region in the $(\overline{x}, s^2)$ space to which the sampling interval d_i is assigned, then D_2 is the cartesian product of (D_{2x}, D_{2s}) and $D_1 = \{(\overline{x}, s^2) \mid \overline{x} \notin S_x, s^2 \notin S_s \text{ and } (\overline{x}, s^2) \notin D_2\}$. Since it is possible to vary the marginal probabilities of the $\overline{X}$ and S^2 charts, one can vary the marginal sampling regions depending on which parameter is of more concern. For example, if one is interested in monitoring the mean more

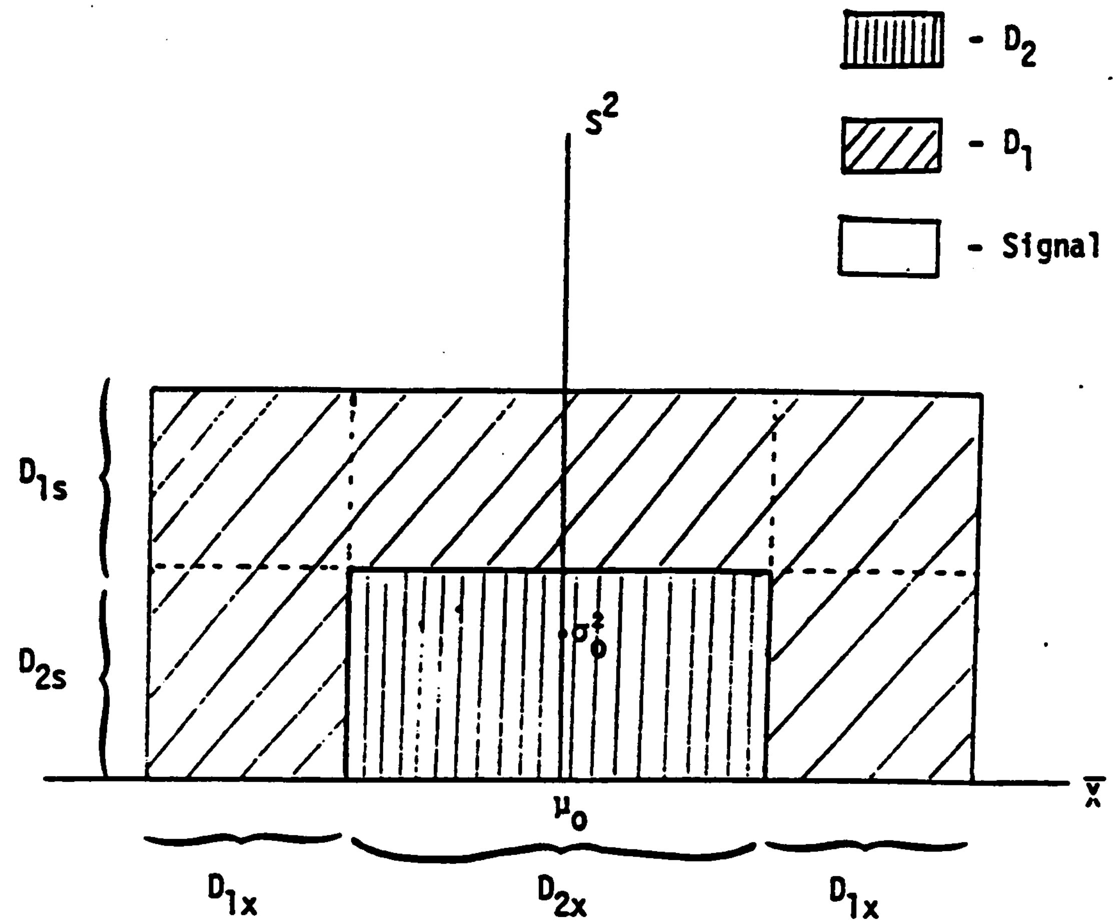


Figure 4. 1. Signal and Sampling Interval Regions for VSI_{sep}

closely, one can increase the probability of the short sampling region for the $\overline{X}$ chart while still keeping the combined probability of D_1 fixed. Note that if, for either of the marginal charts, the probability of the short sampling region is set to zero, the procedure reduces to one which signals if either of the two charts signal, but selects the sampling interval based on only one chart.

iv) VSI_C: a single VSI chart using the sum statistic

The signal function used is the same as for the FSI_C procedure. The sampling interval regions to which the sampling interval d_i is assigned is

$$D_i = \{(\bar{x}, s^2) \mid d_{(\overline{X}, S^2)}(\bar{x}, s^2) = d_i\} \ .$$

These regions are illustrated in Figure 4.2.

4.2 Numerical Results

Since no general theoretically optimal procedure is available for simultaneously monitoring the mean and variance, the four strategies considered in this paper were numerically investigated. The objective was to determine which type of chart performs best for various types of process changes and to determine reasonable ways of choosing the sampling intervals. The criteria for comparing different procedures were the ATS and the adjusted ATS.

The length of the fixed sampling interval is denoted by d and for convenience it is assumed that $d = 1$ time unit. Note that the $ATS = ANSS$ for a FSI chart when the sampling interval is 1. This forces the VSI charts to have $E(R_i) = 1$ when $\underline{\theta} = \underline{\theta}_0$. This causes each of the FSI and VSI procedures to have the same ATS and "false alarm" rate when the process is in control. Thus,

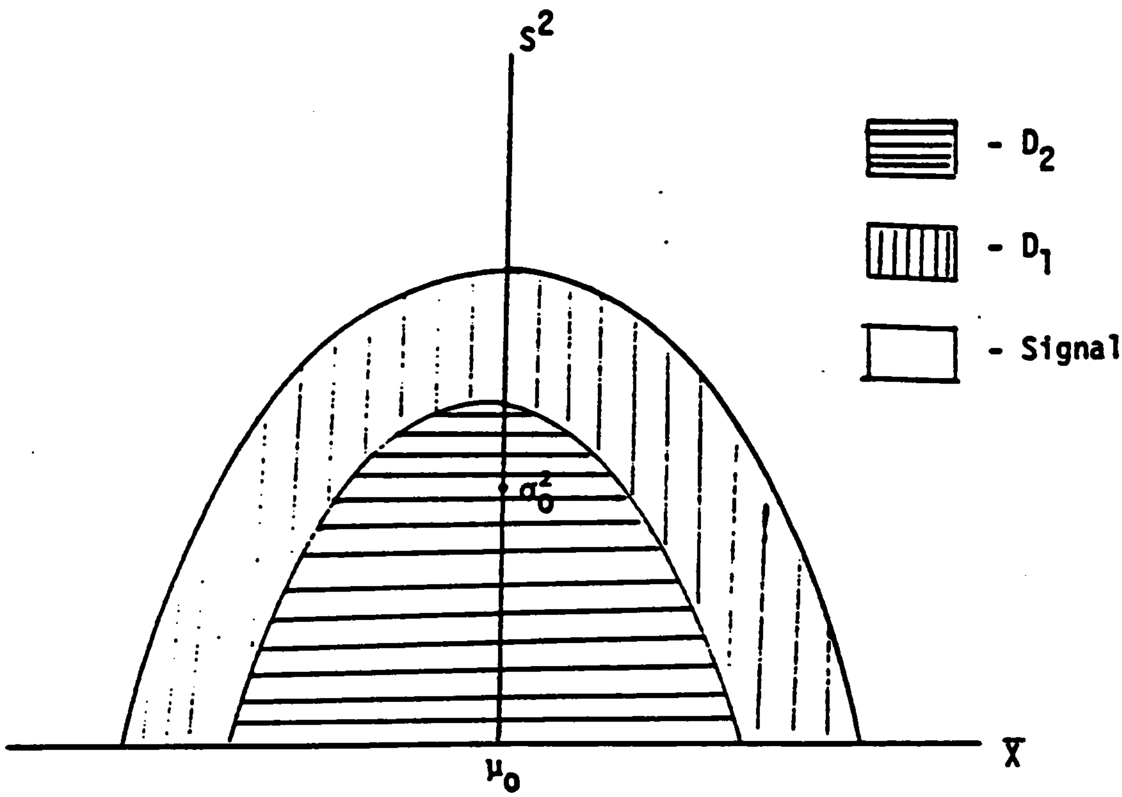


Figure 4. 2. Signal and Sampling Interval Regions for VSI_c

comparing the ATS or the ATS^* of two procedures is analogous to comparing the power of two test procedures when the test size and sample sizes are the same. Further, when a set of only two intervals are used, specifying the values of d_i will fix the probabilities and hence the regions D_i for the VSI chart. Shifts in the mean are measured in terms of standard errors and are denoted by $\delta = \sqrt{n}\,(\mu - \mu_0)/\sigma_0$. Shifts in variability are measured in terms of σ/σ_0. The $ANSS$ was fixed at 100 when the process is in control, and hence, the overall in control signal probability for each procedure is fixed at $\alpha = 0.01$.

Samples of size five were used and tables were constructed for the following shift values in the mean and variance:

δ : 0, 0.1, 0.25, 0.5, 1, 1.1, 1.25, 1.5, 2, 3, 4

and

σ/σ_0 : 1.0, 1.1, 1.25, 1.5, 2, 3, 4 .

For each combination of μ and σ the $ANSS$, the ATS, and the adjusted ATS were computed.

The adjusted time to signal is a more realistic measure of performance for most practical problems since more typically protection against a process going out of control at some unknown future time is needed. However, the properties for this measure are more difficult to assess. That is, properties which uniformly hold true for the ATS do not hold uniformly for the ATS^* . For these reasons, and for the sake of clarity, properties of the ATS and ATS^* for the various procedures are summarized separately. Each table gives the values of the ATS (or the ATS^*) as a function of the standardized mean shift δ and the ratio of the standard deviations to its target value, σ/σ_0.

4.2.1 Sum Versus Separate Charts

The primary purpose of the study was to assess the advantages of using a VSI procedure. Another point of interest was whether the typical approach of using separate charts to monitor the mean and variance was appropriate or whether using the sum statistic was more advantageous. With the aid of FORTRAN programming, the ATS and the ATS^* was computed for the four basic procedures and the results were tabulated. Recall the labels for the four strategies considered were FSI_{Sep}, FSI_C, VSI_{Sep}, and VSI_C. The ATS values for strategies FSI_{Sep} versus FSI_C are given in Table 4.1. Except for σ/σ_0 at or near unity, the ATS for the sum chart is seen to be smaller than for separate charts. For moderate shifts in μ or σ the ATS for the sum chart is considerably smaller than the ATS for separate charts.

Next the separate and sum strategies were compared when the VSI feature was used. There are several choices for sampling intervals in terms of length, symmetry and number of intervals. These will be discussed in detail in sections 4.4 and 4.5. A representative pair of sampling intervals (0.1, 1.9) was picked for this comparison. Thus, VSI chart used only two sampling intervals with the short one set at 0.1 and the long one set at 1.9. The results are presented in Table 4.2. It can be seen that the conclusions that were drawn for the FSI are supported by examining the two strategies using the VSI procedure. Again the sum chart performs better over a wide range of shifts in both μ and σ. When there is little or no shift in the variance, the separate charts procedure does better.

Next the separate and sum strategies were compared in terms of the adjusted ATS. Table 4.3 gives the values of the ATS^* for the FSI_c, VSI_{Sep} and the VSI_c procedures. As before, the variable interval sum statistic is much better than the fixed interval sum statistic except for large shifts. In general, both of the variable sampling procedures (VSI_{Sep} and VSI_c) have a substantially smaller ATS^* for small to moderate shifts in either μ or σ. For quite large shifts, the ATS^* for fixed sampling intervals is smaller than for variable sampling intervals but the difference is not very significant since for such a shift both procedures have a very small ATS^*.

In order to see what effect, if any, the choice of sampling intervals had on the conclusions drawn about the four procedures, they were compared again using the set (0.1, 1.3). This set does not have as long a sampling interval as the set (0.1, 1.9) and so should perform better in terms of the adjusted

Table 4. 1. ATS for FSI_{sep} followed by ATS for FSI_c

δ	σ/σ_0 = 1	1.1	1.25	1.5	2	3	4
0	100	38.49	13.67	4.76	1.87	1.15	1.04
	100	34.59	11.68	4.11	1.72	1.12	1.03
.1	97.87	38.02	13.59	4.75	1.87	1.15	1.04
	98.77	34.25	11.60	4.09	1.71	1.12	1.03
.25	87.78	35.70	13.18	4.69	1.86	1.15	1.04
	92.66	32.55	11.17	4.00	1.70	1.12	1.03
.5	62.58	29.03	11.85	4.50	1.84	1.15	1.04
	74.96	27.46	9.87	3.70	1.64	1.11	1.03
1	24.79	15.34	8.25	3.86	1.77	1.14	1.04
	38.20	15.88	6.56	2.86	1.47	1.08	1.02
1.1	20.48	13.31	7.56	3.71	1.76	1.14	1.04
	32.73	13.99	5.97	2.69	1.43	1.07	1.02
1.25	15.49	10.77	6.60	3.48	1.73	1.14	1.04
	25.85	11.54	5.17	2.46	1.37	1.06	1.02
1.5	9.97	7.62	5.25	3.11	1.67	1.13	1.04
	17.44	8.37	4.06	2.11	1.29	1.05	1.01
2	4.67	4.10	3.38	2.45	1.56	1.12	1.04
	8.28	4.58	2.61	1.61	1.15	1.02	1.01
3	1.73	1.73	1.71	1.61	1.34	1.00	1.03
	2.60	1.86	1.40	1.14	1.03	1.00	1.00
4	1.13	1.16	1.19	1.22	1.18	1.07	1.03
	1.36	1.19	1.08	1.02	1.00	1.00	1.00

Table 4. 2. ATS for VSI_{sep} followed by ATS for VSI_c using (.1,1.9)

	σ/σ_0						
δ	1	1.1	1.25	1.5	2	3	4
0	100	32.00	8.75	2.14	.55	.25	.20
	100	28.25	7.22	1.77	.50	.25	.21
.1	97.58	31.54	8.68	2.13	.55	.25	.20
	98.57	27.91	7.16	1.76	.50	.25	.21
25	86.21	29.25	8.34	2.09	.55	.25	.20
	91.57	26.22	6.82	1.71	.49	.24	.21
.5	58.25	22.75	7.26	1.97	.54	.25	.20
	71.22	21.24	5.79	1.53	.46	.24	.20
1	18.74	10.15	4.47	1.57	.51	.25	.20
	31.15	10.48	3.31	1.05	.39	.23	.20
1.1	14.64	8.42	3.96	1.48	.50	.25	.20
	25.58	8.84	2.89	.95	.37	.22	.20
1.25	10.09	6.31	3.28	1.35	.48	.24	.20
	18.83	6.79	2.34	.83	.34	.22	.20
1.5	5.47	3.89	2.36	1.14	.46	.24	.20
	11.09	4.30	1.63	.64	.30	.21	.19
2	1.74	1.52	1.20	.78	.41	.24	.20
	3.79	1.72	.79	.40	.24	.19	.18
3	.30	.34	.37	.37	.31	.23	.20
	.57	.36	.25	.19	.16	.16	.18
4	.13	.15	.17	.21	.23	.21	.20
	.18	.15	.13	.13	.13	.14	.16

Table 4. 3. ATS* for FSI_{sep}, VSI_{sep} and VSI_c using (.1, 1.9)

	σ/σ_0						
δ	1	1.1	1.25	1.5	2	3	4
		34.09	11.18	3.61	1.22	.62	.53
0		32.08	9.01	2.59	1.16	.94	.91
		28.34	7.51	2.25	1.11	.93	.91
	98.27	33.75	11.10	3.59	1.21	.62	.53
.1	97.49	31.61	8.95	2.59	1.16	.94	.91
	98.48	28.00	7.44	2.24	1.11	.93	.91
	92.16	32.05	10.67	3.50	1.20	.62	.53
.25	86.14	29.33	8.61	2.55	1.16	.94	.91
	91.40	26.32	7.11	2.19	1.11	.93	.91
	74.46	29.96	9.36	3.20	1.14	.61	.53
.5	58.22	22.87	7.56	2.44	1.15	.94	.91
	71.18	21.38	6.10	2.02	1.09	.93	.91
	37.70	15.38	6.06	2.36	.97	.58	.52
1	18.89	10.39	4.83	2.07	1.13	.94	.91
	31.24	10.72	3.71	1.59	1.03	.92	.91
	32.23	13.49	5.47	2.19	.93	.57	.52
1.1	14.83	8.69	4.39	1.99	1.12	.94	.91
	25.71	9.12	3.31	1.50	1.01	.92	.91
	25.35	11.04	4.67	1.96	.87	.56	.51
1.25	10.35	6.63	3.69	1.87	1.11	.93	.91
	19.01	7.11	2.79	1.39	1.00	.92	.91
	16.94	7.87	3.56	1.61	.79	.55	.51
1.5	5.83	4.28	2.81	1.68	1.09	.93	.91
	11.36	4.69	2.13	1.24	.97	.91	.91
	7.78	4.08	2.11	1.11	.65	.52	.51
2	2.27	2.06	1.75	1.37	1.05	.93	.91
	4.24	2.25	1.39	1.06	.94	.91	.91
	2.10	1.36	.90	.64	.53	.50	.50
3	1.03	1.05	1.06	1.05	.98	.96	.91
	1.26	1.07	.98	.93	.91	.91	.91
	.86	.69	.58	.52	.50	.50	.50
4	.92	.93	.93	.94	.94	.92	.91
	.95	.93	.91	.91	.91	.91	.91

ATS , according to Reynolds et al (1988). Also, by the definition of symmetry from section 4.5, the set (0.1, 1.3) is not symmetric about the fixed sampling interval unity, unlike the previous set (0.1, 1.9). Both the notions of length and symmetry shall be studied in depth in sections 4.5 and 4.6.

The values of the ATS for the four basic procedures were plotted against μ for various values of σ. This gives a more comprehensive view of the performance of the four procedures under different conditions. The $VSI_{S_{ep}}$ and the VSI_c used (0.1, 1.3) as the sampling interval set. Figure 4.3 graphs the ATS for the four procedures when there is no shift in the variance. Clearly, using the VSI feature improves the performance of both the separate and sum charts. Also, the VSI_s procedure performs consistently better than the other three procedures. When the variance increases to 1.5, the same pattern can be observed from Figure 4.4, although the differences in the actual performances are reduced. When the variance rises to 3.0, Figure 4.5 shows that the ATS for the FSI_c procedure is converging to one, and the ATS for the VSI_c procedure is converging towards 0.1 which is the length of the short sampling interval.

When the four procedures were evaluated in terms of the adjusted ATS using the same set of sampling intervals, the above conclusions seem to be validated by Figure 4.6 where $\sigma = 1.25$. When the variance increases to 1.5, the pattern looks similar to the previous ones for small values of μ. But for very large values of μ, the procedures using variable sampling intervals suffer and the FSI_c seems to perform the best, in terms of the ATS^* . This can be seen from Figure 4.7.

The graphs just presented were for selected values of shifts in the variance. Their scales were magnified to show details. In order to put the previous results in perspective, the four basic procedures were compared in terms of ratios of their ATS and adjusted ATS . The ratios of the ATS (or the adjusted ATS) of the FSI_c, $VSI_{S_{ep}}$ and the VSI_c procedures, to the ATS (or the adjusted ATS) of the $FSI_{S_{ep}}$ procedure, were computed. The set of sampling intervals used were the same, namely (0.1, 1.3). Table 4.4 presents the ratios using the ATS . It shows that the $VSI_{S_{ep}}$ procedure performs consistently better than the procedures using fixed sampling intervals. The VSI_c procedure is seen to be about five times more efficient than the FSI_c for moderate to large shifts in either of the parameters. Although the VSI_c procedure does better than the $VSI_{S_{ep}}$ when the shift in the variance is large, it does not perform as well as the $FSI_{S_{ep}}$ procedure when there is no shift or a very small shift in the variance.

The ratios computed in terms of ATS^* are presented in Table 4.5. From these results it can be seen that the procedures using variable sampling intervals are less efficient than the procedures using fixed sampling intervals when there is a very large shift in μ or σ or both. However, for a majority of the values of shift, the VSI procedures perform better with the $VSI_{S_{ep}}$ and VSI_c procedures being almost twice as efficient as the FSI_c in several cases. Hence even though the VSI procedures do not seem to perform uniformly better in terms of ATS^* they were still found to be superior for most values of shift in μ and σ.

4.2.2 Sets of Three Sampling Intervals

When the VSI feature is applied, one question that needs to be answered is how many sampling intervals should be used. Clearly the simplest VSI procedure would use only two sampling intervals. The fortuitous result that only two sampling intervals need to be used, to obtain the optimum procedure (in terms of ATS) was proved by Reynolds and Arnold (1987). Recall that an optimum procedure does not exist for the simultaneous monitoring of the mean and variance. However, it was hoped that using two sampling intervals would still give the best results.

A variation of the separate charts procedure, using three sampling intervals, was investigated for the following sampling interval strategy. If both $\overline{X}$ and S^2 are close to target, the longest sampling interval is used; if only one is close to target, a shorter sampling interval is used; and finally, if both are off target but not in the signal region, the shortest sampling interval is used. For this strategy

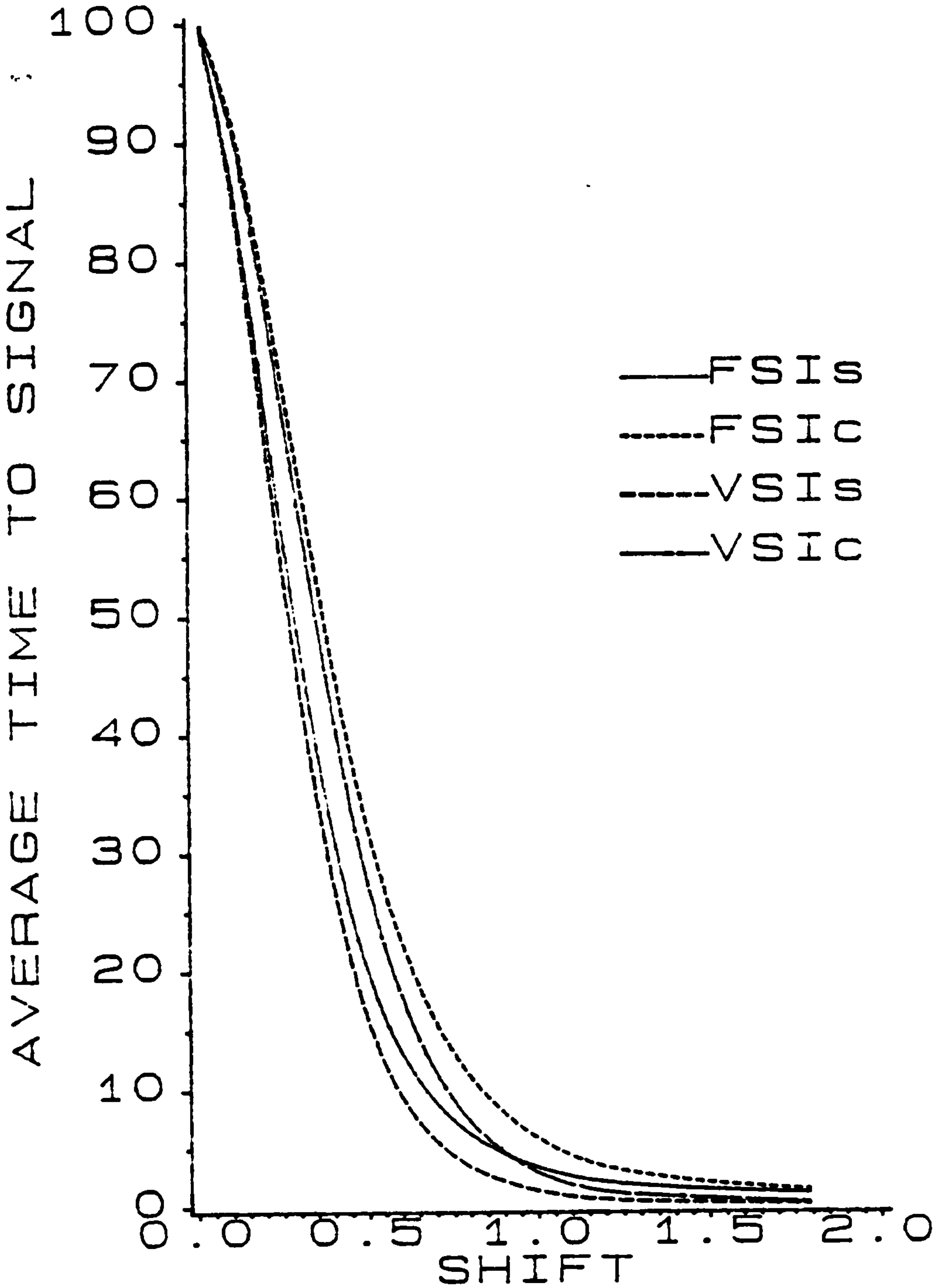


Figure 4. 3. ATS for FSI_c, FSI_{sep}, VSI_{sep} and VSI_c using the set (.1,1.3) when there is no shift in the variance.

