Axiomatic Imaging Theory — Formulate with Fairness & Fun

Proof: Theorem 2 implies $f^2(\sigma) = \langle \mathbf{C}, \sigma \rangle$. **Axiom 4** gives

$$f^2\left(\mathbf{T}\sigma\mathbf{T}^{\mathrm{tr}}\right) = f^2(\sigma) \tag{17}$$

for all orthogonal matrices $\mathbf{T}$. Then

$$\langle \mathbf{C}, \sigma \rangle = \mathrm{Trace}\left(\mathbf{C}^{\mathrm{tr}}\sigma\right) \tag{18}$$
$$= \mathrm{Trace}\left(\mathbf{C}^{\mathrm{tr}}\mathbf{T}\sigma\mathbf{T}^{\mathrm{tr}}\right) \tag{19}$$
$$= \mathrm{Trace}\left(\mathbf{T}^{\mathrm{tr}}\mathbf{C}^{\mathrm{tr}}\mathbf{T}\sigma\right) \tag{20}$$

for all orthogonal matrices $\mathbf{T}$ and all $\sigma \in \bar{\mathcal{P}}$ and by the property that $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$. Since $\mathbf{C}$ is symmetric, there exists an orthogonal matrix $\mathbf{T}_1$ such that

$$\mathbf{T}_1^{\mathrm{tr}}\mathbf{C}^{\mathrm{tr}}\mathbf{T}_1 = \mathbf{D} \tag{21}$$

is diagonal, with diagonal entries $\{d_k, k = 1, \ldots, N\}$. Select any pair of diagonal entries d_k and d_l and let $\mathbf{T}_2$ be the permutation matrix that interchanges the kth and lth rows of the identity matrix and leaves the rest untouched. Applying (20) for $T = T_1$ and $T = T_1T_2$

$$\mathrm{Trace}\left(\mathbf{T}_1^{\mathrm{tr}}\mathbf{C}^{\mathrm{tr}}\mathbf{T}_1\sigma\right) = \mathrm{Trace}\left(\mathbf{T}_2^{\mathrm{tr}}\mathbf{T}_1^{\mathrm{tr}}\mathbf{C}^{\mathrm{tr}}\mathbf{T}_1\mathbf{T}_2\sigma\right) \tag{22}$$

for all σ, which implies that

$$d_k\sigma_{k,k} + d_l\sigma_{l,l} = d_l\sigma_{k,k} + d_k\sigma_{l,l} \tag{23}$$

Let $\hat{f}$ denote the Fourier transform of f, defined as

$$\hat{f}(\omega) = \int_{\mathbf{R}^N} f(x) \cdot \mathrm{e}^{-i\omega\cdot x}dx.$$

For $\hat{p}$, we have, for ω small enough, by a Taylor series expansion around $\omega = 0$ (following Doob [3, p. 140])

$$\hat{p}(\omega) = 1 - \frac{1}{2}\omega^{\mathrm{tr}} \cdot \sigma \cdot \omega + o\left(\|\omega\|^2\right).$$

Since $\hat{q}_n(\omega) = \hat{p}(\omega/\sqrt{n})$, we have

$$\hat{q}_n(\omega) = 1 - \frac{1}{2n}\omega^{\mathrm{tr}} \cdot \sigma \cdot \omega + o\left(\frac{\|\omega\|^2}{n}\right)$$

for any $\omega \in \mathbf{R}^N$, when n is large enough. Then

$$\hat{g}_n(\omega) = \hat{q}_n^n = \left(1 - \frac{1}{2n}\omega^{\mathrm{tr}} \cdot \sigma \cdot \omega + o\left(\frac{\|\omega\|^2}{n}\right)\right)^n.$$

Hence

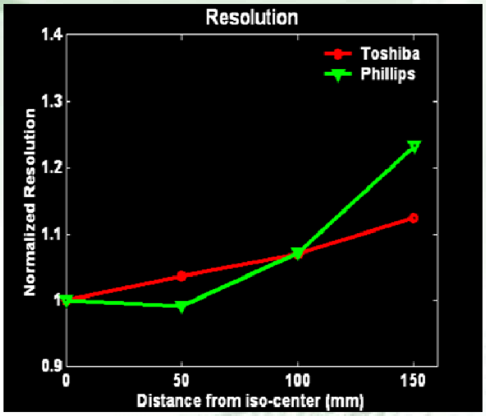
$$\lim_{n\to\infty} \hat{g}_n(\omega) = \mathrm{e}^{-(1/2)\omega^{\mathrm{tr}}\cdot\sigma\cdot\omega} = \widehat{G^\sigma}(\omega) \tag{26}$$

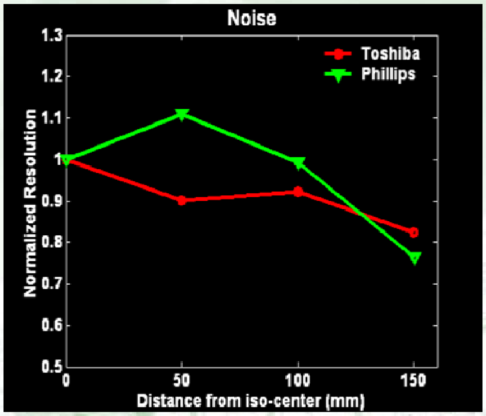
for any $\omega \in \mathbf{R}^N$. By [2, Prop. 8.69], g_n converges to G^σ vaguely in the sense of measure as measure density functions. Since $R[g_n] = R[p]$ for all n, by the continuity **Axiom 2**, $R[p] = R[G^\sigma]$. The conclusion follows immediately. ■

IV. Discussion

*There are many imaging systems. Their performance characterization is important for all applications. Various definitions are introduced for quantification of image resolution, which is the ability of an imaging system to separate two localized signals. In the nonnegative space, we postulated a set of axioms that a good image resolution measure should satisfy, obtained such an image resolution measure, applied our finding in comparing medical CT scanners, and won **a 2004 Herbert M. Stauffer Award**. <u>We believe that imaging theory can be unified using the axiomatic approach</u>.*

From Theory (Axiomatic Resolution Characterization) to Practice (CT Performance Comparison)





Papers by Our Team

Wang G, Li Y: Axiomatic approach for quantification of image resolution. IEEE Signal Processing Letters 6:257-258, 1999 (First paper on axiomatic quantification of image resolution and noise)

O'Sullivan JA, Jiang M, Ma XM, Wang G: Axiomatic quantification of multi-dimensional image resolution. IEEE Signal Processing Letters 9:120-122, 2002 (Extension of the above finding into the multi-dimensional case)

Meinel Jr. JF, Wang G, Jiang M, Frei T, Vannier MW, Hoffman EA: Spatial variation of resolution and noise in multi-slice spiral CT. Academic Radiology 10:607-613, 2003 (First application of the axiomatic resolution theory to comparison of medical CT scanners; 2004 Herbert M. Stauffer Award for Outstanding Basic Science Paper in Academic Radiology, Association of University Radiologists, USA)

Wang G, Jiang M: Axiomatic characterization of nonlinear homomorphic means. Journal of Mathematical Analysis and Applications 303:350-363, 2005 (Axiomatic quantification of means)

Jiang M, Wang G, Ma XM: A general axiomatic system for image resolution quantification. Journal of Mathematical Analysis and Applications 315:462-473, 2006 (Relaxed system for axiomatic quantification of image resolution)