

Relating Understanding of Inverse and Identity to Engagement in Proof in Abstract Algebra

David Plaxco

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M. Wawro, Committee Chair

A. Norton

E. Johnson

J. Wilkins

P. Doolittle

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## ABSTRACT

In this research, I set out to elucidate the relationships that might exist between students' conceptual understanding upon which they draw in their proof activity. I explore these relationships using data from individual interviews with three students from a junior-level Modern Algebra course. Each phase of analysis was iterative, consisting of iterative coding drawing on grounded theory methodology (Charmaz, 2000, 2006; Glaser & Strauss, 1967). In the first phase, I analyzed the participants' interview responses to model their conceptual understanding by drawing on the form/function framework (Saxe, et al., 1998). I then analyzed the participants proof activity using Aberdein's (2006a, 2006b) extension of Toulmin's (1969) model of argumentation. Finally, I analyzed across participants' proofs to analyze emerging patterns of relationships between the models of participants' understanding of identity and inverse and the participants' proof activity. These analyses contributed to the development of three emerging constructs: *form* shifts in service of sense-making, re-claiming, and lemma generation. These three constructs provide insight into how conceptual understanding relates to proof activity.

## **Dedication**

This dissertation is dedicated to the memory Verna Pearl Campbell Hill and Melba Louise Bryant Plaxco, both of whom set an example of motherhood that has passed to my own mother and to my sisters.

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## Chapter 1 - Introduction

*“Je le vois, mais je ne le crois pas!” – Georg Cantor*

Mathematical proof is an important area of mathematics education research that has gained emphasis over recent decades (Coe & Ruthven, 1994; Housman & Porter, 2003; Inglis & Alcock, 2012). The majority of empirical research in proof focuses on individuals’ proof production (Alcock & Inglis, 2008; Weber & Alcock, 2004), individuals’ understanding of or beliefs about proof (Almeida, 2000; Harel & Sowder, 1998; Martin & Harel, 1989), and how students develop notions of proof as they progress through higher-level mathematics courses (Segal, 1999; Selden & Selden, 2010; Tall & Mejia-Ramos, 2012). Researchers have also generated philosophical discussions that explore the purposes of proof (Bell 1976; Dawson, 2006; de Villiers, 1990; Rav, 1999). Much of this discussion centers on the explanatory power of proof (Hanna & Barbeau, 2010; Weber, 2002), with the primary focus being on the techniques and methods involved in a given proof (Hanna & Barbeau, 2010; Thurston, 1996), rather than the development of concepts or definitions (Lakatos, 1976). Few studies, however, use grounded empirical data to explicitly discuss the relationships between an individual’s conceptual understanding and his or her engagement in proof (e.g., Weber, 2005; Weber, 2010).

As an upper-level undergraduate mathematics course, Abstract Algebra provides a suitable context to study students’ engagement in proof. Inverse and identity are fundamental concepts in Abstract Algebra, as is evident from the group axioms (Figure 1.1). Because one prevalent purpose of an Abstract Algebra course is to explore the properties and structure of groups, students will necessarily need to build a strong conceptual understanding of identity and inverse in order to develop more formal and generalized understanding of the group structure. Specifically, students will need to build meaningful understanding of subgroup, conjugation,

normality, and group homo/isomorphism, all of which rely substantially on an understanding of inverse and identity. Within the Abstract Algebra class, relationships between these group-theoretical structures will be developed and supported with formal arguments and proofs, providing multiple opportunities to observe students proving with and about inverse and identity.

A group  $(G, *)$  is a set  $G$  and binary operation  $*$  that satisfy the following properties:

- Associativity (for  $a, b, c$  in  $G$ ,  $a*(b*c) = (a*b)*c$ )
- Closure (for every  $g, h$  in  $G$ ,  $g*h$  is in  $G$ )
- Identity (there exists  $e$  in  $G$  such that, for every  $g$  in  $G$ ,  $g*e = e*g = g$ )
- Inverses (for every  $g$  in  $G$ , there exists  $g^{-1}$  in  $G$  such that  $g*g^{-1} = g^{-1}*g = e$ )

*Figure 1.1. Group axioms*

From this, I focus my research interests toward individual students' engagement in Abstract Algebra proofs that involve or relate to inverse and identity. Specifically, I am interested in students' development of conceptual understanding that occurs chronologically near to students' engagement in proof. This leads to the following research question:

*How does student understanding of inverse and identity relate to student engagement in proof in Abstract Algebra?*

With the remainder of this chapter, I extend this discussion of proof and abstract algebra to motivate this research question and support its value and necessity. This includes a discussion of my personal experiences with proof in graduate level mathematics courses, a provision of quotes from the literature highlighting and supporting a need for this research, and a situation of my theoretical perspective within the literature.

### **Mathematical Proof**

Inherent in the process of proving is the notion that one must show some mathematical relationship that one did not necessarily know before he or she engaged in the activity of proving

(in the case of re-proving, then the relationship can be thought of as taken by the prover to require validation). Each proof begins with the statement of a mathematical relationship, which is either intuitively driven or presented to the individual, and the validity of which is either in question or taken as unknown. The individual then sets out to use specific notions about the concepts involved in the relationship in order to show that the relationship holds within his or her mathematical reality. Once the relationship is shown, there is new potential for the prover to begin to incorporate this new relationship into his or her understanding of the concepts involved (perhaps slowly and over time). Thus, by engaging in the production of a mathematical proof, the individual has the potential to learn about the very concepts about which he or she is proving.

Through my experiences in graduate-level mathematics courses, I have found this to be the case for my own development of understanding. Often, I would follow along with a proof presented during lecture, read a proof in the text, or write a proof for homework and find the results very useful and almost immediately applicable. Other times, I was unable to incorporate the proof's conclusions about the mathematical relationships into my own understanding until days or weeks later, if ever. While I was often able to find valuable techniques and methods in the presented proofs, which is consistent with the discussions in the literature, engaging in proof construction provided different opportunities to develop new understanding – specifically, an understanding of the concepts involved in the proof. These changes in understanding were often subtle and emergent. For example, as I worked to prove that all conformal self-maps of the unit disk were of a certain form, I used dynamic geometry software to explore such maps. In this exploration, I gained a greater sense – not just of the notion of conformal self-maps of the unit disk – but of conformality in general than I had previously understood. This experience helped

me develop my notions of complex-valued functions in new ways by incorporating the intuition I had gained working with the dynamic self-maps.

### **Learning from Proof – Evidence in the Literature**

In my review of the literature, I have found several instances in which researchers either discuss individuals learning from engagement in proof or provide situations that lend themselves to such a discussion. These examples are unique in their amenability to articulate learning from proof at an individual level, rather than an historical account of mathematical development through proof, which is substantially documented (e. g., Dawson, 2006; Lakatos, 1976). Through the following excerpts, I explore existing notions within the literature that connect individuals' engagement in proof with learning, supporting the importance and need for the current research. In each of the first three excerpts, the researchers specifically attribute students' learning to their engagement in proof. In the fourth excerpt, I extend an example from Harel and Sowder (1998) to develop a hypothetical example of learning from proof. Finally, I coordinate these examples toward a focus of my research goals.

In their discussion of students' progression toward more powerful proof approaches, Selden and Selden (2009) describe one type of knowledge (behavioral knowledge) that they claim is critical in coordinating proof activity and conceptual understanding:

In keeping track of students' abilities, it can be helpful to consider whether they can use a concept in constructing proof. More important than being able to articulate definition, students need to use them in proofs, that is, be able to carry out appropriate actions effortlessly in order to leave maximum cognitive resources for other parts of proofs, especially the problem-centered parts. In doing this, students need what we are calling behavioral knowledge, and we suspect this is learned as much from practice at constructing proofs as from abstract definitions.  
(p. 345)

Selden and Selden's (2009) suspicion that procedural knowledge is "learned as much from practice at constructing proofs" (p. 345) provides a clear example of researchers attributing



student learning to engagement in proof. I take this as evidence that further investigation is needed into what kinds of learning might take place as an individual engages in proof.

Larsen (2013) provides another, subtler example of researchers discussing students' learning through proof. In this article, Larsen describes students' development of a minimal set of rules used to generate notions of the group axioms. Larsen explains that students generate the inverse rule for the group by attempting to prove that an early version of the rules could be used to show that Cayley tables maintain the "Sudoku Property" (Latin squares property; p. 720):

The next phase... is launched by asking students whether they could prove the Sudoku property using only their minimal set of rules ... [T]he existence of inverses emerges as a way to justify the cancelation needed to complete the argument. This is an instance in which the instructional design capitalizes on the *systematizing* role of proof (De Villiers, 1990) because the inclusion of the inverse axiom is motivated by the desire to create a system of axioms that is sufficiently powerful to efficiently prove properties that are already apparent to the students. (p. 720)

In this example, Larsen's participants decide to include the inverse rule in their list that eventually becomes the group axioms. Larsen attributes the decision to the students' engagement in proof, citing de Villiers' (1990) notion of the "systematizing role of proof" (p. 720). This discussion helps to validate my current research interests in that Larsen (ibid) credits the students' engagement in proof to their reorganization of the rules for groups.

Weber (2005) discusses three different proof approaches used by students and how students learn from each. According to Weber, procedural and syntactic proof productions tend to afford opportunities for learning about the proving process, whereas "semantic proof productions afford students the opportunity to develop or refine informal representations of mathematical concepts, and use their reasoning with these representations to gain a conviction and understanding of why mathematical theorems are true" (p.358). In this excerpt, Weber's discussion aligns very well with my research interests; indeed, he puts forth the following

questions for future research, “How does reasoning used by undergraduates affect what they will learn from their proof productions? Are the theoretical arguments put forth in this paper correct?” (p. 359). These questions directly support the validity of my research.

With the following discussion, I describe a situation in the literature that I found helpful in thinking about the relationships between proof and understanding. In their seminal work on proof schemes, Harel and Sowder (1998) discuss a student’s (Amy’s) manipulation of a triangle to prove that the sum of the interior angles is 180 degrees. Harel and Sowder point out that “[Amy] *transformed* the triangle and was fully able to *anticipate* the results of the transformations” (p. 259, emphasis in original). In order to illustrate Amy’s work, the authors provide a diagram (Figure 1.2). The authors drew the dotted line AO perpendicular to the base of the triangle and intersecting point A (Figure 1.2, part d). I view this line as critical to the proof because, by applying the *alternate interior angles of a transversal* rule, the central claim that the change in angle is preserved is supported.

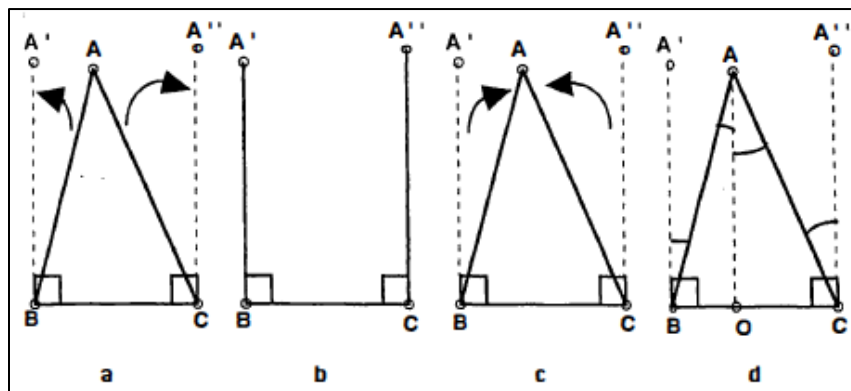


Figure 1.2. Amy’s manipulation of the triangle

According to Harel and Sowder’s discussion, Amy’s transformation was carried out with the anticipation that the action would result in a figure upon which Amy could “apply operations to compensate for the change” (p. 258). Suppose, though, that the operations were carried out in

anticipation that the resulting figure *might* be useful for proving the relationship. It is not out of the question to imagine a series of similar transformations that proved to be unfruitful until this transformation was carried out. Nor is it unrealistic to imagine a student who transformed the triangle as Amy did, and did not draw the line AO. By manipulating the triangle, one might see value in things that he or she perhaps did not see value in before. It might even be the case that the individual did not know about such a line before its construction. But, in acting out that transformation on the figure, one may produce and/or begin to value such a line, providing a new or different view of the triangle resulting from that activity.

To me, this example provides an imaginative illustration of what Weber (2005) describes: an individual engaging in a proof in which informal representations are used to gain a sense of conviction of why the mathematical relationship is true. This example also illustrates the nuances inherent in any relationship between proof and conceptual understanding, warranting a deeper exploration. Balacheff (1986) alludes to these nuances:

Most of the researches [*sic*] on problem solving are centered on heuristics. Thus they forget both the mathematical content - in terms of its psychological complexity - and the situation in which the activity takes place... procedural analysis - pointing out the heuristics patterns - is not sufficient to understand and to explain what is going on. We have to know how the pupils' conceptions are related to the mathematical notions engaged in the problem space. (p. 10)

This quote still holds true with respect to proof research and highlights the importance of investigating these relationships in greater detail, attending to students' developing notions of the mathematical concepts about which they are proving. In the four examples above, each author's discussion focused mostly on the aspects of proof that they discussed. While the notions of behavioral knowledge (Selden & Selden, 2009), the systematizing role of proof (de Villiers, 1990; Larsen, 2013), semantic proof production (Weber, 2005), and transformational proof scheme (Harel & Sowder, 1998) are important and useful, the authors background the

importance of the conceptual understanding and how it both informs and is informed by the student engagement in proof. This motivates a need to generate rich, detailed models of students' understanding concurrently with an investigation of their engagement in proof.

### **A Theoretical Framing**

Proof is often viewed as a coordination of individual activity and social interaction (Harel and Sowder, 1998; Raman, 2003). Cobb and Yackel's (1996) Emergent Perspective provides a useful framework for approaching and coordinating individual student understanding with his or her engagement in the classroom community. Within this framework, the authors describe three different constructs, each with a social and psychological perspective. The authors align their social perspective with an interactionist view (Bauersfeld, Krummheuer, & Voigt, 1988) and their psychological perspective with a radical constructivist view (von Glasersfeld, 1984). The interpretive framework, shown in Figure 1.3, is the primary analytic tool for carrying out analyses within the emergent perspective.

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

*Figure 1.3.* Interpretive framework for the emergent perspective

This framework accounts for the mutual development of classroom practices and norms and individual learning, in turn providing a richer understanding of the development of mathematics in the classroom environment. As a result, an investigation of an individual's mathematical conceptions and engagement in proof activity is informed by an attention to the

social norms, sociomathematical norms, and mathematical practices of the classroom in which these individual aspects of mathematical activity occur as well as by the individual's beliefs about individuals' roles and the nature of mathematical activity in school and mathematical beliefs and values. Based on my interests, in the current research I am consciously choosing to focus on relating individuals' mathematical conceptions (of inverse and identity) and activity (engagement in proof), supporting this investigation with attention to classroom and homework group sociomathematical norms (for proof) and mathematical practices (related to inverse and identity).

Resulting from this discussion is a research project targeted at investigating relationships between the important activity of mathematical proof and individual conceptual understanding of inverse and identity in the rich, proof-intensive context of an abstract algebra class. This investigation is warranted by calls in the literature for investigation of learning from proof (Weber, 2005), focusing on the individual conceptual understanding involved in the proofs (Balacheff, 1986). The results also contribute to the literature by expanding existing documentation of students' understanding of inverse and identity in Abstract Algebra. The investigation is situated within a perspective (Cobb & Yackel, 1996) that adopts the importance of situating individual learning relative to the classroom in which it occurs, acknowledging the broader context in which relationships between proof and conceptual understanding exist.

## Chapter 2 – Literature Review

### Abstract Algebra

In this section, I explore existing research related to inverse and identity, beginning with a general discussion of early development of the children's understanding of additive inverses. I then extend this discussion toward a general discussion of research in Abstract Algebra by discussing researchers' results related to student understanding of group theoretical concepts and relating this to the Abstract Algebraic concepts of inverse and identity. For this discussion, I have chosen to focus on a collection of articles that I group based on the researchers' theoretical framings. I categorize two groups of research articles: researchers whose articles are categorized in the first group use an action, process, object schema (APOS) framework for characterizing student thought; researchers whose work falls in the second group align with the instructional design theory of Realistic Mathematics Education (RME).

**Inverse and identity.** Inverse and identity are two powerful aspects in nearly all mathematical thought. For instance, the notions of identity and inverse are captured in Piaget's INRC (1970) as some of the most basic and fundamental psychological processes an individual might carry out. Inverse and identity are present in addition (and subtraction), multiplication (and division), functions, linear transformations, logic, and so on. I begin by exploring literature focused on early notions of inverse and identity, specifically in the inverse relationship between addition and subtraction. This provides a general background to the more specific notions of inverse and identity in Abstract Algebra. Next, I explore selected research in Abstract Algebra, focusing on researchers' discussions of students' conceptual understanding of group theoretical concepts. This includes a discussion of more general Abstract Algebra concepts, based on each author's emphasis, while maintaining connections to inverse and identity.

In mathematics, inverse and identity must necessarily be defined within a context. That is, any inverse is an inverse *of* an object relative *to* an operation. Similarly, an identity is defined relative to an operation. For example, within arithmetic, there are two fundamental operations: addition and multiplication. Each of these operations can be defined on Real numbers, in their most basic case, combining two numbers to yield a third number. Addition can be thought of as an advanced form of counting in which each of the first two numbers is counted consecutively and the result of the enumeration is recorded as the sum (Thomas & Tall, 2001). Multiplication can be viewed as a counting of one of the first two numbers, repeated for each count of the other number, to yield a third number, recorded as the product. With each of these operations, there exist real numbers 0 and 1 such that the sum of 0 and any number  $x$  is  $x$  and the product of any number  $y$  and 1 is  $y$ . 0 and 1 are then called the additive identity and multiplicative identity, respectively. Further, for any real number,  $x$ , there is a real number,  $-x$ , such that  $x + (-x) = 0$ . Similarly, for any nonzero real number,  $y$ , there is a real number  $1/y$  such that  $y*(1/y) = 1$ . These numbers ( $-x$  and  $1/y$ ) are called the additive and multiplicative inverse, respectively. Of course, the psychological development of these concepts is much more complicated than described here and any thorough discussion is beyond the purposes of this literature review.

Notice, though, that each instance of identity is defined relative to an operation and that each inverse is defined relative to a real number and an operation. I adopt the view that this structure underlies most mathematical thought: for any *action* (or operation) that an individual carries out on a mathematical object, there is another action that will reverse the first action so that the two actions together will not change the object. I take this to be a fundamental quality of human mathematical thought (that abstracted thought is reversible) while acknowledging that there do exist some mathematical actions that are not reversible (e.g., multiplication by 0).

I now discuss a selection of research investigating children's early notions of additive inverses and identity. This informs my perspective by providing insight into the underlying cognitive structures that may support more advanced notions of inverse and identity. Bryant, Christy, and Rendu (1999) quantitatively assessed children's understanding of inverse in the context of the relationship between addition and subtraction using several different types of questions in two studies. In the first study, the authors used the general form  $a + b - b$  to assess the children's understanding of the inverse relationship between addition and subtraction, controlling with problems of the form  $a + a - b$ . These relationships were presented using six different conditions: concrete/identical, concrete/nonidentical, invisible/identical, invisible/nonidentical, word problems, and abstract problems. The concrete conditions relied on columns of blocks to represent specific amounts within the general  $a + b - b$  statement. Invisible conditions were similar to the concrete, except that the blocks were not immediately visible to the participants. The word problem and abstract problem conditions did not rely on physical manipulatives. The second study was similar to the first except the authors also used questions of the form  $a - b + b$  and the form  $a + b - (b + 1)$  and  $a + b - (b - 1)$ . The results provide three main conclusions about student thinking with additive inverses: (a) children understand this relationship as early as five years, (b) children can use this relationship flexibly, and (c) an understanding of additive inverse does not necessarily correspond to addition and subtraction skills. The authors also discussed how the construction of the inversion tasks necessarily provides insight into the children's notions of identity as well. This was evident in their ability to not only identify the pairs of additive inverses, but also recognize the effect of their sum added to the first term.



Rasmussen, Ho, and Bisanz (2003) and Gilmore and Spelke (2008) conducted similar studies with younger participants. Each of these studies supports Bryant et al.'s (1999) findings that children are able to reason with inverse additive relationships of non-numerical quantities in early adolescence. This supports the view discussed earlier that the concept of inverse is fundamental in human mathematical activity and, further, that it does not necessarily rely on specific computational understanding. This provides an interesting background for considering instances of inverse and identity in higher-level mathematics, although it does not necessarily follow that an understanding of inverse is easily developed in every new context. Indeed, this conversation is tempered by Thomas and Tall's (1999) warning that, "generalised arithmetic... can only be used as a secure starting point for algebra if the students have a good sense of its meanings" (p. 597), which contrasts with Carraher, Schliemann, Brizuela, and Earnest's (2006) assertion that there are "compelling reasons for introducing algebra as an integral part of early mathematics" (p. 110) as these authors advocate for inclusion of algebra at earlier stages in the curriculum.

**Research in Abstract Algebra.** Abstract Algebra is widely considered to be an important course in undergraduate mathematics (Hazzan, 1999). Authors often attribute its importance to it frequently being one of the first proof-intensive courses, in which the content develops deductively through proof to generate broader, structural reasoning (Dubinsky, Dautermann, Leron, & Zazkis, 1994), although recent reforms in secondary and undergraduate education have focused on incorporating more proof activity earlier in the curriculum (CCSS, 2010; Harel, 2002; NCTM, 2000). Still, the general unifying power of abstract algebra as well as its ubiquitous use in other branches of mathematics support its importance (Hazzan, 1999).

Education research in Abstract Algebra has used several perspectives to focus on different facets of the subject. The majority of this research has focused on students' understanding of specific concepts within the curriculum in order to build a body of evidence of student understanding and, in turn, inform researchers' instructional techniques and curriculum (e.g., Hazzan, 1999; Larsen & Lockwood, 2013; Leron & Dubinsky, 1995). For example, Simpson and Stehlikova (2006) focus on attentional shifts to analyze one student's development of notions of rings through her investigation of  $Z_{99}$ . In another example of this type of researcher, Hazzan and Leron (1996) investigated what they describe as "students' (mis)use of Lagrange's Theorem," finding that students often believed the converse was true, used a "naïve" version of the converse, and applied the converse inappropriately. Other researchers have focused on students' and instructors' proof in Abstract Algebra (e.g., Hart, 1994; Fukawa-Connelly, 2013; Wheeler & Champion, 2013) – a type of research that I explore in more detail in the Proof section of this chapter. Still other researchers focus on a discussion of the history of Abstract Algebra and its teaching (Katz, 1997; Kleiner, 1996, 2007).

In this review, I discuss several articles within the abstract algebra literature that I chose to group based on the theoretical perspectives of the authors. Authors in the first group of articles use APOS theory to investigate student understanding of Abstract Algebra; the second group consists of research focused on developing curricula based on the instructional design heuristic of guided reinvention. Before discussing each group of articles, I briefly discuss the respective theoretical perspectives of the researchers. With each article, I draw on the researchers' discussion of the conceptual understanding involved in individuals' construction of algebraic concepts.

**Research related to APOS.** The first collection of research articles I discuss centers around the constructivist APOS theory developed by the research group of Asiala, Brown, DeVries, Dubinsky, Mathews and K. Thomas (1996). Several researchers have adopted this theory, both within and outside of Abstract Algebra. I first describe APOS theory, providing an understanding of how it is used in the literature. I then discuss three examples of Abstract Algebra research in this area, having chosen articles that investigate conceptual understanding that is relevant to developing notions of inverse and identity in Abstract Algebra. This includes discussions within the literature of mathematical concepts other than inverse and identity, such as groups, subgroup, normality and cosets, which have important connections to identity and inverse.

As stated, APOS theory refers to a constructivist perspective of epistemology. This perspective relies on the notion of *genetic decomposition*, which Asiala, Dubinsky, Mathews, Morics, and Oktaç (1997) describe as a “model of cognition: that is, a description of specific mental constructions that a learner might make in order to develop her or his understanding of the concept” (p. 242). This model relies on a description of mathematical understanding composed four parts, each of which is developed in support of the others. Asiala, et al (1997) summarize APOS theory by describing each of the four parts and relating each part to the other through the mental operations needed to develop conceptual understanding.

An *action* is a transformation of mathematical objects that is performed by an individual according to some explicit algorithm and is seen by the subject as externally driven. When the individual reflects on the action and constructs an internal operation that performs essentially the same transformation then we say that the action has been *interiorized* to a *process*. When it becomes necessary to perform actions on a process, the subject must *encapsulate* the process as a totality to create an *object*. In many mathematical operations, it is necessary to *de-encapsulate* an object and work with the process from which it came. A *schema* is a coherent collection of processes, objects and other schemas that is invoked to deal with a mathematical problem situation. A schema can be *thematized* to

become another kind of object; a thematized schema can also be unpacked to access the underlying components of the schema. (p. 242, emphasis in original)

The researchers use the four aspects of APOS theory to develop a genetic decomposition of a mathematical concept, which they in turn use to generate instructional sequences in Abstract Algebra. Data is collected during the instructional sequences to analyze to what extent student learning reflects the theorized decomposition, informing both the researchers' genetic decomposition and the instructional sequence. Because one purpose of this literature review is to gain a better understanding of students' development of inverse and identity, I discuss each of the articles with a focus on two aspects: results related to student understanding of algebraic concepts and the authors' genetic decompositions of each concept.

Dubinsky, Dautermann, Leron, and Zazkis (1994) draw on data collected from 24 in-service teachers engagement in a summer institute about group theory. The authors present an outline of what they view as the general development of the concepts of group, subgroup, coset, normality, and quotient group. Although they do not formally describe their framing with the language "APOS theory," the authors begin their article with a description of action, process, object, and schema to describe their perspective. The authors documented the participants' engagement in the abstract algebra course, categorized the data based on the mathematical topics involved, and used the APOS framework to analyze the data towards describing the participants' understanding. While the authors incorporate the APOS framework into their discussion, they do not explicitly state genetic decompositions for any of the concepts they discuss but instead use the framework loosely to provide general descriptions of the concepts.

The authors describe their model of students' development of the group concept as a development from thinking of a group as a set to thinking of a group as a set coordinated with binary operations. Similarly, subgroups are initially described as subsets, which later are

coordinated with the operations corresponding to the larger set. The authors then describe the conclusion of this development as “the encapsulation of two objects, a set and a function (binary operation) coordinated in a pair which may be the student's first real understanding of a group” (p. 292). The authors also describe the reconstruction of this paired concept, organized through isomorphism, so that students understand groups as equivalence classes. With this development of the concept of group, development of identity and inverse is delayed until after a binary operation is defined on the set. Consequently, the participants in the study tended to focus more on the characteristics of the *set* than on the coordination of elements in the set with each other under an operation. For instance, students might conflate  $(\mathbb{R}, +)$  and  $(\mathbb{R}, *)$  because each of these groups is defined using the real numbers as the set of elements. The authors’ discussion focuses more on this confusion than on conceptual understanding once the students did coordinate the set with the operation. The students’ coordination is described as an application of their coming to understand the binary operation as a function on the elements of the set. Further, the authors argue that students’ development of the concept of subgroup may be viewed as a restriction of the function to a subset of the domain. Although the authors do not elaborate, one might draw from this that student understanding of inverse and identity in group theory potentially relies on notions of inverse and identity as they relate to a student’s understanding of function.

Interestingly, the authors do not point out here that restricting the domain of the operation is not sufficient to constitute a subgroup (for instance, a restricted domain might exclude the function’s identity), so it is unclear whether the students found difficulty with this understanding of subgroup. However, the authors later describe students’ eventual understanding of subgroups:

The student comes to realize and use the fact that of the four group properties, associativity is inherited, and that, in addition to closure, it is only necessary to check that the original identity is in the subset and that the inverse of every element of the subset is also in the subset. (p. 281)

The discussion does not include a description of how a student comes to the realization that each of these properties must be satisfied; however, it does provide insight into how students might coordinate inverse and identity when considering subgroups. If, as the authors claim, students' early understanding of subgroup is based on their understanding of restricting a function's domain, then the student would need to coordinate this when checking the group axioms of inverse and identity for the subgroup. For instance, checking the group axiom would require knowing which elements to look for to show that they are included in the subset. The students would need not only to restrict the function to the domain, but also understand that an element's inverse is preserved by the restriction – that it is the same inverse as it was in the larger group.

Asiala et al. (1997) use APOS theory to discuss students' understanding of cosets, normality, and quotient groups. This article outlines a second iteration of data collection and analysis building off of the work discussed in Dubinsky, et al. (1994). The data discussed in this article were drawn from 31 participants engagement in an undergraduate Abstract Algebra class. Similar to the discussion in Dubinsky, et al. (1994), the discussion in this article focuses on the authors' genetic decompositions, supporting this discussion with examples of student work. The authors' discussion approaches normality using the coset definition, rather than the conjugation definition, which might provide more discussion of students' conceptual understanding of inverse, based on the inclusion of inverse notation in conjugation.<sup>1</sup> However, the authors' genetic decomposition of cosets can be leveraged toward a discussion of students' understanding of inverse. In their description, the authors begin with the relation  $gh = hg$ , which they assert is one

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<sup>1</sup> Normality may be defined by conjugation (“ $H$  is a normal subgroup of  $G$  if and only if  $ghg^{-1} \in H$  for all  $g \in G$  and  $h \in H$ .”) or by left and right coset equality (“ $H$  is a normal subgroup of  $G$  if and only if  $gH = Hg$  for all  $g \in G$  and  $h \in H$ .”). These definitions are equivalent. That is, each implies the other.

of the most commonly used, but add that there is “little difference if one of the other relations leading to an equivalent definition is used, such as  $gHg^{-1} \subset H$ ” (p. 296). Accordingly, I will use the latter relation as I continue so that the discussion includes more usage of inverse notation.

The authors describe a stage in which an individual has no understanding of the mathematical meaning of normal followed by a stage in which the individual confuses normality as a property of a subgroup independent of the containing group. Finally, the individual “differentiat[es] between actions in H and G and coordinat[es] these to arrive at normality as an action applied to an object H as a subgroup of a group G” (p. 296). The authors then provide the genetic decomposition for this action, which I include here, altered to reflect my choice of beginning relation ( $gHg^{-1} \subset H$ ).<sup>2</sup>

- (a) The first level is as an action on elements of H and G with little control. This might be expressed as  $[ghg^{-1} \in H], \forall g \in G, h \in H$ . This can become another road to confusing normality with commutativity or it might just stay in this unstructured form.
- (b) The first step from this beginning to a construction of the concept of normality is to differentiate between the elements of G and H. A student at this level might refer to  $[ghg^{-1} \in H], \forall h \in H$  and consider g separately or not at all. The student’s conception at this point is still as an action.
- (c) Once the elements of G and H are differentiated, the action relating to H can be interiorized into a process and encapsulated (as in coset formation) to reach a statement like  $[gHg^{-1} \subset H]$ . Often, although the statement is understood as a process involving the elements of H and a fixed element g, there can be a delay before the student is ready to iterate g over G. This may be the result of not encapsulating the equality  $[gHg^{-1} = H]$ .
- (d) Finally, the relation  $[gHg^{-1} = H]$  is encapsulated as a proposition which is an object that can depend on the parameter g. This parameter is iterated over G to form the process,  $[gHg^{-1} = H], \forall g \in G$ . (p. 296-297)

Throughout this genetic decomposition, one may tend to treat  $g^{-1}$  merely as the inverse of the arbitrary element g in G. However, the authors’ discussion suggests that the student may perceive  $g^{-1}$  in (a), (b), and (c) differently from  $g^{-1}$  in (d). This is because the student’s

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<sup>2</sup> The reverse set inclusion  $gHg^{-1} \supset H$  can be shown by considering  $g = e$ .

encapsulation of the relation would necessitate that the student no longer views  $gHg^{-1}$  as a coset determined by calculating  $ghg^{-1}$  for each  $h$  in  $H$  based on a fixed  $g$ , but rather as a coset that defined by a variable  $g$ . This is a nuanced distinction, but the psychological nature of the symbol “ $g^{-1}$ ” could change in a nontrivial way for the student. This discussion highlights the importance of the distinctions between students’ varying conceptual understanding of inverse, especially in the context of more involved concepts such as cosets.

Brown, DeVries, Dubinsky, and Thomas (1997) explore students’ development of binary operations in rings and groups. The authors discuss students’ responses to the following test question, “If  $\mathbb{R}$  is a ring with identity, denote by  $\mathbb{R}^*$  the set of units in  $\mathbb{R}$  with the multiplication operation from the ring. a) Show that  $\mathbb{R}^*$  is a group” (p. 238). I posit that this question potentially requires students to reason about inverse in at least two ways. First,  $\mathbb{R}^*$ , the set of units in the ring, is defined as the set containing all elements in the ring for which an inverse under the ring operation (rather than the group operation) exists. Second, the students must verify that this subset is a group, which involves determining an operation on the set as well as identifying inverses and an identity. The authors argue that, “after constructing a set and a binary operation, the student must still work to coordinate the various ways in which the binary operation can relate to the set or sets on which it is defined” (p. 202) Further, the authors state that, “several responses indicated confusion on the connection between the set and the binary operation as it related to group properties” (p. 206).

The authors do not indicate that students had any difficulty thinking about the set  $\mathbb{R}^*$ , although some students’ responses, when verifying identity and closure of the group, indicate that they might have believed that  $\mathbb{R}^*$  was equivalent to the entire ring. However, the authors do discuss difficulties they had interpreting students’ verification of the inverse and identity axioms,



stating, “several students did not check that the inverse of an element in  $\mathbb{R}^*$  not only existed in  $\mathbb{R}$ , but was in  $\mathbb{R}^*$ . . . [a] similar situation occurred with the identity for  $\mathbb{R}^*$  in that many students explained why the multiplicative identity of  $\mathbb{R}$  is in  $\mathbb{R}^*$ , but neglected to mention explicitly the fact that it is an identity in  $\mathbb{R}^*$ ” (p. 207). This discussion supports the possibility that the students thought that  $\mathbb{R}^*$  was equivalent to  $\mathbb{R}$ , in which case all inverses and the identity element would already be included in  $\mathbb{R}^*$ . Another plausible explanation is that the students considered  $\mathbb{R}^*$  as a group relative to the group operation, rather than the ring operation. The authors suggest that the students might have felt that these axioms did not require verification. Regardless, this example provides further support for investigation into students’ understanding of inverse and identity.

**RME-based research.** The next collection of research I discuss centers around the recent development of two instructional sequences in Abstract Algebra that use the instructional design heuristic of guided reinvention (Freudenthal, 1991), which is a tenet of the instructional design theory of Realistic Mathematics Education (RME, Freudenthal, 1977). Each of these articles focuses on instructional sequences aimed toward students’ development of abstract algebraic concepts based on emergent models (Gravemeijer, 1998). The first article explores Larsen’s (2013) development of the *Teaching Abstract Algebra for Understanding* (TAAFU) curriculum. The second article details Cook’s (2012) exploration of students’ construction of ring axioms. I first outline the theoretical perspective underlying RME, including a discussion of guided reinvention, which is used by each of the articles discussed here. I then discuss aspects of Larsen’s (2013) TAAFU curriculum aimed at students’ guided reinvention of the group axioms, focusing on the students’ development of notions of inverse and identity and supporting this with excerpts from Larsen (2009) in which he describes a series of teaching experiments used to develop TAAFU. Next, I will discuss students’ notions of inverse and identity developed in

Cook's (2012) teaching experiment.

In recent years, several researchers have adopted the RME instructional design theory (e.g., Gravemeijer, 1999; Larsen, 2013; Rasmussen & Kwon, 2007; Swinyard, 2011; Stephan & Akyuz, 2012; Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). RME adopts a perspective that mathematics, rather than being a static collection of concepts, is a human activity (Freudenthal, 1991). RME promotes students' engagement in mathematical discussion while instructors are viewed as guides of activity to help students develop their own, more meaningful mathematical realities. Accordingly, curricula developed from an RME perspective focus on an iterative process of engaging students in a sequence of tasks and refining the task sequence based on the students' development. Researchers use various design heuristics to develop these sequences, including guided reinvention (Larsen & Zandieh, 2008; Wawro et al., 2012) and emergent models (Freudenthal, 1991, Gravemeijer, 1999, Larsen, 2013).

A researcher uses the guided reinvention heuristic by first developing a sequence through which he or she anticipates students' development of general mathematical ideas from specific experiences. This initial task sequence is often based on thought experiments that anticipate a student's engagement or the researcher's knowledge of the historical development of the intended mathematics (Gravemeijer, 1999). The researcher then iteratively develops the task sequence in order to elicit more meaningful and powerful understanding from students. Throughout this process, the researcher focuses on the students' mathematical activity, attending to activity that anticipates more formal or powerful ways of thinking, and adapting the task sequence to evoke, leverage, and guide future students' activity towards a construction of more powerful and meaningful mathematical realities (Larsen, 2013).

The TAAFU curriculum for the guided reinvention of the group axioms centers on students' formation of a minimal set of rules to describe the symmetries of an equilateral triangle. In this article, Larsen (2013) uses data from in-class implementation of the curriculum to outline the local instructional theory (Gravemeijer, 1999) he developed in designing the curriculum. The curriculum begins with students identifying and symbolizing the symmetries. The students are then asked to analyze combinations of the symmetries. The instructor guides the class in selecting a single set of symbols with which to notate the symmetries as well as notation for the composition of symmetries. Larsen states that, “[p]erhaps the most important shift in the students’ activity occurs when they transition from analyzing combinations geometrically to calculating combinations algebraically using a set of rules that they develop” (p. 4). This analysis process leads to the students’ production of a list of rules to describe the relationships between the symmetries, which they reduce and augment to produce a list of necessary and sufficient relationships to describe the symmetries.

Larsen asserts that the inverse and identity axioms are the most difficult rules for students to generate toward the development of the group axioms. This seems to be an artifact of their engagement in the task sequence by composing transformations in order to generate other transformations. Larsen draws on the students’ prior construction of what are essentially Cayley tables of their transformations to explore the notions of inverse and identity. He provides a more detailed description of this process in a related article (Larsen, 2009). One of the students, Jessica, engages in proving the uniqueness of the identity:

“After initially writing that a group has an identity element, Jessica changed this aspect of her definition, writing that a group has a unique identity element. I asked Jessica if she could prove that the identity must be unique using the rest of her definition. After she worked for about 3 min, I asked her to tell me what she was doing... she was able to express what it would mean to have two identity elements using algebraic symbols ( $s \bullet x = s$  and  $s \bullet y = s$ ) and use this to generate the

equation  $s \cdot x = s \cdot y$ . She also knew that she wanted to be able to say that this equation implies that  $x$  is equal to  $y$ . This led to her second important insight, which is that this follows from the fact that, in the tables they had worked with, no element appeared twice in the same row” (Larsen, 2009, p. 126).

In this excerpt, Jessica develops a relationship between a representation of the identity and an arbitrary element of the group. In order to use the last equation, Jessica must multiply each side by  $s^{-1}$  to eventually yield the equation  $x = y$ . Instead, she used the Sudoku property of the Cayley table to argue that the two elements must be the same. This incident is similar to the incident discussed in Chapter 1. In each of these cases, the coordination of the Cayley table with the specific concepts helped Jessica and Sandra to successfully reason about inverses and identities within the group of symmetries, producing meaningful, more powerful understanding of the two concepts.

Cook (2012) describes a teaching experiment in which he guided two Abstract Algebra students’ reinvention of the ring axioms through their equation solving practices. Cook organizes his discussion relative to a revised version of the emergent model design heuristic, which organizes student activity into situational, referential, general, and formal levels of activity (Gravemeijer, 1999). Cook supplements the original framework with intermediate phases that delineate anticipations of students’ activity for progression to the next level of activity. Cook provided his participants with the equations of the form  $x + a = a + b$  and  $ax = ab$ , which they were to solve with the restriction that the variables in in each equation were elements of specific abstract algebraic rings ( $Z_{12}$ ,  $Z_5$ ,  $Z$ ,  $Z[x]$ , and  $M_2(Z)$ ). These rings were specifically chosen to provide students with a variety of properties with regards to the dimensions of finiteness/infiniteness, integral domain/not an integral domain, field/not a field, and zero divisors/no zero divisors.

Cook provides several instances in which students reason with inverse and identity with respect to both the group operation and the ring operation. For instance, when presented the equation  $x + 3 = 9$  with entries in  $Z_{12}$ , the participants wrote the equation “ $x + 3 - 3 = 9 - 3$ .” When Cook reminded the participants that  $-3$  was not contained in  $Z_{12}$ , the students generated a table of equivalencies for inverses. He also points out that the participants also recognized that  $12$  is the additive identity of  $Z_{12}$ , as they noted in their table. Cook then provided the participants with two multiplication equations:  $5x = 10$  and  $4x = 8$ , pointing out that  $x = 2$  is a solution for both equations, but not a unique solution to  $4x = 8$  in  $Z_{12}$ . The participants developed a different approach for each equation. To solve  $5x = 10$ , the participants right multiplied each side of the equation by  $5$  to yield the equation  $1x = 2$ , but to solve the second equation, they subtracted four from each side and factored the left-hand side, rewriting the equation as  $4(x - 1) = 4$ .

When asked why they didn't use the first method to solve both equations, one participant responded that the method did not work if the number is a factor of the base, explaining “4 times any number does not make 1” (p. 155). It is unclear how the participant knew this or why he thought it was important, but it does highlight an interesting aspect of the nature of the research: in nonstandard situations, students must frequently rely on mathematical thinking contrary to their previous understanding. In this case, the participants were constrained from using multiplicative inverses because  $4$  does not have a multiplicative inverse in  $Z_{12}$ . The participants in this study were already familiar with modulo arithmetic, as their engagement in the activities shows. However, the equations that Cook uses are quite similar in form to the equations used by Bryant, Christy, and Rendu (1999). The participants' in Cook's (2012) article were required to navigate systems in which multiple elements were not invertible (nonzero zero-divisors), which would contrast with the intuitive nature of inverse describe in the literature related to children's

understanding. This is similar to the non-commutability of the symmetries of a triangle explored by participants in Larsen (2013).

These articles inform my research in several ways: they provide examples of the types of student understanding and activity that will likely encounter throughout my research, they provide examples of researchers models of student understanding, and they give examples of research-based Abstract Algebra curricula. Larsen (2009, 2013) and Cook (2012) provide very detailed accounts of students reasoning about Abstract Algebra. Dubinsky, et al.'s (1994) model of subgroup, Brown, et al.'s (1997) discussion of students' development of binary operations, and Asiala, et al.'s (1997) genetic decomposition of normality provide examples of models of student thinking that I will be developing, except that I will focus on modeling individual students' understanding, rather than developing a general, overarching model of the components constituting a concept. The examples of curricula found in the literature provide an understanding of the types of approaches researcher use in trying to teach Abstract Algebra more effectively.

## **Proof**

Educational researchers have clearly established the importance of exploring and discussing students' engagement in and understanding of mathematical proof (Hanna, 2000; Weber, 2010). Indeed, according to Harel and Sowder (2007), "No one questions the importance of proof in mathematics, and in school mathematics" (p. 806). Accordingly, researchers have developed a plethora of approaches to studying proof in mathematics education. Perspectives toward discussing mathematical proof vary in focus from the more abstract philosophical and historical (Balacheff, 2010; Lakatos, 1976; Mancosu, 2001; Thurston, 1995) to the empirical and situational (Fukawa-Connelly, 2012; Selden & Selden, 2003). Foci in proof research range as

well, including psychological dispositions necessary in the development of a more general understanding of proof (Balacheff, 1986), classification of students' values and beliefs about what constitutes proof (Harel & Sowder, 1998), students' and mathematicians' engagement in proof (Savic, 2012; Weber & Alcock, 2004), and students' and mathematicians' reading of proof (Alcock & Inglis, 2008; Alcock & Inglis, 2012; Alcock & Weber, 2005).

Empirical research in proof has been focused on a variety of aspects of proof as well as different groups' development of and understanding of proof. The majority of this research has focused on students' understanding of and engagement in proof in order to improve mathematics education at the secondary and undergraduate level (e.g., Hoyles, 1997; Inglis & Alcock, 2012; Maher, Muter & Kiczek, 2007; Tanguay & Grenier, 2010). For instance, Ellis, Lockwood, William, Dogan, and Knuth (2012) found that students' most common uses of examples in proof were to check the proof, support a general argument, convince, and understand the mathematical relationships involved in the proof. This is representative of a group of researchers who have explored relationships between examples and proof (e.g., Almeida, 2000; Iannone & Nardi, 2007). Other researchers have focused on experts' approaches to proof in order to gain insight into their strategies for proof writing and evaluation (Inglis & Alcock, 2012; Inglis, Mejia-Ramos, Savic, 2012; Weber, & Alcock, 2013). For instance, Weber, Mejia-Ramos, Inglis, and Alcock (2013) have generated an interesting debate over expert mathematicians' skimming practices when reading proof. Still other researchers investigate teachers' understanding of proof and the implications this has towards their teaching (Harel & Rabin, 2010; Knuth, 2002a; 2002b).

Given the expansive and varied collection of research related to proof, I focus my exploration of the literature based on the following goals: familiarizing myself with existing

research investigating students' proof in abstract algebra, finding potentially useful frameworks for analyzing students' engagement in proof activity, investigating an ongoing discussion in the literature concerning the explanatory nature of proof, and generating an operational definition of proof that will provide consistent criteria for identifying participants' engagement in proof.

Working toward these goals, I first discuss a selection of research articles that provide empirical discussions of students engaging in proof in an abstract algebra setting. I then examine three frameworks used in the literature to analyze individuals' engagement in various forms of proof activity. Finally, I explore researchers' perspectives within two overlapping discussions: the first about the fundamental characteristics that constitute proof, which has spawned the second discussion about the explanatory nature of proof. I draw on each discussion to develop my own perspectives related to proof and, in turn, inform my approach in the current research, including an operational definition for identifying proof.

**Proof in Abstract Algebra.** Several researchers have analyzed student proof in abstract algebra (e.g., Harel & Sowder, 1998; Selden & Selden, 1995). In this section, I explore research investigating individuals' engagement in proof about abstract algebraic concepts. From the literature that I reviewed, I have chosen articles that represent a variety of research foci as well as analytic perspectives. In the discussion of each article, I focus on the participants' proof activity, the authors' discussions of the participants' understanding of the algebraic concepts, and pertinent claims that the authors make about the results of their research.

Weber and Alcock (2004) explored four graduate students' and four undergraduate students' productions of five proofs about isomorphism (study 1) as well as four undergraduate students' and four algebraists' responses to a questionnaire about group isomorphism (study 2). In study 1, the authors found that the students' successful proof production depended



dramatically on whether they were a graduate or undergraduate student. Each graduate student successfully proved all five of the statements of isomorphism, whereas the undergraduate students collectively generated only two successful proofs (out of twenty total). The authors claim, however, that the students could correctly answer questions about the facts needed to develop nine more successful proofs based on their responses on a separate test, further claiming that this supports Weber's (2001) previous findings that students might have all of the necessary conceptual understanding needed to generate a proof, but still fail to do so successfully.

The authors then analyzed the students' responses on the protocol. In determining whether  $Q$  and  $Z$  are isomorphic, three of the four doctoral students argued that  $Z$  is cyclic and  $Q$  is not. All four undergraduate students attempted to find a bijection between the two groups. Each group of students responded similarly when determining whether  $Z_p \times Z_q$  is isomorphic to  $Z_{pq}$  ( $p, q$  coprime). The graduate students showed that the two groups were equinumerous and cyclic, whereas the undergraduate students attempted to find bijections between the two groups, none of which were successful (although is possible). Overall, the authors describe the two groups' approaches, showing that the graduate students compared group properties on eighteen of the twenty total tasks. Contrastingly, only one of the undergraduate students attempted to compare group properties (and did so successfully) while nine of these students' approaches attempted to find mappings between the groups and nine other responses were described as "unable to make meaningful progress" (p. 215).

In study 2, the authors discuss algebraists' and undergraduate students' intuitive understanding of isomorphism. The algebraists' discussion of isomorphism generally centered on notions of "sameness" between the two groups, focusing on the potential to re-label or rename the elements of one group so that they aligned directly with the other, including relationships

under the respective group operations. When asked how they would set out to show that two groups are isomorphic, each algebraist described exploring the group in order to gain intuition about them. In contrast to the algebraists, none of the undergraduate students described relabeling of group elements or intuitive notions about groups. When asked about their understanding of isomorphism, all four students presented the formal definition of isomorphism with no intuitive description. Similar to the results in study 1, all four participants describe their approach to determining isomorphism as a sequence of first comparing the order to the groups and then searching for a bijection between the two. This article provides insight into both the conceptual understanding that various individuals tend to have about group isomorphisms as well as their typical approaches to proving about isomorphism.

*Table 2.1 – Larsen and Zandieh’s (2008) reframing of Lakatos’ (1976) activities (p. 209)*

Type of activity	Focus of activity	Outcome if activity
Monster-barring	Counterexample & underlying definitions	Modification or clarification of the definition
Exception-barring	Counterexample & conjecture	Modification of the conjecture
Proof-analysis	Proof, counterexample, & conjecture	Modification of the conjecture & sometimes a definition for a new proof-generated concept

Larsen and Zandieh (2008) utilize Lakatos’ (1976) description of three methods of mathematical discovery to analyze students’ in-class engagement in mathematical activity.

Larsen and Zandieh (2008) adapt three activities – monster-barring, exception-barring, and proof analysis– from Lakatos (1976) and develop a framework describing the focus and outcome of each activity (Table 2.1). The authors use this framework to investigate students’ development of a minimal list of conditions needed to ensure that a subset is a subgroup. The participants in this study were students in a course with a curriculum developed by Larsen, which the authors describe as similar to the TAAFU curriculum (Larsen, 2013). The students were aware that satisfaction of the closure, identity, and inverse axioms was sufficient to show that a subset was a

subgroup and, so, engaged in determining if any of these axioms could also be removed from the set of conditions.

Here, I provide the Larsen and Zandieh's (2008) account of one incident in which students engaged in activity that the authors describe as monster-barring and exception-barring. The students first developed the conjecture that the closure property was sufficient to ensure that the subset was a subgroup. One student (Phil) used the idea of a Cayley table to attempt to prove this conjecture. This proof relied on each element occurring only once in each row of the Cayley table. Phil argued, because the students were assuming closure under the operation and each element occurred only once in a row, that for a given element  $a$ , that element would occur exactly once in the table. Phil argued that the element corresponding to the column in which  $a$  appeared would be the identity element, showing that the set would contain the identity element. Phil argued further that, since the identity element was shown to be in the subset, then it also appeared in each row of the Cayley table and, so, the subset contained inverses. This argument indicates that Phil's understanding of identity included a relationship between elements in the Cayley table and the column corresponding to that element's entry. Further, it seems as though Phil was able to coordinate the concept of inverse to the column corresponding to the identity entry in an element's row. That is, the identity in the Cayley table indicates that the row element and column element are inverses of each other. Larsen and Zandieh point out a "hidden lemma" (p. 211) in Phil's argument: that each element of the subset will appear at least once in each row of the Cayley table. Indeed, Phil's argument assumed *both* that each element in the subset would appear in a row of the Cayley table at most once *and* that every element would show up at least once. Although this is true for finite subsets, this is not necessarily the case if the subset is infinite.

The instructor in the class then provided Phil with a global counterexample – an example which satisfied the hypotheses of the conjecture but for which the conjecture did not hold. The authors describe the students’ reaction as monster-barring because the students initially argued that the presented example was a non-example. Initially, the students provided two arguments barring the counterexample, each of which the instructor defended against: that the counterexample did not use the same group operation and the counterexample did not satisfy the closure hypothesis of the conjecture. After the instructor’s rebuttal, Phil candidly stated that he was “trying to think of a way around it” (p. 212). Applying the authors’ framework, we notice that the students’ arguments against the counterexample focused on the properties of counterexample rather than the conjecture or proof. In Phil’s statement, the “it” to which he refers is the counterexample. A second student (Steve) in the group provides an alternative conjecture: that the students must include inverses in their hypotheses for the conjecture. The authors describe this as exception-barring because Steve modified the conjecture.

Hart (1994) investigated twenty-nine undergraduate and graduate students’ production of six proofs in abstract algebra. Here, I explore Hart’s discussion concerning one of the tasks that asked students to “[p]rove that, if a finite group  $G$  (with identity element  $e$ ) has an even number of elements, then there exists an element  $a \neq e$  in  $G$  such that  $a^2 = e$ ” (p. 54). Hart describes students’ “Reformulation” of a given notation or statement as a process that the students used in order to carry out specific proofs. He provides an example of one such reformulation in the proof of the task mentioned above.

$a^2 = e$  was reformulated as  $a = a^{-1}$  to provide the key to most successful proofs of Proof 5. While this reformulation could have come (syntactically) from simply multiplying both sides by  $a^{-1}$ , it seems more plausible from the context of the problem and students’ work that it was driven by an understanding of inverses, in particular an understanding that any element multiplied by  $a$  which gives the identity must be  $a$ ’s inverse. (p. 60)

In his discussion, Hart explains that this approach was more common among higher-level participants (operationally defined in the article by weighting responses to specific tasks). Hart attributes this reformulation to the participants' understanding of inverses and coordination of this understanding with the concept of identity. Although it is not discussed in the article, one might posit that the participants would also need to coordinate these concepts with their proof activity. That is, the participants would not only need to understand the relationship that enables the reformulation of the equation, but also understand how to leverage the new representation toward the necessary proof. This hypothesized coordination is consistent with findings from the literature (Selden & Selden, 2009; Weber, 2001).

Fukawa-Connelly (2012) explored sociomathematical norms for proof in an abstract algebra classroom consisting of twenty-nine students. He developed a grounded framework describing these sociomathematical norms. This framework consisted of three general types of norms: presenter responsibilities, audience responsibilities, and other sociomathematical norms expected of students regardless of their role at a given moment. The presenter responsibilities included *explaining and defending your work* and *responding to questions*, whereas audience responsibilities consisted of *reading carefully*, *convincing yourself*, and *asking questions*. Overarching norms included *only using peer-validated knowledge*, *working with others*, *taking responsibility for problems*, *high-level ideas should be a focus of discussion*, and *justifying new inferences based upon old ones*.

Fukawa-Connelly describes a classroom episode in which the students discussed a proof that  $(-a)(-b) = ab$  (problem 68e), which relied on the proof of another homework problem, 68b. The episode began with eleven students writing solutions to homework problems on the board. One student, Bill, abandoned his attempt to prove problem 68b at which point the instructor

announced that problems 68b and 68e were unfinished. This led some of the students to work together in their small groups to prove the relationships. One student, John, provided his partner, Kelly, with a draft of a proof of 68e, which she critiqued by pointing out that one line was unnecessary. She then announced to the instructor that John had a proof. John responded by telling the instructor that he was not comfortable enough to present it yet. At this point, Sarah (a student in another group) declared that she thought her group had proved problem 68b. Fukawa-Connelly describes Sarah's presentation of the proof, throughout which she continually credited the other group members with the *high-level ideas*. After Sarah's proof, Bill and John both volunteered to present problem 68e. John's presentation of the proof included a large arrow between two of the lines. When asked about this, John explained that this is where he used the result of 68b. Fukawa-Connelly uses this as an example of the norm that he calls *justifying new assertions based on old ones*. Fukawa-Connelly also describes how John's presentation highlighted the *high-level ideas* within the proof.

Fukawa-Connelly describes Bill and John's eagerness to prove 68e as an example of the norm of *taking responsibility for problems*. Further, John's presentation is described as an example of *explaining and defending your work*. Fukawa-Connelly points out that, even though the instructor told the students that their work on other problems would be easier if they used problem 68b, the students did not use the result until after it was proven. Fukawa-Connelly contends that this is an instance of the students following the classroom norm of *only using peer-validated knowledge* because the students did not use the result of the proof until it was validated.

These articles provide accounts of students engaging in a variety of proof practices – from the (largely unsuccessful) syntactic production, to students attempting to “think of a way

around” and monster-bar counterexamples, to students’ successful reformulation of an equation to develop a proof about inverses, to students’ successful engagement in and adoption of classroom proof norms that valued logical progression of arguments focused on high-level ideas. Each author approached the students’ proof from a different perspective. Whereas Weber and Alcock (2004) focus on semantic and syntactic approaches of individual students, Fukawa-Connelly attends to the normative behaviors and practices of the entire class. Hart’s (1994) perspective was rooted in processes, errors, and misconceptions, with a focus toward classifying students’ ability. This view contrasts sharply with Larsen and Zandieh’s (2008) attention to the types of arguments students developed in defense against counterexamples.

**Analytical Frameworks for Proof.** Researchers have developed a wide variety of frameworks to analyze various aspects of proof. For instance, Bell (1976) categorizes students’ proof production according to varying levels of empirical and deductive responses; Dawkins (2013) adapted Carnap’s (1950) construct of explication to describe students’ development of more formal proofs and representational systems; Larsen and Zandieh (2008) adapted Lakatos’ (1976) proofs and refutations; and Hart (1994) distinguished between four levels of students based on the correctness of their responses to three tasks in order to categorize their proof responses. In this section, I choose to explore four frameworks for analyzing proof and draw on aspects of these frameworks to inform my own analytical perspective. Each of these researchers focuses on a different aspect of proof: what *constitutes* proof for an individual, individuals’ proof *production*, assessing students’ proof *construction*, and modeling the structure of an individual’s argument.

Harel and Sowder (1998) provide an extensive framework for describing how students engage in and think about mathematical proof according to the types of arguments that they use

and find convincing. The *proof scheme* framework is composed of three general categories – External, Empirical, and Analytical (Figure 2.1) – each of which is subdivided based on specific nuances between the arguments that convince the student of a proof’s validity. In support of their proof scheme framework, Harel and Sowder (1998) provide a definition of the process of proving. This process centers on an individual’s observation or conjecture changing from a state of unknowing (doubt) to a state of more (or perhaps less) certainty of truth.

*A conjecture is an observation made by a person who has doubts about its truth. A person's observation ceases to be a conjecture and becomes a fact in her or his view once the person becomes certain of its truth. This is the basis for our definition of the process of proving: By "proving" we mean the process employed by an individual to remove or create doubts about the truth of an observation. The process of proving includes two subprocesses: ascertaining and persuading. Ascertaining is the process an individual employs to remove her or his own doubts about the truth of an observation. Persuading is the process an individual employs to remove others' doubts about the truth of an observation. (Harel & Sowder, 1998, p. 241, emphasis in original)*

According to Harel and Sowder, no person completely displays evidence of exactly one proof scheme. Because of this, students’ proof schemes are generalizations of the types of proof schemes evident to the researcher through their engagement in constructing or reading specific proofs. This points to students’ changing notions of what constitutes ascertaining and persuading.

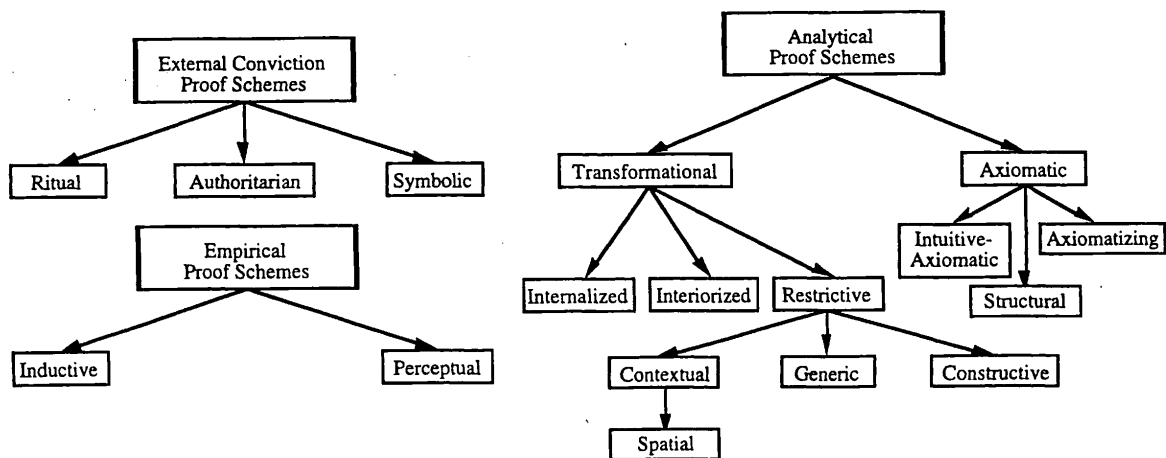


Figure 2.1. Proof Schemes (Harel & Sowder, 1998, p. 245)



Harel and Sowder's (1998) proof schemes help to generally describe one's notions of valid and invalid arguments. As Harel and Sowder point out, however, one's proof scheme often varies depending on any number of factors. Indeed, the authors, in their introduction to the article, state that, "... most importantly, as defined, ascertaining and persuading are entirely subjective and can vary from person to person, civilization to civilization, and generation to generation within the same civilization" (p. 243). The authors go on to say that, "... a person can be certain about the truth of an observation in one situation, but seek additional or different evidence for the same observation in an another situation" (p. 243). One potential point of contention that may rise from this discussion is the authors' claim that, "The kinds of evidence the students may look for are based on whatever conventions are accepted in their class as evidence for a geometric argument" (p. 243). Although this may be ideal, it is not difficult to argue that normative behavior for proof in the classroom frequently does not carry over *cleanly* to individual student thinking. Indeed, Healy and Hoyles (2000) found that, "students simultaneously held two different conceptions of proof: those about arguments they considered would receive the best mark and those about arguments they would adopt for themselves. In the former category, algebraic arguments were popular. In the latter, students preferred arguments that they could evaluate and that they found convincing and explanatory, preferences that excluded algebra" (p. 426).

Weber and Alcock (2004) provide a framework for discussing individuals' semantic and syntactic proof approaches that is quite different from Harel and Sowder's (1998) proof schemes framework. In their research, Weber and Alcock (2004) developed the semantic/syntactic framework to provide a lens for describing individuals' proof production. Notice that, rather than focusing on what one *finds convincing* in a proof or while proving, these authors have shifted the

focus *proof approaches* that individuals use. In this framework, semantic proof production is defined by individuals' use of instantiations of the concepts about which they are proving. Conversely, syntactic proof production relies solely on the manipulation of symbols, carried out generally, with no reference to specific examples of the concepts involved in the proof. As discussed earlier, the authors found that the graduate students and algebraists in their study tended to focus on semantic proof approaches when showing that two groups were isomorphic, gaining intuition about the groups in question, whereas the more novice undergraduate students focused on applying the definition of isomorphism, which necessitated the construction of a group homomorphism – an approach that was generally unsuccessful toward proving or disproving isomorphism.

Weber (2005) extended the semantic/syntactic proof production framework to include the procedural proof approach, in which a student follows a standard algorithm or system for solving that specific type of proof. Weber found that,

Syntactic proof productions afford the opportunity to practice applying rules of inference, including the application of theorems, and possibly the opportunity to develop strategies and heuristics for proof construction, but like procedural proof productions, do not afford the opportunity to consider informal representations of related mathematical concepts. (p. 385)

Here, we see that Weber is focusing more on the aspects of proof that the student learns about, rather than the conceptual understanding that develops by engaging in proof, although he does describe how varied conceptual representations might contribute to better proofs. I explore this discussion in greater detail in the next section.

Selden and Selden (2009) outline a framework for assessing students' proof construction. The framework consists of a *hierarchical structure*, a *construction path*, and *formal-rhetorical* and *problem-centered* parts of mathematical proofs. These different parts of the framework refer

to the process of an ideal prover, who never errs or strays from a correct proof. The *hierarchical structure* denotes the broad form of a proof, outlined as levels of hierarchy – each composed of a specific thought process, the order of which forms a nested chain of reasoning that takes form in the proof. The authors present an example that highlights three levels of hierarchy. The first consists of a rewording of the theorem as hypotheses and conclusion. The second level provides a general outline of how the conclusion is shown. Finally, the third level coordinates the hypotheses with the necessary parts of the second level. This is more clearly shown in Figure 2.2, in which each rectangle represents a different level of the hierarchy. The authors describe a *construction path* as an outline of the step-by-step formation of each line of the proof in the order that the *hierarchical structure* necessitated it. The bracketed numbers in Image X indicate each step of the construction path. The *formal-rhetorical* part of the proof is described generally as the steps in the proof that the student carries out relying on knowledge of general proving techniques and structures. The *problem-centered* parts of the proof are the parts that pertain specifically to the concepts involved in this proof, usually relying on “conceptual knowledge, mathematical intuition, and the ability to bring to mind the ‘right’ resources at the ‘right time’” (p. 7).

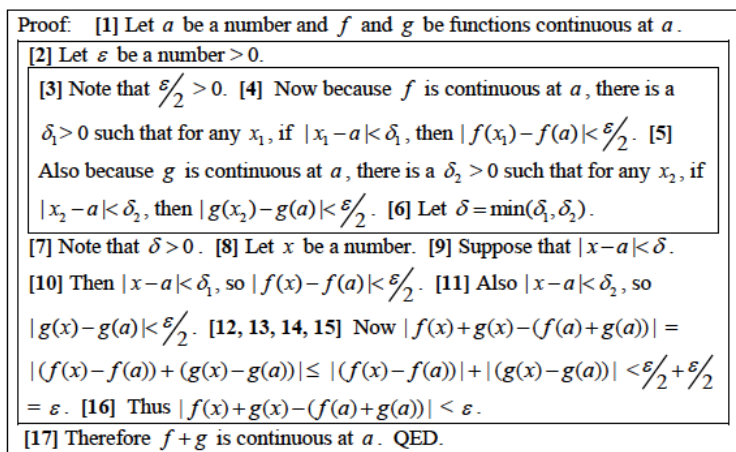


Figure 3. Adding the third level of the hierarchical structure and finishing the proof

Figure 2.2. Hierarchical structure of a proof (Selden & Selden, 2009, p. 5)

When elaborating on the *formal-rhetorical* part of proof with respect to their example,

Selden and Selden describe how “the solution does not arise in any obvious way from intuition about functions,” claiming, “instead, it involves use of  $\epsilon/2$  in the definition of continuity for each  $f$  and  $g$ , a choice of  $\delta$  as the smaller of the resulting two  $\delta$ 's, application of the triangle inequality, and some algebraic rewriting” (p. 7). It could be the case, though, that intuition does prove useful toward finding value in the expression  $|f(x)+g(x) - (f(a)+g(a))|$ . It could be intuition generated from the type of proof strategy that Weber (2005) claims could be learned for specific types of proof. Selden and Selden (2009) go on to discuss the coordination of aspects of proof with students' abilities. In this discussion, the authors claim that it is the coordination of “what a student might be able to do, or not do, and what a proof might call for” that may facilitate students learning proof through construction rather than lecture (p. 7-8).

The authors go on to describe a key point about the way proof is generally viewed - that if a student can prove a certain way in one setting, then they should be able to prove that way in another, similar setting. In so many words, Selden and Selden point out that a student needs a certain level of understanding of a concept to prove about it in a certain way. The way they describe it, though, is by "ability." I can see that as an activity oriented view of understanding - if a researcher witnesses an individual acting in a specific way on/with a representation of a concept (ability to act in a specific way), then this indicates that the individual likely has a certain understanding of that concept. The authors go on to state “a major determiner of the difficulty of constructing proofs seems to be the nature of [a student's] own knowledge and habits of mind” (p. 8). The authors seem to be speaking more about the knowledge of proof construction than the knowledge of the concepts involved in the proof. This relates to the discussion of behavioral knowledge in Chapter 1.

In this section, I discuss Aberdein's (2006a) adaptation of Toulmin's (1969) model of

argumentation. Several researchers have adopted Toulmin’s model of argumentation to document proof (Fukawa-Connelly, 2013; Pedemonte, 2007; Weber, Maher, Powell, Lee, 2008). This analytical tool organizes arguments based on the general structure of *claim*, *warrant*, and *backing*. In this structure, the claim is the general statement about which the individual argues. Data is a general rule or principle that supports the claim and a warrant justifies the use of the data to support the claim. More complicated arguments may use *backing*, which supports the warrant; *rebuttal*, which accounts for exceptions to the claim; and *qualifier*, which states the resulting force of the argument (Aberdein, 2006a). This structure is typically organized into a diagram similar to directed graph, with each part of the argument constituting a node and directed edges emanating from the node to the part of the argument that it supports (Figure 2.3). Some researchers (e.g., Aberdein, 2006b) also represent the Toulmin model by including parenthetical abbreviations of each part of the argument within a transcription of the argument. Informed readers should be able to draw the same information from of each of these organizations of the model.

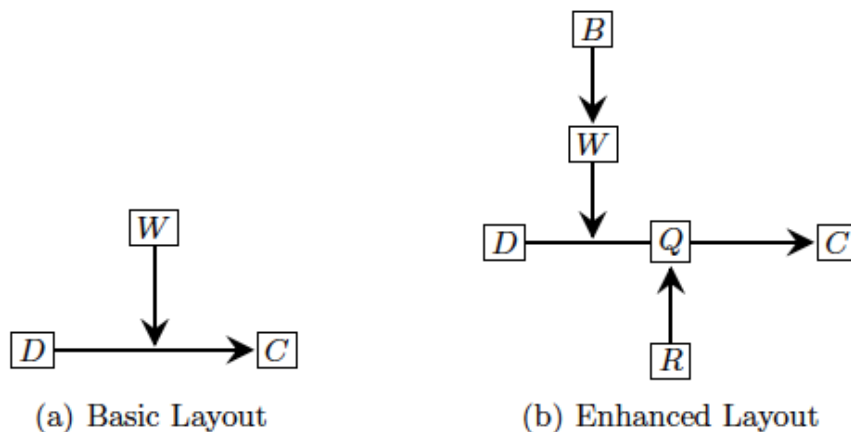


Figure 2.3. Visual representation of Toulmin models (Aberdein, 2006a, p. 211)

Aberdein (2006a) provides a thorough discussion of using Toulmin models to organize proofs, including several examples relating the logical structure of an argument to a Toulmin

model organizing it. Using “layout” to refer to the graphic organization of a Toulmin model, Aberdein includes a set of rules he to coordinate more complicated mathematical arguments in a process he calls *combining layouts*: “(1) treat data and claim as the nodes in a graph or network, (2) allow nodes to contain multiple propositions, (3) any node may function as the data or claim of a new layout, (4) the whole network may be treated as data in a new layout” (p. 213). The first two rules are relatively straightforward – the first focuses on the treatment of the graphical layout, as for the second, one can imagine including multiple data sources in the same data or claim node. The third and fourth rules provide a structure for combining different layouts and rely on organizational principles that Aberdein uses. He provides examples of combined layouts (Figure 2.4).

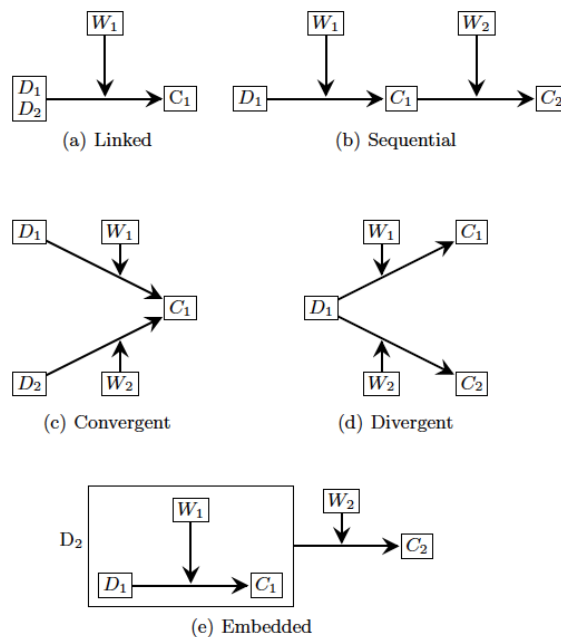


Figure 2.4. Five Ways of Combining Layouts (Aberdein, 2006a, p.214)

Aberdein also discusses Toulmin’s (1979) assertion that mathematical arguments cannot be rebutted, but he instead incorporates lemmas to deal with cases in which the mathematical relationship does not hold. Aberdein, however, argues that mathematical arguments often provide opportunity for rebuttal, stating, “His is an entirely reasonable attitude to take to settled

and formalized mathematical results. However, in the context of informal mathematics, it is equally reasonable to admit the possibility of rebuttal” (p. 220). Instead, Aberdein opts to include rebuttals as lemmas to the data, phrased as conditional statements.

These four frameworks for analyzing proof inform this research by providing varied approaches to viewing students’ engagement in and understanding of proof. Harel and Sowder (1998) provide a foundational framework for describing individuals’ notions of what constitutes proof. Weber and Alcock (2005) concentrate on individuals’ approaches to proof production, focusing on the use of examples versus symbol manipulation. Both Selden and Selden (2009) and Aberdein (2006a) focus on the structure of the argument that is produced. However, Selden and Selden (2009) also work to coordinate students’ types of knowledge within their proof production, focusing on implications for curriculum in a proof classroom, whereas Aberdein’s focus is toward the development and organization of representations that document proof.

**Developing an operational definition of proof.** In this section, I use various researchers’ descriptions and definitions of proof, along with my own views, in order to more concisely describe mathematical activity that I take to be proof. Throughout the literature, a researcher’s explicit statement of what he or she operationally constitutes as a mathematical proof is often missing. Researchers occasionally describe ideal, necessary, or sufficient qualities of a proof, but typically fail to delineate a clear and concise definition of exactly what they take *proof* to mean. Ubiquitous in the literature is the notion that a mathematical proof begins with a statement (referred to as a relationship, observation, conjecture, etc.), proceeds through some chained amount of activity (deduction, argumentation, reasoning, etc.), and terminates in some conclusion about the statement (validation, assertion of truth, rejection, etc.). This is consistent

with the Merriam-Webster dictionary definition of *proof* as “something which shows that something else is true or correct” or “an act or process of showing that something is true” (2013).

Bell (1976) states, “proof is an essentially public activity which follows the reaching of conviction, though it may be conducted internally, against an imaginary potential doubter” (p. 24). He then describes proof’s three senses: (1) *verification* or *justification*, (2) *illumination*, (3) *systematization*. While these three senses describe properties that are generally associated with proof, Bell (1976) leaves open the necessity or sufficiency of any of the three senses and designates these senses as carried by the “mathematical meaning” of proof. de Villiers (1999) extends Bell’s description of proof by providing what he describes as six purposes of proof: verification, explanation, systematization, discovery, communication, and intellectual challenge. Citing de Villiers (1990), Almeida (2000) provides another, somewhat similar description of proof’s four main functions: “(i) verification of the statement, (ii) explanation of the statement, (iii) communication of (i) and (ii) to others, (iv) systematization of the statement into a deductive system” (p. 869, footnotes removed). Almeida and de Villiers each explicate a communicative aspect of proof. That is, paraphrasing Almeida, the “communication of [a verification] and [an explanation] to others” explicitly places the individual’s mathematical proof activity in relation to others in a community. However, each description of proof neglects to explicitly identify the community in which the author intends this communication to take place.

Stylianides (2007) defines what he calls a “conceptualization of the meaning of proof in school mathematics... that can be applied in the context of a classroom community at a given time” (p. 291):

*Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;



2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 291)

Stylianides presents a discussion of his definition centered around seven points, three of which I discuss here. First, he acknowledges that the definition is only one of many possible definitions that one could develop to study proof. This is consistent with Balacheff's (2010) contention that "[m]any theorists have attempted to answer the question of what counts as a proof, from either an epistemological or an educational point of view. However, there is no single, final answer." Stylianides also acknowledges that the classroom community is composed of individual students, each of whom may have a different understanding of the mathematical relationships and arguments developed in the classroom. Accordingly, he points out that he does not imply that every individual in the classroom takes the same view of the proofs under consideration, but focuses on the "statements that can comfortably be assumed and used publicly without further justification" (p. 293). The last of Stylianides' points that I bring up is his statement that "although... the definition is applicable across the whole spectrum of students' mathematical education, the discussion of how the definition could apply at the university level is beyond the scope of this article." (p. 293). Here, I posit that Stylianides may be alluding to the formalization emphasized in undergraduate and graduate mathematics classrooms.

Bell (1976) contends that mathematical proof should provide a sense of "illumination, in that a good proof is expected to convey an insight into why the proposition is true" (p. 24). Similarly, Almeida (2000) and de Villiers (1999) each claim that proofs should explain. These descriptions suggest that these researchers regard proofs as being inherently explanatory. This notion aligns with several other researchers (Hanna 1990; Mancosu, 2001; Steiner, 1978). Weber

(2010) has discussed a perspective regarding the notion of an explanatory proof that situates a proof's explanatory power relative to the proof reader. In his discussion, Weber describes "a proof that explains as a proof that enables the reader of the proof ... to translate the formal argument that he or she is reading to a less formal argument in a separate semantic representation system" (2010, p. 34). This perspective is most clear in his critique of Steiner's discussion of mathematical proof, when he says, "Steiner treats an explanatory proof as a property inherent in the text of the proof rather than an interaction between the proof and its reader" (p. 34). Weber uses this point to draw distinctions between two representational systems that are used in the proof process: formal and informal. Formal representational systems are the signs, notation, and operations that we carry out in abstract thought, whereas informal representational systems largely rely on specific instantiations of concepts used as exemplars. Weber's description of explanatory proofs resonates with Balacheff's (1986) discussion of contradiction. Balacheff (1986) states, "A contradiction does not exist by itself but relative to someone who notices it. It has a witness. So it may happen that it exists for one person and not for another one" (p. 10). Weber (2010) and Balacheff's (1986) perspectives reflect an attention to the context of the proof in question, specifically, to mathematical reality of the prover and the individual reading the proof. This is consistent with Harel and Sowder's (1998) definition of the process of proving, which grounds its perspective in the *individual's* notions of what constitutes proving through the processes of ascertaining and persuading.

Considering this discussion, I see value in Weber's (2010) assertion that a proof conveys explanation only to the degree to which an individual is able to understand the proof as an explanation. Further, with this view, a proof cannot independently convey verification, systematization, discovery, communication, or intellectual challenge. For instance, one can only

view proof as an intellectual challenge according to the degree that an intellect engages with the proof and finds it challenging. Thus, I suggest extending Weber's emphasis on the individual to the other characteristics of proof discussed in the literature. Drawing on this, I feel that it is important to consider a proof relative to the mathematical reality of the individual engaging with (producing, reading, assessing, etc.) the proof. Drawing on both Stylianides' (2007) emphasis of the community in which a proof is communicated as well as Harel and Sowder's (1998) discussion of the process of persuading, I find value in acknowledging the community in which the proof is communicated.

From this, I describe a view of proof informed by this discussion. A proof first requires a statement of a mathematical relationship. An individual engages in proof (becomes a prover) when he or she applies mathematical understanding to support and validate or reject and refute the mathematical relationship. Oral and written artifacts through which the prover communicates support for his or her asserted relationships are viewed as evidence that the individual is engaging or has engaged in proof. Through this communication, the prover is contributing to the establishment of a community of proof constituted by the prover and the audience. The artifacts of proof that the prover communicates may depend on his or her view of what argument would be most convincing within the community of proof in which he or she shares the proof. Accordingly, it is not assumed that the prover would produce the same proof in a different community. From this, in order to identify instances of proof, I will document instances in which a mathematical relationship is stated and participants communicate mathematical arguments in support of or against the relationship. I will focus on the verbal, written, and gestural artifacts of the communication, the structure of the arguments produced, and the intended audience of the proof to provide a clearer documentation of the proof.

**Conclusion.** In my investigation of the proof literature, I have familiarized myself with existing research investigating students' proof in abstract algebra, explored researchers' frameworks for analyzing students' engagement in proof activity, investigated an ongoing discussion in the literature concerning the researchers' notions of the explanatory nature of proof, and generated an operational definition of proof that provides consistent criteria for identifying participants' engagement in proof. This discussion informs my line of research by providing familiarity with other researchers' foci and situating my interests and perspective within the community. The operational definition of engagement in proof allows me to identify instances of proof within the data I collect. The literature provides several frameworks for my approach, organization, and coordination of this data. Together, these inform my methodological approach toward resolving my research question.

### **Chapter 3 - Methods**

The research methods herein reflect my notions of how students develop mathematical concepts. The view adopted necessarily relies on both collective and individual construction of mathematical knowledge. That is, students actively construct their mathematical understanding by engaging in classroom discourse and practices. I view this process as a co-construction of mathematical meaning through interaction with the instructor and peers – necessarily an interactive process that informs an individual’s construction of mathematical meaning. It is also taken that engagement in this process changes depending on the situation in which the student is engaged. In order to explore the connections between a student’s conceptual understanding and his or her engagement in proof and proof-related activity, I first built a rich model of his or her conceptual understanding. Next, I identified the instances in which I infer the student engages in the activity of proving. I then searched for relationships between the student’s conceptual understanding and the proof activity in which they engaged. Throughout my analysis, I focused on each student’s engagement in the interview as the primary sources informing the model of their current understanding of the mathematics. This perspective acknowledges and respects the corresponding development of the classroom, homework group, and researcher-participant interaction, but focuses on the individual interview as the setting for analysis while using the former settings to help situate the results and provide meaningful context to the student’s engagement in mathematical activity.

I begin this chapter by describing the broad research setting and participants. Next, I outline the methods of data collection, including a description of the various settings and the logistical organization I implemented in each setting. I support the rationale for each data source

and collection method relative to my philosophical perspective and research goals. Finally, I describe the approaches I used to analyze the data.

### **Setting and Participants**

Data were collected in a Junior-level introductory Abstract Algebra course, entitled *Modern Algebra*. The course met twice a week, for one hour and fifteen minutes per meeting, over fifteen weeks. The course instructor was an assistant professor in the mathematics department who is a researcher in undergraduate mathematics education. The curriculum used in the course was *Teaching Abstract Algebra for Understanding* (TAAFU) (Larsen, 2013). TAAFU, an inquiry-oriented, RME-based curriculum, relies on Local Instructional Theories (LITs) that anticipate students' development of conceptual understanding of ideas in group theory. The curriculum is intended to allow students to feel as though the mathematics developed is a product of their own informal ways of thinking. Because of this, the students engage frequently in the practice of building and communicating mathematical arguments based on their current ways of understanding. This provided ample opportunities to observe students' declaration of their meaningful mathematical understanding as well as engagement in proof and proof-related activity.

The classroom format consisted of students working in small groups, periodically breaking into whole-class discussion. Members from two of the small groups in the class served as the student participants for this study. At the beginning of the semester, I solicited volunteers willing to participate in the study (by being on camera during class, participating in individual interviews, and participating in homework groups). Fourteen students from the class volunteered to be video recorded during class. Of these fourteen, nine students volunteered for all four aspects of the data collection. The instructor helped me place the class into groups so that two

groups were made up entirely of the nine volunteers who consented to all aspects of the data collection. I refer to these two groups as Group 1 and Group 2 (G1 and G2), which had 5 and 4 students, respectively. The students from G1 and G2 (the “participants”) comprise the selection of students interviewed in the series of individual interviews, and G1 and G2 were also the two homework groups. From these nine students, the current study focuses on three students from G1 (Violet, Tucker, and John) who worked well together during class, establishing a healthy rapport, and participated in the majority of the out-of-class individual interviews and homework groups.

### **Data Collection**

Individual interviews and collection of homework assignments from the participants allowed assessment of individuals’ conceptual understanding by directly eliciting the participant’s personal views about inverse and identity. These points of data collection also engaged each participant in tasks that evoked various aspects of his or her conceptual understanding that might not have been elicited elsewhere. Classroom observations allowed insight into how individual students engage in the TAAFU curriculum. By the nature of the curriculum, the students engaged in the production of in-depth argumentation to support their construction of abstract algebraic concepts. Qualitative descriptions of the development of the classroom mathematics and in the homework group sessions informed protocol development and supported working models of student understanding.

**Individual data collection.** The main data source in which participants engage individually were participant interviews (Appendix A, B, C). I conducted individual interviews with each student from G1 and G2 throughout the semester. The three individual interviews (forty-five to ninety minutes each) took place at the beginning, middle, and end of the semester, respectively. These interviews were semi-structured (Bernard, 1988) and used a common

interview protocol so that each participant was asked the same questions as the others. Unplanned follow-up questions were asked during the interview to probe students' descriptions and assertions. The goal for each interview was to evoke the participants' discussion of inverse and identity and engage them in proof activity that involved inverse and identity. I developed initial protocols for these interviews, which were then discussed and refined with fellow mathematics education researchers to ensure that the prompts in the protocols afforded insight into participants' conceptual understanding as well as ample opportunities for participants to engage in meaningful proof activity.

Each interview began by prompting the student to both generally describe what "inverse" and "identity" meant to them and also to formally define the two mathematical concepts. Additional follow-up questions elicited specific details about what the participant means by his/her given statements, figures, etc. The interview protocol then engaged each participant in specific mathematical activity aimed to elicit engagement in proof or proof related activity. Participants were asked to prove given statements, conjecture about mathematical relationships, and describe how he or she might prove a given statement. As with the questions about defining, each of these tasks had planned and unplanned follow-up questions so that all participants were asked at least the same base questions, but their reasoning was thoroughly explored. These types of interview tasks were intended to elicit the students' individual conceptions of identity and inverse as well as engage each participant in proof and proof-related activity.

I conducted the interviews, and another mathematics education researcher assisted me by operating the camera during some interviews during first round and third rounds of interviews. Throughout the interviews I kept field notes documenting participants' responses to each interview task. I also audio and video recorded each of the interviews, and all participant work



and field notes were retained and scanned into a PDF format. The video camera was placed so that the participant and I are each visible in the video, with the primary focus on the participant. I used the camera's internal microphone as well as an external audio recorder (for back-up) to document each interview. This allowed nearly all audible verbal utterances and visible gestures made by the participant and interviewer during the interview to be documented. The scanned copies of the documents provided documentation of the participants' written work.

The second source of individual student data was the participants' homework assignments from the first ten weeks of class because these weeks correspond to the classroom level data collection. Participants' responses on these assignments also informed the development of interview protocols. After the instructor collected the class homework assignments, I scanned the participants' work, saved these documents in PDF format, and returned them to the instructor for grading. These files, along with the audio, video, and participant work from the interviews, are stored on a secure Virginia Tech Mathematics Department hard drive located in a locked room on campus. These files were copied to a secure external hard drive in order for analysis to occur off campus. The individual participant data sources provide documentation of the communicative aspects (utterances, written representations, and gestures) of the participants' evoked conceptions of inverse and identity as well as the participants' engagement in proof and proof-related activity. The individual interviews provided explicit one-on-one interaction with each participant and allowed me to ask clarifying questions that I had in the moment about the participants' mathematical understanding.

**Classroom data collection.** The participants' engagement in whole class and small group discussion elicited their communication of their developing conceptions of inverse and identity. By the nature of the inquiry-oriented classroom, the participants also engaged frequently in proof

and proof-related activity. This included students' development and presentation of conjectures of mathematical relationships as well as arguments that support such relationships and critiques of their peers' arguments. All the while, the instructor mediated and participated in the discussion and development of mathematical arguments. This mutual co-construction of a mathematical community both informs and is informed by the students' individual understanding and mathematical beliefs and values. Consequently, the data collected in the class sessions is an artifact of the specific classroom community in which the students engaged.

Data collected at the classroom level consists of audio/video recordings, photographic documentation of student and instructor work, and detailed field notes. For the first ten weeks of class, three video cameras were used to collect data from specific perspectives. During whole-class discussion, the first camera was trained on the entire class, focused on the "front" of the room – designated by the whiteboard on which the instructor recorded the most substantial amount of mathematical work. During small group work, this camera followed the instructor as she interacted with the students. A second and third camera collected video recordings of G1 and G2, placed in proximity to the group and trained on these students' table throughout the entirety of each class session.

I assumed a passive role in the class, writing field notes of what I perceived to be significant events and interactions (Mulhall, 2003). I also photographically captured the small groups' common work using a still digital camera before the students erased each board. I met with the instructor and reviewed the field notes after each session in order to help build a working model of some of the students' conceptual understanding. The working model helped inform the ongoing development of homework questions and individual interview questions as well as focus my attention toward specific use of language and engagement in mathematical

activity. I purposefully chose the first seven weeks of class because these are the weeks during which the notions of inverse and identity are developed the most within the curriculum. During weeks eleven through fifteen, I recorded with the first and second cameras. The main purpose of this data collection was to collect additional documentation of the participants' understanding of inverse and identity in order to inform the development of the third individual interview protocol. All recordings, photographs, and field notes are saved to the secure hard drive mentioned in the previous section.

**Homework group data collection.** As with the classroom data, the homework group data was intended to afford additional insight into each participant's understanding within the context of a larger community. I met with G1 and G2 outside of the class meeting during weeks three, five, and eleven. These sessions lasted one hour each. Similar to the classroom data, the data from the homework group provides instantiations of specific participants expressing their individual understanding of the mathematical concepts under consideration. Importantly, the instructor was not present during the homework group data collection. In this setting, I captured evidence of the participants engaging in mathematical discourse with each other without the presence of an expert voice. I asked the students to work on their regularly assigned homework problems together as though I were not there. This elicited instances of the participants engaging in mathematical activity within an alternative setting. The homework groups afforded opportunities to observe the participants sharing their ideas with each other in a low-risk situation with relatively few time constraints, allowing their conversations to follow a more organic path. This sensitized me toward some of the difficulties and successes the participants encountered during their coursework, further informing a working model of the participants' understanding and the development of interview protocol.

Each meeting was video recorded in a small classroom on campus with a whiteboard (or chalkboard) so that the students were able to share drawings, sketches, algebraic approaches, etc. in full view of the camera. I took field notes of each meeting and photographed the group's work before it was erased. I also collected and scanned written work produced during the session and returned it to the participants. These scans were saved as PDF files and all data from the homework sessions are saved to the secure hard drive mentioned above.

### **Analysis Methods**

In this section, I discuss my methods for analyzing the data. These methods reflect my research interests as well as my focus within the interpretive framework for the Emergent Perspective. I focused on students' conceptual understanding of inverse and identity as well as their engagement in proof activity. The bulk of the data analysis was retrospective, taking place after the semester. The retrospective analysis of individual participants' mathematical engagement throughout the semester consists of three stages, which I ordered so that each stage built upon the previous stages toward a resolution of the research question. Analysis of individual understanding consisted of an iterative coding process that I used to generate thorough models of the participants' conceptual understanding and engagement in proof and proof-related activity. I carried out this analysis separately for each participant, coordinating each data source chronologically so that the model of each participant's conceptual understanding corresponds with his or her conceptual development over the semester. I then investigated relationships between the participant's conceptual understanding and proof activity, exploring instances in which meaningful interactions between understanding and activity occurred.

**Models of individual students' understanding.** Consistent with the Emergent Perspective, in this research I operationalize participants' conceptual understanding using Saxe et

al.'s (Saxe, Dawson, Fall, & Howard, 1996; Saxe & Esmonde 2005, Saxe et al, 2009) constructs of *form* and *function*. Throughout the literature, *forms* are defined as cultural representations, gestures, and symbols that are adopted by an individual in order to serve a specific *function* in goal-directed activity (Saxe & Esmonde, 2005). Three facets constitute a *form*: a representational vehicle, a representational object, and a correspondence between the representational vehicle and representational object (Saxe & Esmonde, 2005). Saxe focuses on the use of *forms* to serve specific *functions* in goal-directed activity as well as shifts in *form/function* relations and their dynamic connections to goal formation (1992, p. 227). Through this framework, learning is associated with individuals' adoption of new *forms* to serve *functions* in goal-directed activity as well as the development of new goals in social interaction.

Furthermore, Saxe, Dawson, Fall, and Howard (1996) define microgenesis as “dynamic relations between *form* and *function* that emerge over the course of the construction and accomplishment of emergent goals” (p. 133). The authors go on to explicitly describe how one might think of learning using *form/function* relations, saying, “Mathematical development in the form/function framework can be understood as a process of appropriating forms that have been specialized to serve developmentally prior cognitive functions and respecializing them such that they take on new properties” (p. 126). Accordingly, the *form/function* framework provides an appropriate theoretical framing for investigating the ways that individuals' understanding of identity and inverse relates to their engagement in the goal-directed activity of proving.

Saxe (1999) uses micro-, onto-, and sociogenesis to support a discussion of shifting *form/function* relations in two different settings and to articulate how a *form* might be schematized as a vehicle for mathematical meaning. This schematization involves a representational vehicle, a representational object, and a semantic mapping (correspondence)

between vehicle and object (p. 23). Saxe (1999) states that, “inherent in the microgenetic act is a schematization of a correspondence between the latent qualities of the vehicle and object such that one can come to stand for the other” (p. 24). He goes on to say that, “individuals structure cultural forms ... into means for accomplishing representational and strategic goals. This dynamic process allows for the flexibility of forms to serve different functions in activity, in that the same forms may be structured into means for accomplishing different ends” (Saxe, 1999, p. 26). These quotes draw focus toward the ways in which *forms* are able to shift during goal-oriented activity. This aspect of the *form/function* framework informs the focus of the current study by drawing attention to the ways in which participants structure the *forms* and *functions* upon which they draw during proof activity, specifically with regards to the ways in which specific forms might support varied reasoning within different problem contexts.

Saxe and Esmonde (2005), in their investigation of the development of counting in an isolated village culture, provide further explication of the geneses, demonstrating their analytical utility in the context of the *fu* concept. The authors discuss the use of the notion of representational vehicle in microgenesis by describing a tribal elder who counted stones using a cultural-specific technique of counting body parts:

What occurred in this act of counting that enables the elder to treat her representation as a numerical one? In this regard, we find three dimensions of activity, ... One includes the body parts and *fu*, which become in this case a vehicle of representation; another dimension includes the stones, which become in this case an object of representation; the third dimension consists of a correspondence that becomes established between successive body parts and discrete stones – the mathematical basis for the numerical representation. The production of these three dimensions of activity (a vehicle of representation, an object of representation, and a correspondence between the two) constitutes a microgenesis of number. (p. 211-212)

This affords a sense of how the representational vehicle construct might be used to draw out the aspects of *form* in order to support claims about the *forms* that a participant might use.

Specifically, I focus on the representational vehicles that participants adopt for the representational objects of identity and inverse and the correspondence between the vehicle and objects. The last sentence of this excerpt provides an additional discussion of what constitutes microgenesis. Specifically, the vehicle, object, and correspondence had previously been used to describe the emergence of *form* in individuals' goal directed activity (Saxe 1999), rather than the microgenesis itself, whereas microgenesis was a coordination of *form* and *function* in goal-directed activity.

Saxe's use of the *form/function* framework has come to center primarily on investigations into the development of mathematical understanding of individuals within a community, specifically on shifts in *forms* and the *functions* they serve as individuals engage in goal-directed activity. Saxe views this development as occurring through three strands: microgenesis, sociogenesis, and ontogenesis. As previously mentioned, microgenesis relates to the individual's adoption of cultural or material *forms* to serve specific *functions* in their goal directed activity. Sociogenesis relates to the spread of *forms* within a community, and ontogenesis relates to the changes of *form/function* relations for an individual over an extended period of time. These three strands respectively frame discussion of *form/function* shifts in order to draw out the changes that occur (a) as individuals adopt new *forms* to serve specific *functions* and use existing *forms* to serve new *functions*, (b) as these *forms* are taken up within a community, and (c) as *forms*, *functions*, and *form-function* relations change for individuals over an extended period of time. In the current research, I focus my investigation on the microgenesis of students' understanding of identity and inverse; specifically, I target the *forms* upon which the participants draw in order to serve specific *functions* in the moment as well as *form* shifts that occur during proof activity.

The form/function analysis for participants' understanding consisted of iterative analysis similar to Grounded Theory methodology (Charmaz, 2000, 2006; Glaser & Strauss, 1967). This analysis is differentiated from Grounded Theory most basically by the fact that the purpose of this specific analysis was not to develop a causal mechanism for changes in the students' conceptual understanding, but rather that it was used to develop a detailed model of students' conceptual understanding at given moments in time. Analysis of each data source began with a transcription of each interview (Appendix D). Once the transcript was generated, I carried out an iteration of open coding targeted towards incidents in which the concepts of inverse and identity were mentioned or used. In this iteration, I focused on the representational vehicles used for the representational objects of identity and inverse and pulled excerpts that afforded insight into the correspondence that the participant was drawing between the representational vehicle and object in the moment. Along with the open codes I developed rich descriptions of the participants' responses that served as running analytical memos. After the open coding, I carried out a second iteration of axial coding using the constant comparative method, in which open codes were compared with each other and generalized into broader descriptive categories. These categories emerged from the constant comparison of the open codes and were used to organize subsequent focused codes until saturation was reached. Throughout this process, I wrote analytical memos documenting the decisions that I made in forming the focused codes and, in turn, providing an audit trail for the decisions made in the development of the emerging categories. This supports the methodology's reliability (Charmaz, 2006).

The retrospective analysis of student understanding focused on the development of *form* categories and *function* categories. During the analysis of the *forms* upon which a participant drew, I searched for any sign or name that the participant used to refer to inverse or identity,



including pronouns, and noting the antecedent to which each pronoun referred. During analysis of the *functions* that *forms* served for the participant, I focused on the descriptions of action related to the emerging *forms* of identity and inverse. This included the verbs the participant used, the subject and object in the clauses using those verbs, and any contextualization or conditional phrase that the participant used when describing the action of those verbs. For each new code, I incorporated it into the previously generated codes, and compared it with relevant recent data to support or challenge the developing codes and, in turn, more capably characterize the *forms* that a participant was drawing on and the *functions* that those *forms* served at that point in time. This process began with an open and focused coding of Violet's responses to the Interview questions. Following this, I generated open *form/function* codes for Tucker's responses and, during the focused coding, I compared with the open codes from Tucker's data with Violet's focused codes, including them together when necessary. Finally, I carried out this process with open codes from John's Interview data. These results are found in Chapter 4.

**Documenting engagement in proof.** After I developed *form* and *function* codes for modeling the participants' conceptual understanding, I analyzed the transcripts in order to identify instances in which the participants engaged in proof and proof-related activity. I define mathematical proof to be the intellectual engagement in the activity of supporting or refuting mathematical relationships for oneself, the validity of which is unknown (or taken by the prover to be unknown). This definition aligns with Harel and Sowder's (1998) process of ascertaining. In order to document participants' engagement in proof, I focused on their arguments supporting or refuting mathematical relationships within the contexts of the three individual interviews. This aligns with Harel and Sowder's process of persuading. According to Harel and Sowder (1998, p. 241), together, these are the two subprocesses of proving.

In accordance with my operational definition of proof in Chapter 2, in order to identify instances of participants' proof, I focused on instances in which the participant both stated a claim and acted as though the validity of the claim were unknown in order to provide an argument in support of or refuting the claim. I used Toulmin's (1969) model for argumentation to organize and analyze each participant's communication of a proof. In this analysis, the mathematical relationship that the participant had set out to prove was coded as the *claim* within the model. I then documented the participant's *data* and *warrant* for each proof, additionally including *qualifiers*, *backing*, and *rebuttals* as necessary. In coding the various aspects of the Toulmin schemes (which I call nodes), I first separated statements that conveyed a complete thought, initially focusing on complete sentences and clauses. I then reflected on the intention of each statement, focusing on prepositions and conjunctions that might serve to distinguish the intentions of utterances that comprise the sentence or clause. I then compared these utterances to the constructs of data, claim, warrant, backing, qualifier, and rebuttal, focusing on which node in an argument an utterance might comprise.

Throughout this process I constantly and iteratively compared each utterance relative to the overarching argument in order to parse out how the utterance served the argument in relation to other statements within the proof. For each proof, I then generated a working graphic organizer (i.e., a figure with the various nodes and how they are connected), including corresponding transcription highlighting the structure of the participant's argument. I then iteratively refined the graphical scheme to more closely reflect the structure of the argument as the participant communicated it. After this process, I completed a final iteration in which I compared the scheme to the participant's communication of the proof in its entirety to ensure that the model most accurately reflects the participant's engagement in communicating the proof.