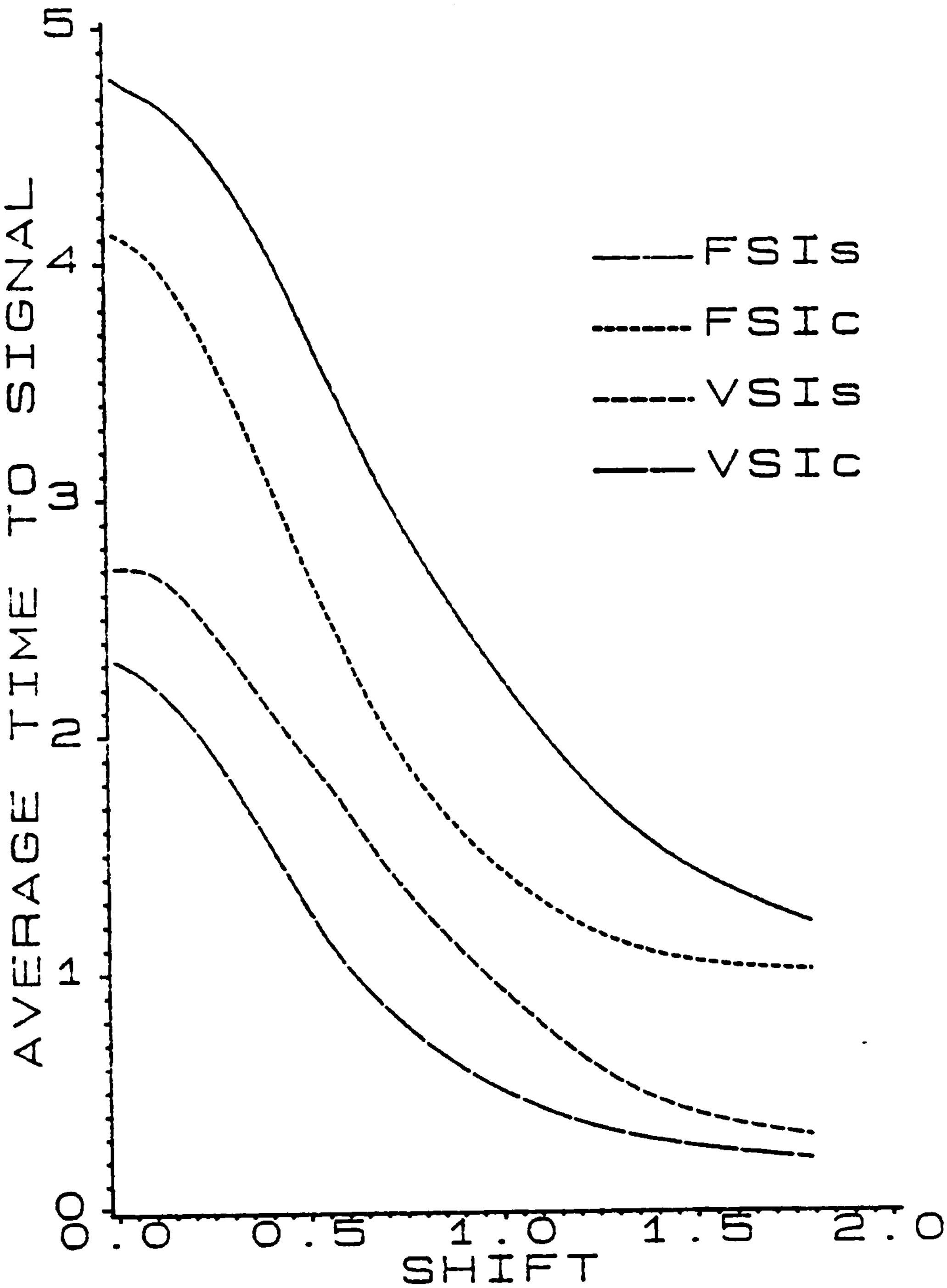


Figure 4. 4. ATS for FSI_{sep} , FSI_c , VSI_{sep} and VSI_c using the set (.1,1.3) when the variance is 1.5

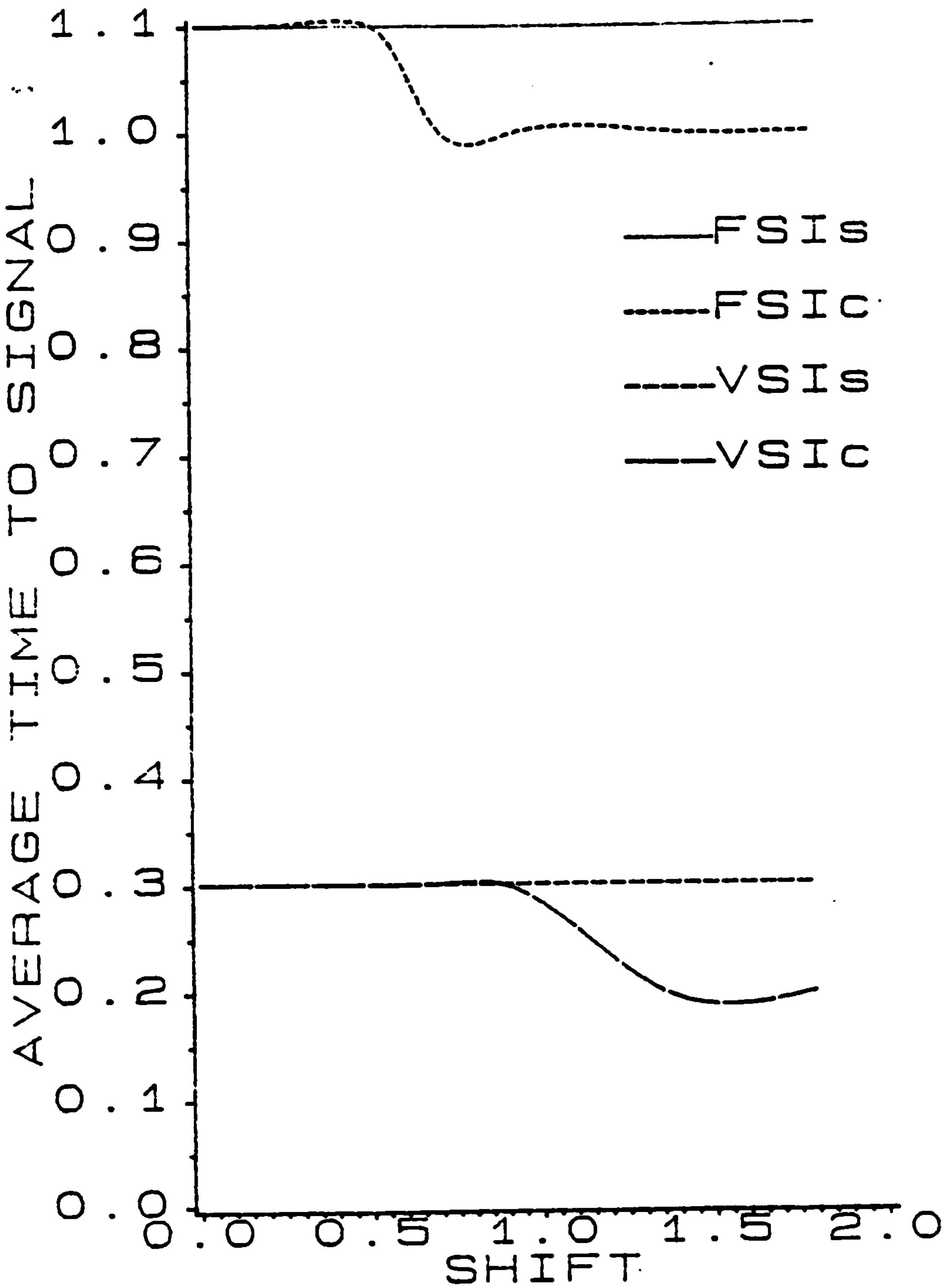


Figure 4. 5. ATS for FSI_{sep} , FSI_c , VSI_{sep} and VSI_c using the set (.1,1.3) when the variance is 3.0

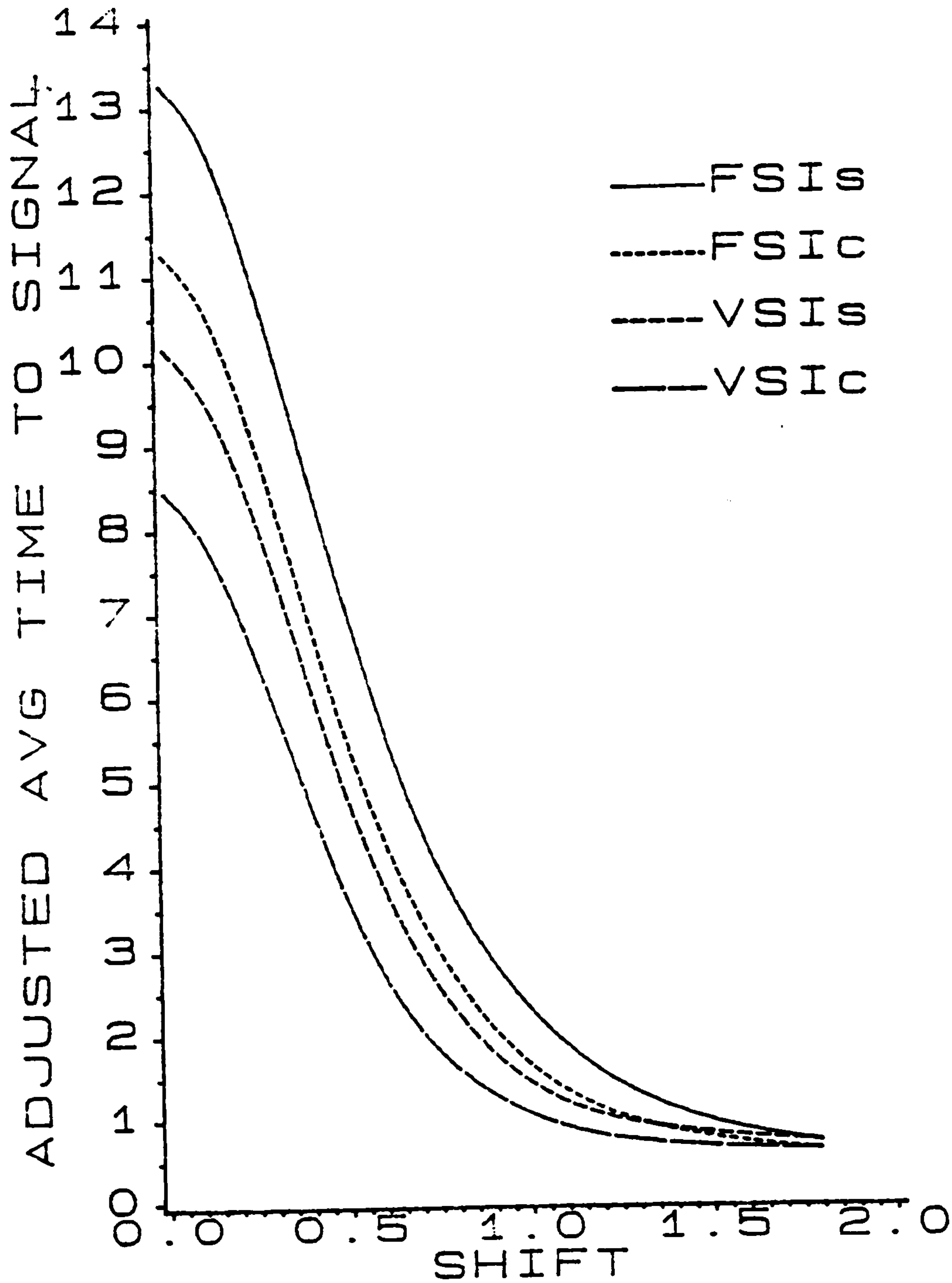


Figure 4. 6. ATS* for FSI_{sep} , FSI_c , VSI_{sep} and VSI_c using the set (.1,1.3) when the variance is 1.25

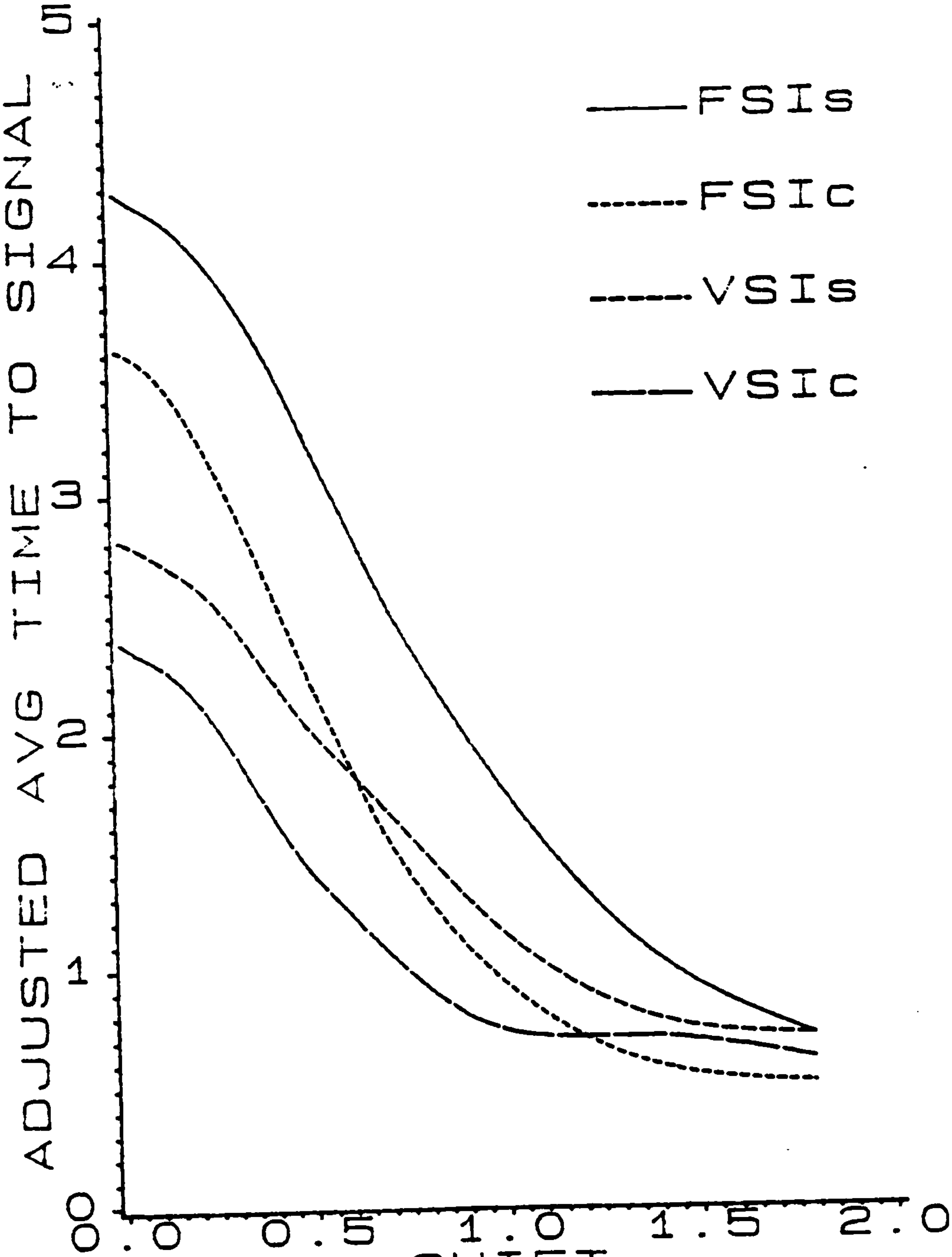


Figure 4. 7. ATS* for FSI_{sep} , FSI_c , VSI_{sep} and VSI_c using the set (.1,1.3) when the variance is 1.5

Table 4. 4. **Ratios of the ATS of FSI_c, VSI_{sep} and VSI_c procedures to the ATS of the FSI_{sep} procedure**

	σ/σ_0						
δ	1	1.1	1.25	1.5	2	3	4
	1.00	.90	.85	.86	.92	.98	1
0	1.00	.83	.64	.45	.30	.22	.20
	1.00	.73	.53	.37	.27	.21	.20
	1.01	.90	.85	.86	.92	.97	.99
.1	1.00	.83	.64	.45	.30	.22	.20
	1.00	.73	.53	.37	.27	.21	.20
	1.06	.91	.85	.85	.91	.97	.99
.25	.98	.82	.63	.45	.29	.22	.20
	1.04	.73	.52	.36	.26	.21	.20
	1.20	.95	.83	.82	.89	.97	.99
.50	.93	.78	.61	.44	.29	.22	.22
	1.14	.73	.49	.34	.25	.21	.20
	1.54	1.04	.80	.74	.83	.94	.98
1	.76	.66	.54	.41	.29	.22	.20
	1.26	.68	.40	.27	.22	.20	.19
	1.60	1.05	.79	.73	.81	.94	.98
1.1	.71	.63	.52	.40	.28	.22	.20
	1.25	.66	.38	.26	.21	.20	.19
	1.75	1.10	.77	.68	.77	.92	.97
1.25	.55	.51	.45	.37	.27	.21	.20
	1.11	.56	.31	.21	.18	.18	.18
	1.77	1.12	.77	.66	.74	.91	.97
1.5	.37	.37	.36	.32	.26	.21	.19
	.81	.42	.23	.16	.17	.17	.18
	1.50	1.10	.82	.71	.77	.91	.97
2	.18	.20	.22	.23	.23	.21	.19
	.33	.21	.14	.12	.12	.15	.16
	1.20	1.02	.9	.84	.74	.93	.97
3	.12	.13	.15	.17	.20	.20	.19
	.16	.13	.11	.10	.11	.13	.15

Table 4. 5. Ratios of the ATS* of the FSI_c, VSI_{sep} and VSI_c procedures to the ATS* of the FSI_{sep} procedure

	σ/σ_0						
δ	1	1.1	1.25	1.5	2	3	4
		.90	.85	.85	.89	.96	.98
0		.84	.68	.61	.85	1.44	1.68
		.75	.57	.53	.81	1.43	1.68
	1.01	.90	.85	.84	.89	.96	.98
.10	1.00	.84	.68	.61	.85	1.44	1.68
	1.00	.75	.57	.53	.81	1.43	1.68
	1.06	.91	.84	.83	.88	.95	.98
.25	.99	.83	.68	.61	.85	1.44	1.68
	1.05	.75	.56	.52	.81	1.43	1.68
	1.20	.95	.82	.80	.85	.94	.98
.50	.94	.80	.67	.61	.86	1.45	1.68
	1.15	.75	.54	.51	.81	1.43	1.68
	1.55	1.04	.78	.70	.76	.90	.96
1	.78	.70	.62	.62	.88	1.46	1.69
	1.29	.72	.48	.47	.81	1.43	1.68
	1.61	1.05	.77	.68	.74	.89	.96
1.1	.74	.68	.62	.62	.89	1.46	1.69
	1.29	.71	.47	.47	.81	1.43	1.68
	1.79	1.10	.75	.62	.67	.86	.95
1.25	.662	.60	.59	.64	.93	1.47	1.69
	1.20	.66	.45	.48	.83	1.44	1.68
	1.86	1.13	.73	.57	.62	.84	.94
1.5	.54	.57	.61	.70	.99	1.49	1.69
	1.02	.62	.48	.54	.88	1.46	1.68
	1.71	1.11	.75	.58	.63	.84	.94
2	.84	.85	.87	.94	1.17	1.55	1.71
	1.02	.87	.80	.84	1.08	1.51	1.70
	1.36	1.04	.83	.73	.74	.87	.95
3	1.46	1.41	1.35	1.31	1.38	1.61	1.73
	1.51	1.41	1.32	1.26	1.33	1.58	1.72

D_3 is the cartesian product of (D_{2x}, D_{2s}) and D_1 is the cartesian product of (D_{1x}, D_{1s}). The region $D_3 = \{(\overline{x}, s^2) \mid \overline{x} \notin S_x, s^2 \notin S_s, (\overline{x}, s^2) \notin D_1 \text{ and } (\overline{x}, s^2) \notin D_3\}$. Thus the following relationships hold:

$$
\begin{aligned}
P(D_{2x})P(D_{2s}) &= P(D_3) \\
P(D_{1x})P(D_{1s}) &= P(D_1) \\
P(D_{1x}) + P(D_{2x}) &= 1 - \alpha_x \\
P(D_{1s}) + P(D_{2s}) &= 1 - \alpha_s
\end{aligned}
$$

Hence, there is a unique set of marginal probabilities for the regions $D_{1x}, D_{2x}, D_{1s}, D_{2s}$ once the probabilities $P(D_3)$, $P(D_1)$ and $\alpha_x = \alpha_s$ are specified. Construction of the sampling interval regions is demonstrated in Figure 4.8.

This variation was investigated using the following sets of sampling intervals: (01., 1.0, 1.9), (0.3, 1.0, 1.7) and (0.5, 1.0, 1.5). When only two sampling intervals are used, Reynolds and Arnold (1987) had established that keeping the two sampling intervals as far apart as possible was optimal under certain conditions. Table 4.6 shows that this is also true for three sampling intervals when the procedures are compared in terms of their adjusted ATS as well.

Next the procedure that used the set (0.1, 1.0, 1.9) was compared to the corresponding separate charts procedures that used only two sampling intervals. Table 4.7 includes the values of the ATS for the $VSI_{S_{ep}}$ charts using the sets (0.1, 1.0, 1.9) and (0.1, 1.9) respectively. It can be seen that the ATS is uniformly smaller when two sampling intervals rather than three are used. This is the same pattern as was found for $\overline{X}$ charts and is fortunate since two sampling intervals yield considerably smaller ATS with only slightly more complexity than fixed sampling interval procedures. When using the ATS^*, this pattern still holds good for small to moderate shifts in both μ and σ. As seen in Table 4.8, the ATS^* for two intervals is smaller than for three intervals, except for large shifts in either μ or σ. When a large shift occurs, the procedure using the set (0.1, 1.9) is at a disadvantage since the shift is likely to have occurred when the sampling interval of length 1.9 was being employed. Thus it takes longer for the shift to be detected and this causes the ATS^* to be large compared to a procedure that uses the set (0.1, 1.0, 1.9), where the shift is less likely to have occurred during the sampling interval of length 1.9.

4.2.3 Symmetry of the Sampling Intervals

A symmetric *VSI* Shewhart chart is defined in Amin (1987) as a Shewhart chart with sampling intervals that satisfy $P(d_i) = P(d_{k+1-i})$ for $i = 1, 2, \ldots, k$ when the process is in control. So, if only two sampling intervals are used, the condition requires that $P(d_1) = P(d_2)$. While comparing the sum and separate charts in section 4.3, a set of symmetric sampling intervals was used initially and then a set of asymmetric sampling intervals was used. This section concentrates on comparing symmetric and asymmetric sampling intervals of different lengths.

Table 4.9 displays the values of the *ATS* for the VSI_c procedure. The sets of sampling intervals used are (.1, 1.9) and (.5, 1.5). It bears out the result that was established by Reynolds and Arnold (1987) , namely, that the further apart the sampling intervals are, the smaller the *ATS* is. Table 4.10 demonstrates the effect of varying the spacing of symmetric sampling intervals on the $VSI_{S_{ep}}$ procedure. As expected, the sampling interval (0.1, 1.9) is preferred to (0.5, 1.5) which in turn, is preferred to (0.9, 1.1). That is, the *ATS* is smaller for sampling intervals more widely spaced. Thus, regardless of which sampling interval was being used, the best procedure in terms of the *ATS* kept the sampling intervals far apart. This held good for symmetric as well as asymmetric sampling intervals.

The same is true for the adjusted *ATS* as well, except when the shift in μ and/or σ is large. This can be seen from Table 4.11. When $\sigma = 2$, the set (0.1, 1.9) is bested by the set (0.5, 1.5), which in turn is bested by the set (0.9, 1.1) when $\sigma \geq 3$. Table 4.12 shows that wider is better for asymmetric sampling intervals as well, by studying the the *ATS* for the VSI_c procedure using the sets (.1, 1.5), (.1, 1.9) and (.1, 4.0). The adjusted *ATS* penalizes procedures that have long sampling

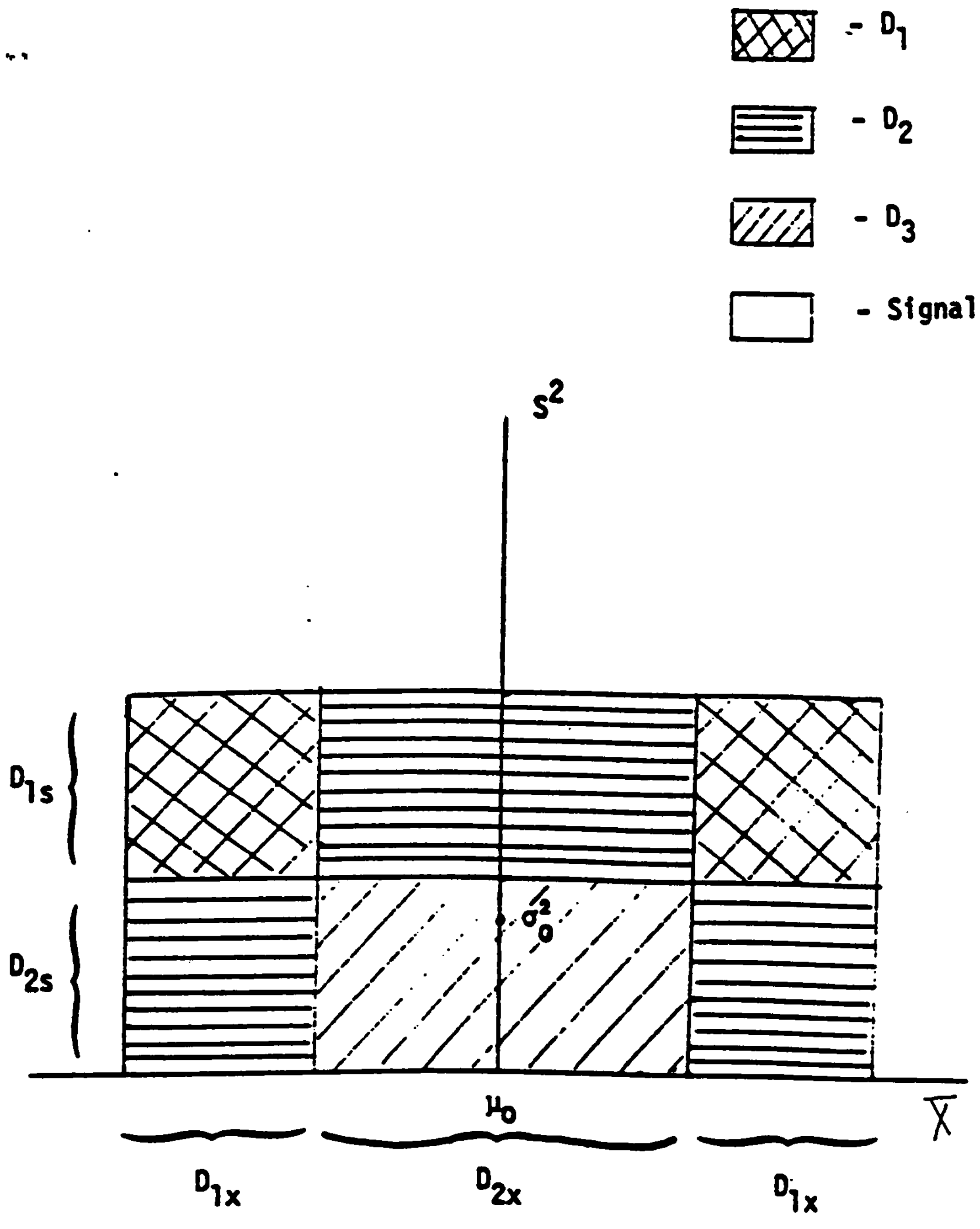


Figure 4. 8. Sampling Interval Regions for the VSI_{sep} procedure using three sampling intervals.

