Three Essays on Taxation and Public Pricing

by

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(ABSTRACT)

This dissertation contains three essays. They are "On Optimal Excise Taxes: Becker's Household-Production Approach", "The Pricing of Public Intermediate Goods Revisited", and "Piecemeal Design and Reform of Commodity Taxes". In the first essay, Becker's household production model is used to investigate optimal excise taxes. We are particularly interested in what can be said about the optimal tax structure from the information of household production activities and the new insights that can be gained into the optimal tax structure by using Becker's approach. In the second essay, we revisit the pricing of publicly produced inputs for competitive downstream industries under a general equilibrium setting and with a general production technology. We emphasize the dependence of the price structure of the private sector upon public prices and explore the consequences on the public pricing rules with such a dependence. In the third essay, we consider the design and reform of commodity taxes with a given general income tax. The major part of the paper investigates tax reforms which are Pareto-improving from the status quo, with and without lump-sum taxes.

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Preface

This dissertation contains three essays on three different but related subjects. The first one, "On Optimal Excise Taxes: Becker's Household-Production Approach", is on optimal taxation. The second one, "The Pricing of Public Intermediate Goods Revisited", is on optimal public pricing. The third one, "Piecemeal Design and Reform of Commodity Taxes", is on the direction of tax reform. The close relationship between the optimal taxation problem and the optimal pricing problem has been known for quite a while. Many results obtained as an optimal taxation problem may well be applied to an optimal pricing problem, and vice versa. Indeed, the celebrated "Ramsey rule" can equally be understood as an optimal taxation principle or an optimal pricing principle depending upon the context involved. In view of the formal similarity between the two types of problems, this should not be surprising. In contrast, the close relationship between the optimal taxation problem and the direction of tax reform is not so obvious. It is in fact a recent discovery. Specifically, one can show that if the status quo does not satisfy the necessary condition of the second-best optimum for any Bergson-Samuelson social welfare function, a Pareto-improving direction of tax reform is feasible. This at the same time means that if a Pareto-improving direction of tax reform is not feasible, then there must exist a Bergson-Samuelson social welfare function to justify the status quo as a second-best optimum. Thus, the direction of tax reform and the optimal taxation problem are also tied together.

The features and the purposes of these essays are left to the introduction section of each essay. Here, we only summarize the contents and the main results.

In the first essay (chapter 1), "On Optimal Excise Taxes: Becker's Household-Production Approach", Becker's household production model is used to investigate optimal excise taxes. Two features of Becker's approach are distinguished. First, unlike the standard labor-leisure analysis, it treats all nonmarket activities symmetrically -- it takes time as well as goods to consume. Second, it regards the household as a producer as well as a consumer and separates household production from household preference. The implications of these two features for optimal taxation are the focus of the paper. In a one-consumer economy with homothetic household production functions, optimal excise taxes have a special structure and can be characterized in terms of the parameters of household production functions. In particular, there is a close relationship to the elasticity of substitution between goods and time. We also incorporate public expenditures in Becker's model and explore how this may affect the optimal tax structure. Finally, we use fixed coefficient household production functions to analyze the connections between Becker's approach and the standard labor-leisure analysis in the context of optimal taxation.

In the second essay (chapter 2), "The Pricing of Public Intermediate Goods Revisited", we revisit the pricing of publicly produced inputs for competitive downstream industries and emphasize the dependence of the price structure of the private sector upon public prices. Ramsey pricing for the private firm is found to be the same as that for the household under a general equilibrium setting and with a general production technology. The information requirements in the public pricing rules derived are not excessive and, in particular, do not involve any knowledge about the production relations of the private sector or the demand relations of privately produced goods.

In the third essay (chapter 3), "Piecemeal Design and Reform of Commodity Taxes", we consider the design and the reform of commodity taxes with a given general income tax. The major part of the paper investigates tax reforms which are Pareto-improving from the status quo, with and without lump-sum taxes. In the case where lump-sum taxes are used to replace commodity taxes, we generalize or modify many results obtained in Dixit (1975). When lump-sum taxes are not available, we make use of the so called "inverse optimum problem" to infer the existence of a variety of feasible Pareto-improving directions. We also interpret the gradient projection algorithm of adjusting tax rates within our setting and connect the work of Tirole and Guesnerie (1981) with that of Ahmad and Stern (1984).

Chapter 1. On Optimal Excise Taxes: Becker's Household-Production Approach

1.1. Introduction

The optimal excise tax problem was first solved by Ramsey (1927) and later extended by Samuelson (1951). Diamond and Mirrlees (1971) provided a recent classic [for surveys of the literature see Bradford and Rosen (1976), Sandmo (1976), Atkinson and Stiglitz (1980) or Auerbach (1985)]. In the tradition of optimal excise taxation, the models used typically employ the standard labor-leisure analysis. In this paper we instead follow Becker (1956) household production to approach the problem.[1]

It probably goes without saying that the most distinctive feature of Becker's approach is his systematic treatment of time, especially the time used in nonmarket activities. This feature distinguishes Becker's approach not only from the standard labor-leisure analysis but also from other models of household production such as Lancaster (1966) and Muth (1966). Becker

1 Atkinson and Stern in conjuction with Gomulka (1980) also used Becker's model to approach taxation, but their concern was different from ours.

(1965) showed that the usual labor-leisure model can be regarded as a special case of his model in which a nonmarket activity called "leisure" uses no goods but time as inputs and all other nonmarket activities, on the other hand, use no time but goods. Becker's approach breaks up the dichotomy of goods versus time in the standard labor-leisure model and treats all nonmarket activities symmetrically -- it takes time as well as goods to consume. As a description of reality, such an approach is certainly more tenable. The implications for optimal taxation with this special feature of Becker's model will be one of the foci of the paper.

Like other approaches of household production, Becker's model is a two-stage formulation: consumer preferences are defined over nonmarket activities, which in turn use goods and time as inputs. It regards the household as a producer as well as a consumer and separates household production from consumer preferences. There needs to be emphasized in the first place that this approach is not a mutually exclusive alternative to the usual one-stage formulation. On the contrary, integrating household production with consumer preferences, one can get a "derived" utility function. If we work directly with the derived utility function, all the usual results derived from the one-stage formulation can still be obtained [see Becker and Michael (1973)]. But with the two-stage formulation to separate household production from consumer preferences, it increases the explanatory power of models -- through household production one can obtain a richer and deeper understanding of consumer behavior.[2] This approach has been proved useful for a variety of other applications [e.g., Becker (1976)]. In this paper, our purpose is to apply the two-stage formulation of Becker's model to the optimal tax problem to see whether it enables us to gain new insights into the optimal tax structure. In particular, we would like to know what can be said about the optimal tax structure from the information of household production activities.

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2 For the strength of the household production approach and the weakness of the standard model, see Becker (1971) and Becker and Michael (1973).

To attack the problem, we conceive a one-consumer economy in which an excise tax is the only tax instrument for a government. With homothetic household production functions, we have the following main results:

(i) If the elasticity of substitution between goods and time is unitary for all household production activities, uniform taxation is optimal. Unlike the previous works which restrict consumer preferences to obtain uniform taxation [e.g., Atkinson and Stiglitz (1972), Sandmo (1974), Sadka (1977) and Deaton (1979)], we derive this result from restrictions on the household production functions.

(ii) Divide all nonmarket activities into two groups: one with elasticity of subtitution between goods and time less than one (inelastic), the other greater than one (elastic). Then the market goods used in the first-group of nonmarket activities should be taxed at a higher rate than those used in the second-group. This is somewhat analogous to the famous inverse elasticity rule obtained in the standard labor-leisure model: tax rates should be higher on price-inelastic market goods than on price-elastic ones. Here, however, it is the the elasticity of substitution in household production between goods and time which is critical.

(iii) Assume that the elasticity of substitution in household production between goods and time is either a constant or a function of the ratio between goods and time. Further, assume that the consumer's marginal utility of income and the shadow price of the government's revenue can be estimated. Then the optimal tax rates on a market good can be calculated only using the parameters of the corresponding household production function. This result stands in stark contrast to the standard one where a system of equations containing the entire substitution terms of Slutsky matrix must be resolved in order to calculate the optimal tax rate of that market good.

One key leading to these three results is this: Becker's household production approach allows one to reduce statements about the marginal rate of substitution in consumption between goods and time into statements about the marginal rate of substitution in household production between goods and time. As a result, it enables us to use the information about household production activities to infer the optimal tax structure.

The paper is organized as follows. In section 1.2, Becker's household production is incorporated into the standard one-consumer model to analyze excise taxes. With homothetic household production functions, we investigate the optimal tax structure and its relationship with the elasticity of substitution in household production between goods and time. Two specific household production functions, CES [Arrow, Chenery, Minhas and Solow (1961)] and Revankar (1971), are applied to calculate the optimal tax rates. This allows us to see the sensitivity of tax rates with respect to changes in the elasticity of substitution between goods and time. We also incorporate public expenditures in Becker's model and explore how this may affect the optimal tax structure. In secton 1.3, we use fixed coefficient household production functions to analyze the connections between Becker's approach and the standard labor-leisure analysis in the context of optimal taxation. It is not surprising to find that Becker's model does provide a generalization. Section 1.4 is the conclusion.

1.2. The model

Consider a one-consumer economy[3] with time endowment T and n market goods $x = (x_1, \ldots, x_n)$. The total time T is divided between "working" t_w and "leisures" $t = (t_1, \ldots, t_n)$. For simplicity producer's prices of x are assumed to be fixed and, without loss of generality, we set them at unity. We also assume a fixed wage rate w for "working" t_w. Without loss of any generality, we choose "worki‑ ‑" as the numéraire and set $w = 1$.

3 For the justifications and the limitations of a one-consumer economy, see Tresch (1981). The one-consumer economy postulate enables us to focus upon the efficiency aspect of public policy.

Consumer preferences are defined over the nonmarket or consumption activities $z = (z_1, \ldots, z_n)$ and represented by the utility function

$$U = u(z). \tag{1}$$

Each consumption activity z_j, $j = 1, \ldots, n$, uses market good x_j and time t_j as inputs, and hence the household production functions are[4]

$$z_j = f^j(x_j, t_j), \quad j = 1, \ldots, n. \tag{2}$$

Substituting (2) into (1) yields the "derived" utility function

$$U = u(f^1(x_1, t_1), \ldots, f^n(x_n, t_n)), \tag{3}$$

which is assumed to possess all the usual properties imposed upon a utility function. It worth noting that if goods and time are dichotomized such that $z_j = x_j$ for $j = 1, \ldots, n\text{-}1$, and $z_n = \sum t_i$, Becker's model is then reduced to the standard labor-leisure analysis.[5] One special feature of Becker's approach is to break up this dichotomy of goods versus time and treats all non-market activities symmetrically -- it takes time as well as goods as inputs in consumption. In the spirit of Becker's model, we assume that $t_j > 0$ for all $j = 1, \ldots, n$. This assumption will be relaxed in section 1.3 when we use fixed coefficient household production functions to discuss the connections between Becker's approach and the standard labor-leisure analysis in the context of optimal taxation.

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4 By our setup, x_j in (2) is a scalar not a vector. This may not be so restrictive as it first seems. In practice, public policy makers may like to apply a uniform tax rate on the market goods used in the same consumption activity after taking administration and information costs into account. If this is so and, further, if all the market goods can be partitioned so as to be used in different consumption activities, then the market goods used in the same consumption activity can be treated as a single good by the Hicksian "composite commodity theorem". Thus n market goods in our model are actually n composite commodities. For details on "composite commodity theorem", see Deaton and Muellbauer (1980) or Varian (1984).

5 In this paper we always let $\sum a_i$ stand for $\sum_{i=1}^{n} a_i$.

From (3), we see that the derived utility function is weakly separable in goods and time used in different consumption activities. Further, unlike the usual separability assumption in the standard labor-leisure analysis, the property here does not require all market goods together to be separable from all leisure time. This is, of course, due to the symmetric treatement of nonmarket activities in Becker's model. The special form of weak separability of the derived utility function (3) will play a critical role in our investigation of optimal excise taxes.

Following Becker (1965), the consumer's problem is to maximize the utility function (1) subject to the production functions (2) as well as the time constraint

$$T = t_W + \sum t_i \tag{4}$$

and the income constraint

$$wt_W = \sum q_i x_i \,, \tag{5}$$

where $q_i = 1 + \theta_i$ is the consumer price of x_i and θ_i is the unit excise tax on x_i. Let $q = (q_1, ..., q_n)$. Since taxes on unearned income bring no distortion, the optimal tax problem becomes straightforward if the amount of unearned income exceeds the tax revenue requirement. To make the problem interesting, we rule out unearned income in Becker's model.

From (5), it is clear that uniform taxation on all the market goods is equivalent to a proportional (labor) income tax. Equations (4) and (5) can be combined into a single constraint

$$wT = \sum (q_i x_i + wt_i), \tag{6}$$

where wT represents the household's "full income", which is not affected by variations in "working" t_w. Full income wT is spent either on market goods or on "leisures".

Forming the Lagrangian for the consumer's optimization program:

$$\ell \equiv u(z) + \alpha[wT - \sum(q_i x_i + wt_i)].$$

Differentiating ℓ with respect to x and t, and setting them equal to zero yields the first order conditions

$$\frac{\partial u}{\partial z_j}\frac{\partial f^j}{\partial x_j} = \alpha q_j\,, \quad j = 1, \ldots, n, \tag{7}$$

and

$$\frac{\partial u}{\partial z_j}\frac{\partial f^j}{\partial t_j} = \alpha w, \quad j = 1, \ldots, n. \tag{8}$$

Equation (8) indicates that the consumer will allocate leisure time such that marginal utility of using time is equalized across all nonmarket activities.

Let C be the exogenously determined tax revenue requirement. The government's problem is to maximize the utility function (1) but keep a balanced budget

$$\sum \theta_i x_i = C. \tag{9}$$

In this optimization process, it needs to take the consumer's behavior into account.

1.2.1. Dual approach

To solve the government's problem, we first adopt a dual approach using consumer prices q as the control variables. Forming the Lagrangian for the government:

$$L \equiv u(z) - \lambda(C - \sum \theta_i x_i).$$

Differentiate L with respect to q and set it equal to zero. Then using (2), (6), (7) and (8) yields

$$\alpha x_j = \lambda(x_j + \sum_i \theta_i \frac{\partial x_i}{\partial q_j}), \quad j = 1, \ldots, n, \tag{10}$$

which is the same formula derived from the standard labor-leisure model [e.g., Auerbach (1985, (5.4))]. By applying the Slutsky equation to the term $\frac{\partial x_i}{\partial q_j}$, one can then obtain the Ramsey rule. It is easy to see that equation (10) can also be derived by working directly with the derived utility function (3). Since we assume that the derived utility function possesses all the usual properties of a utility function, the result (10) has the same formula as that derived from the standard model should come as no surprise. Notice that (10) is a system of equations involving all the cross-price and own-price effects. Hence it is very demanding in the information requirements to resolve the solution.

As mentioned in the introduction, Becker's model is a two-stage formulation -- instead of working directly with a derived utility function, it separates household productions (2) from consumer preferences (1). In the special case where $z_j = x_j$ for $j = 1, \ldots, n\text{-}1$, and $z_n = \sum t_i$, the household production model is reduced to the usual labor-leisure analysis; Becker's two-stage formulation can provide no more information at all. In the general case however, equation (10) normally contains more information. In particular, the information about household production activities. Presumably, more can be said about the optimal tax structure through them. In order to extract this extra information, we appeal to the primal approach (using quantities instead of prices as the control variables). Though the primal approach is less straightforward, it will enable us to see the role of household production in determining the optimal tax structure and, furthermore, to characterize the optimal tax structure in terms of the parameters of the household production functions (2).

1.2.2. Primal approach

We follow Atkinson and Stiglitz (1972), using quantities $x = (x_1, \ldots, x_n)$ and $t = (t_1, \ldots, t_n)$ as the control variables open to the government.[6] First from (7) and (8)

$$R_j = \frac{q_j}{w}, \quad j = 1, \ldots, n, \tag{11}$$

where $R_j \equiv (\frac{\partial f^j}{\partial x_j} / \frac{\partial f^j}{\partial t_j})$, i.e., the marginal rate of substitution between x_j and t_j in the household production of z_j. Note that $R_j = (\frac{\partial u}{\partial x_j} / \frac{\partial u}{\partial t_j})$ too. Thus it also represents the marginal rate of substitution in consumption between x_j and t_j. By Becker's approach, the marginal rate of substitution in consumption between goods and time can indeed be reduced to the marginal rate of substitution in household production between goods and time.[7]

Now dividing (6) by w, from (11) one obtains

$$\sum (R_i x_i + t_i) = T. \tag{12}$$

Since (12) stems from the consumer's budget constraint (6) and the first order conditions (7) and (8), it embodies the consumer's optimization behavior and represents the condition that the consumer is on his offer curve. Comparing (12) with the analogous construct of Atkinson and Stiglitz (1980, (12-32)), one can see that the marginal rate of substitution in consumption

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6 For a different type of primal approach making use of the distance function, see Deaton (1979, 1981). However, this method does not allow us to exploit many properties of a household production function as Atkinson and Stiglitz's does.