An expert in the field then compared and checked the developed Toulmin schemes against transcript of the interview in order to challenge my reasoning for the construction of the scheme. This supports the reliability of the constructions of the Toulmin schemes. The results of this analysis are found in Chapter 5.

**Relating conceptual understanding and proof.** After the development of the *form* and *function* codes and Toulmin schemes modeling the participants' arguments, I interpreted the Toulmin schemes in terms of the *form/function* codes, which is reflected in the discussion of each Toulmin scheme in Chapter 5. In this process, I focused on the participants' use of *forms* and *functions* within nodes of the Toulmin scheme, comparing the roles that specific *forms* and *functions* served in various nodes within the Toulmin model. This analysis also focused on the shifts in which the participants' generated new, related arguments, specifically attending to the shifts in *forms* and *functions* that occurred along with the shifts in argumentation. In these cases, I compared across arguments, looking for similarities and differences between the *forms* upon which the participant drew and the *functions* that the *forms* serve within the respective arguments.

As with the development of the conceptual models, the development of relationships between conceptual understanding and proof centered on an iterative comparison of the patterns emerging across the analyses of the three participants' argumentation. In this comparison, which is found in Chapter 6, I noted differences and similarities in the overall structures of Toulmin models for arguments. Further, I attended to the aspects of *form/function* relations that served consistent roles across similar types of extended Toulmin models. I continuously built and refined hypothesized emerging relationships through constant comparative analysis and analytical memos. Through this process, I characterized constructs that unify the patterns found

between the roles *forms* and *functions* of identity and inverse served across Toulmin schemes for the three participants.

## **Conclusion**

The methods described above provide an informed, straightforward, practical approach toward answering my research question. In order to investigate how student understanding of inverse and identity relates to student engagement in proof, I first used Grounded Theory methodology (Glaser & Strauss, 1967) to build a grounded model of student understanding based on the *forms* of identity and inverse as well as the *functions* these *forms* served for the participants in the moment, then identified and modeled student engagement in proof using Toulmin (1969) models, and finally investigated relationships between the models, also using an iterative, grounded approach. I feel that the methods I have chosen were appropriate, valid, and reflect my epistemological perspective. I also feel that the collected data provide a sufficient sample of students' engagement in proof and discussion of inverse and identity for the purposes of this project.

## **Chapter 4 – Form/Function Analysis of Participant’s Conceptual Understanding of Identity and Inverse**

In this chapter, I use Saxe’s (1996, 2005, 2009) *form/function* analytical framework to analyze data from the three participants’ individual interviews in order to develop a categorical description of the participants’ conceptual understanding of identity and inverse. As discussed, analysis for each participant’s conceptual understanding consisted of an iterative coding process through which open codes served to generate emerging focused codes for the *forms* and *functions* upon which the participants drew during their discussions in the interviews. As expected these *form/function* categories are often different for each participant, although some reflect a consistency across the three participants. I begin with my analysis of Violet’s conceptual understanding of identity and inverse from her responses to the protocols from Interview 1 and Interview 2. I then discuss the analysis of *forms* and the *functions* they served during Tucker’s responses in Interviews 1, 2, and 3. Finally, I discuss analysis of the *form/function* relations reflected by John’s responses during Interviews 1, 2, and 3.

### **Form/Function Analysis of Violet’s Individual Interviews**

Violet participated in the first and second interviews of the semester, though was unable to uphold her appointment for the third interview. Her interviews were chosen for analysis because she communicated clearly during the interviews, was outspoken during class discussions, and worked closely with John and Tucker throughout the semester. From the interviews, 20 *forms* and *functions* of identity and inverse were coded (Table 4.1), with nine of these developed from analysis of Violet’s responses in the first interview, and some of these being supported into the second interview. Throughout the interviews, Violet typically discussed inverses and identities without directly addressing how they corresponded with the operation

under which they might be considered identities or inverses. This became an integral aspect of the *forms* that were developed during analysis, specifically with respect to how Violet attributed action to identity, but also with her discussion of inverses.

*Table 4.1* – Codes from the form/function analysis of Violet’s individual interviews

Identity		Inverse	
Form	Function	Form	Function
Times one <sup>1</sup>	Resemble itself <sup>1</sup>	Letter <sup>1</sup>	Opposite <sup>1</sup>
Symmetry <sup>1</sup>	Transitive <sup>2</sup>	Number <sup>1</sup>	Inverse-inverse <sup>1</sup>
Plus zero <sup>1</sup>	Matching <sup>2</sup>	Triangle <sup>1</sup>	Inverse-generating function <sup>1</sup>
Word <sup>2</sup>	Finding inverse <sup>2</sup>	Generic element <sup>2</sup>	Bring back <sup>2</sup>
Exponential <sup>2</sup>		Exponential <sup>2</sup>	End-operating <sup>2</sup>
Letter <sup>2</sup>			Vanishing <sup>2</sup>

Superscripts indicate the Interview during which evidence of each *form/function* was first coded

**Interview 1.** Analysis of Interview 1 with Violet focused on her responses to most prompts from the protocol (Appendix A), although some are omitted because a discussion of Violet’s responses does not contribute significantly to the development of *form/function* categories. Accordingly, Violet’s responses to questions 1, 2, 3 (a, b, e, g, h), 5, 6, and 7 are the focus of this analysis and are reported here. From the analysis of this interview, three *forms* of identity were coded (“times one,” “symmetry,” and “plus zero”) and these *forms* serve the one coded “resemble itself” *function* of identity; a second “identity as property” *function* of identity is supported as an emerging code, but is not well-developed enough to substantiate a code for a *function* of identity. Analysis of this interview also supports three *forms* of inverse (“letter,” “number,” and “triangle”) and three *functions* of inverse (“opposite,” “inverse-inverse,” and “inverse-generating function”).

**Describing identity – Q1.** Violet’s initial description of identity provides a foundation for how *forms* might serve *functions* for her. Violet begins her response by saying, “I think of identity as anything that makes what you’re looking at resemble itself again” (lines 4-5). From

this statement, it seems that there must exist something (“what you’re looking at” (line 4); later, “what you’re dealing with, the problem you’re looking at, or the object at hand. The shape. Anything” (lines 9-10)) that is made to “resemble itself again” (line 5). This supports the development of a “resemble itself” *function* of identity for Violet. Her initial examples of what the identity is provide instances of representational vehicles that correspond to the representational object of identity within their respective contexts. When asked what she meant by “anything,” Violet responds:

Um, the anything could be like, in multiplication, you multiply by one. That’s the identity. Or like, in a shape, we’re talking about in class right now, like, if you do a rigid motion to it that returns it to itself, it would be an identity of that. (lines 20-22)

In her first example, Violet situates her example by saying “in multiplication.” This provides a context in which the representational vehicle “one” might *function* as an identity. Violet says, “[Y]ou multiply by one. That’s the identity” (line 20). Although Violet does not explain how multiplying by one is an identity (i.e., does not explicitly state what is being multiplied by one) in this instance, it is supported both by her later discussion and by her activity in class that Violet is likely thinking about multiplying another real number by the real number 1. In such a case, the product would equal the other real number, which serves the “resemble itself” *function*. Important in this language is the verb “makes.” Specifically, the number one does not “make” anything. Rather, the activity of multiplying by one would “make” the other real number “resemble itself again.” This reflects the formal, group theoretical definition of identity in the sense that the identity is an element of a set under a group operation. However, Violet’s description subtly differs from the formal definition in that Violet provides a context (“in multiplication”) and describes the identity as when “you multiply by one.” As becomes apparent

in the following discussion of the “resemble itself” *function* of identity, this distinction becomes important for characterizing Violet’s conceptual understanding of identity.

Violet’s second example of identity is slightly different from her first example. She provides a different context (“in a shape,” line 21); accordingly, the representational vehicle for identity is “you do a rigid motion that returns it to itself” (lines 21-22). It seems that the “it” in this quote is a triangle. Similar to the first example, though, Violet uses an action verb to describe the identity. The rigid motion corresponds to the representational object of identity in this case by serving the “resemble itself” *function* with the triangle. It should be noted that it seems that the identity in the first example is (at least partially) the same type of “something” as the “what you’re looking at” (another real number). In contrast, the identity in the second example is “a rigid motion that returns [the triangle] to itself” and the “what you’re looking at” is the triangle. From this, the context affords a slight distinction between the *functions* that the two different *forms* serve. A discussion of Violet’s definition of identity provides insight into this distinction.

In response to Q1aiii, Violet defines identity by saying, “a method that returns the original object back to itself. Where object could be a function or number or shape” (lines 31-32), going on to add that the definition is not limited to these things. The word “method” is important here. It is not limited to operations or set elements, but instead seems to allow the specific instantiations of identity to shift between element, operation, or some combination of them as Violet needs them to. We can see this in Violet’s examples because the “method” changes across her examples. With the triangle, the object is the triangle, not symmetries, and identity is the rotation that returns the triangle to its original orientation. This is interesting because Violet is not talking about the identity symmetry relative to other symmetries – as though it is formally an element in a set of symmetries – but, instead, talks about the identity as a



change that returns the triangle itself to its original orientation. This is important as we consider her other examples of identity. On the other hand, in Violet's multiplicative example, multiplication alone does not constitute a representational vehicle for identity, nor does the number one itself. Rather, it seems the combination of the two affords a unique "method" that serves the "resemble itself" *function* of identity, whereas, the identity symmetry is the unique action on a triangle that serves the "resemble itself" *function*.

Violet follows her definition of identity by elaborating on these two previously generated examples, saying, "if you had, like, three times one equals three, that'd be the identity. Or, if you had a triangle and you did three ninety degree rotations. Or not ninety, what am I saying? One hundred and twenty...that would technically be the identity of that triangle" (lines 48-52). Interestingly, Violet's phrasing leaves open exactly which aspect of the example constitutes the identity in this situation. Her use of the word "that" in each description of the identity in this excerpt could be taken as the real number one, the action of multiplying a number by one, or the entire equation itself. Drawing from Violet's more general statements about identity, there is evidence that Violet is thinking of the identity in the second sense in this case. That is, Violet's definition of an identity as a "method that returns the original object back to itself" (lines 31-32) would support the sense that the "times one" – or multiplication by one – is the representational vehicle that corresponds to identity in this example. Because of this, the "resemble itself" *function* of identity is coded when a *form* of identity fills in the space between a given object and an equivalence relation with the same given object being the other constituent of the equivalence relation (something [Violet acting with a *form* of identity] = something). This analysis supports the development of a "times one" *form* of identity in which the identity is not the real number one contextualized by multiplication, but instead the identity is the action of multiplying by one.

Violet’s second example is distinguished from the first in that the identity reflects an action on a physical object – one that Violet had carried out on a physical example of a triangle several times. Because of this, the identity is the rotation itself, acting on the triangle. Formally, this makes sense, because the intended group from the TAAFU curriculum is the group of symmetries, though Violet’s description does not include composition of this symmetry with other symmetries of the triangle. Conversely, in the first example, the identity element (formally) is the real number one, which is similar in character to the three – the two being the same type of object. In Violet’s written work related to the second example, she draws a labeled triangle and three curved arrows, each labeled  $120^\circ$ . After writing an equal sign, Violet writes arrangements of the letters a, b, and c – used to denote the orientation of the triangle after each rotation – under each labeled arrow (see Figure 4.1). This inscription supports the proposed *function* of identity. The *form* of the identity in this case is constituted by a representational vehicle of three  $120^\circ$  rotations (the actual physical actions of rotating a physical triangle), each inscribed with a labeled curved arrow and a triangular arrangement of letters; this vehicle corresponds to the representational object of identity by returning the triangle to its original orientation. Consequently, the “symmetry” *form* of three  $120^\circ$  rotations serves the “resemble itself” *function* of returning the triangle (Figure 4.1).

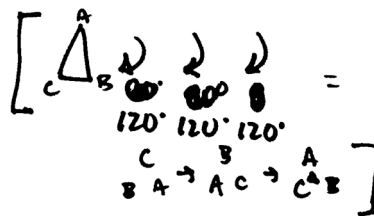


Figure 4.1. Violet’s diagram explaining the  $360^\circ$  rotation

Pressed by the interviewer, Violet follows these examples with a third example of identity:  $x + (-x) = 0$ . This example is characteristically different from the first two. First, this

example draws on the operation of addition and uses the letter  $x$  (both as itself and negated) and number 0. Further, Violet's initial presentation of this example does not have the "original object" on the right-hand side of the equation as the first two examples did – this is not follow the pattern of the "resemble itself" *function* of identity. This does not initially seem problematic for Violet, though she changes the equation  $x + -x = 0$  to read  $x = x$ . The latter equation is more consistent with the "resemble itself" *function* of identity, though it is unclear how Violet is acting with a *form* of identity here. Implicitly, Violet came to the second equation by adding  $x$  to each side of the equation, not dissimilarly from typical algebraic techniques taught in middle and high school. Indeed, perhaps Violet carried out this action with the goal of changing the equation so that  $x$  was on the right-hand side so that the example more consistently aligned with the "resemble itself" *function* of identity. Violet follows this with an explanation of how the first two examples align with her definition of identity. This discussion is consistent with her earlier descriptions of the examples and does not extend the *form/function* analysis.

The interviewer then asks Violet to clarify her previous discussion, focusing on the third example. The interviewer first paraphrases the multiplicative example, with which Violet agrees. The interviewer discusses only the original object and the result – the three and getting back three. Violet follows this with a similar discussion of her second example, focusing on the original object of a triangle and getting back the same triangle. It is not yet clear how Violet is aligning her third example with the first two. Specifically, she states that "x equals itself," a statement which is differentiated from the previous two examples because it does not convey that Violet is acting on the  $x$  in order to make the  $x$  to "resemble itself again." It is possible that Violet views the algebraic manipulation of the equation as the action that serves the "resemble itself" *function* of identity. That is, that the equation does not itself convey a sense of identity

until she manipulates it so that the right-hand side is  $x$  – that she has gotten back  $x$ . This supports the notion that Violet might be trying to rationalize that this third example does indeed convey support a “resemble itself” *function* of identity, though her discussion does not explicitly include a *form* of identity even though 0 is on the right-hand side of the first equation. However, in Violet’s later discussion (during her response to Q2), she explains that the first equation in this third example is different from the first two. This supports a sense that Violet might be trying to rationalize that this third example does indeed convey a *function* of identity, though her discussion does not explicitly include an established *form* of identity even though 0 (formally the additive identity for real numbers) is on the right-hand side of the first equation.

Violet continues by broadening the *function* of identity, incorporating a notion of “property” to describe how she might use identity. Violet states, “like in proofs, you want to prove the identity of some thing, or, I don’t know it’s like basic rule sets- there’s like eight of them or ten or five- a million” (lines 125-127). From this, it seems that “identity as property” is a broader *function* of identity that might be used to prove things. This constitutes an emerging *function* that identity might serve. However, Violet does not describe *how* identity helps in proving. Rather, she describes it as being among a “basis to prove other stuff with” (lines 135-136) adding, “you could use the identity or distributive to rearrange stuff ... to get back what you wanted or manipulate it some way to get something else” (141-142). While these quotes remain general and decontextualized, they do imply Violet might conceptualize *functions* of identity to other situations.

***Describing inverse – Q2.*** Violet is then asked to describe how she thinks about inverse (Q2a). She begins by referring to her third example from the previous task, conveying momentary confusion about whether it is an example of identity or inverse. Before continuing

with this discussion, she explains that inverse “is like something, but you take something else that’s kind of like it, except it’s the opposite” (lines 161-162). This is the first instance in which Violet conveys a potential *function* of inverse: “the opposite.” She then resolves her discussion of her examples of identity, distinguishing the third example from the first two. She initially rephrases “additive identity” to “additive inverse” then mistakenly states that she had previously called it “additive addition.” She continues by explaining that she is uncertain of the name of the relationship that she had previously demonstrated.

Violet then discusses how the  $x$  relates to the negative  $x$ , saying “if  $x$  was the same thing, you would have positive  $x$  and negative  $x$ .” Here, it seems that negative  $x$  is a representational vehicle for the representational object of inverse within a context of addition. However, it is unclear what *function* the inverse is supposed to serve. Instead, Violet seems to focus on the existence of an “opposite” and how to produce it. This supports the development of an “opposite” *function* of inverse in which the existence and production of an “opposite” are the main focus. In this instance, Violet uses a “letter” *form* of inverse (here, “ $-x$ ”) to serve the “opposite” *function*. Violet then elaborates on how she distinguishes between the third example and the first two by explaining, “that’s not really part of that” (line 184). She goes on to say, “Um, this is (points to third example), like, actually adding the opposite back to it to get zero and so, it’s just its self, but here (points to first two examples), you’re not really adding anything to these. Like, you’re just manipulating the same thing. If that makes sense” (lines 189-191). There is a tacit emerging *function* of inverse in the first part of the discussion, specifically, that adding the opposite gets zero, which would align with the formal definition of inverse. Because zero generally represents the additive identity, Violet seems to be conveying a sense that inverses might be used to get an identity. However, Violet has not yet explicitly identified zero as the

additive identity, so it is unclear whether this aspect of the example is an important aspect of the *function* of inverse for her. She distinguishes that example from the others by reiterating that the identity examples do not change the original objects. Violet continues distinguishing the third example, re-naming it the “inverse identity.” This statement supports a sense that Violet might be thinking of the word “identity” as referring to an entire equation in and of itself – much like trigonometric identities – which is supported when Violet calls inverse a “subclass” of identity (line 216).

Violet goes on to say, “So, you're taking something, and you're adding its opposite. That way, they would equal each other” (lines 239-240). It seems important to Violet that an object and an inverse serving the “opposite” *function* have a sort of equivalence. This is interesting because, within the same utterance, Violet states that  $x$  and negative  $x$  are opposite and the same. This supports a sense that the representational vehicle corresponding to an inverse might need to look somewhat similar to the original object, but be different enough to convey a quality of being opposite. In this discussion, the representational vehicles of the letter  $x$  and  $-x$  loosely correspond to the representational object of inverse through the action of addition. This is similar to Violet’s discussion of identity, in which the correspondence between representational vehicle and representational object was mediated by the operation, specifically, that acting with the operation in a specific way served the “resemble itself” *function* of identity. However, unlike with identity, Violet does not outline an expected result of the action of operating with an inverse – in this case, adding – but instead focuses on the notion that “they would equal each other” (lines 239-240).

Asked for other ways she thinks about inverse, Violet describes flipping a triangle. She inscribes this by drawing a labeled triangle, similar to before, a horizontal arrow pointing to the

right, and a second triangle with the b and c switched (see Figure 4.2b). Violet’s description, “if you had the triangle again and you flipped it, I would consider that to be the inverse” (line 256), supports the development of a “triangle” *form* of inverse. This description is also consistent with the “opposite” *function* of inverse because the focus is on the production of an inverse.

Specifically, there is no mention of a purpose or goal for the inverse, which would support a clearer *function* that an inverse might serve; rather, Violet is describing an action that produces a second orientation of the triangle. She does not extend this with any discussion of how the two figures might relate to each other, other than stating that she considers that to be the inverse. Similarly to before, Violet uses a pronoun (“that”) to refer to the example, leaving open to interpretation exactly which aspect of this she views as representing the inverse (i.e., the “opposite” itself or the action required to produce “the opposite”).

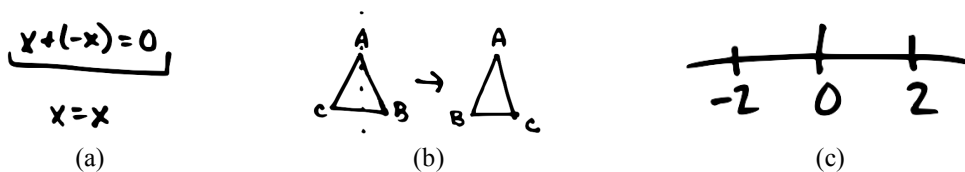


Figure 4.2. Violet’s three examples of inverse

Violet continues discussing her example of inverse, saying, “I would just consider that to be the inverse. That all lumps together in my head [Okay.] for some reason. I don’t know if I’ve ever talked about that and technically called it an inverse, but, I relate them. [Okay.] If I was talking to someone, I would say- I would label that the inverse” (lines 262-265). She continues to use the pronoun “that” to describe the example, leaving open which part of the example she views as a representational vehicle for the inverse. There does not seem to be any sense that Violet is thinking of the inverse of a symmetry, but rather that the second orientation of the triangle is an inverse of the original orientation. Violet goes on to explain that she does not have

any other examples of inverse. Interestingly, she does not provide a multiplicative example of inverse, as she did when providing examples of identity.

Q2aiii prompts the following exchange:

Violet: If you, like, look at a number line (draws Figure 4.2c), I would consider, like, negative two to be the opposite of two. So, I would think that negative two is i- er- inverse of two and that two is the inverse of negative two.

Int: Okay. Um. "And so how does that definition relate to the other ways that you've talked about inverse?"

Violet: Because, here (points to image in Figure 4.2b ) you would be switching the triangle to, like, mirror it to be its opposite. And here (points to image in Figure 4.2a) you would have  $x$  and negative  $x$  and negative  $x$  is  $x$ 's opposite, because they cancel each other out. So, I guess you could kind of put that in the definition if it cancels each other out. (lines 283-293)

The initial definition situates inverse as a result of a process, rather than providing a sense of how a *form* might serve a *function*. This is supported by Violet's pause when stating the definition. Her language initially indicates combining "something" and "its opposite," which is worded such that this combination might result in something, in turn conveying a more succinct *function* of inverse. Instead, Violet pauses and describes how one might "get its opposite." Because of this, the "opposite" *function* of inverse is the only *function* that can be coded from this discussion. Violet follows her definition of inverse with an example in which she describes the number two on a number line and states that negative two is the inverse of two, including a statement that two is also the inverse of negative two. This example echoes the reflective nature of her previous example using the triangle. However, Violet does go on to provide the first indication of a *function* that inverses might serve in combination: "they cancel each other out" (line 292). She discusses this while reflecting on the first example that she generated (" $x + (-x) = 0$ ," which was initially an example of identity). As with identity, Violet describes the concept using a verb.



From this, it stands to reason that she might think about inverse as both an operation and element: something (acting with “an opposite” so that the something is “cancelled out”).

However, Violet admits that she is unsure how her example using the triangle might convey this newly stated property of cancelling out. Further, there is no explicit discussion of what it means to cancel out or what would be the result of this cancellation. However, Violet’s statement that two is also the inverse of negative two is the first indication that Violet thinks of elements as the inverse of their inverse. This is an early instance of an “inverse-inverse” *function* of inverse that characterizes an element as being the inverse of its own inverse element. This *function* emerges more clearly later in this interview and substantially in the second interview.

Violet continues discussing her examples of inverse, elaborating on some of them. This includes a more general statement that positive and negative numbers are inverses, rather than the explicit example of two and negative two. From this, a number and its negative provide a representational vehicle for inverse, corresponding to the representational object by being “opposite.” This constitutes a “number” *form* of inverse in which a negative number is “opposite” an original number. She distinguishes between a general number and its negative and the context of an equation, in which she emphasizes “adding positives and negatives” (lines 312-313). This is the first instance in which Violet explicitly describes an operation through which representational vehicles might correspond to the representational object of inverse. Although she does not elaborate on any type of *function* that inverse might serve in this context, she follows these comments with a discussion of her  $x + (-x) = 0$  example, which might relate to her mentioning that inverses “cancel each other out” (292). She also reiterates that she thinks of the flipped triangle as the opposite of the original triangle, adding that this might not be true, but that this is how she thinks about it. Violet also states that she has learned about inverse as a property

in the past, providing a second potential *function* of inverse similar to that of her description of “the identity property” (lines 133-134), though, as with identity, her description does not support the formation of a new *function* of inverse.

When asked to generate an example of something without an inverse (Q2bii), Violet describes how “zero has no inverse because it is its own inverse” (line 345). She supports this by explaining that zero is neither positive nor negative, but does not connect this with cancellation. It is interesting, though, that Violet, within the same sentence, says that zero has no inverse, but is its own inverse. Violet then attempts to generate an example of a complex number without an inverse, but abandons this attempt. In these examples, Violet is not discussing any operation under which objects might be inverses. Specifically, zero has no multiplicative inverse, but Violet continues to draw on her sense of “opposite,” which she tied to the number line earlier. At face value, it might seem that she is considering zero with respect to addition, but she does not mention addition, only whether zero is positive or negative. Similarly, it is unclear how she is thinking about opposites with complex numbers. It is only later in this line of questioning that Violet mentions cancelling out and is only in the context of positives and negatives cancelling, without mention of addition (under which these numbers might cancel).

***Likert statements – Q3.*** Table 4.2 shows the Likert questions from the Interview 1 protocol, highlighting the prompts used in the analysis of Violet’s conceptual understanding.

*Table 4.2 – Likert statements*

Part	Statement
a*	Zero is an identity.
b*	One is an identity.
c	I think of an identity as anything that I use to keep things unchanged.
d	I think about an identity as a function.
e*	“Inverse” means “negative number.”
f	“Inverse” means “reciprocal.”
g*	Inverses depend on other things, like operations or sets of elements.
h*	I think about an inverse as a function.

\* Indicates Likert statements analyzed in this section

Violet begins her response to Q3a by saying, “right away, I would think that zero is an identity, because you could just say ‘three plus zero equals three,’ and you’re back with what you started, which is what I said was an identity” (lines 401-403). Her response is interesting because she draws on a *form* of identity that she did not use in her earlier discussion of identity with respect to addition. Violet does not explicitly contextualize zero as an identity under addition, but simply begins to add it to three. The phrasing of the question designates the element 0 as an identity, which differs slightly from Violet’s “resemble itself” *function* of identity because the “resemble itself” *function* centers around the action with the element, rather than the element alone. Regardless, Violet seems to be comfortable with this phrasing and quickly assimilates this representational vehicle into a “plus zero” *form* of identity that serves an appropriate “resemble itself” *function* – consistent with her previous examples – though corresponding to addition of real numbers. Violet also compares a  $0^\circ$  rotation with a  $360^\circ$  rotation, stating a preference for  $360^\circ$  because, as she says, she feels that “you would want to do something to it first.”

Similarly, in Violet’s response to Q3b, she agrees that one is an identity, saying, “I would say one is an identity in multiplication, because three times one equals three. So, again, you would be at the same thing” (lines 430-432). In this statement, she explicitly contextualizes the representational vehicle within multiplication before using it in an example. She then extends her response by contrasting one as a multiplicative identity with the notion that one is not an identity under addition. This is important because it seems to be the instance in which Violet is most assertive of a need to contextualize an identity relative to an operation. In her earlier discussion, the operation is not emphasized, taken as given within a context but not explicitly discussed. However, it might be the case in this and her previous response that Violet thinks of zero and one as examples of identity because they align with the structure of the “resemble itself” *function*.

In response to Q3e, Violet explains that, while she does tend to think of negative numbers as inverses, she also regards positive numbers as inverses for negative numbers. This explanation leads to a deeper discussion of a notion that Violet alluded to when generating examples of inverses:

Violet: ...When I think of inverse, I think of a pair. And, the first part of the pair I think of is the negative number, because that's how you cancel out and everything's mostly in positive numbers when you talk about stuff because you can't really have a negative something unless someone took it from you, but- [Uh huh.] any way. I would think of inverse meaning negative number because that's how you would cancel out the positive numbers, but also, the positive number is an inverse of a negative number. [Oh, okay.] So, this statement could be "inverse means positive number." [Oh.] And someone might put that as five, because that's what they first think of when they think of things to cancel each other out in an inverse pair.

Int: Because they might be trying to cancel out negatives. [Mhmm.] Okay.  
Alright, then, "how might you rephrase that so that it more closely aligns with how you think?"

Violet: I would say, "Inverse means a pair that cancels each other out." [Int repeats] But, the whole pair isn't inverse. The- one would be the inverse of the other, if that makes sense. [Oh. Okay.] So an inverse pair in my head I think of you have a number and then the other number is the inverse, but also that number that *was* the number is the inverse of the *other* number if you switch the roles, if that makes sense. (lines 496-513)

This allows her to explain inverses as working in tandem so that each number is the inverse of its own inverse, supporting the development of an “inverse-inverse” *function* of inverse which is characterized by an element serving the *function* of the inverse of its inverse. Violet’s discussion draws on the “number” *form* of inverse, as opposed to the more general language of “something” that she uses earlier in the interview. Throughout this discussion, Violet does not mention any operation under which these numbers might be inverses. Further, Violet uses the word “cancel” four times, which echoes part of her definition of inverse from earlier, although she does not elaborate on what it means to cancel in this context or how she imagines inverses canceling. This supports the broader theme that Violet takes operations as implicit within given contexts,

affording her blended ways of describing identity (as an element and operation simultaneously) and also her tendency to describe inverses as opposites without explicitly discussing the operation under which they might be viewed as such.

Q3g prompts Violet to explicitly discuss inverses in the context of operations and sets of elements. She begins with a reiteration of her inverse pairs discussion, providing an example of a set of real numbers (1, 2, and 3) that did not have inverses within the set. In this example, she is considering inverses to be the negatives of the respective numbers and describes how including negative one in the set would “form an inverse pair” (line 612). Violet mentions the importance of the operation although she does not seem to attend to it when describing inverses, opposites, or cancelling. This is also the second occasion in which she mentions what might be taken as a “cancelling” or “getting back” *function* of inverses when she says, “I also think that it depends on operations, because I said that you need to do a change to something and this would be an operation that you would have to do to get back” (lines 612-614). However, in what seems like a self-correction (lowering her voice and beginning the sentence with “well”), she shifts away from the notion of getting back by alluding to how “to get inverse.” This is the first mention of a sense of “get[ting] back” in relation to inverses. It seems that Violet has some sense that inverses are used to “get back” to something by using an operation, which is language that is colloquially consistent with a notion of cancelling, but it is not clear in this instance. Further, Violet has not yet described any of her previous examples of inverse as “get[ting] back,” instead using that phrase only with identity up to this point.

Violet then explains in response to Q3h that she does not tend to think of inverse as a function, although she is able to. Her explanation of what an inverse function might be can be used to support her later work proving that the two functions are inverses. Specifically, it seems

that she views inverse functions as a means of finding a number's inverse. This is evident by her statement that, "if you have the inverse function, then you could put in a number and spit out what its inverse would be" (lines 628-629). This description of inverse function supports the development of an "inverse-generating function" *function* of inverse in which a function acts as an input-output relation that produces the inverse of a given, input number as its output. It is unclear how Violet is thinking of the number's inverse, although, later, it seems to be the additive inverse.

***Responding to "cheap multiplication" quote – Q5.*** In response to Q5, Violet compares the identity symmetry and the multiplicative identity. This reiterates her previous discussion of identity during her example generating activity. She draws on the "number" and "symmetry" *forms* of identity, though she uses the representational vehicle of  $360^\circ$  for the symmetry identity, rather than the three  $120^\circ$  rotations she previously used. Violet's discussion strongly supports the "resemble itself" *function* of identity. Specifically, Violet says, "with the 360, you don't see anything happening, so you're like, 'Well, did it really happen?' and you can't justify it unless you know that this middle thing happened so I would think that applies to her thing, because that's technically cheap, because unless you know about it, you're really not sure" (lines 828-831). This supports the hypothesized structure of the "resemble itself" *function*. The "symmetry" *form* of identity, which corresponds with her description of the "middle thing" serves the "resemble itself" *function*.

***Multiplicative identity of  $\mathbb{R}\setminus\{0\}$  proof – Q6.*** Violet begins her proof in response to Q6 using her initial example of identity ( $3*1 = 3$ ), drawing on the same reasoning as before to reiterate why this is an example:

Alright. Okay, I would start by saying that we let one to live in the real numbers. And, if you considered, say, three in the real numbers. Um, well, let's just

consider three in this line. And then, now we'll take one that we said was living in the real numbers and we will multiply it by three. And this is obviously three again, which is what I would use as an example for the identity. And you're also getting back a real number, but of course you put in two real numbers, so that's no big surprise. And, that's about as far as I could get, unless I was going to be really detailed and show it a few different times. So, yeah. I would just start by reiterating what we had, which is one in the real numbers. And then I would consider a different real number, which I chose to be three, just as an example. I should probably make it general if I was gonna do a really nice proof, so it could go for all the real numbers. And then I would show that one times any of those real numbers is the real number itself again. (lines 873-883)

Violet describes the example as valid by saying, “you're also getting back a real number, but of course you put in two real numbers, so that's no big surprise” (lines 877-878). This is interesting because Violet explicitly says “you put in two real numbers” (line 878). This conveys a sense of multiplication as a binary operation, which contrasts with Violet’s earlier discussion of identity as a “method.” Particularly, with Violet’s “resemble itself” *function* of identity, the operation and identity element are intrinsically tied so that it treats one element (rather than two) as an input and focuses on that same element being the output, although later in response to Q6d, she describes “performing the identity,” which is more consistent with the “resemble itself” *function* that various *forms* of identity serve for her. She then describes how she might use other real numbers to show that this could work for any real number, adding that her proof should be more general than a few examples. Asked how she might generalize the proof, Violet replaces the three in the original example with the letter  $r$ , which she says could be any real number.

***Proving two given functions are inverses – Q7.*** Violet’s work in response to Q7<sup>3</sup> reflects an extension of the “inverse-generating function” *function* of inverse. She begins by stating that inverse functions should equal each other and writes the two functions on either side of an equal sign:  $f(x) = g(y)$  (Figure 4.3a). She then evaluates the functions at specific values, initially at the

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<sup>3</sup> The prompt poses an ill-formed conjecture to be proven, specifically because it does not indicate that the functions should be shown to be inverses under function composition.

end points of the respective intervals on which the functions are defined (-3 for the first function,  $f(x)$ , and 0 for the second function,  $g(y)$ ). This results in an equation in which the result of evaluating each function does not equal the other ( $f(-3) = 0 \neq g(0) = -3$ ). Formally, this approach is equivalent to the statement  $f(x) - g(y) = 0$ , which would indicate that Violet is thinking similarly to the functions being additive inverses in this context.

(a)

$$f(x) = g(y)$$

$$\sqrt{x+3} = y^2 - 3$$

$$\sqrt{-3+3} = 0^2 - 3$$

$$\sqrt{0} = -3$$

(b)

$$\sqrt{6+3} = 0^2 - 3$$

$$3 = -3$$

$$3 - 3 = 0$$

Figure 4.3. Violet's work related to inverse functions

From her earlier discussion, it could be that Violet is anticipating each function to be the “opposite” of the other when evaluated at specific values. She supports this by expressing surprise that evaluating the functions at the endpoints does not produce opposite numbers. This is supported by Violet's following activity, in which she evaluates  $f(6)$  and  $g(0)$  (which equal 3 and -3, respectively, Figure 4.3b) and notes that the results of these functions are “equal and opposite” (line 997), though Violet does not describe them as inverses under addition. Rather, Violet writes the equation “ $3 - 3 = 0$ .” Through this activity, Violet does not seem to resolve the seemingly arbitrary restrictions to the domains of the functions. Violet's activity throughout her response draws on the “inverse-generating function” *function* of inverse, though slightly differently from in her earlier discussion. Specifically, Violet first evaluates each function to determine the output values and compare them with each other. From her discussion, it seems that if these outputs are opposite, then Violet thinks of the functions as inverses of each other for those values. However, Violet states that she is not very confident in this approach.



**Summary of Interview 1.** In the first interview, Violet’s discussion affords insight into how she thinks about identity and inverse. Violet tends to draw primarily on a “resemble itself” *function* of identity in which the identity is a “method” of acting on an number or shape so that the same number or shape is the result of the “method.” This *function* is served primarily by “times one,” “symmetry,” and “plus zero” *forms* that are contextualized by the number or shape that Violet is considering. This *form/function* relation differs from the formal definition of identity in that Violet tends, especially when dealing with real numbers, to regard the identity as a combination of the identity element and a binary operation without distinguishing between the two. Three *forms* of inverse emerged during the analysis of Interview 1 (“letter,” “number,” and “triangle”) which primarily served the “opposite” *function* of inverse, though the “inverse-inverse” and “inverse-generating function” *functions* of inverse were also developed as well as an emerging “canceling” *function* that cannot be not fully developed through an analysis of Violet’s available discussion. The majority of Violet’s discussion focuses on the production of an inverse from a given object and a comparison of the two. Further, throughout the interview, the operation under which *forms* of inverse might serve *functions* of inverse receives little attention, supporting a sense that Violet is not coordinating inverses in a way that supports the full development of a “canceling” *function* that various *forms* might serve.

**Interview 2.** As with Interview 1, analysis of Violet’s second interview focuses on her responses to specific prompts from the protocol (Appendix B). Violet’s responses to questions 1, 2, 3, 4 (c, d, e, f, g, h), 5, 6, and 7 are the focus of this analysis and are reported here. From the analysis of this interview, three new *forms* of identity were coded (“word,” “exponential,” and “letter”). These *forms* serve the “resemble itself” *function* of identity from the first interview as well as three new *functions* of identity (“transitive,” “matching,” and “finding inverse”).

Analysis of the second interview also supports two new *forms* of inverse (“generic element” and “exponential”) and three new *functions* of inverse (“bring back,” “end-operating,” and “vanishing”).

***Describing identity – Q1a.*** In her general description of identity, Violet says, “I think of the identity of something as something that gets what you're looking at back th- well- Okay. How do I explain this? Okay, when you have an object and you find an identity, when you do that to the object, it gets what you started with back” (lines 6-8). This correspondence requires a context in which “you do” an identity to an object in order to get back that object. Because the goal of this excerpt is to describe how she thinks about identity, *form* and *function* are somewhat vague and general. However, this discussion is quite useful for demonstrating the *function* that the various *forms* of identity might serve. Specifically, identities serve a *function* of “[getting] what you started with back” (line 8). Violet phrases her description of identity using a verb, saying, “when you *do that* to the object” (line 7, emphasis added). This seems to reflect the “resemble itself” *function* of identity from Violet’s first interview. Violet then introduces the notion of an operand. The operand seems to serve as an aspect of the *form/function* relation that enables the identity to act as verb relative to “an object.” This is consistent with the phrase “you do that to the object” (line 8), and is described here as “you perform that on the object, then it gives you back the object” (lines 14-15). Further, this is also consistent with the *forms* and *functions* of identity generated via the analysis of the first interview insofar as the notion of identity is treated as a verb, though Violet was not as focused on describing an “operand” during the first interview. Violet adds a caveat about identity, stating that “you’re doing something, but it’s not actually changing [the object]” (lines 15-16).

When asked for her definition of identity (Q1aiii), Violet responds, “I’m trying to figure out what to call it. I guess, a property that, um, either maps something or, like, takes it back to itself. (3 seconds) With mapping being, like, whatever operand you’re using” (lines 26-28). This definition uses a new noun, “property,” in place of identity, for which she had previously used “something” and “identity.” She also incorporates the new verb “maps” into her definition, adding that this is “with mapping being, like, whatever operand you’re using” (lines 27-28). Consistent with her previous descriptions, the identity “takes it back to itself” (line 27). This supports an analysis that, for Violet in this moment, the *function* of an identity is to perform some action on an object so that the original object is the outcome of the action, which is consistent with the “resemble itself” *function* of identity. Pressed for other ways she thinks about identity, Violet says, “you do something to it and then you get something else back” (line 34). Including the word “else” is a shift from her previous descriptions of identity, which maintained that the result of the identity was getting the original object back, rather than something else.

Responding to Q1b, Violet then provides three specific examples of identity, though they are discussed with varying degrees of explanation, saying, “in multiplication, one where you do, like, two times one is two, so it's the same thing. Or, like, in standard addition, it could be zero. Or in, like, the rotations, where we've been talking about in class, like if you have a triangle, the identity would be three R cause it brings it back to the same mapping” (lines 38-41). The first and third examples explicitly address how the respective *forms* of identity are able to serve the “resemble itself” *function* of the identity, though, even then, the third does so with less fidelity. I say this because, in the first example, Violet begins with two, multiplies it by one, and states that this equals two, adding that this is the same thing (as the original object). With the third example, Violet begins with a triangle and says that three R “brings it back to the same mapping,” rather

than the same triangle. Note that, in each case, Violet explicitly contextualizes the situation in which she sees the “times one” and “symmetry” *forms* as being able to *function* as identities.

Violet’s second example affords less insight into her thinking, because she does not provide an explanation to support her claim that zero is an identity “in standard addition” (line 39). This seems to reflect the “plus one” *function* of identity from Interview 1. It should be noted, though, that with the multiplicative and additive examples, Violet named the numbers “one” and “zero” as identity. This more closely aligns with the formal definitions of multiplicative and additive identities for real numbers than her discussion in the first interview, although her reasoning supporting one as the multiplicative identity is still consistent with the “times one” *form* of identity coded in the first interview. Violet then generates a fourth example, saying, “I guess you could have, like, um, like a function where you do a complete set of things to it and it gets it back, so it would be, like, a function identity or something like that.” However, she does not develop this example as thoroughly as her other examples. It seems to be an afterthought, a sentiment she expresses when she says, “I just thought of it randomly.” From her description, the “function identity” serves as an identity by bringing each element of a set back to itself. This seems to serve the “resemble itself” *function* similarly to her previous examples, except that it is serving this *function* for multiple objects simultaneously.

Across the four examples (one, zero, 3R, and function), Violet’s identities can be viewed as two different types. In the first example, one is an identity that is multiplied by another number so that the product equals the same number again. It is unclear whether Violet thinks of zero in the same way because her example did not explicitly elaborate on *how* zero is an identity under standard addition. Regardless, each of these identities consists of an operation and specific element. The latter two examples, however, have a very different quality to them. Specifically, in

each of these examples, the identity is exclusively an action. For instance, 3R is not combined with any other symmetry. Rather, the triangle is the object that is returned to itself. Similarly, the identity function returns a set of elements each to themselves, as opposed to acting as an identity function under function composition. This distinguishes between two types of examples that Violet is drawing on in this interview: an example in which the identity (an object along with an “operand”) *functions* to return an object of a similar type (i.e., a real number) to itself and examples in which the identity is in and of itself an action on objects of a different type. In a formal treatment of these two examples, the identities are such under the binary operation of composition with other symmetries and functions, respectively.

***Describing inverse – Q2.*** In response to Q2a, Violet provides a general description of what inverse means to her. She describes inverse as “something that brings the element that you’re talking about *back* to the identity” (lines 72-73, her emphasis). In her description, Violet is relating the concept of inverse to the concept of identity, which she had not done in the first interview. Notice that Violet attributes action *to* the inverse by stating that it “brings” another element back to the identity. So, from this description, we can describe a *function* that a given *form* of inverse might serve: bringing another element to the identity. Notice, further, that Violet does not yet explicitly describe how this “bring[ing]” action might play out. However, her elaboration affords a sense of how this happens. Specifically, Violet mentions the “operands...multiplication or addition” (line 80) as well as “inverse functions” (line 81).

Violet defines inverse as, “an element that, when you perform the set operand - I guess whatever you're working under - brings the element back- brings the corresponding element back to the identity” (lines 86-88). Her definition affords a clearer sense of how she conceives of inverse. This is the first instance in this interview in which Violet uses a representational vehicle

other than the word “inverse” to describe how she thinks about inverse (“an element”). She also conveys a means by which an inverse might “bring” an element back to the identity (“perform the operand”). Further, Violet alludes to “the corresponding element.” This supports the following *form/function* analysis: the representational vehicle would be “an element,” which corresponds to the representational object of “inverse” by fulfilling the role of “bring[ing] the corresponding element back to the identity” in a context of when you “perform the set operand.” This analysis contributes to the development of a “generic element” *form* of inverse that serves a “bring back” *function* of inverse.

Violet elaborates on her previous response by providing the context of addition, under which she claims that 3 and -3 are inverses. Violet does not provide any further justification about why these elements might be inverses other than “Because that would be what you were working under.” Specifically, Violet does not explicitly describe how either of these elements might be used to “bring” the other element back to the identity. Further, Violet does not mention the identity in this context. Violet continues her example generating activity by discussing an inverse function, saying, “you could do it with a function. So, like, whatever you put into the first function, wh- if it gave you something, then the inverse function of that function, I guess. I don't- That one's kind of harder to explain” (lines 105-107). This example does not afford much insight into how Violet thinks about inverse functions, though it seems as though Violet is attempting to describe the “inverse-generating function” *function* of inverse. Most evident is that Violet truncates her explanation by beginning a conditional with an antecedent “if” statement without providing a consequent “then” statement and, then, stating “that one’s kind of harder to explain.” When asked to further explain the example, Violet’s response continued to convey an uncertainty about inverse functions.

**Defining group – Q3.** Violet is then asked to define group, which prompts the following

exchange:

Violet: Okay, I would say group is a set of things that hold for, like, four main rules. And those would be associativity and closure and um they have to have an identity and it has to have an inverse. (4 seconds, Int writing)

Int: Okay. So, when you say each of those four things, what do you mean by that? When you said it has to have an inverse, so, what does that mean?

Violet: Um, it has to have something that brings it back to the identity and it has to have th- have an identity element, which brings it back to itself. And, then, it has to have closure, so, like, all the things you do when you perform- using the different elements- when you combine them, like, they have to come back to another element already in the set and not outside the set. [Mm.] And it has to have associativity, so, whatever you do, like, if you do two things in parentheses and then do something else, when you move those, it has to still come out to be the same thing. (lines 126-138)

Violet does not explicitly discuss the binary operation (what she calls an “operand”) when defining group. Nor does she address the operation when explaining each of the group axioms. Rather, she uses the verb “brings” and later uses the verbs such as “perform,” “combine,” and “come back.” This is an important aspect of how representational vehicles correspond with the representational object of identity. The descriptions of inverse as “something that brings it back to the identity” and identity as something that “brings it back to itself” are consistent with the “resemble itself” *function* of identity and “bring back” *function* of inverse, respectively, though they continue to exclude a notion of operation, instead using the more general verb “bring.”

Violet continues by describing associativity and closure for groups, though this discussion does not inform the analysis of her understanding of inverse and identity. When asked what the “it” is in these phrases, Violet says “It would be, an element in the set, like, any of the elements” (line 142), following this comment with an explanation that there would only be one identity element.

While generating examples of groups (Q3d), Violet begins by constructing a table (Figure 4.4) and describing a group of odd and even numbers, saying:

Um, well, you could, like, check associativity. And that would be, like, even plus even plus odd and it's also equal to even plus - even plus odd. Because, you know an even and a- plus an even is is an even and you still have a plus odd. And then you know an even plus an odd is an odd number. So you would just have even plus odd. That's the exact same thing you have over there, so they're both odd. And then you can show that for each of the elements. And then that works. And you know you have an identity, because everything what you have here is even odd even odd, so even brings everything back to itself. So, even would be the identity. And, then you always have something that brings the elements back to even, so that would be the inverse of the- inverse of even would be even because it brings it back to that. And then the inverse of odd- it's whatever brings it back to even, so it would be odd. So, they're self-inverses. And then closure that holds because all these elements are elements that you had in the set and they're not different. So, that's how I would show it's a group. (lines 181-193)

She points out that the table is arranged so that the row and column corresponding to even are identical to the row and column along the edge of the table. Violet uses this to reason that even is the identity element. So, the table provides an organizational structure for the group under the operation so that Violet might identify the identity element of the group. Once she identifies even as the identity, she then explains how she sees “even” as the identity, because it “brings everything back to itself” (line 187). This is consistent with the “resemble itself” *function* of identity and the “word” *form* “even” seems to serve this function. Given Violet’s organization of the group in a table, this also supports a new “matching” *function* of identity in which the row and column corresponding to the identity element are identical to the arrangement of the group elements along the edge of the table.

	even	odd
even	even	odd
odd	odd	even

Figure 4.4. Violet’s group table for even/odd group

Violet goes on to explain how the even/odd group follows her definition of group by “always hav[ing] something that brings the elements back to even” (line 188). This supports a



“finding inverse” *function* of identity in which the location of a *form* of identity in a row (or column) allows Violet to identify the inverse of the element that corresponds to that row (or column). In her explanation of this, Violet supports her reasoning that even is even’s inverse and odd is odd’s inverse. She summarizes this explanation by stating, “they’re self-inverses” (line 191). This is interesting because it is the first time that Violet uses this term. There are good data from the class and homework group videos that show the development of the importance of self-inverses for Violet and her peers throughout the TAAFU curriculum. This example is useful because it explicitly provides two examples of how Violet thinks about inverse. Specifically, the “word” forms of “even” and “odd” serve the “bring back” *function* of inverse. Violet uses the verb “plus” early in her discussion of this group, but relies on more general verbs when supporting her reasoning that odd and even are self-inverses. For instance, she uses the verb “bring” three times: once when generally describing what an inverse would do in this context and once each when describing how even interacts with even and odd with odd. Implicitly, “bring” would describe the result of “add[ing]” the elements of the group together. This is important because, in all of Violet’s examples of inverse thus far, the operation under which elements are being combined with their inverses is either implicit or addressed only to provide a context. Even with multiplication and addition examples, Violet does not explicitly use the respective “operands” to discuss how inverses “bring back” other elements.

Table 4.3 – Likert statements

Part	Statement
a	Zero is an identity.
b	One is an identity.
c	I think of an identity as anything that I use to keep things unchanged.
d	I think about an identity as a function.
e*	“Inverse” means “negative number.”
f*	“Inverse” means “reciprocal.”
g*	Inverses depend on other things, like operations or sets of elements.
h*	I think about an inverse as a function.

\* indicates Likert statements analyzed in this section

*Likert statements – Q4.* Here, I analyze Violet’s responses to Likert statements Q4e through Q4h (Table 4.3), which are the excerpts that afford analysis of how she thinks about inverse. Specifically, in response to Q4e, Violet describes how she thinks of inverses not only as individual elements, but as intrinsically tied to their inverses so that each might serve the same purpose for the other. This reiterates the “inverse-inverse” *function* of inverse developed during analysis of the first interview. However, once again, Violet does not explicitly discuss how the two inverses might be coordinated through an “operand” (in this case, addition) to serve a *function* of inverse for each other. In Violet’s response to Q4f, she explains that she did not previously (in the first interview) think of reciprocals as inverses, but does now, saying “And now I do associate it with inverse. so, that would just be, like, something that brings it back to one er- whatever the identity would be, which in some cases would be one, so, I think of that now, but I didn't always think of that, because I forgot what it meant” (lines 279-282). Again, however, Violet neglects to coordinate an element and its reciprocal as inverses in any meaningful way because she does not mention any operation to contextualize reciprocals as inverses. Unlike her response to Q4e, in this response, Violet alludes to a “number” *form* of identity, “one.” This shows that she is able to think about inverses relative to an identity, though not necessarily with an operation in mind. The differences between these two responses as well as between these responses and her definition convey the context-dependent nature of how Violet thinks about inverse.

In response to Q4g, Violet says, “to have an inverse, you have to know what you want it to go back to, like the identity. So, it would depend on your operation that you're working under” (lines 284-285). This returns to her more general ways of describing inverse, specifically that inverse depends on an operation. However, Violet’s reasoning for why inverses depend on the

operation is mediated by the concept of the identity. Violet then describes the possibility that an inverse might exist, but not be in a set. This shows that Violet is able to consider sets in which an element might not have an inverse. Finally, in her response to Q4h, Violet conveys an uneasiness with the idea of inverse functions, virtually deflecting the statement in the prompt, saying, “I just remember talking about them, but after learning new stuff, I just kind of forget about them” (lines 296-297). This supports the discrepancy between the “inverse-generating function” *function* of inverse and the formal definition of inverse functions under composition.

***Subgroup proof: Integer powers of 2 – Q5.*** In response to Q5, Violet begins by saying,

We want something that's the identity. And we also want something that would be an inverse to bring it back to the identity. (3 seconds) So, (exhales) the identity would just be one, because two raised to one is whatever. And then, we want something that brings it back to one, which would be zero, because two raised to zero is just one. No. Yes. That would work. So, I would say, (writing; mumbling) the identity is equal to one and the inverse would be zero. So it has an identity and an inverse. (lines 321-326)

In this excerpt, Violet draws on a representational vehicle of the number 1 in the exponent that corresponds to the representational object of identity and a representational vehicle of zero in the exponent that corresponds to the representational object of inverse. These representational vehicles constitute “exponential” *forms* of identity and inverse, respectively. Further, these “exponential” *forms* serve the “resemble itself” *function* of identity and “bring back” function of inverse, respectively. The group operation for  $G$  is defined as multiplication although this does not seem to be a factor in Violet’s determining identity and inverse in the subgroup. Instead, Violet seems to focus on the definition of the subset  $H$  and its use of exponential notation to determine the “operand” with which to situate the identity and inverse. Even more, although Violet had described inverse as being specific to each element in a set, she has identified a single inverse for the set  $H$  in much the same way she has identified a single identity element. The

sense that Violet is not focused on the multiplication operation is supported as she continues the proof to show closure and associativity hold by adding elements of the subset  $H$ .

Violet later notices that the inverse she identified (zero) is not an element of  $G$  (even though it is an integer, which makes it a valid exponent of 2 when considering elements in  $H$ ). This realization, however, prompts Violet to begin writing a table to help her organize the set  $H$ . This begins with Violet listing the elements of the set  $H$  along the top and left edges of the table and filling in the products of the elements along the matrix. As before (even/odd example of group), Violet uses the table to organize the elements in the group, which affords her to draw on “matching” *function* of identity. Further, by noticing the location of the identity within a given element’s column or row, Violet is able to identify that element’s inverse, a “finding inverse” *function* of identity. As shown in Figure 4.5, Violet is multiplying elements in the subset in order to complete the table. Notice that she has omitted  $2^0$  from the edges of the table, which is, formally, the identity of the group. Accordingly, she is unable to identify the identity element, because no row matches the edge of the table.

	$2^1$	$2^2$	$2^3$	...
$2^1$	$2^2$	$2^3$	$2^4$	...
$2^2$	$2^3$	$2^4$	$2^5$	...
⋮				

Figure 4.5. Violet’s group table in response to Q5

Violet then attempts to use the table to find the inverses of elements in  $H$ . Because Violet omitted the row and column corresponding to left- and right multiplication by  $2^0$ , she is unable to identify an element in  $H$  that serves the “matching” *function* of identity. Consequently, Violet is unable to identify inverse elements using the “finding inverse” *function* of identity, because she relies on the identity element to serve this *function*. This culminates in Violet abandoning this

approach. Regrettably, the interviewer asks Violet to stop her proof so that they might complete the follow-up questions for this section of the protocol.

***Uniqueness of inverses proof – Q6.*** Violet continues on to Q6 and begins a proof that inverses are unique in a group. The students in the class had completed this proof previously, so Violet had produced a solution to the proof and likely remembered the general approach to the proof. This proof provides great insight into the different *functions* that *forms* of inverse might serve that she has not already demonstrated. In the proof, Violet adopts two letters,  $g^{-1}$  and  $i$ , as primary representational vehicles for inverse, saying:

So, we'd let-  $g$  inverse be an inverse of  $g$ . Um. And (inaud) and I'll say  $i$  be the inverse of another  $g$  in  $G$ . So, then, we would do  $g$  inverse  $g$  is equal to- I'll let  $e$  be the identity. And we know that's equal to the identity. And, also,  $i$  times  $g$  must be equal to the identity as well, because that's an inverse and, by definition it brings it back to the identity. So, then I could do- I would apply  $g$  inverse on the right-hand side of each of the sides. So I would have and then these two cancel out.  $g$  and  $g$  inverse and  $g$  and  $g$  inverse also cancel out. So, you'd have  $g$  inverse, which is equal to  $i$ . And that would be a contradiction, because we assumed that they were two different inverses up here, but they're actually the same inverse.  
(lines 463-471)

Violet then writes the equation  $g^{-1} * g = e = i * g$  (Figure 4.6a). The two equations that Violet described aloud can be viewed as drawing on the “letter” *forms* of inverse to serve the “bring back” *function* of inverse. Violet tacitly applies transitivity to set the two algebraic expressions equal to each other. From this point, Violet applies the cancellation law, which she describes as “apply[ing]  $g$  inverse on the right-hand side of each of the sides” (lines 467-468, Figure 4.6b). This allows her to, in the next sentence, “cancel out” (line 469) the  $g$  from each side of the equation in Figure 4.6b so that she can re-write the equation as  $g^{-1} = i$  (Figure 4.6c, Violet later wrote “ $e$ ” on each side of this equation when explaining her approach). During the right-cancellation, Violet used the “letter” *form*  $g^{-1}$  to cancel  $g$  from equation 2. It is unclear whether

Violet thought that this inverse was any different from the original  $g^{-1}$ , but Violet seems to be comfortable invoking its usefulness.

Figure 4.6 shows three stages of a handwritten proof. Stage (a) shows the equation  $g^{-1}g = e = ig$ . Stage (b) shows  $g^{-1}g(g^{-1}) = ig(g^{-1})$ , with the  $g$  in the first term and the  $g^{-1}$  in the second term crossed out. Stage (c) shows  $(g^{-1}) = (i)e$ , with the  $g$  and  $g^{-1}$  from the previous stage crossed out.

Figure 4.6. Violet’s proof of the uniqueness of inverses

In her discussion, Violet uses the “bring back” *function* of inverse to set two algebraic statements equal to the identity and, so, equal to each other (Figure 4.6a). The latter part of this activity constitutes a “transitive” *function* of identity (here, in the “letter” *form* “ $e$ ”) that Violet has not yet demonstrated in this interview. This is important because it affords Violet an algebraic statement from which to continue the proof. Violet also draws on a “letter” *form* of inverse ( $g^{-1}$ ) to right-end operate on each side of the equation, supporting an “end-operating” *function* of inverse in which her explicit goal is to remove the  $g$ ’s from the right end of each side of the equation. Violet marks through the concatenations of  $g$  and its inverse and re-writes the equation without these concatenations, supporting a “vanishing” *function* of inverse through which an algebraic statement can be re-written with a concatenation of an element and its inverse removed from the statement. Violet’s explanation and later inclusion of the “letter” *form* of identity (“ $e$ ,” Figure 4.6c) supports a sense that she is able to view the changes resulting from the “vanishing” *function* of inverse as mediated by the “bring back” *function* of inverse (to replace the concatenation with a *form* of identity) and the “resemble itself” *function* of identity (so that the *form* of identity need not be written as she re-writes the equation). These new *functions* constitute shifts in the *function* of inverse and identity, though not novel/in-the-moment, as Violet indicated when alluding to her previous experience with this proof, which should provide documentation of an affordance of identity and inverse in Violet’s proving activity.

**Subgroup proof: Normalizer of  $h$  – Q7.** Violet begins her response to Q7 by describing her initial approach to the proof. It is unclear which axiom Violet is attempting to show, but she begins by drawing on the “letter” *form* and the “end-operating” *function* of inverse to operate on the equation “ $g*h*g^{-1} = h$ ” and produce the equation “ $(g^{-1})*g*h*g^{-1} = (g^{-1})*h$ ” (Figure 4.7). From this, Violet marks through the expression “ $(g^{-1})*g$ ” and writes the equation “ $h*g^{-1} = (g^{-1})*h$ .” Unlike before, Violet does not replace the concatenated inverses with a *form* of identity, but instead simply does not write them. This draws on the “vanishing” *function* of inverse, which combines the “end-operating” and “bring back” *function* of inverse as well as the “resemble itself” *function* of identity so that Violet can rewrite the equation having cancelled the inverses without explicitly acknowledging that the identity is used in the cancellation. Violet again draws on the “letter” *form* and the “vanishing” *function* of inverse to right-end operate with  $g$ . In transitioning to the next equation, Violet again does not replace the concatenated inverses with a *form* of identity, but instead copies the letter “ $h$ ” below after marking through “ $g^{-1}*g$ .” Violet described the result of this activity by saying, “that just takes it all to the other side.” It is unclear whether Violet noticed that the right-hand side of the resulting equation was the reversed order of the original equation’s left-hand side.

$$\begin{array}{l}
 h * e = h \\
 g * h * g^{-1} = h \\
 \cancel{(g^{-1}) * g} * h * g^{-1} = \cancel{(g^{-1})} * h \\
 h * g^{-1} * \cancel{g} = \cancel{(g^{-1})} * h * g
 \end{array}$$

Figure 4.7. Violet’s manipulation of the equation  $g*h*g^{-1} = h$

After six minutes of re-reading her work and describing her understanding of the prompt, Violet uses specific numbers in place of  $g$ ,  $g^{-1}$ , and  $h$ . Violet describes 2 and -2 as inverses (presumptively contextualizing the operation as addition) to stand in place of  $g$  and  $g^{-1}$  and

replaces  $h$  with 3. When calculating  $g*h*g^{-1}$ , she multiplies the three numbers, resulting in 12, which, she notices, does not equal 3 as desired. Similar to her activity in response to Q5, this supports the sense that Violet is not attending to the binary operation under which she is combining the elements and the operation under which 2 and -2 are inverses. In terms of the *form/function* categories, Violet is using an additive “number” *form* of inverse to stand in place of the “letter” *forms* given in the prompt. She also seems to be drawing on the “opposite” *function* of inverse developed in the analysis of the first interview without attending to whether these *forms* might also serve the “bring back” *function* of inverse in the context of multiplication.

Later in her discussion, Violet describes how she thinks the identity might act in  $H$ , saying, “you would want an identity that gave you back the little  $h$  so it would be like  $h$  star  $e$  is equal to  $h$ . But I was just thinking, like, I don't know what the operand is, so I was trying to figure out how I would say what I thought the identity would be.” Violet does not seem to use the assumption that an identity element exists in  $G$  and, thus, does not carry out the standard approach of determining whether the identity of  $G$  satisfies the definition of  $H$ . Instead, she describes how an identity of  $H$  would “[give] you back the little  $h$ ” (line 728). This seems to become problematic for Violet based on her statement that she “[doesn’t] know what the operand is.” This is interesting, given that in her proof in response to Q5, she did not use the given operation, except when generating her table.

**Summary of Interview 2.** Analysis of the second interview contributed to the development of several new *forms* and *functions* of identity and inverse and also supported the codes developed in the first interview. There is some evidence that Violet is attending to the binary operation when discussing these concepts, though there are several instances in which Violet draws on an instantiation of what might be an identity or inverse in some contexts with



operations under which that instantiation does not serve their respective *functions* (e.g., Violet's multiplication of additive inverses during her response to Q5). Violet's example generation (Q3) and proof activity (Q5) draw on a table organization that affords two new *functions* of identity ("matching" and "inverse-finding") and provides insight into how inverse and identity might be related for her. Violet's work during Q6 and Q7 also affords a new "transitive" *function* of identity and extends the "resemble itself" *function* of identity as an aspect of the "vanishing" *function* of inverse.

The "inverse-finding" *function* of identity and "vanishing" *function* of inverse demonstrate that Violet has begun to relate the two concepts in meaningful ways that support her proof activity. Specifically, Violet's definition (Q2aiii) of inverse includes the phrase, "brings the corresponding element back to the identity" (lines 87-88), which is not included in her discussion of inverse in the first interview. Her work in the second interview supports two new *functions* of inverse ("bring back" and "vanishing") that provide evidence that she has developed the notions of cancellation she alludes to in the first interview. Violet also draws on a new "end-operating" *function* of inverse, especially when working with "letter" *form* of inverse in response to Q6 and Q7, which was not evident in the first interview.

***Summary of the form/function analysis across Violet's interviews.*** Throughout the two interviews, Violet's discussion of identity centers on language that treats identity as a verb that Violet performed on an element so that the same element is the result of the action. This is evident in each of the *forms* of identity developed through the analysis and the ways in which they serve the "resemble itself" *function* of identity. Her responses in the second interview include an increased emphasis on operation (which might indicate a distinction between an identity element and operation), although there are several instances throughout the interview in

which her language treats identity as both an element and action without contextualizing the identity relative to a specific operation.

Violet's discussion of inverse changes demonstrably across the two interviews. Throughout Interview 1, Violet continually draws on an "opposite" *function* of inverse that focuses on the existence and production of a second element that she describes as an inverse of a first element. Throughout the second interview, Violet extends the *functions* that *forms* of inverse might serve so that inverse is more clearly related to identity. This includes the use of inverses to cancel elements that are operated with other elements, especially when Violet is working with the "letter" *form* of inverse. However, Violet occasionally uses *forms* of inverse without attending to the correspondence between an element, its inverse, and the operation under which they are inverses. For instance, during her response to Interview 2 Q7, Violet draws on the "number" *form* of inverse to test the equation used to define  $H$ . In this process, she shifts from the given "letter" *form* of inverse to the "number" *form* of inverse and substitutes 2 and -2 for  $g$  and  $g^{-1}$ , respectively, and 3 for  $h$ . Although the numbers she uses in the substitution reflect an additive inverse relationship, Violet evaluates the expression by multiplying the three numbers. Further, during her response to Interview 2 Q5, Violet describes exponents of 2 as the identity and inverse, although the operation under which the group is defined is multiplication. However, when drawing on the "letter" *forms* of identity and inverse, as in her responses to Interview 2 Q6 and Q7, this seems less problematic. Specifically, when working with the equation she produces in her response to Q6 as well as the equation used to define  $H$ , Violet is able to manipulate the equation by appropriately drawing on the "end-operating" and "vanishing" *functions* of inverse. However, although this leads to a successful proof in response to Q6, Violet does not leverage the new equations she produces during her response to Q7 to make any claims about the set  $H$ .

## Form/Function Analysis of Tucker’s Individual Interviews

Tucker participated in all three of the individual interviews. His interviews were chosen for analysis because he worked closely with John and Violet throughout the semester, he was very outspoken in class, and he completed responses to most of the protocols during each interview. 29 *form/function* codes for identity and inverse were developed over the analysis of the three interviews (Table 4.4). Some codes (e.g., “operate/same out” and “operate/identity out”) were much more prevalent across the three interviews than others (e.g., “stepping” and “takes back”). Throughout the interviews, Tucker’s activity tended to reflect a sense that he considered identity and inverse elements contextualized by operations. Occasionally, especially in the first interview, Tucker’s discussion of the concepts conveyed a sense that they acted on other elements in an input/output-type relationship. The *forms* upon which Tucker drew shift across the three interviews, focusing on matrices and functions in the first interview and letters and symbols in the second and third interview.

Table 4.4 – Codes from the form/function analysis of Tucker’s individual interviews

Identity		Inverse	
<i>Form</i>	<i>Function</i>	<i>Form</i>	<i>Function</i>
Function <sup>1</sup>	Same in/out <sup>1</sup>	Matrix <sup>1</sup>	Operate/identity out <sup>1</sup>
Matrix <sup>1</sup>	Operate/same out <sup>1</sup>	Number <sup>1</sup>	Takes back <sup>1</sup>
Letter <sup>1</sup>	Inverse-producing <sup>1</sup>	Function <sup>1</sup>	Get back <sup>2</sup>
Number <sup>1</sup>	Matching <sup>2</sup>	Stepping <sup>2</sup>	End-operating <sup>2</sup>
Symmetry <sup>1</sup>	Finding inverse <sup>2</sup>	Letter <sup>2</sup>	Inverse-inverse <sup>2</sup>
Symbol <sup>2</sup>	Transitive <sup>2</sup>	Symbol <sup>2</sup>	Identity as self-inverse <sup>2</sup>
Even <sup>3</sup>		Even/odd <sup>3</sup>	Cancelling <sup>2</sup>
Exponential <sup>3</sup>		Symmetry <sup>3</sup>	Vanishing <sup>2</sup>
			Inverse of a product <sup>2</sup>
			Exponent reducing <sup>3</sup>

Superscripts indicate the Interview during which evidence of each *form/function* was first coded

**Interview 1.** Analysis of the Interview 1 with Tucker focuses on most of his responses to the protocol (Appendix A). I use Tucker’s responses to questions 1, 2, 3 (a, b), 5, 6, 7 to develop the *form/function* categories for Tucker’s understanding of identity and inverse. Throughout

Interview 1, Tucker draws on five *forms* of identity (“function,” “matrix,” “letter,” “number,” and “symmetry”) to serve three *functions* of identity (“same in/out,” “operate/same out,” and “inverse-producing”); he also draw on three *forms* of inverse (“matrix,” “number,” and “function”) to serve two functions (“operate/identity out” and “takes back”).

***Describing identity – Q1.*** In Tucker’s initial response to Q1a, he mentions the identity matrix, then describing identity as a process, saying, “it’s ... the process by which, you know, you get something and then you- it’s like the same process out of like going in as it does coming out” (lines 5-6). This description affords two initial codes for a representational vehicle for identity: identity matrix and process. It also supports an initial code for a “same in/out” *function* of identity in which an identity acts as an input/output process that preserves the object upon which it acts. Tucker then states, “if it was a function, uh, the identity function would just, basically, not change anything about it. And it’ll work with pretty much anything you do” (lines 7-8). This supports the emerging development of a “function” *form* of identity that serves the “same in/out” *function*. Tucker points out in his next statement that this is a general description. Accordingly, the correspondence of these representational vehicles to the representational object of identity seems to be that they each serve the respective *function*. Tucker then states that “the identity matrix is just a matrix, which if you multiply it to another matrix, you’ll get that same matrix out of it” (lines 10-11). This supports the development of a “matrix” *form* of identity in which the representational vehicle of an identity matrix corresponds to the representational object of identity by being multiplied by a second matrix and the product being the second matrix. This excerpt also supports an “operate/same out” *function* of identity through which a *form* of identity (here, “matrix” *form*) is explicitly operated (here, “you multiply it”) with a second element of a similar *form* as the identity (here, “another matrix”) and results in that second element. For each

*function* of identity, Tucker conveys a sense of preservation, or sameness, for which a noun (“process,” “it,” and “matrix”) “comes out” the same as it was before a contextualized action (“process,” “function,” and “multiplication”). The third description helps clarify Tucker’s first two descriptions. Specifically, while it is unclear what Tucker is attributing to a “process” or “function” in the first two descriptions, his third description contextualizes an identity matrix and another matrix as being multiplied. The distinction between the “same in/out” *function* of identity emerges through Tucker’s explicit focus on the operation under which he considers a *form* of identity and another element. Specifically, it is only through this operation that the *form* of identity preserves the other element. In contrast, with the “same in/out” *function* of identity the *form* of identity is not explicitly situated by an operation.

During his response to Q1ai, Tucker uses the phrase “identity property,” which Tucker explains as “saying that there’s an identity *exists*” (lines 17-18, his emphasis). He rephrases this “identity property,” saying:

...the identity property pretty much states that, 'there's a way that you can take this- whatever you have, this whatever it would be, you know- two different things multiplied to each other or whatever that process is and you can find the identity for that. Which, yeah. So, and that's, you know, I think you can use that to apply to I think just, basic operations on a field, I think. Er, something along those lines. (lines 18-22)

It seems that Tucker is struggling to find the words to indicate the mathematical concepts he is trying to describe, relying on the phrases “this whatever it would be,” “different things,” “multiplied to each other,” and “whatever that process is.” Regardless, it seems that the *function* of the “identity property” is to support the existence of an identity, which Tucker describes as being able to be “appl[ied] to ... basic operations on a field” (lines 21-22). In his conclusion to Q1ai, Tucker elaborates on what he means by the word “field,” explaining that he thinks of it “as the set of rules by which things kind of, like, exist in” (line 29). He then provides two examples

of what he means by this<sup>4</sup>:  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . However, he does not mention identity in relation to these two examples, only describing the types of vectors in each space.

In his response to Q1aii, Tucker relates the word “identity” to its semantic meaning and then summarizes how he thinks about it mathematically, saying, “if identity exists, that means there's a way that you can have it- that same function or matrix or whatever, what have you... out again” (lines 76-77). This echoes Tucker’s previous descriptions, drawing on the representational vehicles of function and matrix and describing the situation by saying “you can have it ... out again” (line 77). In response to Q1aiii, Tucker draws on the “matrix” *form* of identity to define identity, admitting that he does not think of this as a formal definition, saying, “I guess you call the identity matrix is, uh, I guess, an n by n matrix, which takes another, separate n by n matrix to itself, when multiplied by it” (lines 82-84). This excerpt is similar to his third description in response to Q1a, drawing on the “matrix” *form* of identity to serve the “operate/same out” *function* of identity, except that he specifies an n x n matrix. Again, Tucker is focused on matrix multiplication as the process. Tucker ascribes the verb “takes” to the “matrix” *form* of identity. He contextualizes this action as occurring “when multiplied by it,” specifically that the identity matrix does not in-and-of-itself act on the other matrix, but that the “tak[ing]” is an outcome of the multiplication, affording it the ability to serve the “operate/same out” *function* of identity.

In response to Q1aiv, Tucker compares “identity matrix” and “identity function,” stating that, “they’re not too different, it’s just that, you know, the identities might look a little different” (lines 105-106). Tucker summarizes this comparison saying, “The identity might look different in each case, but it's pretty much gonna do the same thing of just, take something and, after you apply the identity to it, have the same thing come out” (lines 110-112). This seems to support the

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<sup>4</sup>  $\mathbb{R}^2$  and  $\mathbb{R}^3$  are not fields.

“same in/out” *function* of identity, but is distinguished from Tucker’s previous language by noticing the phrase “after you apply the identity” (lines 111-112). This phrase emphasizes a user-oriented aspect of the activity for Tucker, much like with his description of multiplying with an identity matrix, although he does not distinguish the operation from the identity. In contrast, when drawing on the representational vehicles of “identity process” and “identity function,” the language conveys action inherent in the representational vehicle. This affords a distinction to be drawn between the various *functions* of identity upon which Tucker draws, specifically that, when Tucker ascribes action to a *form* of identity, it is coded as serving the “same in/out” *function* of identity, whereas, if he explicitly differentiates between a type of operation and an identity element, I code this as the “operate/same out” *function* of identity.

In response to Q1b, Tucker returns to his property language, aligning the “identity property” with associativity and commutativity. He introduces two new contexts – scalars and adding and subtracting matrices or vectors – which seem to reflect the two operations of a vector space. He follows this discussion with a new inscription for identity, “*i*,” concatenating it with *A*, and saying,

So. I guess- I the only identity property I think of is that, like, if you have some, you know, uh, not necessarily a scalar, but if you have some, uh, be it or function, or matrix, or what have you, like, we'll call it, like, we'll call it *i* and you- *i* times whate- whatever else you have, so, we'll call it capital A, then equals A, so, *iA* equals A. Where *i* is your identity. (lines 123-126)

The equation Tucker produces supports an emerging “letter” *form* of identity (here, “*i*”) which implicitly serves the “operate/same out” *function* of identity, inscribed as an equation here, because the *A* on the right-hand side of the equation seems to be taken as the same *A* on the left-hand side of the equation. Further, though he uses the word “times” (line 125), it is unclear what type of operation Tucker views the concatenation of the two letters as. It seems from the excerpt

that he views it as matrix multiplication, rather than scalar multiplication, although he does not explicitly state this and includes that  $A$  might also be a function.

$$\left[ \begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array} \right]$$

Figure 4.8. Tucker's augmented matrix

In response to Q1bi, Tucker introduces a generic matrix using the letters a, b, c, d, e, f, g, h, and i and a new inscription of the identity matrix using the standard 3x3 array of zeros and ones (Figure 4.8). This inscription of a “matrix” *form* of identity serves a different *function* from the previous descriptions of the identity matrix. Without using specific vocabulary, Tucker alludes to the process of row-reducing an augmented matrix (with the identity matrix on the right) so that the identity matrix is on the left after some process that Tucker does not explicitly describe. However, Tucker does explain that he views this as a way to use the identity matrix in order to find an inverse matrix. So, the “matrix” *form* of identity, contextualized in an augmented matrix, might serve a new “inverse-producing” *function* of identity through which Tucker might be able to determine the inverse of a given matrix. However, Tucker does not discuss how the inverse matrix is related to the original matrix in this discussion. This “inverse-producing” *function* might indicate a broader relationship that Tucker holds between identity and inverse – that one might be able to use an identity in order to find an inverse – although it is unclear whether Tucker is thinking about this relationship in any other context than an augmented matrix.



*Describing inverse – Q2.* In his response to Q2a, Tucker continues using “function” and “matrix” language, in this case, to describe inverses. Tucker states that, “the way I think about inverses is, when you have the inverse of a function or a matrix and you multiply it by, you know, the matrix or that function, then you will arrive at the identity” (line 149-151). This supports the development of an “operate/identity out” *function* of inverse in which operating an element with a *form* of inverse will result in “the identity.” The “operate/identity out” *function* of inverse reflects the same structure as the “operate/same out” *function* of identity in that each *function* emphasizes the operation as distinct from the inverse or identity, respectively, which warrants a parallel naming structure. Tucker’s language here draws specifically on “matrix” and “function” *forms* and the operation “multiplication” and is much more specific than the general language of the “operate/identity out” *function* of inverse might suggest. However, Tucker’s discussion throughout the rest of the interview supports a sense that this *function* is more general for him than the contexts he provides here. For example, he immediately follows this description with an example, writing “ $A \cdot A^{-1} = I$ ,” and describing “ $I$ ” as the  $n \times n$  identity matrix if  $A$  is an  $n \times n$  matrix (lines 153-154). This example draws on the “matrix” *form* of inverse (written with letters and described as  $n \times n$  matrices) to serve the “operate/identity out” *function* of inverse in the context of matrix multiplication. He goes on to describe how he tends to think about inverse as “one divided by something” (line 155), using one-fourth as an example of the inverse of four, and explaining that “one-fourth is the inverse of four. [Mhmm] Four times one-fourth is one, which is kind of, like, the identity property for everything pretty much multiplied by one” (lines 156-158). This example supports a “number” *form* of inverse, that serves the “operate/identity out” *function* of inverse, as well as a “number” *form* of identity, which is also contextualized by multiplication, although it is unclear what *function* this *form* might serve.

Tucker's response to Q2ai begins by associating inverse with the word "opposite." He follows this with a description of an inverse function, saying,

So, in many ways, it'll look pretty much opposite of, you know, the function, so the inverse of that function will pretty much be almost, like, the reverse of it, so- I don't know, if a function takes something from if it just moves it from one place to the other, is that what a function does, then the inverse will move it from that place to the other. (lines 168-172)

As he describes this, he gestures his hands from one side of his body to the other and back, and then draws an arrow from left to right and another from right to left immediately under it. This supports a "function" *form* of inverse that serves a "takes back" *function* of inverse which, given an original function that maps some type of element from a domain to a codomain, will map the image of element from the codomain to the domain. The "takes back" *function* of inverse tacitly imposes composition of the inverse function with the original function (for which it is an inverse), though it is unclear whether Tucker is contextualizing his activity as function composition.

In response to Q2aiii, Tucker questions whether he would call inverse a property and then says that, "sometimes the inverse doesn't exist" (lines 180-181). He goes on to define inverse by drawing on the "function" *form* of inverse, saying,

If for some inverse, um,  $e$ , when multiplied by its fun- it's correlating function, capital  $E$ , will result in the identity,  $i$ , or whatever. So, if it's the inverse matrix, we'll call that  $A$  to the negative one. We multiply that by it's correlating function  $A$ , correlating matrix  $A$ , then we'll arrive at the identity, so. So, something along the lines of using the terms identity in there. (lines 183-187).

Tucker's definition of inverse is initially phrased using the "function" *form* of inverse (named using letters), which he describes as being multiplied, rather than being composed. Within this definition, Tucker names three different things: the "inverse," "correlating function," and "identity," to which he assigns the letter as " $e$ ," " $E$ ," and " $i$ ," respectively. Tucker explicitly

names the first two objects as functions, which he then describes as being multiplied, but does not describe what type of object the identity is, making it unclear whether he means “the identity” to be the “function” *form* of identity he described in response to Q1. Tucker follows his definition using the “function” *form* of inverse by rephrasing it using the “matrix” *form* of inverse, which reiterates the first example generated in response to Q2a, concluding this definition with a general statement that the notion of identity should be included somewhere in the definition of inverse. Between the two descriptions of inverse that draw on the “function” *form*, there is a shift in the *function* that they serve. This shift reflects the distinction between the “same in/out” and “operate/same out” *functions* of identity in that, with the “takes back” *function* of inverse, action is attributed to the “inverse function.” Conversely, the operation of multiplication that contextualizes the “operate/identity out” *function* distinguishes the action from the various *forms* of inverse that serve this *function*.

**Likert statements – Q3.** Table 4.5 shows the Likert questions used in analysis of Tucker’s responses to the Likert questions. This analysis focuses on parts a and b because Tucker’s responses to these parts were most beneficial toward the development of the *form/function* codes.

Table 4.5 – Likert statements

Part	Statement
a*	Zero is an identity.
b*	One is an identity.
c	I think of an identity as anything that I use to keep things unchanged.
d	I think about an identity as a function.
e	“Inverse” means “negative number.”
f	“Inverse” means “reciprocal.”
g	Inverses depend on other things, like operations or sets of elements.
h	I think about an inverse as a function.

\* indicates Likert statements analyzed in this section

Responding to Q3a, Tucker explains that he initially did not think of 0 as an identity, saying, “No, not at all. Because when you multiply by zero, you're not getting the same thing as you got before. However, I was just strictly thinking about multiplication here” (lines 330-332). He then

describes how 0 might be an identity under addition, finally providing the example “four plus zero being four” (lines 341-342). In this example, Tucker uses “zero” as the “number” *form* of identity to serve the “operate/same out” *function* of identity. This response shows that Tucker is able to think about different operations, but tends to emphasize multiplication over addition.

Responding to Q3b, Tucker provides specific contexts in which he reflects on whether one is an identity:

...for the most part, one is an identity for a lot of different things, because, you know, if you raise it to the one power, you're gonna get the same thing, and if you times it by one, you get the same thing, but, that's not always the case, for instance, if you just, uh, um, like I said, if you add one to it or subtract one to it, you're, uh, gonna get something different. (lines 356-360)

In each of these four contexts, Tucker’s description is stated as an operation involving the “number” *form* of identity “one.” Tucker uses the phrase, “get the same thing” to describe the first two examples and “get something different” to describe the latter two examples. From Tucker’s phrasing of each example, using “it” to describe some object upon which he is operating, it is unclear whether he is drawing on the “same in/out” or “operate/same out” *function* of identity, though it does seem important to Tucker that the operation is changing between the examples. This is the first instance in which Tucker uses exponentiation as an operation to contextualize a “number” *form* of identity. Tucker responds to Q3d saying,

“You know, I- I do like to think of identity as a function, so I think I'm going to hit 5. Um, I think of it as, you know, something that you apply to something else. The identity is, uh, for ins- It's something that you do that you apply to whatever you have- whatever field you're in or whatever and, after you apply it, you have the same thing, so identity- To find to see if it is the identity, you have to actually do the motion of applying to it and seeing if it comes out unchanged. I *do* like to think of it as a function in, in my head.” (lines 408-413)

Tucker uses the phrases, “you apply” and “if it comes out unchanged,” which reflect the “same in/out” *function* of identity. Again, this supports a sense that Tucker is thinking of identity in two

ways: as an object that, when combined with another object in a specific operation, preserves the other object and as a function and as a process itself with an associated input-output quality that preserves the objects in the domain.

**Responding to “cheap multiplication” quote – Q5.** Tucker begins his response to Q5 by recounting the class discussion that led to the instructor’s quote, focusing on whether the  $360^\circ$  rotation is a symmetry. He then explains how he initially thought that the  $360^\circ$  rotation was not a symmetry but changed his mind after considering the motion that occurs during the rotation. In his discussion, Tucker only briefly mentions identity and it seems as though he realizes in this moment that the  $360^\circ$  rotation is an identity, saying, “And if you wanna think about it, it’s kind of like the identity symmetry. (laughs) Because you could apply it to any shape and come out with the same image as before” (lines 584-586). This description is consistent with the “same in/out” *function* of identity. In this context, Tucker describes the  $360^\circ$  rotation as being applied to the “shape,” which is the same before and after the rotation is applied. This supports a “symmetry” *form* of identity, with  $360^\circ$  rotation as the representational vehicle in this case, which serves the “same in/out” *function* of identity when applied to shapes.

**Multiplicative identity of  $\mathbb{R}$  proof – Q6.** Tucker begins his response to Q6 by reading the statement “ $1 \in \mathbb{R}$  is the multiplicative identity for real numbers” aloud, saying,

That hits the nail on the head. I think that's true because, uh, explicitly that word when it says "multiplicative identity," it makes a difference- That's kind of like what I was talking about earlier, like, it wouldn't necessarily be the additive identity because one plus a scalar- another scalar is not gonna equal that same scalar, so, for instance, if we take any number- Let x be an element of the real numbers. To prove it, let's just try it out. So, one times by x is equal to x. Um, let's see, how more can I say about this? (lines 613-619)

He first indicates that he thinks the word “multiplicative” is important, comparing this to his responses to Q3a,b and focusing on one as an identity in the context of multiplication, but not

under addition. Further, Tucker seems to be satisfied that the “number” *form* of identity “one” serves the “operate/same out” *function* of identity. Tucker provides more insight into his proof during the follow-up question when he says,

...if we take this one- if it is the identity, then multiplying it by another number in the real numbers - I just chose an arbitrary one,  $x$  - Should yield you that same number. And- I tried it out. Obviously, it's trivial to try it, but I did it anyway, just to show you one times by any arbitrary  $x$  will yield you  $x$ . Meaning that one is considered a multiplicative identity for real numbers. (lines 630-634)

Tucker distinguishes between the identity and the operation that he is carrying out in the context, which is, again, consistent with the “operate/same out” *function* of identity. Tucker’s response to 6c is the first instance in which Tucker states that an identity must work for every number. This supports a sense that Tucker is able to think about the importance for the identity working for all elements but that it does not seem to be a major aspect of his conceptual understanding.

#### *Interview 1 analysis summary*

Tucker’s discussion throughout the first interview conveys two broad ways that he conceptualizes identity and inverse: as an input/output relation that acts on other objects or elements and as an element that, contextualized by an operation, may be operated with other, similar elements. This is reflected in the distinctions between the “same in/out” and “operate/same out” *functions* of identity as well as between the “operate/identity out” and “takes back” *functions* of inverse. Specifically, the “operate/same out” *function* of identity and “operate/identity out” *function* of inverse are codes when Tucker distinguishes between an element (identity or inverse, respectively) and an operation under which that element might be combined with another. In contrast, the “same in/out” *function* of identity and “takes back” *function* of inverse are coded when Tucker’s language treats the identity and inverse as input/output relations that act on other objects or elements.

**Interview 2.** As with Interview 1, analysis of Tucker’s second interview focuses on his responses to specific prompts from the protocol (Appendix B). In this section, I focus on Tucker’s responses to questions 1, 2, 3, 5, 6, and 7. From the analysis of this interview, I code one new *form* of identity (“symbol”). I also code three new *functions* of identity that this and previously coded *forms* serve: “matching,” “finding inverse,” and “transitive.” Analysis of the second interview also supports three new *forms* of inverse (“stepping,” “letter,” and “symbol”) and five new *functions* of inverse (“get back,” “inverse-inverse,” “cancelling,” “vanishing,” and “inverse of a product”). Tucker also draws heavily on an organization of a “group table” (formally, a Cayley table), which affords the “matching” and “finding inverse” *functions* of identity.

**Describing identity – Q1.** Tucker begins his response to Q1a by commenting on how much the notion of identity has been drilled for him, calling it the “zero operation” and the “boring case,” providing two representational vehicles for identity. He follows these descriptions with a general definition: “You take an element and, if you apply the identity to it, you get the element back, where you started” (lines 12-13). This description reflects the “same in/out” *function* of identity coded during analysis of the first interview, in which an identity is applied to an element in order to get that element back. He provides three specific examples, saying, “If you think about it as a function, it’ll kind of like plussing zero to a number or timesing by one or other things like that- or rotating an object 360 degrees” (lines 13-14). These representational vehicles seem to more directly correspond to the representational object of identity in order to serve the “same in/out” *function* than the “zero operation” and “boring case,” though it is not yet clear how Tucker views the identity as being “applied” to other elements in these cases. However, each of the latter three representational vehicles includes a verb and at least one noun in its description.

At this point, it is unclear how the verb and noun(s) together correspond to identity. Compared to his initial description of identity in Interview 1, Tucker more explicitly addresses the elements and operations involved with identity, but it is unclear during this discussion whether he is viewing the identity elements as distinct and separate from their associated binary operations.

While elaborating on his response (Q1ai), Tucker describes how he thinks of symmetries as functions, saying,

And when I think about functions, I kind of think about, okay, my teachers, like, in elementary school said you have a box and you put something in the box and it spits out something else. So, as a function, you put something in the box and it just goes straight through it and you get the same thing back out. (lines 22-25)

This aligns with the definition of the “same in/out” *function* of identity, specifically, in that Tucker is describing the identity here as something that acts on an element so that the element is preserved through the action. From this description, an identity is not distinct from the operation, but instead acts as an operation in order to serve the “same in/out” *function* of identity. This is supported by the phrases Tucker uses with his examples – “plussing zero to a number,” “timesing by one,” and “rotating an object 360 degrees” – which, in light of his explanation, can be viewed as a combination of operation and element that allow action to be attributed to the element so that, each phrase might serve the “same in/out” *function* of identity.

In response to Q1aai, Tucker spontaneously mentions the notion of inverse, relating it to that of identity, saying, “Obviously, you can think about in- You can think about inverses as well. As in, like, when you have an element and you apply its inverse to it, that'll give you the identity in return” (lines 31-32). It is unclear whether Tucker is viewing the inverse as acting in a similar way as his examples of identity. Specifically, the phrase, “you apply the inverse,” differs slightly from his more specific examples of identity, especially because there is no specific

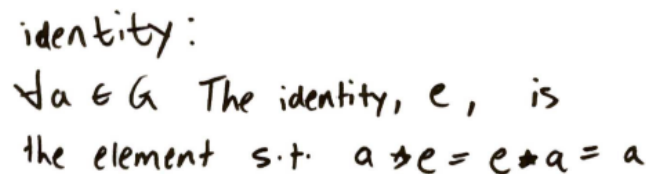


operation that Tucker is describing, supporting the sense that this reflects the “takes back” *function* of inverse.

When asked to provide a definition of identity (Q1a<sub>iii</sub>), Tucker extends his discussion, saying,

I would say, uh, the identity is an element within a group such that any other element in that group, when you apply the operation, the identity to that operation, you'll result in the original element that you started with. So, for instance, if you had some arbitrary element,  $a$ , you apply the identity, let's call it  $e$ , then you'll get  $a$  right back at you. (lines 39-43)

This definition more closely aligns with the “operate/same out” *function* of identity, contextualizing the identity as “an element within a group,” which he does not use a context for identity or inverse during the first interview. He then re-phrases the definition to draw on the “letter” *form* of identity, “ $e$ ,” and continues to use the “apply” language. However, when Tucker writes this definition (Figure 4.9) and reads the written definition aloud, he changes the verbs he uses from “apply” to “star” and from “result” and “get” to “equals.” This more clearly separates the identity element from the operation that allows it to be “applied” to other elements.



identity:  
 $\forall a \in G$  The identity,  $e$ , is  
the element s.t.  $a * e = e * a = a$

Figure 4.9. Tucker’s written definition of identity

Responding to Q1a<sub>iv</sub>, Tucker elaborates on his definition, connecting it to his earlier descriptions by saying “you're really not doing anything to the other elements. Like, you're not changing it in any way” (lines 58-59). This reiterates the importance of preservation that Tucker holds, but is worded slightly differently from the “operate/same out” *function* of identity because it conveys the lack of any action, rather than his previous language of “apply[ing]” an identity. When pressed for more examples of identity (Q1b), Tucker mentions “dividing by one” and “times-ing by the identity matrix,” but does not elaborate on these any more than mentioning

them and alludes to other possible types of identities, but states that he cannot think of other sets in that moment. As before, it is unclear whether Tucker is thinking of the operation and identity element as distinct in each of these examples.

***Describing inverse – Q2.*** Asked about inverse, Tucker indicates that the concept is closely tied with that of identity and states that every element in a group will have an inverse. He then provides a general description for inverse, saying,

I think inverse really ties in with what we talked about with identity, uh in the fact that, um, every element in a group will have an inverse. And that inverse when you apply it to its element- so,  $a$ 's inverse we'll call it minus  $a$ , for instance -  $a$  star minus  $a$  will yield you the identity element. So, it's kinda like, what you have to do to the element to get it back to the identity, type thing. (lines 78-82)

This description aligns with his formal definition of identity. Specifically, Tucker returns to the “apply” language and uses the symbols  $a$  and “minus  $a$ ” (or “ $-a$ ”) to stand for the element he is considering and its inverse, respectively. Tucker’s elaboration on this description uses the phrase “get back,” although in this case, instead of “get[ting] back” the original element, the inverse “get[s] it back” to the identity. This language is consistent with the “letter” *form* of inverse (here contextualized as “minus  $a$ ” and using the operation “star”) and is able to serve the “operate/identity out” *function* of inverse developed during analysis of Interview 1. Tucker then elaborates by describing stepping to the right three paces and then to the left three paces to get “right back to where you started.” This example supports a “stepping” *form* of inverse (“stepping left three paces”) that serves a similar *function* as the “takes back” *function* of inverse. Although not explicitly stated, “where you started” in this example would correspond to the identity.

The inverse to element  $a \in G$  is  
the element  $-a$  s.t.  $a * -a = -a * a = e$   
where  $e \in G$  is the identity.

Figure 4.10. Tucker's written definition of inverse

When asked about other ways he might think of inverse (Q2aii), Tucker states that he associates the concept with “one divided by something” or “to the negative one power,” noting that this “isn't always the case.” He elaborates on this using the example of  $\frac{1}{4}$  and 4, pointing out that these are not inverses “if you talk about, like, addition.” He continues, saying,

I just think of inverses dependent on the operation in a group. So, when you think- when you have groups under the operation of multiplication, the inverses are going to look a lot different than the same set of numbers under the operation of addition. (lines 92-95)

This is interesting because his examples of inverses up to this point have tacitly incorporated the operation, however this supports a sense that Tucker is attentive to the operation under which he is considering identities and inverses. In response to Q2aiii, Tucker begins his formal definition of inverse saying that, “it's gonna use- also this identity element,  $e$ .” This underscores Tucker's connection between the two concepts. He continues, saying, “The inverse to element  $a$  in  $G$  is the element such - we'll call it- we'll call it the element, minus  $a$  - [Okay.] Such that  $a$  star minus  $a$  is equal to minus  $a$  star  $a$  is equal to  $e$ , where  $e$  is the identity element in  $G$ ” (lines 101-103; Figure 4.10). This definition aligns with his formal definition of identity in that it uses the same symbols for an element ( $a$ ), the identity ( $e$ ), and an operation ( $*$ ). Tucker uses the representational vehicle “ $-a$ ” to represent “the inverse to element  $a$ ,” (line 101), which serves the “operate/identity out” *function*. Tucker then provides another illustration of inverse, similar to his three paces example, in which  $a$  is movement to a specific location (the market) and “inverse  $a$ ” is movement “from the market back to home.” Tucker explicitly point out in this example that “home” is the identity.

Following up on Q1 and Q2, the interviewer asks Tucker about his written definitions of identity and inverse, specifically about Tucker's writing each definition as a two-sided identity and inverse, respectively. Tucker explains that it is to ensure that the identity is commutative with every element. This is implied for inverses as well, though Tucker does not explicitly address the commutativity of inverses. When asked to provide more examples of inverse (Q2b), Tucker describes  $\frac{1}{4}$  and 4 as inverses "under the operation of multiplication" and -4 and 4 as inverses "under the operation of addition" (lines 132-133). Each of these is an example of a "number" *form* of inverse serving the "operate/identity out" *function* of inverse. He also alludes to matrices and their inverses, but does not produce an example or describe a type of operation under which he thinks about inverse matrices. Finally, Tucker states that there will be inverses for elements in groups, again, without contextualizing how they might serve a *function* of inverse.

***Describing group – Q3.*** In his response to Q3, Tucker includes a discussion of inverse and identity when defining group as a set of elements and an operation. After describing closure, Tucker says, "there is inverses for each element within that group. And that also shows that there's identity cause otherwise inverses wouldn't make logical sense. Um, so, each element, you can find another element in that group such that you can get back to the identity" (lines 155-157). Although he alludes to an operation when beginning his discussion of group, this description does not quite align with the "operate/identity out" *function* of inverse, supporting an emerging "get back" *function* of inverse. However, it may be the case that Tucker is referring to his previous definition of inverse (Q2aiii) in this description, which did align with the "operate/identity out" *function* of inverse. Interestingly, Tucker does not include an identity "rule" in his description of group, but seems to attribute the existence of an identity to the inverse

“rule,” which he clarifies in response to a follow-up question. Tucker follows this by providing three examples of a group: integers under addition, nonzero real numbers under multiplication, and the symmetries of a triangle. Although he alludes to being able to find the inverse of each element in the group of symmetries, he does not discuss the identities or inverses in the context of these groups.

In his response to Q3b, Tucker repeats his assertion that inverses exist in a group using the “letter” *form of inverse*, which he inscribes as “ $a^{-1}$ ,” whereas he had previously used “minus  $a$ ,” although he does not mention the identity when writing this part of his definition of group. However, when responding to Q3c, Tucker elaborates on his definition of group by relating it with his previous discussion of inverse and identity. He repeats, nearly verbatim his previous definition of inverse. He also clarifies his previous statement about how the existence of inverses includes in it a statement of the existence of identity, saying, “just saying that the inverses exist for each element kind of proves to yourself that, okay, that means there has to be an identity element in group, because part of the definition for inverses includes the identity element” (224-227).

In response to Q3d, Tucker generates a new example of a group, intentionally creating a group using non-mathematical symbols (Figure 4.11). This example generation begins with Tucker listing an operation (swirly) and five elements (smiley, frowny, apple, orange, and star). He then proceeds to construct a Cayley table (or group table) using these five elements in which all elements are listed along the left column and top row of a square array and each cell in the array corresponds to the row element being operated with the column element. Figure 4.11 shows the order in which Tucker completed the table. Tucker states that the star is the identity, provides a relationship between smiley and smiley (equal frowny), and explains that he is filling in the

inverses, likening the task to Sudoku (Figure 4.11c). He then copies the top row and left row in to the column and row corresponding to star (the identity element, Figure 4.11e). As Tucker points out, he is unsuccessful at producing a group with these relationships because, as he points out, he cannot make it satisfy the “Sudoku property” for group tables (Figure 4.11f)<sup>5</sup>. While checking his group table, Tucker questions whether it is possible to construct a group with five elements. He ponders aloud whether he incorrectly filled in the “identities” in the table. Here, he is referring to the assignment of inverses, which he carries out by placing identities in the various rows and columns of the table so that each row and column contains exactly one star (identity).

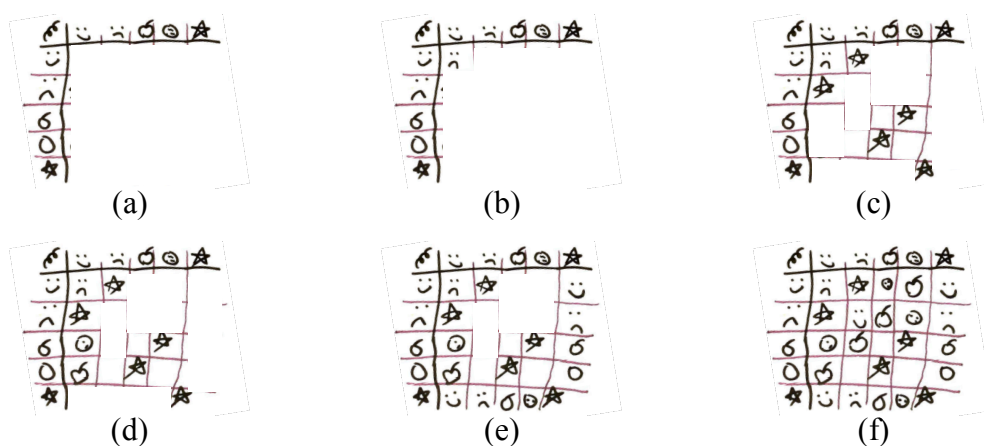


Figure 4.11. Tucker's group table using “symbol” forms

Although his attempt at creating a group was unsuccessful, this activity supports the development of “symbol” forms of identity and inverse (here, “star” is a representational vehicle for identity and each symbol is assigned as an inverse for another symbol) and two new functions of identity: “matching” and “finding inverse.” The “matching” function of identity allows Tucker to match the first row and column to the row and column corresponding to the group's form of identity (change between Figure 4.11d and Figure 4.11e). This reflects the “operate/same out”

<sup>5</sup> His construction of this table does not anticipate that a group of order five has prime order (this is reasonable because it had not been addressed in class) and, so, nontrivial elements must have order five. As a result, Tucker unwittingly assigns products in his table so that smiley has order 3 and, so, his table cannot be completed as a Latin square, preserving Tucker's necessary characteristics of a group table.

