Toward a Rigorous Justification of the Three-Body Impact Parameter Approximation

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(ABSTRACT)

The impact parameter (IP) approximation is a semiclassical model in quantum scattering theory wherein N large masses interact with one small mass. We study this model in one spatial dimension using the tools of time-dependent scattering theory, considering a system of two large-mass particles and one small-mass particle. We demonstrate that the model's predictive power becomes arbitrarily good as the masses of the two heavy particles are made larger by studying the S-matrix for a particular scattering channel. We also show that the IP wave functions can be made arbitrarily close to the full three-body solution, uniformly in time, provided one of the large masses is fixed in place, and that such a result probably will not hold if we allow all the masses to move.

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[1]The Institute for Critical Technology and Applied Science: www.ictas.vt.edu

Dedication

To my mother, Kimberly Bowman. To say that she is the only one who has ever believed in me would perhaps be unfair, but to say that no one would have ever believed in me if not for her ceaseless dedication to helping me be all she thought I could be, is probably not very far from the truth.

And to my fiancée, Lera Oscherov. Thank you for making my time at Virginia Tech so incredible – and sorry it took me so long to work up the nerve to ask you out. I can't wait to write the next chapter of our lives together.

I love and cherish you both.

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Chapter 1

Introduction

Scattering theory provides the mathematical framework for interpreting the results of experiments conducted in particle accelerators. The late J.J. Sakurai has said of scattering theory that it "is impossible to overemphasize the importance of this subject" [20]. Given that scattering experiments have proven to be our primary source of information about the inner workings and constituents of the subatomic world – dating from Ernest Rutherford's seminal "gold foil" experiments of 1909 to the more recent search for the Higgs boson at CERN's Large Hadron Collider – it is difficult to find room to disagree with Sakurai's remark.

In many scattering experiments, one is more or less forced to prepare a system of particles and allow them to interact over a distance scale that is extremely small compared to the observational laboratory apparatus. In light of this, analyzing the nature of the interactions in terms of the *long-term* behaviors of the particles – something that can often be done in terms of a simpler dynamics than that which governs the range of the interactions – becomes very attractive. This is the general approach in scattering theory; the next section is a brief outline of exactly how this is done.

1.1 Multi-particle quantum scattering

In quantum scattering theory, one is typically concerned with the asymptotic ($|t| \to \infty$; i.e., distant past and distant future) behavior of quantum systems. The mathematical setting for the non-relativistic scattering theory of N spinless, distinguishable particles in d dimensions

Table 1.1: Some scattering channels and their associated cluster decompositions for a three-body situation.

Channel	Description	Cluster Decomposition
α_1	① and ③ bound together; ② moving freely.	{① ③,②}
α_2	② and ③ bound together, ① moving freely.	{② ③, ①}
α_3	①, ②, and ③ all moving freely, independently of one another.	{①, ②, ③}

is the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^{Nd})$, the space of square-Lebesgue-integrable, complex-valued functions defined on $\mathbb{R}^{Nd}$. The physical setup is a system of N particles of masses $\mu_1, \ldots, \mu_N$ at locations $r_1, \ldots, r_N \in \mathbb{R}^d$ interacting through pair potentials V_{ij}. In three-body $(N = 3)$[1] quantum scattering, the interacting dynamics is governed by the time-dependent Schrödinger equation, which takes the form

$$i\frac{\partial \psi}{\partial t} = \widetilde{H}\psi = \left(-\sum_{j=1}^{3} \frac{\Delta_j}{2\mu_j} + \sum_{1 \leqslant i < j \leqslant 3} V_{ij}(r_i - r_j) \right) \psi, \tag{1.1}$$

where Δ_j is the Laplacian in the variable r_j. The $\psi = \psi(\mathbf{r}, t) = \psi(r_1, r_2, r_3, t)$ is the wave function, which encodes all the physical information about the system. It is an element of $\mathscr{H}$, and the operators V_{ij} are multiplication operators on this space. In many applications, they represent Coulomb interactions ($V_{ij} \propto |r_i - r_j|^{-1}$), but for our purposes they need only be suitably well-behaved such that $\widetilde{H}$ is a self-adjoint operator defined on an appropriate subspace $D(\widetilde{H}) \subset \mathscr{H}$. Given initial data $\psi(\mathbf{r}, 0)$, the spectral theorem tells us the time evolution of the wave function is given by the action of the *unitary propagator* $U(t) = e^{-i\widetilde{H}t}$ on $\psi(\mathbf{r}, 0)$: $\psi(\mathbf{r}, t) = e^{-i\widetilde{H}t}\psi(\mathbf{r}, 0)$.

Physical reasoning tells us there are a number of possible large-$|t|$ behaviors, which we refer to as *scattering channels*, that we might expect to observe from a three-body system consisting of particles ①, ②, and ③. Some examples include those listed in Table 1.1. As the table indicates, each channel has an associated *cluster decomposition* – a breaking up of the particles into disjoint non-interacting (for large $|t|$) clusters of particles. Of course,

[1] This is the number of particles that will directly concern us in this work.

quantum mechanics only permits us to discuss the *probability* that a three-body system we are investigating will exhibit one of these long-term behaviors.[2] To study these probabilities, we need to build some new machinery.

We first move from the (r_1, r_2, r_3) system to a system of coordinates that allows us to remove the trivial center-of-mass motion from (1.1); Jacobi coordinates (see §XI.5 of [18]) are particularly useful in this regard. Suppose we are interested in the probability that a system which looks like the channel α_1 in the distant past has the same behavior in the distant future. A convenient system of Jacobi coordinates to use in our analysis is

$$x = r_3 - r_1, \qquad y = r_2 - \frac{\mu_1 r_1 + \mu_3 r_3}{\mu_1 + \mu_3}, \qquad R = \frac{\mu_1 r_1 + \mu_2 r_2 + \mu_3 r_3}{M}, \tag{1.2}$$

where $M = \sum_{i=1}^{3} \mu_i$. In the (x, y, R) system, if we ignore any interaction between ① and ② and let $V_i = V_{i3}$, it can be shown that the Hamiltonian in (1.1) becomes

$$\begin{aligned} \widetilde{H} = &-\frac{1}{2}\left(\mu_1^{-1} + \mu_3^{-1}\right)\Delta_x - \frac{1}{2}\left(\mu_2^{-1} + (\mu_1 + \mu_3)^{-1}\right)\Delta_y \\ &+ V_1(x) + V_2\Big(\mu_1(\mu_1 + \mu_3)^{-1}x - y\Big) - \frac{1}{2M}\Delta_R. \end{aligned} \tag{1.3}$$

Now, consider H, which is just (1.3) with the center of mass motion removed; that is,

$$H = -\frac{1}{2}\left(\mu_1^{-1} + \mu_3^{-1}\right)\Delta_x - \frac{1}{2}\left(\mu_2^{-1} + (\mu_1 + \mu_3)^{-1}\right)\Delta_y + V_1(x) + V_2\Big(\mu_1(\mu_1 + \mu_3)^{-1}x - y\Big).$$

If we study the scattering theory associated with H, we are effectively studying a two, as opposed to a three, body problem.

Since we are interested in α_1, we should not expect V_2, the interaction between ② and ③, to be physically relevant for large times. This motivates consideration of the *channel Hamiltonian*

$$H_1 = \underbrace{-\frac{1}{2}\left(\mu_1^{-1} + \mu_3^{-1}\right)\Delta_x + V_1(x)}_{H_1^x} \underbrace{-\frac{1}{2}\left(\mu_2^{-1} + (\mu_1 + \mu_3)^{-1}\right)\Delta_y}_{H_1^y},$$

which is just $H - V_2$. Moreover, the assumption that ① and ③ are bound together for large

[2] We should emphasize that quantum mechanics *also* tells us that a general state can be in a *superposition* of two, or more, of these various "basis" behaviors.

times implies the existence of at least one bound state $\eta_1(x) \in L^2(\mathbb{R}^d, d^d x)$ that satisfies

$$\left(-\frac{1}{2}\left(\mu_1^{-1} + \mu_3^{-1}\right)\Delta_x + V_1(x)\right)\eta_1(x) = E_1\eta_1(x)$$

for some $E_1 < 0$. We are now in a position to define the *channel wave operators* associated with α_1, which are linear operators on the Hilbert space $\mathscr{H}_1 = L^2(\mathbb{R}^d, d^d x) \otimes L^2(\mathbb{R}^d, d^d y)$:

$$\Omega_1^{\pm} = \operatorname*{s-lim}_{t\to\mp\infty} e^{iHt}e^{-iH_1 t}, \tag{1.4}$$

where "s-lim" is the limit in the strong operator topology on $\mathscr{L}(\mathscr{H}_1)$.[3] Of course, part of the program is to show the existence of these strong limits for the self-adjoint operators H and H_1. But if they exist, the wave operators have the the following property: If $\varphi(y)$ is an arbitrary element of $L^2(\mathbb{R}^d, d^d y)$ such that $\psi^{\pm} = \Omega_1^{\pm}\eta_1(x)\varphi(y)$, then $e^{-iHt}\psi^{\pm}$ looks like the product state

$$e^{-iE_1 t}\eta_1(x) \cdot e^{-iH_y t}\varphi(y) = e^{-it(H_1^x + H_1^y)}\varphi(y)\eta_1(x)$$

as $t \to \mp\infty$. This means that in the distant past, the state $e^{-iHt}\psi^+$ – which evolves under the full dynamics H – looks like the particular version of scattering channel α_1 which has the propagation of the center of mass of the ① - ③ cluster with respect to ② described by the specific function $\varphi(y, t) = e^{-iH_y t}\varphi(y)$. The analogous statement holds for $e^{-iHt}\psi^-$ in the distant future.

Elements like $\psi^{\pm}$ – that is, elements that belong to the ranges of the wave operators – are called *scattering states*.

We can proceed in this way to construct wave operators $\Omega_\alpha^{\pm}$ for each of the other scattering channels listed in Table 1.1, modifying the channel Hamiltonian as appropriate. If it just so happens that the absolutely continuous subspace of $\mathscr{H}$ with respect to H can be expressed as

$$\bigoplus_\alpha \operatorname{Ran}\Omega_\alpha^{\pm} = \mathscr{H}_{ac}(H), \quad \text{and} \quad \sigma_{\text{sing}}(H) = \varnothing,$$

the direct sum taken over all channels, then scattering for the three-body system is said to be *asymptotically complete*[4]. Proving asymptotic completeness for a given system has long

[3] According to Barry Simon [22], the odd convention that the limits as $t \to \mp\infty$ are $\Omega^{\pm}$, respectively, is a perversion that it is nonetheless our "moral duty" to preserve! An explanation for this convention can be found in [18]; it has to do with the time-*in*dependent formulation of the theory.

[4] We use $\sigma_{\text{sing}}(H)$ to denote the singularly continuous spectrum of the operator H.

been the holy grail of mathematical quantum scattering theory.

One final object we can define for each channel – an object of particular experimental interest – is the S-matrix, defined as $S_\alpha = (\Omega_\alpha^-)^*\Omega_\alpha^+$. We interpret the quantity

$$P_{\varphi\to\psi} = |(\psi, S_\alpha\varphi)|^2 = \left|\int_{\mathbb{R}^{3d}} \psi(\mathbf{r})\overline{S_\alpha\varphi(\mathbf{r})}\ d\mathbf{r}\right|^2 \tag{1.5}$$

as the probability that a state that looks like $e^{-iH_\alpha t}\varphi$ in the distant past looks like $e^{-iH_\alpha t}\psi$ in the distant future. Thus, the S-matrix relates asymptotic behavior in the past to asymptotic behavior in the future without making explicit reference to the unobservable (for all practical purposes) actual dynamics.

For a more thorough discussion of the general approach to three-particle quantum scattering, the reader is directed to Enss's very edifying work [5].

1.2 The Impact Parameter Approximation

Using the machinery of the last section is nice in theory, but the equations involved turn out to be rather unwieldy in analyzing experimental data [21]. As such, various approximative schemes have been introduced into the theory. One that crops up in a number of places in the chemistry and physics literature (see, for instance, [3], [4], [14]), primarily in the study of atom-ion collisions, is the so-called *impact parameter (IP) approximation*, or charge transfer model. It is based on the fundamental physical intuition that "large" particles can be studied quite well using the techniques of classical mechanics, whereas "small" particles are amenable only to the rules of quantum mechanics.

To get a feel for how the model is employed in practice, consider a three-body system consisting of two large particles and one light particle, as shown in Figure 1.1. It may be helpful to think of the heavy particles (of masses μ_1 and μ_2) as being nuclei, and the light particle (of mass μ_3) as being an electron. Consideration of the *full* dynamics would lead us to the time-dependent Schrödinger equation governed by the Hamiltonian

$$H = -\frac{1}{2\mu_1}\Delta_1 - \frac{1}{2\mu_2}\Delta_2 - \frac{1}{2\mu_3}\Delta_3 + V_1(r_3 - r_1) + V_2(r_3 - r_2) + V(r_2 - r_1).$$

The associated time-dependent Schrödinger equation is a partial differential equation in $3d$

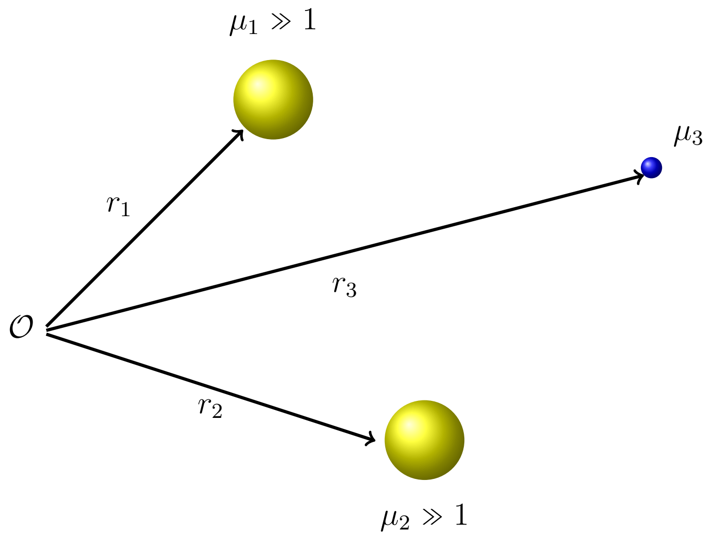

Figure 1.1: Schematic of the problem giving rise to the IP model. We consider two "large" (yellow) particles and a "small" (blue) one. The IP model requires the assumption that there is no interaction between the two large masses.

spatial variables and one time variable, the solution of which is (to put it mildly) a formidable task.

To simplify matters, the IP model assumes that the trajectories of the heavy particles are *known* curves in $\mathbb{R}^d$: $r_1 = r_1(t)$ and $r_2 = r_2(t)$. Physical reasoning tells us this should be a reasonable approximation in certain circumstances, especially if the heavy particles are traveling at a high speed. Ignoring any interaction between μ_1 and μ_2 and dropping their corresponding kinetic energy terms, we obtain the *impact parameter Hamiltonian*

$$H(t) = -\frac{1}{2\mu_3}\Delta_3 + V_1(r_3 - r_1(t)) + V_2(r_3 - r_2(t)).$$

This reduces a three-body problem to a one-body problem, at the cost of introducing time dependence – which is often a technical nuisance – into the Hamiltonian. Nonetheless, the impact parameter Hamiltonian $H(t)$ is a differential operator in only one spatial variable as opposed to three. The curves $r_1(t)$ and $r_2(t)$ are often taken to be straight lines of the form $r_1(t) = v_1 t - a_1$ and $r_2(t) = v_2 t - a_2$.

The widespread use of the IP model in interpreting experimental data begs the question: does the model actually represent, in some appropriate sense, a good approximation to the full dynamics? We attempt to provide a partial answer to this question in the work that follows. First, we provide a brief survey of mathematical results that have been proved about the IP model.

1.3 Past results

The first rigorous mathematical treatment of the IP model seems to have been done in 1980 by Yajima [29]. He proved existence and asymptotic completeness of the wave operators in the N-body case under quite relaxed – and therefore fairly exotic! – assumptions on the interaction potentials, using a clever technique first developed in a paper of Howland [11]. Though Yajima's proof worked for Coulomb potentials, it was not very physical, a shortcoming that was improved upon a few years later by Hagedorn [7]. A year later, Hagedorn also published results [8] pertaining to an IP analog of an important result from quantum dynamics, the RAGE theorem.[5] Wüller [27] proved time-boundedness of the energy in the IP model under some rather mild assumptions about the behavior of the gradient of the potentials; at approximately the same time, Graf [6] published results about a new, geometric proof for the model's asymptotic completeness. A year later [28], Wüller published his own geometric proof of existence and completeness in the IP model.

To our knowledge, the first results about the validity of the approximation – in the sense mentioned at the end of the last section – would not come until 1993, when Ito [12] used time-independent scattering theory to prove that the scattering cross-sections for the full three-body problem approach those supplied by the IP model in dimension $d \geqslant 2$. This paper only treated two-cluster-to-two-cluster scattering; the paper [13] uses similar results for potentially two-cluster-to-three-cluster scattering in spatial dimension $d = 3$. There is no known time-*dependent* treatment of this problem for $d = 1$, however – a gap this work hopes to fill.

[5] Roughly spoken, the RAGE theorem characterizes the difference between bound states and scattering states for a given quantum system. For more detail, see pg. 129 of [26].

1.4 New results

In what follows, we present the rigorous mathematical results we have obtained about whether the three-body ($N = 3$) impact parameter model actually represents a good approximation to the full dynamics in one spatial dimension ($d = 1$). Assuming suitable hypotheses, which will be enumerated and discussed in the following chapters, these can be broken down as follows:

1. Suppose we *fix* one of the heavy masses (at the origin, say). Then the resulting impact parameter wave function becomes a better and better approximation to a solution to the full three-body problem, uniformly in time, as one takes the mass of the remaining (free) heavy mass to infinity. These results are presented in Chapter 2.

2. There is little hope for a uniform wave function result as in (1) if *all* bodies move. An argument showing why this is the case is presented in Chapter 3.

3. Notwithstanding (2), we can show that certain S-matrix elements corresponding to the full dynamics are well approximated by their IP analogs as the masses of the large bodies tend to infinity. This argument is fleshed out in Chapter 4.

Chapter 2

Uniform wave function results

In this chapter, we apply the IP model to a simplified version of a three-body problem in one dimension in which one of the masses is infinite ($\mu_1 = \infty$) and fixed at the origin. We consider the same scattering channel discussed in detail in Chapter 1; namely, ③ bound to ① and ② free as $|t| \to \infty$. A schematic of the setup is shown in Figure 2.1.

Let $v, \epsilon > 0$. We show that the *product* wave function

$$\Psi_a = \Psi_a(x, y, t) = e^{iv^2 t/(2\epsilon)} \widetilde{\varphi}_0(1 + it, 1, \epsilon, vt, v, y) \cdot \psi(x, t), \qquad x, y \in \mathbb{R}, \tag{2.1}$$

which is a solution to the associated impact parameter Schrödinger equation, is an approximate solution to the time-dependent Schrödinger equation governing the system shown in Figure 2.1:

$$i\frac{\partial \Psi_a}{\partial t} = H\Psi_a = \left(-\frac{1}{2}\Delta_x - \frac{1}{2m}\Delta_y + V_1(x) + V_2(x - y)\right)\Psi_a, \tag{2.2}$$

uniformly for $t \in [0, \infty)$ and for physically appropriate choices of the functions $\widetilde{\varphi}_0$ and ψ, with the ψ coming directly from the impact parameter model. In fact, we will be able to conclude that the exact solution (2.1) approximates is given by the following solution to (2.2):

$$\Psi(x, y, t) = e^{-iHt}\Psi_a(x, y, 0) = e^{-iHt}\varphi_0(1, 1, \epsilon, 0, v, y) \cdot [\omega_1^-(1)](x),$$

where the symbols φ_0 and ω_1^- are defined in the following section.

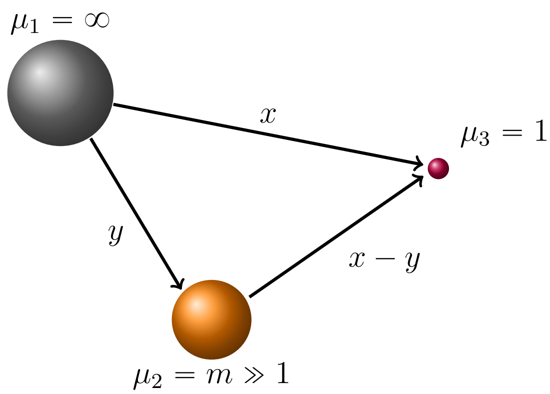

Figure 2.1: Schematic of the case $\mu_1 = \infty$. We suppose $x, y \in \mathbb{R}$, but they are shown here in $\mathbb{R}^2$ for clarity.

2.1 Preliminaries

We define the scaled Fourier transform $\mathscr{F}_\epsilon$ and its inverse $\mathscr{F}_\epsilon^{-1}$ for $\psi \in \mathscr{S}(\mathbb{R})$ (the Schwarz space on $\mathbb{R}$) as follows:

$$[\mathscr{F}_\epsilon \psi](p) = \frac{1}{(2\pi\epsilon)^{1/2}} \int_{\mathbb{R}} \psi(x)\ e^{-ipx/\epsilon} dx,$$
$$[\mathscr{F}_\epsilon^{-1} \psi](x) = \frac{1}{(2\pi\epsilon)^{1/2}} \int_{\mathbb{R}} \psi(\xi)\ e^{ipx/\epsilon} dp.$$

(We will use $\mathscr{F}$ and $\mathscr{F}^{-1}$ for the case $\epsilon = 1$.) Note that for the scaled Fourier transform, the following formula holds for any $\alpha, \beta = 0, 1, 2, \ldots$

$$(ip)^\alpha \frac{d^\beta}{dp^\beta} [\mathscr{F}_\epsilon \psi](p) = \epsilon^{\alpha-\beta} \mathscr{F}_\epsilon \left[\frac{d^\alpha}{dx^\alpha} ((-ix)^\beta \psi(x)) \right].$$

In particular, letting $\alpha = 2$ and $\beta = 0$, we have $-\frac{p^2}{\epsilon^2} [\mathscr{F}_\epsilon \psi](p) = \mathscr{F}_\epsilon \left[\frac{d^2\psi}{dx^2} \right]$. Next, define the *cutoff* function $F(p)$ as follows: Given $v > 0$, choose fixed constants $a, b \in \mathbb{R}$ such that $0 < a < b < v/2$. Construct F such that

- $F \in C^\infty(\mathbb{R})$, with F symmetric about $p = v$.
- $F(p) = 1$ for $p \in [v - a, v + a]$.
- $F(p) = 0$ for $p \notin (v - b, v + b)$.

(See, for instance, §8.1.2 of [24] for an explanation of how such functions might be constructed.) We are now in a position to define the functions $\widetilde{\varphi}_0$ and ψ and motivate them physically.

- $\underline{\widetilde{\varphi}_0(1+it,1,\epsilon,vt,v,y)}$: In [9], Hagedorn constructs certain orthonormal bases

$$\{\varphi_k(A,B,\epsilon,a,\eta,\cdot)\}_{k=0}^{\infty}$$

  of $L^2(\mathbb{R}^d)$ that diagonalize quadratic quantum Hamiltonians. In one dimension, the function $\varphi_0(A,B,\epsilon,a,\eta,y)$ has the form

$$\varphi_0(A,B,\epsilon,a,\eta,y) = \pi^{-1/4}\epsilon^{-1/4}A^{-1/2}\exp\left(BA^{-1}\frac{(y-a)^2}{2\epsilon} + i\eta\frac{y-a}{\epsilon}\right).$$

  It follows that

$$\varphi_0(1+it,1,\epsilon,vt,v,y) = \frac{\pi^{-1/4}\epsilon^{-1/4}}{\sqrt{1+it}}\exp\left(\frac{(y-vt)^2}{2\epsilon(1+it)} + \frac{iv}{\epsilon}(y-vt)\right).$$

  There is ambiguity (± 1) in the choice of the square root of the complex number $1+it$, but since we will be concerned only with $|\varphi_0|^2$ in what follows, we do not need to be concerned with this ambiguity. Now, to obtain $\widetilde{\varphi}_0$, we take

$$\widetilde{\varphi}_0(\cdots,y) = \mathscr{F}_\epsilon^{-1}\left[F(p)[\mathscr{F}_\epsilon\varphi_0](p)\right](y).$$

  Hence, $\widetilde{\varphi}_0(\cdots,y)$ is the back-Fourier transform of the cutoff function F applied to the Fourier transform of $\varphi_0(\cdots,y)$ in p (momentum) space. Note that $\widetilde{\varphi}_0$ removes an arbitrarily small interval about momentum $p=0$. Note also that by Theorem 3.4 of [9], $\Phi(y,t) = e^{iv^2t/2\epsilon}\varphi_0(1+it,1,\epsilon,vt,v,y)$ satisfies the free Schrödinger equation in y corresponding to the Hamiltonian in (2.2); namely,

$$i\epsilon\frac{\partial\Phi}{\partial t} = -\frac{\epsilon^2}{2}\Delta_y\Phi.$$

  The physical intuition behind defining $\widetilde{\varphi}_0$ using this "semiclassical" wave packet is that we treat the particle at y approximately classically in the IP model, given the assumption that its location is described by $y=vt$ with some "wiggle room."

- $\underline{\psi(x,t)}$: Define $H_1 = -\frac{1}{2}\Delta_x + V_1(x)$. For the scattering channel in question, we expect

H_1 to approximately describe the quantum mechanics of the particle at x for large t. We assume there is exactly one normalized bound state, $\eta_1(x)$, of ① and ③, which satisfies $H_1\eta_1 = E_1\eta_1$.

The impact parameter approximation asks us to fix the trajectory of the heavy mass μ_2. We will assume that it travels with constant velocity $v \neq 0$. This gives rise to the IP model Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H(t)\psi = -\frac{1}{2}\Delta_x\psi + V_1(x)\psi + V_2(x - vt)\psi. \tag{2.3}$$

In what follows, we will assume that the potentials $V_i \in C_0^\infty(\mathbb{R})$, such that there exist constants $C_i, a_i > 0$ with

$$|V_i(x)| \leqslant C_i\mathcal{X}(|x| \leqslant a_i). \tag{2.4}$$

By considering the Dyson expansion in the so-called "interaction representation" (see [17]), we can show that a unitary propagator $U(t,s)$ exists for the Hamiltonian $H(t)$ and satisfies

$$\frac{d}{dt}\left(U(t,s)\psi\right) = -iH(t)U(t,s)\psi \tag{2.5}$$

for any $\psi \in D(H(t)) = D(-\Delta_x)$. Let $\psi(x,t) = U(t,0)[\omega_1^-(1)](x)$, where

$$[\omega_1^-(1)](x) = \lim_{t\to\infty}\left[U(0,t)e^{-iE_1t}\eta_1(x)\right]$$

is the channel wave operator associated with the scattering channel we are considering. Because of (2.5), we know that $\psi(x,t)$ satisfies the IP Schrödinger equation (2.3). We claim that, additionally, it looks like the stationary state $e^{-itE_1}\eta_1(x)$ for large times. We now prove this.

Proposition 2.1.1. $\psi(x,t)$ *above satisfies*

$$\left\|\psi(x,t) - e^{-itE_1}\eta_1(x)\right\|_{L^2(\mathbb{R},dx)} \to 0 \qquad \text{as } t \to \infty.$$

Proof. Let $\|\cdot\|_{L^2(\mathbb{R},dx)} = \|\cdot\|$. Fix a $T > 0$. Then by the properties of the unitary propagator,

$$\begin{aligned}\lim_{t\to\infty}\left[\psi(x,T)-U(T,t)e^{-itE_1}\eta_1(x)\right] &= \psi(x,T)-\lim_{t\to\infty}U(T,t)e^{-itE_1}\eta_1(x)\\ &= \psi(x,T)-\lim_{t\to\infty}U(T,0)U(0,t)e^{-itE_1}\eta_1(x)\\ &= \psi(x,T)-U(T,0)[\omega_1^{-}1](x)\\ &= \psi(x,T)-\psi(x,T)=0.\end{aligned}$$

Armed with this, we now employ the Cook's method trick of writing the quantity

$$\psi(x,T)-U(T,t)e^{-itE_1}\eta_1(x)$$

as the integral of its time derivative. Since we have $\frac{d}{dt}U(T,t)=iU(T,t)H(t)$, we have

$$\begin{aligned}&\psi(x,T)-U(T,t)e^{-itE_1}\eta_1(x)=\\ &\qquad\psi(x,T)-e^{-iTE_1}\eta_1(x)-i\int_T^t U(T,s)V_2(x-vs)e^{-isE_1}\eta_1(x)ds.\end{aligned}$$

Taking $t\to\infty$ on both sides of this equation and rearranging, we obtain

$$\psi(x,T)-e^{-iTE_1}\eta_1(x)=i\int_T^\infty U(T,s)V_2(x-vs)e^{-isE_1}\eta_1(x)ds.$$

$$\begin{aligned}\left\|\psi(x,T)-e^{-iTE_1}\eta_1(x)\right\| &\leqslant \int_T^\infty \left\|U(T,s)V_2(x-vs)e^{-isE_1}\eta_1(x)\right\|ds\\ &= \int_T^\infty \left\|V_2(x-vs)\eta_1(x)\right\|ds.\end{aligned}$$

Now we observe that, since $\eta_1(x)$ is an eigenfunction of the Hamiltonian H_1 corresponding to eigenvalue E_1, the corollary to the O'Connor-Combes-Thomas theorem found on pg. 201 of [16] tells us there exist constants $d, C_d>0$ such that

$$|\alpha(x,t)|=|e^{-iE_1t}\eta_1(x)|=|\eta_1(x)|\leqslant C_de^{-d|x|}.$$

Hence (using (2.4)) we can make the following estimate:

$$\begin{aligned}\left\|\psi(x,T) - e^{-iTE_1}\eta_1(x)\right\| &\leqslant \int_T^\infty \left\|C_2\mathcal{X}_{[vs-a_2,vs+a_2]}(x)\cdot C_d e^{-d|x|}\right\| ds \\ &= \int_T^\infty ds \left(\int_{vs-a_2}^{vs+a_2} C_2^2 C_d^2 e^{-2d|x|} dx\right)^{1/2}.\end{aligned}$$

Note that $vs - a_2 > 0$ as long as we choose $s > T \geqslant a_2/v$, in which case we can estimate the x-integral as follows:

$$\begin{aligned}\left\|\psi(x,T) - e^{-iTE_1}\eta_1(x)\right\| &\leqslant C_2 C_d \int_T^\infty \left((2a_2)e^{-2d(vs-a_2)}\right)^{1/2} ds, \quad \text{such that} \\ &\leqslant C_2 C_d \sqrt{2a_2}\int_T^\infty e^{-d(vs-a_2)} ds \\ &\leqslant C_2 C_d \sqrt{2a_2}\, e^{da_2}\int_T^\infty e^{-dvs} ds \\ &= Ke^{-\gamma T} \quad \text{for } K,\gamma > 0.\end{aligned}$$

So we've shown that for all sufficiently large T,

$$\left\|\psi(x,T) - e^{-iTE_1}\eta_1(x)\right\| \leqslant Ke^{-\gamma T}.$$

Letting $T \to \infty$ gives the desired result. □

The magic lemma

Because it will be so useful to us in this chapter, we now state and prove the so-called "magic lemma." Our proof closely mimics the one found in [9]. In Chapter 4, we will use a slightly different version of this lemma.

