ON STRAIN WAVE PROPAGATION IN THERMOELASTIC MEDIA

by

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I. INTRODUCTION

The study of the propagation of elastic waves in solids dates back to the last century. It has been found that in an unbounded isotropic solid two types of elastic waves can be propagated - the dilatational (or longitudinal, irrotational, compressive) and the shear (or transverse, rotational, equivoluminal) waves. For semi-infinite medium with a stress free boundary, it has been found that the surface waves or Rayleigh waves may occur. All the velocities of the propagation of the waves are independent of the frequency but depends on the density and elastic constants of the medium. All the elastic waves do not attenuate in the direction of the propagation of the waves, so that there is no dissipation of energy. A summary of the contributions may be found in a monograph by H. Kolsky (ref. 1).

Various modifications of Hooke's law of stress-strain relations as the basic principles for various theories of visco-elasticity (refs. 1, 2) do yield results of the attenuation of stress waves. Physically the dissipation of energy is due to the transformation of elastic energy into heat energy. This can be shown in a simple descriptive way. When a body is compressed the temperature will rise and when it is expanded the temperature will drop if the body is insulated. Now, in a general way of loading, the stress distribution is not uniform so that some part of the body is heated and some other part is cooled; and due to heat conduction which is an irreversible thermodynamical process, partial heat energy becomes unavailable which accounts for the dissipation of energy. Finally, when a steady state is established, there is a net gain of heat at the loss of a part of elastic energy. In visco-elasticity, however, the heat generated and the effect of heat on the change of strains are not considered in the analysis.

In the theory of elasticity, it is assumed or understood (in fact, it was ignored in the early development of the theory) that there is no thermal effect due to a change of the stress field of the medium which remains in the original isothermal state. The reverse effect that thermal changes in a body will introduce stresses, however, has been known since long ago. Mathematical formulation of the stress-strain-temperature relations was made from different hypotheses by J. M. C. Duhamel in 1838 (ref. 3) and later in 1841 by Frans Neumann (ref. 4); this relationship is sometimes referred as Duhamel-Neumann law. Duhamel-Neumann equations are equations of state and do not yield the change of temperature due to change of stresses.

The heat conduction is interlocked with the change of stress field of the medium. This can also be shown descriptively in a simple way. When heat is applied at some part or boundary of a body, there is heat conduction and a non-uniform temperature field is set up which will introduce a non-uniform field of strain which, in turn, will change somewhat the temperature field. Then the heat conduction rate will change when the temperature field is changed. Problems neglecting the effect of the change of stress field on heat conduction are called thermal stress problems. Then the distribution of temperature is solved as a pure heat conduction problem. With the temperature known, the problem reduces to an ordinary elastical one except that the elastic equation of state is to be replaced by Duhamel-Neumann equation where the temperature is a known function of space coordinates. This is the early theory of thermoelasticity which should not be confused with the present theory of thermoelasticity. Some authors, such as W. Nowacki (ref. 5) and R. J. Knops (ref. 6), still refer to thermal stress problems as thermoelastic problems. Contributions in this field may be found in Melan and Parkus' monograph of thermal stresses (ref. 7) and Gatewood's book of Thermal Stresses which covers many phases of the design problems at elevated temperature (ref. 8). Stresses in a thin plate due to a nucleus of thermoelastic strain has been solved by Nowacki (ref. 9). Tremmel (ref. 10) has made a contribution to general two-dimensional thermal stress problems by using complex variables. The elastic effect of a non-uniform distribution of temperature in an infinite solid by the method of equivalent center of dilatation has been made by Goodier (ref. 11); and in a similar way, the thermoelastic stresses in a semi-infinite solid have been investigated by Mindlin and D. H. Cheng (ref. 12). Thermal stress in a hollow cylinder with temperature dependent elastic thermal constants have been investigated by Trostel (ref. 13). Other contributions of a similar nature can also be found.

From an irreversible thermodynamical point of view, the dissipation of elastic energy is due to the production of entropy. One of the fundamental equations of the present thermoelasticity is the heat conduction coupled with the rate of change of strains due to Biot (ref. 14) and Lessen (ref. 15). Previously, as pointed out by Chadwick (ref. 16), a number of authors had produced thermodynamic analyses leading to the coupled momentum and energy equations which were postulated long ago by Duhamel and Neumann. Although the results obtained by these authors are correct, their methods do not fully acknowledge the irreversible nature of the heat conduction process.

As for thermoelastic boundary value problems, a uniqueness theorem has been proved by Weiner (ref. 17).

The propagation of strain waves is a dynamic thermoelastic problem. Plane thermoelastic waves have been investigated by Lessen (ref. 15) and Deresiewicz (ref. 18). Deresiewicz did not get an explicit form of the solution for arbitrary imposed frequency but he did show that there are two different phase velocities for a given frequency; the one he called an elastic mode and the other a thermal mode. He then proceeded to find the phase velocities and specific losses of energy for both modes of waves at high frequencies and also at low frequencies. He then studied both modes at small coupling before proceeding to the extreme cases of high frequencies and low frequencies. Deresiewicz's results when translated into the notations used in this dissertation are as follows:

At high frequencies:

Phase velocity of E-mode waves, $C_1 = C_T$

Phase velocity of T-mode waves, $C_2 = (2ma^2)^{\frac{1}{2}}$

Specific loss for E-mode waves, $\Delta W/W = 2\pi(C_s^2-C_T^2)/(a^2m)$

Specific loss for T-mode waves, $\Delta W/W = 4\pi$

At low frequencies:

Phase velocity of E-mode waves, $C_1 = C_s$

Phase velocity of T-mode waves, $C_2 = (2ma^2)^{\frac{1}{2}}C_T/C_s$

Specific loss for E-mode waves, $\Delta W/W = 2\pi ma^2(C_s^2-C_T^2)/C_s^4$

Specific loss for T-mode waves, $\Delta W/W = 4\pi$

The above limiting values are for any coupling. Deresiewicz then considered small coupling and then took the limiting values for very weak coupling. The results are translated as below:

At high frequencies:

Phase velocity of E-mode waves, $C_1 = C_T$

Phase velocity of T-mode waves, $C_2 = (2ma^2)^{\frac{1}{2}}C_T/C_s$

Specific loss for E-mode waves, $\Delta W/W = 2\pi(C_s^2-C_T^2)/(ma^2)$

Specific loss for T-mode waves, $\Delta W/W = 4\pi$

At low frequencies:

Phase velocity of E-mode waves, $C_1 = C_s$

Phase velocity of T-mode waves, $C_2 = \left[2ma^2(2-C_s^2/C_T^2)\right]^{\frac{1}{2}}$

Specific loss for E-mode waves, $\Delta W/W = 2\pi ma^2(C_s^2-C_T^2)/(C_s^2C_T^2)$

Specific loss for T-mode waves, $\Delta W/W = 4\pi$

In Lessen's analysis (ref. 15), the specific internal energy, which is a function of entropy and strains, was expanded in a Taylor's series up to the second order terms, and Duhamel-Neumann equation was found necessary. The isentropic and isothermal dilatation wave velocities found by him are different from the ones found by others.

$$\text{Lessen's } C_s^{\,2} = (\lambda + 2\mu)/\rho$$

$$\text{Lessen's } C_T^{\,2} = (\lambda + 2\mu)/\rho \; - B^2/U^2$$

where U is the specific internal energy, B is the second partial derivative with respect to entropy and strain, and the other notations are the same as used in this dissertation. Lessen's phase velocity of elastic mode approaches his isentropic dilatational wave velocity and his phase velocity of thermal mode approaches zero when either the frequency is very low or the second partial derivative of the specific internal energy with respect to entropy is zero at the reference temperature

Another paper on plane thermoelastic waves was written by Chadwick and Sneddon (ref. 19). Harmonic solutions were tried, and then approximations were made for high and low frequencies. The solutions were checked by numerical methods accomplished by high speed electronic computers for certain materials. The phase velocities and the coefficients of attenuation of dilatational waves in aluminum, copper, and iron at high frequencies are listed in the following table in conventional units:

Frequency (sec^{-1})	Phase Velocity	Coefficient of Attenuation (cm^{-1})
1. For aluminum.		
4.66×10^{9}	$1.0177\ C_T$	1.2002
4.66×10^{10}	$1.0175\ C_T$	1.2045×10^{2}
4.66×10^{11}	$1.0091\ C_T$	6.4884×10^{3}
4.66×10^{12}	$1.0002\ C_T$	1.2977×10^{4}
4.66×10^{13}	$1.0000\ C_T$	1.31×10^{4}
2. For copper.		
1.73×10^{9}	$1.0084\ C_T$	3.1521×10^{-1}
1.73×10^{10}	$1.0083\ C_T$	3.1331×10
1.73×10^{11}	$1.0043\ C_T$	1.6407×10^{3}
1.73×10^{12}	$1.0001\ C_T$	3.2640×10^{3}
1.73×10^{13}	$1.0000\ C_T$	3.29×10^{3}
3. For iron.		
1.75×10^{10}	$1.0002\ C_T$	4.3908×10^{-2}
1.75×10^{11}	$1.0002\ C_T$	4.4330
1.75×10^{12}	$1.0001\ C_T$	2.2396×10^{2}
1.75×10^{13}	$1.0000\ C_T$	4.4795×10^{2}
1.75×10^{14}	$1.0000\ C_T$	4.48×10^{2}

Here, C_T stands for isothermal phase velocities in aluminum, copper, and iron, respectively, with a reference temperature of $293^{\circ}K$. The paper also asserts that for a frequency range from less than 10 cycles per second up to about 10^9 cycles per second the phase velocity of the dilatational waves remains a little higher than the isothermal phase velocity.

Earlier in 1930, H. Jeffreys (ref. 20) claimed that the phase velocity of the dilatational waves in solids is the adiabatic velocity just as the sound velocity in the air. Also, he found by thermodynamic concepts without an equation of motion for the solid medium that the velocity of the dilatational waves in the granite layer of the earth's crust is about 1.5 part in a thousand higher than the isothermal phase velocity.

The velocity of Rayleigh waves in a thermoelastic medium has been investigated by Lockett (ref. 21). The phase velocity of the surface waves whose

$$C_T/C_V = 3$$

and

$$\alpha^2 T_R C_T{}^2 C^{-1}(1+\nu)^2(1-\nu)^{-2} = 0.05$$

has been found to be $0.9224\ C_V$ and there is no attenuation in the direction of the propagation of the waves. The temperature is in phase with the vertical displacement, and 180° out of phase with the horizontal displacement.

A review of the recent progress in thermoelasticity has been given by Chadwick (ref. 16).

This dissertation presents a study of the theory of strain wave propagation in a heat-conducting, non-radiating, elastic, homogeneous, isotropic medium. The linear theory of elasticity and a linear macroscopic theory of entropy are applied. General equations of thermoelasticity are stated in tensoral forms before they are applied to a particular problem.

II. NOMENCLATURE

A	Arbitrary constant; amplitude. (Sometimes with subscripts.)
a	Determinant of the metric tensor a_{ij} ; or A thermoelastic constant, $a^2 = k / C_\epsilon$.
a_{ij}	Metric tensor.
a^{ij}	Conjugate tensor.
B	Arbitrary constant (with or without subscript).
B_i	Covariant body force
B^i	Contravariant body force.
b	A thermoelastic constant, $b = \beta^2 T_R (\rho C_\epsilon C_T^2)^{-1}$
C_1	Phase velocity of elastic mode dilatational waves.
C_2	Phase velocity of thermal mode dilatational waves.
C_R	Phase velocity of Rayleigh waves or surface waves.
C_s	Isentropic phase velocity of dilatational waves.
C_T	Isothermal phase velocity of dilatational waves.
C_ϵ	Specific heat at constant strain.

C_V	Phase velocity of equivoluminal or shear waves.
C^*	Complex phase velocity.
C_1^*	Complex phase velocity of elastic mode dilatational waves.
C_2^*	Complex phase velocity of thermal mode dilatational waves.
C_R^*	Complex phase velocity of Rayleigh waves or surface waves.
C_{re}	Real part of complex phase velocity.
C_{im}	Imaginary part of complex phase velocity.
$C_{1(re)}$	Real part of complex phase velocity of elastic mode dilatational waves.
$C_{1(im)}$	Imaginary part of complex phase velocity of elastic mode dilatational waves.
$C_{2(re)}$	Real part of complex phase velocity of thermal mode dilatational waves.
$C_{2(im)}$	Imaginary part of complex phase velocity of thermal mode dilatational waves.
e	Cubic dilatation.
e_{ijk}	Permutation symbol.

f Wave number. $2\pi/f$ is wave-length.

f_1 Wave number of elastic mode dilatational waves.

f_2 Wave number of thermal mode dilatational waves.

f_i Covariant acceleration.

f^i Contravariant acceleration.

g Coefficient of attenuation.

g_1 Coefficient of attenuation of elastic mode.

g_2 Coefficient of attenuation of thermal mode.

H A thermal constant, $H = h / k$.

h Thermal convection coefficient.

h_i Exponential decreasing coefficients. $i = 1, 2, 3$.

i $i = \sqrt{-1}$.

k Thermal conductivity.

m Circular frequency. $m /(2\pi)$ is frequency.

$\tilde{m}$ Dimensionless circular frequency, $\tilde{m} = ma^2 c_T^{-2}$.

n Coefficient of rise (negative) or decay (positive).

$\tilde{n}$ Dimensionless coefficient of rise or decay, $\tilde{n} = na^2 C_T{}^2$.

P_1 , P_2 Functions of m and n.

$\tilde{P}_1$, $\tilde{P}_2$ Functions of $\tilde{m}$ and $\tilde{n}$.

p Complex frequency, $p = m + i\,n$.

Q_1 , Q_2 Functions of m and n.

$\tilde{Q}_1$, $\tilde{Q}_2$ Functions of $\tilde{m}$ and $\tilde{n}$.

q Complex wave number, $q = f + i\,g$.

r Radius — a coordinate.

s Entropy.

T_{ij} Covariant stress tensor.

T^{ij} Contravariant stress tensor.

$T^i{}_j$ Mixed stress tensor.

${}_pT_{(ij)}$ P-components of physical stress tensor.

T_R Reference temperature.

t Time.

U A function of m and n.

$\tilde{U}$ A function of $\tilde{m}$ and $\tilde{n}$.

u_i Covariant displacement.

u^i Contravariant displacement.

V A function of m and n.

$\tilde{V}$ A function of $\tilde{m}$ and $\tilde{n}$.

α Linear coefficient of thermal expansion.

$\alpha_a, \alpha_b, \cdots$ Angles of incident, reflected, and/or refracted waves.

$\alpha'_a, \alpha'_b, \cdots$ Real parts of α_a, α_b, ...

$\alpha''_a, \alpha''_b, \cdots$ Imaginary parts of α_a, α_b, ...

β A thermoelastic constant, $\beta = (3\lambda + 2\mu)\alpha$.

γ_{ij} Covariant strain tensor.

γ^{ij} Contravariant strain tensor.

$\gamma^i_{\ j}$ Mixed strain tensor.

$\gamma_{(ii)}$ Physical strain, change of length per unit length.

$\gamma_{(ij)}$ Physical strain, change of angle.

δ^i_j Kronecker delta having the character of a mixed tensor.

δ_{ij} Kronecker delta having no tensor character.

ϵ_{ijk} Permutation symbol, a relative tensor of weight -1.

ϵ^{ijk} Permutation symbol, a relative tensor of weight $+1$.

Θ Temperature above the reference temperature.

θ Polar angle; colatitude.

θ^* Argument of complex phase velocity.

λ A Lamé constant.

μ A Lamé constant.

ρ Mass density.

Φ Scalar potential function.

ϕ Azimuth.

Ψ_i Vector potential function.

III. GENERAL EQUATIONS OF THERMOELASTICITY.

The general equations of thermoelasticity consist of six equations of strain-displacement relations, six equations of stress-strain relations, three equations of motion and one equation of heat conduction. There are sixteen unknowns involved in these sixteen equations, viz. the three components of displacement, the six components of strain, the six components of stress and the entropy or temperature.

The basic tensor notation used by Synge and Schild in their book of tensor calculus (ref. 22) will be used here. All tensoral quantities are referred to the ordinary physical space of three dimensions so that all indices range from one to three. Unless otherwise noted, all the indices will follow the range and summation convention.

(1) The Equations of Strain-Displacement Relations. It is assumed in the linear theory of elasticity that the components of the displacement vector and their derivatives with respect to the coordinates are infinitesimals of the first order and the squares and products of these infinitesimals are negligible in comparison with their first powers. With this approximation the components of covariant strain tensor γ_{ij} in terms of the covariant displacement tensor u_i are (ref. 23, p.149) :

$$\gamma_{ij} = \tfrac{1}{2}(u_{i|j} + u_{j|i}) \tag{3.1}$$

where the vertical bar denotes covariant differentiation with respect to the unstrained body. The mixed and contravariant strain tensors are

$$\gamma^i_j = a^{im}\gamma_{mj} \tag{3.2}$$

$$\gamma^{ij} = a^{im}a^{jn}\gamma_{mn} \tag{3.3}$$

where a^{ij} is a tensor conjugate to the metric tensor a_{ij} with respect to the unstrained body. The cubical dilitation e is

$$e = \gamma^i_i = a^{im}\gamma_{mi} = u^i_{.|i} \tag{3.4}$$

The physical components of the strain tensor are

$$\gamma_{(ii)} = \gamma_{ii} / a_{ii} \tag{3.5}$$

where the index i is not to be summed, and, for $i \neq j$

$$\gamma_{(ij)} = \left[2\gamma_{ij} - a_{ij}\left(\gamma_{(ii)} + \gamma_{(jj)}\right)\right]\left[a_{ii}a_{jj} - (a_{ij})^2\right]^{-\frac{1}{2}} \tag{3.6}$$

where the indices are also not to be summed. The physical components $\gamma_{(ii)}$ are changes of length per unit length of line elements along the coordinate curves, while $\gamma_{(ij)}$ are the changes of angles between the covariant base vectors $\bar{g}_i$ and $\bar{g}_j$ respectively.