*function* of identity, but carried out for all elements in the group, which is afforded by the table organization. Further, the “operate/identity out” *function* of inverse is reflected in the “finding inverse” *function* of identity, which places the *form* of the identity element in a cell that corresponds to an element’s row and the column of that element’s inverse (change between Figure 4.11b and Figure 4.11c ).

***Subgroup proof: Integer powers of 2 – Q5.*** Tucker begins his response to Q5 by describing how the set  $H$  is a subset of  $G$  and then asking whether the operation of  $H$  is multiplication. The interviewer responds that, because the operation of the group  $G$  is multiplication,  $H$  will inherit this operation. From this, Tucker explains that  $H$  is closed, and that  $H$  is associative, none of which incorporates a discussion of identity or inverse. He then begins his begins his proof to show that  $H$  contains inverses for its elements, saying:

Um, for each element. So, for instance, two's inverse is gonna be one-half, so that would be two to the, um, uh, one-fourth power. No. How do we- Okay. two to the, huh, I'm sorry. Minus two power. Yeah. So, basically, no, wait, wait, wait. Minus one. I don't know what I was thinking. I was trying to think oh you've gotta get rid of the two. Okay. Never mind. Never mind. So, yeah. two's inverse is gonna be one-half so, that's gonna be two to the minus one, which is also in  $Z$ , so- So, yes. (lines 483-488)

This subproof begins with Tucker asserting that the inverse of two is one-half. It seems that this is based on Tucker’s understanding of the real numbers of multiplication, which reflects his previous examples of inverse (Q2a, Interview 1; Q2b, Interview 2) in which he draws on the “number” *form* of inverse to serve the “operate/identity out” *function* of inverse in the context of multiplication. However, it takes Tucker three attempts to correctly identify the element  $2^{-1}$  in  $H$  (elements are represented as  $2^n$ ) that equals one-half.

In response to Q5a, Tucker goes on to describe his approach more generally, saying, “So, basically- is 2 to the  $n$  gonna have an inverse such that- yield you one? Cause one's gonna be the

identity here. [Okay.] Cause well it's under multiplication” (lines 498-500). This statement supports a sense that Tucker is drawing on the “operate/identity out” *function* of inverse by contextualizing the group operation as multiplication and, so also, the “number” *form* “one” of identity. He follows this immediately by saying, “So, let's just call the inverse  $i$ . So, we've got to find something that equals that, so, basically, all we can really do is- well we can divide by two to the  $n$  and- st- to see what  $i$  is. [Mhmm.] And that's gonna equal two to the minus  $n$ . [Okay.] So-  $i$  is equal to two to the minus  $n$ ” (lines 500-502). In this excerpt, Tucker draws on the “letter” *form* of inverse “ $i$ ” to determine which element of  $H$  would be the inverse of a general element,  $2^n$ , of  $H$ . In order to do this, he describes dividing by  $2^n$ , describing this as  $2^{-n}$ . Here, Tucker is drawing on the “operate/identity out” *function* of inverse by beginning with an unknown element and reasoning about the operation (in this case, using division rather than multiplication) in order to determine what the inverse of the given element must be. Tucker’s response to Q5g reiterates his focus on the operation of the group and elaborates on the connection he makes between the reciprocals (of positive integer powers of 2) and negative integer powers of 2.

***Uniqueness of inverses proof – Q6.*** Tucker begins his proof of Q6 by pointing out that he will use the properties of a group while completing his proof. He then rephrases the goal of the proof, saying, “there's only ever going to be one  $g$  to the minus 1. There's not gonna be another way to get you back to the- to the identity.” This affords Tucker to frame the goal of the proof in terms of his “get back” *function* of inverse. He then states that his general approach will be a proof by contradiction. Accordingly, he assumes that there are “two different inverses” of an element  $g$  in the group. He names these inverses  $g^{-1}$  and  $h^{-1}$ . He then draws on his formal definition, the “letter” *form* of inverse (written as  $g^{-1}$  and  $h^{-1}$ ), and the “operate/identity out” *function* of inverse to create the equations  $g * g^{-1} = e$  and  $g * h^{-1} = e$ . Following this, he creates the



equation  $g * g^{-1} = g * h^{-1}$ , reasoning that, since they each equal the identity, then they must also equal each other. This supports the development of a “transitive” *function* of identity in which Tucker is able to generate a new equation in which two statements are set equal based on each of those statements being equal to the identity. From this equation, Tucker, citing the cancellation law, marks through the  $g$  on each side and states that  $g^{-1} = h^{-1}$ . When asked to explain his use of the cancellation law, Tucker points out that he is actually “apply[ing]” the inverse to the left-hand side of the statement on each side of the equation, which gives the identity on each side, in turn, leaving the  $g^{-1}$  and  $h^{-1}$  elements on either side of the equation. This supports the development of a “cancelling” *function* of inverse in which Tucker removes an element and the operation symbol from the same end of both sides of an equation. From his explanation, this is mediated by end-operating on both sides of the equation with a *form* of inverse, which he imagines (tacitly via the “operate/identity out” *function* of inverse) as producing a *form* of identity. This constitutes an “end-operating” *function* of inverse in which Tucker operates on the same end of both sides of an equation with a *form* of inverse. Further, from this, it seems that he draws on the “operate/same out” *function* of identity on each side so that the *form* of identity may be removed from the statement. This is consistent with the explanations of cancellation produced during class as well as the explanations provided by Violet and John.

In response to Q6f, Tucker states that one cannot discuss inverses “without talking about identity.” He then paraphrases his formal definition of inverse and calls the identity “our bridge to set these two equal to each other” (line 650), which supports the “transitive” *function* of identity. Tucker also points out that the cancellation law draws on the notion of identity. In this discussion, he draws on both the “get back” *function* of inverses and the “operate/same out” *function* of identity. When responding to Q6g, Tucker repeats his re-phrasing of the initial

problem statement that drew on the “get back” *function* of inverse, saying, “I guess just thinking about them as the opposite or just, like, the way back t- whatever gets you the identity and just kind of proving that, well, there's only one way to do that, I guess” (lines 668-670). Asked by the interviewer, Tucker explained that he should have shown that the inverse of  $g$  was unique for both sides (as he defined inverse to be two-sided), but that the proof for the other side was very similar to the proof he had already constructed. Tucker then admits that he likely would not have thought about the role of identity in the cancellation law, had the interviewer not asked him to explain the cancellation law.

***Subgroup proof: Normalizer of  $h$  – Q7.*** Tucker begins his proof in Q7 by describing generally how he might show that a subset is a subgroup. This discussion is similar to his discussion in Q5. He begins by trying to show that the set is closed, appropriately setting two arbitrary elements in  $H$  ( $a$  and  $b$ ) and considering the two elements operated together ( $a*b$ ). Further, he generates two equations that reflect  $a$  and  $b$  satisfying the definition of  $H$  ( $a*h*a^{-1} = h$  and  $b*h*b^{-1} = h$ ), drawing on the “letter” *form* of inverse. Tucker states that he thinks he should try to “rearrange” the two equations to produce an equation showing  $a*b$  satisfies the definition of  $H$ . At this point, he seems unsure of how he might carry out such a rearrangement. Tucker describes a strategy in which he applies transitivity to set parts of the two equations equal to each other ( $a*h*a^{-1} = b*h*b^{-1}$ ), this is similar to, although distinct from, the “transitive” *function* of identity because the element  $h$  is replaced in the equations. Tucker then describes “adding” things to each side of the equation, but hesitates to carry this activity through, eventually deciding, along with the interviewer, to move on to prove that  $H$  contains inverses.

Tucker begins by fixing the element  $a$  in  $H$  and re-writing the equations  $a*h*a^{-1} = h$  and  $a*a^{-1} = a^{-1}*a = e$ . He then says, “Now comes the point of realizing, okay, well, what's the

identity in this. Hm. If it exists. If the identity exists. Um, hm. It seems to me like the identity might be something like one” (lines 800-803), later explaining that the group looks “multiplicative in nature, because of the existence of the negative one power” (lines 808-809). During the interim between these quotes, Tucker substitutes the number five for  $g$  and (multiplicative) “number” *form* of inverse one-fifth for  $g^{-1}$  and simplifies the equation as though he were multiplying real numbers. From this, Tucker spends three minutes discussing whether  $g$  inverse exists, eventually reasoning, “you would have an inverse just because  $g$  is in part of the group” (line 817). This prompts an exchange between Tucker and the interviewer in which the interviewer says, “So, real quick,  $H$  is all the elements  $g$  in  $G$  [Mhmm.] such that  $g$  star  $h$  [Right.] star  $g$  inverse equals  $h$ ” (lines 840-841) Tucker responds, “Right. So, that's almost, like, limiting some of them, right? [Okay.] It's just- It's only the elements in  $G$  that- that work like this, type thing” (lines 844-845).

This seems to shift Tucker’s focus from considering whether an inverse exists to determining whether  $a$  inverse satisfies the relationship  $g * h * g^{-1} = h$ , which Tucker says, “If  $a$  inverse is in  $H$ , that means that  $a$  inverse  $h$   $a$  - inverse of an inverse being  $a$  - is also equal to  $h$ ” (lines 857-858), writing the equations  $a^{-1} * h * a = h$  and  $a^{-1} * h * (a^{-1})^{-1} = h$ . Tucker’s statement subtly supports an “inverse-inverse” *function* of inverse in which the inverse of the inverse of an element is the original element. As Tucker explains, “An inverse of an inverse is just gonna be the element” (line 877). This is also reflected in the difference between the two equations that Tucker writes in which  $a$  and  $(a^{-1})^{-1}$  fill the same position.

Tucker returns to discussing for several minutes whether 1 is the identity of  $G$ . In this discussion, he uses  $e$  and 1 as representational vehicles for the identity. The “number” and “letter” *forms* seem to each serve an “operate/same out” *function* of identity, although 1 seems to

be distinguished from  $e$  in that 1 is used with multiplication and  $e$  is more general, as though the operation is unknown. Again, Tucker replaces the letter  $g$  in the equation used to define  $H$  with the number 5 and replaces  $g^{-1}$  with the number  $1/5$ . He points out that “five times anything time one-fifth will give you that number back.” Formally, Tucker is replacing  $(G, *)$  with real numbers under multiplication and subtly commuting the elements so that 5 and  $1/5$  cancel each other out and leave the “number”  $h$ . As he states, this leads him to think that 1 is the identity element in  $G$  since the notation used to define the group “sounds like it’s multiplication.” However, Tucker repeatedly shifts between the two *forms* of identity (1 and  $e$ ) in an attempt to contextualize the situation. Further, he seems to feel that it might not be appropriate to assume that 1 is the identity element. However, this seems to serve to afford Tucker insight into how the proof might be approached.

Through an involved discussion in which the interviewer prompts Tucker, he is able to explicitly shift from phrasing his proof goal as “the identity of  $H$  exists” to saying “if  $e$  is the identity in group  $G$ , then  $e$  must be the identity in  $H$  for  $H$  to be a subgroup.” He then rephrases the second statement as “if the identity is in  $H$  or not.” Though subtle, this difference is nontrivial. Specifically, when asked, Tucker is able determine whether the definition of  $H$  is satisfied. This seems a much less daunting task than showing that an identity element exists. Tucker is able to determine the result of  $e*h*e^{-1}$  by drawing on the “operate/same out” *function* of identity as well as describing the inverse of the identity as the identity, which supports an “identity as self-inverse” *function* of identity contextualized in this instance by replacing the symbol  $e^{-1}$  with the “letter” *form* of identity,  $e$ . Interestingly, Tucker is unable to determine the result of  $1*h*1$ , even though he is taking 1 to be the identity of  $G$  because, as he says, he thinks  $G$  is “multiplicative in nature.” However, having decided to represent the identity of  $G$  with the

number 1, Tucker is still unsure what the result of  $1 * h$  would be. This supports a sense that Tucker is contextualizing the elements of the group  $G$  in different ways. He uses 1 as a representational vehicle for the identity of  $G$ , but 1 can not serve the *function* of identity for Tucker.

Another aspect of this section that it seems that Tucker is not trying to determine whether the identity satisfies the definition of  $H$ . This reflects a sense that Tucker is not considering whether the identity of the group  $G$  is an element of  $H$ , but rather he consistently refers to determining whether the identity of  $H$  “exists, which is indicated by his reaction to the interviewer’s question, “Is  $e$  an element of  $G$  such that  $e \star h \star e \text{ inverse equals } h$ ?” when Tucker asks, “So, that's -that's how you prove that it was in  $H$ ? Is just by, like, looking- Cause we know what it does to little  $h$ ? (line 1057-1058)” This exchange supports the sense that Tucker is initially unsure about how to show that an identity element exists in  $H$  – his explicit goal throughout this activity.

This exchange also seems to bolster Tucker’s activity moving forward to determine whether  $H$  contains the inverses of its elements. Tucker is able to articulate a clear goal for determining this – “Want to show.  $g \text{ inverse star } h \text{ star } g \text{ is equal to } h \dots$  Cause that's what you do when you put in the  $g \text{ inverse}$ ” (lines 1086-1089). Tucker then assumes that  $g * h * g^{-1} = h$  and articulates a goal, saying, “So, we wanna get to there. and I guess you can m- you can kind of like do things to try and get that right side to look like that.” Tucker then says,

So, applying  $g \text{ inverse}$  to both sides would give you  $h \text{ star } g \text{ inverse is equal to } g \text{ inverse star } h$ . And then on the- and then next, you just apply  $g$  to that side. Um, you know, the right side of both these. So, uh, if you apply  $g$  to the right side of both of these, you are just left with  $h \text{ is equal to } g \text{ inverse star } h \text{ star } g$ . Which is what we got right here. (lines 1091-1094)

Tucker describes “applying”  $g$  inverse to the left-end of each side of the equation. In doing so, he re-writes the equation “ $g*h*g^{-1} = h$ ” as “ $h*g^{-1} = g^{-1}*h$ .” He follows this by describing “applying”  $g$  to the right-end of each side of the equation, resulting in the equation “ $h = g^{-1}*h*g$ ,” which Tucker had identified as his goal for this part of the proof. This activity supports a “vanishing” *function* of inverse in which an element and its inverse are described as being operated together and are removed from the algebraic statement.

Tucker goes on to recall that he never completed his proof that the set  $H$  satisfies closure, stating that he is still unsure of how he might go about proving this. After a short exchange in which Tucker and the interviewer discuss the goal of the proof that Tucker previously outlined (rearranging the equation  $(a*b)*h*(a*b^{-1}) = h$  to produce a try equation), the interviewer asks what “what tools” Tucker has that he could use to do that and suggests perhaps using associativity or trying to “un-group stuff” after Tucker mentions inverses and identity. This likely informs Tucker’s approach to the proof by hinting at a helpful strategy that he might use. Tucker continues by rewriting the equation as  $a*b*h*b^{-1}*a^{-1} = h$ , saying, “Well, also, um, if we wanna kind of get rid of the brackets here,  $a$  star  $b$  inverse in brackets if we wanna get rid of those, would be  $b$  inverse star  $a$  inverse” (lines 1166-1167). This supports the formation of an “inverse of a product” *function* of inverse, in which the inverse of two elements operated together is those elements’ inverses operated together in the reverse order as the original elements. Tucker continues by drawing on the “end-operating” and “vanishing” *functions* of inverse several times to eventually produce the equation  $b*h*b^{-1} = a^{-1}*h*a$ . Tucker then draws on his earlier proof that  $H$  contains inverses to point out that the right-hand side of this equation is true, declaring that this means the set  $H$  does satisfy closure.

**Interview 2 analysis summary.** Throughout his discussion in Interview 2, Tucker draws much more heavily than during Interview 1 on *functions* of inverse and identity that reflect a distinction between elements and operations, though there is still some indication that he thinks about the concepts as input/output relations. Tucker's responses also reflect a heavier reliance on *forms* of identity and inverse that are more abstract and generic than his responses during the first interview. This is reflected by his use of "letter" and "symbol" *forms* of identity and inverse, for instance, when defining identity and inverse in response to Q1 and Q2 and group in response to Q3. Tucker also draws on *functions* that afford manipulation of algebraic statements (e.g., the "inverse-inverse," "vanishing," "cancelling," and "inverse of a product" *functions* of inverse and "transitive" *function* of identity) as well as the "matching" and "finding inverse" *functions* of identity that draw on the organization of the "group table." There is evidence, though, that Tucker is not completely comfortable working with abstract groups, for instance, when, in response to Q7, he attempts to use "1" as the identity of the group because he thought the group seems "multiplicative in nature" and was able to interpret " $1 * h * 1$ " more readily than " $e * h * e$ ."

**Interview 3.** Analysis for Interview 3 draws on questions 1-6 from the protocol (Appendix C). Tucker's responses during Interview 3 draw much more heavily on the "operate/same out" *function* of identity and "operate/identity out" *function* of inverse than during Interviews 1 and 2. This reflects a sense that Tucker is focused on thinking of identity and inverse as elements contextualized by an operation, rather than as input/output relations, although he still describes them as such early in the interview. Tucker's activity also supports the development of two new *forms* of identity ("even" and "exponential"), an "even/odd" *form* of inverse, and an "exponent reducing" *function* of inverse. Tucker's discussion throughout the interview draws much more heavily on "letter" and "symbol" *forms* of identity and inverse than

in the first two interviews, relying on other *forms* to provide examples and test conjectures during Interview 3. Tucker also provides two proofs during which he draws “letter” *forms* in a “group table” organization.

***Describing identity – Q1.*** Tucker begins his response to Q1 by describing identity, as a “zero operation” that “sort of maps something to itself” (line 6). He continues, saying, “It could look differently, depending on what set you're in or, what kind of group it is, but, generally, it's gonna take one element and just put it back to itself” (lines 7-9). These descriptions are consistent with the “same in/out” *functions* of identity that centers around the identity acting on an element. When asked for a definition of identity (Q1b), Tucker fixes an element (“*b*”) in a set *G* and draws on the “letter” *form* of identity saying, “I'll make the identity be *e*- *e* in *G* look like *eb* equals to *be* equals *b*. So, basically, *e* doesn't really- *e* just takes *b* and puts it back to itself and the order doesn't really matter for that” (lines 15-17). This definition aligns more closely with the “operate/same out” *function* of identity, although the “letter” *forms* are concatenated (“*eb*” and “*be*”) with no reference to an explicit operation. With the second sentence in the excerpt, Tucker adds to his definition using language indicative of the “same in/out” *function* of identity ascribing the verbs “takes” and “puts” to the letter “*e*.” He also adds that order doesn't matter, which is consistent with his definition of identity in Interview 2. Tucker then provides examples of identity in response to Q1c, saying,

Sure! For instance in, um, the group, I guess the integers under addition, the identity would be zero, because zero plus any number in the integers would give you any number back. And, also, if you wanna do multiplication as your operation, you would have one, because one times anything would be anything. Would give you that element back the same number. (lines 22-25)

These examples draw on additive and multiplicative “number” *forms* to serve the “operate/same out” *function* of identity as Tucker contextualizes each identity element relative to and separate



from the respective operation. He also provides an example drawing on the “symmetry” and (additive and multiplicative) “matrix” *forms* of identity to serve the “same in/out” and “operate/same out” *functions* of identity, respectively, which is also consistent with his discussion throughout the second interview.

***Describing inverse - Q2.*** In response to Q2, Tucker initially describes inverses as working in pairs. He then draws on the “letter” *form* of inverse (“ $a$  to the minus one,” inscribed as “ $a^{-1}$ ”) to serve the “operate/identity out” *function* of inverse, saying,

When you multiply  $a$  by  $a$  minus one (sic)- It doesn't have to be multiplication - it could be addition or anything else - would give you back the identity element in that group, set, what have you. So, I kind of think of it as, just, they work in pairs, cause the inverse of the inverse is that element, and they kinda like are, like, opposites, if you will. (lines 45-49)

Tucker initially contextualizes the inverse using the operation of multiplication, but includes a comment that this operation need not necessarily be multiplication, going on to describe the result of the product as the identity. He then reiterates that he views inverses as pairs, describing the “inverse-inverse” *function* of inverse again and calling inverses “opposites.” Tucker continues his general description of inverse by providing examples, drawing on the “number” *form* of inverse as well as an “even/odd” *form* of inverse to serve the “operate/identity out” *function* of inverse, both of which he contextualizes using the operation of addition. Tucker’s discussion of the “odd/even” *form* of inverse included a description of “even” as the identity, supporting an “even” *form* of identity.

Asked for a definition of inverse (Q2b), Tucker replies, “An inverse element - we'll call it  $i$  - to an element  $a$  works to, um, (clears throat) to be combined with  $a$  (3 seconds) and yield the identity” (lines 68-70). This definition aligns with the “get back” *function* of inverse, rather than the “operate/identity out” *function* of inverse, because Tucker does not contextualize the inverse

with respect to any operation, although he uses a passive verb tense “works to be combined with,” which might be alluding to an operation. When asked why he paused when generating his definition, Tucker explains,

I was wondering if I should've said  $a$  inverse instead of  $i$ . Instead of, like, being confused of which, like- "Wait, should I have made that?" But, yeah. Or a better way to say works to be combined with, cause it could be different types of, um, operations. It could be addition or multiplication or any other weird operation that we can define. So, I was just trying to think of a more general term for saying that. Operated with- Combined with. (lines 76-81)

This clarification aligns Tucker’s more closely to the “operate/identity out” *function* of inverse, because he is contextualizing the identity relative to an operation, rather than ascribing the action to the identity. Tucker also briefly provides two examples using the (multiplicative) “number” *form* of inverse (“five and one-fifth”) and the “matrix” *form* of inverse (“some matrices have inverses,” lines 87-88). Because this discussion is brief, Tucker does not contextualize the examples except to say that five and one-fifth are inverses “under multiplication.” However, Tucker provided several examples during Interview 2 in which these *forms* served the “operate/identity out” *function* of inverse.

Tucker continues by generating an example of a group drawing on the “symbol” *form* of both identity and inverse (Figure 4.12). In doing so, he draws on the “matching” and “finding inverse” *functions* of identity within the context of the group under operation “smiley face.” This process is similar to the example he produces in response to Q4d in Interview 2. Also similar to that example, this is a non-example of a group, although it seems as though Tucker does not notice<sup>6</sup>.

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<sup>6</sup> Using “star” as the identity, Tucker fills in the main diagonal of the table with stars, assigning all nontrivial elements order two in a group of order five. By LaGrange’s theorem, this is not possible. It is apparent that this is a non-example of a group by noting the two triangles in the “square” column.

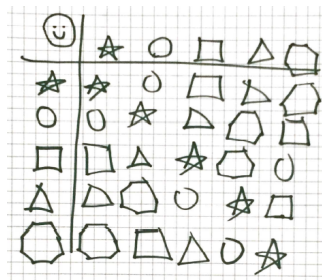


Figure 4.12. Tucker's example of a group

**Describing group – Q3.** Tucker's definition of group more closely aligns with the formal definition of group than did the definition he provided during Interview 2. Particularly, it includes all four group axioms:

...my definition for a group would be, it's a set under a certain operation - addition, multiplication, or some other weird thing - Smiley face for instance- that satisfies four things. Closure, identity, associativity, and inverses. And if those four things are satisfied, then the set is indeed a group (lines 206-209).

Further, when asked what each of the group axioms means, Tucker's responses related to identity and inverse each contextualized the concept relative to the group operation, reflecting the "operate/same out" *function* of identity and "operate/identity out" *function* of inverse, respectively. In response to Q3c, Tucker reiterates his description a group that draws on the "even" *form* of identity and the "even/odd" *form* of inverse, again contextualized with the addition operation.

**Mini-proofs – Q4.** Tucker begins his response to Q4a by rephrasing the statement, saying, "I guess what that is trying to say is that there's not gonna be two identity elements in a group that do the same thing" (lines 299-300). He follows this by stating that he agrees with the statement and asserting that he will prove the statement by contradiction and then draws on two "letter" *forms* of identity ("e" and "i") to suppose that two identities do exist. In doing this, he writes two lines (Figure 4.13a) that reflect his definition of identity, drawing on the "operate/same out" *function* of identity contextualized by the operation "\*" and conveying the

commutativity of the identity. Tucker then applies transitivity to create a new equation, saying, “Since these two things are equal, we can set them equal to each other. So, w- we're gonna have, um- Just do this.  $a \star e$  is equal to  $a \star i$ ” (lines 308-309; Figure 4.13b). From this equation, Tucker draws on the fact that  $G$  is a group to explain that  $a$ 's identity exists, going on to say,

...it's okay for us to- um, add  $a$ 's inverse or multiply  $a$ 's inverse to this on both sides, which would give us- (to self) so, starring  $a$  inverse, starring  $a$  inverse will give us, um, the identity element or  $e$  in this case.  $e \star e$  is equal to let's call this  $i$  star  $i$ . Which, if they are, that means that  $e$  is equal to  $i$ . So, they're the same element. (lines 310-314)

This excerpt coincides with the image in Figure 4.13b, specifically, Tucker's inclusion of “ $*a^{-1}$ ” under each side of the equation, which serves the “operate/identity out” *function* of inverse. This is indicated by Tucker replacing the “ $a$ ” on each side of the equation with the “letter” *form* of identity, “ $e$ ” in the last row of Figure 4.13b. Tucker then draws on the “operate/same out” *function* of both “letter” *forms* of identity (“ $e$ ” and “ $i$ ”) to produce the equation “ $e = i$ .”

$$\begin{aligned} e \star a &= a \star e = a. \\ i \star a &= a \star i = a \end{aligned}$$

(a)

$$\begin{aligned} a^{-1} \star a \star e &= a^{-1} \star a \star i \\ \star a^{-1} & \\ e \star e &= i \star i \quad \underline{e = i} \end{aligned}$$

(b)

Figure 4.13. Tucker's proof of the uniqueness of identity

Initially, in his response to Q4b, Tucker struggles to produce a proof that each element in a group has an inverse. The interviewer explains to Tucker that he is allowed to describe why he thinks this would be true, rather than developing a formal proof. Tucker then begins his description, drawing on the format of the “group table” organization:

...you know this is gonna be true logically, because if  $a$  didn't have an inverse, that means that there would be *no* identity in  $a$ 's column or  $a$ 's row. Which means that- that means- that means something has to appear twice in that row in order to actually fill up all the spaces. Which means that closure would actually fail, because, um. Or closure wouldn't fail, but the Sudoku property would fail, which we know can't happen for groups. (lines 369-373)

In his description, Tucker draws on the “Sudoku property” of the group table to justify why the identity must be in an element’s row or column. The “Sudoku property” (developed in class during the students’ progress in the TAAFU curriculum) is based on the notion that each element of the group should appear in every row and every column of the table exactly once. Tucker’s reasoning centers around whether the identity is in an element’s row or column. This draws on the “finding inverse” *function* of identity, in which the position of a *form* of identity in an element’s row or column indicates the column or row (respectively) of that element’s inverse. Tucker’s response to Q4bi supports this analysis as he explains, “...if one element does not have an inverse, that means that, you know, there's gonna be something missing. Cause uh- we know that there's an identity element...” (lines 395-396).

In response to Q4ciii, Tucker states that he agrees with the statement “every element in the group  $(G, *)$  is an inverse.” His brief explanation draws on the “inverse-inverse” *function* of inverse as he explains, “...as I said earlier, inverses kinda like work in pairs if you will. [Mhmm.] So, to say that  $a$  is an inverse, that would be true to  $a$ 's inverse. So, um, we kinda talked about this earlier. The inverse of  $a$ 's inverse is gonna be equal to  $a$ ” (lines 412-414). He follows this with a brief discussion of repeatedly taking the inverse of an element, describing it as “cyclic between those two elements. ...it's only gonna go back and forth between those two” (lines 416-417) and calling it “inverse inception” (line 419).

Beginning his response to Q4d, Tucker explains what it means for an element of a group to have order two, saying, “to be an order two, would be something that, when you do the operation to itself twice, it's gonna give you the identity” (lines 448-449). He then provides an example of order, drawing on the rotational symmetries of a triangle, and explaining that the order of the element R ( $120^\circ$  rotation of a triangle) is three. He continues by rephrasing what it

means for an element to have order two in terms of inverses, saying, “if an element has order two, that means that it's its own inverse” (line 457). Tucker then begins a proof of the statement by claiming that the given statement is true and creating a group table using letters (Figure 4.14a). He describes the first two elements he places in the table, “ $e$ ” as the identity and “ $a$ ,” as the element of order two, adding, “ $e$ 's its own inverse.  $e$  is order one” (line 465-466).

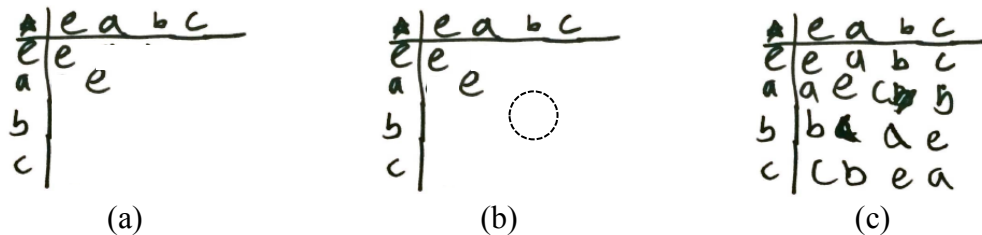


Figure 4.14. Tucker's group table in response to Q4d

After doing this, Tucker includes the letter “ $b$ ” in the table, explaining, “And there's *exactly one*, uh, element that looks like this. So, ther- So, basically,  $b$  times  $b$  will *not* equal  $e$  right here. [Okay.] It will equal some other thing. So, that means that there *has* to be- Actually, Oh hoo!” (469-469). Tucker seems genuinely surprised to realize that there will be an even number of elements in the group. He draws on the “finding inverse” *function* of identity to argue that the identity will not be in the entry corresponding to “ $b*b$ ,” pointing to that position in the group table (Figure 4.14b). Accordingly, Tucker begins to explain that there must be another element in that entry but seems to realize that his prior thinking was incorrect. He goes on to explain his reasoning, saying, “...after you've gotten these two [ $a$  and  $e$ ], there only has to be an even number of elements left. Because they can't be inverses with themselves. Otherwise, you would have just  $b$  times  $b$  would equal  $e$ . It has to be another element  $c$  which is  $b$ 's inverse” (lines 480-482). He then completes the table (Figure 4.14c) with a fourth element,  $c$ . Further clarifying this reasoning in response to 4dii, Tucker says,

...all the other ones need to be inverses with other elements, which means they need to pair up. Which means there'd need to be an even number. Cause if you

pair up, you have to have another buddy. Um, and inverses are unique, so you can't be inverses with two elements, if you will. So, knowing that there needs to be- like, they come in pairs, if you will, for the rest of them. (lines 491-494)

His use of parity, especially his “buddy” phrasing, reflects his discussion in Interview 2 that supported the development of the “inverse-inverse” *function* of inverse. It also echoes his earlier discussion in this interview when he described “inverse inception.”

In his response to Q4e, Tucker continues his use of the group table, in this case, to reason about the number of self-inverses in a group. He begins by drawing a table (Figure 4.15) and writing the letters “e” and “a” along the left and top of the table. He then draws on the “finding inverse” *function* of identity to write two “e’s” in the upper-left corner of the table (Figure 4.15a). Tucker follows this by including an element “b” in the top row, saying, “Now let's say for the sake that *b* is a self-inverse. Saying that there's three. And we're trying to prove that this is a group still” (lines 554-555). Tucker draws on the “finding inverse” *function* of identity to write an “e” in the entry in the *b* row and *b* column (Figure 4.15b) and uses the “matching” *function* of identity to write “a” and “b” in the row and column corresponding to *e* (Figure 4.15c).

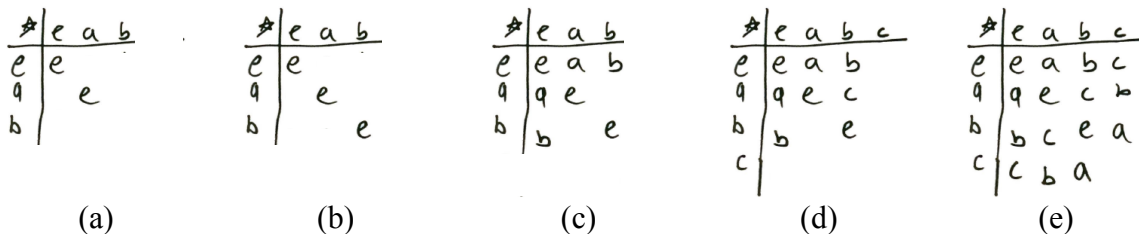


Figure 4.15. Tucker’s group table in response to Q4e

Tucker then says, “And now we have to also prove that- Actually, we can just keep it like this: *e a b*. That's three self-inverses and odd number. And we can prove that this is a group because (8 seconds) uh. (3 seconds) Actually, let's put one more in there” (lines 558-560). Following this, Tucker writes the letter “c” along the top and left columns and in the entry corresponding to the *a* row and *b* column (Figure 4.15d). He continues by filling in the table,

tacitly drawing on the “Sudoku rule,” until creating the table in Figure 4.15e. At this point he says, “It would force- it would force that to equal the identity, actually. Hmm. How to- Now how to prove this. So, basically, um. (8 seconds) You'd ha- it looks like you have to have an even number of self-inverses” (lines 564-566). He goes on to explain that, “As we were working through the problem, um, we realized that  $c$  has to be a self-inverse, because, in order for it to keep the Sudoku rule, um, you'd have to have  $e$  in this last- this last cell right here” (572-574). In this argument Tucker draws on the “finding inverse” *function* of identity to claim that “ $c$  has to be a self-inverse” (his emphasis, line 573).

***Subgroup proof: Normalizer of  $h$  – Q5.*** In his response to Q5, Tucker recognizes the prompt from the second interview but is unable to remember how he approached the proof before. During his initial discussion, Tucker describes how, when proving that a set is a subgroup, he might only need to prove some of the group axioms, saying, “I don't think we need to show identity because showing inverse kinda shows identity. If I'm not mistaken” (lines 626-627). He resolves though that, because he cannot remember whether both are necessary, he will attempt to prove both axioms. When asked which subgroup rule he might be able to prove, Tucker says, “Sure. I can prove that, if  $g$  looks like- if  $g$  works like this, (points to “ $g*h*g^{-1} = h$ ” in problem statement) then  $g$  inverse in  $G$  is going to look like the opposite of this. So it would be, like-” (lines 666-667). After saying this, Tucker writes the equation “ $g^{-1}*h*g = h$ .” After an eighteen-second pause, Tucker says, “I forget what to do, but I remember last time, (scoffs) I took this and I realized, okay, there's inverse elements in  $G$ , so we can solve for this  $h$  (points to “ $h$ ” in the statement “ $g*h*g^{-1}$ ”) right here and prove that it was also, like, the form of- the same form as that. I forget, though, to be honest” (lines 668-670). Asked why he thought the second equation was true, Tucker explains “left-end” operating with  $g$  inverse, saying, “so we can do is-



star  $g$  inverse on both sides on the left side. And we would have the identity- identity element. And I'm gonna just not write that because it would just be that. Star  $g$  inverse is equal to  $g$  inverse star  $h$ . So, well, that also proves that's commutative" (lines 681-683). This excerpt reflects the discussion during Interview 2 that supported the development of the "vanishing" *function* of inverse, except that Tucker is more explicit here regarding his reasoning behind the "vanishing" *function* of inverse, drawing on the "operate/identity out" *function* of inverse and the "operate/same out" *function* of identity to explain his activity. Tucker follows this by right-operating with  $g$  and drawing on the "vanishing" *function* of inverse to produce the equation " $h = g^{-1} * h * g$ ," saying, "So, I guess we just proved that, if the elements  $g$ - these elements are in  $H$ , that means the inverse of those are also in  $H$ " (lines 691-692). However, Tucker goes on to say that he is unsure how to prove that  $H$  is a subgroup of  $G$ , deciding, along with the interviewer, to move on to Q6.

**Order proofs – Q6.** Tucker begins his response to Q6a by rephrasing the statement, saying, "It's just kinda saying that, um, elements and their inverses have the same order... So, if you have an element  $g$  and its order is  $k$ , um, then its inverse element -  $g$  inverse - its order is also  $k$ " (lines 762-765). He then generates an example using the rotational symmetries of a square in order to test the statement, concluding that this example supports the conjecture. This is the first instance in which Tucker explicit describes two symmetries of any shape as inverses of each other, which supports the development of a "symmetry" *form* of inverse, in this case,  $R$  and  $3R$  as inverse rotational symmetries of a square. Tucker also inscribes the identity element of the symmetries of a square using the "letter" *form* of identity, inscribed as " $e$ " and the inscriptions " $R^4$ " and " $12R$ ," which are representational vehicles for the "symmetry" *form* of identity.

Tucker then alludes to his earlier discussion about how inverses work in pairs and writes the equation " $g \cdot g^{-1} = e$ ," saying, "we know that *that's* true" (line 781). Tucker follows this by writing the equation " $g^k = e$ " and then writing the equation " $g \cdot g^{-1} = e$ ." The latter equation seems to have been generated using the "transitive" *function* of identity. Tucker describes "apply[ing]  $g$ 's inverse to both sides" (lines 784-785) of this equation and writes the equation " $g^{-1} = g^{-1}g^k$ ." This reflects the "vanishing" *function* of inverse. After a twenty second pause, Tucker laughs and says, "I feel like I'm close, but I don't know how to get from here to-" (line 786), then writes " $(g^{-1})^k = e$ ."

Tucker explains that he is trying to prove that the given statement is true, but is having difficulty. After a long pause, Tucker says, "Yeah, I'm not sure. I just kinda wanna like- well I wanna like multiply this by  $g$  inverse  $k$  times" (lines 799-800). Asked if this was possible, Tucker responds,

So, if we times it by  $g$  inverse, times it by  $g$  inverse. That's essentially, like, taking away one of the  $g$ 's from here. So, it would be kinda like  $g$  to the  $k$  minus one. And this is just gonna equals  $g$  to-  $g$  inverse. And then, if we just keep on doing that, um, like  $k$  more- er like  $k$  more times, if you will, we would eventually come up with, like,  $g$  to the zero, which is like- Would that be the identity or would that not be? I'm trying to think if that would be or not. That- does that make sense?  
(lines 805-810)

Again, Tucker operates on both sides of the equation, although, rather than drawing on the "vanishing" *function* of inverse, he re-writes the statement " $g^k$ " as " $g^{k-1}$ ." Tucker also anticipates carrying out this action  $k$  times so that the result is  $g^0$ , although he is unsure whether  $g^0$  is the identity element. This is the first time during the interviews that Tucker has carried out exponent arithmetic using the "letter" *form* of inverse and supports the development of an "exponent reducing" *function* of inverse in which operating with an element's inverse might reduce the

exponent of the original algebraic statement (here “ $g^k$ ”) by one each time the inverse is operated with the algebraic statement.

Tucker goes on to rephrase how he imagines this action occurring and describes the process that he is using in the proof, saying,

So, essentially, if you have  $k$  number of  $g$ 's, you pretty much like just taking away one taking away one taking away one. You know what I mean? So- Cause taking away would be the same like- Oh! Okay, so on the very last step you would have like  $g$  to the one is equal to so- Before this. Before this right here. Before this right here, you'd have  $g$  to the one is equal to- uh,  $g$  inverse to the  $k$  minus one right here. And then you do that one more time- star  $g$  inverse and you'd have  $e$  is equal to  $g$  inverse to the  $k$ . (lines 816-821)

Tucker explains his reasoning behind his subtraction of exponents, describing it as “taking away one” (line 817). He then describes the last step in this process as “ $g$  to the one is equal to- uh,  $g$  inverse to the  $k$  minus one” (line 820, his emphasis) and writes the equation “ $g = (g^{-1})^{k-1}$ ”. This allows Tucker to end-operate with the inverse of  $g$  and, thus, draw on the “operate/identity out” and “exponent reducing” *functions* of inverse to produce the equation “ $e = (g^{-1})^k$ ,” saying, “you do that one more time- star  $g$  inverse and you'd have  $e$  is equal to  $g$  inverse to the  $k$ ” (lines 820-821). When asked whether  $g^0$  is the identity, Tucker says, “Right, right. Cause I was thinking well I have  $g$  to the zero. What does that mean? And I wanted to show that, well, that's just equal to  $g$  star  $g$  inverse” (lines 850-851). He goes on to say, “Instead of writing  $g$  to the zero, I- I think it's better to write it as  $g$  star  $g$  inverse is equal to the identity. [Okay.] But, that would actually be the identity” (lines 855-857). This supports a new “exponential” *form* of identity in which an element raised to the zero power might be viewed as equivalent to the identity element.

However, when asked if  $g^0$  is the identity element, Tucker responds, “I think it's better to write it as  $g$  star  $g$  inverse is equal to the identity. But, that would actually be the identity (lines 856-857), which indicates he does not prefer this representation.

Tucker begins his response to Q6b by writing the equation " $g^p = g^k$ ." He then says, "So, this can really look like  $g$  to the  $p$  plus one is equal to  $g$  to the  $k$ , which means that  $p$  plus one is equal to  $k$  or  $p$  is equal to  $k$  minus one" (lines 1006-1008). He then hesitates, adding, "Okay. I think I made like an assumption back there. [What's that?] My assumption was that you could write it as  $g$  to the  $p$  times  $g$  is equal to  $e$ . I was kinda assuming that. I let  $g$  to the  $p$  equal  $g^p$  inverse." (lines 1009-1011). Tucker's discussion implies that his initial equation assumes  $g^p$  is the inverse of  $g$  and, so, may serve the "operate/identity out" *function* of inverse, producing an equation that could then allow Tucker to draw on the "transitive" *function* of identity. From Tucker's argument, he then concluded that the exponents of  $g$  on either side of the equation must also be equal. This is confirmed by Tucker's following explanation as he says,

In one straight line, you'd have, like, okay. We know for  $g$ -  $g$ 's inverse times by  $g$  would equal the identity. [Mhmm.] Okay and so we're saying okay well let's assume that we can write  $g$  inverse as  $g$  to the  $p$  power. Um, that would mean that  $g$  to the  $p$  power dotted with  $g$  is equal to  $e$ . And we already know- we already let  $g$  to the  $k$  equal  $e$ . So that means that  $g$  to the  $k$  is equal to  $g$  to the  $p$  times  $g$ . So, we can set these things two equal to each other. So  $g$  to the  $p$  times  $g$ . That we can re-write- re-write as  $g$  to the  $p$  plus one. Because you have  $g$  to the  $p$ th power plus another  $g$  or times another  $g$ . So you just add another one to that. [Okay.] Is equal to  $g$  to the  $k$ . Which would- which would then say that okay, well, if  $g$  to the  $p$  plus one is equal to  $g$  to the  $k$ , that means that  $p$  plus one must equal  $k$ . And  $p$  would just be  $k$  minus one. (lines 1020-1028)

***Interview 3 analysis summary.*** Tucker's responses during Interview 3 reflect an ability to think much more abstractly than he did in the first and second interviews. Tucker draws on "letter" and "symbol" *forms* of identity and inverse much more heavily and also relies on the definitions to explain his reasoning more often than in the first two interviews. Tucker also refers to an operation more frequently throughout interview three than in the first two interviews. This supports a sense that Tucker regards the context in which he is considering the inverse and identity when manipulating algebraic statements and equations. For instance, Tucker often uses

the verbs “multiply,” “add,” “apply,” and “star” when describing acting with elements. However, it is often the case that he uses these interchangeably, which might indicate that the name of operation itself is as not important as the notion that he is operating on elements, although he often corrects himself after switching operations.

Tucker’s proofs in response to Q6 afford insight into a new “exponent reducing” *function* of inverse and “exponential” *form* of identity. Specifically, Tucker is able to reduce an element’s exponent by operating with the element’s inverse as well as interpret the result of this action if it is carried out several times. For instance, Tucker determined that operating on  $g^k$   $k$  times would result in  $g^0$ . Although this reflects the standard exponent rules for multiplication with real numbers and so, might seem familiar to Tucker, he describes operating individual factors of  $g$  and  $g^{-1}$  in order to reduce the power to zero. Further, although he is initially unable to interpret  $g^0$  in terms of a group element, he anticipates the result of operating with  $g^{-1}$   $k$  times and reverses the last operation in order to imagine carrying it out, invoking his definition of inverse to verify that the result of the broader process is indeed the identity.

***Summary of the form/function analysis Tucker’s across interviews.*** As should be expected, Tucker’s discussion and use of identity and inverse changes drastically across the three interviews. Specifically, Tucker draws more heavily on *functions* that treated inverse and identity as an input/output relationship during the first interview than in the second and more in the second than the third. Similarly, Tucker’s reliance on “number,” “function,” and “matrix” *forms* diminishes from interview to interview while his use of “letter” and “symbol” *forms* increases. The *functions* coded in the later interviews tend to rely on the functions coded in earlier interviews. For instance, in describing what is coded as the “vanishing” *function* of inverse, Tucker relies on the “operate/identity out” *function* of inverse and “operate/same out” *function* of

identity. Similarly, the “finding inverse” *function* of identity relies on the “operate/identity out” *function* of inverse.

Interestingly, Tucker provides a proof in Interview 2 that he is unable to reproduce in Interview 3, although his manipulation of the given equation reflects much of the same (and perhaps more sophisticated) activity as that in Interview 2. This might be partially due to the time constraints of Interview 3 (the “normalizer of  $h$ ” prompt was in the middle of the protocol) relative to those of Interview 2 (the prompt was near the end of the protocol). However, there is also evidence that, in both interviews, Tucker might not have a clear understanding of the definition of  $H$  and the set inclusion that would afford him the ability to explicate a goal within each subproof. This ability is supported through his dialogue with the interviewer more so during the second interview than the third.

Additionally, the *functions* of identity and *inverse* upon which Tucker draws often depend on the *form* of identity and inverse with which Tucker is working as well as the context of his work. For instance, Tucker draws on the “end-operating,” “vanishing,” “cancelling,” “inverse-inverse,” and “inverse as a product” *functions* of inverse when manipulating equations that incorporate “number” and “letter” *forms* of identity and inverse. Similarly, the “matching” and “finding inverse” *functions* of identity apply exclusively when Tucker organizes “letter,” “symbol,” and “even” *forms* of identity in a group table. This affords Tucker the ability to reason differently about identity and inverse in different contexts. For instance, Tucker’s production of group tables during his proof activity seems to afford insight into the general structure of groups that satisfy given properties by relying on the Sudoku rule to deductively generate relationships among elements in a group. On the other hand, the *functions* upon which Tucker draws while

manipulating equations afford Tucker the ability to develop arguments about specific elements and sets.

## Form/Function Analysis of John’s Individual Interviews

John participated in all three of the individual interviews. His interviews were chosen for analysis because he worked closely with Tucker and Violet throughout the semester, he was very outspoken in class, and he completed responses to the majority of questions during each interview as well as providing additional proofs that were not in the protocol. 24 *form/function* codes for identity and inverse were developed over the analysis of the three interviews (Table 4.6). Some codes (e.g., “equals” and “letter”) were much more prevalent across the three interviews than others (e.g., “switching variables” and “graph”). Throughout the interviews, John’s activity tended to center around “letter” *forms* serving “equals” *functions* for both identity and inverse.

Table 4.6 – Codes from the form/function analysis of John’s individual interviews

Identity		Inverse	
<i>Form</i>	<i>Function</i>	<i>Form</i>	<i>Function</i>
Letter <sup>1</sup>	Doing nothing <sup>1</sup>	Letter <sup>1</sup>	Equals <sup>1</sup>
Number <sup>1</sup>	Equals <sup>1</sup>	Number <sup>1</sup>	Get identity <sup>1</sup>
Symmetry <sup>3</sup>	Identity as self-inverse <sup>2</sup>	Graph <sup>1</sup>	Switching variables <sup>1</sup>
	Transitive <sup>2</sup>	Paired functions <sup>1</sup>	Sided <sup>2</sup>
	Vanishing <sup>2</sup>	Symmetry <sup>2</sup>	Self-inverse <sup>2</sup>
	Replacement <sup>3</sup>		Cancelling <sup>2</sup>
	Sided <sup>3</sup>		End-operating <sup>2</sup>
			Vanishing <sup>2</sup>
			Inverse-inverse <sup>2</sup>
			Inverse of a product <sup>2</sup>

Superscripts indicate the Interview during which evidence of each *form/function* was first coded

**Interview 1.** During interview 1, John responded to questions 1-6 from the protocol. Of these prompts, his responses to questions 1, 2, and 6 contribute meaningfully to the analysis of his conceptual understanding of identity and inverse. During John’s responses to the Likert questions (Q3), he reiterated his previous discussion of the concepts so that these responses leant support for the existing *form/function* categories, but did not extend them. Also, John’s responses to Q4 and Q5 do not substantially address identity and inverse and, so, are not analyzed in this



section. Analysis of this interview supports the development of eleven *form/function* codes: “letter” and “number” *forms* of identity and inverse; “graph” and “paired functions” *forms* of inverse; “doing nothing” and “equals” *functions* of identity; and “equals,” “get identity,” and “switching variables” *functions* of inverse.

***Describing identity – Q1.*** John begins his response to Q1 by saying, “Identity? As nothing, almost. [Okay.] It’s kind of just like, when you add zero or multiply by one, you’re not doing anything” (lines 4-5). The notion of “not doing anything” supports an emerging “doing nothing” *function* of identity. John phrases each general description using a verb – “add,” “multiply,” and “(not) doing” – which supports a sense that John might be thinking of the identity as an action. He then generally conveys that he feels the identity is not very helpful for describing things mathematically and says, “it doesn’t seem like anything” (lines 9-10). Asked to elaborate, John goes on to describe identity as a “pattern,” referring to the multiplication tests he and his sister took in elementary school and likening this to the group tables for the symmetries of a triangle that he and his classmates produced in class. However, John does not specifically describe which symmetry he views in a way that is consistent with serving a “doing nothing” *function* of identity.

John defines identity by stating, “it doesn’t do anything to the thing” (line 28). He then uses the letters  $x$  and  $y$  as representational vehicles for the additive and multiplicative identities, respectively. John draws a distinction between the two types of identities as he describes the *functions* of  $x$  and  $y$ : “ $a$  plus  $x$  equals  $a$ ,” and “ $a$  times  $y$  equals  $a$ ” (lines 29-30). He adds that, “the whole point of it is it doesn’t do anything” (lines 30-31). These representational vehicles correspond to the representational object of identity through the pattern in Figure 4.16. We can view this correspondence of each of the vehicles as constituting a “letter” *form* of identity that

seems to reflect the “doing nothing” *function* to the letter *a* because it is preserved to the right of the word “equals.” Because the “doing nothing” *function* of identity is contextualized in equation form, drawing on a specific operation in each case, John’s discussion supports a more specific “equals” *function* of identity in which specific *forms* of identity (here, the “letter” *form*) are operated with another element on one side of an equation whereas the other side of the equation is comprised of that same element (Figure 4.16).

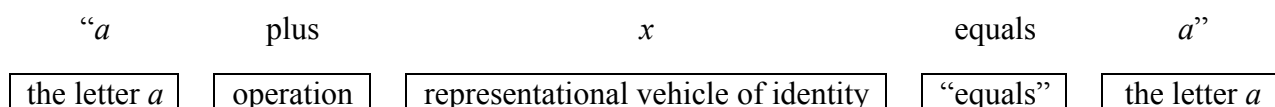


Figure 4.16. Graphical depiction of the “equals” *function* of identity

John goes on to describe identity “more so in symbols, than in words,” saying, “if, in a given number system, if *x* is the additive identi- like, if we use like the real number, zero, is the additive identity, so *a* plus zero equals *a* and one times *a* equals *a*. Just, whatever, given number in a system makes those true, if that makes sense” (lines 39-42). In this excerpt, John again draws on zero and one as representational vehicles for the representational object of identity. These vehicles correspond to *x* and *y* from John’s previous examples, respectively, constituting “number” *forms* of identity (zero and one) that serve the “equals” *function* of identity in the contexts of addition and multiplication, respectively. This stands to reason, because John is very familiar with the additive and multiplicative properties of zero and one, which is supported by his description of his prior experiences. John is generally defining identity in two contexts: additive and multiplicative. This reflects the familiarity of standard operations with real numbers. He concludes this response with the statement “*If* there is even an additive and multiplicative identity, that would be my definition of it” (lines 43-44, his emphasis), introducing a sense that an identity might not exist. Perhaps this is dependent on the “given number system” to which John alludes.

John then describes broader circumstances in which “you do something to something but it doesn’t change it” (line 58), drawing on his philosophical notions of humans’ relationship with the earth on a broader timeline and individuals’ affect on the world within their lifetimes, which reflects the “doing nothing” *function* of identity. John also describes an additive identity matrix as a matrix of all zeros and alludes to a multiplicative identity matrix. However, he does not provide enough information in order to gain insight into how either might correspond to identity or serve the “doing nothing” *function* of identity. Notice that, throughout his discussion, John explicitly contextualizes identity within addition and multiplication. This supports a sense that John is aware that identities might change depending on an operation. Addition and multiplication serve as a facet of the correspondence between representational vehicles and the representational object of identity to constitute the various *forms* of identity.

***Describing inverse – Q2.*** When asked to generally describe the concept of inverse, John says, “Cancellation? It makes nothing” (line 107). He then provides the first example of inverse, writing “ $a + x = 0$ ,” and goes on to question whether a multiplicative inverse exists. John then quickly recants his initial response, saying, “Maybe I just don’t know what inver- Oooh! I just misformed what inverse meant. I’m just not sure what inverse means. [Okay.] Um, it’s one over  $x$  isn’t it? Is the inverse of  $x$ . Gotcha. Okay, that changes things” (lines 113-115). He then contrasts “minus  $x$ , like whatever it is to equal zero when you’re adding” with “one over  $x$ ,” concluding that “one over  $x$  is the inverse of  $x$ ” (lines 120-122). This wavering response reflects a sense that John might not be as sure in how he thinks about inverse as how he thinks about identity, which he contextualized with both addition and multiplication. John’s follow-up response supports the notion that he might be aware of different types of inverses, but is uncertain, saying, “Like, I don’t- additive inverse versus multiplicative inverse- Like, I could be

wrong. Yeah, I'm probably wrong, like I'd go and Google it, like- [Okay] like, at this point in the conversation" (lines 129-131).

John then describes what he means by "additive inverse" and "multiplicative inverse," saying, "for any  $a$ , there is an  $x$  that  $a$  plus  $x$  equals zero. So, like, the  $x$  would be the additive inverse of  $a$  in that given number system. And, I guess, for every  $a$  there's an  $x$  or  $y$  that equals one when you multiply it, so it equals negative  $a$  in this and one over  $a$  in this" (lines 141-144). These descriptions reflect the generally accepted definitions rather closely. This excerpt provides two potential representational vehicles that might correspond with inverse: "negative  $a$ " and "one over  $a$ ," which he also represents with the letters  $x$  and  $y$ , respectively, which supports the development of a "letter" *form* of inverse. Further, the equations upon which he draws follow a pattern that resembles John's "equals" *function* for identity, with the exception that the right-hand side of the equal sign is the corresponding identity under an operation, rather than the original letter or number. Accordingly, this supports what I call the "equals" *function* of inverse (Figure 4.17). However, it is unclear here whether John is drawing any correspondence between the *forms* of inverses and the *forms* of identity (although he later makes this explicit).

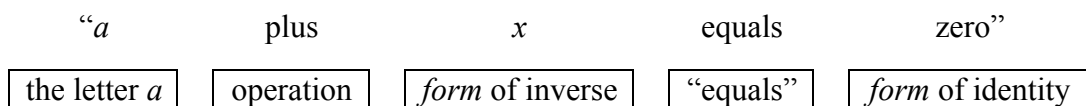


Figure 4.17. Graphical depiction of the "equals" *function* of inverse

Through his further elaboration, John sketches a general relationship between inverse and identity, saying, "Like, just thinking about- still cancelation, and getting back to your, almost, your identities, like as a way- yeah I guess, like, given in the real numbers, adding an inverse gets you back to the identity and multiplying by the inverse gets you back to the multiplicative identity" (lines 155-158). This description reflects John's prior equations that led to the "equals" *function* of inverse. He elaborates on this relationship by defining inverse as, "the number you

add or multiply a value by to get to the additive or multiplicative identity, respectively” (line 165-166). These excerpts more clearly communicate John’s earlier, vaguer descriptions of inverse as “cancellation,” supporting the formation of a more general “get identity” *function* of inverse defined by a general sense of getting an identity, though not written as an equation. Again, John includes both additive and multiplicative types of inverses in his definition, which contradicts his initial uncertainty. John then explicitly draws the hypothesized relationship (from the previous paragraph) between his definition of inverse and his definition of identity, saying, “just kind of, it does- If we go back to the definition of identity, it’s nothing, basically. [Mhmm] And cancelation, like making it nothing” (lines 171-172).

John follows this by generating three examples of inverse in response to Q2b. The first of these examples is identical to the “letter” *form* he previously used when describing his changing notions of inverse:  $a$  plus  $x$  equals zero. With his second example, however, John says, “the inverse of root two is just like one over root two” (lines 185-186). Notice that, unlike in his previous discussion, John does not explicitly state what kind of inverse this is (multiplicative), though it does reflect his earlier statement that inverse means reciprocal. Further, this constitutes a “number” *form* of inverse, though John does not contextualize it by an operation in this instance. With his third example, John attempts to generate a pair of complex numbers that multiply to equal one, eventually declaring  $i$  and  $-i$  to be inverses. John says that his intention “was trying to think of like a something plus  $i$  times another something with something  $i$ ” (lines 188-189). Asked if he meant to add these numbers together, John says that he had intended to multiply them, continuing by saying, “Yeah.  $i$  times negative  $i$  equals one” (line 202), which draws on the “equals” *function* of inverse and supports his claim that the two complex numbers are inverses. Interestingly, John had chosen the only pair of complex numbers that are both

additive and multiplicative inverses. When recounting other times that John had learned about inverse, John lists trigonometric functions and describes tilting graphs  $90^\circ$ . John seems to be describing the graphs of inverse functions under composition, except that the geometric transformation he demonstrates is a  $90^\circ$  rotation, rather than a reflection over the graph of  $y = x$ . During this discussion, he draws a parabolic curve on unlabeled axes and rotates his paper  $90^\circ$  clockwise, mentioning  $\sqrt{x}$  and  $x^2$ . This supports the development of “graph” and “paired functions” *forms* of inverse, neither of which is contextualized by an operation. The “graph” *form* of inverse is constituted by a representational vehicle of a graph that corresponds to the representational object of inverse by being a  $90^\circ$  rotation of a graph of its inverse. The “paired functions” *form* of inverse is constituted by a representational vehicle of a function (in these cases, a polynomial or trigonometric function), which corresponds to the representational object of inverse by being paired with a second function. These pairs of functions reflect inverse functions under composition, although John does not mention or describe this, supporting a sense that he is recalling these pairs of functions as being called inverses from memory.

Asked to generate an example of something without an inverse, John states that he doesn't think that zero has a multiplicative inverse in the real number system, supporting this by pointing out that, “There's nothing you can multiply by zero to get one (line 230).” This example draws on the “get identity” *function* of inverse, specifically, that John views the number zero as an object that cannot be multiplied by any other number to get the multiplicative identity. John follows this example by describing how he checked to confirm that his previous example of inverse –  $\sqrt{x}$  and  $x^2$  – was indeed an example. In order to check this, John says, “I was just making sure that the root of  $x$  times  $x$  squared equaled  $x$ ” (lines 230-231). In doing so, he says, “that equals  $x$  to the five halves, which doesn't follow the rule” (line 237). John had anticipated

the product of the two functions to be  $x$ , which seems to be the “rule” to which he is referring. This is the first instance in which John uses the representational vehicle  $x$  to stand for the identity in the “equals” *function* of inverse – the right-hand side of the equation. Further, John is multiplying the two functions, rather than composing them, supporting a sense that this “paired functions” *form* of inverse might not align with the formal relationship under which these functions might be considered inverses (function composition). Accordingly, John seems surprised when the product of  $\sqrt{x}$  and  $x^2$  is not  $x$  as he had mentioned.

In dealing with this contradiction, John says, “sine to the negative one times sine equals one- wait, yeah, sine to the zero, which is one” (lines 237-238). Here, it seems that John is comparing two other functions that he believes are inverses. Again, John multiplies two functions, however, he interprets the product as “sine to the zero, which is one” (line 238). It is unclear whether this is the expected outcome, especially because he had earlier checked to see if the product of  $\sqrt{x}$  and  $x^2$  was  $x$ . Interestingly, one served as a multiplicative identity in previous examples. However, it seems to be the case that John might not see this as confirming that sine and  $\text{sine}^{-1}$  are inverses, because he begins graphing the sine function and using specific values on that graph to graph  $\text{sine}^{-1}$ . He then recalls several trigonometric functions and, with the help of the interviewer, pairs trigonometric functions and their inverses. John does not elaborate on the pairs of trigonometric functions, but instead graphs  $x^{-2}$  on the same graph he had used to graph  $x^2$ , saying, “With my definition, that’s what the inverse would be. But I don’t remember- I just actually don’t remember inverses very well” (line 254-255). With this statement, John directly addresses the inconsistencies between the “graph” and “paired functions” *forms* of inverse and the “equals” *function* of inverse he seems to anticipate the *forms* serving, attributing the inconsistencies to not remembering them well.

When asked to discuss his work during the inverse function discussion, John describes graphing  $x^2$  and  $x^{-2}$ , rotating the graph of  $x^2$ , and alludes to switching the  $x$  and  $y$  variables. Pertaining to the latter, John says, “you just switch  $y$  and  $x$  for that one. Like,  $x$  squared equals  $y$  and  $x$  equals  $y$  squared, which that means if you continue on that one,  $y$  equals root  $x$ ” (lines 271-272). This lends itself to the emergence a new “switching variables” *function* of inverses in which John switches the  $x$  and  $y$  variable in order to generate an inverse. He demonstrates this *function* using  $y = x^2$ , which he algebraically manipulates by switching the variables and solving for  $y$  in order to generate the equation  $y = x^2$ . This seems to confirm his earlier assertion that  $x^2$  and  $\sqrt{x}$  are inverses, though he states that he is less certain about his definition of inverse after his example generating activity. John continues by describing how he was comparing the graphs of  $x^2$  and  $x^{-2}$  to verify that they were inverses according to his previous definition.

This section of the interview concludes with John describing how he is now less certain about inverses in this context. It seems that this is because of the discrepancies between his earlier multiplicative definition and the examples of inverse function upon which he is drawing. Specifically, John does not coordinate the representational vehicles of functions to the representational object of identity through the operation of function composition. Rather the representational vehicle of the “graph” *form* of inverse corresponds to the representational object of inverse through  $90^\circ$  rotation. Similarly, the representational vehicles for the “paired functions” *form* of inverse correspond to the representational object of inverse through multiplication. Further, John does not describe any *form* of identity that relates to the “graph” *form* of inverse, which prevents any discussion that would lend insight into how this *form* might serve any of established *functions* of inverse. In contrast, John’s discussion reflects an attempt to make it so the “paired functions” *form* of inverse might serve the “equals” *function* of inverse. Specifically,



John multiplies functions with their inverse and describes wanting the product to be  $x$ . Here,  $x$  takes the place of the *form* of identity in the “equals” *function* of inverse, although John does not describe it as such. .

***Multiplicative identity of  $\mathbf{R}$  proof – Q6.*** In his response to Q6, which asks participants to prove that one is the multiplicative identity for the nonzero real numbers under multiplication, John begins by saying he has “no idea how to prove that” (lines 706-707) and does not complete a formal proof. In his response, he writes an equation ( $a \cdot x = a$ ) similar to the one he used to define identity (“ $a$  times  $y = a$ ”) using the “letter” *form* of identity to serve the “equals” *function* of identity. He then explains that his goal would be to prove that one is  $a$ , but he doesn’t know “how to go about proving it” (line 733) and rephrases this by substituting one for  $a$ , saying “I know that one times  $x$  equals  $x$ ” (line 734). This supports his earlier discussion (Q1) in which he drew on “number” *forms* of identity to serve the “equals” *function* of identity. John continues his response by saying, “I just have always *looked* at it as an axiom, I don’t know if it is one. Like, I haven’t ever thought about it at all. Like, I’ve always accepted it, which is what you generally tend to do with axioms” (lines 783-785). This supports a sense that the “number” *form* of identity 1, especially with respect to multiplication of real numbers, is deeply engrained as an identity.

***Interview 1 analysis summary.*** Throughout his discussion during Interview 1, John draws heavily on “number” and “letter” *forms* of identity and inverse. For the most part, these *forms* tend to serve their respective “equals” *functions* (Figure 4.16, Figure 4.17), which center around an equation structure that contains an element, an operation, an equals sign (or the word “equals” for verbal instances), and a *form* of identity. The structure of “equals” *function* of inverse also contains a *form* of inverse. However, during more general descriptions of the concepts, John’s language reflects the “do nothing” *function* of identity and the “get identity”

*function* of inverse. These *functions* reflect a conception of identity and inverse that closely aligns with the “equals” *functions* and it might be the case that John is colloquially using the words “do” and “get” to describe “operate (with)” and “equals.” However, the “do nothing” and “get identity” *functions* are distinguished from the “equals” *function* in that they do not explicitly refer to an operation that one “does” or uses to “get,” whereas, the “equals” *function* does. Three *form/function* codes were also developed during analysis of John’s discussion of inverse functions. However, throughout John’s discussion drawing on the “graph” *form* of inverse, there is no evidence of a *function* of inverse that it might serve except that rotating a graph  $90^\circ$  clockwise produces the graph of the inverse function. John’s discussion of the “paired functions” *form* of inverse draws on the “equals” *function* of inverse as he multiplies functions together to equal either one or  $x$  in different instances. Further, John describes a method of generating inverse functions that constitutes the “switching variables” *function* of inverse. Across the data supporting the development of all three of these *form/function* codes, John’s discussion seems tentative and inconsistent, which is evident from his statement, “I just actually don’t remember inverses very well” (line 255).

**Interview 2.** During interview 2, John responded to questions 1-7 from the protocol. All of these contribute meaningfully to the analysis of his conceptual understanding of identity and inverse except 4e, 4f, and 4h. Analysis of this interview supports the development of eleven new *form/function* codes: the “symmetry” *form* of inverse; “identity as self-inverse,” “transitive,” and “vanishing” *functions* of identity; and “sided,” “self-inverse,” “cancelling,” “end-operating,” “vanishing,” “inverse-inverse,” and “inverse of a product” *functions* of inverse. The majority of the new codes emerged through analysis of John’s proofs, whereas his broad descriptions of identity and inverse supported the codes developed during analysis of Interview 1.

**Describing identity – Q1.** John begins his response to Q1 by describing identity, saying, “an element in a set when added to any other element - or operated on with any other element- out- the outcome is that element ...  $e$  is the identity.  $a \cdot e$  equals  $a$ .  $e \cdot a$  equals  $a$ ” (lines 7-9). This description reflects the “equals” *function* served by identity in the first interview. However, John’s language includes the more general notion of “an element in a set” rather than his prior use of specific numbers and letters. John also changes his description mid-sentence to replace the words “added to” with the phrase “operated on with.” As with the generic noun “element in a set,” the verb “operate” decontextualizes the identity in his description, whereas his description in the first interview relied heavily on the contexts of addition and multiplication. John then says, “like, so,  $e$  is the identity.  $a \cdot e$  equals  $a$ .  $e \cdot a$  equals  $a$ ” (line 9) and writes This statement reflects the equation John used during the first interview that precipitated the “equals” *function* of identity. Specifically, the first equation explicitly reflects the format [the letter  $a$ ] [operation] [*form* of identity] [“equals”] [the letter  $a$ ]. The second equation switches the position of the *form* of identity and the letter  $a$  on the left-hand side of the equation. Further, John uses the word “dot” in place of the operation, which can also be viewed as more general than John’s responses in the first interview.

Asked to provide examples of identity (Q1b), John describes five examples:

John: This set is added- Like, the integers under addition it would be zero. Uh, the integers under multiplication is one. Uh, zero for the uh, symmetries of a triangle.

Int: Any others?

John: Sure. Uh. I don't know enough about matrixes (sic) other than just to say the additive one would be zero- the zero matrix of a given, like, (moves two pointed index fingers in a downward semicircle) thing under addition. Given matrix (inaudible) assume a matrix 1 gives for multiplicative- in square matrices at least. I'm not a hundred percent sure, though.

Int: What do you mean by "the matrix one"?

John: The- there all just ones. The matrix where they're just ones, but I don't think that works. I'm not sure. I don't know anything about matrices. (lines 38-53)

With each example, he does not explicitly support *why* these are examples of identity, focusing on the representational vehicle and its correspondence with the representational object for each example. For instance, he says, “I don't know enough about matrixes [sic] other than just to say the additive one would be zero- the zero matrix of a given, like, thing under addition.” With each example except the symmetries of a triangle, John contextualizes the identity by stating an operation (in this case, “under addition”) and a set (here, matrices). These examples reiterate the “number” *form* of identity (both multiplicative and additive) and introduce two new representational vehicles: “symmetries of a triangle” and “matrices” (both under addition and multiplication). With each of the new representational vehicles, John describes their correspondence to the representational object to varying degrees. Because of this, I do not yet code these as *forms* of identity.

Comparing these examples affords insight into how each representational vehicle corresponds with the representational object of identity in these cases. Specifically, there seems to be a need for the representational vehicle to correspond with a set and operation in order to constitute a *form* of identity. For instance, John provides the example, “the integers under addition, [the identity] would be zero.” First, John lists a set and operation, then states the representational vehicle for identity. Together, these three facets correspond to constitute a “number” *form* of identity. In other words, John seems to require that the representational vehicle have a context under which it may take the *form* of identity. However, it seems that all three facets of the correspondence are not necessary. For instance, John’s example of “zero for the, uh, symmetries of a triangle” (line 39), does not provide an operation (formally, composition). It might be the case that John assumes the operation is implied with the interviewer, based on the interviewer’s presence during class, when the symmetries were developed.

*Describing inverse – Q2.* In response to Q2a, John says, “Um, as an element you can add t- the element you add to another element to return to the identity. [Okay.] Um. [To return to the identity...] Yeah. Get to the identity. I don't know if it's retur- we return” (lines 60-62). As with his description of identity, John uses the word “element” to generally describe inverse. He also uses the verb “add” to describe the operation under which an element might be an inverse and the verb “return” and phrase “to get to” to describe the result of the addition. This definition is consistent with the “get identity” *function* of inverse coded during analysis of the first interview. He rephrases “add” to “operate on,” which seems to generalize the context of this broader description.

While elaborating on his description of inverse, John says, “It's doesn't matter which side you're on- is on regar-, like, but no- yeah, that's wrong. Never mind it's not- left or right. Like, in some sets you can have different identities if you're doing it to the left or the right side of an element” (lines 73-75). This excerpt introduces an important aspect of John's conception of inverses: that left- and right- inverses might be different elements. John is introducing a contextual distinction in which a *form* of inverse might serve a *function* of inverse in one context, but not in another. Specifically, drawing on the “equals” *function* of inverse to interpret his description, he is asserting that it might be the case that [*form* of inverse] [operation] [element] [“equals”] [corresponding *form* of identity] is true, whereas [element] [operation] [*form* of inverse] [“equals”] [corresponding *form* of identity] is not true. This contrasts with his definition of identity, which John defined with two equations, one with a *form* of identity on the left side of the element, the other on the right. Formally, John has described identity as being commutative and inverse as not being commutative, though John cannot recall the word “commutative” in this moment. Further, it should be noted that John's assertion is formally invalid in the context of

groups, as he demonstrates later in the interview during his response to Q7. However, in this moment, John's description supports the formation of a "sided" *function* of inverse, in which the left- and right- inverse of an element are not necessarily the same element.

Further elaborating on his description of inverse, John explains that he believes the inverse depends on the notation that he is using. This provides a correspondence between different representational vehicles and the representational object of inverse as depending on the notation. This aligns with and supports the analysis of the correspondence of the representational vehicle with the representational object of identity. When asked to explain his description of left- and right- inverses, John discusses checking to see if the symmetries of a triangle would provide an example of what he was trying to describe, saying, "seeing if that worked, but they're like, mostly self inverses or- [you've gotta R plus F-] and F plus R. [F plus R] Like, the flips are self inverses, so, that doesn't work. And then the rotations work either way, like the R and two R." John points out that the elements "F plus R," "R plus F," and "F" are self-inverses and that the rotations commute with their inverses as well. Implied in the first part of the argument is that self-inverses – elements that are their own inverse – commute with their inverse. This caveat seems self-evident to John. This discussion supports the development of a "self-inverse" *function* of inverse, in which the same *form* serves as both an element and its inverse. Because the symmetries of the triangle commute with their inverses, it seems that John views this as a non-example of what he was trying to describe. John adds, "I just don't know enough about sets," which indicates that he still feels an example exists, though he probably is unable to generate such an example. Further, this activity provides some insight into how symmetries of triangle constitute a "symmetry" *form* of inverse with which he writes letters as the symmetries and combines them with a plus sign. John uses the representational vehicles of letters to stand for

specific symmetries of triangle: R, F, 2R, F+R, and R+F. John acts as though it is given that R and 2R are inverses and that F, F+R, and R+F are self-inverses. However, John does not explicitly write or say anything to indicate how he knows these symmetries are inverses or the *function* that they might serve. Because the goal of this activity is to determine whether any of the symmetries do not commute with their inverses, John is using the symmetries of the triangle in comparison to their inverses. Because of this, the inverses are serving the specific goal of determining whether a given known pair commute with each other.

In response to Q2b, John provides three examples of inverse: “reciprocal” (real numbers under multiplication), “negative of the element” (in addition; real numbers, integers), and “each element in a group.” The first two examples are similar the “number” *forms* coded with John’s discussion of identity and consist of a reference to specific sets of numbers and an operation. The third example is less developed as John only mentions that the elements are in a group, which may or may not entail a set or operation related to these elements, as with the other examples. Further, John does not demonstrate how these representational vehicles might correspond to the representational object of inverse or a *function* that they might serve.

***Describing group – Q3.*** In response to Q3, John’s definition of group is surprising, especially because it does not align with the definition of group presented and discussed in class, in which groups were defined by the four standard group axioms: associativity, existence of a two-sided identity, existence of two-sided inverses, and closure under the operation<sup>7</sup>. John defines group by saying,

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<sup>7</sup> Note: The definition of group presented and discussed in class used the four standard group axioms: associativity (for all  $a, b, c \in G$ ,  $(a*b)*c = a*(b*c)$ ), existence of a two-sided identity (for all  $g \in G$ , there exists  $e \in G$  such that  $g*e = e*g = g$ ), existence of two-sided inverses (for all  $g \in G$ , there exists  $g^{-1} \in G$  such that  $g*g^{-1} = g^{-1}*g = e$ ), and closure under the operation (for all  $g, h \in G$ ,  $g*h \in G$ ).

John: Uh, a group is a set of. (to self) Does a group have to be non-empty? I think it does. (aloud) a non-empty set- that adheres to the following. One: is associative. Unlike mathematicians. Um, unless I just totally mis- (inaudible) that word. [Two:] There exists- so what was it. Exists some  $x$  and  $y$  in the set such that  $a \cdot x = b$  and  $y \cdot a = b$ . Aah. [Three:] And is binary. I don't know if that's covered by anything else in here, but I it's a thing.

Int: Okay. Um, so say a bit more about that. What do you mean by each of these (points at paper) kind of things.

John: The associative- Associativity just means that, uh, for  $a \cdot b \cdot c$  equals  $a \cdot (b \cdot c)$ , which equals  $(a \cdot b) \cdot c$ . And then, with this, this just means, that it's closed. I guess that kind of covers that then. Um, just means that it's closed and also guarantees the existence of an identity and inverses. Um, because, if- Oh, no. I did this wrong.  $a \cdot b$ - No. I did it right. I'm just dumb. I just can't read my own things. Um, just saying that, like, it guarantees the existence of identities, because if  $a$  and  $b$  are equal, then there still exists some  $x$  that will get you- that won't change it, which would be the identity. Um, if  $b$  is the identity, then the  $x$  would have to be the inverse, so, again choose both of those. Um, which we learned in class those are separate things, but I think one rule's easier than two rules, so I was just trying to think of- I'm really lazy, so I just try to think of writing the least amount possible. (lines 136-157)

John's definition includes a rule about associativity, for which he writes equations that reflect the formal definition:  $a \cdot b \cdot c = (a \cdot b) \cdot c = a \cdot (b \cdot c)$ . John mentions that a group is binary, though he is unsure if it should be included as a rule. Also, rather than describing individual identity, inverse, and closure axioms, John writes, "for any  $a$  and  $b$ , exists some  $x$  and  $y$  in the set such that  $a \cdot x = b$  and  $y \cdot a = b$ " (Figure 4.18; John's original description of this axiom excluded the phrase "for any  $a$  and  $b$ ," though he added it later). John continues by stating he thinks that having this rule give closure, but later changes his mind and adds closure as a third rule. He also describes how this second rule implies the identity and inverse axioms and initially. Notice that this definition draws on the same "letter" *forms* that John used to describe identity and inverse in response to Q1 and Q2, though without explicitly serving coded *functions* for the concepts. However, John goes on to claim that this rule, "guarantees the existence of identities, because if  $a$  and  $b$  are equal, then



there still exists some  $x$  that will get you- that won't change it, which would be the identity. Um, if  $b$  is the identity, then the  $x$  would have to be the inverse, so, again choose both of those" (lines 152-154).

Defn  
 Is NON EMPTY set that  $R$  adheres to the following.

1. Is associative
2. For any  $a$  and  $b$ ,  
~~exists~~ exists some  $x$  and  $y$  in the set such that  
 $a \cdot x = b$  and ~~exists~~  $y \cdot a = b$
- 3 Binary??; Closed

Figure 4.18. John's written work in response to Q3

John's definition includes two equations, each using letters, a dot, and an equal sign and each equation containing  $a$  as one of the elements on the left-hand side and  $b$  as the element on the right-hand side. The primary difference between the two equations is that, in the first,  $a$  is positioned to the left of the dot, with  $x$  to the immediate right of the dot; in contrast, the second equation positions  $a$  to the immediate right of the dot and  $y$  to the left of the dot. In his explanation of his claim, John chooses values for  $b$  so that the equations reflect the "equals" functions of identity and inverse. Specifically, although he does not write this, his first choice of  $a = b$  would produce the equations  $a \cdot x = a$  and  $y \cdot a = a$  (or, alternately,  $b \cdot x = b$  and  $y \cdot b = b$ ), which reflects the equations John produced in his definition of identity. Similarly, letting  $b$  equal the identity in the first equation would reflect the equation he provided while defining inverse.

I raise two points related to John's claims and explanation. First, John does not demonstrate that the element that acts as a left- or right- identity for  $a$  acts as an identity for every element in the group, which his definition of identity stipulated. Second, John does not demonstrate that the left- and right- identity are equal. Rather, he states that  $x$  and  $y$  have to be

the identity. This seems to draw on some notion of the uniqueness of identity, though John does not explicitly address this. Further, while these equations are not inconsistent with his prior descriptions, John does not explicitly write an equation for the left-inverse when defining the concept. Rather, earlier in the interview, he alludes to the left- and right- inverses being different elements, which he explicitly states here. In further explaining his claim that the second rule guarantees the existence of an identity, John draws on the “doing nothing” *function* of identity coded in the first interview, saying, “if  $a$  and  $b$  are equal and you're starting with  $a$  and end with  $a$  [Oh, okay.] the identity does nothing” (lines 192-193). He then repeats his argument that this second rule guarantees inverse elements.

John provides four examples of groups (includes the real numbers and integers under addition and the symmetries of a triangle and square) and alludes to other possible examples of groups that he describes as “mostly just redundant copies of similar things to that” (line 207). Notice that these examples reflect some of the *forms* John used when providing examples of identity and inverse. Specifically, John provides an operation for each example of a group with real numbers, but not for the groups of symmetries. In his discussion of subgroups (Q3e), John provides a specific example saying, “the rotations of an equilateral triangle is a subgroup of the symmetries.” John goes on to state that a set containing the identity and a self-inverse is also a subgroup. John also points out that the identity element is a self-inverse, which supports the development of an “identity as self-inverse” *function* of identity in which a *form* of identity might serve the “self-inverse” *function*.

***Likert statements – Q4.*** Table 4.7 shows the statements used in analysis of John’s responses to the Likert questions. I focus on parts a, b, c, d, g, and h because John’s responses to these parts were most beneficial toward the development of the *form/function* codes.

Table 4.7 – Likert statements

Part	Statement
a*	Zero is an identity.
b*	One is an identity.
c*	I think of an identity as anything that I use to keep things unchanged.
d*	I think about an identity as a function.
e	“Inverse” means “negative number.”
f	“Inverse” means “reciprocal.”
g*	Inverses depend on other things, like operations or sets of elements.
h	I think about an inverse as a function.

\* indicates Likert statements analyzed in this section

In response to the Q4a, John described being able to think of anything as an identity depending on the set and operation, saying, “I could say anything's an identity for any set I make up.

[Mhmm.] Like, and any operation” (lines 277-278). This supports a sense that John is able to think about identity more abstractly than the specific contexts that he has used in the interview.

In response to Q4b, John reiterates his initial example of the “number” *form* of identity under the operation of multiplication, but describes it as such in the set of “the real numbers  $n$ - without zero” (line 303), which is different from his earlier example in which he used all real numbers.

Responding to Q4c, John provides evidence of a new *function* of identity when he says, “we used identities when we were dealing with, like, integrating and stuff to make it- er- I guess that was probably more likely differentiating to make things easier to differentiate by just multiplying by compl- like, complex fractions that just equaled one” (lines 312-315). In this excerpt, John describes multiplying by one in a *form* that might not look like one – “complex fractions.” This hearkens to typical high school algebraic practices for simplifying and rewriting expressions.

In response to Q4d, John says, “I don't think that's how I think at all, because I can't- I don't exactly know what it's saying, so it's low on the list. Um, I think about it more as an element, I guess, not really as a function. Like it's defined within a function, but, not a function itself. It's a combination of an element and a function almost” (lines 332-335). This relates to the correspondence between representational vehicles and the representational objects of identity and

inverse. Specifically, in some cases, such as when John draws on “number” *forms*, he explicitly states the operation whereas, in other cases, such as the context of symmetries of a triangle, he does not state an operation, which John describes as “a combination an element and a function” (line 335). This supports a sense that John is not considering symmetry composition as the operation under which symmetries might be combined and also reflects his discussion during Interview 1 in which he multiplies functions, rather than composing them. John’s response to the seventh Likert question affords deeper insight into how the operation (and set) informs his understanding of inverse. In response to Q4g, he describes these aspects as “context” for inverses, saying, “You can't have an inverse without an operation. Um, and the elements matter, too, I guess. ... Context matters when it comes to these things” (lines 369-371). This underscores an emphasis on the operation under which specific *forms* of inverse might serve *functions* of inverses. This also supports a sense that, in the absence of the appropriate operation, specific *forms* of identity or inverse might not be able to serve a *function* of identity or inverse, respectively.

***Subgroup proof: Integer powers of 2 – Q5.*** In response to Q5, John begins by stating that, though he might not be certain, in order to prove a subset is a subgroup he must show that “it contains the identity” and “it contains the inverse” (line 434). John goes on to say, “Um, we know identity of reals under multiplication is one, just from my knowledge of the real numbers. ... Two: zero is an integer. Two to the zero equals one. So,  $H$  contains the identity” (lines 444-446). Here, he draws on the “number” *form* of identity, which he has said is one for real numbers throughout the interview, citing the group operation of multiplication. He represents one as  $2^0$  in order to verify that the set  $H$  contains the multiplicative identity. John quickly moves to showing that  $H$  contains inverses, saying, “Inverse. Um, so two to the  $n$  times two to the negative  $n$  equals

one. Um, negative  $n$  is an integer. So,  $H$  contains all inverses” (lines 449-450). Here, John draws on the “equals” *function* of identity, using the equation [two to the  $n$ ] [times] [two to the negative  $n$ ] [equals] [one], writing  $2^n \cdot 2^{-n} = 1$ .

When explaining his work, John says, “[ $H$  and  $G$ ] have the same identity. And the elements have the same inverse- Like, the same inverse, I guess. So, well, they have to since the operation stays the same and the identity stays the same. Uh, so you just have to prove that it contains all the inverses for your elements, it contains the identity and- I don't know if that means that it closed or not” (lines 478-482). This reiterates John's previous statements when defining subgroups. When describing how identity informed his approach to the proof (Q5f), John points out that he had known that one would be the identity, twice saying that he was thinking about the context. When describing how inverse informed his approach (Q5g), John says, “inverse in this set would just be to the negative power of whatever we're working with... you add them together to get zero. ... yeah, just, like, g- what do you do to get back to one” (lines 534-539). Here, John's explanation focuses on the additive relationship of the exponents, then shifts to “get[ting] back to the one,” which he had identified as the multiplicative identity. This language draws on the language used with the “get identity” *function* of inverse identified during analysis of the first interview. Accordingly, along with his initial work, this excerpt provides a sense of how the “get identity” and “equals” *functions* of inverse inform his approach in this proof.

***Uniqueness of inverses proof – Q6.*** In beginning his proof in response to Q6, John draws on the “letter” *forms* of identity and inverse. This stands to reason, as the protocol was written using this notation. John describes the identity as  $e$  and defines an element  $g$  and represents its inverse as  $g^{-1}$  and  $a$ . He then draws on the “equals” *function* of inverse to generate two equations:

$g * g^{-1} = e$  and  $g * a = e$ . John follows this by setting the left-hand side of each equation equal, supporting a “transitive” *function* of identity in which John is able to set two algebraic statements equal to each other if they each equal a *form* of identity. John then hesitates and points out that the protocol does not specify whether the inverse of  $g$  is a left- or right- inverse, supporting the sense that the “sided” *function* of inverse is informing his activity in this moment. However, his concern seems to be ameliorated, as John says, “No, we just have the cancellation law that does it automatic- let's just do it” (lines 582-583). He spends several seconds writing and then pauses to describe the proof he has just written. During this explanation, John says, “you have the cancellation law, which let's you get rid of the  $g$ 's and so you get  $g$  to the negative one equals  $a$ ” (lines 588-589). When asked about his use of the cancellation law, John says, “we learned a rule (scoffs) in class saying that if they're on the same side, you can get rid of them both, like you can cancel the two  $g$ 's out” (lines 614-615). From this excerpt, it does not seem that John attributes the cancellation law to the notion of inverse.

When asked how the identity informed his approach (Q6f), John says, “I know the identity is, uh, whatever, like, you add an element and its inverse and you get the identity, so it's necessary for me to, like, show- to set these two things equal to one another” (lines 642-644). This statement reflects the “get identity” *functions* of inverse. Notice that John uses the verb “add” in this excerpt, which is consistent with his previous discussion, though he tends to correct this with the word “operate.” Interestingly, John focuses on how the identity afforded him to create the new equation from the first two, which supports a “transitive” function of identity, named so because the identity mediates the use of transitivity to set two statements equal to each other. Asked how he might complete the proof without the cancellation law, John left-operates by the inverse of  $g$ , saying, “I would just do whatever the- the left in- since I was, like, playing

with the right - I would do whatever with the left inverse- an arbitrary left inverse of  $g$ , eh, which would cancel the  $g$ 's out and would still let you get these equal to one another, which is why the cancellation law works” (lines 657-660). This shows that John is able to describe the role of inverse in cancellation. He demonstrates his explanation by defining a left-inverse of  $g$ , naming it  $b$ , and saying, “we got  $b$  star- you can just do that to the front of both since you're doing both things on the same like normal, then these would cancel out...” (lines 670-671). This supports a “cancelling” *function* of inverse. In serving this *function*, a *form* of inverse is operated with an element that is positioned on the far-right or far-left of statements on both sides of an equation. This results in a cancellation of the element in which John rewrites or describes the previous expression with the cancelled element excluded. In this instance, John did not include any discussion of the identity element or the role it plays in the cancellation. Interestingly, this is the first instance during this interview in which an inverse does not *function* to yield a *form* of identity.

***Subgroup proof: Normalizer of  $h$  – Q7.*** Early in John’s response to Q7, he says, “Um, so, you can s- like, this is just saying that you're allowed to like- since this is the inverse of this this equals the identity if they're next to each other, but they're not next to each other, but since this- since- they do equal the identity because  $h$  is on either side, it's um- Crap is  $h$  just the identity?” (lines 746-750). He initially seems perturbed and unsure of how to proceed, but describes (without much clarity) the “equals” *function* of inverses. It seems that John is initially describing commuting  $g$  or  $g^{-1}$  with  $h$  in the statement, “ $g*h*g^{-1}$ ,” and using the “equals” *function* of inverse with the “letter” *form* of inverse. However, John seems to conclude that the entire left-hand side of the equation would then equal the identity because of the statement, “is  $h$  just the identity?” (line 750). John then says, “It works works really well if [ $h$  is] the identity” (line 758),

and mentions his previous discussion of subgroups composed of the identity and a self-inverse. He shifts his discussion, saying, “if it’s commutative, then it works really well” (line 761). This excerpt supports that John is trying to make sense of the definition of the set  $H$ , though he seems to find this process difficult, saying, “What’s this mean? So- have no idea. ... I’m just trying to get a picture in a group I know of what this, like, is saying” (lines 768-769). John then describes using elements from the group of symmetries of a triangle to, “get, like, a feel for, like, what works and what doesn’t” (line 775). He writes the equation given in the definition of  $H$  with the elements  $g$  and  $g^{-1}$  replaced by  $F$  in the first line and by  $R$  and  $2R$ , respectively, in the second line. He then writes the equation  $R+h = h+R$  under the second line (Figure 4.19). He then says, “And I don’t really know- How this is supposed to look-  $R$  star  $h$  equals  $h$  star  $R$ - By just adding  $R$  to the end of each- But that doesn’t tell me anything” (line 776-777). From this, it seems that John is using “symmetry” *form of inverse* (inscribed with letters) in place of the more general, given “letter” *form*. As John describes, he is “adding  $R$  to the end of each” (line 777), which draws on the “symmetry” *form* to serve a *function* of inverse similar to, but distinct from the “cancelling” *function* of inverse. Specifically, this equation does not contain the same element on the far-right or far-left of each side of the equation. Rather, John right-operates on each side of the equation with  $R$ , which he has described as the inverse of  $2R$ . This supports the formation of a new “end-operating” *function* of inverse, in which an element is removed from the end of a statement on one side of an equation and its inverse is concatenated on the same end of the statement on the opposite side of the equation. This *function* allows John to produce the third equation from the second.



$$F \circ h \circ F = h$$

$$R \circ h \circ R = h$$

$$R \circ h = h \circ R$$

Figure 4.19. Replacing  $g$  and  $g^{-1}$  with letters representing symmetries of a triangle

After several seconds of silence, John says,

Let's pretend  $g$  is the identity. Then it definitely works, so the identity exists. Okay, so, I know the identity exists 'cause if we let  $g, h$  be in  $G$ , um, let  $g$  be the identity of  $G$ , then you get  $g \star h \star g$  to the negative one equals  $h$  and since those are the- the inverse of the identity is the identity and this is the identity, we get  $h$  equals  $h$ - definition of identity. So, we know the identity exists in the set. (lines 781-785)

While saying this, John writes three lines:  $g \star h \star g^{-1} = h$ ,  $e \star h \star e = h$ , and  $h = h$ . He then marks through the  $e$ 's in the second equation. This excerpt shows an important shift from John's previous activity. Similar to before, John is replacing the  $g$  in the given equation. However, John changes from specific elements in a known and familiar group (the symmetries of a triangle) to the generic "letter" form of identity,  $e$ . He also says, "the inverse of the identity is the identity," which reflects the "identity as self-inverse" function of identity. John's process of marking through the letters  $e$  in the second equation support the development of a "vanishing" function of identity in which a form of identity (here, "letter" form) may be removed from a statement in which it is operated with other elements. This function of identity can be viewed as an implied aspect of the "cancelling" function of inverse, specifically in cases when John cancels an element by operating with its inverse and rewrites the statement with the elements removed, rather than writing an identity in place of the canceled elements.

After his subproof related to the identity, John turns to discussing whether "all the inverses are in there," claiming, "They are." He says, "But, like, if it works for  $g$  over here- and  $g$  to the negative one I assume you can just switch them and then  $g$  to the negative- the inverse of  $g$

to the negative one is  $g$ ” (lines 786-789). Asked, why he assumes that the two elements can be “switched,” John replies, “Ah, actually, you can't assume it. ... You can't because it's not commutative” (lines 794, 798). This leads John to continue replacing the  $g$  in the given equation with symmetries of a triangle and confirms that they are not commutative. He then explains why he thinks commutativity is important, saying, “if it's commutative, it obviously works, 'cause you just, like, move it. Like, if it's commutative, it works for all elements of  $G$ , right? Yeah. If it's commutative, 'cause you can just move it over and those cancel. So,  $h$  equals  $h$ . But, we don't know that about our group” (lines 812-814). In this excerpt, John draws on the “letter” *form* of elements serving the “cancelling” *function* of inverse once the letters are rearranged in the statement.

John continues replacing the  $g$  in the given equation with “letter” *forms* of the symmetries of a triangle. When asked about his activity, John says, “I keep getting  $g$  plus  $h$  equals  $h$  plus  $g$ ” (line 878). He then asks himself in a whisper, “What's the inverse of  $h$ ?” (line 884) and draws on the “end-operating” *function* of inverse, saying, “what happens if I use the inverse of  $h$ ? Which has to exist, 'cause it's a group. Then you just get  $g$  equals  $h$  star  $g$  star inverse  $h$ ” (lines 889-890). This conclusion seems consistent (to John) with his other work on this problem until he says, “I can't do that. Well, I- Because that's assuming that left- and right- inverses are the same” (lines 892-893). This statement reflects the “sided” *function* of inverse. Because of this, it seems that John is drawing on the “sided” *function* of inverse to question the validity of applying the “end-operating” *function*. When asked to describe his thinking, John says,

When I did this, I assumed left and right inverses are the same and like, I don't know why that- like,  $g$  to the one- I don't even n- like- I don't know whether that  $g$  to the one equals- like,  $h$ - I don't know how its inverse is, so I don't know how to operate  $g$  with it to get- to cancel. Is the issue- Like, so, it's like a left inverse or a right inverse 'cause I'm assu- like-” (lines 898-901)

This supports the analysis that the “sided” *function* of inverse is causing John to question the validity of applying the “end-operating” *function*, specifically, right-operating on both sides of the equation “ $g*h*g^{-1} = h$ ” with  $g$  to yield the equation “ $g*h = h*g$ .” Further, this situation is distinct from John’s activity in response to Q6 in that John cannot draw on the “cancelling” function of inverse because the statements on either side of the equation do not have the same element in the far-right or far-left position. In other words, because the equation does not satisfy the format of containing the same element on the same end of the statements on either side of the equation, it cannot serve the “cancelling” function.

The interviewer then asks John, “What if, hypothetically, like, in this case- in  $G$ - left and right were the same inverse?” (lines 903-904). John responds, saying, “Is that always true? That could be a thing that's always true, and I'm just dumb” (line 906). This reflects an uncertainty in the “sided” *function* of inverse. At this point in the interview, John leaves for class and returns about 90 minutes later to conclude the session. Upon his return, John states, “I was stuck at  $g$  star  $h$  equals  $h$  star  $g$  ... Just, I don't know where to go from there. Also, I did think about one thing. It wasn't actually about this [Int: Oh, okay.] in the mean time. It was about the inverse thing, and they are- if a group do- they are left and right inverses, like, [Int: Okay, y-] they have to be equal” (lines 965-967). This leads to John discussing why he now believes that left- and right-inverses must be the same (Figure 4.20), contradicting the previous “sided” *function* of inverse. John begins his discussion saying, “like since it's associative, you can group it like  $a$  dot  $b$ , which, since  $a$  is the inverse, this equals the identity, so it all equals  $c$ . And you could also group it like  $b$  dot  $c$ , which equals the identity, so it equals  $a$ . And since those two are equivalent statements,  $c$  equals  $a$ ” (lines 1014-1017). He later clarifies that, “ $a$  would be the left inverse and  $c$  would be the right inverse of  $b$ , under- any old operation” (lines 1042-1043). As John is

describing this process, he writes two lines, using the “letter” *form* of inverse (in darker, thicker ink):

$$\begin{array}{l} (a \cdot b) \cdot c = c \\ a \cdot (b \cdot c) = a \end{array}$$

Figure 4.20. Applying associativity to argue about left- and right-inverses

In his discussion, the “letter” *forms* of inverse, written as  $a$  and  $c$ , serve the “equals” *function* of inverse and John then draws on the “equals” *function* of identity to justify the “vanishing” *function* of identity and produce the right-hand side of each equation. John justifies being able to do this with both groupings by referring to the associativity of the group. He goes on to say, “I thought that left and right inverses were different, but they're not in groups” (line 1033). John reiterates his argument supporting his changing notion of inverse and then describes what prompted him to approach the problem using the statement “ $a*b*c$ ,” saying,

So, I just started with, like, left and right inverse. And, like- 'cause I assumed it was, like, similar- almost similar form to that- (points to upper-right corner of page) but it actually isn't at all. ... since grouping things differently changes how it- changes, like, what it means (air quotes), but like, not actual- like, not what it-like, how you would do it, but not, like, what it means. [Okay.] So, like, those will just cancel and give you  $c$  and  $a$ . I don't really know how I think most of the time. It just kind of happens. (lines 1129-1139)

This provides insight into how the protocol statement informed his approach as well as how John thinks about the *function* of the inverse in this case. Specifically, he says, “those will just cancel and give you  $c$  and  $a$ .” The word “cancel” supports the development of a “vanishing” *function* of inverse. This *function* reflects a combination of the “equals” *function* of inverse and the “vanishing” *function* of identity. In his initial description of his approach, John explicitly draws on the “equals” *function* of inverse and the “equals” and “vanishing” *functions* of identity. Specifically, he explains the process of operating an element with its identity, which produces an identity element (“equals” *function* of inverse) that, because the identity operated with any other

element produces that element (“equals” *function* of identity), it can be left unwritten “vanishing” *function* of identity). However, here, he uses the word “cancel” to contract the entire process into a single word. The distinction between this *function* and the others is supported when John later writes the line, “ $(a*b)*c = a*(b*c)$ ” and crosses through the “ $(a*b)$ ” on the left and “ $(b*c)$ ” on the right as he did with the identity element  $e$  during his subproof that  $H$  contains the identity. Interestingly, the earlier coded “cancelling” *function* of inverse seems to reflect simultaneously drawing on the “vanishing” *function* for concatenations of an element and its inverse when they are each on the same end of both sides of an equation.