7 By using other models of household production to approach public goods, Sandmo (1973) was also able to simplify the Samuelsonian condition: the summation of marginal rate of substitution in consumption is reduced to the summation of marginal rate of substitution in household production. However, since other models of household production do not treat nonmarket activities symmetrically with respect to goods and time, they can not obtain the result here.

between goods and time is now replaced by the marginal rate of substitution in household production between goods and time. This is one of the keys allowing us later to characterize the optimal tax structure in terms of the parameters of the household production functions (2).

From (6) and (9)

$$wT - \sum(x_i + wt_i) = C, \tag{13}$$

which is just another way to write the government's budget balance condition.

With the consumer constrainted on his offer curve and a balanced budget, the Lagrangian for the government's program is:

$$\mathscr{L} \equiv u(z) + \lambda[wT - \sum(x_i + wt_i) - C] + \mu[T - \sum(R_i x_i + t_i)].$$

Differentiating $\mathscr{L}$ with respect to x and t, the first order conditions are

$$\frac{\partial u}{\partial z_j}\frac{\partial f^j}{\partial x_j} = \lambda + \mu R_j + \mu x_j \frac{\partial R_j}{\partial x_j}, \quad j = 1, \ldots, n, \tag{14}$$

and

$$\frac{\partial u}{\partial z_j}\frac{\partial f^j}{\partial t_j} = \lambda w + \mu + \mu x_j \frac{\partial R_j}{\partial t_j}, \quad j = 1, \ldots, n. \tag{15}$$

Note that $\frac{\partial R_i}{\partial x_j} = \frac{\partial R_i}{\partial t_j} = 0$ for $i \neq j$. This is due to the special form of weak separability of the derived utility function (3).

Substituting (7) and (8) into (14) and (15) respectively and eliminating μ, and then using $w = \frac{q_j}{R_j}$ and $q_j = 1 + \theta_j$ yields

$$\frac{\theta_j}{1+\theta_j} = \frac{\lambda - \alpha}{\lambda} \frac{x_j \dfrac{\partial R_j}{\partial t_j} - \dfrac{x_j}{R_j} \dfrac{\partial R_j}{\partial x_j}}{1 + x_j \dfrac{\partial R_j}{\partial t_j}}, \quad j = 1, \dots, n. \tag{16}$$

Observe carefully how this result relies critically upon the two special features of Becker's model. First, the result depends upon the symmetric treatment of nonmarket activities with many different kinds of "leisures". But this property only is not enough since we also need $\frac{\partial R_i}{\partial x_j} = \frac{\partial R_i}{\partial t_j} = 0$ for $i \neq j$ which won't necessarily be true without the weak separability of the derived utility function. Secondly, a derived utility function will possess weak separability with the two-stage formulation. But this property only is not enough either since (11) may not even be defined without time as inputs in all nonmarket activities. Thus, for instance, it won't work for either form of following derived utility functions:

$$u(x_1, \dots, x_n, t_1, \dots, t_n); \; u(f(x_1, \dots, x_k), g(x_{k+1}, \dots, x_n), \textstyle\sum t_i).$$

We do need both features of Becker's model -- the symmetric treatment of nonmarket activities and the two-stage formulation -- to lead us to equation (16).

In appendix A, we calculate the right hand side of (16). Let $f^j_1 \equiv \frac{\partial f^j}{\partial x_j}$, $f^j_2 \equiv \frac{\partial f^j}{\partial t_j}$, $f^j_{12} \equiv \frac{\partial^2 f^j}{\partial t_j \partial x_j}$, $f^j_{11} \equiv \frac{\partial^2 f^j}{\partial x_j^2}$ and $f^j_{22} \equiv \frac{\partial^2 f^j}{\partial t_j^2}$. By (a3) of appendix A, (16) becomes

$$\frac{\theta_j}{1+\theta_j} = \frac{\lambda - \alpha}{\lambda} \frac{x_j t_j [f^j_{11}(f^j_2)^2 - 2 f^j_1 f^j_2 f^j_{12} + f^j_{22}(f^j_1)^2]}{x_j t_j f^j_{22}(f^j_1)^2 - x_j t_j f^j_1 f^j_2 f^j_{12} - t_j f^j_1 (f^j_2)^2}, \quad j = 1, \dots, n. \tag{17}$$

Equation (17) can be used to check a conventional result. If $f^j_{22} \to \infty$ (i.e., a completely inelastic conditional derived demand for t_j) for all j, then $\frac{\theta_j}{1+\theta_j} = \frac{\lambda - \alpha}{\lambda}$, $j = 1, \dots n$. Uniform taxation is optimal. But $f^j_{22} \to \infty$ for all j implies that labor supply becomes completely inelastic.[8] Thus we have found a sufficient condition for a conventional result. It is worth noting that the sufficient condition here is based upon properties of household production not consumer preferences.

In general, little more can be said from equation (17). However, much insight can be gained if one makes the additional assumption that the household production functions (2) are homothetic. This allows us to characterize the optimal tax structure in terms of the parameters of the household production functions and, in particular, to see its relationship with the elasticity of substitution in household production between goods and time.

1.2.3. Homothetic household production functions

Let f^j be a homothetic function, then it can be written as

$$f^j(x_j, t_j) = F(g^j(x_j, t_j)),$$

where $F' > 0$ and g^j is linearly homogeneous. By this property,

$$\frac{\dfrac{\partial f^j}{\partial x_j}}{\dfrac{\partial f^j}{\partial t_j}} = \frac{\dfrac{\partial g^j}{\partial x_j}}{\dfrac{\partial g^j}{\partial t_j}}.$$

Thus the marginal rate of substitution R_j remains the same regardless of whether the household production function f^j is homothetic or linear homogeneous. From equation (16), it is then clear that there is no more restrictive to assume that the household production functions (2) are linear homogeneous than to assume that they are homothetic. Hence, without loss of any generality, we only work with linearly homogeneous household productions in what follows.

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8 $u^j_{22} \equiv \dfrac{\partial\left(\dfrac{\partial u}{\partial z_j}\dfrac{\partial f^j}{\partial t_j}\right)}{\partial t_j} = \dfrac{\partial u}{\partial z_j}\dfrac{\partial^2 f^j}{\partial t_j^2} + \dfrac{\partial^2 u}{\partial z_j^2}\left(\dfrac{\partial f^j}{\partial t_j}\right)^2$. Thus $f^j_{22} \to \infty$ implies $u^j_{22} \to \infty$. If this is true for all j, the demand for "leisures" (and hence the supply of labor) becomes completely inelastic.

With linearly homogeneous f^j, by Euler's theorem

$$x_j f_1^j + t_j f_2^j = f^j . \tag{18}$$

Differentiating (18) with respect to x_j and t_j respectively

$$f_{11}^j = -\frac{t_j}{x_j} f_{12}^j ; \quad f_{22}^j = -\frac{x_j}{t_j} f_{12}^j . \tag{19}$$

Substituting (19) into (17) and using (18) yields[9]

$$\frac{\theta_j}{1+\theta_j} = \frac{\lambda - \alpha}{\lambda} \frac{1}{\dfrac{x_j f_1^j}{f^j} + \sigma_j \dfrac{t_j f_2^j}{f^j}} , \quad j = 1, \ldots, n. \tag{20}$$

where σ_j is the elasticity of substitution between good x_j and time t_j in the household production activity f^j.[10] Note that $\frac{x_j f_1^j}{f^j}$ is the "contribution" share of the market good x_j in consumption activity z_j and $\frac{t_j f_2^j}{f^j}$ is that of time t_j . They can also be understood as the partial output elasticity of consumption activity z_j with respect to good x_j and time t_j respectively.

Equation (20) has some special structure. To begin with, note that with linearly homogeneous f^j, the "contribution" shares of both the market good x_j and time t_j are all functions of the ratio $\frac{t_j}{x_j}$,[11] which in turn (by the consumer's optimization behavior) depends only upon the excise tax θ_j and the parameters of the household production function f^j. Assume that the elasticity of substitution σ_j is a constant or a function of $\frac{t_j}{x_j}$. Further, assume that the consumer's marginal utility α of income and the shadow price λ of the government's revenue can

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9 With distortionary taxes, if all the consumption activities z are normal, then $\lambda > \alpha$; however, if z are not all normal, it is possible that $\lambda < \alpha$ [Atkinson and Stern (1974)]. But by equation (20), we see that with homothetic household production functions and with positive tax revenue, $\lambda > \alpha$ must be true.

10 With linearly homogeneous f^j, $\sigma_j = \dfrac{f_1^j f_2^j}{f^j f_{12}^j}$.

11 When f^j is linearly homogeneous, the marginal products and the average products are all functions of $\frac{t_j}{x_j}$.

be estimated. Then by simply knowing the parameters of the household production function f^j, one can calculate the optimal tax θ_j using (20) [see the quantitative analysis later]. Since $\frac{\lambda - \alpha}{\lambda}$ is a common factor in (20), one may arbitraily choose a tax, say θ_1, to obtain

$$\frac{\dfrac{\theta_j}{1+\theta_j}}{\dfrac{\theta_1}{1+\theta_1}} = \frac{\dfrac{x_1 f_1^1}{f^1} + \sigma_1 \dfrac{t_1 f_2^1}{f^1}}{\dfrac{x_j f_1^j}{f^j} + \sigma_j \dfrac{t_j f_2^j}{f^j}}, \quad j = 2, \ldots, n. \tag{21}$$

Equation (21) establishes θ_j as a function of θ_1 for all $j \neq 1$ and is independent of $\frac{\lambda - \alpha}{\lambda}$. Thus by the same arguments about (20), the optimal relative tax structure can be determined only based upon the household production functions (2).[12]

The power of equation (20) can be most appreciated if one compares it with the standard result (10). There, the assumption that the household production functions (2) are homothetic does not allow us to simplify equation (10) at all. Hence, with the same assumption that α and λ can be estimated, one still needs to solve a system of equations involving all the cross-price and own-price effects to determine the optimal tax rate θ_j. Since homothetic production functions are fairly general and, moreover, since different consumption activities can be approximated by different homothetic functional forms, the homothetic assumption is not very restrictive in Becker's model. Consequently, equation (20) should be very useful in empirical studies.[13]

The intuition behind equation (20) can be understood as follows. The imposition of excise taxes changes the relative shadow prices of different consumption activities, causing substitutions among them. But since the derived utility function (3) is weakly separable in goods

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[12] With "working" t_w as the numéraire and setting $w = 1$, θ_1 can no longer be normalized. Thus although the optimal relative tax structure is independent of consumer preferences (1), the optimal tax structure is not. What equation (21) does is to reduce the optimal tax structure into a single parameter system. As soon as θ_1 is known, the rest of tax rates will ensue.

[13] For empirical studies dealing with the measurement and valuation of nonmarket activities, see the survey by Murphy (1980).

and time used in different consumption activities, the marginal rate of substitution between goods and time for any consumption activity will be independent of this change. In other words, as tax rates change, the relative level of consumption activities z will change, but the isoquant map for any z_j remains the same. The homotheticity of household production further implies that the marginal rate of substitution R_j will be independent of the level of z_j and dependent only upon the time intensity $\frac{t_j}{x_j}$. But $\frac{t_j}{x_j}$ in turn (by the consumer's optimization behavior) depends only upon the tax θ_j and is independent of all other taxes θ_i for $i \neq j$. In fact, as long as θ_j remains constant, the changes of z_j, $j=1, \ldots, n$, will not change the ratio $\frac{t_j}{x_j}$. Thus the substitutions among consumption activities z caused by taxes by themselves bring no tax distortion between goods and time. The only distortionary effect caused by θ_j is the "pure" substitution between x_j and t_j or the change in the ratio $\frac{t_j}{x_j}$. This is is the basic reason leading to the special structure of equation (20). On the one hand, the value of $\frac{\lambda - \alpha}{\lambda}$ relies upon the whole tax structure and consumer preferences (1), and is related to the non-distortionary substitutions among consumption activities z. On the other hand, the distortionary substitution effect caused by θ_j depends only upon the parameterrs of the household production function f^j. Since the substitutions among consumption activities by themselves are non-distortionary, it leads to no tax discrimination among goods, as shown by the common factor $\frac{\lambda - \alpha}{\lambda}$ in (20). Any tax discrimination among goods must come from different distortionary substitution effects between goods and time within different household production activities. This explains why the optimal relative tax structure (21) can be determined only based upon the household production functions (2).

There is an alternative intuition behind the result. Because the derived utility function (3) is weakly separable, the consumer's budgeting process can be decomposed into two stages [see Deaton and Muellbauer (1980) or Varian (1984)]. In the first stage, "full income" is allocated among different consumption activities. This determines a level of z_j to produce for each j. In the second stage, goods are combined with time within each consumption activity. This determines a time intensity $\frac{t_j}{x_j}$ for each z_j. In general, the decision of the second stage will be dependent upon that of the first stage. But with homothetic household production functions, they become independent. In general, the choice of $\frac{t_j}{x_j}$ in the second stage will depend upon $\frac{t_i}{x_i}$ for $i \neq j$. But with the special form of weak separability of the derived utility function (3), they also become independent. Thus if one ignores the income effects of taxation (which are captured by $\frac{\lambda - \alpha}{\lambda}$ in (20)), the two independences above enable the government to choose an optimal tax θ_j to determine an optimal time intensity $\frac{t_j}{x_j}$ only based upon the information about the household production function f^j.

The two-stage budgeting process allows the consumer's program be decomposed into many subprograms. If the proportion of "full income" spent on a consumption activity is known, the allocation of goods and time within the consumption activity will only depend upon the relative prices of goods and time used in that consumption activity [see, again, Deaton and Muellbauer (1980) or Varian (1984)]. Equation (20) provides an analogous outcome for the government's program. Namely, if the consumer's marginal utility α of income and the shadow price λ of the government's revenue are known, then the optimal tax rate levied on the market goods used in a consumption activity can be determined only based upon the information about that consumption activity and is independent of not only all other consumption activities but also consumer preferences. Thus given α and λ, the government's program is virtually decomposed into many subprograms. This allows one to determine "partial" optimal tax rates based upon much less demanding "local" information.

Now let's turn to consider the relationship between the optimal tax structure and the elasticity of substitution in household production between goods and time. Equation (20) immediately implies that if the elasticity of substitution $\sigma_j = 1$ for all j, uniform taxation will be optimal. Thus if the class of Cobb-Douglas functions or its monotonic transformation is a suitable approximation to all the household production functions (2), uniform excise taxes should be adopted. In contrast to the existing literature, the result here is based upon restrictions on household productions not on consumer preferences.

From (20) one can obtain another result. If the substitution in household production between goods and time is elastic ($\sigma_j > 1$), $\frac{\theta_j}{1+\theta_j} < \frac{\lambda - \alpha}{\lambda}$; but if inelastic ($\sigma_j < 1$), $\frac{\theta_j}{1+\theta_j} > \frac{\lambda - \alpha}{\lambda}$. This result confirms the usual intuition. Before taxing, there exists an optimal time intensity for every consumption activity. An increased tax on good x_j causes the substitution in production between x_j and t_j and, consequently, distorts the time intensity $\frac{t_j}{x_j}$. The extent of tax distortion brought about by this substitution can be thought related to the magnitude of $\frac{x_j f_1^j}{f^j} + \sigma_j \frac{t_j f_2^j}{f^j}$. The unitary elasticity of substitution provides, in some sense, a mark to dichotomize the extent of tax distortions into larger or smaller ones. Therefore, it also dichotomizes the optimal tax rates into higher or lower ones with $\frac{\lambda - \alpha}{\lambda}$ serving as the benchmark. There are two extreme cases. If $\sigma_j \to \infty$ (i.e., x_j and t_j are perfect substituts), then $\theta_j = 0$ regardless of the contributuion shares of good or time. On the other hand, if $\sigma_j = 0$ (i.e., x_j and t_j are perfect complements), then the contribution share of time plays no role at all in the optimal tax structure.

