INCOMPLETE VARIABLE DESIGNS IN MULTIVARIATE EXPERIMENTS

by

Irene Patricia Monahan, B.Sc., M.S.

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

DOCTOR OF PHILOSOPHY

in

Statistics

March 1961

Blacksburg, Virginia

TABLE OF CONTENTS

CHAPTER PAGE

I. INTRODUCTION

The purpose of this investigation is the development of methods of estimation and tests of hypotheses in multivariate experiments in which a different subset of the variables under study is observed in each group of experimental units. Wilks (1932b) considered estimation of the parameters of a bivariate normal distribution when some observations on each of the variables were missing. He derived maximum likelihood estimates of the unknown parameters when the following parameters are known: (1) means, (2) means and correlation, (3) variances and correlation. Matthai (1951) and Edgett (1956) extended Wilks' method to the case of a trivariate normal population. Rao (1950) proposed a modification of Wilks' Λ criterion for tests of significance in a multivariate analysis of dispersion when observations on one of the variables are missing. He also suggested (1955) the substitution of cell averages for the missing data when only a few values are absent.

All of these procedures can be applied in situations where observations on some of the variables are missing more or less by accident. For example, Matthai discusses the problem of estimating the maximum femoral length of skeletons.

One can obtain for some skeletons measurements including and excluding spines. For others the measurement including the spine is missing. Matthai's work can also be applied to sample surveys when it is too costly or inconvenient to obtain enough measurements on a particular variable. As a further example one may think of the administration of two examinations to a group of college students over a considerable lapse of time. Some students who took the first test will have left college before the second test is given. Hence, some measurements on the second variable, that is the second test score, are missing.

In the present study we shall be concerned with experiments where variables are missing not by accident, but by design. As an example encountered frequently in educational or sociological research consider the construction of parallel test forms. One phase in the standardization of such tests is the estimation of correlations between parallel forms. If three or more such forms are required, as is frequently the case for tests to be applied on the national level, estimation of correlation coefficients would necessitate the application of all forms to a representative standardization group. The application of more than two forms to the same student may, however introduce errors, for recall, learning, or fatigue may seriously influence the results. A given student in the standardization group may receive only two or three tests, and the experiment should be designed in such a way that equal information is available on all means and variances, and also on correlations on all test forms.

As a further example, we can think of two populations such as students in large universities and students in small liberal arts colleges. By taking a random sample from each population, one might want to determine whether students in the two types of schools react in the same way on several criteria such as achievement drive, attitude toward science, and interest in research. For reasons of economy, limitations in time, or the learning or fatigue effect, it may be preferable to give only two tests to each student. If K sets of two tests are used in the study, the sample from each population would be divided at random into K equal parts. Each subdivision would receive a particular pair of tests.

The purpose of the present study will be the development of methods of estimation of parameters and tests of hypotheses on the basis of such intentionally incomplete sets of data. To facilitate the handling of rather general situations we will assume the "general linear model" for multivariate analysis,

$$E(X'M) = A\,\xi\,M$$

(see, for example, Roy (1957)) where X' is a matrix of order (N x p) which contains all observations; A, of order (N x m) is the "design matrix" that indicates which parameter affects each observation; ξ is a matrix of order (m x p), where m is the number of different parameters affecting each variable; and p is the number of different variables. The matrix M, of order (p x u), was introduced by Roy for the purpose of allowing given linear combinations of variables in the model. It is particularly useful in the present case since, by a suitable array of ones and zeros in the matrix M, we can indicate whether a particular variable is observed or not in a given group of subjects. It will be recalled that simple and multiple regression and analysis of variance and covariance are special cases of this general linear model. (For more detail see Roy (1957) or Kempthorne (1952)).

In accordance with custtomary assumptions made in these models, we will assume that the variance-covariance matrix of the elements in a given row of X' is Σ(p x p), the same for all rows, if these rows are complete. If, by the choice of M, there are blanks in a given row of X', the variance-covariance matrix of the terms in such a truncated row will be the corresponding truncation of Σ. If, for example, variables 1, 2, and 3 out of 5 variables are contained in a row of X', the variance-covariance matrix of the elements of that row will be represented by the first three rows and columns of Σ. We will have to estimate the elements of this joint Σ. Different rows in X' are assumed to be independent.

In addition to estimating the parameters, ξ and Σ, in the general linear model, we may wish to test some hypotheses on certain combinations of these parameters. Such a hypothesis is conveniently and compactly stated in the form

$$H_o: C\xi = 0$$

$$H_a: C\xi \neq 0 \quad ,$$

where C is a predetermined "hypothesis matrix", usually an array of ones, minus ones, and zeros. The usual tests for equality of treatment effects, and tests on a subset of regression weights, are special cases of this "general linear hypothesis". In Chapter IV we shall derive a likelihood ratio test for this general linear hypothesis $C\xi = 0$ where C is an arbitrary matrix.

In research studies in advertising or psychological studies of propaganda, one is often interested in comparing the effect of treatments on different socio-economic classes. If many different tests of beliefs and opinions are used in the study, it would frequently be advantageous to give no more than two tests to a given subject. Applying the results contained in this dissertation, one could compare the effect of treatments on the different socio-economic classes and also estimate test reliabilities.

We will now present a detailed example in order to clarify the notation. Let there be p variables. An observation in the j'th variable is assumed to satisfy the following model:

$$E(x_{rsj}) = \mu_j + \alpha_{rj} + \beta_{sj}$$

where α_{1j}, α_{2j} is an effect due to socio-economic class (the subscript 1 could refer to lower, the subscript 2 to upper class) on the j'th variable (e.g., attitude to a vocation as measured by a questionnaire); β_{1j}, β_{2j}, β_{3j} may be an effect due to a particular "treatment" on the j'th variable. For example, treatment 1 may refer to informative literature, treatment 2 may be a film showing, and treatment 3 may be a speech. The subscript j would denote different types of attitude measurement (questionnaire, behavior rating, projective tests, etc.). Since each of these is measured on some arbitrary scale, the effects on them would have different numerical values. The "parameter matrix" would then have the form

$$\xi = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_p \\ \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1p} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2p} \\ \beta_{11} & \beta_{12} & \cdots & \beta_{1p} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2p} \\ \beta_{31} & \beta_{32} & \cdots & \beta_{3p} \end{bmatrix}$$

It is merely an array of all possible "effects". (μ is some "general" effect which is usually included for ease of computation--it can be left out).

We may now take a sample of, say, 1800 subjects from each of the two classes. In each group, 600 will be subjected to one of the three "treatments". Now suppose that there are 4 variables in which we are interested (the above mentioned questionnaire and a projective test could be two of them). These attitude measurements may take considerable time to perform, and the treatment effect may wear off before all measurements are completed. Thus, of the 600 subjects in each subsample, 100 may receive tests 1 and 2, 100 may receive tests 1 and 3, etc., finally 100 may receive tests

3 and 4. We will say that all persons receiving tests 1 and 2 constitute a "group".

Note that a "group" by this definition contains both socio-economic classes (300 from each class) and will contain all three treatment effects, so that, in a given group, 100 subjects will come from a given socio-economic class and have been subjected to a given treatment. There will be, altogether, 600 subjects in such a "group"; one "group" differs from another only in that another pair of tests (variables) is applied to its members.

Thus, for the first group

$$\mathcal{E}\begin{bmatrix} \underline{x}_{111} & \underline{x}_{112} \\ \underline{x}_{121} & \underline{x}_{122} \\ \underline{x}_{131} & \underline{x}_{132} \\ \underline{x}_{211} & \underline{x}_{212} \\ \underline{x}_{221} & \underline{x}_{222} \\ \underline{x}_{231} & \underline{x}_{232} \end{bmatrix} = \begin{bmatrix} \underline{j} & \underline{j} & \underline{0} & \underline{j} & \underline{0} & \underline{0} \\ \underline{j} & \underline{j} & \underline{0} & \underline{0} & \underline{j} & \underline{0} \\ \underline{j} & \underline{j} & \underline{0} & \underline{0} & 0 & \underline{j} \\ \underline{j} & \underline{0} & \underline{j} & \underline{j} & \underline{0} & \underline{0} \\ \underline{j} & \underline{0} & \underline{j} & \underline{0} & \underline{j} & \underline{0} \\ \underline{j} & \underline{0} & \underline{j} & \underline{0} & \underline{0} & \underline{j} \end{bmatrix} \xi \overset{M_1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

where $\underline{x}_{rsj}$ denotes the vector of all 100 observations on test j for subjects of socio-economic class r who were subjected to treatment s. The symbol $\underline{j}$ denotes a column vector of 100 ones. ξ is the parameter matrix.

It will be noted that this is of the form $E(X'M) = A\xi M$ stated above. Also, the only difference between the first group and any other groups is in the matrix M; (for example, in the second group, the 1 in the second column of M would be in the third row); all other expressions (especially the whole matrix A) remain the same.

Suppose we wish to test the hypothesis that there is no difference between the two socio-economic classes. This can be stated as

$$[0 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0] \quad \xi = 0 \ ,$$

and C is merely the above row vector. Suppose we wish to test the hypothesis that there is no difference between the three "treatments". This can be written as

$$\begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \xi = 0 \quad ,$$

and here C is the above (2 x 6) matrix. In a similar way, any conceivable linear hypothesis on the parameters can be reduced to the general form $C\xi = 0$.

As another example, in the assessment of the value of educational films, one might want to determine the optimum amount of material that should be presented in any one film. Suppose that two films containing different quantities of information are to be tested on random samples from populations of individuals of different educational background: population one, adults with college degrees, population two, adults with high school diplomas, and population three, adults without high school diplomas. Suppose that five achievement tests are arranged in the following sets of two: (12), (23), (34), (45), and (15). Then one-fifth of the adults with college degrees, one-fifth of the adults with high school diplomas, and one-fifth of the adults without high school diplomas would receive a given pair of tests. The set of all subjects from the three populations who take a particular pair of tests would constitute a "group". Half of each group would receive a given one of the two treatments, that is, the showing of one of the films. This can be reduced to the general linear form in a manner analogous to that used in the previous example.

In the problem of standardizing five "parallel forms" of a test, one may proceed as follows. If five tests are administered in sets of two, one would obtain equal information on all variances and covariances by using the ten pairs of digits which can be formed from the first five digits. Five tests may be grouped in ten "balanced" sets of three by using the arrangements (123), (125), (145), (234), (345), (124), (134), (135), (235), and (245). With such a design, each test is administered to six different groups of students, and each pair of tests to three different groups. Thus we have "balanced incomplete sets of variables" where the array of variables (tests, in this case) bears an analogy to the array of treatments in a univariate balanced incomplete block design.

When standardized essay examinations are scored by readers, obviously each reader can not grade every examination paper. With a scheme such as the one presented above, each paper would be graded by two or three readers. By means of the methodology developed in this thesis, one could obtain estimates of reader reliability and also test whether various student populations are performing equally well.

For a physical problem where the results of this research might be applied, we could consider rocket experimentation. Because of limitations in the amount of radio transmitting equipment which a rocket can carry, we are often unable to study at once all variables in which we are interested.

With an incomplete variable design, one could obtain data at a given time on chamber pressure and temperature, in a second instant, on pressure and fuel flow rate, and in a third instant, on temperature and fuel flow rate.

If in addition to the three response variables mentioned above, one were interested in studying six other variables such as acceleration, angular deflection, etc., then the experiment could be designed so as to have twelve "groups" of recordings at a given time in which a different triplet of variables could be measured in each. Within each "group" one could investigate the effect of concomitant variables such as wind velocity, humidity, etc. By setting up a factorial design, one could obtain estimates of variance components, interactions, and means (for example, the mean temperature effect due to different fuels and fuel mixtures) and analyze the effect on each of the variables due to main factors and interactions.

In Chapter VII we shall analyze a demonstration study in which three variables are observed in three groups of two. In this artificial example we shall investigate the effect of a concomitant variable t on the three response variables. Since this analysis is discussed at some length, we might well think of some situations where it could be applied.

The concomitant variable that comes immediately to mind is of course time. We could imagine that laboratory animals are subjected to intense heat or cold for varying lengths of time and that certain physiological measurements such as blood pressure, body temperature, and respiration rate are taken immediately after the stimulus is removed. If the measurements in which the physiologist is interested can not all be taken simultaneously, then an incomplete variable design would be useful.

Or for a similar experiment in psychology, we could suppose that human subjects are exposed to intense light for 0, 1, 2,...,14 seconds and are then asked to distinguish between complementary color pairs. The experimenter would be interested in presenting three series of pairs and requiring the subject to state rapidly whether the members of each pair are the same or different. (The number of correct answers would be the respondent's score.) Since this must be done before the blinding effect of the light has worn off, there would be an advantage in presenting only two color series, rather than three, to each subject.

These examples may show the kind of situations in which such "incomplete variable" designs are useful. They occur most frequently where repetition of tests creates learning or fatigue effects, or where the time available to take measurements is not sufficient to observe all variables in which one may be interested.

In all of these experiments, we decide in advance to divide the random sample into groups of equal size in such a way that there is linkage between the groups. Any feasible experimental design (completely randomized, randomized block, factorial, etc.) may be employed, provided that the same design is used in each group. The reader should be careful to distinguish between the experimental design and the design of the variables, a phrase which we have introduced to describe the arrangement of the response variables in groups. Some selected "variable designs" are discussed in detail in Chapter V.

II. MAXIMUM LIKELIHOOD CONDITIONS

2.1 The Likelihood Function

Let a random vector $\underline{x}_i'$ (1 x p) be distributed by the multivariate normal law with expectation $\underline{\mu}_i'$ and dispersion matrix Σ, so that the probability density function of $\underline{x}_i'$ is given by

$$(2.1.1) \qquad f(\underline{x}_i')=(2\pi)^{-p/2}|\Sigma|^{-\frac{1}{2}} \exp[-\tfrac{1}{2}(\underline{x}_i'-\underline{\mu}_i')\Sigma^{-1}(\underline{x}_i-\underline{\mu}_i)] \quad .$$

If we take N independent observations each distributed according to (2.1.1), we obtain for the logarithm of the likelihood function

$$(2.1.2) \qquad L(X') = -\frac{Np}{2}\log 2\pi - \frac{N}{2}\log|\Sigma| -\tfrac{1}{2}\sum_{i=1}^{N}(\underline{x}_i'-\underline{\mu}_i')\Sigma^{-1}(\underline{x}_i-\underline{\mu}_i)$$

where X' (N x p) is the matrix of observations, whose rows are the $\underline{x}_i'$ above.