John continues his proof that  $H$  is a subgroup of  $G$  by returning to show that  $H$  contains the inverses of its elements. In his approach, John begins with the equation  $g*h*g^{-1} = h$  and draws on the “end-operating” and “vanishing” *functions* of inverse to produce the equation “ $g*h = h*g$ .” John then re-reads the problem statement substituting the latter equation for the former as he reads, saying, ““Prove or disprove is [mumbles] the set all  $g$  such that’  $g$  star  $h$  equals  $h$  star  $g$  ‘is a subgroup of  $G$ ”” (lines 1182-1183). John goes on to say, “the inverse of  $h$  and the identity of  $G$  are in  $H$ ,” adding, “when you’re using inverses, it’s commutative for all groups. ... like, since  $h$  is fixed, it has a fixed inverse. Um, so, if  $g$  is the inverse of  $h$ —” (lines 1202-1208). It seems that John is using his recent proof that “if you’re using inverses, it’s commutative” in combination with his rephrasing of the definition of  $H$  to support his claim that the inverse of  $h$  is included in  $H$ . John follows this discussion by reviewing his previous work and returning to the symmetries of a triangle. He lists the “letter” *forms* of the symmetries of a triangle and says, “I’m just gonna through and see if it- what happens when you do it for all of these.” It seems that he has substituted  $F+2R$  for each instance of the letter  $h$  in the equation  $g*h*g^{-1} = h$  and iteratively substitutes each symmetry and its corresponding inverse for  $g$  and  $g^{-1}$ , respectively. He concludes

this by saying that all rotations work, but that elements with flips did not, which leads him to think that the  $H$  might be a subgroup of  $G$ .

After this, John again explains that if  $G$  is commutative, then  $H$  is the entire group. He describes using the presumed commutativity to switch the positions of  $g$  and  $h$  and then says, “ $g$  and then the  $g$  prime- um, the  $g$  to the negative one and  $g$  just cancel and  $h$  equals  $h$ ” (1326-1327). This reflects the “vanishing” *function* of inverses. He then repeats the previous process of substituting each symmetry into the equation, except that he uses the equation  $g*h = h*g$  and substitutes  $R$  for  $h$ . From this, John reiterates his belief that the conjecture is true - that  $H$  is a subgroup of  $G$ . Next, John says, “I can make that  $h$  star  $g$  prime-  $g$  to the negative one equals  $g$  to the negative one star  $h$ ” (line 1350). He describes his reasoning using the “end-operating” *function* of inverse twice with the equation  $g*h = h*g$ , with  $g^{-1}$  on the left and then on the right, and writes the equation “ $h*g^{-1} = g^{-1}*h$ .” John continues to use the “end-operating” *function* of inverse to manipulate the equation, eventually writing the equation  $e = h^{-1}*g^{-1}*h*g$ . At this moment, he sarcastically cheers and states that he does not know what he has done.

John then says, “No, what was I trying to do? I'm trying to prove- or disprove- that all the inverses of  $g$  are also in  $H$  what does it mean? ... Now what does it take for that to be- oh. It takes for this to be true. Awesome. Okay. I did it. I think” (lines 1402-1406). After he says this, he writes  $h = g^{-1}*h*g$ , and reiterates his argument, adding, “we know all the  $g$ 's are in there, because that's what it is. And, if the- and if this is true, then, like,  $t$ -  $g$  to the negative one is a- in- the inverse of  $g$  is a- part of b- a part of  $H$  as well. Because this is basically the same thing as this, just- 'cause the inverse of  $g$ 's inverse is  $g$ ” (lines 1412-1415). Throughout this excerpt, John points back and forth between the equations the  $g*h*g^{-1} = h$  and  $h = g^{-1}*h*g$ . As he says, “the inverse of  $g$ 's inverse is  $g$ ,” he points to the  $g$  in the equation  $h = g^{-1}*h*g$ . The latter statement

supports the development of an “inverse-inverse” *function* of inverse in which an element is viewed as an inverse to its own inverse. This is the first time that John had explicitly mentioned this aspect of inverse. John goes on to say,

So, it shows the identity does exist in  $H$ , which is necessary for a subgroup. Another thing is that its inverse- has to be included. ... here I manipulated the form to look like  $h$  equals  $g$  to the negative- the inverse of  $g$  star  $h$  star  $g$ , which is also- since-  $g$  star- uh,  $g$  to the negative one's' inverse is  $g$ , this holds, So, all inverse of  $g$  are a part of the- a part of  $H$ , so all the inverses exist in  $H$ . (lines 1446-1455)

John continues his proof that  $H$  is a subgroup of  $G$  by attempting to show that  $H$  is closed under the operation. This subproof, which is formally invalid, follows the same pattern as his “inverses” subproof, begins with the equation  $(g_1 * g_2) * h * (g_1 * g_2)^{-1} = h$ . John then rewrites “ $(g_1 * g_2)^{-1}$ ” as “ $g_2^{-1} * g_1^{-1}$ ” and says, “I mean, if you go one way, you have to go back the same way, so it's  $g_2$  inverse star  $g_1$  inverse” (line 1524-1525). This supports an “inverse of a product” *function* of inverse, in which the inverse of two elements operated together is those elements’ inverses operated together in the reverse order as the original elements. He continues manipulating the equation using the “end-operating” *function* of inverse to produce the equation  $h = (g_1 * g_2)^{-1} * h * (g_1 * g_2)$ , pointing out that this equation is the, “Same form as original” (line 1539).

Finally, John concludes the interview describing how identity and inverse informed his approach to the proof (Q7g,h). Of the identity, John says, “It was important in canceling everything for these two. Like, wouldn't work if I didn't have a clue what identity was...” (lines 1621-1622). Of inverses, John says, “I used a lot of inverses. Just, knowing how they interact with their- they interact. And, like, here, knowing how they, like, switched sides, you know, like, 'cause these are both going to the same side, 'cause that's how switching works...” (lines 1629-1631). He goes on to say, “I thought left and right inverses were different in groups, now I know

they're the same. So that was nice..." (lines 1661-1662). Though most of his discussion is more general, John's description of how his thinking about inverse changed is important when considering shifts in the *functions* that inversed might serve for John. Specifically, John proved his "sided" *function* of inverse to be invalid, which allowed him to make progress with his "inverse" subproof.

**Interview 2 analysis summary.** John's responses to the protocols in Interview 2 are very enlightening. Analysis of these responses generated ten new *functions* of identity and inverse and one new *form* of inverse, with most of these *functions* originating from John's activity drawing on "letter" *forms* of identity and inverse, specifically to manipulate equations in order to accomplish specific goals. For instance, in his response to Q6, John draws on the "transitive" *function* of identity to generate new equations in which each side of the equation also equals the identity. Throughout Interview 2, when John draws on the "vanishing" *functions* of identity and inverse as well as the "cancelling," and "end-operating" *functions* of inverse it is typically to manipulate an equation that is given or that he has produced.

During his responses to Q1, Q2, and Q3, John emphasizes an aspect of identity and inverse that he had not mentioned during Interview 1: left- and right- identity and inverse. John's discussion reflects an understanding that is, for the most part, consistent with the formal notions of the concepts, though there are some instances in which John's discussion affords insight into nuanced distinctions between his thinking and a formal treatment of constructs. John's "sided" *function* of inverse is inconsistent with a formal treatment of inverses in groups. There are instances in which John shifts the *form* upon which he draws in a given situation. For instance, during his response to Q7, John replaces elements represented with generic letters ( $g, h, g^{-1}$ ) with letter representations of the symmetries of a triangle. In this moment, he is drawing on the "end-



operating” *function* of inverse. After shifting back to using the “letter” *form*, John continues to draw on the “end-operating” *function* of inverse, although he expresses uncertainty about whether he is able to do this, with reasoning that seems based on the “sided” *function* of inverse.

**Interview 3.** During interview 3, John responded to all the first six questions from the protocol and all of these responses contribute meaningfully to the analysis of his conceptual understanding of identity and inverse. Analysis of this interview supports the development of two *form/function* codes: the “symmetry” *form* of identity and the “replacement” *function* of identity. For the most part, John’s discussion throughout this interview supports existing *form/function* codes and affords insight into how he draws on various *forms* and *functions* of identity and inverse during proof activity. For example, during his response to Q3, John describes a non-standard group axiom and provides two impromptu proofs explaining how he understands this axiom implies the existence of an identity element and an inverse for each element in the group. In these proofs, John draws almost exclusively on the “letter” *forms* of identity and inverse to serve several different *functions* as he manipulates equations, depending on the goal of his activity.

**Describing identity – Q1.** When asked to describe identity, John begins by referring to “an identity element,” calling it “an element of a set, when combined with other elements, leaves the elements unchanged” (lines 5-7). This description remains general, without naming a specific type of element or operation. John elaborates by alluding to a given operation, rephrasing his description of identity saying, “it doesn’t do anything to the elements,” adding, “necessary for a group” (lines 11-12). These descriptions draw on the “do nothing” *function* of identity developed during analysis of Interviews 1 and 2. John goes on to state that, in a group, the identity is unique, adding that with different operations, there could be different identities. Responding to

Q1b, John produces a definition of identity that aligns almost exactly with his initial description.

He then provides three examples of identity, saying,

Um, for under addition- or- nor- usual addition for the integers, zero. And under normal multiplication for the integers, one. Um. (3 seconds) For, like, the rotations of a tri- or the symmetries of a triangle, just, the initial, like, the initial mapping. (chuckling) I guess, like, depending on how you want to, like, name it. Like, 3R, I guess is one name. 2F. Zero. They're all symmetrical. (lines 41-45)

The first two – zero “under ... usual addition for the integers,” and one “under normal multiplication for the integers,” – align with the “number” *form* of identity from John’s first two interviews, contextualized by an operation. In his description of the third example, however, John describes the “initial mapping” of a triangle as an identity for the symmetries of a triangle, inscribing this as “3R” and “2F,” and “zero.” This supports a “symmetry” *form* of identity, although, John does not contextualize this example relative to any operation for this *form* of identity, as he did with the first two examples.

***Describing inverse - Q2.*** In John’s initial description of inverse, he relies on specific types of inverse, saying, “As a generalized term for, uh, like, ne-gation when using addition. And reciprocal when using multiplication. Or, at least, like using those in a broad sense” (lines 57-58). He then adds, “generally it gets you, well, always it gets you - at least for all that we've been doing - it gets you back to the identity. Since the identity is just, kind of nothing, since it leaves things unchanged. That's why it'd be, like, negation” (lines 60-62). John goes on to define inverse, saying, “for any element, the inverse is an element you can combine with the first that returns you to the identity for a given operation” (lines 75-76). Each of these reflects the “get identity” *function* of inverse. He then provides four examples of inverse, two drawing on the “number” *form* of inverse and two using the “letter” *form* of inverse. With each of the “number” *forms* of inverse, John provides an operation (addition and multiplication) and describes the

inverse (“a number’s negative” and “one over that number”). For the examples using the “letter” *form* of inverse, John contextualizes the inverses based on the “notation” being used, saying the inverse is “negative  $a$ ” and “ $a$  to the negative one” for additive notation and multiplicative notation, respectively. Though he contextualizes each example with an operation or notation, John does not describe how or why each example is an inverse, but simply names each example the inverse. Consequently, these examples do not support a code for any *function* the *forms* might serve.

***Describing group – Q3.*** When defining group, John describes the same axioms he had described in the second interview: an associativity rule and a second rule that I call the “left/right rule.” John describes the left/right rule by saying, “for all  $a, b$  in  $G$  - this meaning  $G$  is the set - there exists some  $x, y$  in  $G$  such that  $a$  star  $x$  equals  $b$  and  $y$  star  $a$  equals  $b$ ” (lines 97-99). With this rule, John claims and then describes deriving the existence of an identity element within the group and an inverse for each element in the group. The left/right rule is defined using a generic “letter” *form* with an operation star. Briefly, John’s approach for leveraging the left/right rule to show the existence of an identity is to let  $a = b$ , and his approach for showing that an inverse exists for each element is to let  $b$  equal the identity. With this description, John necessarily relies on the derivation of the identity element before inverse elements. Accordingly, he completes two subproofs, first setting out to prove that (assuming the left/right rule) a two-sided identity element exists, then showing that each element has a two-sided inverse element. Throughout John’s proof, he uses the “letter” *form* for every element in the group.

To begin his first subproof, John begins with the equations “ $b*x = y*b = b$ .” It seems here, that the  $x$  and  $y$  each act partially as an identity. Specifically,  $x$  acts as a right identity and  $y$  acts as a left identity. John uses the language of “left identity” and “right identity” to describe

how he is thinking about the subproof. He then says, “then I just manipulated it to make it so  $x$  equals  $y$  so it's one identity element” (line 123). This statement tacitly assumes the notion that the group identity should be a two-sided identity, which is consistent with John’s definition in Interview 2 (Q1aiii), though he had not, up to this point in the third interview, explicitly discussed this. John’s discussion supports a new “sided” *function* of identity in which *form* of identity serves a *function* of identity for only one side of an element. It is important to note that, as stated, John does not seem to question whether the left and right identity are the same element in a group, but this *function* of identity is a consequence of the left/right rule potentially allowing  $x$  and  $y$  to be different elements. Accordingly, the “sided” *function* of identity is only coded during this proof. However, John has difficulty showing that the left- and right- identities are equal when asked to do so by the interviewer (this part of the interview lasts 13 minutes).

Further, John does not demonstrate that  $x$  and  $y$  act as identity for every element in the group (as he had defined identity). Throughout this subproof, though, John draws on the equations  $b*x = b$  and  $y*b = b$  to replace the letter  $b$  in equations. This constitutes part of his manipulation of the equation  $b*x = y*b$  with the expressed goal of showing that  $x = y$ , and supports the formation of a “replacement” *function* of identity in which a letter (in this case,  $b$ ) is replaced with an equivalent concatenation of that letter and a letter representing a left- or right-identity with the symbol “\*” between the two letters (in this case,  $b*x$  or  $y*b$ ). Further, John draws on the “end-operating” *function* to augment the equation with the elements  $x$  and  $y$ . John manipulates the equation  $b*x = y*b$  to yield the equation  $b*x = b*y$ . This process uses the “end-operating” and the “replacement” *functions* of identity, but also tacitly draws on the equation “ $y*x = x$ ” to serve the “replacement” *function*. John then uses the left/right rule to produce the equation “ $q*b = y$ .” He follows this by left-end operating on the equation  $b*x = b*y$  with the

letter  $q$  to produce the equation  $q*b*x = q*b*y$  and uses the equation  $q*b = y$  to serve “replacement” *function*, generating the equation  $y*y = y*x$ . Interestingly, though John does not describe it as such,  $q$  is serving the “equals” *function* of inverse in the equation “ $q*b = y$ ” and in the equation  $q*b*x = q*b*y$ . Finally, John produces the equation  $y = x$  from the equation  $y*y = y*x$ , a manipulation for which John provides no explanation, though it is consistent with the “cancelling” *function* of inverse as well as the “identity as self-inverse,” “vanishing,” and the “equals” *functions* of identity.

John then moves to the inverse subproof related to the left/right rule, which he describes as letting  $b$  equal the identity in the equation  $a*x = y*a = b$ , and showing  $x$  equals  $y$ . As John points out, this subproof is similar to his subproof from Q7 in the second interview. John begins with the statement  $x*a*y$  and invokes the associativity of the group to rewrite the statement first as  $(x*a)*y$ , then as  $x*(a*y)$ . He then draws on the “equals” *function* of inverse followed by “vanishing” *function* of identity to show that the statement  $x*a*y$  is equal to both  $x$  and  $y$  and generate the equation  $x = y$  (formally, this uses transitivity, though John does not say this).

**Mini-proofs – Q4.** In response to Q4a, John alludes to manipulating equations to prove that the identity element is unique, though he does not go through this proof. He then quickly responds to Q4b by citing the definition of a group. His response to Q4c is more thorough in that John provides the justification “if it has an inverse element, it's the inverse of its inverse” (line 376-377) to support his response. This reflects the “inverse-inverse” *function* of inverse. Interestingly, in response to Q4d, John uses the phrase “get back to the identity” outside of the context of inverse, when discussing elements’ order, which might support a sense that the “get identity” *function* of inverse plays a role in how John conceptualizes order. However, he does not seem to draw an explicit connection between the notions of order and inverse. He provides a

counterexample to the given statement using the group  $Z_{12}$  as an example of a finite group with an even number of elements and exactly one element of order 2. John's language during this example generation is consistent with the "number" *form*, though the operation and set seem to cause the correspondence of numbers and the representational object of inverse to differ from John's prior discussion, specifically, 6 serves the "self-inverse" *function*, unlike before. John temporarily questions his counterexample when describing the order of zero (the identity element), but then concludes "zero is order 1." In response to Q4e, John quickly provides two counterexamples: the trivial group containing only the identity and the integers under addition. John points out that each of these examples has one self-inverse. In each example, this element is the identity, which John explicitly says when discussing the first example. In this case, the identity is serving the "identity as self-inverse" *function* of identity.

***Subgroup proof: Normalizer of h – Q5.*** John's proof in response to Q5 is similar to his proof in response to the same prompt during Interview 2 Q7, though much shorter and more concise. John even points out that, "this is way easier than I did it last time" (line 510). Unlike his prior proof, John does not describe or try to show that the set  $H$  satisfies closure. However, John's identity subproof is almost identical to that in the second interview, drawing on the "equals" *function* of identity to reason that the equation " $e*h*e = h$ " is true. John's inverse subproof takes slightly longer, mostly due to his struggling to phrase his justification of how the equation " $g^{-1}*h*g = h$ " supports the existence of inverses in the subgroup, describing it by saying it "has the same form" (written) as the equation " $g*h*g^{-1} = h$ ," which is similar to his justification during Interview 2.

However, John's manipulation of the equation " $g*h*g^{-1} = h$ " is different from his response in Interview 2. Specifically, he does not draw on a specific example of a group to test

manipulations of the equation. Further, John seemingly performs only two manipulations of the given equation, compared to the several minutes he spends testing this process during Interview 2. In his response to Interview 3 Q5, John first produces the equation “ $g * h * g^{-1} = g * g^{-1} * h$ ,” writing it immediately under the given equation, then writes the equation “ $g^{-1} * h * g = h$ .” The first manipulation of the given equation indicates that John (left) end-operates only on the right-hand side of the given equation with the statement “ $g * g^{-1}$ .” This is consistent with the “vanishing” *function* of inverse, though the change from the first equation to the second generates a concatenation of an element and its inverse, rather than removing such a concatenation. This supports an “un-vanishing” *function* of inverse through which an algebraic statement is concatenated with an element operated with a *form* of its inverse. Although John does not support this change in the equation with any explanation, it stands to reason from the “vanishing” *function* of inverse and the “equals” *functions* of inverse and identity that the “un-vanishing” *function* of inverse does not change the truth-value of the equation. Specifically, concatenating a statement with an element operated with its inverse is consistent with concatenating with the identity, which is, in turn, equivalent to the original statement. However, it is unclear whether John is reasoning about the manipulation of the equation in this way.

**Order proofs – Q6.** John begins his response to Q6 by testing the specific example of 4 in the group  $Z_{12}$ . With this example, John is using the “number” *form* of inverse (here, 4 and 8) and calculating sums of each element with itself, describing the order of each. He concludes this activity, saying, “So, I do think it holds. I don't know how to go about proving this” (lines 649-650). After two minutes of writing with little discussion, John says, “if  $gk$  is the identity, so  $g$  to the  $k$  star  $g$  to the negative one  $k$  equals  $g$  to the negative one  $k$ ” (lines 655-656, Figure 4.21a). This draws on the “letter” *form* to serve the “end-operating” *function* of inverse, though in a

distinct way from John's previous activity. Specifically, he is now operating not with a single element, but with “ $g$  to the negative one  $k$ ” (line 656), which he writes as “ $(g^{-1})^k$ .” John follows this saying,

I just wanna, like, combine them. But, I don't know, like- Does that work? ... So,  $g$  to the  $k$  star  $g$  to the negative one  $k$  equals  $e$  to the  $k$  with a question mark over the equal sign. I just wanna like- manipulate it using my normal multiplicative-multiplicatives and stuff. [Hmm.] Which, gah, it has to be commutative. Does it? No. Because, they're gonna be touching in the middle. So, like, I don't know how to write it out, but, it does hold. (laughs) Cause it's gonna be  $ggg$ - however many  $g$ 's and then in the middle it's gonna be  $g g$  to the negative one  $g$  to the negative one  $g$  to the negative one  $g$  to the negative one. And then, so these will mean  $e$ . And then this will collapse- It'll just collapse in on itself and equal  $e$ . (lines 658-668)

In this excerpt, John questions how he might manipulate the equation.  $g^k * (g^{-1})^k = (g^{-1})^k$  (Figure 4.21a). He generates the equation in Figure 4.21b and focuses on “the middle” of the left-hand side, at which point, he describes how, “they’re gonna be touching in the middle” (line 664). He then anticipates that the statement will “collapse in on itself and equal  $e$ ” (line 668) and produces the statement in Figure 4.21c. He follows this with a discussion of how he is convinced of the proof, but feels that he must still prove it. He states that, “ $e$  to the  $k$  equals  $e$ ” (line 672). This seems to draw on the “self-inverse” *function* of identity. He continues by claiming, “if there are equal numbers of  $g$  and  $g$  to the negative one, then they will all cancel one another” (lines 682-683), and clarifies this by saying, “so if they're an equal number of  $g$ 's and  $g$  to the negative one's and they're operating on each other, even if it's not commutative, the middle, like, the middle  $g$  and  $g$  to the negative one will cancel and then it'll just collapse in and cancel- they'll all cancel. Until you just- you get  $e$ ” (lines 692-695). This seems to draw on the “get identity” *function* of inverse and the “vanishing” *function* of identity as well as a counting argument related to the number of  $g$  and  $g$  inverse elements.



$$g^k * (g^{-1})^k = (g^{-1})^k$$

$$e^k = e = (g^{-1})^k$$

$$e^k = (g^{-1})^k$$

$$g^k * (g^{-1})^k = e^k = e$$

$$g * g^{-1} = e$$

$$g^{-1} * g = e$$

(a)
(b)
(c)

Figure 4.21. John proving that  $(g^{-1})^k = e$ , given that  $g^k = e$

In response to Q6b, John begins with the equations “ $g^k = e$ ” and “ $g * g^{-1} = e$ .” He then sets the left-hand sides of the two equations equal to each other (“transitive” function of identity) and end-operates with the inverse of  $g$ . John then draws on the “equals” *function* of inverse and “vanishing” *function* of identity to produce the equation  $g^{k-1} = g^{-1}$ . He follows this with a counting argument to explain why it makes sense to him that the inverse of  $g$  would have order  $k-1$ . After John seems satisfied with his proof, he considers the case where  $k = 1$ , which he interprets as  $g$  having to be the identity element. He then reasons that the general solution he produced should be modified to include this case and draws on the “identity as self-inverse” *function* of identity to explain that, in this case,  $p = 1$  is a solution.

**Interview 3 analysis summary.** John’s discussion throughout Interview 3 is very useful in supporting the *form/function* codes generated during analysis of Interview 2. Specifically, John’s proof activity centers around algebraic manipulation of equations drawing on the “letter” *forms* of identity and inverse. John’s impromptu proofs also afford a new *function* of identity in which John replaces an element in an algebraic statement with a concatenation of that element and a *form* of identity. This *function* seemed useful to John as he attempted to prove that left- and right- identities are equivalent.

**Summary of the analysis of John’s conceptual understanding.** Across the three interviews, John draws relatively consistently on “number” and “letter” *forms* of identity and inverse. He tends to draw on other *forms*, for instance, symmetries of a triangle, when example

generating and testing conjectures. Although the most frequently coded *functions* of inverse and identity were the “equals” *functions* throughout the three, John tends to use more diverse *functions* of inverse and identity with each interview. Specifically, his activity focuses more on manipulating algebraic statements in order to re-write them in different ways. Consequently, the eleven *functions* coded during analysis of Interviews 2 and 3 afford these ways of operating.

Many of the new *functions* that emerge in the second and third interview incorporate the “equals” *functions* of inverse and identity. For instance, the “vanishing” *function* of identity in which a *form* of identity is removed from a statement, is a consequence of that *form* of identity serving the “equals” *function* of identity when operated with the other elements in the statement. Similarly, the “vanishing” *function* of inverse reflects the “equals” *function* of inverse as well as the “vanishing” *function* of identity. In this sense, the later *functions* of identity and inverse seem to condense entire processes based on earlier *functions* as single actions that can be carried out without going through the reasoning of the multiple steps.

The development of the “sided” *functions* of identity and inverse are an interesting aspect of John’s conceptual understanding. Specifically, the left/right rule necessitates a situation in which *forms* of identity and inverse are only able to serve *functions* of identity or inverse when operated on the appropriate side of another element. As John points out in Interview 3, he is aware that, generally, the identity “has to work on both sides” (line 275), but, in the context of his rule he says, “this just shows that it works on one [side]” (lines 275-276). This aspect of John’s understanding of identity seems consistent throughout the three interviews. However, during Interview 2, in John’s initial response to Q7, it becomes clear that the “sided” *function* of inverse plays an important role in his proof activity, so much so, that he develops a proof to show that, like the identity, “when you're using inverses, it's commutative for all groups” (lines 1206-

1207). Although he does draw on the “sided” *functions* of identity and inverse during his response to Q3 when explaining how the left/right rule implies the existence of an identity and inverses within a group.

## Chapter 5 - Analysis of Participants' Proofs

The three participants engaged in the production of 27 total proofs (Table 5). I identified these instances by the explicit statement of a relationship or conjecture (often prompted by the interview protocol) that the participant sought to validate or reject through a series of arguments, supporting and explaining their reasoning throughout the proof process. I recorded which protocol the participant was responding to and isolated the blocks of transcript during which the participant was proving as well as explaining their reasoning during follow-up and clarifying question-response exchanges. I then parsed the transcript from each proof to identify distinct arguments that contributed to the participants' communication of their reasoning and iteratively coded the arguments using the Toulmin model of argumentation to construct Toulmin schemes for each argument. I also identified instances in which the participants explained or clarified their statements during their proof production, whether or not these instances were prompted by the interviewer's questions.

Proofs are named according to the first initial of the participants' pseudonyms and the order in which the participant engaged in the proving activity (e.g., Violet's third proof is labeled "V3"). Table 5 summarizes the collection of participants' proofs, identifying the interview during which the participant developed the proof as well as which prompt of the protocol elicited the proof. Because the majority of the proofs consisted of multiple arguments, the arguments are named according to a similar system, which incorporates the proof and argument number (e.g., the first argument of Tucker's ninth proof is named "T9A1"). Some Toulmin schemes I developed during this analysis incorporate Aberdeen's (2006a, p. 7) extensions of the basic Toulmin layout (linked, embedded, etc.). Accordingly, some schemes include multiple nodes of the same type (data, claim, etc.). In order to differentiate between nodes of the same type, I also

developed a naming scheme for these situations. For instance, if a Toulmin model is linked, or contains multiple data that serve the same claim, the data are indexed in order of the participant’s discussion (i.e., Data1, data2). Embedded Toulmin models incorporate an entire Toulmin scheme as a node. I named the sub-nodes within embedded schemes according to the node type, then the index of the parent node (1 if the parent node is not indexed), followed by a decimal point and a second index anticipating the possibility that there might be multiple sub-nodes of the same type. For example, in an embedded, linked Toulmin scheme, the first claim for the Toulmin scheme comprising the second data of the parent model would be indexed as “Claim2.1”. Tables are color coded by node type, with darker colors for children nodes (until the fourth generation). The color code for the Toulmin schemes is as follows: data – blue; claim – green; warrant – red; backing – purple; qualifier – orange; and rebuttal – grey.

*Table 5: Overview of participants’ proofs across interviews; proof naming scheme*

Violet			Tucker			John		
Proof	Interview	Protocol Q	Proof	Interview	Protocol Q	Proof	Interview	Protocol Q
V1	1	6	T1	1	6	J1	1	6
V2	1	7	T2	2	5	J2	2	5
V3	2	5	T3	2	6	J3	2	6
V4	2	6	T4	2	7	J4	2	7
V5	2	7	T5	3	4a	J5	3	3
			T6	3	4b	J6	3	4d
			T7	3	4c	J7	3	4e
			T8	3	4d	J8	3	5
			T9	3	4e	J9	3	6a
			T10	3	5	J10	3	6b
			T11	3	6a			
			T12	3	6b			

### Analyzing Violet’s proofs

Violet generated five proofs across the two interviews in which she participated: two proofs during Interview 1 and three during Interview 2. Her responses to several of the follow-up questions from the protocol constitute arguments that support and/or extend her initial proof approach. Each argument identified in the five proofs affords the construction of a Toulmin

scheme. Table 5.1 provides an overview of Violet’s proofs, including the number of arguments for each proof and type of Toulmin models used for each argument.

*Table 5.1 – Arguments constituting Violet’s proofs*

Proof	Interview	Protocol Q	Number of Arguments	Types of Toulmin Scheme (argument number)
V1	1	6	3	Standard (all)
V2	1	7	3	Standard (all)
V3	2	5	4	Standard (2, 3); Linked (1, 4); Embedded (1)
V4	2	6	1	Embedded, Sequential (1)
V5	2	7	3	Standard (1, 2); Linked, Embedded (3)

**Proof V1: 1 as a multiplicative identity.** During Interview 1, in response to Q6, Violet’s proof (V1) consists of three main arguments, which I have named V1A1, V1A2, and V1A3. The three arguments reflect a similar structure, all of which supporting the same claim, with the first drawing on one specific real number (three), the second alluding to different real numbers, and the third argument replacing the number three with the letter “r.” Violet first reads aloud the statement, “ $1 \in \mathbb{R}$  is the multiplicative identity for the real numbers” (Appendix A), from the student page of the protocol and begins her first argument for the proof by reiterating her initial example of identity, saying,

Okay, I would start by saying that we let one to live in the real numbers. And, if you considered, say, three in the real numbers. Um, well, let's just consider three in this line. And then, now we'll take one that we said was living in the real numbers and we will multiply it by three. And this is obviously three again, which is what I would use as an example for the identity. (lines 873-877)

Together, these constitute a claim, data, and warrant for the proof (Figure 5.1). Violet continues, saying, “And, that's about as far as I could get, unless I was going to be really detailed and show it a few different times,” which qualifies and rebuts for her initial statement. Violet acknowledges that this argument addresses only one real number, rather than all real numbers.

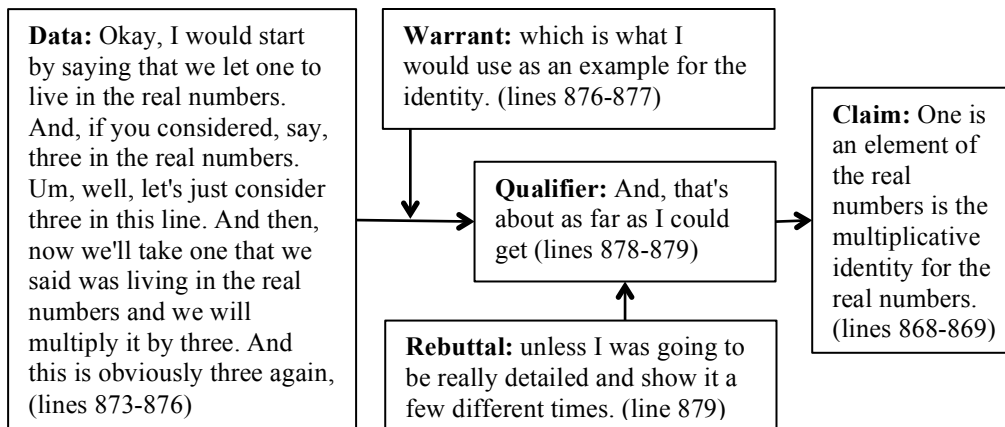


Figure 5.1. V1A1

Violet follows her rebuttal, saying, “So, yeah. I would just start by reiterating what we had, which is one in the real numbers. And then I would consider a different real number, which I chose to be three, just as an example” (lines 879-881). This constitutes the data of a second Toulmin scheme (V1A2, Figure 5.2) in which Violet begins to generalize her previous argument from one specific number to “consider a different real number” (line 881). Apparently, however, this argument is not as general as Violet feels it might be because she qualifies this argument, saying, “I should probably make it general if I was gonna do a really nice proof, so it could go for all the real numbers” (lines 881-882). This supports a sense that Violet feels that her arguments until this point are insufficient and that a formal proof would demonstrate that one is a multiplicative identity for *all* real numbers, rather than some amount of selected real numbers.

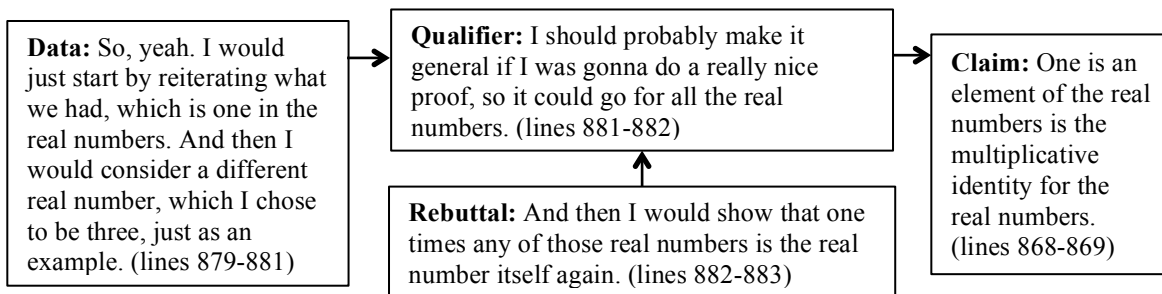


Figure 5.2. V1A2

When asked to explain her qualifier and rebuttal in V1A2, Violet reiterates her argument, instead using the letter “r” in place of the three, saying, “I could say ‘let one live in the real

numbers and let  $r$  be any real number.’ (inaud.) And then I would say, ‘consider  $r$ ’ instead of three. And then we would take one times  $r$ , which is just  $r$ . Cause it’s gonna get you back any real number that you put in. And that would be my proof. Instead of three” (lines 887-890). This constitutes a new argument, (V1A3, Figure 5.3) in which Violet has replaced the inscription of the specific number three with the more general inscription of the letter “ $r$ .” Violet also modifies her written work to reflect this change, marking through the lines in which she has written the number three and replacing the number with the letter “ $r$ .”

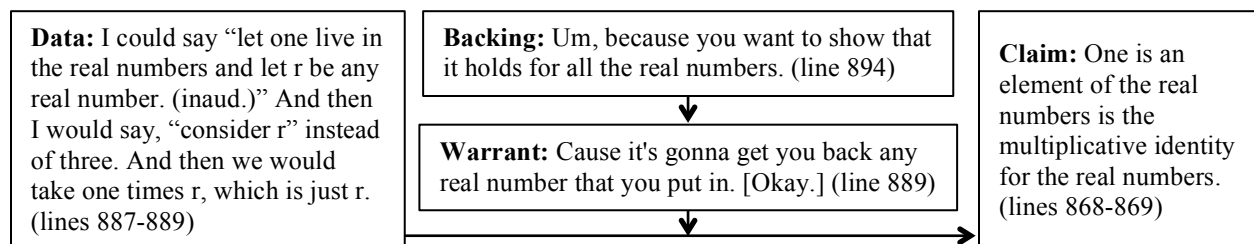


Figure 5.3. V1A3

The shifts between Violet’s arguments reflect an awareness of the need for generality, with each argument becoming less specific than the previous. She moves from one specific number to alluding to using any number and then to the letter “ $r$ ,” which acts in place of a real number within the same argument. This constitutes a shift in *form* of the element being multiplied by one among the three arguments. The *form* of identity, however, remains constant regardless of the *form* of the other element; Violet continues to multiply by the real number one in each argument. It is important to notice that the phrasing of the protocol contrasts with Violet’s discussion up until this point in the interview. Specifically, Violet continually describes identities as acting on the other elements themselves, rather than focusing on the identity as an element, as the prompt might imply. However, the phrase “multiplicative identity” could be seen to reflect the dual, element/operation notions of identity that Violet has drawn upon in her



descriptions of the concept, because it includes the operation “multiplication” and word “identity” within the same phrase.

**Proof V2: Inverse functions.** Violet’s proof in response to Interview 1, Q7 (proof V2) consists of three distinct arguments. In the first argument (V2A1), Violet fails to verify the relationship presented in the protocol, finally qualifying her work by saying, “I’m stuck.” She follows this by altering her approach and seems to validate the statement for at least one case (V2A2). During the follow-up discussion, Violet produces the first iteration of her third argument (V2A3a), in which she explains her reasoning from V2A2 by drawing on her warrant from V2A1. The interviewer then asks Violet to elaborate on this argument, to which Violet responds by augmenting the claim and qualifier in V2A3a, supporting a Toulmin scheme for V2A3b.

Violet begins V2A1 (Figure 5.4) by reading the statement aloud, which is taken as the main claim throughout each argument in this proof. She begins by stating that, “inverse functions would mean that they equal each other” (line 961), and setting the two functions equal to each other. This reflects Violet’s third example of identity in response to Q1 of this interview ( $x + (-x) = 0$ ;  $x = x$ ), which she later adapted as an example of inverse when responding to Q2. Violet then replaces the inscriptions  $f(x)$  and  $g(y)$  with the algebraic statements for the functions,  $\sqrt{x + 3}$  and  $y^2 - 3$ , respectively. Violet then evaluates these functions at the endpoints of their respective domains (-3 and 0), to get  $f(-3) = 0$  and  $g(0) = -3$ . Comparing the two values, Violet describes this as “not what I wanted to happen” (line 968). It seems from the subsequent discussion that Violet anticipates the outcome of the evaluated functions to be either the same number or opposite numbers.

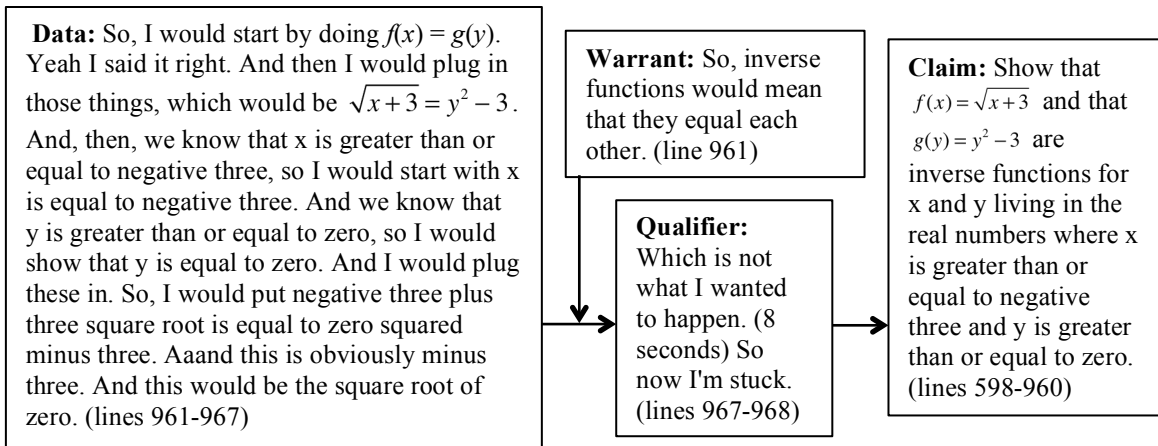


Figure 5.4. V2A1

Asked if she would like to try the proof again, Violet takes 17 seconds and begins a second argument (V2A2), which follows the same approach of setting the two functions equal and evaluating them, although Violet chooses to evaluate  $f(6) = 3$  rather than  $f(-3)$  as she did in V1A1. As before, she evaluates  $g(0)$ , which results in  $g(0) = -3$ . With these values, Violet explains that this, “is what I was hoping for. That way, you could say three minus three is equal to zero” (lines 975-976). Formally, Violet has found values such that the functions are additive inverses when evaluated at these specific values. This supports a sense that Violet is not considering the functions as inverses with respect to composition<sup>8</sup>, but is instead focusing on whether the evaluated functions are equal and/or opposite. Having found such values, Violet positions them in an equation so that they reflect her first example of inverse:  $x + (-x) = 0$ . Violet warrants her work by pointing out that this was her goal throughout her proof activity, qualifies that she had anticipated this relationship to hold for all values in the domain, and rebuts her work by expressing uncertainty whether it should be the case that the relationship holds for all numbers in the domain (Figure 5.5).

<sup>8</sup> The protocol did not specify that the participant is expected to prove that these functions are inverses under composition.

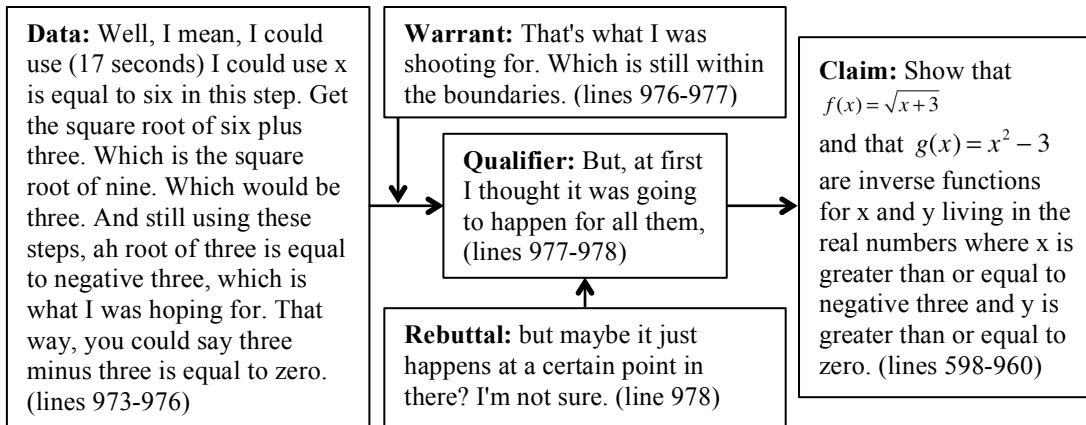


Figure 5.5. V2A2

When asked to explain why she approached her proof this way (Q6a), Violet responds by explaining,

So, to me, that would mean that you would want them to be equal to each other. Like, not necessarily equal, but equal and opposite. So, I would hope for this to be equal to the negative of this or vice versa. And that's what I was trying to get, which is what you get here when you choose  $x$  is equal to 6 and  $y$  is equal to zero. Um, my first thought was that it was going to happen for all of these numbers, but this proves it doesn't. (lines 985-990).

Initially, Violet provides a broad backing to her explanation, which she then rephrases to warrant her use of  $x = 6$  and  $y = 0$  (data). Violet then reiterates the notion that she anticipated this working for all numbers in the domain, adding that her work in V2A1 shows that that is not the case. Together, these statements support a Toulmin scheme for a new argument (V2A3, Figure 5.6), which draws on and extends the first two arguments.

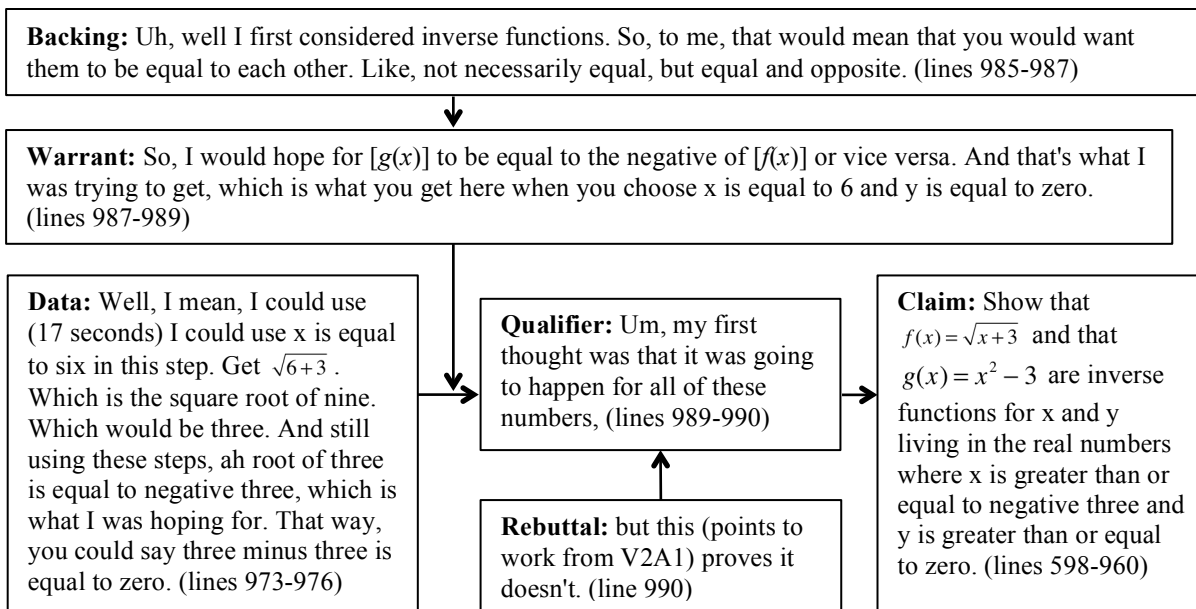


Figure 5.6. V2A3

Violet's work does not address the functions as inverses with respect to function composition, but instead draws on the "opposite" *function* of inverse that centers on an additive notion of "opposite." Violet demonstrates that the functions satisfy inverses in this sense only for the case when  $x = 6$  and  $y = 0$ . Further, she acknowledges that she is unsure whether it is sufficient to show only one case. The Toulmin schemes afford insight into how Violet adapts her argument to make better sense of how the functions  $f$  and  $g$  might be considered inverses. Specifically, Violet changes the values for which she evaluates one function in order for the functions to satisfy her original criteria, which she outlines through her warrant and backing in each argument. On the other hand, Violet's warrant did not change significantly across the three arguments (the functions should be equal or opposite when evaluated for specific values).

**Proof V3: Subgroup proof, integer powers of 2.** Violet produces proof V3 during Interview 2 while responding to Q5: "Prove or disprove:  $H = \{g \in G \mid g = 2n, n \in \mathbf{Z}\}$  is a subgroup of the group  $G$  of non-zero real numbers ( $\mathbf{R} \setminus \{0\}$ ) under regular multiplication." It should be noted that Violet has seen this prompt before, during classwork and, so, was familiar

with it, although she does not fully remember her previous approach during the interview. Violet's proof in response to this prompt consists of four arguments that support Violet's validation of the claim, the majority of her proof activity occurring during the first argument, V3A1. Because the protocol asks for a proof of whether a given set is a subgroup, as Violet points out in her initial discussion (Figure 5.7; Warrant), her proof involves demonstrating the four group axioms<sup>9</sup>. Accordingly, the Toulmin scheme for V3A1 is a linked Toulmin scheme (Aberdein, 2006a) that includes multiple data, each constituted by its own embedded Toulmin scheme, all of which serve together to support validation of the original claim. Violet begins by reading the problem statement from the protocol aloud, which serves as the major claim for the entire proof. She then explains that she needs to show that the four group axioms hold in order for the subset to be a subgroup, constituting the warrant and backing of the overall proof.

Violet then begins verifying the four group rules, first attempting to find an identity and inverse within the subset  $H$ , saying,

So, the identity would just be one, because two raised to one is whatever. And then, we want something that brings it back to one, which would be zero, because two raised to zero is just one. No. Yes. That would work. So, I would say, (writing; mumbling) the identity is equal to one and the inverse would be zero. So it has an identity and an inverse. (lines 323-326)

This excerpt constitutes the first two data of V3A1, each of which can be viewed as an argument in its own right. Specifically, in the first sentence, Violet claims that one is the identity of the group (Claim1.1), supporting this claim with the data, "because two raised to one is whatever" (line 323; Data1.1, Figure 5.7). Similarly, Violet claims that zero is the inverse

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<sup>9</sup> During classroom discussion, the students had determined that it was not necessary to show associativity for subgroups. Violet alludes to this during her discussion, but states that she cannot recall which rules she is able to assume and, so, decides to prove all four.

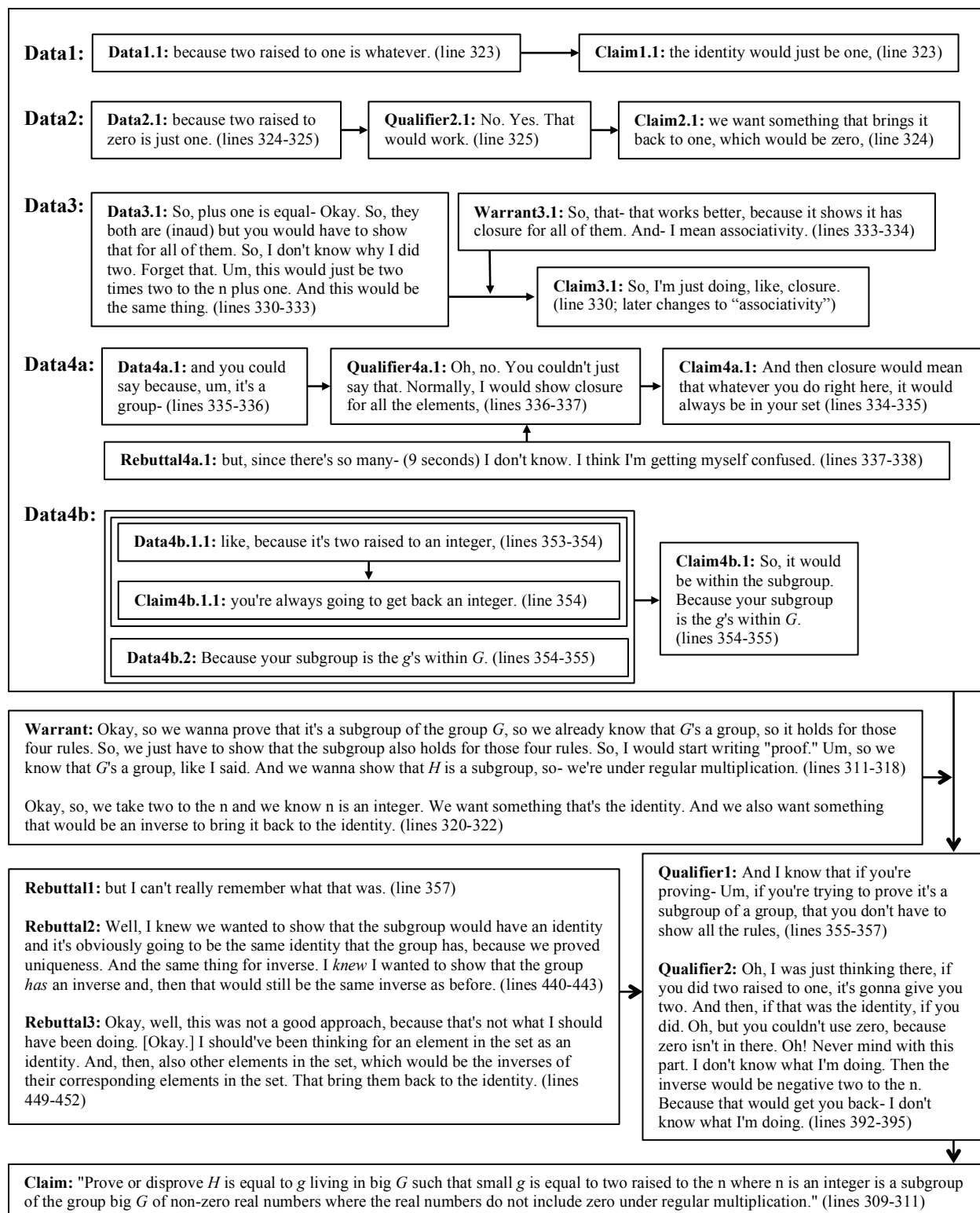


Figure 5.7. V3A1

(Claim2.1) and supports this claim with data2.1, saying, “because two raised to zero is just one” (line 324-325). Violet’s reasoning that the “two raised to one is just whatever” (line 323), though somewhat vague, draws more broadly on the notion that that an identity does not change the object upon which it acts. In this sense, the exponent one would serve as an identity by making the base number, two, resemble itself and, thus, the exponent one is able to serve the “resemble itself” *function* of identity, though Violet does not explicitly say this in her explanation. Similarly, the exponent zero is able to serve the “bring back” *function* of inverse because the result of raising two to the zero power is one.

It is important to note that this “one” Violet refers to in data2.1 is the value of the entire statement “ $2^0$ ,” rather than the exponent “one,” which she claims is the identity. Formally, the identity of the group (which was defined in the prompt as being under the operation of real number multiplication) is the number one. Accordingly, Violet’s argument constituting data2 could be used to formally demonstrate that the subset  $H$  contains the identity of the larger group  $G$ , which Violet is attempting to show in Data1. Overall, Violet’s first two data in V3A1 do not situate the identity and inverse(s) relative to the group operation of multiplication, instead drawing on the definition of  $H$ , which uses exponential notation. This supports a sense that, although she describes the group as “under regular multiplication” (line 315), Violet is not contextualizing the identity and inverse that she found with respect to the group operation.

During her discussion coded as Data3, Violet is attempting to show that the subset satisfies associativity (lines 333-334). She considers the statement “ $2^2 + 2^2 + 1$ ” and groups it as  $(2^2 + 2^2) + 1$  and  $2^n + (2^n + 1)$ , later rewriting these as  $(2^n + 2^n) + 1$  and  $2^n + (2^n + 1)$  and then rewriting them as  $(2^n + 2^n) + 2^n$  and  $2^n + (2^n + 2^n)$ . Violet evaluates each statement, in order to verify that each pair is equivalent. This activity seems to satisfy for Violet that the set  $H$  satisfies

associativity. As with her subarguments coded as Data1 and Data2, Violet’s argument for associativity does not include a discussion of the group operation of multiplication. Rather, Violet uses addition to determine whether the set is associative, beginning with specific numbers that satisfy the definition of set  $H$  and generalizing these by replacing the exponents with the letter  $n$ . In her argument comprising Data4, Violet generates two distinct arguments (coded as Data4a and Data4b) that the set  $H$  is closed. In her discussion coded as Data4a, Violet begins to argue that, “because, um, it’s a group,” but truncates the sentence and focuses on the number of elements in the subset, leading her to conclude that she is confused. When asked to explain her thinking, Violet generates a second argument (coded as Data4b) that focuses on the integer powers of elements in  $H$ , which she claims causes them to remain within the subgroup. Again, Violet does not discuss the group operation of multiplication, which supports a sense that this is not informing her approach.

A handwritten group table for V3. The table is a grid with four columns and four rows. The columns are labeled with powers of 2:  $2^{-1}$ ,  $2^1$ ,  $2^2$ , and  $2^3$ . The rows are labeled with powers of 2:  $2^{-1}$ ,  $2^1$ ,  $2^2$ , and  $2^3$ . The entries in the cells are the sum of the row and column labels. For example, the cell at the intersection of  $2^{-1}$  and  $2^1$  contains  $2^0$ . The table is drawn with a grid of lines, and there are ellipses at the end of the first row and first column to indicate continuation.

$2^{-1}$	$2^1$	$2^2$	$2^3$	...
$2^{-1}$	$2^0$	$2^1$	$2^2$	
$2^1$	$2^2$	$2^3$	$2^4$	
$2^2$	$2^3$	$2^4$	$2^5$	

Figure 5.8. Violet’s group table for V3

When asked to explain what else Violet thinks she needs to show in order to complete the proof, Violet begins generating a group table (Figure 5.8). She lists the elements of  $H$ , beginning with  $2^1$ ,  $2^2$ , and  $2^3$  across the top of the table, then  $2^{-1}$ ,  $2^1$ , and  $2^2$  down the left-most column, describing that each would continue in all both directions (V3A4, Data1; Figure 5.10). At this point, Violet describes a discussion she had with the instructor about this approach, which led Violet to feel that the table would not be sufficient to prove the group axioms for an infinite number of elements, seemingly stalling V3A4. At this point, the interviewer then asks Violet to



discuss her previous work in V3A1, to which Violet responds by rebutting her earlier Claim2.1 from V3A1 (that the exponent “one” is the identity), leading to two new, short arguments: V3A2 and V3A3 (Figure 5.9). In these arguments, Violet draws on the problem statement’s exclusion of the element zero (it is excluded from the group  $G$ , but Violet excludes it from the potential exponents that define  $H$ ) to rebut her earlier claim that “the inverse would be zero” (line 326). She follows this by beginning a second argument in which she claims that “negative two to the  $n$ ” would be the inverse, but abandons this argument, ending the statement that serves as data mid-sentence and qualifying her statement, “I don’t know what I’m doing” (line 395).

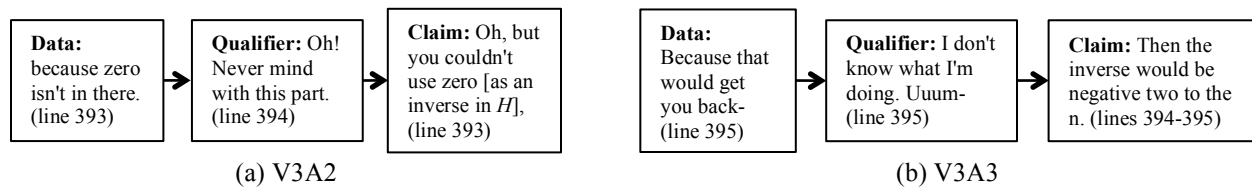


Figure 5.9. V3A2 and V3A3

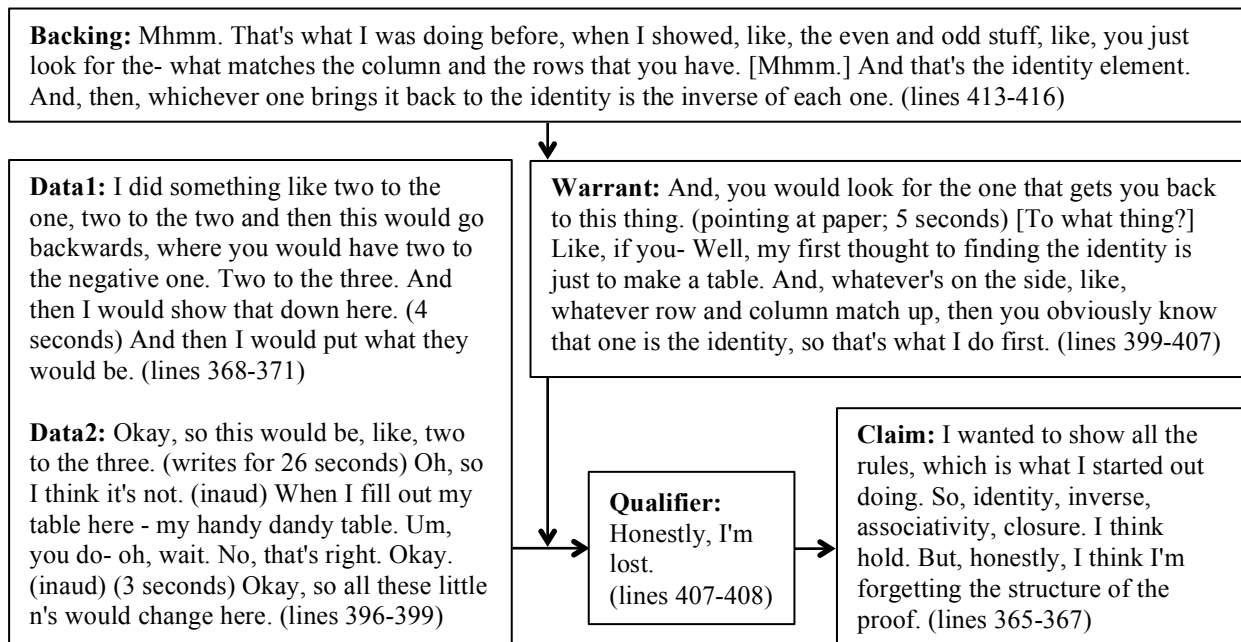


Figure 5.10. V3A4

After this discussion, Violet returns to her previous argument (V3A4) and continues creating the group table by filling it in with the product of the entries in the respective row and

column (e.g., the entry in the  $2^2$  column and  $2^1$  row is  $2^3$ ). This is the first instance during this proof in which Violet has multiplied elements of the set  $H$ . Notice, though, that Violet omitted the element  $2^0$  in the left column and top row. Violet then describes how she might use the group table to identify the identity element and inverse elements (Warrant, V3A4; Figure 5.10). In this discussion, Violet refers to the even/odd group table she had produced earlier in the interview that led to the “matching” and “finding inverse” *functions* of identity (warrant and backing, V3A4; Figure 5.10). However, immediately after describing how she would identify the identity of the group, Violet says, “honestly, I’m lost” (lines 407-408). Although this qualifier abruptly ends V3A4, there is evidence that Violet’s activity constructing this argument, along with arguments V3A2 and V3A3, afford her insight into how she might approach the proof differently.

Specifically, when later asked how identity informed her approach to the proof, Violet responds by describing the first argument she generated, saying,

Okay, well, this was not a good approach, because that's not what I should have been doing. I should've been thinking for an element in the set as an identity. And, then, also other elements in the set, which would be the inverses of their corresponding elements in the set. That bring them back to the identity. (lines 499-452)

In this excerpt, Violet emphasizes the identity as an element in the set and inverses as “other elements in the set” with “corresponding elements” (line 451). This contradicts Claim1.1 and Claim2.1 in V3A1 because, as Violet points out, the identity and inverse that she had previously found are not elements of the set  $H$ . Further, Violet seems to be rebutting that there is a single inverse, using the plural form “inverses” for the first time in proof V3. Violet’s early discussion during V3A1 focused on exponent powers that seem to satisfy the “resemble itself” *function* of identity and “bring back” *function* of

inverse. However, Violet's shift to using the group table afforded her an opportunity to multiply elements of  $H$ , incorporating the group operation for the first time in the proof. Further, Violet is able to draw on the "matching" and "finding inverse" *functions* of identity that allow her to focus on set elements that might serve the *functions* of identity and inverse, rather than exponents that seem to serve those functions.

**Proof V4: Uniqueness of inverse elements.** Violet's proof (V4A1) in response to Interview 2 Q6 consists of one argument modeled by a Toulmin scheme embedded with a sequential Toulmin scheme (here, two Toulmin schemes concatenated so that the claim of the first argument serves as data for the second argument) as the data for the overarching argument. Violet had proven this conjecture before, with her peers and instructor in class. This likely contributes to the conciseness of the proof, though she had also seen the prompt from question Q5 before Interview 2. Violet begins her proof by reading the prompt aloud and rephrasing the conjecture, saying, "So, we wanna show that, for each element  $g$  in the group  $G$ , that the inverse of  $g$  is unique" (lines 461-462). This constitutes a claim for the overarching argument. Violet continues by situating her activity as a proof by contradiction, saying, "So, I would start by assuming that there were *two* inverses. So- A proof by contradiction" (lines 462-463). This is coded as backing within the Toulmin scheme (Figure 5.11) because Violet later draws on this to warrant that her data supports the claim. Violet is providing a context that affords her the ability to generate two "letter" *forms* of inverse for the element  $g$ . In doing so, Violet says, "So, we'd let- (2 seconds)  $g$  inverse be an inverse of  $g$ . (exhales heavily) Um. And (inaud) and I'll say  $i$  be the inverse of another  $g$  in  $G$ " (lines 463-464). Although Violet names " $i$ " the "inverse of another  $g$  in  $G$ " (line 463), she later clarifies that she is thinking of " $g$  inverse" (which she inscribes as  $g^{-1}$ ) and " $i$ " as inverses of the same element,  $g$ , in the group.

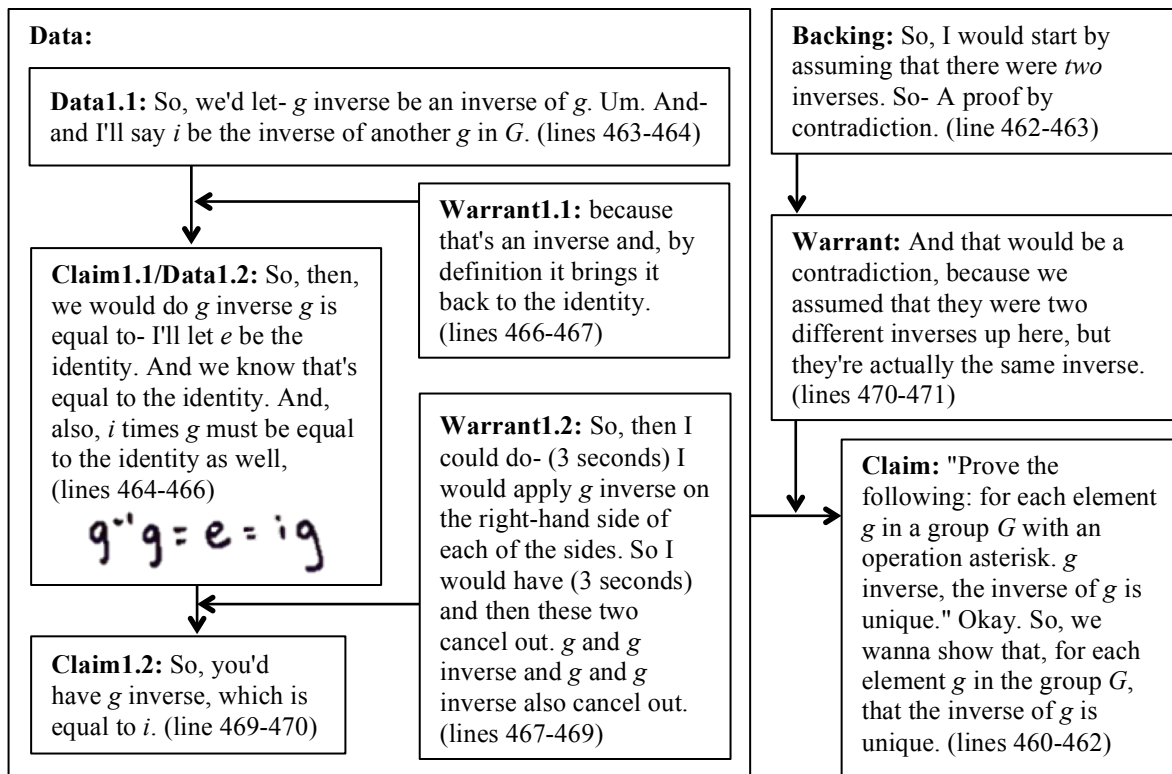


Figure 5.11. V4A1

Having generated the two inscriptions for the inverse of  $g$ , Violet then draws on the definition of identity to warrant the production of the equation, " $g^{-1}g = e = ig$ " (Claim1.1/Data1.2, Figure 5.11). Having constructed this equation, Violet uses this claim as data and draws on the "end-operating" and "vanishing" *functions* of inverse to warrant (Warrant1.2) Claim1.2, saying, "So, then I could do- (3 seconds) I would apply  $g$  inverse on the right-hand side of each of the sides. So I would have (3 seconds) and then these two cancel out.  $g$  and  $g$  inverse and  $g$  and  $g$  inverse also cancel out. So, you'd have  $g$  inverse, which is equal to  $i$ " (lines 467-470). From this, Violet states, "And that would be a contradiction, because we assumed that they were two different inverses up here, but they're actually the same inverse" (lines 470-471). This, supported by her earlier backing that this was a proof by contradiction supports the original claim that the inverse of the element  $g$  is unique in the group  $G$ .

Compared to her earlier proofs, Violet's fourth proof (V4) is relatively straightforward, supported by the fact that the entire protocol, including follow-up questions spans less than eight minutes during the interview as well as Violet's direct approach to the proof. It provides useful insight into how her various *functions* of identity and inverse afford Violet reasoning that produces and warrants change in the equations relating the element and its two presumed inverses. Specifically, Violet's "bring back" *function* of inverse and "transitive" *function* of identity support the production of the initial data from which Violet proceeds. Further, Violet is able to manipulate the equations by drawing on the "end-operating" and "vanishing" *functions* of inverse in order to warrant new claims. Finally, Violet relies on the broader logic of proof by induction to warrant and back her work so that it may support the original claim.

**Proof V5: Subgroup proof, normalizer of h.** Violet begins proof V5 by reading the statement aloud and repeating the group axioms, explaining that it is necessary to show that these axioms hold within the subgroup (warrant, Figure 5.12). This is coded as a warrant for the argument for the first of three arguments Violet produces during this proof. Violet then writes the equation used to define  $H$ , " $g*h*g^{-1} = h$ ," and draws on the "end-operating" and "vanishing" *functions* of inverse to manipulate this equation (data, Figure 5.12). This transcript is coded as data for argument V5A1. However, Violet seems unable to draw on this activity to warrant the data to support the claim in a meaningful (to her) way in this moment. Near the end of this excerpt, Violet seems to be describing closure, although she does not make any assertions about the closure of the subset  $H$ .

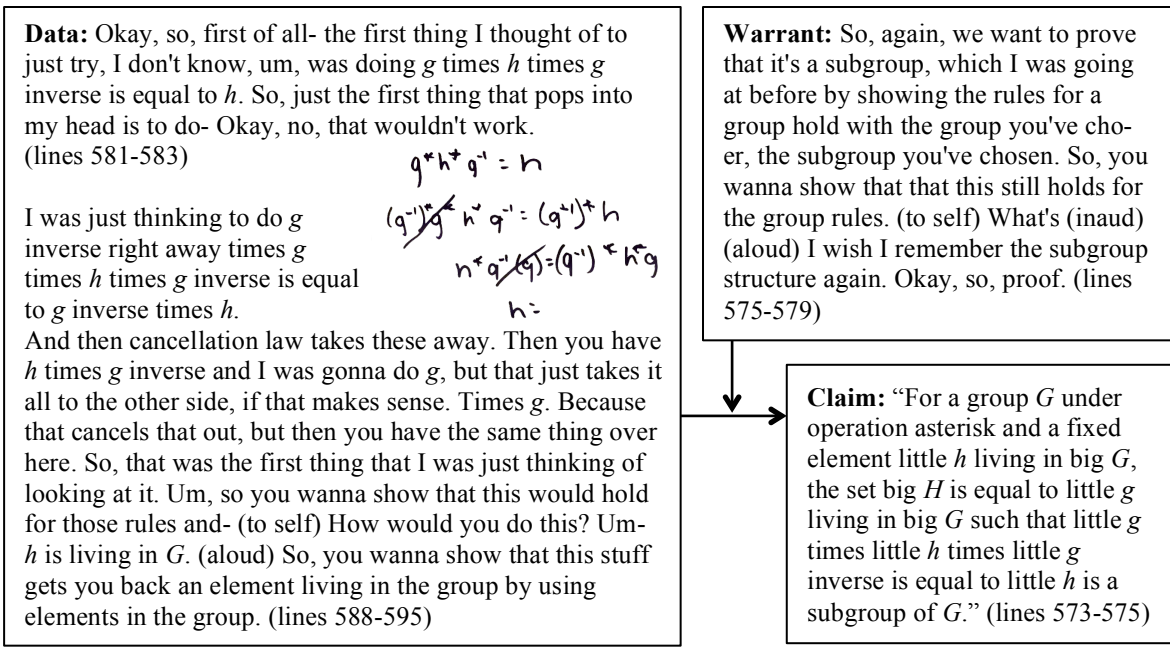


Figure 5.12. V5A1

After several second of silence, Violet reiterates her statement about getting “back and element living in the group” (line 594), saying,

What I want to do by reading it is to show that  $g$  um multiplied by  $h$  multiplied by  $g$  inverse - if that was multiplication, whatever operand that is - that all of these using these elements gets you back another element in- living in  $G$ . [Mhmm.] So, I would probably, just, play around with it to try to see if I could show that that's an element living in  $G$ , like, to get this down to just be  $g$  or  $h$ . (raises pitch of voice, conveying hesitance or uncertainty) (4 seconds) So, I would- I'm just gonna start by doing stuff. I don't really know what I'm trying to get to or what I'm gonna get. (lines 603-609)

In this excerpt Violet calls the group operation “multiplication” and describes a general goal for her activity. However, she does not describe how she might interpret her activity. Further, her description of closure seems to focus on the elements being contained in the group  $G$  and whether the two elements operated together would also be an element of  $G$ . This is distinct from a formal approach to verify the closure of a subset of a group, which considers two arbitrary elements in the subset when operated together. It seems as though

Violet is unable to (or did not know to) fix two such elements of  $H$  as she was able to do with the two inverses she presented in V4, instead manipulating only the given equation.

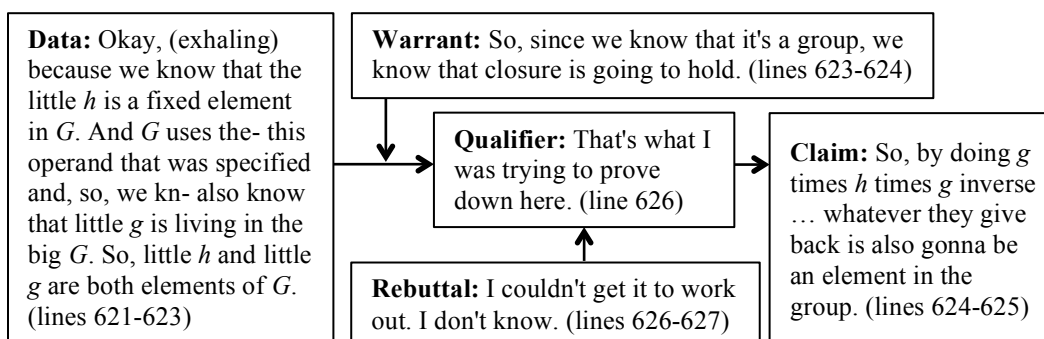


Figure 5.13. V5A2

When asked whether she thinks  $H$  is a subgroup of  $G$ , Violet states that she does. The interviewer then asks her why she believes this, which leads Violet to generate a second argument (Figure 5.13). In this argument, Violet describes  $h$  and  $g$  as “elements of  $G$ ” (line 623). She then points out that, “since we know that it’s a group, we know that closure is going to hold” (lines 623-624). Again, Violet is describing the elements relative to the group  $G$ , rather than the subset  $H$ . This seems to warrant for Violet that the subgroup satisfies closure. However, Violet rebuts the argument, saying, “I couldn’t get it to work out. I don’t know” (lines 626-627). At this point, the interviewer clarifies the prompt, emphasizing with Violet that the element  $h$  is fixed in the group and restating the definition of  $H$ . This seems to make sense to Violet because she finishes the interviewer’s sentence explaining the definition of  $H$ . When asked what she is thinking about, Violet responds by describing the process of replacing the elements  $h$ ,  $g$ , and  $g^{-1}$  with the numbers three, two, and negative two, respectively (Claim1.1, Figure 5.14), claiming, “you would want that to equal three” (lines 649-650). She then evaluates the statement “ $2*3*(-2)$ ” using regular multiplication of real numbers, noticing that this product does not equal the anticipated outcome, but dismisses this and evaluates “ $(-2)*3*2$ .”

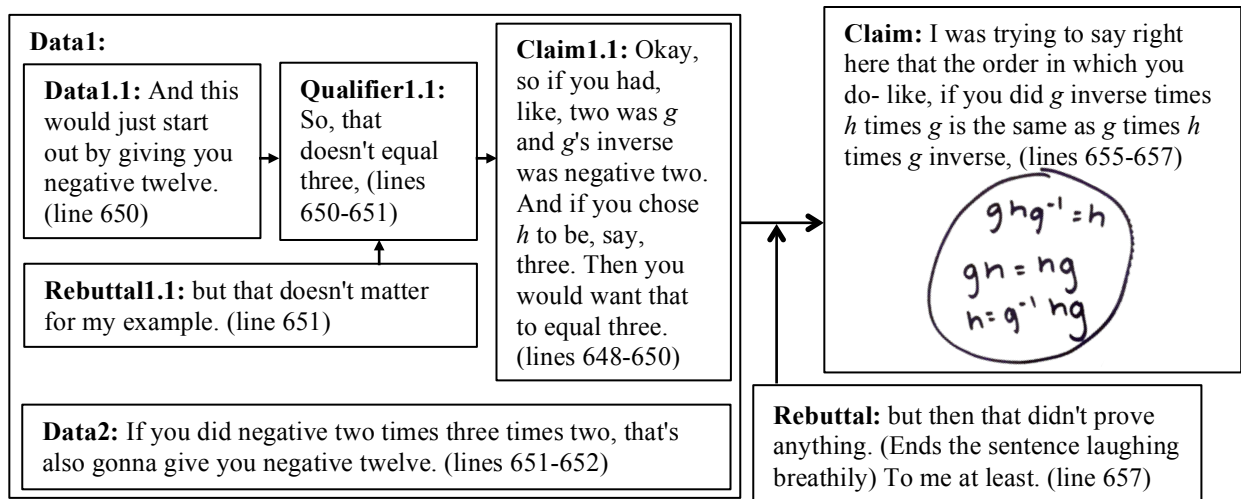


Figure 5.14. V5A3

These processes constitute two data that support a broader claim related to the data for V5A1, which Violet rewrites. Violet’s activity in the excerpt coded as V5A3 shifts from the “letter” *form* of inverse to “number” *form* of inverse. However, Violet uses the additive inverse (negative two) with the operation of multiplication. This explains why the outcome of Violet’s activity differs from her expectation. However, this difference does not seem to bother Violet, likely because she also anticipates the product of the second statement to also equal negative twelve, though she does not explicitly state this. Violet draws on this instantiation to support the more general relationship between  $g*h*g^{-1}$  and  $g^{-1}*h*g$  that she demonstrates with her activity during V5A1. Although this argument seems to verify for Violet that  $g*h*g^{-1} = g^{-1}*h*g$ , she follows her claim with the rebuttal, “but then that didn’t prove anything,” adding, “to me at least” (line 657). This sentiment is consistent throughout the remainder of Violet’s discussion of the proof, resulting in her later stating that she does not feel as though she has proven the conjecture.

Throughout her argument, by beginning with the equation that defines  $H$ , Violet seems to assume that the letter  $g$  is an element of  $H$ , but does not describe it as such at any point during her proof. This is consistent with her statement, “I’m just gonna start by doing stuff. I don’t really



know what I'm trying to get to or what I'm gonna get" (lines 608-609). Violet does not provide a warrant to support how the data resulting from her activity might justify any claims about whether  $H$  is a subgroup of  $G$ . Further, Violet does not seem to consider whether the identity of  $G$  satisfies the definition of  $H$ . Instead, when asked during follow-up questions whether  $H$  contains an identity, Violet generally describes it using nearly identical language to her definition in response to Q1, saying, "It would be (9 seconds) whatever just brought it back to  $h$ ? It would be  $g g$  inverse?" (line 790). She then follows this by saying, "No because it like- It's on a different side still" (line 795). This excerpt supports a sense that Violet views the equation used to define the set  $H$  similarly to the equation she uses to define identity.

**Summary of analysis of Violet's proofs.** Several patterns emerge across Violet's proofs. First, in most of her proofs, Violet replaces one *form* with another. This constitutes *form* shifts in which violet changes one form for another, for which she provides different reasons, depending on the context in which she is working. For instance, in V1A3, Violet shifts from using the number three in the equation " $3 \cdot 1 = 3$ " to the letter  $r$ . As Violet explains, this allows her work to "show that [one is the multiplicative identity] holds for all the real numbers" (line 894), which affords her the ability to make a more general proof. Throughout this proof, Violet consistently uses the operation of multiplication, which allows one to consistently serve the "resemble itself" *function* of identity. During V2, Violet replaces the letters  $x$  and  $y$  in the functions  $f(x)$  and  $g(y)$  with specific numbers, which allows Violet to evaluate the functions at specific values and compare them. However, Violet focuses on showing that the functions are "equal and opposite" (line 987, formally she chooses specific values of  $x$  and  $y$  so that  $f(x) = -g(y)$ ), rather than as inverses under composition, which is understandable, given the prompt does not explicitly ask if

the functions are inverse under composition. This results in Violet thinking that she has produced a proof that demonstrates the claim given in the prompt for these specific values.

In V5, Violet is able to manipulate the equation used to define  $H$  to generate the equation  $h = g^{-1} * h * g$  by drawing on the “letter” *form* of inverse to serve the “end-operating” and “vanishing” *functions* of inverse, but is unsure how this might help her show that  $H$  is a subgroup of  $G$ . Instead, in V5A3, she again shifts *forms* when she replaces the elements  $g$ ,  $h$ , and  $g^{-1}$  with 2, 3, and -2, respectively, in order to gain a better sense of how she might prove that the set  $H$  is a subgroup. However, having replaced  $g$  and  $g^{-1}$  with additive inverses, in calculating the expression  $2 * 3 * (-2)$ , Violet uses the operation of multiplication, which results in -12, rather than 3, which she anticipates being the result. She seems to reconcile this discrepancy by noticing that  $(-2) * 3 * 2$  also equals -12. However, this does not seem to afford Violet any insight into the proof.

Between V3A1 and V3A4, Violet changes from discussing the elements in set  $H$  individually, to organizing them in the group table. This affords Violet the insight to begin to shift the *functions* upon which she wishes to draw from the “resemble itself” *function* of identity and “bring back” *function* of inverse to the “matching” and “finding inverse” *functions* of identity. Specifically, Violet’s initial approach to the proof (V3A1) does not correspond the integer powers of 2 with the operation of real number multiplication when identifying an identity or inverse(s), even though Violet mentions real number multiplication when beginning the proof. Instead, she initially focuses on exponents that seem to serve the “resemble itself” and “bring back” *functions* of identity and inverse, respectively, which leads her to identify a single number, 0, as “the inverse” (line 326) and exponentiation by one as the identity of the subset. In this instance, Violet likely views an aspect of the set definition as serving the necessary *functions* of inverse because she is not attending to the group operation.

However, as Violet continues her proof, she organizes the group elements in a group table, which affords a shift in “functions” upon which she is able to draw and also allows Violet to focus on multiplying elements in the set  $H$ . It seems that in considering the “finding inverse” *function* of identity (which the group table affords), Violet realizes that her previous claim that zero is the inverse is inconsistent with each element in the group having an inverse. This leads to V3A2 and V3A3 in which Violet rejects her claim that 0 is the inverse of the group. Further supporting this notion is Violet’s later explanation that she “should’ve been thinking for an element in the set as an identity. And, then, also other elements in the set, which would be the inverses of their corresponding elements in the set” (lines 450-452). However, although her *function* shift affords Violet this insight into her earlier invalid argument, her construction of the group table omits the element  $2^0$ . Accordingly Violet is unable to identify the identity element because, in the table she produces, no element serves the “matching” *function* of identity.

## Analyzing Tucker’s proofs

Across the three interviews, Tucker produces 12 proofs (Table 5.2) comprised of 36 arguments coded with Toulmin schemes. Tucker produces most of the proofs during the third interview, though his response to Interview 2 Q7 provides the greatest number of arguments of all the proofs. For the most part, Tucker’s proofs involve one argument that comprises Tucker’s primary validation of the given claim from the prompt with one or two additional arguments supporting or explaining part of the main argument.

Table 5.2 – Arguments constituting Tucker’s proofs

Proof	Int.	Protocol Q	Number of Arguments	Types of Toulmin Scheme Used (argument number)
T1	1	6	2	Linked (1); Standard (2)
T2	2	5	2	Linked (all); Embedded (all)
T3	2	6	2	Embedded (all); Sequential (1)
T4	2	7	11	Linked (1, 2, 6); Embedded (3, 6, 9, 10, 11); Sequential (3, 7, 9, 10, 11); Standard (4, 5, 8)
T5	3	4a	1	Embedded; Sequential
T6	3	4b	2	Embedded (all); Sequential (2)
T7	3	4c	1	Embedded
T8	3	4d	3	Embedded (1); Linked (1); Sequential (2, 3)
T9	3	4e	2	Embedded (all); Linked (1); Sequential (2)
T10	3	5	2	Embedded (1); Sequential (2); Divergent (2)
T11	3	6a	3	Linked (1, 2, 3); Embedded (1, 3); Sequential (1, 3)
T12	3	6b	2	Embedded (1); Linked (1); Sequential (all)

**Proof T1: 1 as a multiplicative identity.** In response to Interview 1 Q6 (Appendix A), Tucker produces a proof consisting of two main arguments to verify that 1 is the multiplicative identity for the real numbers. In his first argument (T1A1, Figure 5.15), Tucker begins by describing how the phrasing of the protocol emphasizes that 1 is a “multiplicative identity.” He contrasts this with the notion of an additive identity, which warrants his argument by situating his activity relative to multiplication, rather than a different operation. Tucker then backs this data by describing how, “one plus a scalar- another scalar is not gonna equal that same scalar” (lines 616-617). He continues, saying, “if we take any number- Let  $x$  be an element of the real

numbers. To prove it, let's just try it out. So, one times by  $x$  is equal to  $x$ " (lines 617-618). This excerpt constitutes data for his argument in which he fixes an arbitrary real number, inscribing it with the letter " $x$ ," and verifies that multiplying this number by one satisfies the "operate/same out" *function* of identity. This seems to be sufficient for verifying the statement in the protocol because Tucker says, "Um, let's see, how more can I say about this?" (lines 618-619). However, he goes on to add, "Uh. So, I guess you have  $x$  is equal to  $x$ " (line 619). With this statement, Tucker simplifies the first side of the equation so that his statement becomes the tautology " $x$  is equal to  $x$ ," extending his initial data so that the equation is trivially true. Although it is unclear whether Tucker feels that the second data is necessary, he does immediately say, "therefore, um, one is an identity for  $x$ . I guess" (lines 619-620), which supports a sense that Tucker feels these data are sufficient for the proof.

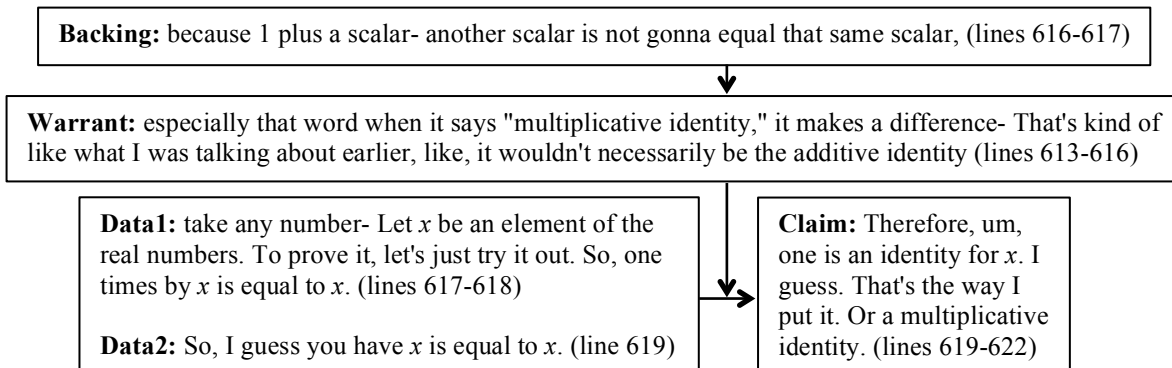


Figure 5.15. T1A1

During the follow-up discussion, when asked how the way he thinks about identity informed his approach to the proof (Q6b), Tucker produces a second argument (T1A2, Figure 5.16) that parallels T1A1 and elaborates on his reasoning in T1A1. Specifically, Tucker describes  $x$  as an arbitrary real number as part of the data of the argument and qualifies his argument by saying "it's trivial to try it" (line 632). He also warrants his argument by rephrasing

his discussion from earlier in the interview (Q1) that led to the development of the “operate/same out” *function* of identity.

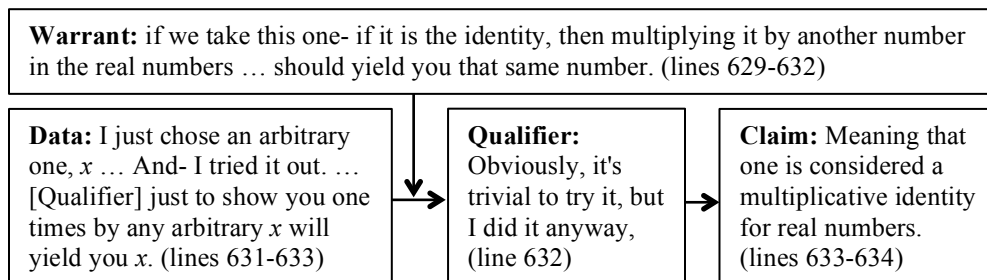


Figure 5.16. T1A2

**Proof T2: Subgroup proof, integer powers of 2.** Tucker produces his second proof (T2) during his response to Interview 2 Q5. Proof T2 consists of two different arguments. The first argument (T2A1, Figure 5.17) consists of a linked, embedded Toulmin scheme in which four data, each composed of a Toulmin scheme in-and-of itself, support a warranted and qualified claim that the set of integer powers of two ( $H = \{2^n \mid n \in \mathbf{Z}\}$ ) form a subgroup of the nonzero real numbers under standard multiplication ( $\mathbf{R}/\{0\}, \cdot$ ). The second argument is a standard Toulmin scheme that parses out Tucker’s discussion during a follow-up question in which he more generally argues that the set  $H$  contains the inverse of each of its elements. Tucker had previously proved that this set is a subgroup during classwork. Accordingly, his proof is relatively straightforward, compared to some of his other proofs.

As stated, the Toulmin scheme for T2A1 contains four embedded data, in order: that the set  $H$  is a subset of  $G$  (coded as Data1), is closed (Data2), is associative (Data3), and contains the inverse of each element (Data4). Tucker begins with an argument that  $H$  is a subset of  $G$ . His data during the production of this argument consists of Tucker sequentially replacing the “ $n$ ” in the exponent of 2 with the numbers 1, 2, and 3 and calculating the result. He then alludes to the continuation of this process, saying, “And so on and so forth” (lines 444-445). He follows this by warranting this data, explaining, “you're not going to get something after doing this that's gonna

be not in the real numbers. You're not going to get some crazy fraction or some crazy, uh, coefficient like that” (lines 446-448).

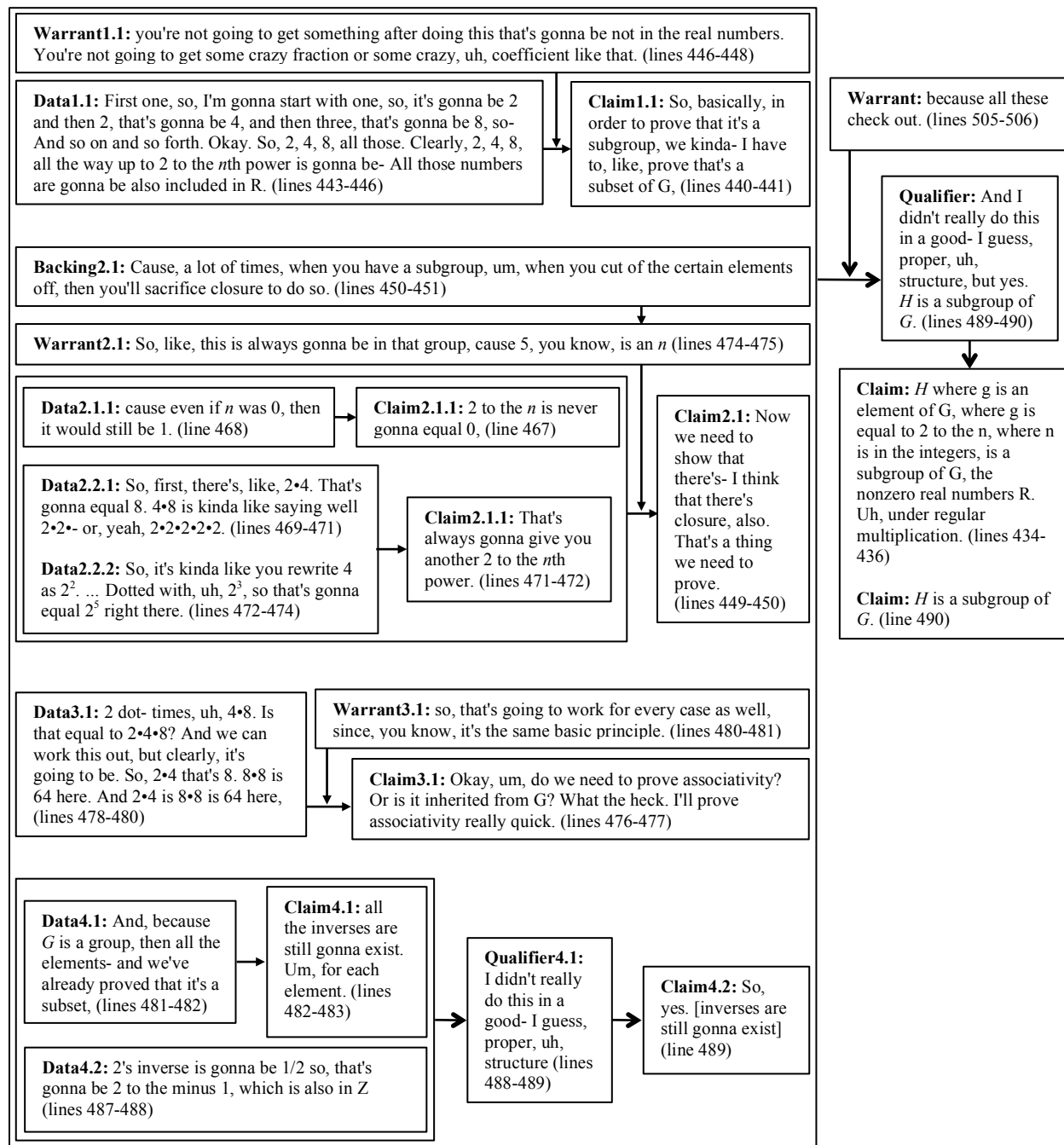


Figure 5.17. T2A1

Tucker then begins a new subargument (Data2) by saying, “Now we need to show that there's- I think that there's closure, also. That's a thing we need to prove” (lines 449-450). He

then describes how, in forming a subgroup, some of a group's elements might be excluded, which might "sacrifice closure" (line 451), and asks, "What is the operation in  $H$ ?" (line 452). The interviewer then states aloud the section of the prompt in which the group  $G$  is described as being under the operation of "regular multiplication," to which Tucker replies, "So, so is  $H$ , I take it?" This discussion leads to the interviewer stating, "Yeah. If it were a subgroup, it would inherit that." This interaction might inform Tucker's proof production, specifically focusing Tucker's activity on the multiplication of elements, rather than the exponentiation used to define  $H$ . Tucker then continues with his proof, providing the initial data that  $2^n$  will never be zero, even if  $n$  is zero, explaining that this would be one. He then discusses the product of two and four, stating that it is eight and describes the product of four and eight. During this description, Tucker writes the product and then re-writes it as integer powers of two, explaining that their product is "two times two times- (more quickly) two times two times two" (line 471). From this, he claims that this process will "always give you another two to the  $n^{\text{th}}$  power" (lines 471-472). He then warrants his data and restates the claim, saying, "... this is always gonna be in that group, cause five, you know, is an  $n$ , so it's gonna have closure as well" (lines 474-475).

Tucker follows this by asking whether he needs to prove that the set  $H$  satisfies associativity (Data3), concluding that he will show it anyway, saying, "what the heck" (line 477). He goes on to calculate the products  $(2 \cdot 4) \cdot 8$  and  $2 \cdot (4 \cdot 8)$ , concluding that these each equal 64. He then warrants this data by saying, "so, that's going to work for every case as well, since, you know, it's the same basic principle" (lines 480-481). With this statement, Tucker seems to be generalizing the specific case that he has calculated, though the claim and its generality seem relatively self-evident to Tucker. He then goes on to generate a subargument that the set contains the inverse of each of its elements. This argument constitutes Data4 of the larger argument that



the set is a subgroup of  $G$  and is modeled by a linked, embedded Toulmin scheme. Tucker begins by pointing out that  $G$  is a group and that  $H$  is a subset of the group (Data4.1) and states that the inverses of the elements exist (Claim4.1). He then describes the inverse of 2 as  $\frac{1}{2}$  and renaming it as  $2^{-1}$ , saying, “2's inverse is gonna be 1/2 so, that's gonna be 2 to the minus 1, which is also in  $Z$ ” (lines 487-488, coded as Data4.2). As with his discussion comprising Data2 and Data3, Tucker focuses on a specific instantiation (in this case, 2) in order to develop his argument. However, he qualifies his argument, saying, “I didn't really do this in a good- I guess, proper, uh, structure” (lines 488-489), which suggests that he does not feel that this proof is the best possible proof he could have constructed.

When asked to clarify his subargument for inverses in T2A1, Tucker produces a second argument (T2A2, Figure 5.18) that reflects a more general statement of his first argument that inverses exist in the set  $H$ . In his discussion, Tucker says,

I wanted just to, kind of, like, prove that each element in  $H$ - all these 2 to the powers- are basi- do- are these all still gonna have, um, inverses? That's what I'm trying to prove. And, basically- So, basically- is 2 to the  $n$  gonna have an inverse such that- yield you one? Cause one's gonna be the identity here. Cause well it's under multiplication. So, let's just call the inverse  $i$ . So, we've got to find something that equals that, so, basically, all we can really do is- well we can divide by 2 to the  $n$  and- st- to see what  $i$  is. And that's gonna equal 2 to the minus  $n$ . So-  $i$  is equal to 2 to the minus  $n$ . So, basically, whenever you have an element such, like, 2 to the 18, then its inverse is gonna be 2 to the minus 18. To find its inverse. And that's gonna exist for every element. And minus 18 is still in the integers, so it's, you know, within the group. (lines 496-505)

Tucker begins by stating a broad claim that he is trying to prove that the set  $H$  contains the inverse of each element. He then rhetorically asks, “is 2 to the  $n$  gonna have an inverse such that- yield you one?” (lines 498-499), which provides a specific goal and serves as a claim for his ensuing justification. Before providing data for this argument, however, Tucker situates his claim

by pointing out that one is the identity because he is considering the group under multiplication, which serves as a backing for his argument.

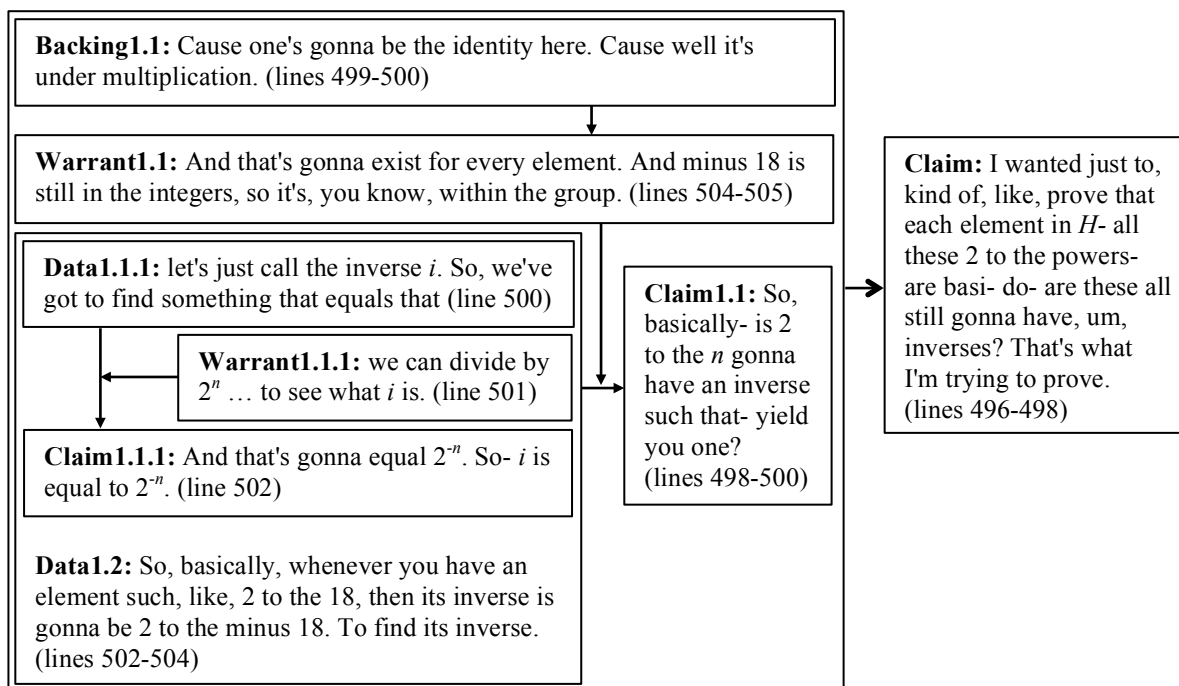


Figure 5.18. T2A2

He then provides an argument that serves as Data1.1 for the overarching argument, calling the inverse of  $2^n$  “ $i$ ” and stating that the inverse will be found by dividing by  $2^n$  to warrant the claim that  $i$  equals  $2^{-n}$ . Tucker follows this by providing a specific example using  $2^{18}$ , which serves as Data1.2. He explains that the inverse of  $2^{18}$  is  $2^{-18}$ , which echoes his argument in Data1.1. He then provides a warrant for these data by saying, “And that's gonna exist for every element. And minus 18 is still in the integers, so it's, you know, within the group” (lines 504-505). This warrant justifies both the generality of Data1.1 and that the specific inverse he describes in Data1.2 satisfies the definition of the set  $H$ .