From (10) let $\frac{\partial x_i}{\partial q_j} = 0$ for all $i \neq j$, then it yields the famous inverse elasticity rule:

$$\frac{\theta_j}{1+\theta_j} = \frac{\lambda - \alpha}{\lambda} \frac{1}{\varepsilon_j}, \quad j = 1, \dots, n, \tag{22}$$

where $\varepsilon_j = -\frac{\partial x_j}{\partial q_j} \frac{q_j}{x_j}$, i.e., the own-price elasticity of demand for x_j. By (22), if the demand for x_j is elastic ($\varepsilon_j > 1$), $\frac{\theta_j}{1+\theta_j} < \frac{\lambda - \alpha}{\lambda}$; but if inelastic ($\varepsilon_j < 1$), $\frac{\theta_j}{1+\theta_j} > \frac{\lambda - \alpha}{\lambda}$. Hence with the assumption that cross price effects are zero or household production functions are homothetic, $\frac{\lambda - \alpha}{\lambda}$ becomes a benchmark for optimal tax rates. Those markets goods with inelastic demand or inelastic substitution with time in household production should be taxed higher than this benchmark; otherwise taxed lower. Suppose there comes out a public policy which desires to dichotomize all market goods into two groups such that one levied with a higher tax rate and the other with a lower tax rate.[14] With appropriate assumptions, the crite-

[14] In contrast to the "two-class" tax rule [Mirrlees (1975)], this may be dubbed as the "two-category" tax rule.

rion of the dichotomy can then be based either upon demand elasticities or upon elasticities of substitution in household production between goods and time. However, assuming homothetic household productions seems much more tenable than assuming zero cross effects of demand.

The analyses so far are qualitative: in the following we turn to quantitative analyses. For the sake of concreteness, we calculate optimal excise taxes for two specific linearly homogeneous household production functions. The first has a constant elasticity of substitution, the so called CES production function [Arrow, Chenery, Minhas and Solow (1961)]. The second is called the Revankar production function [Revankar (1971)]; its elasticity of substitution is variable and depends upon the ratio between goods and time.

CES Production Function

The CES production function has the following functional form:

$$z_j = A_j[\gamma_j x_j^{-\rho_j} + (1-\gamma_j)t_j^{-\rho_j}]^{-\frac{1}{\rho_j}}, \tag{23}$$

where $A_j > 0$, $0 < \gamma_j < 1$, and $\rho_j > -1$. It is known that the CES production function has

$$\sigma_j = \frac{1}{1+\rho_j}. \tag{24}$$

From (23), calculating f_1^j and f_2^j to substitute into (20)

$$\frac{\theta_j}{1+\theta_j} = \frac{\lambda-\alpha}{\lambda} \frac{1}{\dfrac{1}{1+(\dfrac{1-\gamma_j}{\gamma_j})(\dfrac{x_j}{t_j})^{\rho_j}} + \sigma_j \dfrac{1}{1+(\dfrac{\gamma_j}{1-\gamma_j})(\dfrac{t_j}{x_j})^{\rho_j}}}. \tag{25}$$

By the consumer's optimization behavior

$$\frac{t_j}{x_j} = \left(\frac{1-\gamma_j}{\gamma_j}\right)^{\sigma_j}(1+\theta_j)^{\sigma_j}. \tag{26}$$

Substituting (26) into (25) yields

$$\frac{\theta_j}{1+\theta_j} = \frac{\lambda-\alpha}{\lambda}\,\frac{1+\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\sigma_j}(1+\theta_j)^{-\rho_j\sigma_j}}{1+\sigma_j\left(\frac{1-\gamma_j}{\gamma_j}\right)^{\sigma_j}(1+\theta_j)^{-\rho_j\sigma_j}}. \tag{27}$$

Because (27) is not a closed form, a numerical method is used to calculate $\frac{\theta_j}{1+\theta_j}$. In order to focus on the sensitivity of tax rates to changes in the elasticity of substitution between goods and time, we set $\gamma_j = 0.5$ and arbitrarily let $\frac{\lambda-\alpha}{\lambda} = 0.1$. The results are as follows:

σ_j	=	0 -- 0.01	0.09	0.20	0.25	0.40	0.50	0.67	0.77	1.00
$\frac{\theta_j}{1+\theta_j}$	=	0.18	0.17	0.16	0.15	0.14	0.13	0.12	0.11	0.10
σ_j	=	1.11	1.43	1.67	2.20	2.86	3.33	5.00	10.0	20.0	100 -- ∞
$\frac{\theta_j}{1+\theta_j}$	=	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00

So as the elasticity of substitution σ_j increases from 0 to ∞, the optimal tax rate $\frac{\theta_j}{1+\theta_j}$ decreases from 0.18 to 0.

By the inverse elasticity rule (22), there exists an inverse relationship betwen the optimal excise tax rates and the own-price elasticities. For the CES household production function, the above numerical calculation also indicates an inverse relationship but between the optimal excise tax rates and the elasticities of substitution in household production between goods and time. The intuition behind both results can be put properly in Sandmo's words: "[M]inimize the deviation from the nondistortive, pre-tax allocation" [Sandmo (1976, p.46)].

Revankar Production Function

With $\frac{f_1^j}{f_2^j} = a_j + b_j \frac{t_j}{x_j}$, the Revankar production function has the following functional form:

$$z_j = [a_j x_j^{1+b_j} + (1+b_j) t_j x_j^{b_j}]^{\frac{1}{1+b_j}}, \tag{28}$$

where $b_j > 0$ and $\frac{t_j}{x_j} > -\frac{a_j}{b_j}$. One can show that

$$\sigma_j = 1 + \frac{a_j}{b_j} \frac{x_j}{t_j}. \tag{29}$$

Note that σ_j is not a constant but changes as $\frac{x_j}{t_j}$ changes, and the rate of change is a constant, i.e., $\frac{a_j}{b_j}$. Further, if $a_j = 0$, $\sigma_j = 1$. Under this restriction, a Revankar function is reduced to a Cobb-Douglas function.

From (28), calculating f_1^j and f_2^j to substitute into (20)

$$\frac{\theta_j}{1+\theta_j} = \frac{\lambda - \alpha}{\lambda} \frac{1}{\dfrac{1}{1 + \dfrac{1}{a_j(\frac{x_j}{t_j}) + b_j}} + (1 + \frac{a_j}{b_j}\frac{x_j}{t_j}) \dfrac{1}{a_j(\frac{x_j}{t_j}) + (1+b_j)}}. \tag{30}$$

By the consumer's optimization behavior

$$\frac{t_j}{x_j} = \frac{1 + \theta_j - a_j}{b_j}. \tag{31}$$

Substituting (31) into (30) yields

$$\frac{\theta_j}{1+\theta_j} = \frac{1}{1 + \dfrac{1}{\frac{\lambda - \alpha}{\alpha}(1 - \frac{a_j}{1+b_j})}}. \tag{32}$$

By (32), one can see that there is an "almost" inverse relationship between the optimal excise tax rate $\frac{\theta_j}{1+\theta_j}$ and $\frac{a_j}{b_j}$. If we restrict attention to the class of Revankar functions in which either a_j or b_j is the same for all j, then by (29) and (32) the greater the response of σ_j to changes in $\frac{x_j}{t_j}$ (i.e., the higher $\frac{a_j}{b_j}$), the smaller $\frac{\theta_j}{1+\theta_j}$ should be. At least in these cases there will be an inverse relationship between the optimal tax rates and the elasticities of substitution in household production between goods and time.

1.2.4. Public expenditures

In the previous discussions, only the revenue side of the government's budget is considered. The expenditure side is completely ignored. This is an unsatisfactory approach because the need to tax is simply derived from the need to provide public services. In this subsection, we explore the consequence of this omission on the optimal tax structure by assuming that the given tax revenue C is accompanied by a given (exogenously determined) pattern of public expenditures.

All public expenditures enter utility directly in the standard model. In contrast, only some public expenditures enter consumer preferences (1) directly by the household production approach [an example is defense, which can be consumed without other inputs of market goods or time]. Others will enter utility indirectly through the household production functions (2) [an example is highway facilities; the consumer must combine some market good (a car) and transit time with this public expendiure to produce a consumption activity called "travel"]. To incorporate the former type of public expenditures, the utility function (1) can be rewritten as

$$U = u(z; z_0), \tag{1'}$$

where z_0 is a vector and represents those public expenditures entering consumer preferences directly. For the second type of public expenditures, their impacts can be approximated by factor-augmenting effects on household production activities [for instance, with the improvement of highway facilities, the same car and amount of time enable the consumer to travel further and with more comfort]. After taking these types of public expenditures into account, the household production functions (2) can be rewritten as

$$z_j = f^j(\kappa_{1j} x_j, \kappa_{2j} t_j), \quad j = 1, \ldots, n, \tag{2'}$$

where κ_{1j} and κ_{2j} are, respectively, the goods-augmenting and the time-augmenting factors. From the viewpoint of consumer choice, z_0 and $(\kappa_{1j}, \kappa_{2j})$ are all parameters. For a given pattern of public expenditures, there will be a corresponding z_0 and $(\kappa_{1j}, \kappa_{2j})$, $j=1, \ldots, n$.

Assuming that the household production functions (2′) are homothetic, then formula (20) remains the same.[15] By our previous understanding about equation (20), only the values of α or λ may change with the replacement of (1) by (1′). Thus the optimal relative tax structure remains unchanged. However, the replacement of (2) by (2′) is different. The factor-augmenting types of public expenditures can cause changes in the optimal relative tax structure. We illustrate this by again using CES and Revankar production functions. Assume that the public expenditures have augmented good x_j by a factor κ_{1j} and time t_j by κ_{2j}, then the optimal tax formulae (27) and (32) become respectively[16]

$$\frac{\theta_j}{1+\theta_j} = \frac{\lambda - \alpha}{\lambda} \frac{1 + (\frac{\kappa_{1j}}{\kappa_{2j}})^{\rho_j \sigma_j} (\frac{1-\gamma_j}{\gamma_j})^{\sigma_j} (1+\theta_j)^{-\rho_j \sigma_j}}{1 + \sigma_j (\frac{\kappa_{1j}}{\kappa_{2j}})^{\rho_j \sigma_j} (\frac{1-\gamma_j}{\gamma_j})^{\sigma_j} (1+\theta_j)^{-\rho_j \sigma_j}}, \tag{33}$$

and

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15 This assertion can be verified as follows. First, let $f^j(\kappa_{1j}x_j, \kappa_{2j}t_j) = g^j(x_j, t_j)$, then equaton (17) holds with f^j replaced by g^j. Next, regard $\kappa_{1j}x_j$ and $\kappa_{2j}t_j$ as new variables, then equation (17) again holds with x_j and t_j replaced by $\kappa_{1j}x_j$ and $\kappa_{2j}t_j$ respectively (here all derivatives are with respect to $\kappa_{1j}x_j$ and $\kappa_{2j}t_j$). Finally, one can verify that equation (20) holds.

16 (26) and (31) change respectively to $\frac{t_j}{x_j} = (\frac{1-\gamma_j}{\gamma_j})^{\sigma_j}(1+\theta_j)^{\sigma_j}(\frac{\kappa_{1j}}{\kappa_{2j}})^{\rho_j\sigma_j}$ and $\frac{t_j}{x_j} = \frac{1+\theta_j - (\frac{\kappa_{1j}}{\kappa_{2j}})a_j}{b_j}$.

$$\frac{\theta_j}{1+\theta_j} = \frac{1}{1 + \cfrac{1}{\frac{\lambda-\alpha}{\alpha}\left[1 - \left(\frac{\kappa_{1j}}{\kappa_{2j}}\right)\frac{a_j}{1+b_j}\right]}} . \tag{34}$$

Compared with (27) and (32), it is obvious that if $\frac{\kappa_{1j}}{\kappa_{2j}} = 1$, we have the same optimal relative tax structure as before. This makes sense. As noted before, the optimal tax θ_j depends only upon the parameters of f^j given the values of λ and α. By the consumer's optimization behavior, $1 + \theta_j$ will be equal to the marginal rate of substitution R_j, and so the relative marginal product $(\frac{\partial f^j}{\partial x_j} / \frac{\partial f^j}{\partial t_j})$. But with $\frac{\kappa_{1j}}{\kappa_{2j}} = 1$, the relative marginal product remains unchanged for any given time intensity $\frac{t_j}{x_j}$.[17] Therefore no efficiency can be gained by modifying the previous relative tax structure. But if $\frac{\kappa_{1j}}{\kappa_{2j}} \neq 1$, there may then exist efficiency gain by having a different relative tax structure, as shown by (33) and (34).[18]

In Becker's model, goods-augmenting effects are equivalent to increases in the productivity of working time, and time-augmenting effects are equivalent to increases in the productivity of consumption time. If public expenditures bring about equal advances in the productivity of working and consumption time (i.e., $\frac{\kappa_{1j}}{\kappa_{2j}} = 1$), the same optimal relative tax structure without considering the expenditure side of the government's budget remains optimal. But if public expenditures bring about unequal advances (i.e., $\frac{\kappa_{1j}}{\kappa_{2j}} \neq 1$), then the omission of public expenditures from the model may lead to a sub-optimal tax structure.

Public expenditures will affect the productivity of consumption activities, so will many other variables such as age, climate, education, family size, and socio-political factors. These are called "environmental variables" in Becker's approach. Basically, the problem of optimal

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17 Without any loss of generality, we can assume that f^j is linearly homogeneous. Hence (2′) can be rewritten as $z_j = \kappa_{1j} f^j(x_j, \frac{\kappa_{2j}}{\kappa_{1j}} t_j)$. With $\frac{\kappa_{2j}}{\kappa_{2j}} = 1$, $z_j = \kappa_{1j} f^j(x_j, t_j)$. This leads to our claim.

18 However, note that if $\sigma_j = 1$ ($\rho_j = 0$ in CES production functiuon or $a_j = 0$ in Revanker production function), the optimal relative tax structure becomes independent of $\frac{\kappa_{1j}}{\kappa_{2j}}$. This can be seen easily from (33) and (34).

excise taxes is to choose a tax structure (and hence a price structure) to "control" consumer behavior so as to achieve the least tax distortion in a second-best world. But it is well known from empirical studies, consumer behavior can not be entirely be explained by changes in relative market prices. Quite a portion may be due to changes in environmental variables. To keep the tax structure optimal, it is important to account for these changes. Through the household production functions, the effects of changes in environmental variables can be systematically incorporated into the model [see Becker (1971) and Becker and Michael (1976)]. This provides some advantage of using Becker's model to approach optimal excise taxes.

1.3. Fixed coefficient household production functions

In this section, we use fixed coefficient household production functions to analyze the connections between Becker's approach and the standard labor-leisure analysis in the context of optimal taxation.

Let's assume that the household production functions (2) are the following type:

$$x_j = a_j z_j \,; \quad t_j = b_j z_j \,, \quad j = 1, \ldots, n. \tag{35}$$

We restrict $t_j > 0$ for all j in section II. Here we relax the assumption and allow $t_j = 0$ for some j (i.e., $b_j = 0$).

Substituting (35) into the consumer's Lagrangian ℓ gives

$$\ell = u(z) + \alpha[wT - \sum (q_i a_i + wb_i) z_i].$$

Differentiating ℓ with respect to z, the first order conditions are

$$u_j = \alpha(q_j a_j + wb_j), \quad j = 1, ..., n, \tag{36}$$

where $u_j \equiv \frac{\partial u}{\partial z_j}$.