(2) <u>The Equations of Stress-Strain Relations</u>. The total strain can be thought to consist of the sum of the free thermal strain and the elastic strain produced by the resistance of the medium to the thermal expansion (ref. 24, p.359), thus the three different tensoral forms of the stress-

strain relations for the continuous, elastic, homogeneous, isotropic medium are

$$\gamma^i_j = \alpha\Theta\delta^i_j + \frac{1}{2\mu}T^i_j - \frac{\lambda}{2\mu(3\lambda+2\mu)}T^r_r\delta^i_j \tag{3.7}$$

$$\gamma^{ij} = \alpha\Theta a^{ij} + \frac{1}{2\mu}T^{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}T^r_r a^{ij} \tag{3.8}$$

$$\gamma_{ij} = \alpha\Theta a_{ij} + \frac{1}{2\mu}T_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}T^r_r a_{ij} \tag{3.9}$$

where α is the coefficient of thermal expansion, Θ the temperature above the reference temperature T_R, δ^i_j the Kronecker delta, and λ and μ are Lamé elastic constants. The stress tensor can be expressed in terms of strain tensors by solving for them from the above equations.

$$T^i_j = 2\mu\,\gamma^i_j + \lambda\gamma^r_r\delta^i_j - \beta\Theta\delta^i_j \tag{3.10}$$

$$T^{ij} = 2\mu\,\gamma^{ij} + \lambda\gamma^r_r a^{ij} - \beta\Theta a^{ij} \tag{3.11}$$

$$T_{ij} = 2\mu\,\gamma_{ij} + \lambda\gamma^r_r a_{ij} - \beta\Theta a_{ij} \tag{3.12}$$

where

$$\beta = (3\lambda+2\mu)\alpha \tag{3.13}$$

There are eight convenient sets of physical components of stress for solving boundary value problems in elasticity involving general coordinates (ref. 25). The parallel components or p-components of the stress vector (for an element of volume referred to the convariant base triad) along the covariant base vectors are stated below:

$$T^{ij} = {}_pT_{(ij)}(a^{ii}/a_{jj})^{\frac{1}{2}} \tag{3.14}$$

where the indices i and j are not to be summed.

(3) The Equations of Motion. The covariant and contravariant forms of the equations of motion are

$$T^{i}_{\cdot j|i} + B_j = \rho f_j \tag{3.15}$$

$$T^{ij}_{\cdot\cdot|i} + B^j = \rho f^j \tag{3.16}$$

where B_j and B^j are the covariant and contravariant tensors of body force, respectively, f_j and f^j are the covariant and contravariant tensors of acceleration, respectively, and ρ is the mass desity of the medium. Expressed in terms of displacement, they are

$$\mu a^{ir}(u_{r|ji} + u_{j|ri}) + (\lambda a^{rs}u_{r|sj} - \beta\Theta_{|j}) + B_j = \rho \ddot{u}_j \tag{3.17}$$

$$\mu a^{ir}a^{js}(u_{r|si} + u_{s|ri}) + (\lambda a^{rs}u_{r|si} - \beta\Theta_{|i})a^{ij} + B^j = \rho \ddot{u}^j \tag{3.18}$$

(4) The Equation of Heat Conduction. The equation of heat conduction for a thermally isotropic medium is

$$k\, a^{mn}\Theta_{|mn} = \dot{q} \tag{3.19}$$

where k is the thermal conductivity and q is the heat generated per unit volume.

(5) The Entropy of an Elastic Element. Macroscopic entropy defined for a medium in non-equilibrium state is used. Van Kampen (ref. 26) has shown that the macroscopic entropy of an irreversible thermodynamic system

agrees with various microscopic entropies of an irreversible thermodynamic system in observable extent.

The two basic assumptions in thermodynamics of irreversible processes are:

(a) The system can be broken into infinitesimal pieces each of which may be treated as reversible thermodynamic systems;

(b) Definitions of the thermodynamic coordinates remain valid provided the non-equilibrium state is not too far from equilibrium.

The entropy of an elastic element has been found (ref. 14) to be

$$s = C_\epsilon \ln(1 + \frac{\Theta}{T_R}) + \beta e \tag{3.20}$$

and, when the variation of temperature is small in comparison with the reference temperature, the approximation is

$$s = C_\epsilon \Theta / T_R + \beta e \tag{3.21}$$

where s is the entropy, C_ϵ is the specific heat capacity at constant strain, T_R is the reference temperature, and Θ is the temperature above the reference temperature. By the definition of entropy

$$s = q / T_R \tag{3.22}$$

and the following relation is obtained:

$$q = C_\epsilon \Theta + T_R \beta e \tag{3.23}$$

Substitution of the above equation in Eq. (3.19) yields

$$k\, a^{mn}\theta_{|mn} = c_\epsilon \dot{\theta} + \beta T_R \dot{u}^i_{.\,|i} \tag{3.24}$$

which is the equation that couples the temperature field with the displacement field or strain field. In terms of entropy, this equation is

$$a^{mn}s_{|mn} - \beta a^{mn} u^i_{.\,|imn} = (c_\epsilon / k)\dot{s} \tag{3.25}$$

(6) Resolution of the Displacement Vector into Components. By Helmholtz's theorem, a vector can be, in general, resolved into an irrotational part and a solenoidal part. Let the displacement vector u_i be expressed as the sum of the gradient of a scalar function Φ and the curl of a vector function Ψ_i ; thus, in rectangular Cartesian coordinates

$$u_i = \Phi_{,i} + e_{ijk} \Psi_{k,j} \tag{3.26}$$

where

$$\Psi_{k,k} = 0 \tag{3.27}$$

and e_{ijk} is the permutation symbol. Transforming e_{ijk} into general coordinate systems, we obtain ϵ^{ijk} which is a relative tensor of weight $+1$ and ϵ_{ijk} which is a relative tensor of weight -1. The determinant of the metric tensor

$$a = \text{determinant} \left| a_{ij} \right| \tag{3.28}$$

is a relative invariant of weight 2. In general coordinates, the covariant and contravariant forms of Eq. (3.26) are respectively

$$u_i = \Phi_{|i} + \sqrt{a}\, a_{mi}\, \epsilon^{mjk}\, \Psi_{k|j} \tag{3.29}$$

$$u^i = a^{ij}\, \Phi_{|j} + \sqrt{a}\, \epsilon^{ijk}\, \Psi_{k|j} \tag{3.30}$$

Eq. (3.27) in general coordinates is

$$\Psi^{k}_{\cdot\,|k} = 0 \tag{3.31}$$

or

$$a^{ij}\, \Psi_{j|i} = 0 \tag{3.32}$$

Substituting u^i of Eq. (3.30) in Eq. (3.24) and using the fact that Euclidean space is a flat space, the following equation is obtained:

$$k a^{mn} \Theta_{|mn} = c_\epsilon \dot{\Theta} + \beta T_R\, a^{mn}\, \dot{\Phi}_{|mn} \tag{3.33}$$

Thus only the scalar function Φ is coupled with the temperature Θ.

Substitution of u_i of Eq. (3.29) in Eq. (3.17) yields the result

$$(2\mu+\lambda)\, a^{ij}\, \Phi_{|ij} - \rho \ddot{\Phi} = \beta \Theta \tag{3.34}$$

$$\mu\, a^{jk}\, \Psi_{i|jk} - \rho \ddot{\Psi}_i = 0 \tag{3.35}$$

These are wave equations. Associated with Ψ_i is the shear wave of which the phase velocity is

$$c_V = \sqrt{\frac{\mu}{\rho}} \tag{3.36}$$

It is independent of the temperature of the medium. Associated with the scalar function Φ is the dilatational wave of which the phase velocity depends upon the thermal condition. The isothermal phase velocity of the dilatational wave is

$$c_T = \sqrt{\frac{2\mu + \lambda}{\rho}} \tag{3.37}$$

Another constant of interest is the isentropic phase velocity of the dilatational wave. Solving Eq. (3.21) for the temperature Θ and substituting in Eq. (3.34), the following equation is obtained:

$$\left[c_T^2 + \beta^2 T_R/(\rho c_\epsilon)\right] a^{ij}\,\Phi_{|ij} - \ddot{\Phi} = s\beta T_R/(\rho c_\epsilon) \tag{3.38}$$

By setting the entropy s to a constant, the isentropic phase velocity of a dilatational wave is obtained

$$c_s = (1 + b)^{\frac{1}{2}}\, c_T \tag{3.39}$$

where

$$b = \beta^2 T_R/(\rho c_\epsilon c_T^2) \tag{3.40}$$

The constant b is positive, therefore the isentropic phase velocity is greater than the isothermal phase velocity.

(7) <u>Stresses and Strains in Terms of the Scalar and Vector Functions.</u>

Since the order of covariant differentiation in flat space is immaterial, the covariant strain tensor may be written in the following way:

$$\gamma_{ij} = \Phi_{|ij} + \tfrac{1}{2}\sqrt{a}\,\left(a_{mi}\,\epsilon^{mnk}\,\Psi_{k|nj} + a_{mj}\,\epsilon^{mnk}\,\Psi_{k|ni}\right) \tag{3.41}$$

The cubical dilitation e and the temperature Θ are

$$e = a^{mn}\Phi_{|mn} \tag{3.42}$$

$$\Theta = \frac{\rho}{\beta}\left(c_T^{2} a^{mn}\Phi_{|mn} - \ddot{\Phi} \right) \tag{3.43}$$

Substitution of Eqs.(3.41) to (3.43) in Eq.(3.12) yields the covariant stress tensor in terms of the scalar function Φ and the vector function Ψ_i as follows:

$$\begin{aligned} T_{ij} = {} & 2\mu(\Phi_{|ij} - a^{mn}\Phi_{|mn}a_{ij}) + \rho\ddot{\Phi}\, a_{ij} \\ & + \mu\sqrt{a}\,(a_{mi}\,\epsilon^{mnk}\Psi_{k|nj} + a_{mj}\,\epsilon^{mnk}\Psi_{k|ni}) \end{aligned} \tag{3.44}$$

IV. PROPAGATION OF PLANE WAVES IN AN INFINITE MEDIUM.

For plane waves, rectangular Cartesian coordinates will be used. The metric tensor is equal to the Kronecker delta and the conjugate tensor is precisely the metric tensor itself; therefore there is no distinction between contravariant and covariant components. Since the components of the metric tensor are all constants, all Christoffel symbols vanish, and the covariant differentiation is the same as partial differentiation. The governing equations, then, reduce to

$$a^2\Theta_{,ii} - \dot{\Theta} = (\beta T_R / c_\epsilon)\dot{\Phi}_{,ii} \tag{4.1}$$

$$c_T^2\Phi_{,ii} - \ddot{\Phi} = (\beta/\rho)\,\Theta \tag{4.2}$$

$$c_V^2\Psi_{i,jj} - \ddot{\Psi}_i = 0 \tag{4.3}$$

where the comma in the index line denotes ordinary partial differentiation and

$$a^2 = k/c_\epsilon \quad , \tag{4.4}$$

which should not be confused with the determinant of the metric tensor given by Eq.(3.28).

The shear wave will propagate exactly as in an elastic medium and will not be discussed here. To arrive at the dilatational wave equation, the temperature Θ will be eliminated from Eqs.(4.1) and (4.2) to get the governing equation on Φ:

$$a^2 C_T{}^2 \Phi_{,iijj} - (a^2 \ddot{\Phi} + c_s{}^2 \dot{\Phi})_{,ii} + \dddot{\Phi} = 0 \qquad (4.5)$$

Let the direction of wave propagation be parallel to the x^1 or x-axis and assume a harmonic solution in complex form:

$$\Phi = A \exp i(pt + qx) \qquad (4.6)$$

where the constants A, p, and q are, in general, complex numbers. Substitution of Eq.(4.6) in Eq.(4.5) yields the result that the constants p and q must satisfy the following equation:

$$a^2 C_T{}^2 q^4 - (a^2 p^2 - iC_s{}^2 p) q^2 - ip^3 = 0 \qquad (4.7)$$

Solving for q^2,

$$q^2 = (2a^2 C_T{}^2)^{-1} p \left[a^2 p - iC_s{}^2 \pm \sqrt{(a^2 p - iC_s{}^2)^2 + 4ia^2 C_T{}^2 p} \right] \qquad (4.8)$$

Let

$$p = m + in \qquad (4.9)$$

$$q = f + ig \qquad (4.10)$$

where m, n, f, and g are real numbers. The physical meaning of these four constants are as follows:

(a) The scalar function Φ contains the factor $\exp(-nt)$ when it is expanded in terms of real constants. When n is positive, the amplitude of the dilatational plane wave decreases with time; when n is negative, the amplitude increases with time. Thus, the constant n is the coefficient of decay or rise of the wave.

(b) The scalar function, when expanded in terms of real constants, also contains the factor $\exp(-gx)$. Hence, when g is positive, the amplitude of the wave decreases as the wave travels along the positive direction of x-axis. If the wave travels in the negative direction of x-axis and the constant g is positive, the amplitude would increase as the wave travels in a medium. But, since this is against natural phenomenon, the positive g should not be associated with regressive waves. This constant, g, is called the coefficient of attenuation. Positive g will be associated with progressive waves; and negative g associated with regressive waves.

(c) The scalar function contains also sines and cosines of the argument $(mt + fx)$. This is the argument of all wave functions. m is the circular frequency, and $\frac{1}{2}m/\pi$ is the frequency. Since frequency is always positive, only positive values will be taken for the constant m.

(d) $2\pi/f$ is the wave length of the dilatational waves. m/f is the phase velocity or the velocity of wave propagation. When f is positive the wave is progressive; and when f is negative the wave is regressive. The wave length is considered positive too. Therefore, in calculating the wave length, the absolute value of f should be used.

Substituting the complex numbers given in Eqs. (4.9) and (4.10) in Eq. (4.8), the following equation may be obtained:

$$f^2 - g^2 + 2ifg = \frac{m + in}{2a^2C_T^{\,2}}\left[ma^2 + i(na^2 - C_s^{\,2}) \pm \sqrt{U + iV}\,\right] \qquad (4.11)$$

where

$$U = (m^2-n^2)a^4 - C_s^4 + 2na^2(2C_T^2 - C_s^2) \tag{4.12}$$

$$V = 2ma^2\left[na^2 + (2C_T^2 - C_s^2)\right] \tag{4.13}$$

The square root of $(U-iV)$ can be expressed in trigonometrical form

$$\sqrt{U+iV} = \sqrt[4]{U^2+V^2}\left[\cos(\tfrac{1}{2}\tan^{-1}V/U) + i\sin(\tfrac{1}{2}\tan^{-1}V/U)\right] \tag{4.14}$$

The left hand side of Eq.(4.11) will be represented by $(P+iQ)$ and the subscripts 1 and 2 will be used for P and Q to denote the pairs of numbers corresponding to the plus and minus square roots of $(U+iV)$ respectively; thus

$$P_1 = (2a^2C_T^2)^{-1}\left[(m^2-n^2)a^2 + nC_s^2 + \sqrt[4]{U^2+V^2}\left\{m\cos(\tfrac{1}{2}\tan^{-1}V/U) - n\sin(\tfrac{1}{2}\tan^{-1}V/U)\right\}\right] \tag{4.15}$$

$$Q_1 = (2a^2C_T^2)^{-1}\left[2mna^2 - mC_s^2 + \sqrt[4]{U^2+V^2}\left\{n\cos(\tfrac{1}{2}\tan^{-1}V/U) + m\sin(\tfrac{1}{2}\tan^{-1}V/U)\right\}\right] \tag{4.16}$$

and

$$P_2 = (2a^2C_T^2)^{-1}\left[(m^2-n^2)a^2 + nC_s^2 - \sqrt[4]{U^2+V^2}\left\{m\cos(\tfrac{1}{2}\tan^{-1}V/U) - n\sin(\tfrac{1}{2}\tan^{-1}V/U)\right\}\right] \tag{4.17}$$

$$Q_2 = (2a^2C_T^2)^{-1}\left[2mna^2 - mC_s^2 - \sqrt[4]{U^2+V^2}\left\{n\cos(\tfrac{1}{2}\tan^{-1}V/U) + m\sin(\tfrac{1}{2}\tan^{-1}V/U)\right\}\right] \tag{4.18}$$

Subscripts 1 and 2 will also be applied to f and g to bear the same meaning; thus

$$f_1^2 - g_1^2 + 2if_1g_1 = P_1 + iQ_1 \tag{4.19}$$

$$f_2^2 - g_2^2 + 2if_2g_2 = P_2 + iQ_2 \tag{4.20}$$

Solving for f's and g's , the following results may be obtained:

$$f_1^2 = \tfrac{1}{2}(P_1 + \sqrt{P_1^2 + Q_1^2}) \tag{4.21}$$

$$g_1^2 = \tfrac{1}{2}(-P_1 + \sqrt{P_1^2 + Q_1^2}) \tag{4.22}$$

and

$$f_2^2 = \tfrac{1}{2}(P_2 + \sqrt{P_2^2 + Q_2^2}) \tag{4.23}$$

$$g_2^2 = \tfrac{1}{2}(-P_2 + \sqrt{P_2^2 + Q_2^2}) \tag{4.24}$$

where only the positive square roots have been taken since f's and g's are real numbers. There are two distinct phase velocities of the dilatational waves; the squares of these are

$$C_1^2 = 2m^2(P_1 + \sqrt{P_1^2 + Q_1^2})^{-1} \tag{4.25}$$

$$C_2^2 = 2m^2(P_2 + \sqrt{P_2^2 + Q_2^2})^{-1} \tag{4.26}$$

The P's and Q's are functions of m and n , so that the phase velocities of the dilatational waves, C_1 and C_2, are functions of the frequency, m, and the coefficient of decay or rise, n. The elastic dilatational wave has only one phase velocity which is independent of the frequency. This is a noticeable difference between thermoelastic and elastic dilatational waves.