By virtue of the group given in the proof, the majority of Tucker’s activity draws on the “number” *form* of identity and inverse, contextualized multiplicatively. Tucker frequently shifts between representing elements of  $H$  with exponents and the evaluation of that number. For

instance, when determining whether the set  $H$  contains inverses, Tucker initially focuses on the number two and its multiplicative inverse,  $\frac{1}{2}$ , before re-writing the  $\frac{1}{2}$  as an integer power of two. He admits that this proof is not quite formal but is able to explain his thinking more generally in T2A2 by shifting to representing the inverse with, first, the “letter” *form* of inverse,  $i$ , then the “letter” *form* of inverse,  $-n$ , in the exponent on the number 2. Throughout this shift, Tucker contextualize the representational vehicles of inverses as multiplication of real numbers, allowing the *forms* to serves the needed “operate/identity out” *function* of inverse so that the product of  $2^n$  and its inverse is the multiplicative identity for real numbers, 1.

**Proof T3: Uniqueness of inverses.** Tucker produces his third proof (T3) in response to Interview 2 Q6 to show the uniqueness of a group element’s inverse. The Toulmin analysis of T3 consists of two arguments, the first argument (T3A1) an embedded, sequential scheme and the second (T3A2) an embedded scheme. His first argument centers around the claim that the inverse element is unique and draws on the cancellation law to support his activity. Tucker produces his second argument in response to a follow-up question in which the interviewer asked about the process of cancellation. Tucker begins this response by restating the claim and broadly situating his activity by describing the group axioms, which serves as backing for his activity. He also rephrases the claim, colloquially explaining, “there's only one way to skin a cat in this case. In this unique case!” (lines 588-589). Tucker then frames his proof as a proof by contradiction, which serves to warrant his data. Following this, he begins a justification that serves as data to support his broader argument (Figure 5.19), saying,

So, let  $g$  be an element in  $G$ . So, suppose that  $g$  star  $g$  inverse is equal to- uh, we're gonna call the identity  $e$ .  $e$  where  $e$  is the identity. And  $g$  star  $h$  inverse is equal to  $e$ . So, basically, then we could set these two sides equal to each other, since they're both equal to the identity. So,  $g$  star  $g$  inverse is equal to  $g$  star  $h$  inverse. And, now, here we can pretty much, like, use the ca- cancelation law to kinda prove that, okay, um, these are the same on this side, so, you can pretty

much go like that and say  $g$  inverse is equal to  $h$  inverse using the cancellation law. Which, I think in class she said we can just write that instead of going through the, like, process, but- (lines 590-601)

The embedded argument begins with Tucker describing  $g^{-1}$  and  $h^{-1}$  as inverses of  $g$ . He draws on the “operate/identity out” *function* of inverse to generate two equations in which  $g^{-1}*g$  and  $h^{-1}*h$  equal the identity element, which Tucker names  $e$ . He then sets the two expressions equal to each other, creating the equation  $g^{-1}*g = h^{-1}*h$  that serves a both Claim1.2 and Data1.1, warranting this by using the “transitive” *function* of identity, saying, “we could set these two sides equal to each other, since they're both equal to the identity” (Warrant1.1, lines 596-597). From this equation, Tucker applies the cancellation law, which warrants the claim that  $g^{-1} = h^{-1}$ . Altogether, this embedded, sequential argument serves as data to support Tucker’s original claim that inverse elements are unique, warranted and backed by his earlier statements.

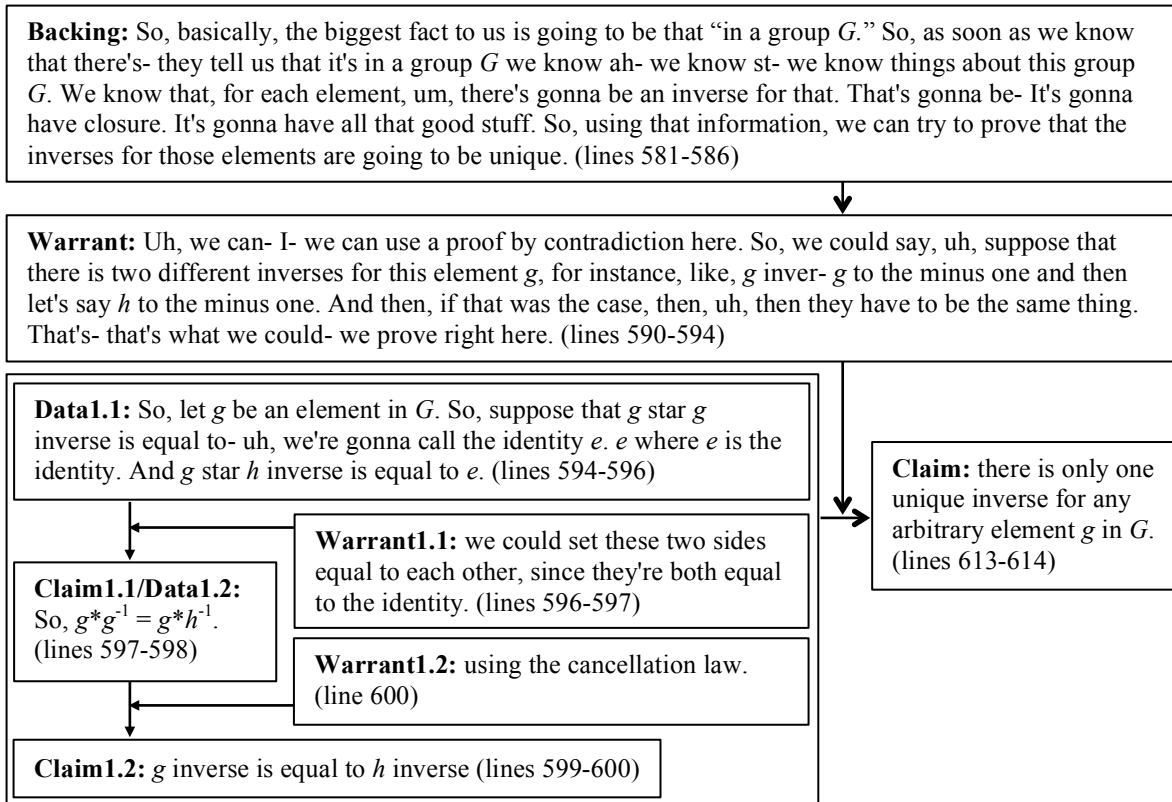


Figure 5.19. T3A1

When asked to explain the statement “we can just write [“cancellation law”] instead of going through the, like, process” (line 601), Tucker constructs a new argument (T3A2, Figure 5.20), explaining, “you try to get rid of  $g$ ” (line 606). He describes this process, constituting an embedded argument that serves as data for T3A2. In this argument, Tucker begins with Data1.1, explaining, “You’d apply the inverse to that. On both sides, so, basically, alright, well,  $g$  inverse times that. So that’s gonna get rid of it. Gives you  $e$ , the identity” (lines 606-608). In this excerpt, Tucker draws on the “end-operating” and “operate/identity out” *functions* of inverse to explain that “[applying] the inverse” to “both sides” of the equation results in the identity. He then warrants this data, saying, “And the identity times any element is gonna get you th- that element back towards you” (lines 608-609). Warrant1.1 draws on the “operate/same out” *function* of identity to support the use of Data1.1. Tucker concludes the embedded argument with the claim, “And you’ll end up with this stuff being cancelled out.”

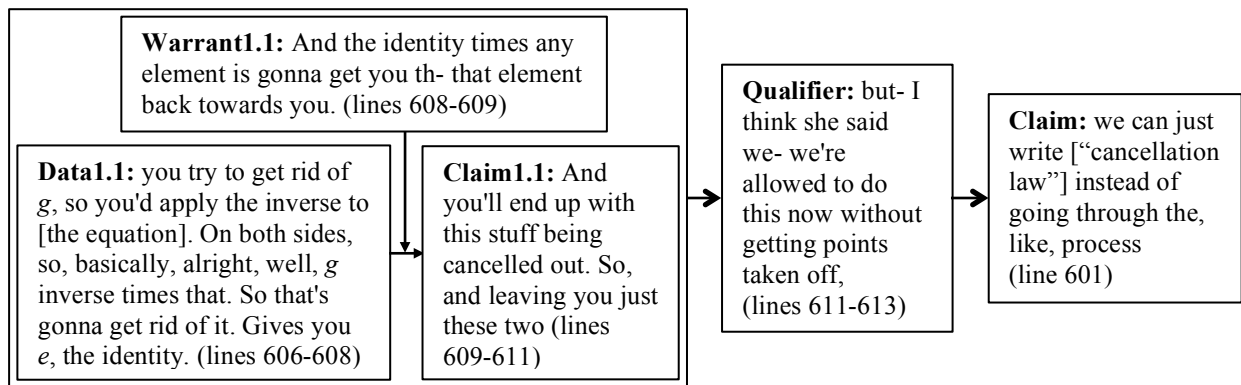


Figure 5.20. T3A2

In his production of proof T3, Tucker draws on the “operate/identity out” and “end-operating” *functions* of inverse and the “operate/same out” and “transitive” *functions* of identity. Analysis of this proof affords insight into the “cancelling” *function* of inverse. Specifically, Tucker is able to draw on prior *functions* of identity and inverse to produce an equation and justify manipulation of the equation as well as carry out this process as a single activity in which

the underlying justification need not necessarily be discussed. During the follow-up questions, Tucker expresses confidence in his proof, which supports a sense that Tucker is comfortable using the “cancellation” *function* of inverse in his proof as well as producing an argument that does not draw on the cancellation law, but still serves to support the original claim.

**Proof T4: Subgroup proof – normalizer of  $h$ .** Tucker produces his fourth proof (T4) during Interview 2 in response to Q7, which asks participants to “Prove or disprove the following: for a group  $G$  under operation  $*$  and a fixed element  $h \in G$ , the set  $H = \{g \in G : g*h*g^{-1} = h\}$  is a subgroup of  $G$ .” Tucker’s response can be viewed as consisting of eleven different arguments (one overarching argument and ten subarguments) that together contribute toward his entire proof that the set  $H$  is a subgroup of the group  $G$ . As with T2, Tucker completes this proof by verifying the subgroup rules for the set  $H$ , saying, “So, I guess we just kind of like, prove it one step at a time, type of thing” (line 733). This quote, referring to the subgroup rules, serves as a warrant for each “one step” to serve as data for the broader subgroup proof. Accordingly, the Toulmin scheme for T4A1 (Figure 5.21), which is Tucker’s initial and concluding argumentation about proving that a set is a subgroup, can be viewed as an overarching argument for which the remaining arguments (T4A2-T4A11) serve as data and the previous quote serves as a warrant (Warrant1) for the claim “So, we’re trying to prove that  $H$  is a subgroup of  $G$ ” (line 729). For instance, T4A2 outlines Tucker’s argument that the set  $H$  is a subset of  $G$ , allowing T4A2 to be viewed as an embedded Toulmin scheme within the linked structure of T4A1 – more specifically, as Data1 in T4A1. Tucker’s response to Q7 includes several detours from proving one subgroup rule to another. As a result, the arguments composing the four linked data in T1A1 are staggered (Figure 5.21) with earlier arguments ending without verifying the subgroup rule for Data2, Data3, and Data4.

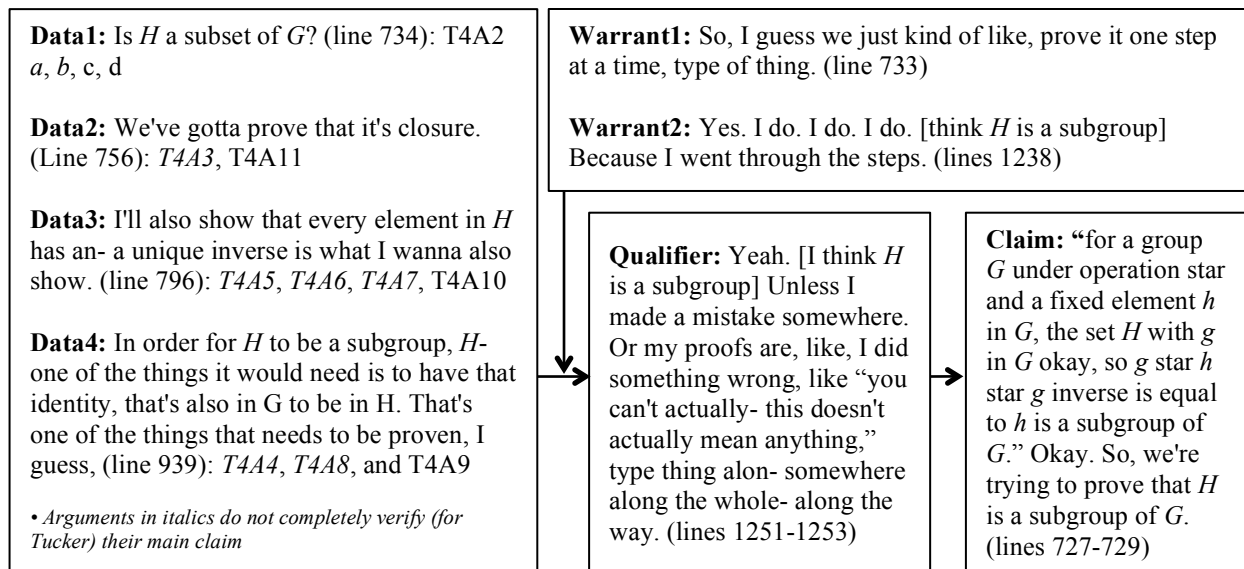


Figure 5.21. T4A1

In the first subargument that Tucker produces after describing his approach (T4A2, Figure 5.22) he demonstrates that the set  $H$  is a subset of the group  $G$ . T4A2 is modeled with a linked Toulmin scheme in which Tucker provides four similar statements that each serve as data for the claim “ $H$  is a subset of  $G$ ” (line 471). Although Tucker initially focuses on the equation used to define  $H$ , after 50 seconds of mostly silence and Tucker reading the equation used to define  $H$  aloud, he abruptly says, “wait a minute. It says that  $g$  is in  $G$ , right? Okay. Yeah. (laughs) I don't know why that was taking forever. Okay. So, yeah.  $H$  is a subset of  $G$ , cause  $H$  is just all these elements  $g$ ” (lines 740-742). Tucker elaborates on this data twice, rephrasing it slightly (Data2, Data3) and then points out that “we don't actually have to actually, like, prove anything here. We can just take it from the- what they give us,” (lines 746-747), which warrants the data's support of the claim.

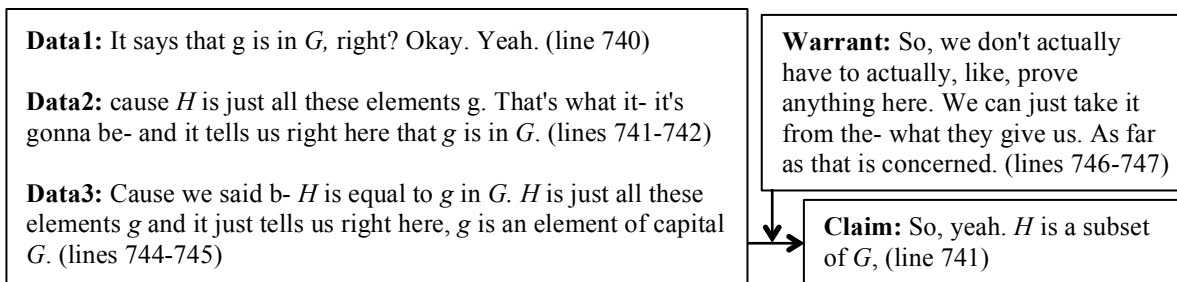


Figure 5.22. T4A2

Tucker then describes the next subgroup rule, saying, “We've gotta prove that it's closure” (line 756) going on to rephrase this as “have to prove that all of those star themselves will yield you back another one” (line 762), which constitute Claim1 and Claim2 of T4A3 (Figure 5.23), respectively. He follows this by describing his approach, saying,

$a$  be an element of  $H$  and  $b$  be an element of  $H$ , then by definition of the set,  $a$  star  $h$  star  $a$  inverse is equal to  $h$  and  $b$  star  $h$  star  $b$  inverse is equal to  $h$ . And it looks like we can just, like, set these equal to each other. So,  $a$  star  $h$  star  $a$  inverse is equal to  $b$  star  $h$  star  $b$  inverse. See, what we're trying to do here is prove that  $a$  star  $b$  is also in  $H$  to prove that there's closure. That's what you've got to do to prove closure. So- In order to prove- I'm just gonna write this out. Trying to show that  $a$  star  $b$  is an element  $H$ , which means  $a$  star  $b$  (mumbles, writing) - I'm just gonna need to write star between that - star  $h$  star  $a$  star  $b$  inverse is equal to  $h$ . So that's what we gotta, um, we need to prove. Um- Okay. I think it would be easiest to rearrange (points to  $a*h*a^{-1} = b*h*b^{-1}$ ) this so that we get that (points to  $(a*b)*h*(a*b)^{-1} = h$ ). Hm. That's a good question. I don't know how we're supposed to do that, though. (lines 764-775)

This excerpt provides transcript of the entirety of T4A3, except for Tucker's initial claims.

Notice that Tucker begins by fixing two elements of  $H$ ,  $a$  and  $b$ . These serve as initial data (Data1) in a sequence in which Tucker draws on the definition of the set to warrant (Warrant1) the generation of two equations:  $a*h*a^{-1} = h$  and  $b*h*b^{-1} = h$  (Claim1/Data2). These equations then serve as data in order for Tucker to generate a third equation ( $a*h*a^{-1} = b*h*b^{-1}$ ) comprised of the left-hand sides of the first two equations set equal to each other, serving as a second claim that is warranted by the phrase “it looks like we can just set these two equal to each other” (lines



766-767). Together, these data, warrants, and claims follow the structure of a sequential argument that serves as data toward supporting the claim that  $H$  satisfies closure.

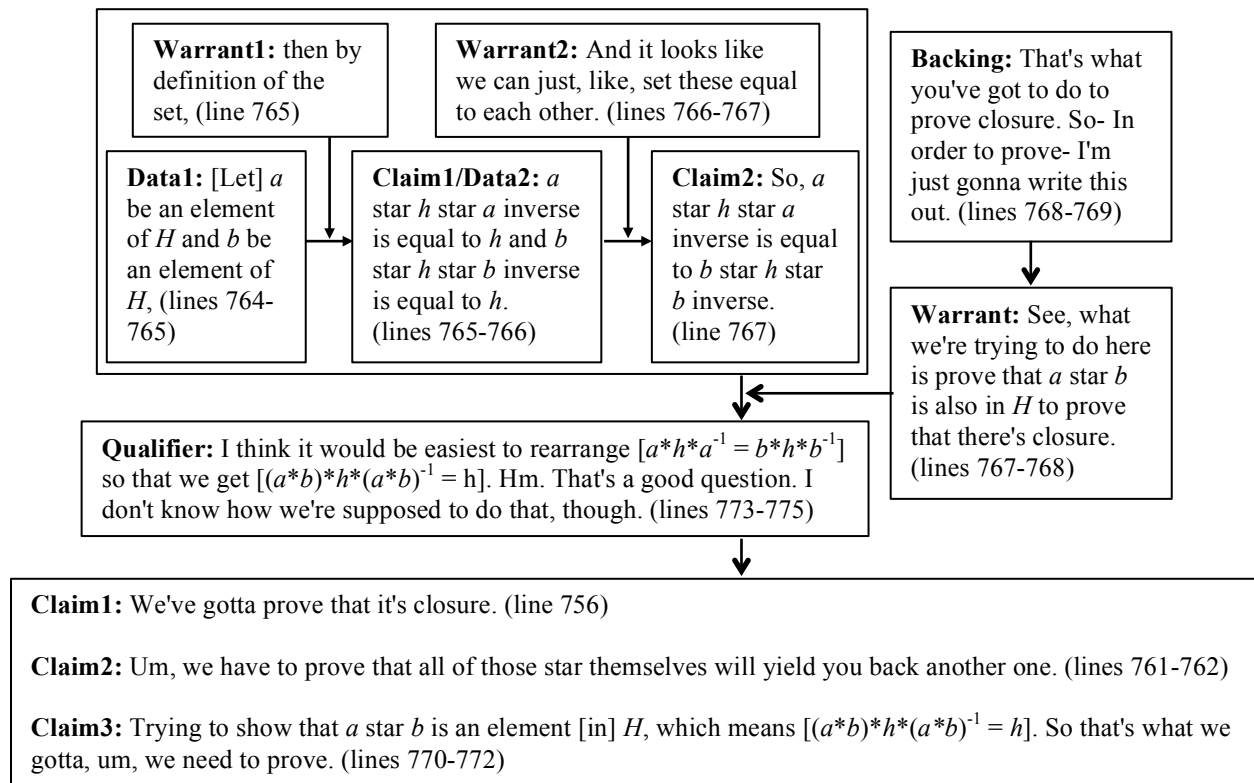


Figure 5.23. T4A3

Tucker follows his initial sequential argument by rephrasing the claim to reflect the definition of the set  $H$ . Specifically, similarly to Data2/Claim1, Tucker replaces  $g$  and  $g^{-1}$  with the algebraic statement “ $(a*b)$ ,” yielding the equation “ $(a*b)*h*(a*b)^{-1} = h$ .” This shifts the goal of the proof from a more general description of closure to an algebraic framing that reflects the data that Tucker has produced. He then provides a series of statements that qualify, warrant, and back the embedded argument to support this new claim. First, he alludes to rearranging the equation “ $a*h*a^{-1} = b*h*b^{-1}$ ” to “get that” while pointing to the written “ $(a*b)*h*(a*b)^{-1} = h$ ,” but admits that he is unsure how he might do this, which qualifies his argument. Tucker then provides a warrant for how his data might serve the claim that elements in  $H$  satisfy closure, saying, “See, what we're trying to do here is prove that  $a$  star  $b$  is also in  $H$  to prove that there's

closure” (lines 767-768). He adds, “That's what you've got to do to prove closure. So- In order to prove- I'm just gonna write this out” (lines 768-769), which serves as backing for this warrant.

Tucker’s argument then stalls as he again expresses his uncertainty in the qualifier, saying, “This is a little trickier. I'm open to suggestions here” (line 784).

The interviewer then asks Tucker to describe any other things he might need to show for the proof, which prompts Tucker to discuss the need for an inverse for each element of  $H$  to exist, saying,

I'll also show that every element in  $H$  has an- a unique inverse is what I want also show. So, I guess- What would the inverses for these elements would look something like this. So, we're saying let  $a$  be in- oops - let  $a$  be in  $H$ . So,  $a$  star  $h$  star  $a$  inverse is equal to  $h$ . Suppose  $a$  inverse is in  $H$  such that  $a$  star  $a$  inverse is equal to  $a$  inverse star  $a$  is equal to the identity. Hm. Now- Now comes the point of realizing, okay, well, what's the identity in this. (lines 795-800)

In this excerpt, Tucker fixes an element,  $a$ , in  $H$  and begins describing the inverse element of  $a$ , which reflects the definition of inverse he provided earlier in the interview (Interview 2, Q2biii). This leads Tucker to question what the identity element in  $H$  might be, the discussion of which yields a new argument (T4A4, Figure 5.24). He goes on to surmise that the identity element is 1, saying, “It seems to me like the identity might be something like one. Cause we have- okay, one'll work” (lines 801-802). This serves as the claim of a new argument (T4A4) in which Tucker tries to determine whether 1 is the identity of  $H$  by substituting the number 5 for the letter  $g$  and “five to the negative one, so one-fifth” for  $g^{-1}$  in the equation used to define  $H$  and saying “five  $h$  divided by five is equal to  $h$ . And then five star one would be equal to five” (lines 804-805). This serves as data to support the claim, which Tucker warrants by pointing out the multiplicative notation used in the prompt.

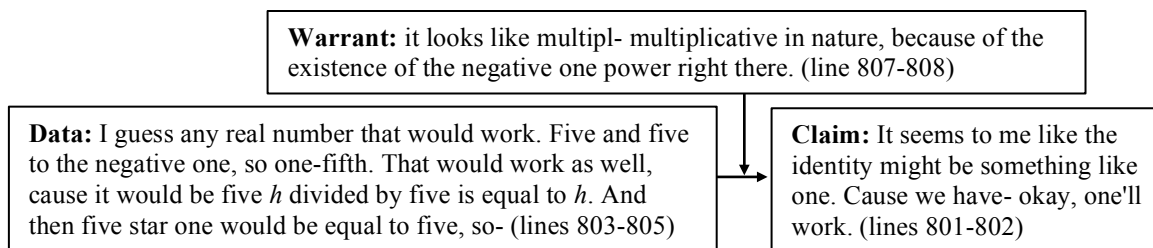


Figure 5.24. T4A4

Notable in T4A4 is Tucker’s blending of two different *forms*, specifically, the “number” *form* of identity, “1,” and elements in the letter *form*, which are typically used along with an identity represented by the letter “ $e$ .” This shift indicates that Tucker is having difficulty with the notation used in the prompt. Specifically, Tucker’s focus on the multiplicative notation with more abstract elements represented as letters leads him blend representational vehicles indicative of the “letter” and “number” *forms* of identity and inverse. In order to deal with this, he replaces the  $g$  and  $g^{-1}$  with the real number 5 and the multiplicative inverse of five, respectively. This affords him to be able to view 5 and  $5^{-1}$  (which he rephrases as  $1/5$  and “divided by 5”) as serving the “vanishing” *function* of inverse and 1 as serving the “operate/same out” *function* of identity. This is reflected in Data (Figure 5.24) when Tucker says, “five  $h$  divided by five is equal to  $h$ . And then five star one would be equal to five” (lines 804-805).

Tucker then returns to discussing whether each element of  $H$  has an identity by appealing to the definition of  $H$ , saying,

And I guess inverses do have to kinda exist in this ca- Just by- Just by looking at this. (points to top-left of page) I mean, it's- it's showing that, okay,  $H$  is equal to all the  $g$  in  $G$  such that  $g$  star  $h$  star  $g$  inverse is equal to  $h$ . So it just kind of tells us right there that  $g$  inverse exists. (lines 810-813)

This constitutes a new argument (T4A5, Figure 5.25) in which Tucker claims “inverses do have to kinda to exist in this” (lines 810-811). He supports this claim by providing the data, “Just by looking at this. I mean, it's- it's showing that, okay,  $H$  is equal to all the  $g$  in  $G$  such that  $g$  star  $h$  star  $g$  inverse is equal to  $h$ ” (lines 811-812) and warrants the data’s support of the claim, saying,

“So it just kind of tells us right there that  $g$  inverse exists” (lines 812-813). In this argument, Tucker seems to justify the existence of  $g^{-1}$  by its inclusion in the definition of  $H$ , supporting a sense that Tucker might not be focused on whether  $H$  contains inverse elements, which is further supported by Tucker’s next argument.

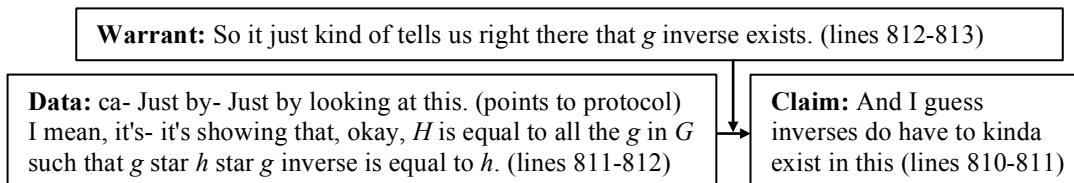


Figure 5.25. T4A5

After the statement from which T4A5 is taken, Tucker immediately begins a new argument for the existence of inverses for elements in  $H$  based on the inverse group axiom for  $G$  (T4A6, Figure 5.26). Tucker begins this argument by appealing to the fact that  $G$  is a group (Data1.1) to support the claim “every element in  $G$  has an inverse” (lines 815-816, Claim1.1). He follows this by reiterating “ $H$  consists of the l- lower-case  $g$ , which is in  $G$ ” (line 815) in order to support the claim “[For] all those you would have an inverse” (Claim, lines 815-816). Tucker adds, “we really don’t need to work it out, except we just use the definition they give us” (lines 817-818), which warrants Data1 and Data2’s support for the Claim, before finally providing a third data (Data3) that the “letter” *form* of inverse is used in the definition of  $H$ .

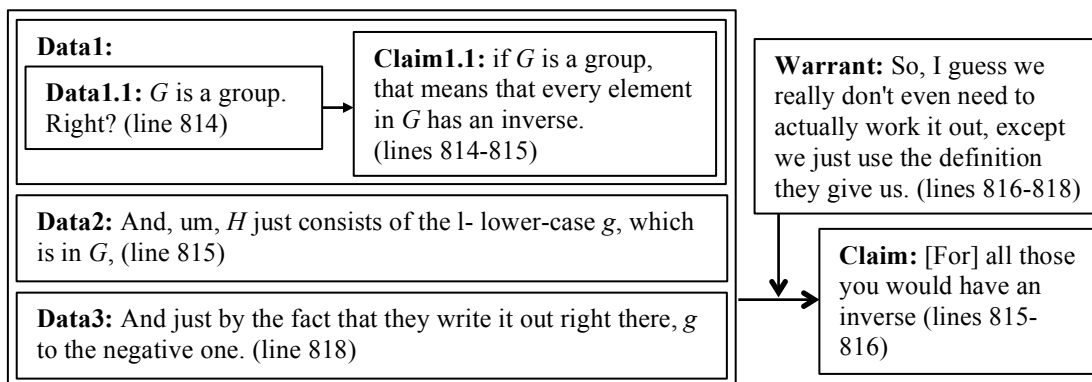


Figure 5.26. T4A6

Together, T4A5 and T4A6 support a sense that Tucker is not thinking about the inverse subgroup rule for the set  $H$  as determining whether  $H$  contains the inverse of each of its elements. Rather, Tucker's arguments are consistent with demonstrating that, for a given element in  $H$ , the inverse of that element merely exists – not that it exists and also satisfies the definition of  $H$ , or exists in  $H$ . Specifically, Tucker appeals to existence of inverses in  $G$  and “the fact that they write it out” as data in support of his claim that “you would have an inverse” (line 816). Further, during T4A5, the excerpt of transcript coded as Warrant only mentions the existence of  $g$  inverse. However, Tucker does describe  $a$  inverse as “in  $H$ ” when introducing the subgroup rule, saying, “Suppose  $a$  inverse is in  $H$  such that  $a$  star  $a$  inverse is equal to  $a$  inverse star  $a$  is equal to the identity” (lines 798-799).

There is also evidence to suggest that Tucker might not think about the definition of  $H$  as a restriction of the set  $G$ , as though the set  $H$  is equivalent to  $G$ . This is consistent with Tucker's statement that, “ $H$  just consists of the l- lower case  $g$ , which is in  $G$ ” (line 815). Accordingly, the interviewer, anticipating this, attempts to clarify the definition of  $H$ , focusing on the fixed nature of the element  $h$  and the restriction of the set to only the elements of  $G$  that satisfy the given equation. In response to this discussion, Tucker agrees and describes how he's thinking about what it means for the elements in  $H$  to have inverses, saying, “So, [ $H$  is] almost, like, limiting some of them, right? It's just- It's only the elements in  $G$  that- that work like this, type thing. So, if  $H$  is only those elements that work” (lines 743-745). He goes on to describe how elements in  $H$  “still have inverses,” but does not mention whether those inverses are also elements of  $H$ . However, when asked to explain his written work, specifically the line containing the statement, “ $a^{-1} \in H$ ,” Tucker says, “If  $a$  inverse is in  $H$ , that means that  $a$  inverse  $h$   $a$  - inverse of an inverse being  $a$  - is also equal to  $h$ ” (lines 856-857), writing the equation “ $a^{-1} * h * a = h$ .”

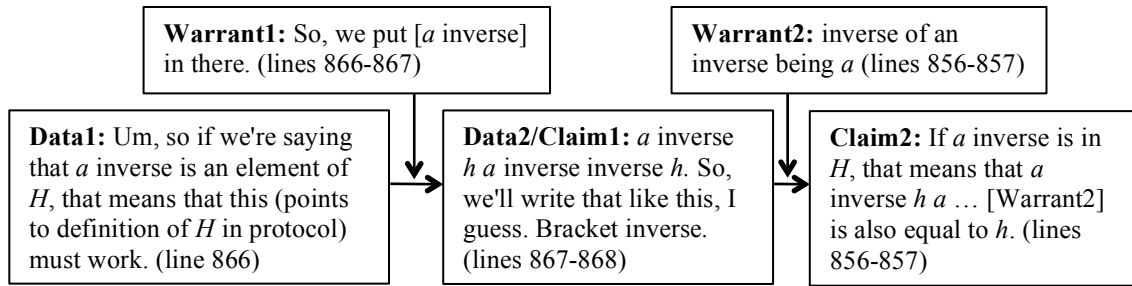


Figure 5.27. T4A7

When asked to explain his statement, Tucker supports the claim (coded as Claim2 in Figure 5.27) with an argument modeled using a sequential Toulmin scheme (T4A7, Figure 5.27). Tucker begins his explanation by describing  $a^{-1}$  as satisfying the equation in the definition of  $H$ . This is similar to his activity in T4A3 because he replaces the letter “g” in the equation used to define  $H$  with the symbol “ $a^{-1}$ .” This serves as warrant (Warrant1) for Tucker to change the equation “ $a^{-1} * h * a = h$ ” so that it looks like “ $a^{-1} * h * (a^{-1})^{-1} = h$ ,” which Tucker does by including the parentheses and -1 exponents in the same equation he has already written. The connection between this equation and the equation from Tucker’s original claim is supported by the phrase, “inverse of an inverse being an inverse” (lines 856-857), which reflects the “inverse-inverse” *function* of inverse and serves as a warrant to justify that “ $a^{-1} * h * a = h$ ” and  $a^{-1} * h * (a^{-1})^{-1} = h$ ” are consistent equations.

However, Tucker does not generate a new goal from these equations as he does in T4A3. Rather, Tucker says, “if  $a$  inverse is the inverse of  $a$ , then  $a$  star  $a$  inverse is equal to  $a$  inverse star  $a$  is equal to  $e$ , the identity. Right?” (lines 883-884). This begins an exchange in which Tucker the identity element, calling it “ $e$ ” and discusses it with the interviewer. Asked where the “ $e$ ” came from, Tucker responds, “I’m just calling it,  $e$  is the identity. But, um, I’m trying to figure out what  $e$  would equal in this case, and m- that’s where I’m kind of stuck” (lines 889-890). Tucker then describes  $e$  as the identity in  $H$ , going on to say, “If it doesn’t exist that means  $H$  can’t be a subgroup, but. I’m thinking that the identity is gonna be equal to one” (lines 914-

915). At this point, the interviewer again reiterates the definition of  $H$ , focusing on the fact that it is a subset of  $G$  and works “under the same operation star” (line 919), as Tucker has mentioned. Throughout this excerpt, Tucker agrees, periodically saying, “mhhh” and “yeah.” The interviewer then asks, “so, what’s the identity in  $H$ ?” (line 920) to which Tucker responds “It’s also gonna be the identity in  $G$ . I guess.” However, when asked if the identity of  $G$  is in  $H$ , Tucker says, “So, obviously, we know that  $G$  has an identity. Um, and we’ve already proved that  $H$  is a subset, so- In order for  $H$  to be a subgroup,  $H$ - one of the things it would need is to have that identity, that’s also in  $G$  to be in  $H$ . That’s one of the things that needs to be proven, I guess.” (lines 937-940). He then adds, “So, if  $e$  is the identity in group  $G$ , then  $e$  must be the identity in  $H$  for  $H$  to be a subgroup” (lines 941-942).

After this, when asked how he might find out whether the identity exists, Tucker again returns to describing the identity as 1. This leads to the following exchange:

Int: So, if  $e$  in  $G$  - that’s the identity of  $G$  - [Mhhh.] then  $e$  needs to be in  $H$ , for  $H$  to-

Tucker: Mhhh. If  $e$  is the identity in  $G$ , then  $e$  must be the identity in  $H$  for  $H$  to be a subgroup.

Int: And you don’t know whether it’s in  $H$ ?

Tucker: (13 seconds) Mm. I’m with you just- can we give an examp- like, let’s say that  $e$  is equal to one.

Int: Okay. I don’t know that one’s in  $G$ .

Tucker: Right. That’s what I’m saying, like, you don’t know if one is in  $G$ , so it’s like-

Int: I know  $e$  is in  $G$ , cause you just (points to paper) said it, right?

Tucker: (chuckles) Yeah.

Int: Anyway. Okay, so suppose it’s one. What would you do? Like, how would you figure if one was the identity?

Tucker: I’d say- I guess I’d kinda come to a, um, a block in the road. If I prove it’s the identity here, we have no way to prove it’s the identity in-

Int: How would you prove it’s the identity there? (lines 970-993)

Tucker responds by providing an argument in which he initially substitutes the number one for the letter  $g$  in the equation  $g*h*g^{-1} = h$ , saying, “Well, I’d just say that, okay well,

one star  $h$  star one - cause one to the negative one is one - is equal to  $h$ ” (lines 996-997). This constitutes an argument (T4A8, Figure 5.28) in which Tucker uses the equation  $1 * h * 1^{-1} = h$  (serving as Data) to generate the equation  $1 * h * 1 = h$  (which serves as a Claim). Tucker provides a warrant for the change in the equation by saying, “one to the negative one is one” (lines 996-997), which seems to be based on Tucker’s reasoning with real numbers, rather than reasoning with inverses. Tucker then qualifies his argument, saying “I guess you don’t know what one star  $h$  does, though... in my head that- it just would work, with just, like, normal numbers, but it prob- it might not, actually” (lines 997-999). With this statement, Tucker is indicating that, even though he has described thinking about one as the identity element, it does not serve any *function of* identity in this context.

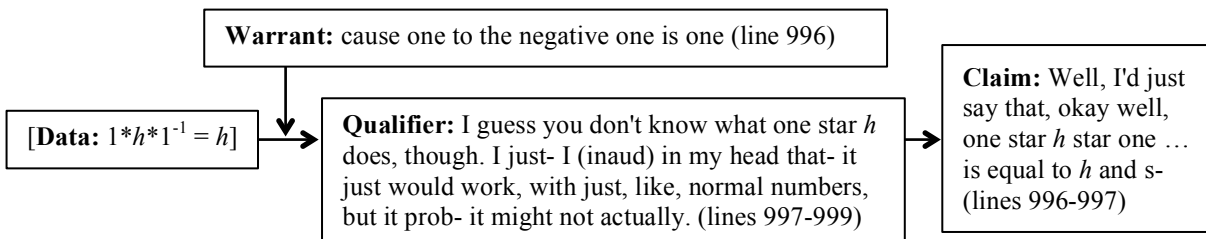


Figure 5.28. T4A8

In response to Tucker’s argument, the interviewer returns to discussing the “letter” *form* of the identity, asking “Well, what about  $e$ ?” (line 1001). Tucker responds,

Well, we know what- what  $e$  would do.  $e$  would just take every element and do nothing to it, pretty much, in  $G$ , and it would do the same in  $H$  if it was the identity. So- Hm. Yeah. I'm not sure what you would do. (lines 1003-1005)

This prompts a discussion in which the interviewer first asks Tucker what would happen if he tried to do with  $e$  what he had just tried with the number one (i.e., substitute  $e$  for the letter “ $g$ ” in the equation  $g * h * g^{-1} = e$ ). Tucker responds by returning to the equation  $a * a^{-1} = e$ , indicating that he might not have understood the suggestion. Following this, the



interviewer contrasts Tucker’s descriptions of one as an identity element and  $e$  as an identity element, which culminates with the interviewer asking, “So, what if we put  $e$  in here for  $g$ ?” (line 1043). Tucker responds, “If we put  $e$  in there for  $g$ . Uh, well,  $e$  times-  $e$  star  $h$  would equal  $h$  star  $e$  -  $e$ 's inverse, which is itself- would just equal  $h$ , so  $h$  is equal to  $h$ ” (lines 1046-1047).

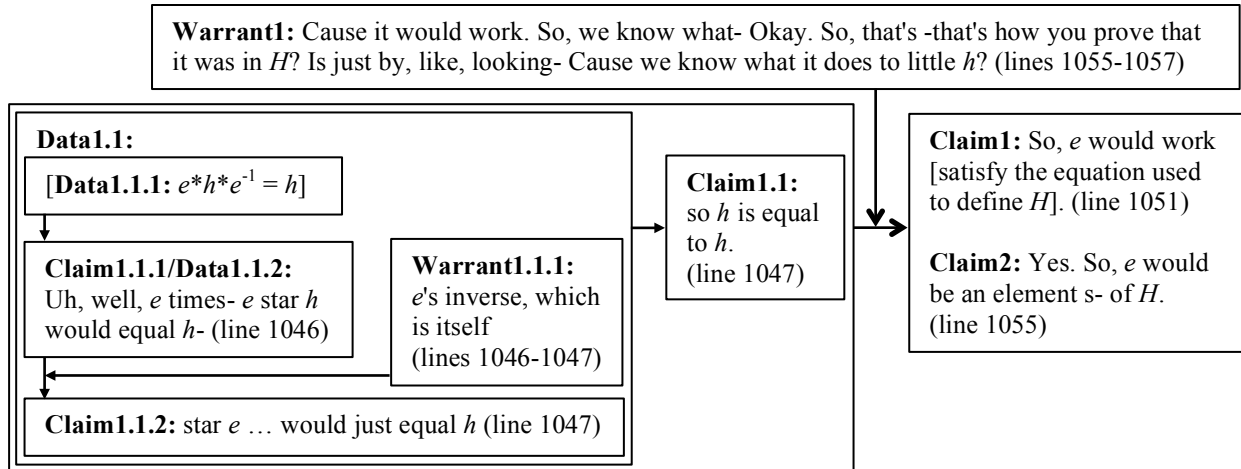


Figure 5.29. T4A9

This excerpt constitutes an argument that can be modeled with an embedded Toulmin scheme (Figure 5.29). The equation  $e * h * e^{-1} = h$  serves as the initial data (Data1.1.1) that Tucker evaluates by reading from left-to-right, first pointing out that “ $e$  star  $h$  would equal  $h$ ” (line 1046), then operating the result of this with  $e$ , warranting the change from  $e^{-1}$  to  $e$ , by saying “ $e$ 's inverse, which is itself” (lines 1046-1047), and claiming that the result is  $h$ . In this sequential argument, which serves as data (Data1.1), Tucker tacitly draws on the “operate/same out” *function* of identity and explicitly draws on the “identity as self-inverse” *function* of identity to support the changes in the equation that lead to the claim “so  $h$  is equal to  $h$ ” (line 1047). Tucker follows this by adding, “So,  $e$  would work” (line 1051), which constitutes a main claim (Claim1) for the overall argument. The interviewer then points to the definition of  $H$  in the prompt and asks, “Is  $e$  an element of  $G$  such that  $e$  star  $h$  star  $e$  inverse equals  $h$ ?” (1053). Tucker responds,

Yes. So,  $e$  would be an element  $s$ - of  $H$ . Cause it would work. So, we know what- Okay. So, that's -that's how you prove that it was in  $H$ ? Is just by, like, looking- Cause we know what it does to little  $h$ ? (lines 1055-1057)

This response is telling, supporting a sense that, up to this point, Tucker has not focused on whether an identity element satisfies the definition of the set  $H$ . Rather, his consistent phrasing of his goal as determining whether the identity of  $H$  “exists” prevented meaningful progress toward proving that the set  $H$  contains the identity of  $G$ , even though Tucker has stated  $H$  will contain an identity if it is a subgroup.

Following T4A9, Tucker returns to discussing inverse elements, saying, “So once you know that, okay,  $e$  is also the identity in  $H$ , you can then prove that there are inverses in  $H$  or-” (lines 1065-1066). After a brief exchange in which Tucker rephrases his argument from T4A5, which focuses on the fact that “ $g^{-1}$ ” is written in the definition of  $H$ , Tucker reads over his work and says, “I- you know what I might do actually?” (line 1078). He then begins an explanation, but pause and restarts in order to explain his thinking more clearly, saying,

So, right now, we have  $g$  star  $h$  star  $g$  inverse is equal to  $h$ . We want to get to somewhere that looks like- ... Want to show.  $g$  inverse star  $h$  star  $g$  is equal to  $h$ . In order for the inverse of  $g$  to satisfy this (points to definition of  $H$ ) right here. Cause that's what you do when you put in the  $g$  inverse. (lines 1084-1086).

With this excerpt begins a new argument (T4A10, Figure 5.30) with which he attempts to show that the set  $H$  contains inverses of its elements. He begins with the equation used to define  $H$ , saying, “right now, we have  $g$  star  $h$  star  $g$  inverse is equal to  $h$ ” (line 1085), which serves as initial data (Data1.1) for the argument. He then describes wanting to show that  $g^{-1} * h * g = h$ , which reflects the Claim from T4A7, and serves as the claim in the new argument (Claim1). He supports this claim by explaining that this goal means that  $g^{-1}$  satisfies the given equation, saying, “Cause that's what you do when you put in the  $g$  inverse” (line 1087). This warrants the claim by reflecting Tucker’s activity in T4A7 in which he replaces  $g$  in the equation used to

define  $H$  with an element's inverse ( $a^{-1}$  in T4A7; here,  $g^{-1}$ ) and draws on the “inverse-inverse” *function* of inverse to rewrite the equation ( $a^{-1}*h*a = h$  in T4A7;  $g^{-1}*h*g = h$  here). This constitutes a shift in Tucker's description of what it would mean for the set  $H$  to contain inverse elements, drawing on his prior argument, and anticipating a manipulation of the definition of  $H$  to result in the same equation.

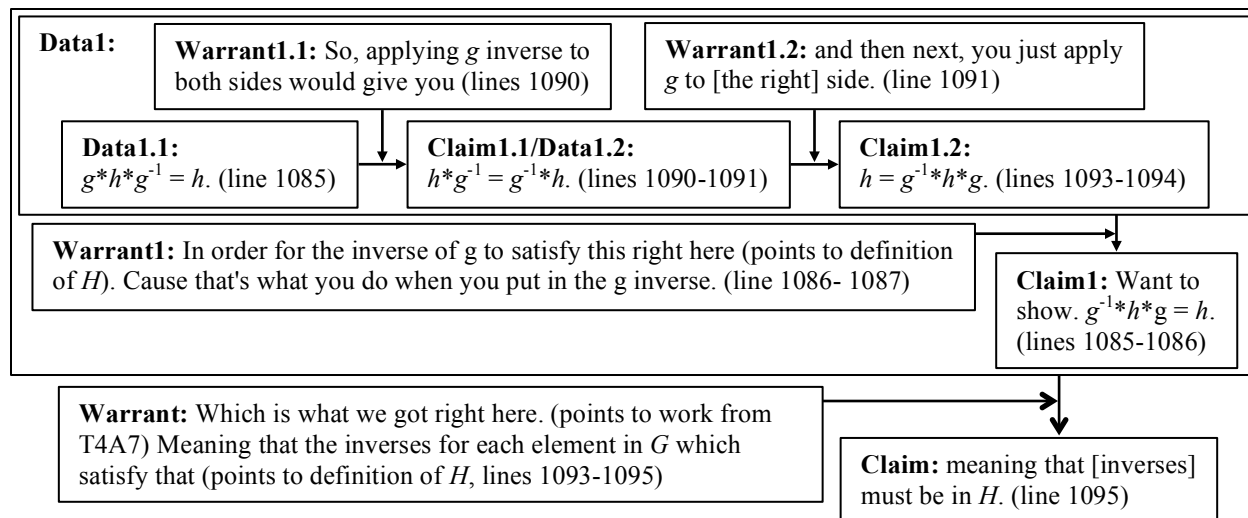


Figure 5.30. T4A10

Tucker then continues, explaining how he might manipulate the first equation so that it looks like the second equation. This process constitutes a subargument that is modeled (Data1, Figure 5.30) with an embedded, sequential Toulmin scheme. Tucker begins by left-operating with  $g^{-1}$ , saying, “let's apply the  $g$  inverse to that. So, applying  $g$  inverse to both sides would give you  $h$  star  $g$  inverse is equal to  $g$  inverse star  $h$ ” (lines 1089-1091). This process comprises a warrant that draws on the “end-operating” and the “vanishing” *functions* of inverse to support the claim that a new equation (Claim1.1/Data1.2) can be produced. This equation then serves as data as Tucker describes right-operating with the element  $g$  to produce the equation  $h = g^{-1}*h*g$  (Claim1.2). Similar to the left-operation with  $g^{-1}$ , this draws on the “end-operating” and “vanishing” *functions* of inverse to warrant the new claim. However, this action also subtly

draws on the “inverse-inverse” *function* of inverse in that Tucker is using the element  $g$  as the inverse of its own inverse in order to cancel the  $g^{-1}$  on the right end of the left-hand side of the equation. Tucker then interprets the result of this activity, saying, “Which is what we got right here. Meaning that the inverses for each element in  $G$  which satisfy that (points to definition of  $H$ ), mean that must be in  $H$ ” (lines 1093-1095), which comprises a warrant and claim for the overarching argument that  $H$  contains the inverses of its elements. T4A10 leverages Tucker’s prior work in T4A7, which he was previously unable to do.

Tucker produces his final argument in support of T4 (T4A11, Figure 5.31) after he reminds the interviewer that he had not quite completed showing that  $H$  satisfies closure earlier in the interview (T4A3). Tucker initially indicates that he is still unsure of how to prove that the set satisfies closure. The interviewer asks Tucker what tools he might have that he could use, to which Tucker replies, “existence of inverses and identity. I don't know” (line 1148). The interviewer reminds Tucker that he has associativity then suggests, “Or you could just un-group stuff” (line 1154). This might provide Tucker with some insight into a possible approach to the proof, as he then says, “And, like, kinda like modify this right side as well, or- (4 seconds) Okay, so, if we bring that over-” and begins writing. He then starts his argument by saying, “So, we know that  $a$  works and  $b$  works” (line 1163), which serves as the first data (Data1) in support of the claim, “We wanna show that  $a*b$  works” (line 1164, Claim). It seems that Tucker is using the word “works” here to indicate that they satisfy the definition of  $H$ . This sense is supported by Tucker generating the equations  $a*h*a^{-1} = h$  and  $b*h*b^{-1} = h$ . Tucker then provides an argument that reflects his anticipated goal during T3A3 (Qualifier), which he had described by saying, “I think it would be easiest to rearrange [ $a*h*a^{-1} = b*h*b^{-1}$ ] so that we get [ $(a*b)*h*(a*b)^{-1} = h$ ]” (lines 773-774).

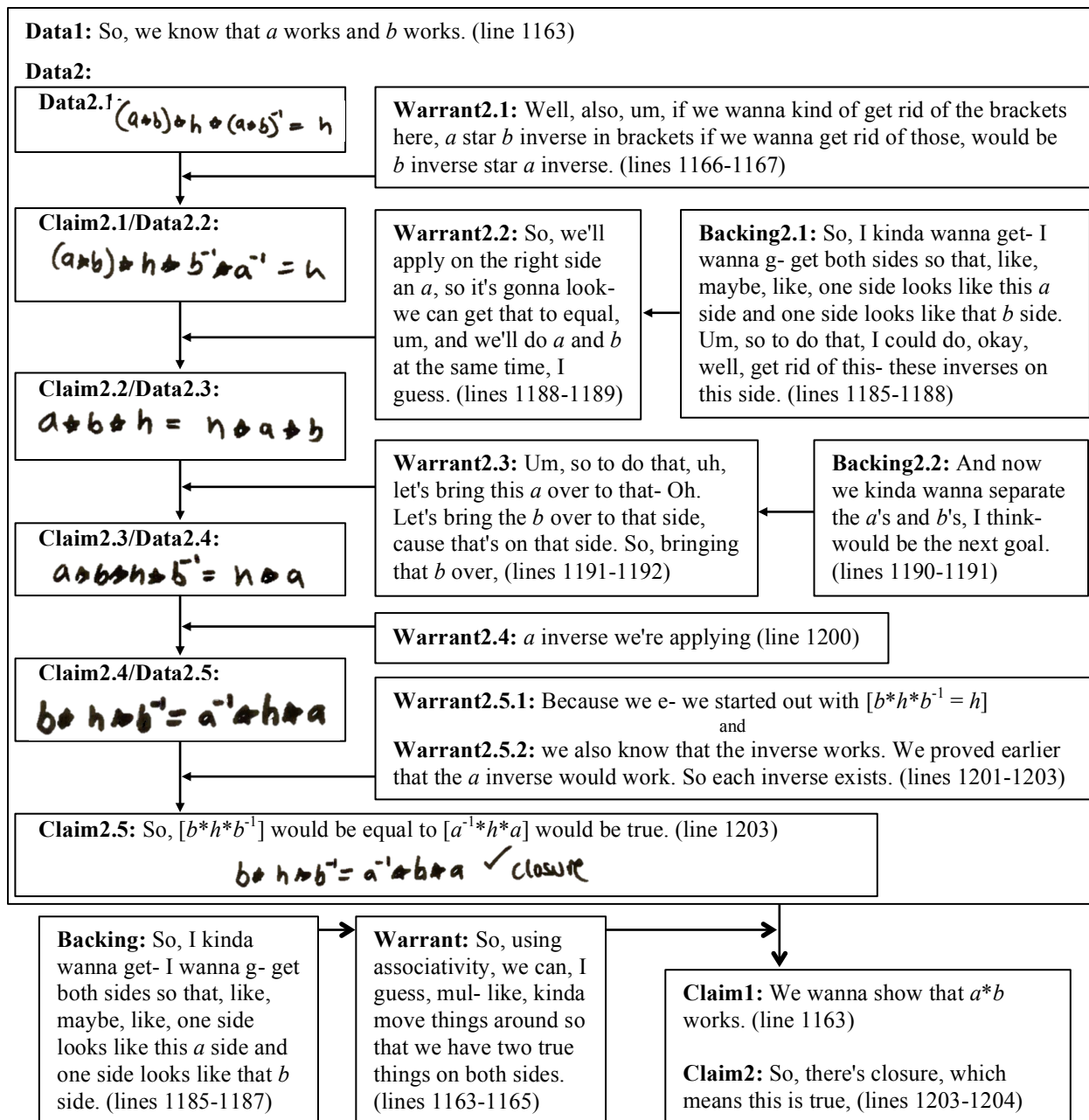


Figure 5.31. T4A11

Tucker describes how he expects to approach the proof, saying, “So, using associativity, we can, I guess, mul- like, kinda move things around so that we have two true things on both sides” (lines 1163-1165), which serves as a warrant for Tucker’s overarching argument. He then draws on the “inverse of a product” *function* of inverse to warrant (Warrant2.2) rewriting the equation  $(a*b)*h*(a*b)^{-1} = h$  (Data2.1) as  $(a*b)*h*b^{-1}*a^{-1} = h$  (Claim2.1/Data2.2). Tucker

draws on this new equation to serve as data for developing a new equation, first backing his activity by describing his reasoning for changing the equation, saying,

So, I kinda wanna get- I wanna g- get both sides so that, like, maybe, like, one side looks like this  $a$  side and one side looks like that  $b$  side. Um, so to do that, I could do, okay, well, get rid of this- these inverses on this side. (lines 1185-1188)

This supports a strategy that it seems Tucker anticipates helping to eventually generate the desired equation. Tucker generates the new equation by drawing on the “end-operating” and “vanishing” *functions* of inverse to simultaneously remove the  $a^{-1}$  and  $b^{-1}$  from the right end of the left-hand side of the equation and concatenate  $a$  and  $b$  on the right end of the right-hand side of the equation, resulting in the new equation  $a*b*h = h*a*b$  (Claim2.2/Data2.3). In doing so, he warrants his activity by saying, “So, we’ll apply on the right side an  $a$ , so it’s gonna look- we can get that to equal, um, and we’ll do  $a$  and  $b$  at the same time, I guess” (lines 1188-1189).

Tucker continues by drawing on the same *functions* of inverse to remove the  $b$  that he had just concatenated on the right-hand side of the equation and concatenate  $b^{-1}$  on the left-hand side of the equation, essentially undoing part of his activity in which he generated the equation in (Claim2.2/Data2.3). He explains,

And now we kinda wanna separate the  $a$ 's and  $b$ 's, I think- would be the next goal. Um, so to do that, uh, let's bring this  $a$  over to that- Oh. Let's bring the  $b$  over to that side, cause that's on that side. So, bringing that  $b$  over... (lines 1191-1192)

This serves as warrant and backing for Tucker to generate the equation  $a*b*h*b^{-1} = h*a$  (Claim2.3/Data2.4). Tucker immediately follows Claim2.3 by saying, “ $a$  inverse we’re applying” (line 1200), which warrants the equation  $b*h*b^{-1} = a^{-1}*h*a$ . Throughout this entire data-claim sequence, new equations are generated by Tucker’s manipulation of the previous equation, drawing primarily on the “end-operating” and “vanishing” *functions* of inverse, the “inverse of a product” *function* of inverse, and also, implicitly, the

“inverse-inverse” *function* of inverse. Having generated the equation  $b*h*b^{-1} = a^{-1}*h*a$ ,

Tucker interprets his work, saying,

Because we e- we started out with this and we also know that the inverse works. We proved earlier that the  $a$  inverse would work. So each inverse exists. So, that would be equal to that would be true. So, there's closure, which means this is true. (lines 1201-1204)

In the first part of this excerpt, Tucker provides two warrants (Warrant2.5.1, Warrant2.5.2) that support the claim that the equation  $b*h*b^{-1} = a^{-1}*h*a$  is true, seemingly drawing on the equation  $b*h*b^{-1} = h$  in Data1 and T4A10, which supports the equation  $h = a^{-1}*h*a$ . However, Tucker does not explicitly draw on these two equations or any sense of transitivity, instead saying “we started out with this” (lines 1201-1202) while pointing to the statement  $b*h*b^{-1}$  and “ $a$  inverse would work. So each inverse exists” (lines 1202-1203), while pointing to the statement  $a^{-1}*h*a$ . Tucker finally concludes, “So, there’s closure, which means this is true” (lines 1203-1204), which serves as a second claim for the overall argument, hence, concluding proof T4, that the set  $H$  is a subgroup of  $G$ .

Tucker’s proof that the set  $H$  is a subgroup of  $G$  involves several subarguments that provide insight into how the ways in which he thinks about identity and inverse relative to subgroups inform his proof approach. Most notable is Tucker’s initial struggle determining whether the subset  $H$  satisfies the inverse and identity group axioms. Specifically, Tucker’s focus on whether the identity of  $H$  exists constrained meaningful progress in determining whether  $H$  is a subgroup. After the interviewer’s intervention in which he and Tucker discussed whether the identity of  $G$  satisfies the definition of the set  $H$ , Tucker’s approach shifted so that he could successfully carry out the strategies that he alluded to in his earlier arguments for proving the closure and inverses subgroup rules (T4A3 and T4A7). In each of these cases, Tucker’s prior

arguments served to support his approach. In the case of proving that inverses are contained in  $H$ , Tucker's previous activity of substituting  $a^{-1}$  for the element  $g$  in the definition of  $H$  (T4A7) provided an equation that Tucker is able to interpret as meaning that inverses exist in the set  $H$ . Similarly, while proving that  $H$  satisfies closure, Tucker describes in his initial argument (T4A3) how he is thinking that he should try to manipulate the equation  $(a*b)*h*(a*b)^{-1} = h$  (also generated by replacing  $g$  in the equation used to define  $H$ ) in order to generate the equation  $a*h*a^{-1} = b*h*b^{-1}$ . Although he begins T4A11 with this goal in mind, Tucker produces the equation  $b*h*b^{-1} = a^{-1}*h*a$ , which he is able to interpret by drawing on his activity in T4A10 in which he proved that inverses exist in  $H$ . Interestingly, Tucker's previous proof activity serves as either warrant or backing in each of these cases, supporting a sense that Tucker is able to incorporate these new results in order to reason about new arguments.

This highlights an important aspect of Tucker's proof activity: reframing a claim or goal in such a way that he might draw on his conceptual understanding to anticipate achieving this new goal through meaningful use of his available *functions* of identity and inverse. One aspect of this reframing is interpreting a broader goal in terms of a *form* or *forms* so that they might serve the necessary *functions* for achieving the goal. For instance, in T4A4 and T4A8, Tucker focuses on the "multiplicative nature" of the definition of  $H$ , which leads him to think one would be the identity in the group. Tucker then replaces the letter  $g$  with the number five and  $g^{-1}$  with the multiplicative inverse of five, shifting from the "letter" *form* to the (multiplicative) "number" *form* which is consistent with the one satisfying the identity. However, when Tucker returns to the "letter" *form*, one is no longer able to serve the *function* of identity because, as Tucker says, "I guess you don't know what one star  $h$  does, though... in my head that- it just would work, with just, like, normal numbers, but it prob- it might not, actually" (lines 997-999). It is not until



after the conversation in which the interviewer suggests using  $e$  in place of one (a *form* shift) that the identity can serve any meaningful *function*, affording Tucker the ability to simplify the equation  $e * h * e^{-1} = h$  by drawing on the “identity as self-inverse” and “operate/same out” *functions* of identity. Even then, however, Tucker has some difficulty realizing that this means that  $e$  satisfies the definition of  $H$ , although once he realizes this, he is able to continue his proof that  $H$  satisfies the inverse and closure subgroup rules.

**Proof T5: Uniqueness of identity in a group.** Tucker produces his fifth proof (T5) in response to Interview 3 Q4a (Appendix C), which asks the participant to determine whether the statement “The identity element ( $e$ ) of a group ( $G, *$ ) is unique”<sup>10</sup> is true or false and justify his or her answer. Tucker’s proof consists of a single argument (T5A1, Figure 5.32) modeled with an embedded, sequential Toulmin scheme. He begins his proof by reading the prompt and restating it in his own words, constituting a claim for the argument. Tucker then states that he will produce a proof by contradiction and explains what he means by this, saying, “I’m gonna say, okay, well, what if there is two identity elements in a group. And then I’m gonna use that to show that, well, these two must be the exact same thing, then” (lines 302-304), which serves as a warrant for the argument. He then supposes the existence of two identity elements in the group, which he names “ $e$ ” and “ $i$ ” and draws on the “operate/same out” *function* of identity to generate two equations, saying, “ $e$  star an arbitrary element  $a$  is equal to  $a$  star  $e$  is equal to  $a$ . And, similarly,  $i$  star  $a$  is equal to  $a$  star  $i$  is equal to  $a$ ” (lines 306-308). From this, Tucker generates the equation “ $a * e = a * i$ ” (Claim1.1/Data1.2), warranting this equation with the statement, “since these two things are equal, we can set them equal to each other” (line 308).

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<sup>10</sup> This statement is true. A more concise proof than Tucker’s would be to suppose two elements each act as the identity and consider the element  $e_1 * e_2$ , noticing that, by definition of identity (for all  $g \in G$ ,  $e * g = g * e = g$ ),  $e_1 = e_1 * e_2 = e_2$ .

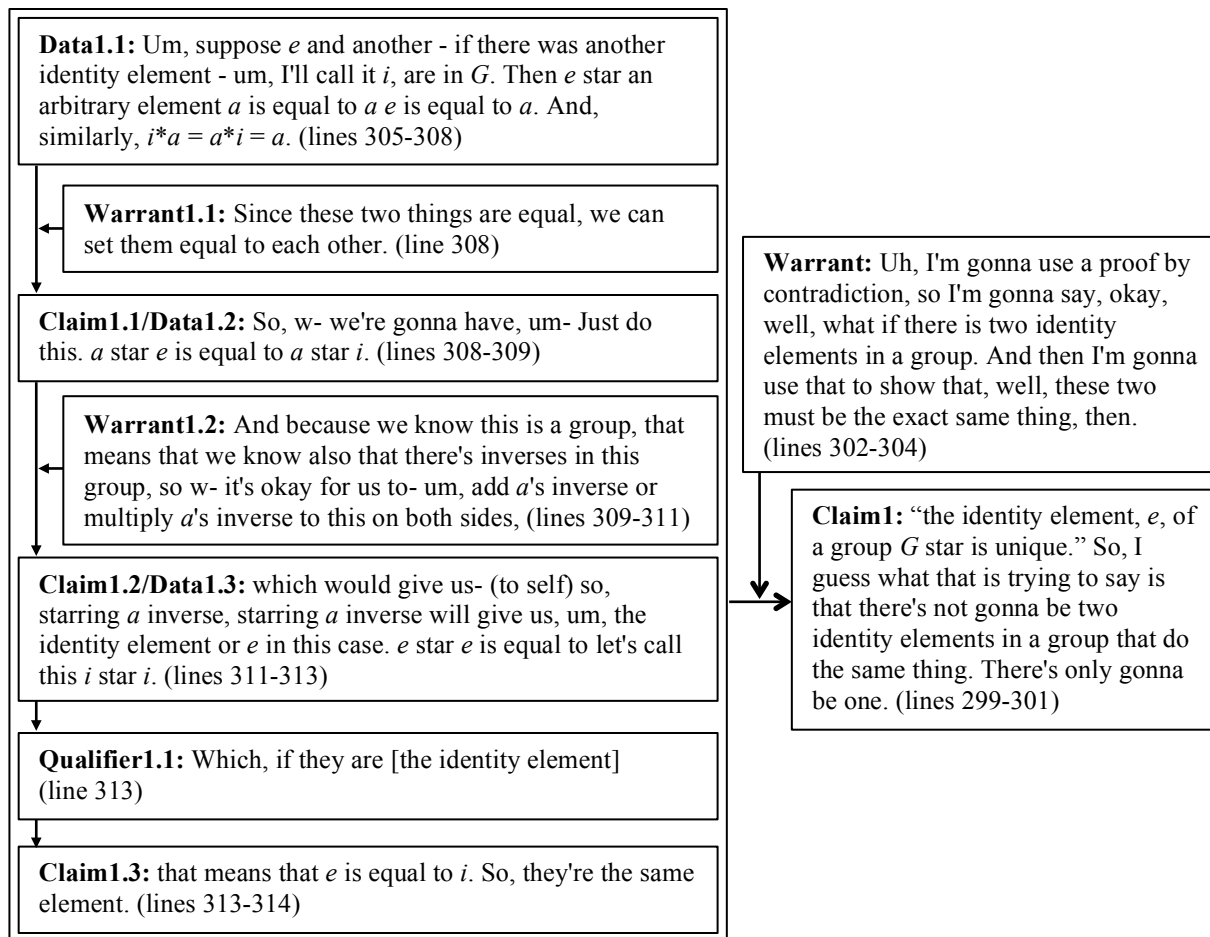


Figure 5.32. T5A1

Tucker continues his argument by citing that the group contains inverses to justify operating on both sides of the equation with the inverse of  $a$ , saying, "it's okay for us to- um, add  $a$ 's inverse or multiply  $a$ 's inverse to this on both sides" (lines 301-311). This excerpt serves to warrant Claim1.2/Data1.3 in which Tucker describes "starring  $a$  inverse" with each side of the equation to produce the new equation " $e*a = i*i$ ." This equation then serves as data for Tucker's final claim in this Toulmin scheme (Claim1.3) in which he says, "that means that  $e$  is equal to  $i$ . So, they're the same element" (lines 313-314), qualifying this claim by saying, "Which, if they are [the identity element]" (line 313). Tucker's proof draws on the "operate/same out" *function* of identity as well as the "end-operating" and "operate/identity out" *functions* of inverse.

Interestingly, Warrant1.2 reflects T3A2 in which Tucker explains the cancellation law, though he does not describe the process as cancellation during this proof.

Throughout T5, Tucker exclusively draws on the “letter” *form* of identity, representing it with both “*e*” and “*i*.” Tucker generates the beginning equation for the proof from the two equations that draw on his definition of identity, which reflects the “operate/same out” *function* of identity. Tucker is also able to draw on the “cancelling” *function* of inverse to produce a copy of the identity on each side of the equation, concatenating each with the existing identities on either side. Interestingly, Tucker chooses each representation of identity so that it matches the existing letter on either side before he draws on the “operate/same out” *function* of identity to rewrite each side as a single copy of the respective identity element. Tucker’s familiarity with the proof likely affords a more direct approach to the proof than if he had not seen it before. Accordingly, T5 does not provide much insight into a dynamic approach (as with T4). For instance, Tucker began with a goal, which he could anticipate achieving and seemed to understand which equations he would need to begin with in order to accomplish this goal. However, it is important to notice how the various *functions* of identity upon which Tucker draws throughout the proof serve to warrant changes in the equation he produces, which is a consistent emerging theme throughout Tucker’s proofs.

**Proof T6: Every element in a group has an inverse element.** Tucker generates his sixth proof (T6) in response to Interview 3 Q4b, to support his assertion that the statement, “Every element in a group  $(G, *)$  has an inverse element”<sup>11</sup> is true. Proof T6 is comprised of two arguments modeled by an embedded Toulmin scheme and an embedded, sequential Toulmin scheme, respectively. It should be stated that Tucker’s initial response to prove the statement

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<sup>11</sup> This statement is true, by definition of a group.

stalled, prompting the interviewer to state that Tucker need not necessarily produce a formal proof. This stands to reason based on the fact that the statement can be verified via proof by definition (of a group). Accordingly, Tucker likely does not think about the two arguments he produces as a formal proof, but rather an informal proof explaining his reasoning about group elements' inverses. Accordingly, Tucker likely feels that this argument would be insufficient for validating his claim in some communities. However, Tucker was a good sport about the difficulty he found proving the statement, and his arguments afford sufficient insight into Tucker's reasoning about identity and inverse and how he constructs arguments with them. Thus, an analysis of these arguments is warranted.

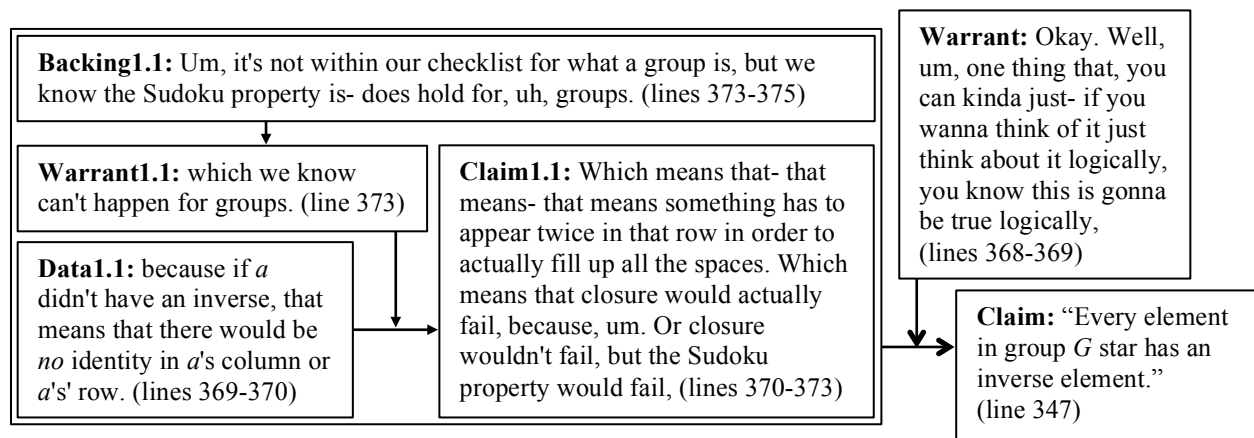


Figure 5.33. T6A1

Tucker's first argument (T6A1, Figure 5.33) begins with him reading the statement aloud from the protocol. After a short discussion about the expectations of whether Tucker should produce a formal proof, Tucker appeals to logic, saying, “one thing that, you can kinda just- if you wanna think of it just think about it logically, you know this is gonna be true logically” (lines 368-369), which serves as a warrant for the rest of his argument. He continues, saying, “because if *a* didn't have an inverse, that means that there would be *no* identity in *a*'s column or *a*'s' row” (lines 370-371). Drawing on the structure of the group tables that Tucker has produced during his coursework as well as the “finding inverse” *function* of identity, this statement serves

as data to support Tucker’s claim, in the embedded Toulmin scheme that “something has to appear twice in that row in order to actually fill up all the spaces. Which means that ... the Sudoku property would fail” (lines 370-373). Tucker then provides further support for his argument, saying, “which we know can't happen for groups. Um, it's not within our checklist for what a group is, but we know the Sudoku property is- does hold for, uh, groups” (lines 373-375), labeled as Warrant1.1 and Backing1.1.

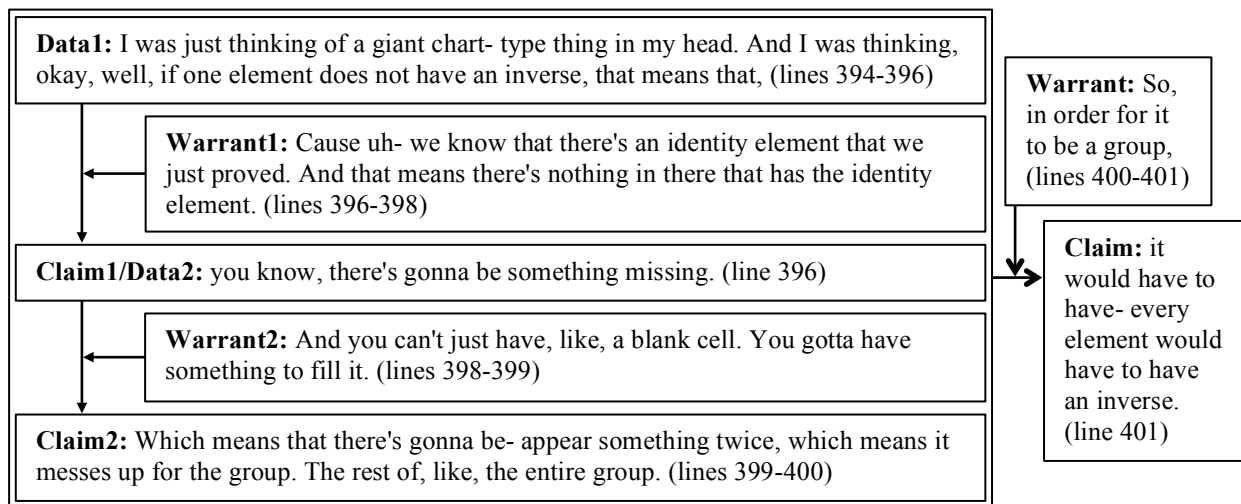


Figure 5.34. T6A2

In response to the follow-up question 4bi Tucker generates a second argument supporting the given statement (T6A2, Figure 5.34). T6A2 is modeled with an embedded sequential Toulmin scheme that serves as data for the broader argument. Tucker’s reasoning during T6A2 generally reflects Data1.1 and the beginning of Claim1.1 of T6A1. Specifically, Tucker again draws on the structure of a group table (Data1), focuses on whether a row or column is missing the identity element (Claim1/Data2, Warrant1) which would leave a blank cell (Warrant2), and that, in order to complete the table, an element must appear twice in it (Claim2). Unlike his reasoning in T6A1, however, Tucker does not explicitly draw on the Sudoku property (from class, that every element appears in each row and each column exactly once), but only refers to each cell needing to be filled and, subtly, that each element appears at *most* once in each column

or row. Throughout T6, Tucker draws on the “finding inverse” *function* of identity to argue about the existence of a given element’s inverse. Specifically, Tucker is concerned with whether the identity element can be located within a given row or column. This affords insight into an aspect of Tucker’s understanding of groups that affords him the ability to reason about specific elements in relation to others.

**Proof T7: Every element in a group is an inverse element.** Tucker’s seventh proof (T7) consists of one argument (T7A1, Figure 5.35), modeled with an embedded Toulmin scheme. Tucker produces this argument in response to the statement, “Every element in a group  $(G, *)$  is an inverse element” (Interview 2, Q4c)<sup>12</sup>, which he claims is true. He follows his claim by warranting his argument, referring to his previous statement in response to Q2 that inverses “work in pairs” (line 412). Tucker then produces a subargument that functions as data for the broader argument, beginning with a claim (Claim1), saying, “So, to say that  $a$  is an inverse, that would be true to  $a$ 's inverse” (lines 412-413). He then supports this claim by explaining, “The inverse of  $a$ 's inverse is gonna be equal to  $a$ ” (lines 313-314) and warrants this claim by, again, referring to the parity of inverses. Finally, Tucker provides backing for the statement by describing repeatedly taking the inverse, saying, “if you [take its inverse] again, it's gonna give you back to  $a$ . So, it's almost like it's cyclic between those two elements” (lines 615-616).

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<sup>12</sup> This statement is true. Every element of a group is the inverse of its own inverse.

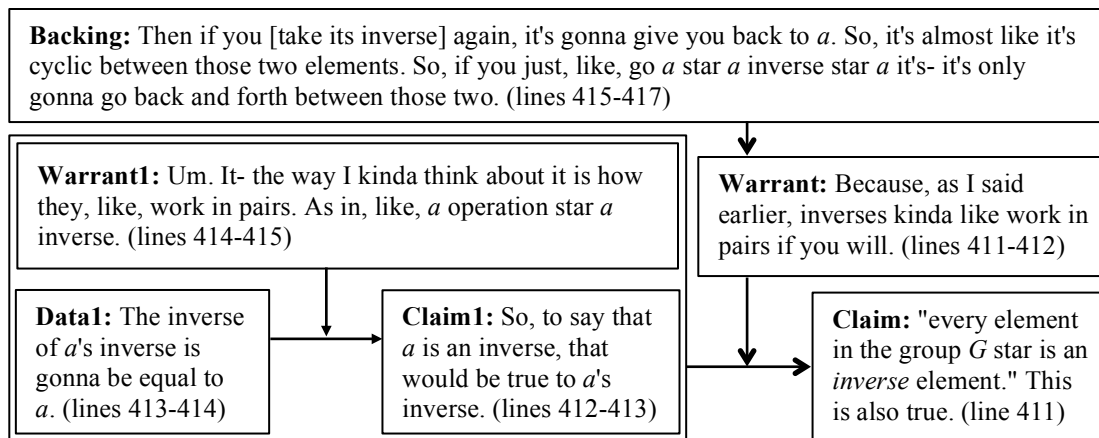


Figure 5.35. T7A1

Throughout Tucker's argument he draws on the "inverse-inverse" *function* of inverse, which demonstrates the importance of this *function* of inverse. He also provides an interesting discussion of how he thinks as about the process of taking an inverse as cyclic. This supports a sense that Tucker is comfortable reflecting on the "inverse-inverse" *function* of inverse beyond the initial statement that an element is the inverse of its inverse, anticipating the result of considering the inverse of an inverse of an inverse... as many times as necessary. In turn, this reflects a deeper meaning that Tucker draws from the "inverse-inverse" *function* of inverse.

**Proof T8: Finite groups with 1 element of order 2 contain an odd number of elements.** Tucker produces proof T8 in response to Interview 3 Q4d. This proof consists of three arguments, the first (T8A1) an embedded, linked scheme, the second (T8A2) and third (T8A3) are sequential Toulmin schemes. Tucker begins his response to the statement, "If a finite group contains exactly one element of order 2, then the group contains an odd number of elements," (Appendix C)<sup>13</sup> by explaining what the word "order" means. He then rephrases what it means to

<sup>13</sup> This statement is true. Tucker's argument addresses the crux of the relationship that the identity and element of order two are the only self-inverses in the group. The remaining element in the group can be paired with their inverses to generate a parity argument for the order of the group.

be an element of order two in terms of inverse, saying, “if an element has order two, that means that it's its own inverse” (line 457). After reading back over the prompt, Tucker says,

I'm gonna say true. and I'll explain why. So, this is true because, if  $a$ 's its own inverse and there's an inverse for every element within this group, that means that- (begins creating group table) So you ha- we know it's go- we know there's gonna be the identity. So, this- We know there's gonna be the identity and that's its own inverse. And  $a$  is its own inverse, as well, so  $e$ 's its own inverse.  $e$  is order one. [00:41:54.628] And let's put  $a$  next. And  $a$  right here.  $a$  is also its own inverse. So,  $e$  is also down here. And there's *exactly one*, uh, element that looks like this. So, ther- So, basically,  $b$  times  $b$  will *not* equal  $e$  right here. It will equal some other thing. So, that means that there *has* to be- Actually, Oh hoo! This is an even number of elements. Never mind. (lines 462-470)

Tucker first asserts that the statement is true, which serves as the claim of T8A1, truncates an initial argument midsentence, and begins creating a group table using the element  $e$  and calling it the identity (Data1). He then includes the element  $a$  in the table, repeating the statement that “ $a$  is also its own inverse” (line 467), which serve as Data2.1 in an embedded scheme that constitutes Data2 in the broader linked scheme. He explains that  $e$  is in the entry corresponding to the  $a$  row and  $a$  column (Figure 5.36, Claim2.1). After placing the two  $e$ 's in the table, Tucker reiterates that “there's exactly one, uh, element that looks like this” (lines 467-468, Data3), referring to  $a$  being the only element of order two. Tucker then includes a third element,  $b$ , in the table and points to the empty space at the entry corresponding to the  $b$  column and  $b$  row, saying, “So, basically,  $b$  times  $b$  will *not* equal  $e$  right here. It will equal some other thing” (line 468, Claim1). Tucker follows this by saying, “So, that means that there *has* to be- Actually, Oh hoo! This is an even number of elements. Never mind” (lines 469-470). This comment constitutes a qualifier and rebuttal. The first sentence from this excerpt is coded as a qualifier because it seems that, in this moment, Tucker realizes that his argument is leading to a contradiction to his original claim. This is supported by Tucker's immediate rebuttal that such a



group contains an even number of elements to which he later adds, “I don't think I was counting identity when I was thinking of elements” (line 472).

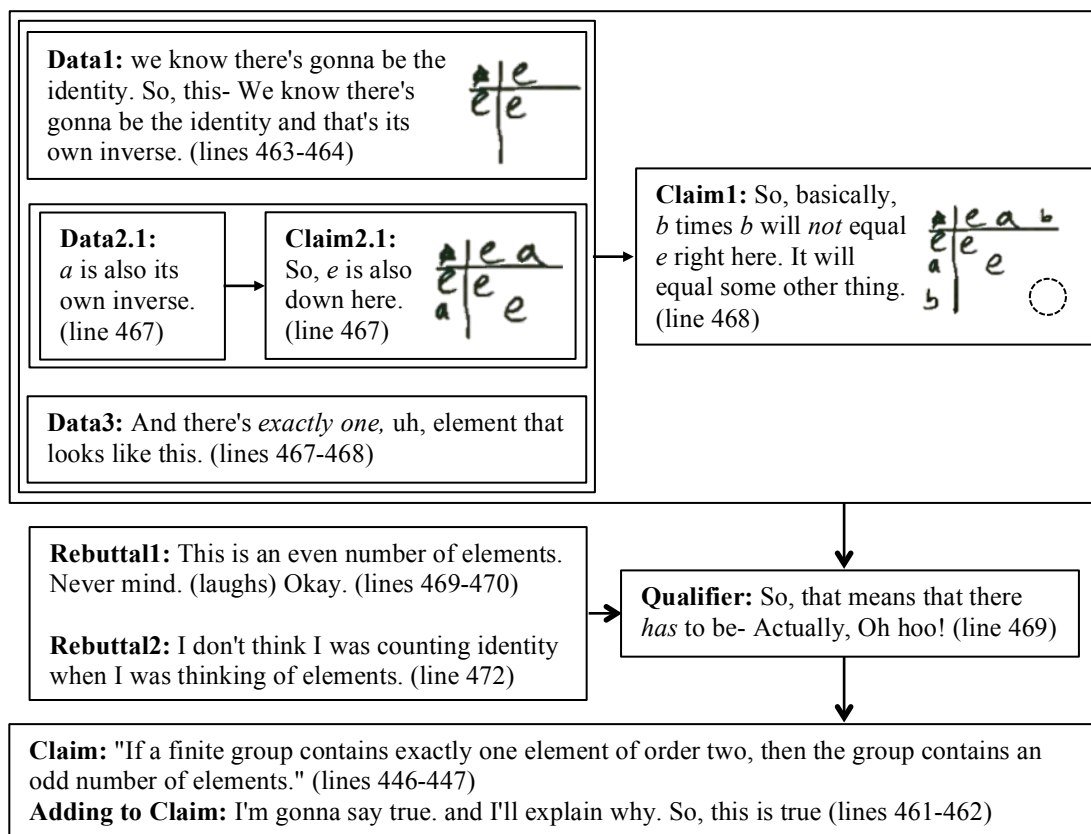


Figure 5.36. T8A1

Tucker then explains that he will “just work it out” and continues completing the table, initially describing aloud the elements that he is filling into the table but remaining silent for approximately the last thirty seconds. T8A2 consists of Tucker’s spoken words during his construction of the group table along with images of the table as Tucker wrote elements in the specific entries (Figure 5.37a, b). Although there are two Toulmin schemes in Figure 5.37, together, these schemes afford insight into Tucker’s reasoning as he is constructing his table. Further, based on Tucker’s construction of group tables during Interviews 2 and 3, writing an entry in the table at any given moment is dependent on the other elements in the table at that time (i.e., Sudoku property, “matching” *function* of identity, “finding inverse” *function* of identity).

Accordingly, T8A2 organizes Tucker’s spoken reasoning for construction of the group table in tandem with his corresponding written work and, so, the two Toulmin schemes are grouped together as a single argument, although they do not together form a single Toulmin scheme.

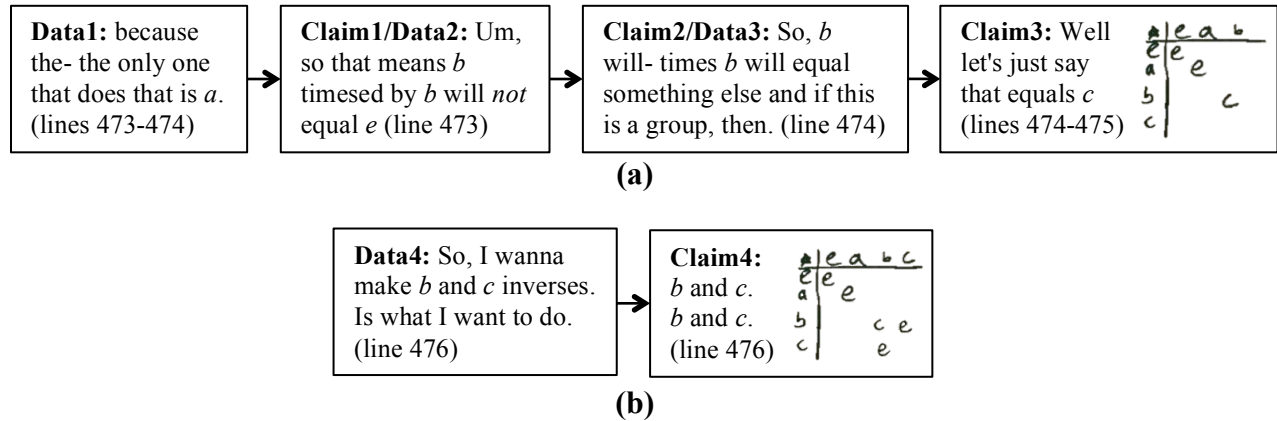


Figure 5.37. T8A2

The first Toulmin scheme produced (T8A2, Figure 5.37a) organizes Tucker’s reasoning for including a fourth element in the table as the result of operating  $b$  with itself. Tucker begins his line of reasoning by pointing out that  $a$  is the only element in the table that has  $e$  in the entry corresponding to operating  $a$  with itself. This serves as data to support Claim1 that  $b$  operated with itself is not the identity element, which in turn, serves as data supporting the claim that the entry corresponding to the  $b$  row and  $b$  column is a different element (Claim2). Because of this, Tucker introduces a fourth element,  $c$ , and claims that this is the result of operating  $b$  with itself. Tucker follows this chain of reasoning by saying, “So, I wanna make  $b$  and  $c$  inverses. Is what I want to do” (line 476). After saying this, Tucker writes the letter  $e$  in the entry corresponding to the  $c$  row and  $b$  column as well as the entry corresponding to the  $b$  row and  $c$  column.

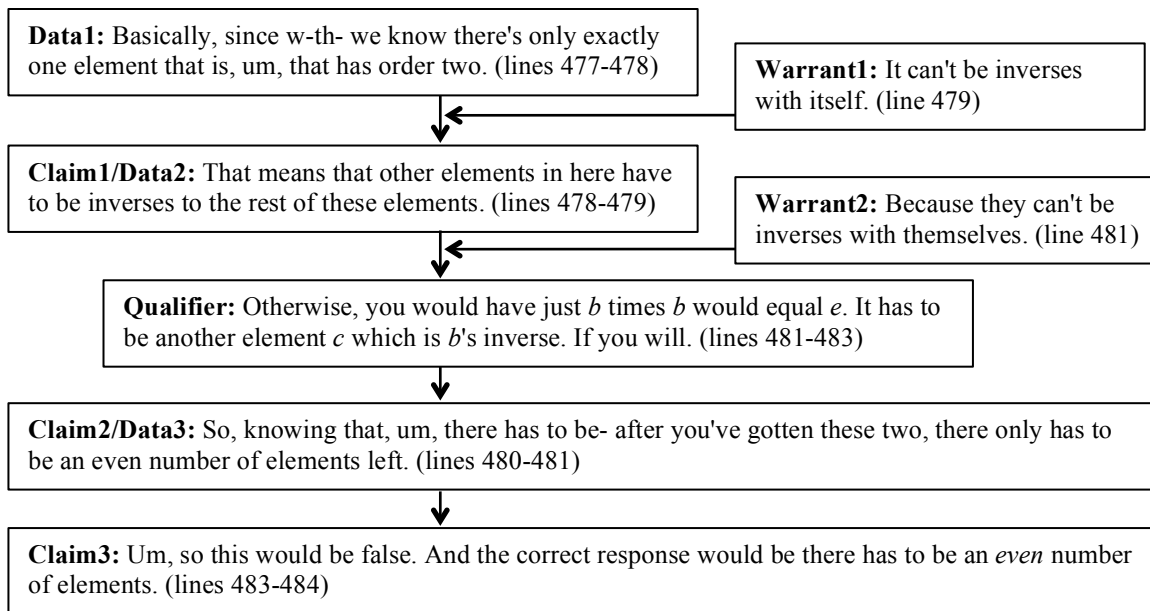


Figure 5.38. T8A3

Following this, Tucker produces a third argument (T8A3, Figure 5.38) modeled with a sequential Toulmin scheme. He begins by reiterating, “there’s only exactly one element that is, um, that has order two” (line 478, Data1). This statement supports the claim (Claim1), “That means that other elements in here have to be inverses to the rest of these elements” (lines 478-479), which Tucker warrants by saying, “It can’t be self-inverses with itself” (line 481). Tucker extends this argument by claiming, “So, knowing that, um, there has to be- after you've gotten these two, there only has to be an even number of elements left” (lines 480-481), which he warrants similarly to Warrant1 supporting Claim1, saying, “Because they can’t be inverses with themselves” (line 481). Tucker goes on to qualify Claim2 so that it serves as Data3 for the final claim, saying,

Otherwise, you would have just  $b$  times  $b$  would equal  $e$ . It has to be another element  $c$ , which is  $b$ 's inverse If you will. Um, so this would be false. And the correct response would be there has to be an *even* number of elements. (lines 481-482)

This final Data3/Qualifier/Claim3 argument draws on the “inverse-inverse” *function* of inverse to generate a parity argument for the remaining elements in the group, specifically focusing on

the elements  $b$  and  $c$ , in turn affording a counting argument incorporating the identity and element of order two as the first two elements (“after you’ve gotten these two,” line 480) and then each new element in the group counted along with its inverse, and preserving the parity of the group.

Tucker’s initial argument (T8A1) draws on the “inverse finding” *function* of identity, along with the Sudoku property, to construct a table for a group satisfying the constraints given in the prompt. Through this construction, Tucker realizes that his original intuition about the claim is mistaken as he finds that he needs an additional element to act as the inverse of  $b$  in the table. In recognizing this, Tucker couches his new claim that the group should contain an even number of elements until he completes filling in the table. He is then able to generate a new argument to support his new claim (T8A3). In this argument, Tucker draws on the “inverse-inverse” *function* of inverse to reason about the parity of the elements in the group. This argument is informed by his earlier construction of the group table, specifically in Claim1/Data2, Warrant1, Warrant2, and the Qualifier. By drawing on the “letter” *form* as elements in the group table, Tucker is able to generalize the reasoning he develops so that any inclusion of a new element will also necessitate the inclusion of its inverse, affording a broader intuition about the structure of groups that satisfy the given properties (finite with exactly one element of order 2).

**Proof T9: Number of self-inverses is even.** Tucker generates his ninth proof in response to Interview 3 Q4e, which, similarly to the other prompts in Q4, asks the participant to state whether they feel a statement is true or false and explain their reasoning, in this case, the statement “For a group  $(G, *)$ , the number of ‘self-inverses’ is even. (Here, “self-inverse” means

an element  $g \in G$  such that  $g * g = e$ .)” (Appendix C)<sup>14</sup>. In this proof, Tucker produces two arguments, the first modeled with an embedded, sequential Toulmin scheme and the second modeled with a sequential scheme. As with T8, the first argument in this proof (T9A1) centers on the generation of a group table to reason about the elements in a group. The second argument focuses on truth of the initial statement and Tucker’s reasoning behind his assertion that it is true.

Tucker begins by creating a table with two elements,  $e$  and  $a$ , naming them both self-inverses and writing the letter  $e$  in entries corresponding to  $e * e$  and  $a * a$  (Figure 5.39, Data1). He continues by including a third element ( $b$ ) and declaring that it is also a self-inverse (Data2.1), saying, “if these are all self-inverses, that means that  $e$  is just gonna go down the diagonal like that” (lines 555-556) Drawing on the “finding inverse” *function* of identity, this excerpt serves as Claim2.1 in the Toulmin scheme embedded as Data2/Claim1. Tucker then says, “And now we have to also prove that- Actually, we can just keep it like this:  $e a b$ . That’s three self-inverses an odd number. (8 seconds) Uh. (3 seconds) Actually, let’s put one more in there” (lines 558-559). In this excerpt Tucker begins to claim that he can continue filling in the table, but hesitates, deciding instead to include a fourth element in the table without giving immediate reasoning (although he does provide some explanation during T9A2). Tucker’s statement seems to be based on his construction of the table up until this point, which supports coding as Claim2, but also leads to his continuation of producing the table with a fourth element included, justifying its use as Data3.

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<sup>14</sup> The given statement is false. The integers under addition, nonzero real numbers under multiplication, and any group of odd order are counterexamples. In Tucker’s proof he tacitly assumes that the group contains more than one self-inverse. Were this constraint included in the conjecture, his proof would be valid for finite groups. However, Tucker does not address this at any point during his proof.

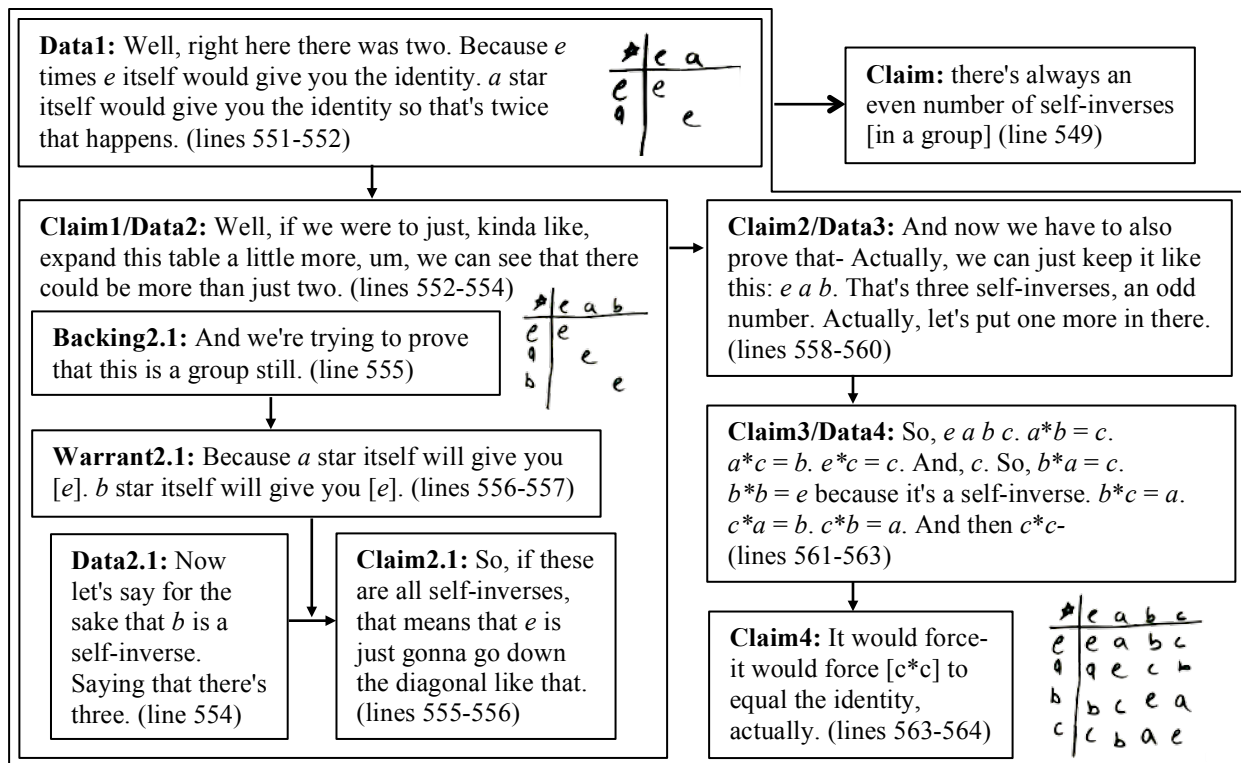


Figure 5.39. T9A1

Tucker continues filling in the table, first listing the elements “ $e, a, b, c,$ ” aloud and periodically saying aloud a three letter sequence with the word “star” between the first and second letter and the phrase “is gonna give us,” “is gonna be,” or “is” between the second and third letter. Only in one instance does Tucker provide reasoning for the equations, saying “ $b$  star  $b$  is  $e$  because it’s a self-inverse” (line 562). This process ends with Tucker saying, “It would force- it would force that to equal the identity, actually” (lines 564-565). This statement constitutes a fourth claim that allows Tucker to complete the construction of the group table, writing  $e$  as the entry in to the  $c$  row and  $c$  column.

From this activity, Tucker seems convinced that the statement in the prompt is true, saying, “Hmm. How to- Now how to prove this. So, basically, um. (8 seconds) You'd ha- it looks like you have to have an even number of self-inverses” (lines 564-566). This statement constitutes Tucker’s affirmation of the original statement in the prompt based on his work in

T9A1, which leads him to restate the claim that there are an even number of self-inverses and begin a new argument (T9A2, Figure 5.40) to support the statement. Tucker begins by saying, “Because, if you didn't- if you- wha- what I was trying to do first was say, okay if we have three self-inverses, like right here” (lines 566-567), which constitutes initial data (Data1.1) for an embedded sequential Toulmin scheme, which constitutes the first data (Data1) in a larger embedded, sequential scheme. Tucker begins his construction with an odd number of self-inverses, which, it seems, he uses to test the original claim.