Table 4. 6. ATS* for VSI_{sep} using the sampling interval sets (.1,1,1.9), (.3, 1, 1.7) and (.5, 1, 1.5)

	σ/σ_0				
δ	1	1.1	1.25	1.5	2
		33.83	10.22	3.01	1.13
0		34.72	10.84	3.25	1.15
		35.63	11.48	3.51	1.18
	86.50	31.07	9.78	2.96	1.12
.25	86.64	31.95	10.39	3.20	1.14
	86.80	32.85	11.02	3.46	1.18
	59.61	24.54	8.61	2.81	1.11
.50	60.13	25.39	9.19	3.04	1.13
	60.66	26.26	9.78	3.29	1.17
	20.97	11.72	5.57	2.33	1.07
1	21.67	12.38	6.02	2.52	1.08
	22.39	13.06	6.49	2.74	1.11
	12.19	7.73	4.26	2.06	1.07
1.25	12.78	8.26	4.64	2.23	1.04
	13.38	8.80	5.03	2.42	1.07
	3.09	2.54	1.96	1.39	.95
2	3.30	2.74	2.13	1.48	.94
	3.52	2.96	2.32	1.59	.95

Table 4. 7. ATS for VSI_{sep} Using sampling interval sets (.1, 1.0, 1.9) and (.1, 1.9)

	σ/σ_0						
δ	1	1.1	1.25	1.5	2	3	4
0	100	34.00	10.27	2.92	.91	.48	.41
	100	32.00	8.75	2.14	.55	.25	.20
.10	97.69	33.54	10.20	2.91	.91	.48	.41
	97.58	31.54	8.68	2.13	.55	.25	.20
2.5	86.79	31.24	9.82	2.86	.91	.47	.41
	86.21	29.25	8.34	2.09	.55	.25	.20
.50	59.87	24.69	8.64	2.71	.89	.47	.41
	58.25	22.75	7.26	1.20	.54	.25	.20
1	21.11	11.78	5.54	2.20	.84	.47	.40
	18.74	10.15	4.47	1.57	.51	.25	.20
1.1	16.96	9.97	4.97	2.08	.83	.47	.40
	14.64	8.42	3.96	1.48	.50	.25	.20
1.25	12.28	7.75	4.20	1.91	.80	.46	.40
	10.09	6.31	3.28	1.35	.48	.24	.20
1.5	7.36	5.12	3.14	1.63	.76	.46	.40
	5.47	3.89	2.36	1.14	.46	.24	.20
2	3.04	2.43	1.79	1.16	.68	.45	.40
	1.74	1.53	1.20	.78	.41	.24	.20
3	.99	.87	.75	.62	.51	.42	.39
	.30	.34	.37	.37	.31	.23	.20
4	.63	.55	.47	.41	.39	.39	.38
	.13	.15	.18	.21	.23	.21	.20

Table 4. 8. ATS* for VSI_{sep} using sampling interval sets (.1, 1.0, 1.9) and (.1, 1.9)

	σ/σ_0						
δ	1	1.1	1.25	1.5	2	3	4
0		33.83	10.22	3.01	1.13	.76	.72
		32.08	9.01	2.59	1.16	.94	.91
.10	97.39	33.36	10.15	3.00	1.13	.76	.72
	97.49	31.61	8.95	2.59	1.16	.94	.91
.25	86.50	31.07	9.78	2.96	1.12	.76	.72
	86.14	29.33	8.61	2.55	1.16	.94	.91
.50	59.61	24.54	8.61	2.81	1.11	.76	.72
	58.22	22.87	7.56	2.44	1.15	.94	.91
1	20.96	11.72	5.57	2.33	1.07	.76	.72
	18.89	10.39	4.83	2.07	1.13	.94	.91
1.1	16.84	9.93	5.01	2.22	1.06	.76	.72
	14.83	8.69	4.34	1.99	1.12	.94	.91
1.25	12.19	7.73	4.26	2.06	1.04	.76	.72
	10.35	6.63	3.69	1.87	1.11	.93	.91
1.5	7.32	5.15	2.81	1.81	1.01	.76	.54
	5.83	4.28	3.25	1.68	1.09	.93	.91
2	3.09	2.54	1.96	1.39	.95	.75	.54
	2.27	2.06	1.75	1.37	1.05	.93	.91
3	1.12	1.07	1.01	.94	.83	.74	.53
	1.03	1.05	1.06	1.05	.98	.93	.91
4	.77	.78	.78	.78	.76	.73	.53
	.92	.93	.93	.94	.94	.92	.91

intervals when the shift in μ and/or σ is large. Table 4.13 gives the values of the adjusted ATS for the VSI_c procedure using the sets (.1, 1.1), (.1, 1.5) and (.1, 3.0). When the shifts in the mean and variance are 1.5 or less, using a wide set of sampling intervals is profitable. However, as the shifts get larger, the procedure using the set (0.1, 3.0) loses to the procedure using the set (0.1, 1.5), in terms of the adjusted ATS. The latter in turn loses to the procedure using the set (0.1, 1.1) when the shift in the mean is 3 or greater and the shift in the variance is 2 or greater. Table 4.14 presents the values of the adjusted ATS for the $VSI_{S_{ep}}$ procedures when sets of sampling intervals (0.1, 1.1), (0.1, 1.3) and (0.1, 1.5) are used. Again, the procedure using the set (0.1, 1.5) has the smallest ATS^* values for small to moderate shifts in the mean and variance. However, for values of $\mu > 2$ and $\sigma > 2$ the procedure using the set (0.1, 1.1) is the best.

4.2.4 Varying the Marginal Probabilities

With the overall signal probability held constant for separate charts, one could still vary the separate signal probabilities and the separate sampling interval probabilities corresponding to regions D_{ix} and D_{is}. In this study, the separate chart signal probabilities were held constant and equal. The separate chart procedures were then studied by varying sampling interval regions on the separate charts which enables us to place varying emphasis on the respective individual charts.

When the decision rule is to use the minimum of the sampling intervals assigned by the $\overline{X}$ and S^2 charts, the only time the long sampling interval is used is when both charts assign the long sampling interval. Since $P(D_2) = P(D_{2x}) \cdot P(D_{2s})$, different combinations of $P(D_{2x})$ and $P(D_{2s})$ can result in the same value of $P(D_2)$.

Even though the mean and variance are simultaneously monitored, there may be circumstances where it is desirable to monitor the mean more closely than the variance, or vice versa. This can easily be accomplished with the $VSI_{S_{ep}}$ procedure by varying the marginal probabilities of the indi-

Table 4. 9. ATS for VSI_c using the sets (.1, 1.9) and (.5, 1.5)

	σ/σ_0						
δ	1	1.1	1.24	1.5	2	3	4
0	100	31.07	9.20	2.80	1.04	.63	.57
	100	28.25	7.22	1.77	.50	.25	.21
.10	98.66	30.73	9.13	2.80	1.04	.63	.57
	98.57	27.91	7.16	1.76	.50	.25	.21
.25	92.01	29.04	8.75	2.72	1.03	.63	.57
	91.49	26.22	6.82	1.71	.49	.24	.21
.50	72.88	24.01	7.60	2.50	.99	.63	.57
	71.22	21.24	5.79	1.53	.46	.24	.20
1	34.28	12.88	4.75	1.85	.87	.61	.56
	31.15	10.48	3.31	1.05	.39	.23	.20
1.1	28.76	11.13	4.26	1.73	.84	.60	.56
	25.58	8.84	2.89	.95	.37	.22	.20
1.25	21.95	8.90	3.60	1.55	.80	.59	.56
	18.83	6.79	2.34	.83	.34	.22	.20
1.5	13.92	6.11	2.71	1.30	.74	.58	.56
	11.09	4.30	1.63	.64	.30	.21	.19
2	5.79	2.99	1.60	.94	.66	.56	.55
	3.79	1.72	.79	.40	.24	.19	.18
3	1.47	1.03	.76	.61	.55	.54	.54
	.57	.36	.25	.19	.16	.16	.18
4	.70	.61	.55	.52	.52	.52	.55
	.18	.15	.13	.13	.13	.14	.16

Table 4. 10. **ATS for VSI_{sep} using sampling interval sets (.1, 1.9) (.5, 1.5) and (.9, 1.1)**

	σ/σ_0						
δ	1	1.1	1.25	1.5	2	3	4
	100	32.00	8.75	2.14	.55	.25	.20
0	100	34.89	10.94	3.30	1.14	.65	.58
	100	37.77	13.13	4.47	1.72	1.05	.95
	97.58	31.54	8.68	2.13	0.55	.25	.20
.10	97.71	34.42	10.86	3.29	1.14	.65	.58
	97.83	37.30	13.05	4.46	1.72	1.05	.95
	86.21	29.25	8.34	2.09	.55	.25	.20
.25	86.91	32.12	10.49	3.25	1.13	.65	.58
	87.61	34.99	12.64	4.40	1.72	1.05	.95
	58.25	22.75	7.26	1.97	.54	.25	.20
.50	60.18	25.54	9.30	3.09	1.12	.65	.58
	62.10	28.33	11.34	4.22	1.70	1.05	.95
	18.74	10.15	4.47	1.57	.51	.25	.20
1	21.43	12.46	6.15	2.59	1.07	.64	.58
	24.11	14.76	7.83	3.60	1.63	1.04	.95
	14.64	8.42	3.96	1.48	.50	.24	.20
1.1	17.24	10.59	5.56	2.47	1.06	.64	.58
	19.83	12.77	7.16	3.46	1.62	1.04	.95
	10.09	6.31	3.28	1.35	.48	.24	.20
1.25	12.49	8.29	4.76	2.30	1.04	.64	.58
	14.89	10.27	6.23	3.24	1.59	1.04	.95
	5.47	3.89	2.36	1.14	.46	.24	.20
1.5	7.47	5.55	3.64	2.01	1.00	.64	.58
	9.47	7.21	4.93	2.89	1.54	1.04	.95
	1.74	1.52	1.20	.78	.41	.23	.20
2	3.04	2.67	2.17	1.52	.92	.63	.57
	4.35	3.82	3.13	2.27	1.43	1.03	.95
	.30	.34	.37	.37	.31	.23	.20
3	.94	.96	.97	.92	.77	.61	.57
	1.57	1.58	1.57	1.47	1.23	1.00	.94
	.13	.15	.17	.21	.23	.21	.20
4	.58	.60	.63	.66	.65	.59	.57
	1.02	1.05	1.06	1.11	1.07	.98	.94

Table 4. 11. ATS* for VSI_{sep} using the sampling interval sets (.1, 1.9), (.5, 1.5) and (.9, 1.1)

δ \ σ/σ_0	1	1.1	1.25	1.5	2	3	4
0		32.08	9.01	2.59	1.16	.94	.91
		34.61	10.76	3.23	1.15	.71	.65
		37.29	12.67	4.03	1.31	.64	.54
1	97.49	31.61	8.95	2.59	1.16	.94	.91
	97.34	34.14	10.69	3.22	1.15	.71	.65
	97.34	36.83	12.59	4.02	1.30	.64	.54
.25	86.14	29.33	8.61	2.55	1.16	.94	.91
	86.55	31.84	10.32	3.18	1.15	.71	.65
	87.11	34.51	12.18	3.97	1.30	.64	.54
.50	58.22	22.87	7.56	2.44	1.15	.94	.91
	59.84	25.29	9.14	3.03	1.14	.71	.65
	61.61	27.86	10.89	3.79	1.28	.64	.54
1	18.89	10.39	4.83	2.07	1.13	.94	.91
	21.19	12.27	6.03	2.54	1.09	.71	.65
	23.65	14.31	7.38	3.17	1.22	.64	.54
1.1	14.83	8.69	4.34	1.99	1.12	.94	.91
	17.02	10.42	5.45	2.43	1.08	.70	.65
	19.37	12.32	6.72	3.03	1.20	.63	.54
1.25	10.35	6.63	3.69	1.87	1.11	.93	.91
	12.31	8.15	4.66	2.26	1.06	.70	.65
	14.43	9.82	5.79	2.82	1.17	.63	.54
1.5	5.83	4.82	2.81	1.68	1.09	.93	.91
	7.35	5.44	3.57	1.99	1.03	.70	.65
	9.03	6.77	4.49	2.46	1.13	.63	.54
2	2.27	2.06	1.75	1.37	1.05	.93	.91
	3.01	2.65	2.15	1.53	.96	.69	.65
	3.92	3.39	2.71	1.85	1.02	.62	.54
3	1.03	1.05	1.06	1.05	.98	.96	.91
	1.02	1.03	1.03	.97	.82	.68	.64
	1.17	1.17	1.16	1.06	.82	.59	.54
4	.92	.93	.93	.94	.94	.92	.91
	.69	.71	.73	.74	.72	.66	.64
	.62	.65	.68	.70	.67	.57	.53

Table 4. 12. ATS for VSI_{sep} using the sampling interval sets (.1, 1.5) (.1, 1.9) and (.1, 4.0)

	σ/σ_0						
δ	1	1.1	1.25	1.5	2	3	4
	100	33.24	9.49	2.42	.65	.29	.24
0	100	32.00	8.75	2.14	.55	.25	.20
	100	29.84	7.57	1.73	.43	.20	.16
	97.64	32.77	9.42	2.42	.65	.29	.24
.10	97.58	31.54	8.68	2.13	.55	.25	.20
	97.50	29.37	7.50	1.72	.43	.20	.16
	86.51	30.45	9.06	2.38	.64	.29	.24
.25	86.21	29.25	8.34	2.09	.55	.25	.20
	85.77	27.17	7.20	1.70	.43	.20	.16
	59.03	23.86	7.92	2.24	.63	.29	.24
.50	58.25	22.75	7.26	1.20	.54	.25	.20
	57.07	20.93	6.24	1.59	.42	.20	.16
	19.70	10.92	4.95	1.80	.59	.29	.24
1	18.74	10.15	4.47	1.57	.51	.25	.20
	17.36	8.99	3.77	1.26	.40	.20	.16
	15.54	9.12	4.41	1.70	.58	.29	.24
1.1	14.64	8.42	3.96	1.48	.50	.25	.20
	13.37	7.39	3.32	1.19	.39	.19	.16
	10.87	6.91	3.67	1.55	.57	.29	.24
1.25	10.09	6.31	3.28	1.35	.48	.24	.20
	9.01	5.46	2.73	1.08	.38	.19	.16
	6.05	4.34	2.67	1.31	.54	.29	.24
1.5	5.47	3.89	2.36	1.14	.46	.24	.20
	4.69	3.28	1.93	.91	.36	.19	.16
	2.02	1.77	1.39	.91	.48	.28	.23
2	1.74	1.53	1.20	.78	.41	.24	.20
	1.38	1.22	.96	.62	.32	.19	.16
	.37	.41	.44	.44	.36	.27	.23
3	.30	.34	.37	.37	.31	.23	.20
	.24	2.7	.29	.30	.24	.18	.16
	.15	.17	.21	.25	.27	.25	.23
4	.13	.15	.18	.21	.23	.21	.20
	.12	.13	.15	.17	.19	.17	.16

Table 4. 13. ATS* for VSI_c using the sampling interval sets (.1, 1.1), (.1, 1.5) and (.1, 3.0)

	σ/σ_0						
δ	1	1.1	1.25	1.5	2	3	4
		31.93	9.44	2.71	.94	.60	.56
0		29.30	7.92	2.25	.97	.76	.73
		27.46	7.34	2.55	1.57	1.42	1.41
	98.26	31.58	9.36	2.69	.93	.60	.56
.10	98.34	28.96	7.85	2.24	.97	.76	.73
	98.92	27.12	7.28	2.54	1.57	1.42	1.41
	91.86	29.87	8.96	2.62	.92	.60	.56
.25	91.48	27.26	7.50	2.18	.96	.76	.73
	91.57	25.47	6.98	2.50	1.57	1.42	1.41
	73.36	24.77	7.75	2.38	.89	.59	.56
.50	71.78	22.25	6.43	2.00	.94	.75	.73
	70.66	20.61	6.04	2.35	1.55	1.42	1.41
	35.39	13.34	4.75	1.72	.78	.58	.56
1	32.39	11.34	3.87	1.51	.87	.74	.73
	30.10	10.31	3.85	1.98	1.5	1.41	1.40
	29.84	11.52	4.23	1.60	.76	.58	.55
1.1	26.83	9.67	3.43	1.42	.86	.74	.73
	24.62	8.79	3.49	1.91	1.49	1.41	1.40
	22.92	9.19	3.54	1.42	.73	.57	.55
1.25	20.04	7.56	2.87	1.30	.84	.74	.73
	18.06	6.91	3.02	1.81	1.48	1.41	1.40
	14.62	6.24	2.61	1.18	.68	.56	.55
1.5	12.17	4.99	2.14	1.12	.81	.74	.73
	10.73	4.68	2.44	1.68	1.46	1.41	1.40
	5.99	2.93	1.48	.85	.61	.55	.55
2	4.57	2.30	1.30	.90	.76	.73	.73
	4.19	2.51	1.80	1.53	1.43	1.40	1.40
	1.35	.93	.71	.60	.55	.55	.54
3	1.16	.93	.81	.75	.73	.73	.73
	1.68	1.54	1.46	1.42	1.40	1.40	1.40
	.66	.60	.57	.55	.55	.55	.54
4	.78	.75	.74	.73	.73	.73	.73
	1.44	1.42	1.41	1.40	1.40	1.40	1.40

Table 4. 14. ATS* for VSI_{sep} Listed in Order Corresponding to Sampling Interval Sets (.1, 1.1), (.1, 1.3) and (.1, 1.5)

	σ/σ_0						
δ	1	1.1	1.25	1.5	2	3	4
		35.84	11.28	3.22	1.03	.61	.56
0		34.05	10.07	2.78	.98	.68	.65
		33.10	9.52	2.64	1.03	.76	.74
	97.32	35.36	11.20	3.21	1.03	.61	.56
.10	97.32	33.58	10.00	2.77	.98	.68	.65
	97.36	32.63	9.45	2.63	.103	.76	.74
	86.80	33.02	10.81	3.17	1.02	.61	.56
.25	86.41	31.25	9.63	2.73	.98	.68	.65
	86.25	30.32	9.19	2.60	1.02	.76	.74
	60.63	26.31	9.56	3.00	1.01	.61	.56
.50	59.42	24.61	8.46	2.59	.97	.68	.65
	58.81	23.76	7.98	2.47	1.02	.76	.74
	22.04	12.83	6.22	2.47	.96	.61	.56
1	20.41	11.52	5.40	2.14	.94	.68	.65
	19.63	10.93	5.08	2.06	.99	.76	.74
	17.77	10.89	5.60	2.35	.95	.61	.56
1.1	16.22	9.69	4.84	2.04	.93	.68	.65
	15.50	9.16	4.55	1.97	.98	.76	.74
	12.88	8.48	4.75	2.17	.93	.61	.56
1.25	11.51	7.44	4.08	1.89	.91	.68	.65
	10.90	7.00	3.84	1.83	.97	.76	.74
	7.65	5.60	3.57	1.88	.90	.60	.56
1.5	6.60	4.80	3.05	1.65	.89	.68	.65
	6.17	4.50	2.88	1.61	.94	.76	.73
	2.96	2.57	2.04	1.39	.83	.60	.56
2	2.46	2.17	1.76	1.26	.84	.67	.65
	2.31	2.06	1.71	1.26	.90	.76	.73
	.84	.87	.88	.83	.70	.59	.56
3	.82	.84	.86	.83	.74	.66	.65
	.88	.90	.91	.89	.82	.75	.73
	.58	.59	.61	.63	.62	.57	.56
4	.65	.66	.67	.69	.68	.65	.64
	.74	.75	.76	.77	.77	.74	.73

vidual charts. That is, varying $P(D_{1x})$ and $P(D_{1s})$ while holding the overall signal probability at the constant value desired to control the false alarm rate. Table 4.15 illustrates the effect on the *ATS* of varying $P(D_{2s})$ using values of .994987, .8 and .497493. The first value corresponds to using only the $\overline{X}$ chart to assign overall sampling intervals and the last value corresponds to using only the S^2 chart. Of course any value between these extremes could be used. The effect is that if $P(D_{2s})$ is larger the chart detects shifts in μ faster and shifts in σ slower. However, either chart can still signal an out of control process.

This strategy was also studied in terms of the adjusted *ATS* . Some of the results are presented in Table 4.16. The values of the marginal probability of the S^2 chart was set at $P(D_{2s}) = 0.497493$, .6 and .703562. Note that the last strategy gives equal weight to the mean and the variance in terms of sampling intervals. The second strategy detects shifts in the variance faster since it corresponds to $P(D_{2x}) = 0.825$. This can be seen from Table 4.16 where the sampling interval used is (0.1, 1.9). The first strategy has also been studied in terms of *ATS* in Table 4.15. It is best at detecting shifts in the variance, whereas the strategy with $P(D_{2s}) = .703562$ is best at detecting shifts in the mean.

4.3 Comments

Four basic procedures for simultaneously monitoring the mean and variance for a normally distributed process were investigated. Fixed sampling intervals were used in two procedures and variable sampling intervals in the remaining two procedures. The primary measures of comparison were the average time to signal *(ATS)* and the adjusted average time to signal *(ATS*)*. The procedures were compared when their *ATS* values were equal when $\theta = \theta_0$ and their *ANSS* values were equal for all θ.

Table 4. 15. ATS for VSI_{sep} Using (.1, 1.9) and Listed in Order of P (D_{2s}) values of .994987, .8 and .497493

	σ/σ_0						
δ	1	1.1	1.25	1.5	2	3	4
	100	35.92	11.67	3.63	1.24	.68	.59
0	100	32.73	9.19	2.31	.61	.28	.23
	100	31.10	8.87	2.21	.59	.27	.22
	97.51	35.37	11.58	3.61	1.24	.68	.59
.10	97.55	32.25	9.12	2.30	.61	.28	.23
	97.87	31.70	8.81	2.21	.59	.27	.22
	85.80	32.71	11.10	3.55	1.23	.68	.59
.25	86.01	29.86	8.76	2.26	.61	.28	.23
	87.78	29.77	8.54	2.18	.59	.27	.22
	57.15	25.17	9.59	3.32	1.21	.67	.59
.50	57.72	23.12	7.60	2.13	.60	.28	.23
	62.58	24.20	7.68	2.09	.58	.27	.22
	17.44	10.78	5.73	2.60	1.12	.67	.58
1	18.10	10.12	4.62	1.69	.56	.27	.23
	24.79	12.79	5.35	1.79	.56	.27	.22
	13.44	8.84	5.04	2.43	1.09	.66	.58
1.1	14.05	8.35	4.08	1.59	.54	.27	.23
	20.48	11.10	4.90	1.72	.55	.27	.22
	9.07	6.52	4.11	2.19	1.06	.66	.58
1.25	9.59	6.22	3.36	1.44	.53	.27	.23
	15.49	8.98	4.28	1.62	.55	.27	.22
	4.74	3.89	2.88	1.81	1.00	.65	.58
1.5	5.11	3.78	2.39	1.20	.50	.27	.22
	9.97	6.36	3.40	1.45	.53	.27	.22
	1.40	1.43	1.38	1.19	.86	.63	.58
2	1.57	1.44	1.19	.82	.44	.26	.22
	4.67	3.42	2.19	1.14	.49	.27	.22
	.24	.29	.38	.50	.60	.59	.56
3	.27	.31	.36	.38	.33	.25	.22
	1.73	1.45	1.11	.75	.42	.26	.22
	.12	.13	.17	.25	.41	.53	.55
4	.13	.14	.17	.21	.24	.23	.22
	1.13	.97	.77	.57	.37	.25	.22

Table 4. 16. **ATS* for VSI_{sep} using (.1, 1.9) and listed in order of P (D_{2s}) values of .497493, .6 and .703562**

δ	σ/σ_0 = 1	1.1	1.25	1.5	2	3	4
0		32.09	9.05	2.58	1.10	.86	.84
		31.61	8.74	2.48	1.12	.90	.88
		32.08	9.01	2.59	1.16	.94	.91
.10	97.69	31.70	9.00	2.57	1.10	.86	.84
	97.53	31.17	8.68	2.48	1.12	.90	.88
	97.49	31.61	8.95	2.59	1.16	.94	.91
.25	87.61	29.76	8.72	2.54	1.10	.86	.84
	86.50	29.00	8.36	2.45	1.11	.90	.88
	86.14	29.33	8.61	2.55	1.16	.94	.91
.50	62.41	24.20	7.86	2.45	1.09	.86	.84
	59.24	22.82	7.38	2.34	1.11	.90	.88
	58.22	22.87	7.56	2.44	1.15	.94	.91
1	24.61	12.78	5.52	2.16	1.07	.86	.84
	20.12	10.70	4.81	2.00	1.08	.90	.88
	18.89	10.39	4.83	2.07	1.13	.94	.91
1.1	20.31	11.10	5.08	2.09	1.07	.86	.84
	15.97	9.02	4.34	1.93	1.08	.90	.88
	14.83	8.69	4.34	1.99	1.12	.94	.91
1.25	15.32	8.97	4.46	1.98	1.06	.86	.84
	11.32	6.96	3.71	1.81	1.07	.90	.88
	10.35	6.63	3.69	1.87	1.11	.93	.91
1.5	9.80	6.35	3.58	1.81	1.04	.86	.84
	6.52	4.57	2.85	1.64	1.05	.90	.88
	5.83	4.28	2.81	1.68	1.09	.93	.91
2	4.50	3.42	2.37	1.50	1.00	.86	.84
	2.55	2.21	1.79	1.34	1.01	.90	.88
	2.27	2.06	1.75	1.37	1.05	.93	.91
3	1.55	1.44	1.29	1.11	.94	.85	.83
	1.04	1.05	1.05	1.02	.95	.89	.88
	1.03	1.05	1.06	1.05	.98	.96	.91
4	.96	.96	.95	.93	.88	.84	.83
	.89	.90	.91	.92	.91	.89	.88
	.92	.93	.93	.94	.94	.92	.91

The relative performances of the procedures are much easier to assess when the comparison is based upon the ATS. For this comparison, variable sampling with two widely spaced intervals is uniformly and substantially better than the fixed sampling interval procedures. However, from a practical point of view, the adjusted time to signal is a more realistic comparison though not as straightforward.

In general, both of the variable sampling procedures (VSI_{Sep} and VSI_C) have a substantially smaller ATS^* for small to moderate shifts in either μ or σ. For quite large shifts, the ATS^* for fixed sampling intervals is smaller than for variable sampling intervals but the difference is not of a large magnitude, since for such a shift both procedures have a very small ATS^*. In applying the VSI to either strategy, several decisions have to be made. One is the number of sampling intervals to be used. This was investigated using sets of three and two sampling intervals and it was found that using only two sampling intervals was preferable to using three. Fortunately, this was found to be true for sum as well as separate charts.

Another decision to be made is the width of the sampling interval. It was found that wider is always better in terms of ATS. However, in terms of ATS^*, very wide sampling intervals can cause the FSI procedures to beat the VSI procedures for large shifts in μ or σ. One must use some judgment concerning the longer sampling interval. For general performance, a reasonable size for d_2 is obtained when d_1 and d_2 are chosen to be approximately symmetric about the usual fixed sampling interval d. Here the longest sampling interval is only twice the usual fixed sampling interval and yet substantially smaller values of the ATS^* are obtained for small and moderate shifts in the process. If one is quite concerned about large shifts, the sampling intervals (.1, 1.5) seem to be a good compromise to behave quite well for all values of μ and σ.

When comparing the two variable sampling interval procedures (VSI_{Sep} and VSI_C) neither one is clearly better than the other. The VSI_{Sep} procedure, which uses separate charts, performs very well and is quite flexible. By varying the marginal sampling interval probabilities one can cause the procedure to be relatively more sensitive to either μ or σ while still allowing either chart to signal.

Using separate charts for each parameter also has the advantage that it immediately suggests the parameters most likely out of control.

The single chart based on the VSI_C procedure tends to have a smaller ATS^* than the VSI_{Sep} procedure except when σ is on target and the shift is in the mean only. It has the advantage of requiring that only one control chart be monitored. If a signal is obtained, the quantities U_t and V_t could be used as diagnostic tools to determine which parameter most likely shifted.

If one is in the position that monitoring the mean is of primary interest but a large shift in σ is also of interest, then a reasonable choice would be to use VSI_{Sep} with the marginal sampling interval probabilities chosen so that a signal can be obtained from either chart but the sampling intervals are based only on $\overline{X}$. This would enable one to monitor the mean with high power for detection of shifts and yet also signal for large shifts in the variance.

In summary, if one expects small to moderate shifts in either μ or σ and wants to detect them quickly, then any VSI procedure is superior to the corresponding FSI procedure. When the shift is large, choosing a VSI procedure with the appropriate set of sampling intervals will still lead to quicker detection. Hence, if the general nature of the shift is known, using the appropriate VSI procedure will give better results.

Chapter V

V. Monitoring Multiple Means

Chapter Four concentrated on the problem of monitoring the mean and variance simultaneously using VSI procedures. It dealt with a univariate, multiparameter problem. This chapter considers the problem of measuring several variables, and hence is a multivariate, multiparameter problem.

Let the quality variables of interest be $\underline{X} = (X_1, X_2, \ldots, X_m)$ and $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_m)$. It is assumed that the underlying distribution of X is multivariate normal with mean $\underline{\mu}$ and known dispersion matrix Σ . Any change in the process which indicates a deterioration in quality is assumed to be reflected by a change in $\underline{\mu}$. The target value of the mean is represented by $\underline{\mu}_0$. Thus the problem under consideration is that of monitoring $\underline{\mu}$ when Σ is known and constant.