The other difference between thermoelastic and elastic dilitational waves is that the former attenuates while the later does not attenuate.

To find the physical significance of the elastic and thermal modes of dilitational waves, consider the extreme small coupling. The isentropic phase velocity approaches the isothermal phase velocity at smaller aund smaller coupling (see Eq. 3.39). The state will be considered steady for this special case. There are two ways to find the solutions for this case of very small coupling and steady state:

(1) By setting $C_s = C_T$ and $n = 0$ in Eqs. (4.15) to (4.18) to find the solutions; or

(2) By setting $\beta = 0$ and solving Eqs. (4.1) and (4.2).

(1) When $C_s = C_T$, and $n = 0$, Eqs. (4.15) to (4.18) reduce to the following equations:

$$P_1 = m^2 / C_T^2 \tag{4.27}$$

$$Q_1 = 0 \tag{4.28}$$

$$P_2 = 0 \tag{4.29}$$

$$Q_2 = - m a^{-2} \tag{4.30}$$

Then, by Eqs(4.25) and (4.26), the two phase velocities of dilatational waves are found.

$$C_1{}^2 = C_T{}^2 \tag{4.31}$$

$$C_2{}^2 = 2ma^2 \tag{4.32}$$

By Eqs.(4.21) to (4.24), the constants f_i and g_i ($i = 1, 2$) are:

$$f_1{}^2 = m^2/C_T{}^2 \tag{4.33}$$

$$g_1{}^2 = 0 \tag{4.34}$$

$$f_2{}^2 = \tfrac{1}{2}m/a^2 \tag{4.35}$$

$$g_2{}^2 = \tfrac{1}{2}m/a^2 \tag{4.36}$$

(2). When the constant β approaches zero the Eqs.(4.1) and (4.2) reduce to the following equations:

$$a^2\Theta_{,ii} - \dot{\Theta} = 0 \tag{4.37}$$

$$C_T{}^2\Phi_{,ii} - \ddot{\Phi} = 0 \tag{4.38}$$

In this extreme case, the temperature and the dilatational waves are not coupled; each of the two kinds of waves propagates with a velocity different from the other. Eq.(4.38) is exactly the same as the wave equation of elastic dilatational wave; the phase velocity is C_T . To solve Eq.(4.37), let

$$\Theta = B \exp i(mt + qx) \tag{4.39}$$

where B is an arbitrary complex constant, and m and q are the same as given in Eqs. (4.9) and (4.10). Substitution of the above equation in Eq. (4.37) yields the following equation:

$$a^2(f^2 - g^2) + i(m - 2a^2fg) = 0 \tag{4.40}$$

This equation is satisfied when

$$f^2 = g^2 = \tfrac{1}{2}m/a^2 \tag{4.41}$$

Then the square of thermal phase velocity is $2ma^2$.

Comparing the results obtained by the above two methods, it is seen that, in the extreme case, C_1 equals the elastic dilatational phase velocity and C_2 equals the thermal phase velocity. For this reason, the waves with phase velocity C_1 will be called elastic modes or simply E-modes and the waves with phase velocity C_2 will be called thermal modes or T-modes.

It is to be noted here that associated with each mode of thermoelastic dilatational wave there is a thermal wave propagating with the same phase velocity as the associated dilatational wave except a phase difference, since, by Eq. (4.2), the temperature Θ is equal to a combination of derivatives of the scalar function Φ.

V. CHARACTERISTICS OF DILATATIONAL WAVES

The solutions of the preceding chapter show two characteristics of thermoelastic dilatational waves. First, the waves attenuate as they propagate in the medium. Second, there are two modes of thermoelastic dilatational waves associated with each other, having different phase velocities and different coefficients of attenuation.

Moreover, the phase velocities, the coefficients of attenuation, and the wave length are dependent on frequency and the coefficient of rise or decay.

These characteristics are not associated with elastic waves.

In this chapter, approximate solutions will be derived from the complete solutions of the preceding chapter in order to show how the phase velocities, the coefficients of attenuation, and the wave length of the two modes of thermoelastic dilatational waves vary with the frequency and the coefficient of rise or decay.

Let

$$\tilde{m} = ma^2/C_T^{\,2} \tag{5.1}$$

$$\tilde{n} = na^2/C_T^{\,2} \tag{5.2}$$

$$\tilde{U} = U/C_T^{\,4} \tag{5.3}$$

$$\tilde{V} = V/C_T^{\,4} \tag{5.4}$$

$$\tilde{P}_i = P_i C_T^2/m^2 \tag{5.5}$$

$$\tilde{Q}_i = Q_i C_T^2/m^2 \tag{5.6}$$

where the index takes the value 1 or 2. The quantities $\tilde{m}$, $\tilde{n}$, $\tilde{U}$, $\tilde{V}$, $\tilde{P}_i$, $\tilde{Q}_i$, are dimensionless. Eqs. (4.12) to (4.18) may be expressed in terms of these dimensionless quantities as follows :

$$\tilde{U} = \tilde{m}^2 - \tilde{n}^2 - 2\tilde{n}(1-b) - (1+b)^2 \tag{5.7}$$

$$\tilde{V} = 2\tilde{m}(1-b+\tilde{n}) \tag{5.8}$$

$$\sqrt{\tilde{U} + i\tilde{V}} = \sqrt[4]{\tilde{U}^2+\tilde{V}^2}\,\left(\cos\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} + i\,\sin\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} \right) \tag{5.9}$$

$$\tilde{P}_1 = \tfrac{1}{2}\tilde{m}^{-2}\left[\tilde{m}^2-\tilde{n}^2+\tilde{n}(1+b)+\sqrt[4]{\tilde{U}^2+\tilde{V}^2}\,(\tilde{m}\cos\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} - \tilde{n}\sin\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U})\right] \tag{5.10}$$

$$\tilde{Q}_1 = \tfrac{1}{2}\tilde{m}^{-2}\left[2\tilde{m}\tilde{n} - \tilde{m}(1+b)+\sqrt[4]{\tilde{U}^2+\tilde{V}^2}\,(\tilde{n}\cos\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} + \tilde{m}\sin\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U})\right] \tag{5.11}$$

$$\tilde{P}_2 = \tfrac{1}{2}\tilde{m}^{-2}\left[\tilde{m}^2-\tilde{n}^2+\tilde{n}(1+b)-\sqrt[4]{\tilde{U}^2+\tilde{V}^2}\,(\tilde{m}\cos\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} - \tilde{n}\sin\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U})\right] \tag{5.12}$$

$$\tilde{Q}_2 = \tfrac{1}{2}\tilde{m}^{-2}\left[2\tilde{m}\tilde{n} - \tilde{m}(1+b)-\sqrt[4]{\tilde{U}^2+\tilde{V}^2}\,(\tilde{n}\cos\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} + \tilde{m}\sin\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U})\right] \tag{5.13}$$

where b is a dimensionless constant given by Eq.(3.40).

A survey of some metallic and non-metallic materials (see Appendix) shows that the constant b is positive and less than unity. For non-metallic materials, b is very small. In C. G. S. unit system, the constant a is very small and the constant C_T is very big. Therefore, the variable $\tilde{m}$ (which is always positive) is much smaller than unity even the corresponding frequency is as high as hundreds megacycles per second for many materials. Theoretically, however, $\tilde{m}$ may vary from zero to plus infinity. Regardless of physical possibility, the variable $\tilde{n}$ will be let to vary from minus infinity to plus infinity in the following analysis.

In order to show how the solutions vary with frequency, it will be convenient to expand the following expressions into power series of $\tilde{m}$:

$$\sqrt[4]{\tilde{U}^2 + \tilde{V}^2} \quad , \quad \cos\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} \quad , \quad \sin\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U}$$

which appear in Eqs. (5.9) to (5.13). Moreover, when the series converge fast in certain region of ($\tilde{m}$, $\tilde{n}$), only few terms of the series need be taken for good approximations. For this reason, the following analysis is divided into sections according to the magnitudes of the variables $\tilde{m}$ and $\tilde{n}$. When $\tilde{n}$ is zero, the wave propagation is in steady state; and when $\tilde{n}$ is different from zero, the wave propagation is in unsteady state.

I. Steady state. In this case, $\tilde{n}$ is zero and Eqs. (5.7) and (5.8) reduce to the following equations :

$$\tilde{U} = \tilde{m}^2 - (1+b)^2 \tag{5.14}$$

$$\tilde{V} = 2\,\tilde{m}\,(1-b) \tag{5.15}$$

Tables 1 and 2 show how $\tilde{U}$, $\tilde{V}$, and $\tan^{-1}\tilde{V}/\tilde{U}$ change with frequency in the media of carbon steel and granite.

Table 1. Variation of $\tan^{-1}\tilde{V}/\tilde{U}$ with frequency for carbon steel.

Frequency cycles/sec.	$\tilde{m}$	$\tilde{U}$	$\tilde{V}$	$\tilde{V}/\tilde{U}$	$\tan^{-1}\tilde{V}/\tilde{U}$
....					
44	10^{-10}	-1.0229	1.977×10^{-10}	-1.9×10^{-10}	Approx. 180°
....					
....					
4.4×10^{6}	10^{-5}	-1.0229	1.977×10^{-5}	-1.9×10^{-5}	Approx. 180°
4.4×10^{8}	10^{-3}	-1.0229	1.977×10^{-3}	-0.00193	$179^\circ\ 53'$
4.4×10^{9}	0.01	-1.0228	0.01977	-0.0193	$178^\circ\ 54'$
7.93×10^{9}	0.01807	-1.0226	0.0357	-0.0349	178°
2.59×10^{11}	0.591	-0.6736	1.168	-1.732	120°
4.45×10^{11}	1.0114	0	2.0004		90°
7.61×10^{11}	1.732	1.9771	3.4242	1.732	60°
2.44×10^{13}	56.6	3202.5	111.7	0.0349	2°
4.4×10^{14}	1000	10^{6}	2×10^{3}	2×10^{-3}	$1'$
4.4×10^{16}	10^{5}	10^{10}	2×10^{5}	2×10^{-5}	Approx. 0°
....					
....					
....					
....					

Table 2. Variation of $\tan^{-1}\tilde{V}/\tilde{U}$ with frequency for granite.

Frequency cycles/sec.	$\tilde{m}$	$\tilde{U}$	$\tilde{V}$	$\tilde{V}/\tilde{U}$	$\tan^{-1}\tilde{V}/\tilde{U}$
.....					
22	10^{-11}	-1.00222	2×10^{-11}	-2×10^{-11}	Approx. 180°
.....					
.....					
2.2×10^{7}	10^{-5}	-1.00222	2×10^{-5}	-2×10^{-5}	Approx. 180°
2.2×10^{9}	10^{-3}	-1.00222	2×10^{-3}	-2×10^{-3}	179° 53'
2.2×10^{10}	0.01	-1.00212	0.01998	-0.01992	178° 51'
3.84×10^{10}	0.01746	-1.00192	0.03492	-0.0349	178°
1.27×10^{12}	0.579	-0.66698	1.15671	-1.732	120°
2.23×10^{12}	1.00111	0	2.0		90°
3.82×10^{12}	1.732	1.9976	3.46015	1.732	60°
1.26×10^{14}	57.2	3270.8	114.4	0.0349	2°
2.2×10^{15}	10^{3}	10^{6}	2×10^{3}	2×10^{-3}	1'
2.2×10^{17}	10^{5}	10^{10}	2×10^{5}	2×10^{-5}	Approx. 0°
.....					
.....					
.....					
.....					

In the following analysis, the frequencies will be divided into three ranges according to the angle of $\tan^{-1}\tilde{V}/\tilde{U}$. Frequency range I will be such that the angle approximately 180° . This range covers frequencies from zero (theoretically speaking) up to hundreds megacycles per second. Most of the practical applications will be in this range. Frequency range II will be such that the angle $\tan^{-1}\tilde{V}/\tilde{U}$ is between 180° and 0°. This is the ultra-high frequency range. Frequency range III will be such that the angle $\tan^{-1}\tilde{V}/\tilde{U}$ is approximately 0° .

(A) Frequency range I. In this range of frequency, $\tilde{m}$ is much smaller than unity and is as small as to the order of 10^{-11} when the frequency is as low as seismic waves. The value of $\cos\frac{1}{2}\tan^{-1}\tilde{V}/\tilde{U}$ is almost zero and the value of $\sin\frac{1}{2}\tan^{-1}\tilde{V}/\tilde{U}$ is approximately unity.

$\cos\frac{1}{2}\tan^{-1}\tilde{V}/\tilde{U}$ and $\sin\frac{1}{2}\tan^{-1}\tilde{V}/\tilde{U}$ are functions of $\tan^{-1}\tilde{V}/\tilde{U}$ which, in turn, is a function of $\tilde{m}$. They are expanded into power series of $\tilde{m}$ as follows :

$$\cos\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} = \tilde{m}(1-b)(1+b)^{-2} + \tfrac{1}{2}\tilde{m}^3(1-b)(1+b)^{-6}\left[2(1+b)^2 - 3(1-b)^2\right]$$
$$+ 1/8\ \tilde{m}^5(1-b)(1+b)^{-10}\left[8(1+b)^4-36(1+b)^2(1-b)^2+31(1-b)^4\right] + \ldots \quad (5.16)$$

$$\sin\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} = 1 - \tfrac{1}{2}\tilde{m}^2(1-b)^2(1+b)^{-4} + 1/8\ \tilde{m}^4(1-b)^2(1+b)^{-8}\cdot$$
$$\left[11(1-b)^2-8(1+b)^2\right] - 1/16\ \tilde{m}^6(1-b)^2(1+b)^{-12}\left[69(1-b)^4\right.$$
$$\left.- 88(1-b)^2(1+b)^2 + 24(1+b)^4\right] + \ \ldots.. \quad (5.17)$$

The expression $\sqrt[4]{\tilde{U}^2 + \tilde{V}^2}$ is also expanded into power series of $\tilde{m}$.

$$\sqrt[4]{\tilde{U}^2 + \tilde{V}^2} = (1+b) + \tfrac{1}{2}\tilde{m}^2(1+b)^{-3}\left[2(1-b)^2-(1+b)^2\right] - 1/8\,\tilde{m}^4(1+b)^{-7}\cdot$$

$$\left[12(1-b)^4-12(1-b)^2(1+b)^2 + (1+b)^4\right] + 1/16\,\tilde{m}^6(1+b)^{-11}\left[56(1-b)^6\right.$$

$$\left. - 84(1-b)^4(1+b)^2 + 30(1-b)^2(1+b)^4 - (1+b)^6\right] + \ldots\ldots\ldots \qquad (5.18)$$

The $\tilde{P}$'s and $\tilde{Q}$'s are then obtained as power series of $\tilde{m}$ as follows:

$$\tilde{P}_1 = (1+b)^{-1} + \tilde{m}^2 b(1-b)(1+b)^{-5} + 1/16\,\tilde{m}^4(1-b)(1+b)^{-9}\left[3(1+b)^4\right.$$

$$\left. - 10(1+b)^2(1-b)^2 + 7(1-b)^4\right] + \ldots\ldots\ldots \qquad (5.19)$$

$$\tilde{Q}_1 = -\tilde{m}b(1+b)^{-3} - 1/16\,\tilde{m}^3(1+b)^{-7}\left[(1+b)^4 - 6(1+b)^2(1-b)^2 + 5(1-b)^4\right]$$

$$- 1/32\,\tilde{m}^5(1+b)^{-11}\left[(1+b)^6 - 15(1+b)^4(1-b)^2 + 35(1+b)^2(1-b)^4\right.$$

$$\left. - 21(1-b)^6\right] + \ldots\ldots\ldots \qquad (5.20)$$

$$\tilde{P}_2 = b(1+b)^{-1} - \tilde{m}^2 b(1-b)(1+b)^{-5} - 1/16\,\tilde{m}^4(1-b)(1+b)^{-9}\left[3(1+b)^4\right.$$

$$\left. - 10(1+b)^2(1-b)^2 + 7(1-b)^4\right] + \ldots\ldots\ldots \qquad (5.21)$$

$$\tilde{Q}_2 = \tilde{m}^{-1}(1+b) + \tilde{m}b(1+b)^{-3} + 1/16\,\tilde{m}^3(1+b)^{-7}\left[(1+b)^4 - 6(1+b)^2(1-b)^2\right.$$

$$\left. + 5(1-b)^4\right] + 1/32\,\tilde{m}^5(1+b)^{-11}\left[(1+b)^6 - 15(1+b)^4(1-b)^2\right.$$

$$\left. + 35(1+b)^2(1-b)^4 - 21(1-b)^6\right] + \ldots\ldots\ldots \qquad (5.22)$$

(a) Reduction to the uncoupled case. Setting b to zero, the following results can be obtained :

$$\tilde{P}_1 = 1 \tag{5.23}$$

$$\tilde{Q}_1 = 0 \tag{5.24}$$

$$\tilde{P}_2 = 0 \tag{5.25}$$

$$\tilde{Q}_2 = \tilde{m}^{-1} \tag{5.26}$$

In dimensional forms, they are :

$$P_1 = m^2 C_T^{-2} \tag{5.27}$$

$$Q_1 = 0 \tag{5.28}$$

$$P_2 = 0 \tag{5.29}$$

$$Q_2 = m\, a^{-2} \tag{5.30}$$

Then, by substituting Eqs. (5.27) to (5.30) in Eqs. (4.21) to (4.26), the following results are obtained:

$$f_1^{\,2} = m^2 C_T^{-2} \tag{5.31}$$

$$g_1^{\,2} = 0 \tag{5.32}$$

$$f_2^{\,2} = \tfrac{1}{2}\, m\, a^{-2} \tag{5.33}$$

$$g_2^{\,2} = \tfrac{1}{2}\, m\, a^{-2} \tag{5.34}$$

$$c_1^{\,2} = c_T^{\,2} \tag{5.35}$$

$$c_2^{\,2} = 2\, m\, a^2 \tag{5.36}$$

These are, as expected, exactly the solutions of uncoupled elastic and thermal wave equations —— Eqs. (4.37) and (4.38).