Lemma 2.1.1 (The magic lemma)**.** Suppose there exist $p, q \in \mathbb{R}$ such that

$$\left\|\left(i\epsilon^q \frac{\partial}{\partial t} - H(\epsilon)\right)\psi_{\mathrm{a}}\right\| \leqslant \mu(t)\epsilon^p$$

for a measurable function $\mu(t)$ and some "approximate" solution ψ_a to the time-dependent Schrödinger equation associated with the Hamiltonian $H(\epsilon)$. Then there exists an *exact* solution $\psi = \psi(t)$ to

$$i\epsilon^q \frac{\partial \psi}{\partial t} = H(\epsilon)\psi$$

which satisfies

$$\|\psi_{\mathrm{a}}(t) - \psi(t)\| \leqslant \epsilon^{p-q} \int_0^t \mu(s) ds.$$

Proof. Let $\psi_a(t)$ be given. Then $\psi(t) = e^{-iH(\epsilon)t/\epsilon^q}\psi_a(0)$ is an exact solution to the Schrödinger equation

$$i\epsilon^q \frac{\partial \psi}{\partial t} = H(\epsilon)\psi.$$

The unitarity of $e^{-iH(\epsilon)t/\epsilon^q}$ and the fundamental theorem of calculus allow us to write

$$\begin{aligned}
\left\|e^{-iH(\epsilon)t/\epsilon^q}\psi_a(0) - \psi_a(t)\right\|_{L^2(dx)} &= \left\|\psi_a(0) - e^{iH(\epsilon)t/\epsilon^q}\psi_a(t)\right\|_{L^2(dx)} \\
&= \left\|\int_0^t \frac{\partial}{\partial s}\left(\psi_a(0) - e^{iH(\epsilon)s/\epsilon^q}\psi_a(s)\right) ds\right\| \\
&\leqslant \int_0^t \left\|\left(-\frac{iH(\epsilon)}{\epsilon^q} e^{iH(\epsilon)s/\epsilon^q}\psi_a(s) - e^{iH(\epsilon)s/\epsilon^q}\frac{\partial \psi_a}{\partial s}\right)\right\| ds \\
\text{(after factoring out } i\epsilon^{-q}) &= \epsilon^{-q}\int_0^t \left\|\left(i\epsilon^q \frac{\partial \psi_a}{\partial s} - H(\epsilon)\right)\psi_a(s)\right\| ds \\
&\leqslant \epsilon^{p-q}\int_0^t \mu(s)\; ds.
\end{aligned}$$

□

Hence, in point of fact, the wave function $e^{-iH(\epsilon)t/\epsilon^q}\psi_a(0)$ is the exact solution we are looking for.

2.2 Finite times

We want to use the magic lemma to show there is an exact solution $\Psi(x, y, t)$ to the full three-body Schrödinger equation that is asymptotic to the wave function given in (2.1). On any finite, positive-time interval (of the form $t \in [0, T]$ for some $T > 0$), we can do this as follows: We write

$$\left\|\left(i\frac{\partial}{\partial t} - H\right)\Psi_a\right\| = \left\|\left(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta_x + \frac{1}{2m}\Delta_y - V_1(x) - V_2(x-y)\right)\Psi_a\right\|$$

and note that it will behoove us to use the triangle inequality to write

$$\left\| \left(i\frac{\partial}{\partial t} - H \right) \Psi_a \right\| \leqslant \left\| \left(i\frac{\partial}{\partial t} - H \right) e^{iv^2t/2\epsilon} (\widetilde{\varphi}_0(y,t) - \varphi_0(y,t))\psi(x,t) \right\| \tag{2.6}$$

$$+ \left\| \left(i\frac{\partial}{\partial t} - H \right) e^{iv^2t/2\epsilon} \varphi_0(y,t)\psi(x,t) \right\|. \tag{2.7}$$

We now individually try to control each of the pieces on the right. The first piece becomes

$$\left\| \left(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta_x + \frac{1}{2m}\Delta_y - V_1(x) - V_2(x-y) \right) e^{iv^2t/2\epsilon} (\widetilde{\varphi}_0(y,t) - \varphi_0(y,t))\psi(x,t) \right\|.$$

We will look first at

$$\left\| (V_1(x) + V_2(x-y)) e^{iv^2t/2\epsilon} (\widetilde{\varphi}_0(y,t) - \varphi_0(y,t))\psi(x,t) \right\|.$$

Since both V_1 and V_2 are C^∞ functions with compact support, we can use Hölder's inequality to write

$$\begin{aligned}
&\left\| (V_1(x) + V_2(x-y)) e^{iv^2t/2\epsilon} (\widetilde{\varphi}_0(y,t) - \varphi_0(y,t))\psi(x,t) \right\|_{L^2(dxdy)} \\
&\qquad \leqslant \|V_1(x) + V_2(x-y)\|_\infty \|(\widetilde{\varphi}_0(y,t) - \varphi_0(y,t))\psi(x,t)\|_{L^2(dxdy)} \\
&\qquad \leqslant K \|\psi(x,t)\|_{L^2(dx)} \|\widetilde{\varphi}_0(y,t) - \varphi_0(y,t)\|_{L^2(dy)} \\
&\qquad \leqslant K \|\widetilde{\varphi}_0(y,t) - \varphi_0(y,t)\|_{L^2(dy)},
\end{aligned}$$

where $K = \|V_1(x) + V_2(x-y)\|_\infty$. We now go to momentum space to bound the remaining norm:

$$\begin{aligned}
\|\widetilde{\varphi}_0(y,t) - \varphi_0(y,t)\|_{L^2(dy)} &= \left\| \mathscr{F}_\epsilon^{-1} \Big[F(p)\mathscr{F}_\epsilon[\varphi_0](p,t) - \mathscr{F}_\epsilon[\varphi_0](p,t) \Big](y) \right\|_{L^2(dy)} \\
&= \|F(p)\widehat{\varphi}_0(p,t) - \widehat{\varphi}_0(p,t)\|_{L^2(dp)} \\
&= \|(F(p)-1)\widehat{\varphi}_0(p,t)\|_{L^2(dp)} \\
&= \int_{\mathbb{R}} |F(p)-1|^2 |\varphi_0(1, 1+it, 1/m, v, -vt, p)|^2 dp.
\end{aligned}$$

The function $F(p)-1$ is nonzero exactly on the set $(-\infty, v-a)\cup(v+a,\infty)$. We investigate the above norm by breaking up the integral in the following way:

$$\int_{\mathbb{R}} = \int_{-\infty}^{v-b} + \int_{v-b}^{v-a} + \int_{v+a}^{v+b} + \int_{v+b}^{\infty} = 2\int_{v+a}^{v+b} + 2\int_{v+b}^{\infty} = I_{\text{in}} + I_{\text{out}},$$

where we have used the fact that $|\varphi_0(1, 1+it, 1/m, v, -vt, p)|^2$ is even about $p = v$. On the interval $(v+a, v+b)$, we have $|F(p)-1|^2 \leqslant 1$, so we can write

$$I_{\text{in}} \leqslant 2\int_{v+a}^{v+b} |\varphi_0(1, 1+it, \epsilon, v, -vt, p)|^2\ dp \leqslant 2(b-a)\ |\varphi_0(1, 1+it, \epsilon, v, -vt, v+a)|^2.$$

Hence $\boxed{I_{\text{in}} \leqslant 2(b-a)\ \pi^{-1/2}\ \epsilon^{-1/2}\ e^{-a^2/\epsilon}}$; now to estimate I_{out}. On the interval $(v+b,\infty)$, the function $|F(p)-1|^2 = 1$. After making the change of variables $u = \epsilon^{-1/2}(p-v)$, we see that I_{out} becomes

$$\begin{aligned} I_{\text{out}} = 2\int_{v+b}^{\infty} |\varphi_0(1, 1+it, \epsilon, v, -vt, p)|^2 dp &= 2\pi^{-1/2}\epsilon^{-1/2}\int_{v+b}^{\infty} \exp\left(-\frac{(p-v)^2}{\epsilon}\right) dp \\ &= 2\pi^{-1/2}\int_{b\epsilon^{-1/2}}^{\infty} e^{-u^2} du = \operatorname{erfc}(b\epsilon^{-1/2}). \end{aligned}$$

We can use the standard asymptotic expansion (see, for example, [1]) of the complementary error function $\operatorname{erfc}(x)$ to write

$$\boxed{I_{\text{out}} = \frac{1}{b}\sqrt{\frac{\epsilon}{\pi}}\ e^{-b^2/\epsilon} + \mathcal{O}\left(\sqrt{\epsilon}\ e^{-b^2/\epsilon}\right).}$$

We also need to control

$$\begin{aligned} &\left\| (\widetilde{\varphi}_0(y,t) - \varphi_0(y,t)) \left(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta_x\right) \psi(x,t) \right\|_{L^2(dxdy)} \\ &\qquad = \left\| (\widetilde{\varphi}_0(y,t) - \varphi_0(y,t)) \left(V_1(x) + V_2(x-vt)\right) \psi(x,t) \right\|_{L^2(dxdy)} \\ &\qquad\qquad \leqslant K \left\| \widetilde{\varphi}_0(y,t) - \varphi_0(y,t) \right\|_{L^2(dy)}, \end{aligned}$$

and we just saw how to bound such a term above. The piece

$$\left\| \left(i\frac{\partial}{\partial t} + \frac{\epsilon}{2}\Delta_y \right) e^{iv^2t/2\epsilon}(\widetilde{\varphi}_0(y,t) - \varphi_0(y,t))\psi(x,t) \right\|_{L^2(dxdy)}$$
$$= \left\| \left(i\frac{\partial}{\partial t} - \frac{p^2}{2\epsilon} \right) e^{iv^2t/2\epsilon}(F(p)-1)\widehat{\varphi}_0(p,t) \right\|_{L^2(dp)}$$
$$= \left\| (F(p)-1) \left(i\frac{\partial}{\partial t} - \frac{p^2}{2\epsilon} \right) e^{iv^2t/2\epsilon}\widehat{\varphi}_0(p,t) \right\|_{L^2(dp)},$$

where we have used the fact that $\epsilon = 1/m$ and that $-\dfrac{p^2}{\epsilon^2} = \dfrac{d^2}{dy^2}$, turns out to be identically zero: Since

$$\widehat{\varphi}_0(p,t) = e^{-iv^2t/\epsilon}\ \pi^{-1/4}\ \epsilon^{-1/4}\ \exp\left(-\frac{(1+it)(p-v)^2}{2\epsilon} - ivt\frac{(p-v)}{\epsilon} \right)$$
$$= e^{-iv^2t/\epsilon}\ \pi^{-1/4}\ \epsilon^{-1/4}\ \exp\left(-\frac{(p-v)^2}{2\epsilon} \right) \exp\left(-it\frac{(p-v)^2}{2\epsilon} \right) \exp\left(-ivt\frac{(p-v)}{\epsilon} \right), \quad (2.8)$$

we have

$$\left(i\frac{\partial}{\partial t} - \frac{p^2}{2m} \right) e^{-iv^2t/2\epsilon}(F(p)-1)\widehat{\varphi}_0(p,t)$$
$$= \left[i\left(-\frac{iv^2}{2\epsilon} \right) + i\left(-i\frac{(p-v)^2}{2\epsilon} \right) + i\left(-iv\frac{(p-v)}{\epsilon} \right) - \frac{p^2}{2m} \right] e^{-iv^2t/2\epsilon}(F(p)-1)\widehat{\varphi}_0(p,t)$$
$$= \frac{1}{2\epsilon}\left(v^2 + p^2 - 2pv + v^2 + 2pv - 2v^2 - p^2 \right) e^{-iv^2t/2\epsilon}(F(p)-1)\widehat{\varphi}_0(p,t) = 0.$$

Now we need to try to control (2.7). We do this by first writing $V_2(x-y) = (V_2(x-y) - V_2(x-vt)) + V_2(x-vt)$. Then

$$\left\| \left(i\frac{\partial}{\partial t} - H \right) e^{iv^2t/2\epsilon}\varphi_0(y,t)\psi(x,t) \right\| = \left\| \left[\left(i\frac{\partial}{\partial t} + \frac{\epsilon}{2}\Delta_y \right) \left(e^{iv^2t/2\epsilon}\varphi_0(y,t) \right) \right] \psi(x,t) \right.$$
$$+ \left[\left(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta_x - V_1(x) - V_2(x-vt) \right) \psi(x,t) \right] e^{iv^2t/2\epsilon}\varphi_0(y,t)$$
$$\left. + (V_2(x-y) - V_2(x-vt)) e^{iv^2t/2\epsilon}\varphi_0(y,t)\psi(x,t) \right\|$$

The first and second terms are zero, so we need only estimate

$$\| (V_2(x-y) - V_2(x-vt))\varphi_0(y,t)\psi(x,t) \|_{L^2(dxdy)}. \quad (2.9)$$

The fundamental theorem of calculus tells us that

$$|V_2(x-y) - V_2(x-vt)| = \left| \int\limits_{x-y}^{x-vt} \nabla V_2 \cdot dl \right| \leqslant |y - vt| \, \|\nabla V_2\|_\infty .$$

The supremum exists because V_2 is C^∞. Hence, using Hölder's inequality and the fact that $\|\psi(x,t)\|_{L^2(dx)} = 1$, we have

$$\|(V_2(x-y) - V_2(x-vt))\varphi_0(y,t)\psi(x,t)\|_{L^2(dxdy)} \leqslant \|\nabla V_2\|_\infty \, \|(y-vt)\ \varphi_0(y,t)\|_{L^2(dy)} .$$

Note that

$$\begin{aligned}\|(y-vt)\ \varphi_0(1+it,1,\epsilon,vt,v,y)\|_{L^2(dy)} &= \left(\int_{\mathbb{R}} \frac{|y-vt|^2}{\sqrt{1+t^2}}\ \pi^{-1/2}\ \epsilon^{-1/2}\ \exp\left(-\frac{|y-vt|^2}{\epsilon(1+t^2)} \right) \right)^{1/2} \\ &= \left(\pi^{-1/2} \cdot \epsilon^{-1/2} \cdot \epsilon^{1/2}\ \cdot (1+t^2)^{1/2} \int_{\mathbb{R}} \epsilon\ (1+t^2)^{1/2}\ w^2\ e^{-w^2}\ dw \right)^{1/2} \\ &= K\ \epsilon^{1/2} (1+t^2)^{1/2}.\end{aligned}$$

Putting this all together, we note that there exist constants $A_0, A_1, A_2 > 0$ such that

$$\begin{aligned}\left\| \left(i\frac{\partial}{\partial t} - H(\epsilon) \right) \Psi_a(x,y,t,\epsilon) \right\|_{L^2(dxdy)} &\leqslant A_0\ \epsilon^{-1/2}\ e^{-a^2/\epsilon} + A_1\ \epsilon^{1/2}\ e^{-b^2/\epsilon} + \mathcal{O}\left(\epsilon^{1/2}\ e^{-b^2/\epsilon} \right) \\ &\quad + A_2\ \epsilon^{1/2}(1+t^2)^{1/2}. \quad (2.10)\end{aligned}$$

We can therefore apply the magic lemma and integrate (2.10) in time to show that there exists an exact solution Ψ to the Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\Delta_x \Psi - \frac{1}{2m}\Delta_y \Psi + V_1(x)\Psi + V_2(x-y)\Psi$$

such that for all t with $t \in [0,T]$ for some $T < \infty$, we have

$$\begin{aligned}\|\Psi - \Psi_a\|_{L^2(dxdy)} &\leqslant T\left(A_0\ \epsilon^{-1/2}\ e^{-a^2/\epsilon} + A_1\ \epsilon^{1/2}\ e^{-b^2/\epsilon} + \mathcal{O}\left(\epsilon^{1/2}\ e^{-b^2/\epsilon} \right) \right) \\ &\quad + A_2 \epsilon^{1/2} \left(T\sqrt{1+T^2} + \sinh^{-1} T \right). \quad (2.11)\end{aligned}$$

But this expression shows us we have a problem with *infinite* times, so we will need to treat $t \in (T, \infty)$ separately.

2.3 Infinite times

We'd now like to take a look at what the magic lemma tells us for infinite times. To this end, we rewrite

$$\left\| \left(i\frac{\partial}{\partial t} - H \right) \Psi_a \right\| \leqslant \left\| \left(i\frac{\partial}{\partial t} - H \right) e^{iv^2t/2\epsilon} \ \widetilde{\varphi}_0(y,t) \ \Big(\psi(x,t) - \alpha(x,t) \Big) \right\| \tag{2.12}$$

$$+ \left\| \left(i\frac{\partial}{\partial t} - H \right) e^{iv^2t/2\epsilon} \ \widetilde{\varphi}_0(y,t) \ \alpha(x,t) \right\|. \tag{2.13}$$

Recall that the wave function $\alpha(x,t) = e^{-iE_1 t} \ \eta_1(x)$ solves the time-dependent Schrödinger equation

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta_x - V_1(x) \right) \alpha(x,t) = 0.$$

We can bound the right hand side of (2.12) by

$$\left\| \Big(\psi(x,t) - \alpha(x,t) \Big) \left[\left(i\frac{\partial}{\partial t} + \frac{\epsilon}{2}\Delta_y \right) e^{iv^2t/2\epsilon} \ \widetilde{\varphi}_0(y,t) \right] \right\|_{L^2(dxdy)} \tag{2.14}$$

$$+ \left\| e^{iv^2t/2\epsilon} \ \widetilde{\varphi}_0(y,t) \left[\left(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta_x - V_1(x) - V_2(x-y) \right) \Big(\psi(x,t) - \alpha(x,t) \Big) \right] \right\|_{L^2(dxdy)}. \tag{2.15}$$

The first piece is identically zero by an earlier calculation. We can simplify (2.15) by noting that

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta_x - V_1(x) - V_2(x-y) \right) \Big(\psi(x,t) - \alpha(x,t) \Big) = \left(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta_x - V_1(x) \right) \psi$$

$$- \left(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta_x - V_1(x) \right) \alpha - V_2(x-y)(\psi - \alpha) = V_2(x-vt)\psi - V_2(x-y)(\psi - \alpha). \tag{2.16}$$

We can therefore use Hölder's inequality and Proposition 1 to rewrite the norm in (2.15) as

$$
\begin{aligned}
&\left\| e^{iv^2t/2\epsilon}\ \widetilde{\varphi}_0(y,t) \right\|_{L^2(dy)} \left\| V_2(x-vt)(\psi-\alpha) - V_2(x-y)(\psi-\alpha) + V_2(x-vt)\alpha \right\|_{L^2(dx)} \\
&\leqslant \|V_2\|_\infty \|\psi-\alpha\|_2 + \|V_2\|_\infty \|\psi-\alpha\|_2 + \|V_2(x-vt)\alpha(x,t)\|_{L^2(dx)} \\
&\leqslant C_0 e^{-\gamma t} + \|V_2(x-vt)\ \alpha(x,t)\|_{L^2(dx)}\,,
\end{aligned}
$$

which holds for all sufficiently large t. To estimate $\|V_2(x-vt)\ \alpha(x,t)\|_{L^2(dx)}$, we write

$$
\begin{aligned}
\|V_2(x-vt)\ \alpha(x,t)\|^2_{L^2(dx)} = \int_{\mathbb{R}} |V_2(x-vt)|^2\ \eta_1^2(x)\ dx &\leqslant \int_{\mathbb{R}} C_2 \cdot \mathcal{X}_{\{|x-vt|\leqslant a_2\}} \cdot e^{-2c|x|}\ dx \\
&= C_2 \int_{vt-a_2}^{vt+a_2} e^{-2c|x|}\ dx \\
&\leqslant C_2 e^{-2c|vt-a_2|}.
\end{aligned}
$$

It follows that we can bound (2.12) by the quantity

$$
\boxed{C_0 e^{-\gamma t} + C_2 e^{-2c|vt-a_2|}}
$$

which is integrable in t as desired. Now we need to look at (2.13):

$$
\begin{aligned}
&\left\| \left(i\frac{\partial}{\partial t} - H \right) e^{iv^2t/2\epsilon} \widetilde{\varphi}_0(y,t)\ \alpha(x,t) \right\|_{L^2(dxdy)} \\
&\leqslant \left\| \left(i\frac{\partial}{\partial t} + \frac{1}{2m}\Delta_y \right) e^{iv^2t/2\epsilon} \widetilde{\varphi}_0(y,t) \right\|_{L^2(dy)} \\
&+ \left\| \widetilde{\varphi}_0(y,t) \left(i\frac{\partial}{\partial t} + \frac{1}{2}\Delta_x - V_1(x) - V_2(x-y) \right) \alpha(x,t) \right\|_{L^2(dxdy)} \\
&= \left\| V_2(x-y)\ \widetilde{\varphi}_0(y,t)\ \eta_1(x) \right\|_{L^2(dxdy)} \\
&= \left\| V_2(x-y)\ \mathscr{F}_\epsilon^{-1}\Big[F(p)\ \varphi_0(1, 1+it, \epsilon, v, -vt, p) \Big](y)\ \eta_1(x) \right\|_{L^2(dxdy)}.
\end{aligned}
$$

We attack the estimation of the above norm by considering it in two disjoint regions of $\mathbb{R}^2$: the so-called classically *allowed* and classically *forbidden* regions, defined with respect to where we placed b in defining the cutoff function $F(p)$. To this end, we define

$$\mathcal{X}_{\mathrm{a}}(x,y,t) = \begin{cases} 1 & \text{if } |y-vt| \leqslant bt \text{ and } |x-y| \leqslant a_2, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\mathcal{X}_{\mathrm{f}}(x,y,t) = \begin{cases} 1 & \text{if } |y-vt| > bt \text{ and } |x-y| \leqslant a_2, \\ 0 & \text{otherwise.} \end{cases}$$

So, we are interested in estimating

$$\left\| V_2(x-y)\ \mathscr{F}_{\epsilon}^{-1}\big[F(p)\varphi_0(1,1+it,\epsilon,v,-vt,p)\big](y)\ \eta_1(x)\ \big(\mathcal{X}_{\mathrm{a}}(x,y,t)+\mathcal{X}_{\mathrm{f}}(x,y,t)\big)\right\|_{L^2(dxdy)}$$

It will be notationally convenient to write

$$\Upsilon(y,t) = e^{iv^2t/(2\epsilon)}\ \mathscr{F}_{\epsilon}^{-1}\Big[F(p)\ \varphi_0(1,1+it,\epsilon,v,-vt,p)\Big](y)$$

so that we are actually trying to estimate

$$\left\| V_2(x-y)\ \Upsilon(y,t)\ \eta_1(x)\ \big(\mathcal{X}_{\mathrm{a}}(x,y,t)+\mathcal{X}_{\mathrm{f}}(x,y,t)\big)\right\|_{L^2(dxdy)}.$$

We'll break this up using the triangle inequality and work with the classically allowed and forbidden regions individually.

2.3.1 The classically allowed region

The estimation of the L^2 norm for the classically allowed region goes as follows:

$$\begin{aligned}
&\| V_2(x-y)\ \Upsilon(y,t)\ \eta_1(x)\ \mathcal{X}_{\mathrm{a}}(x,y,t)\|_{L^2(dxdy)} \\
&\qquad = \left(\int_{\mathbb{R}} dx \int_{\mathbb{R}} dy\ |V_2(x-y)|^2 \big|\widetilde{\varphi}_0(1+it,1,\epsilon,vt,v,y)\ \eta_1(x)\ \mathcal{X}_{\mathrm{a}}(x,y,t)\big|^2\right)^{\frac{1}{2}} \\
&\qquad\qquad \leqslant C_2 \left\| \widetilde{\varphi}_0(1+it,1,\epsilon,vt,v,y)\ \eta_1(x)\ \mathcal{X}_{\mathrm{a}}(x,y,t)\ \mathcal{X}_{\{|x-y|\leqslant a_2\}}\right\|_{L^2(dxdy)},
\end{aligned}$$

where we have again used the fact that $V_2 \in C_0^\infty$ to write $|V_2(x-y)| \leqslant C_2 \cdot \mathcal{X}_{\{|x-y|\leqslant a_2\}}$ for some $C_2, a_2 > 0$. So, in computing this norm, we are interested in the region where $|y-vt| \leqslant bt$

and $|x-y| \leqslant a_2$. This lies inside the region where $vt - bt - a_2 \leqslant x \leqslant vt + bt + a_2$. Since $|\eta_1(x)| \leqslant C_d e^{-d|x|}$ and $\|\widetilde{\varphi}_0\|_{L^2(dy)} \leqslant \|\varphi_0\|_{L^2(dy)} = 1$, we have

$$\begin{aligned}
\|V_2 \cdot \eta_1 \cdot \Upsilon \cdot \mathcal{X}_{\mathrm{a}}\|_{L^2(dxdy)} &\leqslant C_2 \cdot C_d \left\| \mathcal{X}_{\mathrm{a}} \cdot \mathcal{X}_{\{|x-y| \leqslant a_2\}} \cdot e^{-d|x|} \cdot \left(e^{d|x|}\ \eta_1(x)\right) \cdot \widetilde{\varphi}_0(1+it, 1, \epsilon, vt, v, y) \right\|_{L^2(dxdy)} \\
&\leqslant C_2 \left\| \mathcal{X}_{\mathrm{a}} \cdot \mathcal{X}_{|x-y| \leqslant a_2} \cdot e^{-d|x|}\ \widetilde{\varphi}_0(1+it, 1, \epsilon, vt, v, y) \right\|_{L^2(dxdy)} \\
&\leqslant C_2 \left(\int_{vt-bt-a_2}^{vt+bt+a_2} e^{-2d|x|}\ dx \right)^{\frac{1}{2}} \left(\int_{(v-b)t}^{(v+b)t} dy\ |\widetilde{\varphi}_0(1+it, 1, \epsilon, vt, v, y)|^2 \right)^{\frac{1}{2}} \\
&\leqslant C_2 \cdot \sqrt{2(bt+a_2)} \cdot e^{-d|vt-bt-a_2|} \cdot 1 \\
&\leqslant K t e^{-d_0 t},
\end{aligned}$$

as long as t is taken to be sufficiently large.