Assume that the process is monitored by taking a random sample at each sampling point. The statistic computed from the ith sample is represented by $\underline{\overline{X}}^{(i)} = \{\overline{X}_1^{(i)}, \overline{X}_2^{(i)}, \ldots, \overline{X}_m^{(i)}\}$. If the chart signals after a sample is observed, then either the process is stopped or it continues onto the next sample while an assignable cause is sought. It is also assumed that a Shewhart-type chart is being used, so that the decision made after a sample is taken depends only on the present sample and not on any previous samples.

5.1 Separate Charts and a Chi-Squared Chart

There are two possible ways of using the statistic $\overline{X}$ to monitor $\underline{\mu}$. One obvious way would be to use m separate charts based on $\overline{X}_1$, $\overline{X}_2$, ..., $\overline{X}_m$, respectively. Here the cost of maintaining several charts is offset by the fact that the variable that is out of control is easily located. The problem of maintaining several charts could be overcome by combining the information into a single univariate statistic. The statistic $Y = m(\overline{X} - \underline{\mu}_0)'\Sigma^{-1}(\overline{X} - \underline{\mu}_0)$ has a chi-squared distribution with m degrees of freedom. If there is a shift in any of the means, the statistic will have a non-central chi-squared distribution with non- centrality parameter $\lambda = m(\underline{\mu} - \underline{\mu}_0)'\Sigma^{-1}(\underline{\mu} - \underline{\mu}_0)$. The control limits for a chart based on this statistic would be set at $\{0, \chi^2_{m,1-\alpha}\}$. One disadvantage to this method is that if the procedure signals, it is not immediately clear which means are out of control.

If the VSI feature is to be applied using m separate charts, there are several signal functions and sampling interval functions that are viable. In this chapter, the following two signal functions are studied:

i) The procedure would signal if any of the individual charts signalled.

ii) The signal function is set up using a score function that depends on the sampling interval regions.

The above two procedures will be explained in turn. Consider the signal function (i). If $\alpha = P$(signal for the control procedure), then the probability that any of the m charts signal is denoted by α_m where $\alpha_m = 1 - (1 - \alpha)^{1/m}$. The control limits for the jth chart would be set at $\{\mu_j - z_{1-\alpha_m}\sigma_{jj}/\sqrt{n}\,, \mu_j + z_{1-\alpha_m}\sigma_{jj}/\sqrt{n}\,\}$, where σ_{jj} is the jth diagonal element of the matrix Σ .

If only two sampling intervals are used, the in- control region within each chart is subdivided into regions D_{1i} and D_{2i} as defined at (3.2.1). If the jth statistic falls in region D_1 , a sampling interval of length d_1 is assigned. As before, a decision rule has to be set up to specify the sampling interval to be used in order to avoid conflicts. The sampling interval function should be a function of the intervals of the separate charts. One choice for a decision rule is to use the minimum of all of the sampling intervals. This is a conservative strategy, since it uses the long sampling interval only if all the charts assigned the long sampling interval. This is referred to as the S1 strategy. A variation of this strategy would be to use a procedure with the same signal function that assigned the short sampling interval only if at least two charts assign the short sampling interval. This is referred to as the S2 strategy. In general, the S_h strategy uses the sampling interval d_1 if and only if at least h charts specify the short sampling interval. Note that each of the above procedures would have to use different D_l as defined in (3.1.3) in order to yield the same ATS in control.

Signal function (ii) was expected to make the signal function for the procedure using separate charts more sensitive. For a set of m charts, let δ_{ij} be the score for the jth chart at the ith sample. δ_{ij} is a function of the sample statistic and can take on values of 0, 1 or m, as follows:

$$\begin{aligned} \delta_{ij} &= 0 \quad \text{if} \quad \overline{X}_j^{(i)} \; \varepsilon \; D_{2j} \\ \delta_{ij} &= 1 \quad \text{if} \quad \overline{X}_j^{(i)} \; \varepsilon \; D_{2j} \\ \delta_{ij} &= m \quad \text{if} \quad \overline{X}_j^{(i)} \; \varepsilon \; (D_{2j} \cup D_{1j})^c \end{aligned}$$

Define δ_i to be $\sum_{j=1}^{m} \delta_{ij}$, the total score at the ith sample. Thus if all the charts assign the short sampling interval or if any of the charts signal, the sum δ_i takes on a value of at least m. The following decision rule was used:

$$\begin{array}{ll} \delta_i \geq m & \text{signal} \\ 0 < \delta_i < m & \text{use } d_1 \\ \delta_i = 0 & \text{use } d_2 \end{array}$$

Thus this procedure is simply the S1 strategy with a modified signal function. Let $p_0 = P\{\delta_{ij} = 0\}$, $p_1 = P\{\delta_{ij} = 1\}$ and $p_2 = P\{\delta_{ij} = m\}$. As the number of variables increase, p_0 has to increase to achieve the desired probability $P[D_2]$. Once this probability is fixed, the probability of a signal can be expressed as a polynomial function of p_2 of degree m. This can be solved for p_2 if the overall signal probability is fixed.

Consider the simple case for $m = 3$. If a set of three charts is being used, the procedure would signal if all the three charts assigned the short sampling interval or if any of the charts signalled. The probability associated with the former event is p_1^3, and the probability of the latter is $1-(1-p_2)^3$. Since the two events are disjoint, the probability that the overall procedure signals is

$$\begin{aligned} P(\text{signal}) &= 1-(1-p_2)^3 + p_1^3 \\ &= 1-(1-p_2)^3 + (1-p_0-p_2)^3, \quad \text{since } p_0+p_1+p_2 = 1 \end{aligned}$$

But $p_0 = P_{\mu_0}[D_2]^{1/3}$. Hence, by fixing the overall signal probability and the probability of assigning the long sampling interval, when the process is in control, the polynomial can be solved in terms of p_2.

This technique can be generalized to signal if any b out of the k charts assign the short sampling interval. Obviously, b cannot be less than two, or else the long sampling interval would never be used.

5.2 A Combination Approach

Lucas (1982) added Shewhart limits to cusum charts and thereby combined the key features of both procedures. The cusum procedure uses information from the past samples and hence is quick to detect small shifts. The addition of Shewhart control limits modified the procedure so it could detect a large shift from the mean, for any single observation. Similarly, the chi-squared chart condenses the information from all the charts but may not be sensitive to a shift in only one mean. So the chi-squared chart was combined with the separate charts in order to improve the performance of both procedures. The procedure that was studied had a signal function that depended on the chi-squared chart as well as the separate Shewhart charts and a sampling interval function that depended on the chi-squared chart alone. Hence, such a procedure would signal if any of the separate charts or the chi-squared chart signalled but would pick the sampling interval based only on the chi-squared chart.

Note that the chi-squared statistic is not independent of the m statistics upon which the separate charts are based, even if the individual variables are themselves independent. Hence, when using the combination approach, an upper bound was found for the probability of an overall signal as follows:

$$\begin{aligned} P(\text{overall signal}) &= P(\text{at least one of the separate charts signals or the chi-squared chart signals}) \\ &\leq P(\text{at least one of the separate charts signals}) + P(\text{the chi-squared chart signals}) \end{aligned}$$

If the probability of a signal for the combined procedure is fixed at α , the probability of a signal for a chi-squared chart can be taken to be $\alpha/2$. Further, by setting the probability of a signal for any of the separate charts at α_m, it can be seen that

$$\alpha/2 = 1 - (1 - \alpha_m)^m \text{ or } \alpha_m = 1 - (1 - \alpha/2)^{1/m} \tag{5.1}$$

when the m variables are assumed to be independent.

5.3 *Methodology and Results*

A sampling interval of length unity was used for all the FSI charts and this forced $E(R_i) = 1$ for all the VSI charts. The m variables were assumed to be independent and for convenience, the covariance matrix Σ was taken to be the identity matrix. The set of sampling intervals was restricted to a set of two sampling intervals symmetric about the fixed sampling interval. This greatly reduces the complexity of the VSI procedure and can be partially justified by recalling that a set of two sampling intervals spaced as far apart as possible was found to lead to the optimum procedure, in terms of ATS, for Shewhart $\overline{X}$ and cusum charts by Reynolds et al (1987) and Amin et al (1988), respectively. In terms of adjusted ATS, the best procedure also used two sampling intervals, but reduced the length of the longer sampling interval.

The theorem stated in Section 3.2 holds true for the case of monitoring the means of several variables when the variables in question are independent. Since optimal procedures have been found to exist for univariate $\overline{X}$ charts, it is expected that numerically, at least, the *VSI* feature should result in an increase in efficiency. In a summary of the previous sections, the basic procedures studied in this chapter are:

1) maintaining m separate charts, one for each parameter. This is the typical approach used in practice, but the properties are difficult to evaluate when the statistics used in the separate charts are correlated. Several variations on this procedure, using different sampling intervals and/or signal functions are investigated.

2) combining the information in the sample into a chi-squared statistic (since the variances are assumed to be known).

3) The combined procedure was studied through simulation, since its properties could not be evaluated exactly.

The shift in the means was measured in terms of the resulting non- centrality parameter. The values of the non-centrality parameter that were picked for this study ranged from .05 to 50. The performance of the separate charts depends on the value of μ in addition to the non-centrality parameter. In order to cover various ways in which the shifts could take place, the following types of shifts were considered:

i. One mean shifted and the other means remained on target.

ii. Equal shifts in all the means.

iii. Equal shifts in approximately half the means and no shifts in the rest.

iv. There was a linear shift in all the means (i.e., if the first chart shifted by g units, then the second was assumed to shift by 2g units, the third by 3g, etc).

The S1, S2 and S3 strategies were applied to sets of variables ranging from two to twenty-five. All three strategies were compared to using the chi-squared statistic. Although the adjusted ATS was considered to be a more realistic measure of performance, most of the comparisons were done in terms of both the ATS as well as the adjusted ATS.

Table 5.1 shows the results of a study involving a set of two variables. The values of the ATS for the chi-squared, S1 and S2 procedures are displayed. The S3 strategy could not be used since only two variables are involved. The table includes ten different values of the non-centrality parameter and three types of shifts in the means. The VSI feature enhanced both the chi-squared and the separate charts procedures. The chi-squared chart did consistently better than the S1 strategy when all the means experienced the same shift or if the shift was linear. But if the shift was in only one of the means, the S1 strategy did better than the chi-squared chart for larger values of the non-centrality parameter (larger than one). This holds for both the ATS as well as the adjusted ATS as can be seen from the next table. Table 5.2 presents the values of the adjusted ATS for the chi-squared, S1 and S2 strategies for the same set of two variables. Again the VSI feature showed a uniform improvement over the FSI procedure. It was hoped that the S2 strategy would increase the advantages of the S1 strategy. Unfortunately, its performance was found to be uniformly worse as compared to both the S1 as well as the chi-squared charts. This is true in terms of the ATS and the adjusted ATS and can be seen from Table 5.1 and Table 5.2 respectively.

This is further substantiated by studying Table 5.3 and Table 5.4 which display the values of the ATS and the adjusted ATS respectively, for a set of five charts with eight different values of the non-centrality parameter. The adjusted ATS resulting from a shift occurring in three out of five means was found to be close to those resulting from a linear shift. Again, the chi-squared chart was found to be more sensitive for small and moderate shifts, and it was especially good at detecting a small shift spread over all the means. The S3 strategy was found to perform even worse than the S2 strategy. These conclusions were substantiated by studying the ATS and the adjusted ATS values for a set of 10 variables with eight different non-centrality parameters (refer to Table 5.5 and Table 5.6 respectively). It can also be seen that the separate charts procedure is markedly more efficient when the shift is in one mean only. When the number of variables was increased to twenty-five, the same pattern was observed for the chi-squared, S1, S2 and S3 strategies, in terms of both the ATS and the adjusted ATS. This is shown in Tables 5.7 and 5.8.

Table 5. 1. ATS values for the chi-squared (X) and separate (Sep) charts using FSI (F), VSI (V) and modifying the sampling interval function (S2), when m = 2.

NCP	Shift	X (F)	X (V)	Sep (F)	S1	S2
0	one			100	100	100
	all	100	100	100	100	100
	linear			100	100	100
.1	one			81.57	79.25	80.21
	all	80.80	78.47	81.76	79.54	80.49
	linear			81.76	79.43	80.39
0.2	one			68.05	64.25	65.85
	all	67.09	63.30	68.89	65.04	66.59
	linear			68.59	64.75	66.32
.3	one			57.78	53.02	55.05
	all	56.88	52.14	59.17	54.29	56.23
	linear			58.66	53.83	55.80
0.5	one			43.39	37.64	40.13
	all	42.78	37.04	45.62	39.54	41.89
	linear			44.79	38.83	41.24
1	one			24.79	18.74	21.46
	all	24.76	18.65	27.84	21.00	23.48
	linear			26.65	20.12	22.69
1.5	one			16.22	10.75	13.29
	all	16.44	10.82	19.27	12.70	14.92
	linear			18.04	11.91	14.26
5	one			3.47	1.06	2.34
	all	3.66	1.10	4.80	1.41	2.11
	linear			4.20	1.25	2.17
20	one			1.05	.11	.66
	all	1.06	.11	1.15	.12	.14
	linear			1.10	.12	.22
50	one			1	.10	.63
	all	1	.10	1	.10	.10
	linear			1	.10	.11

Table 5. 2. **ATS* values for chi-squared(X) and separate(Sep) charts using FSI (F) and VSI (V) procedures and modifying the sampling interval function (S2), when m = 2.**

NCP	Shift	X (F)	X (V)	Sep(F)	S1	S2
	one			89.52	88.65	89.18
0.5	all	89.02	88.14	88.74	88.74	89.27
	linear			88.71	88.71	89.24
	one			81.07	79.18	80.14
.10	all	80.30	78.41	81.37	79.47	80.41
	linear			81.26	79.37	80.31
	one			67.55	64.21	65.78
0.2	all	66.59	63.26	68.39	65.00	66.53
	linear			68.09	64.71	66.26
	one			57.28	53.01	55.00
0.3	all	56.38	52.13	58.67	54.23	56.18
	linear			58.16	53.81	55.75
	one			42.89	37.67	40.11
.5	all	42.28	37.08	45.12	39.58	41.88
	linear			44.29	38.87	41.22
	one			24.29	18.89	21.5
1	all	24.26	18.80	27.34	21.15	23.54
	linear			26.15	20.27	22.75
	one			15.72	10.99	13.38
1.5	all	15.94	11.07	18.77	12.94	15.05
	linear			17.54	12.16	14.38
	one			2.97	1.66	2.57
5	all	4.40	2.30	4.30	2.02	2.58
	linear			3.70	1.86	2.56
	one			.55	.91	.94
20	all	.69	.93	.65	.92	.92
	linear			.60	.92	.93
	one			.50	.91	.91
50	all	.50	.91	.50	.91	.91
	linear			.50	.91	.91

Table 5. 3. **ATS values for the chi-squared(X) and separate(Sep) charts using FSI (F) and VSI (V) procedures and modifying the sampling interval function (S2 and S3), when m = 5**

NCP	Shift	X(F)	X(V)	S1	S2	S3
	one			100	100	100
0	all	100	100	100	100	100
	three			100	100	100
	linear			100	100	100
	one			54.67	55.42	56.34
.5	all	58.33	52.66	58.87	59.35	60.26
	three			58.14	58.66	59.58
	linear			58.19	58.71	59.63
	one			31.46	32.61	33.76
1	all	38.20	31.15	38.6	39.15	40.36
	three			37.23	37.89	39.11
	linear			37.32	37.97	39.18
	one			25.70	19.27	21.71
1.5	all	26.91	19.84	35.74	26.94	28.71
	three			33.63	25.32	27.26
	linear			33.75	25.42	27.34
	one			18.31	12.48	14.86
2	all	19.95	13.33	28.52	19.60	21.25
	three			26.18	17.96	19.81
	linear			26.31	18.05	19.89
	one			1.97	2.94	3.50
5	all	6.02	2.32	4.58	4.65	5.32
	three			3.73	4.01	4.75
	linear			3.78	4.05	4.76
	one			.13	.55	.72
20	all	1.17	.13	.21	.21	.22
	three			.18	.17	.23
	linear			.17	.18	.23
	one			.10	.5	.66
50	all	1	.10	.10	.10	.10
	three			.10	.10	.10
	linear			.10	.10	.10

Table 5. 4. **ATS* values for the chi-squared(X) and separate(Sep) charts using FSI(F) and VSI(V) procedures and modifying the sampling interval function (S2 and S3), when m = 5**

NCP	Shift	X(F)	X(V)	Sep(F)	S1	S2	S3
	one			80.80	78.20	78.55	79.04
.2	all	78.64	75.91	82.04	79.41	79.68	80.17
	three			81.84	79.21	79.49	79.98
	linear			81.85	79.22	79.51	80.00
	one			59.66	54.66	55.41	56.31
0.5	all	57.83	52.66	64.23	58.86	59.33	60.23
	three			63.43	58.13	58.65	59.55
	linear			63.48	58.18	58.7	59.60
	one			37.62	31.54	32.66	33.78
1	all	37.70	31.24	46.13	38.68	39.21	40.40
	three			44.51	37.31	37.96	39.14
	linear			44.61	37.39	38.04	39.22
	one			25.20	19.43	20.67	21.77
1.5	all	26.41	20.01	35.24	27.10	27.58	28.81
	three			33.13	25.48	26.11	27.35
	linear			33.25	25.57	26.20	27.44
	one			17.81	12.70	13.96	14.96
2	all	19.45	13.56	28.02	19.82	20.22	21.41
	three			25.68	18.18	18.77	19.96
	linear			25.81	18.27	18.85	20.04
	one			4.42	2.48	3.05	3.69
5	all	5.52	2.84	10.79	5.08	5.14	5.75
	three			8.76	4.24	4.48	5.14
	linear			8.85	4.28	4.52	5.15
	one			.59	.92	.95	.96
20	all	.67	.92	1.41	1.01	1.00	1.01
	three			1.00	.96	.96	.98
	linear			1.03	.96	.97	.98
	one			.50	.91	.91	.91
50	all	.50	.91	.52	.91	.91	.91
	three			.50	.91	.91	.91
	linear			.51	.91	.91	.91

Table 5. 5. **ATS values for the chi-squared(X) and separate(Sep) charts using FSI(F) and VSI(V) procedures and modifying the sampling interval function (S2 and S3), when m = 10**

NCP	Shift	X(F)	X(V)	Sep(F)	S1	S2	S3
	one			100	100	100	100
0	all	100	100	100	100	100	100
	five			100	100	100	100
	linear			100	100	100	100
	one			71.93	67.35	67.45	68.36
.5	all	69.06	64.03	76.92	72.17	71.99	72.20
	five			76.35	71.62	71.48	71.78
	linear			76.52	71.78	71.62	71.90
	one			50.13	43.80	44.21	46.09
1	all	50.07	43.02	61.79	54.39	54.06	54.37
	five			60.34	53.06	52.83	53.41
	linear			60.75	53.44	53.18	53.68
	one			35.31	28.71	29.38	29.38
1.5	all	37.67	29.99	51.15	42.24	41.80	41.80
	five			48.93	40.32	40.04	40.04
	linear			49.54	40.86	40.53	40.53
	one			25.56	19.31	20.14	22.55
2	all	29.19	21.52	43.28	33.54	33.01	33.33
	five			40.49	31.27	30.96	32.00
	linear			41.25	31.88	31.51	32.34
	one			6.50	3.15	4.00	5.57
5	all	9.47	4.45	20.69	11.00	10.35	10.50
	five			17.02	8.89	8.62	10.12
	linear			17.90	9.40	9.04	10.08
	one			2.20	.57	1.12	2.22
10	all	3.27	.81	9.52	2.84	2.46	2.48
	five			6.79	1.93	1.82	2.95
	linear			7.36	2.13	1.96	2.64
	one			1.38	.24	.66	2.07
15	all	1.85	.30	5.60	1.05	.89	.88
	five			3.72	.66	.62	1.42
	linear			4.08	.74	.67	1.48

Table 5. 6. **ATS* values for the chi-squared(X) and separate(Sep) charts using FSI(F) and VSI(V) procedures and modifying the sampling interval function (S2 and S3) when m = 10**

NCP	Shift	X(F)	X(V)	Sep(F)	S1	S2	S3
	one			93.66	92.87	92.85	92.95
0.1	all	91.94	90.98	93.77	93.19	93.15	93.21
	five			93.93	93.15	93.12	93.18
	linear			93.94	93.16	93.13	93.19
	one			71.43	67.31	67.42	68.32
0.5	all	68.56	64.01	76.42	72.13	71.96	72.17
	five			75.85	71.58	71.44	71.74
	linear			76.02	71.74	71.59	71.87
	one			49.63	43.83	44.23	46.08
1	all	49.57	43.07	61.29	54.41	54.09	54.39
	five			59.84	53.08	52.86	53.43
	linear			60.25	53.46	53.21	53.70
	one			34.81	28.81	29.45	31.72
1.5	all	37.17	30.09	50.65	42.32	41.88	42.21
	five			48.43	40.40	40.13	40.95
	linear			49.04	40.94	40.62	41.29
	one			25.06	19.46	20.25	22.57
2	all	28.69	21.69	42.78	33.67	33.14	33.46
	five			39.99	31.40	31.10	32.11
	linear			40.75	32.02	31.65	32.46
	one			6.00	3.57	4.29	5.62
5	all	8.97	4.89	20.19	11.38	10.75	10.90
	five			16.52	9.27	9.02	10.43
	linear			17.40	9.78	9.44	10.43
	one			1.70	1.22	1.52	2.11
10	all	2.77	1.47	9.02	3.44	3.11	3.12
	five			6.29	2.55	2.45	3.42
	linear			6.86	2.74	2.60	3.18
	one			.88	.97	1.09	1.48
10	all	1.35	1.04	5.10	1.77	1.64	1.63
	five			3.22	1.39	1.36	1.94
	linear			3.58	1.46	.141	1.72

Table 5. 7. ATS values for the chi-squared(X) and separate(Sep) charts using FSI(F) and VSI(V) procedures and modifying the sampling interval function (S2 and S3) when m = 25.

NCP	Shift	X(F)	X(V)	Sep(F)	S1	S2	S3
	one			100	100	100	100
0	all	100	100	100	100	100	100
	twelve			100	100	100	100
	linear			100	100	100	100
	one			83.99	80.88	80.80	81.36
.5	all	80.09	76.26	88.06	85.04	84.63	84.49
	twelve			87.88	84.86	84.43	94.36
	linear			87.94	84.91	84.49	84.40
	one			66.16	61.06	61.86	62.73
1	all	65.06	58.93	78.43	73.13	72.38	72.14
	twelve			77.85	72.56	71.80	71.76
	linear			78.02	72.73	71.97	71.87
	one			38.09	31.92	34.12	34.90
2	all	44.51	36.41	63.84	55.49	54.27	53.87
	twelve			62.32	54.06	52.92	53.05
	linear			62.75	54.47	53.33	53.28
	one			9.52	5.69	8.03	8.30
5	all	17.78	10.61	39.35	27.67	25.92	25.30
	twelve			35.84	24.91	24.12	24.47
	linear			36.78	25.65	24.67	24.63
	one			2.78	.95	2.79	2.67
10	all	6.23	2.22	22.18	10.96	9.44	8.90
	twelve			17.97	8.53	9.30	9.08
	linear			18.97	9.12	9.31	8.83
	one			1.58	.34	2.55	2.06
15	all	3.16	.73	14.47	5.09	4.06	3.71
	twelve			10.66	3.47	4.96	4.34
	linear			11.47	3.83	4.61	3.92

Table 5. 8. ATS* values for the chi-squared(X) and separate(Sep) charts using FSI(F) and VSI(V) procedures and modifying the sampling interval function (S2 and S3) when m = 25

NCP	Shift	X(F)	X(V)	S(F)	S1	S2	S3
	one			93.64	92.72	92.54	92.68
.2	all	90.84	89.50	94.39	93.50	93.32	93.27
	twelve			94.36	93.46	93.29	93.24
	linear			94.37	93.47	93.30	93.25
	one			83.49	80.83	80.75	81.29
0.5	all	79.59	76.21	87.56	84.98	84.57	84.44
	twelve			87.38	84.80	84.37	84.31
	linear			87.44	84.85	84.43	84.35
	one			65.66	61.05	61.83	62.69
1	all	64.56	58.93	77.93	73.10	72.37	72.13
	twelve			77.35	72.53	71.78	71.75
	linear			77.52	72.70	71.96	71.86
	one			37.59	31.99	34.13	34.88
2	all	44.01	36.5	63.33	55.52	54.33	53.93
	twelve			61.82	54.10	52.97	53.10
	linear			62.25	54.51	53.38	53.34
	one			9.02	6.00	8.09	8.33
5	all	17.28	10.92	38.85	27.87	26.16	25.56
	twelve			35.34	25.12	24.35	24.69
	linear			36.28	25.86	24.90	24.86
	one			2.28	1.51	2.69	2.61
10	all	5.73	2.77	21.68	11.37	9.92	9.40
	twelve			17.47	8.96	9.69	9.48
	linear			18.47	9.54	9.72	9.27
	one			1.08	1.03	1.84	1.66
15	all	2.66	1.40	13.97	5.64	4.68	4.36
	twelve			10.16	4.05	5.40	4.84
	linear			10.98	4.40	5.11	4.49
	one			.71	.94	1.59	1.38
20	all	1.53	1.08	9.75	3.28	2.72	2.54
	twelve			6.52	2.30	3.68	3.06
	linear			7.16	2.50	3.29	2.71

The procedure using the score δ_i to determine the signal function was investigated using sets of five and ten variables and the sampling intervals (.1, 1.9). Table 5.9 presents the values of the ATS when five variables are used and $b = 4$. This is referred to as the MS(4/5) strategy. As explained previously, the MS(4/5) strategy would produce an additional signal if any four out of the five charts assigned a short sampling interval. It can be seen that when using the VSI procedure, the ATS for such a procedure is much less than the ATS of even the chi-squared procedure, once the non-centrality parameter gets to be moderately large (above one half). Table 5.10 presents the ATS values for a set of ten charts with $b = 8$, the MS(8/10) strategy. Thus an additional signal will be produced if any eight or more charts assign the short sampling interval. Again this signal function is better than the S1 strategy when the shift in the non-centrality parameter is moderate to large. Table 5.11 gives the values of the adjusted ATS for the MS(4/5) strategy. This strategy improved the FSI_t procedure also.