Since the trace of a scalar equals the scalar, we can write

$$\sum_{i=1}^{N}(\underline{x}_i'-\underline{\mu}_i')\Sigma^{-1}(\underline{x}_i-\underline{\mu}_i)=\sum_{i=1}^{N}\mathrm{tr}(\underline{x}_i'-\underline{\mu}_i')\Sigma^{-1}(\underline{x}_i-\underline{\mu}_i) \quad .$$

Now

$$\sum_{i=1}^{N}\mathrm{tr}(\underline{x}_i'-\underline{\mu}_i')\Sigma^{-1}(\underline{x}_i-\underline{\mu}_i)=\sum_{i=1}^{N}\mathrm{tr}\Sigma^{-1}(\underline{x}_i-\underline{\mu}_i)(\underline{x}_i'-\underline{\mu}_i') \quad ,$$

since tr AB = tr BA. Hence the last term of (2.1.2) can be written

$$-\tfrac{1}{2}\sum_{i=1}^{N}(\underline{x}_i'-\underline{\mu}_i')\Sigma^{-1}(\underline{x}_i-\underline{\mu}_i) =$$

$$-\tfrac{1}{2}\ \mathrm{tr}\ \Sigma^{-1}\sum_{i=1}^{N}(\underline{x}_i-\underline{\mu}_i)(\underline{x}_i'-\underline{\mu}_i') \quad .$$

Now if we denote the matrix of expectations by μ (with rows μ_i'), we can write

$$\sum_{i=1}^{N}(\underline{x}_i-\underline{\mu}_i)(\underline{x}_i'-\underline{\mu}_i') = (X-\mu)(X'-\mu') \quad .$$

This matrix product written in detail is

$$\begin{bmatrix} x_{11}-\mu_1 & x_{21}-\mu_1 & \cdots & x_{N1}-\mu_1 \\ x_{12}-\mu_2 & x_{22}-\mu_2 & \cdots & \cdot \\ \vdots & \vdots & & \vdots \\ x_{1p}-\mu_p & x_{2p}-\mu_p & \cdots & x_{Np}-\mu_p \end{bmatrix} \begin{bmatrix} x_{11}-\mu_1 & x_{12}-\mu_2 & \cdots & x_{1p}-\mu_p \\ x_{21}-\mu_1 & x_{22}-\mu_2 & \cdots & \cdot \\ \vdots & \vdots & & \vdots \\ x_{N1}-\mu_1 & x_{N2}-\mu_2 & \cdots & x_{Np}-\mu_p \end{bmatrix} =$$

$$\begin{bmatrix} \sum_{i=1}^{N}(x_{i1}-\mu_1)^2 & \sum_{i=1}^{N}(x_{i1}-\mu_1)(x_{i2}-\mu_2) \cdots & \sum_{i=1}^{N}(x_{i1}-\mu_1)(x_{ip}-\mu_p) \\ \sum_{i=1}^{N}(x_{i2}-\mu_2)(x_{i1}-\mu_1) & \sum_{i=1}^{N}(x_{i2}-\mu_2)^2 \quad \cdots & \cdot \\ \vdots & \vdots & \vdots \\ \sum_{i=1}^{N}(x_{ip}-\mu_p)(x_{i1}-\mu_1) & \sum_{i=1}^{N}(x_{ip}-\mu_p)(x_{i2}-\mu_2) \cdots & \sum_{i=1}^{N}(x_{ip}-\mu_p)^2 \end{bmatrix}$$

which is the sum over all N observations of the vector product

$$(\underline{x}_i-\underline{\mu}_i)(\underline{x}_i'-\underline{\mu}_i') = \begin{bmatrix} x_{i1}-\mu_1 \\ x_{i2}-\mu_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{ip}-\mu_p \end{bmatrix} [x_{i1}-\mu_1, x_{i2}-\mu_2 \ldots x_{ip}-\mu_p]$$

Thus we can write the logarithm of the likelihood function in the form

$$(2.1.3) \quad L(X') = -\frac{Np}{2}\log 2\pi - \frac{N}{2}\log|\Sigma| - \tfrac{1}{2}\operatorname{tr}\Sigma^{-1}(X-\mu)(X'-\mu') \quad .$$

Here and in the sequel we follow the notation used in Roy (1957).

2.2 Lemmas

We shall need the following lemmas for differentiation of the likelihood function and subsequent derivations. <u>Definition</u>: The derivative of a scalar quantity "with respect to a matrix" will be understood to mean the matrix which is obtained if we differentiate the scalar with respect to each term of the matrix and write the result in an array corresponding to that in the original matrix. With this understanding we have, for an arbitrary A and an arbitrary X subject to existence of the following expressions:

Lemma 1. $\dfrac{\partial \operatorname{tr} AXA'}{\partial A} = AX' + AX$.

Proof. $$\frac{\partial \operatorname{tr} AXA'}{\partial a_{ij}} = \frac{\partial}{\partial a_{ij}} \sum_{k=1}^{m} \sum_{\gamma=1}^{n} \sum_{\delta=1}^{n} a_{k\gamma} x_{\gamma\delta} a_{k\delta}$$

$$= \sum_{\delta=1}^{n} x_{j\delta} a_{i\delta} + \sum_{\gamma=1}^{n} a_{i\gamma} x_{\gamma j}$$

$$= (AX')_{ij} + (AX)_{ij} \quad .$$

The lemma follows. In particular, if X is symmetric

$$\frac{\partial \operatorname{tr} AXA'}{\partial A} = 2AX \quad .$$

Lemma 2. $\dfrac{\partial \log |A|}{\partial A} = (A^{-1})'$.

Proof. $$|A| = \sum_{\ell=1}^{n} a_{i\ell} \operatorname{cof}(a_{i\ell}) \quad .$$

Hence

$$\frac{\partial |A|}{\partial a_{ij}} = \operatorname{cof} a_{ij} \quad .$$

Now

$$\frac{\partial \log |A|}{\partial a_{ij}} = \frac{1}{|A|} \operatorname{cof}(a_{ij}) = a^{ji}$$

and the lemma is proved.

This lemma states that if we differentiate log $|A|$ with respect to the element in the i'th row and j'th column in A, the result will be a^{ji}. Care must be taken to employ appropriate notation if a symmetric matrix is involved. As long as we understand a_{ij} to be the element in the i'th row and j'th column, regardless of any functional relationship that a_{ij} may have to one or more other elements in the matrix, the above lemma can be employed. However if we understand a_{ij} in a symmetric matrix to mean that distinct element which occurs in the i'th row and j'th column and in the j'th row and i'th column, we may write $a_{ij} = a_{ji} = a$ and then

$$(2.2.1) \qquad \frac{\partial \log|A|}{\partial a} = \frac{\partial \log|A|}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial a} + \frac{\partial \log|A|}{\partial a_{ji}} \frac{\partial a_{ji}}{\partial a} \quad .$$

Hence, using this form of the partial derivative, we may write

$$(2.2.2) \qquad \begin{aligned} \frac{\partial \log|A|}{\partial a} &= 2a^{ij} \qquad && i < j \\ &= a^{ii} && i = j \end{aligned}$$

where $a_{ij} = a_{ji} = a$.

<u>Definition</u>: For the purpose of the following lemma, differentiation of a matrix with respect to a scalar will be understood to mean the matrix which is obtained if we differentiate each term of the original matrix with respect to the given scalar quantity and write the resulting form in the same manner as the original matrix. (Some authors prefer to write the result in transposed form. We will not follow that practice here.) With this understanding we have

Lemma 3. $$\frac{\partial A^{-1}}{\partial x} = - A^{-1} \frac{\partial A}{\partial x} A^{-1}$$

Proof can be found in elementary textbooks on matrices such as, for example, Browne (1958), the simplest form being

$$0 = \frac{\partial AA^{-1}}{\partial x} = \frac{\partial A}{\partial x} A^{-1} + A \frac{\partial A^{-1}}{\partial x}$$

from which the lemma follows.

Lemma 4. $$\frac{\partial \mathrm{tr} A^{-1}X}{\partial A} = - (A^{-1}XA^{-1})' \quad .$$

Proof. $$\frac{\partial \mathrm{tr} A^{-1}X}{\partial a^{ij}} = \frac{\partial}{\partial a^{ij}} \sum_{k=1}^{n} \sum_{\ell=1}^{n} a^{k\ell} x_{\ell k} = x_{ji} \quad .$$

By lemma 3 we have

$$\frac{\partial a^{ij}}{\partial a_{k\ell}} = -(A^{-1} \frac{\partial A}{\partial a_{k\ell}} A^{-1})_{ij} \quad .$$

Now

$$\frac{\partial \mathrm{tr} A^{-1}X}{\partial a_{k\ell}} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \mathrm{tr} A^{-1}X}{\partial a^{ij}} \frac{\partial a^{ij}}{\partial a_{k\ell}}$$

$$= -\sum_{i=1}^{n} \sum_{j=1}^{n} (A^{-1} \frac{\partial A}{\partial a_{k\ell}} A^{-1})_{ij} x_{ji}$$

$$= - \mathrm{tr}\, A^{-1} \frac{\partial A}{\partial a_{k\ell}} A^{-1} X$$

$$= - \mathrm{tr} \frac{\partial A}{\partial a_{k\ell}} A^{-1}XA^{-1} \quad .$$

Now if we understand $a_{k\ell}$ to mean the element in the k'th row and ℓ'th column of A, regardless of possible functional relationship between the elements of the matrix, we find that

$\frac{\partial A}{\partial a_{k\ell}}$ has a one in the $(k\ell)$ position and zeros elsewhere. The only contribution this element makes to the trace is its product with the element in the ℓ'th row and k'th column of $A^{-1}XA^{-1}$. Hence

$$-\mathrm{tr}\,\frac{\partial A}{\partial a_{k\ell}}\,A^{-1}XA^{-1} = (A^{-1}XA^{-1})_{\ell k}$$

and the lemma is proved.

If both A and X are symmetric and if a_{ij} denotes that quantity which occurs both in the i'th row and j'th column and in the j'th row and i'th column, we can write the lemma as

$$\begin{aligned} \frac{\partial \mathrm{tr} A^{-1}X}{\partial a_{ij}} &= -\,2(A^{-1}XA^{-1})_{ij} \qquad & i < j \\ &= -(A^{-1}XA^{-1})_{ii} \qquad & i = j \end{aligned}$$

in complete analogy to the corollary to lemma 2.

Lemma 5. $\quad \frac{\partial \mathrm{tr} XA}{\partial X} = A'$.

Proof. $$\frac{\partial \mathrm{tr} XA}{\partial X} = \frac{\partial \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\, a_{ji}}{\partial x_{k\ell}}$$

$$= a_{\ell k} = (A')_{k\ell}$$

and the lemma is proved.

All these lemmas represent familiar and trivial results. They were stated here merely in an attempt to unify notation throughout this dissertation.

2.3 Maximum Likelihood Estimates

The general linear model in multivariate analysis is

$$(2.3.1) \qquad E(X') = A\xi$$

where $A(N \times m)$ is a design matrix and $\xi(m \times p)$ is a matrix of unknown parameters. (The reader will recall that the design matrix in multivariate analysis is exactly the same as that in univariate analysis. The parameter matrix in the multivariate case differs form that in the univariate case in that it has p columns corresponding to the p variables being measured.) The variance of the j'th row vector of X' is Σ. Hence the logarithm of the likelihood of the observations for this model may be written

$$(2.3.2) \qquad L(X') = -\frac{Np}{2}\log 2\pi - \frac{N}{2}\log|\Sigma| - \tfrac{1}{2}\operatorname{tr}\Sigma^{-1}(X-\xi'A')(X'-A\xi) \quad .$$

Expanding the last term of (2.3.2), we obtain

$$\begin{aligned}(2.3.3) \qquad & -\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}(X-\xi'A')(X'-A\xi) \\ &= -\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}(XX'-XA\xi-\xi'A'X'+\xi'A'A\xi) \\ &= -\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}XX'+\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}XA(A'A)^{-1}A'X' \\ &\quad -\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}XA(A'A)^{-1}A'X'+\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}XA\xi \\ &\quad +\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}\xi'A'X'-\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}\xi'A'A\xi \quad ,\end{aligned}$$

where the second and third terms were added and subtracted,

$$(2.3.4) \qquad = -\tfrac{1}{2}\,\mathrm{tr}\,\Sigma^{-1}\,[XX' - XA(A'A)^{-1}A'X']$$

$$-\tfrac{1}{2}\,\mathrm{tr}\,\Sigma^{-1}\,[XA(A'A)^{-1} - \xi'](A'A)[(A'A)^{-1}A'X' - \xi]$$

Only the last term of (2.3.4) involves ξ. Thus the maximum likelihood estimate of ξ is given by

$$(2.3.5) \qquad \hat{\xi} = (A'A)^{-1}A'X'$$

since all the terms of the logarithm of the likelihood are negative and this solution makes the last term equal to zero. Thus, the log-likelihood function in Ω, with ξ replaced by $\hat{\xi}$, is

$$(2.3.6) \qquad L(\Omega|\xi=\hat{\xi}) = -\frac{Np}{2}\log 2\pi - \frac{N}{2}\log|\Sigma|$$

$$-\tfrac{1}{2}\,\mathrm{tr}\,\Sigma^{-1}\,[XX' - XA(A'A)^{-1}A'X']$$

Now let

$$(2.3.7) \qquad E = XX' - XA(A'A)^{-1}A'X'$$

The symbol E has been used for this matrix since it can be seen that the diagonal elements represent the "sums of squares due to error" for the j'th variable in a univariate analysis (j=1,2,...,p), see, e.g., Heck (1958). The likelihood function can then be written

$$(2.3.8) \qquad L(\Omega|\xi=\hat{\xi}) = -\frac{Np}{2}\log 2\pi - \frac{N}{2}\log|\Sigma| - \tfrac{1}{2}\,\mathrm{tr}\,\Sigma^{-1}E \quad .$$

The maximum likelihood estimate for Σ can be found by setting

$$\frac{\partial L(\Omega|\xi=\hat{\xi})}{\partial\Sigma^{-1}} = 0 \quad .$$

By lemmas 2 and 5,

$$\frac{\partial \log|\Sigma|}{\partial\Sigma^{-1}} = - \frac{\partial \log|\Sigma^{-1}|}{\partial\Sigma^{-1}} = - \Sigma$$

and

$$\frac{\partial \operatorname{tr}\Sigma^{-1}E}{\partial\Sigma^{-1}} = E \quad .$$

Hence

$$\frac{\partial L(\Omega|\xi=\hat{\xi})}{\partial\Sigma^{-1}} = \frac{N}{2}\Sigma - \tfrac{1}{2} E \quad .$$

Setting this expression equal to zero we obtain for the maximum likelihood estimate of the dispersion matrix

$$(2.3.9) \qquad \hat{\Sigma} = \frac{1}{N} E$$

which is of course an exact analogue for the univariate maximum likelihood estimate of variance. The maximum of the log-likelihood function in Ω is then

$$(2.3.10) \qquad \log L(\hat{\Omega}) = - \frac{Np}{2} \log 2\pi - \frac{N}{2} \log |E| + \frac{Np}{2} \log N - \frac{Np}{2} \quad .$$

2.4 The Likelihood Function for Incomplete Variable Designs

Now let us generalize this approach to the situation where the observations can be subdivided into K groups each consisting of n_i observations. We shall assume however that the parameter matrix ξ is the same for all K groups and that the dispersion matrix Σ is the same for each observation vector. The matrix X' can be thought of as partitioned into K submatrices X_i' in the following manner.