2.3.2 The classically forbidden region

In this section, we will investigate

$$\left\| V_2(x-y) \Upsilon(y,t)\ \eta_1(x)\ \mathcal{X}_{\mathrm{f}}(x,y,t) \right\|_{L^2(dxdy)}.$$

We first write Υ as the inverse transform of its Fourier transform:

$$\Upsilon(y,t) = \frac{1}{2\pi\epsilon} \int_{\mathbb{R}} dp'\ e^{ip'y/\epsilon} \int_{\mathbb{R}} d\xi\ e^{-ip\xi/\epsilon}\ e^{iv^2t/(2\epsilon)}\ \mathscr{F}_\epsilon^{-1}\Big[F(p)\ \varphi_0(1, 1+it, \epsilon, v, -vt, p)\Big](\xi).$$

If we evaluate this at $t = 0$, we get

$$\Upsilon(y,0) = \frac{1}{2\pi\epsilon} \int_{\mathbb{R}} dp\ e^{ipy/\epsilon} \int_{\mathbb{R}} d\xi\ e^{-ip\xi/\epsilon}\ \mathscr{F}_\epsilon^{-1}\Big[F(p)\ \varphi_0(1, 1, \epsilon, v, 0, p)\Big](\xi).$$

Since taking the second derivative of this expression with respect to y brings down a factor of $-p^2/\epsilon^2$ and $\Upsilon(y,t)$ has propagator

$$U_\Upsilon(t,0) = \exp\left(\frac{i\epsilon t}{2} \frac{d^2}{dy^2}\right),$$

we see that

$$\Upsilon(y,t) = U_\Upsilon(t,0)\ \Upsilon(y,0) = \frac{1}{\sqrt{2\pi\epsilon}}\int_{\mathbb{R}} dp\ e^{ipy/\epsilon}\ e^{-ip^2t/(2\epsilon)}\int_{\mathbb{R}} d\xi\ e^{-ip\xi/\epsilon}\ \mathscr{F}_\epsilon^{-1}\Big[F(p)\ \varphi_0(1,1,\epsilon,v,0,p)\Big](\xi)$$
$$= \frac{1}{\sqrt{2\pi\epsilon}}\int_{\mathbb{R}} dp\ e^{ipy/\epsilon}\ e^{-ip^2t/(2\epsilon)}\ F(p)\ \varphi_0(1,1,\epsilon,v,0,p).$$

Now, we make several changes of variables. Let $z = p - v$. Then we find

$$\Upsilon(y,t) = \frac{1}{\sqrt{2\pi\epsilon}}\int_{\mathbb{R}} dz\ e^{i(v+z)y/\epsilon}\ e^{-i(v+z)^2t/(2\epsilon)}\ F(v+z)\ \varphi_0(1,1,\epsilon,0,0,z)$$
$$= \frac{1}{\sqrt{2\pi\epsilon}}e^{-iv^2t/(2\epsilon)}\ e^{ivy/\epsilon}\int_{\mathbb{R}} dz\ \exp\left[\frac{i}{\epsilon}\left(z(y-vt) - \frac{z^2t}{2}\right)\right]\ F(v+z)\ \varphi_0(1,1,\epsilon,0,0,z).$$

Now, we let $\mu = z/\sqrt{\epsilon}$. Then since $dz = \sqrt{\epsilon}\ d\mu$, the above becomes

$$\frac{1}{\sqrt{2\pi}}e^{-iv^2t/(2\epsilon)}\ e^{ivy/\epsilon}\int_{\mathbb{R}} d\mu\ \exp\left[\frac{i}{\epsilon}\left(\sqrt{\epsilon}\ \mu(y-vt) - \frac{\epsilon\mu^2t}{2}\right)\right]\cdot F(v+\sqrt{\epsilon}\mu)\cdot\epsilon^{-1/4}\ \varphi_0(1,1,1,0,0,\mu).$$

Reducing and rewriting slightly, we have

$$\Upsilon(y,t) = (2\pi)^{-1/2}\ \epsilon^{-1/4}\ e^{i\epsilon^{-1}v^2t}\ e^{i\epsilon^{-1}v(y-vt)}\int_{\mathbb{R}} d\mu\ e^{i\epsilon^{-1/2}[\mu(y-vt)-\epsilon^{1/2}\mu^2t/2]}\cdot F(v+\sqrt{\epsilon}\mu)\ \varphi_0(1,1,1,0,0,\mu).$$

This invites us to define the following two functions of μ:

$$f(\mu) = \mu(y-vt) - \frac{1}{2}\epsilon^{1/2}\mu^2 t, \qquad g(\mu) = F(v+\sqrt{\epsilon}\mu)\ \varphi_0(1,1,1,0,0,\mu).$$

Since F is supported on the set $\{p \in [v-b, v+b]\}$ and $p = v + \epsilon^{1/2}\mu$, we have $v - b \leqslant v + \epsilon^{1/2}\mu \leqslant v + b$, or $-\epsilon^{-1/2}b \leqslant \mu \leqslant \epsilon^{-1/2}b$. So the integral above can be written

$$\int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu\ e^{i\epsilon^{-1/2}f(\mu)}\ g(\mu) = \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu\ \left(f'(\mu)\ e^{i\epsilon^{-1/2}f(\mu)}\right)\left(\frac{g(\mu)}{f'(\mu)}\right).$$

We can now integrate by parts, letting

$$u = \frac{g(\mu)}{f'(\mu)} \Rightarrow du = \frac{g'(\mu)}{f'(\mu)} - \frac{g(\mu)f''(\mu)}{(f'(\mu))^2}, \qquad dv = f'(\mu)\ e^{i\epsilon^{-1/2}f(\mu)}d\mu \Rightarrow v = -i\epsilon^{1/2}e^{i\epsilon^{-1/2}f(\mu)}.$$

Using the fact that $f'(\mu) = y - vt - \epsilon^{1/2}\mu t$ and $f''(\mu) = -\epsilon^{1/2}t$, we find that

$$\int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu\ e^{i\epsilon^{-1/2}\left(\mu(y-vt)-\epsilon^{1/2}\mu^2 t/2\right)} F(v+\sqrt{\epsilon}\mu)\ \varphi_0(1,1,1,0,0,\mu)$$
$$= i\epsilon^{1/2} \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} e^{i\epsilon^{-1/2}f(\mu)} \left(\frac{g'(\mu)}{y - vt - \epsilon^{1/2}\mu t} + \epsilon^{1/2} t \frac{g(\mu)}{(y - vt - \epsilon^{1/2}\mu t)^2} \right) d\mu.$$

We'll need to do one more integration by parts to get the decay in time we need. The first integral above looks like

$$i\epsilon^{1/2} \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} \left(f'(\mu)\ e^{i\epsilon^{-1/2}f(\mu)}\right) \left(\frac{g'(\mu)}{(f'(\mu))^2}\right)\ d\mu$$
$$= -i\epsilon^{1/2}(-i\epsilon^{1/2}) \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu\ e^{i\epsilon^{-1/2}f(\mu)} \left(\frac{g''(\mu)}{(f'(\mu))^2} - \frac{2f''(\mu)g'(\mu)}{(f'(\mu))^3} \right)$$
$$= -\epsilon \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu\ e^{-i\epsilon^{-1/2}f(\mu)} \left(\frac{g''(\mu)}{(y - vt - \epsilon^{1/2}\mu t)^2} + 2\epsilon^{1/2} t \frac{g'(\mu)}{(y - vt - \epsilon^{1/2}\mu t)^3} \right)$$

We pause for a moment to define I_1 and I_2 as follows:

$$I_1 = -\epsilon \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu\, e^{-i\epsilon^{-1/2}f(\mu)} \frac{g''(\mu)\cdot \mathcal{X}_{\mathrm{f}}(x,y,t)}{(y - vt - \epsilon^{1/2}\mu t)^2}, \quad I_2 = -2\epsilon^{3/2} t \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu\, e^{-i\epsilon^{-1/2}f(\mu)} \frac{g'(\mu)\cdot \mathcal{X}_{\mathrm{f}}(x,y,t)}{(y - vt - \epsilon^{1/2}\mu t)^3}.$$

Similarly, the second integral looks like

$$i\epsilon t \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} \left(f'(\mu) e^{i\epsilon^{-1/2} f(\mu)}\right) \left(\frac{g(\mu)}{(f'(\mu))^3}\right) d\mu$$

$$= -\epsilon^{3/2} t \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} e^{i\epsilon^{-1/2} f(\mu)} \left(\frac{g'(\mu)}{(y - vt - \epsilon^{1/2}\mu t)^3} + 3\epsilon^{1/2} t \frac{g(\mu)}{(y - vt - \epsilon^{1/2}\mu t)^4}\right).$$

And we will define

$$I_3 = -\epsilon^{3/2} t \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu\, e^{i\epsilon^{-1/2} f(\mu)} \frac{g'(\mu) \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)}{(y - vt - \epsilon^{1/2}\mu t)^3}, \quad I_4 = -3\epsilon^2 t^2 \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu\, e^{i\epsilon^{-1/2} f(\mu)} \frac{g(\mu) \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)}{(y - vt - \epsilon^{1/2}\mu t)^4}.$$

Note that

$$g'(\mu) = \epsilon^{1/2} F' \varphi_0 + F \varphi_0' \qquad \text{and} \qquad g''(\mu) = \epsilon F'' \varphi_0 + 2\epsilon^{1/2} F' \varphi_0' + F \varphi_0''.$$

Since $F \in C_0^\infty(\mathbb{R})$, there exist positive constants D_n so that $\left\|F^{(n)}\right\|_\infty \leqslant D_n$ for every $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also, a computation tells us there exist constants $C_n > 0$ such that $\left\|\varphi_0^{(n)}\right\|_\infty \leqslant C_n$ for all $n \in \mathbb{N}_0$. We will need these estimates in what follows.

We have that

$$\Upsilon(y,t) \cdot \mathcal{X}_{\mathrm{f}}(x,y,t) = (2\pi)^{-1/2}\ \epsilon^{-1/4}\ e^{i\epsilon^{-1} v^2 t}\ e^{i\epsilon^{-1} v(y - vt)} \sum_{n=1}^{4} I_n,$$

so the triangle inequality gives

$$\left|\Upsilon(y,t) \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)\right| \leqslant \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{4} \epsilon^{-1/4}\ |I_n|.$$

We now take a look at each of the terms in the sum separately.

- $\mathbf{I_1}$: After taking the absolute values inside the integral and using the triangle inequality, we see that

$$|I_1| \leqslant \epsilon \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu \, \frac{|g''(\mu)| \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)}{(y - vt - \epsilon^{1/2}\mu t)^2} \leqslant \epsilon^2 \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu \, \frac{|F''(v + \sqrt{\epsilon}\mu)||\varphi_0(1,1,1,0,0,\mu)|}{|y - vt - \epsilon^{1/2}\mu t|^2} \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)$$
$$+ 2\epsilon^{3/2} \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu \frac{|F'(v + \sqrt{\epsilon}\mu)||\varphi_0'(1,1,1,0,0,\mu)|}{|y - vt - \epsilon^{1/2}\mu t|^2} \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)$$
$$+ \epsilon \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu \frac{|F(v + \sqrt{\epsilon}\mu)||\varphi_0''(1,1,1,0,0,\mu)|}{|y - vt - \epsilon^{1/2}\mu t|^2} \cdot \mathcal{X}_{\mathrm{f}}(x,y,t).$$

We use the fact that for each $n \in \mathbb{N}$ we have $|\varphi_0^{(n)}| \leqslant C_n$ to write

$$|I_1| \leqslant C_0\epsilon^2 \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu \frac{|F''(v + \sqrt{\epsilon}\mu)|}{|y - vt - \epsilon^{1/2}\mu t|^2} \cdot \mathcal{X}_{\mathrm{f}}(x,y,t) + 2C_1\epsilon^{3/2} \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu \frac{|F'(v + \sqrt{\epsilon}\mu)|}{|y - vt - \epsilon^{1/2}\mu t|^2} \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)$$
$$+ C_2\epsilon \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu \frac{|F(v + \sqrt{\epsilon}\mu)|}{|y - vt - \epsilon^{1/2}\mu t|^2} \cdot \mathcal{X}_{\mathrm{f}}(x,y,t).$$

We now change the integration variable to z to remove all ϵ's from the integrals. We then have $d\mu = \epsilon^{-1/2}dz$, so

$$|I_1| \leqslant C_0\epsilon^{3/2} \int_{-b}^{b} dz \frac{|F''(v + z)|}{|y - vt - zt|^2} \cdot \mathcal{X}_{\mathrm{f}}(x,y,t) + 2C_1\epsilon \int_{-b}^{b} dz \frac{|F'(v + z)|}{|y - vt - zt|^2} \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)$$
$$+ C_2\epsilon^{1/2} \int_{-b}^{b} dz \frac{|F(v + z)|}{|y - vt - zt|^2} \cdot \mathcal{X}_{\mathrm{f}}(x,y,t).$$

Now, consider the interval $[-b, -a]$. By Taylor's theorem and the fact that $F^{(n)}(v-b) =$

$F^{(n)}(v+b) = 0$ for all $n \in \mathbb{N}$, we have for $z \in [-b, -a]$

$$F^{(n)}(v+z) = \cancel{F^{(n)}(v-b)} + \cancel{F^{(n+1)}(v-b)}(z+b) + \int_{-b}^{z} (z-w)F^{(n+2)}(v+w)dw$$
$$= \int_{-b}^{z} (z-w)F^{(n+2)}(v+w)dw.$$

Hence for $z \in [-b, -a]$, we have

$$|F^{(n)}(v+z)| \leqslant \int_{-b}^{z} |z-w||F^{(n+2)}(v+w)|dw \leqslant D_{n+2} \int_{-b}^{z} |z-w|dw = \frac{1}{2}D_{n+2}|z+b|^2,$$

where $D_k = \sup_{w \in [-b,-b+\delta]} |F^{(k)}(v+w)| < \infty$. A similar argument allows us to obtain an analogous result for $z \in [a, b]$, only there we have

$$|F^{(n)}(v+z)| \leqslant \frac{1}{2}D_{n+2}|z-b|^2.$$

After observing that $F^{(n)} = \delta_{n,0}$ on $[-a, a]$ for $n \in \mathbb{N}$, it follows that

$$\begin{aligned}
|I_1| \leqslant & \frac{C_0 D_4}{2}\epsilon^{3/2} \left[\int_{-b}^{-a} dz \frac{|z+b|^2 \cdot \mathcal{X}_\mathrm{f}}{|(y-vt)-zt|^2} + \int_{a}^{b} dz \frac{|z-b|^2 \cdot \mathcal{X}_\mathrm{f}}{|(y-vt)-zt|^2} \right] \\
& + C_1 D_3 \epsilon \left[\int_{-b}^{-a} dz \frac{|z+b|^2 \cdot \mathcal{X}_\mathrm{f}}{|(y-vt)-zt|^2} + \int_{a}^{b} dz \frac{|z-b|^2 \cdot \mathcal{X}_\mathrm{f}}{|(y-vt)-zt|^2} \right] \\
& + \frac{C_2 D_2}{2}\epsilon^{1/2} \left[\int_{-b}^{-a} dz \frac{|z+b|^2 \cdot \mathcal{X}_\mathrm{f}}{|(y-vt)-zt|^2} + \int_{-a}^{a} dz \frac{\mathcal{X}_\mathrm{f}}{|(y-vt)-zt|^2} + \int_{a}^{b} dz \frac{|z-b|^2 \cdot \mathcal{X}_\mathrm{f}}{|(y-vt)-zt|^2} \right].
\end{aligned}$$

We now exploit the fact that $|y - vt| > bt$ when $\mathcal{X}_\mathrm{f}(x, y, t) \neq 0$: For positive times $t > 0$, we have

$$|(y-vt)-zt| \geqslant |y-vt| - |zt| > bt - |z|t = (b-|z|)t.$$

Since we are working in the classically forbidden region, we're *outside* the region where

$\widetilde{\varphi}_0 = F \cdot \varphi_0 \neq 0$, and that's exactly the region where $p \notin [v-b, v+b]$, so $(b-|z|)t = (b-|p-v|)t > 0$. Note that

$$|(b-|z|)t|^2 = \begin{cases} t^2|b+z|^2, & z \in [-b,-a], \\ t^2|b-z|^2, & z \in [a,b]. \end{cases}$$

Therefore, we can write

$$\begin{aligned} |I_1| &\leqslant \frac{1}{2}C_0 D_2 \epsilon^{3/2}\left[\frac{b-a}{t^2} + \frac{b-a}{t^2}\right] + C_1 D_1 \epsilon \left[\frac{b-a}{t^2} + \frac{b-a}{t^2}\right] \\ &+ \frac{1}{2}C_2 D_0 \epsilon^{1/2}\left[\frac{b-a}{t^2} + \frac{1}{t^2}\frac{2a}{b(b-a)} + \frac{b-a}{t^2}\right] \\ &= \frac{\epsilon^{3/2}}{t^2}\cdot C_0 D_2 (b-a) + \frac{\epsilon}{t^2}\cdot 2C_1 D_1 (b-a) + \frac{\epsilon^{1/2}}{t^2}\left(C_2 D_0 (b-a) + \frac{2a}{b(b-a)}\right) \\ &= K_0 \frac{\epsilon^{3/2}}{t^2} + K_1 \frac{\epsilon}{t^2} + K_2 \frac{\epsilon^{1/2}}{t^2}. \end{aligned}$$

- $\mathbf{I_2}$: Again, we bring the absolute values inside the integral and find that

$$\begin{aligned} |I_2| &\leqslant 2\epsilon^{3/2} t \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu\, \frac{|g'(\mu)| \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)}{|y - vt - \epsilon^{1/2}\mu t|^3} \\ &\leqslant 2C_0 \epsilon^2 t \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu\, \frac{|F'(v+\sqrt{\epsilon}\mu)| \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)}{|y - vt - \epsilon^{1/2}\mu t|^3} + 2C_1 \epsilon^{3/2} t \int_{-b/\sqrt{\epsilon}}^{b/\sqrt{\epsilon}} d\mu \frac{|F(v+\sqrt{\epsilon}\mu)| \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)}{|y - vt - \epsilon^{1/2}\mu t|^3}. \end{aligned}$$

Changing the integration variable to z gives

$$|I_2| \leqslant 2C_0 \epsilon^{3/2} t \int_{-b}^{b} dz\, \frac{|F'(v+z)| \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)}{|y - vt - zt|^3} + 2C_1 \epsilon t \int_{-b}^{b} dz \frac{|F(v+z)| \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)}{|y - vt - zt|^3}.$$

We can again invoke Taylor's theorem as we did above on the interval $[-b,-a]$ to get that, for $z \in [-b,-a]$,

$$|F^{(k)}(v+z)| \leqslant D_{k+3} \int_{-b}^{z} (z-w)^2 dw = \frac{1}{3} D_{k+3}|z+b|^3, \quad \text{where } D_j = \frac{1}{2} \sup_{z \in [-b,-a]} |F^{(j)}(v+z)| < \infty.$$

For $z \in [a, b]$, we find

$$|F^{(k)}(v+z)| \leqslant \frac{1}{3} D_{k+3} |z-b|^3.$$

The estimates for $|y - vt - zt|$ obtained above still hold: i.e., $|y - vt - zt|^3 \geqslant |bt - |z|t|^3$. And we have

$$|(b-|z|)t|^3 = \begin{cases} t^3|b+z|^3, & z \in [-b,-a], \\ t^3|z-b|^3, & z \in [a,b]. \end{cases}$$

Putting all of these estimates together, we can write

$$\begin{aligned}
|I_2| &\leqslant 2C_0\epsilon^{3/2} t \left[\int_{-b}^{-a} dz \, \frac{\frac{1}{3}D_4|z+b|^3}{t^3|b-|z||^3} + \int_a^b dz \, \frac{\frac{1}{3}D_4|z-b|^3}{t^3|b-|z||^3} \right] \\
&\quad + 2C_1\epsilon t \left[\int_{-b}^{-a} dz \, \frac{\frac{1}{3}D_3|z+b|^3}{t^3|b-|z||^3} + \int_{-a}^{a} dz \, \frac{1}{t^3|b-|z||^3} + \int_a^b dz \, \frac{\frac{1}{3}D_3|z-b|^3}{t^3|b-|z||^3} \right] \\
&\leqslant \frac{2}{3} C_0 D_4 \frac{\epsilon^{3/2}}{t^2} \cdot 2(b-a) + \frac{2}{3} C_1 D_3 \frac{\epsilon}{t^2} \cdot 2(b-a) + 2C_1 \frac{\epsilon}{t^2} \cdot \frac{(2b-a)a}{(b-a)^2 b^2} \\
&= J_0 \frac{\epsilon^{3/2}}{t^2} + J_1 \frac{\epsilon}{t^2} + J_2 \frac{\epsilon^{1/2}}{t^2}.
\end{aligned}$$

- $\mathbf{I_3}$: The integral I_3 is very similar to I_2, so we don't have to do very much work here.

$$|I_3| = \frac{1}{2}|I_2| \leqslant \frac{1}{2}\left(J_0 \frac{\epsilon^{3/2}}{t^2} + J_1 \frac{\epsilon}{t^2} + J_2 \frac{\epsilon^{1/2}}{t^2} \right).$$

- $\mathbf{I_4}$: Bringing absolute values inside the integral, changing the integration variable to z, and using the same bound for $|\varphi_0|$ used above, we obtain

$$|I_4| \leqslant 3C_0\epsilon^{3/2} t^2 \int_{-b}^{b} dz \, \frac{|F(v+z)| \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)}{|y-vt-zt|^4}.$$

We can argue using Taylor's theorem as above that

$$|F(v+z)| \leqslant \frac{D_4}{24} \begin{cases} |z+b|^4 & \text{for } z \in [-b,-a], \\ |z-b|^4 & \text{for } z \in [a,b]. \end{cases}$$

It follows that, since we still have $|(y - vt) - zt| \geqslant |b - |z||$,

$$
\begin{aligned}
|I_4| &\leqslant 3C_0 \epsilon^{3/2} t^2 \left[\int_{-b}^{-a} dz \; \frac{D_4}{24} \frac{|z+b|^4}{t^4 |b - |z||^4} + \int_{-a}^{a} dz \; \frac{1}{t^4 |b - |z||^4} + \int_{a}^{b} dz \; \frac{D_4}{24} \frac{|z-b|^4}{t^4 |b-|z||^4} \right] \\
&\leqslant 3C_0 \frac{\epsilon^{3/2}}{t^2} \left(\frac{D_4}{12}(b-a) + \frac{2(a^3 - 3a^2 b + 3ab^2)}{3b^3 (b-a)^3} \right) = L \frac{\epsilon^{3/2}}{t^2}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&|\Upsilon(y,t) \cdot \mathcal{X}_{\mathrm{f}}(x,y,t)| \\
&\quad \leqslant t^{-2} (2\pi)^{-1/2} \left[\left(K_0 + \frac{3}{2} J_0 + L \right) \epsilon^{5/4} + \left(K_1 + \frac{3}{2} J_1 \right) \epsilon^{3/4} + \left(K_2 + \frac{3}{2} J_2 \right) \epsilon^{1/4} \right].
\end{aligned}
$$

It follows from Hölder's inequality that, for all sufficiently large T,

$$
\begin{aligned}
&\left\| V_2(x-y) \; e^{iv^2 t/(2\epsilon)} \mathscr{F}_{\epsilon}^{-1} \Big[F(p) \; \varphi_0(1, 1+it, \epsilon, v, -vt, p) \Big](y) \; \eta_1(x) \right\|_{L^2(dxdy)} \\
&\qquad = \| \Upsilon(y,t) \; V_2(x,y) \; \eta_1(x) \, (\mathcal{X}_{\mathrm{a}} + \mathcal{X}_{\mathrm{f}}) \|_{L^2(dxdy)} \\
&\qquad \leqslant \| \Upsilon \cdot \mathcal{X}_{\mathrm{a}} \cdot V_2 \cdot \eta_1 \| + \| \Upsilon(y,t) \cdot \mathcal{X}_{\mathrm{f}} \|_{\infty} \, \|V_2\|_{\infty} \, \|\eta_1(x)\|_{L^2(dx)} \\
&\leqslant K t e^{-d_0 t} + t^{-2} (2\pi)^{-1/2} \left[\left(K_0 + \frac{3}{2} J_0 + L \right) \epsilon^{5/4} + \left(K_1 + \frac{3}{2} J_1 \right) \epsilon^{3/4} + \left(K_2 + \frac{3}{2} J_2 \right) \epsilon^{1/4} \right].
\end{aligned}
$$

So, for large times, we have the estimate

$$
\begin{aligned}
\left\| \left(i \frac{\partial}{\partial t} - H(\epsilon) \right) \Psi_a(x,y,t) \right\| &\leqslant C_0 e^{-\gamma t} + C_2 e^{-2c|vt - a_2|} + K t e^{-d_0 t} + t^{-2} (2\pi)^{-1/2} \Bigg[\left(K_0 + \frac{3}{2} J_0 + L \right) \epsilon^{5/4} \\
&\qquad + \left(K_1 + \frac{3}{2} J_1 \right) \epsilon^{3/4} + \left(K_2 + \frac{3}{2} J_2 \right) \epsilon^{1/4} \Bigg] \\
&= C_0 e^{-\gamma t} + M e^{-2cvt} + K t e^{-d_0 t} + t^{-2} \mathcal{O}(\epsilon^{1/4}).
\end{aligned}
$$

Integrating this in time on (T, ∞), the magic lemma gives

$$
\|\Psi - \Psi_a\| \leqslant \frac{e^{-\gamma T}}{\gamma} + \frac{K}{d^2} \, (1 + dT) \, e^{-dT} + \frac{1}{T} \mathcal{O}(\epsilon^{1/4}). \tag{2.17}
$$

2.3.3 Putting it all together

We have two sets of estimates for $\|\Psi - \Psi_a\|$ – one given by (2.11), the other by (2.17). We need to pick a time T to switch from one set to the other and still get the decay in ϵ that we need. Suppose we have $T = |\log\epsilon|$. (Recall that ϵ is a constant we are allowing to be small.) Then the magic lemma gives

$$\begin{aligned}\|\Psi - \Psi_a\| \leqslant & |\log\epsilon| \left(A_0\ \epsilon^{-1/2}\ e^{-a^2/\epsilon} + A_1\ \epsilon^{1/2}\ e^{-b^2/\epsilon} + \mathcal{O}\left(\epsilon^{1/2}\ e^{-b^2/\epsilon}\right)\right) \\ & + A_2\left(|\log\epsilon|\sqrt{1+|\log\epsilon|^2} + \sinh^{-1}|\log\epsilon|\right) \\ & + \frac{e^{-\gamma|\log\epsilon|}}{\gamma} + \frac{K}{d^2}\left(1+d|\log\epsilon|\right)e^{-d|\log\epsilon|} + \frac{1}{|\log\epsilon|}\mathcal{O}(\epsilon^{1/4}).\end{aligned}$$

Taking $\epsilon \to 0$ on the right hand side, we obtain the desired result of zero! Hence our proposed Ψ_a is a good approximation, uniformly in time for $t \in [0,\infty)$, to an exact solution to the three-body problem. The calculation for $t \in (-\infty, 0]$ should be basically identical.

Chapter 3

A negative result for uniformity

In Chapter 2, we considered a model in which the mass of one of the large particles was *actually* infinite. Of course, we would like to obtain a similar result about the uniform closeness of the wave functions in time for a case where both large masses are actually described by (say) localized Gaussian wave packets – that is, a case where *both* large masses are allowed to *move*. However, as discussed in Chapter 1, there is little hope for finding a result of this form if all the masses are allowed to move. This is the problem we now examine by considering an *exactly solvable* two-particle system.

3.1 Problem setup

3.1.1 The IP solution

Let's consider a model consisting only of two particles of masses $\mu_1 = \epsilon^{-1}$ and $\mu_3 = 1$ in one dimension. It's governed by the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi = \left(-\frac{\epsilon}{2}\frac{\partial^2}{\partial r_1^2} - \frac{1}{2}\frac{\partial^2}{\partial r_3^2} + V(r_3 - r_1)\right)\psi. \tag{3.1}$$

Let's apply the IP approximation to this problem, assuming $v_1 = 0$ and making the simplifying assumption that $r_1(t) = 0$. The IP Hamiltonian is just

$$H(t) = -\frac{1}{2}\frac{\partial^2}{\partial r_3^2} + V(r_3).$$

We will continue to study the channel where μ_3 is bound to μ_1, so let us suppose there exists a bound state α satisfying

$$H\alpha = \left(-\frac{1}{2}\frac{\partial^2}{\partial r_3^2} + V(r_3)\right)\alpha(r_3) = E\alpha(r_3). \tag{3.2}$$

Observe that since α solves the above equation,

$$\begin{aligned} |\alpha''| &= 2|V - E||\alpha| \\ &\leqslant 2(\|V\|_\infty + |E|)|\alpha| \\ &\leqslant 2C_\gamma(\|V\|_\infty + |E|)e^{-\gamma|r_3|} \\ &\leqslant 2C_\gamma(\|V\|_\infty + |E|), \end{aligned}$$

so α'' is bounded, and since H is self-adjoint, α can be chosen to be real. To obtain the full IP model solution, we will multiply the bound state α by a Gaussian in r_1, localized in position (and momentum) space near zero, simply because we're interested in something that solves the free Schrödinger equation in r_1:

$$\left(i\epsilon\frac{\partial}{\partial t} + \frac{\epsilon^2}{2}\Delta_1\right)\varphi_1 = 0 \implies \varphi_1(r_1, t) = \varphi_0\left(1 + it, 1, \epsilon, 0, 0, r_1\right),$$

So, the IP solution has the particularly simple form

$$\boxed{\psi(r_1, r_3, t) = \varphi_0\left(1 + it, 1, \epsilon, 0, 0, r_1\right) \cdot e^{-iEt}\alpha(r_3)}$$

3.1.2 The exact solution

But this model can be solved *exactly*! Since $M = 1 + \epsilon^{-1} = \dfrac{1+\epsilon}{\epsilon}$ and $\mu = \dfrac{\epsilon^{-1}}{1+\epsilon^{-1}} = \dfrac{1}{1+\epsilon}$, in center of mass coordinates $R = \dfrac{\epsilon^{-1} r_1 + r_3}{\epsilon^{-1}+1} = \dfrac{r_1 + \epsilon r_3}{1+\epsilon}$ and $x = r_3 - r_1$,

$$\begin{aligned} H &= -\frac{\epsilon}{2(1+\epsilon)}\frac{\partial^2}{\partial R^2} - \frac{\epsilon+1}{2}\frac{\partial^2}{\partial x^2} + V(x) \\ &= -\frac{\epsilon}{2(1+\epsilon)}\frac{\partial^2}{\partial R^2} + \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x) - \frac{\epsilon}{2}\frac{\partial^2}{\partial x^2}\right). \end{aligned}$$

Since this Hamiltonian separates, we want to construct a solution $\psi(x, R, t) = \mathcal{R}(R, t)X(x)$ to the Schrödinger equation

$$\left(i\frac{\partial \psi}{\partial t} + \frac{\epsilon}{2(1+\epsilon)}\frac{\partial^2}{\partial R^2}\right)\mathcal{R}(R,t) - \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x) - \frac{\epsilon}{2}\frac{\partial^2}{\partial x^2}\right)X(x) = 0.$$

By the discussion on pgs. 15-16 of [16], for small enough ϵ, our assumption (3.2) on α implies the existence of a unique eigenvector $\eta(x) = \alpha(x) + O(\epsilon)$ such that

$$\left(-\frac{1}{2}(1+\epsilon)\frac{\partial^2}{\partial x^2} + V(x)\right)\eta(x) = E'\eta(x).$$

Following [9], it is physically reasonable to set

$$\mathcal{R}(R,t) = \varphi_0\left(1 + \frac{it}{1+\epsilon}, 1, \epsilon, 0, 0, R\right) = \varphi_0\left(1 + \frac{it}{1+\epsilon}, 1, \epsilon, 0, 0, \frac{r_1 + \epsilon r_3}{1+\epsilon}\right).$$

It can then be easily checked that the product wavefunction

$$\boxed{\Psi(r_1, r_3, t) = e^{-iE't}\varphi_0\left(1 + \frac{it}{1+\epsilon}, 1, \epsilon, 0, 0, \frac{r_1 + \epsilon r_3}{1+\epsilon}\right)\eta(r_3 - r_1)}$$

solves (3.1). Note that we have localized the wave packet describing R about the origin. This is only an approximation, but it *should* be a more and more reasonable approximation as we take $\epsilon \to 0$.