(b) Solutions for lower frequencies. At lower frequencies in the frequency range I, $\tilde{m}$ is so small that approximations can be made by taking only the first term of the series; thus

$$\tilde{P}_1 = (1+b)^{-1} \tag{5.37}$$

$$\tilde{Q}_1 = -\,\tilde{m}\, b\, (1+b)^{-3} \tag{5.38}$$

$$\tilde{P}_2 = b\, (1+b)^{-1} \tag{5.39}$$

$$\tilde{Q}_2 = \tilde{m}^{-1}(1+b) \tag{5.40}$$

or,

$$P_1 = m^2\, (1+b)^{-1}\, c_T^{-2} \tag{5.41}$$

$$Q_1 = -\, m^3\, a^2\, b\, (1+b)^{-3}\, c_T^{-4} \tag{5.42}$$

$$P_2 = m^2\, b\, (1+b)^{-1}\, c_T^{-2} \tag{5.43}$$

$$Q_2 = m\, a^{-2}\, (1+b) \tag{5.44}$$

Then, the following results may be obtained:

$$f_1^{\,2} = m^2 (1+b)^{-1} c_T^{-2} \tag{5.45}$$

$$g_1^{\,2} = \tfrac{1}{4} m^4 a^4 b^2 (1+b)^{-5} c_T^{-6} \tag{5.46}$$

$$f_2^{\,2} = \tfrac{1}{2} m a^{-2} (1+b) \tag{5.47}$$

$$g_2^{\,2} = \tfrac{1}{2} m a^{-2} (1+b) \tag{5.48}$$

$$c_1^{\,2} = (1+b) c_T^{\,2} = c_s^{\,2} \tag{5.49}$$

$$c_2^{\,2} = 2 m a^2 (1+b)^{-1} \tag{5.50}$$

The phase velocity of S-mode dilatational waves at low frequencies is practically independent of frequency and is equal to the isentropic phase velocity. The phase velocity of T-mode is much smaller than that of S-mode, and varies to the square root of frequency. The coefficient of attenuation of S-mode is very small while that of T-mode is larger. For S-mode, the coefficient of attenuation is proportional to the square of frequency; and for T-mode, it is proportional to the square root of frequency.

(c) Solutions for higher frequencies. Within frequency range I, $\tilde{m}$ is still a small positive number. Hence the following approximations may be taken:

$$f_1^{\,2} = m^2 (1+b)^{-1} c_T^{-2} + \tfrac{1}{4} m^4 a^4 c_T^{-6} b (4-3b)(1+b)^{-5} \tag{5.51}$$

$$g_1^{\,2} = \tfrac{1}{4}\, m^4 a^4 C_T^{-6} b^2 (1+b)^{-5} - \tfrac{1}{4}\, m^6 a^8 C_T^{-10} (2-5b+b^2) b^2 (1+b)^{-9} \qquad (5.52)$$

$$f_2^{\,2} = \tfrac{1}{2}\, ma^{-2}(1+b) + \tfrac{1}{2}\, m^2 C_T^{-2} b (1+b)^{-1} \qquad (5.53)$$

$$g_2^{\,2} = \tfrac{1}{2}\, ma^{-2}(1+b) - \tfrac{1}{2}\, m^2 C_T^{-2} b (1+b)^{-1} \qquad (5.54)$$

$$C_1^{\,2} = (1+b) C_T^{\,2} - \tfrac{1}{4}\, m^2 a^4 C_T^{-2} b (4-3b)(1+b)^{-3} \qquad (5.55)$$

$$C_2^{\,2} = 2\, ma^2 (1+b)^{-1} - 2\, m^2 a^4 C_T^{-2} b (1+b)^{-3} \qquad (5.56)$$

The phase velocity of E-mode waves decreases when frequency increases. The rate of decreasing is, however, very slow. The phase velocity of T-mode waves inreases with frequency, and the rate of increasing is relatively higher. Take carbon steel as an example, the numerical values of the squares of the phase velocities of both modes versus circular frequency are shown in the following table:

m	$C_1^{\,2}$	$C_2^{\,2}$
1 sec.$^{-1}$	$(1.0114 - 1.44 \times 10^{-25})\ C_T^{\,2}$	$(0.253 - 1.02 \times 10^{-17})$ cm^2/sec^2
10^2 "	$(1.0114 - 1.44 \times 10^{-21})\ C_T^{\,2}$	$(25.3 - 1.02 \times 10^{-13})$ cm^2/sec^2
10^4 "	$(1.0114 - 1.44 \times 10^{-17})\ C_T^{\,2}$	$(2.53 \times 10^3 - 1.02 \times 10^{-9})$ cm^2/sec^2
10^6 "	$(1.0114 - 1.44 \times 10^{-13})\ C_T^{\,2}$	$(2.53 \times 10^5 - 1.02 \times 10^{-5})$ cm^2/sec^2
10^8 "	$(1.0114 - 1.44 \times 10^{-9})\ C_T^{\,2}$	$(2.53 \times 10^7 - 1.02 \times 10^{-1})$ cm^2/sec^2

The coefficient of attenuation of E-mode is much smaller than that of T-mode; they both increase with frequency. The coefficient of E-mode is approximately proportional to the square of frequency; and the coefficient of attenuation of T-mode is approximately proportional to the square root of frequency. The following table is calculated for carbon steel:

m (sec.$^{-1}$)	g_1^2 (cm.$^{-2}$)	g_2^2 (cm.$^{-2}$)
1	$1.15 \times 10^{-41} - 2.81 \times 10^{-66}$	$3.95 - 1.42 \times 10^{-14}$
10^2	$1.15 \times 10^{-33} - 2.81 \times 10^{-54}$	$3.95 \times 10^2 - 1.42 \times 10^{-10}$
10^4	$1.15 \times 10^{-25} - 2.81 \times 10^{-42}$	$3.95 \times 10^4 - 1.42 \times 10^{-6}$
10^6	$1.15 \times 10^{-17} - 2.81 \times 10^{-30}$	$3.95 \times 10^6 - 1.42 \times 10^{-2}$
10^8	$1.15 \times 10^{-9} - 2.81 \times 10^{-18}$	$3.95 \times 10^8 - 1.42 \times 10^2$

(B) <u>Frequency range II.</u> In this range, the angle $\tan^{-1}\tilde{V}/\tilde{U}$ varies from approximately 180° to approximately 0°. For most materials, this range covers approximately 10^9 to 10^{14} cycles per second. The angle $\tan^{-1}\tilde{V}/\tilde{U}$ is 90° when the order of frequency is about 10^{11} to 10^{12} cycles per second. For this special case,

$$\tilde{m} = 1 + b \qquad (5.57)$$

$$P_1 = \frac{1}{2}\left[1 + \sqrt{1-b^2}\;(1-b)^{-1}\right] m^2 C_T^{-2} \tag{5.58}$$

$$Q_1 = \frac{1}{2}\left[-1 + \sqrt{1-b^2}\;(1-b)^{-1}\right] m^2 C_T^{-2} \tag{5.59}$$

$$P_2 = \frac{1}{2}\left[1 + \sqrt{1-b^2}\;(1-b)^{-1}\right] m^2 C_T^{-2} \tag{5.60}$$

$$Q_2 = \frac{1}{2}\left[-1 + \sqrt{1-b^2}\;(1-b)^{-1}\right] m^2 C_T^{-2} \tag{5.61}$$

where

$$m = \tilde{\alpha}\, C_T^{\,2}\, a^{-2} = C_T^{\,2}\, a^{-2}\,(1+b)\,. \tag{5.62}$$

Hence,

$$f_1^{\,2} = \frac{1}{4} m^2 C_T^{-2} (1+b)^{-1}\left[(1+b) + \sqrt{1-b^2} + 2\sqrt{1+b}\right] \tag{5.63}$$

$$g_1^{\,2} = \frac{1}{4} a^{-4} C_T^{\,2} (1+b)\left[-(1+b) - \sqrt{1-b^2} + 2\sqrt{1+b}\right] \tag{5.64}$$

$$f_2^{\,2} = \frac{1}{4} m^2 C_T^{-2} (1+b)^{-1}\left[(1+b) - \sqrt{1-b^2} + 2\sqrt{1+b}\right] \tag{5.65}$$

$$g_2^{\,2} = \frac{1}{4} a^{-4} C_T^{\,2} (1+b)\left[-(1+b) + \sqrt{1-b^2} + 2\sqrt{1+b}\right] \tag{5.66}$$

The constant b is a positive number smaller than unity, therefore the above results may be expanded into power series of b; thus

$$f_1^{\,2} = m^2 C_T^{-2}\left(1 - \tfrac{1}{2}b + 5/16\, b^2 - 9/32\, b^3 + 9/256\, b^4 + \ldots\ldots\right) \tag{5.67}$$

$$g_1^{\,2} = 1/16\, a^{-4} C_T^{\,2} b^2\left(1 + 3/2\, b + 11/16\, b^2 + 13/32\, b^3 + \ldots\ldots\right) \tag{5.68}$$

$$f_2^{\,2} = \tfrac{1}{2} m^2 C_T^{-2}\left(1 + 1/8\, b^2 - 1/16\, b^3 + 11/128\, b^4 + \ldots\ldots\ldots\right) \tag{5.69}$$

$$g_2^{\,2} = 1/8\, a^{-4} C_T^{\,2}\left(1 + b - 3/8\, b^2 - 5/16\, b^3 - 5/128\, b^4 + \ldots\right) \tag{5.70}$$

The squares of the phase velocities of the two modes are:

$$c_1^{\,2} = c_T^{\,2}(1 + \tfrac{1}{2}b - 5/16\ b^2 - 1/32\ b^3 - 1/128\ b^4 + \ldots\ldots) \qquad (5.71)$$

$$c_2^{\,2} = 2\ c_T^{\,2}(1 - 1/8\ b^2 + 1/16\ b^3 - 9/128\ b^4 + \ldots\ldots\ldots) \qquad (5.72)$$

Since the constant b is a small positive number (see Appendix), the following approximations may be made:

$$c_1^{\,2} = (1 - \tfrac{1}{2}b)\ c_T^{\,2} \qquad (5.73)$$

$$c_2^{\,2} = 2\ c_T^{\,2} \qquad (5.74)$$

$$g_1^{\,2} = 1/16\ a^{-4}\ c_T^{\,2}\ b^2 \qquad (5.75)$$

$$g_2^{\,2} = 1/8\ a^{-4}\ c_T^{\,2}\ (1+b) \qquad (5.76)$$

At this high frequency, the T-mode phase velocity is higher than the E-mode phase velocity. The coefficients of attenuation of both modes are high at high frequencies; that of T-mode is much higher than that of E-mode.

For carbon steel, this frequency, given by Eq. (5.62) is 4.45×10^{11} cycles per second (see Table 1).

Due to the fast change of the angle $\tan^{-1}\tilde{V}/\tilde{U}$, the change of the phase velocity as well as the coefficient of attenuation is big. For an numerical example, the phase velocities and coefficients of attenuation for carbon steel at frequencies 2.59×10^{11} , 4.45×10^{11}, and 7.61×10^{11} cycles per second (see Table 1) are listed below:

Frequency (cycles/sec.)	2.59×10^{11}	4.45×10^{11}	7.61×10^{11}
C_1	$0.93368\ C_T$	$1.00284\ C_T$	$1.11873\ C_T$
C_2	$0.88432\ C_T$	$1.41422\ C_T$	$1.38698\ C_T$
g_1	$5.77 \times 10^5\ \text{cm}^{-1}$	$2.2 \times 10^4\ \text{cm}^{-1}$	$1.66 \times 10^5\ \text{cm}^{-1}$
g_2	$1.65 \times 10^6\ \text{cm}^{-1}$	$5.3 \times 10^6\ \text{cm}^{-1}$	$8.07 \times 10^5\ \text{cm}^{-1}$

(c) Frequency range III. In this exceedingly high frequency range, $\tilde{m}$ is much bigger than $(1+b)$, and $\tilde{V}$ is much smaller than $\tilde{U}$, so that the following approximate values of $\tilde{P}$'s and $\tilde{Q}$'s may be taken:

$$\tilde{P}_1 = 1 \tag{5.77}$$

$$\tilde{Q}_1 = \tfrac{1}{2}(1-3b)\tilde{m}^{-1} \tag{5.78}$$

$$\tilde{P}_2 = b\,\tilde{m}^{-2} \tag{5.79}$$

$$\tilde{Q}_2 = -\tfrac{1}{2}(3-b)\,\tilde{m}^{-1} \tag{5.80}$$

or

$$P_1 = m^2 C_T^{-2} \tag{5.81}$$

$$Q_1 = \tfrac{1}{2}(1-3b)\,m\,a^{-2} \tag{5.82}$$

$$P_2 = b\, a^{-4}\, c_T^{\ 2} \tag{5.83}$$

$$Q_2 = -\tfrac{1}{2}\,(3-b)\, m\, a^{-2} \tag{5.84}$$

Then, the following approximate solutions are found:

$$f_1^{\ 2} = m^2 c_T^{\ -2} \tag{5.85}$$

$$g_1^{\ 2} = \tfrac{1}{4}\,(1-3b)^2\, a^{-4}\, c_T^{\ 2} \tag{5.86}$$

$$f_2^{\ 2} = \tfrac{1}{2}\,(3-b)\, m\, a^{-2} \tag{5.87}$$

$$g_2^{\ 2} = \tfrac{1}{2}\,(3-b)\, m\, a^{-2} \tag{5.88}$$

$$c_1^{\ 2} = c_T^{\ 2} \tag{5.89}$$

$$c_2^{\ 2} = 2\,(3-b)^{-1}\, m\, a^2 \tag{5.90}$$

It has been assumed that, in this frequency range, the value of $m\, a^2$ is much greater than $c_T^{\ 2}$. Thus, the T-mode phase velocity exceeds the S-mode phase velocity when the frequency is exceedingly high.

II. Unsteady state. When the constant n is different from zero, the amplitude is changing with time. When n is positive, the amplitude decreases with time; and the period during which n is positive is called the decay period. On the contrary, during the rise period, n is negative.

In unsteady state, the solutions for a certain medium vary with two parameters, m and n . Re-arranging Eqs. (5.7) and (5.8):

$$\tilde{U} = -\left[(1-b) + \tilde{n}\right]^2 - (4b - \tilde{m}^2) \tag{5.91}$$

$$\tilde{V} = 2\tilde{m}\left[(1-b) + \tilde{n}\right] \tag{5.92}$$

Some special cases are discussed below.

(A) Period of Rise. During the period of rise, $\tilde{n}$ is negative. Particularly, if

$$\tilde{n} = -(1-b) \tag{5.93}$$

$$\tilde{m}^2 = 4b \tag{5.94}$$

then both $\tilde{U}$ and $\tilde{V}$ are zero and

$$\tilde{P}_1 = \tilde{P}_2 = -\tfrac{1}{4} b^{-1}(1 - 3b) \tag{5.95}$$

$$\tilde{Q}_1 = \tilde{Q}_2 = -\tfrac{1}{4} b^{-\frac{1}{2}}(3 - b) \tag{5.96}$$

And the solutions are :

$$f_1^{\,2} = f_2^{\,2} = m^2 C_T^{-2}(9/16 + 3/64\, b - 1/128\, b^2 + \ldots\ldots) \tag{5.97}$$

$$\varepsilon_1^{\,2} = \varepsilon_2^{\,2} = C_T^{-2}(1 - 3/4\, b + 3/16\, b^2 - 1/32\, b^3 + \ldots\ldots) \tag{5.98}$$

$$c_1^{\,2} = c_2^{\,2} = C_T^{\,2}(16/9 - 4/27\, b + 1/27\, b^2 + \ldots\ldots\ldots\ldots) \tag{5.99}$$

Thus, for this particular case, there is no distinction between E-mode and T-mode; there is only one kind of dilatational wave.