(2.4.1)
$$X' = \begin{bmatrix} x_{11}^{(1)} & x_{12}^{(1)} & \cdots & x_{1p}^{(1)} \\ x_{21}^{(1)} & x_{22}^{(1)} & \cdots & x_{2p}^{(1)} \\ \vdots & \vdots & & \vdots \\ x_{n_1 1}^{(1)} & x_{n_1 2}^{(1)} & \cdots & x_{n_1 p}^{(1)} \\ \hdashline x_{11}^{(2)} & x_{12}^{(2)} & \cdots & x_{1p}^{(2)} \\ x_{21}^{(2)} & x_{22}^{(2)} & \cdots & x_{2p}^{(2)} \\ \vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots \\ x_{n_2 1}^{(2)} & x_{n_2 2}^{(2)} & \cdots & x_{n_2 p}^{(2)} \\ \hdashline \vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots \\ \hdashline x_{11}^{(K)} & x_{12}^{(K)} & \cdots & x_{1p}^{(K)} \\ x_{21}^{(K)} & x_{22}^{(K)} & \cdots & x_{2p}^{(K)} \\ \vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots \\ x_{n_K 1}^{(K)} & x_{n_K 2}^{(K)} & \cdots & x_{n_K p}^{(K)} \end{bmatrix}$$

The superscript for each element in X' refers to the group number; the first subscript, to the observation number within the group; the second subscript, to the variable number. The general linear model for the i'th group is

$$E(X_i') = A_i\xi \tag{2.4.2}$$

where $A_i(n_i \times m)$ is the design matrix for the i'th group. Substituting $A_i\xi$ for $A\xi$ and n_i for n in (2.3.2), we obtain the log-likelihood function for the i'th group

$$L(X_i') = -\frac{n_i p}{2}\log 2\pi - \frac{n_i}{2}\log|\Sigma| - \tfrac{1}{2}\,\mathrm{tr}\,\Sigma^{-1}(X_i-\xi' A_i')(X_i'-A_i\xi) \quad . \tag{2.4.3}$$

Let us suppose however that in the i'th group the actual observations are not X_i' but $Y_i'=X_i'M_i$ where M_i (of order $p \times u$, $u \leq p$) is usually a matrix of ones and zeros which selects from the p variables in the X_i' matrix those variables which are to be measured in the i'th group. In this dissertation we are considering the situation where only a subset of size u of the variables is observable in the i'th group. For example, M_i for group one might be of the form

$$M_1' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & \ldots & 0 \end{bmatrix} .$$

If this matrix is substituted in $Y_1' = X_1'M_1$ where X_1' is the first submatrix of (2.4.1), we have for the matrix of actual observations in the first group

$$\begin{bmatrix} x_{11}^{(1)} & x_{12}^{(1)} & x_{14}^{(1)} \\ x_{21}^{(1)} & x_{22}^{(1)} & x_{24}^{(1)} \\ \vdots & \vdots & \vdots \\ x_{n_1 1}^{(1)} & x_{n_1 2}^{(1)} & x_{n_1 4}^{(1)} \end{bmatrix} .$$

The matrix Y_i consists of certain columns of the original matrix X. In the sequel we will denote the elements of this matrix by $y_{ij}^{(K)}$, and it should be noted that to each $y_{ij}^{(K)}$ there corresponds an element of the X matrix which may however not have the same subscripts.

In this example we assumed that only variables 1, 2, and 4 are observed in the first group. The general argument however allows for an arbitrary matrix M_i and to this extent it is equivalent to the post-matrix M used by Roy (1957) in his extension of the general linear hypothesis.

Since in the i'th group

(2.4.4) $$Y_i' = X_i'M_i$$

the matrix of expectations for the observations in the i'th group is

(2.4.5) $$E(Y_i') = E(X_i')M_i = A_i\xi M_i$$

and the covariance matrix for each row vector is

(2.4.6) $$\mathrm{var}(\underline{y}_{ij}') = M_i' \ \mathrm{var}(\underline{x}_{ij}')M_i = M_i'\Sigma M_i \quad .$$

The covariance matrices are the same for all observation row vectors within the i'th group, but they differ from group to group since in general the M_i are all different. Substituting $A_i\xi M_i$ for $A_i\xi$, $M_i'\Sigma M_i$ for Σ, and u for p in (2.4.3), we obtain the log-likelihood function for Y_i'.

(2.4.7) $$L(Y_i') = -\frac{n_i u}{2}\log 2\pi - \frac{n_i}{2}\log|M_i'\Sigma M_i| - \tfrac{1}{2}\,\mathrm{tr}\,(M_i'\Sigma M_i)^{-1}(Y_i-M_i'\xi'A_i')(Y_i'-A_i\xi M_i) \quad .$$

The log-likelihood function for the whole sample of $N = \sum_{i=1}^{K} n_i$ observations

(2.4.8) $$L(Y') = -\frac{Nu}{2}\log 2\pi - \tfrac{1}{2}\sum_{i=1}^{K} n_i \log|M_i'\Sigma M_i| - \tfrac{1}{2}\,\mathrm{tr}\sum_{i=1}^{K}(M_i'\Sigma M_i)^{-1}(Y_i-M_i'\xi'A_i')(Y_i'-A_i\xi M_i)$$

is obtained by summing over the K groups. Each M_i is of the same order (p x u) so that it is assumed that the same number of variables is observed in each group.

In the next section we shall consider the special case where the design matrix and the number of sample observations is the same for all K groups, that is, where $A_i=A$, $n_i=n$, and $N=Kn$. The variable design matrices M_i, however, will still be assumed to differ from group to group.

2.5 Maximum Likelihood Equations for ξ and Σ

To find the maximum likelihood estimate of the matrix ξ, we shall equate to zero the partial derivative of (2.4.8) with respect to ξ. First let

$$(2.5.1) \qquad U_i = M_i'\Sigma M_i$$

and let

$$(2.5.2) \qquad Q_i = U_i^{-1}(Y_i-M_i'\xi'A_i')(Y_i'-A_i\xi M_i) \quad .$$

Now by the chain rule for differentiation we have

$$(2.5.3) \qquad \frac{\partial \mathrm{tr}Q_i}{\partial \xi_{\nu\delta}} = \sum_{\beta=1}^{n}\sum_{\alpha=1}^{u} \frac{\partial \mathrm{tr}Q_i}{\partial(Y_i'-A_i\xi M_i)_{\beta\alpha}} \; \frac{\partial(Y_i'-A_i\xi M_i)_{\beta\alpha}}{\partial \xi_{\nu\delta}} \quad .$$

Using lemma 1 from section 2.2 we have

$$(2.5.4) \qquad \frac{\partial \mathrm{tr}Q_i}{\partial(Y_i'-A_i\xi M_i)} = 2(Y_i'-A_i\xi M_i)U_i^{-1}$$

and hence

$$(2.5.5) \qquad \frac{\partial \operatorname{tr} Q_i}{\partial (Y_i' - A_i \xi M_i)_{\ell\alpha}} = 2 \sum_{\beta=1}^{u} y_{\ell\beta}^{(i)} \, u_{(i)}^{\beta\alpha} - 2 \sum_{\mu=1}^{m} \sum_{\gamma=1}^{p} \sum_{\beta=1}^{u} a_{\ell\mu}^{(i)} \xi_{\mu\gamma} m_{\gamma\beta}^{(i)} u_{(i)}^{\beta\alpha} \quad .$$

In this and in ensuing formulas, subscripts denote elements of a matrix and superscripts denote elements of the inverse. If the matrix itself has a subscript, this subscript will appear in the element as a superscript in parentheses and in the element of an inverse as a subscript in parentheses. For instance, the element in the ℓ'th row and α'th column of the matrix M_i is denoted by $m_{\ell\alpha}^{(i)}$; the element in the ℓ'th row and α'th column of U_i^{-1}, by $u_{(i)}^{\ell\alpha}$. An exception occurs in the observation matrices Y. Here $y_{\ell\alpha}$ stands for the element in the ℓ'th row and α'th column of Y'. The inconsistency is due to the usage introduced by Roy (1957) who defines the model as $E(Y') = A\xi M$.

Next we obtain the second factor in the product in (2.5.3), that is

$$(2.5.6) \qquad \frac{\partial (Y_i' - A_i \xi M_i)_{\ell\alpha}}{\partial \xi_{\nu\delta}} = - \frac{\partial (A_i \xi M_i)_{\ell\alpha}}{\partial \xi_{\nu\delta}} = - \frac{\partial}{\partial \xi_{\nu\delta}} \sum_{\gamma=1}^{p} \sum_{\mu=1}^{m} a_{\ell\mu}^{(i)} \xi_{\mu\gamma} m_{\gamma\alpha}^{(i)} = - a_{\ell\nu}^{(i)} m_{\delta\alpha}^{(i)} \quad .$$

Substituting (2.5.5) and (2.5.6) into (2.5.3) we obtain

(2.5.7)
$$\frac{\partial \mathrm{tr} Q_i}{\partial \xi_{\nu\delta}} = -2\sum_{\ell=1}^{n}\sum_{\alpha=1}^{u}\Big[\sum_{\beta=1}^{u} y_{\ell\beta}^{(i)} u_{(i)}^{\beta\alpha} a_{\ell\nu}^{(i)} m_{\delta\alpha}^{(i)} - \sum_{\mu=1}^{m}\sum_{\gamma=1}^{p}\sum_{\beta=1}^{u} a_{\ell\mu}^{(i)} \xi_{\mu\gamma} m_{\gamma\beta}^{(i)} a_{\ell\nu}^{(i)} m_{\delta\alpha}^{(i)} u_{(i)}^{\beta\alpha}\Big]$$

$$= -2\sum_{\ell=1}^{n}\sum_{\alpha=1}^{u}\Big[\sum_{\beta=1}^{u} a_{\ell\nu}^{(i)} y_{\ell\beta}^{(i)} u_{(i)}^{\beta\alpha} m_{\delta\alpha}^{(i)} - \sum_{\mu=1}^{m}\sum_{\gamma=1}^{p}\sum_{\beta=1}^{u} a_{\ell\nu}^{(i)} a_{\ell\mu}^{(i)} \xi_{\mu\gamma} m_{\gamma\beta}^{(i)} u_{(i)}^{\beta\alpha} m_{\delta\alpha}^{(i)}\Big]$$

$$= -2[A_i' Y_i' U_i^{-1} M_i']_{\nu\delta} + 2[A_i' A_i \xi M_i U_i^{-1} M_i']_{\nu\delta} \quad .$$

Summing over the groups, we have

(2.5.8)
$$\frac{\partial \mathrm{tr} \sum_{i=1}^{K} Q_i}{\partial \xi_{\nu\delta}} = -2\sum_{i=1}^{K}(A_i' Y_i' U_i^{-1} M_i')_{\nu\delta} + 2\sum_{i=1}^{K}(A_i' A_i \xi M_i U_i^{-1} M_i')_{\nu\delta} \quad .$$

Now $\sum_{i=1}^{K} Q_i$ is the only factor in (2.4.8) which involves ξ. To obtain an equation system for $\hat{\xi}$, the maximum likelihood estimate of ξ, we must equate (2.5.8) to zero and replace ξ by $\hat{\xi}$.

Any solution, $\hat{\xi}$, which satisfies this equation system, will produce a stationary value of the likelihood function. At this time no attempt has been made to determine under what conditions the solution will produce a maximum. The equation system is so complex that an explicit solution can only be determined by numerical iterative techniques to be described later. A first guess to start the iterative method will be based upon a heuristic method which will approximate a maximum likelihood solution. If the iterative process converges it is to be expected that the resulting $\hat{\xi}$ actually produces a maximum, for two reasons. First, the initial guess is close to the maximum, and second, all known iterative methods on matrices converge to an extremum, rather than to one of the saddle points, provided that the characteristic roots of certain matrix expressions are different. What these complicated discriminant matrices are has not been determined. Thus, the claim that the iterative procedure will produce the maximum likelihood estimates is, mathematically speaking, a conjecture, just as in the case of many other maximum likelihood estimates which are obtained by iterative methods (e.g., the Probit regression analysis).