Table 5. 9. **ATS values for the chi-squared(X) and separate(sep) charts using the the FSI(F) and VSI(V) procedures and modifying the signal function (MS), when m = 5.**

NCP	Shift	X(F)	X(V)	Sep(F)	S1	MS(4/5)
	one			81.30	78.26	77.28
.2	all	79.14	75.96	82.54	79.46	78.96
	half			82.34	79.26	78.68
	linear			82.35	79.28	78.70
	one			60.16	54.67	52.03
.5	all	58.33	52.66	64.73	58.87	57.85
	half			63.93	58.14	56.86
	linear			63.98	58.19	56.92
	one			38.12	31.46	27.80
1	all	38.20	31.15	46.63	38.60	37.19
	half			45.01	37.23	35.35
	linear			45.11	37.32	35.44
	one			4.92	1.97	1.86
5	all	6.02	2.32	11.29	4.58	3.80
	half			9.26	3.73	2.99
	linear			9.35	3.78	3.07
	one			1.88	.39	.39
10	all	2.26	.44	4.68	.94	.72
	half			3.55	.71	.6
	linear			3.60	.72	.62
	one			1.09	.13	.13
20	all	1.17	.13	1.09	.21	.17
	half			1.50	.18	.16
	linear			1.53	.17	.16

Table 5. 10. **ATS values for the chi-squared(X) and separate(sep) using the FSI(F) and VSI(V) procedures and modifying the signal function (MS), when m = 10.**

NCP	Shift	X(F)	X(V)	Sep(F)	S1	MS(8/10)
	one			88.35	86.09	86.09
0.2	all	85.65	83.11	89.46	87.20	87.20
	half			89.33	87.08	87.08
	linear			89.37	87.12	87.12
	one			71.93	67.35	67.35
0.5	all	69.06	64.03	76.92	72.17	72.17
	half			76.35	71.62	71.62
	linear			76.52	71.78	71.78
	one			50.13	43.80	43.79
1	all	50.07	43.02	61.79	54.39	54.39
	half			60.34	53.06	53.06
	linear			60.75	53.44	53.44
	one			6.47	3.15	3.14
5	all	9.47	4.45	20.69	11.00	11.00
	half			17.02	8.89	8.89
	linear			17.90	9.40	9.40
	one			2.20	.57	.57
10	all	3.27	.81	9.51	2.84	2.84
	half			6.78	1.93	1.93
	linear			7.35	2.13	2.12
	one			1.13	.15	.15
20	all	1.36	.17	3.74	.52	.52
	half			2.44	.32	.32
	linear			2.67	.36	.36

Table 5. 11. **ATS* values for the chi-squared(X) and separate(Sep) charts using FSI (F) and VSI (V) procedures & modifying the signal function (MS), when m = 10.**

NCP	Shift	C(F)	C(V)	S(F)	S(V)	MS(F)	MS(4/5)
	one			59.66	54.66	56.71	52.02
0.5	all	57.83	52.66	64.23	58.86	63.09	57.85
	three			63.43	58.13	62.01	56.85
	linear			63.48	58.18	62.07	56.91
	one			37.62	31.54	33.07	27.88
1	all	37.70	31.24	46.13	38.68	44.39	37.26
	three			44.41	37.31	42.19	35.43
	linear			44.61	37.39	42.29	35.51
	one			17.81	12.70	14.25	10.39
2	all	19.45	13.56	28.02	19.82	25.78	18.32
	three			25.68	18.18	22.78	16.25
	linear			25.81	18.27	22.90	16.34
	one			4.42	2.48	4.06	2.35
5	all	5.52	2.84	10.79	5.08	8.70	4.29
	three			8.76	4.24	6.73	3.49
	linear			8.85	4.28	6.93	3.56
	one			1.38	1.09	1.42	1.09
10	all	1.76	1.15	4.18	1.65	2.89	1.41
	three			3.05	1.42	2.32	1.29
	linear			3.10	1.43	2.47	1.32
	one			.59	.92	.6	.92
20	all	.67	.92	1.41	1.01	.92	.95
	three			1.00	.96	.94	.95
	linear			1.03	.96	.91	.95

To evaluate the combination approach, a simulation study was run for a set of ten variables. Since the S1 strategy worked best when all the shift was in one of the means only and the chi-squared chart was good at detecting a small shift spread over all the means, these two procedures were compared to the combination approach. It was found that the four categories of shift considered fell into two distinct groups, the first one containing the case where all the shift was in one of the means only and the remaining three categories clumped to form the second group. Hence the number of categories studied were reduced to two, namely, the shift taking place in one or in all of the means.

The control limits on the individual charts were set so that the overall ANSS in control was approximately one hundred. Samples were drawn from a multivariate normal distribution and the number of signals was recorded. This is simply a run of Bernoulli trials with $p =$ P(signal). The variance associated with the estimate of p is pq/n . Since p is very small, a very large sample size is required to estimate it accurately. If $n = 10^6$ the standard deviation of an estimate of p is of the order of 10^{-4}. Hence, to be reasonably confident of the results, a million samples were drawn each time.

Since only an upper bound could be found for the signal probability of the combination approach, an initial value of $\alpha = .01$ was used and the value of α_m was calculated from equation (5.1). The value of α sets the control limit for the chi-squared chart. The probability α_m is required to set the control limits for the separate charts such that the $ANSS$ of the combination approach is 100 in control. This was achieved through iteration by experimenting with different values of α. Once the value of α that yielded an $ANSS$ of 100 was obtained, the ATS and the ATS^* could be calculated. Since the sampling interval function is a function of the chi-squared chart alone, the cut-off point for the region that assigned the long sampling interval was set at $\chi^2_{m,.495}$ when the process was in control. Using this, the ATS and the adjusted ATS values were obtained.

Next simulations were run for different values of the non-centrality parameter. Table 5.12 shows that this procedure strikes a balance between the separate charts and the chi- squared chart. The

ATS for the combination approach is always less than the ATS of at least one of the other two procedures. Thus if the variables can be assumed to be independent, and one is unaware of the nature of the shift that will take place, the safest procedure to use, among those studied here, is the combination approach.

5.4 The Effect of Correlations

So far, it had been assumed that the k variables on which the charts are based are independent. This is not always a realistic assumption. Hence the underlying distribution of the quality varibles was changed by assuming that the covariance matrix was different from identity. Then the effect on the performances of the three basic procedures listed in section 5.3 was studied. The study used two different approaches. They were:

i. There was correlation in the data, but the charts were set up as if the variables were independent. Hence the robustness of these procedures to correlation was actually studied.

ii. There was correlation in the data, and this was incorporated into the chi-squared statistic. Hence it was assumed that the nature of the correlation was known. There is no practical way to use the correlation in the separate charts, but it can be used in the combination approach through the chi-squared statistic.

For approach (i), a set of ten variables was used and a million samples were drawn for each value of the shift. Such a large number of replications were required not only to guarantee precision, but also to be able to detect small differences between procedures. All the variances were assumed to be one, and the sampling interval (.1, 1.9) was used throughout. A set of six different correlations

Table 5. 12. Simulation results from comparing a combination(com) of the chi-squared and S1 procedures.

NCP	Type	ATS C(V)	ATS S1	ATS Com	ATS* C(V)	ATS* S1	ATS* Com
.2	One	85.57	86.09	84.20	85.48	86.02	84.12
	Ten		87.20	84.99		87.13	84.92
.5	One	65.99	67.35	64.04	65.95	67.32	64.02
	Ten		72.17	66.88		72.13	66.85
2	One	22.29	19.31	19.25	22.43	19.46	19.40
	Ten		33.54	24.42		33.67	24.59
5	One	4.64	3.15	3.43	5.06	3.57	3.84
	Ten		11.00	5.33		11.38	5.77
10	One	.85	.57	.67	1.49	1.22	1.28
	Ten		2.84	.93		3.44	1.59
20	One	.18	.15	.18	.95	.92	.93
	Ten		.52	.18		1.29	.96

was studied, where two out of the ten variables were assumed to have pairwise correlations of -.75, -.50, -.25, 0.25, 0.50, and 0.75, respectively.

Figure 5.1 graphs the adjusted ATS against correlation when there is no shift and it can be seen that the adjusted ATS is roughly the same for equal negative and positive correlations for all three procedures. Further, the chi-squared chart seems to be affected the most by large correlations. When there is a shift of 0.5 in the mean of the variable that is correlated with another, Figure 5.2 shows the same results. The adjusted ATS was also plotted against correlation when there is a shift of 2.0 spread over all the means. Figure 5.3 shows that the graph is symmetric about a correlation of zero though the differences between the three procedures is reduced. Since negative correlations seem to give the same values of ATS as positive correaltions and because the dispersion matrix has to be non-negative definite, the types of correlations used in further investigations were restricted. The following cases were studied:

a) Five of the ten variables were assumed to all have pairwise correlations of 0.25, 0.50 and 0.75 respectively.

b) All of the ten variables were assumed to have pairwise correlations of 0.25, 0.50 and 0.75 respectively.

When there is a shift in all ten means, and the variables are still assumed to be independent, the graph of adjusted ATS against shift is shown in Figure 5.4. The correlation was increased to 0.25 for five out of ten variables, and the shift was restricted to one mean only, where the mean corresponded to one of the correlated variables. For this situation, the ATS values are graphed against shift in Figure 5.5. Figure 5.6 plots the values of the adjusted ATS against shift when there is a correlation of 0.25 among five of the ten variables and the shift is spread over all the means. When the three basic procedures are compared it can be seen that the chi-squared chart has a high false alarm rate, when the variables are assumed to be independent. The combined chart is also similarly affected, but the separate charts seem relatively robust.

When the pairwise correlations are further increased to 0.50 for five out of the ten variables, Figure 5.7 plots the values of the ATS against shift for the three basic procedures when the shift is in one mean only, and the mean corresponds to a correlated variable. Figure 5.8 displays the graph of the adjusted ATS plotted against shift when the shift is in all the means and the correlation is 0.5 for five variables. When the variables are assumed to be independent, the performance of the chi-squared and the combined charts is further worsened. This degree of correlation affects even the separate charts by making them relatively insensitive to shifts in the means.

Increasing the pairwise correlations to 0.75 for five out of ten variables affected the three basic procedures in much the same way, but to a greater extent. This can be seen from Figure 5.9 which displays the graph of ATS values plotted against shift when the shift is in one mean only, again, corresponding to a correlated variable. Figure 5.10 plots the same graph, but uses the adjusted ATS values instead of the ATS values.

The above conclusions were substantiated by looking at the same sets of pairwise correlations among all the ten variables. Figure 5.11 plots the values of the ATS when the shift is in all the means and the correlation is 0.50. Figure 5.12 studies the effect on the ATS when the shift is in one mean only and shows that the performance of the separate charts procedure improves dramatically. Next the effect of a correlation of 0.50 on the adjusted ATS is displayed in Figure 5.13, when the shift is in all the means. Finally, Figure 5.14 displays the graph of the adjusted ATS when the shift is in the mean of a correlated variable and the correlation is 0.50.

If the nature of the correlation is known, it can be incorporated in the chi-squared statistic. The properties of the separate charts become extremely complicated and so the correlation was not used while calculating the ATS or the adjusted ATS for the separate charts. For approach (ii), Figure 5.15 displays the graph of the ATS plotted against shift when there is a correlation of 0.50 in five variables and the shift is spread over all the means. The chi-squared chart that incorporates the correlation does not suffer from a high false alarm rate and is clearly better than the three basic procedures. This is borne out by studying the same graph, but using the adjusted ATS values instead, as shown in Figure 5.16. The combination approach can also profit from using the correlation structure in the chi-squared statistic. Figure 5.17 displays the graph of ATS plotted against shift when there is a correlation of 0.25 in five variables and the shift is in all the means. Both the chi-squared and the combined charts are greatly improved. They perform better than the separate charts while still maintaining an $ANSS$ value of 100 when the process is in control. Figure 5.18 shows the same graph, but uses the adjusted ATS values. It can be seen that the chi-squared chart works best when it can incorporate the correlations.

5.5 Comments

The problem of monitoring several parameters simultaneously is a common one. Several different approaches can be used. A few reasonable ones were studied in terms of their ATS and adjusted ATS. The best procedure to use in any given situation depended on the kind of shift that the process was expected to experience. It was found that using separate charts led to an advantage when the shift was in one of the means only. In general, for uncorrelated variables, it was found that using a procedure whose sampling interval function depended only on the chi-squared chart but signalled if any of the individual charts signalled, was the best among all the procedures examined. This procedure worked reasonably well no matter what form the shift took, if the variables being monitored were independent.

When the variables are correlated, none of the three basic procedures studied were found to be very robust. The combined chart which performed well for independent variables had a high false alarm rate. The separate charts were found to be resistant to false alarms when the amount of correlation among the variables was moderate, but became less sensitive to shifts in the mean as the correlation increased. However, when correlation was incorporated into the chi- squared chart and hence the combination approach, they both did much better than the separate charts.

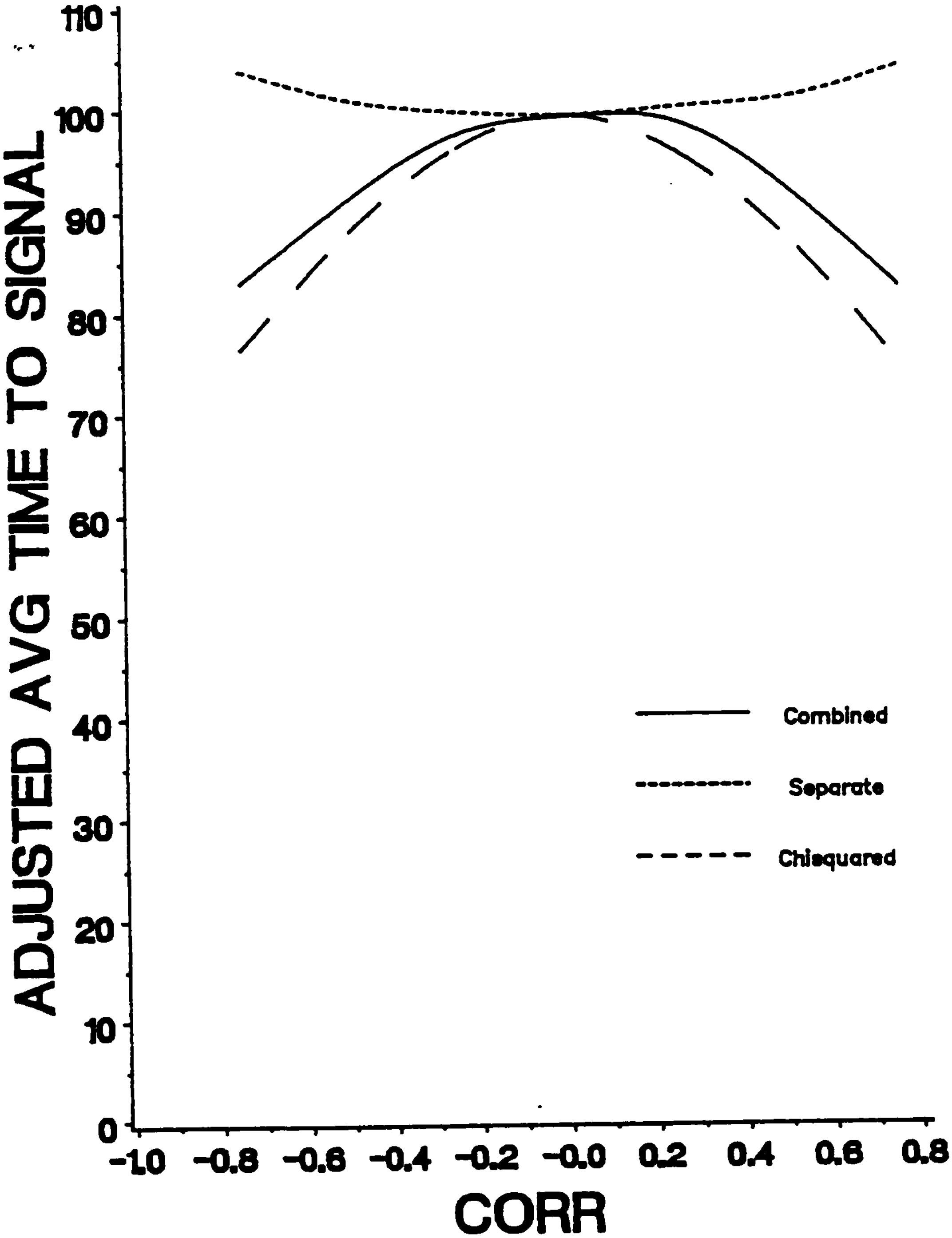


Figure 5. 1. Graph of the adjusted ATS versus correlation when shift = 0.

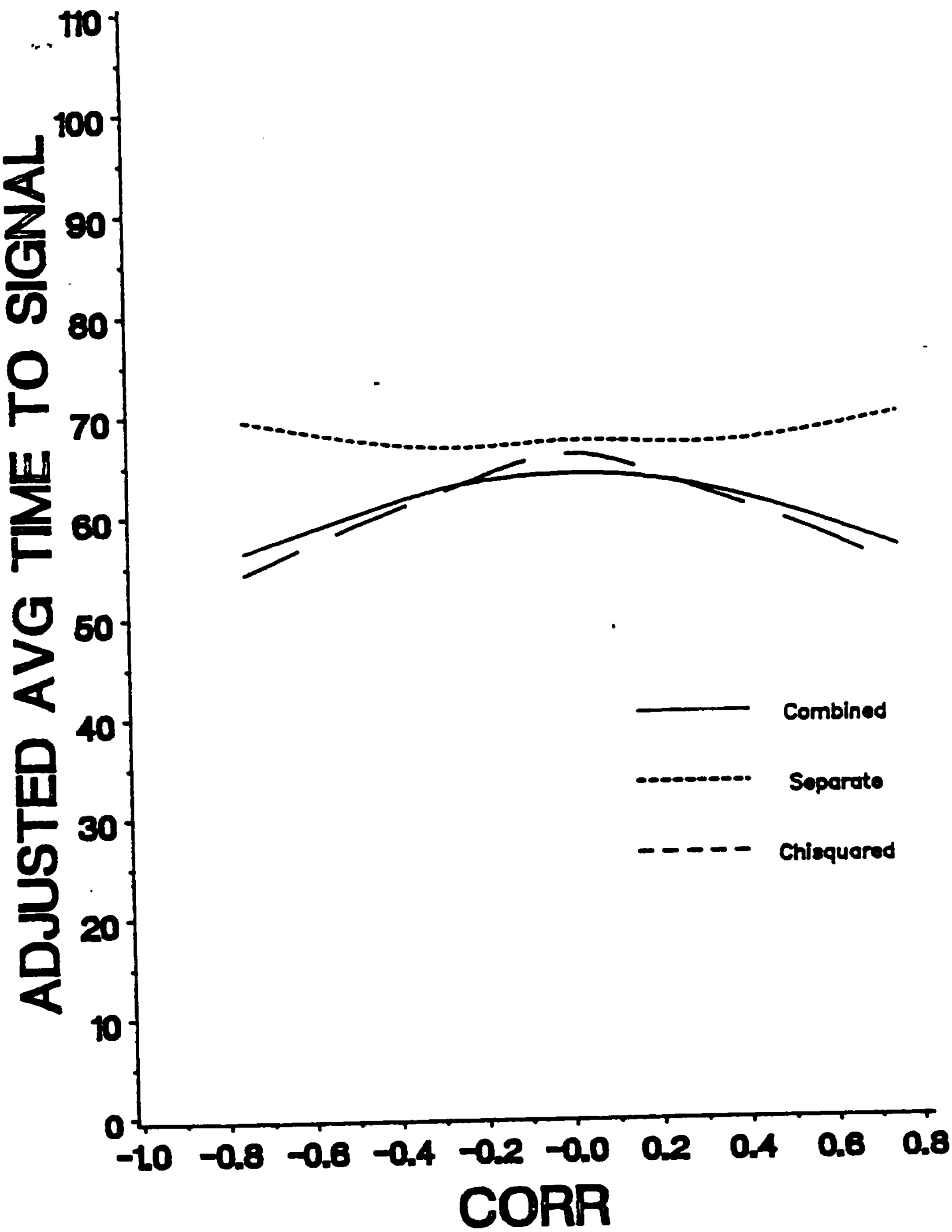


Figure 5. 2. Graph of ATS* versus correlation when shift = 0.5 in one mean only.

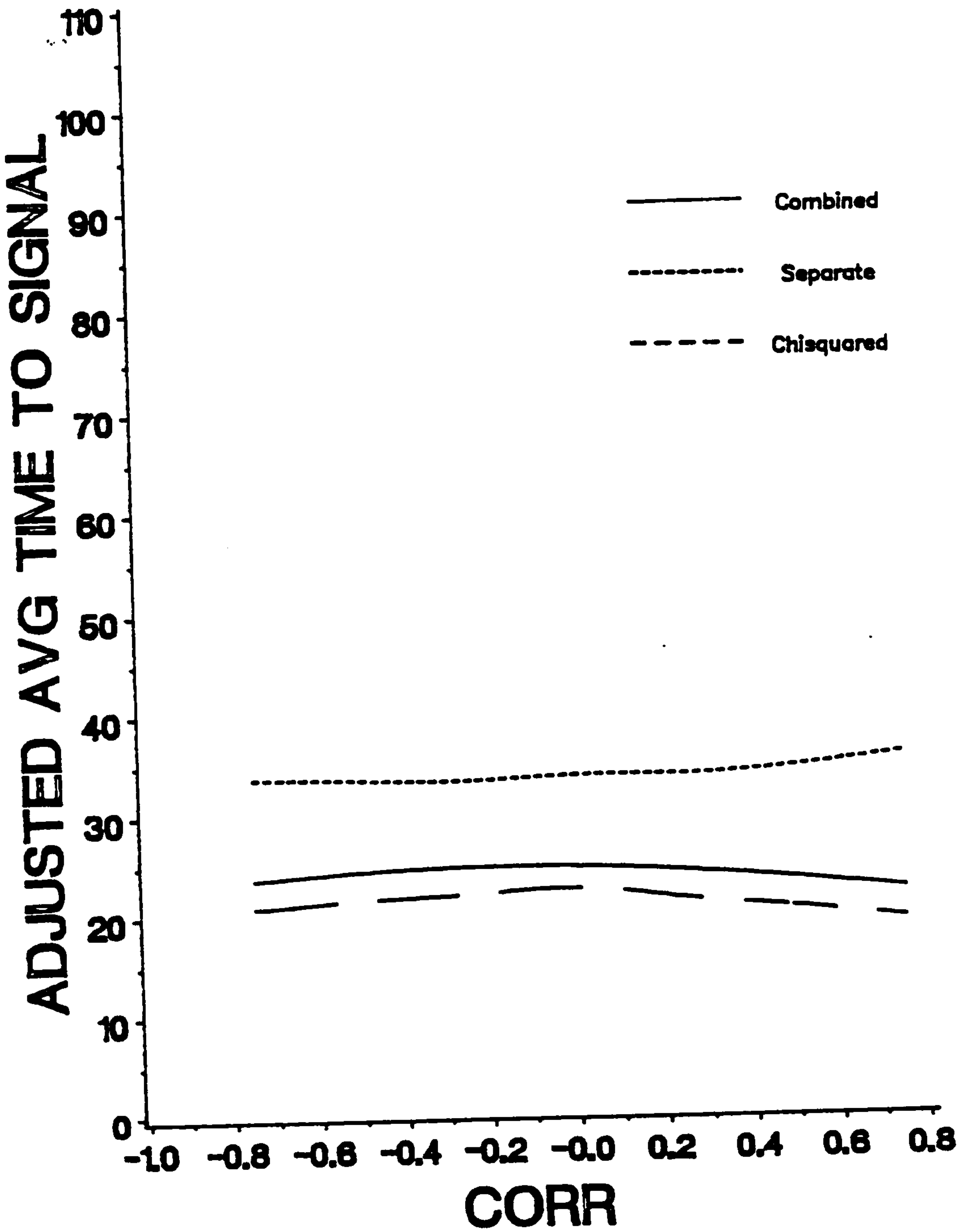


Figure 5. 3. Graph of ATS* versus correlation when shift = 2 and spread over all the means.

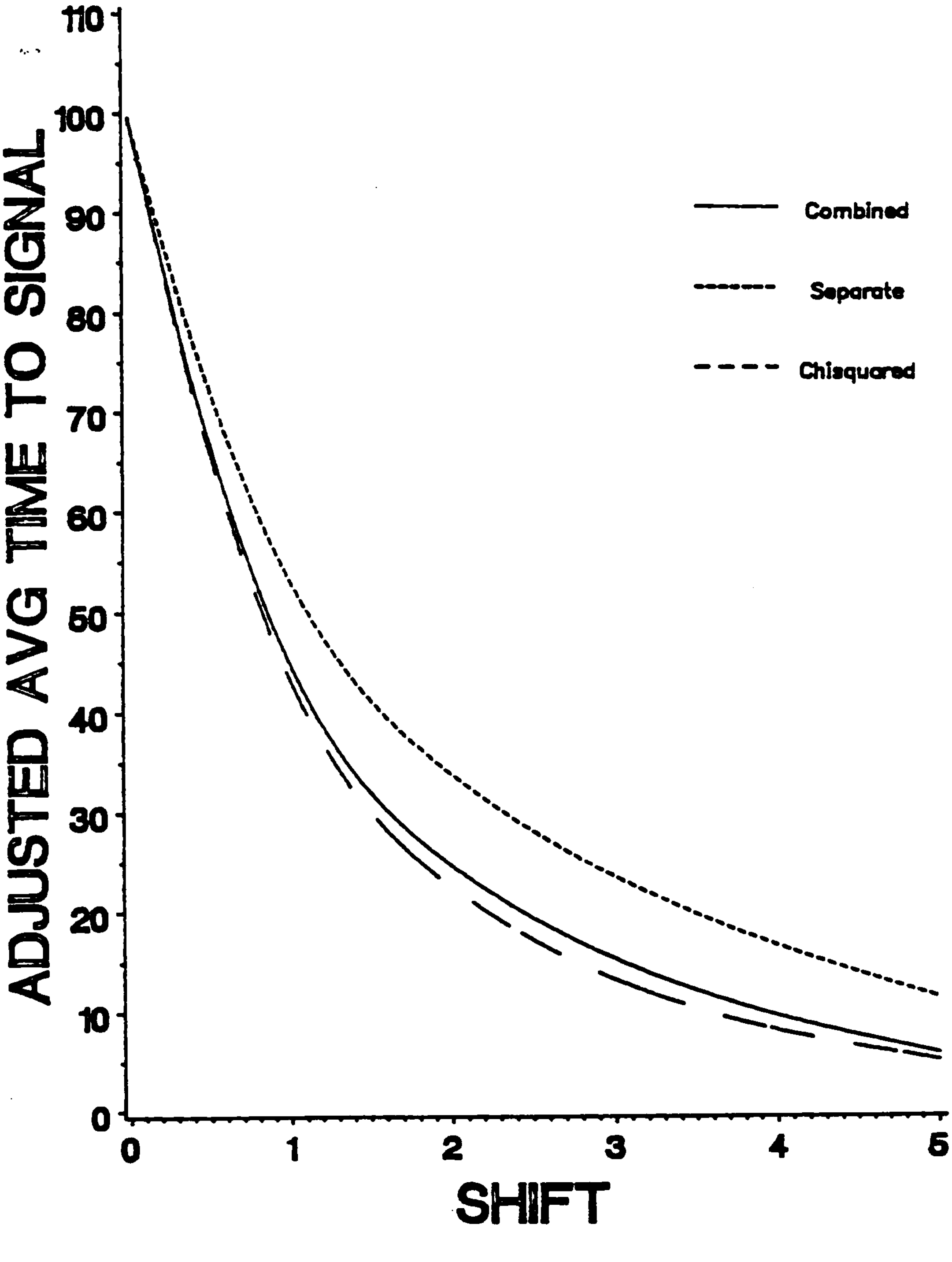


Figure 5. 4. Graph of ATS* and shift in all means when correlation = 0.