3.1.3 What we want to bound

Following the lead of Chapter 2, we are interested in showing that

$$\Big\| e^{-iEt}\varphi_0\left(1+it,1,\epsilon,0,0,r_1\right)\alpha(r_3) \\ -e^{-iE't}\varphi_0\left(1+\frac{it}{1+\epsilon},1,\epsilon,0,0,\frac{r_1+\epsilon r_3}{1+\epsilon}\right)\eta(r_3-r_1)\Big\|_{L^2(dr_1dr_3)} \tag{3.3}$$

goes to zero uniformly in t as $\epsilon \to 0$. We use the triangle inequality to bound the above norm from above by a sum of three pieces, where all norms henceforth are assumed to be norms in $L^2(dr_1dr_3)$ unless otherwise stated:

$$\begin{aligned}(3.3) \leqslant &\left\|\varphi_0\left(1+\frac{it}{1+\epsilon},1,\epsilon,0,0,\frac{r_1+\epsilon r_3}{1+\epsilon}\right)(\alpha(r_3-r_1)-\eta(r_3-r_1))\right\| \\ &+\left\|\left(e^{-iEt}\varphi_0(1+it,1,\epsilon,0,0,r_1)-e^{-iE't}\varphi_0\left(1+\frac{it}{1+\epsilon},1,\epsilon,0,0,\frac{r_1+\epsilon r_3}{1+\epsilon}\right)\right)\alpha(r_3)\right\| \\ &+\left\|\varphi_0\left(1+\frac{it}{1+\epsilon},1,\epsilon,0,0,\frac{r_1+\epsilon r_3}{1+\epsilon}\right)(\alpha(r_3)-\alpha(r_3-r_1))\right\| \\ &= P_1(\epsilon,t)+P_2(\epsilon,t)+P_3(\epsilon,t).\end{aligned}$$

We now demonstrate that the norm P_2 above does *not* go to zero uniformly in time.

3.1.4 An attempt at bounding P_2

It can be shown that

$$\varphi_0\Big(1+\frac{it}{1+\epsilon},1,\epsilon,0,0,\frac{r_1+\epsilon r_3}{1+\epsilon}\Big) = (1+\epsilon)^{1/2}\varphi_0\left(1+it+\epsilon,\frac{1}{1+\epsilon},\epsilon,-\epsilon r_3,0,r_1\right)$$

If we let $A_1 = 1+it(1+\epsilon)^{-1}$, $B_1 = 1$, $A_2 = 1+it+\epsilon$, and $B_2 = (1+\epsilon)^{-1}$, we find that $\overline{A}_iB_i + A_i\overline{B}_i = 2$, putting us in the position to use Proposition 4 of [10] when we set about trying to estimate

$$
\begin{aligned}
P_2(\epsilon) &= \Big\| \big(e^{-iEt}\varphi_0(1+it,1,\epsilon,0,0,r_1) \\
&\qquad -(1+\epsilon)^{1/2}e^{-iE't}\varphi_0\left(1+it+\epsilon,(1+\epsilon)^{-1},\epsilon,-\epsilon r_3,0,r_1\right)\big)\,\alpha(r_3)\Big\| \\
&= \int_{\mathbb{R}} dr_1 \int_{\mathbb{R}} dr_3 \left| e^{-iEt}\varphi_0(1+it,1,\epsilon,0,0,r_1) - (1+\epsilon)^{1/2}e^{-iE't}\varphi_0\left(1+it+\epsilon,(1+\epsilon)^{-1},\epsilon,-\epsilon r_3,0,r_1\right)\right|^2 |\alpha(r_3)|^2 \\
&= \int_{\mathbb{R}} dr_3 |\alpha(r_3)|^2 \int_{\mathbb{R}} dr_1 \left| e^{-iEt}\varphi_0(1+it,1,\epsilon,0,0,r_1) - (1+\epsilon)^{1/2}e^{-iE't}\varphi_0\left(1+it+\epsilon,(1+\epsilon)^{-1},\epsilon,-\epsilon r_3,0,r_1\right)\right|^2.
\end{aligned}
$$

Define

$$
f_c(r_1,t) = e^{-iEt}\varphi_0(1+it,1,\epsilon,0,0,r_1), \qquad f_d(r_1,r_3,t) = e^{-iE't}\varphi_0\left(1+it+\epsilon,(1+\epsilon)^{-1},\epsilon,-\epsilon r_3,0,r_1\right).
$$

Then

$$
\begin{aligned}
P_2(\epsilon) &= \int_{\mathbb{R}} dr_3 |\alpha(r_3)|^2 \int_{\mathbb{R}} dr_1 \left| f_c(r_1,t) - (1+\epsilon)^{1/2} f_d(r_1,r_3,t)\right|^2 \\
&= \int_R dr_3 |\alpha(r_3)|^2 \left\| f_c(r_1,t) - (1+\epsilon)^{1/2} f_d(r_1,r_3,t)\right\|^2_{L^2(dr_1)}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\left\| f_c(r_1,t) - (1+\epsilon)^{1/2} f_d(r_1,r_3,t)\right\|^2_{L^2(dr_1)} &= \left\langle f_c - (1+\epsilon)^{1/2} f_d, f_c - (1+\epsilon)^{1/2} f_d\right\rangle \\
&= \|f_c\|^2 + (1+\epsilon)\|f_d\|^2 - 2(1+\epsilon)^{1/2}\operatorname{Re}\langle f_c, f_d\rangle \\
&= 2+\epsilon - 2(1+\epsilon)^{1/2}\operatorname{Re}\langle f_c, f_d\rangle.
\end{aligned}
$$

So

$$
P_2(\epsilon) = \int_{\mathbb{R}} dr_3 |\alpha(r_3)|^2 \left(2+\epsilon-2(1+\epsilon)^{1/2}\operatorname{Re}\langle f_c,f_d\rangle\right) = \int_{\mathbb{R}} f_\epsilon(r_3)dr_3.
$$

Proposition 4 of [10] allows us to conclude

$$
\langle f_c, f_d\rangle = e^{-i(E'-E)t}\left(\frac{2(1+\epsilon)}{2+\epsilon(2+\epsilon+it)}\right)^{1/2}\exp\left(-\frac{1}{2}\epsilon r_3^2\frac{1}{(2+\epsilon(2+\epsilon+it))}\right) \to 1 \quad \text{as } \epsilon \to 0.
$$

Hence

$$
\lim_{\epsilon\to 0} f_\epsilon(r_3) = |\alpha(r_3)|^2 \lim_{\epsilon\to 0}\left(2+\epsilon-2(1+\epsilon)^{1/2}\operatorname{Re}\langle f_c,f_d\rangle\right) = 0.
$$

Since, for any $\epsilon > 0$ and for all $t \in \mathbb{R}$,

$$|\langle f_c, f_d \rangle| = \sqrt{\frac{2(1+\epsilon)}{\sqrt{t^2\epsilon^2 + (2+\epsilon(2+\epsilon))^2}}} \exp\left(-\frac{\epsilon r_3^2(2+\epsilon(2+\epsilon))}{2(t^2\epsilon^2 + (2+\epsilon(2+\epsilon))^2)}\right) \leqslant 1,$$

we also have the (r_3) pointwise bound

$$\begin{aligned} |f_\epsilon(r_3)| &\leqslant |a(r_3)|^2 \left(2 + \epsilon + 2(1+\epsilon)^{1/2} |\mathrm{Re}\langle f_c, f_d\rangle|\right) \\ &\leqslant |a(r_3)|^2 \left(2 + \epsilon + 2(1+\epsilon)^{1/2} |\langle f_c, f_d\rangle|\right) \\ &\leqslant |a(r_3)|^2 \left(2 + \epsilon + 2(1+\epsilon)^{1/2}\right), \end{aligned}$$

so $f_\epsilon \in L^1(dr_3)$ for all ϵ. Since

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} dr_3 |\alpha(r_3)|^2 \left(2 + \epsilon + 2(1+\epsilon)^{1/2}\right) = 4 \int_{\mathbb{R}} dr_3 |\alpha(r_3)|^2 = \int_{\mathbb{R}} dr_3 \lim_{\epsilon \to 0} \left(|\alpha(r_3)|^2 (2 + \epsilon + 2(1+\epsilon)^{1/2})\right),$$

we can invoke the generalized dominated convergence theorem (see, for instance, [19], pg. 89):

$$\lim_{\epsilon \to 0} P_2(\epsilon) = \lim_{\epsilon \to 0} \int_{\mathbb{R}} h_\epsilon(r_3) dr_3 = \int_{\mathbb{R}} \left(\lim_{\epsilon \to 0} h_\epsilon(r_3)\right) dr_3 = 0. \tag{3.4}$$

However, these results are *not uniform* in time – an assertion which is the cornerstone of this chapter.

Proposition 3.1.1. *The result* (3.4) *is not uniform in time.*

Proof. The quantifiers for time uniformity are as follows:

$$(\forall \eta > 0)(\exists \delta > 0)(\forall t \in \mathbb{R})(\forall \epsilon \in (0, \delta))(P(t, \epsilon) < \eta).$$

The negation of this statement is:

$$\boxed{(\exists \eta > 0)(\forall \delta > 0)(\exists t \in \mathbb{R})(\exists \epsilon \in (0, \delta))(P(t, \epsilon) \geqslant \eta)}$$

To simplify notation, let $A(\epsilon) = 2 + 2\epsilon + \epsilon^2$. Then for any $t \in \mathbb{R}$, we have

$$\begin{aligned} P(t, |t|^{-1}) &= \int_{\mathbb{R}} |\alpha(r_3)|^2 \left|2 + |t|^{-1} - 2(1 + |t|^{-1})^{1/2} \operatorname{Re}\langle f_c, f_d\rangle\right| dr_3 \\ &\geqslant \int_{\mathbb{R}} |\alpha(r_3)|^2 \left(2 + |t|^{-1} - 2(1 + |t|^{-1})^{1/2} |\operatorname{Re}\langle f_c, f_d\rangle|\right) dr_3 \\ &\geqslant \int_{\mathbb{R}} |\alpha(r_3)|^2 \left(2 + |t|^{-1} - 2(1 + |t|^{-1})^{1/2} |\langle f_c, f_d\rangle|\right) dr_3 \\ &= \int_{\mathbb{R}} |\alpha(r_3)|^2 \left(2 + |t|^{-1} - \frac{2\sqrt{2}(1 + |t|^{-1})}{(A(|t|^{-1})^2 + 1)^{1/4}} \exp\left(-\frac{r_3^2(2 + 2|t|^{-1} + |t|^{-2})}{2(2|t| + 2 + |t|^{-1} + 1}\right)\right) dr_3 \\ &= \int_{\mathbb{R}} \mathcal{F}_t(r_3) dr_3. \end{aligned}$$

It is easy to see that $\mathcal{F}_t(r_3) \to 2\left(1 - \sqrt[4]{\frac{4}{5}}\right)|\alpha(r_3)|^2 = \mathcal{F}(r_3)$, pointwise in r_3, as $t \to \infty$. Since α is square-integrable, $\mathcal{F}$ is integrable. Note also that there exist positive constants C and a such that

$$|\mathcal{F}_t(r_3)| \leqslant Ce^{-2a|r_3|}\left(2 + |t|^{-1} + 2(1 + |t|^{-1})^{1/2}\right) = \mathcal{G}_t(r_3),$$

and that

$$\lim_{t\to\infty} \int_{\mathbb{R}} G_t(r_3) dr_3 = \lim_{t\to\infty} (2 + |t|^{-1} + 2(1 + |t|^{-1})^{1/2}) \int_{\mathbb{R}} Ce^{-2a|r_3|} dr_3 = \frac{8C}{a} = \int_{\mathbb{R}} \lim_{t\to\infty} G_t(r_3) dr_3.$$

We can therefore once again invoke the generalized dominated convergence theorem (see [19]) and conclude that

$$\limsup_{t\to\infty} P(t, |t|^{-1}) \geqslant \lim_{t\to\infty} \int_{\mathbb{R}} \mathcal{F}_t(r_3) dr_3 = 2\left(1 - \sqrt[4]{\frac{4}{5}}\right) \|\alpha\|^2_{L^2(dr_3)} \approx 0.109 \|\alpha\|^2_{L^2(dr_3)} > 0.$$

Hence we can take (say) $\eta = \frac{1}{10} \|\alpha\|^2_{L^2(dr_3)}$. Then, for any given $\delta > 0$, we need only find a $|t|$ so large that (1) $\epsilon = |t|^{-1} \in (0, \delta)$ and (2) $P(t, |t|^{-1}) > \frac{1}{10} \|\alpha\|^2_{L^2(dr_3)}$ (such a t always exists, by the definition of the lim sup). This proves that $P(\epsilon, t)$ does not go to zero uniformly in time. □

3.2 Some remarks

We remark that the crucial difference between the approach we have just taken and the approach taken in the last chapter is that μ_1, which is *initially* located at the origin, has in this chapter been allowed to move. More precisely, it has been described by a Gaussian wave packet which, though initially localized near the origin, spreads in time. The calculation in Proposition 3.1.1 tells us that, by removing the "anchored" infinite mass and placing both large masses on a equal footing, we have introduced enough bedlam into the system to make a uniform result like the one obtained in Chapter 2 unattainable.

In light of this, it seems reasonable to attempt a weaker result about the S-matrix elements in the situation where all three masses are allowed to move. This is the mantle we take up in the next chapter.

Chapter 4

Results for the S-matrix

4.1 A word on notation

In this chapter, we will need to work with several norms on various Hilbert and Banach spaces. Note that $\|\cdot\|_p$, $1 \leqslant p \leqslant \infty$, will always refer to the L^p norm on $\mathbb{R}$ with respect to Lebesgue measure in the appropriate variable, which should be clear from context. We use $\|\cdot\|_{\mathscr{L}(X,Y)}$ to denote the space of bounded linear operators from X to Y, where both X and Y are normed vector spaces. The symbol $\mathcal{X}$ is used to denote an indicator function.

4.2 Statement and formulation

We once again study the one-dimensional scattering theory for a system of two large particles and one light particle, as depicted in Figure 4.2, removing Chapter 2's restriction that μ_1 have infinite mass but setting $\mu_1 = \mu_2 = \epsilon$. In the (r_1, r_2, r_3) system, the governing Hamiltonian has the form

$$\widetilde{H} = -\frac{\epsilon}{2}\Delta_1 - \frac{\epsilon}{2}\Delta_2 - \frac{1}{2}\Delta_3 + V_1(r_3 - r_1) + V_2(r_3 - r_2), \tag{4.1}$$

and the dynamics are controlled by the associated time-dependent Schrödinger equation $i\frac{\partial}{\partial t}\Psi = \widetilde{H}\Psi$. We will restrict our attention to distant-past and distant-future behavior characterized by the scattering channel defined by the cluster decomposition $\{C_1, C_2\}$ shown in Figure 4.1. Note that this is the same cluster decomposition discussed in the introduction!

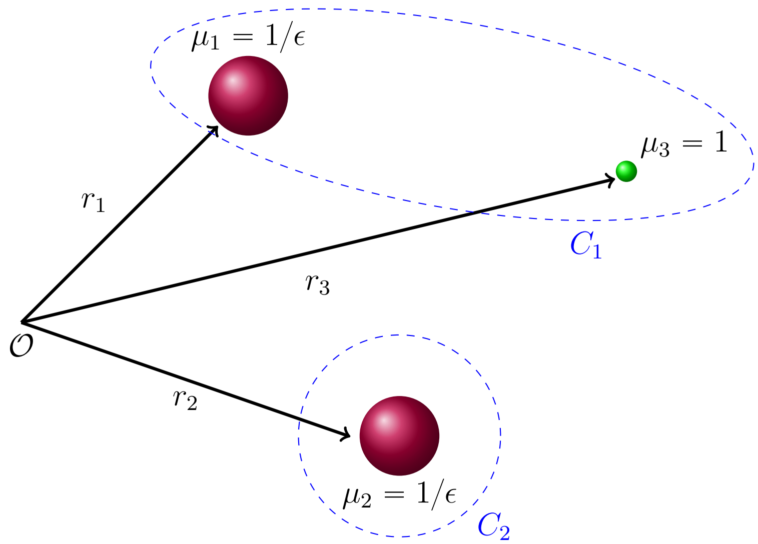

Figure 4.1: The cluster decomposition we will study for both distant-past and distant-future behavior.

We therefore are concerned with estimating the S-matrix elements defined by

$$S = (\Omega_1^-)^* \Omega_1^+,$$

where, as in (1.4), we have

$$\Omega_1^\pm = \operatorname*{s-lim}_{t\to\mp\infty} e^{iHt} e^{-iH_1 t},$$

these operators expressed in a suitable set of Jacobi coordinates (defined shortly). We will suppose that both V_1 and V_2 are elements of $C_0^\infty(\mathbb{R})$, the infinitely differentiable functions of compact support. As such, there exist four positive constants C_1, R_1 and C_2, R_2 such that $|V_1| \leqslant C_1 \mathcal{X}(|x| \leqslant R_1)$ and $|V_2| \leqslant C_2 \mathcal{X}(|x| \leqslant R_2)$.

We will find it convenient to work with two *different* sets of clustered Jacobi coordinates – one which is appropriate for the full three-body treatment of the channel under consideration, and one that is more naturally suited to the impact parameter model.

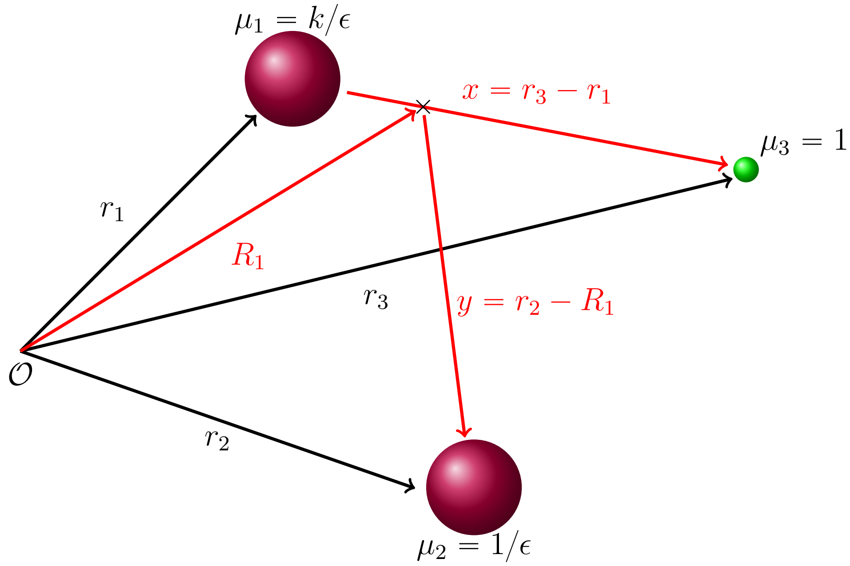

Figure 4.2: The clustered Jacobi coordinate setup for the full three-body analysis. We constrain these particles to live in one spatial dimension, but they are shown here in two for clarity. Note that the red vector R_1 locates the center of mass of the cluster C_1 consisting of particles 1 and 3.

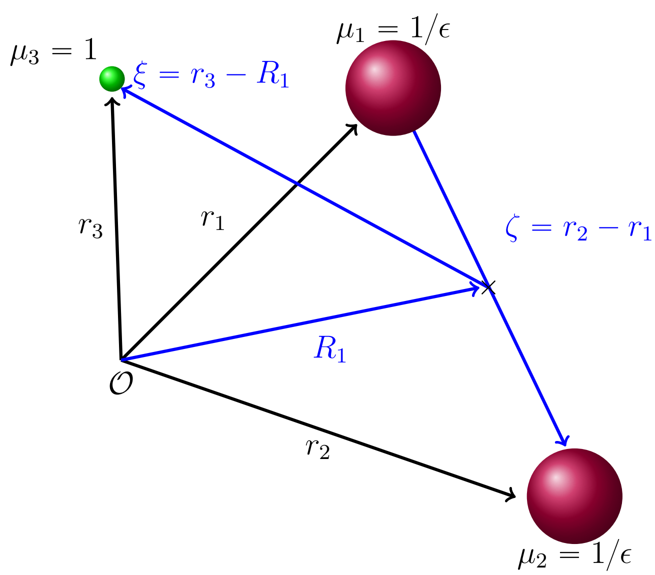

Figure 4.3: The Jacobi coordinate setup for the IP analysis. The blue vector R_1 locates the center of mass of the cluster C_1 consisting of particles 1 and 2.

4.2.1 Jacobi coordinates for the full three-body problem

The standard construction of the channel wave operators for the channel of interest (see, e.g., [18], [5], or [25]) involves the Jacobi coordinate system shown in Figure 4.2. This system allows us to easily remove the cluster-center-of-mass relative coordinate from the potentials that appear in the channel Hamiltonian. The coordinates x and y were listed previously in (1.2); for the particle masses shown in figure, those formulas become

$$x = r_3 - r_1, \qquad y = r_2 - \frac{r_1 + \epsilon r_3}{1+\epsilon}. \tag{4.2}$$

The corresponding Hamiltonian governing the full system (with the kinetic energy of the center of mass removed) is therefore given by

$$H(x,y) = -\frac{1}{2}\left(1+\epsilon\right)\Delta_x - \frac{\epsilon}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right)\Delta_y + V_1(x) + V_2\left(\frac{x}{1+\epsilon} - y\right),$$

and the channel Hamiltonian we are interested in is obtained by just dropping V_2:

$$H_1(x,y) = H_1 = \underbrace{-\frac{1}{2}\left(1+\epsilon\right)\Delta_x + V_1(x)}_{H_1^x} \underbrace{-\frac{\epsilon}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right)\Delta_y}_{H_1^y}. \tag{4.3}$$

We assume that there is a single, nondegenerate bound state $\eta_1 \in L^1(\mathbb{R}, dx)$ of H_1^x with corresponding energy $E_1^\epsilon < 0$; i.e.,

$$\left(-\frac{1}{2}(1+\epsilon)\Delta_x + V_1(x)\right)\eta_1 = E_1^\epsilon \eta_1.$$

We remind the reader that a corollary to the O'Connor-Combes-Thomas theorem ([16], pgs. 198-201) furnishes positive constants C_d and d such that $|\eta_1(x)| \leqslant C_d e^{-d|x|}$.

We now construct the channel wave operators

$$\Omega_1^\pm = \operatorname*{s-lim}_{t\to\mp\infty} e^{iHt}e^{-iH_1 t}, \qquad \Omega_1^\pm : L^2(\mathbb{R}, dx)\otimes L^2(\mathbb{R}, dy).$$

These strong limits exist for our C_0^∞ potentials by, for instance, Theorem XI.35 of [18]. The associated S-matrix element, which helps supply probabilistic information about incoming and outgoing asymptotic states, is given by $S_{11} = (\Omega_1^-)^*\Omega_1^+$.

4.2.2 Jacobi coordinates for the impact parameter model

In the impact parameter model, trivialization of the motion of the massive particles ① and ② allows us to consider a single-particle scattering problem involving a particle of unit mass located by the vector r_3 interacting with the *time-dependent* potential

$$V(r_3, t) = V_1(r_3 - v_1 t) + V_2(r_3 - v_2 t),$$

where we have made the classical approximations $r_1 \mapsto v_1 t$ and $r_2 \mapsto v_2 t$. We take the standard approach of assuming $v_1 \neq v_2$ to deny ourselves the easy "out" of transitioning to a moving reference frame and trivializing the time dependence out of the problem. We also label the heavy particles in such a way that $v_2 > v_1$, implying $v > 0$.

If we choose to work with the clustered Jacobi coordinate system (ξ, ζ) depicted in Figure 4.3, then we are making the change of variables

$$\zeta = r_2 - r_1, \qquad \xi = r_3 - \frac{r_1 + r_2}{2}.$$

The Hamiltonian $\widetilde{H}$ written in terms of these coordinates can be shown to have the form

$$H(\zeta, \xi) = -\epsilon \Delta_\zeta - \frac{\epsilon + 2}{4} \Delta_\xi + V_1\left(\xi + \frac{\zeta}{2}\right) + V_2\left(\xi - \frac{\zeta}{2}\right). \tag{4.4}$$

The utility of the coordinate system in Figure 4.3 can now be seen – the ζ variable is trivialized in the (ξ, ζ) system under the IP model, whereas neither of the variables in (4.2) would be trivialized. As we take $\epsilon \to 0$, the coordinate $\zeta = r_2 - r_1$ approaches $v_2 t - v_1 t = vt$. So, the correct form for the impact parameter Hamiltonian is actually

$$H(t) = -\frac{1}{2}\Delta_\xi + V_1\left(\xi + \frac{vt}{2}\right) + V_2\left(\xi - \frac{vt}{2}\right). \tag{4.5}$$

This is the Hamiltonian of a single particle of unit mass located at ξ in an external, time-dependent potential given by $V(\xi, t) = V_1\left(\xi + \frac{vt}{2}\right) + V_2\left(\xi - \frac{vt}{2}\right)$. Our assumptions on V_1 and V_2 satisfy the hypotheses of Theorem 2.1 of Graf's work [6] (which is based on the results in Yajima's paper [30]), so $H(t)$ generates a unitary propagator $U(t, s)$. The same theorem gives us the existence of a unitary propagator $U_1(t,s)$ generated by the channel Hamiltonian

$$H_1(t) = -\frac{1}{2}\Delta_\xi + V_1\left(\xi + \frac{vt}{2}\right).$$

We then define the *impact parameter wave operators* $\omega_1^\pm$ as follows:

$$\omega_1^\pm = \operatorname*{s-lim}_{t\to\mp\infty} U(0,t)U_1(t,0)e^{-iv\xi/2}, \qquad \omega_1^\pm : L^2(d\xi) \to L^2(d\xi).$$

(The presence of the phase $e^{-iv\xi/2}$.) The existence of the wave operators $\omega_1^\pm$ for our well-behaved potentials is guaranteed by Proposition 20.1 of [15]. Using these wave operators, we can of course define the *impact parameter S-matrix element* $\sigma_{11}(\xi)$ for this channel, given by

$$\sigma_{11}(\xi) = (\omega_1^-)^*\omega_1^+.$$

We now suppose there is a single, nondegenerate eigenvalue $E_1 < 0$ with corresponding eigenvector $\alpha_1 \in L^1(\mathbb{R}, d\xi)$ for the eigenvalue problem

$$\left(-\frac{1}{2}\Delta_\xi + V_1(\xi)\right)\alpha_1(\xi) = E_1\alpha(\xi).$$

Then, for every $t \in \mathbb{R}$, the function $\alpha_1\left(x + \frac{vt}{2}\right)$ satisfies

$$H_1(t)\alpha\left(\xi + \frac{vt}{2}\right) = \left(-\frac{1}{2}\Delta_\xi + V_1\left(\xi + \frac{vt}{2}\right)\right)\alpha_1\left(\xi + \frac{vt}{2}\right) = E_1\alpha\left(\xi + \frac{vt}{2}\right).$$

Of course, if this is the case, then

$$U_1(t,0)e^{-iv\xi/2}\alpha_1(\xi) = e^{-iE_1t}e^{-iv^2t/8}e^{-iv\xi/2}\alpha_1\left(\xi + \frac{vt}{2}\right). \tag{4.6}$$

Following the lead of Chapter 2, we seek a wave packet in ζ localized in position space around $\zeta = vt$ that satisfies the free time-dependent Schrödinger equation

$$\left(i\epsilon\frac{\partial}{\partial t} + \frac{\epsilon^2}{2}(2)\Delta_\zeta\right)\varphi_0(\zeta) = 0,$$

which can be read off from (4.4). This gives us a reduced mass of $\frac{1}{2}$, so our wave packet should be localized in momentum space about $\frac{v}{2}$. Let

$$\varphi_0(\zeta,t) = \exp\left(\frac{iv^2t}{4\epsilon}\right)\varphi_0\left(1+2it,1,\epsilon,vt,\frac{v}{2},\zeta\right)$$
$$\implies \varphi_0(\zeta) = \varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\zeta\right).$$

It follows that the impact parameter scattering operator σ_{11} acts on elements of the Hilbert space $L^2(d\xi d\zeta) = L^2(d\xi)\otimes L^2(d\zeta)$ as the tensor product $\sigma_{11}(\xi)\otimes I$.

4.2.3 Coordinate transformation

Naturally, when we carry out our calculations, we will need to choose one system – either (x,y) or (ξ,ζ) – with which to work. We will therefore find the following variable transformations useful: To go from (x,y) to (ξ,ζ), it can be shown that

$$x(\xi,\zeta) = \xi + \frac{\zeta}{2}, \qquad y(\xi,\zeta) = \frac{1}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right)\zeta - \frac{\epsilon}{1+\epsilon}\xi. \tag{4.7}$$

In the reverse direction, we have

$$\zeta(x,y) = y + \frac{\epsilon}{1+\epsilon}x, \qquad \xi(x,y) = \frac{1}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right)x - \frac{y}{2}. \tag{4.8}$$

Proposition 4.2.1. *The variable transformation given in* (4.7) *and* (4.8) *is unitary for every* ϵ.

Proof. We compute the Jacobian matrix:

$$J(x,y) = \begin{pmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} \\ \frac{\partial\zeta}{\partial x} & \frac{\partial\zeta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right) & -\frac{1}{2} \\ \frac{\epsilon}{1+\epsilon} & 1 \end{pmatrix},$$

from which it follows that

$$\det J(x,y) = \frac{2+\epsilon}{2+2\epsilon} + \frac{\epsilon}{2+2\epsilon} = \frac{2+2\epsilon}{2+2\epsilon} = 1.$$

□

4.2.4 Wave functions

We need to choose a sensible element of $L^2(\mathbb{R}, dy)$ to characterize the time evolution of the y coordinate for the full three-body model. For small ϵ, the y coordinate is approximately $r_2 - r_1$, so it seems physically reasonable that y should be highly localized around $y = v_2 t - v_1 t = vt$. This function should also satisfy the free time-dependent Schrödinger equation in the y variable, which can be read off from (4.3):

$$i\frac{\partial}{\partial t}\varphi(y,t) = H_1^y \varphi(y,t) = -\frac{\epsilon}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right)\Delta_y \varphi(y,t). \tag{4.9}$$

This describes the free propagation of a "particle" at y of (reduced) mass $\gamma = \frac{1+\epsilon}{2+\epsilon}$. Hence, if our particle has (approximately) the velocity vt, it should have momentum γv. In light of these considerations, [9] suggests it might be reasonable to use

$$\varphi(y,t) = \varphi_0^\epsilon(y,t) = \exp\left(\frac{i\gamma v^2 t}{2\epsilon}\right)\varphi_0\left(1 + it\gamma^{-1}, 1, \epsilon, vt, \gamma v, y\right).$$

However, as in Chapter 2, there is the issue of removing momenta in some arbitrarily small interval around $p = 0$. Therefore we define a *new* cutoff function $F(p)$ (where we are once again using p as the momentum coordinate for y) with the following properties: Choose fixed constants $a, b \in \mathbb{R}$ such that $0 < a < b < v/2$. Construct F such that

- $F \in C^\infty(\mathbb{R})$, with F symmetric about $p = v/2$.
- $F(p) = 1$ for $p \in [v/2 - a, v/2 + a]$.
- $F(p) = 0$ for $p \notin (v/2 - b, v/2 + b)$.