It should be noted that U_i involves the unknown elements of Σ which we will estimate in the sequel and replace by $\hat{\Sigma}$. Thus U_i will be replaced by $\hat{U}_i$. We then have the first set of equations which the maximum likelihood estimates of ξ and Σ must satisfy,

$$\sum_{i=1}^{K} A_i'A_i\hat{\xi}M_i\hat{U}_i^{-1}M_i' = \sum_{i=1}^{K} A_iY_i'\hat{U}_i^{-1}M_i' \quad . \tag{2.5.9}$$

If for simplification we assume that the experimental design matrix is the same for each group, that is,

$$A_i = A \quad (\text{and } n_i = n) \tag{2.5.10}$$

we have

$$A'A\hat{\xi}\sum_{i=1}^{K} M_i\hat{U}_i^{-1}M_i' = A'\sum_{i=1}^{K} Y_i'\hat{U}_i^{-1}M_i' \tag{2.5.11}$$

and hence

$$\hat{\xi} = (A'A)^{-1}A'\sum_{i=1}^{K} Y_i'\hat{U}_i^{-1}M_i'\Big(\sum_{i=1}^{K} M_i\hat{U}_i^{-1}M_i'\Big)^{-1} \quad . \tag{2.5.12}$$

We assume here that $A'A$ is non-singular, that is that the rank of A is equal to the number of columns in A. If $A'A$ is singular, the customary methods of constraints or elimination or addition of columns in A may be employed to obtain a non-singular matrix which can be used in the above formula. These methods can be found in standard textbooks on design such as Kempthorne (1952), Rao (1952), and Roy (1957).

We cannot however solve (2.5.12) until after we have a maximum likelihood estimate of Σ, which we will now proceed to find. In the following derivations we will understand differentiation with respect to $u_{k\ell}$, $\sigma_{\gamma\delta}$, etc. to imply differentiation with respect to an element of the matrices concerned, regardless of functional relations between the elements in symmetric matrices. The same problem was solved in terms of the corollaries to lemmas 2 and 4, where $u_{k\ell}$ stands for the elements in the k'th row and ℓ'th column and in the ℓ'th row and k'th column of U. The final matrix equations for the estimation of ξ and Σ are of course the same regardless of which of the two alternative definitions (of differentiation of a symmetric matrix with respect to a non-diagonal element) is used in the derivation.

Now to find the maximum likelihood estimate for Σ, we shall need to differentiate the last two terms of (2.4.8) with respect to Σ. To find the derivative of the next to the last term, we again employ the chain rule for differentiation of a function of several variables which in turn are functions of several variables. First we have

$$\text{(2.5.13)} \qquad \frac{\partial \log|U_i|}{\partial \sigma_{\gamma\delta}} = \sum_{k=1}^{u} \sum_{\ell=1}^{u} \frac{\partial \log|U_i|}{\partial u_{k\ell}^{(i)}} \frac{\partial u_{k\ell}^{(i)}}{\partial \sigma_{\gamma\delta}} \quad .$$

From lemma 2 we have

$$\frac{\partial \log|U_i|}{\partial u_{k\ell}^{(i)}} = u_{(i)}^{k\ell} \tag{2.5.14}$$

since U_i is symmetric. Now

$$u_{k\ell}^{(i)} = \sum_{\alpha=1}^{p} \sum_{\beta=1}^{p} m_{\alpha k}^{(i)} \sigma_{\alpha\beta} m_{\beta\ell}^{(i)} \tag{2.5.15}$$

and hence

$$\frac{\partial u_{k\ell}^{(i)}}{\partial \sigma_{\gamma\delta}} = m_{\gamma k}^{(i)} m_{\delta\ell}^{(i)} \quad . \tag{2.5.16}$$

Substituting (2.5.14) and (2.5.16) into (2.5.13), we obtain

$$\frac{\partial \log|U_i|}{\partial \sigma_{\gamma\delta}} = \sum_{k=1}^{u} \sum_{\ell=1}^{u} u_{(i)}^{k\ell} m_{\gamma k}^{(i)} m_{\delta\ell}^{(i)} \tag{2.5.17}$$

$$= (M_i U_i^{-1} M_i')_{\gamma\delta} \quad .$$

Therefore summing over the K groups, we have

$$\frac{\partial \sum_{i=1}^{K} \log|U_i|}{\partial \Sigma} = \sum_{i=1}^{K} M_i U_i^{-1} M_i' \tag{2.5.18}$$

where

$$U_i = M_i' \Sigma M_i \quad .$$

Now consider the last term of the likelihood equation and let

$$(2.5.19) \qquad Z_i = (Y_i - M_i'\xi' A_i')(Y_i' - A_i \xi M_i) \quad .$$

From lemma 4 in section 2.2 we have

$$\frac{\partial \operatorname{tr} U_i^{-1} Z_i}{\partial U_i} = - U_i^{-1} Z_i U_i^{-1}$$

If we set

$$(2.5.20) \qquad B_i = U_i^{-1} Z_i U_i^{-1}$$

we can write

$$(2.5.21) \qquad \frac{\partial \operatorname{tr} U_i^{-1} Z_i}{\partial u_{k\ell}^{(i)}} = - b_{k\ell}^{(i)} \quad .$$

By the chain rule for differentiation, we have

$$(2.5.22) \qquad \frac{\partial \operatorname{tr} U_i^{-1} Z_i}{\partial \sigma_{\gamma\delta}} = \sum_{k=1}^{u} \sum_{\ell=1}^{u} \frac{\partial \operatorname{tr} U_i^{-1} Z_i}{\partial u_{k\ell}^{(i)}} \frac{\partial u_{k\ell}^{(i)}}{\partial \sigma_{\gamma\delta}} \quad .$$

Substituting (2.5.16) and (2.5.21) in (2.5.22), we obtain

$$\frac{\partial \operatorname{tr} U_i^{-1} Z_i}{\partial \sigma_{\gamma\delta}} = - \sum_{k=1}^{u} \sum_{\ell=1}^{u} b_{k\ell}^{(i)} m_{\gamma k}^{(i)} m_{\delta \ell}^{(i)}$$

$$= -(M_i B_i M_i')_{\gamma\delta}$$

Hence, summing over the K groups, we have

$$(2.5.23) \qquad \frac{\partial \operatorname{tr} \sum_{i=1}^{K} U_i^{-1} Z_i}{\partial \Sigma} = - \sum_{i=1}^{K} M_i B_i M_i'$$

where Z_i and B_i are given in equations (2.5.19) and (2.5.20). From (2.5.18) and (2.5.23) we can write the derivative of the log-likelihood equation (2.5.8).

$$\frac{\partial L(Y')}{\partial \Sigma} = -\frac{n}{2}\sum_{i=1}^{K} M_i U_i^{-1} M_i' + \tfrac{1}{2}\sum_{i=1}^{K} M_i B_i M_i' \quad . \tag{2.5.24}$$

Equating (2.5.24) to zero and capping matrices which involve parameters we obtain the second relation for the estimation of ξ and Σ,

$$n\sum_{i=1}^{K} M_i \hat{U}_i^{-1} M_i' = \sum_{i=1}^{K} M_i \hat{B}_i M_i' \quad . \tag{2.5.25}$$

We will substitute for $\hat{\xi}$ (which is implied in $\hat{B}_i$ in (2.5.25)) the expression in (2.5.12). Then, introducing the following notation

$$\hat{U}_i = M_i' \hat{\Sigma} M_i \tag{2.5.26}$$

$$\hat{V} = \sum_{i=1}^{K} M_i \hat{U}_i^{-1} M_i' \tag{2.5.27}$$

$$\hat{W} = \sum_{i=1}^{K} Y_i' \hat{U}_i^{-1} M_i' \tag{2.5.28}$$

and

$$\hat{P}_i = Y_i - M_i' \hat{V}^{-1} \hat{W}' A(A'A)^{-1} A' \tag{2.5.29}$$

we obtain the following function, $\phi(\hat{\Sigma})$, which the maximum likelihood estimates of Σ must satisfy

$$\phi(\hat{\Sigma}) = \hat{V} - \frac{1}{n} \sum_{i=1}^{K} M_i \hat{U}_i^{-1} \hat{P}_i \hat{P}_i' \hat{U}_i^{-1} M_i' = 0 \quad . \tag{2.5.30}$$

Note that no element of $\hat{\xi}$ is present in (2.5.30).

III. SOLUTION FOR THE MAXIMUM LIKELIHOOD ESTIMATE OF DISPERSION

3.1 Development of Iterative Techniques

Our next problem is to obtain $\hat{\Sigma}$ from the matrix equation (2.5.30). Since $\hat{\Sigma}$ and $\phi(\hat{\Sigma})$ are symmetric matrices of order p, each contains p(p+1)/2 distinct elements. Hence (2.5.30) is actually a system of p(p+1)/2 distinct equations in p(p+1)/2 unknowns. For the solution of these equations, we use a method which is found in most numerical analysis textbooks and which is a generalization of the familiar Newton method for the solution of one equation in one unknown. According to the Newton formula, the first approximation X_1 for the root of the equation $f(x) = 0$ is

$$x_1 = x_o - f(x_o)/f'(x_o) \tag{3.1.1}$$

where x_o is an initial trial value for the solution. The analogous formula for the solution of two equations $f(x,y)=0$ and $g(x,y)=0$ is

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_o \\ y_o \end{bmatrix} - \begin{bmatrix} \frac{\partial f}{\partial x}\Big|_{x_o,y_o} & \frac{\partial f}{\partial y}\Big|_{x_o,y_o} \\ \frac{\partial g}{\partial x}\Big|_{x_o,y_o} & \frac{\partial g}{\partial y}\Big|_{x_o,y_o} \end{bmatrix}^{-1} \begin{bmatrix} f(x_o,y_o) \\ g(x_o,y_o) \end{bmatrix} . \tag{3.1.2}$$

This is of course obtained from (3.1.1) by replacing the scalars x_1, x_o, and $f(x_o)$ by the corresponding vectors for the two variable case and also replacing the reciprocal of the derivative of the function $f(x)$ by the inverse of the matrix of partial derivatives of the functions $f(x,y)$ and $g(x,y)$. The generalization to the formula for the solution of n equations in n unknowns is obvious.

In our case, the $p(p+1)/2$ functions can be ordered by taking distinct elements from successive rows of the matrix

(3.1.3)
$$\phi(\Sigma) = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdot & \cdots & \phi_{1p} \\ & \phi_{22} & \cdot & \cdots & \phi_{2p} \\ & & \phi_{33} & \cdots & \cdot \\ & & & & \vdots \\ & & & & \phi_{pp} \end{bmatrix} .$$

The ordered functions can be written as elements of a column vector

(3.1.4)
$$\underline{\phi} = \begin{bmatrix} \phi_{11} \\ \phi_{12} \\ \vdots \\ \phi_{1p} \\ \phi_{22} \\ \vdots \\ \phi_{2p} \\ \phi_{33} \\ \vdots \\ \phi_{pp} \end{bmatrix}$$

which is analogous to the vector in (3.1.2) containing the functions f and g. The distinct elements of the matrix

(3.1.5)
$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdot & \cdots & \sigma_{1p} \\ & \sigma_{22} & \cdot & \cdots & \sigma_{2p} \\ & & \sigma_{33} & \cdots & \cdot \\ & & & & \vdots \\ & & & & \sigma_{pp} \end{bmatrix}$$

ordered in the same way, can also be written as a column vector

(3.1.6)
$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \vdots \\ \sigma_{1p} \\ \sigma_{22} \\ \vdots \\ \sigma_{2p} \\ \sigma_{33} \\ \vdots \\ \sigma_{pp} \end{bmatrix} .$$

We now define

$$(3.1.7) \qquad \nabla\phi = \begin{bmatrix}
\frac{\partial\phi_{11}}{\partial\sigma_{11}} & \frac{\partial\phi_{11}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{11}}{\partial\sigma_{1p}} & \frac{\partial\phi_{11}}{\partial\sigma_{22}} & \cdots & \frac{\partial\phi_{11}}{\partial\sigma_{2p}} & \frac{\partial\phi_{11}}{\partial\sigma_{33}} & \cdots & \frac{\partial\phi_{11}}{\partial\sigma_{pp}} \\
\frac{\partial\phi_{12}}{\partial\sigma_{11}} & \frac{\partial\phi_{12}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{12}}{\partial\sigma_{1p}} & \frac{\partial\phi_{12}}{\partial\sigma_{22}} & \cdots & \frac{\partial\phi_{12}}{\partial\sigma_{2p}} & \frac{\partial\phi_{12}}{\partial\sigma_{33}} & \cdots & \frac{\partial\phi_{12}}{\partial\sigma_{pp}} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial\phi_{1p}}{\partial\sigma_{11}} & \frac{\partial\phi_{1p}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{1p}}{\partial\sigma_{1p}} & \frac{\partial\phi_{1p}}{\partial\sigma_{22}} & \cdots & \frac{\partial\phi_{1p}}{\partial\sigma_{2p}} & \frac{\partial\phi_{1p}}{\partial\sigma_{33}} & \cdots & \frac{\partial\phi_{1p}}{\partial\sigma_{pp}} \\
\frac{\partial\phi_{22}}{\partial\sigma_{11}} & \frac{\partial\phi_{22}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{22}}{\partial\sigma_{1p}} & \frac{\partial\phi_{22}}{\partial\sigma_{22}} & \cdots & \frac{\partial\phi_{22}}{\partial\sigma_{2p}} & \frac{\partial\phi_{22}}{\partial\sigma_{33}} & \cdots & \frac{\partial\phi_{22}}{\partial\sigma_{pp}} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial\phi_{2p}}{\partial\sigma_{11}} & \frac{\partial\phi_{2p}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{2p}}{\partial\sigma_{1p}} & \frac{\partial\phi_{2p}}{\partial\sigma_{22}} & \cdots & \frac{\partial\phi_{2p}}{\partial\sigma_{2p}} & \frac{\partial\phi_{2p}}{\partial\sigma_{33}} & \cdots & \frac{\partial\phi_{2p}}{\partial\sigma_{pp}} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial\phi_{pp}}{\partial\sigma_{11}} & \frac{\partial\phi_{pp}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{pp}}{\partial\sigma_{1p}} & \frac{\partial\phi_{pp}}{\partial\sigma_{22}} & \cdots & \frac{\partial\phi_{pp}}{\partial\sigma_{2p}} & \frac{\partial\phi_{pp}}{\partial\sigma_{33}} & \cdots & \frac{\partial\phi_{pp}}{\partial\sigma_{pp}}
\end{bmatrix} .$$

Each column of (3.1.7) consists of distinct elements of the derivative of the matrix $\phi(\Sigma)$ with respect to a particular parameter element $\sigma_{k\ell}$ of the symmetric matrix Σ.