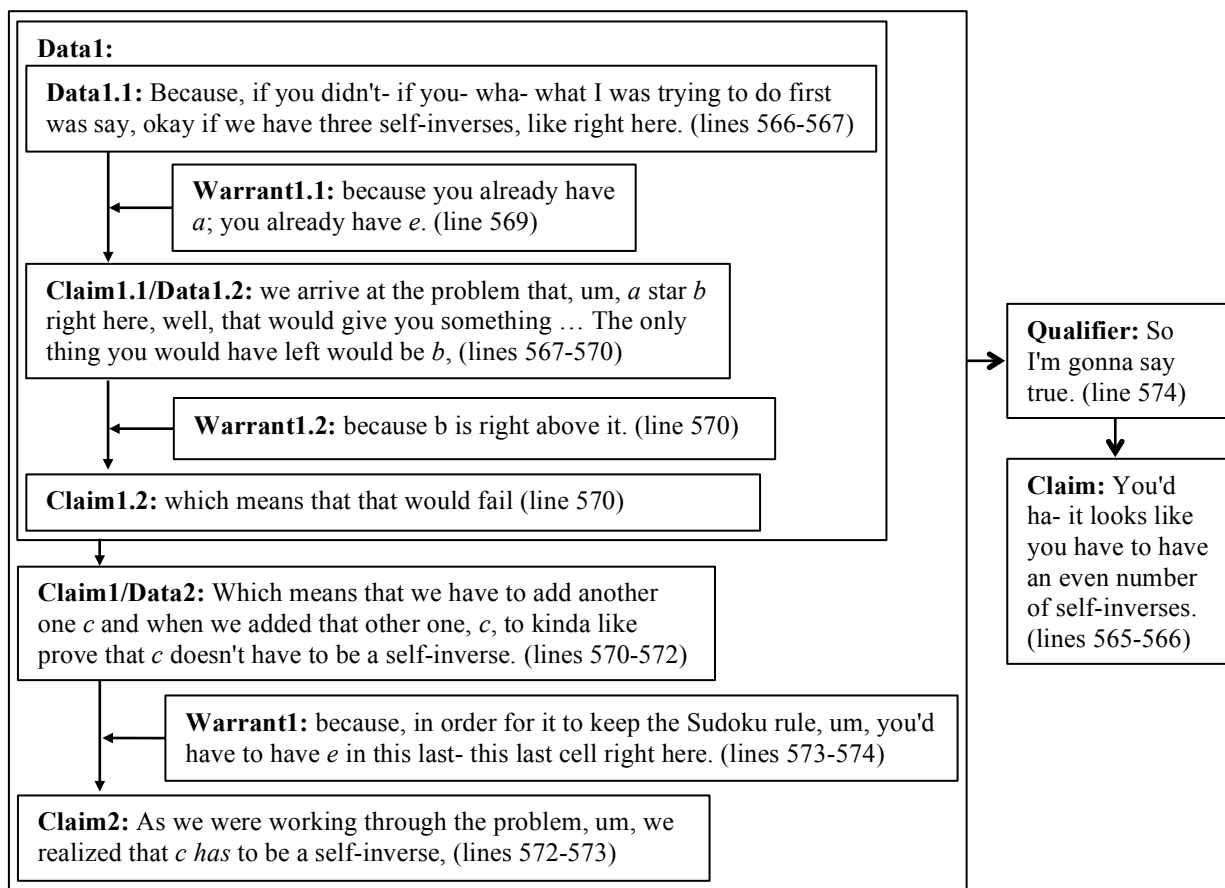


Figure 5.40. T9A2

Tucker then explains how he determined the entry in the *a* row and *b* column of the table with the statement,

Um, we h- we arrive at the problem that, um, *a* star *b* right here, well, that would give you something- you- the only thing you have left- because you already have

$a$ ; you already have  $e$ . The only thing you would have left would be  $b$ , which means that that would fail because  $b$  is right above it. (lines 567-570)

This excerpt constitutes two claim/data and two warrants in the embedded sequential Toulmin scheme. Tucker first points out that there is a “problem” determining the entry, then warrants the claim that the entry must be  $b$ , saying, “because you already have  $a$ ; you already have  $e$ ” (line 569, Warrant1.1). He then points out that “ $b$  is right above it” (line 570), which warrants a new claim (Claim1.2) that placing  $b$  in this position “would fail” (line 570). Together, this embedded, sequential argument serves as data for Tucker’s next claim (Claim1/Data2) that “we have to add another one  $c$  and when we added that other one,  $c$ , to kinda like prove that  $c$  doesn't have to be a self-inverse” (lines 570-572). This excerpt affords insight into Tucker’s reasoning behind Claim2/Data3 in T9A1 when he decided to “put one more in there” (line 560). Further, Tucker further supports a sense that he is testing the original statement that the number of self-inverses is even by saying that he still intends to “kinda like prove that  $c$  doesn't have to be a self-inverse” (line 571-572). Tucker continues the argument, explaining, “As we were working through the problem, um, we realized that  $c$  has to be a self-inverse, because, in order for it to keep the Sudoku rule, um, you'd have to have  $e$  in this last- this last cell right here” (lines 572-574). This excerpt constitutes a final claim (Claim2) that the element  $c$  must also be a self-inverse and warrants the claim (Warrant2) by drawing on the Sudoku rule for group tables. Tucker then qualifies the original statement, saying, “So, I’m gonna say true” (line574).

Tucker’s generation of the group table in T9 reflects his activity in T8. Specifically, he begins with the identity element and the same element,  $a$ , which he describes as having order 2, and draws primarily on the “matching” and “inverse finding” *functions* of identity and the Sudoku rule for groups. Tucker tacitly assumes that there are at least two self-inverses in the group, which necessitates that any finite group Tucker generates will contain an even number of



self-inverses as he eventually claims. It seems that Tucker is unaware that he has imposed an additional constraint on the group that he intends to produce. Further, Tucker only briefly alludes to the group potentially being infinite, though he does not describe how this might relate to the conjecture. This seems to be a limitation of the group table approach on which Tucker has drawn throughout the interviews (T6, T8, and T9). However, as with T8, Tucker’s use of the “letter” *form* allows him to interpret the results of his activity at each step and make decisions moving forward in the construction of the group. Accordingly, Tucker’s generalization, though incorrect for the given prompt would be a valid argument for finite groups with at least two self-inverses.

**Proof T10: Subgroup proof – the normalizer of  $h$ .** Tucker produces his tenth proof (T10) in response to Interview 3 Q5, which prompted the participants to “Prove or disprove the following: for a group  $G$  under operation  $*$  and a fixed element  $h \in G$ , the set  $H = \{g \in G : g * h * g^{-1} = h\}$  is a subgroup of  $G$ ” (Appendix C). Although Tucker constructed fourteen arguments in response to the same prompt during Interview 2 (which addressed each subgroup rule that he mentioned) and he recognizes the current prompt from that interview, he states during this interview that he can not remember his earlier proof, except for his work showing that the set  $H$  contains inverses. Consequently, T10 consists of three arguments; the first is a broader argument about proving that sets are subgroups in which Tucker argues that showing the set contains inverses negates the need to show that it contains an identity element, and the latter two arguments focus on the inverse of each element in  $H$ .

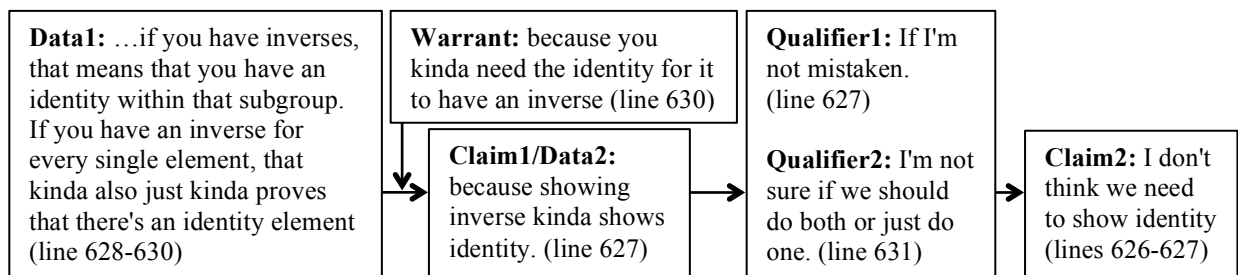


Figure 5.41. T10A1

Tucker's first argument (T10A1, Figure 5.41) is modeled with a sequential Toulmin scheme. During his initial discussion of the prompt Tucker describes what he needs to show in order to prove that the set  $H$  is a subgroup of  $G$ . Tucker begins this argument by stating, "I don't think we need to show identity because showing inverse kinda shows identity. If I'm not mistaken" (lines 626-627). This constitutes an initial claim (Claim2) supported with Claim1/Data2 and qualified with Qualifier1. Tucker then supports Claim1/Data2, explaining,

Well, if you have inverses, that means that you have an identity within that subgroup. If you have an inverse for every single element, that kinda also just kinda proves that there's an identity element because you kinda need the identity for it to have an inverse. (lines 628-630)

In this excerpt, coded as Data1, Tucker clarifies that he thinks of inverses and identity as intrinsically related and then warrants this connection to support Claim1/Data2 by saying, "you kinda need the identity for it to have an inverse" (line 630). This argument reflects a discussion from class in which the students were asked to reduce the group axioms to form a collection of rules for determining whether a set is a subgroup. In that discussion, it was determined that showing the existence of inverses as well as the closure of the set under the operation precludes the need to demonstrate that the set contains an identity element. It seems that Tucker is recounting this argument, although he does not include that the set is closed under the operation. Further, this discussion about the relationship between the identity and inverse subgroup rules is consistent with the "operate/same out" *function* of identity and the "operate/identity out" *function* of inverse. Specifically, because the "operate/identity out" *function* of inverse includes the identity as the result of operating an element with its inverse, it necessitates the existence of an identity in order for *forms* of inverse to serve the *function*. This reflects the sequence in which Tucker developed his proof in T4, especially with respect to the way in which Tucker's first

argument that  $H$  contains inverses leads to his discussion of what the identity of the group is in which he shifts between using  $1$  and  $e$  as the identity.

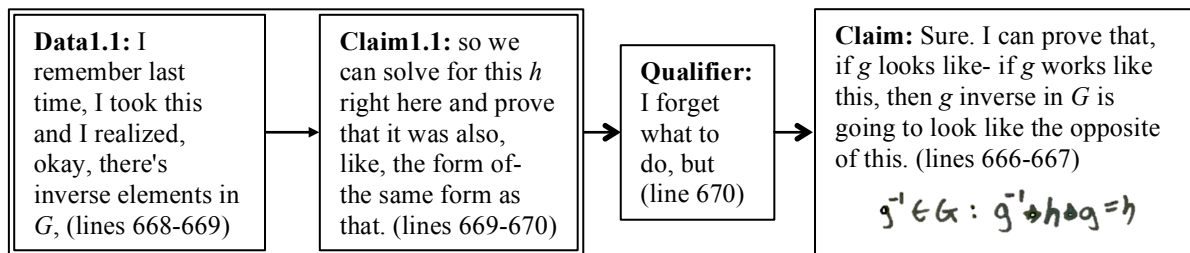


Figure 5.42. T10A2

After Tucker struggles to begin his proof, the interview asks Tucker if there is any part of the proof that he feels he could do. Tucker responds,

Sure. I can prove that, if  $g$  looks like- if  $g$  works like this, then  $g$  inverse in  $G$  is going to look like the opposite of this. So it would be, like, (writes for 8 seconds) Wait wait wait wait. (writes for 18 seconds) I forget what to do, but I remember last time, I took this and I realized, okay, there's inverse elements in  $G$ , so we can solve for this  $h$  right here and prove that it was also, like, the form of- the same form as that. (lines 666-670)

This excerpt constitutes the second argument in T10 (T10A2, Figure 5.42), which is modeled with an embedded Toulmin scheme. Tucker begins by claiming that the inverse of  $g$  will look like the opposite of the equation used to define the set  $H$ , writing “ $g^{-1} \in G : g^{-1} * h * g = h$ .” He qualifies this statement, saying that he does not remember what to do. After this, Tucker explains that, because the group  $G$  contains inverse elements, “we can solve for this  $h$  right here (points to  $h$  in  $g * h * g^{-1}$ , then his written statement “ $g^{-1} \in G : g^{-1} * h * g = h$ ”) and prove that it is also, like, the form of- the same form as that (points to definition of  $H$  in prompt)” (lines 669-670). This excerpt constitutes data that supports the initial claim that  $g^{-1}$  “works like  $[g^{-1} * h * g]$ ,” though it can be parsed into an embedded data/claim scheme in which the existence of inverse elements affords Tucker the ability to change the equation  $g * h * g^{-1} = h$  so that he can solve for  $h$ , which would produce the equation  $g^{-1} * h * g = h$ .

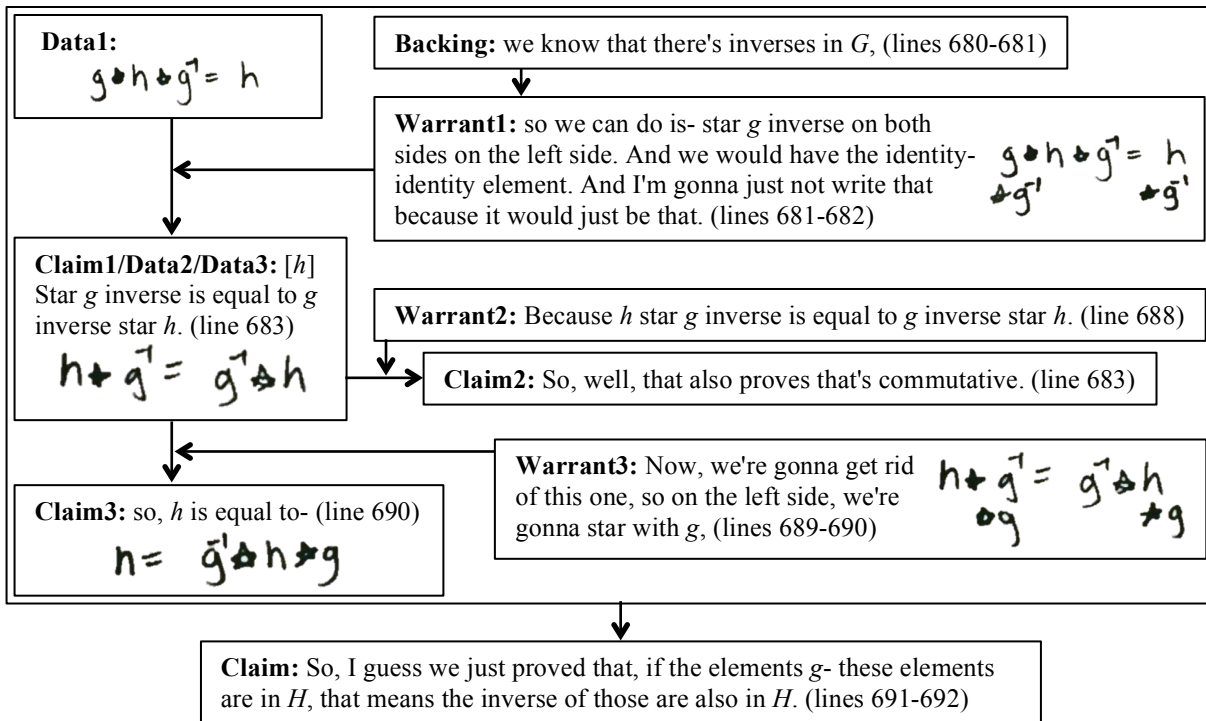


Figure 5.43. T10A3

During follow-up questions to Tucker’s discussion coded as T10A2, the interviewer asks Tucker the question “How do you think- Like, why do you think that this is true (points to  $g^{-1} \in G : g^{-1} * h * g = h$ ) from that? (points to  $g * h * g^{-1} = h$ )” (line 676). With this question, the interviewer is essentially asking Tucker to explain his reasoning behind Claim1.1 in T10A2.

Tucker’s response constitutes a new argument (T10A3, Figure 5.43), which he begins by saying,

Well, we could- um basically, we know that there's inverses in  $G$ , so we can do is- star  $g$  inverse on both sides on the left side. And we would have the identity- identity element. And I'm gonna just not write that because it would just be that. Star  $g$  inverse is equal to  $g$  inverse star  $h$ . So, well, that also proves that's commutative. (lines 680-683)

Tucker begins by recalling that inverses exist in the group  $G$ , a similar statement to Data1.1 in T10A2, although in this case the statement is used to support Tucker’s explicit manipulation of the equation, rather than a more general statement as in T10A2. Tucker then draws on the “end-operating” and “vanishing” *functions* of inverse to manipulate the equation, warranting the claim (Claim1/Data2/Data3) that  $h * g^{-1} = g^{-1} * h$ . Immediately after writing this, Tucker claims that “So,

well, that also proves that's commutative" (line 683, Claim2). The interviewer then asks Tucker what he means by this statement, which Tucker responds, saying, "Um, these two elements. Because  $h$  star  $g$  inverse is equal to  $g$  inverse star  $h$ " (line 688), which warrants Claim1/Data2's support of Claim2. Returning to his original argument, Tucker again draws on the "end-operating" and "vanishing" *functions* of inverse (right-end operating with  $g$  in this case) to manipulate the equation in Claim1/Data2, which warrants (Warrant2) a new claim (Claim2) that  $h = g^{-1} * h * g$ . Together this linked argument serves as data for a broader claim, because Tucker concludes by claiming "So, I guess we just proved that, if the elements  $g$ - these elements are in  $H$ , that means the inverse of those are also in  $H$ " (lines 691-692). During the follow-up questions, Tucker explains that he still does not remember his previous proof, supporting a sense that he might not be very confident in the major claim of T10A3.

Tucker's algebraic manipulation in T10A3 is nearly identical to that in his T4A10, drawing on the "end-operating," "vanishing," and "inverse-inverse" *functions* of inverse. However, Tucker's discussion of why this process shows that inverses exist in the set  $H$  differs from his discussion in Interview 2. Specifically, Tucker describes his goal as "solving for" the  $h$  in the statement  $g * h * g^{-1}$  (Claim1.1, T10A2). In doing so, he generates the equation  $g^{-1} * h * g = h$ , which seems to be sufficient evidence for showing that  $H$  contains inverses, although he does not explain why this is the case. Specifically, he does not mention anything similar to his reasoning from T4A7 in which he substituted  $a^{-1}$  into the equation  $g * h * g^{-1} = h$  and interpreted  $a^{-1} * h * (a^{-1})^{-1} = h$  by drawing on the "inverse-inverse" *function* of inverse to rewrite the equation as  $a^{-1} * h * a = h$ . However, Tucker does mention that the equation  $g^{-1} * h * g = h$  has "the same form" as the equation  $g * h * g^{-1} = h$ , which might be an allusion to that reasoning, though it is unclear whether Tucker is thinking about it in this way.

**Proof T11: Proving  $g^k = e$  implies  $(g^{-1})^k = e$ .** Tucker produces his eleventh proof (T11) in response to Interview 3 Q6a, which asks participants to (for a group,  $(G, *)$ ) “Prove or disprove the following statement: For  $g \in G$ , if  $g^k = e$ , then  $(g^{-1})^k = e$ ” (Appendix C). In his response, Tucker produces four arguments modeled by: two a linked Toulmin schemes (T11A1, T11A2); a linked, embedded scheme in which one embedded scheme is sequential and the other embedded scheme is standard (T11A3); and a sequential scheme (T11A4). Together, T11A2 and T11A3 constitute Tucker’s formal proof of the statement, though T11A1 (in which Tucker tests the conjecture with an example) and T11A4 (Tucker’s explanation of how identity informed his approach to the proof; Interview 3, Q6av) provide useful insight into how Tucker is reasoning about the conjecture.

Tucker begins by rephrasing the statement in terms of an element’s order, saying, “Basically, um, I guess it kinda deals with order a little bit. It's just kinda saying that, um, elements and their inverses have the same order is, kinda like what it's pretty much saying” (lines 762-764). He then tests the conjecture with the element R and its inverse  $3R$  from the rotational symmetries of a square, which constitutes the first argument of T11 (T11A1, Figure 5.44). Tucker begins by saying, “R has order four because if we rotate a square four times, we get back to where we started” (lines 769-771). This is coded as the first data (Data1) in the linked Toulmin scheme. Tucker has generated an example of an element with a known (to Tucker) order, 4, and inverse,  $3R$ . He then proceeds to determine the order of  $3R$ , saying, “Now, we have to find out what we have to raise three R to in order to, um, get back to what we started” (lines 772-773). Tucker describes each rotation of  $3R$  until he has calculated four rotations, which yields twelve R and then states that twelve R is equal to  $e$ , which he indicated as the identity in Data1. From this, Tucker explains, “Um, so, those look like they have the same order for this one

case, just to kind of think about that. But, l- I wanna try to prove that for every case now” (lines 776-776), which constitutes a claim and qualifier for this example generating activity.

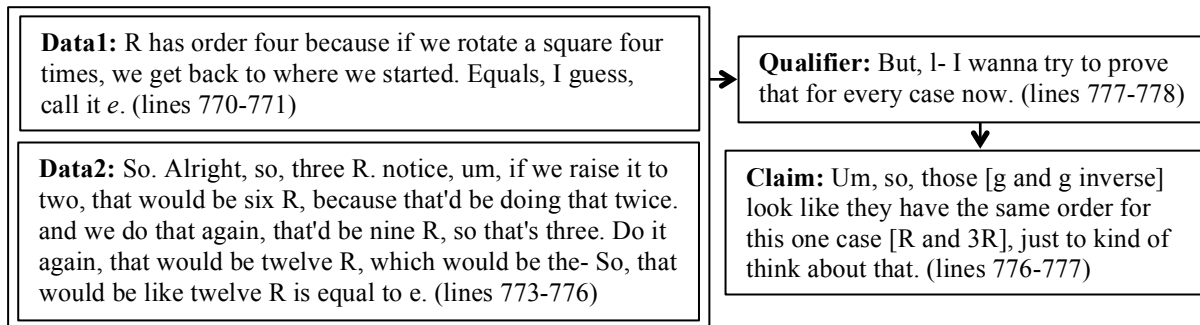


Figure 5.44. T11A1

Tucker’s second argument (T11A2, Figure 5.45) consists of his initial attempts to prove the conjecture. He begins by writing the equation “ $g * g^{-1} = e$ ” and saying, “So we know *that’s* true” (line 781) and then writing “ $g^k = e$ ” and saying, “We also know that  $g$  to the  $k$  is equal to  $e$ ” (line 781). These two statements constitute Data1 and Data2, respectively, for Tucker’s argument. Tucker then tacitly draws on the “transitive” *function* of identity to generate the equation “ $g * g^{-1} = g^k$ ,” and then writes “ $g^{-1} = g^{k-1}$ ” below it before marking through the second line without explaining how he generates the second line or why he erases it. He then states, “If we apply  $g$ ’s inverse to both sides, we would get  $g$  inverse is equal to  $g$  to the  $k$  [writes “ $g^{-1} = g^k$ ”]  $g$  inverse  $g$  to the  $k$  [includes “ $g^{-1}$ ” to the right of the equals sign, which changes his equation to “ $g^{-1} = g^{-1} g^k$ ”]” (lines 784-785), which constitutes a warrant and claim for this argument, specifically, Tucker draws on the “end-operating” *function* of inverse to warrant the claim that “ $g^{-1} = g^{-1} g^k$ .” Finally, he qualifies his activity, saying, “I feel like I’m close, but I don’t know how to get from here to- So, that’s what we wanna show” (lines 786-787) and writing the equation “ $(g^{-1})^k = e$ .”

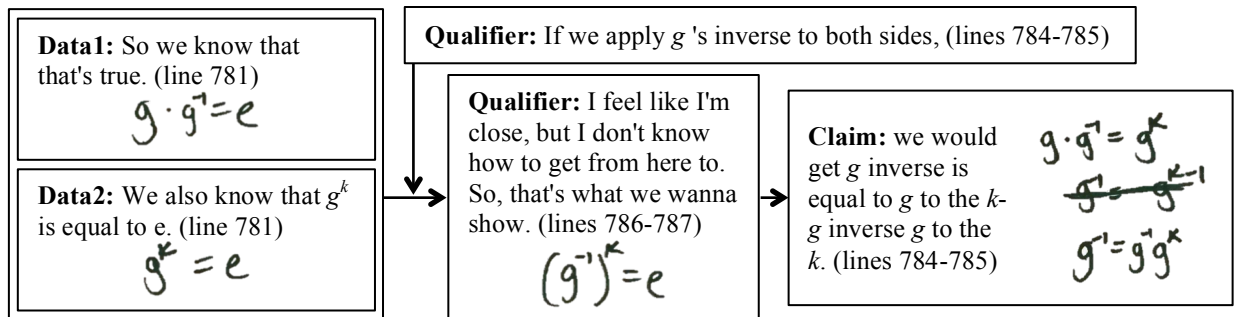


Figure 5.45. T11A2

Asked whether Tucker thinks that the statement is valid or invalid (6avii), Tucker responds that he believes it is valid but is having difficulty developing an approach. After a brief exchange in which Tucker describes his thinking by saying,

I think maybe, actually, using this (points to  $g^k = e$ ) and somehow reaching this (points to  $(g^{-1})^k = e$ ). Like, thinking about how- how to do that. Yeah, I'm not sure. I just kinda wanna like- well I wanna like multiply this (points to  $g^k = e$ ) by  $g$  inverse  $k$  times" (lines 798-800)

This excerpt serves as a warrant for a new linked argument (T11A3, Figure 5.46) in which each claim is represented as an embedded Toulmin scheme - the first a linked scheme (Data1), the second a standard scheme (Data2). The interviewer then asks Tucker if he can do what he described, to which Tucker responds by generating the first embedded argument (Data1).



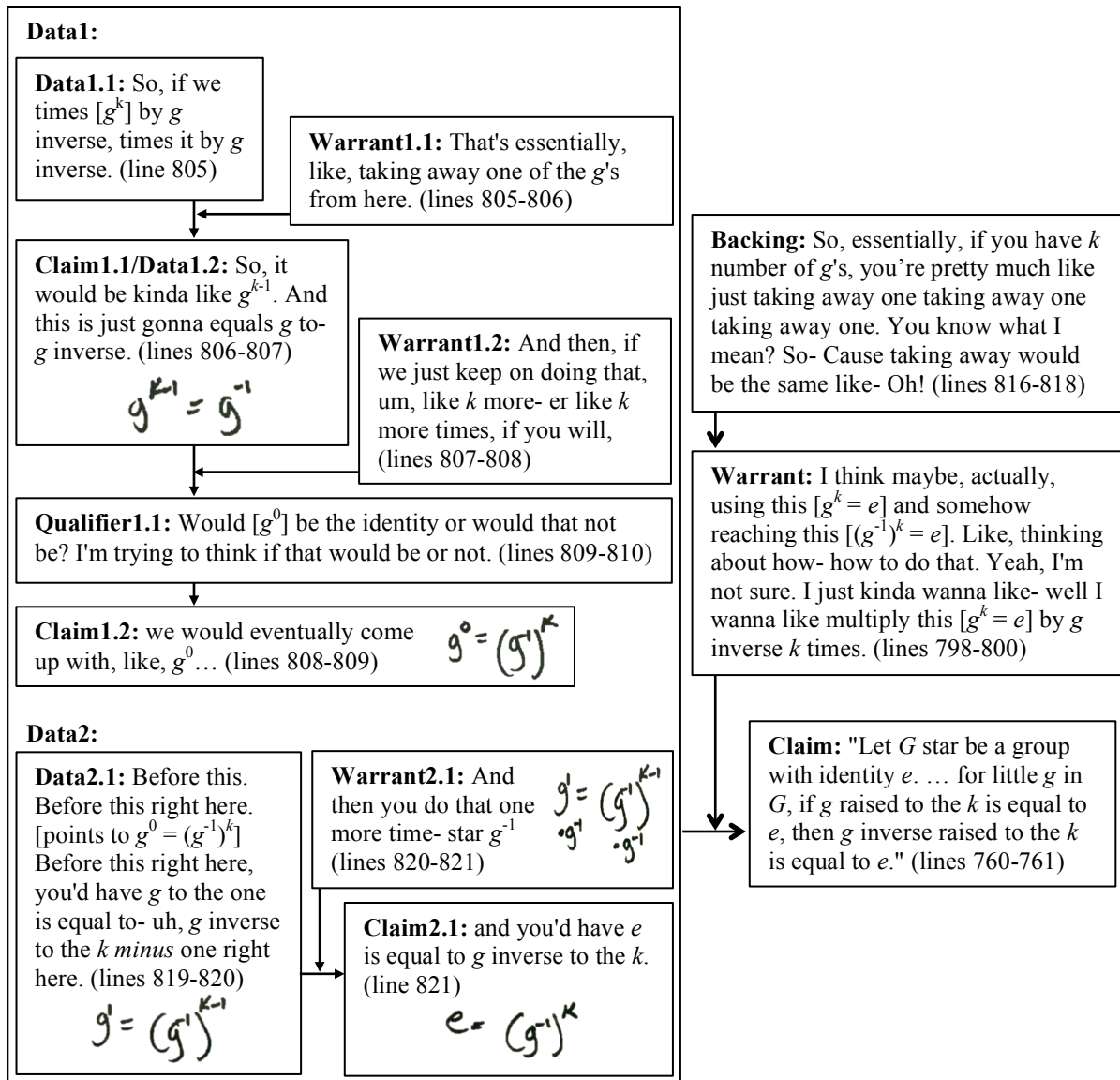


Figure 5.46. T11A3

Tucker begins by describing a repeated process of end-operating with  $g^{-1}$  (Data1.1), saying, “So, if we times it by  $g$  inverse, times it by  $g$  inverse” (line 805), which serves as initial data for a linked argument (Data1.1). He warrants this, saying, “That's essentially, like, taking away one of the  $g$ 's from here” (805-806), which draws on the “vanishing” *function* of inverse, in order to support the claim that “So, it would be kinda like  $g$  to the  $k$  minus one. And this is just gonna equals  $g$  to-  $g$  inverse” (lines 806-807, Claim1.1/Data1.2). As he says this, Tucker writes the equation “ $g^{k-1} = g^{-1}$ ,” which is the same equation he marked through in T11A2. This equation

serves as data for Tucker's next claim (Claim1.2/Data1.3), which he first warrants by again drawing on the "end-operating" *function* of inverse, though describing that process as occurring " $k$  more times" (line 808). Tucker continues, saying, "we would eventually come up with, like,  $g$  to the zero, which is like- Would that be the identity or would that not be? I'm trying to think if that would be or not" (lines 808-810). This excerpt constitutes a final claim in the sequential Toulmin scheme, which Tucker writes as " $g^0 = (g^{-1})^k$ ," as well as a qualifier for the claim. Interestingly, Tucker anticipates the result of his repeated use of the "end-operating" and "vanishing" *functions* of inverse but is uncertain about how to interpret the result of this activity. Specifically, he questions whether  $g^0$  is the identity element, because Tucker does not yet recognize this inscription as a representational vehicle for identity.

Asked to explain Claim1.1/Data1.2, Tucker begins by reiterating Warrant1.1 (backing) but pauses mid-sentence, excitedly exclaiming, "Oh!" (line 800). He then begins a new sub-argument, which serves as a second data (Data2) for the overarching argument, saying,

Before this. Before this right here. Before this right here, you'd have  $g$  to the one is equal to- uh,  $g$  inverse to the  $k$  minus one right here. And then you do that one more time- star  $g$  inverse and you'd have  $e$  is equal to  $g$  inverse to the  $k$ . Which would prove that that is true. (lines 819-822)

This excerpt begins with Tucker pointing out that in the last step of the process he describes (Warrant2.1), he would have the equation " $g^1 = (g^{-1})^{k-1}$ ." He then draws on the "end-operating" *function* of inverse again to warrant a new conclusion to the process, writing " $*g^{-1}$ " under each side of the equation " $g^1 = (g^{-1})^{k-1}$ " and then generating the new equation " $e = (g^{-1})^k$ ," which constitutes Claim2.1 of the embedded Toulmin scheme. Tucker then immediately interprets this argument as validating the conjecture, thus supporting the original claim in the broader argument.

In T11A1, Tucker is able to interpret the conjecture from the prompt in terms of order, which, in turn, affords him the ability to generate an example of the conjecture drawing on the “letter” *forms* of symmetries of a square rather than the more generic letter *forms* used in the prompt. From this activity, Tucker gains a sense that the conjecture is valid and sets out to prove it. In T11A3, Tucker is able to anticipate the result of repeatedly end-operating with  $g^{-1}$ , which he was unsure of during T11A2. This indicates a shift in how Tucker is able to draw on the “end-operating” *function* of inverse, making it more general than end-operating with the inverse a single time. However, although Tucker successfully anticipates the outcome of this process, he is unable to immediately interpret  $g^0$  as being equivalent to the identity element. He further confirms this during the follow-up questions, saying, “Well, I was thinking well, wait a minute. If you, um, minus it  $k$  times you're gonna get  $g$  to the zero. And I was thinking, like, what does that even mean?” (lines 917-918). Through his discussion in Data2 of T11A3, Tucker is able to imagine the next-to-last step in the process, which affords him a more meaningful (apparent from his excitement) interpretation of his activity, which he also supports during the follow-up discussion, saying, “And that's what my kind of like revelation was that it's not really  $g$  raised to the zero, per se. It's just, um, kinda like, i- it's-  $g$ - it's zero left over  $g$ 's kinda like thing” (lines 922-923).

**Proof T12: Proving  $g^k = e$  for some positive  $k$  implies  $g^{-1} = g^p$  for some positive  $p$ .**

Tucker’s twelfth proof (T12), which he generates in response to Interview 3 Q6b, consists of an argument modeled with a sequential Toulmin scheme containing a linked data (T12A1) and one represented by an embedded scheme (T12A2). The prompt for Q6b introduces a group  $(G, *)$  with identity, states that  $g^k = e$ , and asks participants, “Is it possible to write  $g^{-1}$  as  $g^p$  for some positive number  $p$ ?” (Appendix C). Tucker begins by reading the prompt and writing the

equation “ $g^{-1} \cdot g = e$ .” He then rhetorically asks, “So, so is it possible to write this (points to “ $g^{-1} \cdot g = e$ ”) as (writes “ $g^p \cdot g = e$ ”).” These equations constitute the first data (Data1.1) in a linked Toulmin scheme that serves as data in a broader sequential Toulmin scheme (T12A1, Figure 5.47). Tucker then says, “And we know that  $g$  to the  $k$  is equal to  $e$ . So, we can kinda, like, put these two things equal to each other. So, we'd have  $g$  to the  $p$  times  $g$  is equal to  $g$  to the  $k$ ” (lines 1003-1005). This excerpt completes the embedded scheme by providing a second data (Data1.2), which Tucker attributes to the prompt; a warrant for the use of these two data, in which Tucker supports the generation of a new equation (Claim1/Data2); and the statement of the new equation, which Tucker then writes as “ $g^p \cdot g = g^k$ .”

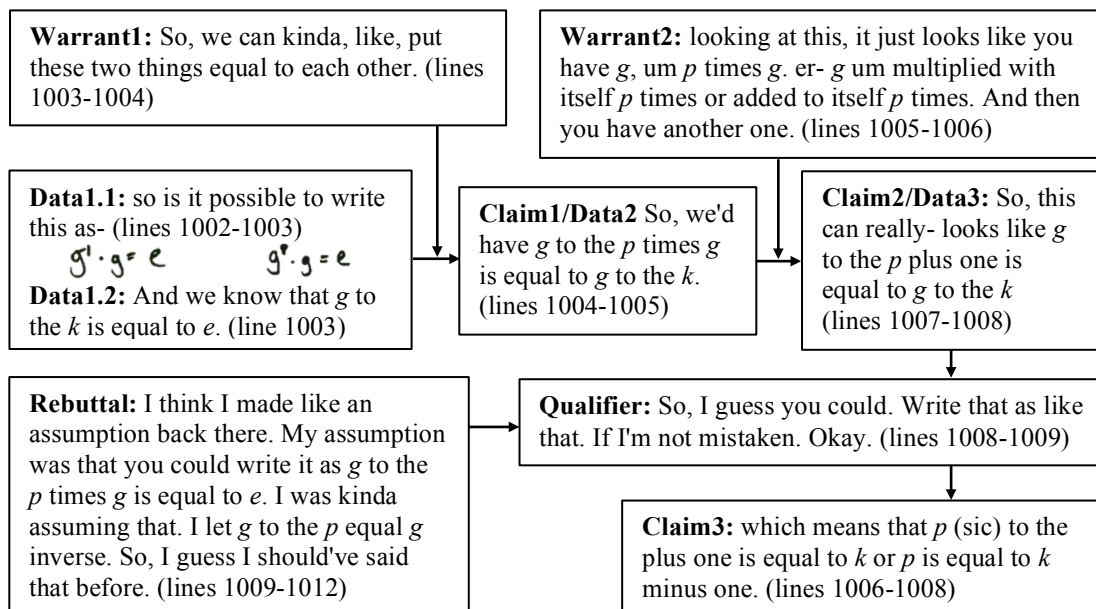


Figure 5.47. T12A1

From this equation, Tucker describes  $g^p$  as “ $g$  um multiplied with itself  $p$  times or added to itself  $p$  times,” and then points out that there is another  $g$  on the left-hand side of the equation, which serves as Warrant2 in the sequential Toulmin scheme for the claim (Claim2/Data3) that “ $g$  to the  $p$  plus one is equal to  $g$  to the  $k$ .” From this, Tucker directly claims (Claim3) that this “means that  $p$  to the plus one is equal to  $k$  or  $p$  is equal to  $k$

minus one” (lines 1006-1008). Tucker then qualifies his argument, saying “I guess you could. Write that as like that. If I’m not mistaken” (lines 1008-1009) and rebuts his proof by pointing out that he began his proof by assuming that such a  $p$  exists, adding that he should have said that before. When asked to explain his proof in “one straight line,” Tucker responds by producing a nearly identical argument, only including the initial data that he assumes that such a  $p$  exists before writing the equation “ $g^p \cdot g = e$ ,” and omitting the qualifier and rebuttal.

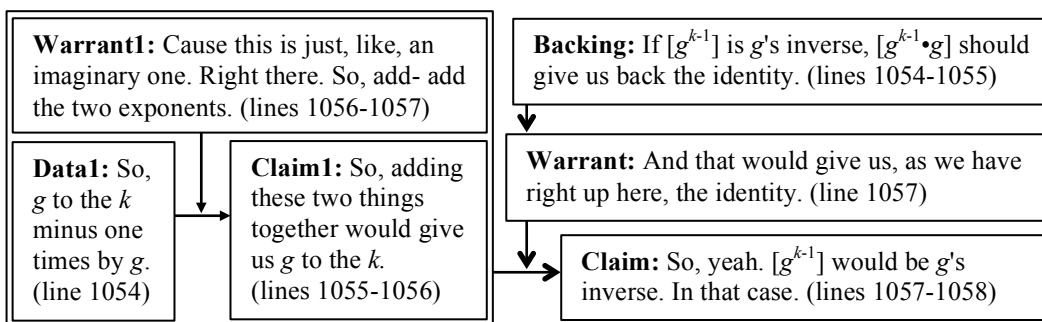


Figure 5.48. T12A2

During follow-up question to this proof, the interviewer asks Tucker whether he thinks  $g^{k-1}$  is the same thing as  $g^{-1}$ . Tucker’s response provides a second argument for T12 (T12A2, Figure 5.48) modeled with an embedded Toulmin scheme. He begins by repeating the element “ $g$  to the  $k$  minus one times by  $g$ ,” which serves as the data in an embedded scheme (Data1) and explaining that, “If it is  $g$ 's inverse, this should give us back the identity” (lines 1054-1055). This statement provides an overarching goal for the argument in the form of a conditional statement that Tucker is able to test, justifying its coding as a backing for the argument. Following the backing, Tucker claims that “adding these two things together would give us  $g$  to the  $k$ ,” (lines 1055-1056), warranting this with the statement, “Cause this is just, like, an imaginary one. Right there. So, add- add the two exponents” (lines 1056-1057). Tucker then points to the equation “ $g^k = e$ ” in the

prompt, saying, “And that would give us, as we have right up here, the identity” (line1057), which, supported by his original backing, warrants the claim, “So, yeah.  $[g^{k-1}]$  would be  $g$ 's inverse. In that case” (lines 1057-1058).

Throughout T12, Tucker draws on the “end-operating” *function* of inverse to manipulate the equation he generated by first drawing on the “operate/identity out” *function* of inverse and the given criterion that  $g^k = e$ . This activity consistently draws on the “letter” *forms* of identity and inverse without any shifts in *form*. Tucker realizes that he initially assumes the claim that he wishes to prove, but this does not seem problematic. Further, Tucker’s final claim in T12A1 is tacitly based on the equivalence of exponents for the same element on either side of the equation ( $g^{p+1} = g^k$  implies  $p+1 = k$ ), from which Tucker is then able to solve for  $p$  using common algebraic reasoning for real numbers. However, this seems to be an unproblematic conclusion to Tucker. Assuming the claim seems to have afforded Tucker a strategy for determining an alternate representation for  $g^{-1}$ . Specifically, by creating the equation with  $g^p$  operated with  $g$ , Tucker has inserted the letter  $p$  into the equation so that he is able to manipulate the exponent, rather than an approach that would not have  $p$  already in an exponent of  $g$ , which would necessitate solving an equation for  $g^{-1}$  in terms of  $g^k$ .

**Summary of analysis of Tucker’s proofs.** Across his twelve proofs, Tucker regularly draws on *functions* of identity and inverse to serve as warrants in his argumentation. In most cases, this is in service of a broader goal that Tucker has described or alluded to. In other cases, Tucker draws on *functions* of identity and inverse to explore situations, in turn gleaning insight into specific situations about which he can then develop arguments. The *functions* upon which Tucker draws often depend on the *forms* with which he is working. For instance, when

generating proofs in which he is manipulating equations, Tucker consistently draws on the “letter” and “number” *forms* to serve the “end-operating,” “cancelling,” “vanishing,” and “operate/identity out” *functions* of inverse as well as the “operate/same out,” “transitive,” and “vanishing” *functions* of identity. Typically, Tucker carries out these manipulations with a specific goal in mind, although, in a few instances, his activity seems more exploratory (e.g., T11). In contrast, Tucker’s proofs in which he uses the group table rely on the “finding inverse” and “matching” *functions* of identity as well as the Sudoku property of groups during primarily exploratory activity – resulting in Tucker interpreting the results of his construction at each step in the process of constructing the table (e.g., T8).

There are also instances in which Tucker begins with a specific goal and must draw on a specific *form* in order to interpret that goal. For instance, in T4A3 and T4A7, Tucker begins by naming the closure and inverse subgroup rules, respectively, as something necessary to show, but must draw on the “letter” *form* and the definition of the set  $H$  in order to interpret those goals in the context of the problem, although he is initially unable to achieve the goals. In the case of identity (T4A5, T4A6), this proves problematic as Tucker attempts to argue that the identity exists in  $H$ , because he is unable to generate a *form* of identity that is consistent with the definition of the set without a prompting discussion from with the interviewer. Specifically, Tucker shifts between the (multiplicative) “number” *form* of identity, “1” and the “letter” *form* of identity, “ $e$ .” However, Tucker is unable to draw on either of these *forms* of identity to serve any *function* of identity without prompting from the interviewer. In contrast, in T4A7, Tucker is able to draw on the “inverse-inverse” *function* of inverse after substituting  $a^{-1}$  for  $g$  in the equation used to define  $H$  to produce the equation “ $a^{-1} * h * a = h$ ,” although he is unsure how this might help him to show that  $H$  contains the inverse of each of its elements. He is unable to leverage this

equation until after he says, “So, right now, we have  $g \star h \star g$  inverse is equal to  $h$ . We want to get to somewhere that looks like- ... Want to show.  $g$  inverse star  $h$  star  $g$  is equal to  $h$ ” (lines 1084-1086). In this moment, Tucker has shifted the goal of his proof activity so that his initial data and his claim share the same “letter” *form* upon which he is also able to draw to serve the necessary *functions* to reach his claim.

In contrast, the given statement to be proven in Interview 3 Q6a (prompted T10) reflects initial data and claim that share the same “letter” *form*. Rather than shifting the *form* to generate a proof of the statement, Tucker’s initial reaction is to shift to the “symmetry” *form* in order to test the conjecture (T10A1). However, as Tucker shifts back to the “letter” *form* in order to generate his proof, it is unclear how or whether Tucker’s example testing informed the proof other than helping convince him that the given statement is true. Instead Tucker draws on the “exponent-reducing” *function* of inverse to change the given equation. He then anticipates carrying out this process  $k$  number of times, but is unsure whether the result of this anticipated process ( $g^0$ ) is the identity. This leads to Tucker analyzing his anticipated process and reversing the last step, which affords him an opportunity to draw on the “operate/identity out” *function* of inverse and verify that  $g^0$  is the identity, in turn, validating the original claim.



## Analyzing John's Proofs

Across John's interviews, he produces ten proofs (Table 5.3) consisting of a total of 25 arguments. The majority of these proofs occur during John's third interview, including an impromptu proof that John provides while explaining his definition of group. The majority of John's proofs consist of a single argument, which often contains embedded arguments that serve to draw out finer detail about the how John's arguments might be constructed. For the proofs in which several arguments are used, it is often the case that John poses a question regarding whether his approach is valid, leading to the production of new arguments in order to resolve his issue. These typically serve to support John's activity generating a broader argument for the proof.

*Table 5.3 – Arguments constituting John's proofs*

Proof	Interview	Protocol Q	Number of Arguments	Types of Toulmin Scheme (argument number)
J1	1	6	3	Standard (1); Embedded (2); Linked (3)
J2	2	5	1	Embedded; Linked
J3	2	6	2	Embedded (1); Sequential (1); Linked (2)
J4	2	7	7	Standard (1, 4, 6); Linked (1, 5); Embedded (2, 3, 5, 7); Sequential (2, 3, 5, 7); Divergent (5)
J5	3	3	4	Linked (1); Embedded (all); Sequential (all)
J6	3	4d	1	Linked; Embedded; Sequential
J7	3	4e	1	Linked
J8	3	5	1	Linked; Embedded; Sequential
J9	3	6a	4	Standard (1); Linked (4); Embedded (2, 3, 4); Sequential (2)
J10	3	6b	1	Embedded; Sequential; Linked

**Proof J1: 1 as a multiplicative identity.** John produces his first proof in response to Interview 1 Q6 (Appendix A), in which participants are asked to prove that one is the multiplicative identity for real numbers. John's response (J1) is comprised of three arguments: two arguments modeled with a standard Toulmin scheme (J1A1, J1A3) and one argument modeled with an embedded Toulmin scheme (J1A2). Overall, John's response indicates that he

feels that the statement is true, although he has difficulty generating a proof. This is indicated by John's initial appeal to authority (J1A1), as he says, "Um, somebody told me a while ago that it was true. There. The internet" (line 719). This is modeled using a standard Toulmin scheme (Figure 5.49) in which this statement serves as the lone Data directly supporting the given Claim, without warrant, backing, qualification, or rebuttal. The lack of any additional support for the claim indicates that John is likely thinking of the statement as taken or axiomatic, which his earlier discussion in response to Interview 1, Q1 and later discussion in response to this prompt support.

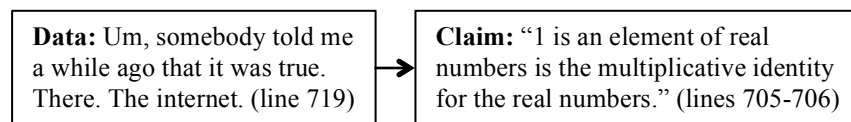


Figure 5.49. J1A1

When asked by the interviewer what it means to be a multiplicative identity (Q6, f), John responds,

Um, you multiply a number by the multiplicative identity and you still get that number. So, just, like,  $a$  times  $x$  equals  $x$ .  $a$  would be the multiplicative identity.  $a$  is mult. id. ...  $a$  would be the multiplicative identity. I have no idea about how to go about proving the one is  $a$ , though. Other than the fact that I know that one times  $x$  is  $x$ . (lines 730-734)

John initially responds to the interviewer's question and then connects this to the statement given in the prompt, which, together, constitute a second argument (J1A2, Figure 5.50), which is modeled with an embedded Toulmin scheme with the original statement serving as the primary claim of the argument. John's explanation of what it means to be a multiplicative identity can be modeled as an embedded argument represented with a standard Toulmin scheme. John's description of multiplicative identity serves as a warrant (Warrant1.1) to justify the relationship between his following statements. He describes the equation  $a$  times  $x$  equals  $x$ , which reflects

the structure of the “equals” *function* of identity (Figure 4.16) and serves as data (Data1.1) for the claim that “ $a$  would be the multiplicative identity” (Claim1.1, line 731).

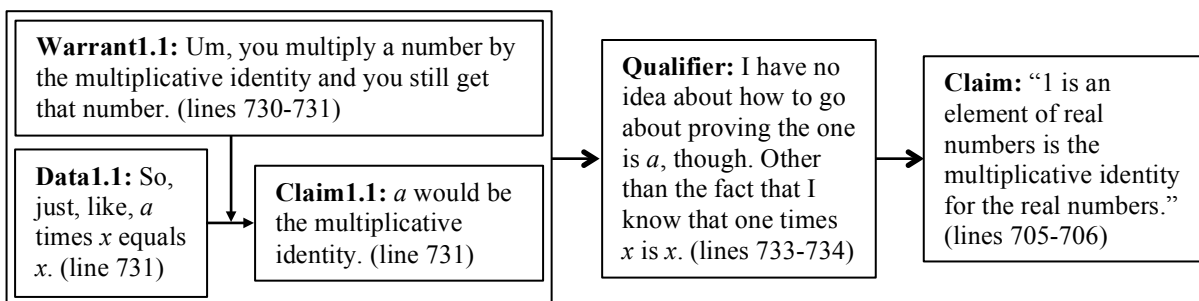


Figure 5.50. J1A2

John then connects this description of the multiplicative identity back to the original prompt, saying, “I have no idea about how to go about proving the one is  $a$ , though. Other than the fact that I know that one times  $x$  is  $x$ ” (lines 733-734). This serves as a qualifier in the argument in the sense that John verifies he is unable to draw on his description of multiplicative identity in order to prove the given statement in the prompt. In other words, in this moment, John is unable to provide any warrant that supports how the “equals” *function* of identity might be used to prove the given claim. However, immediately following his discussion coded as J1A2, John begins describing how the claim might be supported using addition, saying,

Except multiplication is shorthand for addition- like, repetitive addition. So, could you use that to continue? Like, if two  $x$  equals  $x$  plus  $x$  then one  $x$  equals  $x$  plus nothing... Like, for any  $x$ , that’s just true if you’re using, like, multiplication as repetitive addition. Wait. I just can’t think of how the symbols work, or how the, like- I don’t know how to prove this at all. (line 734-738)

This excerpt constitutes a new argument (J1A3, Figure 5.51) that can be viewed as John attempting to produce a warrant for J1A2. As with J1A2, the original statement from the protocol serves as the claim for J1A3. John begins the excerpt by describing an alternative way of representing multiplication as repeated addition, which serves to warrant how his following activity might help to prove the claim. He then relates two examples of what he is describing,

saying, “Like, if two  $x$  equals  $x$  plus  $x$  then one  $x$  equals  $x$  plus nothing” (lines 735-736), which serves as Data1 in the model. With this statement, John has shifted from using the letter “ $a$ ” that he uses in J1A2 to using specific numbers in an equation for which the right-hand side is phrased as addition, the latter equation drawing the “number” *form* of identity. Following this, John states a general claim that the structure in his examples is “just true if you’re using, like, multiplication as repetitive addition” (lines 736-737). This statement provides backing for his original warrant in the argument, although this argument also seems to stall as John qualifies his argument with the statement, “Wait. I just can’t think of how the symbols work, or how the, like- I don’t know how to prove this at all” (line 737-738).

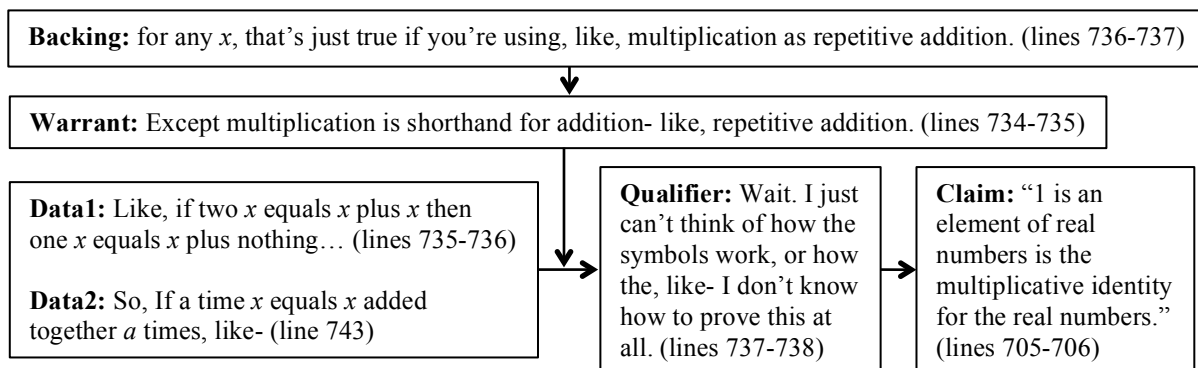


Figure 5.51. J1A3

When asked how he might show that one is the multiplicative identity (Q6g), John responds by providing an equation that reflects the two equations in Data1, with a product on the left-hand side and addition on the right-hand side, but replaces the specific numbers with the letter “ $a$ .” This constitutes a second data for J1A3, which reflects Data in J1A2, except that the right-hand side is phrased additively, rather than reflecting the “equals” *function* of identity. However, John abandons this approach, first questioning whether he should use the letter  $b$ , rather than  $a$ , then briefly mentioning summation notation without describing how it might relate to the proof.

John’s second and third arguments afford interesting insight into how his understanding of identity informs his proof approach. Specifically, John shifts from using a “letter” *form* of identity (“*a*”) with his “equals” *function* of identity (J1A2) to generally describe multiplication additively while drawing on “number” *forms* (J1A3). This shift affords the potential development of a warrant for J1A2 that could justify how the “equals” *function* of identity using the “letter” *form* of identity might serve to support the claim that the “number” *form* 1 is the multiplicative identity. Drawing on “number” *forms*, John is able to represent multiplication as repeated addition using the instantiations of two and one. John then shifts again – from the “number” *form* back to the “letter” *form* – while still drawing on the additive expansion of multiplication on the right-hand side of the equation. However, John is unable to connect this back to the data in J1A2 in order to provide a warrant for that argument.

**Proof J2: Subgroup proof, integer powers of 2.** John generates his second proof (J2, Figure 5.52) in response to Interview 2 Q5, which asks the participants to “Prove or disprove:  $H = \{g \in G \mid g = 2^n, n \in \mathbb{Z}\}$  is a subgroup of the group  $G$  of non-zero real numbers ( $\mathbb{R} \setminus \{0\}$ ) under regular multiplication” (Appendix B), which he had previously proven as an assignment for class. Accordingly, his proof is relatively straightforward, modeled with a single Toulmin scheme (J2A1), though it is comprised of four embedded schemes – one for each subgroup rule that John describes – each of which serves as data in a linked scheme. John begins his response by recalling having seen the prompt during coursework and then naming the subgroup rules that he feels he need to prove, saying,

And, so, it contains the inverse, it contains the identity. I think we have to prove that it is closed. And- I think there’s. ‘All  $g$  in  $G$ ,’ Okay. So, it’s definitely contained. It is a- at least a subset. Since all its elements have to be in  $G$ . So, we don’t have to prove it’s a subset. At least I don’t think. I’m gonna go with you don’t... (lines 434-441)

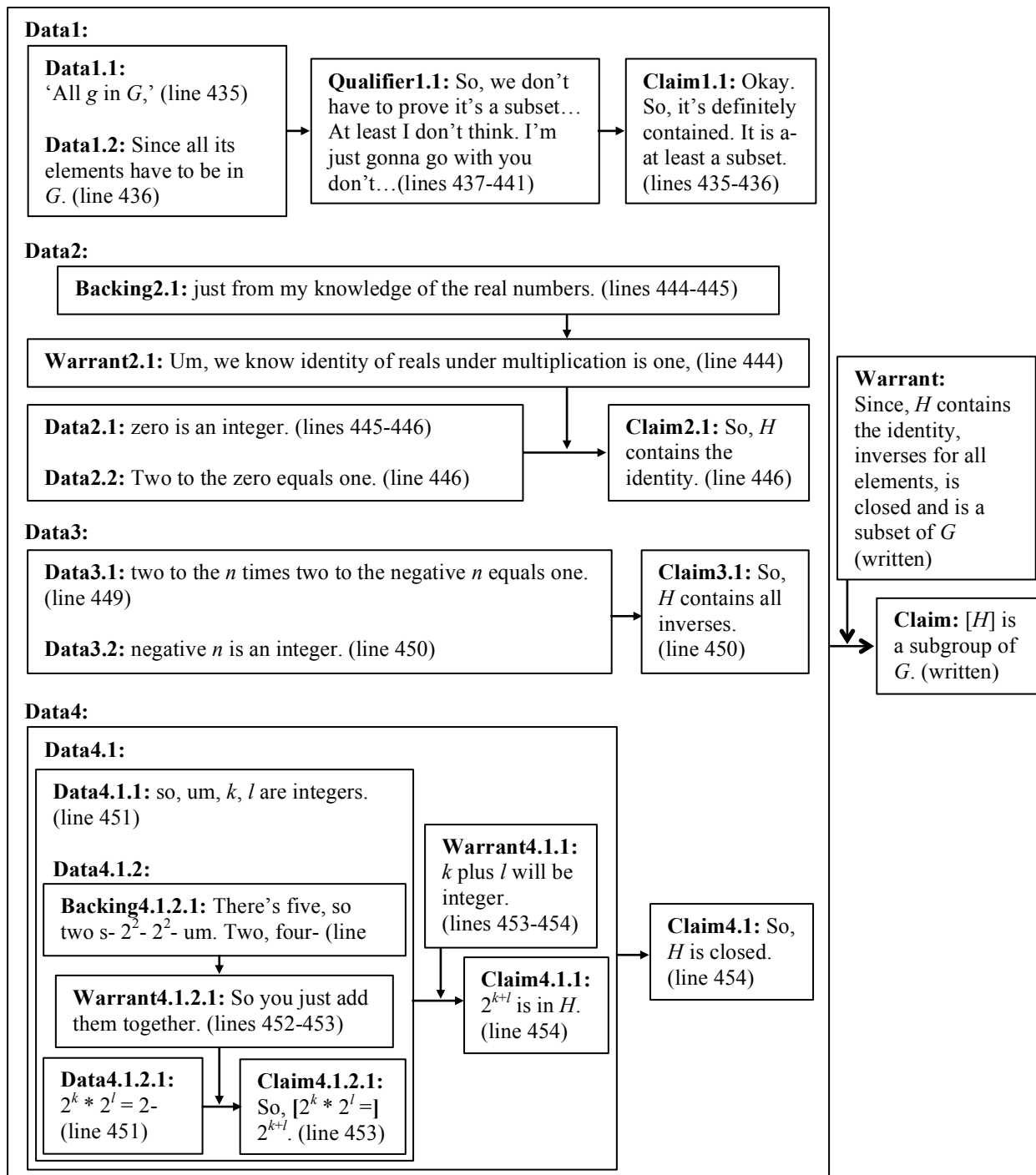


Figure 5.52. J2A1

This excerpt outlines John's approach to proving the broader claim that  $H$  is a subgroup of  $G$  and provides transcript for his first sub-argument, modeled as an embedded standard Toulmin scheme that serves as Data1 in J2A1. John begins this argument by reading the phrase "All  $g$  in

$G$ ” from the prompt, which serves as data (Data1.1) supporting John’s claim, “Okay. So, it’s definitely contained. It is a- at least a subset” (lines 435-436). He then rephrases Data1.1, which serves as a second data (Data1.2) and then qualifies his argument, saying, “So, we don’t have to prove it’s a subset. At least I don’t think. I’m gonna go with you don’t...” (lines 437-441).

John then produces an argument that the set  $H$  contains the identity element, which serves as Data2 within the larger Toulmin scheme, saying, “Identity. Um, we know identity of reals under multiplication is one, just from my knowledge of the real numbers... Two: zero is an integer. Two to the zero equals one. So,  $H$  contains the identity” (lines 443-446). John begins this subargument by stating that he knows that the identity is one, which serves as a warrant (Warrant2.1) for the argument, and provides a backing for this warrant (Backing 2.1) by appealing to his “knowledge of the real numbers” (lines 444-445). John then points out that “zero is an integer,” (lines 445-446), adding, “Two to the zero equals one” (line 446). These two statements serve as data (Data2.1 and Data2.2) to support his ensuing claim (Claim2.1) that “ $H$  contains the identity” (line 446). John continues by constructing an argument (Data3) that the set  $H$  contains inverses, saying, “Inverse. Um, so two to the  $n$  times two to the negative  $n$  equals one. Um, negative  $n$  is an integer. So,  $H$  contains all inverses” (line 449-450). The first two sentences in this excerpt provide two data (Data3.1 and Data3.2), which immediately precede John’s claim that, “ $H$  contains all inverses” (line 450). The conciseness of the argument reflects a sense that John feels the data he provides are sufficient support the claim without any further justification as well as the fact that John had previously constructed this proof and, so might be responding from memory. Data3.1 reflects the equation used to define the “equals” *function* of inverse, by positioning two elements operated with each other on one side of an equation with a *form* of identity on the other side of the equation, although John does not discuss this in any way.

In his last subargument (Data4), John sets out to show that the set  $H$  is closed. He begins by naming two integers using the letters  $k$  and  $l$ , which serves as Data4.1.1. He then considers the product " $2^k * 2^l$ ," pausing after saying, "equals 2-," which serves as Data 4.1.2.1, after which he asks, "How's that work?" (line 452). This leads John to calculate the product " $2^2 * 2^2$ ," which he calculates to be  $2^4$ , then saying, "So you just add them together" (lines 452-453). This constitutes an example generation that serves as backing for the warrant that exponents are added together, in turn allowing John to support his following claim that  $2^k * 2^l = 2^{k+l}$ . He then points out that  $k$  plus  $l$  is an integer, which serves as Warrant4.1.1 to support the claim " $2^{k+l}$  is in  $H$ ," Altogether, this process serves as Data4.1 that affords John to make the claim that " $H$  is closed" (line 454). Together, the four data support the claim given in original statement for John, which he writes, "Since,  $H$  contains the identity, inverses for all elements, is closed and is a subset of  $G$ , it is a."

**Proof J3: Uniqueness of an element's inverse.** John generates his third proof in response to Interview 2 Q6, which asks participants to "Prove the following: for each element  $g$  in a group  $G$  with operation  $*$ ,  $g^{-1}$  (the inverse of  $g$ ) is unique" (Appendix B). As with Interview 2 Q5, John proved this statement during class and, so, is familiar with the statement and likely drew on his prior experiences when producing his proof during the interview, indicated by the linearity of his argument. Analysis of John's proof draws out two arguments, the first of which he constructs to verify the statement given in the prompt (J3A1), which is modeled with an embedded, linked Toulmin scheme. In the second argument (J3A2), John explains (in response to a follow-up question) the cancellation law that he uses as a warrant in J3A1.



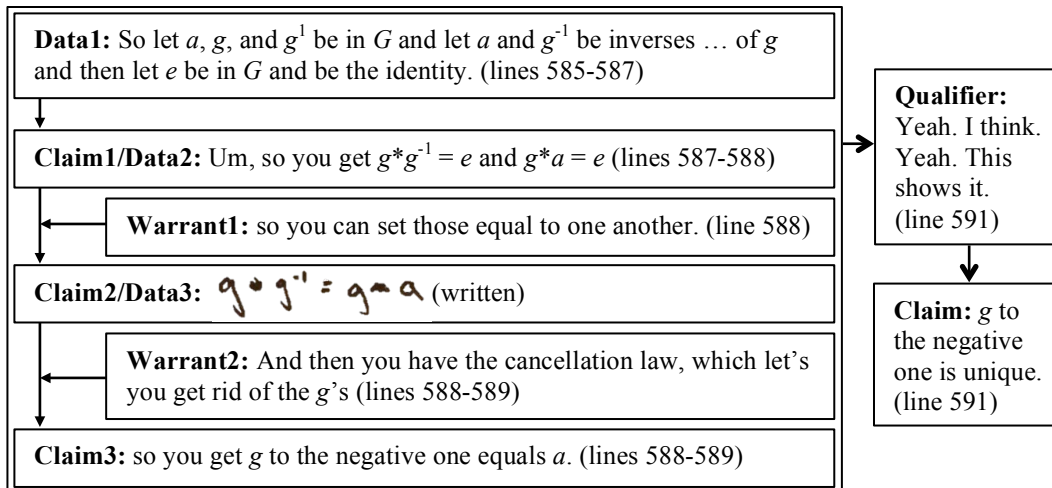


Figure 5.53. J3A1

John begins J3A1 (Figure 5.53) by fixing three elements ( $g$ ,  $g^{-1}$ , and  $a$ ) in  $G$  and describing  $g^{-1}$  and  $a$  as inverses of  $g$ ; he then lets  $e$  be the identity in  $G$ . This serves as the first data in a sequential Toulmin scheme, supporting John's generation of two equations –  $g * g^{-1} = e$  and  $g * a = e$  – which reflect the equation used to define the “equals” *function* of inverse and serve as Claim1 following the initial definition of the elements and then serves as Data2 which supports the generation of the equation  $g * g^{-1} = g * a$ . John warrants Claim2 by saying “so you can set those equal to each other,” although he does not explain why he thinks this. That being said, from the development of this equation, it seems that John draws on the “transitive” *function* of identity. John then says, “And then you have the cancellation law, which let's you get rid of the  $g$ 's” (lines 588-589), providing a warrant for the new equation “ $g^{-1} = a$ .” Altogether, this sequence of data/claim pairs serve as the data for John's overarching claim “ $g$  to the negative one is unique” (line 591), which he qualifies saying, “Yeah. I think. Yeah. This shows it” (line 591).

John's use of the cancellation law in J3A1 reflects the “cancelling” *function* of inverse. When asked to explain the cancellation law during follow-up questions, John provides an argument (J3A2, Figure 5.54), which is represented with a sequential Toulmin scheme with one linked node. John begins J3A2 by fixing a “letter” *form* of inverse,  $b$ , which he specifies as a

left-inverse of  $g$ . This constitutes the first of two linked data (Data1.1) in the first node of the sequential scheme. He then re-writes the equation  $g * g^{-1} = g * a$  from J3A1, constituting Data1.2 in the linked scheme. Together, these data serve as Data1 to support a new equation,  $b * g * g^{-1} = b * g * a$ , that serves as Claim1/Data2, then providing a warrant this claim, saying, “you can just do that to the front of both since you’re doing both things on the same like normal” (lines 670-671). John then explains, “then these would just cancel out” (lines 671-672), which serves to warrant his statement “so you’re just left with  $g$  negative one equals  $a$  again” (line 672).

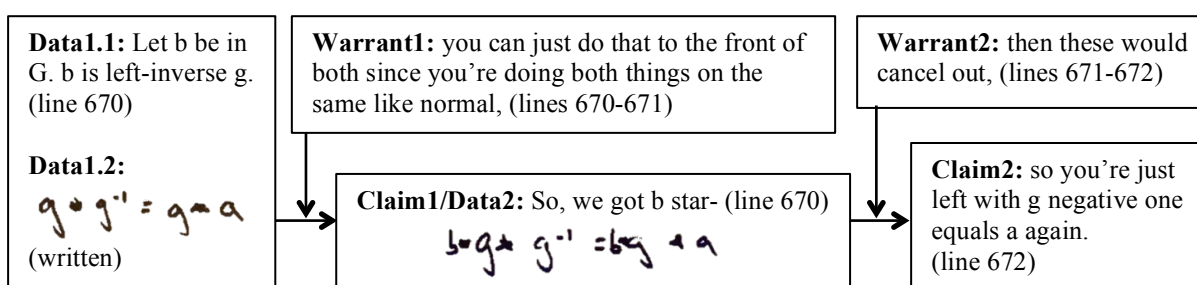


Figure 5.54. J3A2

John’s proof reflects the general approach of the proof discussed during class, which is also similar to the proofs generated by Tucker and Violet. J3A1 draws on the “letter” *forms* of identity and inverse, and several *functions* of identity and inverse. For instance, John tacitly draws on the “equals” *function* of inverse to generate the two equations in Claim1/Data2 of J3A1. He also implicitly draws on the “transitive” *function* of identity to set the respective left-hand sides of those equations equal. John then explicitly draws on the “cancelling” *function* of inverse to rewrite this equation with the  $g$ ’s removed. These *functions* of identity or inverse afford the three data-claim shifts, either implicitly or explicitly, so that John can show that the inverse of  $g$  is unique. Further, in Claim1/Data2 of J3A2, John specifies that  $b$  is a left-inverse, which reflects the “non-commutative” *function* of inverse before describing the cancellation of  $b * g$  on each side of the equation, which aligns with the “vanishing” *function* of inverse. Again,

as with J3A1, the various *functions* of inverse support the data-claim shifts along the sequential Toulmin scheme.

**Proof J4: Subgroup proof – first proof that the normalizer of  $h$  is a subgroup.** John produces his fourth proof (J4) in response to Interview 2 Q7, which prompts participants, “Prove or disprove the following: for a group  $G$  under operation  $*$  and a fixed element  $h \in G$ , the set  $H = \{g \in G : g*h*g^{-1} = h\}$  is a subgroup of  $G$ ” (Appendix B). John’s proof can be viewed as consisting of 7 arguments (one overarching argument and 6 related subarguments) that together contribute toward his entire proof that the set  $H$  is a subgroup of the group  $G$ . Similarly to his approach in J2, John completes this proof by verifying the subgroup rules for the set  $H$ , which he describes during his response as needing to show that the identity exists, inverses are contained in  $H$ , and that the set  $H$  is closed (John earlier described that associativity is from the larger group, lines 477-478). The structure constitutes an overarching argument (J4A1, Figure 5.55) that  $H$  is a subgroup of  $G$  in which the other arguments comprising J4 can be viewed as linked data in J4A1.

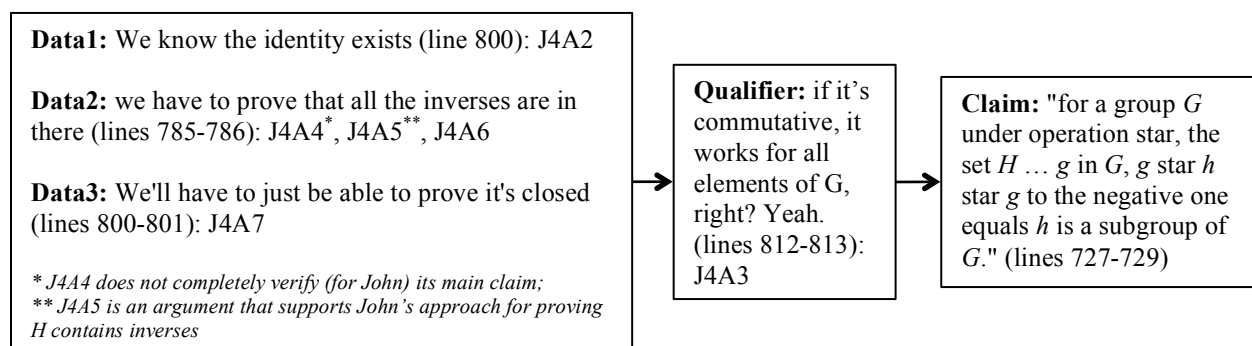


Figure 5.55. J4A1

After reading the prompt several times, John briefly mentions commutativity, softly speaking in truncated sentences. At one point he states, “If it's commutative it works really well” (line 761), but does not support this statement. He then begins writing and says, “I'm just trying

to get a picture in a group I know of what this, like, is saying” (line 769). He goes on to mention the names of elements in the group of symmetries of a triangle, saying,

I'm using self inverses, just (unaud; begins whispering to self)  $2R$  equals  $h$ . Like, just trying to get, like, a feel for, like, what works and what doesn't. And I don't really know- How this is supposed to look.  $R$  star  $h$  equals  $h$  star  $R$ - By just adding  $R$  to the end of each- But that doesn't tell me anything. (lines 774-777)

This excerpt is followed by several seconds of silence as John continues to write while speaking inaudibly to himself. It seems that John is testing the definition of the set  $H$  with specific elements in the group of symmetries of a triangle, though he scratches through most of his work. However, he does leave the equations “ $R*h*2R = h$ ” and “ $R*h = h*R$ ,” which are consistent with the excerpt above. From this, it seems that John is drawing on the “symmetry” *form* of inverse to serve the “end-operating” *function* of inverse to manipulate the equation.

John then abruptly says, “Let's pretend  $g$  is the identity. Then it definitely works, so the identity exists” (line 781). He then expounds on this statement by producing an argument coded as the first subargument for J4 (J4A2, Figure 5.56) toward proving that  $H$  is a subgroup of  $G$  by showing that  $H$  contains the identity element of  $G$ .

I know the identity exists 'cause if we let  $g, h$  be in  $G$ , um, let  $g$  be the identity of  $G$ , then you get  $g$  star  $h$  star  $g$  to the negative one equals  $h$  and since those are the the inverse of the identity is the identity and this is the identity, we get  $h$  equals  $h$ - definition of identity. (lines 782-784)

This argument is modeled with an embedded, linked Toulmin scheme. John first provides the statement “I know the identity exists” (line 782), which serves as a claim (Claim). He then states, “cause if we let  $g, h$  be in  $G$ , um, let  $g$  be the identity of  $G$ ” (lines 782-783), which serves as initial data (Data1), supporting the claim “then you get  $g$  star  $h$  star  $g$  to the negative one equals  $h$ ” (line 783, coded as Claim1/Data2). From these statements, it seems that John is treating the letter  $g$  as the identity in the equation used to define  $H$ . John then says, “and since those are the-

the inverse of the identity is the identity and this is the identity” (lines 783) which serves as a warrant (Warrant2) that validates that Claim1/Data2 supports the new claim “we get  $h$  equals  $h$ ” (line 784). John then says, “definition of identity” (line 784), which serves as a backing (Backing2) for Warrant2. Altogether, the embedded sequential Toulmin scheme serves as data supporting the overall claim that the identity exists.

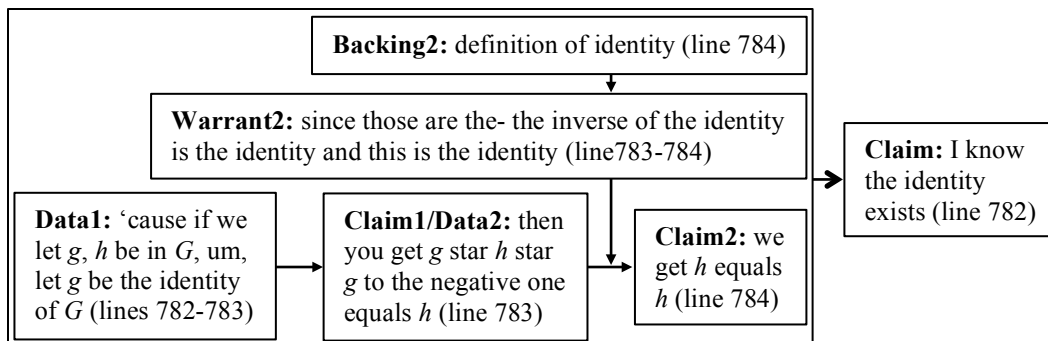


Figure 5.56. J4A2

John’s initial statement, “Let’s pretend  $g$  is the identity” (line 781) affords insight into how John thinks about the identity subgroup rule. John does not question the existence of the identity in the set  $H$ , but instead first imagines if the element in the equation used to define  $H$  is the identity. It seems as though a part of this imagining includes allowing that element to serve the *functions* of identity within the equation. It seems that this process should result in a valid conclusion, which is reflected first by John’s statement, “then it definitely works” (line 781) and later in the excerpt coded as Claim2 when John states “we get  $h$  equals  $h$ ” (line 784). In this argument, John uses the letter  $g$  as the identity, which is interesting because, up until this point, John primarily uses the symbol  $e$  as the representational vehicle for “letter” *form* of identity. In the excerpt coded as Warrant2,  $g$  to serves the “equals” and “identity as self-inverse” *functions* of identity in order for John to interpret the algebraic expression  $g*h*g^{-1}$ . This supports the sense that the symbol that he uses for the identity is flexible because John does not need to shift the representational vehicle from  $g$  to  $e$  in order for that element to be an identity.

Following J4A2, John returns to writing and says, “So f plus R plus R plus f- You can't move that around, can you? Plus that equals r- Let's just see. Okay, good. So, it's not commutative. So. I don't know where to go from there” (lines 802-804). From this, it seems that John is determining that the symmetries of a triangle are not commutative, presumably in order to generate more examples, although he abandons this line of reasoning. When asked to explain what he is doing and why he feels he does not know where to go from here, John says, “I'm just trying to think of groups where it's not commutative, because if it's commutative, it obviously works, 'cause you just, like, move it” (lines 811-812). He then elaborates on this statement by producing an argument in which he describes a consequence of the group  $G$  being commutative, claiming that, if  $G$  is commutative, then the set  $H$  is the entire group. This argument (J4A3, Figure 5.57) is modeled with an embedded sequential Toulmin scheme.

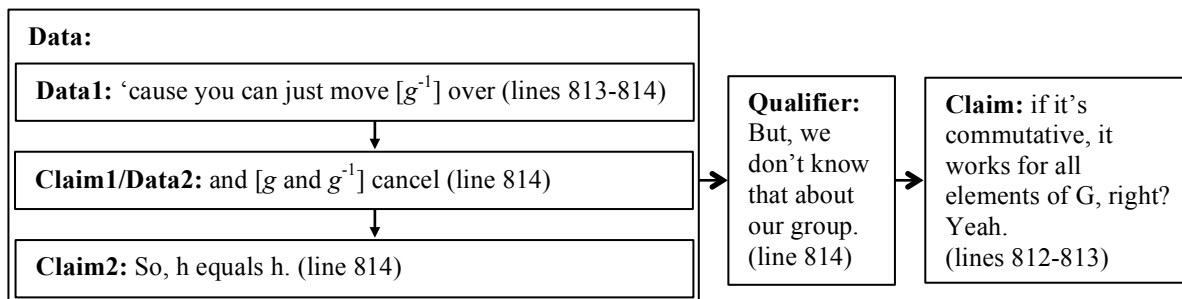


Figure 5.57. J4A3

J4A3 begins with the claim, “if it's commutative, it works for all elements of  $G$ , right? Yeah” (lines 812-813, Claim). John supports this by saying, “If it's commutative, 'cause you can just move it (points to  $g^{-1}$  in “ $g * h * g^{-1} = h$ ”) over and those cancel” (lines 813-814). This statement can be further coded as a sequential Toulmin scheme in which the phrase “cause you can just move it over” (lines 813-814) serves as initial data (Data1), which supports the claim “and those cancel” (line 814, Claim1/Data2). This claim in turn serves as data for the statement

“So,  $h$  equals  $h$ ” (line 814, Claim2). John then explains, “But, we don't know that about our group” (line 814), which serves as a qualifier for the argument.

Similarly to J4A2, the result of John's activity is the equation  $h = h$ , which seems to serve an important role for John as he determines whether his arguments are valid, verifying that the assumptions he makes satisfy the equation used to define  $H$ . It seems that John generated this proof with the intention of removing the elements  $g$  and  $g^{-1}$  from the equation, which he was only able to do once the two elements were adjacent. In this sense, it seems that John anticipates these elements being able to serve the “vanishing” *function* of inverse. Accordingly, assuming that the group is commutative affords John the ability to carry this process out. This is further evidenced by John's discussion leading up to J4A3 in which he says, “I'm just trying to think of groups where it's not commutative, because if it's commutative, it obviously works, 'cause you just, like, move it” (lines 811-812). Thus, this argument supports John's proof construction by informing his example generation. Specifically, John focuses on the group of symmetries of a triangle as he tries to develop a sense of whether the set  $H$  is a subgroup.

John continues by substituting more elements from the group of symmetries of a triangle for  $g$  and their respective inverses for  $g^{-1}$  in the equation  $g * h * g^{-1} = h$ . During this time he does not describe his activity, but instead explains that he wishes he knew more examples of non-commutative groups. When asked what he is doing, John replies, “I just added the inverse of  $f$  plus  $R$  instead of  $f$  plus  $R$  to both sides- ... to make it look like (inaud) keep getting it to look like, which is.  $f$ , like- I keep getting  $g$  plus  $h$  equals  $h$  plus  $g$ ” (lines 873-878). This seems to reflect John's earlier example generation activity in that he is drawing “symmetry” *forms* of inverse to serve the “end-operating” *function* of inverse in order to manipulate an equation. He then interprets the result of this action by shifting to the “letter” *form* used in the prompt.

Following this, John begins silently writing equations using the letters  $g$  and  $h$ , beginning with the equation “ $g \cdot h \cdot g^{-1} = h$ ,” and generating the equation “ $g \cdot h = h \cdot g$ ,” which presumably draws on the “end-operating” *function* of inverse. John then says, “I’m just thinking, like, I was like, what happens if I use the inverse of  $h$ ? Which has to exist, 'cause it's a group. Then you just get  $g$  equals  $h$  star  $g$  star inverse  $h$ ” (lines 889-890). He goes on to say, “Oh! That was assum- I can't do that. Well, I- Because that's assuming that left- and right- inverses are the same” (lines 892-893). This activity constitutes a new argument (J4A4, Figure 5.58) that can be modeled with a sequential Toulmin scheme.

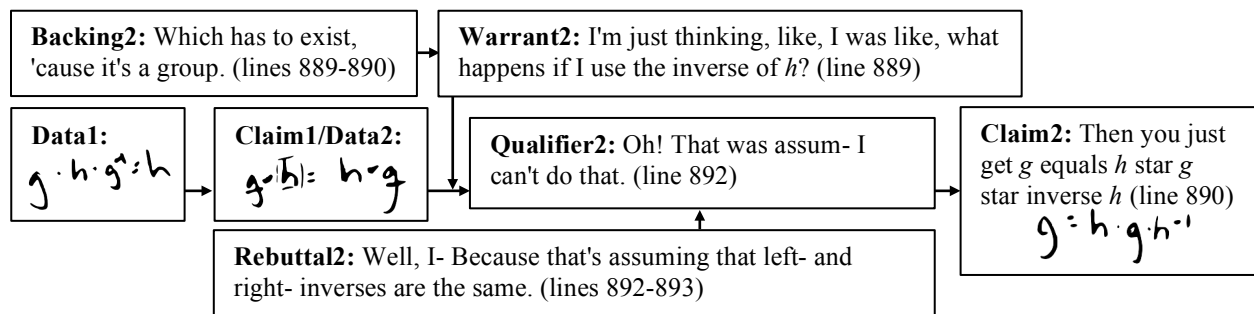


Figure 5.58. J4A4

In J4A4, the first two equations John writes serve as a data-claim pair, the second of which serves, in turn, as data for the third equation that John discusses producing. Accordingly, these equations are coded as Data1 and Claim1/Data2, respectively. John’s statement “I’m just thinking, like, I was like, what happens if I use the inverse of  $h$ ?” (line 889, Warrant2), which he backs with the statement “Which has to exist, 'cause it’s a group” (lines 889-890), serves as a warrant for him to produce the third equation. He then describes the result of this process, saying, “Then you just get  $g$  equals  $h$  star  $g$  star inverse  $h$ ” (line 890, Claim2). John then qualifies this statement, saying, “Oh! That was assum- I can't do that” (line 892, Qualifier2) and explains why he says this by adding, “Well, I- Because that's assuming that left- and right- inverses are the same” (lines 892-893, Rebuttal2), which serves as a rebuttal to the statement.



In this argument, John attempts to draw on the “letter” *form* of the inverse of  $h$  to serve the “end-operating” *function* of inverse so that he may manipulate the equation, but he abandons his activity based on his qualifier and rebuttal. This reflects the “sided” *function* of inverse in that John is attending to the end of each side of the equation on which he is end-operating but is unable to justify whether this is valid. The interviewer asks John what would happen if the left- and right- inverses of  $h$  were always the same. John replies, “Is that always true? That could be a thing that’s always true, and I’m just dumb” (line 906). This provides a sense that John is uncertain whether his distinction between left- and right- inverses is necessary.

At this point in the interview, John leaves to go to class, but he and the interviewer agree to continue the interview after class. When John returns, he explains that he had thought about the problem he was experiencing with right- and left- inverses while he was away and is able to prove that they are the same element. This leads to a new argument (J4A5, Figure 5.59), modeled with an embedded, divergent, sequential Toulmin scheme, in which John argues that “the left- and right- inverses in a group, they have to be equivalent” (line 986, coded as Claim).

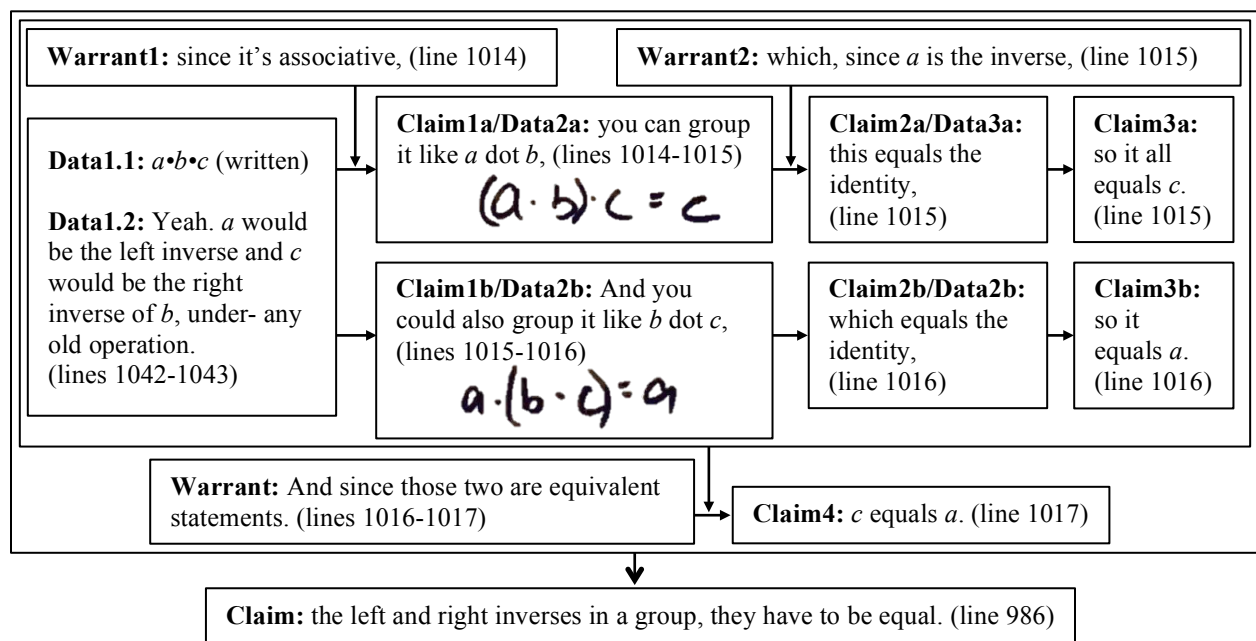


Figure 5.59. J4A5

John begins J4A5 by writing the equation “ $a \cdot b \cdot c$ ,” which serves as initial data (Data1.1) and later explains that  $a$  is the left-inverse of  $b$  and  $c$  is the right-inverse of  $b$ , which serves as Data1.2. Together, these data support two sequential arguments: that the expression  $a \cdot b \cdot c$  equals  $c$  and that the expression  $a \cdot b \cdot c$  also equals  $a$ . Drawing on the associativity of the group (coded as Warrant1), John explains, “you can group it like  $a$  dot  $b$ ” (lines 1014-1015, Claim1a/Data2a) and writes  $(a \cdot b) \cdot c$ , which serves as a claim supported by the initial data. This statement then serves as data for the claim “this equals the identity” (line 1015, Claim2a/Data3a), which John warrants (Warrant2) by pointing out that  $a$  is the inverse, which reflects the “equals” *function* of inverse. This then serves as data as John then claims that the expression  $a \cdot b \cdot c$  equals  $c$  (Claim3a). John continues by considering the initial data again, which serves to support the claim “And you could also group it like  $b$  dot  $c$ ” (lines 1015-1016, Claim1b/Data2b), tacitly drawing on the associativity of the group as in Warrant1. This then serves as data for the claim that  $b \cdot c$  equals the identity (Claim2b/Data3b), which serves as data for the claim that  $a \cdot b \cdot c$  equals  $a$ . Altogether, this divergent, sequential argument serves as data for John’s claim that “ $a$  equals  $c$ ” (line 1017, coded as Claim4), which John supports by pointing out that the two expressions  $(a \cdot b) \cdot c$  and  $a \cdot (b \cdot c)$  are equivalent (coded as Warrant). This argument, in turn, serves as data to supports John’s overarching claim that the left- and right- inverses must be equivalent.

J4A5 serves an important role in John’s proof activity in that it validates the manipulations to the equation he developed in J4A4. Accordingly, this supports a shift in the *functions* that *forms* of inverse are able to serve. Specifically, J4A5 seems to contradict a need to draw on the “sided” *function* of inverse. Throughout the follow-up discussion to J4A5, John emphasizes the importance of associativity for showing that left- and right- inverses are equivalent. This reflects an interesting interaction between the group axioms, for John, in which

the associativity axiom informs his understanding of inverse, especially given his use of the left/right rule when defining groups.

After a discussion in which John clarifies J4A5, he spends several minutes reading over his work in this proof from the part of the interview before he left for class in order to remind himself of his previous work and remember what parts of the subgroup proof he had shown. As he reads over his work, John returns to the equations he had produced during J4A4. He then recalls that he was trying to prove that the set  $H$  contains inverses, saying,

No, what was I trying to do? I'm trying to prove- or disprove- that all the inverses of  $g$  are also in  $h$ . What does it mean? So, the inverse of  $g$  is ...  $g$  to the negative one or  $g$ - inverse  $g$ . Now what does it take or that to be- oh. It takes for this to be true. Awesome. Okay. I did it. (lines 1402-1406)

In this excerpt, John seems to realize that an equation he had previously written in A4 supports that  $H$  contains the inverse of each of its elements. When asked what happened, John writes three equations and produces a new argument (J4A6, Figure 5.60) that supports that “all the inverses exist in  $H$ ” (line 1455), which is modeled with a standard Toulmin scheme.

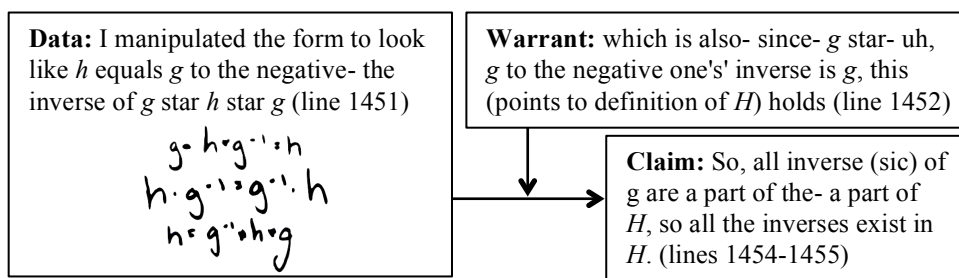


Figure 5.60. J4A6

John begins his argument by referring to the three equations he had written, saying, “I manipulated the form to look like  $h$  equals  $g$  to the negative- the inverse of  $g$  star  $h$  star  $g$ ” (line 1451), which serves as the initial data (Data) for the argument. He then explains, “which is also- since-  $g$  star- uh,  $g$  to the negative one's inverse is  $g$ , this (points to definition of  $H$ ) holds” (line 1452), which serves as warrant that supports the claim “So, all inverse (sic) of  $g$  are a part of the-

a part of  $H$ , so all the inverses exist in  $H$ . (lines 1454-1455, coded as Claim). John's manipulation of the equation used to define  $H$  seems to draw on the "end-operating" *function* of inverse first to produce the equation " $h * g^{-1} = g^{-1} * h$ " then to generate the equation " $h = g^{-1} * h * g$ ." This is the same type of manipulation that John argues he is unable to use in J5A4 because he is uncertain whether left- and right- inverses are the same element. However, having developed an argument that left- and right- inverses are the same element (J4A5), John is able to "end-operate" with  $g^{-1}$  on the left side of  $g$  to generate the second equation and with  $g$  on the right side of  $g^{-1}$  produce the final equation. He is then able to use this equation to support his desired claim that  $H$  contains the inverses of its elements.

Following J4A6, John states that he needs to prove that  $H$  satisfies closure, saying, "I'm trying to figure out if any, like,  $g_1$  star  $g_2$ - is that uh another  $g$ - is that in  $G$  (sic)?" (lines 1504-1505). He then generates a new argument (J4A7, Figure 5.61), which is modeled with an embedded, sequential Toulmin scheme. John begins his argument by writing the equation  $(g_1 * g_2) * h * (g_1 * g_2)^{-1} = h$  (coded as Data1) and saying "I'm just trying to see if that's true. I guess is what I'm trying to do at the end of the day" (lines 1508-1509, coded as Warrant). This quote serves as a warrant for John's ensuing activity by supporting his work toward showing that  $H$  is closed. From this, John produces a second equation (Claim1/Data2)  $g_1 * g_2 * h * g_2^{-1} * g_1^{-1} = h$ , which he supports by saying, "So, I just, like, expanded that" (line 1511, coded as Warrant1). This seems to draw on the "inverse of a product" *function* of inverse in that John removes the parentheses from the expression  $(g_1 * g_2)^{-1}$ , commutes the order of the elements operated together, and places a -1 exponent on each. When asked how he did this, John supports his statement, saying, "I mean, if you go one way, you have to go back the same way, so it's  $g_2$  inverse star  $g_1$  inverse" (line 1524-1525, coded as Backing1).

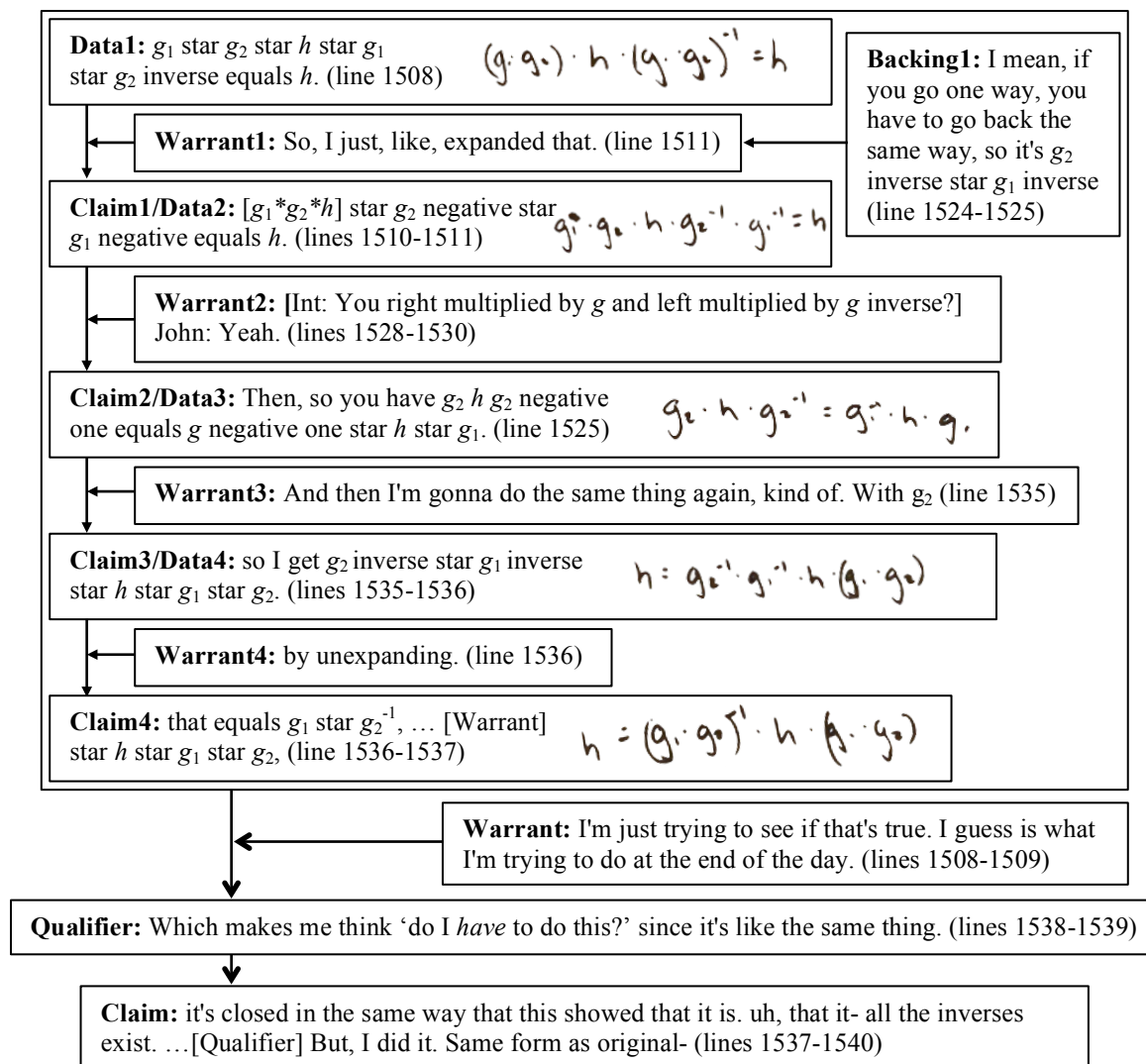


Figure 5.61. J4A7

Claim1/Data2 then serves as data when John further uses it to produce the third equation  $g_2 * h * g_2^{-1} = g_1^{-1} * h * g_1$ , which serves as Claim2/Data3. At this point, the interviewer asks, “You right multiplied by  $g$  and left multiplied by  $g$  inverse?” (line 1528), which John agrees with, saying, “Yeah” (line 1530). This exchange is coded as Warrant2 because of John’s agreement, although the interviewer provides it. John then states, “And then I’m gonna do the same thing again, kind of. With  $g_2$ ” (line 1535, Warrant3), which serves to warrant his generation of the equation  $h = g_2^{-1} * g_1^{-1} * h * (g_1 * g_2)$ , which is coded as Claim3/Data4. John then produces the equation  $h = (g_1 * g_2)^{-1} * h * g_1 * g_2$ , which serves as Claim4, explaining his reasoning by saying, “by

unexpanding” (line 1536, coded as Warrant4). Finally, John says of the last equation that it “shows that it's closed in the same way that this showed that it is. uh, that it- all the inverses exist. Which makes me think ‘do I *have* to do this?’ since it's like the same thing. But, I did it. Same form as original” (lines 1537-1540). This excerpt serves as both an overarching claim for his argument that  $H$  is closed and a qualifier for that claim in which John questions whether it was necessary to prove closure, relating his process here to his argument that  $H$  contains inverses.

In this argument, John begins by replacing  $g$  and  $g^{-1}$  in the equation used to define  $H$  with the expressions  $(g_1 * g_2)$  and  $(g_1 * g_2)^{-1}$ , respectively, resulting in the equation  $(g_1 * g_2) * h * (g_1 * g_2)^{-1} = h$ . This reflects his initial approaches to proving that  $H$  contains the identity element and inverses for each element. He proceeds to manipulate this equation by drawing on the “end-operating” *function* of inverse, resulting in the equation  $h = (g_1 * g_2)^{-1} * h * (g_1 * g_2)$ . It should be noted that John does not formally prove that  $H$  satisfies closure and is right in comparing his process here to the process he used to show that  $H$  contains inverses. This is because his argument here can be viewed as showing that, if the element  $g_1 * g_2$  is an element of  $H$ , then its inverse is also an element of  $H$ . This reflects an aspect of J4A6 that contributes to John’s difficulty explaining why he feels his work shows that inverses exist in  $H$ . In that argument, John appeals to the “form” of the equation  $h = g^{-1} * h * g$  being the same as the “form” of the equation used to define  $H$ . Similarly, in this argument, John’s statement that is coded as Warrant reflects a sense that he might not anticipate the result of his work. This can be partially attributed to John’s production of the elements  $g_1$  and  $g_2$  without ever describing them as elements of  $H$ . Without first defining these elements as such, the initial equation with which John begins seems like an assumption that  $g_1 * g_2$  is already an element of  $H$ . Accordingly, when John states that he intends to “see if that’s

true” (lines 1508-1509), although he may tacitly think that  $g_1$  and  $g_2$  are elements of  $H$ , he has no other data that reflects the same *form* as the equation he produces so that he might be able to determine whether the equation is “true.” Instead he is left to appeal to what he calls the “form” of the equation in that it reflects the same pattern as the equation used to define  $H$ . In light of this, his argument that the set  $H$  contains inverses (J4A6) seems similarly invalid.