Next, for the convenience of analysis, we assume that there exists a consumption activity z_k such that $a_k = 0$ and $b_k = 1$, i.e., no goods but time used as inputs in the kth consumption activity. With this assumption, from (36)

$$u_k = \alpha w. \tag{37}$$

Now by (35), (36) and (37), the consumer's budget constraint (6) can be rewritten as

$$u_k T - \sum u_i z_i = 0.$$

So the government's Lagrangian $\mathscr{L}$ becomes

$$\mathscr{L} = u(z) + \lambda[wT - \sum(a_i + wb_i)z_i - C] + \mu[u_k T - \sum u_i z_i].$$

Differentiating $\mathscr{L}$ with respect to z and setting it equal to zero yields the first order conditions

$$u_j = \lambda(a_j + wb_j) + \mu u_j(1 + \sum_i \frac{u_{ij} z_i}{u_j} - \frac{u_{kj} T}{u_j}), \quad j \neq k, \tag{38}$$

and

$$u_k = \lambda w + \mu u_k(1 + \sum_i \frac{u_{ik} z_i}{u_k} - \frac{u_{kk} T}{u_k}). \tag{39}$$

Define

$$H^j \equiv -\left(\sum_i \frac{u_{ij} z_i}{u_j} - \frac{u_{kj} T}{u_j}\right),\ j \neq k;\ H^k \equiv -\left(\sum_i \frac{u_{ik} z_i}{u_k} - \frac{u_{kk} T}{u_k}\right).$$

Using (37) and the definition of H^k, from (39)

$$\mu = \frac{\lambda - \alpha}{\alpha} \frac{1}{H^k - 1}. \tag{40}$$

By (36), (40) and the definition of H^j, (38) yields

$$\frac{\theta_j}{1+\theta_j} = 1 - \frac{1}{\dfrac{\lambda}{\alpha} \dfrac{a_j + wb_j}{a_j} \dfrac{1}{1 + \dfrac{\lambda - \alpha}{\alpha} \dfrac{H^j - 1}{H^k - 1}} - \dfrac{wb_j}{a_j}},\quad j \neq k. \tag{41}$$

It is well known that if the demand for "leisure" ($\sum t_i$) (or the supply of labor (t_w)) is completely inelastic, then uniform taxation is optimal. This result can be derived with some restrictions on the fixed coefficient household production functions (35). Let $x_j = z_j$ for $j \neq k$ and $t_k = T - t_w$ (i.e., $a_j = 1$ and $b_j = 0$ for $j \neq k$; $a_k = 0$ and $b_k = 1$). Then if $u_{kk} \to \infty$ (i.e., a completely inelastic demand for z_k), $H^k \to \infty$. So from (41)

$$\frac{\theta_j}{1+\theta_j} = \frac{\lambda - \alpha}{\lambda},\quad j \neq k.$$

Uniform taxation is optimal. The result is not surprising, for, with the assumption that $u_{kk} \to \infty$ and the above restrictions on the household production functions, the model is reduced to the standard labor-leisure one with completely inelastic labor supply. However, without the restriction that $a_j = 1$ and $b_j = 0$ for $j \neq k$ but only $u_{kk} \to \infty$, (41) yields

$$\frac{\theta_j}{1+\theta_j} = 1 - \frac{1}{\frac{\lambda}{\alpha} + (\frac{\lambda}{\alpha} - 1)\frac{b_j}{a_j}}, \quad j \neq k.$$

Unless $\frac{b_j}{a_j}$ for all $j \neq k$ are equal to each other, uniform taxation is not optimal any more; the optimal tax rates are also governed by the technology coefficients of the household production functions. The result clearly indicates two things. First, Becker's model generalizes the standard labor-leisure analysis in the context of optimal taxation. This is a logical outcome since Becker's model itself is a generalization of the standard labor-leisure analysis. Second, it is important to take the technology condition of household production into account in determining the optimal tax structure. Ignoring them or being insensitive to their changes may very likely lead to a sub-optimal solution.

1.4. Conclusion

In this paper it is demonstrated that by using Becker's model one can gain new insights into the optimal tax problem. Two features of Becker's model -- the symmetric treatment of non-market activities and the two-stage formulation -- have been proved crucial in the derivation of a variety of new and refined results. As emphasized in the introduction, Becker's approach is not a mutally exclusive alternative to the standard labor-leisure model. Instead it is a refinement, and hence it allows richer findings. In particular, one can use the information about household production activities to infer the optimal tax structure. Further, the effects of environmental variables on the optimal tax structure can be systematically incorporated into the model through the household production functions. It seems that this line of approach deserves to be pursued further.

In this paper, we have begun the exploration of household production on the optimal tax structure by using the simplest possible versions of Becker's model and of excise taxes.

Further investigations might consider elaborations of these models such as relaxing the assumption of fixed producer prices in the excise tax model or considering joint production in Becker's model.

Finally, the optimal tax problem has a well-known close relationship with the optimal public pricing problem.[19] With reinterpretation and some minor changes, the results in this paper may well add insights to the latter problem.

[19] Comparing Boiteux (1956) with Ramsey (1927), one can see that there are a lot of formal similarities between the optimal public pricing problem and the optimal taxation problem.

Chapter 2. The Pricing of Public Intermediate Goods Revisited

2.1. Introduction

Most public enterprises (or public utilities) sell their outputs to the private firm as well as to the household. Those sold to the household are for final consumption and enter the household's utility directly. On the other hand, those sold to the private firm are used as inputs to further production and hence enter the household's utility indirectly. Based upon this observation of "directly" versus "indirectly", it seems intuitively that the pricing rule for the private firm should somehow be governed differently from that for the household in a second-best world.[20] In a pioneer paper, Feldstein (1972) discussed the pricing of publicly produced inputs for competitive downstream industries and found that Ramsey pricing for public intermediate goods is structurally the same as that for public final goods.[21] Later, Spencer and Brander

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20 In a first-best world, marginal cost pricing is the optimal rule for both the household and the private firm.

21 Efficient pricing with the budget balance on a public enterprise is often referred to as Ramsey pricing. The terminologies of public intermediate goods or public final goods are following Feldstein

(1983) also reached the same conclusion. However, Feldstein's model makes a fixed coefficient assumption on the production technology, and Spencer and Brander's is a partial equilibrium analysis. It is certainly of great interest both theoretically and practically to know whether the pricing rule for the private frim is still the same as that for the household under a more general setting. The purpose of this paper is to address this issue. As a by-product, we derive the optimal customer-class pricing rules for a public enterprise.

In the analysis of optimal public pricing, the price structure of the private sector is usually taken as given. The focus is on how the exogenously determined pricing behavior of the private sector (e.g., monopoly distortion) should be taken into account in setting public prices. As Bös (1985, p.139) put it: "[The] model aimed at showing the accommodation of public pricing to given pricing structures of the private economy" [see also Sheshinski (1986)]. Following Feldstein's and Spencer and Brander's models, we explicitly consider the dependence of the price structure of the private sector upon public prices in this paper.[22] It is a natural setup with the scenario of downstream industries. We explore the consequences on the public pricing rules with such a dependence.

Section 2.2 introduces the model of the analysis and considers a private sector with a price structure depending upon public prices. Under a general equilibrium setting and with a general production technology, we seek to answer whether the optimal pricing rule for the private firm is the same as that for the household. Section 2.3 derives some useful formulae and expresses the public pricing rules in terms of price elasticities of demand for publicly produced goods. It will be found that the information requirements in these pricing rules are not excessive and, in particular, need not involve knowledge about the production relations

(1972). As mentioned by Feldstein himself, it would more correctly be referred to as "publicly produced" intermediate or final goods.

22 For incorporating the monopoly distortion of the private sector in public pricing within this setting, see Spencer and Brander (1983).

of the private sector or the demand relations of privately produced goods. Section 2.4 is the conclusion.

2.2. The model

We assume a one-consumer economy and focus upon the efficiency aspect of public pricing.

Consider an economy with many identical consumers and a fixed number of private firms. There is a monopolistic public enterprise (or public utility) producing goods $z = z^h + z^f$ in otherwise competitive markets with $z^h = (z_1^h, ..., z_m^h)$ sold to the household as final goods and $z^f = (z_1^f, ..., z_m^f)$ to the private firm as intermediate goods. They are charged $p^h = (p_1^h, ..., p_m^h)$ and $p^f = (p_1^f, ..., p_m^f)$ respectively; p_i^h is not necessarily equal to p_i^f for any $i = 1, ..., m$. We assume that it is infeasible or highly costly to resell z between households and private firms for technical or legal reasons.

Consumer preferences are represented by the utility function $u(x, z^h, L_x + L_z)$, where $x = (x_1, ..., x_n)$ are the net outputs produced by private firms; L_x is the labor supplied to private firms and L_z to the public enterprise. The household makes no difference between labor supplied to private firms and that to the public enterprise. The utility function u is assumed to be nonsatiated and strictly quasi-concave. Without loss of generality, we choose labor as the numéraire.

Let Y_j be the production possibility set for the jth private firm and $Y = \sum_j Y_j$ is the aggregate production possibility set, which is assumed to be compact and strictly convex. A typical production plan in Y is (x, z^f, L_x) , where z^f and L_x are used as inputs to produce x. The production transformation for the public enterprise is $L_z = G(z^h + z^f)$, where G is assumed to exhibit increasing returns to scale. One can show that if production takes place under scale economies, deficits will occur with marginal cost pricing.[23] Thus if an increasing returns to scale public enterprise has to balance the budget on its own, then the public prices charged the household or the private firm must deviate from marginal cost in a second-best world. The question is: What is the optimal departures from marginal cost pricing and, in particular, should there be a uniform rule govern both types of customers or should it be different?

Under the standard assumptions that the household's objective is to maximize utility and the objective of the private firm to maximize profit, we have the following programs.

For the household:

$$\max_{\{x,\, z^h,\, L_x + L_z\}} \quad u(x, z^h, L_x + L_z) \qquad \text{subject to} \quad qx + p^h z^h = L_x + L_z + \pi, \tag{A}$$

where $q = (q_1, \ldots, q_n)$ are the prices for x, and π is the profit made by private firms and distributed to households. Since labor is used as the numéraire in our model, the wage rate $w = 1$.

For the private firm:[24]

$$\max_{\{x,\, z^f,\, L_x\}} \quad qx - p^f z^f - L_x \qquad \text{subject to} \quad (x, z^f, L_x) \in Y. \tag{B}$$

From program (A), one can derive the household's demand for $z^h = z^h(q, p^h, \pi)$ and the indirect utility function $v = v(q, p^h, \pi)$; from program (B), the profit function $\pi = \pi(q, p^f)$ and the derived demand of the private firm for $z^f = z^f(q, p^f)$. One can also derive the market equilibrium

23 Actually, strict local increasing returns to scale are a sufficient and necessary condition to have deficits under marginal cost pricing [Baumol (1976, 1977) and Panzar and Willig (1977)].

24 Maximizing aggregate profits over the aggregate production set is equivalent to maximizing each firm's profit over its own production set [e.g., Varian (1984)].

price vector q^* for x given p^h and p^f from programs (A) and (B). In a general equilibrium setting, q^* normally depends upon both p^h and p^f. We assume that it is unique for any given (p^h, p^f).[25]

Let $\pi^*(p^h, p^f) = p^h z^h(q^*, p^h, \pi(q^*, p^f)) + p^f z^f(q^*, p^f) - G(z^h(q^*, p^h, \pi(q^*, p^f)) + z^f(q^*, p^f))$ and $v^*(p^h, p^f) = v(q^*, p^h, \pi(q^*, p^f))$, where π^* is the public enterprise's profit function and v^* is the indirect utility function; both have taken account of the dependence of q^* upon p^h and p^f. The public enterprise is assumed to face the following program:

$$\begin{aligned} &\max_{\{p^h, p^f\}} v^*(p^h, p^f) \\ &\text{subject to} \\ &\pi^*(p^h, p^f) = 0, \end{aligned} \tag{C}$$

where $\pi^* = 0$ is the budget balance condition on the public enterprise.[26]

Proposition 1: $\frac{\partial v^*}{\partial p^h} = -\alpha z^h$ and $\frac{\partial v^*}{\partial p^f} = -\alpha z^f$, where α is the household's marginal utility of income.

Proof: By Roy's identity and Hotelling's lemma, $\frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^\ell} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^\ell} = 0$ for $\ell = h,f$.

Hence, $\frac{\partial v^*}{\partial p^h} = \frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^h} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^h} + \frac{\partial v}{\partial p^h} = \frac{\partial v}{\partial p^h} = -\alpha z^h$ and

$\frac{\partial v^*}{\partial p^f} = \frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^f} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^f} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial p^f} = \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial p^f} = -\alpha z^f$. Q.E.D.

In the proof, we use the properties that $\frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^\ell} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^\ell} = 0$ for $\ell = h,f$. Thus changes in the market equilibrium prices q^* due to changes in p^h or p^f bring no effect at all on the household's utility.

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25 Consider the following example with the utility function $u = \ln(x - a) + \ln(z^h - b) + \ln(1 - L_x - L_z)$ and the production function $x = (z^f)^{\frac{1}{4}}(L_x)^{\frac{1}{4}}$. One can show that $q^* = \frac{8a + 2\sqrt{16a^2 + 10(1 - bp^h)(p^f)^{-\frac{1}{2}}}}{5(p^f)^{-\frac{1}{2}}}$.

26 In program (C), all economic agents (the household, the private firms and the government) satisfy their budget constraints and all markets but one are in equilibrium, hence the last market (labor) must also be in equilibrium. This is due to Walras' Law [see Diamond and Mirrlees (1971)]. Furthermore, since the household does not differentiate between labor supplied to the private firms and that to the public enterprise, there is no need to treat the market for L_x and that for L_z separately.

With the above proposition at hand, it is easy to derive the optimal public pricing rule from the first order conditions of program (C):

$$\frac{1}{z_i^h}\frac{\partial \pi^*}{\partial p_i^h} = \frac{1}{z_j^h}\frac{\partial \pi^*}{\partial p_j^h} = \frac{1}{z_i^f}\frac{\partial \pi^*}{\partial p_i^f} = \frac{1}{z_j^f}\frac{\partial \pi^*}{\partial p_j^f} = \frac{\alpha}{\lambda}, \quad i, j = 1, \ldots, m, \tag{1}$$

where λ is the Lagrange multiplier associated with the constraint of program (C). This necessary condition for optimality requires marginal profit yields of price changes to be proportional to output levels, with z_i^h and z_i^f, $i = 1, \ldots, m$, treated as different products. It is clear that the pricing rule for public intermediate goods is not different from that for public final goods in this formulation. As mentioned in the introduction, this result is not so intuitively obvious as it may seem, for public final goods enter the household's utiltiy directly and public intermediate goods indirectly.

Equation (1) can be regarded as a generalization of Version I of Ramsey pricing in Baumol and Bradford (1970). In fact, if $p^h = p^f = p$ is required, i.e., no price discrimination between the household and the private firm, then the optimal pricing rules become identical to Version I of Ramsey pricing in Baumol and Bradford (1970, (2a)):[27]

$$\frac{1}{z_i}\frac{\partial \pi^*}{\partial p_i} = \frac{1}{z_j}\frac{\partial \pi^*}{\partial p_j}, \quad i, j = 1, \ldots, m. \tag{2}$$

Compared with equation (1), we see that it is no longer necessary to differentiate between those sold to the household z^h and those sold to the private firm z^f ; only the aggregate $z = z^h + z^f$ matters in the optimal pricing rule.

Remark 1. We assume the aggregate production set Y to be strictly convex. This assumption makes the solution from program (B) to be unique. It also faciliates the derivation of Propo-

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27 With the restriction $p^h = p^f = p$, $v^*(p^h, p^f) = v^*(p) = v(q^*, p, \pi(q^*, p))$. Following a proof similar to that of Proposition 1, one can then derive $\frac{dv^*}{dp} = -\alpha(z^h + z^f) = -\alpha z$.

sition 1.[28] However, this assumption excludes constant returns to scale production technology from consideration. To take care of the constant returns to scale case, one can assume that the production technology of Y is captured by some parameter vector γ such that Y exhibits constant returns to scale if $\gamma =$; and decreasing returns to scale otherwise.[29] Denote the aggregate production set Y with γ technology by $Y(\gamma)$. If $\gamma \rightarrow \gamma_0$ implies $Y(\gamma) \rightarrow Y(\gamma_0)$, then one can approximate the technology of constant returns to scale as close as possible by that of decreasing returns to scale. Alternatively, one may assume that the derivatives of v^* with respect to p^h and p^f are continuous in γ. With this continuity property, Proposition 1 holds when $\gamma = \gamma_0$ [see appendix B].

Remark 2. Immediately after Proposition 1, we made a remark that changes in market equilibrium prices q^* due to changes in p^h or p^f do not change the household's utility. However, since both z^h and z^f are functions of q^*, changes in market equilibrium prices q^* due to changes in p^h or p^f can affect the profit of a public enterprise. With a balanced budget imposed upon the public enterprise, this effect needs to be taken into account when calculating the marginal profit yields of price changes $\frac{\partial \pi^*}{\partial p^h}$ or $\frac{\partial \pi^*}{\partial p^f}$ in equation (1). The magnitude of this effect may be significant and hence can play an important role in the optimal pricing.

Remark 3. In general, market equilibrium prices depend upon both public prices charged the household and the private firm. But even in a general equilibrium setting, there may exist some sufficient conditions such that $\frac{\partial q^*}{\partial p^h} = 0$ or $\frac{\partial q^*}{\partial p^f} = 0$. Here, we would like to discuss one of them -- the nonsubstitution theorem [Samuelson (1951)].[30] Let's assume that there are n private firms and that each produces one good $x_j > 0$, $j = 1, \ldots, n$, with the technology exhibiting

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28 If the profit function $\pi(q^*, p^f)$ is identically zero, which is possible with Y only exhibiting convex, then the indirect utility function $v(q^*, p^h, \pi(q^*, p^f))$ becomes $v(q^*, p^h)$. We can no longer use the properties that $\frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^\ell} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^\ell} = 0$ for $\ell = h, f$. To drive at Proposition 1 in this situation, one needs to make some additional assumptions [see appendix B].