The final form we choose for $\varphi(y,t)$ is therefore

$$\varphi(y,t) = \tilde{\varphi}_0^\epsilon(y,t) = \exp\left(-\frac{i\gamma v^2 t}{2\epsilon}\right)\mathscr{F}_\epsilon^{-1}\left[F(p)\varphi_0\left(1, 1 + it\gamma^{-1}, \epsilon, \gamma v, -vt, p\right)\right](y).$$

Note that, by Plancherel's Theorem, $\|\widetilde{\varphi}_0^\epsilon(y,t)\| \leqslant 1$, since

$$\begin{aligned}\|\widetilde{\varphi}_0^\epsilon(y,t)\|_2 &= \left\|F(p)\varphi_0\left(1, 1+it\gamma^{-1}, \epsilon, \gamma v, -vt, p\right)\right\|_2 \\ &\leqslant \left\|\varphi_0\left(1, 1+it\gamma^{-1}, \epsilon, \gamma v, -vt, p\right)\right\|_2 \\ &= 1.\end{aligned}$$

We will often use this fact in what follows. We should also remark that $\widetilde{\varphi}_0^\epsilon(y,t)$ is still a solution of the time-dependent Schrödinger equation (4.9). To see why, we go to momentum space.

$$\begin{aligned}&\left\|\left(i\frac{\partial}{\partial t} - H_1^y\right)\widetilde{\varphi}_0^\epsilon(y,t)\right\|_{L^2(dy)} \\ &= \left\|\left(i\frac{\partial}{\partial t} + \frac{\epsilon}{2\gamma}\Delta_y\right)\exp\left(-\frac{i\gamma v^2 t}{2\epsilon}\right)\mathscr{F}_\epsilon^{-1}\left[F(p)\varphi_0\left(1, 1+it\gamma^{-1}, \epsilon, \gamma v, -vt, p\right)\right](y)\right\|_{L^2(dy)} \\ &\qquad = \left\|F(p)\left(i\frac{\partial}{\partial t} - \frac{p^2}{2\gamma\epsilon}\right)\exp\left(-\frac{i\gamma v^2 t}{2\epsilon}\right)\varphi_0\left(1, 1+it\gamma^{-1}, \epsilon, \gamma v, -vt, p\right)\right\|_{L^2(dp)}\end{aligned}$$

If we let $\omega = \frac{\gamma v^2}{2\epsilon}$ and $\widehat{\psi}(t) = \varphi_0\left(1, 1+it\gamma^{-1}, \epsilon, \gamma v, -vt, p\right)$, then after dropping $F(p)$ this norm is

$$\leqslant \left\|ie^{-i\omega t}\left(-i\omega\widehat{\psi}(t) + \widehat{\psi}'(t)\right) + \frac{p^2}{2\gamma\epsilon}e^{-i\omega t}\widehat{\psi}(t)\right\|.$$

Since

$$\widehat{\psi}(t) = \pi^{-1/4}\epsilon^{-1/4}\exp\left(-\left(1+\frac{it}{\gamma}\right)\frac{(p-\gamma v)^2}{2\epsilon} - \frac{itv}{\epsilon}(p-\gamma v)\right),$$

this becomes

$$= \left\|e^{-i\omega t}\widehat{\psi}(t)\left(\frac{\gamma v^2}{2\epsilon} - \frac{v^2\gamma}{2\epsilon} + \frac{p^2}{2\gamma\epsilon}\right) - \frac{p^2}{2\gamma\epsilon}e^{-i\omega t}\widehat{\psi}(t)\right\| = 0,$$

as desired. In what follows, we will use $\widetilde{\varphi}_0^\epsilon(y)$ to denote $\widetilde{\varphi}_0^\epsilon(y,0)$.

In light of the above discussion, it seems reasonable, given our consideration of a situation in which the IP model is applicable, to allow our channel scattering operator S_{11} to act on the product wave function

$$\widetilde{\varphi}_0^\epsilon(y)\eta_1(x).$$

Similarly, from the impact parameter perspective, we should be interested in the action of σ_{11} on

$$\varphi_0(\zeta)\alpha_1(\xi).$$

It is the crucial connection between these two functions that will be explored in the remainder of this chapter.

4.3 Statement of the result

We are now in a position to state the central result of this chapter. Note first that since $\|\varphi_0(\zeta)\|_{L^2(d\zeta)} = 1$,

$$\langle \varphi_0(\zeta)\alpha_1(\xi), \sigma_{11}[\alpha_1(\xi)\varphi_0(\zeta)]\rangle_{L^2(d\xi d\zeta)} = \langle \alpha_1(\xi), \sigma_{11}[\alpha_1(\xi)]\rangle_{L^2(d\xi)} .$$

Theorem 4.3.1. *The quantum amplitudes corresponding to the IP model for the three-body scattering problem in* (4.1) *are good approximations to the full three-body amplitudes in the sense that*

$$\left| \langle \eta_1(x)\widetilde{\varphi}_0^\epsilon(y), S_{11}[\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)]\rangle_{L^2(dxdy)} - \langle \alpha_1(\xi), \sigma_{11}[\alpha_1(\xi)]\rangle_{L^2(d\xi)} \right| \to 0 \qquad (4.10)$$

as $\epsilon \to 0$.

The proof of this theorem will require the result of the following proposition. It is clear from the definitions of S and σ that (4.10) is equivalent to (4.11).

Proposition 4.3.1. *If*

$$\left\| \Omega_1^\pm[\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)](x(\zeta,\xi), y(\zeta,\xi)) - \varphi_0(\zeta)\omega_1^\pm[\alpha_1(\xi)] \right\|_{L^2(d\zeta d\xi)} \to 0 \quad \text{as } \epsilon \to 0,$$

then the difference in the amplitudes

$$\left| \left\langle \Omega_1^-[\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)], \Omega_1^+[\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)] \right\rangle_{L^2(dxdy)} - \left\langle \omega_1^-[\alpha_1(\xi)], \omega_1^+[\alpha_1(\xi)] \right\rangle_{L^2(d\xi)} \right| \qquad (4.11)$$

also goes to zero as $\epsilon \to 0$.

Proof. The notation $\Omega_1^\pm[\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)](x(\zeta,\xi), y(\zeta,\xi))$ simply indicates that we are expressing the function $\Omega_1^\pm[\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)]$ in terms of ξ and ζ. We use the fact that the variable transformation $(\zeta, \xi) \to (x, y)$ is unitary, together with the fact that the channel wave operators are norm-preserving. (They are, in fact, partial isometries, as shown in, for instance, [18].) To simplify notation for the proof, let $\Omega_1^\pm(\cdot,\cdot) = \Omega_1^\pm[\eta_1(x)\widetilde{\varphi}_0(y)](\cdot,\cdot)$ and

$\omega_1^\pm(\xi,\zeta) = \varphi_0(\zeta)\omega_1^\pm[\alpha_1(\xi)]$. Then Cauchy-Schwarz gives

$$\begin{aligned}
&\left|\left\langle \Omega_1^-(x,y), \Omega_1^+(x,y)\right\rangle_{x,y} - \left\langle \omega_1^-(\xi), \omega_1^+(\xi)\right\rangle_\xi\right| \\
&\qquad = \left|\left\langle \Omega_1^-(x,y), \Omega_1^+(x,y)\right\rangle_{x,y} - \left\langle \omega_1^-(\xi,\zeta), \omega_1^+(\xi,\zeta)\right\rangle_{\xi,\zeta}\right| \\
&= \Big|\left\langle \Omega_1^-(x(\zeta,\xi), y(\zeta,\xi)), \Omega_1^+(x(\zeta,\xi), y(\zeta,\xi))\right\rangle_{\xi,\zeta} - \left\langle \omega_1^-(\xi,\zeta), \Omega_1^+(x(\zeta,\xi), y(\zeta,\xi))\right\rangle_{\zeta,\xi} \\
&\qquad + \left\langle \omega_1^-(\xi,\zeta), \Omega_1^+(x(\zeta,\xi), y(\zeta,\xi))\right\rangle_{\zeta,\xi} - \left\langle \omega_1^-(\xi,\zeta), \omega_1^+(\xi,\zeta)\right\rangle_{\xi,\zeta}\Big| \\
&\qquad = \Big|\left\langle \Omega_1^-(x(\zeta,\xi), y(\zeta,\xi)) - \omega_1^-(\xi,\zeta), \Omega_1^+(x(\zeta,\xi), y(\zeta,\xi))\right\rangle_{\zeta,\xi} \\
&\qquad\quad + \left\langle \omega_1^-(\xi,\zeta), \Omega_1^+(x(\zeta,\xi), y(\zeta,\xi)) - \omega_1^+(\xi,\zeta)\right\rangle_{\zeta,\xi}\Big| \\
&\qquad \leqslant \left\|\Omega_1^+(x(\zeta,\xi), y(\zeta,\xi))\right\| \left\|\Omega_1^-(x(\zeta,\xi), y(\zeta,\xi)) - \omega_1^-(\xi,\zeta)\right\| \\
&\qquad\quad + \left\|\omega_1^-(\xi,\zeta)\right\| \left\|\Omega_1^+(x(\zeta,\xi), y(\zeta,\xi)) - \omega_1^+(\xi,\zeta)\right\| \\
&\qquad = \left\|\Omega_1^-(x(\zeta,\xi), y(\zeta,\xi)) - \omega_1^-(\xi,\zeta)\right\| + \left\|\Omega_1^+(x(\zeta,\xi), y(\zeta,\xi)) - \omega_1^+(\xi,\zeta)\right\|.
\end{aligned}$$

□

Hence, showing that $\left\|\Omega_1^\pm(x(\zeta,\xi), y(\zeta,\xi)) - \omega_1^\pm(\xi,\zeta)\right\|_{L^2(d\xi d\zeta)} \to 0$ as $\epsilon \to 0$ is sufficient. This now becomes our goal: to prove

$$\left\|\Omega_1^\pm[\eta_1(x)\widetilde{\varphi}_0(y)](x(\zeta,\xi), y(\zeta,\xi)) - \varphi_0(\zeta)\omega_1^\pm[\alpha_1(\xi)]\right\|_{L^2(d\zeta d\xi)} \to 0 \quad \text{as } \epsilon \to 0. \tag{4.12}$$

We provide the proof only for the "distant future" wave operators Ω_1^- and ω_1^-; the proof for Ω_1^+ and ω_1^+ is completely analogous.

4.3.1 Preliminary remarks

Before embarking on the proof, we pause to explain its structure and strategy. Let $\beta > 0$ be a given arbitrary positive real number. Choose a finite time $T > 0$ such that

$$T > \max\left\{\frac{R_2}{2}, \frac{R_2}{v}, \frac{1}{dv}\log\left(\frac{8C_d e^{dR_2}\sqrt{2R_2}}{dv\beta}\right), T_1\right\}, \tag{4.13}$$

where T_1 is a positive time such that the integral of the L^1 function

$$C_2 C_d e^{6dR_2}\sqrt{4\left(R_2+\frac{4}{t}(b+\delta)\right)}\exp\left(-8dt\left(v-(b+\delta)\right)\right)<\frac{\beta}{4}.$$

(T_1 exists by, e.g., Proposition 1.12 on pg. 65 of [23].) Note that for this T,

$$\begin{aligned}\Omega_1^- = \operatorname*{s-lim}_{t\to\infty} e^{iHt}e^{-iH_1t} &= \operatorname*{s-lim}_{t\to\infty} e^{iH(t+T)}e^{-iH_1(t+T)}\\ &= e^{iTH}\left(\operatorname*{s-lim}_{t\to\infty} e^{iHt}e^{-iH_1t}\right)e^{-iTH_1}\\ &= e^{iTH}\Omega_1^- e^{-iTH_1}.\end{aligned}$$

(This is just the so-called "intertwining" property for the wave operators.) Therefore, (4.12) is equivalent to the requirement that

$$\begin{aligned}&\left\|e^{iTH}\Omega_1^- e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - e^{iTH}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)\right.\\ &\qquad\left.+e^{iTH}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - \varphi_0(\zeta)[\omega_1^-(\alpha_1)](\xi)\right\|\\ &=\left\|e^{iTH}(\Omega_1^- - I)e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) + e^{iTH}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - \varphi_0(\zeta)[\omega_1^-(\alpha_1)](\xi)\right\|\\ &\qquad\to 0 \quad \text{as } \epsilon\to 0.\end{aligned}$$

By the triangle inequality and the unitarity of the variable transformation, it will be sufficient to show there exists a $\rho > 0$ such that $0 < \epsilon < \rho$ implies

$$\left\|e^{iTH}(\Omega_1^- - I)e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)\right\|_{L^2(dxdy)} < \frac{\beta}{2} \tag{4.14}$$

and

$$\left\|e^{iTH}e^{-iTH_1}\eta_1(x(\xi,\zeta))\widetilde{\varphi}_0^\epsilon(y(\xi,\zeta)) - \varphi_0(\zeta)[\omega_1^-(\alpha_1)](\xi)\right\|_{L^2(d\xi d\zeta)} < \frac{\beta}{2}. \tag{4.15}$$

We will begin by considering (4.14). Since (as we will show) this piece concerns infinite-time behavior, we will henceforth refer to the norm in (4.14) as the **infinite times** piece. We will similarly refer to (4.15) as the **finite times** piece. We begin by controlling the infinite times piece.

4.4 Infinite times

We first observe that

$$\begin{aligned} e^{iTH}(\Omega_1^- - I)e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) &= \lim_{t\to\infty} e^{iTH}e^{itH}e^{-itH_1}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - e^{iTH}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) \\ &= \lim_{t\to\infty}\left[e^{i(t+T)H}e^{-i(t+T)H_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - e^{iTH}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)\right] \\ &= \lim_{t\to\infty}\left[e^{itH}e^{-itH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - e^{iTH}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)\right] \\ &= \lim_{t\to\infty}\int_T^t \frac{d}{ds}\left(e^{isH}e^{-isH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)\right)ds, \end{aligned}$$

where in the last step we have used the fundamental theorem of calculus. Since

$$\begin{aligned} \frac{d}{ds}\left(e^{isH}e^{-isH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)\right) &= iHe^{isH}e^{-isH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - ie^{isH}H_1e^{-isH_1} \\ &= ie^{isH}(H-H_1)e^{-isH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) \\ &= ie^{isH}V_2\left(\frac{x}{1+\epsilon}-y\right)e^{-isH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y), \end{aligned}$$

we have the obvious bound

$$\begin{aligned} \left\|e^{iTH}(\Omega_1^- - I)e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)\right\|_{L^2(dxdy)} &\leqslant \int_T^\infty \left\|e^{isH}V_2\left(\frac{x}{1+\epsilon}-y\right)e^{-isH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)\right\|_{L^2(dxdy)} ds \\ &= \int_T^\infty \left\|V_2\left(\frac{x}{1+\epsilon}-y\right)e^{-isH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)\right\|_{L^2(dxdy)} ds \\ &= \int_T^\infty \left\|V_2\left(\frac{x}{1+\epsilon}-y\right)\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,s)\right\|_{L^2(dxdy)} ds. \end{aligned}$$

So, switching back to t for the time variable, we would now like to investigate the t-integrability of the norm

$$\begin{aligned} &\left\|V_2\left(\frac{x}{1+\epsilon}-y\right)\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,t)\right\|_{L^2(dxdy)} \\ &= \left\|V_2\left(\frac{x}{1+\epsilon}-y\right)\eta_1(x)\mathscr{F}_\epsilon^{-1}\left[F(p)\varphi_0\left(1,1+i\left(\frac{2+\epsilon}{1+\epsilon}\right)t,\epsilon,\frac{1+\epsilon}{2+\epsilon}v,-vt,p\right)\right](y)\right\|_{L^2(dxdy)} \\ &\qquad = \left\|V_2\left(\frac{x}{1+\epsilon}-y\right)\eta_1(x)\mathscr{F}_\epsilon^{-1}\left[F(p)\varphi_0\left(1,1+\frac{it}{\gamma},\epsilon,\gamma v,-vt,p\right)\right](y)\right\|_{L^2(dxdy)}. \end{aligned}$$

We will do this by considering it separately on two disjoint regions of y-space. Fix a $\delta > 0$ such that $\frac{v}{2} - b > \delta$, and let

$$\mathcal{A}(t) = \left\{ y \in \mathbb{R} : \left| y - \frac{vt}{2\gamma} \right| < \frac{t}{\gamma}(b+\delta) \right\}$$

be the open set which defines the "classically allowed region." Define its complement $\mathcal{A}^c(t)$ to be the "classically forbidden region." Then certainly, for any t,

$$\left\| V_2 \left(\frac{x}{1+\epsilon} - y \right) \eta_1(x) \widetilde{\varphi}_0^\epsilon(y,t) \right\|_{L^2(dxdy)} = \left\| V_2 \left(\frac{x}{1+\epsilon} - y \right) \eta_1(x) \widetilde{\varphi}_0^\epsilon(y,t) \Big(\mathcal{X}_{\mathcal{A}(t)}(y) + \mathcal{X}_{\mathcal{A}^c(t)}(y) \Big) \right\|_{L^2(dxdy)},$$

so we can break this up using the triangle inequality and bound each piece individually.

4.4.1 The classically allowed region

The classically allowed region is easiest:

$$\begin{aligned} &\left\| V_2 \left(\frac{x}{1+\epsilon} - y \right) \eta_1(x) \widetilde{\varphi}_0^\epsilon(y,t) \mathcal{X}_{\mathcal{A}(t)}(y) \right\|^2 \\ &\quad = \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \left| V_2 \left(\frac{x}{1+\epsilon} - y \right) \right|^2 |\eta_1(x)|^2 |\widetilde{\varphi}_0(y,t)|^2 \mathcal{X}_{A(t)}(y) \\ &\quad \leqslant C_2^2 C_d^2 \int_{\mathbb{R}} e^{-2d|x|} dx \int_{\mathbb{R}} |\widetilde{\varphi}_0(y,t)|^2 \mathcal{X} \left(\left| \frac{x}{1+\epsilon} - y \right| < R_2 \right) \mathcal{X}_{A(t)}(y) dy. \end{aligned}$$

The product of characteristic functions $\mathcal{X}\left(\left|\frac{x}{1+\epsilon} - y\right| < R_2\right) \mathcal{X}_{A(t)}(y)$ can be rewritten, since it is nonzero only when

$$-R_2 \leqslant \frac{x}{1+\epsilon} - y \leqslant R_2 \qquad \text{and} \qquad -\frac{t}{\gamma}(b+\delta) \leqslant y - \frac{vt}{2\gamma} \leqslant \frac{t}{\gamma}(b+\delta).$$

This amounts to the requirement that

$$\theta_1 = (1+\epsilon) \left(\frac{vt}{2\gamma} - \frac{t}{\gamma}(b+\delta) - R_2 \right) \leqslant x \leqslant (1+\epsilon) \left(\frac{vt}{2\gamma} + \frac{t}{\gamma}(b+\delta) + R_2 \right) = \theta_2,$$

so we can say

$$\left\| V_2\left(\frac{x}{1+\epsilon} - y\right)\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,t)\mathcal{X}_{\mathcal{A}(t)}(y)\right\|^2 \leqslant C_2^2 C_d^2 \int_{\theta_1}^{\theta_2} e^{-2d|x|}dx \int_{\mathbb{R}} |\widetilde{\varphi}_0^\epsilon(y,t)|^2 dy$$
$$\leqslant C_2^2 C_d^2 \int_{\theta_1}^{\theta_2} e^{-2d|x|}dx.$$

As long as we make sure $\theta_1 > 0$ – which we have done by assuming (4.13) – the x integral can be estimated as

$$\int_{\theta_1}^{\theta_2} e^{-2d|x|}dx \leqslant 2\left(R_2 + \frac{t}{\gamma}(b+\delta)\right)(1+\epsilon)\exp\left(-2d(1+\epsilon)\left(\frac{vt}{2\gamma} - \frac{t}{\gamma}(b+\delta) - R_2\right)\right).$$

We therefore have, after taking square roots,

$$\left\| V_2\left(\frac{x}{1+\epsilon} - y\right)\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,t)\mathcal{X}_{\mathcal{A}(t)}(y)\right\|_{L^2(dxdy)}$$
$$\leqslant C_2 C_d e^{d(1+\epsilon)R_2}\sqrt{2\left(R_2 + \frac{t}{\gamma}(b+\delta)\right)(1+\epsilon)}\exp\left(-2d(1+\epsilon)\left(\frac{vt}{2\gamma} - \frac{t}{\gamma}(b+\delta) - R_2\right)\right).$$

Provided we restrict our attention to $\epsilon < 1$ – which is certainly physically reasonable, since we want the heavy particles to be (at least) heavier than ③, which has unit mass! – we can say

$$\left\| V_2\left(\frac{x}{1+\epsilon} - y\right)\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,t)\mathcal{X}_{\mathcal{A}(t)}(y)\right\|_{L^2(dxdy)}$$
$$\leqslant C_2 C_d e^{6dR_2}\sqrt{4\left(R_2 + \frac{4}{t}(b+\delta)\right)}\exp\left(-8dt\left(v - (b+\delta)\right)\right).$$

Since T satisfies (4.13), the integral $\int_T^\infty dt$ of this norm can be made smaller than $\frac{\beta}{4}$. Our remaining task is to show that we can make the classically forbidden piece less than $\frac{\beta}{4}$.

4.4.2 The classically forbidden region

We need to show

$$\left\| V_2\left(\frac{x}{1+\epsilon}-y\right)\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,t)\mathcal{X}_{\mathcal{A}^c(t)}(y)\right\|_{L^2(dxdy)} \in L^1(\mathbb{R},dt). \tag{4.16}$$

We first write

$$\begin{aligned}
&\left\| V_2\left(\frac{x}{1+\epsilon}-y\right)\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,t)\mathcal{X}_{\mathcal{A}^c(t)}(y)\right\|^2_{L^2(dxdy)} \\
&\quad = \int_{\mathbb{R}} dx|\eta_1(x)|^2 \int_{\mathbb{R}} dy \left| V_2\left(\frac{x}{1+\epsilon}-y\right)\right|^2 |\widetilde{\varphi}_0^\epsilon(y,t)|^2 \mathcal{X}_{A(t)}(y) \\
&\quad \leqslant \|V_2\|_\infty^2 \int_{\mathbb{R}} dx|\eta_1(x)|^2 \int_{\mathbb{R}} dy|\widetilde{\varphi}_0^\epsilon(y,t)| \cdot \left(|\widetilde{\varphi}_0^\epsilon(y,t)|\mathcal{X}_{A(t)}(y)\right) \\
&\quad \leqslant \|V_2\|_\infty^2 \|\widetilde{\varphi}_0^\epsilon\|_\infty \int_{\mathbb{R}} dx\ |\eta_1(x)|^2 \int_{\mathbb{R}} dy\ |\widetilde{\varphi}_0^\epsilon(y,t)|\mathcal{X}_{A(t)}(y) \\
&\quad = \|V_2\|_\infty^2 \|\widetilde{\varphi}_0^\epsilon\|_\infty \int_{\mathbb{R}} dy\ |\widetilde{\varphi}_0^\epsilon(y,t)|\mathcal{X}_{A(t)}(y).
\end{aligned}$$

The norm $\|\widetilde{\varphi}_0^\epsilon\|_\infty$ is finite by Theorem IX.28 of [17]: since $\widetilde{\varphi}_0^\epsilon(y,t) \in D(-\Delta_y)$, we know it is bounded and continuous, and that there exists a positive constant $M > 0$, independent of ϵ and t, such that

$$\begin{aligned}
\|\widetilde{\varphi}_0^\epsilon\|_\infty &\leqslant \|-\Delta_y\widetilde{\varphi}_0^\epsilon(y,t)\| + M\,\|\widetilde{\varphi}_0^\epsilon(y,t)\| \\
&\leqslant \left\|p^2F(p)\varphi_0(1,1+i\gamma^{-1}t,\epsilon,\gamma v,-vt,p)\right\| + M \\
&\leqslant \left(\frac{v}{2}+b\right)^2 + M.
\end{aligned}$$

So, we would like to show that

$$\int_{\mathbb{R}} dy\ |\widetilde{\varphi}_0^\epsilon(y,t)|\mathcal{X}_{A(t)}(y) \in L^1([T,\infty),dt)$$

for the T we have chosen. We choose first to estimate

$$|\widetilde{\varphi}_0^\epsilon(y,t)|\mathcal{X}_{A^c}(y,t).$$

We first write $\widetilde{\varphi}_0^\epsilon$ as the inverse of its Fourier transform. Since

$$\varphi_0^\epsilon(y,t) = \exp\left(\frac{i\gamma v^2 t}{2\epsilon}\right)\varphi_0\left(1+it\gamma^{-1},1,\epsilon,vt,\gamma v,y\right),$$

we have

$$\begin{aligned}\widetilde{\varphi}_0^\epsilon(y,t) &= \exp\left(-\frac{i\gamma v^2 t}{2\epsilon}\right)\mathscr{F}_\epsilon^{-1}\left[F(p)\varphi_0\left(1,1+it\gamma^{-1},\epsilon,\gamma v,-vt,p\right)\right]\\ &= (2\pi\epsilon)^{-1}e^{-i\gamma v^2 t/(2\epsilon)}\int_{\mathbb{R}} dp\, e^{ipy/\epsilon}\int_{\mathbb{R}} dy' e^{-ipy'/\epsilon}\mathscr{F}_\epsilon^{-1}\left[F(p')\varphi_0\left(1,1+it\gamma^{-1},\epsilon,\gamma v,-vt,p'\right)\right](y').\end{aligned}$$

Now, since $\widetilde{\varphi}_0^\epsilon(y,t)$ evolves under the dynamics captured by e^{-itH_y}, we have (formally)

$$e^{-itH_y}e^{ipy/\epsilon} = \exp\left(\frac{it\epsilon}{2\gamma}\frac{d^2}{dy^2}\right)e^{ipy/\epsilon} = \left(I+\frac{it\epsilon}{2\gamma}\frac{d^2}{dy^2}+\cdots\right)e^{ipy/\epsilon} = e^{-itp^2/(2\gamma\epsilon)},$$

so we have

$$\begin{aligned}\widetilde{\varphi}_0^\epsilon(y,t) &= e^{-itH_y}\widetilde{\varphi}_0^\epsilon(y) =\\ &(2\pi\epsilon)^{-1}e^{-i\gamma v^2 t/(2\epsilon)}\int_{\mathbb{R}} dp\; e^{ipy/\epsilon}e^{-itp^2/(2\gamma\epsilon)}\int_{\mathbb{R}} dy' e^{-ipy'/\epsilon}\mathscr{F}_\epsilon^{-1}\left[F(p)\varphi_0\left(1,1,\epsilon,\gamma v,0,p'\right)\right](y')\\ &= (2\pi\epsilon)^{-1/2}e^{-i\gamma v^2 t/(2\epsilon)}\int_{\mathbb{R}} e^{ipy/\epsilon}e^{-itp^2/(2\gamma\epsilon)}F(p)\varphi_0(1,1,\epsilon,\gamma v,0,p)dp.\end{aligned}$$

We now make several changes of variables. First, let $z = p - \gamma v$. Then the above becomes

$$\begin{aligned}\widetilde{\varphi}_0^\epsilon(y,t) &= (2\pi\epsilon)^{-1/2}\int_{\mathbb{R}} e^{i(z+\gamma v)y/\epsilon}e^{-it(z+\gamma v)^2/(2\gamma\epsilon)}F(z+\gamma v)\varphi_0(1,1,\epsilon,0,0,z)dz\\ &= (2\pi\epsilon)^{-1/2}\exp\left(\frac{iv\gamma}{\epsilon}(y-vt)\right)\int_{\mathbb{R}} e^{iyz/\epsilon}e^{-itz^2/(2\gamma\epsilon)}e^{-itvz/\epsilon}F(z+\gamma v)\varphi_0(1,1,\epsilon,0,0,z)dz.\end{aligned}$$

To simplify notation, let $\phi = \frac{v\gamma}{\epsilon}(y-vt)$. We make the change of variables $\mu = \epsilon^{-1/2}z$, and note that $\varphi_0(1,1,\epsilon,0,0,z) = \epsilon^{-1/4}\varphi_0(1,1,1,0,0,\mu)$. Then

$$\begin{aligned}\widetilde{\varphi}_0^\epsilon(y,t) &= (2\pi)^{-1/2}\epsilon^{-1/4}e^{i\phi}\int_{\mathbb{R}} e^{iy\epsilon^{-1/2}\mu}e^{-it\mu^2/(2\gamma)}e^{-itv\epsilon^{-1/2}\mu}F(\epsilon^{1/2}\mu+\gamma v)\varphi_0(1,1,1,0,0,\mu)d\mu\\ &= (2\pi)^{-1/2}\epsilon^{-1/4}e^{i\phi}\int_{\mathbb{R}}\exp\left[i\epsilon^{-1/2}\left(\mu(y-vt)-\frac{\epsilon^{1/2}}{2\gamma}\mu^2 t\right)\right]F(\epsilon^{1/2}\mu+\gamma v)\varphi_0(1,1,1,0,0,\mu)d\mu.\end{aligned}$$

Define functions f and g as follows:

$$f(\mu) = \mu(y - vt) - \frac{\epsilon^{1/2}}{2\gamma}\mu^2 t \qquad \text{and} \qquad g(\mu) = F(\epsilon^{1/2}\mu + \gamma v)\varphi_0(1,1,1,0,0,\mu).$$

Note that since the cutoff function F is supported only on the interval $\frac{v}{2} - b \leqslant p \leqslant \frac{v}{2} + b$, we can restrict the integral to

$$\frac{v}{2} - b \leqslant \epsilon^{1/2}\mu + \gamma v \leqslant \frac{v}{2} + b, \quad \text{or} \quad \epsilon^{-1/2}\left(v\left(\frac{1}{2} - \gamma\right) - b\right) \leqslant \mu \leqslant \epsilon^{-1/2}\left(v\left(\frac{1}{2} - \gamma\right) + b\right).$$