Let us denote by $\phi^{[k\ell]}$ the matrix of derivatives of $\phi(\Sigma)$ with respect to the parameter $\sigma_{k\ell}$. (In general, we shall let $Q^{[k\ell]}$ denote the partial derivative of the matrix Q with respect to the parameter $\sigma_{k\ell}$ so that if Q is symmetric,

$$Q^{[k\ell]} = \frac{\partial Q}{\partial \sigma_{k\ell}} + \left(\frac{\partial Q}{\partial \sigma_{k\ell}}\right)' \qquad k \neq \ell$$

$$Q^{[kk]} = \frac{\partial Q}{\partial \sigma_{kk}}$$

where $\frac{\partial Q}{\partial \sigma_{k\ell}}$ represents the partial derivative of Q with respect to the element in the k'th row and ℓ'th column of Σ when $\sigma_{k\ell}$ and $\sigma_{\ell k}$, $k\neq\ell$, are treated as distinct elements.)

Starting with an initial estimate Σ_o for the solution of equation (2.5.30), we compute $\phi(\Sigma_o)$. We also compute the $p(p+1)/2$ matrices $\phi^{[k\ell]}(\Sigma_o)$ for each $(k\ell)$ combination. The ordered distinct elements of each of these matrices form a column of the matrix (3.1.7), whose value at the initial estimate we denote by ϕ_o. Let $\underline{\sigma_o}$ represent the vector of distinct elements of Σ_o arranged as in the vector given by (3.1.6) and let $\underline{\phi_o}$ represent the vector (3.1.4) evaluated at the initial estimate. Then the vector of first approximations $\underline{\sigma_1}$ to the roots of the maximum likelihood equation is given by the Newton formula

(3.1.9) $$\underline{\sigma_1} = \underline{\sigma_o} - (\nabla\phi_o)^{-1}\,\underline{\phi_o}$$

which is completely analogous to the formula (3.1.2) for the solution of two equations in two unknowns. Successive approximations are obtained by repeating the process.

It remains to find an explicit expression for $\phi^{[k\ell]}(\Sigma)$ which is the derivative with respect to $\sigma_{k\ell}$ of $\phi(\Sigma)$, a sum of products of several matrices. Products of matrices are differentiated in the same way as products of scalars, except that the order of multiplication must be preserved. From lemma 3 in section 2.2 we have

$$U_i^{-1[k\ell]} = -\, U_i^{-1} U_i^{[k\ell]} U_i^{-1} \quad . \tag{3.1.10}$$

Hence we may write the derivative of the first term (2.5.27) of $\phi(\Sigma)$ as

$$V^{[k\ell]} = -\sum_{i=1}^{K} M_i U_i^{-1} U_i^{[k\ell]} U_i^{-1} M_i' \quad .$$

But, from (2.5.1),

$$U_i^{[k\ell]} = M_i' \Sigma^{[k\ell]} M_i \tag{3.1.11}$$

where $\Sigma^{[k\ell]}$ is the matrix of derivatives of Σ with respect to $\sigma_{k\ell}$. Hence

$$V^{[k\ell]} = -\sum_{i=1}^{K} M_i U_i^{-1} M_i' \Sigma^{[k\ell]} M_i U_i^{-1} M_i' \quad . \tag{3.1.12}$$

The individual elements of $V^{[k\ell]}$ are quite easily obtained. For it follows from (3.1.12) and the definition of $V^{[k\ell]}$ as the partial derivative of V with respect to the parameter $\sigma_{k\ell}$ that

$$(3.1.13) \qquad V_{rs}^{[k\ell]} = -\sum_{i=1}^{K}(M_iU_i^{-1}M_i')_{rk}(M_iU_i^{-1}M_i')_{s\ell}$$

$$-\sum_{i=1}^{K}(M_iU_i^{-1}M_i')_{r\ell}(M_iU_i^{-1}M_i')_{sk} \qquad k\neq\ell$$

$$V_{rs}^{[kk]} = -\sum_{i=1}^{K}(M_iU_i^{-1}M_i')_{rk}(M_iU_i^{-1}M_i')_{sk}$$

Thus if we write the K matrices $M_iU_i^{-1}M_i'$ explicitly, we multiply in each matrix the element in the r'th row and k'th column by the element in the s'th row and ℓ'th column and add these products over all K matrices to obtain $\partial V_{rs}/\partial\sigma_{k\ell}$. If $k=\ell$, $V_{rs}^{[kk]} = \partial V_{rs}/\partial\sigma_{kk}$. If $k\neq\ell$, we multiply in each $M_iU_i^{-1}M_i'$ matrix the element in the r'th row and ℓ'th column by the element in the s'th row and k'th column to obtain $\partial V_{sr}/\partial\sigma_{k\ell}$. Then $V_{rs}^{[k\ell]} = \partial V_{rs}/\partial\sigma_{k\ell} + \partial V_{sr}/\partial\sigma_{k\ell}$.

Now from the definitions (2.5.27) and (2.5.28) we have

$$(3.1.14) \qquad F_i^{[k\ell]} = -M_i'(V^{-1}W'^{[k\ell]}A + V^{-1[k\ell]}W'A)(A'A)^{-1}A'$$

where

$$(3.1.15) \qquad W'^{[k\ell]}A = -\sum_{i=1}^{K}M_iU_i^{-1}(M_i'\Sigma^{[k\ell]}M_i)U_i^{-1}Y_iA \qquad .$$

The term in the r'th row and s'th column of this matrix may be obtained from

$$(3.1.16) \qquad (W'^{[k\ell]}A)_{rs} = -\sum_{i=1}^{K}(M_iU_i^{-1}M_i')_{rk}(M_iU_i^{-1}Y_iA)_{s\ell}$$

$$-\sum_{i=1}^{K}(M_iU_i^{-1}M_i')_{r\ell}(M_iU_i^{-1}Y_iA)_{s\ell} \qquad k\neq\ell$$

$$(W'^{[kk]}A)_{rs} = -\sum_{i=1}^{K}(M_iU_i^{-1}M_i')_{rk}(M_iU_i^{-1}Y_iA)_{sk} \quad .$$

From lemma 3 in section 2.2 we have

$$(3.1.17) \qquad V^{-1[k\ell]} = -V^{-1}V^{[k\ell]}V^{-1} \quad .$$

The second term of (2.5.30) is a sum of products of six matrices, the first and last of which are analogous to scalar constants. Differentiation of this term is comparable to differentiation of a product of four scalar functions, pre-multiplied and post-multiplied by scalar constants. Thus the second term of $\phi^{[k\ell]}(\Sigma)$ may be written

$$(3.1.18) \qquad -\frac{1}{n}\Big(\sum_{i=1}^{K}M_iU_i^{-1[k\ell]}P_iP_i'U_i^{-1}M_i'$$

$$+ M_iU_i^{-1}P_iP_i'U_i^{-1[k\ell]}M_i'$$

$$+ M_iU_i^{-1}P_i^{[k\ell]}P_i'U_i^{-1}M_i'$$

$$+ M_iU_i^{-1}P_iP_i'^{[k\ell]}U_i^{-1}M_i'\Big) \quad .$$

If we let

(3.1.19) $$R_i^{\{k\ell\}} = P_i^{[k\ell]}P_i'$$

and

(3.1.20) $$T_i^{\{k\ell\}} = U_i^{[k\ell]}U_i^{-1}P_iP_i' \quad ,$$

then (3.1.18) can be written more simply as

(3.1.21) $$= \frac{1}{n}\sum_{i=1}^{K} M_iU_i^{-1}(-T_i^{\{k\ell\}}-T_i^{\{k\ell\}'}+R_i^{\{k\ell\}}+R_i^{\{k\ell\}'})U_i^{-1}M_i' \quad .$$

Hence the derivative of the function $\phi(\Sigma)$ given in (2.5.30) may be written

(3.1.22) $$\phi^{[k\ell]}(\Sigma) = V^{[k\ell]}+ \frac{1}{n}\sum_{i=1}^{K} M_iU_i^{-1}[(T_i^{\{k\ell\}}+T_i'^{\{k\ell\}}) - (R_i^{\{k\ell\}}+R_i'^{\{k\ell\}})]U_i^{-1}M_i'$$

where $V^{[k\ell]}$, $R_i^{\{k\ell\}}$, and $T_i^{\{k\ell\}}$ are defined in (3.1.12), (3.1.19), and (3.1.20) respectively.

3.2 Computational Forms

We may use (2.5.29) and (3.1.14) to obtain a simpler computational form for $R_i^{\{k\ell\}}$. Thus

(3.2.1) $$\begin{aligned} R_i^{\{k\ell\}} = P_i^{[k\ell]}P_i' &= \{-M_i'(V^{-1}W'^{[k\ell]}A+V^{-1[k\ell]}W'A)(A'A)^{-1}A'\} \\ &\quad \{Y_i'-A(A'A)^{-1}A'WV^{-1}M_i\} \\ &= -(M_i'V^{-1})(W'^{[k\ell]}A)(A'A)^{-1}(A'Y_i') \\ &\quad +(M_i'V^{-1})V^{[k\ell]}V^{-1}(W'A)(A'A)^{-1}(A'Y_i') \end{aligned}$$

$$+ (M_i'V^{-1})(W'^{[k\ell]}A)(A'A)^{-1}(A'W)(V^{-1}M_i)$$

$$- (M_i'V^{-1})V^{[k\ell]}V^{-1}(W'A)(A'A)^{-1}(A'W)(V^{-1}M_i) \quad .$$

Let

$$X_i = (A'A)^{-1}A'Y_i' \quad . \tag{3.2.2}$$

This is the matrix of standard least squares estimates of the parameters in the i'th group. The matrix $R_i^{\{k\ell\}}$ can then be written

$$\begin{aligned} R_i^{\{k\ell\}} = & -(M_i'V^{-1})(W'^{[k\ell]}A)X_i + (M_i'V^{-1})V^{[k\ell]}V^{-1}(W'A)X_i \\ & + (M_i'V^{-1})(W'^{[k\ell]}A)(A'A)^{-1}(A'W)V^{-1}M_i \\ & - (M_i'V^{-1})V^{[k\ell]}V^{-1}(W'A)(A'A)^{-1}(A'W)(V^{-1}M_i) \end{aligned} \tag{3.2.3}$$

The matrix products $W'A$ and $W'^{[k\ell]}A$ can also be expressed in terms of the least squares estimates in the i'th group. For, from the definitions (2.5.28), (3.1.15), and (3.2.2) we can write

$$(A'A)^{-1}A'W = \sum_{i=1}^{K} X_i U_i^{-1} M_i' \tag{3.2.4}$$

and

$$W'^{[k\ell]}A(A'A)^{-1} = -\sum_{i=1}^{K} M_i U_i^{-1}(M_i'\Sigma^{[k\ell]}M_i)U_i^{-1}X_i' \tag{3.2.5}$$

Hence $R_i^{\{k\ell\}}$ can be expressed in terms of the least squares estimates in the i'th group as

$$(3.2.6)\qquad R_i^{\{k\ell\}} = M_i'V^{-1}[\sum_{j=1}^{K} M_jU_j^{-1}(M_j'\Sigma^{[k\ell]}M_j)U_j^{-1}X_j'](A'A)X_i$$

$$+M_i'V^{-1}V^{[k\ell]}V^{-1}(\sum_{j=1}^{K} M_jU_j^{-1}X_j')(A'A)X_i$$

$$-M_i'V^{-1}[\sum_{j=1}^{K} M_jU_j^{-1}(M_j'\Sigma^{[k\ell]}M_j)U_j^{-1}X_j'](A'A)$$

$$(\sum_{j=1}^{K} X_jU_j^{-1}M_j')V^{-1}M_i$$

$$-M_i'V^{-1}V^{[k\ell]}V^{-1}(\sum_{j=1}^{K} M_jU_j^{-1}X_j')(A'A)(\sum_{j=1}^{K} X_jU_j^{-1}M_j')V^{-1}M_i \quad .$$

Now let

$$(3.2.7)\qquad F = \sum_{i=1}^{K} M_iU_i^{-1}X_i'$$

$$(3.2.8)\qquad F^{[k\ell]} = -\sum_{i=1}^{K} M_iU_i^{-1}(M_i'\Sigma^{[k\ell]}M_i)U_i^{-1}X_i' \quad .$$

Then, substituting in (3.2.6) we obtain

$$(3.2.9)\qquad R_i^{\{k\ell\}} = -M_i'V^{-1}F^{[k\ell]}(A'A)X_i+M_i'V^{-1}V^{[k\ell]}V^{-1}FA'AX_i$$

$$+M_i'V^{-1}F^{[k\ell]}(A'A)F'V^{-1}M_i$$

$$-M_i'V^{-1}V^{[k\ell]}V^{-1}F(A'A)F'V^{-1}M_i \quad .$$

Let

$$S_i^{\{k\ell\}} = M_i'V^{-1}V^{[k\ell]}V^{-1}F(A'A) \tag{3.2.10}$$

and

$$E_i^{\{k\ell\}} = M_i'V^{-1}F^{[k\ell]}(A'A) \quad . \tag{3.2.11}$$

Then $R_i^{\{k\ell\}}$ can be written

$$\begin{aligned} R_i^{\{k\ell\}} &= -E_i^{\{k\ell\}}X_i + S_i^{\{k\ell\}}X_i \\ &\quad + E_i^{\{k\ell\}}F'V^{-1}M_i - S_i^{\{k\ell\}}F'V^{-1}M_i \quad . \end{aligned} \tag{3.2.12}$$

If we combine (3.1.10), (3.1.11), and (3.1.20), we can write the matrix $T_i^{\{k\ell\}}$ as

$$T_i^{\{k\ell\}} = M_i'\Sigma^{[k\ell]}M_iU_i^{-1}P_iP_i' \quad . \tag{3.2.13}$$

The $\{k\ell\}$ superscript has been used for T_i, R_i, E_i, and S_i to emphasize the fact that a different matrix must be computed for each $(k\ell)$ combination.