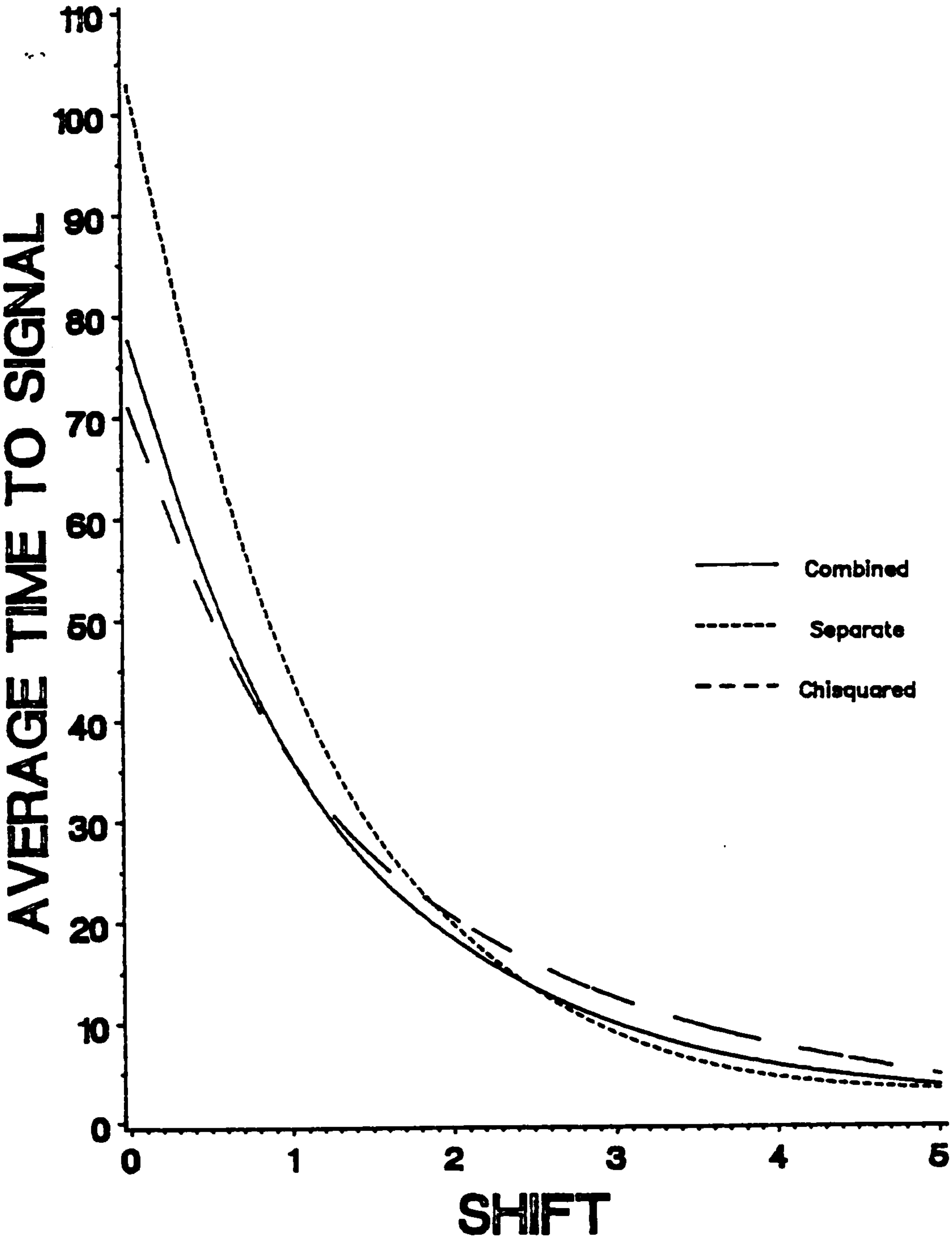


Figure 5. 5. Graph of ATS and shift in one mean when corr = .25 in five of the ten variables.

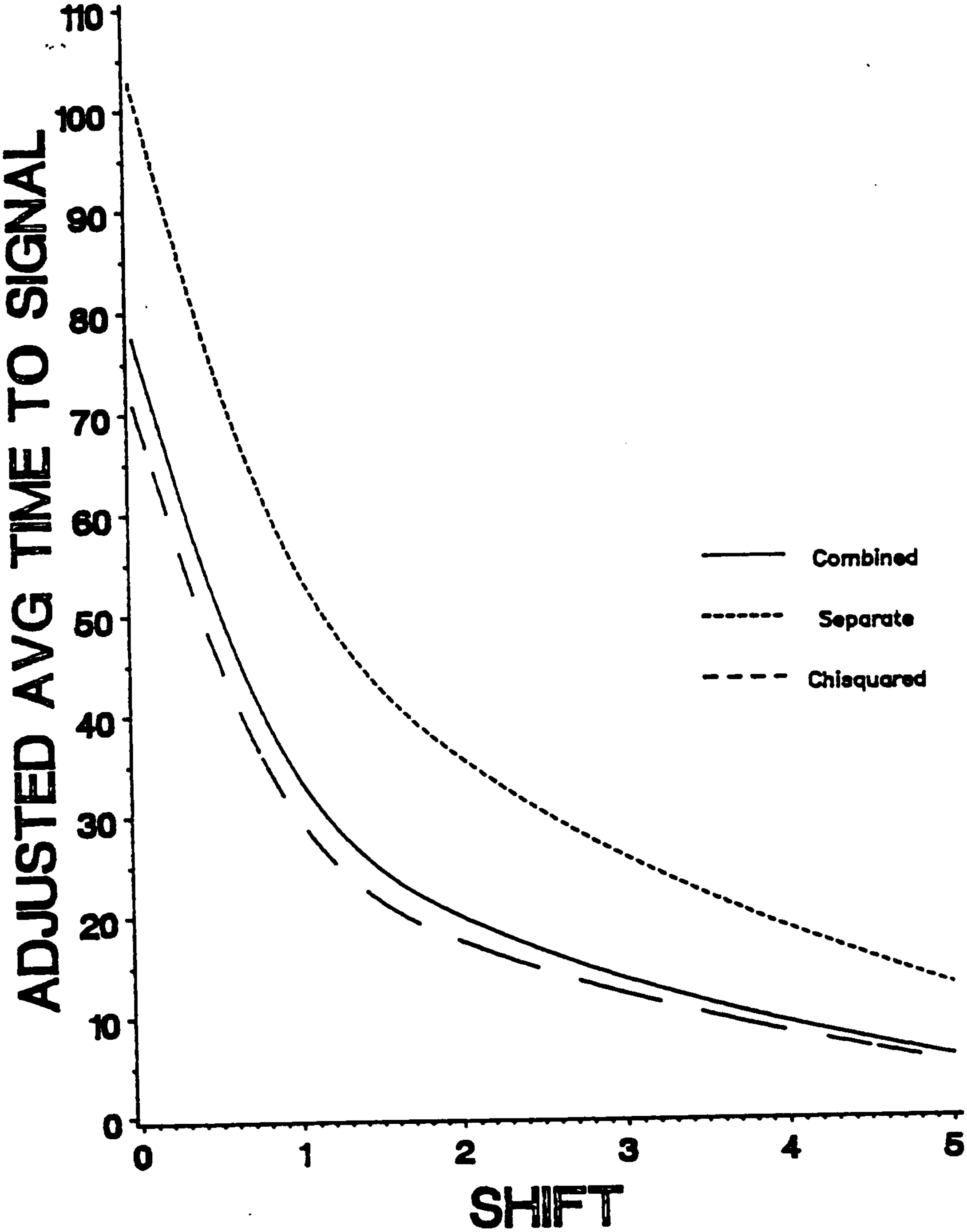


Figure 5. 6. Graph of ATS* and shift in all means when corr = .25 in five of the ten variables.

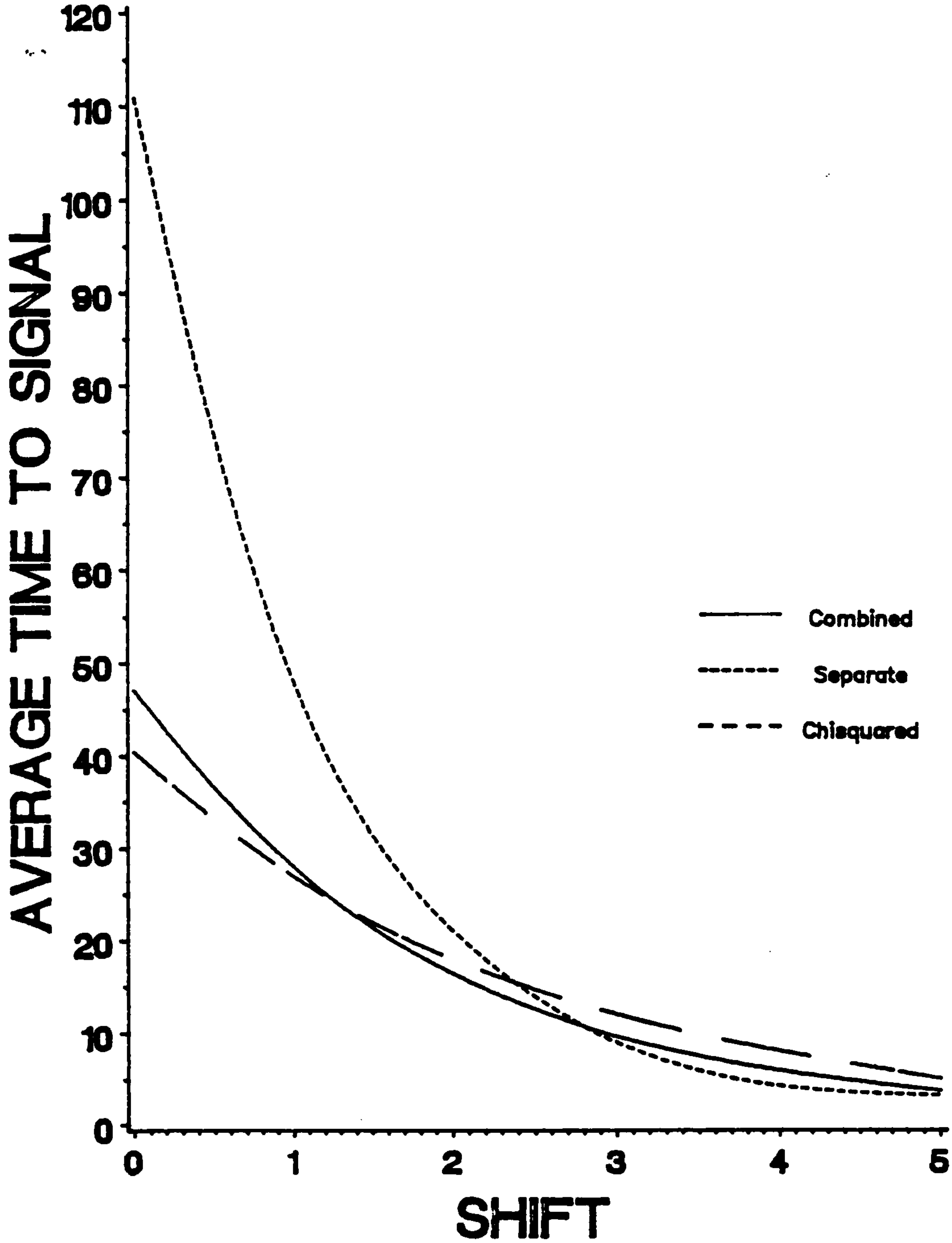


Figure 5. 7. Graph of ATS and shift in one mean when corr = .5 in five of the ten variables.

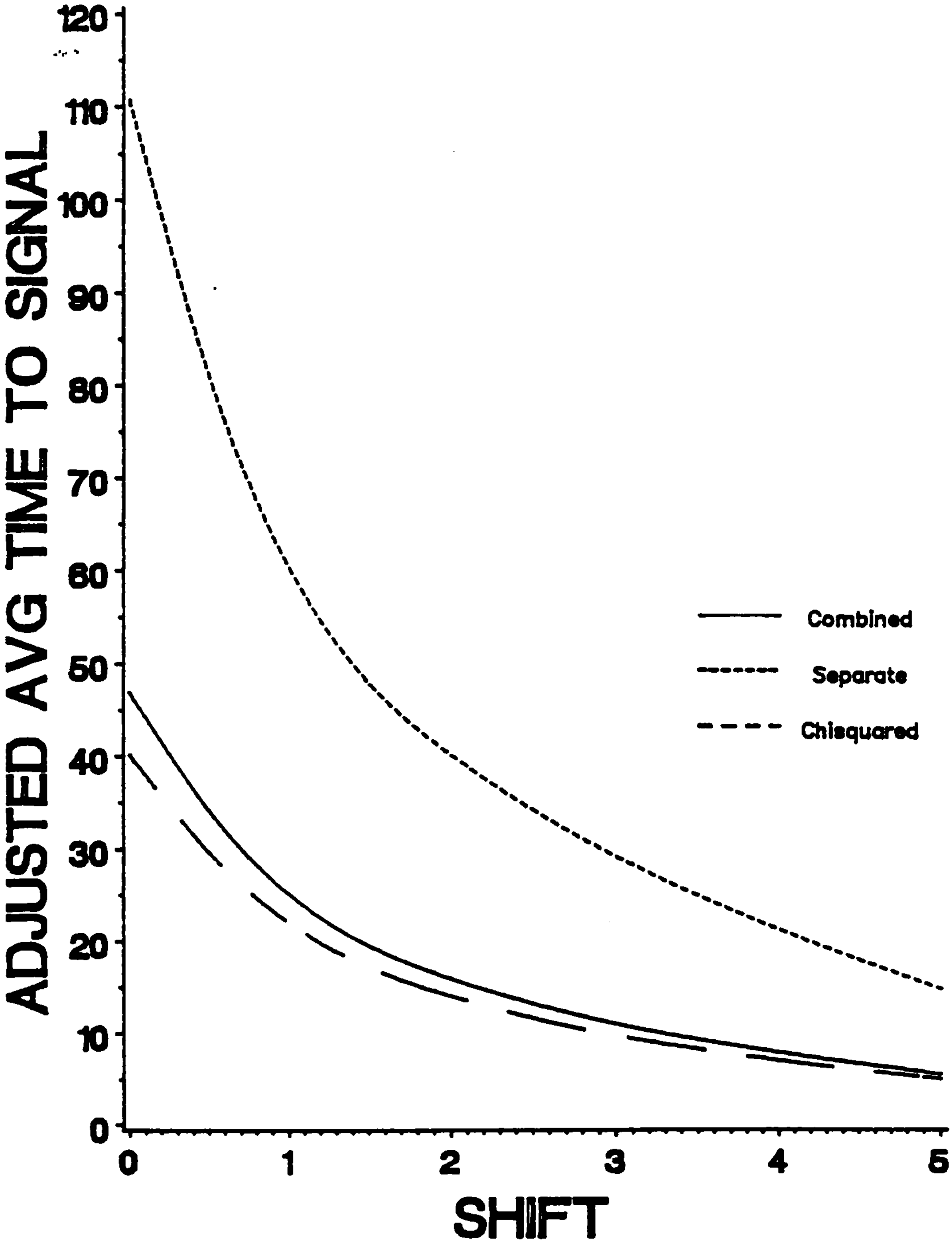


Figure 5. 8. Graph of ATS* and shift in all means when corr = .5 in five of the ten variables.

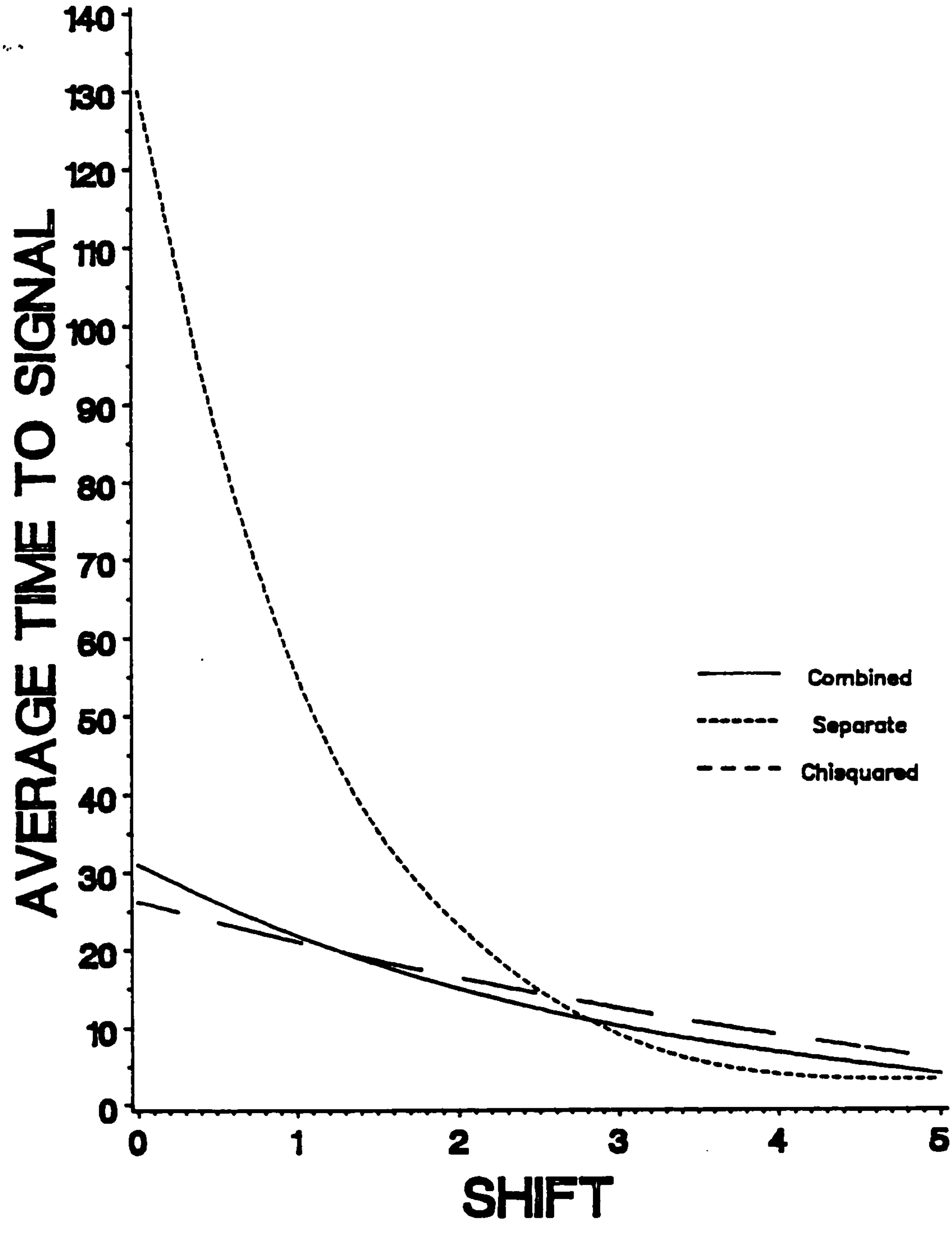


Figure 5. 9. Graph of ATS and shift in one mean when corr = .75 in five of the ten variables.

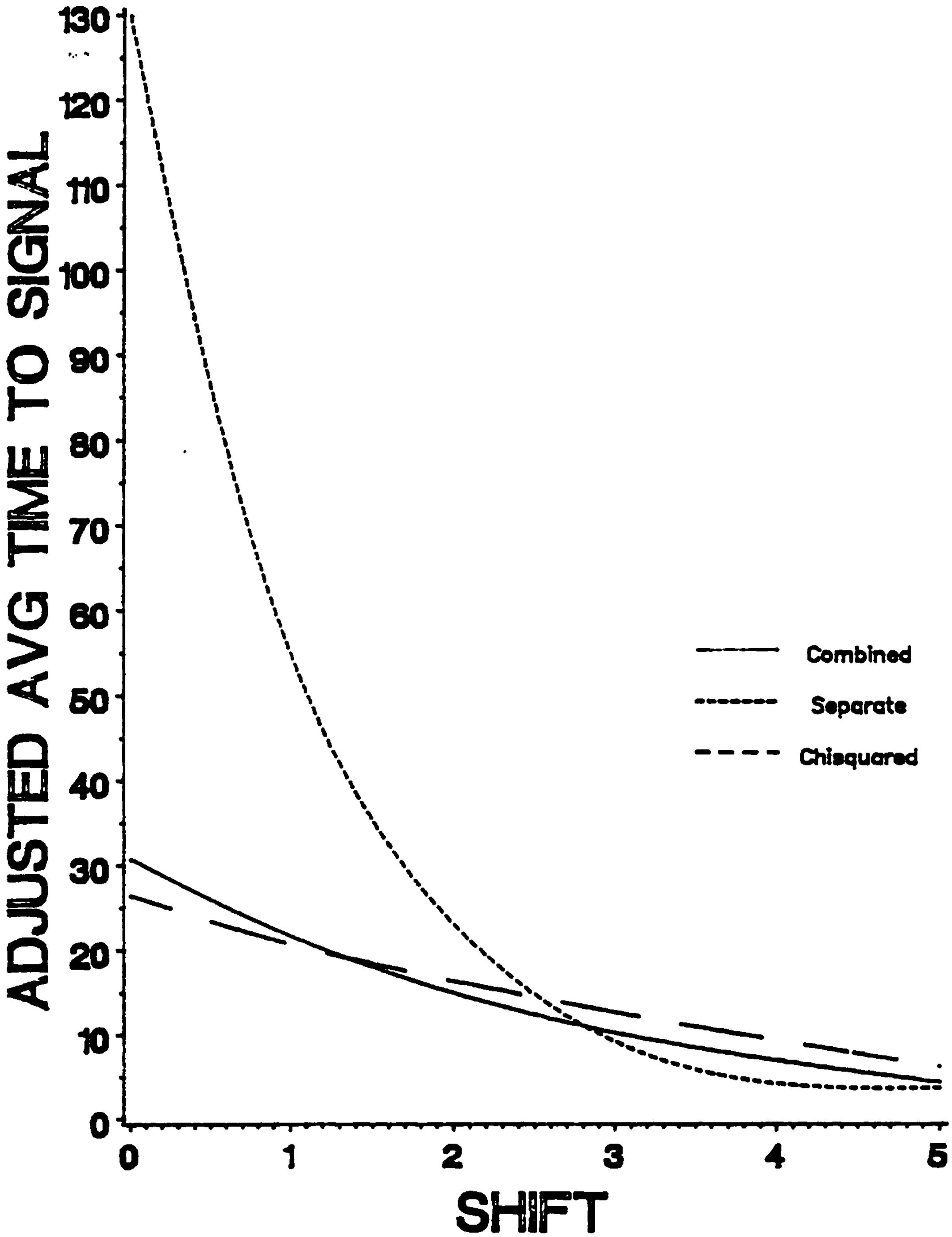


Figure 5. 10. Graph of ATS* and shift in one mean when corr = .75 in five of the ten variables.

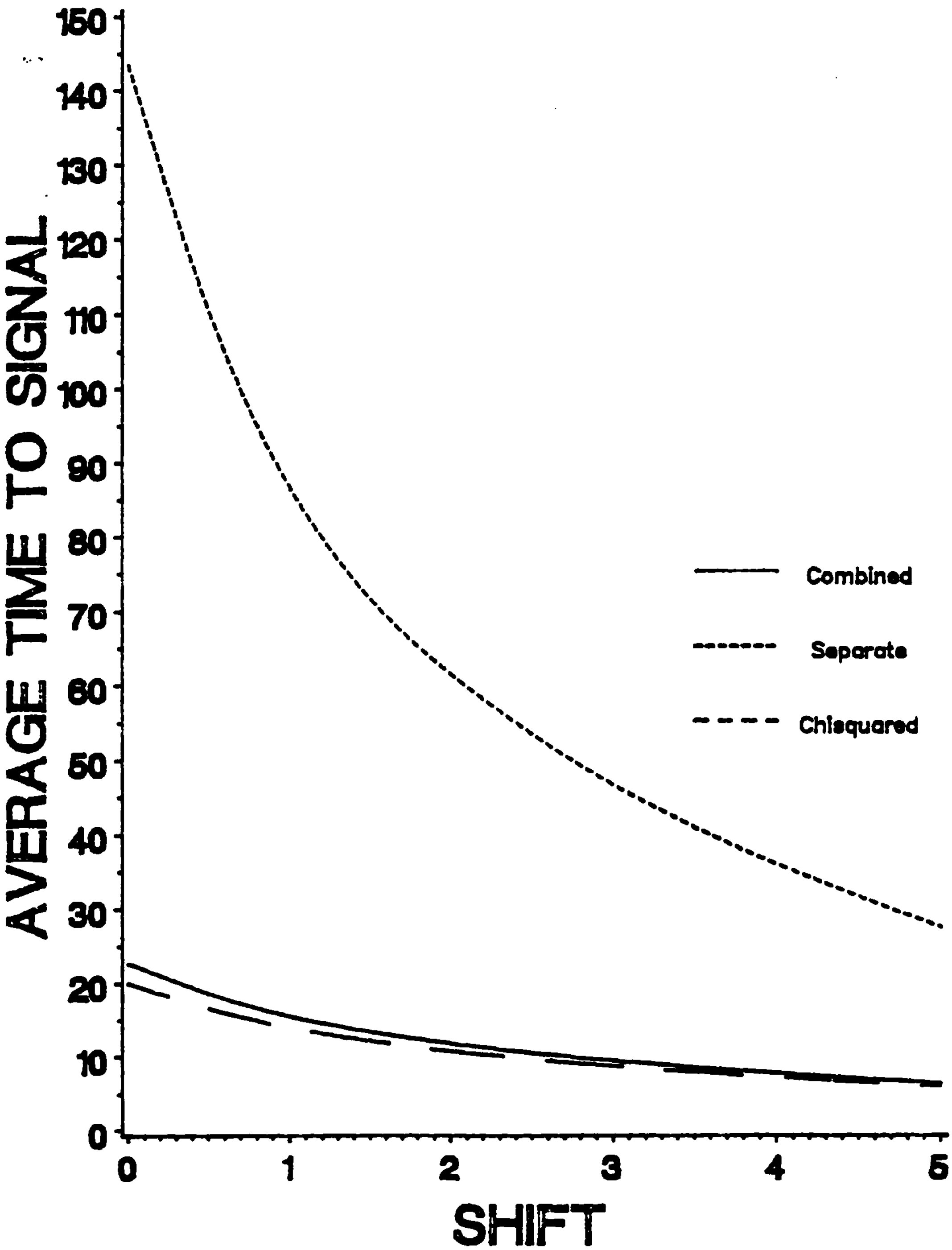


Figure 5. 11. Graph of ATS and shift in all means when corr = .5 in all ten variables.