Let $\beta_\pm = \beta_\pm(\epsilon) = \epsilon^{-1/2}\left(v\left(\frac{1}{2} - \gamma\right) \pm b\right)$. Then

$$\begin{aligned} \widetilde{\varphi}_0^\epsilon(y,t) &= (2\pi)^{-1/2}\epsilon^{-1/4}e^{i\phi}\int_{\beta_-}^{\beta_+} e^{i\epsilon^{-1/2}f(\mu)}g(\mu)d\mu \\ &= (2\pi)^{-1/2}\epsilon^{-1/4}e^{i\phi}\int_{\beta_-}^{\beta_+}\left(f'(\mu)e^{i\epsilon^{-1/2}f(\mu)}\right)\frac{g(\mu)}{f'(\mu)}d\mu. \end{aligned} \tag{4.17}$$

The term inside parentheses inside the integrand has been arranged to be an exact derivative (modulo a constant), so we are now able to integrate by parts. If we let

$$u = \frac{g(\mu)}{f'(\mu)} \Rightarrow du = \frac{g'(\mu)}{f'(\mu)} - \frac{g(\mu)f''(\mu)}{(f'(\mu))^2} = h(\mu)$$

and

$$dv = f'(\mu)\ e^{i\epsilon^{-1/2}f(\mu)}d\mu \Rightarrow v = -i\epsilon^{1/2}e^{i\epsilon^{-1/2}f(\mu)}$$

and use the fact that $F(\frac{v}{2} \pm b) = 0$, we find that the integral in (4.17) can be written as

$$\begin{aligned} i\epsilon^{1/2}\int_{\beta_-}^{\beta_+} e^{i\epsilon^{-1/2}f(\mu)}h(\mu)d\mu &= i\epsilon^{1/2}\int_{\beta_-}^{\beta_+}(-i\epsilon^{1/2})\left(i\epsilon^{-1/2}f'(\mu)e^{i\epsilon^{-1/2}f(\mu)}\right)\frac{h(\mu)}{f'(\mu)}d\mu \\ &= \epsilon\int_{\beta_-}^{\beta_+}\left(i\epsilon^{-1/2}f'(\mu)e^{i\epsilon^{-1/2}f(\mu)}\right)\frac{h(\mu)}{f'(\mu)}d\mu. \end{aligned}$$

These stationary phase calculations permit us to get as much decay in t as we need; we will need to go *three* (tedious) steps further. The next integration by parts gives

$$u = \frac{h(\mu)}{f'(\mu)} \implies du = \frac{h'(\mu)}{f'(\mu)} - \frac{h(\mu)f''(\mu)}{(f'(\mu))^2} = j(\mu).$$

It follows that

$$\begin{aligned}\epsilon \int_{\beta_-}^{\beta_+} \left(i\epsilon^{-1/2} f'(\mu) e^{i\epsilon^{-1/2} f(\mu)}\right) \frac{h(\mu)}{f'(\mu)} d\mu &= -\epsilon \int_{\beta_-}^{\beta^+} e^{i\epsilon^{-1/2} f(\mu)} \frac{j(\mu)}{f'(\mu)} d\mu \\ &= -\epsilon \int_{\beta_-}^{\beta^+} (-i\epsilon^{1/2}) \left(i\epsilon^{-1/2} f'(\mu) e^{i\epsilon^{-1/2} f(\mu)}\right) \frac{j(\mu)}{f'(\mu)} d\mu \\ &= i\epsilon^{3/2} \int_{\beta_-}^{\beta^+} \left(i\epsilon^{-1/2} f'(\mu) e^{i\epsilon^{-1/2} f(\mu)}\right) \frac{j(\mu)}{f'(\mu)} d\mu.\end{aligned}$$

For the next iteration, let

$$u = \frac{j(\mu)}{f'(\mu)} \implies du = \frac{j'(\mu)}{f'(\mu)} - \frac{j(\mu)f''(\mu)}{(f'(\mu))^2} = k(\mu).$$

Then

$$\begin{aligned}i\epsilon^{3/2} \int_{\beta_-}^{\beta_+} \left(i\epsilon^{-1/2} f'(\mu) e^{i\epsilon^{-1/2} f(\mu)}\right) \frac{h(\mu)}{f'(\mu)} d\mu &= -\epsilon^2 \int_{\beta_-}^{\beta^+} (-i\epsilon^{1/2}) \left(i\epsilon^{-1/2} f'(\mu) e^{i\epsilon^{-1/2} f(\mu)}\right) \frac{j(\mu)}{f'(\mu)} d\mu \\ &= -\epsilon^2 \int_{\beta_-}^{\beta^+} \left(i\epsilon^{-1/2} f'(\mu) e^{i\epsilon^{-1/2} f(\mu)}\right) \frac{k(\mu)}{f'(\mu)} d\mu.\end{aligned}$$

We will need to go one step further. Let

$$u = \frac{k(\mu)}{f'(\mu)} \implies du = \frac{k'(\mu)}{f'(\mu)} - \frac{k(\mu)f''(\mu)}{(f'(\mu))^2}.$$

We can compute $k'(\mu)$ in terms of g and f'. Our life is made simpler by observing that $f'''(\mu) = 0$, since

$$f'(\mu) = y - vt - \epsilon^{1/2}\gamma^{-1}\mu t \implies f''(\mu) = -\epsilon^{1/2}\gamma^{-1} t \implies f'''(\mu) = 0.$$

After an egregious and not particularly enlightening computation, we obtain

$$du = \sum_{n=0}^{4} L_n \left(\frac{t\epsilon^{1/2}}{\gamma}\right)^n \frac{g^{(4-n)}(\mu)}{(f'(\mu))^{n+4}} d\mu$$

for some (uninteresting) constants $L_n \in \mathbb{Z}$. Of course, we also have

$$dv = i\epsilon^{-1/2} f'(\mu)\ e^{i\epsilon^{-1/2} f(\mu)} d\mu \implies v = e^{i\epsilon^{-1/2} f(\mu)}.$$

So this last integration by parts allows us to write

$$\widetilde{\varphi}_0^{\epsilon}(y,t) = -(2\pi)^{1/2}\epsilon^{-1/4} e^{i\phi}\epsilon^2 \int_{\beta_-}^{\beta_+} e^{i\epsilon^{-1/2} f(\mu)} \sum_{n=0}^{4} L_n \left(\frac{t\epsilon^{1/2}}{\gamma}\right)^n \frac{g^{(4-n)}(\mu)}{(f'(\mu))^{n+4}} d\mu.$$

Therefore, after "crashing through"[1] the integral with absolute values, we obtain

$$|\widetilde{\varphi}_0^{\epsilon}(y,t)| \mathcal{X}_{A^c(t)}(y) \leqslant (2\pi)^{1/2}\epsilon^{7/4} \sum_{n=0}^{4} \int_{\beta_-}^{\beta^+} L_n \left(\frac{t\epsilon^{1/2}}{\gamma}\right)^n \frac{|g^{(4-n)}(\mu)|}{|f'(\mu)|^{n+4}} \mathcal{X}_{A^c(t)}(y) d\mu.$$

To control each of these integrals, we will need the following expression for the mth derivative of g:

$$g^{(m)}(\mu) = \sum_{k=0}^{m} \binom{m}{k} \epsilon^{k/2} F^{(k)}(\epsilon^{1/2}\mu + \gamma v)\varphi_0^{(m-k)}(1,1,1,0,0,\mu).$$

A computation using $\varphi_0(1,1,1,0,0,\mu) = \pi^{-1/4}e^{-\mu^2/2}$ tells us that $\left\|\varphi_0^{(n)}\right\|_\infty \leqslant 3\pi^{-1/4}$ for all $n \in \{0,1,\ldots,4\}$. Therefore, since $p = \epsilon^{1/2}\mu + \gamma v$, we can say

$$|g^{(4-n)}(\mu)| \leqslant 3\pi^{-1/4} \sum_{k=0}^{4-n} \binom{4-n}{k} \epsilon^{k/2} |F^{(k)}(p)|.$$

Making the change of integration variable back to p and substituting in for the actual value of $f'(\mu) = y - \frac{pt}{\gamma}$, we find there exist positive constants $L_{n,k}$ (independent of ϵ and t) such that

$$|\widetilde{\varphi}_0^{\epsilon}(y,t)| \mathcal{X}_{A^c(t)}(y) \leqslant \sum_{n=0}^{4} \sum_{k=0}^{4-n} L_{n,k}\epsilon^{7/4}\epsilon^{(k+n)/2} \int_{\frac{v}{2}-b}^{\frac{v}{2}+b} \frac{t^n}{\gamma^n} \frac{|F^{(k)}(p)|}{|y - \frac{pt}{\gamma}|^{n+4}} \mathcal{X}_{A^c(t)}(y) dp. \tag{4.18}$$

[1] It is our pleasure to thank John Rossi for introducing us to this charming mathematical colloquialism.

There are a total of fifteen (15) terms in the above double sum, all of which are suitably well-behaved in the following sense: Note that for any choice of n and k, the function $|F^{(k)}(p)|$ is, by definition of F, able to "donate" another characteristic function to the integrand:

$$\frac{t^n}{\gamma^n}\int_{\frac{v}{2}-b}^{\frac{v}{2}+b}\frac{|F^{(k)}(p)|}{|y-\frac{pt}{\gamma}|^{n+4}}\mathcal{X}_{A^c(t)}(y)dp = \frac{t^n}{\gamma^n}\int_{\mathbb{R}} dp\frac{|F^{(k)}(p)|}{|y-\frac{pt}{\gamma}|^{n+4}}\mathcal{X}_{A^c(t)}(y)\mathcal{X}\left(\left|p-\frac{v}{2}\right|\leqslant b\right).$$

Now, we rescale the y and p in the denominator by factoring out a t/γ:

$$\begin{aligned}\frac{t^n}{\gamma^n}\int_{\mathbb{R}} dp\frac{|F^{(k)}(p)|}{|y-\frac{pt}{\gamma}|^{n+4}}\mathcal{X}_{A^c(t)}(y)\mathcal{X}\left(\left|p-\frac{v}{2}\right|\leqslant b\right)\\ = \frac{t^n}{\gamma^n}\int_{\mathbb{R}} dp\frac{|F^{(k)}(p)|}{|\frac{\gamma y}{t}-p|^{n+4}(\frac{t}{\gamma})^{n+4}}\mathcal{X}_{A^c(t)}(y)\mathcal{X}\left(\left|p-\frac{v}{2}\right|\leqslant b\right)\\ = \left(\frac{\gamma^4}{t^4}\right)\int_{\mathbb{R}} dp\frac{|F^{(k)}(p)|}{|\frac{\gamma y}{t}-p|^{n+4}}\mathcal{X}_{A^c(t)}(y)\mathcal{X}\left(\left|p-\frac{v}{2}\right|\leqslant b\right).\end{aligned}$$

The product $\mathcal{X}_{A^c(t)}(y)\mathcal{X}\left(\left|p-\frac{v}{2}\right|\leqslant b\right)$ satisfies

$$\begin{aligned}\mathcal{X}_{A^c(t)}(y)\mathcal{X}\left(\left|p-\frac{v}{2}\right|\leqslant b\right)\\ = \mathcal{X}\left(\left|p-\frac{v}{2}\right|\leqslant b\right)\left[\mathcal{X}\left(\frac{\gamma y}{t}\leqslant\frac{v}{2}-(b+\delta)\right)+\mathcal{X}\left(\frac{\gamma y}{t}\geqslant\frac{v}{2}+(b+\delta)\right)\right].\end{aligned}$$

Using the fact that the set $|p-v/2|\leqslant b$ is the same as $p-v/2\leqslant b$ *and* $p-v/2\geqslant -b$, this product can be written as

$$\begin{aligned}&\mathcal{X}_{A^c(t)}(y)\mathcal{X}\left(\left|p-\frac{v}{2}\right|\leqslant b\right)\\ &= \mathcal{X}\left(\frac{\gamma y}{t}-p\leqslant-\delta\right)\mathcal{X}\left(\frac{\gamma y}{t}+p\leqslant v-\delta\right)+\mathcal{X}\left(\frac{\gamma y}{t}-p\geqslant\delta\right)\mathcal{X}\left(\frac{\gamma y}{t}+p\geqslant v+\delta\right)\\ &\leqslant \mathcal{X}\left(\frac{\gamma y}{t}-p\leqslant-\delta\right)+\mathcal{X}\left(\frac{\gamma y}{t}-p\geqslant\delta\right).\end{aligned}$$

Let $\Xi(z)=\mathcal{X}\left(z\leqslant-\delta\right)+\mathcal{X}\left(z\geqslant\delta\right)$. We have just shown that

$$\frac{\gamma^4}{t^4}\int_{\frac{v}{2}-b}^{\frac{v}{2}+b}\frac{|F^{(k)}(p)|}{|y-\frac{pt}{\gamma}|^{n+4}}\mathcal{X}_{A^c(t)}(y)dp = \frac{\gamma^4}{t^4}\int_{\mathbb{R}} dp\ \frac{|F^{(k)}(p)|}{|\frac{\gamma y}{t}-p|^{n+4}}\Xi\left(\frac{\gamma y}{t}-p\right).$$

We now make the observation that the integrand above looks like a convolution of the function $|F^{(k)}(p)|\in L^1(\mathbb{R},dp)$ with the function $\Xi(p)/|p|^{n+4}\in L^1(\mathbb{R},dp)$. (The presence of Ξ

is crucial for this inclusion!) This implies that

$$H_{n,k}(z) = \left(\frac{\Xi(p)}{|p|^{n+4}} * |F^{(k)}(p)| \right)(z) = \int_{\mathbb{R}} dp \frac{|F^{(k)}(p)|}{|z-p|^{n+4}} \Xi(z-p) \in L^1(\mathbb{R}, dz)$$

for every allowed choice of n and k. By Young's inequality ([17], pg. 32), the integral of $H_{n,k}(\gamma y/t)$ with respect to y is finite:

$$\int_{\mathbb{R}} H_{n,k}\left(\frac{\gamma y}{t}\right) dy = \frac{t}{\gamma} \int_{\mathbb{R}} H_{n,k}(z) dz = \frac{t}{\gamma} \left\| H_{n,k} \right\|_1 < \infty.$$

Hence

$$\begin{aligned}\int_{\mathbb{R}} dy \ |\widetilde{\varphi}_0^{\epsilon}(y,t)| \mathcal{X}_{A^c(t)}(y) &\leqslant \sum_{n=0}^{4} \sum_{k=0}^{4-n} L_{n,k} \epsilon^{7/4} \epsilon^{(k+n)/2} \int_{\mathbb{R}} dy \int_{\frac{v}{2}-b}^{\frac{v}{2}+b} \frac{t^n}{\gamma^n} \frac{|F^{(k)}(p)|}{|y - \frac{pt}{\gamma}|^{n+4}} \mathcal{X}_{A^c(t)}(y) dp \\ &\leqslant \frac{\gamma^3}{t^3} \epsilon^{7/4} \sum_{n=0}^{4} \sum_{k=0}^{4-n} \epsilon^{(k+n)/2} L_{n,k} \left\| H_{n,k} \right\|_1 < \infty.\end{aligned}$$

So $|\widetilde{\varphi}_0^{\epsilon}(y,t)| \mathcal{X}_{A^c(t)}(y) \in L^1(dy)$ as desired. It follows that

$$\begin{aligned}\left\| V_2\left(\frac{x}{1+\epsilon} - y\right) \eta_1(x) \widetilde{\varphi}_0^{\epsilon}(y,t) \mathcal{X}_{A^c(t)}(y) \right\|_{L^2(dxdy)} &\leqslant \|V_2\|_\infty \|\widetilde{\varphi}_0^{\epsilon}\|_\infty^{1/2} \left(\int_{\mathbb{R}} dy \ |\widetilde{\varphi}_0^{\epsilon}(y,t)| \mathcal{X}_{A(t)}(y) \right)^{1/2} \\ &\leqslant \|V_2\|_\infty \|\widetilde{\varphi}_0^{\epsilon}\|_\infty^{1/2} \frac{1}{t^{3/2}} \gamma^{3/2} \epsilon^{7/8} \left(\sum_{n=0}^{4} \sum_{k=0}^{4-n} \epsilon^{(k+n)/2} L_{n,k} \left\| H_{n,k} \right\|_1 \right)^{1/2} \in L^1([T,\infty), dt)\end{aligned}$$

for *any* $T > 0$. Hence

$$\begin{aligned}&\int_T^\infty dt \ \left\| V_2\left(\frac{x}{1+\epsilon} - y\right) \eta_1(x) \widetilde{\varphi}_0^{\epsilon}(y,t) \mathcal{X}_{A^c(t)}(y) \right\|_{L^2(dxdy)} \\ &\qquad \leqslant \int_T^\infty dt \ \|V_2\|_\infty \|\widetilde{\varphi}_0^{\epsilon}\|_\infty^{1/2} \frac{\gamma^{3/2}}{t^{3/2}} \epsilon^{7/8} \left(\sum_{n=0}^{4} \sum_{k=0}^{4-n} \epsilon^{(k+n)/2} L_{n,k} \left\| H_{n,k} \right\|_1 \right)^{1/2} = \frac{2}{\sqrt{T}} \epsilon^{7/8} \mathscr{I}_\epsilon,\end{aligned}$$

where the constant $\mathscr{I}_\epsilon$ contains no negative powers of ϵ. This clearly can be made smaller that $\beta/4$ for any T by choosing a sufficiently small ϵ.

4.4.3 The infinite times piece – a recap

Pick a time $T > 0$ that satisfies (4.13), and let ϵ_0 be chosen such that $2T^{-1/2}\hbar^{7/8}\mathscr{I}_\epsilon < \frac{\beta}{4}$ for all $0 < \epsilon < \epsilon_0$. Then

$$\begin{aligned}\left\|e^{iTH}(\Omega_1^- - I)e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)\right\|_{L^2(dxdy)} &\leqslant \int_T^\infty \left\|V_2\left(\frac{x}{1+\epsilon} - y\right)\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,t)\right\| dt \\ &\leqslant \int_T^\infty \left\|V_2\left(\frac{x}{1+\epsilon} - y\right)\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,t)\mathcal{X}_{A(t)}(y)\right\| dt \\ &\quad + \int_T^\infty \left\|V_2\left(\frac{x}{1+\epsilon} - y\right)\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,t)\mathcal{X}_{A^c(t)}(y)\right\| dt \\ &< \frac{\beta}{4} + \frac{\beta}{4} = \frac{\beta}{2},\end{aligned}$$

as desired.

4.5 Finite times

We would now like to control (4.15), the finite times piece: Given any $\beta > 0$, we want to show that

$$\left\|e^{iTH}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - \varphi_0(\zeta)\omega_1^-[\alpha_1(\xi)]\right\|_{L^2(d\xi d\zeta)} < \frac{\beta}{2},$$

where of course it is understood that $x = x(\xi,\zeta)$ and $y = y(\xi,\zeta)$. We first use the triangle inequality to say

$$\begin{aligned}&\left\|e^{iTH}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - \varphi_0(\zeta)\omega_1^-[\alpha_1(\xi)]\right\| \\ &\qquad \leqslant \left\|e^{iTH}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - \varphi_0(\zeta)U(0,T)U_1(T,0)\alpha_1(\xi)\right\| \\ &\qquad\qquad + \left\|\varphi_0(\zeta)U(0,T)U_1(T,0)\alpha_1(\xi) - \varphi_0(\zeta)\omega_1^-[\alpha_1(\xi)]\right\| = N_1 + N_2.\end{aligned}$$

We first try to control the norm N_2. Since

$$\begin{aligned}\omega_1^-[\alpha_1(\xi)] &= \lim_{t\to\infty} U(0,t)U_1(t,0)\alpha_1(\xi) \\ &= \lim_{t\to\infty}\left[U(0,t)e^{-iE_1t}e^{-iv^2t/8}e^{-iv\xi/2}\alpha_1\left(\xi + \frac{vt}{2}\right)\right],\end{aligned}$$

we have

$$\begin{aligned}
N_2 &= \left\| \left(U(0,T)U_1(T,0)\alpha_1(\xi) - \omega_1^-[\alpha_1(\xi)] \right) \varphi_0(\zeta) \right\|_{L^2(d\xi d\zeta)} \\
&= \left\| U(0,T)U_1(T,0)\alpha_1(\xi) - \omega_1^-[\alpha_1(\xi)] \right\|_{L^2(d\xi)} \\
&= \left\| U(0,T)U_1(T,0)\alpha_1(\xi) - \lim_{t\to\infty} [U(0,t)U_1(t,0)\alpha_1(\xi)] \right\|_{L^2(d\xi)} \\
&= \lim_{t\to\infty} \| U(0,T)U_1(T,0)\alpha_1(\xi) - U(0,t)U_1(t,0)\alpha_1(\xi) \|_{L^2(d\xi)} \\
&= \lim_{t\to\infty} \left\| \int_T^t \frac{d}{ds} [U(0,s)U_1(s,0)\alpha_1(\xi)]\, ds \right\|_{L^2(d\xi)} \\
&= \lim_{t\to\infty} \left\| \int_T^t (iU(0,s)H(s)U_1(s,0)\alpha_1(\xi) - iU(0,s)H_1(s)U_1(s,0)\alpha_1(\xi))\, ds \right\|_{L^2(d\xi)} \\
&= \lim_{t\to\infty} \left\| \int_T^t U(0,s)(H(s) - H_1(s))U_1(s,0)\alpha_1(\xi) ds \right\|_{L^2(d\xi)} \\
&\leqslant \lim_{t\to\infty} \int_T^t \left\| V_2\left(\xi - \frac{vt}{2}\right) U_1(s,0)\alpha_1(\xi) \right\|_{L^2(d\xi)} ds \\
&= \lim_{t\to\infty} \int_T^t \left\| V_2(\xi - vt/2) e^{-iE_1 t} e^{-iv^2 t/8} e^{-iv\xi/2} \alpha_1(\xi + vt/2) \right\|_{L^2(d\xi)} ds \\
&= \lim_{t\to\infty} \int_T^t ds \left(\int_{\mathbb{R}} |V_2(\xi)|^2 |\alpha_1(\xi + vs)|^2 d\xi \right)^{1/2} \\
&\leqslant \lim_{t\to\infty} C_d \int_T^t ds \left(\int_{-R_2}^{R_2} e^{-2d|\xi+vs|} d\xi \right)^{1/2} \\
&\leqslant C_d \lim_{t\to\infty} \int_T^t ds \left(\int_{vs-R_2}^{vs+R_2} e^{-2d|u|} du \right)^{1/2} \\
&\leqslant C_d e^{dR_2} \sqrt{2R_2} \lim_{t\to\infty} \int_T^t e^{-dvs} ds \\
&= \frac{C_d e^{dR_2} \sqrt{2R_2}}{dv} e^{-dvT}.
\end{aligned}$$

In making the above estimate, we have used the fact that $v > 0$ and that $T > \frac{R_2}{v}$ by way of (4.13) on page 51. The above expression is therefore less than $\frac{\beta}{6}$.

We now return to N_1. To control this norm, we will need the following unitary propagators, all of which exist by Theorem 20.1 of [15]:

$$\begin{aligned}
W(t,s) &\quad \text{is generated by } -\frac{1}{2}(1+\epsilon)\Delta_x - \frac{\epsilon}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right)\Delta_y + V_1(x) + V_2\left(\frac{x}{1+\epsilon} - vt\right),\\
W_x^\epsilon(t,s) &\quad \text{is generated by } H_x^\epsilon = -\frac{1}{2}(1+\epsilon)\Delta_x + V_1(x) + V_2\left(\frac{x}{1+\epsilon} - vt\right),\\
W_x(t,s) &\quad \text{is generated by } -\frac{1}{2}\Delta_x + V_1(x) + V_2\left(x - vt\right),\\
W_y(t,s) &\quad \text{is generated by } H_1^y = -\frac{\epsilon}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right)\Delta_y.
\end{aligned}$$

We first make use of $W(t,s)$. Note that

$$\begin{aligned}
N_1 = \left\| e^{iTH} e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - U(0,T)U_1(T,0)\alpha_1(\xi)\varphi_0(\zeta)\right\| \\
\leqslant \left\| e^{iTH} e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - W(0,T)e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y)\right\| \\
+ \left\| W(0,T)e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y) - U(0,T)U_1(T,0)\alpha_1(\xi)\varphi_0(\zeta)\right\| = N_3 + N_4. \quad (4.19)
\end{aligned}$$

We first bound the N_3 term. First, note that

$$N_3 = \left\| e^{iTH}\left[e^{-iE_1T}\eta_1(x)\widetilde{\varphi}_0(y,T)\right] - W(0,T)\left[e^{-iE_1T}\eta_1(x)\widetilde{\varphi}_0(y,T)\right]\right\|.$$

We would now like to use a *finite-times* variant of Lemma 2.1.1, the magic lemma, for potentially time-dependent Hamiltonians. We now state and prove this lemma, since we will need it a few times in the remaining calculations.

Lemma 4.5.1 (The magic lemma for finite times)**.** Suppose $H(t)$ generates the unitary propagator $U(t,s)$ and $H_1(t)$ generates $U_1(t,s)$. Suppose there exists a vector φ such that, for $t \in [0,T]$,

$$\left\|\left(i\frac{\partial}{\partial t} - H(t)\right)U_1(t,T)\varphi\right\| \leqslant \mu(\epsilon,t) \quad (4.20)$$

for some bounded, t-measurable function μ. Then

$$\|U(t,T)\varphi - U_1(t,T)\varphi\| \leqslant T\sup_{t\in I}|\mu(\epsilon,t)|.$$

Proof. Let $\psi(t) = U_1(t,T)\varphi$, and let $f(t) = U(t,T)\psi(T) - \psi(t)$. Then, provided (4.20) is satisfied, the fundamental theorem of calculus gives, for any $t < T$, that $f(T) - f(t) =$

$-f(t) = \psi(t) - U(t,T)\psi(T) = \int_t^T f'(s)ds$. It follows that

$$\begin{aligned}\|\psi(t) - U(t,T)\psi(T)\| = \|U(T,t)\psi(t) - \psi(T)\| &= \left\| \int_t^T \frac{d}{ds}\left(U(T,s)\psi(s) - \psi(T)\right) ds \right\| \\ &= \left\| \int_t^T \left(\left[-\frac{d}{ds} U(T,s) \right] \psi(s) - U(T,s) \frac{d\psi}{ds} \right) ds \right\| \\ &= \left\| \int_t^T \left(iU(T,s)H(s)\psi(s) - U(T,s)\frac{d\psi}{ds} \right) ds \right\| \\ &\leqslant \int_t^T \left\| iU(T,s) \left(i\frac{\partial \psi}{\partial s} - H(s)\psi(s) \right) \right\| ds \\ &= \int_t^T \left\| \left(i\frac{\partial \psi}{\partial s} - H(s)\psi(s) \right) \right\| ds \\ &\leqslant \int_t^T \mu(s) ds \leqslant T \sup_{t\in I} |\mu(\epsilon, t)|.\end{aligned}$$

But $\|\psi(t) - U(t,T)\psi(T)\| = \|U_1(t,T)\psi - U(t,T)\psi\| = \|U(t,T)\psi - U_1(t,T)\psi\|$, so this is the desired result. □

We would now like to apply the lemma to N_3 with $e^{-i(t-T)H}$ in place of $U(t,T)$ and $W(t,T)$ in place of $U_1(t,T)$. By the triangle inequality,

$$\begin{aligned}&\left\| e^{iTH}\left[e^{-iE_1T}\eta_1(x)\widetilde{\varphi}_0(y,T)\right] - W(0,T)\left[e^{-iE_1T}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,T)\right]\right\| \\ \leqslant &\left\| \left(i\frac{\partial}{\partial t} + \frac{1}{2}(1+\epsilon)\Delta_x + \frac{\epsilon}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right) - V_1(x) - V_2\left(\frac{x}{1+\epsilon} - vt\right)\right) W(t,T)\left[e^{-iE_1T}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,T)\right]\right\| \\ &\qquad + \left\| \left(V_2\left(\frac{x}{1+\epsilon} - y\right) - V_2\left(\frac{x}{1+\epsilon} - vt\right)\right) W(t,T)\left[e^{-iE_1T}\eta_1(x)\widetilde{\varphi}_0^\epsilon(y,T)\right]\right\|.\end{aligned}$$

The first of these norms is zero by definition of $W(t,s)$. Since $W(t,s)$ can be realized as $W_x^\epsilon(t,s) \otimes W_y(t,s) = W_x^\epsilon(t,s) \otimes e^{-iH_1^y t}$, we are able to write the second (nonzero) norm as

$$\left\| \left(V_2\left(\frac{x}{1+\epsilon} - y\right) - V_2\left(\frac{x}{1+\epsilon} - vt\right)\right) \widetilde{\varphi}_0^\epsilon(y,t) W_x^\epsilon(t,T)\left[e^{-iE_1T}\eta_1(x)\right]\right\|. \tag{4.21}$$

The fundamental theorem of calculus gives

$$V_2\left(\frac{x}{1+\epsilon} - y\right) - V_2\left(\frac{x}{1+\epsilon} - vt\right) = \int_{\frac{x}{1+\epsilon}-vt}^{\frac{x}{1+\epsilon}-y} V'(s)ds.$$

Therefore,

$$\left|V_2\left(\frac{x}{1+\epsilon}-y\right)-V_2\left(\frac{x}{1+\epsilon}-vt\right)\right| \leqslant \left|\frac{x}{1+\epsilon}-y-\left(\frac{x}{1+\epsilon}-vt\right)\right| \|V_2'\|_\infty = |y-vt|\,\|V_2'\|_\infty .$$