29 We rule out the possibility that Y exhibits increasing returns to scale since it is incompatible with competitive markets.

30 The arguments here follows Varian (1984).

constant returns to scale. Since the production technology is constant returns to scale, the cost function $c_j(q, p^f, x_j)$ of each private firm can be rewritten as $g_j(q, p^f)x_j$, where g_j is some function of q and p^f. Furthermore, in equilibrium the profit of each private firm will be zero, i.e., $q_jx_j - g_j(q, p^f)x_j = 0$, $j = 1, ..., n$. With $x_j > 0$, $j=1, ..., n$, we get $q_j - g_j(q, p^f) = 0$, $j = 1, ..., n$. Thus the market equilibrium prices q^* are independent of p^h and only depend upon p^f. In theoretical or empirical studies, much trouble can be saved if one knows in advance that the nonsubstitution theorem is validly applicable.

2.3. Optimal pricing rules

Equation (1) can be transformed into more useful formulae by expressing optimal public prices in terms of price elasticities of demand for publicly produced goods. We consider the case of a public enterprise with a single product. First, define $y^f(p^h, p^f) = z^f(q^*, p^f)$ and $y^h(p^h, p^f) = z^h(q^*, p^h, \pi(q^*, p^f))$, where y^f and y^h are respectively the derived demand of the private firm and the household's demand for publicly produced goods after taking the dependence of q^* upon p^f and p^h into account. Let $\varepsilon^{fh} = \frac{\partial y^f}{\partial p^h}\frac{p^h}{y^f}$, $\varepsilon^{hf} = \frac{\partial y^h}{\partial p^f}\frac{p^f}{y^h}$, and $\varepsilon^{\ell\ell} = \frac{\partial y^\ell}{\partial p^\ell}\frac{p^\ell}{y^\ell}$, $\ell = f,h$. From equation (1), one can obtain

$$\frac{\alpha - \lambda}{\lambda} = \varepsilon^{hh}\theta^h + \frac{b}{1-b}\varepsilon^{fh}\theta^f, \tag{3}$$

and

$$\frac{\alpha - \lambda}{\lambda} = \varepsilon^{ff}\theta^f + \frac{1-b}{b}\varepsilon^{hf}\theta^h, \tag{4}$$

where $\theta^\ell = (p^\ell - \frac{\partial G}{\partial z})/p^\ell$, $\ell = f,h$, and $b = \frac{p^f z^f}{p^h z^h + p^f z^f}$. The variable θ^ℓ indicates the extent of the departure from marginal cost pricing; the variable b the proportion of revenue contribution from private firms. Equations (3) and (4) are pretty similar to equations (2.6) and (2.7) in Feldstein (1972), though our model has some differences from Feldstein's.[31]

Equations (3) and (4) can be solved for θ^f and θ^h :

$$\theta^{f} = \frac{\lambda - \alpha}{\lambda} \frac{\varepsilon^{hh} - \frac{1-b}{b}\varepsilon^{hf}}{\varepsilon^{fh}\varepsilon^{hf} - \varepsilon^{ff}\varepsilon^{hh}}, \tag{5}$$

and

$$\theta^{h} = \frac{\lambda - \alpha}{\lambda} \frac{\varepsilon^{ff} - \frac{b}{1-b}\varepsilon^{fh}}{\varepsilon^{fh}\varepsilon^{hf} - \varepsilon^{ff}\varepsilon^{hh}}. \tag{6}$$

Once again, the optimal pricing rules for public final goods and public intermediate goods are essentially the same in structure.

The ratio of optimal price differentials is

$$\frac{\theta^{f}}{\theta^{h}} = \frac{\varepsilon^{hh} - \frac{1-b}{b}\varepsilon^{hf}}{\varepsilon^{ff} - \frac{b}{1-b}\varepsilon^{fh}}. \tag{7}$$

If the cross-price elasticities of demand for publicly produced goods are zero, we obtain the celebrated "inverse elasticity" rule -- a larger departure from marginal cost pricing for the good with a lower own-price elasticity. Otherwise, the inverse elasticity rule needs to be modified by the cross-price elasticities of demand, which are further weighted by the revenue contribution of the private firm relative to that of the household. Note that there exists some symmetry in this weighting. It should be emphasized that the so called cross and own price elasticities of demand here are based upon y^f and y^h, not z^f and z^h. Thus they include the

[31] An obvious difference: Feldstein's model is a many-consumer setting; here, a one-consumer setting.

possible effects of changes in market equilibrium prices or in equilibrium profits due to changes in public prices.

When a public enterprise produces more than one product, the general formulae of pricing rules in terms of price elasticities of demand become much more complicated. However, the basic ideas embodied in equation (7) are carried over. For instance, let there be two publicly produced goods and define $\varepsilon_{ij}^{\ell\vartheta} = \frac{\partial y_i^{\ell}}{\partial p_j^{\vartheta}} \frac{p_j^{\vartheta}}{y_i^{\ell}}$ and $b_i^{\ell} = \frac{p_i^{\ell} z_i^{\ell}}{p_1^h z_1^h + p_2^h z_2^h + p_1^f z_1^f + p_2^f z_2^f}$, where $\ell,\vartheta = f,h$; $i,j = 1,2$. In the case of zero cross effects between the household's demand for z^h and the derived demand of the private firm for z^f, equation (1) yields

$$\frac{\theta_1^f}{\theta_2^f} = \frac{\varepsilon_{22}^{ff} - \frac{b_2^f}{b_1^f} \varepsilon_{21}^{ff}}{\varepsilon_{11}^{ff} - \frac{b_1^f}{b_2^f} \varepsilon_{12}^{ff}}, \tag{8}$$

and

$$\frac{\theta_1^h}{\theta_2^h} = \frac{\varepsilon_{22}^{hh} - \frac{b_2^h}{b_1^h} \varepsilon_{21}^{hh}}{\varepsilon_{11}^{hh} - \frac{b_1^h}{b_2^h} \varepsilon_{12}^{hh}}. \tag{9}$$

On the other hand, if the cross effects between the demand for z_1 and the demand for z_2 are zero, then equation (1) yields

$$\frac{\theta_1^f}{\theta_1^h} = \frac{\varepsilon_{11}^{hh} - \frac{b_1^h}{b_1^f} \varepsilon_{11}^{hf}}{\varepsilon_{11}^{ff} - \frac{b_1^f}{b_1^h} \varepsilon_{11}^{fh}}, \tag{10}$$

and

$$\frac{\theta_2^f}{\theta_2^h} = \frac{\varepsilon_{22}^{hh} - \dfrac{b_2^h}{b_2^f}\varepsilon_{22}^{hf}}{\varepsilon_{22}^{ff} - \dfrac{b_2^f}{b_2^h}\varepsilon_{22}^{fh}}. \tag{11}$$

It is obvious that all of these four formulae are structurally identical to equation (7).

Feldstein (1972) emphasized that the information requirements in the public pricing rules must not be excessive if they are to be of practical use. Although our model is a general setup, the information requirements in the optimal pricing formulae derived are fairly reasonable. In particular, since all the relevant information about the private sector can be summarized by the market equilibrium prices q^* and the equilibrium profit $\pi(q^*, p^f)$, the pricing rules need not involve knowledge about the production relations of the private economy or the demand relations of privately produced goods. This permits great informational efficiency.

2.4. Conclusion

We revisited the pricing of public intermediate goods in a general equilibrium setup and with a general production technology, and explored the consequences on the public pricing rules with the dependence of the price structure of the private sector upon public prices. It was found that changes in market equilibrium prices due to changes in public prices do not change household's utility. On the other hand, changes in market equilibrium prices due to changes in public prices can affect the profit of a public enterprise through changes in the household's demand for public final goods or changes in the derived demand of the private firm for public intermediate goods. With a balanced budget restriction on a public enterprise, these effects on the profit may play a significant role in public pricing.

Even in a general setting, Ramsey pricing for public intermediate goods is not different from that for public final goods. Nonetheless, since cross and own price elasticities of demand and the revenue contribution from the household relative to that from the private firm are expected to be different, it is usually desirable to practice price discrimination between them.

Chapter 3. Piecemeal Design and Reform of Commodity Taxes

3.1. Introduction

This paper is about "piecemeal" tax design and tax reform. Tax design, stictu sensu, means that a new tax system can be conceived and installed from scratch; to write the tax law "de novo on 'a clean sheet of paper'" [Feldstein (1976a, p.77)]. Tax reform, in contrast, starts from a sub-optimal existing tax system and aims to improve it with respect to a social welfare function or the Pareto criterion. Tax reform, as a rule, falls short of the global optimum of tax design. This could be due to various reasons [see Diewert (1978)]. One possibility is that information about relevant parameters is usually limited to a neighborhood of the existing position.

In "piecemeal" tax design, there are two groups of taxes. The first group consists of taxes which can be freely designed or redesigned. The second group consists of given taxes which are immutable, for whatever reasons. In "piecemeal" tax reform, one starts with an initial tax system, partitioned into two groups. The taxes in the first group can be changed, to some extent, whereas any change of the taxes in the second group is excluded.

There are many possible setups to analyze piecemeal tax design or tax reform. However, since taxation on the uses and the sources of income is most important to a government's revenue, we assume a tax system consisting of two different kinds of taxes -- commodity taxes and an income tax. The former represents a tax on the uses of income and the latter a tax on the sources of income. The intention of this paper is to analyze the design and the reform of commodity taxes with a given general income tax.

In the field of tax design, most studies concentrate upon commodity taxes or income tax. Ramsey (1927) and Diamond and Mirrlees (1971), among others, considered the design of commodity tax in the absence of any other taxes. Mirrlees (1971) and Sheshinski (1972) pioneered the design of income tax in the absence of other taxes. Atkinson and Stiglitz (1976) and Atkinson (1977) worked on the design of a tax system consisting of both commodity taxes and an income tax. Our piecemeal design of commodity taxes can be viewed as a case between tax design and tax reform. On the one hand, it recognizes the existing tax system (i.e., income tax). On the other hand, part of the tax system (i.e., commodity taxes) can be installed from scratch. Stiglitz and Dasgupta (1971) provided a notable example of piecemeal tax design within a different setting. Given a profit tax, they found that the optimal taxation rule for commodity taxes depends critically upon whether the given tax rate on pure profits is equal to or less than one. If this tax rate is less than one, the optimal rule for "piecemeal" commodity taxes will be different from that for commodity taxes without the presence of profit tax. The tax design of a particular tax depends upon other given taxes. This is an obvious message conveyed in their analysis. It is also the general conclusion derived from the theory of the second-best analysis [Lipsey and Lancaster (1956)].

The tax reform literature at large is two-fold. One branch of the literature emphasizes individual gains or losses during the process of tax reform and discusses the optimal way of reforming an existing tax structure (immediate, partial, postponed or phased-in enactment of reform) and balancing between welfare gains of tax reform and undesirable effects of the reform-induced arbitrary redistribution of income [e.g., Feldstein (1976a and 1976b) and Zodrow (1981, 1985)]. The other branch focuses on the direction of tax reform and aims at identifying the welfare-improving adjustment of tax structure [e.g., Corlett and Hague (1953), Dixit (1975) and Guesnerie (1977)]. Our analysis of piecemeal tax reform concentrates upon the second type of question.

For an accurate comparison, we reinterpret the usual analysis of commodity taxes by themselves as the case of "isolated" analysis in our model, by presuming that the presence of a given income tax has been ignored. With all the political and institutional constraints encountered in reality, both "piecemeal tax design" and "piecemeal tax reform" of commodity taxes should be more relevant than "isolated" ones for practical purposes.

First, the piecemeal design of commodity tax is our focus (section 3.2). In a many-consumer economy, the many-person Ramsey rule [Diamond (1975)] needs to be amended. The direction of the amendment critically depends upon whether the goods considered are complementary with or substitutive to leisure. When designing a tax in isolation, variations in tax rates will only affect the tax revenue of that particular tax. But when designing a tax piecemeally, variations in tax rates may affect the tax revenues of all other given taxes as well as those of the tax to be designed. With a balanced budget imposed on the government's welfare maximization program, these "spillover" effects need to be taken into account in the optimal taxation rules. In our setup, "spillover" effects are captured by the cross-effect between the uses of income and the sources of income, and hence between demands for goods and supply of labor. This is the reason why the relationship between goods and leisure (substitutes, complements or neutrals) plays a critical role in the characterization of optimal conditions for our piecemeal tax design.

In section 3.3 we turn to the piecemeal reform of commodity taxes, with and without lump-sum taxes. In the case where nondistortionary lump-sum taxes are used to replace commodity taxes, we follow the work of Dixit (1975) closely but with two generalizations. First, it is in the setting of a many-consumer, not one-consumer economy. Second, instead of considering tax reform in isolation, the reform of commodity taxes is considered with a given income tax. As a result, we extend or modify some well-established results obtained in Dixit. When lump-sum taxes are not available, we make use of the so called "inverse optimum problem" [see Ahmad and Stern (1984)] to infer the existence of a variety of feasible Pareto-improving directions. The relationship between the feasibility of Pareto-improving directions and the inverse optimum problem was first discovered by Guesnerie (1977) and later elaborated in Diewert (1978), Dixit (1979), and Ahmad and Stern (1984) among others. We also interpret the gradient projection algorithm of adjusting tax rates within our setting and connect the work of Tirole and Guesnerie (1981) with that of Ahmad and Stern (1984).

3.2. Piecemeal tax design

We adopt a model based upon Diamond and Mirrlees (1971) but with labor as the only factor. It has the advantage of allowing us to see more clearly the role that labor supply plays in our "piecemeal" tax design and tax reform analysis.

Let $N=\{1, ..., H\}$ be the society with generic member h. Labor is the only factor of production. It is measured in some efficiency unit. This implicitly implies that the members of society may differ in hourly productivities and, consequently, in hourly wages. We choose one efficiency unit of labor as the numéraire. Individual h supplies labor L^h ; aggregate labor supply is $L=\sum_h L^h$.[32] Besides labor, there are n market goods, indexed by $k=1, ..., n$, with

[32] In this paper $\sum a^h$ stands for $\sum^H a^h$ and $\sum a_i$ stands for $\sum^n a_i$. All the functions are assumed to be continuously differentiable. Weaker assumptions can of course be made for some functions. Unless

supplies denoted by $y = (y_1, ..., y_n)$ and demands by $x = (x_1, ..., x_n)$, where $x_j = \sum_h x_j^h$, $j = 1, ..., n$. The aggregate production of the private sector is governed by a convex cost function $F(y)$. Social feasibility consists of two requirements: First, market clearing $x = y$, and secondly, a social production constraint $L = F(y) + \overline{R}$, where $\overline{R}$ is the fixed tax revenue needed for some exogenously determined government activities.[33] F and $\overline{R}$ actually represent the labor input requirements from the private sector and the government respectively. The private sector obtains labor inputs through markets and the government through tax instruments. The production of the private sector is assumed to take place under constant returns to scale or else pure profits are taxed at 100%.[34] Without loss of generality, we assume that the 100% profit tax is imposed at the production side and no pure profit is distributed to consumers.

The hth member's preferences are represented by the utility function $u^h(x^h, L^h)$, which is assumed to be nonsatiated and strictly quasi-concave. Suppose that, besides the 100% profit tax, the tax system of society consists of two different kinds of taxes -- commodity taxes and an income tax. Let $\theta = (\theta_1, ..., \theta_n)$ be the unit commodity taxes on goods x and let T, a function of gross labor income, be the (labor) income tax schedule. Note that the tax treatment of the uses of income and that of the sources of income are not symmetric. This merely reflects the usual practice. Throughout the paper, the income tax schedule T is, for some exogenous reasons, treated as fixed. The government can only introduce or change commodity taxes. With this setup, the hth consumer faces the following program:

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unclear, we do not notationally differentiate between a vector (or a matrix) and its transpose. Let b be a vector, then $b >> 0$ means $b_i > 0$ for all i, $b > 0$ means $b_i \geq 0$ for all i with $b \neq 0$, and $b \geq 0$ means $b_i \geq 0$ for all i.

33 Note that with $x = y$ and a balanced budget for the government, $L = F(y) + \overline{R}$ must be true according to Walras' Law [see Diamond and Mirrlees (1971)]. However, in the tax reform analysis later, it may be necessary that $L > F(x) + \overline{R}$. This corresponds to the temporary production inefficiency described by Guesnerie (1977).