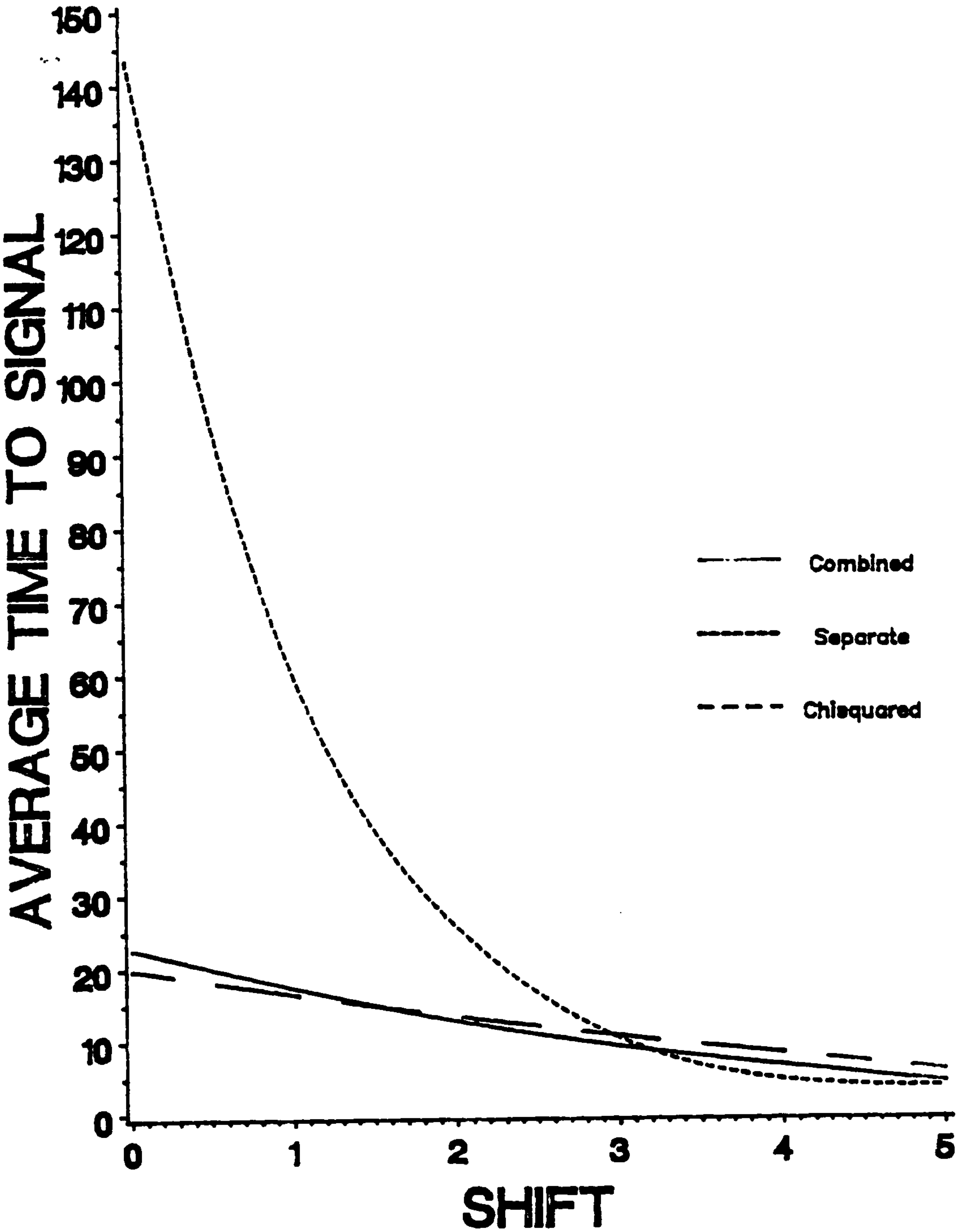


Figure 5. 12. Graph of ATS and shift in one mean when corr = .5 in all ten variables.

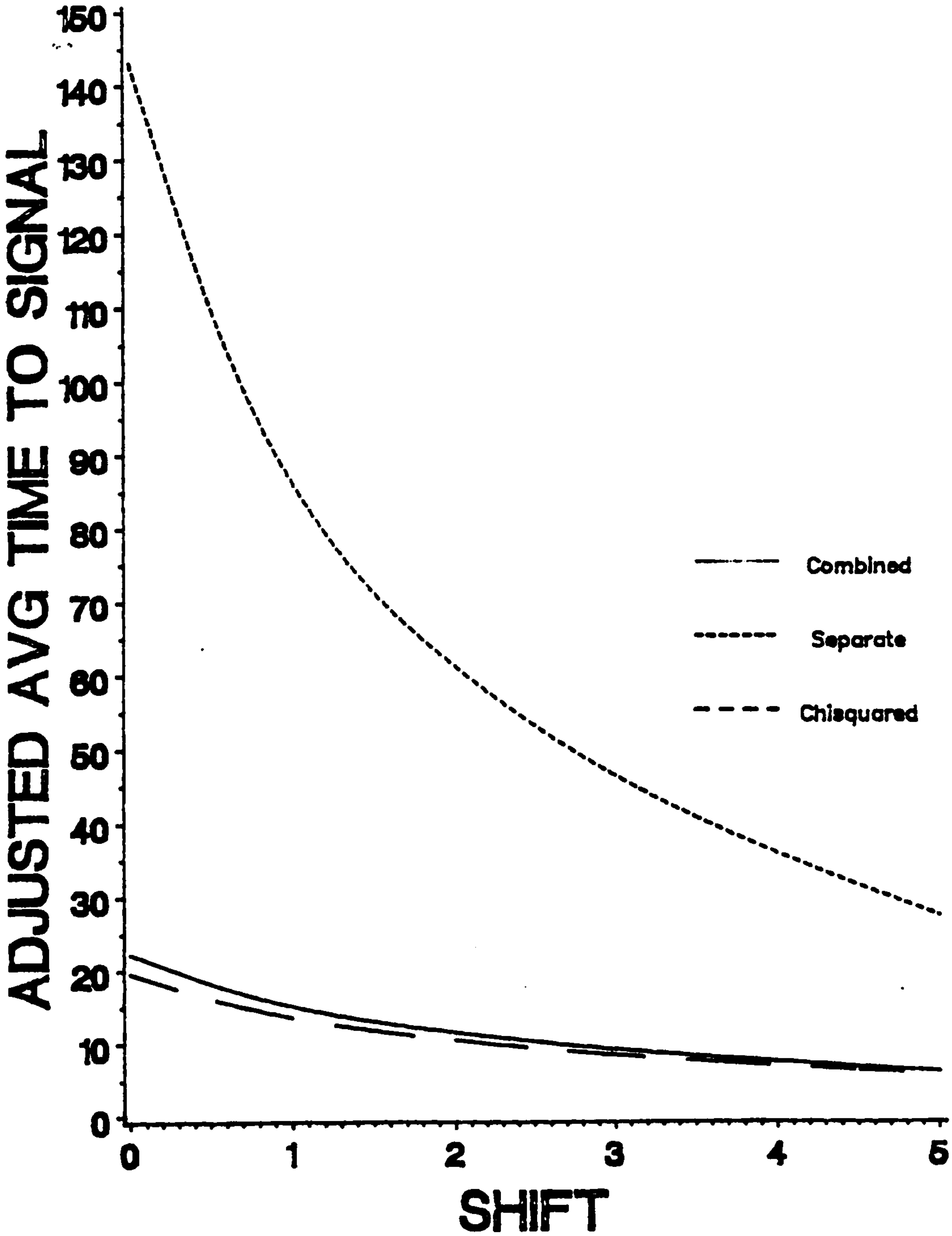


Figure 5. 13. Graph of ATS* and shift in all means when corr = .5 in all ten variables.

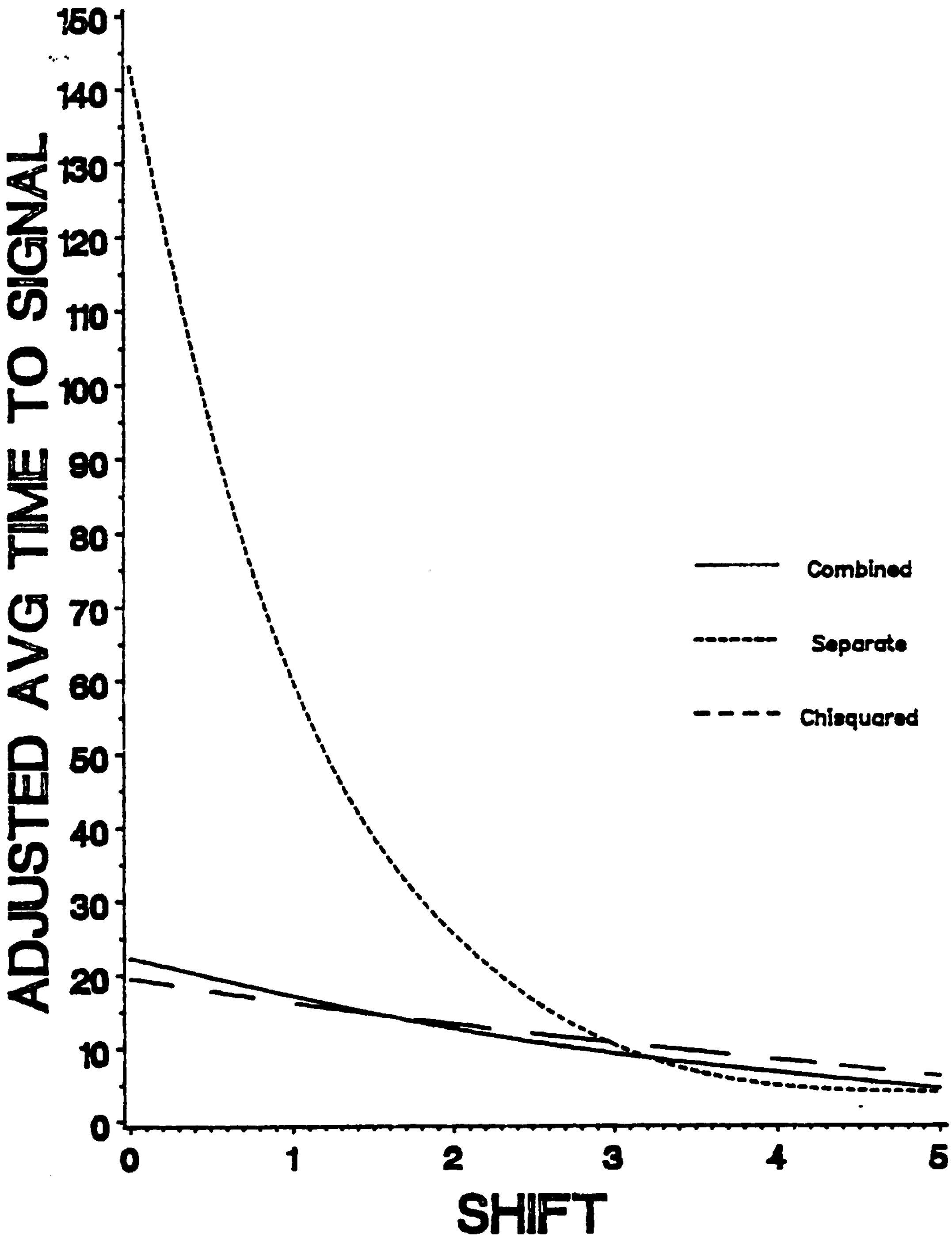


Figure 5. 14. Graph of ATS* and shift in one mean when corr = .5 in all ten variables.

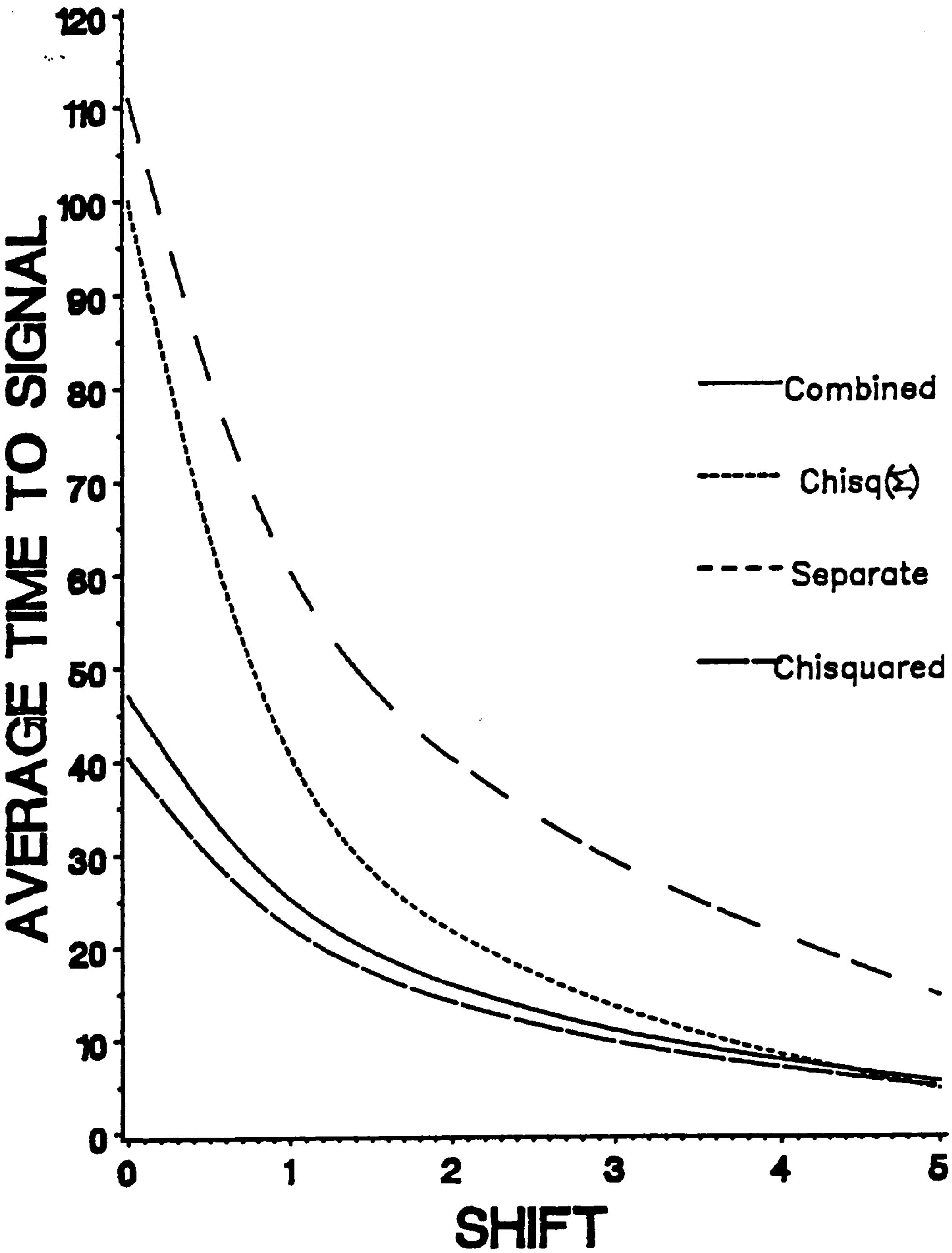


Figure 5. 15. Graph of ATS and shift in all means when corr = .5 in five variab les.

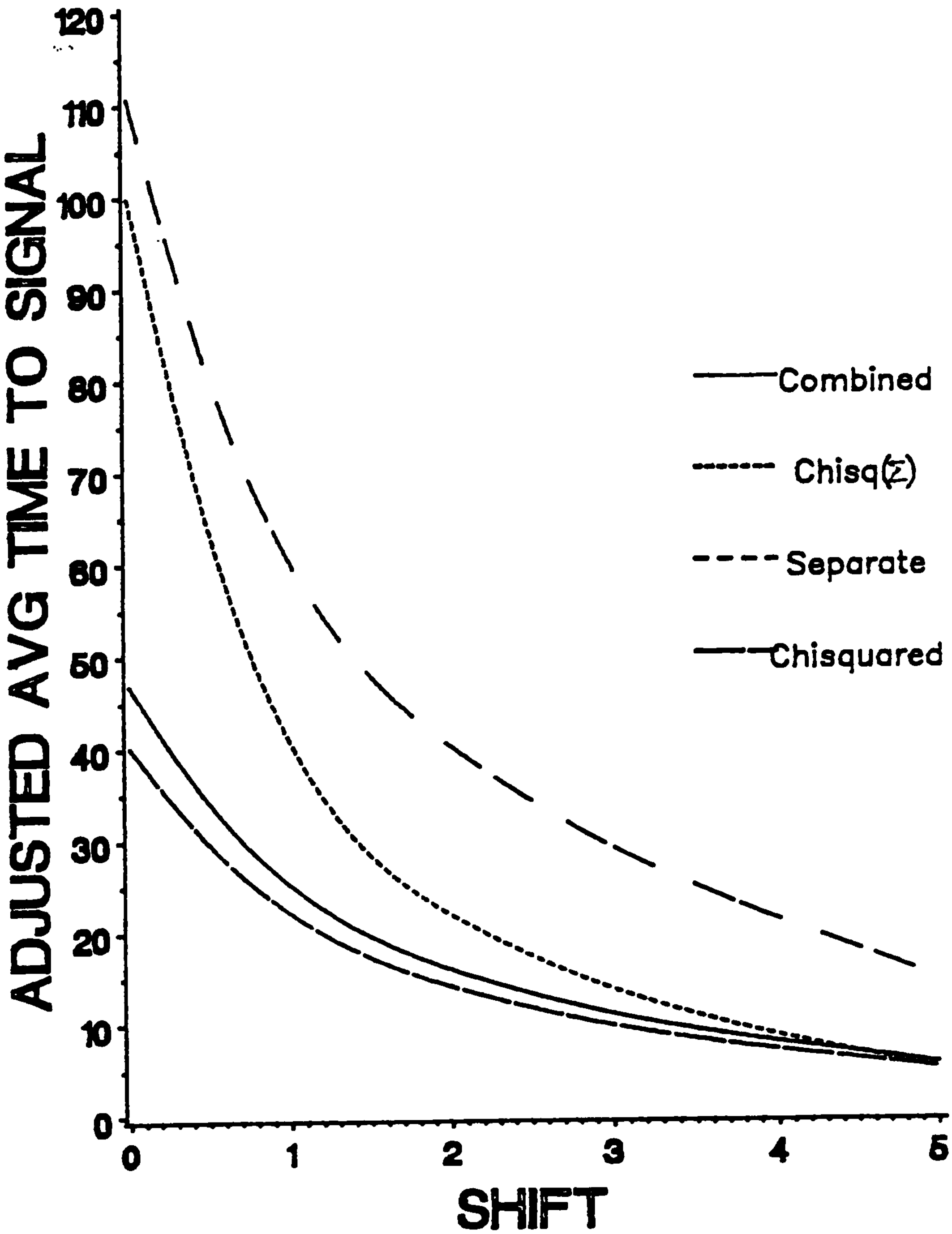


Figure 5. 16. Graph of ATS* and shift in all means when corr = .5 in five variables.

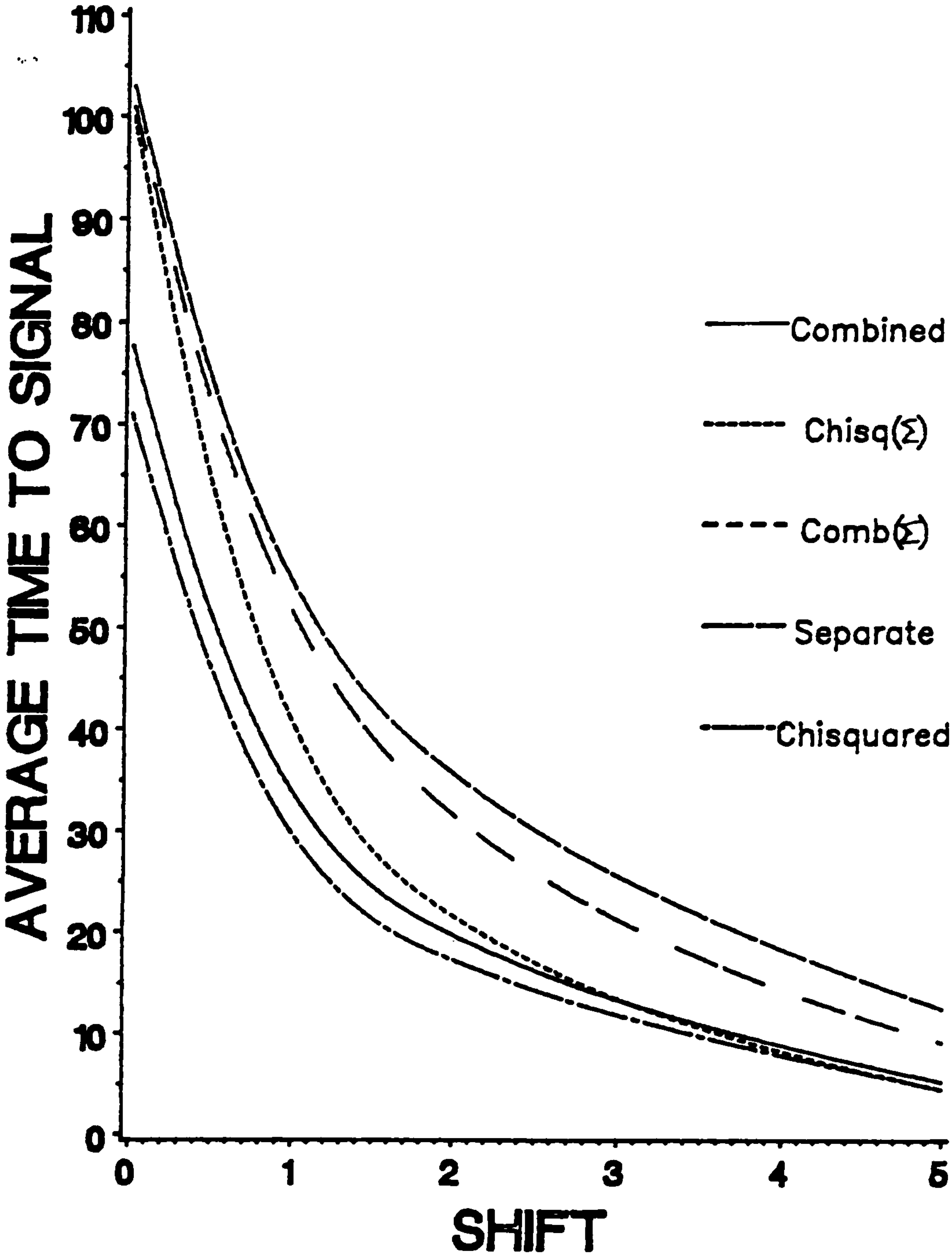


Figure 5. 17. Graph of ATS and shift in all means when corr = .25 in five variables.

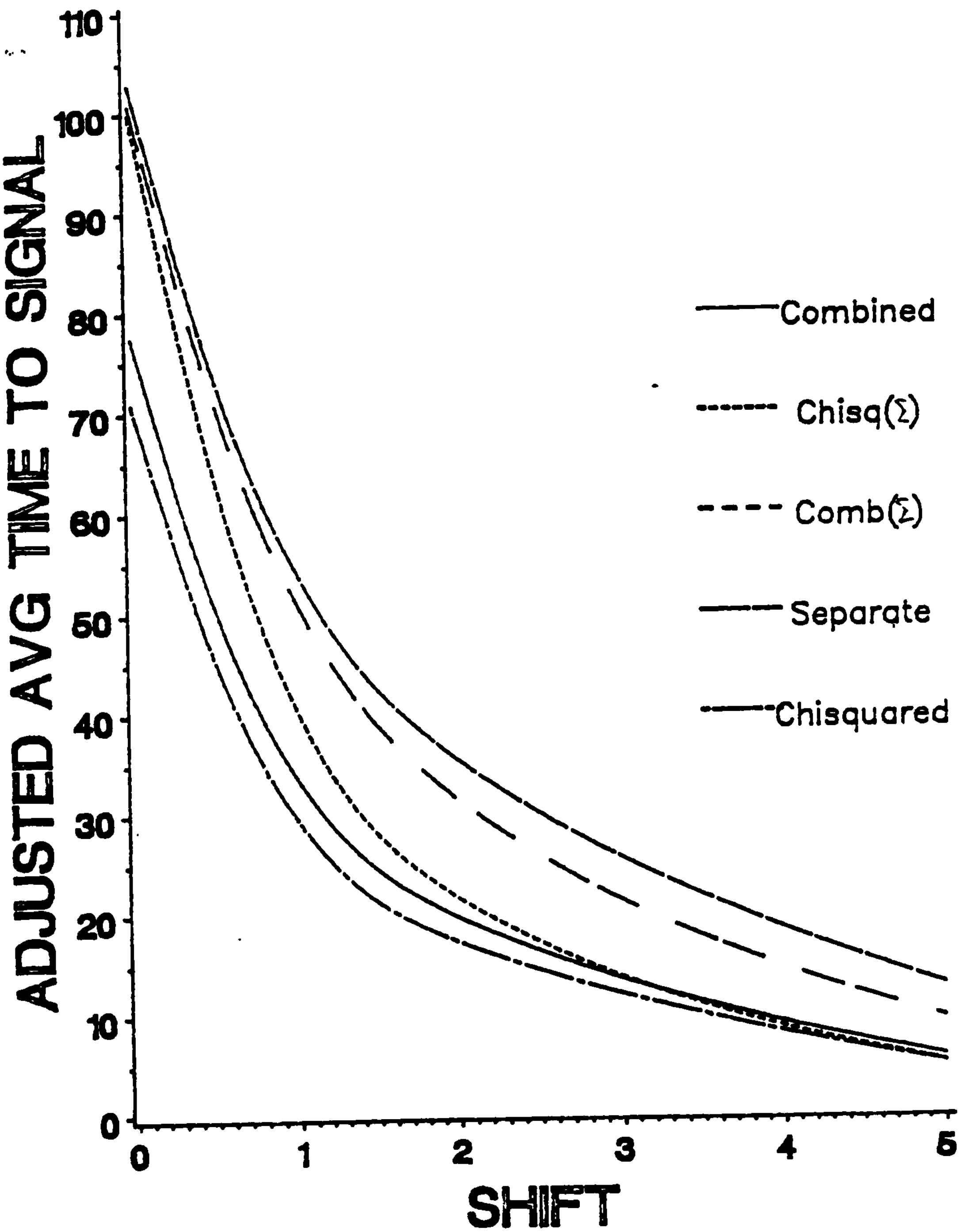


Figure 5. 18. Graph of ATS* and shift in all means when corr = .25 in five variables.

Chapter VI

VI. Conclusions and Future Research

The basic idea for monitoring several parameters have been presented in this dissertation. Chapter Four emphasized the simultaneous monitoring of the mean and variance. Chapter Five extended the variable sampling interval procedure to a multivariate means chart. These investigations lead us to believe that any reasonable fixed interval sampling strategy can be substantially improved by using variable sampling intervals in a reasonable way.

Extensions to more than two parameters in a univariate problem deserves further investigation. In all the results obtained so far, it was assumed that the shift in any of the parameters remained constant. It is possible that there may be a continuous drift in the parameters. Then, the reaction time of the *VSI* chart to a drift in any of the parameters needs to be studied. Amin (1987) studied the effect of a gradual shift in the process mean on a two-sided $\overline{X}$ chart using the *VSI* procedure. He showed that although adding the *VSI* feature did not decrease the length of the initial period of no detection, the *VSI* chart detected the drift faster than the *FSI* chart once the process mean got far away from the target mean.

Another interesting problem would be to create economic models for these procedures. So far, they have been compared in terms of time alone. But it may be expensive to implement some of the procedures studied, since the operator has to monitor the chart continuously, instead of at scheduled points. Hence, it may be worthwhile to set up a model which somehow takes the cost factor into account as well.

When monitoring multiple means and the correlation among the variables is unknown, an obvious extension seems to be to estimate the correlations and use a Hotelling's T^2 statistic instead of a chi-squared statistic to monitor the means. If the correlations are re-estimated at every sampling point, using only the current sample, the calcualted statistics will be independent from sample to sample. However, the sample size is usually too small to allow a good estimte of the correlation matrix. Alternatively, the correlations can be estimated from a sequence of preliminary samples or a single large initial sample. The initial sample can then be updated by successive samples. This would cause the control limits to change at each sampling point. Further, Ghosh et al (1981) have pointed out that for the univariate case, when the control limits of the $\overline{X}$ chart are based on an estimate of σ^2, then the t-statistic computed from samples using the same estimates of σ^2, will no longer be independent. Hence the distribution of the run length is no longer geometric and the ARL cannot be characterized by the probability of a signal at a given point. Further Ghosh et al (1981) showed, using numerical integration, that if a small sample size is used to estimate σ^2 the ARL as well as the variance of the run length distribution, increases.

Hence further investigations need to be directed, first of all, at determining the sample size and the method required to get "good" estimates of the correlation. Once these estimates are obtained, an effort to compute the ATS can be made.

Often runs rules are used along with Shewhart charts in order to make them more sensitive to small changes in the parameter. This variation on a Shewhart chart could be used in conjunction with the *VSI* feature in future studies. The signal function would have to be a function of past samples as well as the current sample. This would make the properties more complicated to calculate.