So, writing the norm in (4.21) as an integral, we have by Fubini's theorem

$$\begin{aligned}
&\left\|\left(V_2\left(\frac{x}{1+\epsilon}-y\right)-V_2\left(\frac{x}{1+\epsilon}-vt\right)\right)\widetilde{\varphi}_0^\epsilon(y,t)W_x^\epsilon(t,T)\left[e^{-iE_1T}\eta_1(x)\right]\right\|^2 \\
&= \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \left|V_2\left(\frac{x}{1+\epsilon}-y\right)-V_2\left(\frac{x}{1+\epsilon}-vt\right)\right|^2 |\widetilde{\varphi}_0^\epsilon(y,t)|^2 \left|W_x(t,T)\left[e^{-iE_1T}\eta_1(x)\right]\right|^2 \\
&\leqslant \|V_2'\|_\infty \int_{\mathbb{R}} dy|y-vt|^2|\widetilde{\varphi}_0^\epsilon(y,t)|^2 \int_{\mathbb{R}} dx \left|W_x^\epsilon(t,T)\left[e^{-iE_1T}\eta_1(x)\right]\right|^2 \\
&= \|V_2'\|_\infty \int_{\mathbb{R}} dy|y-vt|^2|\widetilde{\varphi}_0^\epsilon(y,t)|^2 \left\|W_x^\epsilon(t,T)\left[e^{-iE_1T}\eta_1(x)\right]\right\|_{L^2(dx)}^2 .
\end{aligned}$$

By unitarity of $W_x^\epsilon(t,T)$ and the fact that η_1 is normalized, this just becomes

$$\begin{aligned}
&\|V_2'\|_\infty \int_{\mathbb{R}} |y-vt|^2|\widetilde{\varphi}_0(y,t)|^2 dy \leqslant \|V_2'\|_\infty \int_{\mathbb{R}} |y-vt|^2|\varphi_0(y,t)|^2 dy \\
&= \|V_2'\|_\infty \int_{\mathbb{R}} |y-vt|^2 \left|\varphi_0\left(1+i\left(\frac{2+\epsilon}{1+\epsilon}\right)t, 1, \epsilon, vt, \frac{1+\epsilon}{2+\epsilon}v, y\right)\right|^2 dy \\
&= \|V_2'\|_\infty \pi^{-1/2}\epsilon^{-1/2}\left(1+\left(\frac{2+\epsilon}{1+\epsilon}\right)^2 t^2\right)^{-1/2} \int_{\mathbb{R}} |y-vt|^2 \exp\left(-\frac{(y-vt)^2}{\epsilon\left(1+\left(\frac{2+\epsilon}{1+\epsilon}\right)^2 t^2\right)}\right) dy.
\end{aligned}$$

This is an integral of the form $\int_{\mathbb{R}} u^2 e^{-u^2/\alpha^2} du = \frac{1}{2}\pi^{1/2}\alpha^3$ for $\alpha = \epsilon^{1/2}\left(1+\left(\frac{2+\epsilon}{1+\epsilon}\right)^2 t^2\right)^{1/2}$, so we conclude that

$$\begin{aligned}
&\left\|\left(V_2\left(\frac{x}{1+\epsilon}-y\right)-V_2\left(\frac{x}{1+\epsilon}-vt\right)\right)\widetilde{\varphi}_0(y,t)W_x^\epsilon(t,T)\left[e^{-iE_1T}\eta_1(x)\right]\right\|^2 \\
&\leqslant \frac{1}{2}\|V_2'\|_\infty \epsilon^{-1/2}\pi^{-1/2}\left(1+\left(\frac{2+\epsilon}{1+\epsilon}\right)^2 t^2\right)^{-1/2} \cdot \pi^{1/2}\epsilon^{3/2}\left(1+\left(\frac{2+\epsilon}{1+\epsilon}\right)^2 t^2\right)^{3/2} \\
&= \frac{1}{2}\|V_2'\|_\infty \epsilon\left(1+\left(\frac{2+\epsilon}{1+\epsilon}\right)^2 t^2\right).
\end{aligned}$$

Taking square roots on both sides gives

$$\left\| \left(i\frac{\partial}{\partial t} - H \right) W(t,T) \left[e^{-iE_1 T} \eta_1(x) \widetilde{\varphi}_0(y,T) \right] \right\|_{L^2(dxdy)} \leqslant \epsilon^{1/2} \left(\frac{1}{2} \|V_2'\|_\infty \left(1 + \left(\frac{2+\epsilon}{1+\epsilon} \right)^2 t^2 \right) \right)^{1/2} = \mu(\epsilon, t).$$

The function μ is continuous in t, and

$$\sup_{t \in I} |\mu(\epsilon,t)| = \epsilon^{1/2} \left(\frac{1}{2} \|V_2'\|_\infty \left(1 + \left(\frac{2+\epsilon}{1+\epsilon} \right)^2 T^2 \right) \right)^{1/2} \to 0 \quad \text{as } \epsilon \to 0,$$

so Lemma 4.5.1 gives $N_3 \to 0$ as $\epsilon \to 0$ as desired. We can therefore choose an ϵ that will make N_3 less than $\frac{\beta}{6}$.

Once more applying the fact that $W(t,s) = W_x^\epsilon(t,s) \otimes e^{-iH_1^y t}$, we are left with trying to figure out how to bound

$$\begin{aligned} N_4 &= \left\| W(0,T) e^{-iTH_1} \eta_1(x) \widetilde{\varphi}_0^\epsilon(y) - U(0,T) U_1(T,0) \alpha_1(\xi) \varphi_0(\zeta) \right\| \\ &= \Big\| e^{-iTE_1^\epsilon} \left[W_x^\epsilon(0,T) \eta_1(\cdot) \right](x) \widetilde{\varphi}_0^\epsilon(y) \\ &\qquad - e^{-iTE_1} e^{-iv^2T/8} \left[U(0,T) e^{-iv\cdot/2} \alpha_1 \left(\cdot + \frac{vT}{2} \right) \right](\xi) \varphi_0(\zeta) \Big\|. \end{aligned} \tag{4.22}$$

The brunt of the work lies here, and this piece is therefore worthy of its own subsection.

4.5.1 Bounding N_4

We begin by using the triangle inequality to break N_4 up into several pieces. It will be convenient here to introduce the following notation pertaining to the semiclassical wave packets, since part of our strategy will be to show that the Gaussian in ζ is well approximated by the Gaussian in y for small values of ϵ. Recall that $\widetilde{\varphi}_0^\epsilon(y) = \mathscr{F}_\epsilon^{-1} \left\{ F(p) \varphi_0 \left(1, 1, \epsilon, \left(\frac{1+\epsilon}{2+\epsilon} \right) v, 0, p \right) \right\}(y)$. We let $\varphi_0^\epsilon(y) = \varphi_0(1, 1, \epsilon, 0, \gamma v, y)$ and observe that $\gamma v \to \frac{v}{2}$ as $\epsilon \to 0$. The triangle inequality applied to (4.22) gives:

$$
\begin{aligned}
N_4 = &\left\| e^{-iTE_1^\epsilon}\left[W_x^\epsilon(0,T)\eta_1(\cdot)\right](x)\widetilde{\varphi}_0^\epsilon(y) - e^{-iTE_1}e^{-iv^2T/8}\left[U(0,T)e^{-iv\cdot/2}\alpha_1\left(\cdot+\frac{vT}{2}\right)\right](\xi)\varphi_0(\zeta)\right\| \\
\leqslant &\left\| e^{-iTE_1^\epsilon}\Big(W_x^\epsilon(0,T)\eta_1(x) - W_x(0,T)\eta_1(x)\Big)\widetilde{\varphi}_0^\epsilon(y)\right\|_{L^2(dxdy)} \\
&+\left\| e^{-iTE_1^\epsilon}[W_x(0,T)\eta_1(\cdot)](x)\left(\widetilde{\varphi}_0^\epsilon(y) - \varphi_0^\epsilon(y)\right)\right\|_{L^2(dxdy)} \\
&+\left\| e^{-iTE_1^\epsilon}[W_x(0,T)\eta_1(\cdot)](x)\left(\varphi_0^\epsilon(y) - \varphi_0\left(1,1,\epsilon,0,\frac{v}{2},y\right)\right)\right\|_{L^2(dxdy)} \\
&+\left\| e^{-iTE_1^\epsilon}[W_x(0,T)\eta_1(\cdot)]\left(\xi+\frac{\zeta}{2}\right)\Big\{\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},y(\zeta,\xi)\right)\right. \\
&\qquad\left. - e^{-iv\xi/2}e^{iv\zeta/(2\epsilon)}\varphi_0(1,1,\epsilon,0,0,\zeta-\epsilon\xi)\Big\}\right\|_{L^2(d\xi d\zeta)} \\
&+\left\| e^{-iTE_1^\epsilon}[W_x(0,T)\eta_1(\cdot)]\left(\xi+\frac{\zeta}{2}\right)\Big\{e^{-iv\xi/2}e^{iv\zeta/(2\epsilon)}\varphi_0\left(1,1,\epsilon,0,0,\zeta-\epsilon\xi\right)\right. \\
&\qquad\left. - e^{-iv\xi/2}\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\zeta\right)\Big\}\right\|_{L^2(d\xi d\zeta)} \\
&+\left\| \left\{[W_x(0,T)\eta_1(\cdot)]\left(\xi+\frac{\zeta}{2}\right) - [W_x(0,T)\eta_1(\cdot)](\xi)\right\}\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\zeta\right)\right\|_{L^2(d\xi d\zeta)} \\
&+\left\| e^{-iTE_1^\epsilon}\Big\{[W_x(0,T)\eta_1(\cdot)](\xi)\right. \\
&\qquad\left. - U(0,T)\left[e^{-iv^2T/8}e^{-iv\xi/2}\eta_1\left(\cdot+\frac{vT}{2}\right)\right](\xi)\right\}\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\zeta\right)\Bigg\|_{L^2(d\xi d\zeta)} \\
&+\left\| \left(e^{-iTE_1^\epsilon} - e^{-iTE_1}\right)U(0,T)\left[e^{-iv^2T/8}e^{-iv\xi/2}\eta_1\left(\cdot+\frac{vT}{2}\right)\right](\xi)\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\zeta\right)\right\|_{L^2(d\xi d\zeta)} \\
&+\left\| e^{-iTE_1}e^{-iv^2T/8}U(0,T)\left[e^{-iv\xi/2}\left(\eta_1\left(\cdot+\frac{vT}{2}\right)\right.\right.\right. \\
&\qquad\left.\left.\left. -\alpha_1\left(\cdot+\frac{vT}{2}\right)\right)\right](\xi)\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\zeta\right)\right\|_{L^2(d\xi d\zeta)}.
\end{aligned}
$$

Each of the above nine (9) norms will be bounded by one in a series of lemmas. We work sequentially, using the first lemma to bound what is perhaps the most difficult piece.

Lemma 4.5.2.

$$
\left\| e^{-iTE_1^\epsilon}\Big(W_x^\epsilon(0,T)\eta_1(x) - W_x(0,T)\eta_1(x)\Big)\widetilde{\varphi}_0^\epsilon(y)\right\|_{L^2(dxdy)} \leqslant \left\|W_x^\epsilon(0,T)\eta_1(x) - W_x(0,T)\eta_1(x)\right\|_2
$$

is small for small ϵ.

Proof. We will once again use Lemma 4.5.1 by finding a bounded measurable function $\mu(\epsilon, t)$ such that

$$\left\| \left(i\frac{\partial}{\partial t} - H_x^\epsilon(t) \right) W_x(t,T)\eta_1(x) \right\|_2 \leqslant \mu(\epsilon, t), \tag{4.23}$$

from which it follows that

$$\| W_x^\epsilon(0,T)\eta_1(x) - W_x(0,T)\eta_1(x)\|_2 \leqslant T \sup_{t\in I} |\mu(\epsilon,t)|.$$

The norm in (4.23) is

$$\begin{aligned}
&\left\| \left(i\frac{\partial}{\partial t} - H_x^\epsilon \right) W_x(t,T)\eta_1(x) \right\|_2 \\
&\quad = \left\| \left[i\frac{\partial}{\partial t} + \frac{1}{2}(1+\epsilon)\Delta_x - V_1(x) - V_2\left(\frac{x}{1+\epsilon} - vt\right) \right] W_x(t,T)\eta_1(x) \right\|_2 \\
&= \left\| \left[-\frac{1}{2}\Delta_x + V_1(x) + V_2(x-vt) + \frac{1}{2}(1+\epsilon)\Delta_x - V_1(x) - V_2\left(\frac{x}{1+\epsilon} - vt\right) \right] W_x(t,T)\eta_1(x) \right\|_2 \\
&\quad \leqslant \frac{\epsilon}{2} \left\| -\Delta_x W_x(t,T)\eta_1(x) \right\| + \left\| \left(V_2(x-vt) - V_2\left(\frac{x}{1+\epsilon} - vt\right) \right) W_x(t,T)\eta_1(x) \right\|_2 .
\end{aligned}$$

We first control the second norm. By Hölder's inequality,

$$\begin{aligned}
&\left\| \left(V_2(x-vt) - V_2\left(\frac{x}{1+\epsilon} - vt\right) \right) W_x(t,T)\eta_1(x) \right\|_2 \\
&\quad \leqslant \left\| V_2(x-vt) - V_2\left(\frac{x}{1+\epsilon} - vt\right) \right\|_\infty \| W_x(t,T)\eta_1(x)\|_2 \\
&\quad = \left\| V_2(x-vt) - V_2\left(\frac{x}{1+\epsilon} - vt\right) \right\|_\infty = \left\| V_2(x) - V_2\left(\frac{x}{1+\epsilon}\right) \right\|_\infty .
\end{aligned}$$

By the uniform continuity of V_2 on $\mathbb{R}$, for any $\beta > 0$ there exists a real δ with $0 < \delta < 2R_2$ such that $\left|V_2(x) - V_2\left(\frac{x}{1+\epsilon}\right)\right| < \frac{\beta}{16T}$ provided $|x - \frac{x}{1+\epsilon}| = \frac{\epsilon}{1+\epsilon}|x| < \delta$. Take

$$\epsilon_1 \leqslant \min\left\{ 1, \epsilon_0, \frac{\delta}{2R_2 - \delta} \right\}, \tag{4.24}$$

(recall the definition of ϵ_0 given in 4.4.3) and note that

$$2R_2 \frac{\epsilon}{1+\epsilon} < \delta, \quad \text{iff} \quad 2R_2\epsilon < \delta(1+\epsilon), \quad \text{iff} \quad \epsilon < \frac{\delta}{2R_2 - \delta}.$$

Now, if $|x| \geqslant 2R_2$ and $\epsilon < \epsilon_1$, we know $|x|/(1+\epsilon) > R_2$, so $V_2(x) = V_2(x/(1+\epsilon)) = 0$. On the other hand, if $|x| < 2R_2$, then

$$\left| x - \frac{x}{1+\epsilon} \right| = \frac{\epsilon}{1+\epsilon}|x| < \frac{\epsilon}{1+\epsilon} 2R_2 < \delta,$$

which gives $|V_2(x) - V_2(x/(1+\epsilon))| < \frac{\beta}{16T}$ as desired.

To bound the first term, we use the methods outlined in §X.12 of [17] – the Dyson expansion in the "interaction representation" – to place a uniform bound on $W_x(t,T)$ as an operator from the second Sobolev space $H^2 = H^2(\mathbb{R}) = D(-\Delta_x)$ to H^2. This will be sufficient, since

$$\|-\Delta_x W_x(t,T)\eta_1(x)\|_2 \leqslant \|-\Delta_x\|_{\mathscr{L}(H^2,\mathscr{H})} \|W_x(t,T)\|_{\mathscr{L}(H^2)} \|\eta_1(x)\|_{H^2} = C\,\|W_x(t,T)\|_{\mathscr{L}(H^2)}$$

for some $C > 0$, provided the bound on $W_x(t,T)$ exists, a fact we now demonstrate. Let $V(x,t) = V_1(x) + V_2(x - vt)$, let $H_0 = -\Delta_x$, and define $\widetilde{V}(x,t) = e^{iH_0t}V(x,t)e^{-iH_0t}$. This is a bounded, self-adjoint operator on $\mathscr{H}$ that generates a unitary propagator $\widetilde{U}(t,s)$ by Theorem X.69 in [17]. Since $V(x,t)$ satisfies the hypotheses of Theorem X.71 in [17], we know

$$W_x(t,T) = e^{-itH_0}\widetilde{U}(t,T)e^{iTH_0}.$$

Taking $\|\cdot\|_{\mathscr{L}(H^2)}$ on both sides, we obtain

$$\|W_x(t,T)\|_{\mathscr{L}(H^2)} \leqslant \left\|\widetilde{U}(t,T)\right\|_{\mathscr{L}(H^2)}. \tag{4.25}$$

So, showing the bound on $W_x(t,T)$ is equivalent to showing the bound for $\widetilde{U}(t,T)$. However, by the Dyson series for $\widetilde{U}(t,T)$, we have

$$\widetilde{U}(t,T) = I - i\int_T^t \widetilde{V}(x,t_1)dt_1 - \int_T^t dt_1 \int_T^{t_1} \widetilde{V}(x,t_2)\widetilde{V}(x,t_1)dt_2 + \cdots \tag{4.26}$$

As a result, questions about $\widetilde{U}(t,T)$ reduce to questions about $\widetilde{V}(x,t)$. So we need to estimate

$$\begin{aligned}
\left\|\widetilde{V}(x,t)\right\|_{\mathscr{L}(H^2)} &= \left\|e^{iH_0t}V(x,t)e^{-iH_0t}\right\|_{\mathscr{L}(H^2)} \\
&= \left\|(H_0+1)e^{iH_0t}V(x,t)e^{-iH_0t}(H_0+1)^{-1}\right\|_{\mathscr{L}(\mathscr{H})} \\
&= \left\|(-\Delta_x+1)V(x,t)(-\Delta_x+1)^{-1}\right\|_{\mathscr{L}(\mathscr{H})}.
\end{aligned}$$

by the unitarity of e^{itH_0} as an element of $\mathscr{L}(\mathscr{H})$ and the fact that it commutes with both H_0+1 and its inverse. Let $\psi \in \mathscr{H}$. Then

$$\begin{aligned}
&\left\|(-\Delta_x+1)V(x,t)(-\Delta_x+1)^{-1}\psi\right\|_{\mathscr{H}} \\
&\qquad \leqslant \left\|-\Delta_x[V(x,t)(-\Delta_x+1)^{-1}\psi]\right\|_{\mathscr{H}} + \left\|V(x,t)(-\Delta_x+1)^{-1}\psi\right\|_{\mathscr{H}} \\
&\qquad \leqslant \left\|-V''\right\|_\infty \left\|(-\Delta_x+1)^{-1}\right\|_{\mathscr{L}(\mathscr{H},H^2)} \left\|\psi\right\|_{\mathscr{H}} + 2\left\|\nabla V(x,t)\nabla[(-\Delta_x+1)^{-1}\psi]\right\|_{\mathscr{H}} \\
&\qquad\quad + \left\|V\right\|_\infty \left\|\Delta_x[(-\Delta_x+1)^{-1}\psi]\right\|_{\mathscr{H}} + \left\|V\right\|_\infty \left\|(-\Delta_x+1)^{-1}\right\|_{\mathscr{L}(\mathscr{H},H^2)} \left\|\psi\right\|_{\mathscr{H}}.
\end{aligned}$$

Obviously, the first and last terms on the right hand side of the above expression are taken care of. For the second, we use Hölder's inequality and the Plancherel theorem to write

$$\begin{aligned}
\left\|\nabla V(x,t)\nabla[(-\Delta_x+1)^{-1}\psi]\right\|_{\mathscr{H}} &= \left\|(i\nabla V(x,t))(-i\nabla[(-\Delta_x+1)^{-1}\psi])\right\|_{\mathscr{H}} \\
&\leqslant \left\|V'\right\|_\infty \left\|\frac{p}{p^2+1}\hat{\psi}\right\|_{L^2(\mathbb{R},dp)} \\
&\leqslant \left\|V'\right\|_\infty \left\|\frac{p}{p^2+1}\right\|_\infty \left\|\hat{\psi}\right\|_{L^2(\mathbb{R},dp)} \\
&= \frac{1}{2}\left\|V'\right\|_\infty \left\|\psi\right\|_{\mathscr{H}}.
\end{aligned}$$

We take a similar approach to the third term:

$$\begin{aligned}
\left\|V\right\|_\infty \left\|\Delta_x[(-\Delta_x+1)^{-1}\psi]\right\|_{\mathscr{H}} \leqslant \left\|V\right\|_\infty \left\|\frac{p^2}{p^2+1}\hat{\psi}\right\|_{L^2(\mathbb{R},dp)} &\leqslant \left\|V\right\|_\infty \left\|\frac{p^2}{p^2+1}\right\|_\infty \left\|\hat{\psi}\right\|_{L^2(\mathbb{R},dp)} \\
&= \left\|V\right\|_\infty \left\|\psi\right\|_{\mathscr{H}}.
\end{aligned}$$

Therefore, for any $\psi \in \mathscr{H}$, we have

$$\begin{aligned}
\left\|(H_0+1)V(x,t)(H_0+1)^{-1}\psi\right\|_{\mathscr{L}(\mathscr{H})} \leqslant \Big(&\left\|V''\right\|_\infty \left\|(H_0+1)^{-1}\right\|_{\mathscr{L}(\mathscr{H},H^2)} + \frac{1}{2}\left\|V'\right\|_\infty \\
&+ \left\|V\right\|_\infty + \left\|V\right\|_\infty \left\|(H_0+1)^{-1}\right\|_{\mathscr{L}(\mathscr{H},H^2)}\Big) \left\|\psi\right\|_{\mathscr{H}} = K\left\|\psi\right\|_{\mathscr{H}}.
\end{aligned}$$

Hence $\left\|\widetilde{V}(x,t)\right\|_{\mathscr{L}(H^2)}$ has uniform bound K. Taking $\|\cdot\|_{\mathscr{L}(H^2)}$ on both sides of (4.26), we obtain

$$\begin{aligned}\left\|\widetilde{U}(t,T)\right\|_{\mathscr{L}(H^2)} &\leqslant \sum_{n=0}^{\infty} \frac{|t-T|^n}{n!}\left(\sup_{t\in[0,T]}\left\|\widetilde{V}(x,t)\right\|_{\mathscr{L}(H^2)}\right)^n \\ &\leqslant \sum_{n=0}^{\infty}\frac{T^n}{n!}K^n = e^{KT}.\end{aligned}$$

It follows from (4.25) that $\|W_x(t,T)\|_{\mathscr{L}(H^2)} \leqslant e^{TK}$ on $[0,T]$, so

$$\frac{\epsilon}{2}\left\|H_0 W_x(t,T)\eta_1(x)\right\|_{\mathscr{H}} \leqslant \frac{\epsilon}{2}e^{KT}\left\|H_0\right\|_{\mathscr{L}(H^2,\mathscr{H})} \to 0 \quad \text{as } \epsilon \to 0.$$

Hence there is a $\epsilon_2 > 0$ such that $\epsilon < \epsilon_2$ will make $\frac{\epsilon}{2}e^{KT}\left\|H_0\right\|_{\mathscr{L}(H^2,\mathscr{H})} < \frac{\beta}{16T}$.

Putting this result together with the result in (4.24), we can say that

$$\left\|\left(i\frac{\partial}{\partial t} - H_x^\epsilon(t)\right)W_x(t,T)\eta_1(x)\right\|_{L^2(dx)} \leqslant \left\|V_2(x) - V_2(x/(1+\epsilon))\right\|_\infty + \frac{\epsilon}{2}e^{KT}\left\|H_0\right\|_{\mathscr{L}(H^2,\mathscr{H})}.$$

It follows from Lemma 4.5.1 that

$$\begin{aligned}&\left\|\left(W_x^\epsilon(0,T)\eta_1(x) - W_x(0,T)\eta_1(x)\right)\widetilde{\varphi}_0(y)\right\|_{L^2(dxdy)} \\ &\qquad\qquad \leqslant T\left\|V_2(x) - V_2(x/(1+\epsilon))\right\|_\infty + \frac{\epsilon}{2}Te^{KT}\left\|H_0\right\|_{\mathscr{L}(H^2,\mathscr{H})},\end{aligned}$$

and if we take $\epsilon < \min\{\epsilon_1, \epsilon_2\}$, we will have

$$\left\|\left(W_x^\epsilon(0,T)\eta_1(x) - W_x(0,T)\eta_1(x)\right)\widetilde{\varphi}_0(y)\right\|_{L^2(dxdy)} < T\left(\frac{\beta}{16T} + \frac{\beta}{16T}\right) = \frac{\beta}{8}.$$

□

Lemma 4.5.3.

$$\left\|e^{-iTE_1^\epsilon}\left[W_x(0,T)\eta_1(\cdot)\right](x)\left(\widetilde{\varphi}_0^\epsilon(y) - \varphi_0^\epsilon(y)\right)\right\|_{L^2(dxdy)} \to 0 \quad \text{as } \epsilon \to 0.$$

Proof. The "∼" is necessary for the (stationary phase) infinite time arguments, but for finite times it's just a nuisance. The unitary of W_x and $e^{-iTE_1^\epsilon}$ and normalization of η_1 mean the above norm is just $\|\widetilde{\varphi}_0^\epsilon(y) - \varphi_0^\epsilon(y)\|_{L^2(dy)}$. We can go to momentum space in the y variable to bound this norm:

$$\begin{aligned}\|\widetilde{\varphi}_0^\epsilon(y) - \varphi_0^\epsilon(y)\|_{L^2(dy)} &= \left\| \mathscr{F}_\epsilon^{-1}\Big[F(p)\mathscr{F}_\epsilon[\varphi_0^\epsilon(p)] - \mathscr{F}_\epsilon[\varphi_0^\epsilon(p)]\Big](y)\right\|_{L^2(dy)} \\ &= \|F(p)\widehat{\varphi}_0^\epsilon(p) - \widehat{\varphi}_0^\epsilon(p)\|_{L^2(dp)} \\ &= \|(F(p)-1)\widehat{\varphi}_0^\epsilon(p)\|_{L^2(dp)} \\ &= \int_{\mathbb{R}} |F(p)-1|^2 \left|\varphi_0\left(1,1,\epsilon,\left(\frac{1+\epsilon}{2+\epsilon}\right)v,0,p\right)\right|^2 dp.\end{aligned}$$

The function $F(p)-1$ is nonzero exactly on the set $(-\infty, \frac{v}{2}-a) \cup (\frac{v}{2}+a, \infty)$. Hence we investigate the above norm by breaking up the integral in the following way:

$$\int_{\mathbb{R}} = \int_{-\infty}^{v/2-b} + \int_{v/2-b}^{v/2-a} + \int_{v/2+a}^{v/2+b} + \int_{v/2+b}^{\infty}.$$

On the intervals $(v/2-b, v/2-a)$ and $(v/2+a, v/2+b)$, we have $|F(p)-1|^2 \leqslant 1$. Note that if we consider only

$$\epsilon < \frac{4a}{v-2a}, \tag{4.27}$$

the peak of $|\varphi_0(1,1,\epsilon,\left(\frac{1+\epsilon}{2+\epsilon}\right)v,0,p)|^2$ lies inside the interval $(\frac{v}{2}, \frac{v}{2}+a)$, so we can say that

$$\begin{aligned}&\int_{v/2-b}^{v/2-a} |F(p)-1|^2 \left|\varphi_0\left(1,1,\epsilon,\left(\frac{1+\epsilon}{2+\epsilon}\right)v,0,p\right)\right|^2 dp \\ &\qquad \leqslant \int_{v/2-b}^{v/2-a} \left|\varphi_0\left(1,1,\epsilon,\left(\frac{1+\epsilon}{2+\epsilon}\right)v,0,p\right)\right|^2 dp \\ &\qquad \leqslant (b-a)\left|\varphi_0\left(1,1,\epsilon,\left(\frac{1+\epsilon}{2+\epsilon}\right)v,0,\frac{v}{2}-a\right)\right|^2 \\ &\qquad\qquad = \pi^{-1/2}\epsilon^{-1/2}(b-a)\exp\left(-\frac{(2a(2+\epsilon)+v\epsilon)^2}{4\epsilon(2+\epsilon)^2}\right).\end{aligned}$$

Similarly,

$$\int_{v/2+a}^{v/2+b} |F(p)-1|^2 \left|\varphi_0\left(1,1,\epsilon,\left(\frac{1+\epsilon}{2+\epsilon}\right)v,0,p\right)\right|^2 dp$$
$$\leqslant (b-a)\left|\varphi_0\left(1,1,\epsilon,\left(\frac{1+\epsilon}{2+\epsilon}\right)v,0,\frac{v}{2}+a\right)\right|^2$$
$$= \pi^{-1/2}\epsilon^{-1/2}(b-a)\exp\left(-\frac{(-2a(2+\epsilon)+v\epsilon)^2}{4\epsilon(2+\epsilon)^2}\right).$$

Both of these quantities vanish as $\epsilon \to 0$. Now, on the intervals $(-\infty, \frac{v}{2}-b)$ and $(\frac{v}{2}+b, \infty)$, the function $|F(p)-1|^2 = 1$, so we are left with two integrals we can evaluate in terms of the complementary error function:

$$\int_{v/2+b}^{\infty} \left|\varphi_0\left(1,1,\epsilon,\left(\frac{1+\epsilon}{2+\epsilon}\right)v,0,p\right)\right|^2 dp = \pi^{-1/2}\epsilon^{-1/2}\int_{v/2+b}^{\infty} \exp\left(-\frac{\left(p-\frac{(1+\epsilon)v}{(2+\epsilon)}\right)^2}{\epsilon}\right)dp$$
$$= \frac{1}{2}\operatorname{erfc}\left(\epsilon^{-1/2}\frac{2b(2+\epsilon)-v\epsilon}{2(2+\epsilon)}\right)$$
$$= O\left(\frac{e^{-x(\epsilon)^2}}{x(\epsilon)}\right) \quad \text{as } \epsilon \to 0.$$

where in the last step we have set $x = x(\epsilon) = \epsilon^{-1/2}\frac{2b(2+\epsilon)-v\epsilon}{2(2+\epsilon)}$ ($\to \infty$ as $\epsilon \to 0$) and used the standard asymptotic expansion (see, for example, [1]) of the complementary error function. A similar result holds for the integral $\int_{-\infty}^{v/2-b}$, and the result is proved. Hence there exists an □

Lemma 4.5.4.

$$\|[W_x(0,T)\eta_1(\cdot)](x)\,(\varphi_0^\epsilon(y)-\varphi_0(y))\|_{L^2(dxdy)} \to 0 \quad \text{as } \epsilon \to 0.$$

Proof. The idea here is that we would rather work with

$$\varphi_0(y) = \varphi_0\left(1,1,\epsilon,0,\frac{v}{2},y\right)$$

as opposed to the slightly more cumbersome

$$\varphi_0^\epsilon(y) = \varphi_0\left(1, 1, \epsilon, 0, \left(\frac{1+\epsilon}{2+\epsilon}\right)v, y\right).$$

We will try to control the square of this norm, writing it first as an inner product:

$$\begin{aligned}\|\varphi_0^\epsilon(y) - \varphi_0(y)\|^2 &= \langle \varphi_0^\epsilon(y) - \varphi_0(y), \varphi_0^\epsilon(y) - \varphi_0(y)\rangle_{L^2(dy)} \\ &= 1 + 1 - 2\,\mathrm{Re}\,\langle \varphi_0^\epsilon(y), \varphi_0(y)\rangle_{L^2(dy)} \\ &= 2 - 2\,\mathrm{Re}\left[\int_{\mathbb{R}} \pi^{-1/2}\epsilon^{-1/2}e^{-y^2/\epsilon}\exp\left(\frac{-ivy}{2(2+\epsilon)}\right)dy\right] \\ &= 2\left(1 - \exp\left(-\frac{\epsilon v^2}{16(2+\epsilon)^2}\right)\right),\end{aligned}$$

which obviously vanishes as we make ϵ small. □

Lemma 4.5.5.

$$\left\|\left[W_x(0,T)\eta_1(\cdot)\right]\left(\xi + \frac{\zeta}{2}\right)\left(\varphi_0(1,1,\epsilon,0,\frac{v}{2},y(\zeta,\xi)) - e^{-iv\xi/2}e^{iv\zeta/(2\epsilon)}\varphi_0(1,1,\epsilon,0,0,\zeta-\epsilon\xi)\right)\right\|_{L^2(d\xi d\zeta)} \to 0 \quad \text{as } \epsilon \to 0.$$

Proof. Hereafter, we choose to work in the (ξ, ζ) system of coordinates, so we need to write

$$\left[W_x(0,T)\eta_1(\cdot)\right](x)\,\varphi_0(y)$$

as

$$\left[W_x(0,T)\eta_1(\cdot)\right]\left(\xi + \frac{\zeta}{2}\right)\varphi_0\left(\frac{1}{2}\left(\frac{2+\epsilon}{1+\epsilon} - \frac{\epsilon}{1+\epsilon}\xi\right)\right).$$

As we proceed, we would like to work with

$$\left[W_x(0,T)\eta_1(\cdot)\right]\left(\xi + \frac{\zeta}{2}\right)\varphi_0\left(\zeta - \epsilon\xi\right)$$

instead of

$$\left[W_x(0,T)\eta_1(\cdot)\right]\left(\xi + \frac{\zeta}{2}\right)\varphi_0\left(\frac{1}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right)\zeta - \frac{\epsilon}{1+\epsilon}\xi\right).$$

(This is tantamount to keeping the leading order in both ξ and ζ in the expression for $y(\xi, \zeta)$.)