When the frequency is very small such that

$$m^2 \ll 4\,b \tag{5.100}$$

and $\tilde{n}$ is the same as that given by Eq. (5.93), then

$$\cos\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} = 0 \tag{5.101}$$

$$\sin\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} = 1 \tag{5.102}$$

The following approximations may be made:

$$\tilde{P}_1 = -\,(\,1 - b^{\frac{1}{2}} - b\,)\,\tilde{m}^{-2} \tag{5.103}$$

$$\tilde{Q}_1 = -\tfrac{1}{2}\,(\,3 - 2b^{\frac{1}{2}} - b\,)\,\tilde{m}^{-1} \tag{5.104}$$

$$\tilde{P}_2 = -\,(\,1 + b^{\frac{1}{2}} - b\,)\,\tilde{m}^{-2} \tag{5.105}$$

$$\tilde{Q}_2 = -\tfrac{1}{2}\,(\,3 + 2b^{\frac{1}{2}} - b\,)\,\tilde{m}^{-1} \tag{5.106}$$

or,

$$P_1 = -\,(\,1 - b^{\frac{1}{2}} - b\,)\,a^{-4}\,c_T^{\,2} \tag{5.107}$$

$$Q_1 = -\tfrac{1}{2}\,(\,3 - 2b^{\frac{1}{2}} - b\,)\,a^{-2}m \tag{5.108}$$

$$P_2 = -\,(\,1 + b^{\frac{1}{2}} - b\,)\,a^{-4}\,c_T^{\,2} \tag{5.109}$$

$$Q_2 = -\tfrac{1}{2}\,(\,3 + 2b^{\frac{1}{2}} - b\,)\,a^{-2}\,m \tag{5.110}$$

Then, the following approximate solutions are obtained:

$$f_1^{\,2} = 1/16\,(3-2b^{\frac{1}{2}}-b)^2(1-b^{\frac{1}{2}}-b)^{-1}\,a^2\,C_T^{\,-2} \tag{5.111}$$

$$g_1^{\,2} = (1-b^{\frac{1}{2}}-b)\,a^{-4}\,C_T^{\,2} \tag{5.112}$$

$$f_2^{\,2} = 1/16\,(3+2b^{\frac{1}{2}}-b)^2(1+b^{\frac{1}{2}}-b)^{-1}\,a^2\,C_T^{\,-2} \tag{5.113}$$

$$g_2^{\,2} = (1+b^{\frac{1}{2}}-b)\,a^{-4}\,C_T^{\,2} \tag{5.114}$$

$$C_1^{\,2} = 16\,(1-b^{\frac{1}{2}}-b)(3-2b^{\frac{1}{2}}-b)^{-2}\,C_T^{\,2} \tag{5.115}$$

$$C_2^{\,2} = 16\,(1+b^{\frac{1}{2}}-b)(3+2b^{\frac{1}{2}}-b)^{-2}\,C_T^{\,2} \tag{5.116}$$

For carbon steel, the squares of the two phase velocities at low frequencies but with high coefficient of rise are shown below:

$$C_1^{\,2} = 1.82196\,C_T^{\,2} \tag{5.117}$$

$$C_2^{\,2} = 1.70923\,C_T^{\,2} \tag{5.118}$$

When the frequency is very high such that

$$a^2 \gg 4\,b \tag{5.119}$$

and $\tilde{R}$ is the same as that given by Eq. (5.93), then

$$\cos\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} = 1 \tag{5.120}$$

$$\sin\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} = 0 \tag{5.121}$$

And, by substituting the above values into Eqs. (5.10) to (5.13), the following equations are obtained:

$$\tilde{P}_1 = 1 - \tilde{M}^{-2}(1-b) \tag{5.122}$$

$$\tilde{Q}_1 = -\tilde{M}^{-1}(2-b) \tag{5.123}$$

$$\tilde{P}_2 = -\tilde{M}^{-2}(1-b) \tag{5.124}$$

$$\tilde{Q}_2 = -\tilde{M}^{-1} \tag{5.125}$$

In dimensional forms, they are

$$P_1 = m^2 C_T^{-2} - a^{-4} C_T^{\,2}(1-b) \tag{5.126}$$

$$Q_1 = -a^{-2} m(2-b) \tag{5.127}$$

$$P_2 = -a^{-4} C_T^{\,2}(1-b) \tag{5.128}$$

$$Q_2 = -a^{-2} m \tag{5.129}$$

For the special case that

$$m\, a^2 = C_T^{\,2} \tag{5.130}$$

Eqs. (5.126) to (5.129) become the following equations:

$$P_1 = b\, a^{-4}\, C_T^{\,2} \tag{5.131}$$

$$Q_1 = -a^{-4}\, C_T^{\,2}\,(2-b) \tag{5.132}$$

$$P_2 = - a^{-4} C_T^{\,2} (1-b) \tag{5.133}$$

$$Q_2 = - a^{-4} C_T^{\,2} \tag{5.134}$$

Then, the following results are obtained:

$$f_1^{\,2} = \tfrac{1}{2} a^{-4} C_T^{\,2} \left(b + \sqrt{4 - 4b + 2b^2} \right) \tag{5.135}$$

$$g_1^{\,2} = \tfrac{1}{2} a^{-4} C_T^{\,2} \left(-b + \sqrt{4 - 4b + 2b^2} \right) \tag{5.136}$$

$$f_2^{\,2} = \tfrac{1}{2} a^{-4} C_T^{\,2} \left(-1 + b + \sqrt{2 - 2b + b^2} \right) \tag{5.137}$$

$$g_2^{\,2} = \tfrac{1}{2} a^{-4} C_T^{\,2} \left(1 - b + \sqrt{2 - 2b + b^2} \right) \tag{5.138}$$

For carbon steel,

$$f_1^{\,2} = a^{-4} C_T^{\,2} \tag{5.139}$$

$$g_1^{\,2} = 2.13 \times 10^{13} \ \text{cm.}^{-2} \tag{5.140}$$

$$f_2^{\,2} = 0.2088\, a^{-4} C_T^{\,2} \tag{5.141}$$

$$g_2^{\,2} = 2.58 \times 10^{13} \ \text{cm.}^{-2} \tag{5.142}$$

The phase velocities of the two modes of dilatational waves are

$$C_1 = C_T \tag{5.143}$$

$$C_2 = 2.19089\, C_T \tag{5.144}$$

The above analysis shows that the phase velocity of dilatational wave at low frequencies may exceed the isentropic phase velocity during the period of rise. This result may be applied to explain certain seismic wave phenomena and will be discussed later in a separate section.

(B) Period of decay. During the period of decay, $\tilde{n}$ is positive. Consider the special case where

$$\tilde{n} = 1 - b \tag{5.145}$$

$$\tilde{m} \ll 1 - b \tag{5.146}$$

For this special case,

$$\cos\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} = \tfrac{1}{2}\,\tilde{m}\,(1-b)^{-1} \tag{5.147}$$

$$\sin\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} = 1 \tag{5.148}$$

$$\sqrt[4]{\tilde{U}^2 + \tilde{V}^2} = 2\,(1-b) + \tfrac{1}{2}\,\tilde{m}^2(1-b)^{-1} - \ldots\ldots\ldots\ldots \tag{5.149}$$

Then the following approximate equations may be obtained:

$$\tilde{P}_1 = -\,\tilde{m}^{-2}\,(\,1 - 3b + 2b^2\,) \tag{5.150}$$

$$\tilde{Q}_1 = \tilde{m}^{-1}\,(\,2 - 3b\,) \tag{5.151}$$

$$\tilde{P}_2 = \tilde{m}^{-2}\,(\,1 - b\,) \tag{5.152}$$

$$\tilde{Q}_2 = -\,\tilde{m}^{-1} \tag{5.153}$$

In dimensional forms, they are

$$P_1 = -a^{-4} c_T^{2} (1-3b+2b^2) \tag{5.154}$$

$$Q_1 = m a^{-2} (2-3b) \tag{5.155}$$

$$P_2 = a^{-4} c_T^{2} (1-b) \tag{5.156}$$

$$Q_2 = -m a^{-2} \tag{5.157}$$

Then the following approximate results are obtained:

$$f_1^{2} = \tfrac{1}{4} m^2 c_T^{-2} (2-3b+2b^2)(1-3b)^{-1} \tag{5.158}$$

$$g_1^{2} = a^{-4} c_T^{2} (1-3b) \tag{5.159}$$

$$f_2^{2} = a^{-4} c_T^{2} (1-b) \tag{5.160}$$

$$g_2^{2} = a^{-4} c_T^{2} (1-b) \tag{5.161}$$

$$c_1^{2} = 4 (1-3b)(2-3b+2b^2)^{-2} c_T^{2} \tag{5.162}$$

$$c_2^{2} = (1-b)^{-1} c_T^{-2} m^2 a^4 \tag{5.163}$$

Hence, the phase velocity of E-mode at low frequencies during a fast decay period is approximately equal to the isothermal dilatational phase velocity. The phase velocity of T-mode under the same condition is very low and is proportional to the frequency.

If the frequency is very high such that

$$\tilde{m} \gg 1 - b \tag{5.164}$$

If $\tilde{m}$ is the same as before (Eq. 5.145), then, approximately

$$\cos\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} = 1 \tag{5.165}$$

$$\sin\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} = 2\,\tilde{m}^{-1}(1 - b) \tag{5.166}$$

$$\sqrt[4]{\tilde{U}^2 + \tilde{V}^2} = \tilde{m} + 4\,\tilde{m}^{-1}(1-b)^2 \tag{5.167}$$

Then the following approximate equations may be obtained:

$$\tilde{P}_1 = 1 \tag{5.168}$$

$$\tilde{Q}_1 = \tilde{m}^{-1}(2-3b) \tag{5.169}$$

$$\tilde{P}_2 = \tilde{m}^{-2}(1-b) \tag{5.170}$$

$$\tilde{Q}_2 = -\tilde{m}^{-1} \tag{5.171}$$

In dimensional forms, the above equations are:

$$P_1 = m^2\,C_T^{-2} \tag{5.172}$$

$$Q_1 = m\,a^{-2}(2-3b) \tag{5.173}$$

$$P_2 = a^{-4}\,C_T^{\,2}(1-b) \tag{5.174}$$

$$Q_2 = -m\,a^{-2} \tag{5.175}$$

Then, the following approximate solutions are obtained:

$$f_1^{\,2} = m^2\, c_T^{-2} \tag{5.176}$$

$$g_1^{\,2} = \tfrac{1}{4}\, a^{-4}\, c_T^{\,2}\, (2-3b)^2 \tag{5.177}$$

$$f_2^{\,2} = \tfrac{1}{2}\, m\, a^{-2} \tag{5.178}$$

$$g_2^{\,2} = \tfrac{1}{2}\, m\, a^{-2} \tag{5.179}$$

$$c_1^{\,2} = c_T^{\,2} \tag{5.180}$$

$$c_2^{\,2} = 2\, m\, a^2 \tag{5.181}$$

Thus, the phase velocity of E-mode at very high frequencies during the rapid decaying period is equal to the isothermal dilatational wave velocity. The phase velocity of T-mode is very low and is proportional to the square root of the frequency.

(c) Exceedingly high rate of rise or decay. When the absolute value of n is exceedingly high while the frequency is relatively low, the following approximate equations may be made:

$$\cos\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} = \tilde{m}\,/\,\tilde{n} \tag{5.182}$$

$$\sin\tfrac{1}{2}\tan^{-1}\tilde{V}/\tilde{U} = 1 \tag{5.183}$$

$$\sqrt[4]{\tilde{U}^2 + \tilde{V}^2} = \left|\tilde{n}\right| \cdot \left(1 + \tilde{m}^2/\tilde{n}^2\right) \tag{5.184}$$

Then, if $\tilde{n}$ is positive, the following approximate equations may be found:

$$\tilde{P}_1 = -\tilde{n}^2 \tilde{m}^{-2} \tag{5.185}$$

$$\tilde{Q}_1 = 2\tilde{n}\tilde{m}^{-1} \tag{5.186}$$

$$\tilde{P}_2 = \tfrac{1}{2}\tilde{n}\tilde{m}^{-2}(1+b) \tag{5.187}$$

$$\tilde{Q}_2 = -\tfrac{1}{2}\tilde{m}^{-1}(1+b) \tag{5.188}$$

If $\tilde{n}$ is negative, the values of $\tilde{P}_1$ and $\tilde{P}_2$, and $\tilde{Q}_1$ and $\tilde{Q}_2$ are interchanged respectively (i.e., for example, $\tilde{P}_1$ becomes $\tilde{P}_2$ and $\tilde{P}_2$ becomes $\tilde{P}_1$). In dimensional forms, they are

$$P_1 = -n^2 C_T^{-2} \tag{5.189}$$

$$Q_1 = 2\,n\,m\,C_T^{-2} \tag{5.190}$$

$$P_2 = \tfrac{1}{2}\,n\,a^{-2}(1+b) \tag{5.191}$$

$$Q_2 = -\tfrac{1}{2}\,m\,a^{-2}(1+b) \tag{5.192}$$

Then the following results may be obtained:

$$f_1^{\,2} = m^2 C_T^{-2} \tag{5.193}$$

$$g_1^{\,2} = n^2 C_T^{-2} \tag{5.194}$$

$$f_2^{\ 2} = \tfrac{1}{2}\, n\, a^{-2}\,(1+b) \tag{5.195}$$

$$g_2^{\ 2} = \tfrac{1}{2}\, n\, a^{-2}\,(1+b) \tag{5.196}$$

$$c_1^{\ 2} = c_T^{\ 2} \tag{5.197}$$

$$c_2^{\ 2} = 2\, n^{-1} m^2 a^2 (1+b) \tag{5.198}$$

Therefore, when the rate of rise or decay is exceedingly high, one of the dilatational modes propagates with a velocity equal to isothermal dilatational wave velocity. This result may explain why many experimental data of elastic wave velocity excited by detonation of some chemicals were reported in agreement with isothermal rather than isentropic velocity (see, for example, ref. 1).

III. Phase velocity of seismic wave. Usual seismic dilatational wave velocity in the continental earth crust is about 5 to 6 kilometers per second (ref. 28, p. 253). However, the recorded seismic wave velocities from the underground atomic explosion of September 19, 1957, in Nevada (ref. 29) are quite different from the usual ones; some are as high as 6.85 to 8.7 kilometers per second and some others are as low as 0.305 to 0.37 kilometer per second. The recorded frequencies are, as the usual seismic wave (ref. 28, p. 25), from 3.7 to 9.1 cycles per second. A table of the distinct seismic waves from that underground explosion is shown below:

Table 3. Distinct Seismic Waves Recorded From The Underground Atomic Explosion of September 19, 1957 in Nevada. (Taken from reference 29.)

Time of Arrival at Station No.1		Apparent Velocity	Direction of Arrival	Frequency
min.	sec.	m/sec.		cps
1	23.51	8,700	N 65°30' W	3.8
1	40.52	6,850	N 65°50' W	3.7
5	50.56	350	S 25°20' W	9.1
14	2.91	8,090	N 18°20' E	4.5
14	28.66	7,480	N 34°10' E	6.2
28	48.0	370	N 77°20' E	4.3
34	56.52	305	N 87°30' E	8.3
39	34.06	7,660	N 47°40' W	6.7

The site of explosion was N 65°52' W to the site of observation, and at a distance of 5°22' geocentric or an arc length of 594.6 kilometers. Apparently there were reflections and refractions before the waves reached the observation station; and this explains those different directions of arrival.

These unusual seismic wave velocities, which cannot be explained as Grossling (ref. 29) said in his paper, can now be explained by using the thermoelastic wave theory. Conceivably, the underground atomic explosion was a fast rising source of seismic waves. If the coefficient of rise is as big as given by Eq. (5.93), then the phase velocities are given by Eqs. (5.115) and (5.116). Assuming that the thermoelastic constants for the earth crust near the site of atomic explosion and the site of observation are as follows:

$$c_T = 6 \text{ km/sec.} \tag{5.199}$$

$$b = 0.001 \tag{5.200}$$

$$a^2 = 0.015 \text{ cm}^2\text{/sec.} \tag{5.201}$$

then, if the coefficient of rise is

$$n = -(1-b)\, a^{-2}\, c_T^{\,2} = -2.4 \times 10^{13} \text{ sec}^{-1}; \tag{5.202}$$

the phase velocities of the two modes of dilatational wave are

$$c_1 = 8.11 \text{ km/sec.} \tag{5.203}$$

$$c_2 = 7.96 \text{ km/sec.} \tag{5.204}$$

The above theoretical figures are fairly close to the recorded data from the underground atomic explosion.

If the coefficient of rise is twice as large as the one given in Eq. (5.202), c_1 will be higher and c_2 will be lower. If the thermoelastic constants as well as the reference temperature are the same as before, then

$$c_1 = 8.48 \text{ km/sec.} \tag{5.205}$$

$$c_2 = 4.81 \text{ km/sec.} \tag{5.206}$$

It seems possible to find a value of the coefficient of rise greater than -4.8×10^{13} sec^{-1} and less than -2.4×10^{13} sec^{-1} such that the ratio of the phase velocities of the two modes is

$$c_1/c_2 = 8{,}700/6{,}850 = 1.27 \tag{5.207}$$

If the reference temperature and the thermoelastic constants of the earth's crust at the place where the underground atomic explosion took place and the observations were made are such that

$$c_1 = 8.7 \text{ km/sec.} \tag{5.208}$$

then the first two arrived waves (see Table 3, lines 1 and 2) may be explained as the elastic and the associated thermal modes.

The extra low phase velocities recorded from the underground atomic explosion may be explained by thermal mode.

As to the direction of arrival of the recorded seismic waves, the reflection and refraction (see next chapter) of thermoelastic waves may be used for the explanation.

VI. REFLECTION AND REFRACTION OF THERMOELASTIC WAVES.

Similar to reflection and refraction of elastic wave, incidence of thermoelastic wave on a boundary will also cause reflection and refraction. In this chapter, reflection and refraction of plane thermoelastic wave at a plane boundary will be studied.

Rectangular Cartesian coordinates are used in the analysis, and the coordinate axes are so orientated that the yz-plane coincides with the plane boundary and the xy-plane is parallel to the direction of plane wave propagation.

Temperature and stress are given by Eqs. (3.43) and (3.44) in terms of the scalar potential function, Φ, of dilatational wave and vector potential function, Ψ_i, of shear wave. In rectangular Cartesian coordinate system, they reduce to the following equations:

$$\beta\,\Theta = (2\mu+\lambda)\Phi_{,kk} - \rho\ddot{\Phi} \tag{6.1}$$

$$T_{ij} = 2\mu(\Phi_{,ij} - \Phi_{,kk}\delta_{ij}) + \rho\ddot{\Phi}\delta_{ij} + \mu(e_{imn}\Psi_{n,mj} + e_{jmn}\Psi_{n,mi}) \tag{6.2}$$

In the following analysis of reflection and refraction of thermoelastic wave, it has been found that the complex form of the scalar and vector potential functions are more convenient. Although the complex circular frequency, p , and the complex wave number, q , have been used in the analysis in the preceding two chapters, the oscillatory and non-oscillatory parts of the wave motion have been separated. In other words the potential function, Φ , has been considered as

$$\Phi = A \exp\left[i(mt+fx) - (nt+gx) \right] \qquad (6.3)$$

or, in real forms to show explicitly the oscillatory and non-oscillatory parts of the wave motion,

$$\Phi = e^{-(nt+gx)} \left[A_1 \cos(mt+fx) - A_2 \sin(mt+fx) \right] \qquad (6.4)$$

where A_1 and A_2 are real and imaginary parts of the constant A. The oscillatory part propagates at the speed of m/f and the non-oscillary part propagates at the speed of n/g. The non-oscillatory wave is actually a pulse or a shock, while the oscillatory part is the ordinary wave observed, for example, as the seismographic records (see, for example, ref. 29). The E-mode and T-mode dilatational waves are both oscillatory waves.