The product P_iP_i' in the expression (2.5.30) for $\phi(\Sigma)$ may also be rewritten for computational simplification. From (2.5.29) we have

$$\begin{aligned} P_iP_i' &= \{Y_i - M_i'V^{-1}W'A(A'A)^{-1}A'\}\{Y_i' - A(A'A)^{-1}A'WV^{-1}M_i\} \\ &= Y_iY_i' - M_i'V^{-1}W'A(A'A)^{-1}A'Y_i' \\ &\quad - Y_iA(A'A)^{-1}A'WV^{-1}M_i + M_i'V^{-1}W'A(A'A)^{-1}A'WV^{-1}M_i \quad . \end{aligned} \tag{3.2.14}$$

Using (3.2.2), (3.2.4), and (3.2.5) we can write

$$(3.2.15)\quad P_iP_i' = Y_iY_i' - M_i'V^{-1}\left(\sum_{j=1}^{K} M_jU_j^{-1}X_j'\right)(A'A)X_i$$
$$- X_i'(A'A)\left(\sum_{j=1}^{K} X_iU_i^{-1}M_i'\right)V^{-1}M_i$$
$$+ M_i'V^{-1}\left(\sum_{j=1}^{K} M_jU_j^{-1}X_j'\right)(A'A)\left(\sum_{j=1}^{K} X_jU_j^{-1}M_j'\right)V^{-1}M_i \quad .$$

Then, substituting (3.2.7) and (3.2.8) in (3.2.15) we obtain a computational form for P_iP_i' expressed essentially in terms of the group least squares estimates of the parameters.

$$(3.2.16)\qquad P_iP_i' = Y_iY_i' - M_i'V^{-1}F(A'A)X_i$$
$$- X_i'(A'A)F'V^{-1}M_i + M_i'V^{-1}F(A'A)F'V^{-1}M_i \quad .$$

Now let

$$(3.2.17)\qquad G_i = M_i'V^{-1}F$$

and

$$H_i = G_i(A'A)X_i \quad .$$

Then

$$(3.2.18)\qquad P_iP_i' = Y_iY_i' - H_i - H_i' + G_i(A'A)G_i' \quad .$$

The reader will note from an examination of the definitions (3.2.7), (3.2.2), and (2.5.27) that the maximum likelihood estimate of the parameter matrix given by (2.5.12) can be expressed as

(3.2.19) $$\hat{\xi} = \hat{F}'\hat{V}^{-1} \quad .$$

3.3 Initial Estimates

The initial estimates $\underline{\sigma_o}$ may be obtained from an analysis of variance carried out separately for each of the groups. For each group an estimate of u variances and $\binom{u}{2}$ covariances may be made from the "within" dispersion matrix, that is, from the matrix of error sums of squares and sums of products divided by an appropriate number. Usually the error matrix is divided by error degrees of freedom. However in our case, since we are trying to obtain the maximum likelihood estimate, we recommend that the elements of the error matrix for each group be divided by n, the number of observations in each group. If each pair of variables occurs together only once in the same group, the non-diagonal elements of the group error matrices, divided by n, provide initial estimates for the covariances. A combined estimate of each variance may be obtained by averaging or, what is the same thing, pooling the separate estimates obtained from the groups in which a particular variable is measured.

In Chapter V, we shall discuss variable designs in which certain pairs of variables never occur together in the same group. Point estimation of the covariance between two variables is of course impossible unless the variables are measured in the same group. The implicit equations and the iterative procedure for the estimation of Σ do not involve these non-estimable covariances. This fact may be illustrated by the following example which may easily be generalized to any p and u.

Example. Four variables in three groups of two.

Group	Variables
(1)	12
(2)	23
(3)	34

In this design, the variable pairs 13, 14, 24 do not occur together in the same group. The matrix of initial estimates is

$$\Sigma_o = \begin{bmatrix} \sigma_{11} & \sigma_{12} & * & * \\ \sigma_{12} & \sigma_{22} & \sigma_{23} & * \\ * & \sigma_{23} & \sigma_{33} & \sigma_{34} \\ * & * & \sigma_{34} & \sigma_{44} \end{bmatrix}$$

where asterisks have been used to indicate the non-estimable covariances. The maximum likelihood equation for this design is

$$\phi(\hat{\Sigma}) = \begin{bmatrix} \phi_{11} & \phi_{12} & o & o \\ \phi_{12} & \phi_{22} & \phi_{23} & o \\ o & \phi_{23} & \phi_{33} & \phi_{34} \\ o & o & \phi_{34} & \phi_{44} \end{bmatrix} = 0 \quad .$$

The Newton iterative formula (3.1.9) is

$$\underline{\sigma_1} = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{33} \\ \sigma_{34} \\ \sigma_{44} \end{bmatrix} - \begin{bmatrix} \frac{\partial\phi_{11}}{\partial\sigma_{11}} & \frac{\partial\phi_{11}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{11}}{\partial\sigma_{44}} \\ \frac{\partial\phi_{12}}{\partial\sigma_{11}} & \frac{\partial\phi_{12}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{12}}{\partial\sigma_{44}} \\ \frac{\partial\phi_{22}}{\partial\sigma_{11}} & \frac{\partial\phi_{22}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{22}}{\partial\sigma_{44}} \\ \frac{\partial\phi_{23}}{\partial\sigma_{11}} & \frac{\partial\phi_{23}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{23}}{\partial\sigma_{44}} \\ \frac{\partial\phi_{33}}{\partial\sigma_{11}} & \frac{\partial\phi_{33}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{33}}{\partial\sigma_{44}} \\ \frac{\partial\phi_{34}}{\partial\sigma_{11}} & \frac{\partial\phi_{34}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{34}}{\partial\sigma_{44}} \\ \frac{\partial\phi_{44}}{\partial\sigma_{11}} & \frac{\partial\phi_{44}}{\partial\sigma_{12}} & \cdots & \frac{\partial\phi_{44}}{\partial\sigma_{44}} \end{bmatrix}^{-1} \begin{bmatrix} \phi_{11} \\ \phi_{12} \\ \phi_{22} \\ \phi_{23} \\ \phi_{33} \\ \phi_{34} \\ \phi_{44} \end{bmatrix} .$$

Thus four variances and three covariances can be estimated for this design. The matrices Σ_o and $\phi(\Sigma_o)$ each have seven distinct elements and the matrix of derivatives $\nabla\phi_o$ is a square matrix of order seven.

3.4 The Computational Procedure

The iterative procedure for obtaining the maximum likelihood solution for the equation (2.5.30) is outlined below in sufficient detail to enable a reader acquainted with the elements of statistics and matrix manipulation to follow the instructions mechanically. The reader will note that matrix products are grouped in such a way that only matrices of small dimensions are used repeatedly in the calculations. For example, the matrix Y_i', a large (n x u) matrix, is always pre-multiplied by the transpose of the design matrix, A'(m x n). When a long matrix expression after a summation sign begins with the product $A'Y_i'$, an inexperienced computer might factor the A' outside of the summation sign, obtain a succession of n-rowed matrices, add them together, and then pre-multiply by the matrix A'. The sensible procedure of course is to obtain the products $A'Y_i'$ first, since the resulting matrices are of relatively small dimensions. Moreover, the products $A'Y_i'$ are computed in the process of obtaining the group error matrices for the initial estimate of dispersion.

The steps in the computational procedure are as follows:

1. Obtain the K matrices $A'Y_i'$(m x u) for groups $i=1,2,\ldots,K$ where Y_i'(n x u) is the matrix of n observations

for the u variables measured in the i'th group and $A'(m \times n)$ is the transpose of the common design matrix for all groups.

The columns of $A'Y_i'$ represent the right hand side of the normal equations for the estimation of variable one (first column of $A'Y_i'$), variable two (second column of $A'Y_i'$),... varible u (u'th column of $A'Y_i'$) in the i'th group. For instance, in multiple regression the elements of $A'Y_i'$ are $\sum_j y_{j\alpha}^{(i)}$, $\sum_j x_{1j}y_{j\alpha}^{(i)}$, $\sum_j x_{2j}y_{j\alpha}^{(i)}$, etc. where $y_{j\alpha}^{(i)}$ is the j'th observation in the i'th group on variable α and α takes u different values corresponding to the u variables present in group i. For example, if the i'th group contains variables 3, 4, and 5, α would take the indices 3, 4, and 5 corresponding to the first, second, and third column of $A'Y_i'$. It will be noted that the superscript i does not occur with the concomitant variables x_1, x_2,... . This is due to the fact that the values x_{1j}, x_{2j}, etc. must be the same for each group. For example, if t (time) and t^2 are two concomitant variables and if measurements are taken at time 0,1,2,...,20 in the first group, measurements at the same times must be taken in the other groups, so that x_{1j} (or t) takes the values 0,1,2,...,20 and x_{2j} (or t^2) takes the values 0,1,4,...,400 in each group. In an analysis of variance situation, the α'th column of $A'Y_i'$

represents a column of treatment totals, block totals, etc. for the α'th variable over the elements in the i'th group.

2. Compute $(A'A)^{-1}$, a square matrix of order m. $A'A$ is the matrix of coefficients in the left side of the normal equations for the estimation of the m parameters for each of the u variables in the i'th group.

3. Obtain the K(m x u) matrices

$$X_i = (A'A)^{-1}A'Y_i' \qquad i=1,2,\ldots,K$$

by multiplying the matrix $A'A$ used in step 2 by each of the $A'Y_i'$ matrices obtained in step 1. X_i is the matrix of least squares estimates of the parameters in the i'th group.

4. To obtain the diagonal elements (ii) of Σ_o (the matrix of initial estimates of variances and covariances), add the "error" or "within" sums of squares from groups which contain the i'th variable and divide the result by nr, where n is the number of observations in each group and r is the number of groups containing the i'th variable. That is, nr is the total number of observations on the i'th variable. The use of n instead of n_e (the degrees of freedom due to error) is recommended here, since we are attempting to obtain an initial estimate of the maximum likelihood solution.

5. To obtain the non-diagonal elements (ij) of Σ_o, add

the error sums of products from groups which contain variables i and j simultaneously and divide the result by $n\lambda$ where n is the number of observations in each group and λ is the number of groups containing the pair of variables i and j. That is, $n\lambda$ is the total number of experimental units for which both variables i and j are measured.

6. If the pair of variables i and j never occurs in the same group, enter an asterisk in the (ij) position of Σ_o to indicate that the covariance between the variables i and j is non-estimable.

7. Obtain K submatrices $U_i(u \times u)$ of Σ_o by selecting, for each i, the rows and columns of Σ_o corresponding to the u variables contained in the i'th group.

8. Compute for groups $i=1,2,\ldots,K$ the $u \times u$ matrices U_i^{-1} by inverting the matrices obtained in step 7.

9. Obtain for each of the groups $i=1,2,\ldots,K$ the $p \times u$ matrix $M_iU_i^{-1}$, where p is the total number of variables and u is the number of variables in the i'th group. M_i is a $p \times u$ matrix which has a single 1 in each column and all other entries equal to zero. The 1's occur in those rows which correspond to the variables contained in the i'th group. For example if the i'th group contains variables 3, 4, and 5, M_i

would have a 1 in the third row and first column, the fourth row and second column, and the fifth row and third column, and would have zeros elsewhere.

Consequently, $M_i U_i^{-1}$ will be a p x u matrix which has zeros in those rows which correspond to variables <u>not</u> contained in the i'th group. The remaining u rows will contain the elements of U_i^{-1}, row by row.

10. Obtain for each of the groups i=1,2,...,K the p x p matrix $M_i U_i^{-1} M_i'$. This matrix has rows and columns of zeros corresponding to the variables missing in the i'th group. The u x u submatrix U_i^{-1} occupies the remaining positions.

11. Compute the p x p matrix

$$V = \sum_{i=1}^{K} M_i U_i^{-1} M_i'$$

by adding corresponding elements of each of the matrices obtained in step 10. If the pair of variables i and j never occurs together in the same group, V will have a zero in the (ij) position.

12. Calculate V^{-1}.

13. Compute the u x m matrices $U_i^{-1} X_i'$ by multiplying the matrices U_i^{-1} obtained in step 8 by the transpose of the corresponding matrices X_i obtained in step 3.

14. Compute the p x m matrix

$$F = \sum_{i=1}^{K} M_i U_i^{-1} X_i' \quad .$$

Insert the u x m matrix $U_i^{-1}X_i'$ in the u rows of a p x m matrix corresponding to the u variables contained in the i'th group. Fill in zeros in the remaining rows and sum the p x m matrices over the K groups.

15. Compute $V^{-1}F$(p x m) .

16. Compute the u x m matrices

$$G_i = M_i' V^{-1} F$$

by selecting from the p x m matrix product $V^{-1}F$ the u rows corresponding to the u variables measured in the i'th group.

17. Compute the u x u matrices

$$H_i = G_i (A'A) X_i$$

whose factors were found in steps 2, 3, and 16.

18. Compute the u x u matrices

$$P_i P_i' = Y_i Y_i' - H_i - H_i' + G_i (A'A) G_i' \quad .$$

The product $Y_i Y_i'$ is the matrix of total sums of squares and products for the i'th group which were obtained previously in the process of calculating the error sums of squares and products for the i'th group.

19. Compute the u x u matrices $U_i^{-1}(P_i P_i')$ whose factors

were obtained in steps 8 and 17.