Note that, by the definition of the φ_0's, it is easy to show

$$\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\zeta-\epsilon\xi\right)=e^{-iv\xi/2}e^{iv\zeta/(2\epsilon)}\varphi_0\left(1,1,\epsilon,0,0,\zeta-\epsilon\xi\right).$$

Let $g(x)=[W_x(0,T)\eta_1(\cdot)](x)$. Then we want to control

$$\begin{aligned}
&\left\|e^{-iTE_1^\epsilon}g\left(\xi+\frac{\zeta}{2}\right)\left[\varphi_0(\zeta-\epsilon\xi)-\varphi_0\left(\frac{1}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right)\zeta-\frac{\epsilon}{1+\epsilon}\xi\right)\right]\right\|_{L^2(d\xi d\zeta)}\\
&\qquad=\left\|g\left(\xi+\frac{\zeta}{2}\right)\left[\varphi_0(\zeta-\epsilon\xi)-\varphi_0\left(\frac{1}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right)\zeta-\frac{\epsilon}{1+\epsilon}\xi\right)\right]\right\|_{L^2(d\xi d\zeta)}\\
&\qquad=\int_{\mathbb{R}}d\xi\int_{\mathbb{R}}d\zeta\left|g\left(\xi+\frac{\zeta}{2}\right)\right|^2\left|\varphi_0(\zeta-\epsilon\xi)-\varphi_0\left(\frac{1}{2}\left(\frac{2+\epsilon}{1+\epsilon}\right)\zeta-\frac{\epsilon}{1+\epsilon}\xi\right)\right|^2\\
&\quad=\int_{\mathbb{R}}d\zeta\int_{\mathbb{R}}d\xi|g(\xi)|^2\left|\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\zeta-\frac{\epsilon}{1+\epsilon}\xi\right)-\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\left(1+\frac{\epsilon}{2}\right)\zeta-\epsilon\xi\right)\right|^2,
\end{aligned}$$

where we have used translation invariance of the integral in ξ: $\xi\mapsto\xi-\zeta/2$. Now, we use the fact that

$$\varphi_0(A,B,\epsilon,a,\eta,bx-y)=b^{-1/2}\varphi_0\left(\frac{A}{b},bB,\epsilon,\frac{y+a}{b},b\eta,x\right)$$

and, after switching the integration order, we obtain

$$\begin{aligned}
\int_{\mathbb{R}}d\xi|g(\xi)|^2\Bigg[\int_{\mathbb{R}}d\zeta\Bigg|&\varphi_0\left(1,1,\epsilon,\frac{\epsilon}{1+\epsilon}\xi,\frac{v}{2},\zeta\right)\\
&-\left(1+\frac{\epsilon}{2}\right)^{-1/2}\varphi_0\left(\frac{2}{2+\epsilon},1+\frac{\epsilon}{2},\epsilon,\frac{2\epsilon\xi}{2+\epsilon},\frac{v}{2}\left(1+\frac{\epsilon}{2}\right),\zeta\right)\Bigg|^2\Bigg].
\end{aligned}$$

The ζ integral is just the inner product of the difference between

$$f(\xi,\zeta)=\varphi_0\left(1,1,\epsilon,\frac{\epsilon}{1+\epsilon}\xi,\frac{v}{2},\zeta\right)$$

and

$$f_1(\xi,\zeta)=\left(1+\frac{\epsilon}{2}\right)^{-1/2}\varphi_0\left(\frac{2}{2+\epsilon},1+\frac{\epsilon}{2},\epsilon,\frac{2\epsilon\xi}{2+\epsilon},\frac{v}{2}\left(1+\frac{\epsilon}{2}\right),\zeta\right).$$

So we want to try to control

$$\int_{\mathbb{R}}d\xi|g(\xi)|^2\left(1+\frac{1}{1+\epsilon/2}-2\operatorname{Re}\langle f,f_1\rangle_{L^2(d\zeta)}\right).$$

Both f and f_1 satisfy the hypotheses of Proposition 4 of [10], so we can use it to obtain

$$\operatorname{Re}\langle f, f_1\rangle = \sqrt{\frac{8}{8+\epsilon(4+\epsilon)}}\exp\left(-\frac{\epsilon v^2}{8(8+\epsilon(4+\epsilon)} - \frac{\epsilon^2\xi^2}{2(1+\epsilon)^2(8+\epsilon(4+\epsilon)}\right) * \cos\left(\frac{v\epsilon(4+\epsilon)\xi}{2(1+\epsilon)(8+\epsilon(4+\epsilon))}\right).$$

It is therefore clear that $\operatorname{Re}\langle f, f_1\rangle \to 1$ as $\epsilon \to 0$, implying that the ξ integrand $|g(\xi)|^2(1 + \frac{1}{1+\epsilon/2} - 2\operatorname{Re}\langle f, f_1\rangle) \to 0$ as $\epsilon \to 0$. What's more,

$$|g(\xi)|^2 \left|1 + \frac{1}{1+\epsilon/2} - 2\operatorname{Re}\langle f, f_1\rangle\right| \leqslant |g(\xi)|^2 \left(1 + \frac{1}{1+\epsilon/2} + 2|\operatorname{Re}\langle f, f_1\rangle|\right) \leqslant 4|g(\xi)|^2 \in L^1(d\xi),$$

so we can invoke the generalized dominated convergence theorem (see [19], pg. 89) to conclude that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}} d\xi |g(\xi)|^2 \left(1 + \frac{1}{1+\epsilon/2} - 2\operatorname{Re}\langle f, f_1\rangle\right) = \int_{\mathbb{R}} d\xi \lim_{\epsilon\to 0}\left(|g(\xi)|^2\left(1 + \frac{1}{1+\epsilon/2} - 2\operatorname{Re}\langle f, f_1\rangle\right)\right) = 0,$$

which is what we wanted to show. □

Lemma 4.5.6.

$$\left\| [W_x(0,T)\eta_1(\cdot)]\left(\xi + \frac{\zeta}{2}\right) e^{-iv\xi/2} e^{iv\zeta/(2\epsilon)}\Big(\varphi_0(1,1,\epsilon,0,0,\zeta-\epsilon\xi) - \varphi_0(1,1,\epsilon,0,0,\zeta)\Big)\right\|_{L^2(d\xi d\zeta)} \to 0 \quad \text{as } \epsilon \to 0.$$

Proof. Retaining the same definition of g used in the previous lemma, we are interested in controlling

$$\begin{aligned}
&\left\|[W_x(0,T)\eta_1](\xi+\zeta/2)e^{-iv\xi/2}e^{iv\zeta/(2\epsilon)}\Big(\varphi_0(1,1,\epsilon,0,0,\zeta-\epsilon\xi) - \varphi_0(1,1,\epsilon,0,0,\zeta)\Big)\right\|^2_{L^2(d\zeta d\xi)} \\
&= \left\|g(\xi+\zeta/2)\Big(\varphi_0(1,1,\epsilon,0,0,\zeta-\epsilon\xi) - \varphi_0(1,1,\epsilon,0,0,\zeta)\Big)\right\|^2_{L^2(d\zeta d\xi)} \\
&= \int_{\mathbb{R}} d\zeta \int_{\mathbb{R}} d\xi |g(\xi+\zeta/2)|^2 |\varphi_0(1,1,\epsilon,0,0,\zeta-\epsilon\xi) - \varphi_0(1,1,\epsilon,0,0,\zeta)|^2.
\end{aligned}$$

We perform the variable shift $\xi \mapsto \xi - \frac{\zeta}{2}$ and obtain

$$= \int_{\mathbb{R}} d\xi |g(\xi)|^2 \left[\int_{\mathbb{R}} d\zeta \left| (1+\epsilon/2)^{-1/2} \varphi_0 \left(\frac{2}{2+\epsilon}, 1+\frac{\epsilon}{2}, \epsilon, \frac{2\epsilon\xi}{2+\epsilon}, 0, \zeta \right) - \varphi_0 (1,1,\epsilon,0,0,\zeta) \right|^2 \right]$$
$$= \int_{\mathbb{R}} d\xi |g(\xi)|^2 \left(1 + \frac{2}{2+\epsilon} - 2\,\mathrm{Re}\,\langle h, h_1 \rangle_{L^2(d\zeta)} \right).$$

Again, Proposition 4 of [10] tells us that

$$\mathrm{Re}\,\langle h, h_1 \rangle = \sqrt{\frac{8}{8+\epsilon(4+\epsilon)}} \exp\left(-\frac{v^2\epsilon}{8(8+\epsilon(4+\epsilon))} - \frac{2\epsilon\xi^2}{8+\epsilon(4+\epsilon)} \right) \cos\left(\frac{v(4+\epsilon)\xi}{8+\epsilon(4+\epsilon)} \right).$$

Hence the norm has more or less the same form as the norm we bounded above using the dominated convergence theorem, so we should be good. □

Lemma 4.5.7.

$$\left\| \left\{ [W_x(0,T)\eta_1(\cdot)] \left(\xi + \frac{\zeta}{2} \right) - [W_x(0,T)\eta_1(\cdot)](\xi) \right\} \varphi_0 \left(1,1,\epsilon,0,\frac{v}{2},\zeta \right) \right\|_{L^2(d\xi d\zeta)} \to 0 \quad \text{as } \epsilon \to 0.$$

Proof. Again, we let $g(x) = [W_x(0,T)\eta_1(\cdot)](x)$, so we are really interested in the small-ϵ behavior of

$$\left\| \left(g\left(\xi + \frac{\zeta}{2} \right) - g(\xi) \right) e^{-iv\xi/2} e^{iv\zeta/(2\epsilon)} \varphi_0 (1,1,\epsilon,0,0,\zeta) \right\|_{L^2(d\xi d\zeta)}$$
$$= \left\| \left(g\left(\xi + \frac{\zeta}{2} \right) - g(\xi) \right) \varphi_0 \left(1,1,\epsilon,0,\frac{v}{2},\zeta \right) \right\|_{L^2(d\xi d\zeta)}. \quad (4.28)$$

It will be helpful to recall the following characterization of the domain of the Laplacian operator, $D(-\Delta_x) = H^2(\mathbb{R}) = H^2$, as seen from position (x) space:

> $D(-\Delta_x)$ is the set of all $\psi \in L^2(\mathbb{R}, dx)$ such that $\nabla_x \psi$ exists and is an element of $L^2 \cap \mathrm{AC}(\mathbb{R})$ whose a.e. derivative $\Delta_x \psi$ is an element of $L^2(\mathbb{R}, dx)$.

Here, of course, $\mathrm{AC}(\mathbb{R})$ is the set of all absolutely continuous functions defined on the real line, and it is this fact we will need in the proof. The propagator $W_x(t,s)$ is generated by $-\frac{1}{2}\Delta_x + V_1(x) + V_2(x - vt)$, a self-adjoint operator whose domain is H^2. The propagator therefore maps H^2 into H^2, and since $\eta_1 \in H^2$ by assumption, it follows that $g(x) = [W_x(0,T)\eta_1](x) \in H^2$, so g' is absolutely continuous on $\mathbb{R}$ and an element of L^2. Moreover, if we Taylor expand $g(\xi+\zeta/2)$ around $\zeta=0$, we get (see [2])

$$g\left(\xi+\frac{\zeta}{2}\right)=g(\xi)+\frac{\zeta}{2}g'(\xi+\theta(\zeta)).$$

So the square of the norm in (4.28) becomes

$$\begin{aligned}\int_{\mathbb{R}} d\zeta \int_{\mathbb{R}} d\xi \left|g\left(\xi+\frac{\zeta}{2}\right)-g(\xi)\right|^2 \left|\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\zeta\right)\right|^2 \\ =\frac{1}{2}\int_{\mathbb{R}} d\zeta|\zeta|^2\left|\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\zeta\right)\right|^2\int_{\mathbb{R}} d\xi\left|g'(\xi+\theta(\zeta))\right|^2.\end{aligned}$$

We can easily translate ξ and obtain

$$\begin{aligned}\left\|\left(g\left(\xi+\frac{\zeta}{2}\right)-g\left(\xi\right)\right)\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\zeta\right)\right\|^2_{L^2(d\xi d\zeta)} \\ \leqslant \frac{1}{2}\left\|g'\right\|_{L^2(d\xi)}\int_{\mathbb{R}} d\zeta|\zeta|^2\left|\varphi_0\left(1,1,\epsilon,0,\frac{v}{2},\zeta\right)\right|^2 \\ =\frac{1}{2}\pi^{-1/2}\epsilon^{-1/2}\left\|g'\right\|_{L^2(d\xi)}\int_{\mathbb{R}} d\zeta|\zeta|^2 e^{-\zeta^2/\epsilon}=\frac{1}{4}\epsilon\left\|g'\right\|_{L^2(d\xi)}.\end{aligned}$$

Taking square roots on both sides gives something that is $O(\epsilon^{1/2})\to 0$ as $\epsilon\to 0$ as desired. □

Lemma 4.5.8.

$$\left\|[W_x(0,T)\eta_1(\cdot)](\xi)-U(0,T)\left[e^{-iv^2T/8}e^{-iv\xi/2}\eta_1\left(\cdot+\frac{vT}{2}\right)\right](\xi)\right\|_{L^2(d\xi)}=0.$$

Proof. We observe that $W_x(t,s)$ is generated by (after relabelling x as ξ) $H_W=-\frac{1}{2}\Delta_\xi+V_1(\xi)+V_2(\xi-vt)$ and that $U(t,s)$ is generated by $H_U=-\frac{1}{2}\Delta_\xi+V_1(\xi+\frac{vt}{2})+V_2(\xi-\frac{vt}{2})$. Hence there is a Galilean boost involved in going from W_x to U. We observe that U and W_x have a common domain and make the claim that, for any vector $\psi\in L^2(\mathbb{R},d\xi)$, we have

$$U(t,0)\psi=e^{-itv^2/8}e^{-iv\xi/2}\mathscr{T}\left(\frac{vt}{2}\right)W_x(t,0)\psi, \tag{4.29}$$

where $\mathscr{T}(vt/2)$ is the operator that just adds $vt/2$ to x:

$$\mathscr{T}\left(\frac{vt}{2}\right)\psi(x) = \psi\left(x + \frac{vt}{2}\right).$$

This operator is clearly unitary, since it takes only a shift of integration variable to show that

$$\langle \mathscr{T}(vt/2)\psi, \mathscr{T}(vt/2)\varphi\rangle = \int_{\mathbb{R}} \psi\left(x + \frac{vt}{2}\right)\overline{\varphi\left(x + \frac{vt}{2}\right)}dx = \int_{\mathbb{R}} \psi(x)\overline{\varphi(x)}dx = \langle \psi, \varphi\rangle.$$

Hence $\mathscr{T}^{-1} = \mathscr{T}^* = \mathscr{T}(-vt/2)$. If the claim (4.29) is true, then taking adjoints of the operators on both sides gives

$$U(0,t) = W_x(0,t)\mathscr{T}\left(-\frac{vt}{2}\right)e^{iv\xi/2}e^{itv^2/8}.$$

This can obviously be rewritten as

$$U(0,t)e^{-itv^2/8}e^{-iv\xi/2}\mathscr{T}(vt/2) = W_x(0,t),$$

which is what we want to show. We therefore try to prove (4.29). So, suppose $\psi(x,t)$ and $\varphi(x,t)$ are any solutions of

$$i\frac{\partial\psi}{\partial t} = H_W\psi \qquad \text{and} \qquad i\frac{\partial\varphi}{\partial t} = H_U\varphi.$$

To prove (4.29), we need to show that

$$\varphi(x,t) = e^{-itv^2/8}e^{-ivx/2}\psi\left(x + \frac{vt}{2}, t\right) = e^{-itv^2/8}e^{-ivx/2}\mathscr{T}(vt/2)\psi(x,t).$$

First, we compute the time derivative of φ and multiply by i:

$$\begin{aligned} i\frac{\partial}{\partial t}\varphi(x,t) = i\Bigg(&-\frac{iv^2}{8}\varphi(x,t) + \frac{v}{2}e^{-itv^2/8}e^{-ivx/2}\frac{\partial\psi}{\partial x}\left(x + \frac{vt}{2}, t\right) \\ &+ e^{-itv^2/8}e^{-ivx/2}\frac{\partial\psi}{\partial t}\left(x + \frac{vt}{2}, t\right)\Bigg) \\ = \underline{\frac{v^2}{8}\varphi(x,t)} &+ ie^{-itv^2/8}e^{-ivx/2}\frac{\partial\psi}{\partial x}\left(x + \frac{vt}{2}, t\right) + \underline{i\frac{v}{2}e^{-itv^2/8}e^{-ivx/2}\frac{\partial\psi}{\partial t}\left(x + \frac{vt}{2}, t\right)}. \end{aligned}$$

Now we compute the x derivative of ψ:

$$\frac{\partial}{\partial x}\varphi(x,t) = -\frac{iv}{2}\varphi(x,t) + e^{-itv^2/8}e^{-ivx/2}\frac{\partial \psi}{\partial x}\left(x + \frac{vt}{2}, t\right).$$

This of course implies that

$$\begin{aligned}\frac{1}{2}\Delta_x\varphi(x,t) &= -\frac{iv}{4}\left(-\frac{iv}{2}\varphi(x,t) + e^{-itv^2/8}e^{-ivx/2}\frac{\partial \psi}{\partial x}\left(x + \frac{vt}{2}, t\right)\right) \\ &\quad + \frac{1}{2}\left(-\frac{iv}{2}\right)e^{-itv^2/8}e^{-ivx/2}\frac{\partial \psi}{\partial x}\left(x + \frac{vt}{2}, t\right) + \frac{1}{2}e^{-itv^2/8}e^{-ivx/2}\Delta_x\psi\left(x + \frac{vt}{2}, t\right) \\ &= \underline{-\frac{v^2}{8}\varphi(x,t)} - \underline{\frac{iv}{2}e^{-itv^2/8}e^{-ivx/2}\frac{\partial \psi}{\partial t}\left(x + \frac{vt}{2}, t\right)} + \frac{1}{2}e^{-itv^2/8}e^{-ivx/2}\Delta_x\psi\left(x + \frac{vt}{2}, t\right).\end{aligned}$$

After canceling the (underlined) terms they have in common, it follows that

$$\begin{aligned}i\frac{\partial}{\partial t}\varphi(x,t) + \frac{1}{2}\Delta_x\varphi(x,t) &= e^{-itv^2/8}e^{-ivx/2}\left(i\frac{\partial}{\partial t}\psi\left(x + \frac{vt}{2}, t\right) + \frac{1}{2}\Delta_x\psi\left(x + \frac{vt}{2}, t\right)\right) \\ &= e^{-itv^2/8}e^{-ivx/2}\left(V_1\left(x + \frac{vt}{2}\right)\psi\left(x + \frac{vt}{2}, t\right) + V_2\left(x - \frac{vt}{2}\right)\psi\left(x + \frac{vt}{2}, t\right)\right) \\ &= V_1\left(x + \frac{vt}{2}\right)\varphi(x,t) + V_2\left(x - \frac{vt}{2}\right)\varphi(x,t),\end{aligned}$$

which is precisely the time-dependent Schrödinger equation satisfied by φ. This proves the claim. □

Lemma 4.5.9.

$$\left\|\left(e^{-iTE_1^\epsilon} - e^{-iTE_1}\right)U(0,T)\left[e^{-iv^2T/8}e^{-iv\xi/2}\eta_1\left(\cdot + \frac{vT}{2}\right)\right](\xi)\varphi_0\left(1,1,\epsilon,0,0,\zeta\right)\right\|_{L^2(d\xi d\zeta)}$$

goes to zero as $\epsilon \to 0$.

Proof. Some perturbation arguments similar to those hinted at briefly in Chapter 3 – that is, an appeal to the discussion on pgs. 15-16 of [16] – work to show that, for sufficiently small ϵ, we have

$$\eta_1(x) = \alpha_1(x) + O(\epsilon),$$

where $O(\epsilon)$ is to be interpreted with respect to the Hilbert space norm $\|\cdot\|$. Moreover, the eigenenergies E_1^ϵ and E_1 associated with η_1 and α_1, respectively, satisfy

$$E_1^\epsilon = E_1 + O(\epsilon),$$

where now $O(\epsilon)$ is with respect to $|\cdot|$ on $\mathbb{C}$. Indeed, consider the family of operators

$$H(\epsilon) = -\frac{1}{2}(1+\epsilon)\Delta_x + V_1(x) = \underbrace{-\frac{1}{2}\Delta_x + V_1(x)}_{H_0} \underbrace{-\frac{\epsilon}{2}\Delta_x}_{\epsilon H_1}. \tag{4.30}$$

for the real parameter ϵ. Since we assume $V_1 \in C_0^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$, Theorem 10.15 of [17] tells us H_0 is a self-adjoint operator on $D(H_0) = D(-\Delta) = H^2 = D(\epsilon H_1)$ for any ϵ. The resolvent set of H_0 is nonempty, since it is self-adjoint and hence $\sigma(H_0) \subset \mathbb{R}$. By the triangle inequality for the L^2-norm, for any $\psi \in H^2$,

$$\begin{aligned}
\left\|-\frac{1}{2}\Delta_x\psi\right\| &= \left\|-\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + V_1(x)\psi - V_1(x)\psi\right\| \\
&\leqslant \left\|-\frac{1}{2}\Delta_x\psi + V_1(x)\psi\right\| + \|V_1(x)\psi\| \\
&\leqslant \|H_0\psi\| + \|V_1\|_\infty \|\psi\|.
\end{aligned}$$

Hence the perturbation H_1 is H_0-bounded. The lemma on pg. 16 of [16] therefore tells us that $H(\epsilon)$ is an analytic family of type (A) near $\epsilon = 0$, implying $H(\epsilon)$ is an analytic family in the sense of Kato near $\epsilon = 0$. Theorem XII.8 of [16] then tells us that for ϵ sufficiently small, there is an isolated, nondegenerate eigenvalue E_1^ϵ of $H(\epsilon)$ with corresponding analytic eigenvector η_1. Hence – though it has not been necessary to use these relations until now – we actually have

$$\|\eta_1(x) - \alpha_1(x)\| = O(\epsilon) \qquad \text{and} \qquad |E_1^\epsilon - E_1| = O(\epsilon).$$

To apply these ideas, we first do the trivial ζ integral and reduce the problem to a study of

$$\begin{aligned}
&\left\|\left(e^{-iTE_1^\epsilon} - e^{-iTE_1}\right) U(0,T)\left[e^{-iv^2T/8}e^{-iv\xi/2}\eta_1\left(\cdot + \frac{vT}{2}\right)\right](\xi)\right\|_{L^2(d\xi)} \\
&\quad = \left|e^{-iTE_1^\epsilon} - e^{-iTE_1}\right| \left\|U(0,T)e^{-iv^2T/8}e^{-iv\xi/2}\mathscr{T}\left(\frac{vT}{2}\right)\eta_1(\xi)\right\|_{L^2(d\xi)} \\
&\quad = \left|e^{-iTE_1^\epsilon} - e^{-iTE_1}\right| \to 0 \quad \text{as } \epsilon \to 0,
\end{aligned}$$

by the unitarity of all the operators inside the norm. □

Lemma 4.5.10.

$$\left\| U(0,T)\left[e^{-iv^2T/8}e^{-iv\xi/2}\left(\eta_1\left(\cdot+\frac{vT}{2}\right)-\alpha_1\left(\cdot+\frac{vT}{2}\right)\right)\right](\xi)\varphi_0\left(1,1,\epsilon,0,0,\zeta\right)\right\|_{L^2(d\xi d\zeta)}.$$

goes to zero as $\epsilon \to 0$.

Proof. We can again do the ζ integral independently and arrive at

$$\begin{aligned}\left\| U(0,T)\left[e^{-iv^2T/8}e^{-iv\xi/2}\left(\eta_1\left(\cdot+\frac{vT}{2}\right)-\alpha_1\left(\cdot+\frac{vT}{2}\right)\right)\right](\xi)\right\| \\ = \left\|\eta_1\left(\xi+\frac{vT}{2}\right)-\alpha_1\left(\xi+\frac{vT}{2}\right)\right\| \\ = \|\eta_1(\xi)-\alpha_1(\xi)\| = O(\epsilon)\to 0 \quad \text{as } \epsilon\to 0.\end{aligned}$$

by using the same perturbation arguments advanced in the previous lemma. □

4.5.2 The finite times piece – a recap

If T has been chosen to satisfy (4.13) on pg. 51, and if we pick an $\boldsymbol{\epsilon}$ which satisfies (4.27) on pg. 74 and such that each of the 9 norms shown at the beginning of §4.5.1 is less than $\frac{\beta}{54}$ whenever $\epsilon < \boldsymbol{\epsilon}$ – something we have shown we can do – then

$$\begin{aligned}&\left\|e^{iTH}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^{\epsilon}(y)-\varphi_0(\zeta)[\omega_1^-(\alpha_1)](\xi)\right\|_{L^2(d\xi d\zeta)} \\ &\leqslant \left\|U(0,T)U_1(T,0)\alpha_1(\xi)\varphi_0(\zeta)-\varphi_0(\zeta)[\omega_1^-(\alpha_1)](\xi)\right\| \\ &\quad + \left\|e^{iTH}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^{\epsilon}(y)-U(0,T)U_1(T,0)\alpha_1(\xi)\varphi_0(\zeta)\right\| \\ &< \frac{\beta}{6}+\left\|W(0,T)e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^{\epsilon}(y)-U(0,T)U_1(T,0)\alpha_1(\xi)\varphi_0(\zeta)\right\| \\ &\quad + \left\|e^{iTH}e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^{\epsilon}(y)-W(0,T)e^{-iTH_1}\eta_1(x)\widetilde{\varphi}_0^{\epsilon}(y)\right\| \\ &< \frac{\beta}{6}+\frac{\beta}{6}+9\left(\frac{\beta}{54}\right)=\frac{\beta}{2},\end{aligned}$$

as desired.

Chapter 5

Conclusion

Despite the results reported in the previous chapters, much room is left for further research.

In one dimension, perhaps the most obvious next step would be to try to extend these results to other scattering channels and cluster decompositions for the three-body problem. We do not think this would present any significant new challenges, though a slightly different approach would need to be taken to deal with the channel in which all three particles are free.

Ito [12] seems to have included the potential corresponding to the internuclear repulsion in treating the full three-body problem. We think we could introduce a similar term into (4.1) and repeat our analysis with little difficulty. It would also be worthwhile to study the case where ① and ③ are allowed to form more than one bound state.

A relaxation of the rather stringent $C_0^\infty(\mathbb{R})$ requirement for the interaction potentials would be a nice generalization, though we feel significantly more machinery would have to be employed to handle, say, Coulomb potentials. See Yajima's paper [29] for a glimpse of the technical headache that might be required!

A generalization of the time-dependent methods employed in this work to $d \geqslant 2$ would be nice to see. As mentioned in Chapter 1, Ito ([12], [13]) seems to have obtained some results for dimension higher than one, but using (somewhat complicated) time-independent tools. We will like to obtain similar results using time-dependent methods.

It would also be nice to allow for more generality in the evolution of (say) the y coordinate than to use just the zeroth semiclassical wave packet φ_0. We believe this work has laid the foundation for such a calculation, since the φ_k's form a basis for L^2, and computations with φ_k's (at least finitely many of them) should be similar to the calculations in this work involving φ_0, modulo some constants.

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