34 When pure profits are taxed less than 100%, see Stiglitz and Dasgupta (1971) and Dasgupta and Stiglitz (1972) for the tax design problem; Diewert (1978) and Dixit (1979) for the tax reform problem.

$$\max_{\{x^h, L^h\}} u^h(x^h, L^h) \quad \text{subject to } qx^h + T(L^h) = L^h, \tag{A}$$

where $q = (q_1, \ldots, q_n)$ are the consumer prices. Let $p = (p_1, \ldots, p_n)$ be the producer prices, and so $q = p + \theta$. Program (A) is solved by the commodity demand $x^h(q)$ and the labor supply $L^h(q)$, which determine the indirect utility $v^h(q) = u^h(x^h(q), L^h(q))$. Let M^h be a hypothetical lump-sum income for the hth consumer. If in (A), we replace the constraint $qx^h + T(L^h) = L^h$ by $qx^h + T(L^h) = L^h + M^h$, then the envelope theorem (applied at $M^h = 0$) yields $\frac{\partial v^h}{\partial M^h} = \alpha^h$ and $\frac{\partial v^h}{\partial q} = -\alpha^h x^h$, where α^h is the marginal utility of the hth consumer's income. With the nonsatiation assumption on the utility functions, $\alpha^h > 0$ for all $h \in N$.

Consider the dual of program (A):

$$\min_{\{x^h, L^h\}} qx^h + T(L^h) - L^h \quad \text{subject to } u^h(x^h, L^h) = \bar{u}^h. \tag{B}$$

From program (B) one can derive the expenditure function $E^h(q, \bar{u}^h)$. For the time being, we only note that $\frac{\partial E^h}{\partial q} = x^h$. This is also derived using the envelope theorem.

Certain conditions, exhibited in Auerbach (1985), guarantee that goverment's control of the consumer price vector q is equivalent to its control of the commodity tax vector θ. Following common practice, we use the consumer price vector q as the control variable for the government instead of commodity taxes; the choice of any admissible q implicitly determines a commodity tax vector θ. Now let $\overline{W}(v^1, \ldots, v^H)$ be the social welfare function adopted by the society N. It is assumed that $\frac{\partial \overline{W}}{\partial M^h} \geq 0$ for all $h \in N$ and $\frac{\partial \overline{W}}{\partial M^h} > 0$ for some $h \in N$. With q as its control variable, the government's program is

$$\max_{\{q\}} \overline{W}(v^1, \ldots, v^H) \quad \text{subject to } L = F(x) + \overline{R}, \tag{C}$$

where, following Diamond and Mirrlees (1971), we have replaced the output y by the aggregate demand x as the argument of the cost function F. For each consumer price vector q, $x = x(q) = \sum_h x^h(q)$ is aggregate commodity demand and $L = L(q) = \sum_h L^h(q)$ is aggregate labor supply. The government faces the feasibility constraint $L = F(x) + \overline{R}$: Aggregate labor supply L must allow to produce x and to provide the tax revenue $\overline{R}$. Only those q fulfilling this feasiblity requirement are admissible and, as a rule, form a lower dimensional submanifold of Euclidean n-space. Given such a q, a separating hyperplane argument yields a producer price system p such that $x \in \operatorname{argmax}_y\, py - F(y)$. Moreover, for F differentiable at x, $p = \frac{dF}{dy}(x)$. In the absence of boundary problems, this p is unique. Then the commodity tax vector is $\theta = q - p$. Therefore, an admissible consumer price vector q determines a producer price vector p and a commodity tax vector θ. In particular, if consumer prices serve as control variables, then the corresponding commodity tax system can be recovered.

Form the Lagrangian for program (C):

$$\mathscr{L} \equiv \overline{W}(v^1, \ldots, v^H) + \lambda[L - F(x) - \overline{R}].$$

Differentiate $\mathscr{L}$ with respect to q and set the derivative equal to zero. Then using $\frac{\partial v^h}{\partial q} = -\alpha^h x^h$, the consumer's budget constraint, and the profit maximization condition $\frac{dF}{dy} = p$ yield

$$\sum_h \frac{\partial \overline{W}}{\partial M^h} x_k^h = \lambda \sum_h (x_k^h + \theta \frac{\partial x^h}{\partial q_k} + T_h' \frac{\partial L^h}{\partial q_k}), \quad k = 1, \ldots, n, \tag{1}$$

where $T_h' = \frac{dT}{dL^h}$. System (1) gives the necessary condition for the second-best optimum.

Let $R(q) := L(q) - F(x(q))$, i.e., the government's revenue function. Then $-\frac{\partial \overline{W}}{\partial q_k}$ equals the left hand side of (1) and $\lambda \frac{\partial R}{\partial q_k}$ the right hand side of (1) for all k. Following Ahmad and Stern (1984), one may define $\lambda_k = -(\frac{\partial \overline{W}}{\partial q_k} / \frac{\partial R}{\partial q_k})$ and interpret it as the social marginal cost of obtaining an extra efficiency labor unit by raising consumer price q_k. By this definition, equation (1) dictates that $\lambda_k = \lambda$ for all $k=1, \ldots, n$, at the social optimum, i.e., the social marginal cost of obtaining an extra efficiency labor unit from each commodity tax instrument should be equalized. This is the marginal condition one would expect from an optimization problem.[35]

Define $\mu^h = \frac{\partial \overline{W}}{\partial M^h} + \lambda\theta \frac{\partial x^h}{\partial M^h}$. Furthermore, define $\gamma_k = \sum_h \mu^h(\frac{x_k^h}{X_k})$. The variable γ_k is the distributional characteristic of good k when designing commodity taxes "in isolation" [Feldstein (1972); Atkinson and Stiglitz (1976)]. With these definitions and using the Slutsky equation, (1) yields

$$\sum_h \sum_i \theta_i E_{ki}^h = -(1 - \frac{\gamma_k}{\lambda})X_k - \sum_h T_h' \frac{\partial L^h}{\partial q_k}, \quad k = 1, \ldots, n, \tag{2}$$

where $E_{ki}^h = \frac{\partial^2 E^h}{\partial q_i \partial q_k}$, $k,i=1, \ldots, n$.

If one ignores the last term of (2), then the formula is the standard many-person Ramsey rule [Diamond (1975)]. With the last term added, the standard many-person Ramsey rule is obviously modified. Let's assume that the income tax schedule is nondecreasing, i.e., $T_h' \geq 0$. For the sake of interpretation, we are willing to make a convenient assumption that all members of society agree whether a commodity k is complementary with labor ($\frac{\partial L^h}{\partial q_k} < 0$ and hence, $\frac{\partial L}{\partial q_k} < 0$), substitutive to labor ($\frac{\partial L}{\partial q_k} > 0$), or unrelated to labor ($\frac{\partial L}{\partial q_k} = 0$).[36] Unless $\frac{\partial L}{\partial q_k} = 0$, labor supply changes as consumer price q_k varies. With the income tax schedule T fixed, this in turn affects income tax revenues. If $\frac{\partial L}{\partial q_k} > 0$, an increase in q_k can bring in additional income tax revenue. This implies a "positive" spillover effect to the goal of balancing the government's budget, and hence the consumer price q_k should be set higher than without

35 This is an interior solution. We implicitly exclude the possibilities that $\frac{\partial R}{\partial q_k} \leq 0$ or $\frac{\partial R}{\partial q_k} \to \infty$ for some k.

36 This is not a dramatic assumption. In fact, it seems quite natural and appealing to common sense if one follows the household production approach: some goods intrinsically accompany "leisure" activities; others intrinsically accompany "working" activities [see Becker and Michael (1973)].

this effect. Consequently, the reduction in compensated demand for x_k needs to be more than $1 - \frac{y_k}{\lambda}$ at the social optimum, as indicated in equation (2). On the other hand, if $\frac{\partial L}{\partial q_k} < 0$, the situation is just opposite. The standard many-person Ramsey rule remains optimal only if $\frac{\partial L}{\partial q_k} = 0$.

The modification on the many-person Ramsey rule described above can be seen from a different angle by checking equation (1) directly. If $\frac{\partial L}{\partial q_k} > 0$, the social marginal cost λ_k "in ecemeal" is lower than that "in isolation" for every value of q_k; conversely, if $\frac{\partial L}{\partial q_k} < 0$.[37] Now assume that λ_k is an increasing function of q_k.[38] Then compared with the "isolated" design of commodity taxes, the "piecemeal" design will pick a higher value of q_k if $\frac{\partial L}{\partial q_k} > 0$ and a lower value of q_k if $\frac{\partial L}{\partial q_k} < 0$. Namely, it requires to price goods complementary with leisure higher and goods substitutive to leisure lower.

3.3. Piecemeal tax reform

With the setup in section 2 and with an arbitrary tax system (commodity taxes and an income tax schedule) already in place such that $x = y$ and $L = F(y) + \overline{R}$, we consider two possible ways of "piecemeal" tax reform: (i) the replacement of commodity taxes by lump-sum taxes; (ii) the adjustment of the commodity tax structure itself. Our focus is to pinpoint welfare-improving and, in particular, Pareto-improving changes in consumer prices. As Buchanan (1976) and Frey (1976) have emphasized, political feasibility is critical in the practice of tax reform. Members of society reach a consensus on the direction of tax reform much easier if that direction leads to a Pareto-improvement. For this reason and this reason alone, Pareto-

37 We implicitly assume here that an increase in any consumer price q_k (presumably through an increase in tax θ_k) brings about an increase in commodity tax revenue, i.e., $\sum_h (x_k^h + \theta \frac{\partial x^h}{\partial q_k}) > 0$ for all k in our analysis. T s is a conventional assumption.

38 If there exists an λ_k such that it is a decreasing function of q_k, all the tax revenue should be collected through the tax instrument θ_k. We then have a corner solution and $\lambda_k = \lambda$ for all k is no longer optimal. We rule out this situation.

improving price changes deserve special emphasis. We confine our analysis to infinitesimal changes in consumer prices.

3.3.1. Tax reform with lump-sum taxes

A lump-sum tax is nondistortionary in the sense that it only brings about income effects. But it is well known that an arbitrary move from distortionary taxes toward nondistortionary ones may actually reduce welfare [see Atkinson and Stiglitz (1980)]. Hence, our task here is to identify the "right" directions of tax reform. As can be imagined, many old results obtained in a one-consumer-"isolated" setting no longer hold in a many-consumer-"piecemeal" setting. We will not elaborate on these "negative" results, however. Instead we only discuss the "positive" ones we have found.

First, assume that the government adopts the following non-uniform lump-sum taxation rule to replace commodity taxes:

$$dG^h = -x^h d\theta - \theta dx^h - T_h' dL^h - x^h dp, \quad h \in N, \tag{3}$$

where G^h is the lump-sum tax levied on the hth consumer. This scheme is most easy to understand if $dp=0$, i.e., fixed producer prices. Then (3) simply means that lump-sum taxes are changed to keep each individual's tax payment constant. If $dp \neq 0$, the scheme in addition needs to consider expenditure changes on goods x due to changes in producer prices, i.e., $x^h dp$. Note that the constraint of program (C) remains satisfied with changes in G^h following equation (3). This can be verified through the consumers' budget constraints. Thus totally differentiating $L = F(x) + \overline{R}$. /e have $(p\frac{\partial x}{\partial q} - \frac{\partial L}{\partial q})dq = 0$. By Proposition 1 of Guesnerie (1977), tight equilibrium is implicitly preserved.[39] Therefore, equation (3) is the only constraint that needs to be taken into account during the process of tax reform.

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[39] We implicitly assume the local controllability of the production sector through changes in producer prices. For the precise meaning and approach, see Guesnerie (1977) and Weymark (1979). The terminology of "tight" equilibrium is following Guesnerie (1977). A "nontight" equilibrium is better known as an equilibrium with free disposability of output.

After including the lump-sum tax G^h, the commodity demand is $x^h = x^h(q, G^h)$ and the labor supply is $L^h = L^h(q, G^h)$. Thus $dx^h = \frac{\partial x^h}{\partial q} dq - \frac{\partial x^h}{\partial M^h} dG^h$ and $dL^h = \frac{\partial L^h}{\partial q} dq - \frac{\partial L^h}{\partial M^h} dG^h$. By equation (3) and the Slutsky equation one can then obtain

$$\theta dx^h + T_h' dL^h = \frac{(\theta E_{qq}^h + T_h' L_q^h) dq}{\Omega^h}, \quad h \in N \tag{4}$$

where $L_q^h = \frac{\partial L^h(q, \bar{u}^h)}{\partial q}$, $\Omega^h = 1 - \theta \frac{\partial x^h}{\partial M^h} - T_h' \frac{\partial L^h}{\partial M^h}$, and E_{qq}^h is an $n \times n$ matrix with ijth element E_{ij}^h.

Using (3) and the facts $\frac{\partial v^h}{\partial M^h} = \alpha^h$ and $\frac{\partial v^h}{\partial q} = -\alpha^h x^h$, we have

$$du^h = \alpha^h (\theta dx^h + T_h' dL^h), \quad h \in N. \tag{5}$$

Equation (5) indicates that any consumer's welfare will be improved by the tax reform if, in response to the reform, the consumer changes his (her) consumption and labor supply such that this change would lead to an increase of tax revenue under the old tax system. This result is analogous to the one derived for the "isolated" reform of commodity taxes in a one-consumer economy [e.g., Auerbach (1985, (8.4))].

Substituting (4) into (5), one obtains

$$du^h = \alpha^h \frac{(\theta E_{qq}^h + T_h' L_q^h) dq}{\Omega^h}, \quad h \in N. \tag{6}$$

Equation (6) is a generalization of Dixit (1975, (7)).

In the following it is assumed that $\Omega^h > 0$ for all $h \in N$, i.e., an increase in total tax payments due to a small increase in income is less than the increase in income. Let $T_h'' = \frac{dT_h'}{dL^h}$. We also assume that $0 \le T_h' < 1$ and $T_h'' \ge 0$. Thus the income tax schedule is nonregressive. Further, we assume $q > 0$, $\theta > 0$, and $p > 0$ for the remainder of this subsection. The assumption is not necessary for all the results, but it simplifies exposition.

With a general income tax schedule T, some standard properties of expenditure functions or compensated demands may no longer hold. For later use, we establish the next two lemmas.

Lemma 1. The expenditure function $E^h(q, \bar{u}^h)$ is concave in q.

Proof: Immediate consequence of the definition of concavity. Q.E.D.

With E^h concave in q, E^h_{qq} is negative semi-definite. Like Dixit (1975), we will in fact assume that it is negative definite. This assumption is made for convenience. It is not a necessity within our context.[40]

Since $T_h'' \geq 0$, $qx^h + T(L^h) - L^h$ is a convex function in (x^h, L^h). Then with a strictly quasi-concave u^h, the solution from program (B) is unique. Thus to any q and h, the solution of (B) associates a unique labor supply $L^h(q)$ and, consequently, a unique marginal income tax rate $T_h'(L^h(q))$. Let $\omega^h = 1 - T_h'(L^h(q))$ and consider the following program, a local linearization of the objective function of program (B):

$$\begin{aligned} &\min_{\{x^h, L^h\}} \; qx^h - \omega^h L^h \\ &\text{subject to } u^h(x^h, L^h) = \bar{u}^h. \end{aligned} \tag{D}$$

It is obvious that the solution $(\hat{x}^h, \hat{L}^h)$ from program (D) is the same as the solution (x^h, L^h) from program (B). Thus we have identities $\hat{x}^h(q, \omega^h, \bar{u}^h) = x^h(q, \bar{u}^h)$ and $\hat{L}^h(q, \omega^h, \bar{u}^h) = L^h(q, \bar{u}^h)$. Since $qx^h - \omega^h L^h$ is linear in (q, ω^h), both $\hat{x}^h$ and $\hat{L}^h$ are homogeneous of degree zero in (q, ω^h). Let $\hat{L}^h_{\omega^h}$ and $\hat{L}^h_q$ be the derivatives of compensated labor supply $\hat{L}^h(q, \omega^h, \bar{u}^h)$ with respect to ω^h and q respectively, and let $\Delta = \hat{L}^h_{\omega^h} T_h''$. Then we have:

Lemma 2. $L^h_q = \dfrac{1}{1 + \Delta} \hat{L}^h_q$ and $qE^h_{qq} = \dfrac{1 - T_h'}{1 + \Delta} \hat{L}^h_q$.

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[40] Let z be a non-zero vector in n-dimensional Euclidean space. It is a rare coincidence that the quadratic form $zE^h_{qq}z$, evaluated at equilibrium values, is equal to zero for all consumers even if E^h_{qq} is only negative semi-definite.

Proof: The results are derived by differentiating the identities $\hat{L}^h(q, \omega^h, \bar{u}^h) = L^h(q, \bar{u}^h)$ and $\hat{x}^h(q, \omega^h, \bar{u}^h) = x^h(q, \bar{u}^h)$ with respect to q, and using the homogeneity of $\hat{L}^h$ and $\hat{x}^h$. Q.E.D.