20. Compute the p x p matrix

$$\phi(\Sigma) = V - \frac{1}{n}\sum_{i=1}^{K} M_i (U_i^{-1} P_i P_i') U_i^{-1} M_i' \quad .$$

The first term of this expression was calculated in step 11. The i'th matrix product in the second term is obtained by pre-multiplying the matrix U_i^{-1} computed in step 8 by the matrix $U_i^{-1}P_iP_i'$ found in step 18, placing the resulting u x u matrix in the u rows and columns of a p x p matrix corresponding to the u variables contained in the i'th group, and filling in zeros in the remaining positions. The sum of these K matrices of order p, divided by n, is the second term of $\phi(\Sigma)$.

21. Construct the vector

$$\underline{\phi} = \begin{bmatrix} \phi_{11} \\ \phi_{12} \\ \vdots \\ \phi_{1p} \\ \phi_{22} \\ \vdots \\ \phi_{2p} \\ \vdots \\ \phi_{pp} \end{bmatrix}$$

by selecting distinct elements in order from the p successive rows of the symmetric matrix $\phi(\Sigma)$.

22. Obtain the K(u x u) matrices $M_i'\Sigma^{[k\ell]}M_i$, for fixed k and ℓ, beginning with k=1, ℓ=1 for the first iterative loop. If $k\neq\ell$ these matrices are computed for the following combinations $[k\ell]$: (1,1), (2,2)...(p,p) and for those pairs (k,ℓ), $k\neq\ell$, which correspond to pairs of variables k and ℓ which occur at least once in the same group.

$\Sigma^{[kk]}$ is a p x p matrix with a 1 in the k'th diagonal position and zeros elsewhere. $\Sigma^{[k\ell]}$ $(k\neq\ell)$ is a p x p matrix with 1's in the $k\ell$ and ℓk positions and zeros elsewhere. $M_i'\Sigma^{[kk]}M_i$ has a single 1 in the main diagonal and zeros elsewhere. The position of this 1 is equal to the rank which the k'th variable has in the i'th group (ordered from low to high). For example, if variables 2, 4, and 5 occur in group i, $M_i'\Sigma^{[22]}M_i$ will have a 1 in the first row and column and zeros elsewhere; $M_i'\Sigma^{[44]}M_i$ will have a 1 in the second row and column and zeros elsewhere; $M_i'\Sigma^{[55]}M_i$ will have a 1 in the third row and column and zeros elsewhere. All other matrices of the type $M_i'\Sigma^{[kk]}M_i$ will be null matrices.

When $k\neq\ell$ and i refers to any group which does not contain both variables k and ℓ simultaneously, $M_i'\Sigma^{[k\ell]}M_i$ is a null matrix. When i denotes a group which contains both variables k and ℓ, it is a matrix with 1's in two symmetric off-diagonal positions and zeros elsewhere. In order to find the positions in which the 1's occur, rank the variable numbers in the i'th group from smallest to largest. The rank of the k'th variable number is then equal to the row number and the rank of the ℓ'th variable number, to the column number of the position in which one of the "1" elements occurs. To obtain the position of the other "1" element, interchange the row and column numbers.

In the preceding example,

$$M_i'\Sigma^{[24]}M_i = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_i'\Sigma^{[25]}M_i = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_i'\Sigma^{[45]}M_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

All other matrices for this group i would be null matrices.

23. Compute the u x u matrices

$$T_i^{\{k\ell\}} = M_i'\Sigma^{[k\ell]}M_i U_i^{-1} P_i P_i'$$

for groups i=1,2,...,K. Since $M_i'\Sigma^{[k\ell]}M_i$ is a factor of $T_i^{\{k\ell\}}$, certain matrices may be disregarded, for when $k = \ell$ and i refers to a group which does not contain the variable $k=\ell$, or when $k\neq\ell$ and i refers to any group which does not contain both variables k and ℓ simultaneously, $T_i^{\{k\ell\}}$ will be a null matrix.

$T_i^{\{kk\}}$ has a single row of non-zero elements taken from $U_i^{-1}P_iP_i'$ which was obtained in step 19. To find the appropriate row, look at the single non-zero element of $M_i'\Sigma^{[kk]}M_i$ which was obtained in step 22; the row in which the 1 occurs corresponds to the non-zero row of $T_i^{\{kk\}}$.

$T_i^{\{k\ell\}}$ has two rows of non-zero elements taken from $U_i^{-1}P_iP_i'$ provided that the pair of variables k, ℓ occurs in group i. To find the appropriate rows, look at the two non-zero elements in $M_i'\Sigma^{[k\ell]}M_i$ which was obtained in step 22; the rows in which the two 1's occur correspond to the non-zero rows of $T_i^{\{k\ell\}}$. However the two rows must be interchanged, i.e., the row corresponding to variable ℓ in $M_i'\Sigma^{[k\ell]}M_i$ (and hence in $U_i^{-1}P_iP_i'$) would take the place corresponding to variable k in $T_i^{[k\ell]}$ and vice versa.

24. Compute the p x p matrix

$$V^{[k\ell]} = -\sum_{i=1}^{K} M_iU_i^{-1}(M_i'\Sigma^{[k\ell]}M_i)U_i^{-1}M_i' \quad .$$

When $k=\ell$, this matrix is a sum over all groups which contain the variable $k=\ell$. When $k\neq\ell$, the only non-zero terms in the sum are those coming from the groups in which the pair (k,ℓ) occurs. If the variable k never occurs in the same group with variables ℓ and m, $V^{[kk]}$ will have a zero element in the (ℓ,m) position. If the pair of variables k and ℓ never occurs in the same group with variable m, all the elements of the m'th row and column of $V^{[k\ell]}$ will be zero.

The individual terms under the sum are as follows:

$M_iU_i^{-1}(M_i'\Sigma^{[k\ell]}M_i)U_i^{-1}M_i'$ is a p x p symmetric matrix whose elements are the squares and cross products of the elements in that column of $M_iU_i^{-1}$ which corresponds to the position of variable k in group i. Of course if variable k does not occur in group i, this will be a null matrix.

In order to obtain the terms $M_iU_i^{-1}(M_i'\Sigma^{[k\ell]}M_i)U_i^{-1}M_i'$ for $k\neq\ell$, let us denote that column of $M_iU_i^{-1}$ which corresponds to the position of the k'th variable in group i by $\underline{z}_k$, and that which corresponds to the position of the ℓ'th variable in group i by $\underline{z}_\ell$. Then the above matrix equals $\underline{z}_k\underline{z}_\ell'+\underline{z}_\ell\underline{z}_k'$.

25. Compute the p x m matrix

$$F^{[k\ell]} = -\sum_{i=1}^{K} M_iU_i^{-1}(M_i'\Sigma^{[k\ell]}M_i)U_i^{-1}X_i' \quad .$$

The factor $U_i^{-1}X_i'$(u x m) was obtained in step 13. When this is pre-multiplied by $M_i'\Sigma^{[k\ell]}M_i$, the result is either a null matrix or a matrix with one or two non-zero rows corresponding to the one or two rows of $M_i'\Sigma^{[k\ell]}M_i$ which contain a one. This matrix should be pre-multiplied by U_i^{-1}. The i'th term of the sum is then found by inserting the u x m matrix $U_i^{-1}(M_i'\Sigma^{[k\ell]}M_i)U_i^{-1}X_i'$ in the u rows of a p x m matrix corresponding to the u variables contained in the i'th group and supplying zeros in the remaining p - u rows.

26. Compute the u x m matrices

$$S_i^{\{k\ell\}} = M_i'V^{-1}V^{[k\ell]}V^{-1}F(A'A)$$

The factors of this product were obtained in steps 2, 12, 14, and 24.

27. Compute the u x m matrices

$$E_i^{\{k\ell\}} = M_i'V^{-1}F^{[k\ell]}A'A$$

The factors of this product were obtained in steps 2, 12, and 25.

28. Calculate the u x u matrices

$$R_i^{\{k\ell\}} = -E_i^{\{k\ell\}}X_i+S_i^{\{k\ell\}}X_i$$
$$+E_i^{\{k\ell\}}F'V^{-1}M_i-S_i^{\{k\ell\}}F'V^{-1}M_i$$

whose factors were found in the directly preceding steps and in steps 3, 12, and 14.

29. Compute the p x p matrices

$$\phi^{[k\ell]}(\Sigma) = V^{[k\ell]} + \frac{1}{n}\sum_{i=1}^{K} M_i U_i^{-1}[T_i^{\{k\ell\}} + T_i'^{\{k\ell\}} - (R_i^{\{k\ell\}} + R_i'^{\{k\ell\}})]U_i^{-1}M_i'$$

The terms of this expression were computed in steps 9, 22, 24, and 28.

30. Construct the vector

$$\underline{\phi}^{[k\ell]} = \begin{bmatrix} \phi_{11}^{[k\ell]} \\ \phi_{12}^{[k\ell]} \\ \vdots \\ \phi_{1p}^{[k\ell]} \\ \phi_{22}^{[k\ell]} \\ \vdots \\ \phi_{2p}^{[k\ell]} \\ \vdots \\ \phi_{pp}^{[k\ell]} \end{bmatrix}$$

by taking distinct elements in order from successive rows of $\phi^{[k\ell]}(\Sigma)$.

31. With the vectors $\underline{\phi}^{[k\ell]}$ as columns, construct the square matrix

$$\phi_o = [\underline{\phi}^{[11]}\ \underline{\phi}^{[12]} \ldots \underline{\phi}^{[1p]}\ \underline{\phi}^{[22]} \ldots \underline{\phi}^{[2p]} \ldots \underline{\phi}^{[pp]}]$$

If all pairs occur together at least once in the design, this matrix is of order $p(p+1)/2$. This number is reduced by the number of combinations (k,ℓ) which never occur in the same group.

32. Construct a vector $\underline{\sigma}_o$ from the distinct elements of the matrix Σ_o of initial estimates.

$$\underline{\sigma}_o = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \vdots \\ \sigma_{1p} \\ \sigma_{22} \\ \vdots \\ \sigma_{2p} \\ \vdots \\ \sigma_{pp} \end{bmatrix}$$

The elements must be taken in an order corresponding to the elements of the vector $\underline{\phi}$ and the rows and columns of ϕ. This order has been determined arbitrarily by taking distinct elements in order from successive rows of the matrices Σ_o, $\phi(\Sigma)$ and $\phi^{[k\ell]}(\Sigma)$.

33. Solve $(\nabla\phi_o)\underline{X} = \phi_o$ for $\underline{X}$.

34. Using the solution $\underline{X} = \underline{X}_o$, obtain a vector $\underline{\sigma}_1$ of

new estimates from

$$\underline{g}_1 = \underline{g}_o - \underline{X}_o$$

Successive approximations $\underline{g}_2$, $\underline{g}_3$, $\underline{g}_4$, etc. are obtained by repeating the entire process until convergence is reached. When the approximations begin to converge, the same $\nabla \phi$ matrix, say $\nabla \phi_i$, may be used in succeeding iterations. In this case, the instruction in step 30 after the i'th iteration would require the solution of

$$(\nabla \phi_i)\underline{X} = \underline{\phi}_i \text{ for } \underline{X} = \underline{X}_i$$

$$(\nabla \phi_i)\underline{X} = \underline{\phi}_{i+1} \text{ for } \underline{X} = \underline{X}_{i+1}$$

$$\vdots$$

$$(\nabla \phi_i)\underline{X} = \underline{\phi}_t \text{ for } \underline{X} = \underline{X}_t$$

IV. TESTING OF HYPOTHESES

4.1 The General Linear Hypothesis

To test the null hypothesis $C\xi=0$ in the most general linear model (2.3.1), we maximize the likelihood function (2.3.2) subject to the condition $C\xi=0$.

In the last term of the likelihood function, we have

$$\operatorname{tr} \Sigma^{-1}(X-\xi'A')(X'-A\xi) = \operatorname{tr}(X'-A\xi)\Sigma^{-1}(X-\xi'A') \quad .$$

By lemma 1 in section 2.2

$$\frac{\partial \operatorname{tr}(X'-A\xi)\Sigma^{-1}(X-\xi'A')}{\partial(X'-A\xi)} = 2(X'-A\xi)\Sigma^{-1}$$

whence

$$\frac{\partial \operatorname{tr}(X'-A\xi)\Sigma^{-1}(X-\xi'A')}{\partial(X'-A\xi)_{k\alpha}} = 2\sum_{j=1}^{p} x_{kj}\sigma^{j\alpha} - 2\sum_{j=1}^{m}\sum_{\beta=1}^{p} a_{kj}\xi_{j\beta}\sigma^{\beta\alpha}$$

where x_{kj} is the kj element of X'. Also

$$\frac{\partial(X'-A\xi)_{k\alpha}}{\partial\xi_{\ell\gamma}} = -\frac{\partial}{\partial\xi_{\ell\gamma}}\sum_{j=1}^{m} a_{kj}\xi_{j\alpha} \begin{array}{l} = 0 \text{ if } \alpha\neq\gamma \\ = -a_{k\ell} \text{ if } \alpha=\gamma \end{array} \quad .$$

Using the chain rule for differentiation, we have

$$\frac{\partial \operatorname{tr}(X'-A\xi)\Sigma^{-1}(X-\xi'A')}{\partial\xi_{\ell\gamma}} =$$

$$\sum_{k=1}^{N}\sum_{\alpha=1}^{p} \frac{\partial \operatorname{tr}(X'-A\xi)\Sigma^{-1}(X-\xi'A')}{\partial(X'-A\xi)_{k\alpha}} \frac{\partial(X'-A\xi)_{k\alpha}}{\partial\xi_{\ell\gamma}} =$$

$$-2\sum_{k=1}^{N}\sum_{j=1}^{p} x_{kj}\sigma^{j\gamma}a_{k\ell} + 2\sum_{k=1}^{N}\sum_{j=1}^{m}\sum_{\beta=1}^{p} a_{kj}a_{k\ell}\xi_{j\beta}\sigma^{\beta\gamma}$$

whence

$$\text{(4.1.1)}\quad \frac{\partial\ \mathrm{tr}(X'-A\xi)\Sigma^{-1}(X-\xi'A')}{\partial\xi_{\ell\gamma}} = -2(A'X'\Sigma^{-1})_{\ell\gamma}+2(A'A\xi\Sigma^{-1})_{\ell\gamma}\ .$$