The complex phase velocity is defined, similar to the real phase velocity, as the ratio of the complex frequency to the complex wave number; thus

$$C^* = p / q \qquad (6.5)$$

where C^* is the complex velocity. If C_{re} and C_{im} are real and imaginary parts of C^* then

$$C^* = C_{re} + i\, C_{im} \qquad (6.6)$$

When the phase velocity is complex, it implies that the wave attenuates. When the complex phase velocity is known, the real phase velocity and the coefficient of attenuation of the oscillatory part of the wave can be determined uniquely; and the same is true, conversely. By the definition of the complex phase velocity,

$$C_{re} + i\, C_{im} = (m + i\, n)\, (f + i\, g)^{-1} \tag{6.7}$$

Hence,

$$C_{re} = (mf + ng)\, (f^2 + g^2)^{-1} \tag{6.8}$$

$$C_{im} = (nf - mg)\, (f^2 + g^2)^{-1} \tag{6.9}$$

and

$$f = (m\, C_{re} + n\, C_{im})\, (C_{re}^{\ 2} + C_{im}^{\ 2})^{-1} \tag{6.10}$$

$$g = (n\, C_{re} - m\, C_{im})\, (C_{re}^{\ 2} + C_{im}^{\ 2})^{-1} \tag{6.11}$$

The real phase velocity of the oscillatory part is

$$C = m\, (C_{re}^{\ 2} + C_{im}^{\ 2})\, (m\, C_{re} + n\, C_{im})^{-1} \tag{6.12}$$

Thus, either the complex phase velocity, C^*, or the real phase velocity C, with the coefficient of attenuation, g, may be used to describe the wave motion.

The relationship between the real and imaginary phase velocity is more direct in case of steady state; thus

$$c = (c_{re}^{2} + c_{im}^{2})\, c_{re}^{-1} = \left| c^{*} \right| \tan \theta^{*} \qquad (6.13)$$

where $\left| c^{*} \right|$ is the modulus and θ^{*} is the argument of the complex phase velocity. The coefficient of attenuation for the steady state case is related to the complex wave number and the complex phase velocity in the following way:

$$g = -m / \left| c^{*} \right| \sin \theta^{*} = - \left| q \right| \sin \theta^{*} \qquad (6.14)$$

In the following analysis, c_1^{*} and c_2^{*} are used to denote the complex phase velocities of the elastic mode and the thermal mode of the plane dilatational waves. The analysis will be very similar to elastic waves, except that there is a thermal boundary condition to be satisfied in addition to the stress and/or displacement boundary conditions. This additional boundary condition can always be satisfied because the plane dilatational wave has two modes so that there is an additional reflected and/or refracted wave.

(1) Reflection of a plane dilatational wave at a plane boundary.

The analysis will be the same whether the incident wave is a T-mode or E-mode dilatational wave train. For definiteness, let the incident wave be a E-mode. Let the scalar potential function of the incident wave be Φ_a, and angle of incidence be α_a; thus

$$\Phi_a = A_a \exp i(tp + xq_a \cos\alpha_a + yq_a \sin\alpha_a) \tag{6.15}$$

where

$$q_a = p / c_1^* \tag{6.16}$$

Let the boundary be thermally insulated and stress free. Then the equations of boundary conditions are:

$$\theta_{,1} = 0 \quad \text{at} \quad x = 0 \tag{6.17}$$

$$T_{11} = 0 \quad \text{at} \quad x = 0 \tag{6.18}$$

$$T_{12} = 0 \quad \text{at} \quad x = 0 \tag{6.19}$$

Enlightened by the elastic solution, assume the reflected waves are two dilatational modes and a shear wave. Their potential functions are as follows:

$$\Phi_b = A_b \exp i(tp - xq_b \cos\alpha_b + yq_b \sin\alpha_b) \tag{6.20}$$

$$\Phi_c = A_c \exp i(tp - xq_c \cos\alpha_c + yq_c \sin\alpha_c) \tag{6.21}$$

$$\Psi_1 = \Psi_2 = 0 \tag{6.22}$$

$$\Psi_3 = A_d \exp i(tp - xq_d \cos\alpha_d + yq_d \sin\alpha_d) \tag{6.23}$$

where

$$q_b = p / c_1^* \tag{6.24}$$

$$q_c = p / c_2^* \tag{6.25}$$

$$q_d = p / C_V \tag{6.26}$$

The value of C_V is given by Eq. (3.36); and the values of C_1^* and C_2^* may be determined by using Eq. (4.7).

$$q_b^2 q_c^2 = -i p^3 a^{-2} C_T^{-2} \tag{6.28}$$

$$q_b^2 + q_c^2 = (a^2 p^2 - i p C_s^2) a^{-2} C_T^{-2} \tag{6.29}$$

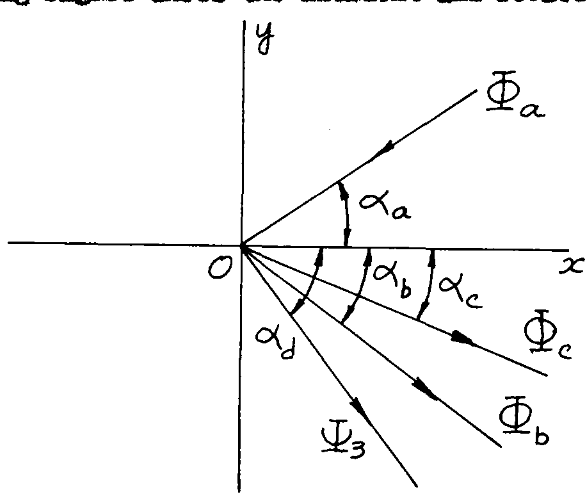
Hence

$$(C_1^*)^2 (C_2^*)^2 = i p a^2 C_T^2 \tag{6.30}$$

$$(C_1^*)^{-2} + (C_2^*)^{-2} = (a^2 - i p^{-1} C_s^2) a^{-2} C_T^{-2} \tag{6.31}$$

The values of C_1^* and C_2^* may be obtained by solving the above two equations, or may be obtained by substituting the values of f and g obtained in Chapter V in Eqs. (6.8) and (6.9) for a particular problem.

The following figure shows the incident and reflected waves:



Substituting the functions Φ_a, Φ_b, Φ_c, and Ψ_d, in Eqs. (6.17) to (6.18), the following equations may be obtained:

$$
\begin{aligned}
(2\mu+\lambda)\Big[& A_a\, q_a^3 \cos\alpha_a \exp i(tp + yq_a \sin\alpha_a) \\
& - A_b\, q_b^3 \cos\alpha_b \exp i(tp + yq_b \sin\alpha_b) \\
& - A_c\, q_c^3 \cos\alpha_c \exp i(tp + yq_c \sin\alpha_c)\Big] \\
- \rho p^2 \Big[& A_a\, q_a \cos\alpha_a \exp i(tp + yq_a \sin\alpha_a) \\
& - A_b\, q_b \cos\alpha_b \exp i(tp + yq_b \sin\alpha_b) \\
& - A_c\, q_c \cos\alpha_c \exp i(tp + yq_c \sin\alpha_c)\Big] = 0
\end{aligned}
\qquad (6.32)
$$

$$
\begin{aligned}
2\mu \Big[& A_a\, q_a^2 \sin^2\alpha_a \exp i(tp + yq_a \sin\alpha_a) \\
& + A_b\, q_b^2 \sin^2\alpha_b \exp i(tp + yq_b \sin\alpha_b) \\
& + A_c\, q_c^2 \sin^2\alpha_c \exp i(tp + yq_c \sin\alpha_c) \\
& + A_d\, q_d^2 \sin\alpha_d \cos\alpha_d \exp i(tp + yq_d \sin\alpha_d)\Big] \\
- \rho\, p^2 \Big[& A_a \exp i(tp + yq_a \sin\alpha_a) \\
& + A_b \exp i(tp + yq_b \sin\alpha_b) \\
& + A_c \exp i(tp + yq_c \sin\alpha_c)\Big] = 0
\end{aligned}
\qquad (6.33)
$$

$$A_d q_d^2(\sin^2\alpha_d - \cos^2\alpha_d) \exp i(tp + yq_d \sin\alpha_d)$$

$$+ 2 A_a q_a^2 \sin\alpha_a \cos\alpha_a \exp i(tp + yq_a \sin\alpha_a)$$

$$- 2 A_b q_b^2 \sin\alpha_b \cos\alpha_b \exp i(tp + yq_b \sin\alpha_b)$$

$$- 2 A_c q_c^2 \sin\alpha_c \cos\alpha_c \exp i(tp + yq_c \sin\alpha_c) = 0 \qquad (6.34)$$

The above three equations must be satisfied for all values of time, t, and all values of the coordinate y. If all the arguments of the exponential functions in the above three equations are the same, then the exponential function can be factorised. Therefore, in order to satisfy the boundary conditions, the following three equations must be satisfied:

$$q_a \sin\alpha_a = q_b \sin\alpha_b$$

$$= q_c \sin\alpha_c$$

$$= q_d \sin\alpha_d \qquad (6.35)$$

Then, after having factorised the exponential functions, Eqs. (6.32) to (6.34) can be satisfied by choosing proper values for the constants A_b, A_c, A_d. In other words, these three equations becomes the equations for solving the unknown constants.

The above conditions may also be written in the following form:

$$\frac{\sin\alpha_a}{C_1^*} = \frac{\sin\alpha_b}{C_1^*} = \frac{\sin\alpha_c}{C_2^*} = \frac{\sin\alpha_d}{C_V} \tag{6.36}$$

The angle α_b is obtained immediately; thus

$$\alpha_b = \alpha_a \tag{6.37}$$

The angles α_c and α_d must be complex. For numerical computation, the two complex equations for determining these two complex angles will be expressed in real forms. Let

$$\alpha_c = \alpha'_c + i\,\alpha''_c \tag{6.38}$$

$$\alpha_d = \alpha'_d + i\,\alpha''_d \tag{6.39}$$

$$C_1^* = C_{1(re)} + i\,C_{1(im)} \tag{6.40}$$

$$C_2^* = C_{2(re)} + i\,C_{2(im)} \tag{6.41}$$

Substituting Eqs. (6.38) to (6.41) in the last two of Eqs. (6.36) and equating the real and imaginary parts, the following four real equations are obtained:

$$C_{2(re)}\sin\alpha_a = C_{1(re)}\sin\alpha'_c\cosh\alpha''_c - C_{1(im)}\cos\alpha'_c\sinh\alpha''_c \tag{6.42}$$

$$C_{2(im)}\sin\alpha_a = C_{1(re)}\cos\alpha'_c\cosh\alpha''_c + C_{1(im)}\sin\alpha'_c\cosh\alpha''_c \tag{6.43}$$

$$C_V \sin\alpha_a = C_{1(re)} \sin\alpha'_c \cosh\alpha''_d - C_{1(im)} \cos\alpha'_d \sinh\alpha''_d \tag{6.44}$$

$$0 = C_{1(im)} \sin\alpha'_d \cosh\alpha''_d + C_{1(re)} \cos\ {}_d \sinh\ {}_d \tag{6.45}$$

Hence, the angle of reflection of the same mode as the incident mode is equal to the angle of incidence; the angle of reflection of the different mode and the angle of reflection of shear waves are dependent on the frequency and the coefficient of rise or decay of the incident wave.

If the incident wave is at very low frequency and at steady state then, by substituting the values of f's and g's of Eqs. (5.45) to (5.48) in Eqs. (6.8) and (6.9), the real and imaginary parts of the phase velocities of the two dilatational modes are as follows:

$$C_{1(re)} = (1+b)\ C_T \tag{6.46}$$

$$C_{1(im)} = \tfrac{1}{2} bma^2 (1+b)^{-3/2}\ C_T^{-1} \tag{6.47}$$

$$C_{2(re)} = 2^{\frac{1}{2}}\ a\ (1+b)^{-\frac{1}{2}} \tag{6.48}$$

$$C_{2(im)} = -2^{\frac{1}{2}}\ a\ (1+b)^{-\frac{1}{2}} \tag{6.49}$$

The value of $C_{1(re)}$ is much greater than the other three C's of the above equations. Hence, the following approximate solutions of Eqs. (6.42) to (6.45) are obtained as follows:

$$\alpha'_c = \alpha''_c = \alpha''_d = 0 \tag{6.50}$$

$$\sin\alpha_d = C_V\, C_s^{-1} \sin\alpha_a \tag{6.51}$$

Thus the reflected T-mode is approximately perpendicular to the boundary plane and the angle of reflection for reflected shear wave is slightly smaller than the same angle by elastic wave theory.

If the incident wave is a T-mode, then, in Eqs. (6.42) to (6.45), the constants $C_{1(re)}$ and $C_{1(im)}$ would be $C_{2(re)}$ and $C_{2(im)}$ respectively, and vice versa. Now, if the incident T-mode wave be at low frequencies, and if the state is steady, the angles of reflection will be complex numbers and, approximately

$$\alpha'_c \cong \alpha'_d \cong 45^\circ \tag{6.52}$$

$$\cosh\alpha''_c \cong \sinh\alpha''_c \tag{6.53}$$

$$\cosh\alpha''_d \cong \sinh\alpha''_d \tag{6.54}$$

Then the amplitudes can be solved. For incident E-mode, the equations for solving the amplitudes are:

$$(2\mu+\lambda)(A_a q_a^3\cos\alpha_a - A_b q_b^3\cos\alpha_b - A_c q_c^3\cos\alpha_c)$$
$$-\rho p^2 (A_a q_a\cos\alpha_a - A_b q_b\cos\alpha_b - A_c q_c\cos\alpha_c) = 0 \tag{6.55}$$

$$2\mu(A_c + A_a + A_b + A_d)q_a^2\sin^2\alpha_a - \rho p^2(A_a + A_b + A_c) = 0 \tag{6.56}$$

$$A_d q_d^2(\sin^2\alpha_d - \cos^2\alpha_d) + 2q_a\sin\alpha_a(A_a q_a\cos\alpha_a - A_b q_b\cos\alpha_b - A_c q_c\cos\alpha_c) = 0 \tag{6.57}$$

(2) Reflection of plane shear wave at a plane boundary. Reflection of SH wave (i. e. shear wave with direction of particle vibration parallel to the boundary) is the same as that of elastic wave. Thus, only the reflection of SV wave (i. e. shear wave with direction of particle vibration perpendicular to that of SH wave) will be studied here. The x and y components of the vector potential functions of both the incident and reflected shear waves are zero, so that, in writing Ψ_ζ, the subscript "ζ" will be omitted. Let the potential function for the incident SH wave be

$$\Psi_a = A_a \exp i(tp + xq_a \cos\alpha_a + yq_a \sin\alpha_a) \tag{6.58}$$

and the potential functions of reflected SH wave and two modes of dilatational waves be respectively

$$\Psi_b = A_b \exp i(tp - xq_b \cos\alpha_b + yq_b \sin\alpha_b) \tag{6.59}$$

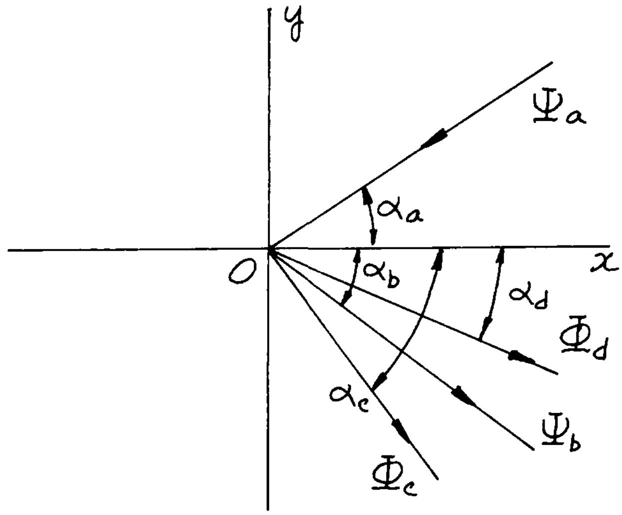
$$\Phi_c = A_c \exp i(tp - xq_c \cos\alpha_c + yq_c \sin\alpha_c) \tag{6.60}$$

$$\Phi_d = A_d \exp i(tp - xq_d \cos\alpha_d - yq_d \sin\alpha_d) \tag{6.61}$$

The angles of reflection are, similar to the analysis of section (1), to be determined by the following equations:

$$\frac{\sin\alpha_a}{C_V} = \frac{\sin\alpha_b}{C_V} = \frac{\sin\alpha_c}{C_1^*} = \frac{\sin\alpha_d}{C_2^*} \tag{6.62}$$

The angle of incidence and the angles of reflection are show in the following figure:



In Eqs. (6.58) and (6.59), the complex frequency has been used for shear wave in case the incident shear wave is from a rising (decaying) source. Substituting Eq. (6.58) in Eq. (4.3), the following equation is obtained:

$$c_v^2 \, q_a^2 - p^2 = 0 \tag{6.63}$$

Hence,

$$c_v^2(f^2 - g^2) - (m^2 - n^2) = 0 \tag{6.64}$$

$$c_v^2 \, f \, g - m \, n = 0 \tag{6.64a}$$

where the subscript a has been dropped. Thus, for unsteady state, the coefficient of attenuation of shear wave is not zero. For steady state, the coefficient of attenuation is zero.