The most notable aspect of this proof is John’s development of J4A5 in which he develops an argument showing that left- and right- inverses must be equal. John’s question of whether he is able to operate with an element on both the left and right sides of its inverse impedes his ability to manipulate the equation used to define  $H$  while generating J4A3. However, although John describes that he wishes to show that  $H$  contains inverses, he does not describe his goal for manipulating the equation or how this might help him show that the set contains inverses. John does not articulate an algebraic expression from his manipulation of equations to support his final argument that inverses exist in  $H$  until after producing J4A5 and reviewing his previous work for several minutes. At this point, John asks the question, “What does it mean?” (line 1403-1404) and continues, “Now what does it take for that to be- oh. It takes for this to be true” (lines 1405, pointing to the equation “ $h = g^{-1}*h*g$ ”). In this moment, John seems to realize that, for  $g^{-1}$  to be an element of  $H$ , it must satisfy the equation  $g*h*g^{-1} = h$ , which he recognizes as  $h = g^{-1}*h*g$ . Even then, however, John only supports that  $g^{-1}$  is an element of  $H$  by describing it as having the same “form” as the left-hand side of the equation used to define  $H$ .

Another interesting aspect of J4 is John’s production of J4A3 in which he argues that, if  $G$  is commutative, then  $H$  is all elements in  $G$ . This informs John’s example generation as he tries to gain a sense of what the problem statement means. Specifically, John chooses to use elements of the group of symmetries of a triangle when testing which sets satisfy the definition

of  $H$ . As John points out, this is the only non-commutative group that he is able to generate, which might limit the productivity of his attempts to better understand the proof requested from the prompt.

**Proof J5: The left/right rule.** While responding to Interview 3 Q3, John produces two impromptu proofs to explain a single alternate group rule (the left/right rule) that he claims supports that both an identity element and inverse elements exist in the group, which he phrases, “I’m trying to prove that, for all  $a, b$  in  $G$ . That \*if\* for all  $a, b$  in  $G$ , there exists some  $x$  and  $y$  such that  $a \cdot x$  equals  $b$  and  $y \cdot a$  equals  $b$  proves that an identity and inverses exist” (line 204-206). John goes on to clarify that he is able to use his alternate group rule to prove that a unique identity exists in the group and each element in the group has a unique inverse. In service of this, he produces four arguments: the first argument generally outlines his approach to showing that the identity and inverses exist, the second and third arguments together are John’s proof showing that a unique identity exists in the group, and the fourth argument is John’s proof that left- and right-inverses are equal. I call this rule the “left/right rule” because an important aspect of the rule is the sidedness of the elements  $x$  and  $y$ , in that John does not assume, for a given  $a$  and  $b$ , that the  $x$  and  $y$  that satisfy the equations in the rule are the same element. This is a subtlety inherent in John’s discussion that becomes clearer as he describes the existence of identity and inverses.

In John’s first argument of this proof (J5A1, Figure 5.62), he introduces the left/right rule and describes his general reasoning for showing how it is equivalent to the identity and inverse axioms for groups, saying, “Um, and then this- so, this, like, proves the exis- if this is true, it has an identity and inverses. And each element has an inverse” (lines 112-113), which serves as the overarching claim for the argument (Claim). He then provides an anecdote about needing to



prove this statement for his instructor in order to use it on a test. When asked what this proof is like, John says,

What? The proof? Um, so start- So, we assume that this is true. And then, so I um- The first thing I did was assume  $a$  equals  $b$ . So,  $x$  and  $y$  are acting as right and left identities. And then I just manipulated it to make it so  $x$  equals  $y$  so it's one identity element. And once you know the identity exists, you let, uh,  $b$  equal the identity. So,  $x$  and  $y$  have to exist as the inverses for  $a$ . So there has to- Since they *act* as the inverse, the inverse exists. (lines 121-125)

This excerpt begins with John assuming that the left/right rule is true, which is coded as the first data (Data1) in a linked Toulmin scheme. He then describes two different parts of the proof, which constitute two additional data for the overarching argument (Data2 and Data3). Each of these data can be viewed as an embedded argument (sequential and standard, respectively). In the first part, John begins by assuming  $a$  equals  $b$ , which serves as initial data (Data2.1) for the embedded argument.

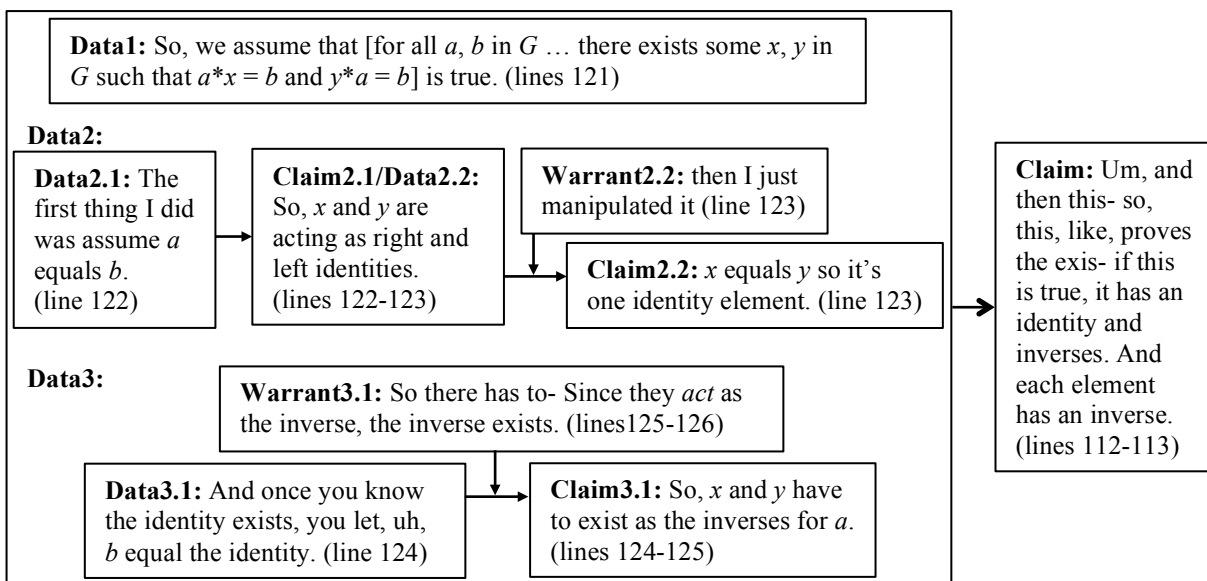


Figure 5.62. J5A1

He then states, “So,  $x$  and  $y$  are acting as right and left identities” (lines 122-123), which is not supported with any further explanation, but serves as a new claim (Claim2.1/Data2.2) following the statement in Data 2.1. This claim then serves as data as John continues by explaining, “And then I just manipulated it to make it so  $x$  equals  $y$  so it's one identity element”

(line 123). This sentence serves as a new claim (Claim2.2) that  $x$  and  $y$  are equal warranted by John's allusion to "manipulat[ing] it" (as shown in J5A2, the "it" is an equation generated from the assumptions he makes in the argument).

John then outlines a second argument that supports the existence of inverses in the group (Data3). He begins by saying, "And once you know the identity exists, you let, uh,  $b$  equal the identity. So,  $x$  and  $y$  have to exist as the inverses for  $a$ " (line 124). This is coded as a data-claim pair (Data3.1, Claim3.1) that John supports by saying, "So there has to- Since they *act* as the inverse, the inverse exists" (lines 125-126), which serves as a warrant for the claim that  $x$  and  $y$  exist as inverses (Warrant3.1). Later, in J5A4, John discusses a process of showing that the left- and right-inverses must be equal, which reflects a parallel structure to Warrant2.2 and Claim2.2.

Interestingly, in each of the embedded arguments, John describes the elements  $x$  and  $y$  as "acting like" the type of element he wishes to show exists (Claim2.1/Data2.2, Warrant3.1), which is followed by a claim that they exist. This supports a sense that the "equals" *functions* of identity and inverse inform John's approach. The initial data for the subarguments comprising Data2 and Data3 are the assumptions that  $b$  equals  $a$  and  $b$  equals the identity, respectively. So, in each argument, John chooses  $b$  specifically so that the equations from the left/right rule reflect the structure of the respective "equals" *functions*. Further, John's statement regarding the equality of the left- and right-identity (and, later, left- and right-inverse) supports a sense that this is an important aspect of identity and inverse for John, specifically, that the "sided" *functions* of identity and inverse are false in a group, though he draws on them during J5 because he does not assume such an equality when proving the left/right rule implies an identity and inverses exist.

When asked to explain how his rule supports the existence of an identity, John develops an argument in which he manipulates an equation with the expressed goal of showing  $x = y$  from

the initial equation  $x \cdot b = b \cdot y$ . Broadly, he tries to achieve this through a two-stage process: in the first stage, he attempts to produce an equation of the form  $b \cdot x = b \cdot y$ ; in the second stage, he works from  $b \cdot x = b \cdot y$  to generate the equation  $x = y$ . Initially, John carries out both stages, admitting beforehand that he does not fully remember his prior proof. Accordingly, after carrying out both stages, he notices an error in the first stage and attempts to complete the first stage two more times. Throughout this cyclical process, John primarily mumbles to himself in incomplete sentences and marks through much of his work, affording little insight into the entire development of the argument. Analyzed herein are the final iteration of the first stage (J5A2) and the only iteration of the second stage (J5A3), though J5A2 chronologically occurred after J5A3.

John begins the argument J5A2 (Figure 5.63) by letting  $a$  equal  $b$  (coded as Data1) and using the left/right rule to generate the equation  $b \cdot x = y \cdot b = b$  (coded as Claim1/Data2) as he had described in J5A1, saying “So, if we let  $a$  equal  $b$ , we have  $b$  dot  $x$  equals  $y$  [dot  $b$ ] ... equals  $b$ ” (lines 147-148). This quote is drawn from the first iteration of the first stage and the remaining data is drawn from the final iteration. At the beginning of the final iteration, John explains that he is trying to manipulate the equation  $b \cdot x = y \cdot b$  with the goal of showing  $b \cdot x = b \cdot y$ , saying, “So, I need to get this side down to  $b$  dot  $x$  without changing the meaning of that side. Oh. So, that can become  $b$ - or  $b$  dot  $y$ ” (lines 222-223). This serves as an overarching claim (Claim) for the argument, which is modeled with an embedded, sequential Toulmin scheme.

After stating his goal for the proof, John says, “Putting the  $y$  at the end. ... So, we get  $b$  dot  $x$  dot  $y$  equals  $y$  dot  $b$  dot  $y$ ” (lines 223-224), which serves as a claim (comprised of a new equation, Claim2/Data3) that is warranted with end-operating by the letter  $y$  (Warrant2). After this, John spends 33 second writing the remainder of the proof after which he explains his work aloud, saying,

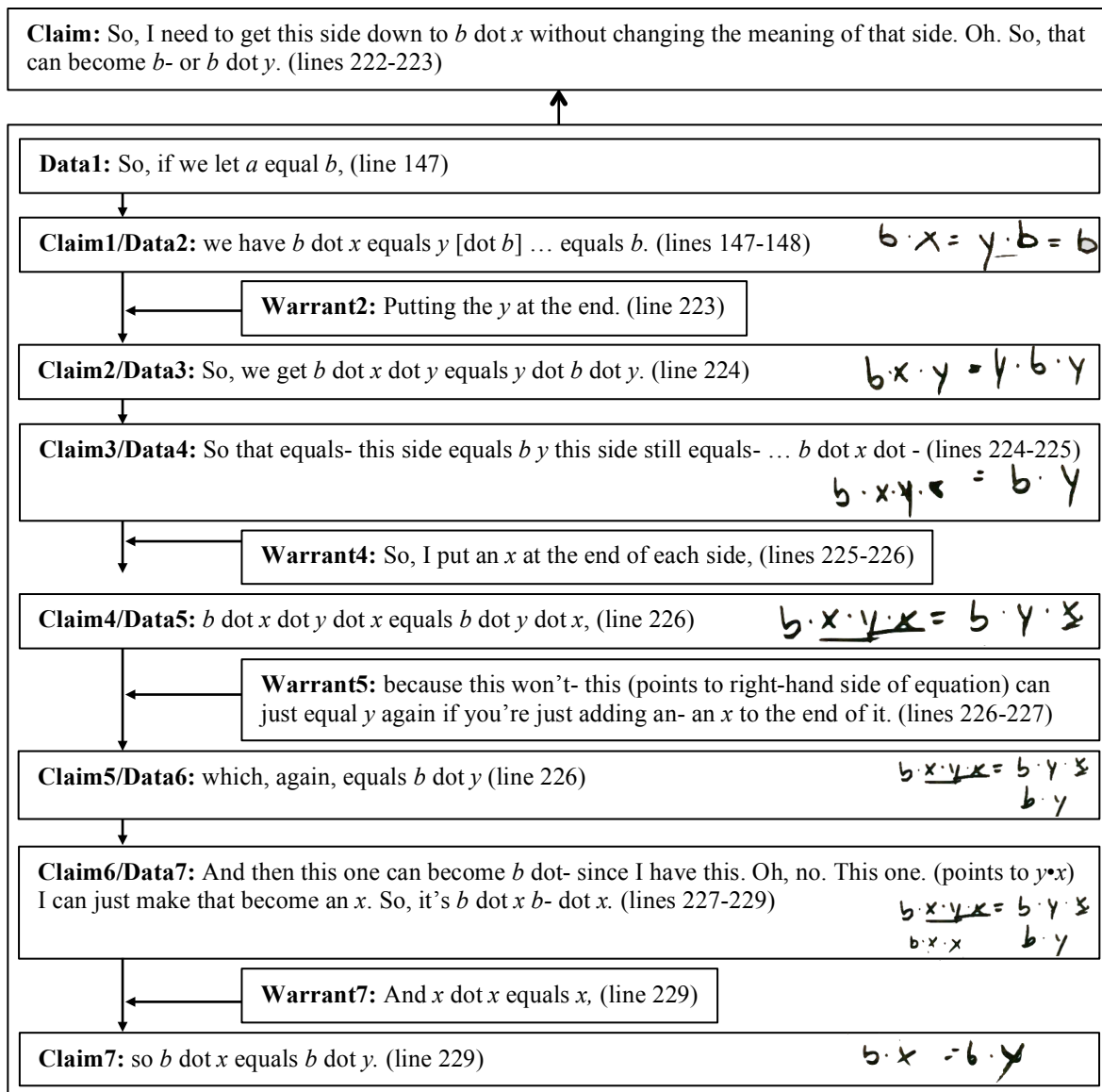


Figure 5.63. J5A2

So, we get  $b$  dot  $x$  dot  $y$  equals  $y$  dot  $b$  dot  $y$ . So that equals- this side equals  $b y$  this side still equals- Ow. Oh, yeah.  $b$  dot  $x$  dot - So, I put an  $x$  at the end of each side, so  $b$  dot  $x$  dot  $y$  dot  $x$  equals  $b$  dot  $y$  dot  $x$ , which, again, equals  $b$  dot  $y$  because this won't- this can just equal  $y$  again if you're just adding an- an  $x$  to the end of it. And then this one can become  $b$  dot- since I have this. Oh, no. This one. I can just make that become an  $x$ . So, it's  $b$  dot  $x$   $b$ - dot  $x$ . And  $x$  dot  $x$  equals  $x$ , so  $b$  dot  $x$  equals  $b$  dot  $y$ . (lines 224-229)

In this explanation, the equation in Claim2/Data3 equation then serves as data to support a new equation (coded as Claim3/Data4) as John says, “So that equals- this side equals  $b y$  this side still equals- Ow. Oh, yeah.  $b$  dot  $x$  dot” (lines 224-225). Although he does not audibly support this

claim with reasoning, the difference between the written equations in Data2/Claim3 and Data3/Claim4 is that the left-most letter “ $y$ ” is removed from the expression “ $y \bullet b \bullet y$ ” on the right-hand side of the equation in Data2/Claim3. This reflects the “replacement” *function* of identity in that the concatenation  $y \bullet b$  is replaced by the letter  $b$ , presumably based on the equation  $y \bullet b = b$  in Data1.

John then says, “So, I put an  $x$  at the end of each side, so  $b$  dot  $x$  dot  $y$  dot  $x$  equals  $b$  dot  $y$  dot  $x$ ” (lines 225-226). This is coded as a warrant (Warrant4) and claim because John is right end-operating with  $x$ , which supports the new equation “ $b \bullet x \bullet y \bullet x = b \bullet y \bullet x$ ” (Claim4/Data5). He then reasons that the right-hand side of the equation can be rewritten as “ $b \bullet y$ ,” saying, “because this won’t- this can just equal  $y$  again if you’re just adding an- an  $x$  to the end of it” (lines 226-227), which serves as a warrant (Warrant5) for John having written “ $b \bullet y$ ” under the right-hand side of the equation. This seems to reflect  $x$  simultaneously serving the “vanishing” *function* of identity and the (right) “sided” *function* of identity in concatenation with the letter  $y$ . Formally, however, it is not valid to use Warrant5 to justify the change he makes in this moment because John has not demonstrated, nor did he assume, that  $x$  is a right-identity for all elements in the group. Rather, John’s assumption is that  $x$  serves as a right-identity for  $b$ , which does not immediately support  $x$  as being a right-identity for any other elements (though it can be shown)<sup>15</sup>. John continues by changing the same equation, writing “ $b \bullet x \bullet x$ ” under the left-hand side (Claim6/Data7) and finally saying, “And  $x$  dot  $x$  equals  $x$ ,” (Warrant7, line 229), which serves as a warrant for the final claim in the argument, “so  $b$  dot  $x$  equals  $b$  dot  $y$ ” (Claim7, line

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<sup>15</sup> Formally, in order to show that the left- and right- identities are equal, one must first show that the left-identity of  $a$  is a left-identity for all elements in the group and that the right-identity is a right-identity for all elements in the group. John’s left/right rule can be used to show this: Given  $a \bullet e = a$ , fix  $g \in G$ . By the left/right rule, there exists  $r \in G$  such that  $r \bullet a = g$ . So,  $g \bullet e = r \bullet a \bullet e = r \bullet a = g$ . Similarly, given  $i \bullet a = a$ , fix  $h \in G$ . By the left/right rule, there exists  $s \in G$  such that  $a \bullet s = h$ . So,  $i \bullet h = i \bullet a \bullet s = a \bullet s = h$ . Letting  $e$  and  $i$  be the right- and left- identities, respectively, notice that, because  $i$  and  $e$  are, respectively, left- and right- identities for every element in the group,  $i = i \bullet e = e$ . Thus,  $e$  is a unique (two-sided) identity element for all elements in the group.

229). Each of these new claims seems to be justified by the same reasoning in Warrant5 and, so, neither is formally valid.

As discussed, during the interview, John's first approach to showing the left/right rule implies the existence of a unique identity begins with Data1 from J5A2 ( $b \bullet x = y \bullet b = b$ ) and concludes with the claim for J5A3 ( $x = y$ ). However, in correcting his original approach, John marks through several lines of equations and constructs a new argument leading up to the equation  $b \bullet x = b \bullet y$ , explaining that the rest of his original work is correct. Because of this, data for John's production of J5A3 (Figure 5.64) are drawn from his original work, which occurs chronologically before J5A2. Formally, John's claim in J5A3 shows that both  $x$  and  $y$  serve as right-identities for  $b$  and his claim in J5A3 shows that  $x$  and  $y$  must be equivalent and, hence, a unique identity element (if  $x = y$  and  $x \bullet b = b \bullet y = b$  then  $x \bullet b = b \bullet x = b$ , so  $x$  is a two-sided identity).

J5A3 is modeled with an embedded, sequential Toulmin scheme with a linked data in the sequence. J5A3 begins with the equation " $b \bullet y = b \bullet x$ " (Data1.1). He also implicitly draws on the left/right rule to generate an element  $q$ , which he describes by saying, "there also exists some  $q$  that- so,  $q$  dot  $b$  equals um, what do I want it to equal. I think I want it to equal  $y$ .  $q$  dot  $b$  equals  $y$ " (lines 187-188), and writes the equation " $q \bullet b = y$ ," which serves as data (Data1.2). It should be noted that, although John does not describe  $q$  as the inverse of  $b$ ,  $q$  is serving the "equals" *function* of inverse in the equation  $q \bullet b = y$ . It seems that Data1.1 supports the generation of a new equation that John writes as " $q \bullet b \bullet y = q \bullet b \bullet x$ " (Claim1/Data2) without providing any explanation, although it reflects (left) end-operating both sides of the equation  $b \bullet y = b \bullet x$  with the element  $q$ . This new equation then serves as data to support a new equation that John generates " $y \bullet y = y \bullet x$ " (Claim2/Data3), first warranting his activity by referring to the equation in Data1.2.

John then says, “and since it’s the-” (line 190), which serves as a warrant (Warrant3) to support the new equation “ $y = x$ ” (Claim3). It seems that, in the truncated sentence in Warrant3, John might be appealing to one or both elements  $x$  and  $y$  as serving the “equals” *function* of identity in concatenation with an element other than  $b$ , which the change in the equations from Claim2/Data3 to Claim3 supports. If this were the case, it would also be the same invalid reasoning that John uses to warrant Claim5/Data6, Claim6/Data7, and Claim7 in J5A2. Finally, John interprets the result of this process, saying, “So, there’s the identity” (line 190), which serves as the overarching claim (Claim) for J5A3.

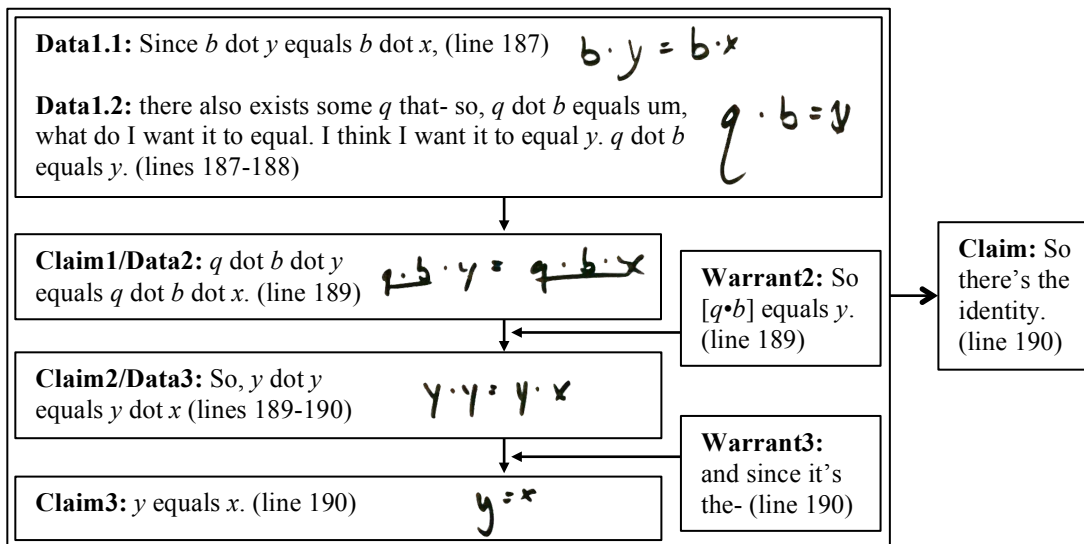


Figure 5.64. J5A3

John goes on to provide another argument (J5A4, Figure 5.65) related to the left/right rule in which he sets out to show that the left/right rule implies that left- and right- inverses are equal. As John explains, this proof relies on the existence of an identity in the group. As John mentions during the interview, this argument reflects the argument he produced during J4A3. John begins his proof by assuming that “ $x \cdot a$  and  $a \cdot y$  both equal the identity element” (line 304), adding, “and we know it’s associative” (line 305), which constitute initial data for the proof (Data1). John then generates (presumably based on his reference to associativity in Data1) the

equation “ $x \bullet (a \bullet y) = (x \bullet a) \bullet y$ ” (written as such). He then states, “And if  $y$  does not equal  $x$ , that is not true” (lines 306-307), which serves to qualify his activity, seemingly anticipating the outcome of his proof.

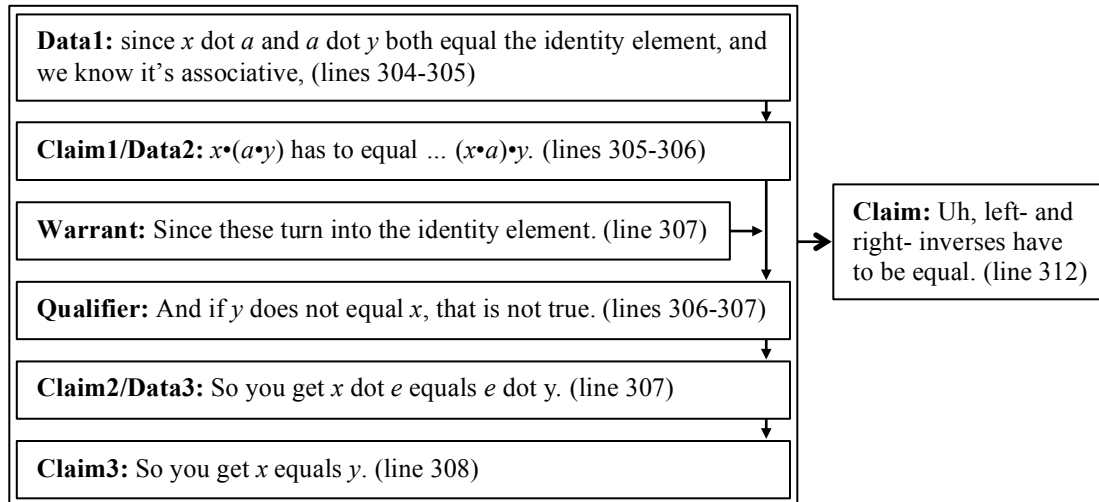


Figure 5.65. J5A4

Following this, John warrants his next claim, saying, “Since these turn into the identity element” (line 307, Warrant) while pointing at each pair of in the parentheses, which draws on the “equals” *function* of inverse. John then declares, “So you get  $x$  dot  $e$  equals  $e$  dot  $y$ ” (line 307), which reflects replacing the expressions “ $(a \bullet y)$ ” and “ $(x \bullet a)$ ” with the “letter” *form* of the identity element,  $e$  (Claim2/Data3)). Finally, John says, “So you get  $x$  equals  $y$ ” (line 308), which serves as the final claim in the embedded sequence (Claim3). Though John did not audibly warrant the final claim, the change from the equation in Claim2/Data3 to that in Claim3 reflects the “vanishing” *function* of identity of each side of the equation. When asked “So, then, that was the proof to show that-?” (line310), John explains “Uh, left- and right- inverses have to be equal” (line 312). This serves as the overarching claim (Claim) for the entire argument.

J5 affords interesting insight into how John’s conceptual understanding relates to his approach in this proof. First, John’s desire to use the left/right rule reflects an attitude (which



John expresses) toward the group axioms that they should be as minimal as possible. However, throughout the three interviews John only directly refers to the left/right rule when describing groups (Interview 2 Q3, Interview 3 Q3) as compared to his subgroup proofs during which he proves that a subset contains the identity element and the inverse of each element in the set. An argument could be made that the “sided” *function* of inverse, especially in J4A4 and J4A5, stems from this aspect of John’s understanding of the group structure. For instance, J5A4 reflects John’s argument in J4A5, which would indicate that he might view the two as related. However, in each of these arguments, he draws on the associativity of the group, which, along with the existence of a left- and right- inverse, is sufficient for showing that left- and right- inverses are equal. Although it seems that the connection might be that John is attuned to the possibility that left- and right- inverses might not be equal because of the left/right rule, during his explanation in Interview 2 (Q2), he attributes this phenomenon to non-commutativity.

In J5A2, John’s use of  $x$  as a right-identity for  $y$  and  $x$  reflects an understanding that a *form* of identity should serve the *functions* of identity when concatenated with any element in the group. However, given his choice of the left-right rule as an alternative to the identity and inverse axioms as well as his assumption that  $x$  and  $y$  serve as right- and left- identities of  $b$ , respectively, he must first show that  $x$  and  $y$  also satisfy the definition of right- and left- identities, again respectively, for every element in order to use them in this argument. Further, John again uses  $x$  and  $y$  in this way during J5A3. The fact that these elements serve *functions* of identity universally reflects that John understands them to do so, even when he is trying to prove their existence and uniqueness.

**Proof J6: Odd order of a finite group with exactly one element of order 2.** John produces his sixth proof in response to Interview 3 Q4d, which is a true/false statement with the

prompt “If a finite group contains exactly one element of order 2, then the group contains an odd number of elements.” John’s response of consists of one argument that is modeled with a linked, embedded Toulmin scheme (J6A1, Figure 5.66), in which he generates a single counterexample to the given conjecture. John begins his argument by generating a specific example of a group with an element of order 2, saying,

Um, I think it just means how many times you have to operate it with itself to get back to an identity. So, two ti- two of them. So, like, in  $\mathbf{Z}$  mod 12, 6. So, I don’t think that’s true. Counterexample  $\mathbf{Z}$  mod 12. No. Hol- yeah. Because nothing else has an order of 2, yet it has an even number. It has 12 elements. And it is finite, cause it only has 12 elements. (lines 404-407)

This excerpt begins with an initial claim in which John states what he thinks it means for an element to have order two, supporting this with a specific example (Claim1, Data1).

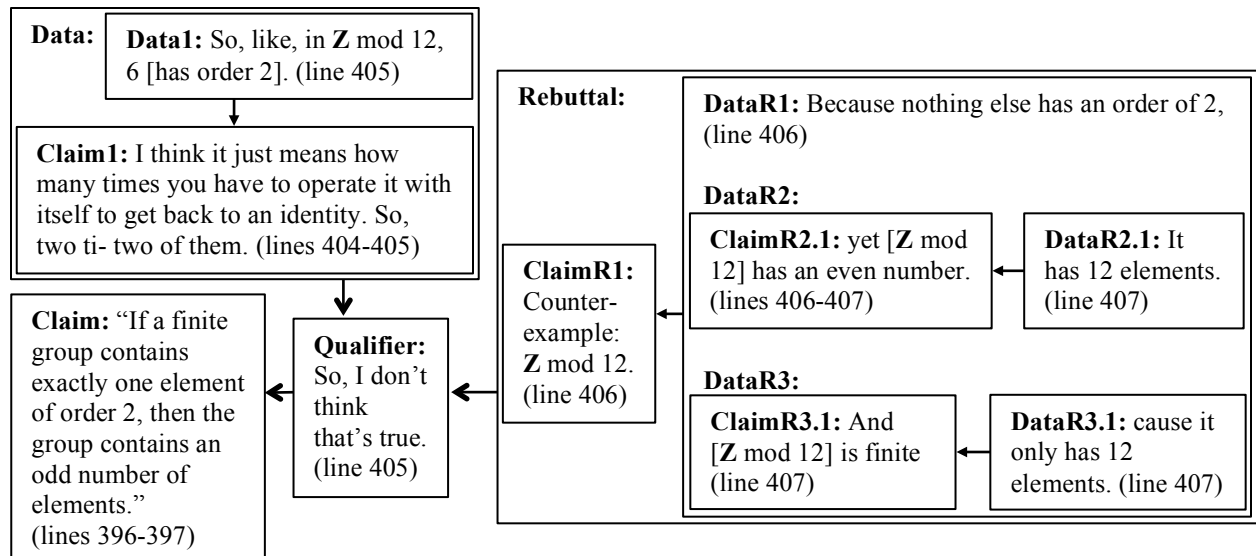


Figure 5.66. J6A1

Producing this example immediately leads John to provide a qualifier (Qualifier), contradicting the main claim of the argument, and point out that  $\mathbf{Z}$  mod 12 is a counterexample to the claim (ClaimR1), which serves as the claim of an argument embedded in the rebuttal. ClaimR1 is supported by John’s description of  $\mathbf{Z}$  mod 12, which is coded as three linked data in the rebuttal (DataR1, DataR2, DataR3), two of

which are coded as embedded data-claim schemes. These data verify that  $\mathbf{Z} \bmod 12$  satisfies the criteria given in the prompt. John states that there are no other elements of order 2 (DataR1), and claims that  $\mathbf{Z} \bmod 12$  has an even number of elements (ClaimR2.1) and is finite (ClaimR3.1), each supported with the data that  $\mathbf{Z} \bmod 12$  contains 12 elements (DataR2.1, DataR3.1).

In this argument, John is able to draw on the example of the integers modulo 12 to generate an appropriate counterexample to the given statement. This example informs John's approach by precluding a constructive proof (such as Tucker's) to rebut the claim because John is satisfied with a single counterexample. Although he does not use the word "inverse," John's language describing order reflects the "get identity" *function* of inverse while drawing on the "number" *form* in which the correspondence between representational vehicles of numbers and the representational objects of identity and inverse is contextualized by modular arithmetic. This provides John with an example that he can verify satisfies the constraints of the given statement.

**Proof J7: Even number of self-inverses.** John produces his seventh proof (J7), in response to Interview 3 Q4e. He provides a single argument, modeled here with a linked Toulmin scheme (J7A1, Figure 5.67). In this argument, John disproves the given claim that, "For a group  $G$  star the number of self-inverses is even" (Appendix C) using two (correct) counterexamples – the trivial group and the integers under addition (Data1 and Data2). John's brief response supports a sense that he is confident that these groups are counterexamples (Warrant). The main finding from this proof about how John's conceptual understanding relates to his proof activity is that he is able to produce these counterexamples and verify that each

contains a single self-inverse – the identity element – which reflects the “identity as self-inverse” *function* of identity informing his example generation of the groups.

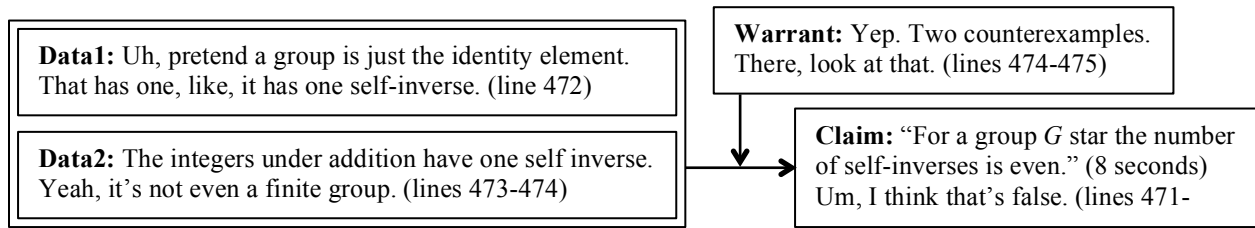


Figure 5.67. J7A1

**Proof J8: Subgroup proof – second proof that the normalizer of  $h$  is a subgroup.**

John produces his eighth proof (J8) in response to Interview 3 Q5, which has an identical prompt to Interview 2 Q7, asking the participants to, “Prove or disprove the following: for a group  $G$  under operation  $*$  and a fixed element  $h \in G$ , the set  $H = \{g \in G : g*h*g^{-1} = h\}$  is a subgroup of  $G$ .” John recalls the prompt from Interview 2 and compares how he now perceives the difficulty of the proof relative to the first time he proved it, remarking that it seems much easier this time. Accordingly, John’s response is much more organized, resulting in a proof modeled with a single linked, embedded Toulmin scheme (J8A1, Figure 5.68). John first reads the prompt aloud, which serves as the main claim (Claim) for J8A1. The Toulmin scheme contains three linked data to support this claim, each corresponding to a subgroup rule that John describes (associativity, identity, and inverse). Unlike his response to Interview 2 Q7, however, John does not mention needing to show that the set  $H$  satisfies closure.

John begins his proof by describing whether  $H$  satisfies associativity, saying, “I’m just gonna say associativity. Okay. So, associativity carries from  $G$ –” (lines 506-507). This serves as the first data (Data1) in the linked Toulmin scheme. John then provides a short subargument that the identity of  $G$  exists within the set  $H$ , saying, “Identity exists because in a subgroup, the identity's the same, so and  $e$  star  $h$  star  $e$  equals  $h$  is true” (lines 508-509). In this excerpt, the

initial phrase “identity exists” (line 508) serves as a claim (Claim2.1). John then explains that “in a subgroup, the identity’s the same” (line 508), which serves as a warrant to validate how the phrase “ $e$  star  $h$  star  $e$  equals  $h$  is true” (lines 508-509, coded as Data2.1) supports Claim2.1. Altogether, this subargument constitutes a second Data (Data2) supporting the original claim that  $H$  is a subgroup (Claim).

Following this, John provides an argument in which he explains why he thinks the set  $H$  contains the inverse of each of its elements. He begins by saying, “Like, I know it exists, cause you just, like, it's there. Like, it's gonna act the same way. Because, if  $g$  acts that way, like, you just switch them” (lines 514-515). This statement provides a broad sense of why John feels inverses exist in the set and serves as backing (Backing) for his later reasoning. John then writes a series of equations that constitute an argument that can be modeled with a sequential Toulmin scheme and serves as data for John’s claim that inverses are contained in  $H$ . In this subargument, John first writes the equation used to define  $H$  (coded as Data3.1.1), which supports the generation of the equation “ $g*h*g^{-1} = g*g^{-1}*h$ .” John does not describe why he feels this new equation can be written, though, as discussed in, reflects the reverse process of the “vanishing” *function* of inverse, which contributed to the development of the “un-vanishing” *function* of inverse. Specifically, John concatenates the left end of the right-hand side of the equation with the element  $g$  operated with its inverse.

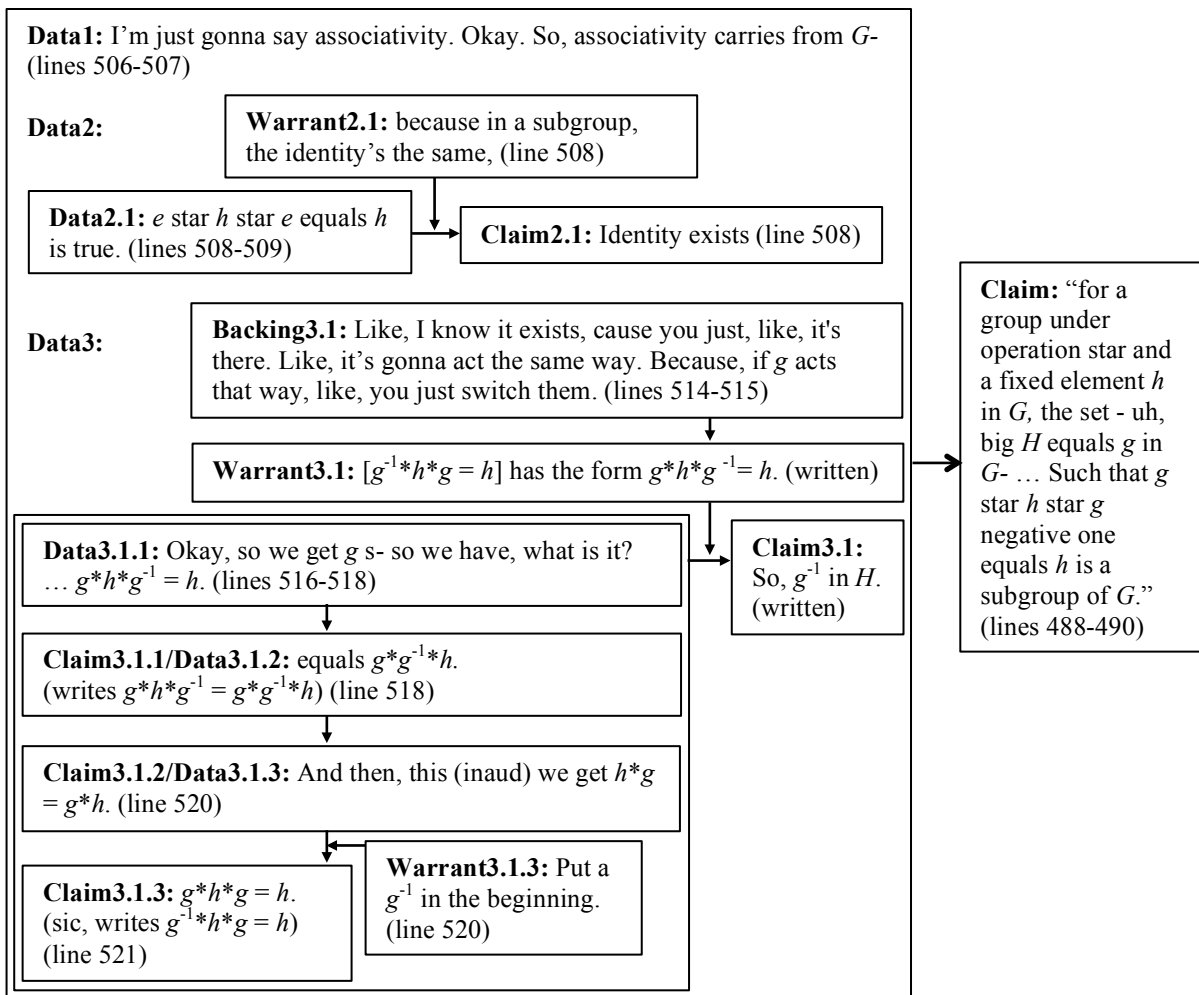


Figure 5.68. J8A1

John then writes the equation " $h*g = g*h$ ," which, as with his previous step, he does not explain. This equation serves as data to support a new equation when John says, "Put a  $g$  to the negative one in the beginning.  $g$  star  $h$  star  $g$  equals  $h$ " (lines 520-521). This excerpt constitutes a warrant (Warrant3.1.3) that reflects the "end-operating" *function* of inverse to justify the new equation " $g^{-1}*h*g = h$ ," which John then writes (Claim3.1.3). Following this, John writes "Has the form  $g*h*g^{-1}$ . So,  $g^{-1}$  in  $H$ " under the equation " $g^{-1}*h*g = h$ ," which serves as Warrant3.1 and Claim3.1, to complete John's argument. This reasoning is similar to J5A6 in which he also appeals to the "form" of the definition of  $H$ . John then states that he thinks his work proves the

given statement that  $H$  is a subgroup of  $G$ , although he does not generate an argument supporting the closure of  $H$ , as he had in J4.

Throughout most of J8, it is unclear which *functions* of identity and inverse upon which John is drawing. Few excerpts are coded as warrants, which reflects the lack of explanation in his argument. In his subargument supporting the existence of the identity in  $H$ , John replaces the elements  $g$  and  $g^{-1}$  in the equation defining  $H$  with the letter  $e$  without any explanation and simply states that the equation is true. Although the *functions* of identity upon which he might be drawing can be inferred, John's lack of explanation reflects that, for John, the truth of the equation  $e * h * e = h$  is a given. This supports a sense that the verification of this subargument is somewhat trivial. Further, he provides only one warrant that reflects the "end-operating" *function* of inverse (Warrant3.1.3), which was critical in his development of his first proof that the normalizer of  $h$  is a subgroup. The parallel structure of this proof, with the exception that he does not include an argument supporting closure, and the general lack of reasoning to support his claims, supports a sense that John might be relying on his memory of his previous proof (J4) to show that  $H$  contains inverses.

**Proof J9: Proving  $g^k = e$  implies  $(g^{-1})^k = e$ .** John produces his ninth proof (J9) in response to Interview 3 Q6a, which asks participants to "Prove or disprove the following statement: For  $g \in G$ , if  $g^k = e$ , then  $(g^{-1})^k = e$  for a given group  $(G, *)$ ." Analysis of John's proof generated four arguments modeled by a standard Toulmin scheme (J9A1), two embedded Toulmin schemes (J9A2, J9A3), and a linked Toulmin scheme with an embedded argument as a qualifier (J9A4). J9A1 reflects John's initial activity in reaction to the prompt in which he generates an equation. In J9A2, John produces a claim about the equation from J9A1, which he initially questions but reaffirms. This affords John insight in order to develop a new argument

(J9A3) in which he explains why he believes his claim in J9A2 is true. Finally, John draws on his result in J9A3 to support the statement given in the prompt.

John begins his response by quietly writing while periodically mumbling to himself, mentioning  $\mathbf{Z} \bmod 12$  and saying, “Right now, I’m just looking at it with- for myself and seeing what it means” (lines 631-632) and later, when asked what is happening, adding “Um, I’m just seeing if I can find a counter example” (line 634). Following this, he writes “ $4 + 4 + 4 = 0$ ” and “ $8 + 8 + 8 = 0$ ” and says, “So, four plus four plus four equals zero. And three plus- Oh, no. It’s inverse would be 8. And three. Okay. So, that works” (lines 641-643). This seems to lead John to try to prove the claim as he later states, “Okay. So, I do think it holds. I don’t know how to go about proving this. I don’t know why one example makes me think it holds, but, you know” (lines 649-651). Following this, John writes for several seconds while mumbling to himself and then says aloud, “Um, if  $g^k$  the identity, so  $g$  to the  $k$  star  $g$  to the negative one  $k$  equals  $g$  to the negative one  $k$ . Seems like it’d be useful maybe. I don’t know. It’s the most meaningful thing I can make from that” (lines 655-657). This excerpt is coded as argument J9A1 (Figure 5.69), the first sentence of which serves as a data-claim pair, which is then qualified, supporting a sense that John produces the equation without necessarily anticipating how he might use it to prove the given statement in the prompt. Although he does not describe how he generated the equation “ $g^k * (g^{-1})^k = (g^{-1})^k$ ,” it seems that John manipulated the given equation  $g^k = e$  by (right) end-operating with the expression “ $(g^{-1})^k$ ,” although it is unclear whether John thinks of this as a single element or as an inverse of any element.

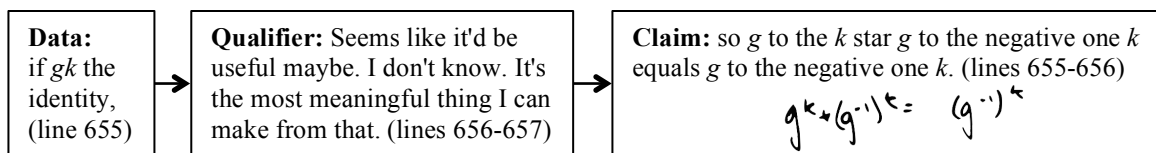


Figure 5.69. J9A1



John then say, “Gah, I just wanna, like, combine them. But, I don't know, like- (writing for 13 seconds) Does that work? Which is it- is it just not true? Is there any time where- Is there any time that's not true?” (lines 658-660). He goes on to describe a second equation that he wrote during the pause in the excerpt, saying,

So,  $g$  to the  $k$  star  $g$  to the negative one  $k$  equals  $e$  to the  $k$  with a question mark over the equal sign. I'm tryin- Like, if that holds, then- then this holds. Is it to prove that, like- I just wanna- since- I just wanna like- manipulate it using my normal multiplicat- multiplicatives and stuff. Which, gah, it has to be commutative. Does it? No. Because, they're gonna be touching in the middle. (lines 660-664)

This excerpt constitutes a new argument (J9A2, Figure 5.70) modeled with an embedded Toulmin scheme. In this argument, John generates the equation “ $g^k * (g^{-1})^k = e^k$ ,” which serves as the claim in the embedded Toulmin scheme (Claim1). John indicates that he is unsure whether the equation is true by writing a question mark over the equals sign, which qualifies his claim (Qualifier). He then provides a conditional statement that serves as the main claim (Claim) in the argument, explaining that, if the equation he generated is true, the conjecture provided in the prompt must also be true. Following this, John indicates a strategy that he thinks could be productive toward resolving whether the equation he generates is true, which serves as a warrant for the overarching argument. Finally, John questions whether the equation “ $g^k * (g^{-1})^k = e^k$ ” is only valid if the elements are commutative, which serves as a qualifier (Qualifier1) for the equation, resolving his concern by pointing out that “they’re gonna be touching in the middle” (line 664), which serves as data (Data1) supporting Claim1.

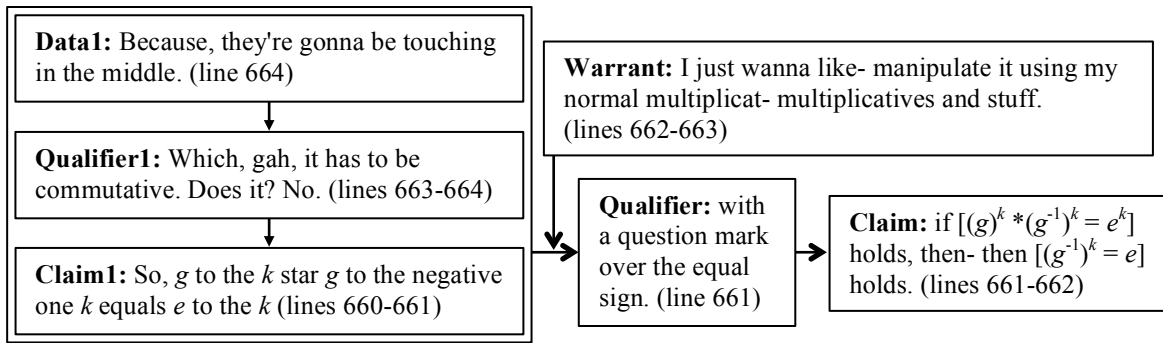


Figure 5.70. J9A2

This activity seems to afford John insight into how he might approach showing that “ $g^k * (g^{-1})^k = e$ ” as is evidenced by his statement that the equation does hold, which serves as the main claim (Claim) for a new argument J9A3 (embedded Toulmin scheme, Figure 5.71), although he first qualifies this claim, saying, “I don’t know how to write it out,” (lines 664-665). John continues by describing the expression “ $g^k * (g^{-1})^k$ ,” pointing out that “it’s gonna be  $g, g, g$ - however many  $g$ ’s and then in the middle it’s gonna be  $g, g^{-1}, g^{-1}, g^{-1}, g^{-1}$ ” (lines 665-667). This serves as data (Data1) in the embedded Toulmin scheme. John continues by saying, “so these will mean  $e$ ” (line 667), which serves as a warrant (Warrant1) supporting the claim (Claim1) that “[ $g^k * (g^{-1})^k$ ] will colla- It’ll just collapse in on itself and equal  $e$ ” (lines 667-668). Together, this subargument serves as data supporting the larger claim that  $g^k * (g^{-1})^k = e$ .

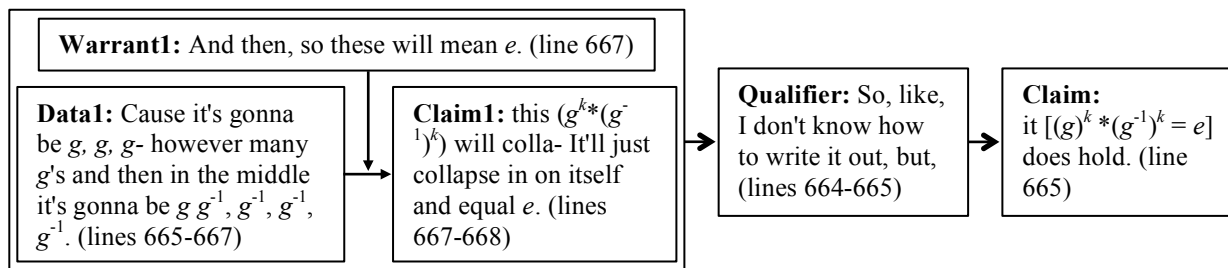


Figure 5.71. J9A3

John then draws on his claim from J9A3 in order to complete his proof that if  $g^k = e$ , then  $(g^{-1})^k = e$ , which constitutes J9A4 (Figure 5.72) and is modeled with a linked Toulmin scheme with an embedded argument comprising one of the data. John begins his argument by saying, “if

they're an equal number of  $g$ 's and  $g$  to the negative one's and they're operating on each other," (lines 692-693) which serves as initial data (Data1.1) for the embedded argument. He then qualifies and warrants his impending claim, saying, "even if it's not commutative" (line 693, coded as Qualifier1.1) and adding, "the middle, like, the middle  $g$  and  $g$  to the negative one will cancel" (lines 693-694, coded as Warrant1.1). Finishing his statement, John states, "and then it'll just collapse in and cancel- they'll all cancel. Until you just- you get  $e$ " (lines 694-695), which serves as the claim of the embedded argument (Claim1.1). He then adds, "And since  $g$  to the  $k$  star  $g$  to the negative one to the  $k$  equals  $e$  and  $gk$ -  $g$  to the negative one to the  $k$  equals  $g$  to the negative one to the  $k$ " (lines 695-696), which is coded as Data2 because it contributes additional information to his previous statement. Together, the two data support the overarching claim (Claim) that " $e$  equals  $g$  to the negative one to the  $k$ " (line 696).

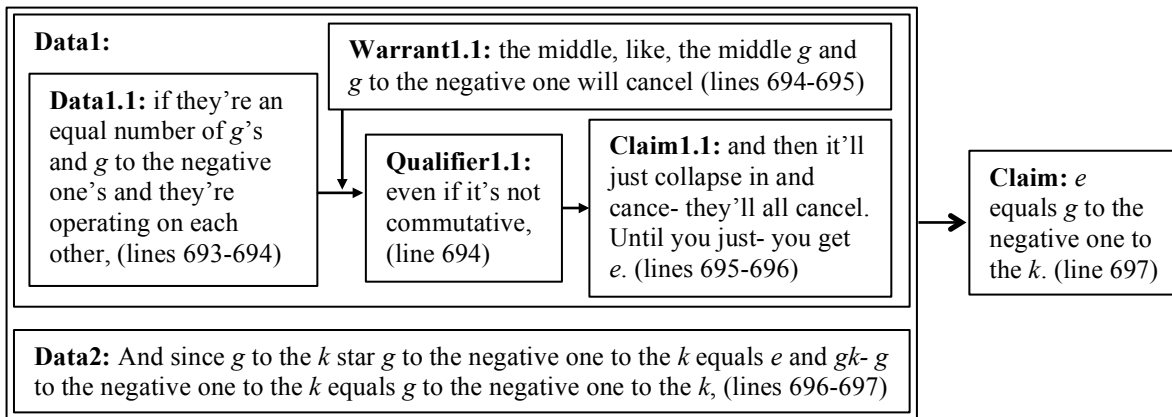


Figure 5.72. J9A4

The series of the four arguments analyzed in J9 affords insight into how John is able to leverage his initial intuition about how he might prove the claim provided in the prompt to generate an argument. First, he begins by generating the equation  $g^k * (g^{-1})^k = (g^{-1})^k$ , which reflects (right) end operating on both sides of the given equation  $g^k = e$  with the expression  $(g^{-1})^k$ . This shifts the goal of the proof from determining that  $(g^{-1})^k = e$  to determining whether  $g^k * (g^{-1})^k = e^k$ . John initially doubts this, wondering aloud whether commutativity is necessary for the equation

to be true. It seems that, initially, John anticipates commuting the  $k$  number of  $g$ 's and  $g^{-1}$ 's so that each would pair with a corresponding inverse, producing  $k$  expressions " $g * g^{-1}$ " operated together. This would allow John to draw on the "equals" *function* of inverse for each pair, in turn, producing  $k$   $e$ 's. This would explain why John initially mentions  $e^k$ . However, John is initially hesitant to commute the elements, saying, "it has to be commutative" (line 663).

Although John has twice demonstrated that elements commute with their inverses, explicitly saying so during Interview 2, he does not draw on this aspect of his understanding of inverse here. It further seems that, by concentrating on the "middle" of the expanded expression, John is able to anticipate drawing on the "vanishing" *function* of inverses to eliminate the expression  $g * g^{-1}$  from the original expression, which is supported by his description, "it'll just collapse in and cancel- they'll all cancel. Until you just- you get  $e$ " (lines 694-695). This seems to afford a shift in the right-hand side of the equation from  $e^k$  in J9A2 to  $e$  in J9A3.

However, considering an expansion of the exponential notation provides John an opportunity to focus on the specific elements,  $g$  and  $g^{-1}$ , in the middle of the expansion he describes, affording him the opportunity to build an argument about the expression  $g^k * (g^{-1})^k$ , specifically, that this expression will "just collapse in on itself and equal  $e$ " (lines 667-668). In order to do this, John's argument draws first on the "equals" *function* of inverse to change the " $g * g^{-1}$ " at the center of the expanded expression to  $e$ . Also, although John does not explicitly explain his reasoning behind the collapse he describes, it tacitly draws on the "vanishing" *function* of identity, because, were the  $e$  at the center of the expression to vanish, the elements  $g$  and  $g^{-1}$  would again be adjacent at the center of the expression. Importantly in this proof, it seems that the elements in the initial equation John produces are unable to serve a *function* of inverse. Rather, it is not until John considers the expansion of the exponential notation that the

individual elements  $g$  and  $g^{-1}$  are able to serve the “equals” and “vanishing” *functions* of inverse. Accordingly, John’s ability to reorganize the algebraic expression affords him the opportunity to draw on *functions* of inverse in a more productive way.

**Proof J10: Proving  $g^k = e$  for some positive  $k$  implies  $g^{-1} = g^p$  for some positive  $p$ .**

John produces his tenth proof (J10) in response to Interview 3 Q6b, which prompts the participants, “Let  $(G, *)$  be a group with identity  $e$  and let  $g^k = e$  for some positive integer  $k$ . Is it possible to write  $g^{-1}$  as  $g^p$  for some positive number  $p$ ?” (Appendix C). J10 consists of a single argument (J10A1) that is modeled with an embedded, sequential Toulmin scheme (Figure 5.73). John begins by saying, “ $g$  star  $g$  to the negative one [equals]  $e$  because they’re inverses. And  $g$  star  $k$  equals  $e$ . So,  $g$  star  $g$  to the negative ones equals  $g$  star  $k$ ” (lines 741-742). The statement “ $g$  star to the negative one [equals]  $e$ ” (line 741, coded as Claim1.1.1) is an initial claim that John supports, saying, “because they’re inverses” (lines 741-742, coded as Data1.1.1), which, together serves as Data1.1. John adds “ $g$  star  $k$  equals  $e$ ” (line 742; sic, although he writes this as  $g^k = e$ ), which is coded as Data1.2 because John uses this, along with Data1.1, to support the claim (Claim 1/Data2) that “ $g$  star  $g^{-1}$  equals  $g$  star  $k$ ” (line 742; sic, writes  $g \cdot g^{-1} = e = g^k$ ).

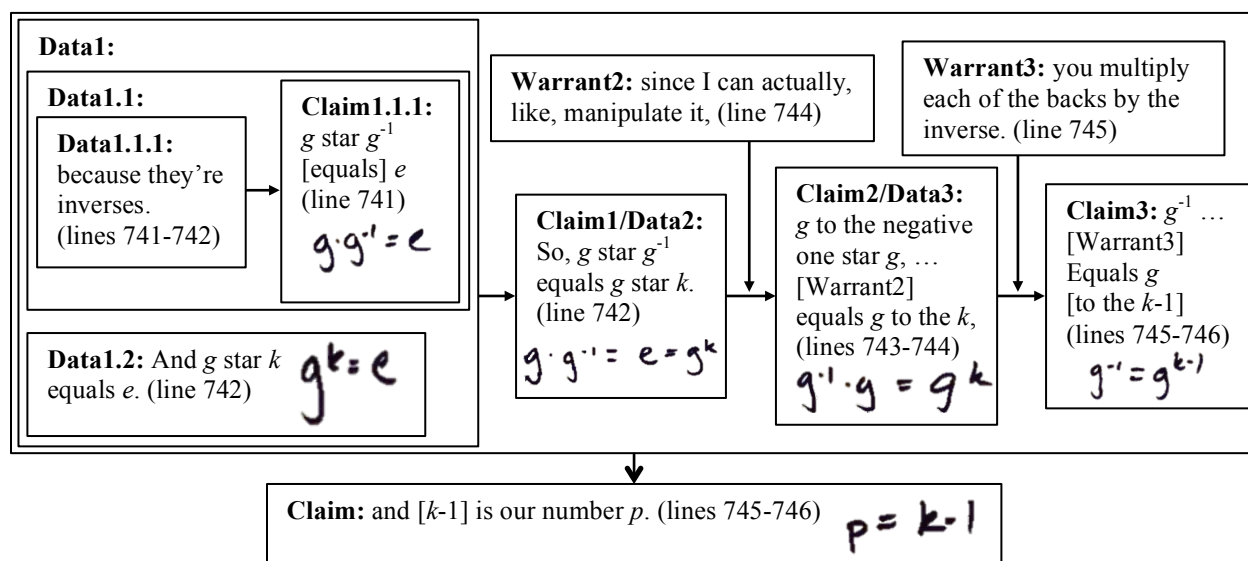


Figure 5.73. J10A1

John continues, saying, “ $g$  to the negative one star  $g$ , since I can actually, like, manipulate it, equals  $g$  to the  $k$ ” (lines 743-744). In this statement, John provides a new equation (Claim2/Data3) and provides a warrant for that equation (Warrant2). Claim2/Data3 reflects the equation in Claim1/Data2 with the elements  $g$  and  $g^{-1}$  commuted on the left-hand side. John then says, “so  $g$  to the negative one- you multiply each of the backs by the inverse- equals  $g$  times  $k$  to the negative one and that is our number  $p$ ” (lines 745-746) which serves as a final claim in the embedded sequence (Claim3) and is warranted by John describing (right) operating with  $g^{-1}$  (Warrant3).

In J10, John’s work first draws on the “transitive” *function* of identity to generate an equation, then on the “end-operating” *function* of inverse to manipulate this equation in order to verify the given claim. In Warrant3 and Claim3 John’s (right) end-operation with  $g^{-1}$  reduces the exponent of  $g$  by one. This reflects a sense that John is able to imagine  $g^{-1}$  operating with a single factor of  $g^k$ , which allows it to serve the “vanishing” *function* of inverse. Near the beginning of his response to the prompt, John states that he feels this proof is easy, which indicates that anticipating the goal of his activity is not problematic.

**Summary of analysis of John’s proofs.** Across his ten proofs, John draws most often on the “letter” *forms* of identity and inverse to serve their various respective *functions*. On some occasions during his proof activity, John replaces more generic *forms* of group elements with more specific elements in a different *form*. For instance, during his fourth argument in J2, John replaces the letters  $k$  and  $l$  in the expression  $2^k * 2^l$  with the number 2 and evaluates  $2^2 * 2^2$ . This seems to be done in order to remind himself of the pattern for adding exponents when multiplying real numbers. Similarly, during J4, John replaces the elements in a generic “letter” *form*  $g$ ,  $h$ , and  $g^{-1}$  with letters representing the “symmetry” *form* in order to gain a better sense of

whether he should be trying to prove whether  $H$  is a subgroup. He specifically chooses this form based on his reasoning in J4A2, indicating that John is drawing on his proof activity to inform his example generation. John also draws on the “number” *form* when generating counterexamples in J5 and J6, specifically drawing on the integers and the integers modulo 12. These allow John to generate counterexamples that satisfy the given constraints in the prompt and contradict the claim.

John is typically able to express a goal for his proof activity that is different from but supports an original goal. For instance, in J4, John first claims that he believes  $H$  contains the identity immediately after saying, “Let’s pretend  $g$  is the identity” (line 781). With this statement, the letter  $g$  is then able to serve the “equals” and “identity as self-inverse” *functions* of identity. John’s pretense that  $g$  is the identity shifts the goal of his activity from determining whether  $H$  contains the identity to interpreting the validity of an equation in which an element is able to serve specific *functions*. Later in J4, John is unsure about how to prove that  $H$  contains inverses. After asking the question, “What does this mean? ... Now what does it take or that to be- oh” (lines 1402-1405), he points out that it requires  $g^{-1}*h*g = h$  to be true. John then warrants his argument that that  $H$  contains inverses (J4A6) by drawing on the “inverse-inverse” *function* of inverse to explain that  $g^{-1}*h*g = h$  holds the same form as  $g*h*g^{-1} = h$ . Although he is unable to articulate this in terms of  $g^{-1}$  satisfying the definition of  $H$ , this argument reflects having carried out a substitution of the element  $g^{-1}$  that is similar to his activity in which he “pretends”  $g$  is the identity. This would afford the inverse in the expression serving the necessary *function* so that John is able to interpret its validity, which he does by comparing its form with the equation he generates by manipulating  $g*h*g^{-1} = h$ .

Another important aspect of John's proof activity that occurs in several proofs is his production of additional arguments to justify whether he is able to carry out a specific action. This is most clear in J4A4 and J4A6. While trying to prove that inverses exist in  $H$ , John manipulates the equation  $g*h*g^{-1} = h$  by drawing on the "end-operating" *function* of inverse. He then questions whether this is a valid action, explaining that he is unsure whether an inverse he is using is a left- or right- inverse, because they might not be equal, which reflects the "sided" *function* of inverse. Accordingly, he abandons his approach. When asked what he would do if it were the case that left- and right- inverses are the same, John supposes that this could always be the case. This leads to John's production of J4A5 through which he is able to draw on associativity to verify that left- and right- inverses are equivalent, in turn, leading him to reject the "sided" *function* of inverse that was informing his approach to the proof that  $H$  is a subgroup.

In J9, although the prompt asks the participant to show that  $(g^{-1})^k = e$ , John shifts the goal of his proof to showing that  $g^{k*}(g^{-1})^k = e$ . This shift is mediated by his activity in which he (right) end-operates the given equation  $g^k = e$  with the expression  $(g^{-1})^k$  in J9A1 to produce the equation  $g^{k*}(g^{-1})^k = (g^{-1})^k$ . This leads John to produce a new argument in which he provides a conditional statement that, if  $g^{k*}(g^{-1})^k = e$ ,  $(g^{-1})^k = e$ , which reflects the transitivity of  $(g^{-1})^k = g^{k*}(g^{-1})^k = e$ . Further, John first attends to the "middle" of the expansion of the exponential notation after posing the question about whether the group needed to be commutative. This seems to motivate his production of J9A2, focusing again on the "middle" of the expansion containing the individual elements,  $g$  and  $g^{-1}$ , which are able to serve the "vanishing" *function* of inverse whereas the elements  $g^k$  and  $(g^{-1})^k$  could not serve.



## Chapter 6 – Conclusions and Future Work

In this chapter I discuss the results of this research, namely the insight this work affords toward answering the research question, “How does student understanding of inverse and identity relate to student engagement in proof in Abstract Algebra?” In Chapter 4, I used Saxe’s *form/function* analytical framework (Saxe, Dawson, Fall, & Howard, 1996; Saxe & Esmonde, 2005) to characterize three participants’ conceptual understanding of identity and inverse. In Chapter 5, I then analyzed the participants’ proofs using Toulmin’s (1969) model of argumentation. These analyses of Violet, Tucker, and John afford insight into how students’ conceptual understanding relates to their mathematical proof. In this chapter, I synthesize important aspects that emerged across the *form/function* and Toulmin analyses of the three participants’ individual interviews. In my discussion, I provide more general descriptions of the results and include examples from the data that support the broader claims before returning to extend the general discussion and situate it within the literature. This underscores an important aspect of these results, namely that the relationships between conceptual understanding and proof activity discussed here emerged from an examination of analyses of conceptual understanding and proof specifically contextualized with respect to identity and inverse in Abstract Algebra. The broader discussion of each of the three constructs highlights the shared characteristics across the specific examples identified in the participants’ responses and can be thought to extend more generally to proof activity in other content domains of mathematics, although constructs might occur differently in those contexts.

Throughout the Toulmin schemes of the participants’ proofs, *functions* of inverse and identity as well as the correspondence between representational object and representational vehicle for a given *form* typically contributed as warrants and backings to the argument.

Similarly, it was most often the case that representational vehicles of *forms* served as parts of data and claims that situated the goal-directed activity. These characteristics of the participants' arguments highlight an important aspect of these proofs that likely extends beyond this content, although one could imagine different patterns emerging in other content domains (e.g., geometry or analysis). This connection between the *form/function* framework and the Toulmin schemes for the participants' proofs provides a basis for discussing the three general construct that emerged from the analysis across the two models of the participants' responses: *form* shifts in service of sense making, re-claiming, and lemma generation. I define *form* shifts in service of sense-making as instances in which an individual, while engaged in mathematical proof, replaces one representational vehicle with another in order (which typically serves as data) to explore the implications of a given claim. I define re-claiming as instances in which an individual attempts to rephrase a claim and its hypotheses using a form that is able to serve *functions* that were previously unable to be served in order to support the original claim. Lemma generation occurs when an individual, in an attempt to provide a warrant for an intended claim, seems unsure whether he or she is able to draw on specific reasoning and develops an argument in support (or refutation) of the warrant (which, in turn, typically informs the *functions* that specific *forms* are able to serve for that individual).

These constructs help answer the question of how conceptual understanding of inverse and identity relates to engagement in proof activity in Abstract Algebra by providing insight into how each aspect (conceptual understanding/proof activity) informs the other. The first two constructs – *form* shifts in service of sense-making and re-claiming – lend insight into how conceptual understanding might inform proof activity, whereas, lemma generation reflects how proof activity informs conceptual understanding. After discussing each of these three constructs

that emerged from the data, I highlight the limitations of the study, contributions that it provides to the field of mathematics education, and implications for future research.

### **Form Shifts in Service of Sense-Making.**

Throughout the interviews, the three participants each engaged in *form* shifts in order to make sense of a scenario with varying levels of success. In these cases, each participant's understanding of inverse and identity – specifically, the *forms* upon which each participant was able to draw in the moment – informed his or her proof activity by affording an alternative representation of the mathematical situation in which he or she was working. The primary indicator of a *form* shift is a change in representational vehicle during goal directed activity. Along with each change in representational vehicle, a change in the correspondence between representational vehicle and representational object also occurs. This is an important aspect of this mathematical activity because it affords insight into how the *forms* upon which an individual draws might inform the progression of proof activity.

Within the data, more productive *form* shifts occurred when the correspondence between the representational vehicle and representational object in the new *form* carried parallels to the correspondence in the original *form*. This often afforded connections to be drawn between the *forms* with greater fidelity, in turn allowing participants to more clearly describe how their *form* shifting activity was situated relative to the broader goal of the proof. However, it was also often the case that correspondence in the new *form* did not reflect the correspondence in the original form. In these cases, the *form* shifts were less productive toward the development of the proof. Another important aspect of a *form* shift is the affordance of different *functions* that the new *form* might be able to serve. It was often the case that after a *form* shift, a participant was able to carry out more productive activity than before the *form* shift. In some cases, the participant was then

able to shift the *form* back to the original and interpret his or her activity relative to the original *form* or draw on the original *form* to serve the same *functions* as the second *form*. The differences between the two *forms*' vehicle-object correspondence and available *functions* contribute to the degree of success that the *form* shift affords.

For example, during V5A3, Violet shifts from the “letter” *form* to the “number” *form* by replacing  $g$ ,  $h$ , and  $g^{-1}$  with 2, 3, and -2, respectively in order to make sense of the set  $H$ . While drawing on the “letter” *form* to serve the “end-operating” and “vanishing” *functions* of inverse, Violet appropriately manipulates the given equation used to define  $H$  ( $g*h*g^{-1} = h$ ). However, she is unable to leverage the resulting equation ( $h = g^{-1}*h*g$ ) to support that  $H$  is a subgroup. After making the *form* shift, Violet multiplies the numbers  $2*3*(-2)$ , which results in -12. She explains that this is not what she had expected the product to be. This is attributed to her mixing the operation of multiplication with additive inverses. Although Violet is able to draw on the “letter” *form* of inverse to serve appropriate *functions* of inverse, her *form* shift does not afford her to draw on similar *functions*, namely because she calculates the product of the numbers, rather than manipulating the equation as she did with the “letter” *form* of inverse. Further, Violet’s correspondence between representational vehicle and representational object in the “number” *form* is inconsistent with Violet’s expectation. Accordingly, her *form* shift is unsuccessful for Violet as she explains, “but then that didn’t prove anything” (line 657).

Similar to Violet’s *form* shift, Tucker shifts from the “letter” *form* of identity to the “number” *form* of identity when he replaces the number 1 for the letter  $g$  in the same equation used to define  $H$ . He reasons that the multiplicative notation used in the definition of  $H$  leads him to think that 1 is the identity in  $H$ . In doing so, Tucker rewrites the equation as  $1*h*1 = h$ , explaining that  $1^{-1}$  equals 1. However, from this *form* shift, the “number” *form* of identity is

unable to serve any of the necessary *functions* of identity to afford Tucker insight into how this might inform the broader proof. Specifically, Tucker does not view the “number” *form* as serving the “operate/same out” *function* of identity. Further, shifting back to the “letter” *form* of identity after a prompting discussion with interviewer, Tucker is similarly unable to draw on the “letter” *form* of identity to serve the “operate/same out” or “vanishing” *function* of identity, although it does serve the “identity as self-inverse” *function* of identity as Tucker rewrites the equation as “ $e*h*e = h$ .” In this case, the two *forms* reflect consistent *functions* and, thus, do not afford further progress toward verifying that the identity exists in  $H$ .

In his response to the same prompt, John replaces the “letter” *form* with the “symmetry” *form* in order to better make sense of the subset  $H$ . As he explains, he does this because he wants to use a non-commutative group. From the available data, it seems that John appropriately draws on the “symmetry” *form* of inverse to serve the “end-operating” *function* of inverse. He then carries out this same type of equation manipulation with the “letter” *form*. However, this leads to his concern over whether the left- and right- inverses are the same, which initially seems to contradict the generality of his approach, because the symmetries of a triangle have equal left- and right- inverses and, so, do not serve the “sided” *function* of inverse. This process involves two *form* shifts: the first from the “letter” *form* to “symmetry” *form* and the second shift back to the “letter” *form*. The first *form* shift seems to have afforded John to draw on the “end-operating” *function*, which then carried back into his manipulation of the “letter” *form* after the second shift. However, John is unsure how general this process is.

Students often encounter situations in Abstract Algebra in which a more abstract representation presents a challenge for determining the structure of a group with given constraints, as was the case in Interview 2 Q7. The participants’ activity in these examples

affords insight into the general aspects involved in producing a *form* shift toward making sense of a more abstract situation. Specifically, a *form* shift necessarily involves comparisons to be drawn from a given *form* to the target *form* with respect to the correspondence between representational vehicle and representational object comprising each *form*. For example, in Violet's argument, the correspondence between the "number" *form* of inverse, -2, and the representational object of inverse is inconsistent with the correspondence between the "letter" *form* of inverse,  $g^{-1}$ , and the representational object of inverse. Namely, Violet does not coordinate -2 as an inverse with respect to multiplication. This points to a broader aspect of *form* shifts that can be problematic when an individual is trying to make sense of a situation with which he or she is unfamiliar. Further, the "letter" *form* of inverse serves an appropriate "end-operating" *function* of inverse, whereas the "number" *form* does not. In this sense, the *form* shift does not afford sense making because it serves no new *functions* of inverse and, indeed, serves fewer *functions* of inverse.

A *form* shift seems to allow Tucker to draw on the "inverse-inverse" *function* of inverse with the "number" *form* of identity, 1, under multiplication, as well as with the "letter" *form*,  $e$ . However, this shift does not afford further sense making because this seems to be the only *function* of identity that either *form* can serve. Namely, there is little correspondence between either of the representational vehicles and the representational object of identity. This lack of correspondence in either *form* restricts any insight the *form* shift might afford. John's *form* shift initially seems successful in that the "symmetry" *form* of inverse is able to serve the "end-operating" *function* of inverse. However, shifting back to the given "letter" *form*, the John is initially unable to reason that the "end-operating" *function* of inverse can also be served by the

“letter” *form* of inverse. In this we see a problematic aspect of *form* shifts in that a new *function* is initially afforded by the new *form*, but this does not carry back to the original *form*.

These aspects of *form* shifts in service of sense-making – the ability to draw on different *functions* and the similarities and differences between the original and new vehicle-object correspondences – situate well with broader discussions found in the literature of example use and changes in representation and/or notation during proof. For instance, the types of *form* shifts discussed in this section reflect a semantic proof approach (Weber & Alcock, 2004; Weber, 2005) because the participants’ *form* shifts moved from more the formal and general “letter” *form* to specific instantiations using the “number” *form* (for Violet and Tucker) and the “symmetry” *form* (for John) and, occasionally, back to the “letter” *form*. Such shifts from more specific *forms* to more general *forms* also reflect aspects of Mason & Pimm’s (1984) generic examples. Specifically, the authors discuss how students often do not see specific examples as providing insight into more general relationships. This seems to have been the case more so with Violet and Tucker than with John, although he also initially thought that the example upon which he drew after his *form* shift did not generalize. Further, this discussion addresses Balacheff’s (1986) concern that researchers shift their focus away from the “heuristics” of students’ activity and toward “how the pupils’ conceptions are related to the mathematical notions engaged in the problem space” (p. 10). A focus on the correspondence between representational vehicle and representational object across *form* shifts and the differences in *functions* that the two *forms* are able to serve affords a perspective that draws attention to specific aspects of students’ conceptual understanding in order to better understand their activity. This should help researchers to better characterize the semantic proof approaches that students generate and gain insight into students’ difficulties finding more general relationships from their activity with more specific *forms*.

## Re-Claiming

Re-claiming (Figure 6.1) is the process of reframing an existing claim in a way that affords an individual the ability to draw on a specific *form* of identity or inverse and the *functions* that this *form* might be able to serve. In this study's data, it was often the case that re-claiming occurred when a participant was asked to prove or disprove a general statement and, in response, interpreted the general statement using a specific *form* to produce a new claim in terms of this *form*. An important part of successfully re-claiming is the consistency between the original claim and new claim. The individual must also be able to interpret any possible hypotheses or assumptions of the original claim with respect to the new *form* upon which they draw. Once the individual generates appropriate initial data from the given hypotheses and assumptions, he or she is then able to draw on the new *form* to serve specific *functions*, which affords meaningful argumentation toward the new claim. Finally, after supporting the new claim, the individual should be able to provide a warrant for how or why this claim supports the original claim.

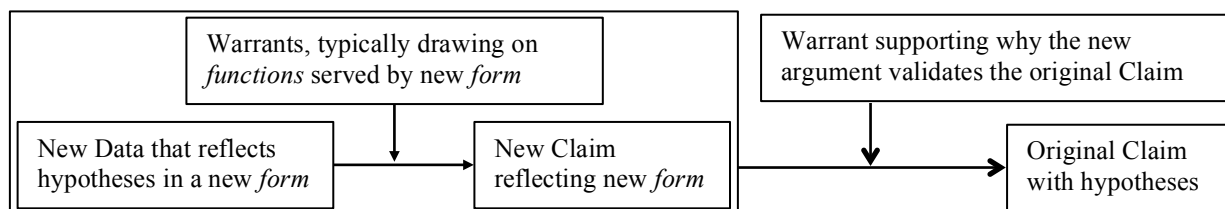


Figure 6.1. Toulmin scheme reflecting the general structure of re-claiming

For example, Tucker's proof that a finite group with exactly one element of order two contains an even number of elements (T8) affords insight into a type of re-claiming in which Tucker's activity is not necessarily anticipatory of the new claim he will make<sup>16</sup>. In his proof,

<sup>16</sup> The given statement in the prompt states that such a group has an odd number of integers, which is false. The re-claiming that takes place in this example is not Tucker's decision to rebut the given claim and prove that such a group has an even number of elements. Rather, it is



Tucker constructs a group table with which he generates elements using the “letter” *forms* of inverse and identity and organizes the elements of the group. A necessary aspect of this activity toward a valid argument is that Tucker’s group table reflects the constraints of the given statement; specifically that  $G$  is finite and contains exactly one element of order 2. Through his construction process, Tucker is able to draw on the “matching” and “inverse finding” *functions* of identity, the “inverse-inverse” *function* of inverses, and the Sudoku rule for group tables to develop insight into the parity of the group after counting the identity and the element of order 2. This exploratory construction allows Tucker to produce a new claim about the specific group he produces by drawing on the *functions* served by the “letter” *form* in the group table, which he is then able to generalize for all groups.

Another example of re-claiming is that, in all three participants’ responses to Interview 2 Q6, in which they were able to prove that the inverse of a given element is unique through a proof by contradiction, re-claiming the proof by reframing the claim from “for each element  $g$  in a group  $G$  with operation  $*$ ,  $g^{-1}$  (the inverse of  $g$ ) is unique,” to “ $g^{-1} = i$ ” (Violet), “ $g^{-1} = h^{-1}$ ” (Tucker), and “ $g^{-1} = a$ ” (John). All three participants, having previously proved this statement during classwork, provide a proof by contradiction (Figure 6.2) during the interviews in which they draw on the “letter” *forms* of inverse and identity to assume that there exist two inverses for a given element  $g$  in the group (which affords the production of initial data using the “letter” *form*), generate two equations using a *function* of inverse that reflects the definition (“bring back” for Violet, “operate/identity out” for Tucker, and “equals” for John), and draw on the “cancelling” *function* of inverse to produce an equation contradicting the original assumption that the inverses are not equal. In service of this proof, the participants each conclude that this

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Tucker’s claim that the group in the table must have an even number of elements, which he then generalizes to all groups that satisfy the given hypotheses.

proved the statement that an element's inverse is unique. These proofs reflect all aforementioned aspects of re-claiming. Specifically, participants reinterpret the general claim by generating initial data in a specific *form* based on the original claim (in this case, being a proof by contradiction, they each draw on the “letter” *form* of identity to necessarily produce data contradictory to the original claim). They then draw on available *functions* of identity and inverse that this *form* serves in order to generate a new claim that reflects the same *form* as their restatement of the initial data. Finally, each participant interprets this claim to argue that it supports the original conjecture.

Still other proofs afford insight into how re-claiming might be more or less successful. For instance, in response to Interview 2 Q7 (proving that the normalizer of  $h \in G$  is a subgroup of  $G$ ), the three participants re-claim with varying levels of success. For instance in V5A3, though Violet is able to appropriately manipulate the equation so that it could be viewed as demonstrating that  $H$  contains inverses, she does not interpret her activity in any way that would support the original claim. This can be attributed to the fact that she does not begin with any assumption that the equation  $g*h*g^{-1} = h$  is true (or that the truth of this meant  $g$  is an element of  $H$ ), nor does she interpret the new equation to draw any meaningful connections with the subgroup rules. Similarly, in John's initial approach (J4A4) to the same part of the same proof, he manipulates the equation  $g*h*g^{-1} = h$  to generate the new equation  $h = g^{-1}*h*g$  but is initially unable to interpret his work in terms of the subgroup rules (indeed, he even questions whether the production of the new equation is valid). However, John resolves this issue later in the interview with the statement, “What does it mean? So, the inverse of  $g$  is ...  $g$  to the negative one or  $g$ - inverse  $g$ . Now what does it take or that to be- oh. It takes for this to be true” (lines 1402-1405). In this excerpt, John generates a new claim (J4A6) that, in order for the set  $H$  to contain

inverses for each of its elements, it must be the case that “ $h = g^{-1} * h * g$ .” However, John’s reasoning that this equation “has the same form” (written) neglects an appropriate interpretation of the subgroup rule that would begin with the initial data that  $g$  is an element of  $H$  and follow with the manipulation of the equation that John has carried out.

Similarly, John’s proof that the same set  $H$  is closed<sup>17</sup> (J4A7) does not begin with the initial conditions that  $g_1$  and  $g_2$  are elements of  $H$ . Because of this, John does not establish initial data that draw on the “letter” *form* of inverse and identity (namely, the equations  $g_1 * h * g_1^{-1} = h$  and  $g_2 * h * g_2^{-1} = h$ ) that can serve to validate the equation he claims should be true. Accordingly, John again appeals to the “form” of a new equation that he produces. In contrast, in T4A11, Tucker appropriately re-claims the closure subgroup rule by assuming that  $a$  and  $b$  are elements in  $H$  and writing the equations  $h = a * h * a^{-1}$  and  $h = b * h * b^{-1}$ . From this, he then claims that  $(a * b)$  must also be an element of  $H$ , replaces  $g$  in the original equation used to define  $H$ , is able to draw on the necessary *functions* of inverse and identity as well as the fact that he had already shown that inverses exist in  $H$ , and interprets the results of this activity by explaining how it supports the claim that  $H$  is closed.

A sense of the various facets of re-claiming as defined in this study can be drawn from this discussion. Specifically, in re-claiming, it is not sufficient, to only reframe a claim. Rather, one must likely also reframe its related (often hidden) hypotheses. These aspects of reclaiming reflect the frequently taught proof mantras of “what do I know?” and “what do I want to show?”

In the context of the *form/function* framework, these restated hypotheses serve as initial data

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<sup>17</sup> The closure subgroup rule states that, for any two elements in the subset, these elements satisfy closure under the group operation. Thus, an appropriate re-claim of the closure subgroup rule must necessarily include the initial data that two elements (represented in a new *form*) are contained in the subset and conclude with a verification (again, in the new *form*) that these elements operated together (the “product” of the two elements) must also satisfy the definition of the subset. T4A11 includes an appropriate re-claiming of the closure subgroup rule.

(drawing on a specific *form* of identity or inverse) in an argument in which the participant is able to draw on the *form* of identity or inverse with which the data is reframed to serve appropriate *functions* of identity and inverse in support of the new claim. The individual must then reason that this new argument supports the original claim. The discussion of participant's responses from this research during which they re-claimed a given statement provides successful and unsuccessful examples in which each of these aspects of reclaiming was carried out. For instance, Violet (in V5A3) and John (in V4A7) do not generate new data from the hidden hypotheses of the identity and closure subgroup rules, respectively.

Re-claiming provides a second type of proof activity in which the individual's conceptual understanding (*forms* upon which an individual draws and the *functions* that these *forms* are able to serve) informs the individual's proof approach. Specifically, the access to a *form* that is able to serve specific *functions* affords the individual an opportunity to generate a meaningful argument that he or she would likely not have been able to produce without re-claiming the initial statement. This activity is not necessarily an inherent necessity of a given conjecture, but rather depends on the individual's understanding in the moment. As with *form* shifts in service of sense-making, this reflects the importance of Balacheff's (1986) call to focus on students' understanding when considering their proof activity.

### **Lemma Generation**

Lemma generation is the process through which an individual, in an attempt to provide a warrant for a specific data-claim relation, questions the validity of the warrant he or she intends to use and develops a new argument supporting or refuting the validity of their intended warrant. This can be viewed as a process of warrant analysis in which the individual might question the *forms* of identity and inverse in the data or the *functions* that these *forms* might be able to serve.

Accordingly, lemma generation involves the individual's reflection on and analysis of the *form* of identity or inverse upon which he or she is drawing and the *functions* that this form might serve in a given context toward providing a warrant for a claim. In these instances, the individual is able to develop his or her own conceptual understanding through proof activity, constituting a way that engaging in proof activity informs conceptual understanding.

For example, in his proof that the normalizer of  $h$  is a subgroup of  $G$ , John questions whether left- and right- inverses are the same element. In other words, he is unsure whether the “letter” *form* of inverse is able to serve the “end-operating” *function* of inverse when trying to manipulate the equation  $g*h*g^{-1} = h$ . This leads John to develop a new argument (J4A5) through which he is able to prove that left- and right-inverses are equivalent in a group, drawing on the associativity of the group. In turn, this informs John's proof approach by affording the “letter” *form* of inverse to serve the “end-operating” *function* of inverse, which contradicts the “sided” *function* of inverse upon which John had previously been drawing.

During J4, John develops another lemma (J4A3) that, if the group  $G$  is commutative, then  $H$  is the entire group  $G$ . He develops this lemma early in the proof when he describes switching the positions of  $g^{-1}$  and  $h$  in the equation  $g*h*g^{-1} = h$ , saying, “you can just move it over and those cancel” (lines 813-814). John's discussion during the interview supports a sense that, rather than first supposing that the group is commutative, he first thought to commute the elements  $h$  and  $g^{-1}$  (anticipating that he could then “cancel”  $g*g^{-1}$ , which would reflect the “vanishing” *function* of inverse). If this is the case, John's lemma development reflects the sense that he anticipates the results of acting in a specific way and determines what qualities of the group must be necessary in order to warrant such actions.

Another instance of lemma generation from the data occurs during Tucker's response to Interview 3 Q6a in his proof (T11A3) that  $g^k = e$  implies  $(g^{-1})^k = e$ . Specifically, Tucker anticipates the result of iteratively end-operating on both sides of the given equation with  $g^{-1}$  " $k$  times" to be  $g^0$ . However, after writing the equation  $g^0 = (g^{-1})^k$ , Tucker is unsure how to interpret the expression  $g^0$ ; specifically, he is unsure whether this expression is the same as the identity (T11A3, Qualifier1.1). Because of this, he is unable to provide a warrant explaining why his previous activity supports his intended claim (he had provided a warrant earlier in his argument). During his explanation of his thinking in the moment, Tucker says, "So, essentially, if you have  $k$  number of  $g$ 's, you're pretty much like just taking away one taking away one taking away one. You know what I mean? So- Cause taking away would be the same like- Oh!" (lines 816-818). At this point, he reflects on his anticipated action, analyzing the imagined process so that he is able to reverse the process to the next to last step.

Lemma generation reflects an important aspect of proof activity that affords individuals the opportunity to inform their own understanding and, indeed, learn through their proof activity. Specifically, analysis of whether one is able to warrant a desired claim data (either given or generated by the individual) provides the individual with an opportunity to shift the types of *functions* that specific *forms* of inverse or identity might serve. Or, as in Tucker's case, reflect on the *functions* upon which he or she draws to analyze the unintended consequences of functions. The process of lemma generation reflects Lakatos' (1976) discussion of proof analysis in which individuals reflect on their proof approach rather than focusing on a local or global counterexample. However, in the current research, participants' lemma generation was not a consequence of encountering a specific counterexample to a conjecture, but instead a focus on the validity of the argument that a participant intended to build. The participants' activity would

reflect a more general consideration of the *functions* that specific *forms* might serve in the moment. This is an important aspect of such warrant and *function* analysis: the willingness and openness to question one's own mathematical activity in a way that doesn't preclude the possibility that one's current ways of understanding might be wrong. This reflects a healthy skepticism of the reasoning upon which the individual relies - a skepticism that I argue should be supported with the curiosity to hypothesize alternate ways of understanding.

### **Constraints**

The current research was constrained by several factors. First, my focus on three students' responses to individual interview protocols limits analysis of the relationships between conceptual understanding and proof activity, warranting further analysis of different participants' conceptual understanding and proof activity. Also, although this analysis was informed by the broader contexts of the classroom environment, the focus on the individual interview setting affords insight into a specific community of proof in which argumentation develops differently than in other communities. For instance, the structure of the interview setting necessitated that participants developed their arguments solely on their own understanding in the moment and for the audience of a single interviewer. My early observations of and reflections on the development of argumentation in the classroom and homework groups included the mutual development of argumentation in which participants' argumentation was informed by their interactions. Accordingly, analysis of the classroom and homework group data is warranted.

Further, the TAAFU curriculum through which the participants developed their understanding of identity and inverse is unique in the approach that it takes toward the development of the various concepts in abstract algebra. This necessarily informs the *form/function* categories developed through analysis of the participants' discussions. For

instance, the curriculum focused on students developing broader notions of groups from the initial example of the symmetries of a triangle. This is reflected in the participants' initial discussions of identity and inverse as processes in-and-of themselves (as with Violet and Tucker) and, later, ability to draw on the symmetries of a triangle when trying to think of an example of non-commutative groups (as with John). Collecting data with students from different types of instructional settings would likely afford insight into different *form/function* categories for students' understanding of identity and inverse as well as different relationships between proof and conceptual understanding.

### **Contributions**

This research contributes to the field by drawing on the *form/function* framework to characterize students' conceptual understanding of inverse and identity. This affords insight into the *forms* upon which students participating in the TAAFU curriculum might draw as well as the various *functions* that these *forms* are able serve. This research also contributes to the field by providing several examples of how Aberdein's (2006a) extension of Toulmin's (1969) model of argumentation might be used to analyze proofs in an Abstract Algebra context. Further, this research draws attention to three aspects of relationships between individuals' conceptual understanding and proof activity. Specifically, this research draws on the *form/function* framework and Toulmin model of argumentation to describe the various factors involved in students' engagement in *form* shifts, re-claiming, and lemma generation.

These results situate well among the work of contemporary mathematics education researchers, specifically those focused on research in undergraduate education. For instance, Zazkis, Weber, and Mejia-Ramos (2014) have developed three constructs that also draw on Toulmin schemes to model students proofs in which the researchers focus on students



development of formal arguments from informal arguments. These constructs provide interesting parallels with the three aspects of relationships between conceptual understanding and proof activity developed in the current research. For instance, Zazkis, Weber, and Mejia-Ramos (2014) describe the process of rewarranting, in which an individual relies on the warrant of an informal argument to generate a warrant in a more formal argument. This reflects some aspects of *form* shifts in service of sense making. Specifically, as stated, a *form* shift to a specific case might afford *functions* that can be carried back to a more general case. However, the current research focuses more on the aspects of conceptual understanding that might inform such activity.

Further, this work provides a lens and language that researchers and instructors might be able to use in their reflection on students' proof activity and how their students' conceptual understanding might inform such activity. For instance, in the future I anticipate focusing on my own students' shifting *forms* and *functions* as well as the ways in which these shifts occur, attending to the ways in which correspondences between representational vehicle and representational object may or may not align. This will allow me to make better sense of students' proof activity and provide opportunities to challenge students' thinking so that it might be more productive. I also anticipate drawing on these experiences while implementing inquiry-oriented curricula, especially TAAFU, so that I can anticipate the ways in which students develop more general notions from the specific examples, including additional examples that could be beneficial for students in their practice of *form* and *function* shifts and proof generation.

### **Future Research**

Moving forward from this research, I intend to analyze the data from other participants' individual interviews in order to develop more *form* and *function* codes for identity and inverse, affording deeper insight into the various *form/function* relations students in this class developed.

Such analysis should also explore the proof activity of the other participants in the study, which would provide a larger sample of proof activity, in turn affording new and different insights into the relationships between mathematical proof and conceptual understanding. I also intend to analyze the sociomathematical norms and classroom math practices within the classroom. This will afford insight into the sociogenesis and ontogenesis of *forms* and *functions* at the classroom and small group levels in order to support and extend the individual analyses – which are focused on microgenesis – in the current research. This research also warrants further investigation and extension of the constructs of *form* shifts in service of sense making, re-claiming, and lemma generation. Specifically, how might different students’ proof activity inform and extend the constructs developed in the current research and in what other ways might conceptual understanding relate to proof activity? Future investigation should also explore teaching approaches and techniques that supports students’ more productive engagement in *form* shifts in service of sense making, re-claiming, and lemma generation during proof production as well as the ways that students might develop these practices in and out of the classroom.

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## Appendix A – Interview 1 Protocol

- 1) (a) “Different people think about concepts in mathematics differently. I’m curious to know: How do *you* think about identity?”
  - i) “Say a bit more about that.”
  - ii) “Do you have other ways that you think about identity?”
  - iii) “Do you have a *definition* of identity?”
  - iv) “How does that definition relate to the other way(s) in which you think about identity?”

(b) Could you give a few examples of identity?

  - i) “When are times that you can remember learning about or talking about identity in mathematics classes?”
  
- 2) (a) “Similar to the last question, I’m curious to know: How do *you* think about inverse?”
  - i) “Say a bit more about that.”
  - ii) “Do you have other ways that you think about inverse?”
  - iii) “What do you think the *definition* of inverse is?”
  - iv) “How does that definition relate to the other way(s) in which you think about inverse?”

(b) Could you give a few examples of inverse?

  - i) “When are times that you can remember learning about or talking about inverse in mathematics classes?”
  - ii) “Can you give an example of something without an inverse?”
  - iii) “Why do you think this is such an example?”
  - iv) “What does it mean to be an inverse in this example?”
  
- 3) The participants are handed a list of Likert-scale questions that reads, “For each of the following statements, indicate the extent to which the statement connects with how *you* think. Please read each of the statements out loud and elaborate on your responses.”

*Follow-ups for each statement. Skip if redundant:*

  - i) “Could you explain why you chose that number?”
  - ii) “How might you rephrase the question so that it more closely aligns with how you think?” Or to a statement that you would agree with?
  - iii) “Is there any other way that you think about this statement?”
  - a) Zero is an identity.
  - b) One is an identity.
  - c) I think of an identity as anything that I use to keep things unchanged.
  - d) I think about an identity as a function.
  - e) “Inverse” means “negative number.”
  - f) “Inverse” means “reciprocal.”
  - g) Inverses depend on other things, like operations or sets of elements.
  - h) I think about an inverse as a function.

- 4) (a) “A lot of the discussion in class so far has centered on symmetry. Can you give me one or two examples of a symmetry?”
- “What about this example makes it a symmetry?”
  - “How is this example related to the discussions from class?”
  - “What part of your example do you think is most important for showing symmetry?”
- If participant generates more than one example:*
- “How are these two examples the same?”
  - “How are these two examples different?”
- (b) “What does the word ‘symmetry’ mean to you?”
- “How does what you just said relate to the example(s) you generated?”
  - “Can you use your example(s) to explain how something might not be a symmetry?”
- 5) “The other day in class, Dr. Johnson said, ‘It’s like multiplying by one. It’s cheap, but it’s still multiplying. Just because it doesn’t change it doesn’t mean that we don’t consider it multiplication.’”
- “What do you think Dr. Johnson meant by this statement?”
    - “Do you remember why this came up in class?”
    - “How does this seem related to what you were working on?”
  - “How do you think this statement relates to the symmetries?”
    - “Would you say that this is how you think about some of the symmetries?”
    - “How is this different from how you think about symmetries?”
    - “How is a 360 rotation the same/different than multiplying by one?”
- 6) “This question relates to what Dr. Johnson said in class the other day. Prove the following statement: ‘ $1 \in \mathbb{R}$  is the multiplicative identity for the real numbers.’”
- “Why did you approach this proof this way?”
  - “How did the way you think about identity inform your approach to this proof?”
  - “Do you think that this proves the statement?”
  - “On a scale from one to ten, how confident are you in this proof?”
  - “On a scale from one to ten, how much does this proof verify the relationship for you?”
- Follow-ups if participant has difficulty beginning the proof:*
- “What does it mean to be a multiplicative identity?”
  - “How *might* you show that 1 is a multiplicative identity?”
  - “Where would you start to show that this is true?”
- 7) “Show that  $f(x) = \sqrt{x + 3}$  and  $g(y) = y^2 - 3$  are inverse functions for  $x, y \in \mathbb{R}, x \geq -3, y \geq 0$ .”
- “Why did you approach this problem this way?”
  - “How did the way you think about inverse inform your approach to this problem?”
  - “On a scale from one to ten, how confident are you in your response?”
  - “On a scale from one to ten, how much does this response verify that  $f$  and  $g$  are inverse functions?”

## Appendix B – Interview 2 Protocol

- 1) (a) “Different people think about concepts in mathematics differently. I’m curious to know: How do *you* think about identity?
  - i) “Say a bit more about that.”
  - ii) “Do you have other ways that you think about identity?”
  - iii) “Do you have a *definition* of identity?”
  - iv) “How does that definition relate to the other way(s) in which you think about identity?”
- (b) Could you give a few examples of identity?
- 2) (a) “Similar to the last question, I’m curious to know: How do *you* think about inverse?
  - i) “Say a bit more about that.”
  - ii) “Do you have other ways that you think about inverse?”
  - iii) “What do you think the *definition* of inverse is?”
  - iv) “How does that definition relate to the other way(s) in which you think about inverse?”
- (b) Could you give a few examples of inverse?
- 3) “In this course, the class has done a lot of work with groups. Could you define group for me and give a few examples of groups?”
  - a) “Say a bit more about that.”
  - b) “What do you mean when you say \_\_\_\_\_?” (identity, inverse, closure, associativity)
  - c) “How does this relate to the ways that you described inverse and identity before?”
  - d) “Can you think of any more examples of groups?”
  - e) “What does it mean to be a subgroup?”
- 4) The participants are handed a list of Likert-scale questions that reads, “For each of the following statements, indicate the extent to which the statement connects with how *you* think. Please read each of the statements out loud and elaborate on your responses.”