Amin (1987) studied a Shewhart $\overline{X}$ chart with warning limits using the *VSI* procedure and showed that the *VSI* improved the performance of the chart. The work that has been done using multivariate Shewhart charts could also be extended to multivariate cusum charts.

Chapter Four dealt with the case of monitoring the mean and variance of a single variable simultaneously, whereas Chapter Five dealt with the case of monitoring multiple means when the covariance matrix was constant. If the covariance matrix is allowed to vary, then the problem of monitoring the vector of means as well as the covariance matrix arises. This would be the multivariate generalization of the problem dealt with in Chapter Four.

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APPENDIX

Calculation of the Efficiencies of the Four Basic Procedures in Terms of ATS and ATS*.

Note: The subroutines used were modifications of IMSL subroutines.

DIMENSION C(11),D(7)

DOUBLE PRECISION NCUTY,NFY,N,NCUTX,MU1,MU0,A_H,O_Z

DATA C,D/0.0,0.1,0.25,0.5,1.0,1.1,1.25,1.5,2.0,3.0,4.0,1.0,1.1,

C1.25,1.5,2.0,3.0,4.0/

(C specifies 11 values of shift in the mean and D specifies 7 values of shift in the variance.)

ALPH=.005012562893D0

(ALPH is the signal probability of the $\overline{X}$ and S^2 charts.)

PZ=ALPH/2.0

(The PZth quantile will yield the lower cutoff points for the signal regions of the $\overline{X}$ and S^2 charts.)

CALL MDNRIS (PZ,YZ,IER)

MU0=0.0

SIGMA0=1.0

(The target values of μ and σ are set at 0 and 1.)

N=5.0

(A sample of size five is used at each sampling point.)

DO 200 I=1,11

MU1=C(I)/(N**0.5)

```
DO 201 J = 1,7
SIGMA1 = D(J)
NFY = YZ*SIGMA0/SIGMA1 + (N**0.5)*(MU0-MU1)/SIGMA1
FY = -YZ*SIGMA0/SIGMA1 + (N**0.5)*(MU0-MU1)/SIGMA1
CALL MDNORD (NFY,P1)
CALL MDNORD (FY,P2)
PCH = 1.0D0-ALPH
DF = 4.0D0
CALL MDCH8 (PCH,DF,X,IER)
CS = X*(SIGMA0**2)/(SIGMA1**2)
CALL MDCH (CS,DF,PC1,IER)
ANSS1 = 1.0/(1-(P2-P1)*PC1)
ATS1 = ANSS1
AATS1 = ANSS1-0.5
```

(AATS1 is the adjusted ATS for the FSI_s procedure.)

```
DF2 = 5.0
PCH2 = .99
CALL MDCH8 (PCH2,DF2,X2,IER)
CS2 = X2*(SIGMA0**2)/(SIGMA1**2)
PNONC = ((MU1-MU0)**2)*N
```

(PNONC is the non-centrality parameter for the combined statistic.)

```
CALL MDCHN (CS2,DF2,PNONC,PC2,IER)
ANS2 = 1.0/(1-PC2)
ATS2 = ANS2
AATS2 = ANS2-0.5
```

(AATS2 is the adjusted ATS for the FSI_c procedure.)

```
PMIN = .851781182D0
```

(The PMINth quantile will yield the upper cutoff point for the D_2 region for the $\overline{X}$ chart.)

CALL MDNRIS (PMIN,YMIN,IER)

XK = YMIN*SIGMA0/(N**0.5)

CUTX = (XK + MU0-MU1)*(N**0.5)/SIGMA1

CALL MDNORD (CUTX,PX1)

NCUTX = (-XK + MU0-MU1)*(N**0.5)/SIGMA1

CALL MDNORD (NCUTX,PX2)

P2X = PX1-PX2

ALF1 = 1.0/ANSS1

P1X = 1.0-P2X-(1.0-(P2-P1))

CUT = .703562364D0

(CUT is $P(D_{2r})$.)

CUT1 = .2914250731D0

(CUT1 is $P(D_{1r})$.)

CALL MDCH8 (CUT,DF,S,IER)

SQ = S*(SIGMA0**2)/(SIGMA1**2)

CALL MDCH (SQ,DF,P2S,IER)

P1S = 1.0-P2S-(1-PC1)

P3D1 = P1X*P1S + P1X*P2S + P2X*P1S

(P3D1 is the probability of using the short sampling interval.)

P3D2 = P2X*P2S

(P3D2 is the probability of using the long sampling interval.)

ATS3 = ANSS1*(0.5*(P1X*P1S + P1X*P2S + P2X*P1S) + 1.5*P2X*P2S)/(1.0-ALF1)

AATS3 = (0.5*0.5*(CUT1*CUT1 + 2.0*CUT1*CUT) + 1.5*1.5*CUT*CUT)/(2.0*

C(0.5*(CUT1*CUT1 + 2.0*CUT1*CUT) + 1.5*CUT*CUT)) + ANSS1*(0.5*(P1X*P1S +

CP1X*P2S + P2X*P1S) + 1.5*P2X*P2S)

(AATS3 is the adjusted ATS for the VSI_r procedure.)

P3K1 = .99D0

(P3K1 is the probability of the in-control region for the combined statistic.)

```
CALL MDCH8 (P3K1,DF2,X3,IER)
CS3=X3*(SIGMA0**2)/(SIGMA1**2)
CALL MDCHN (CS3,DF2,PNONC,PC3,IER)
P3K2=.495D0
```

(P3K2 is the probability of the D_2 region for the combined statistic.)

```
PO3=P3K1-P3K2
CALL MDCH8 (P3K2,DF2,X4,IER)
CS4=X4*(SIGMA0**2)/(SIGMA1**2)
CALL MDCHN (CS4,DF2,PNONC,PC4,IER)
P4D1=PC3-PC4
P4D2=PC4
```

(P4D2 is the probability that the long sampling interval is assigned.)

```
WID1=0.1*(P4D1)+1.9*(P4D2)
ALF2=1.0/ANS2
ATS4=ANS2*(WID1)/(1.0-ALF2)
AATS4=(PO3*(0.1*0.1+1.9*1.9))/(2.0*PO3*(0.1+1.9))+ANS2*WID1
```

(AATS4 is the adjusted ATS for the VSI_c procedure.)

```
EFF1=ATS2/ATS1
EFF2=ATS3/ATS1
EFF3=ATS4/ATS1
```

(EFF calculates the efficiencies of the ATS in terms of the ATS of the FSI_t procedure.)

```
AEF1=AATS2/AATS1
AEF2=AATS3/AATS1
AEF3=AATS4/AATS1
```

(AEF calculates the efficiencies of the ATS* in terms of the ATS* of the FSI_t procedure.)

```
201 CONTINUE
200 CONTINUE
STOP
```

```
END
```

Calculation of the ATS and ATS* for the VSI_s Procedure Using Three Sampling Intervals, (0.1, 1.0, 1.9).

```
DIMENSION C(11),D(7)
REAL*8N,MU1,MU0
DOUBLE PRECISION NFY,NCUTX,NCUTY,A_H,O_Z
DATA C,D/0.0,0.1,0.25,0.5,1.0,1.1,1.25,1.5,2.0,3.0,4.0,
C1.0,1.1,1.25,1.5,2.0,3.0,4.0/
```

(C specifies 11 values of shift in the mean and D specifies 7 values of shift in the variance.)

```
ALPH=.005012562893D0
```

(ALPH is the signal probability of the $\overline{X}$ and S^2 charts.)

```
PZ=ALPH/2.0
CALL MDNRIS (PZ,YZ,IER)
MU0=0.0
SIGMA0=1.0
N=5.0
DO 200 I=1,11
MU1=C(I)/(N**0.5)
DO 201 J=1,7
SIGMA1=D(J)
NFY=YZ*SIGMA0/SIGMA1+(N**0.5)*(MU0-MU1)/SIGMA1
FY=-YZ*SIGMA0/SIGMA1+(N**0.5)*(MU0-MU1)/SIGMA1
CALL MDNORD (NFY,P1)
CALL MDNORD (FY,P2)
```

```
PCH = 1.0D0-ALPH
DF = 4.0D0
```

(DF are the degrees of freedom of the chi-squared statistic on which the the S^2 chart is based.)

```
CALL MDCH8 (PCH,DF,X,IER)
CS = X*(SIGMA0**2)/(SIGMA1**2)
CALL MDCH (CS,DF,PC1,IER)
ANSS1 = 1.0/(1-(P2-P1)*PC1)
ATS1 = ANSS1
AATS1 = ANSS1-0.5
PMIN = .748746859D0
```

(The PMINth quantile will yield the upper cutoff point for the D_2 region for the $\overline{X}$ chart.)

```
CUT = .497493719D0
CUT1 = .497493719D0
```

(CUT is $P(D_{2s})$ and CUT1 is $P(D_{1s})$.)

```
CALL MDNRIS (PMIN,YMIN,IER)
XK = YMIN*SIGMA0/(N**0.5)
CUTX = (XK + MU0-MU1)*(N**0.5)/SIGMA1
CALL MDNORD (CUTX,PX1)
NCUTX = (-XK + MU0-MU1)*(N**0.5)/SIGMA1
CALL MDNORD (NCUTX,PX2)
P2X = PX1-PX2
```

(P2X is the probability that the $\overline{X}$ chart assigns the long sampling interval.)

```
ALF1 = 1/ANSS1
P1X = 1.0-P2X-(1.0-(P2-P1))
CALL MDCH8 (CUT,DF,S,IER)
SQ = S*(SIGMA0**2)/(SIGMA1**2)
CALL MDCH (SQ,DF,P2S,IER)
P1S = 1.0-P2S-(1-PC1)
```

(P1S is the probability that the S^2 chart assigns the short sampling interval.)

```
P3D1=P1X*P1S+P1X*P2S+P2X*P1S
P3D2=P2X*P2S
ATS3=ANSS1*(0.1*(P1X*P1S)+(P1X*P2S+P2X*P1S)+1.9*P2X*P2S)/(1.0-ALF1
C)
AATS3=(0.1*0.1*(CUT1*CUT1+2.0*CUT1*CUT)+1.9*1.9*CUT*CUT)/(2.0*
C(0.1*(CUT1*CUT1+2.0*CUT1*CUT)+1.9*CUT*CUT))+ANSS1*(0.1*(P1X*P1S+
CP1X*P2S+P2X*P1S)+1.9*P2X*P2S)
201 CONTINUE
200 CONTINUE
STOP
END
```

Calculation of the ATS and ATS* for the VSI$_s$ Procedure when the Marginal Probabilities are Varied.

```
DIMENSION C(11),D(7),P(3),T(3),U(3),V(3),W(3)
REAL*8N,MU1,MU0
DOUBLE PRECISION NFY,NCUTX,A_H,O_Z
DATA C,D,P,T,U/0.0,0.1,0.25,0.5,1.0,1.1,1.25,1.5,2.0,3.0,4.0,
C1.0,1.1,1.25,1.5,2.0,3.0,4.0,.748746859,.809375,
C.997493719,.994987437,.8,.497493719,0,.194987437,.497493719/
```

(C contains 11 values of shifts in the mean and D contains 7 values of shifts in the variance.

The Pth quantile will yield the cutoff point for the D_2 region of the $\overline{X}$ chart.

T is $P(D_{2s})$ and U is $P(D_{1s})$.)

```
ALPH=.005012562893D0
```

(ALPH is the probability of the signal regions for both the $\overline{X}$ and the S^2 charts.)

```
PZ=ALPH/2.0
CALL MDNRIS (PZ,YZ,IER)
MU0=0.0
SIGMA0=1.0
```

(The target value of the mean is 0 and the target value of the variance is 1.)

```
N=5.0
```

(The sample size used is five.)

```
DO 200 I=1,11
MU1=C(I)/(N**0.5)
DO 201 J=1,7
SIGMA1=D(J)
NFY=YZ*SIGMA0/SIGMA1+(N**0.5)*(MU0-MU1)/SIGMA1
FY=-YZ*SIGMA0/SIGMA1+(N**0.5)*(MU0-MU1)/SIGMA1
CALL MDNORD (NFY,P1)
CALL MDNORD (FY,P2)
PCH=1.0D0-ALPH
DF=4.0D0
CALL MDCH8 (PCH,DF,X,IER)
CS=X*(SIGMA0**2)/(SIGMA1**2)
CALL MDCH (CS,DF,PC1,IER)
ANSS1=1.0/(1-(P2-P1)*PC1)
ATS1=ANSS1
AATS1=ANSS1-0.5
```

(AATS1 is the adjusted ATS for the FSI_1 procedure.)

```
DO 300 K=1,3
PMIN=P(K)
CUT=T(K)
```

```
CUT1 = U(K)
CALL MDNRIS (PMIN,YMIN,IER)
XK = YMIN*SIGMA0/(N**0.5)
CUTX = (XK + MU0-MU1)*(N**0.5)/SIGMA1
CALL MDNORD (CUTX,PX1)
NCUTX = (-XK + MU0-MU1)*(N**0.5)/SIGMA1
CALL MDNORD (NCUTX,PX2)
P2X = PX1-PX2
```

(P2X is the probability that the $\overline{X}$ chart assigns the long sampling interval.)

```
ALF1 = 1.0/ANSS1
P1X = 1.0-P2X-(1.0-(P2-P1))
CALL MDCH8 (CUT,DF,S,IER)
SQ = S*(SIGMA0**2)/(SIGMA1**2)
CALL MDCH (SQ,DF,P2S,IER)
P1S = 1.0-P2S-(1-PC1)
```

(P1S is the probability that the S^2 chart assigns the short sampling interval.)

```
P3D1 = P1X*P1S + P1X*P2S + P2X*P1S
```

(P3D1 is the probability that the short sampling interval is assigned.)

```
P3D2 = P2X*P2S
ATS3 = ANSS1*(.1*(P1X*P1S + P1X*P2S + P2X*P1S) + 1.9*P2X*P2S)/(1.0-ALF1)
V(K) = ATS3
```

(V(K) is the ATS for the procedure that uses the P(K)th quantile to calculate $P(D_{2x})$ and the probability T(K) to calculate $P(D_{2s})$.)

```
AATS3 = (0.1*0.1*(CUT1*CUT1 + 2.0*CUT1*CUT) + 1.9*1.9*CUT*CUT)/(2.0*
C(0.1*(CUT1*CUT1 + 2.0*CUT1*CUT) + 1.9*CUT*CUT)) + ANSS1*(0.1*(P1X*P1S +
CP1X*P2S + P2X*P1S) + 1.9*P2X*P2S)
W(K) = AATS3
```

(W(K) is the ATS* for the procedure that uses the P(K)th quantile to calculate $P(D_{2x})$ and the probabilities T(K) and U(K) to calculate $P(D_{2r})$ and $P(D_{1r})$ respectively.)

```
 300 CONTINUE
 201 CONTINUE
 200 CONTINUE
STOP
END
```

Calculation of the ATS and the ATS* Using the S3 Strategy.

```
REAL*8 N,MU1(5),MU0(5)
DOUBLE PRECISION FY(5),P1(5),NFY(5),P2(5),NCUTX(5),PX2(5),
CATS1(85),AATS1(85),ANSS1(85),SIG(85),YZ,PZ,PMIN,YMIN,ATS3(85),
CAATS3(85),PO1,P1R(85),P2R(85),PX1(5),CUTX(5),ALF1(85),P11X(5),
CP22X(5),PR1(85),PR2(85),PR3(85),PR4(85),PR5(85),PR6(85),SING(5),
CDV1(85),DV2(85),DV3(85),DV4(85),DV5(85),DV6(85),DV7(85),DV8(85),
CDV9(85),DV10(85)
DATA MU0/0,0,0,0,0/
```

(The target value for $\underline{\mu}$ is set at $\underline{0}$)

```
PZ = 0.998995976D0
```

(The PXth quantile will yield the cutoff point for the signal region of an individual chart.)

```
CALL MDNRIS (PZ,YZ,IER)
N = 5.0
```

(The sample size used is five.)

```
PMIN = .749497988D0
```

(The PMINth quantile will yield the upper cutoff point for the D_2 region of an individual chart.)

```
CALL MDNRIS (PMIN,YMIN,IER)
```

```
XK = YMIN/(N**0.5)
DO 200 I = 1,85
(85 different values of shift are specified.)
SIG(I) = 1.0D0
(SIG(I) is the probability that none of the individual charts signal.)
(PR2 - PR6 are the probabilities that exactly one of the charts assign the short sampling interval.)
PR1(I) = 1.0D0
PR2(I) = 1.0D0
PR3(I) = 1.0D0
PR4(I) = 1.0D0
PR5(I) = 1.0D0
PR6(I) = 1.0D0
(DV1 - DV10 are the probabilities that exactly two of the charts assign the short sampling interval.)
DV1(I) = 1.0D0
DV2(I) = 1.0D0
DV3(I) = 1.0D0
DV4(I) = 1.0D0
DV5(I) = 1.0D0
DV6(I) = 1.0D0
DV7(I) = 1.0D0
DV8(I) = 1.0D0
DV9(I) = 1.0D0
DV10(I) = 1.0D0
READ(5,111) (MU1(J),J = 1,5)
111 FORMAT (5(F11.9,1X))
DO 201 J = 1,5
(A set of five charts is used.)
NFY(J) = YZ + (N**0.5)*(MU0(J)-MU1(J))
```

```
FY(J) = -YZ + (N**0.5)*(MU0(J)-MU1(J))
CALL MDNORD (NFY(J),P1(J))
CALL MDNORD (FY(J),P2(J))
SIG(I) = (P1(J)-P2(J))*SIG(I)
SING(J) = P1(J)-P2(J)
(The probability that the jth chart does not signal is SING(J).)
CUTX(J) = (XK + MU0(J)-MU1(J))*(N**0.5)
CALL MDNORD (CUTX(J),PX1(J))
NCUTX(J) = (-XK + MU0(J)-MU1(J))*(N**0.5)
CALL MDNORD (NCUTX(J),PX2(J))
P22X(J) = PX1(J)-PX2(J)
(The probability of the jth chart assigning the long sampling interval is P22X(J).)
P11X(J) = SING(J)-P22X(J)
PR1(I) = P22X(J)*PR1(I)
IF (J-2) 300,500,700
300  PR2(I) = P11X(J)*PR2(I)
PR3(I) = P22X(J)*PR3(I)
PR4(I) = P22X(J)*PR4(I)
PR5(I) = P22X(J)*PR5(I)
PR6(I) = P22X(J)*PR6(I)
DV1(I) = P11X(J)*DV1(I)
DV2(I) = P11X(J)*DV2(I)
DV3(I) = P11X(J)*DV3(I)
DV4(I) = P11X(J)*DV4(I)
DV5(I) = P22X(J)*DV5(I)
DV6(I) = P22X(J)*DV6(I)
DV7(I) = P22X(J)*DV7(I)
DV8(I) = P22X(J)*DV8(I)
```

```
DV9(I) = P22X(J)*DV9(I)
DV10(I) = P22X(J)*DV10(I)
GO TO 201
500  PR2(I) = P22X(J)*PR2(I)
PR3(I) = P11X(J)*PR3(I)
PR4(I) = P22X(J)*PR4(I)
PR5(I) = P22X(J)*PR5(I)
PR6(I) = P22X(J)*PR6(I)
DV1(I) = P11X(J)*DV1(I)
DV2(I) = P22X(J)*DV2(I)
DV3(I) = P22X(J)*DV3(I)
DV4(I) = P22X(J)*DV4(I)
DV5(I) = P11X(J)*DV5(I)
DV6(I) = P11X(J)*DV6(I)
DV7(I) = P11X(J)*DV7(I)
DV8(I) = P22X(J)*DV8(I)
DV9(I) = P22X(J)*DV9(I)
DV10(I) = P22X(J)*DV10(I)
GO TO 201
700  IF (J-4) 303,505,707
303  PR2(I) = P22X(J)*PR2(I)
PR3(I) = P22X(J)*PR3(I)
PR4(I) = P11X(J)*PR4(I)
PR5(I) = P22X(J)*PR5(I)
PR6(I) = P22X(J)*PR6(I)
DV1(I) = P22X(J)*DV1(I)
DV2(I) = P11X(J)*DV2(I)
DV3(I) = P22X(J)*DV3(I)
```

```
DV4(I)=P22X(J)*DV4(I)
DV5(I)=P11X(J)*DV5(I)
DV6(I)=P22X(J)*DV6(I)
DV7(I)=P22X(J)*DV7(I)
DV8(I)=P11X(J)*DV8(I)
DV9(I)=P11X(J)*DV9(I)
DV10(I)=P22X(J)*DV10(I)
GO TO 201
505 PR2(I)=P22X(J)*PR2(I)
PR3(I)=P22X(J)*PR3(I)
PR4(I)=P22X(J)*PR4(I)
PR5(I)=P11X(J)*PR5(I)
PR6(I)=P22X(J)*PR6(I)
DV1(I)=P22X(J)*DV1(I)
DV2(I)=P22X(J)*DV2(I)
DV3(I)=P11X(J)*DV3(I)
DV4(I)=P22X(J)*DV4(I)
DV5(I)=P22X(J)*DV5(I)
DV6(I)=P11X(J)*DV6(I)
DV7(I)=P22X(J)*DV7(I)
DV8(I)=P11X(J)*DV8(I)
DV9(I)=P22X(J)*DV9(I)
DV10(I)=P11X(J)*DV10(I)
GO TO 201
707 PR2(I)=P22X(J)*PR2(I)
PR3(I)=P22X(J)*PR3(I)
PR4(I)=P22X(J)*PR4(I)
PR5(I)=P22X(J)*PR5(I)
```

```
PR6(I) = P11X(J)*PR6(I)
DV1(I) = P22X(J)*DV1(I)
DV2(I) = P22X(J)*DV2(I)
DV3(I) = P22X(J)*DV3(I)
DV4(I) = P11X(J)*DV4(I)
DV5(I) = P22X(J)*DV5(I)
DV6(I) = P22X(J)*DV6(I)
DV7(I) = P11X(J)*DV7(I)
DV8(I) = P22X(J)*DV8(I)
DV9(I) = P11X(J)*DV9(I)
DV10(I) = P11X(J)*DV10(I)
201 CONTINUE
ANSS1(I) = 1.0/(1-SIG(I))
ATS1(I) = ANSS1(I)
```

(The ATS for the FSI_t procedure.)

```
AATS1(I) = ANSS1(I)-0.5
```

(The ATS* for the FSI_t procedure.)

```
ALF1(I) = 1.0/ANSS1(I)
P2R(I) = PR1(I) + PR2(I) + PR3(I) + PR4(I) + PR5(I) + PR6(I) + DV1(I) + DV2(I) +
```

(P2R is the probability that the long sampling interval is assigned.)

```
CDV3(I) + DV4(I) + DV5(I) + DV6(I) + DV7(I) + DV8(I) + DV9(I) + DV10(I)
P1R(I) = SIG(I)-P2R(I)
ATS3(I) = ANSS1(I)*(0.1*P1R(I) + 1.9*P2R(I))/(1.0-ALF1(I))
```

(The ATS for the S3 procedure.)

```
PO1 = .495D0
AATS3(I) = (0.1*0.1*PO1 + 1.9*1.9*PO1)/(2.0*(0.1*PO1 + 1.9*PO1)) +
CANSS1(I)*(0.1*P1R(I) + 1.9*P2R(I))
```

(The ATS* for the S3 procedure.)

```
200 CONTINUE
STOP
END
```

Calculation of the Modified Signal (M) Strategy with m = 5 and b = 4

```
REAL*8 N,MU1(5),MU0(5)
DOUBLE PRECISION FY(5),P1(5),NFY(5),P2(5),NCUTX(5),PX2(5),NIG(85),
CATS1(85),AATS1(85),ANSS1(85),SIG(85),YZ,PZ,PMIN,YMIN,ATS3(85),
CAATS3(85),PO1,PM1(85),PM2(85),PX1(5),CUTX(5),ALF(85),SING(5),
CPD1(5),DEP(5,5),PD2(5),MEG(5),ORM(85)
DATA MU0/0,0,0,0,0/
N=5.0D0
```

(A sample size of five is used.)

```
PZ=0.999122515D0
```

(The PZth quantile will yield the upper cutoff point for the signal region of an individual chart.)

```
CALL MDNRIS (PZ,YZ,IER)
PMIN=.934401228D0
```

(The PMINth quantile will yield the upper cutoff point for the D_2 region of an individual chart.)

```
CALL MDNRIS (PMIN,YMIN,IER)
XK=YMIN/(N**0.5)
DO 200 I=1,85
PM2(I)=1.0D0
```

(PM2 is the probability that all five charts assign the long sampling interval.)

```
SIG(I)=1.0D0
```

(SIG is the probability that none of the j statistics are in the signal region.)

```
NIG(I)=1.0D0
```

(NIG is the probability that all five charts assign the short sampling interval.)

ORM(I) = 0.0D0

(ORM is the probability that four out of the five charts assign the short sampling interval.)

```
READ(5,*) (MU1(J),J = 1,5)

WRITE(6,*) (MU1(J),J = 1,5)

DO 201 J = 1,5

NFY(J) = YZ + (N**0.5)*(MU0(J)-MU1(J))

FY(J) = -YZ + (N**0.5)*(MU0(J)-MU1(J))

CALL MDNORD (NFY(J),P1(J))

CALL MDNORD (FY(J),P2(J))

SING(J) = P1(J)-P2(J)
```

(SING(J) is the probability that the jth statistic is in the in-control region.)

```
CUTX(J) = (XK + MU0(J)-MU1(J))*(N**0.5)

CALL MDNORD (CUTX(J),PX1(J))

NCUTX(J) = (-XK + MU0(J)-MU1(J))*(N**0.5)

CALL MDNORD (NCUTX(J),PX2(J))

PD2(J) = PX1(J)-PX2(J)
```

(PD2(J) is the probability that the jth statistic is in the region D_2 .)

```
PD1(J) = SING(J)-PD2(J)

SIG(I) = SING(J)*SIG(I)

NIG(I) = PD1(J)*NIG(I)

PM2(I) = (PX1(J)-PX2(J))*PM2(I)
```

(PM2 is the probability that the long sampling interval is assigned.)

```
MEG(J) = 1.0D0

DO 202 K = 1,5
```

(Since there are five ways in which four out of five charts choose the short sampling interval.)

```
DEP(J,K) = 0.0D0

IF (K-J) 1,2,1
```

```
1  DEP(J,K) = PD1(J)
GO TO 121
2  DEP(J,K) = PD2(J)
121  MEG(J) = DEP(J,K)*MEG(J)
202  CONTINUE
ORM(I) = MEG(J) + ORM(I)
201  CONTINUE
ALF(I) = 1.0D0-SIG(I) + NIG(I) + ORM(I)
PM1(I) = 1.0D0-ALF(I)-PM2(I)
ANSS1(I) = 1.0D0/ALF(I)
ATS1(I) = ANSS1(I)
AATS1(I) = ANSS1(I)-0.5
(The ATS* for the MS(3/5) strategy using fixed sampling intervals.)
ATS3(I) = ANSS1(I)*(0.1*PM1(I) + 1.9*PM2(I))/(1.0-ALF(I))
PO1 = .495D0
AATS3(I) = (0.1*0.1*PO1 + 1.9*1.9*PO1)/(2.0*(0.1*PO1 + 1.9*PO1)) +
CANSS1(I)*(0.1*PM1(I) + 1.9*PM2(I))
(The ATS* for the MS(3/5) strategy using variable sampling intervals.)
200  CONTINUE
STOP
END
```

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