The amplitudes are to be determined by the following equations:

$$(2\mu+\lambda)\left[A_c q_c^3 \cos\alpha_c + A_d q_d^3 \cos\alpha_d\right.$$

$$\left. - \rho p^2 (A_c q_c \cos\alpha_c + A_d q_d \cos\alpha_d) \right] = 0 \qquad (6.65)$$

$$2\mu(- A_a q_a^2 \cos\alpha_a \sin\alpha_a - A_b q_b^2 \cos\alpha_b \sin\alpha_b$$

$$- A_c q_c^2 \sin^2\alpha_c - A_d q_d^2 \sin^2\alpha_d)$$

$$- \rho p^2 (A_a + A_b + A_c + A_d) = 0 \qquad (6.66)$$

$$2 A_c q_c^2 \cos\alpha_c \sin\alpha_c + 2 A_d q_d^2 \cos\alpha_d \sin\alpha_d$$

$$- A_a q_a^2(\cos^2\alpha_a - \sin^2\alpha_a) - A_b q_b^2(\cos^2\alpha_b - \sin^2\alpha_b) = 0 \qquad (6.67)$$

The boundary has been assumed thermally insulated.

If the boundary convects heat, then Eq. (6.17) will be changed to the following equation:

$$\Theta_{,1} - H\Theta = 0 \qquad \text{at} \quad x = 0 \qquad (6.68)$$

where H is the ratio of thermal convection coefficient to thermal conductivity. Then Eq. (6.65) is to be replaced by the following equation:

$$(2\mu+\lambda)\left[A_c q_c^2(H - i\, q_c \cos\alpha_c) + A_d q_d^2(H - i\, q_d \cos\alpha_d)\right]$$

$$- \rho p^2 \left[A_c(H - i\, q_c \cos\alpha_c) + A_d(H - i\, q_d \cos\alpha_d)\right] = 0 \qquad (6.69)$$

(3) Reflection and refraction of plane dilatational waves at a plane interface between two media.

Let both the two media be heat-conducting, non-radiating, elastic homogeneous, and isotropic. Also, let the two media be connected rather than just in contact. There are six conditions of continuity, namely the continuity of normal and tangential displacements, the continuity of normal and tangential stresses, the continuity of temperature, and the continuity of heat conduction so that there is no accumulation of heat at the boundary.

For definiteness, let the incident wave be the E-mode dilatational wave, and let the potential function of the incident wave be

$$\Phi_a = A_a \exp i(tp + xq_a \cos\alpha_a + yq_a \sin\alpha_a) \tag{6.70}$$

There must be three reflected waves and three refracted waves so that the six conditions of continuity can be satisfied. Let the potential functions of the three reflected waves —— an E-mode, a T-mode, and a shear wave —— be respectively as follows:

$$\Phi_b = A_b \exp i(tp - xq_b \cos\alpha_b + yq_b \sin\alpha_b) \tag{6.71}$$

$$\Phi_c = A_c \exp i(tp - xq_c \cos\alpha_c + yq_c \sin\alpha_c) \tag{6.72}$$

$$\Psi_d = A_d \exp i(tp - xq_d \cos\alpha_d + yq_d \sin\alpha_d) \tag{6.73}$$

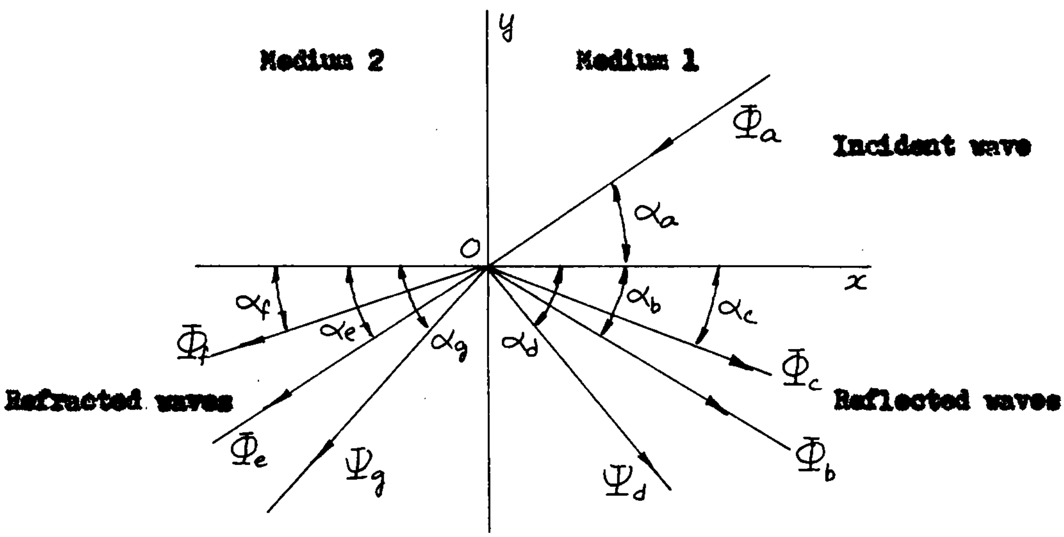
Also, let the potential functions of the three refracted waves —— an E-mode, a T-mode, and a shear wave —— be respectively the following:

$$\Phi_e = A_e \exp i(tp + xq_e \cos\alpha_e + yq_e \sin\alpha_e) \tag{6.74}$$

$$\Phi_f = A_f \exp i(tp + xq_f \cos\alpha_f + yq_f \sin\alpha_f) \tag{6.75}$$

$$\Psi_g = A_g \exp i(tp + xq_g \cos\alpha_g + yq_g \sin\alpha_g) \tag{6.76}$$

where Ψ_d and Ψ_g are understood the x^3 or z components of the vector functions Ψ_{1_d} and Ψ_{1_g} respectively. The incident angle, the angles of reflection, and the angles of refraction are shown in the following figure:



The angles of reflection and the angles of refraction must satisfy the following equations:

$$\frac{\sin\alpha_a}{C_1^*} = \frac{\sin\alpha_b}{C_1^*} = \frac{\sin\alpha_c}{C_2^*} = \frac{\sin\alpha_d}{C_V} = \frac{\sin\alpha_e}{C_1'^*} = \frac{\sin\alpha_f}{C_2'^*} = \frac{\sin\alpha_g}{C_V'} \tag{6.77}$$

where C_1^*, C_2^*, and C_V are respectively the phase velocities of E-mode, T-mode, and shear waves in medium 1, and $C_1'^*$, $C_2'^*$, and C_V' are respectively the phase velocities of E-mode, T-mode, and shear waves in medium 2.

In rectangular Cartesian coordinates, the displacement vector is, by reducing either Eq. (3.29) or (3.30),

$$u_i = \Phi_{,i} + e_{ijk} \Psi_{k,j} \tag{6.78}$$

The six conditions of continuity at the interface of the two media are

$$u_1 = u'_1 \tag{6.79}$$

$$u_2 = u'_2 \tag{6.80}$$

$$T_{11} = T'_{11} \tag{6.81}$$

$$T_{12} = T'_{12} \tag{6.82}$$

$$\Theta = \Theta' \tag{6.83}$$

$$k\,\Theta_{,x} = k'\,\Theta'_{,x} \tag{6.84}$$

where the unprimed quantities refer to medium 1, and the primed quantities refer to medium 2. Substitution of the potential functions in the above six equations yields the following six equations by which the amplitudes of the reflected and refracted waves can be determined in terms of the amplitude of the incident wave:

$$A_a q_a \cos\alpha_a - A_b q_b \cos\alpha_b - A_c q_c \cos\alpha_c + A_d q_d \sin\alpha_d$$

$$= A_e q_e \cos\alpha_e + A_f q_f \cos\alpha_f + A_g q_g \sin\alpha_g \tag{6.85}$$

$$(A_a + A_b + A_c) q_a \sin\alpha_a - A_d q_d \cos\alpha_d = (A_e + A_f) q_a \sin\alpha_a$$

$$+ A_g q_g \cos\alpha_g \tag{6.86}$$

$$2\mu \left[q_a^2 \sin^2\alpha_a (A_a + A_b + A_c) + A_d q_a q_d \sin\alpha_a \cos\alpha_d \right] - \rho p^2 (A_a + A_b +$$

$$+ A_c) = 2\mu' \left[q_a^2 \sin^2\alpha_a (A_e + A_f) - A_g q_a q_g \sin\alpha_a \cos\alpha_g \right]$$

$$- \rho' p^2 (A_e + A_f) \tag{6.87}$$

$$2\mu q_a \sin\alpha_a (A_a q_a \cos\alpha_a - A_b q_b \cos\alpha_b - A_c q_c \cos\alpha_c) - \mu A_d q_d^2 (\cos^2\alpha_d -$$

$$- \sin^2\alpha_d) = 2\mu' q_a \sin\alpha_a (A_e q_e \cos\alpha_e + A_f q_f \cos\alpha_f)$$

$$- \mu' A_g q_g^2 (\cos^2\alpha_g - \sin^2\alpha_g) \tag{6.88}$$

$$(2\mu + \lambda)(A_a q_a^2 + A_b q_b^2 + A_c q_c^2) - \rho p^2 (A_a + A_b + A_c)$$

$$= (2\mu' + \lambda')(A_e q_e^2 + A_f q_f^2) - \rho' p^2 (A_e + A_f) \tag{6.89}$$

$$k(2\mu + \lambda)(A_a q_a^3 \cos\alpha_a - A_b q_b^3 \cos\alpha_b - A_c q_c^3 \cos\alpha_c) - k\rho p^2 (A_a q_a \cos\alpha_a +$$

$$+ A_b q_b \cos\alpha_b + A_c q_c \cos\alpha_c) = k'(2\mu' + \lambda')(A_e q_e^3 \cos\alpha_e + A_f q_f^3 \cos\alpha_f)$$

$$- k' \rho' p^2 (A_e q_e \cos\alpha_e + A_f q_f \cos\alpha_f) \tag{6.90}$$

The complex wave numbers are related to the complex frequency and phase velocities as follows:

$$q_b = q_a = p / C_1^* \qquad (6.91)$$

$$q_c = p / C_2^* \qquad (6.92)$$

$$q_d = p / C_V \qquad (6.93)$$

$$q_e = p / C'^*_1 \qquad (6.94)$$

$$q_f = p / C'^*_2 \qquad (6.95)$$

$$q_g = p / C'_V \qquad (6.96)$$

(4) Reflection and refraction of plane shear wave at a plane interface between two media.

The reflected waves will be a shear wave, an E-mode, and a T-mode dilatational wave; the refracted waves will also be a shear wave, an E-mode, and a T-mode dilatational wave. The analysis will be similar to the above section.

If the incident shear wave is a SH wave, the solution will be the same as elastic solution.

VII. THERMOELASTIC SURFACE WAVES.

One type of plane waves propagating in a semi-infinite medium with amplitude decreasing exponentially with the depth measured from the boundary is called surface waves or Rayleigh's waves. The phase velocity of elastic surface waves is independent of frequency(see, for example, ref. 1 or ref.6). In this chapter, thermal effect will be considered and such waves will be called thermoelastic surface waves.

Rectangular Cartesian coordinates are used in the analysis, and the coordinate axes are so orientated that

(a) the xz-plane coincides with the boundary,

(b) the postive y-axis points towards the interior of the medium, and

(c) the x-axis is in the direction of wave propagation.

The directions of motion of particles of the medium are assumed, as elastic surface waves, parallel to the xy-plane.

The displacement vector, u_i , will be resolved into an irrotational part and a solenoidal part (see Section 6, Chapter III) in the following mathematical anaylsis. Physically, however, the surface waves cannot be resolved into the above mentioned two parts. In other words, if a scalar potential function and a vector potential function are used to describe the displacement, they must be given the same phase velocity.

Let

$$u_i = \Phi_{,i} + e_{ijk}\Psi_{k,j} \tag{7.1}$$

and try harmonic solutions. Assume

$$\Phi = A \exp (ipt + iqx - hy) \tag{7.2}$$

This scalar function Φ must satisfy the following governing equation:

$$a^2 c_T^2 \Phi_{,iijj} - (a^2 \ddot{\Phi} + c_s^2 \dot{\Phi})_{,ii} + \dddot{\Phi} = 0 \tag{7.3}$$

which is the same as Eq. (4.5). Hence, the following equation may be obtained:

$$a^2 c_T^2 (q^2 - h^2)^2 - (a^2 p^2 - i\, p c_s^2)(q^2 - h^2) - i\, p^3 = 0 \tag{7.4}$$

The above equation is similar to Eq. (4.7), hence the solutions may be readily obtained, thus

$$q^2 - h_1^2 = P_1 + i\, Q_1 \tag{7.5}$$

$$q^2 - h_2^2 = P_2 + i\, Q_2 \tag{7.6}$$

where the P's and Q's are the same as given in Eqs. (4.15) to (4.18).

Therefore, the scalar potential function, , is of the following form:

$$\Phi = A_1 \exp (ipt + iqx - h_1 y) + A_2 \exp (ipt + iqx - h_2 y) \tag{7.7}$$

The component of displacement in x^3 or z direction is zero, so that the x^1 and x^2 or x and y components of the vector potential function, Ψ_i , must vanish. Let the x^3 or z component of the vector potential function be

$$\Psi_3 = A_3 \exp\,(ipt + iqx - h_3 y) \tag{7.8}$$

The governing equation for Ψ_3 is

$$c_V{}^2 \Psi_{3,ii} - \ddot{\Psi}_3 = 0 \tag{7.9}$$

Substitution of Eq. (7.8) in Eq. (7.9) yields the following equation:

$$q^2 - h_3{}^2 = p^2 / c_V{}^2 \tag{7.10}$$

If the boundary is stress free and thermally insulated, then the boundary conditions are:

$$T_{22} = 0 \qquad \text{at } y = 0 \tag{7.11}$$

$$T_{12} = 0 \qquad \text{at } y = 0 \tag{7.12}$$

$$\Theta_{,2} = 0 \qquad \text{at } y = 0 \tag{7.13}$$

Now, in rectangular Cartesian coordinates,

$$T_{22} = \rho \ddot{\Phi} - 2\mu \Phi_{,11} - 2\mu \Psi_{3,12} \tag{7.14}$$

$$T_{12} = 2\mu \Phi_{,12} + \mu (\Psi_{3,22} - \Psi_{3,11}) \tag{7.15}$$

$$\Theta_{,2} = (2\mu + \lambda)(\Phi_{,112} + \Phi_{,222}) - \rho \ddot{\Phi}_{,2} \tag{7.16}$$

Substitution of Eqs. (7.14) to (7.16) in Eqs. (7.11) to (7.13) yields the following three equations:

$$(2\mu q^2 - \rho p^2)\, A_1 + (2\mu q^2 - \rho p^2)\, A_2 + 2\, i\, \mu\, q\, h_3\, A_3 = 0 \tag{7.17}$$

$$2\, i\, q\, h_1\, A_1 \;+\; 2\, i\, q\, h_2\, A_2 \;-\; (h_3^{\,2} + q^2)\, A_3 = 0 \tag{7.18}$$

$$\left[(2\mu+\lambda)(q^2 - h_1^{\,2}) - \rho p^2\right] h_1 A_1 + \left[(2\mu+\lambda)(q^2 - h_2^{\,2}) - \rho p^2\right] h_2 A_2 = 0 \tag{7.19}$$

For non-trivial solution, the determinant of the coefficients of A's must vanish which is an equation for determining the complex wave number, q. The complex phase velocity of thermoelastic surface wave is

$$c_R^{*} = p / q \tag{7.20}$$

If A_2 is equal to zero, Eq. (7.19) reduces to

$$(2\mu+\lambda)(q^2 - h_1^{\,2}) - \rho p^2 = 0 \tag{7.21}$$

and Eqs. (7.17) and (7.18) reduce to the following equations which are, as expected, the same equations for elastic surface waves:

$$\left[(2\mu+\lambda)(h_1^{\,2} - q^2) + 2\mu q^2\right] A_1 + 2i\mu q h_3 A_3 = 0 \tag{7.22}$$

$$2iqh_1 A_1 - (h_3^{\,2} + q^2)\, A_3 = 0 \tag{7.23}$$

VIII. PROPAGATION OF SPHERICAL WAVES IN AN INFINITE MEDIUM.

The spherical polar coordinates are used here; the radius r , the azimuth ϕ , and the colatitude θ are denoted by x^1 , x^2 , and x^3 respectively. The three non-vanishing components of metric tensor are:

$$a_{11} = 1 \tag{8.1}$$

$$a_{22} = r^2 \tag{8.2}$$

$$a_{33} = r^2 \sin^2\theta \tag{8.3}$$

and the three nonvanishing components of the conjugate tensor are:

$$a^{11} = 1 \tag{8.4}$$

$$a^{22} = r^{-2} \tag{8.5}$$

$$a^{33} = r^{-2} \sin^{-2}\theta \tag{8.6}$$

The physical components of the displacement vector in the directions of the base vectors g_2 and g_3 are zero and the component in the radial direction is a function of r and time t only. The components of the vector potential function Ψ_i are all zero for this spherical symmetrical case.