With the above two lemmas at hand, we can now prove some results concerning the Pareto-improving changes in consumer prices. First, consider a reduction of all consumer prices following $dq = -a\theta$, where a is a positive scalar, then from (6)

$$du^h = \alpha^h \frac{-a\theta E^h_{qq}\theta - aT_h' L^h_q \theta}{\Omega^h}, \quad h \in N. \tag{7}$$

If one ignores the income tax (namely, tax reform "in isolation" instead of "in piecemeal") and if, indeed, E^h_{qq} is negative definite, then (7) implies $du^h > 0$ for any consumer. An analogous result in the setting of a one-consumer economy can be found in Dixit (1975, Theorem 1). In contrast, a "piecemeal" tax reform with $dq = -a\theta$ need not be welfare-improving for each consumer. The hth consumer's welfare gain or loss depends upon how the term $T_h' L^h_q \theta$ behaves compared with $\theta E^h_{qq}\theta$ in equation (7). Now let $\theta = bq$, where b is a scalar with $0 < b < 1$, i.e., a uniform taxation before the tax reform. Then, by Lemma 2, (7) becomes

$$du^h = \alpha^h \frac{-a\theta E^h_{qq}\theta - ab\dfrac{T_h'}{1-T_h'} qE^h_{qq}q}{\Omega^h}, \quad h \in N. \tag{8}$$

With $dq = -a\theta$ and $\theta = bq$, $dq = -abq$, i.e., a proportional reduction of all consumer prices. The sign of $ab\dfrac{T_h'}{1-T_h'}$ is positive, therefore

Proposition 1. A proportional reduction of all consumer prices starting from a uniform commodity tax structure, with lump-sum taxes changed according to equation (3), is Pareto-improving.

The condition of uniform commodity taxation under the initial tax system is essential for the conclusion of Proposition 1 in the many-consumer case, while in the one-consumer case such a condition is not necessary to arrive at the conclusion of the proposition.

Suppose the producer price system p is fixed. Then $dq = d\theta$. Suppose that in addition, one starts with a uniform commodity tax structure ($\theta = bq$). Then a proportional reduction of all consumer prices ($dq = -abq$) is equivalent to a proportional reduction of commodity taxes ($d\theta = -a\theta$). Hence the following is a corollary of the previous proposition.

Proposition 2. A proportional reduction of a uniform commodity tax structure, with producer prices fixed and with lump-sum taxes changed to keep each individual's tax payment constant, is Pareto-improving.

Without an income tax, Proposition 2 is the analogue of Dixit (1975, Theorem 2). However, Dixit's result was derived without fixed producer prices in a one-consumer economy; ours in a many-consumer economy with fixed producer prices.[41] Producer prices remain unchanged in two possible cases: (a) Tax reform leaves the aggregates x and L unchanged; there is only a reallocation among consumers; (b) The technology exhibits constant returns to scale and tax reform changes only the scale of production without altering the product mix, i.e., the output x before tax reform is changed to an output of the form rx, $r > 0$, after tax reform.

Keep the assumption of uniform taxation, i.e., $\theta = bq$. Then by Lemma 2, (6) yields

$$du^h = \alpha^h \frac{[b(1 - T_h') + T_h']L_q^h dq}{\Omega^h}, \quad h \in N. \tag{9}$$

Note that the bracket $[b(1 - T_h') + T_h']$ is a positive term. As an immediate consequence, one obtains

41 The assumption of fixed producer prices is necessary here, for Dixit's equation (19) does not hold in a many-consumer setting.

Proposition 3. Suppose that initially there is a uniform commodity tax and that being net complementary with or net substitutive to leisure is an intrinsic property of a good. Then a sufficient condition for a Pareto-improvement is that consumer prices be increased on goods complementary with leisure or decreased on goods substitutive to leisure, with lump-sum taxes changed according to equation (3).

This result generalizes a similar one in Dixit (1975, Theorem 5) and is true only for a small perturbation (the first small move destroys the uniform tax structure).

Next assume that the given income tax schedule T is proportional or linear, i.e., $T_h' = T'$ for all $h \in N$. By Lemma 2, (6) yields

$$du^h = \alpha^h \frac{(\theta + \frac{T'}{1-T}q)E^h_{qq}dq}{\Omega^h}, \quad h \in N. \tag{10}$$

Proposition 4. Suppose the given income tax is proportional or linear. Then a reduction of consumer prices following the rule $dq = -a(\theta + \frac{T'}{1-T'}q)$, with lump-sum taxes changed according to equation (3), is Pareto-improving.

Proposition 4 is true for any commodity tax structure and hence can be applied to reform commodity taxes sequentially.

So far we have investigated the question of whether the consumers' welfare can be increased, given a revenue requirement for the government. Propositions 1 to 4 provide various conditions for Pareto-improvements, while each individual's tax payment remains fixed (if producer prices are fixed). We can also turn the problem around and ask whether government revenue can be increased, given the utility profile of society at the existing tax system. If the answer is yes, then, returning revenues to consumers in a lump-sum fashion can be Pareto-improving. In the following, we re-examine the previous setup from the revenue generating point of view. In Proposition 5 to 8 below, tax revenue is increased, whereas each individual's utility level is kept constant.

Let's assume that the society is at some utility profile $\bar{u} = (\bar{u}^1, ..., \bar{u}^H)$ before tax reform. Totally differentiating $R(q) = L(q) - F(x(q))$ at $\bar{u}$ and using $\frac{dF}{dy} = p$, we obtain

$$dR = \sum_h (L_q^h - pE_{qq}^h)dq. \tag{11}$$

Suppose $p = bq$, which implies $\theta = (1 - b)q$, i.e., a uniform commodity tax structure. Then by Lemma 2, (11) changes to

$$dR = \sum_h [1 - b(1 - T_h')]L_q^h dq. \tag{12}$$

Since the bracket $[1 - b(1 - T_h')]$ is a positive term, consequently one obtains

Proposition 5. Suppose there is a uniform commodity tax and being net complementary with or net substitutive to leisure is an intrinsic property of a good. Then increasing consumer prices on goods complementary with leisure or decreasing consumer prices on goods substitutive to leisure, with lump-sum taxes changed to keep each individual's utility constant, increases government revenue.

Proposition 5 is very similar to Proposition 3 except that the lump-sum taxation scheme adopted is different from that of equation (3). This proposition can also be regarded as an extension of Dixit (1975, Theorem 5).

By Lemma 2, (11) also yields

$$dR = \sum_h \left(\frac{q}{1 - T_h'} - p \right) E^h_{qq} dq. \tag{13}$$

Let $p = bq$ and $dq = -aq$. This leads to

Proposition 6. A proportional reduction of all consumer prices starting from a uniform commodity tax structure, with lump-sum taxes changed to keep each individual's utility constant, increases government revenue.

The policy prescription in Proposition 6 is exactly the same as that in Proposition 1 except for different lump-sum taxation schemes.

If $T_h' = T'$, i.e., a proportional or linear income tax, then from equation (13) we have

Proposition 7. If the given income tax is proportional or linear, a reduction of consumer prices following the rule $dq = -a(\frac{q}{1 - T'} - p)$, with lump-sum taxes changed to keep each individual's utility constant, increases government revenue.

Proposition 7 is true for any given commodity tax structure and hence it can be applied to reform commodity taxes sequentially.

Finally, we consider the case where the only commodity taxes are trade taxes and world prices $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)$ are constant for the society N. Thus both consumers and producers are facing the same domestic prices $q = \bar{p} + t$, where $t = (t_1, \ldots, t_n)$ are the trade taxes. Domestic outputs y are given by $\frac{dF}{dy}(y) = q$ and so imports are $x - y$. By trade balance, exports in terms of efficiency labor units must be $\bar{p}(x - y)$. We also assume a completely non-specialized equilibrium as in Dixit (1975). To be consistent with this assumption, we assume $\frac{d^2F}{dy^2}$ to be positive definite, which implies strict convexity of the cost function F. The society now faces the following constraint:

$$L = F(y) + \overline{p}(x - y) + R. \tag{14}$$

From the profit maximization condition $\frac{dF}{dy} = q$, we have $dy = (\frac{d^2F}{dy^2})^{-1}dq$, where $(\frac{d^2F}{dy^2})^{-1}$ is the inverse of $\frac{d^2F}{dy^2}$ and is also positive definite. Now assume uniform trade taxes, i.e., $\overline{p} = bq$, or equivalently $t = \frac{1-b}{b}\overline{p}$ with $0 < b < 1$. Then totally differentiating (14) at $u = \overline{u}$ and using $dy = (\frac{d^2F}{dy^2})^{-1}dq$ yields, by Lemma 2,

$$dR = \sum_h \left(\frac{1}{1 - T_h'} - b\right) q E^h_{qq} dq + (b - 1) q \left(\frac{\partial^2 F}{\partial y^2}\right)^{-1} dq. \tag{15}$$

Let $dq = -aq$, i.e., a proportional reduction of domestic prices. Since $(\frac{1}{1 - T_h'} - b)$ is positive and $(b - 1)$ negative, (15) leads to

Proposition 8. Assume a small country with completely non-specialized equilibrium and $\frac{d^2F}{dy^2}$ positive definite. Then a proportional reduction of domestic prices starting from uniform trade taxes with lump-sum taxes to keep each individual's utility constant, increases government revenue.

This proposition is the counterpart of Dixit (1975, Theorem 4) within our setting.

So far we have considered tax reforms consisting in particular types of commodity tax reductions, accompanied by changes of non-uniform lump-sum taxes. Our commodity and income taxes are anonymous in the sense that they are based on observable transactions only and not on exogenous, possibly unobservable, individual characteristics; therefore they are relatively easy to administer. In contrast, non-uniform lump-sum taxes are difficult to administer and rarely used. Nonetheless, it is of theoretical and practical interest to learn what could be achieved by a modest use of this instrument. Notice that most of our analysis is local, whereas the use of lump-sum taxes in other contexts usually requires global analysis (for instance, lump-sum transfers needed to reach the conclusion of the "second welfare theorem").[42]

Suppose now the best of all worlds where arbitrary individual-oriented lump-sum taxes are available to the government without any informational or administrative difficulties. What is achievable? By the "second welfare theorem", a first best optimum can be achieved if appropriate lump-sum taxes are imposed and no other taxes. However, the existing income tax schedule is exempt from elimination or alteration in our piecemeal tax reform setting. Hence, intuitively, one should not replace commodity taxes completely by lump-sum taxes in accordance with the general theory of the second-best. This can be seen precisely by including lump-sum taxes G^h, $h \in N$, as instruments in program (C) to get $1 - \frac{\mu^h}{\lambda} = T_h' \frac{\partial L^h}{\partial M^h}$ for all h. Using this fact, (2) yields

$$\sum_h [\sum_i \theta_i E^h_{ik} + T_h' L^h_k] = 0, \quad k = 1, \ldots, n, \tag{16}$$

where L^h_k is the kth component of L^h_q. It is clear that the solution $\theta = 0$ will normally not satisfy equation (16).

In the analysis above, we find that there are ways to reform a uniform commodity tax structure which lead to a Pareto-improvement. As a matter of fact, even if the government's objective is simply to improve social welfare (not necessarily Pareto-improving), a uniform taxation is normally not optimal either. This can be seen by checking equation (16). Substituting $\theta = bq$ (i.e., a uniform tax structure) into the left hand side of (16) and using Lemma 2, one gets $\sum_h [b(1 - T_h') + T_h'] L^h_k$. This term will normally not be zero for all k. The suboptimality of uniform taxation is an old result. It can be dated back all the way to Ramsey (1927) and Corlett and Hague (1953).

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[42] A uniform lump-sum taxation scheme should be easier to administer than a non-uniform one. Unfortunately, such a uniform design leads us nowhere in a many-consumer setting.

3.3.2. Tax reform without lump-sum taxes

In this subsection, we employ the first-order approximation of functions to approach the direction of tax reform. This method fails when there are zero gradients [see Diewert (1978)]. To avoid this difficulty, we exclude zero gradients from consideration. This is a fairly mild restriction within our context.

3.3.2.1. Welfare-improving directions

When lump sum taxes are not available, one gets from differentiation of the social welfare function $\overline{W}$ and government revenue function R

$$d\overline{W} = \sum_i \Big(- \sum_h \frac{\partial \overline{W}}{\partial M^h} x_i^h \Big) dq_i \, ; \; dR = \sum_i \Big[\sum_h \Big(x_i^h + \theta \frac{\partial x^h}{\partial q_i} + T_h' \frac{\partial L^h}{\partial q_i} \Big) \Big] dq_i . \tag{17}$$

Let $\nabla\overline{W}$ and ∇R be the gradients of $\overline{W}$ and R respectively. The kth component of $\nabla\overline{W}$ is $\frac{\partial \overline{W}}{\partial q_k}$ and that of ∇R is $\frac{\partial R}{\partial q_k}$. It is clear that $\{dq \mid dR \geq 0\}$ is the set of feasible directions of consumer price changes. If we write dR in another way so that $dR = (-p\frac{\partial x}{\partial q} + \frac{\partial L}{\partial q})dq$, then the set $\{dq \mid dR \geq 0\}$ corresponds to the feasible directions of consumer prices described in Guesnerie (1977). In the following, two possible cases for the starting point of the existing tax system are discussed. In the first case, consumer prices are adjusted along the gradient of the social welfare function. In the second case, we apply the so-called gradient projection algorithm. It may be helpful to think of the problem in geometric terms for both cases. The geometric interpretation of tax reform was first noted by Dixit (1975).

(i) $\nabla\overline{W} \cdot \nabla R \geq 0$. In this case, setting changes in consumer prices along the gradient of the objective function, i.e., $dq = \nabla\overline{W}$, will improve welfare in the most preferred direction of society. Note that during the process of tax reform, the government may need to adopt a surplus budget policy. Otherwise there will be an aggregate production inefficiency, for $L > F(x) + \bar{R}$. It is worth noting that while it is feasible to have welfare-improving and aggregate production efficiency at the same time, this is not the most desirable direction of tax reform. This phenomenon has been accounted for in some detail by Dixit (1979).

(ii) $\nabla\bar{W} \cdot \nabla R < 0$. In this case, the gradient projection algorithm projects the gradient $\nabla\bar{W}$ on the hyperplane $dR = 0$ and finds a direction $\vec{OD}$ for the adjustment of consumer prices. We get $\vec{OD} = \nabla\bar{W} - \frac{\nabla\bar{W} \cdot \nabla R}{||\nabla R||^2} \nabla R$, where $||\nabla R||$ is the Euclidean norm of ∇R [Tirole and Guesnerie (1981, Proposition 1)]. It can be checked that $\vec{OD} \cdot \nabla\bar{W} > 0$ and hence $d\bar{W} > 0$ if $\nabla\bar{W}$ and ∇R are not collinear. Otherwise $\nabla\bar{W} = -c\nabla R$ for some positive scalar c and $\vec{OD} = 0$. Note that, by equation (1), $\nabla\bar{W} = -c\nabla R$ is necessary for the starting point to be optimal. However, it is not sufficient, unless $\bar{W}$ is concave in consumer prices q and R is convex.

Now let $\bar{\lambda} = -\frac{\nabla\bar{W} \cdot \nabla R}{||\nabla R||^2}$. Since $\nabla\bar{W} \cdot \nabla R < 0$, $\bar{\lambda} > 0$. Without loss of generality, we may choose $\bar{\lambda} = 1$ and normalize $\nabla\bar{W}$ in efficiency labor units. From the formula of $\vec{OD}$ derived, we then have, for $\vec{OD} \neq 0$,

$$dq_k \gtreqless 0 \quad \text{iff} \quad -\frac{\partial\bar{W}}{\partial q_k} \lesseqgtr \frac{\partial R}{\partial q_k}, \qquad k = 1, \ldots, n, \tag{18}$$

where $-\frac{\partial\bar{W}}{\partial q_k}$ can be interpreted as the marginal cost of raising consumer price q_k in terms of social welfare loss and $\frac{\partial R}{\partial q_k}$ the marginal benefit in terms of tax revenue gain. By equation (18), the gradient projection algorithm dictates the following rule. If $\nabla\bar{W}$ and ∇R are not collinear and the marginal cost is greater than the marginal benefit, the consumer price should be adjusted downward; otherwise adjusted upward. The magnitude of adjustment in each consumer price is exactly the difference between the marginal benefit and the marginal cost brought about from changes in the corresponding consumer price. Thus the gradient projection algorithm of adjusting consumer prices to increase welfare turns out to be an elaboration based upon one of the most basic principles of Economics.[43] Ahmad and Stern (1984) pointed out that if there exist i and j such that $\lambda_i \neq \lambda_j$, i.e., the social marginal cost of obtaining an extra efficiency labor unit by raising the consumer price q_i is different from that by raising consumer price q_j, then a welfare-improving move is possible. The gradient projection algorithm also indicates the welfare-improving possibilities, but in addition it specifies that move as the difference between $\frac{\partial R}{\partial q_k}$ and $-\frac{\partial \overline{W}}{\partial q_k}$ for all $k=1, ..., n$. As explained in Tirole and Guesnerie (1981), the gradient projection algorithm leads to the "best small improvement" [Dixit (1979)] and the "optimal tax perturbation" [Diewert (1978)] in the case where feasible consumer price changes are confined to the set $\{dq \mid dR = 0\}$. In fact, since $\nabla\overline{W} \cdot \nabla R < 0$, the statement remains true even if we relax consumer price changes to the feasible set $\{dq \mid dR \geq 0\}$. This is due to Proposition 3 of Dixit (1979).