*Follow-ups for each statement. Skip if redundant:*

  - i) “Could you explain why you chose that number?”
  - ii) “How might you rephrase the question so that it more closely aligns with how you think?” Or to a statement that you would agree with?
  - iii) “Is there any other way that you think about this statement?”
  - a) Zero is an identity.
  - b) One is an identity.
  - c) I think of an identity as anything that I use to keep things unchanged.
  - d) I think about an identity as a function.
  - e) “Inverse” means “negative number.”
  - f) “Inverse” means “reciprocal.”
  - g) Inverses depend on other things, like operations or sets of elements.
  - h) I think about an inverse as a function.

- 5) “Prove or disprove:  $H = \{g \in G \mid g = 2^n, n \in \mathbb{Z}\}$  is a subgroup of the group  $G$  of non-zero real numbers  $(\mathbb{R} \setminus \{0\})$  under regular multiplication.”
- “Why did you approach this problem this way?”
  - “What did you mean when you wrote \_\_\_\_\_?”
  - “Do you think that your work proves/disproves the statement?”
  - “On a scale from one to ten, how confident are you in this proof?”
  - “On a scale from one to ten, how much does this proof verify the relationship for you?”
  - “How did the way you think about identity inform your approach to this proof?”
  - “Similarly, how did the way you think about inverse inform your approach to this proof?”

*Follow-ups if participant has difficulty beginning the proof:*

- “Do you think the statement is true or false?” (If this is unclear, re-word the question as: “Do you think you should prove this statement or disprove it?”)
  - “What does it mean to be a subgroup?”
  - “How might you show that  $H$  satisfies the criteria for a subgroup?”
- 6) “Prove the following: for each element  $g$  in a group  $G$  with operation  $*$ ,  $g^{-1}$  (the inverse of  $g$ ) is unique.”
- “Why did you approach this proof this way?”
  - “What did you mean when you wrote \_\_\_\_\_?”
  - “Do you think that your work proves the statement?”
  - “On a scale from one to ten, how confident are you in this proof?”
  - “On a scale from one to ten, how much does this proof verify the relationship for you?”
  - “How did the way you think about identity inform your approach to this proof?”
  - “Similarly, how did the way you think about inverse inform your approach to this proof?”

*Follow-ups if participant has difficulty beginning the proof:*

- “What does it mean for an inverse to be unique?”
  - “Where would you start to show that this is true?”
- 7) “Prove or disprove the following: for a group  $G$  under operation  $*$  and a fixed element  $h \in G$ , the set  $H = \{g \in G : g * h * g^{-1} = h\}$  is a subgroup of  $G$ .”
- “Why did you approach this problem this way?”
  - “What did you mean when you wrote \_\_\_\_\_?”
  - “Do you think that this proves/disproves the statement?”
  - “On a scale from one to ten, how confident are you in this proof?”
  - “On a scale from one to ten, how much does this proof verify the relationship for you?”
  - “How did the way you think about identity inform your approach to this proof?”
  - “Similarly, how did the way you think about inverse inform your approach to this proof?”

*Follow-ups if participant has difficulty beginning the proof:*

- “Do you think that this set  $H$  is a subgroup of  $G$ ?”
- “Where would you start to show that this is true/false?”
- “What does it mean to be a subgroup?”
- (i) (If participant’s response to previous question includes the identity/inverse axioms for groups, ask the relevant follow-up questions below.)
- “How might you show that  $H$  does/does not meet the criteria you just described?”

- l) “What role, if any, do you think the identity plays in determining if  $H$  is a subgroup of  $G$ ?”
  - m) “What role, if any, do you think inverses play in determining if  $H$  is a subgroup of  $G$ ?”
- 8) “Charlie, a student from another class, writes the identity of the group of symmetries of the triangle as ‘OR.’ Why do you think Charlie represents it this way?” (*“Why do you think it might have made sense to Charlie to write the identity this way?”*)
- a) “Would you say that this is an acceptable way to notate the identity of this group? Explain why you feel this way.”
  - b) “Are there any other ways you can think of representing this element?”
  - c) “Is there a way to represent this element that is *most correct*? That is, is there a ‘best’ way to represent the identity of this group?”
  - d) “Which way of representing this element do you most prefer?”

### Appendix C – Interview 3 Protocol

1. “Throughout the semester, we have discussed various mathematical ideas. I’m curious to know: How do you think about identity?
  - a. “Do you have other ways that you think about identity?”
  - b. “Do you have a *definition* of identity?”
  - c. “Could you give a few examples of identity?”
2. “Similar to the last question, I’m curious to know: How do you think about inverse?
  - a. “Do you have other ways that you think about inverse?”
  - b. “What do you think the *definition* of inverse is?”
  - c. “Could you give a few examples of inverse?”
3. “In this course, the class has done a lot of work with groups. Could you define group for me?”
  - a. “What do you mean when you say \_\_\_\_\_?” (identity, inverse, closure, associativity)
  - b. “How does this relate to the ways that you described inverse and identity before?”
  - c. “Could you give a few examples of groups?”
4. “For each of the following statements, indicate whether you feel that the statement is true or false. Please read each of the statements out loud and elaborate on your responses. For instance, if you feel that a statement is true, describe generally how you might go about proving it. Conversely, if you believe a statement to be false, support your response with some reasoning.”
  - i. “Why did you choose ‘True’/‘False’?”
  - ii. “What do you think the *key idea* is for showing that this statement is ‘True’/‘False’?”
  - iii. “How confident are you that this statement is ‘True’/‘False’?”

*If participant struggles:*

  - iv. “Which way are you leaning?”
  - v. “What are you thinking about?”
  - vi. “What about the statement makes you think that it might be ‘True’ or ‘False’?”
  - vii. “Do you not *know* whether the statement is true, or are you just unable to *prove* it right now?”
  - a. The identity element ( $e$ ) of a group  $(G, *)$  is unique.
  - b. Every element in a group  $(G, *)$  has an inverse element.
  - c. Every element in a group  $(G, *)$  is an inverse element.
  - d. If a finite group contains exactly one element of order 2, then the group contains an odd number of elements.
  - e. For a group  $(G, *)$ , the number of “self-inverses” is even. (Here, “self-inverse” means an element  $g \in G$  such that  $g * g = e$ .)
5. “Prove or disprove the following: for a group  $G$  under operation  $*$  and a fixed element  $h \in G$ , the set  $H = \{g \in G : g * h * g^{-1} = h\}$  is a subgroup of  $G$ .”

- a. “Why did you approach this problem this way?”
- b. “What did you mean when you wrote \_\_\_\_?”
- c. “Do you think that this proves/disproves the statement?”
- d. “On a scale from one to ten, how confident are you in this proof?”
- e. “On a scale from one to ten, how much does this proof verify the relationship for you?”
- f. “How did the way you think about identity inform your approach to this proof?”
- g. “Similarly, how did the way you think about inverse inform your approach to this proof?”

*Follow-ups if participant has difficulty beginning the proof:*

- h. “Do you think that this set  $H$  is a subgroup of  $G$ ?”
- i. “Where would you start to show that this is true/false?”
- j. “What does it mean to be a subgroup?”

*(If participant’s response to previous question includes the identity/inverse axioms for groups, ask the relevant follow-up questions below.)*

- k. “How might you show that  $H$  does/does not meet the criteria you just described?”
- l. “What role, if any, do you think the identity plays in determining if  $H$  is a subgroup of  $G$ ?”
- m. “What role, if any, do you think inverses play in determining if  $H$  is a subgroup of  $G$ ?”

6. a) “Let  $(G, *)$  be a group with identity  $e$ . Prove or disprove the following statement: For  $g \in G$ , if  $g^k = e$ , then  $(g^{-1})^k = e$ .”
  - i. “Why did you approach this proof this way?”
  - ii. “Do you think that this proves the statement?”
  - iii. “On a scale from one to ten, how confident are you in this proof?”
  - iv. “On a scale from one to ten, how much does this proof verify the relationship for you?”
  - v. “How did the way you think about identity inform your approach to this proof?”
  - vi. “Similarly, how did the way you think about inverse inform your approach to this proof?”

*Follow-ups if participant has difficulty beginning the proof:*

- vii. “Do you think that the given statement is valid or invalid? That is, would you choose to prove or disprove the statement?”
- viii. “What approach might you take to begin such a proof (or to disprove the statement)?”
- ix. “What examples can you think of that might help demonstrate the stated relationship?”

- b) “Let  $(G, *)$  be a group with identity  $e$  and let  $g^k = e$  for some positive integer  $k$ . Is it possible to write  $g^{-1}$  as  $g^p$  for some positive number  $p$ ?”
  - i. “Why did you approach this problem this way?”
  - ii. “Do you think that this proves that such a  $p$  always exists in this type of situation?”
  - iii. “On a scale from one to ten, how confident are you in your work?”

- iv. “On a scale from one to ten, how much does your work verify that this is/is not possible?”
- v. “How did the way you think about identity inform your response?”
- vi. “Similarly, how did the way you think about inverse inform your response?”

*Follow-ups if participant has difficulty beginning the proof:*

- vii. “Do you think that it is or is not possible?”
- viii. “Do you think that such a  $p$  always exists?”
- ix. “What examples can you think of that might help demonstrate the stated relationship?”

7. “Prove the following: If each nonidentity element of a group  $(G, *)$  has order 2, then  $(G, *)$  is abelian.”

- a. “Why did you approach this proof this way?”
- b. “Do you think that this proves the statement?”
- c. “On a scale from one to ten, how confident are you in this proof?”
- d. “On a scale from one to ten, how much does this proof verify the relationship for you?”
- e. “How did the way you think about identity inform your approach to this proof?”
- f. “Similarly, how did the way you think about inverse inform your approach to this proof?”

*Follow-ups if participant has difficulty beginning the proof:*

- g. “How are you thinking about the statement?”
- h. “What approach might you take to begin such a proof?”
- i. “Consider  $a$  and  $b$  are elements of  $G$ . Do you think that it might be helpful to think about the element  $a*b$ ?”



## Appendix D – Transcript of Tucker, Interview 2

1 [00:00:48.638]

2 Int: So, “Different people think about concepts in mathematics differently.” [(Chuckles) Okay.]  
3 “I’m curious to know: how do you think about identity?”

4  
5 Tucker: Identity. Okay, well-

6  
7 Int: Oh, by the way, feel free to use markers.

8  
9 Tucker: Alright. Awesome. Well, now that we’ve kind of, like, drilled the notion of identities and  
10 everything, I think I- my perspective’s kind of changed, it’s basically it’s the- it’s kind of like the  
11 zero operation (air quotes), the- the just like, the- it’s like a boring case, where it just kind of like-  
12 You take an element and, if you apply the identity to it, you get the element back, where you  
13 started, so- If you think about it as a function, it’ll kind of like plussing zero to a number or  
14 timesing by one or other things like that. Or rotating an object 360 degrees. And different groups  
15 also have different elements.

16  
17 [00:01:37.788]

18 Int: Okay. Alright, cool. So, um, “just say a bit more about that.” You- you- it seemed like you  
19 have two different kind of things, you said, like, [Mhmm.] as a function or-

20  
21 Tucker: Well, I mean, we kind of, like, just, like, think about, um, symmetries as functions as  
22 well. [Uh huh.] So- And when I think about functions, I kind of think about, okay, my teachers,  
23 like, in elementary school said you have a box and you put something in the box and it spits out  
24 something else. [Uh huh.] So, as a function, you put something in the box and it just goes straight  
25 through it and you get the same thing back out, so- [Okay.] So, it could either be, like, plussing  
26 one to- I’m sorry. Plussing zero to something or timesing by one, but- (clicks tongue) [Okay.]

27  
28 [00:02:17.080]

29 Int: Alright. Um, do you have any other ways you think about identity besides those, uh-?

30  
31 Tucker: Obviously, you can think about in- You can think about inverses as well. As in, like,  
32 when you have an element and you apply its inverse to it, that’ll give you the identity [Okay.] in  
33 return. [Int mumble while writing] I know I’m gonna slip up saying identity and inverse mixed  
34 up, [Yeah.] cause I always for some reason.

35  
36 Int: Yeah, I- well, you should see me typing this stuff, it’s- (both laugh) Alright. Alright, so, um-  
37 So, “Do you have a definition of identity?”

38  
39 Tucker: A definition of identity? I guess when we talked about groups and symmetries, I would  
40 say, uh, the identity is an element within a group such that any other element in that group, when  
41 you apply the operation, the identity to that operation, you’ll result in the original element  
42 [Okay.] that you started with. So, for instance, if you had some arbitrary element,  $a$ , you apply  
43 the identity, let’s call it  $e$ , then you’ll get a right back at you. [Okay.] That thing. [Could you write  
44 that out for me?] Yeah, sure. So- (8 seconds, begins writing) so, for all  $a$  in a group, let’s call it,

45 just,  $G$ , um, (8 seconds) um, is the element such that  $a$ - well, the operation under  $G$  will be star-  $a$   
46 times star is equal to  $e$  star  $a$  is equal to  $a$ .

47  
48 [00:04:07.370]

49 Int: Okay. So, for every  $a$  in  $G$ , is that the-

50  
51 Tucker: Yeah, for all  $a$  in  $G$  - any arbitrary element  $a$  in  $G$  - the identity,  $e$ , is the element such  
52 that  $a$  star  $e$  is equal  $e$  star  $a$  is equal to  $a$ .

53  
54 Int: Okay. Great. Thanks. Alright, so, "How does that definition relate to the other ways, um, in  
55 which you think about identity?"

56  
57 Tucker: Other ways and this definition? [Mhmm.] Um- (clicks tongue 5x) it just kind of shows  
58 that, like, you're really not doing anything to the other elements. Like, you're not changing it in  
59 any way. Um, so it's kind of, like, the way I see it as, like, plussing zero, or, like, I guess- I feel  
60 as- I feel almost as, like, as almost an empty operation is how I kind of think about it, type of  
61 thing.

62  
63 Int: Okay. Cool. Alright, um, so, I think you've already done this, but "could you give a few  
64 examples-" uh, so plussing zero, multiplying by one, rotating 360 degrees. Do you have any  
65 other examples you (inaud)?

66  
67 Tucker: Um, I mean, they have the opposite of those, are like dividing by one, um (clicks tongue  
68 6x, laughs) I guess- (clicks tongue 7x) trying to think of other ones. Times-ing by the identity  
69 matrix, I guess, with matrices. (3 seconds) Um, that's all I can think about with th- with the sets  
70 right now. I'm trying to think of other colorful- colorful looking sets, but I can't really think of  
71 them. [Okay.] But, yeah.

72  
73 [00:05:45.055]

74 Int: Alright. Well, if you think of them, [Mm.] just let me know. Like, uh, along the way, if you-  
75 [Yeah.] Alright, so, um, so "similar to this last question, [Mhmm.] I'm curious to know how you  
76 think about inverse."

77  
78 Tucker: How I think about inverse? I think inverse really ties in with what we talked about with  
79 identity, uh in the fact that, um, every element in a group will have an inverse. And that inverse  
80 when you apply it to its element- so,  $a$ 's inverse we'll call it minus  $a$ , for instance -  $a$  star minus  $a$   
81 will yield you the identity element. [Okay.] So, it's kinda like, what you have to do to the  
82 element to get it back to the identity, type thing. [Okay.] And if you think about it, kind of like  
83 almost directions. It's like, okay, if you move- if you step to the right three paces, the inverse of  
84 that will be stepping left three paces, because that gets you right back to where you started.  
85 [Okay, cool.]

86  
87 Int: (writes for 7 seconds) Alright, so, um, so "do you other ways that you think about inverse?"

88  
89 Tucker: Um, like I said, inverse, I always kind of thought of inverse as one divided by something  
90 [Mhmm.] or to the negative one power, but, um, that isn't always the case, because, like, one

91 fourth won't be the inverse of four if you talk about, like, addition. Minus four would be the  
92 inverse of four. [Okay.] So, I, um, I guess I just think of inverses dependent on the operation in a  
93 group. [Okay.] So, when you think- when you have groups under the operation of multiplication,  
94 the inverses are going to look a lot different than the *same* set of numbers under the operation of  
95 addition. They're going to have different inver- [Hm.] inverses and such.

96

97 [00:07:23.210]

98 Int: Gotcha. Okay. Cool. Alright. So, um, "what do you think the definition of inverse is?"

99

100 Tucker: Okay. Um, it's gonna use- also this identity element,  $e$ , so- [Okay.] Uh. (begins writing  
101 on student page; ) The inverse to element  $a$  in  $G$  is the element such - we'll call it- we'll call it the  
102 element, minus  $a$  - [Okay.] Such that  $a$  star minus  $a$  is equal to minus  $a$  star  $a$  is equal to  $e$ , where  
103  $e$  is the identity element in  $G$ .

104

105 Int: Okay. Cool. That's very clear. [Yeah.] Alright, and that seems to relate back to what you  
106 were saying, like, what you have to do to get an element back to the identity [Yeah, yeah. Like-]  
107 so this is the thing you do to get back that (trails off).

108

109 Tucker: If- If the identity's kind of like home, and operation  $a$  goes you to, like, I don't,  
110 somewhere else, like the market, then inverse  $a$  is going from market back to home. [Okay.]  
111 Type thing.

112

113 [00:08:35.183]

114 Int: Cool. Alright, um, so, so in each of these, you have, like,  $a$  star  $e$  and then  $e$  star  $a$  [Mhmm.]  
115 and then  $a$  star minus  $a$  and minus  $a$  star  $a$ . [Yeah.] Can you talk about that for a minute?

116

117 Tucker: Why I have to show both, type of thing? [Yeah, sure.] Okay, well, the reason I say  $a$  star  
118  $e$  is equal to  $e$  star  $a$  so that, basically, to kind of cover the bases, type thing to show that it works  
119 both ways. Like, it's not just going to be, like, it's- it's commutative, or- you know. So, basically,  
120 it's not just gonna work for one situation [Okay.] where  $a$  star  $e$  is going to work, but  $e$  star  $a$   
121 doesn't necessarily work. That means  $e$  would be the identity for that. There would just be- And  
122 it's- it's- I think it's kind of rare that things aren't really commutative- groups are co- I could be  
123 wrong, but I- It's pretty rare, but, you just have to kind of, like, make that case to show that it is  
124 [Okay.] type thing. I th- The only thing I can think of that aren't really commutative is, like, I  
125 guess matrices and stuff [Mm.] but, I still have to think of other things that wouldn't work with  
126 that, so- [Okay.] That just kind of, like, covers your bases.

127

128 [00:09:40.878]

129 Int: Alright, cool. So- "So, could you give a few examples of inverses?" [Mhmm.] You already  
130 mentioned, like,  $1/4$  and-

131

132 Tucker: Yeah,  $1/4$  and  $4$  under the operation of multiplication. [Uh huh.] Um, and then negative  
133  $4$  and  $4$  under the operation of addition. And, I guess- let's think about- (clicks tongue) In-  
134 Matrices and their inverse matrices- I can't think of any- it would take time [Yeah.] to think of an  
135 example right now, but those would be- work. Um, (taps bottom of table) yeah. I think, um,  
136 yeah, there's gonna be inverses for each element if it's- if it's defined as a group, so-

137  
138 Int: Okay. Any other examples?  
139  
140 Tucker: (exhales) Let's see. Again, I can't think of any fun- more exciting sets or anything, but-  
141 [Okay.] (Tucker laughs) I guess I could make one up, too. It doesn't have to be, like, the sets of  
142 all real numbers, either. (inaud; clicks tongue)  
143  
144 [00:10:42.112]  
145 Int: Alright, well, if you think of any, you can just mention them to me, and I'll go back and right  
146 them. Okay. Alright, so, um, moving on a bit, [Mhmm.] In this course, the class has done a lot of  
147 work with groups. Could you define group for me and [Mhmm.] give a few examples of groups?  
148  
149 Tucker: Yeah. There's three- I believe there's three, um, specifications, um, for a set to be a  
150 group. So, basically- I- I- I'll first kind of talk about what a group is. A group is, like, a set of  
151 numbers or just elements, [Mhmm.] under which there's also an operation such that, uh, there's  
152 closure. So, basically, if you- there's something in the group and then you apply an operation to  
153 another element in that group, you're result is going to yield you another element in that group,  
154 it's not going to be something that's outside the group. [Okay.] Um, so that's closure. Also, that  
155 there is inverses for each element within that group. And that also shows that there's identity  
156 cause otherwise inverses wouldn't make logical sense. Um, so, each element, you can find  
157 another element in that group such that you can get back to the identity. [Uh huh.] Um, and the  
158 third one- Oooh. The third one. (chuckles) Closure, inverses, and oh my gosh. There's one- I- i-  
159 it's obvious, I just can't think of it. It's not identity, because that's under, pretty much talk about  
160 inverses, but it's- Is there a third one? I think there- [You said there was.] I- a- I think there is,  
161 but I can't think of it right now.  
162  
163 [00:12:24.907]  
164 Int: Okay. Well, do you mind writing out the things that you've been describing.  
165  
166 Tucker: Yeah. So, um- Oh! Associative. It's also associative. [Okay.] That's the third one. It's  
167 like the one you didn't really think of cause most everything is associative. But, still gotta check.  
168 (taps table and clicks pen; mumbles inaud. "*I think, but I'm not a hunderd percent*"?) But,  
169 (chuckles) basically, a group, we'll call (begins writing) let  $G$  be a group- So,  $G$  could be-  $G$   
170 could be, I guess, um, all the integers under addition, type thing, so it might look something like  
171 this. (points to paper) [Okay.] Or, um, it could be all the inverses under multiplication, but you  
172 have to- without zero, [Okay.] otherwise zero kind of makes things messy, cause you have zero  
173 times everything is zero, so- [Yeah.] Um, and the reason why it's eas- usually easier just to say  
174 the all real numbers is cause, if you just kind of limit from, like, you know, negative three to  
175 three for instance, um, you're not going to have closure, because three plus three- two elements  
176 in the group will yield you something not in the group. [Mm.] Like six or something. So- Um,  
177 yeah. It just- So, also kind of like the rotations of a triangle type thing would be a group,  
178 because, you know, if you think about it, there's six elements in the rotations- er not just rotation,  
179 I'm sorry - symmetries of a triangle. There's six elements in the symmetry of a triangle and all  
180 those six, when you, you know, apply them together will give you another one of the six and all  
181 those six you can find their inverses, and they're all associative, so that checks out as a group. So,  
182 does that kind of make sense, though? [Sure.] (inaud)

183  
184 [00:14:31.420]  
185 Int: Sure. Um, (clicks tongue) You did a really good job explaining and answered all the  
186 questions ahead of time. (both chuckle) Okay. So. So, sa- before you had said closure, inverses  
187 and associative. Could you just write out for each of those three things- [Mhmm.] Um, kind of  
188 just, like, what you mean by each of those words?  
189  
190 Tucker: Yeah. So, closure, I guess is kind of hard to visualize, (begins writing as he speaks) I  
191 guess, but each element in that group, when operated on with another element- any other  
192 element- will yield you- will always yield you one of the elements in the- the group. It's not  
193 going to yield you anything outside [Okay.] the group, so, if your groups was just the real  
194 numbers, it's not going to give you some crazy number. It's gonna be another real number.  
195 [Okay.] (writes more; 8 seconds) Okay. Um, inverses- I kind of really already talked about that  
196 [Yeah.] So- So, basically for all  $a$  in  $G$ , I guess just to say it really quickly,  $a$  inverse exists,  
197 where (inaud) is the inverse of  $a$ . And, lastly, associativity, basically, in, like, lay man's terms,  
198 you can move parentheses around, so  $a$  star  $b$  star  $c$  is equal to  $a$  star  $b$  star  $c$  .  
199  
200 [00:16:59.605]  
201 Int: Oh, okay. Alright, cool. So, um, when you say- you said, "like we said before here," so,  
202 (pointing at student page and reading) for every  $a$  in  $G$ ,  $a$  inverse exists [Mhmm.] where  $a$   
203 inverse is the inverse of  $a$ . [Yeah.] And you said, "like before," so, before you wrote (turns over  
204 student page from question 2) um, this line here [Mhmm.] so,  $a$  star minus  $a$ . [Yeah.] Is that the  
205 same thing?  
206  
207 Tucker: Yeah, so basically- what, you mean the minus  $a$  and the  $a$  inverse?  
208  
209 Int: Yeah, sure, I mean-  
210  
211 Tucker: Yeah. Yeah. They're the same thing. Basically, If I- If I'm saying that the  $a$ - this is the  
212 inverse of  $a$ , that means that  $a$  inverse  $a$ -  $a$  times the inverse  $a$  or star, whatever [Mhmm.] is  
213 going to yield you the identity, and vice versa. [Okay.] So, if- I'm- If we're saying that this is the  
214 inverse, then that's what it means, pretty much. [Alright, cool.] Y- You can go on later if you  
215 wanted to fill this out more and say, um, where the inverse can be defined as  $a$  star  $a$  inverse is  
216 equal to  $a$  star inverse- or whatever, yeah.  
217  
218 Int: Okay, cool. Alright, and at the beginning of this question, when you were talking about  
219 inverses, [Mhmm.] you mentioned identity [Mhmm.] as, like, part of [Right.] your [definition of  
220 inv-] discussion of inverses, yeah. So, like, how does that play into what you're saying there?  
221  
222 [00:18:16.143]  
223 Tucker: Yeah. If you're saying that, okay, each element has an inverse, you don't need to further,  
224 you know, prove that the elem- there's an, um, identity element. Because, just saying that the  
225 inverses exist for each element kind of proves to yourself that, okay, that means there has to be  
226 an identity element in group, because part of the definition for inverses includes the identity  
227 element. So, for instance, if  $a$  inverse, like we just said, was the inverse of  $a$ , [Mhmm.] that  
228 means that  $a$  star  $a$  inverse is equal to the identity element. So, that pr- that kind of proves itself

229 right there that the identity element exists if there is an inverse for each element.

230

231 Int: Oh, okay. Alright. Cool. Thanks, that was very clear. Um, so you gave some examples of  
232 groups. You talked about integers under addition, [Mhmm.] symmetries of a triangle- [Yeah.]  
233 Um, can you think of any more examples of groups?

234

235 Tucker: Well, grou- I mean, you can be like cr- you can almost create your own group, too, with  
236 just symbols. Like, smiley face and, like, apple type things, so long as, like, you can prove that,  
237 okay, here's is going to be the inverse of smiley face is equal to frowny face type thing. [Mhmm.]  
238 You can just, like, make your own. It doesn't really need to be, like, mathematical at all. [Okay.]  
239 To be honest, like, so long as you can, like, show that all these things hold true. Like, if you have  
240 all the, like, smiley face, frowny face, like apple, orange type at the top. [Uh huh.] Then within  
241 that table, if you kind of think of it as a table, you're going to have, you know, smiley face,  
242 frowny face, apple, orange. You're not going to have, like, a weird potato, you know [Oh, okay.]  
243 in that table, you know? So- that would prove that that has closure, and then you can prove that  
244 everything has an inverse, so- I mean, it doesn't have to be, like, (lowers pitch of voice) oh, the  
245 set of integers under (voice changes back to normal) anything, it could be anything. [Okay.] It's  
246 just a huge variety.

247

248 [00:20:01.398]

249 Int: Alright, cool. Can- can you show me what that table would look like? I'm curious.

250

251 Tucker: I can actually. (picks up marker and begins writing) You know what? Yeah. So,  
252 basically, the operation under this is going to be- think of something fun - like, swirly, like that.  
253 [(Int chuckles) Okay.] Okay? So, if I- Smiley face, frowny face, um, (clicks tongue) I don't  
254 know. Apple. This is going to be an apple. [Okay.] That's just an orange. [Okay. (chuckles)]  
255 Kind of boring looking. And, um, let's think of- I need- Now I have to think of, like, what would  
256 be the identity for these. So, uh, identity is just going to be equal to star, something boring.  
257 [Okay.] So, basically, (laughs) happy face happy face is going to equal, uh, frowny face. I know.  
258 (chuckles) I'm- I'm just going to see if this works. (laughs) [Okay.] So- Okay, I'm going to do my  
259 inverses first. Yeah. So, those are the inverses, so- Duh, duh, duh. Uh, you kind of do, like, the  
260 Sudoku rule, almost, [Okay.] type thing, to figure it out, but, uh- So, I also need to have an  
261 orange and an apple. Okay, so I need to do orange, apple, and happy face. I guess this is the  
262 identity element, so we can just, kind of go down the list with this, too. Yeah. That's supposed to  
263 be over there. (chuckles)

264

265 Int: Uh, um, let's use (grabs marker and begins drawing on student page) red to keep the columns  
266 separated.

267

268 [00:21:56.320]

269 Tucker: There you go. (laughs as Int draws)

270

271 Int: Is that right?

272

273 Tucker: Yeah. That's perfect. [Okay.] Nice. Okay. (chuckles) And duh, duh! Now's where I've  
274 got to think about it a little more. (laughs) So, each row needs to have that. So, yeah, just kind of

275 like playing the Sudoku game. So, frowny face plus frowny face is going to equal happy face.  
276 (Chuckles) Frowny face plus apple is going to equal- uh, let's call it orannn- apple. Yeah.  
277 Frowny face plus orange is equal- equal to orange. I'm actually not sure this is going to work.  
278 We'll just find out. So, apple plus happy face is going to equal orange. Apple plus frowny face is  
279 going to equal apple. (raises pitch as he says apple, as though asking a question) Apple plus  
280 apple is going to equal. (sighs heavily) I don't think it's going to work. I think I made a just  
281 couple of mistakes, cause, like, the Sudoku rule isn't playing out well.

282  
283 [00:23:21.405]

284 Int: Mhmm. (Tucker chuckles) I noticed in this column, it broke the Sudoku rule.

285  
286 Tucker: Ooo, yeah. Oh, I was missing a- What was I missing? Frowny face, happy face, apple,  
287 orange, star. I had that there, but it wasn't (inaud) here. Okay. This is going to get changed into-  
288 (laughs) Okay. This- This is kind of like a puzzle. This is kind of confusing. (chuckles, then  
289 laughs) Isn't it, though? What's- Okay. Sorry. (14 seconds) [Mm (around 8th second)] Oh I think  
290 this should be an orange and that should be an apple. I just need to switch these two.

291  
292 Int: Oh, but you have an orange here.

293  
294 [00:24:44.643]

295 Tucker: Ah. Yeah, yeah, yeah. Hm. (chuckles, five second pause, laughs) That's cr- huh- do you  
296 think it's possible? With five? Kinda sorted out like this? [Mhmm.] You think it is? [Mhmm.]  
297 Did I mess up with my identities? (quickly) No. (pauses for ten seconds, moving pen down the  
298 columns, then chuckles, 5 seconds) Do you know how to do this? (Chuckles; Int nods)  
299 [Uummm.] This was going so well. Hm.

300  
301 Int: I tell you what. Let's not s- [Tucker mumbles inaud] Yeah, we're almost half-way done. Um-  
302 [Tucker mumbles more inaud] Let's come back to this. Okay?

303  
304 Tucker: I'm going to bring this home. Can I go back to my room with this? (laughs)

305  
306 Int: Okay. Well, I ca- I can make you a copy. How about that?

307  
308 Tucker: Okay. That'll be awesome.

309  
310 [00:26:11.927]

311 Int: Alright. Um- (Tucker laughs) Alright, cool. [Tucker mumbles inaud] So, there's other groups  
312 out there.

313  
314 Tucker: Yes! (laughing) There's other- and you can make up your own if you are smarter than  
315 me. [Yeah.] (Tucker chuckles) Yeah. Like, they don't have to be mathematical is what I'm trying  
316 to prove right here.

317  
318 Int: Yeah. I mean, I- I feel you there. (Tucker chuckles)

319  
320 Tucker: (looks at page) Oh. Okay.

321  
322 Int: (9 seconds) Mm. (points at page with pen; 5 seconds) Yeah. (mumble inaud) So- We'll do  
323 that later. [Okay.] Alright.  
324  
325 Tucker: Pretty fun puzzle, though. (both laughs) It's almost like- like a little Rubik's Cube  
326 [Yeah.] slash Sudoku.  
327  
328 [00:27:06.918]  
329 Int: Yeah, yeah, yeah. So, um- "So, what does it mean to be a subgroup?"  
330  
331 Tucker: To be a subgroup? [Yeah.] Basically, um, if we're taking this table right here, and you  
332 were to, like, to chop off- let's say, a couple of elements right here, they would still, like, work as  
333 a group. So, for instance, and a better example than this would be, um, the symmetries of a  
334 triangle. [Mhmm.] That's a group, right? But if you have th- just the rotations of the triangle, that  
335 is going to be a subgroup, because, um, all those- the ele- uh, the three elements of the rotations  
336 of a triangle are also in the six elements of- are all part of the six elements of the symmetries of a  
337 triangle, [Okay.] yet they still form a group. R, 2R, and 3R will still form a group, cause they all  
338 have closure, inverses, and associativity.  
339  
340 Int: Okay. Alright, cool. Alright, so, um- Great. (mumbles to self) Okay. Alright, cool. Alright.  
341 So, um, let's move on from this. [Okay.] And, um- And, I don't know if you remember from last  
342 time, but there's these questions- [Oh, yeah, yeah.] Okay, so- Let's um- Last time, I had three  
343 follow-ups for each question, right? [Mhmm.] So- Er- for each statement. So, "for each of the  
344 following statements, indicate the extent to which the statement connects with how *you* think.  
345 Alright, so please read each of the statements out loud and elaborate on your response."  
346  
347 Tucker: Agree or disagree?  
348  
349 Int: Yeah. [Okay.] Um, before, I asked you, like, could you explain why you chose that number?  
350 [Mhmm.] How might you rephrase the question so that it more closely aligns with how you  
351 think? Um, or to a statement that you- you agree with? And then, is there any other way that you  
352 think about the statement? Alright? [Great.] So, just keep those questions in mind. Let's- um,  
353 there's a lot more behind these [Mm.] this time, so let's not spend too much time, you know?  
354  
355 Tucker: Yeah, yeah.  
356  
357 Int: Okay. Alright, so, for each statement, just read it aloud.  
358  
359 [00:29:03.235]  
360 Tucker: Here we go. "Zero is an identity." So, (chuckles) It's, um- It's very unfinished. It's kind  
361 of, like, not a full sentence. Uh, um, do you want me to say what would be a better definition?  
362 So, zero is an identity for the group, um, integers under addition. Is in- is the identity element.  
363 [Okay.] That would be a better definition for that. So, I said 2, because, yeah it is for a certain  
364 group, [Yeah] but not always.  
365  
366 Int: Okay.



367  
368 Tucker: "One is an identity." Well, this is, like, the same thing here. [Okay.] Well, this is, like,  
369 the same thing right here. [Okay.] Also kind of like and unfinished sentence. One if an identity  
370 for a group, um, integers under multiplication, minus zero- without zero.  
371  
372 "I think of an identity as anything that I use to keep things unchanged." (repeating) Anything I  
373 use to keep unchanged. (clicks tongue) Um, yeah. I- I agree. Wait. Yeah. Identity, okay. I was  
374 like, wait, inverse? Identity. Okay, yeah. That- that's- that's- That's kind of like how I think about  
375 it. I think of identity as an element which, when applied to another element, will not change that  
376 element, so, yeah. That's-  
377  
378 I- "I think about an identity as a function." I- I usual- I do think of it as a function, um, most of  
379 the time, I think. [Okay.] Um. It's kind of like how I said, like, it's like a function that doesn't do  
380 anything, pretty much or doesn't change what you have originally, so- Um, almost all the time I  
381 like to think of it this way.  
382  
383 Int: Okay. So, why the four and not the five?  
384  
385 [00:30:44.880]  
386 Tucker: Because, I- I don't think about it, like, as a function, per se. Kind of like,  $f$  of  $x$  type  
387 thing. I think of it more as, like, um, an applica- er- like, you apply to something, but, not really  
388 so much as a process of- as a function. Cause, I mean, groups can be not functions, as we've kind  
389 of shown with our crazy table, yet, you know, they'll have identities in them as well. [Okay.] But,  
390 it's a good way to think about identities, I think.  
391  
392 Int: Okay. Cool. Yeah. Good.  
393  
394 Tucker: Check over the next one?  
395  
396 Int: Yeah. [Okay.] Sorry. There's nothing (inaud)  
397  
398 Tucker: So, "inverse means, negative number." Um, I'm gonna go with two in this one, cause  
399 while, um, that's- you know, not necessarily true, it is true in some cases. Inverse could be the  
400 negative numbers if we're talking about a- addition- groups under addition. In which case, yeah.  
401 The inverses are going to be the negative of the elements. [Okay.]  
402  
403 "Inverse means reciprocal." (raises pitch of voice as though asking a question) I mean- kind of,  
404 like this very first one. Yeah. Th- Th- That's true under the process of multiplication, but- So, 2  
405 on that one.  
406  
407 "Inverses depend on other things, like operations and sets of elements." Yes, that's very much-  
408 how I think. (both chuckle) So- That's perfect sense, I think I would say.  
409  
410 Int: Perfect sense?  
411  
412 Tucker: Well- (mumbles) ...other thing- I mean, it's not very pretty., but it makes sense to me,

413 how's that?

414

415 “I think about an inverse as a function. Um, yeah, I'm gonna go with the four again here. [Okay.]  
416 Because this- mm- it's closer to a five than the last one, though, because always thought of  
417 inverses as, like, a process. Which, a function is a, I guess, a process. Um, as a process to which  
418 gets you back to the identity. That's how I've always thought of inverses. [Okay.] And, function,  
419 I also think of as a process type thing, like a- like a box you throw something in. [Yeah.] Some-  
420 Something else comes out.

421

422 Int: Okay. Great. Fantastic. You're doing a great job. You're very clear with your explanations.  
423 (Tucker chuckles) Um- Yeah.

424

425 Tucker: I try to be. I mean, I don't- I don't know if I sound too clear, but.

426

427 Int: No, no. You're doing great. Um- You explain your- (in a softened voice) I like how you  
428 explain your thinking very clearly. (Tucker laughs) Um, okay, so, for the next three questions.  
429 [Mhmm.] There's gonna be a statement at the top of the paper. And, then, so, f- Uh, it'll ask you  
430 to prove something or not p- Prove or disprove, or something like that, right. So, for each  
431 problem, just go ahead and read it- read it out loud and then, um, I guess just do your thing.

432

433 [00:33:25.405]

434 Tucker: Okay, so we're “proving or disprove that  $H$  where  $g$  is an element of  $G$ , where  $g$  is equal  
435 to 2 to the  $n$ , where  $n$  is in the integers, is a subgroup of  $G$ , the nonzero real numbers  $R$ . Uh,  
436 under regular multiplication.” So that means without zero?

437

438 Int: Yeah, so, the real numbers, take out zero.

439

440 Tucker: Okay. Under multiplication. Alright. Cool. So, basically, in order to prove that it's a  
441 subgroup, we kinda- I have to, like, prove that's a subset [Mm.] of  $G$ , so- Let's see what we have-  
442 uh I'm just gonna write this out. So, I like to write out, kind of like, sometimes the first few  
443 elements. [Okay.] That helps me visualize a little better, so. First one, so, I'm gonna start with  
444 one, so, it's gonna be 2 and then 2, that's gonna be 4, and then three, that's gonna be 8, so- And so  
445 on and so forth. Okay. So, 2, 4, 8, all those. Clearly, 2, 4, 8, all the way up to 2 to the  $n$ th power  
446 is gonna be- All those numbers are gonna be also included in  $R$ . [Okay.] So, I mean th- you're  
447 not going to get something after doing this that's gonna be not in the real numbers. You're not  
448 going to get some crazy fraction or some crazy, uh, coefficient like that. [Okay.] Um, so that  
449 kinda li- That checks off right there. Uh. Now we need to show that there's- I think that there's  
450 closure, also. That's a thing we need to prove. Cause, a lot of times, when you have a subgroup,  
451 um, when you cut of the certain elements off, then you'll sacrifice closure to do so. But. So. So,  
452  $H$  is just all these elements right here. What is the operation in  $H$ ?

453

454 Int: It's under, um- So,  $G$  is a group under regular multiplication. [Mhmm.] Real number  
455 multiplication.

456

457 Tucker: So, so is  $H$ , I take it?

458

459 Int: Um, well, yeah. You're asked to prove or disprove whether  $H$  is a subgroup.  
460  
461 Tucker: Okay. So, all these elements, I mean, the operation is multiplication? With  $G$ ? Just like  
462  $G$ ?  
463  
464 [00:35:37.295]  
465 Int: Yeah. If it were a subgroup, it would inherit that. [Okay.] As a group it would inherit that.  
466  
467 Tucker: Okay. Cool. So, yeah. So, um, you're not gonna get zero here, ever. So that kinda- Cause  
468 2 to the  $n$  is never gonna equal 0, cause even if  $n$  was 0, then it would still be 1. [Okay.] So, um,  
469 yeah. You're always gonna have closure and this goes on for infinity, so- Okay. So, first, there's,  
470 like, 2 times 4. That's gonna equal 8. 4 times 8 is kinda like saying well 2 times 2 times- or, yeah,  
471 2 times 2 times- (quickly) 2 times 2 times 2. [Mhmm.] That's always gonna give you another 2 to  
472 the  $n$ th power. [Okay.] So, it's kinda like you rewrite 4 as 2 squared. And, I don't know why I  
473 wrote star there. [Int laughs] Just get used to- [Force of habit? Yeah.] Yeah. Dotted with, uh, 2  
474 cubed, so that's gonna equal 2 to the 5th [Okay.] right there. So, like, this is always gonna be in  
475 that group, cause 5, you know, is an  $n$ , so it's gonna have closure as well. [Okay.] So- Closure,  
476 check. Okay, um, do we need to prove associativity? Or is it inherited from  $G$ ? (clicks tongue;  
477 scoffs) What the heck. I'll prove associativity really quick. So, basically, um, if you wanna do  
478 this, we could do 2 dot- times, uh, 4 times 8. Is that equal to 2 times 4 times 8? And we can work  
479 this out, but clearly, it's going to be. So, 2 times 4 that's 8. 8 times 8 is 64 here. And 2 times 4 is 8  
480 times 8 is 64 here, so, that's going to work for every case as well, since, you know, it's the same  
481 basic principle. So, yes. Associativity? Yes. Okay. And, because  $G$  is a group, then all the  
482 elements- and we've already proved that it's a subset, that means that all the inverses are still  
483 gonna exist. Um, for each element. So, for instance, 2's inverse is gonna be  $1/2$ , so that would be  
484 2 to the, um, uh,  $1/4$  power. No. How do we- Okay. 2 to the, huh, I'm sorry. Minus 2 power.  
485 Yeah. (laughs) So, basically, (pauses 8 seconds) no, wait, wait, wait. Minus one. I don't know  
486 what I was thinking. I was trying to think (lowers pitch of voice) oh you've gotta get rid of the  
487 two. [Int laughs] Okay. Nevermind. Nevermind. So, yeah. 2's inverse is gonna be  $1/2$  so, that's  
488 gonna be (lowers volume of voice) 2 to the minus 1, which is also in  $Z$ , so- So, yes. And I didn't  
489 really do this in a good- I guess, proper, uh, structure, but (Int chuckles) yes.  $H$  is a subgroup of  
490  $G$ . (writes for 8 seconds).  
491  
492 [00:38:56.998]  
493 Int: So, I didn't quite follow the scratching out the [Mhmm.] 2 to the minus 2, so, could you talk  
494 me through that again?  
495  
496 Tucker: Yeah. Basically, um, what- I wanted just to, kind of, like, prove that each element in  $H$ -  
497 all these 2 to the powers- are basi- do- are these all still gonna have, um, inverses? That's what  
498 I'm trying to prove. And, basically- So, basically- is 2 to the  $n$  gonna have an inverse such that-  
499 yield you one? Cause one's gonna be the identity here. [Okay.] Cause well it's under  
500 multiplication. So, let's just call the inverse  $i$ . So, we've got to find something that equals that, so,  
501 basically, all we can really do is- well we can divide by 2 to the  $n$  and- st- to see what  $i$  is.  
502 [Mhmm.] And that's gonna equal 2 to the minus  $n$ . [Okay.] So-  $i$  is equal to 2 to the minus  $n$ . So,  
503 basically, whenever you have an element such, like, 2 to the 18, then its inverse is gonna be 2 to  
504 the minus 18. To find its inverse. And that's gonna exist for every element. And minus 18 is still

505 in the integers, so it's, you know, within the group. [Okay.] So,  $H$  is a subgroup of  $G$  because all  
506 these check out.

507  
508 Int: Alright, cool. "So, why did you approach, uh, this problem this way?"

509  
510 Tucker: As in, like, how I ran out the elements and stuff? [Mhmm.] Um, I- I guess it helps you to  
511 kinda like, think about what it's- it looks like, cause it's not really apparent at first to say, okay, 2  
512 to the  $n$ . Um, and it just kind of helps to write out, okay, so that's gonna be 2, 4, 8, and so on and  
513 so forth. So- And from there you can just kinda, like, look at those, like, and say, okay, well  
514 obviously that's inverse is  $1/2$ , that's  $1/4$ , that's  $1/8$ . And then you g- kinda just notice a pattern  
515 right there.  $1/2$  is 2 to the minus 1.  $1/4$  is 2 to the minus 2. [Okay.] Um,

516  
517 Int: Cool. [Yeah.] Alright. So, uh, "do you think that you work, uh, proves or disproves this  
518 statement?"

519  
520 Tucker: Uh, I think it proves. [Okay.] Under multiplication.

521  
522 Int: "So, on a scale of 1 to 10, how confident are you in this proof?"

523  
524 Tucker: That this is a subgroup? I'm not confident that this is the best- uh that this proof will pass  
525 as, like, with good structure and stuff like that. [Mhmm.] But, I- I'm confident that  $H$  is a  
526 subgroup. [Okay.] A hundred percent confident  $H$  would be a subgroup.

527  
528 [00:41:25.993]

529 Int: Alright. So, you're not confident-?

530  
531 Tucker: I'm not confident that my proof is up to standards. Like, it could have been written out  
532 with a better, uh, I guess, litera- literature structure, type thing. [Okay.] E- e-, but yeah. I'm- I'm  
533 almost positive that  $H$  is a subgroup of  $G$ . [Okay.] Unless I missed something really- (laughs and  
534 sits back in chair rubbing hair.) [Um-] I could have done better narrative structure, I guess, but-

535  
536 Int: (laughs) "Narrative structure"

537  
538 Tucker: (chuckles) I- I- I like- I like to go through proofs kind of like steps, like being, like,  
539 okay, well, this- this checks out, this checks out, this checks out. [Yeah.] Instead of saying  
540 (lowers pitch of voice) we need to show that this and that [Okay.] (brings voice to normal tone.  
541 Cause I don't like writing, to be honest. (both laugh) So-

542  
543 Int: Alright. So, "on a scale from 1 to 10, how much does this proof verify the relationship for  
544 you?"

545  
546 Tucker: Um, ten.

547  
548 Int: Yeah. You had said a hundred percent. I figured that'd be 10 out of 10. [Yeah. (Chuckles)]  
549 We're all mathematicians. (Tucker laughs) Um, so, "how did the way that you think about  
550 identity inform your approach to this proof?"

551  
552 Tucker: The way I think about identity? [Mhmm.] Um, well, this group that they're kinda talking  
553 about is under the operation of multiplication, so I had to think about okay, well, under this  
554 operation, in this group, what's gonna be my identity? And it's gonna be one in this case. And  
555 then later on, when we're inverses for each element, that kind of comes into play, like, What do  
556 you have to get you to one? What do you need to do to each operation? So- kind of have to  
557 realize, like, if you didn't know what one was, then you wouldn't be able to ever find the identity.  
558 [Okay.] If you thought that it was zero, then I would say, you wouldn't be able to because, you  
559 know, the identity would just be zero- the- I'm sorry, the inverse would be zero [Okay.] and then  
560 it'd just be-

561  
562 Int: Did you mean that before, too? Inverse here? (points to top-left of student page)

563  
564 Tucker: Yeah. (laughs) Sorry.

565  
566 Int: No, it's fine. Um, so then- so then- I guess a similar question. "How did the way you think  
567 about inverse inform your approach to this proof?"

568  
569 Tucker: Well, the way I- the way we're thinking about it here is kinda like to the, I guess, the  
570 reciprocal of these, because, again, we're under multiplication, so it's kinda gonna be, um, we  
571 gotta think about more like fractions type thing. [Okay.] So. (pause) Yeah, so. 2 and 1/2. 4 and  
572 1/4. [Alright, cool.] That would be just, like, negative powers, I guess.

573  
574 Int: Alright. Anything else? [Shakes head.] You good? I'm good. Moving along. (Tucker  
575 chuckles)

576  
577 [00:43:58.417]

578 Tucker: Okay. So, now we have to prove that, for each element  $g$  in a group  $G$  with operation  
579 star,  $g$  inverse, the inverse of  $g$  is unique. Doesn't really tell you much about it, does it? [I'm  
580 sorry?] That doesn't really tell you much about it, does it? Um. [About?] The group. [Oh. Okay.]  
581 So, basically, we're just trying to prove that inverses are unique in a group. So, basically, the  
582 biggest fact to us is going to be that in a group  $G$ . So, as soon as we know that there's- they tell  
583 us that it's in a group  $G$  we know ah- we know st- we know things about this group  $G$ . We know  
584 that, for each element, um, there's gonna be an inverse for that. That's gonna be- It's gonna have  
585 closure. It's gonna have all that good stuff. So, using that information, we can try to prove that  
586 the inverses for those elements are going to be unique. [Okay.] Um, (3 seconds) so proving that-  
587 there's only- there's only ever going to be one  $g$  to the minus 1. There's not gonna be another way  
588 to get you back to the- to the identity. Like, there's only one way to skin a cat in this case.  
589 (laughs) [Okay.] In this unique case! (laughs) Okay. So, if I'm going to prove this, I'd say that.. I  
590 guess we'll just use their element  $g$ . (begins writing) So, let  $g$  be in  $G$ . Um. Uh, we can- I- we can  
591 use a proof by contradiction here. So, we could say, uh, suppose that there is two different  
592 inverses for this element  $g$ , for instance, like,  $g$  inver-  $g$  to the minus one and then let's say  $h$  to  
593 the minus one. And then, if that was the case, then, uh, then they have to be the same thing.  
594 [Okay.] That's- that's what we could prove right here. So, let  $g$  be an element in  $G$ . (continues  
595 writing) So, suppose that  $g$  star  $g$  inverse is equal to- uh, we're gonna call the identity  $e$ . [Okay.]  
596  $e$  where  $e$  is the identity. And  $g$  star  $h$  inverse is equal to  $e$ . So, basically, then we could set these

597 two sides equal to each other, since they're both equal to the identity. [Okay.] So,  $g$  star  $g$  inverse  
598 is equal to  $g$  star  $h$  inverse. And, now, here we can pretty much, like, use the ca- cancelation law  
599 [Mhmm.] to kinda prove that, okay, um, these are the same on this side, so, you can pretty much  
600 go like that and say  $g$  inverse is equal to  $h$  inverse using the cancellation law. Which, I think in  
601 class she said we can just write that instead of going through the, like, process, but-

602  
603 [00:47:13.062]

604 Int: Well, what would that process be like?

605  
606 Tucker: Well, you'd basically, um, you try to get rid of  $g$ , so you'd apply the inverse to that.  
607 (breathy laugh) [Okay.] On both sides, so, basically, alright, well,  $g$  inverse times that. So that's  
608 gonna get rid of it. Gives you  $e$ , the identity. And the identity times any element is gonna get you  
609 th- that element back towards you. And you do it for that side; do it for this side. And you'll end  
610 up with this stuff (traces finger up and down) being cancelled out. [Oh, okay.] So, and leaving  
611 you just these two but- (taps finger back and forth left and right on page) Thin- I think she said  
612 we- we're allowed to do this now without getting points taken off, so- [Okay. I won't take off  
613 points either.] Thank you. (both laugh) So, then we have  $g$  inverse equals  $h$  inverse. Thus, there  
614 is only one unique inverse for any arbitrary element  $g$  in  $G$ .

615  
616 Int: Alright great. Four minutes exactly. [Nice!] Quick work. (both laugh) Okay. So, um- So,  
617 "Why did you approach the proof this way?"

618  
619 Tucker: Um, I guess it's- It's not really (inaud) a proof by contradiction, so much as it is just kind  
620 of showing that, okay, let's- let's say- let's say it's not unique type of thing. So, let's say that, um-  
621 it's kind of a proof by contradiction, but you're not saying (lowers pitch of voice) well, this can't  
622 equal. Um, so, basically, we're just saying, well let's suppose that it wasn't unique. Let's suppose  
623 that there's two elements that will give you the identity. And the way I approached this is- I  
624 think, makes more sense to me, is saying, okay, well if that's true, then they have to be equal. It  
625 just kind of makes so much just logical sense type thing. It'd be a lot harder to prove that nothing  
626 else out there will give you that, you know, type thing. [Mhmm.] Cause then you have to try  
627 every other element type thing, [Mhmm.] you know? [Yeah.] Whereas this, you only have to try  
628 **two** ones and say that they both equal to each other and say, okay, then these must be the same.  
629 [Mhmm.] Rather than just every other single one, so-

630  
631 [00:49:19.695]

632 Int: Okay. Alright. Cool. Um, "So do you think that your work proves the statement?"

633  
634 Tucker: I do. Um, yes.

635  
636 Int: Okay. "On a scale from one to ten, how confident are you in the proof?"

637  
638 Tucker: Ten. [Ten?] Yes.

639  
640 Int: And, "On a scale from one to ten, how much does this proof verify the relationship for you?"

641  
642 Tucker: Ten.

643  
644 Int: Okay. Alright. “So, how did the way you think about identity inform your approach to this  
645 proof?”  
646  
647 Tucker: Um, well, I guess the  $i$ - the identity is crucial because we're talking about inverses and  
648 you can't talk about inverses without talking about identity. Um, we started out by saying that  $g$   
649 star  $g$  inverse is going to equal the identity as does  $g$  star  $h$  inverse [Mm.] is equal to the identity.  
650 So, this is kinda like our bridge to set these two equal to each other. [Okay.] That's what it means  
651 to do that, so- In that case it's- you can't, like- you can't not use the identity in this proof, in that  
652 case.  
653  
654 Int: Okay. Cool.  
655  
656 Tucker: And also the cancellation law, you'll be using the identity and stuff like that. [How's  
657 that?] Well, um, basically how I talked about. We need to get rid of these  $g$ 's on both sides and  
658 it's- you can't just really, unless you do the cancellation, like, you can't just take that out like I  
659 did. You have to say, well that means times-ing my its inverse  $j$ -  $g$  inverse will yield you  $e$  the  
660 identity. [Mhmm.] And the property of the identity is such that- means that  $e$  star  $g$  inverse is  
661 gonna equal  $g$  inverse, so- You'll be using using the properties of the identity in the cancellation  
662 law, then, that way.  
663  
664 Int: Um, “Similarly, how did the way you think about inverse inform your approach to the  
665 proof?”  
666  
667 Tucker: The way I think about inverse is- (3 seconds) I guess (breathy laugh) I think of them as  
668 unique in a group, [Mhmm.] so I go in knowing that, okay, they must be unique. Um, I guess just  
669 thinking about them as the opposite or just, like, the way back  $t$ - whatever gets you the identity  
670 and just kind of proving that, well, there's only one way to do that, I guess. Is what we're- what  
671 we're proving.  
672  
673 [00:51:32.590]  
674 Int: Okay. Only one way to skin a cat, huh?  
675  
676 Tucker: Yeah. In this case. (both chuckle)  
677  
678 Int: Alright. Um. Earlier in the interview, when you- when we first started talking about inverses  
679 you wrote, um - I think it's right here - you wrote  $a$  star minus  $a$  equals minus  $a$  star  $a$  equals  $e$ .  
680  
681 Tucker: Mhmm. And I didn't do it here? [Yeah.] Uh- I- Uh- (breathy laugh) I guess I should  
682 have, but I just kind of wanted to save space and time. [Oh, okay.] But, yeah, I should have  
683 wroten  $g$  star  $g$  inverse is equal to  $g$  star  $g$  is equal to  $e$  the identity.  
684  
685 Int: One more time?  
686  
687 Tucker: I should have actually wroten that- well, I'll do it right here (writes for three seconds) I  
688 should have wrote it more like that. [Okay.] Just to be clear. To show that they're commutative.

689 Otherwise,  $g$  s- if this was the only the case for this direction, type thing. If this was only the case  
690 for that direction - so,  $g$  first and then  $g$  inverse - [Uh huh.] then this wouldn't actually truly be  
691 the i- the inverse for  $g$ .

692  
693 Int: Okay.

694  
695 Tucker: So it has to be, like, both ways, I think.

696  
697 Int: Okay. Would this be the only thing you'd need to write, then, er- (pointing at top-left of  
698 paper)

699  
700 Tucker: Yes. [Okay.] And same down here. Right here. (pointing near center)  $g$  star  $h$  inverse is  
701 equal to  $h$  sta-  $h$  inverse star  $g$  is equal to  $e$ . [Okay.] Just proves that, um, those two are  
702 commutative, or work both ways.

703  
704 Int: Alright. Also, um, so, this is kind of a funny question, but, if- if I hadn't asked you to- to  
705 explain why it's okay or why you use the cancellation law here, [Mhmm.] would you still have  
706 said that you thought about the identity right here? That's a funny question, but (inaud, Tucker  
707 begins responding)

708  
709 Tucker: (breaths deeply) I guess, if I wasn't- if I just kind of, like, wasn't thinking of it, wasn't  
710 explaining it, I probably wouldn't have thought about it, cause I just would be looking right here.  
711 [Uh huh.] Like, oh! Just cancel that. But I didn't realize that the identity is part of the whole  
712 process for cancelling that. [Okay.] I didn't realize it in that way. [Okay.] But. If- i- I mean the  
713 cancellation law is- it's really just gonna save you time, cause it's super tedious to write that out  
714 [Yeah.] in more lines [Yeah.] but- though, the identity would come into play for that.

715  
716 [00:54:01.187]

717 Int: I guess it just saves you from needing to think about the identity [Exactly.] on the way.  
718 [Yeah.] Okay. Alright, cool! Alright. You're doing- You're doing awesome. Trucking along.  
719 We've got six minutes and an hour. I don't know if you need to be somewhere or- Um, or what.

720  
721 Tucker: Uh, I- I'm fine until, yeah. I don't need to be going anywhere.

722  
723 Int: Okay. I don't want to, like, hold you up or anything. Cause this question might run a little  
724 [Okay.] longer than those. [Is this the last one?] No. There's one more after that, but, um, you  
725 know, if we don't get to it, we do get to it.

726  
727 [00:54:34.050]

728 Tucker: Okay. So, "prove or disprove the following: for a group  $G$  under operation star and a  
729 fixed element  $h$  in  $G$ , the set  $H$  with  $g$  in  $G$  okay, so  $g$  star  $h$  star  $g$  inverse is equal to  $h$  is a  
730 subgroup of  $G$ ." Okay. So, we're trying to prove that  $H$  is a subgroup of  $G$ . So,  $h$  is a fixed  
731 element in  $G$ .  $h$  is not changing. The only thing that's changing is, I guess the  $g$ 's? (question mark  
732 because inflection in voice shifted, raising pitch) Okay, so. Hm. (8 seconds, then clicks tongue)  
733 Interesting. (10 seconds) So,  $h$  is an element in  $G$ . The set  $H$  is equal to- (whispering by the end;  
734 6 seconds) So, I guess we just kind of like, prove it one step at a time, type of thing. Um.



735 ([00:55:48.325] begins writing) Is  $H$  a subset of  $G$ ? So, is everything in  $H$  also going to be in  $G$ ?  
736 Is kind of wh- I guess we can prove first and foremost. So,  $H$  is just  $g$  in  $G$  such that  $g$  star  $h$  star  
737  $g$  inverse is equal to  $h$ . (3 seconds) Hm. Interesting. What's that gonna- I'm trying to fish out  
738 what this is gonna look like, you know? [Mhmm.] As like- If- With real numbers and stuff.  
739 (scoffs; 12 seconds) Well- (5 seconds) If  $h$  is an element of  $G$ , that means that  $g$  star  $h$  star  $g$   
740 inverse or- is equal to  $h$ . That means that these two  $g$ 's kinda have to be in  $G$  as well. Probably if-  
741 Hmm. And, wait a minute. It says that  $g$  is in  $G$ , right? [Int scoffs] Okay. Yeah. (laughs) I don't  
742 know why that was taking forever. Okay. So, yeah.  $H$  is a subset of  $G$ , cause  $H$  is just all these  
743 elements  $g$ . That's what it- it's gonna be- and it tells us right here that  $g$  is in  $G$ . [Okay.  
744 [00:57:13.713]] That's a lot of  $g$ 's, sorry. [Int laughs; Golly  $g$ .] (Tucker, repeating) Golly  $g$ . (both  
745 chuckle) So, yes. [Okay.] Um- [And why is that?] Cause we said  $H$  is equal to  $g$  in  $G$ .  $H$  is just  
746 all these elements  $g$  [Mhmm.] and it just tells us right here,  $g$  is an element of capital  $G$ . [Okay.]  
747 So, we don't actually have to actually, like, prove anything here. We can just take it from the-  
748 what they give us. As far as that is concerned. [Okay. Cool.] I was trying to think about it right  
749 here, like, alright (lowers voice) would it need to always need to be still in that? [Int laughs]  
750 (inaud) like, wait it just tells it right there. (chuckles) Okay, so, yeah.  $H$  is gonna be a subset of  $G$   
751 because it's consists of all the elements  $g$  in  $G$ .

752  
753 Int: Okay.

754  
755 [00:57:58.402]  
756 Tucker: Okay, so. Next we gotta prove that, um- (clicks tongue for 3 seconds; lowers voice)  
757 We've gotta prove that it's closure. Okay. (clicks tongue quickly for 1 second) Um, to prove that  
758 there's closure- (3 seconds) Mm. Let's see how are- How are we gonna do this? So, we gotta  
759 prove that- (2 seconds) we're gonna just need to prove that  $g$  star  $g$  is in  $G$  and  $g$  star  $g$  inverse is  
760 in  $G$ . (inflection raises at end, almost in an inquisitive way.) Hm. (4 seconds) So, we're just-  
761 We're trying to prove that- All these numbers  $g$  cause there could be- there's obviously gonna be-  
762 it could be infinite many numbers of this (inaud) that will result in this being true. [Mhmm.] Um,  
763 we have to prove that all of those star themselves will yield you back another one. [Okay.] So,  
764 I've gotta think of a way to do this, but, um, I'm trying to think of a way. Okay, so. (3 seconds) I  
765 could say, let  $g$ - let  $a$  be an element of  $H$ . Like, I'm- I'm gonna try it this way just try it out.  $a$  be  
766 an element of  $H$  and  $b$  be an element of  $H$ , then by definition of the set,  $a$  star  $h$  star  $a$  inverse is  
767 equal to  $h$  and  $b$  star  $h$  star  $b$  inverse is equal to  $h$ . And it looks like we can just, like, set these  
768 equal to each other. So,  $a$  star  $h$  star  $a$  inverse is equal to  $b$  star  $h$  star  $b$  inverse. See, what we're  
769 trying to do here is prove that  $a$  star  $b$  is also in  $H$  to prove that there's closure. That's what  
770 you've got to do to prove closure. So- In order to prove- I'm just gonna write this out. (continues  
771 writing; int chuckles) Trying to show that  $a$  star  $b$  is an element  $H$ , which means  $a$  star  $b$   
772 (mumbles) - I'm just gonna need to write star between that - star  $h$  star  $a$  star  $b$  inverse is equal to  
773  $h$ . So that's what we gotta, um, we need to prove. Um- (clicks tongue; chuckles) Okay. (four  
774 seconds) I think it would be easiest to rearrange (points to paper, center-middle) this so that we  
775 get that (taps same point on paper). Hm. (8 seconds) That's a good question. I don't know how  
776 we're supposed to do that, though.

777  
778 Int: What're you thinking about doing?

779  
780 [01:01:47.503]

781 Tucker: Um, I was thinking about some- somehow rearranging the two sides that we set equal to  
782 each other. [Mhmm.] Using that to lead you to here. [Okay.] By, like, okay we'll add this to that  
783 side and add that to that side, but I don't know. (3 seconds) Kinda like get  $h$  on one side and then-  
784 gettin' there, but I'm not sure how we're supposed to do that. (2 seconds; glances around room)  
785 Hm. (5 seconds) This is a little trickier. I'm open to suggestions here. (both laugh) I know this is  
786 what I want to show in order to prove closure. (Okay.) I know this is- how I'm trying to show it  
787 somehow, but, to be honest, I'm not sure how I'm supposed to do that.

788  
789 Int: Um, Okay. Well. Let's put a- Let's bookmark this. [Mhmm.] What else- So, you want to  
790 show closure. You- closure. You want to show this thing. What else would you want to do to - to  
791 try to pro- Are y- It seems like you're trying to prove this [Right] instead of disprove it. Um.  
792 (Tucker laughs nervously; Int joins in laughter) [I guess. Yeah.] Another twist, huh? [Yeah.] Um,  
793 so what else would you try to show to see if  $H$  is a subgroup of  $G$ ?

794  
795 [01:03:15.605]  
796 Tucker: I'll also show that every element in  $H$  has an- a unique inverse is what I want also show.  
797 [Okay.] (5 seconds) So, I guess- What would the inverses for these elements would look  
798 something like this. (begins writing near bottom of page) So, you're saying let  $a$  be in- oops - let  $a$   
799 be in  $H$ . So,  $a$  star  $h$  star  $a$  inverse is equal to  $h$ . Suppose  $a$  inverse is in  $H$  such that  $a$  star  $a$   
800 inverse is equal to  $a$  inverse star  $a$  is equal to the identity. Hm. Now- Now comes the point of  
801 realizing, okay, well, what's the identity in this. (5 seconds) Hm. (9 seconds) If it exists. If the  
802 ide- identity exists. Um, hm. (8 seconds) It seems to me like the identity might be something like  
803 one. Cause we have- okay, one'll work. (4 seconds) And, when you apply any other number that  
804 will work, like, let's- if you have, like- I guess any real number that would work. Five and five to  
805 the negative one, so one-fifth. That would work as well, cause it would be five  $h$  divided by five  
806 is equal to  $h$ . And then five star one would be equal to five, so- I feel like one would be the  
807 identity in this case. Cause it- this- the def- kinda like how they- what they show us is the, um,  
808 the kind of trying to prove to us that, uh, it's very com- it looks like multipl- multiplicative in  
809 nature, because of the existence of the negative one power [Mhmm.] right there. So, it seems like  
810 it's almost multiplica- multiplicative in nature, so- which leads you to think the identity is going  
811 to be equal to one. (3 seconds) Ah- yeah. In that case- And I guess inverses do have to kinda  
812 exist in this ca- Just by- Just by looking at this. (points to top-left of page) I mean, it's- it's  
813 showing that, okay,  $h$  is equal to all the  $g$  in  $G$  such that  $g$  star  $h$  star  $g$  inverse is equal to  $h$ . So it  
814 just kind of tells us right there that  $g$  inverse exists. AND, okay. Also another way to do that is,  $G$   
815 is a group. Right? So, that means that, if  $G$  is a group, that means every element in  $G$  has an  
816 inverse. [Mhmm.] And, um,  $H$  just consists of the l- lower-case  $g$ , which is in  $G$ , which means  
817 that all those you would have an inverse just because  $g$  is in part of the group. [Mhmm.] So, I  
818 guess we really don't even need to actually work it out, except we just use the definition they  
819 give us. [Okay.] And just by the fact that they write it out right there,  $g$  to the negative one.  
820 Would that be enough to prove that inverses exist? For every element? (9 seconds; glances  
821 repeatedly from paper to interviewer) Or could you not just say that, just because they write out  $g$   
822 to the minus one. (7 seconds) I think it would work. You understand what I'm trying to say?

823  
824 [01:07:40.803]  
825 Int: I think so. Um, I'm a little confused, though. (4 seconds) So, yeah. So,  $g$ 's an element of the  
826 group  $G$ . [Mhmm.] Alright. [Because. Yeah.] Because  $G$ 's a group,  $g$  inverse exists [Mhmm.] in

827 *G*. [Right.] Right? [Mhmm.] Um, so when you say- when you say “inverses” exist here,  
828 [Mhmm.] what does that mean in the context of this whole prob- of this problem? Like- What  
829 does that mean to you? That inverses exist?  
830  
831 Tucker: (8 seconds) I guess it means that- (4 seconds; scoffs) It means that it's a group, I guess.  
832  
833 Int: What's the “it”?  
834  
835 [01:08:37.692]  
836 Tucker: *H*. *H* is a group. [Okay.] Cause *H* is just all the elements *g* in *G*. And we know that,  
837 okay, if *G* is a group, then, uh, inverses exist for each element. [Okay.] So, you can kinda almost,  
838 like, um, just take that kinda inherited from the group *G*- to say that-  
839  
840 Int: So, real quick, *H* is all the elements *g* in *G* [Mhmm.] such that  $g \star h$  [Right.]  $\star g$  inverse  
841 equals  $h$ .  
842  
843 [01:09:08.478]  
844 Tucker: Right. So, that's almost, like, limiting some of them, right? [Okay.] It's just- It's only the  
845 elements in *G* that- that work like this, type thing. [Okay.] So, if *H* is only those elements that  
846 work, but *G* me- has *every* element, not just the ones that work in this special case, that means,  
847 um- It's kinda like say that- Okay. Trying to think about how to say this. (4 seconds) You know,  
848 *G* wor- *G*'s a group, so that means, *every* element. And then you have *H*, which is just, like, only  
849 some of those elements, [Okay.] but it still means that every element has an inverse. [Okay.] And  
850 you're just taking a few of those, which means they still have inverses. You're not taking  
851 anything *larger* than *G*, you know what I mean? [Yeah.] So-  
852  
853 Int: Okay, so, *a*- you said *a* is in *H*. [Mhmm.] Right? So, let *a* in *H*- Let *a* be an element of *H*.  
854 [Mhmm]  $a \star a$  inverse equals  $h$ . [Mhmm.] You said so  $a \star a$  inverse equals  $h$ . And now you're  
855 saying *a* inverse is in *H*. [Right.] A- What's going on there? I-  
856  
857 Tucker: So, basically, that would- If *a* inverse is in *H*, that means that *a* inverse  $\star a$  - inverse of  
858 an inverse being *a* - is also equal to  $h$ .  
859  
860 Int: What was that last part? Befor-  
861  
862 Tucker: The inverse of an inverse? Or no?  
863  
864 Int: Yeah. That.  
865  
866 [01:10:40.117]  
867 Tucker: Um, so if we're saying that *a* inverse is an element of *H*, that means that this must work.  
868 So, we put then (“that in”) there. Okay.  $a$  inverse  $\star a$  inverse *inverse*  $h$ . So, we'll write that like  
869 this, I guess. Bracket inverse.  
870  
871 Int: Oh, o- OKAY. So, the inverse of- *of* the inverse of *a*.  
872

873 Tucker: Mhmm. [Okay.] Right, right. Inverse twice.  
874  
875 Int: But you had it written as  $a$ ? [Yeah.] And, are those two things the same, or-?  
876  
877 Tucker: Yes. [Okay.] An inverse of an inverse is just gonna be the element.  
878  
879 Int: Okay. Alright, so, uh- Sorry. I d- [Mhmm.] You can write it how you had it again, if you  
880 want. [Okay. That's about right.] Um- Okay, so, what now?  
881  
882 Tucker: (chuckles) We're tr- Now- now we prove that if this is the inverse, that means that  $a$   
883 inverse star  $a$  is equal to the identity. (10 seconds) Hm. (6 seconds) Correct. I believe. (writes for  
884 14 seconds, bottom of page) So, if  $a$  inverse is the inverse of  $a$ , then  $a$  star  $a$  inverse is equal to  $a$   
885 inverse star  $a$  is equal to  $e$ , the identity. (2 seconds) Right?  
886  
887 Int: What happened to one? [Um-] And where did  $e$  come from?  
888  
889 [01:12:27.913]  
890 Tucker: I'm just calling it,  $e$  is the identity. but, um, I'm trying to figure out what  $e$  would equal  
891 in this case, and m- that's where I'm kind of stuck. I kept thinking, well, what is the identity. I'm  
892 thi- I was thinking before that it would just be one, because it's kind of sounds like it's  
893 multiplication. [Mhmm.] Cause that would work for all, like, the multiplication parts. Like, five  
894 times anything times one-fifth will give you- that number back. Understand?  
895  
896 Int: I think so. Where's  $e$ ?  
897  
898 Tucker: What do you mean "where is  $e$ ?"  
899  
900 Int: Like, what i- where did this come from?  
901  
902 Tucker: Oh, I'm just using that as the identity.  
903  
904 Int: Mokay. (Tucker writes for five seconds) the identity?  
905  
906 Tucker: In  $H$ .  
907  
908 Int: In  $H$ ?  
909  
910 Tucker: Yeah.  
911  
912 Int: Okay. What is the identity in  $H$ ?  
913  
914 [01:13:30.240]  
915 Tucker: Well, that's the thing. If it- If it doesn't exist that means  $H$  can't be a subgroup, but. I'm  
916 thinking that the identity is gonna be equal to one.  
917  
918 Int: So,  $H$  is a set of elements in  $G$ . [Mhmm.] Right? [Yeah.] These- these elements in  $G$  satisfy

919 [Mhmm.] that equation. [Mhmm.] Right? So, it's just a bunch of stuff from  $G$ . [Mhmm.] And,  
920 under the same operation star, [Yeah.] since  $H$  is a subset and we're asking whether it's a  
921 subgroup, we use the same operation. [Mhmm.] So, then- so what's an identity in  $H$ ?

922  
923 Tucker: It's also gonna be the identity in  $G$ . I guess.

924  
925 Int: Why do you say that?

926  
927 Tucker: Cause is a subset of  $G$ .

928  
929 Int: You think that's true?

930  
931 Tucker: Mhmm. You're not gonna change the identity if just- by, like, making the group smaller  
932 type thing.

933  
934 Int: Okay. Is the identity of  $G$  in  $H$ ?

935  
936 [01:14:47.663]

937 Tucker: I feel like- I feel like it'd have to be. If  $G$  is a group. So,  $G$  is a group, correct? (Int nods)  
938 So, obviously, we know that  $G$  has an identity. [Okay.] um, and we've already proved that  $H$  is a  
939 subset, so- In order for  $H$  to be a subgroup,  $H$ - one of the things it would need is to have that  
940 identity, that's also in  $G$  to be in  $H$ . [Oh, okay.] That's one of the things that needs to be proven, I  
941 guess, so- let's just say if that- for the identity for- in group  $G$ - [Alright, I was gonna give you  
942 more. (hands new sheet of paper to Tucker.)] Yeah. So, if  $e$  is the identity in group  $G$ , then  $e$   
943 must be the identity in  $H$  for  $H$  to be a subgroup.

944  
945 Int: Okay.

946  
947 Tucker: I'm just trying to- I'm just wondering *if* the identity *is* in  $H$  or not.

948  
949 Int: Oh! Okay.

950  
951 Tucker: I wasn't quite sure, which is why I didn't want to call it one or anything like that.

952  
953 Int: Oh, okay.

954  
955 Tucker: So, I decided to call it  $e$ .

956  
957 Int: Okay. That makes sense. alright.

958  
959 Tucker: W- I still don't know if it exists or not, but-

960  
961 Int: How would you find that out?

962  
963 [01:15:55.903]

964 Tucker: I don't know. (laughs) I'm trying to think, because- I try to, like, like, think of examples

965 of what it *might* be. And that's why I came up with one. Like, it might be one, because, one is the  
966 identity under multiplication. [Mhmm.] And this *seems* like it's multiplication in its nature.  
967 [Okay.] With, like, the negative one powers and stuff. [Oh, okay.] So, it *seems* like it's  
968 multiplication in nature, which means, like, it *might* be one, but I don't know. How am I exactly  
969 gonna prove that. (begins writing; 9 seconds) So.  
970  
971 Int: So, if  $e$  in  $G$  - that's the identity of  $G$  - [Mhmm.] then  $e$  needs to be in  $H$ , for  $H$  to-  
972  
973 Tucker: Mhmm. If  $e$  is the identity in  $G$ , then  $e$  must be the identity in  $H$  for  $H$  to be a subgroup.  
974  
975 Int: And you don't know whether it's in  $H$ ?  
976  
977 Tucker: (13 seconds) Mm. I'm with you just- can we give an examp- like, let's say that  $e$  is equal  
978 to one.  
979  
980 Int: Okay. I don't know that one's in  $G$ .  
981  
982 Tucker: Right. That's what I'm saying, like, you don't know if one is in  $G$ , so it's like-  
983  
984 Int: I know  $e$  is in  $G$ , cause you just (points to paper) said it, right?  
985  
986 Tucker: (chuckles) Yeah.  
987  
988 Int: Anyway. Okay, so suppose it's one. What would you do? Like, how would you figure if one  
989 was the identity?  
990  
991 Tucker: I'd say- I guess I'd kinda come to a, um, a block in the road. If I prove it's the identity  
992 here, we have no way to prove it's the identity in-  
993  
994 Int: How would you prove it's the identity there?  
995  
996 [01:17:37.202]  
997 Tucker: Well, I'd just say that, okay well, one star  $h$  star one - cause one to the negative one is  
998 one - is equal to  $h$  and s- I guess you don't know what one star  $h$  does, though. [Right.] I just- I  
999 (chuckles; inaud) in my head that- it just would work, with just, like, normal numbers, but it  
1000 prob- it might not actually.  
1001  
1002 Int: Okay. [So-] Well what about  $e$ ?  
1003  
1004 Tucker: Well, we know what- what  $e$  would do.  $e$  would just take every element and do nothing  
1005 to it, pretty much, in  $G$ , [Okay.] and it would do the same in  $H$  if it was the identity. So- (12  
1006 seconds) Hm. (11 seconds) Yeah. I'm not sure what you would do.  
1007  
1008 Int: Well, what if you tried to do with the  $e$  what you tried to do with the one?  
1009  
1010 Tucker: What I tried to  $e$  with the one? So, just a s- Where  $a$  is an element of  $H$  by the way.

1011 [Okay.] So- (writes for 27 seconds) So that's what it'd have to behave like for that to be the  
1012 inverse.  
1013  
1014 Int: Is that what you- Wait. What-? How did we get here? I'm confused. I'm sorry. (chuckles)  
1015 We're running long. [Yeah.] I don't know- I don't know if you wanna keep going or-  
1016  
1017 Tucker: Um, I've kind of hit a road block, though, unfortunately.  
1018  
1019 Int: Okay. Alright, well- you- with the- I'll ask you this: [Okay.] so, with the one, you said one  
1020 star  $h$  star one-  
1021  
1022 Tucker: Is equal to  $h$ . [Okay.] But, I was thinking that one would behave like multiplication,  
1023 which star doesn't mean multiplication, though.  
1024  
1025 Int: Yeah. [So-] And, w- uh- one to the negative one isn't always one either, like with addition.  
1026 Like, if the- if that's just the inverse of one-  
1027  
1028 Tucker: Oh. If- This sign doesn't mean inverse, it could mean-  
1029  
1030 Int: Oh, like reciprocal?  
1031  
1032 Tucker: Yeah. Re- I was thinking this sign meant reciprocal, not inverse.  
1033  
1034 Int: Oh. Okay. Alright. Well, what if, instead of- instead of trying one, since we don't know what  
1035 one to  $h$ . [Mhmm. Right.] Right? We know what  $e$  does to  $h$ . You said it just sum- like, what did  
1036 you say.  
1037  
1038 Tucker: What  $e$  does to  $h$ ?  
1039  
1040 Int: Yeah, cause  $h$  is an element of  $g$ , right?  
1041  
1042 Tucker: Right, so-  
1043  
1044 Int: So, what if we put  $e$  in here for  $g$ ?  
1045  
1046 [01:20:35.430]  
1047 Tucker: If we put  $e$  in there for  $g$ . Uh, well,  $e$  times-  $e$  star  $h$  would equal  $h$  star  $e$  -  $e$ 's inverse,  
1048 which is itself [Mhmm.] - would just equal  $h$ , so  $h$  is equal to  $h$ .  
1049  
1050 Int: Okay. [Okay.] So-  
1051  
1052 Tucker: So,  $e$  would work.  
1053  
1054 Int: Is- (pointing to top of paper) Is  $e$  an element of  $G$  such that  $e$  star  $h$  star  $e$  inverse equals  $h$ ?  
1055  
1056 Tucker: Yes. So,  $e$  would be an element s- of  $H$ . Cause it would work. So, we know what- Okay.

1057 So, that's -that's how you prove that it was in  $H$ ? Is just by, like, looking- Cause we know what it  
1058 does to little  $h$ ?  
1059  
1060 Int: Well, we know what things in  $H$  look like- things in  $H$  satisfy this.  
1061  
1062 Tucker: Right. And  $e$  would satisfy that, so- [So. Yeah.]  $e$  must be in  $H$ .  
1063  
1064 Int: Yeah, so to see if  $e$  is in  $H$ , does it satisfy this?  
1065  
1066 Tucker: And it does. Okay. [Yeah?] So once you know that, okay,  $e$  is also the identity in  $H$ , you  
1067 can then prove that there are inverses in  $H$  or- That's what I'm saying. That's what I was trying to  
1068 say with, like, inverses, since you know that the inverses are over here, [Mhmm.] That means  
1069 that they must exist, cause they write them down, you know? Or would that not be enough?  
1070  
1071 Int: Well, this is- this is all in  $G$ , right?  
1072  
1073 Tucker: Right. The inverses satisfy that in  $G$ , though, don't they?  
1074  
1075 Int: (inaud)  
1076  
1077 [01:22:00.303]  
1078 Tucker: The inverses are in  $G$ , so they satisfy this. (points top-middle of paper; 5 seconds) Okay.  
1079 I- you know what I might do actually? I might, um, you'd have this right here and I would say,  
1080 okay, apply  $g$  to that and if  $y$ - and then see if you get the other way around. Apply  $g$  to  $h$ . I'd  
1081 have  $g$  inverse star  $h$ . And then I would apply the, uh- I'm sorry.  
1082  
1083 Int: Can- Can you write it down here? [Yeah. Yeah.] That way I can see it.  
1084  
1085 Tucker: So, right now, we have  $g$  star  $h$  star  $g$  inverse is equal to  $h$ . We want to get to somewhere  
1086 that looks like- we want to get to somewhere that looks like- (mumbles) Want to show.  $g$  inverse  
1087 star  $h$  star  $g$  is equal to  $h$ . (aloud) In order for the inverse of  $g$  to satisfy this (points to top-middle  
1088 of paper) right here. [Okay.] Cause that's what you do when you put in the  $g$  inverse. [Okay.] So,  
1089 we wanna get to there. and I guess you can m- you can kind of like do things to try and get that  
1090 right side to look like that. [Okay] So, what I would do is I would say, okay, well let's- let's apply  
1091 the  $g$  inverse to that. So, applying  $g$  inverse to both sides would give you  $h$  star  $g$  inverse is equal  
1092 to  $g$  inverse star  $h$ . And then on the- and then next, you just apply  $g$  to that side. Um, you know,  
1093 the right side of both these. [Mhmm.] So, uh, if you apply  $g$  to the right side of both of these, you  
1094 are just left with  $h$  is equal to  $g$  inverse star  $h$  star  $g$ . Which is what we got right here. (draws  
1095 arrow to right side of page, about 1/3 from top) [Okay.] Meaning that the inverses for each  
1096 element in  $G$  which satisfy that, mean that must be in  $H$ . [Okay.] Does that make sense?  
1097  
1098 Int: Yeah. So, you started with  $g$  being in big  $H$ , [Mhmm.] right? And you wanted to show that  $g$   
1099 inverse-  
1100  
1101 Tucker: (finishing Int's sentence) -verse is in big  $H$ .  
1102



1103 Int: (along with, but just behind Tucker) in big  $H$ . And you did that?  
1104  
1105 [01:24:01.062]  
1106 Tucker: Right. I did because- Let's- Let's, for instance, leas- okay, let's just ply- try plugging in  $g$   
1107 inverse.  $g$  inverse star  $h$  star  $g$  inverse inverse -  $g$  - is equal to  $h$ . That's what we- it would need to  
1108 look like for  $g$  inverse to work. And that's what we just proved right there. [Okay.] Does that  
1109 make sense?  
1110  
1111 Int: Yeah. And that's the same thing you had written down here, right? (points to bottom of page)  
1112 [Right. I was get- I was]  $a$  inverse inverse (inaud)  
1113  
1114 Tucker: Mhmm. That's what I was trying to get to, but with  $a$ 's.  
1115  
1116 Int: Okay. So, are the- (3 seconds) Wait. Is that it?  
1117  
1118 Tucker: (laughs) So, we've proven that there's closure. Wait, I never pr- proved closure, I don't  
1119 think.  
1120  
1121 Int: Oh, yeah. We book marked it, didn't we?  
1122  
1123 Tucker: We did. We book marked closure, but, I um, I feel like that's what you need to show,  
1124 right? (clicks tongue) Again, I don't know how to show that- all the- elements of  $H$  starred with  
1125 each other would give you back elements in  $H$ . I don't know how y- I would be able to show that.  
1126  
1127 Int: Well, so, you have  $a$  star  $b$  and  $a$  star  $b$  big in- like, inverse on the outside, right? [Mhmm.  
1128 Mhmm.] Is there any way to take that statement and-  
1129  
1130 [01:25:19.880]  
1131 Tucker: And say that, okay, well that means it must work, right?  
1132  
1133 Int: I mean, is there any way to do that?  
1134  
1135 Tucker: (chuckles)  
1136  
1137 Int: Like, can you- can you-?  
1138  
1139 Tucker: So, if  $a$  works- if  $a$  works and  $b$  works, [Uh huh.] that means  $a$  star  $b$  works? That's what  
1140 we're trying to prove?  
1141  
1142 Int: I think that's what you said earlier.  
1143  
1144 Tucker: Yeah. That's what I was trying to prove, but I don't know, using this sentence, you can-  
1145 how you would be able to prove that.  
1146  
1147 Int: Is there anything (clears throat) like, what tools do you have that you could use?  
1148

1149 Tucker: Well, in- existence of inverses and identity. I don't know.  
1150  
1151 Int: You also have associativity. [Okay.] So, you could re-group stuff.  
1152  
1153 Tucker: Okay. I guess.  
1154  
1155 Int: Or you could just un-group stuff.  
1156  
1157 [01:26:07.390]  
1158 Tucker: And, like, kinda like modify this right side as well, or- (4 seconds) Okay, so, if we bring  
1159 that over- (begins writing; 13 seconds) Trying to show- I wan- We want to show that this works.  
1160 [Okay.] That's what we want to show. And we know that  $a$  works and  $b$  works for this.  
1161  
1162 Int: Yeah. Didn't you assume that early on, [Yeah.] like, right (points to paper, center) right here?  
1163  
1164 Tucker: Yeah. So, we know that  $a$  works and  $b$  works. We wanna show that  $a \star b$  works. So,  
1165 using associativity, we can, I guess, mul- like, kinda move things around so that we have two  
1166 true things on both sides. [Okay.] So, we- we can try to move it around so that we have- (8  
1167 seconds) Well, also, um, if we wanna kind of get rid of the brackets here,  $a \star b$  inverse in  
1168 brackets (holds up to fingers) if we wanna get rid of those, would be  $b$  inverse star  $a$  inverse. Uh-  
1169  
1170 Int: Okay. I think I remember that from class.  
1171  
1172 Tucker: Yeah.  
1173  
1174 Int: Right? You said something about a- life's a track or something.  
1175  
1176 Tucker: Yeah. Uh, this, yeah, like, again, like, talking about, like, directions, you have to start  
1177 where you ended up [Uh huh.] in the first one. Um, no one really liked it, though. (chuckles)  
1178  
1179 Int: I beg to differ.  
1180  
1181 Tucker: (laughs) Well, it made sense to me. (laughs)  
1182  
1183 Int: Naw, I'm saying, yeah, I liked it, too. Like, that-  
1184  
1185 [01:27:49.328]  
1186 Tucker: (chuckles) Okay, so this would be, um,  $b$  inverse star  $a$  inverse is equal to  $h$ . So, I kinda  
1187 wanna get- I wanna g- get both sides so that, like, maybe, like, one side looks like this  $a$  side and  
1188 one side looks like that  $b$  side. [Okay.] Um, so to do that, I could do, okay, well, get rid of this-  
1189 these inverses on this side. So, we'll apply on the right side an  $a$ , so it's gonna look- we can get  
1190 that to equal, um, and we'll do  $a$  and  $b$  at the same time, I guess. Star  $h$ . Is equal (chuckling) to  $h$   
1191 star  $a$  star  $b$ . And now we kinda wanna separate the  $a$ 's and  $b$ 's, I think- [Okay] would be the next  
1192 goal. Um, so to do that, uh, let's bring this  $a$  over to that- Oh. Let's bring the  $b$  over to that side,  
1193 cause that's on that side. So, bringing that  $b$  over, we have  $a \star b \star h \star$  - my stars are  
1194 starting to look really cruddy, by the way - [Ah, that's fine.] I apologize. [No, that's cool.] (both

1195 chuckle) star  $b$  inverse is equal to  $h$  star  $a$ -  $h$  star  $a$ . Sorry.  
1196  
1197 Int: Okay. (both chuckle)  
1198  
1199 [01:29:15.718]  
1200 Tucker: Um, and now we wanna get that  $a$  over, so that- the  $a$ 's and  $b$ 's are finally separated and  
1201 we end up with  $b$  star  $h$  star  $b$  inverse is equal to -  $a$  inverse we're applying -  $a$  inverse star  $h$  star  
1202  $a$ . Um, (3 seconds) yeah, that's a true statement right there. Because we e- we started out with  
1203 this and we also know that the inverse works. We proved earlier that the  $a$  inverse would work.  
1204 [Mhmm.] So each inverse exists. So, that would be equal to that would be true. So, there's  
1205 closure, which means this is true, I think. Yes? No? Did I do something terribly wrong?  
1206  
1207 Int: I think so. I- So, you said that you proved that the inverses work.  
1208  
1209 Tucker: Mhmm.  
1210  
1211 Int: I think that's like right here, right? [Yeah. Yeah.] So, like- Oh, yeah. So,  $g$  working meant-  
1212 like, from that, you said that [Right.]  $g$  inverse works.  
1213  
1214 Tucker: Right up here, where, yeah, we proved that  $g$  inverse would work. [Okay.] So, this being  
1215 equal to that-  
1216  
1217 Int: So, if, like, having  $a$  in there, you also know that  $a$  inverse also works. So, then, you can-  
1218  
1219 Tucker: So, we- We started it right up here, with this. (points middle-right of page 1) [Uh huh.]  
1220 And by saying- then we star- then we tried it with  $a$  star  $b$  and we ended up with this. And we  
1221 already know that this right here (middle-right, page 1), even though it looks a little different  
1222 than that (bottom-right, page 2), those are actually equal to each other. [Okay.] Cause we proved  
1223 that right over here.  
1224  
1225 Int: And everything equals.  
1226  
1227 [01:30:56.617]  
1228 Tucker: Yes. So, [Okay.] I guess that means that closure is proved?  
1229  
1230 Int: I- I guess so.  
1231  
1232 Tucker: I mean- (both chuckle)  
1233  
1234 Int: Alright, so, if you- (Tucker writes, bottom-right, page 2; Int laughs) Alright. Alright. So,  
1235 here's the questions. Are you ready? [Yeah.] Alright, so. "Do you think that this proves this  
1236 statement?" That  $H$  is a subgroup of  $G$ .  
1237  
1238 Tucker: Yes. I do. I do. I do. [Int (inaud)] (Tucker laughs) Because I went through the steps.  
1239  
1240 Int: Alright. [Yeah.] So, "on a scale of one to ten, how confident are you in the proof?"

1241  
1242 Tucker: Oh, boy. Geez. (3 seconds; lowers tone of voice) Eight.  
1243  
1244 Int: Eight? And “On a scale of one to ten, how much does this proof- this proof verify the  
1245 relationship for you?”  
1246  
1247 Tucker: Eight. [Eight?] Mhmm. (3 seconds) Nine. I think nine.  
1248  
1249 Int: Nine? (laughs) So, you think it's a subgroup?  
1250  
1251 Tucker: Yeah. Unless I made a mistake somewhere. Or my proofs are, like, I did something  
1252 wrong, like you can't actually- this doesn't actually mean anything, type thing alon- somewhere  
1253 along the whole- along the way.  
1254  
1255 Int: So, what parts of what you did make you that confident about it?  
1256  
1257 Tucker: Well, we know it's a subset. That's a gimme. [Okay.] Cause, right here. (scoffs) All the  
1258 elements  $g$  in  $G$ . Uh, closure, that was probably the hardest one to prove. Um, I was really  
1259 confident. I was- I'm almost, like, I'm a hundred percent confident that this is what we needed to  
1260 prove in order for closure to be, you know, to work.  
1261  
1262 Int: Uh huh. Yeah. I believe you there.  
1263  
1264 [01:32:32.268]  
1265 Tucker: And, um, I think I proved that- I ended up proving that really well by showing that,  
1266 okay, we wanna show that this is, basically, is true. So, in order to do that, we can kinda like  
1267 warp both sides without breaking any rules. So that we get two things that we know are equal.  
1268 So, if we know- we didn't really change anything with this structure here. We just kinda, like,  
1269 moved things over and we showed that, yes, this is- these are two things that are equal. So, that  
1270 means this must be equal. So we've proven that closure would work. [Okay.] So, we proved  
1271 closure, we proved subset, and we also proved that here that every element has an inverse, so-  
1272 Unless I'm missing something, but I think- I think that would show it.  
1273  
1274 Int: Cool. [Mhmm.] So, “how did the way you think about identity inform your approach to this  
1275 proof?”  
1276  
1277 [01:33:33.270]  
1278 Tucker: Uh, it definitely helped a lot with knowing, okay, we gotta know if  $e$  is some identity in  
1279  $G$ , then we have to see that, okay, is that gonna be in  $H$  now? So, we just, pretty much showed-  
1280 we just tried it out pretty much. Say okay, yeah,  $e$  would be in  $H$  because we know what  $e$  does  
1281 to the little  $h$  right here. [Mhmm.] Cause  $h$  is in  $G$  and  $e$  inverse is alwa- is gonna be itself  
1282 because that's just the nature of identities and so, we've proven that, yeah, it would work in  $H$ ,  
1283 so-  
1284  
1285 Int: So, it seemed like y- You explained that very clearly now. It seemed like you, kind of, like,  
1286 got hung up, like, with the one/ $e$  thing- [Mhmm.] Can you, um, can you talk about (inaud)-

1287  
1288 Tucker: That was back when I was thinking of this sign - this inverse sign - was literally meaning  
1289 reciprocal. [Oh, okay.] I didn't know that it was saying, like, um, inverse. [Okay.] I thought that  
1290 was- that symbol meant reciprocal. so, I was think that, if it's reciprocal, that means- um, it's  
1291 gonna be like multipla- multiplication in nature, which means that one is probably gonna be the  
1292 **inverse**. [Okay.] So, I- I didn't realize that that's just actually meaning  $g$ 's inverse, not necessarily  
1293 one over  $g$ . That's what- that's what I got hung up on. [Okay.] So.  
1294  
1295 Int: Alright, cool. And then, how about inverses? "How did your understanding of inverses  
1296 inform your approach to the proof?"  
1297  
1298 [01:34:58.513]  
1299 Tucker: Mm. (clicks tongue) It really just kind of played off my- my idea of identities, I think.  
1300 Once- It's just, kinda really, (inaud) my- I kinda like went off what I had with the identities and  
1301 went from there. Let's see. Yeah. I wanted to just, basically, show that, okay, the inverses would  
1302 need to work for them to exist in  $H$ . Know what I mean? So, for each arbitrary element in  $H$  to  
1303 have an inverse, those inverses need to *work* in  $H$ . They need to be able to prove that this would  
1304 work. So, I'd have to prove *this* when you think about it. And, I *did*. So I know those two things  
1305 are actually equal to each other. [Cool.] So, this works (points middle-left p 2), which means that  
1306 works (points middle-right p 2). And then, yeah.  
1307  
1308 Int: It's a tough one isn't it?  
1309  
1310 Tucker: It was tough. I mean, it's- when I look back, it's really not that hard, but, I don't know, it  
1311 just takes some, like, pushing (inaud) [Yeah.] to really think about it. It looks a lot more  
1312 challenging when you read it like this. Like, it just-  
1313  
1314 Int: Hieroglyphics. Like, math speak.  
1315  
1316 Tucker: Yeah, math speak. And then you just think, like, okay, actually, that's not that hard.  
1317  
1318 Int: Okay. Alright, cool. Do you have time-? I know. It's taken forever. I promise this question  
1319 will be *so* much quicker than the others. [Okay.] Okay? "So, Charlie, a student from another  
1320 class, writes the identity of the group of symmetries of a triangle - so the symmetries of a triangle  
1321 - um, he or she - I can't tell if it's a boy or girl - um, writes the identity of that group, uh, as, zero  
1322  $R$ ." [Mhmm.] (Int writes "0R" on blank page) [Or.] "Or." Yeah, (both chuckle) zero. Alright.  
1323 Um, okay. "Why do you think Charlie represents it this way?"  
1324  
1325 [01:36:49.512]  
1326 Tucker: Um, cause their idea of identities is just, like, the nothing kinda sym- the nothing  
1327 function, type thing. Like, you're not doing anything to that element. When, really, um, I think a  
1328 better way to think about it is you're doing something to it, but it gets you back to that same  
1329 place. [Okay.] Like, we talked about early on- very early on - that symmetries were, like, um,  
1330 rigid motions [Mhmm.] that map a figure to itself. And, really, not rotating at all, you're not- it's  
1331 not a motion, cause you're not actually rotating it. Which is why it's better to write this as three  
1332  $R$ , because you're doing something to it. It's a rigid motion that's not moving the object, *but* it

1333 still gets you back to *exactly* where it was before, so- [Okay.] I guess their notion of identity is a  
1334 little bit different.

1335  
1336 Int: Alright. So, would you say that this is an acceptable way to notate the identity.  
1337

1338 Tucker: I don't really think so, because just by the definition of symmetries. [Okay.] It might be  
1339 with other things besides symmetries, like zero would be a great way to write the identity under  
1340 addition, but not in the case of symmetries, I don't think. [Okay.] Just by the definition of  
1341 symmetry.

1342  
1343 Int: Okay. Um, are there any other ways you can think of representing the identity of that group?  
1344

1345 Tucker: Three R.

1346  
1347 Int: Three R?

1348  
1349 Tucker: Mhmm. This is a triangle, right.

1350  
1351 Int: Yeah, with a triangle. Yeah.  
1352

1353 Tucker: Three R, because that would be (2 seconds) you know, you're still- you're actually doing  
1354 a motion instead of not doing anything to it.

1355  
1356 Int: Okay. Are there any- Are there other ways?

1357  
1358 Tucker: Two F. Another one. [Okay.] Two flips. Um, yeah.  
1359

1360 Int: Alright. Is there a way to represent this element that is *most* correct?  
1361

1362 Tucker: There's no best way. But that's (points at paper) the worst way. (Int laughs) Two F and  
1363 three R, I think, are interchangeable. [Okay.] But, I don't like- I don't like the idea of zero R.

1364  
1365 Int: Alright. Cool. (Tucker laughs)

1366  
1367 \*\*\*\*End of Video