The contravariant component of displacement u^1 , the covariant component u_1 , and the physical component $u_{(1)}$ are the same and is denoted by u_r Eq. (3.29) and Eq. (3.30) reduce to a single equation:

$$u_r = \Phi_{,r} \tag{8.7}$$

Eqs. (3.32) and (3.33) reduce to

$$(a^2/r)(r\,\Theta)_{,rr} - \dot{\Theta} = (\beta T_R/C_\epsilon)(1/r)(r\,\dot{\Phi})_{,rr} \tag{8.8}$$

$$(c_T^{\,2}/r)(r\,\Phi)_{,rr} - \ddot{\Phi} = (\beta/\rho)\,\Theta \tag{8.9}$$

where a^2 is given in Eq. (4.4). Eliminating Θ from the above two equations, the following governing equation on Φ is obtained:

$$(a^2 c_T^{\,2} r^{-1})(r\,\Phi)_{,rrrr} - r^{-1}\left[r(a^2\ddot{\Phi} + c_s^{\,2}\dot{\Phi})\right]_{,rr} + \dddot{\Phi} = 0 \tag{8.10}$$

Assuming a solution

$$\Phi = F(r)\exp(ipt) \tag{8.11}$$

and substituting it into Eq. (8.10), the following equation may be obtained:

$$a^2 c_T^{\,2}(F'''' + 4r^{-1}F''') + (a^2p^2 - ipc_s^{\,2})(F'' + 2r^{-1}F') - ip^3F = 0 \tag{8.12}$$

where the primes denote differentiation with respect to r. If p is a real number, the following equation may be obtained by equating both the real and imaginary parts to zero :

$$c_T^{\,2}(F'''' + 4r^{-1}F''') + p^2(F'' + 2r^{-1}F') = 0 \tag{8.13}$$

$$c_s^{\,2}(F'' + 2r^{-1}F') + p^2F = 0 \tag{8.14}$$

Eq. (8.14) is a generalized Bessel equation whose general solution is

$$F(r) = A\, r^{-\frac{1}{2}}\, J_{\frac{1}{2}}(pr/C_s) + B\, r^{-\frac{1}{2}} J_{-\frac{1}{2}}(pr/C_s) \tag{8.15}$$

where A and B are arbitrary constants and the J's are Bessel functions of the orders as indicated by the subscripts. The condition that the function $F(r)$ also satisfy Eq.(8.13) requires $C_s = C_T$. Hence the solution of (8.15) may be regarded as an approximation for materials like quartz glass whose C_s and C_T are almost the same (see Appendix).

For materials whose C_s and C_T cannot be considered the same, assuming a solution

$$\Phi = A\, r^{-1} \exp\,(ipt + iqr) \tag{8.16}$$

and substituting into Eq. (8.10), the following equation may be obtained:

$$a^2 C_T^2 q^4 - a^2 p^2 q^2 + i C_s^2 p q^2 - i p^3 = 0 \tag{8.17}$$

which is exactly the same as Eq. (4.7). Hence, similar to the plane dilatational waves, the spherical dilatational waves have two distinct modes —— an elastic mode and a thermal mode —— and the phase velocities of both modes are the same as those of plane dilatational waves respectively.

IX. PROPAGATION OF CYLINDRICAL WAVES IN AN INFINITE MEDIUM.

The cylindrical polar coordinates are used here; the radius r, the polar angle θ, and the altitude z are denoted by x^1, x^2, and x^3, respectively. The three non-vanishing components of metric tensor are

$$a_{11} = 1 \tag{9.1}$$

$$a_{22} = r^2 \tag{9.2}$$

$$a_{33} = 1 \tag{9.3}$$

and the three non-vanishing components of the conjugate tensor are

$$a^{11} = 1 \tag{9.4}$$

$$a^{22} = r^{-2} \tag{9.5}$$

$$a^{33} = 1 \tag{9.6}$$

The physical components of the displacement vector in the directions of the base vector $\bar{g}_2$ and $\bar{g}_3$ are zero and the component in the radial direction is a function of r and time t, only. All the components of the vector potential function Ψ_i are zero.

The contravariant component of displacement, u^1, the covariant component, u_1, and the physical component, $u_{(1)}$, are the same and are denoted by u_r. Eq. (3.29) and Eq. (3.30) reduce to the same equation:

$$u_r = \Phi_{,r} \tag{9.7}$$

Eqs. (3.33) and (3.34) reduce to

$$a^2(\Theta_{,rr} + r^{-1}\Theta_{,r}) - \dot{\Theta} = \frac{\beta T_R}{C_\epsilon}(\dot{\Phi}_{,rr} + r^{-1}\dot{\Phi}_{,r}) \tag{9.8}$$

$$c_T^2(\Phi_{,rr} + r^{-1}\Phi_{,r}) - \ddot{\Phi} = \frac{\beta}{\rho}\,\Theta \tag{9.9}$$

where a^2 is given in Eq. (4.4). Eliminating Θ from the above two equations, the following equation may be obtained:

$$a^2 c_T^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\Phi - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)(a^2\ddot{\Phi} + c_s^2\dot{\Phi}) + \dddot{\Phi} = 0 \tag{9.10}$$

Assuming a solution

$$\Phi = F(r)\exp(ipt) \tag{9.11}$$

and substituting into Eq. (9.10), the following equation may be obtained by equating the real and imaginary parts to zero:

$$c_T^2(F'''' + 2r^{-1}F''' - r^{-2}F'' + r^{-3}F') + p^2(F'' + r^{-1}F') = 0 \tag{9.12}$$

$$c_s^2(F'' + r^{-1}F') + p^2 F = 0 \tag{9.13}$$

Eq. (9.13) is a generalised Bessel equation whose solution is

$$F(r) = A\,J_0(pr/c_s) + B\,Y_0(pr/c_s) \tag{9.14}$$

where A and B are arbitrary constants, and J_0 and Y_0 are respectively

Bessel functions of zero order of the first kind and the second kind. The condition that this function $F(r)$ must also satisfy Eq. (9.12) requires that $C_s = C_T$.

For the case where C_s may not be considered close to C_T, assume the following solution:

$$\Phi = A\, r^{-\frac{1}{2}} \exp\,(ipt + iqr) \tag{9.15}$$

and substitute it into Eq. (9.10) to get the following equation:

$$a^2C_T^{\,2}(q^2-\tfrac{1}{4}r^{-2})^2-(a^2p^2-iC_s^{\,2}p)(q^2-\tfrac{1}{4}r^{-2})-ip^3+a^2C_T^{\,2}r^{-4} = 0 \tag{9.16}$$

This is similar to Eq. (4.7) except that this equation contains the variable r. Substituting Eqs. (4.8) and (4.9) into the above equation, the following equation may be obtained:

$$f^2-g^2+2ifg-\tfrac{1}{4}r^{-4} = (2a^2C_T^{\,2})^{-1}(m+in)\left[ma^2+i(na^2-C_s^{\,2})\pm\sqrt{U+iV}\right] \tag{9.17}$$

where

$$U = (m^2-n^2)a^4-C_s^{\,4}+2na^2(C_s^{\,2}-C_T^{\,2})-4a^4C_T^{\,4}r^{-4}(m^2-n^2)(m^2+n^2)^{-2} \tag{9.18}$$

$$V = 2ma^2\left[na^2+(2C_T^{\,2}-C_s^{\,2})+4na^2C_T^{\,4}r^{-4}(m^2+n^2)^{-2}\right] \tag{9.19}$$

The square root of $(U+iV)$ can be expressed in trigonometrical form. Let f_1 and g_1 be the values corresponding to the positive square root, and f_2 and g_2 be the values corresponding to the minus square root of $(U + iV)$, then Eq. (9.17) can be written as the following two equations:

$$f_1^{\,2} - g_1^{\,2} + 2if_1g_1 - \tfrac{1}{4}\,r^{-4} = P_1 + i\,Q_1 \tag{9.20}$$

$$f_2^{\,2} - g_2^{\,2} + 2if_2g_2 - \tfrac{1}{4}\,r^{-4} = P_2 + i\,Q_2 \tag{9.21}$$

where the P's and Q's are of the same forms given by Eqs. (4.15) to (4.18) except the functions U and V are to be replaced by those given by Eqs. (9.18) and (9.19). Solving Eqs. (9.20) and (9.21), the following results are obtained:

$$f_1^{\,2} = \tfrac{1}{2}\left[P_1 + \tfrac{1}{4}\,r^{-4} + \sqrt{(P_1 + \tfrac{1}{4}\,r^{-4})^2 + Q_1^{\,2}}\;\right] \tag{9.22}$$

$$g_1^{\,2} = \tfrac{1}{2}\left[-P_1 - \tfrac{1}{4}\,r^{-4} + \sqrt{(P_1 + \tfrac{1}{4}\,r^{-4})^2 + Q_1^{\,2}}\;\right] \tag{9.23}$$

$$f_2^{\,2} = \tfrac{1}{2}\left[P_2 + \tfrac{1}{4}\,r^{-4} + \sqrt{(P_2 + \tfrac{1}{4}\,r^{-4})^2 + Q_2^{\,2}}\;\right] \tag{9.24}$$

$$g_2^{\,2} = \tfrac{1}{2}\left[-P_2 - \tfrac{1}{4}\,r^{-4} + \sqrt{(P_2 + \tfrac{1}{4}\,r^{-4})^2 + Q_2^{\,2}}\;\right] \tag{9.25}$$

The phase velocities of the two modes of cylindrical dilatational waves are:

$$c_1 = m\,/\,f_1 \tag{9.26}$$

$$c_2 = m\,/\,f_2 \tag{9.27}$$

These solutions as well as the solution of Eq. (9.14) show that the cylindrical dilatational waves are not exactly periodic. The phase velocity varies along the coordinate r. However, at a distance far from the origin, the waves are almost periodic.

X. CONCLUSIONS.

When the production of entropy in an irreversible thermodynamic process is applied to the strain wave propagation in a thermoelastic media, the dilatational wave not only attenuates but also has two distinct modes —— an elastic mode and a thermal mode —— both of which propagate with attenuation. The shear wave, however, does not attenuate.

The phase velocities of the two dilatational modes are dependent on the frequency, and their coefficients of attenuation are also dependent on the frequency. If the state is unsteady, they are all dependent on the coefficient of rise or decay.

In steady state, the phase velocity of elastic mode is very closely equal to the isentropic phase velocity at low frequencies and approaches the isothermal phase velocity at very high frequencies. The phase velocity of thermal mode is much lower than that of elastic mode at low frequencies and becomes higher and higher as the frequency increases and finally, at very high frequencies, exceeds the phase velocity of elastic mode.

During the period of rise when the rate of rise is high, the phase velocity of elastic mode may exceed the isentropic phase velocity at low frequencies and approaches, as the frequency increases and becomes very high, the isothermal phase velocity. The phase velocity of thermal mode may be much higher than in steady state and may exceed the isentropic phase velocity even at low frequencies. At very high frequencies, the phase velocity of thermal mode exceeds that of elastic mode.

During the period of decay, if the rate of decay is high, the phase velocity of elastic mode is almost equal to the isothermal phase velocity at high or low frequencies. The phase velocity of thermal mode is, if the rate of decay is high, much smaller than it is in steady state; and it increases with frequency and exceeds the phase velocity of elastic mode when the frequency is very high.

Upon the incidence of either mode of dilatational wave or SV shear wave, the reflected waves are composed of both modes of dilatational waves and a SV shear wave. The refracted waves also are composed of these three waves.

The surface waves are also dissipative.

The spherical dilatational wave propagates at the same velocity as the plane dilatational wave. The cylindrical dilatational wave approaches the plane dilatational wave when it is far away from the origin.

The phase velocities of the two modes of thermoelastic strain waves are able to explain the unusual phase velocities from an underground atomic explosion.

Thermoelastic strain wave theory is unable to explain if the shear wave will attenuate. The reflection and refraction from the incident SV shear wave may serve to explain the dissipation of seismic SV shear wave. However, the SH type shear wave will never dissipate by either the theory of thermoelasticity or the theory of elasticity. This is an area to be explored which might lead to a modification of the theory of thermoelasticity.

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XIII. Appendix

Thermoelastic Constants of some Materials

	a^2 $cm^2/sec.$	C_T^2 cm^2/sec^2	$\frac{\beta^2 T_R}{\rho C_\epsilon}$ cm^2/sec^2	b Dimensionless
Aluminum, pure	0.888	3.52×10^{11}	7.99×10^{9}	2.27×10^{-2}
Duralumin	0.617	3.39×10^{11}	8.07×10^{9}	2.38×10^{-2}
Lead	0.247	$0.199 \quad 10^{11}$	2.74×10^{9}	1.38×10^{-1}
Iron, pure	0.160	3.52×10^{11}	3.99×10^{9}	1.13×10^{-2}
Carbon steel	0.128	3.53×10^{11}	4.03×10^{9}	1.14×10^{-2}
Cast iron	0.128	1.20×10^{11}	2.72×10^{8}	0.27×10^{-2}
Copper, pure	1.172	1.92×10^{11}	2.92×10^{9}	1.52×10^{-2}
Bronze, 90:10	0.280	1.80×10^{11}	2.88×10^{9}	1.60×10^{-2}
Brass, 60:40	0.109	1.25×10^{11}	1.54×10^{9}	1.23×10^{-2}
Magnesium	0.891	3.11×10^{11}	6.83×10^{9}	2.10×10^{-2}
Nickel, pure	0.144	3.00×10^{11}	3.42×10^{9}	1.14×10^{-2}
Conrete	0.0053	1.14×10^{11}	1.39×10^{8}	1.22×10^{-3}
Rock, Sanstone	0.00800	1.48×10^{11}	1.51×10^{8}	1.02×10^{-3}
Limestone	0.00756	2.83×10^{11}	4.32×10^{8}	1.53×10^{-3}
Granite	0.01460	2.02×10^{11}	2.25×10^{8}	1.11×10^{-3}
Glass, Plate glass	0.00432	3.16×10^{11}	6.11×10^{8}	1.93×10^{-3}
Quartz glass	0.00744	3.60×10^{11}	2.08×10^{6}	5.78×10^{-6}

(Table continued)

(Table continued)

	ρ gm/cm^3	α 1/°C	C_ϵ gm/sec^{2}cm °C	k erg/sec cm °C	ν Dimensionless
Aluminum, pure	2.7	24×10^{-6}	2.50×10^7	2.22×10^7	0.3
Duralumin	2.8	25×10^{-6}	2.59×10^7	1.60×10^7	0.3
Lead	11.3	29×10^{-6}	1.42×10^7	3.50×10^6	0.3
Iron, pure	7.88	12×10^{-6}	3.64×10^7	5.84×10^6	0.3
Carbon steel	7.85	12×10^{-6}	3.63×10^7	4.67×10^6	0.3
Cast iron	7.70	10×10^{-6}	4.20×10^7	5.36×10^6	0.3
Copper, pure	8.93	17×10^{-6}	3.38×10^7	3.97×10^7	0.3
Bronze, 90:10	8.8	18×10^{-6}	3.35×10^7	9.34×10^6	0.3
Brass, 60:40	8.5	19×10^{-6}	3.21×10^7	3.50×10^7	0.3
Magnesium	1.74	26×10^{-6}	1.83×10^7	1.63×10^7	0.3
Nickel, pure	8.8	13×10^{-6}	4.06×10^7	5.84×10^6	0.3
Concrete	2.0	12×10^{-6}	1.76×10^7	9.34×10^4	0.2
Rock, Sandstone	2.2	9×10^{-6}	1.75×10^7	1.40×10^5	0.2
Limestone	2.7	8×10^{-6}	2.16×10^7	1.63×10^5	0.2
Granite	2.7	8×10^{-6}	2.16×10^7	3.15×10^5	0.2
Glass, Plate glass	2.5	8×10^{-6}	1.89×10^7	8.16×10^4	0.22
Quartz glass	2.2	4×10^{-7}	1.57×10^7	1.17×10^5	0.22

(Table continued)

(Table continued)

	E dynes/cm^2	μ dynes/cm^2	λ dynes/cm^2	β dynes/cm^2 °C
Aluminum, pure	7.06×10^{11}	2.72×10^{11}	4.07×10^{11}	4.24×10^{7}
Duralumin	7.06×10^{11}	2.72×10^{11}	4.07×10^{11}	4.42×10^{7}
Lead	1.67×10^{11}	6.42×10^{10}	9.64×10^{10}	1.21×10^{7}
Iron, pure	2.06×10^{12}	7.93×10^{11}	1.19×10^{12}	6.18×10^{7}
Carbon steel	2.06×10^{12}	7.93×10^{11}	1.19×10^{12}	6.18×10^{7}
Cast iron	6.87×10^{11}	2.64×10^{11}	3.96×10^{11}	1.72×10^{7}
Copper, pure	1.28×10^{12}	4.91×10^{11}	7.36×10^{11}	5.42×10^{7}
Bronze, 90:10	1.18×10^{12}	4.53×10^{11}	6.79×10^{11}	5.30×10^{7}
Brass, 60:40	7.86×10^{11}	3.03×10^{11}	4.53×10^{11}	3.75×10^{7}
Magnesium	4.02×10^{11}	1.55×10^{11}	2.32×10^{11}	2.62×10^{7}
Nickel	1.96×10^{12}	7.55×10^{11}	1.13×10^{12}	6.38×10^{7}
Concrete	2.06×10^{11}	8.58×10^{10}	5.72×10^{10}	4.04×10^{6}
Rock, Sandstone	2.94×10^{11}	1.23×10^{11}	8.16×10^{10}	4.41×10^{6}
Limestone	6.87×10^{11}	2.87×10^{11}	1.91×10^{11}	9.16×10^{6}
Granite	4.91×10^{11}	2.04×10^{11}	1.36×10^{11}	6.55×10^{6}
Glass, Plate glass	6.87×10^{11}	2.82×10^{11}	2.21×10^{11}	9.81×10^{6}
Quartz glass	6.87×10^{11}	2.82×10^{11}	2.21×10^{11}	0.49×10^{6}

All the constants in the above table are either calculated based on, or directly taken from the table given in Melan and Parkus' book of thermal stresses (ref. 7, pp. 106-107). The reference temperature, T_R , is 300°K.

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ON STRAIN WAVE PROPAGATION IN THERMOELASTIC MEDIA

by

Kuang Liu Cheng

ABSTRACT

A complete solution of thermoelastic dilatational waves in an elastic, heat conducting, homogeneous, and isotropic medium has been obtained for both the steady and unsteady states. Discussions of the solution for a wide range of frequencies and various values of the coefficient of rise (or decay) have been made and the result has been applied to the explanation of the unusual seismic waves recorded from an underground atomic explosion of September 19, 1957, in Nevada. Reflection and refraction of plane waves at plane boundary and plane interface between two media have been studied. The surface waves have also been studied. The solutions of spherical and cylindrical dilatational waves have been found.