In the piecemeal design of commodity taxes, we found that, compared with "isolated" design of commodity taxes, goods complementary with leisure should be priced higher than goods which are substitutive to leisure. There is an analogue in the piecemeal reform of commodity taxes. Again, for the convenience of interpretation, assume that whether goods are complementary with or substitutive to leisure is intrinsic. From the components of ∇R, we see that whether $\frac{\partial L}{\partial q_k}$ being positive or negative has different effects on the marginal benefit $\frac{\partial R}{\partial q_k}$. If $\frac{\partial L}{\partial q_k} > 0$, then $\frac{\partial R}{\partial q_k}$ "in piecemeal" will be greater than that "in isolation"; if $\frac{\partial L}{\partial q_k} < 0$, the situation is just opposite. Therefore, compared with "isolated" reform of commodity taxes, the gradient projection algorithm requires for consumer prices of goods complementary with leisure a more positive adjustment, and those of goods substitutive to leisure a more negative adjustment. The directions of adjustment of some consumer prices may even be reverse. Only for goods neutral to leisure or labor supply, the gradient projection algorithm for the "piecemeal" reform of commodity taxes dictates the same direction and the same magnitude of adjustment as that for the "isolated" reform of commodity taxes.

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[43] See Tirole and Guesnerie (1981) for a different interpretation.

3.3.2.2. *Pareto-improving directions*

Since the work of Guesnerie (1977), the existence of Pareto-improving directions (PID) of tax reform without lump-sum taxes has attracted a lot of attention from economists [e.g., Diewert (1978), Dixit (1979), Guesnerie (1979), Weymark (1978, 1979, 1981), Ahmad and Stern (1984) and Wibaut (1987)]. In the previous analysis, we discussed welfare-improving tax reform with a given social welfare function (SWF). A Pareto-improving move will normally lead to welfare improvement, but not vice versa. However, as demonstrated in Ahmad and Stern (1984), there is a close relationship between feasible PIDs and SWFs. In particular, one may pose the "inverse optimum proble:" Is there a SWF to support the status quo of the existing tax system as an optimum? In case there is no such SWF, one can then infer indirectly the existence of feasible PIDs. We address this question in the following. Let's first introduce some terminologies. A SWF W which embodies the property that $\frac{\partial W}{\partial M^h} \geq 0$ for all $h \in N$ will be called a Paretian SWF. If $\frac{\partial W}{\partial M^h} > 0$ for all $h \in N$, W will be called a strong Paretian SWF. On the other hand, if it allows $\frac{\partial W}{\partial M^h} < 0$ for some $h \in N$, then W will be simply called a SWF. Obviously, a strong Paretian SWF is a Paretian SWF, and a Paretian SWF is a SWF.

For the sake of concreteness, let's suppose $\nabla\overline{W} \cdot \nabla R < 0$ at the starting point of the existing system and choose $\overline{\lambda} = 1$. It is also assumed that $\frac{dR}{dq} > 0$. This assumption is true if the value of $\sum_h x_i^h$ is greater than that of $\sum_h(\theta \frac{\partial x^h}{\partial q_i} + T_h' \frac{\partial L^h}{\partial q_i})$ for all i [see (17)].[44] We want to know whether there exists a SWF W such that $-\nabla W = \nabla R$, i.e. the SWF W justifies the status quo as a second-best optimum. For convenience, we now regard the government as an additional player in the society N. The greater is the budget surplus, the higher is the government's welfare (presumably, the tax revenue requirement in the next period can be reduced). Since the government can adopt a policy with surplus budget, this eliminates the desirability of temporary production inefficiency during the process of tax reform [see Smith (1983)].

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44 If $\frac{dR}{dq} > 0$ holds, then, from (17), it is obvious that $\nabla\overline{W} \cdot \nabla R \geq 0$ will be ruled out.

Let $\mathscr{R}^r$ represent r-dimensional Euclidean space and $\mathscr{R}^r_+$ be its nonnegative orthant. With the government as a player, the problem of feasible PID can be thought as finding changes in consumer prices such that $du^h \geq 0$ for all $h \in N$ and $dR > 0$. Namely, there exists a solution to the following system:

$$\Phi z \leq 0, \ \frac{dR}{dq} z > 0, \ z \in \mathscr{R}^n, \tag{E}$$

where $z = dq$ and Φ is an $H \times n$ matrix with hith element x_i^h. By the Minkowski-Farkas lemma, system (E) has a solution if and only if the following system (F) has no solution:

$$w\Phi = \frac{dR}{dq}, \ w \in \mathscr{R}^H_+, \tag{F}$$

where $w = (\frac{\partial W}{\partial M^1}, \ldots, \frac{\partial W}{\partial M^H})$ and W is a Paretian SWF. If W happens to be the SWF adopted by the society N, then system (F) is no different from system (1). Suppose (F) has a solution $w \in \mathscr{R}^H_+$. Then $W(v^1, \ldots, v^H) := \sum_h \frac{w^h}{\alpha^h} v^h$ is a Paretian SWF such that the status quo satisfies the necessary condition of the second-best optimum for W. Therefore, if, to the contrary, the status quo does not satisfy the necessary condition of the second-best optimum for any Paretian SWF, then system (F) does not have a solution and, consequently, system (E) has a solution. This proves

Proposition 9 If the status quo does not satisfy the necessary condition of the second-best optimum for any Paretian SWF, a PID is feasible.

The basic idea of Proposition 9 originates in Guesnerie (1977) and Harris (1979), though both of them emphasized that the condition of second-best optimum holds independent of the concept of a SWF [see also Diewert (1978), Dixit (1979), Weymark (1979) and Ahmad and Stern (1984)]. Note that $du^h = 0$ for all $h \in N$ but $dR > 0$ is a PID in our setting. Proposition 9 can be strengthened or modified by using variants of the Minkowski-Farkas lemma. Proofs of the following results are collected in appendix C.

Proposition 10. If the status quo does not satisfy the necessary condition of the second-best optimum for any Paretian SWF, a strictly PID for every member of society and the government is feasible.

Under the same assumption as Proposition 9, Proposition 10 draws a stronger conclusion. Basically, it corresponds to Proposition 4 of Guesnerie (1977). However, since we treat the government as an additional player. tight equilibrium is always preserved. Further, our result allows not only all members of society but also the government to be strictly better off.

Proposition 11. If the status quo does not satisfy the necessary condition of the second-best optimum for any strong Paretian SWF, a PID is feasible.

Proposition 11 arrives at the conclusion of Proposition 9 under a weaker hyphothesis: only strong Paretian SWFs need to be checked. If, on the contrary, the hypothesis of Proposition 9 is strengthened such that no SWF passes the test, then a strong statement about the possible gain in tax revenue can be made:

Proposition 12. If the status quo does not satisfy the necessary condition of the second-best optimum for any SWF, it becomes feasible for the government to extract an additional efficiency labor unit of tax revenue without affecting any individual's welfare.

Unlike other results of this subsection, the next proposition indicates the sign of consumer price changes for a PID. Restrictions on the direction of tax reform were also discussed in Diewert (1978) and Dixit (1979).

Proposition 13. If the marginal benefit of raising each consumer price is greater than the marginal cost for every Paretian SWF at the status quo, then a PID with no decreases in consumer prices is feasible.

The next result considers a PID for a subgroup of consumers.

Proposition 14. If the status quo does not satisfy the necessary condition of the second-best optimum for any SWF which assigns nonnegative marginal valuations to some given individuals' incomes, then a PID with all other individuals' welfare fixed is feasible.

In the above, we specify the conditions for the existence of a variety of feasible PIDs. With given data Φ and $\frac{dR}{dq}$, these theorems require us first to solve a system of linear equalities or inequalities (i.e., $w\Phi = \frac{dR}{dq}$ or $w\Phi \geq \frac{dR}{dq}$) to confirm or reject the existence of some specified feasible PID. If it indicates existence, one can then appeal to the techniques of linear programming, such as described in Ahmad and Stern (1984), to find a corresponding feasible PID.

Recall that Φ has row x^h, $h \in N$, $\frac{\partial v^h}{\partial q} = -\alpha^h x^h$, and z represents a potential dq; so $x^h z = 0$ implies $du^h = 0$ only in an infinitesimal sense. Under the weak assumption that $x^h > 0$ for all h, the system (E) has a solution if and only if the following system has a solution:

$$\Phi z << 0, \ \frac{dR}{dq} z > 0, \ z \in \mathscr{R}^n. \tag{G}$$

In the latter case, one obtains a strictly PID. A modification of this kind is also possible in Propositions 12 and 14.

Finally, note that $\frac{dR}{dq}$ in the "isolated" reform of commodity taxes is different from that in "piecemeal" reform of commodity taxes. Therefore, a feasible PID for the tax reform "in isolation" may not be feasible for that "in piecemeal", and vice versa. This difference is surely critical in practice.

3.4. Conclusion

This paper analyzed the design and the reform of commodity taxes with a given income tax. In a "piecemeal" setting we derived interesting new results and modified several old ones known from the "isolated" setting. Going from "isolated" tax design or tax reform to "piecemeal" ones is obviously a step toward reality. Since policy implications of the former may be substantially different from those of the latter, the step from "isolated" analysis to "piecemeal" analysis becomes essential if one wants to avoid harmful policy prescriptions.

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Appendix A.

This appendix provides the calculation on the right hand side of equation (16) in Chapter 1.

By definition $R_j \equiv \frac{f_1^j}{f_2^j}$, so

$$\frac{\partial R_j}{\partial x_j} = \frac{f_2^j f_{11}^j - f_1^j f_{12}^j}{(f_2^j)^2}, \tag{a1}$$

and

$$\frac{\partial R_j}{\partial t_j} = \frac{f_2^j f_{12}^j - f_1^j f_{22}^j}{(f_2^j)^2}. \tag{a2}$$

Using (a1) and (a2)

$$\frac{x_j \frac{\partial R_j}{\partial t_j} - \frac{x_j}{R_j}\frac{\partial R_j}{\partial x_j}}{1 + x_j \frac{\partial R_j}{\partial t_j}} = \frac{2x_j f_1^j f_2^j f_{12}^j - x_j f_{22}^j (f_1^j)^2 - x_j f_{11}^j (f_2^j)^2}{f_1^j (f_2^j)^2 + x_j f_1^j f_2^j f_{12}^j - x_j f_{22}^j (f_1^j)^2}. \tag{a3}$$

Appendix B.

This appendix proves the claim in Chapter 2 that Proposition 1 holds with $\gamma = \gamma_0$ if the derivatives of v^* with respect to p^h and p^f are continuous in γ.

It is well known that $\pi(q^*, p^f; \gamma_0) = 0$. Since $\frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^h} = 0$ at $\gamma = \gamma_0$, $\lim_{\gamma \to \gamma_0}\frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^h} = 0$. With $\frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^h} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^h} = 0$ for all $\gamma \neq \gamma_0$, $\lim_{\gamma \to \gamma_0}(\frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^h} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^h}) = 0$, and so $\lim_{\gamma \to \gamma_0}\frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^h} = 0$. This implies $\frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^h} = 0$ at $\gamma = \gamma_0$. Hence, $\frac{\partial v^*}{\partial p^h} = \frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^h} + \frac{\partial v}{\partial p^h} = \frac{\partial v}{\partial p^h} = -\alpha z^h$ when $\gamma = \gamma_0$.

Similarly, since $\frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^f} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial p^f} = 0$ at $\gamma = \gamma_0$, $\lim_{\gamma \to \gamma_0}(\frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^f} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial p^f}) = 0$. With $\frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^f} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^f} = 0$ for all $\gamma \neq \gamma_0$, $\lim_{\gamma \to \gamma_0}(\frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^f} + \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial q^*}\frac{\partial q^*}{\partial p^f}) = 0$, and so $\lim_{\gamma \to \gamma_0}\frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^f} = \lim_{\gamma \to \gamma_0}\frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial p^f}$. This implies $\frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^f} = \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial p^f}$ at $\gamma = \gamma_0$. Hence, $\frac{\partial v^*}{\partial p^f} = \frac{\partial v}{\partial q^*}\frac{\partial q^*}{\partial p^f} = \frac{\partial v}{\partial \pi}\frac{\partial \pi}{\partial p^f} = -\alpha z^f$ when $\gamma = \gamma_0$.

Appendix C.

Most of the the lemmas used in the following can be found in Mangasarian (1968), while others are immediate corollaries of the Minkowski-Farkas lemma.

Proof of Proposition 10: Apply the Gordan lemma which asserts that exactly one of the following two systems has a solution:

$$\text{System 1.}\quad \Phi z << 0,\ \frac{dR}{dq} z > 0,\ z \in \mathscr{R}^n;$$
$$\text{System 2.}\quad w\Phi = w_0 \frac{dR}{dq},\ (w, w_0) \neq 0,\ w \in \mathscr{R}^H_+,\ w_0 \in \mathscr{R}_+.$$

Since we rule out zero gradients, $(x^h_1, \ldots, x^h_n) \neq 0$ for all $h \in N$. By the property of Φ and the assumption that $\frac{dR}{dq} > 0$, $w = 0$ if and only if $w_0 = 0$. Since $(w, w_0) \neq 0$ and nonnegative, without loss of generality, we may choose $w_0 = 1$. Q.E.D.

Proof of Proposition 11: Apply the Stiemke lemma which asserts that exactly one of the following two systems has a solution:

$$\text{System 1.} \quad \begin{bmatrix} \Phi \\ -\dfrac{dR}{dq} \end{bmatrix} z < 0, \ z \in \mathscr{R}^n;$$

$$\text{System 2.} \quad w\Phi = \frac{dR}{dq}, \ w >> 0, \ w \in \mathscr{R}^H. \qquad \text{Q.E.D.}$$

Proof of Proposition 12: Apply the Gale lemma which asserts that exactly one of the following two systems has a solution:

$$\text{System 1.} \quad \Phi z = 0, \ \frac{dR}{dq} z = 1, \ z \in \mathscr{R}^n;$$

$$\text{System 2.} \quad w\Phi = \frac{dR}{dq}, \ w \in \mathscr{R}^H.$$

Mathematically, system 2 of Gale's lemma says that $\frac{dR}{dq}$ lies in the linear space spanned by Φ. Q.E.D.

Proof of Proposition 13: Substituting $[\Phi^t, -\mathrm{I}]^t$ for Φ in (E) and (F), we get a corollary of the Minkowski-Farkas lemma. It says that exactly one of the following two systems has a solution:

$$\text{System 1.} \quad \Phi z \leq 0, \ z \geq 0, \ \frac{dR}{dq} z > 0, \ z \in \mathscr{R}^n;$$

$$\text{System 2.} \quad w\Phi \geq \frac{dR}{dq}, \ w \in \mathscr{R}^H_+.$$

Recall that the left hand side of system 2 can be interpreted as the marginal cost of raising consumer prices in terms of social welfare loss and the right hand the marginal benefit in terms of tax revenue gain. Application of the corollary then yields the proposition. Q.E.D.

Proof of Proposition 14: Partitioning Φ into $[\Phi_1, \Phi_2]$ and then substituting $[\Phi_1^t, \Phi_2^t, -\Phi_2^t]^t$ for Φ in (E) and (F), one can derive another corollary of the Minkowski-Farkas lemma. It says that exactly one of the following two systems has a solution:

$$\text{System 1.}\quad \Phi_1 z \le 0,\ \Phi_2 z = 0,\ \frac{dR}{dq} z > 0,\ z \in \mathscr{R}^n;$$

$$\text{System 2.}\quad w_1\Phi_1 + w_2\Phi_2 = \frac{dR}{dq},\ w_1 \ge 0,\ (w_1, w_2) \in \mathscr{R}^H.$$

Note that w_2 is not required to be nonnegative. The proposition follows from this corollary. Q.E.D.

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