Consequently from (2.3.2) we have

$$\text{(4.1.2)}\quad \frac{\partial L(X')}{\partial\xi_{\ell\gamma}} = (A'X'\Sigma^{-1})_{\ell\gamma} - (A'A\xi\Sigma^{-1})_{\ell\gamma}\ .$$

If in the constraint $C\xi=0$ we multiply the rows of C into the γ'th column of ξ, we obtain the conditions which affect $\xi_{\ell\gamma}$, namely

$$\sum_{k=1}^{m} c_{1k}\xi_{k\gamma} = 0$$

$$\sum_{k=1}^{m} c_{2k}\xi_{k\gamma} = 0$$

$$\vdots$$

$$\sum_{k=1}^{m} c_{sk}\xi_{k\gamma} = 0\ .$$

Hence, to maximize the function (2.3.2) subject to the condition $C\xi=0$, we equate to zero the derivative with respect to $\xi_{\ell\gamma}$ of the function

$$L(X')+\lambda_{1\gamma}\sum_{k=1}^{m} c_{1k}\xi_{k\gamma}+\lambda_{2\gamma}\sum_{k=1}^{m} c_{2k}\xi_{k\gamma}+\ \dots\ +\lambda_{s\gamma}\sum_{k=1}^{m} c_{sk}\xi_{k\gamma}\ .$$

From (4.1.2) we see that this yields the conditions

$$\text{(4.1.3)}\quad (A'X'\Sigma^{-1})_{\ell\gamma} - (A'A\hat{\xi}\Sigma^{-1})_{\ell\gamma} + \sum_{i=1}^{s}\lambda_{i\gamma}c_{i\ell} = 0$$

on the maximum likelihood estimate $\hat{\xi}$. There are m x p such equations corresponding to the $\xi_{\ell\gamma}$ elements of the m x p parameter matrix. If we denote the s x m matrix of Lagrange multipliers by Λ, the conditions (4.1.3) can be written

$$(4.1.4) \qquad (A'X'\Sigma^{-1})_{\ell\gamma}-(A'A\hat{\xi}\Sigma^{-1})_{\ell\gamma}+(C'\Lambda)_{\ell\gamma} = 0 \quad .$$

Pre-multiplying by $C(A'A)^{-1}$ and post-multiplying by Σ, we obtain

$$(4.1.5) \qquad C(A'A)^{-1}A'X'-C\hat{\xi}+C(A'A)^{-1}C'\Lambda\Sigma = 0 \quad .$$

But $C\hat{\hat{\xi}} = 0$, hence

$$(4.1.6) \qquad \Lambda = -[C(A'A)^{-1}C']^{-1}C(A'A)^{-1}A'X'\Sigma^{-1} \quad .$$

Substituting this expression for Λ in (4.1.5), we obtain for the maximum likelihood estimate of ξ in ω

$$(4.1.7) \qquad \hat{\hat{\xi}} = (A'A)^{-1}A'X'-(A'A)^{-1}C'[C(A'A)^{-1}C']^{-1}C(A'A)^{-1}A'X' \quad .$$

Then

$$X'-A\hat{\hat{\xi}} = [I-A(A'A)^{-1}A']X' + A(A'A)^{-1}C'[C(A'A)^{-1}C']^{-1}C(A'A)^{-1}A'X' \quad .$$

Using the fact that $I-A(A'A)^{-1}A'$ is an idempotent matrix, we obtain after cancellation of some terms

$$(X-\hat{\hat{\xi}}'A')(X'-A\hat{\hat{\xi}}) = X[I-A(A'A)^{-1}A']X' + XA(A'A)^{-1}C'[C(A'A)^{-1}C']^{-1}C(A'A)^{-1}A'X' \quad .$$

Now as in (2.3.7), let

$$E = X[I-A(A'A)^{-1}A']X' = XX'-XA(A'A)^{-1}A'X'$$

and let

$$(4.1.8) \qquad H = XA(A'A)^{-1}C'[C(A'A)^{-1}C']^{-1}C(A'A)^{-1}A'X' =$$

$$\hat{\xi}'C'[C(A'A)^{-1}C']^{-1}C\hat{\xi}$$

where $\hat{\xi}$ is given by (2.3.5).

Then

$$(4.1.9) \qquad (X-\hat{\hat{\xi}}'A')(X'-A\hat{\hat{\xi}}) = E+H$$

and

$$L(\omega|\xi=\hat{\hat{\xi}}) = -\frac{Np}{2}\log 2\pi - \frac{N}{2}\log|\Sigma|-\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}(E+H) \quad .$$

Using lemmas 2 and 5 we obtain

$$\frac{\partial L(\omega|\xi=\hat{\hat{\xi}})}{\partial\Sigma^{-1}} = \frac{N}{2}\Sigma - \tfrac{1}{2}(E+H)$$

whence

$$(4.1.10) \qquad \hat{\hat{\Sigma}} = \frac{1}{N}(E+H) \quad .$$

The maximum of the log-likelihood in ω is then

$$(4.1.11) \quad L(\hat{\omega}) = -\frac{Np}{2}\log 2\pi - \frac{N}{2}\log|H+E| + \frac{Np}{2}\log N - \frac{Np}{2} \quad .$$

The maximum of the log-likelihood in Ω is given by (2.3.10). Hence the logarithm of the likelihood ratio for testing the hypothesis $C\xi=0$ is

$$\log\lambda = \frac{N}{2}\log|E| - \frac{N}{2}\log|H+E|$$

Let

(4.1.12) $$\Lambda = \lambda^{2/N} = \frac{|E|}{|H+E|} \quad .$$

This is Wilks' lamda statistic for testing the hypothesis $C\xi=0$. If the identity matrix is substituted for C in (4.1.8), H will be recognized as the matrix of regression sums of squares and products for the hypothesis $\xi=0$. The matrix E is of course the array of error sums of squares and sums of products.

4.2 The General Linear Hypothesis for the Incomplete Variable Case

Following the argument used in the preceding section, we can obtain a procedure for testing the hypothesis $C\xi=0$ for the case where we have incomplete sets of variables. If we replace equation (4.1.2) by (2.5.8) we have

(4.2.1) $$\frac{\partial L(Y')}{\partial \xi_{\ell\gamma}} = \sum_{i=1}^{K} [A_i'Y_i'U_i^{-1}M_i']_{\ell\gamma} - \sum_{i=1}^{K} [A_i'A_i\xi M_i U_i^{-1}M_i']_{\ell\gamma} \quad .$$

Instead of (4.1.3) we write for this case

(4.2.2) $$\sum_{i=1}^{K} [A_i'Y_i'U_i^{-1}M_i']_{\ell\gamma} - \sum_{i=1}^{K} [A_i'A_i\xi M_i U_i^{-1}M_i']_{\ell\gamma} + \sum_{i=1}^{s} \lambda_{i\gamma}C_{i\ell} = 0 \quad .$$

Assuming as before that the design matrix is the same in each group, we have the following relation (corresponding to 4.1.4) which the conditional maximum must satisfy.

$$(4.2.3) \qquad \sum_{i=1}^{K} \Lambda' Y_i' \hat{\hat{U}}_i^{-1} M_i' - \Lambda' \Lambda \hat{\hat{\xi}} \sum_{i=1}^{K} M_i \hat{\hat{U}}_i^{-1} M_i' + C' \Lambda = 0 \quad .$$

Pre-multiplying by $C(\Lambda'\Lambda)^{-1}$ we obtain

$$(4.2.4) \qquad \sum_{i=1}^{K} C(\Lambda'\Lambda)^{-1} \Lambda' Y_i' \hat{\hat{U}}_i^{-1} M_i' - C\hat{\hat{\xi}} \sum_{i=1}^{K} M_i \hat{\hat{U}}_i^{-1} M_i' + C(\Lambda'\Lambda)^{-1} C' \Lambda = 0 \quad .$$

Now let

$$X_i = (\Lambda'\Lambda)^{-1} \Lambda' Y_i'$$

and

$$(4.2.5) \qquad \hat{V} = \sum_{i=1}^{K} M_i \hat{\hat{U}}_i^{-1} M_i' \quad .$$

Then equation (4.2.4) becomes

$$(4.2.6) \qquad \sum_{i=1}^{K} C X_i \hat{\hat{U}}_i^{-1} M_i' - C\hat{\hat{\xi}} \hat{\hat{V}} + C(\Lambda'\Lambda)^{-1} C' \Lambda = 0 \quad .$$

But $C\hat{\hat{\xi}} = 0$, hence

$$(4.2.7) \qquad \sum_{i=1}^{K} X_i \hat{\hat{U}}_i^{-1} M_i' + (\Lambda'\Lambda)^{-1} C' \Lambda = 0 \quad .$$

If we solve this expression for Λ and substitute the result in equation (4.2.6) we obtain

$$(4.2.8) \qquad \sum_{i=1}^{K} X_i \hat{\hat{U}}_i^{-1} M_i' - \hat{\hat{\xi}} \hat{\hat{V}} - (\Lambda'\Lambda)^{-1} C' [C(\Lambda'\Lambda)^{-1} C']^{-1} \sum_{i=1}^{K} C X_i \hat{\hat{U}}_i^{-1} M_i' = 0$$

where X_i is the matrix of standard least squares estimates in the i'th group.

Let us introduce

$$(4.2.9) \qquad \hat{\hat{X}}_i = X_i-(A'A)^{-1}C'[C(A'A)^{-1}C']^{-1}CX_i \quad .$$

This expression represents the matrix of standard least squares estimates of the parameters in the reduced model which is applicable if the null hypothesis is true. That is, they are the estimates under H_o in the regression test of significance. Substituting (4.2.9) in (4.2.8) and solving for $\hat{\hat{\xi}}$ we obtain

$$(4.2.10) \qquad \hat{\hat{\xi}} = (\sum_{i=1}^{K} \hat{\hat{X}}_i \hat{\hat{U}}_i^{-1} M_i')\hat{\hat{V}}^{-1}$$

which has precisely the same form as $\hat{\xi}$ given in equation (2.5.12). For, from (2.5.26), (2.5.27), and (3.2.2) we see that $\hat{\xi}$ can be written

$$(4.2.11) \qquad \hat{\xi} = (\sum_{i=1}^{K} X_i \hat{U}_i^{-1} M_i')\hat{V}^{-1}$$

We can therefore say that after the $\hat{\hat{U}}_i$ are obtained, $\hat{\hat{\xi}}$ can be found in the same way as $\hat{\xi}$, except that the standard least squares estimates in each of the i groups must be replaced by the corresponding standard least squares estimates in each of the i groups under the restricted model in which H_o is true. The reader will note that the hypothesis matrix C sets up the same functional form of the hypothesis for each variable, so that there is no ambiguity involved in referring to the standard least squares estimates under H_o for each of the i groups.

The derivative $\frac{\partial L(Y')}{\partial \Sigma}$ is of course the same under the null hypothesis as in the general case. For introducing the constraint and the Lagrangian multipliers merely amounts to adding to the derivative terms

$$\frac{\partial}{\partial \Sigma}\,(\underline{\lambda}_\gamma' C \underline{\xi}_\gamma)$$

which are equal to zero. Consequently the second equation for the maximum likelihood estimates is the same as (2.5.30), except that $\hat{\underline{\xi}}$ must be replaced by $\hat{\hat{\underline{\xi}}}$ in the expression for $\hat{B}_i$, which as shown above merely amounts to replacing the standard least squares estimate X_i in each group by the corresponding standard least squares estimate $\hat{\hat{X}}_i$ for the restricted model in which H_o is true.

In Ω the maximum likelihood estimate $\hat{\Sigma}$ is given implicitly by (2.5.30) with

$$\hat{F}_i = Y_i - M_i' \hat{\xi}' A'$$

In Chapter III (equation 3.2.7) we introduced the notation

$$\hat{F} = \sum_{i=1}^{K} M_i \hat{U}_i^{-1} X_i'$$

and wrote the maximum likelihood estimate of the parameter matrix in (3.2.19) as

$$\hat{\xi} = \hat{F}'\hat{V}^{-1}$$

with $\hat{U}_i$, X_i, and $\hat{V}$ as defined by (2.5.26), (3.2.2), and (2.5.27) respectively.

Correspondingly in ω if we define

$$\hat{\hat{F}} = \sum_{i=1}^{K} M_i \hat{\hat{U}}_i^{-1} \hat{\hat{X}}_i' \tag{4.2.12}$$

where $\hat{\hat{X}}_i$ is given in (4.2.10) and $\hat{\hat{U}}_i = M_i' \hat{\hat{\Sigma}} M_i$, then the maximum likelihood estimate of the parameter matrix can be written

$$\hat{\hat{\xi}} = \hat{\hat{F}}' \hat{\hat{V}}^{-1} \tag{4.2.13}$$

where $\hat{\hat{V}}$ is defined in (4.2.5). The maximum likelihood estimate $\hat{\hat{\Sigma}}$ in ω is given by

$$\phi(\hat{\hat{\Sigma}}) = \hat{\hat{V}} - \frac{1}{n}\sum_{i=1}^{K} M_i \hat{\hat{U}}_i^{-1} \hat{\hat{P}}_i \hat{\hat{P}}_i' \hat{\hat{U}}_i^{-1} M_i' = 0 \tag{4.2.14}$$

where

$$\hat{\hat{P}}_i = Y_i - M_i' \hat{\hat{\xi}}' A' \quad . \tag{4.2.15}$$

Now from (2.4.8)

$$\log L(\hat{\Omega}) = -\frac{Nu}{2}\log 2\pi - \frac{n}{2}\sum_{i=1}^{K}\log |\hat{U}_i| \tag{4.2.16}$$

$$- \tfrac{1}{2} \operatorname{tr} \sum_{i=1}^{K} \hat{U}_i^{-1} \hat{P}_i \hat{P}_i'$$

and

(4.2.17) $$\log L(\hat{\omega}) = -\frac{Nu}{2}\log 2\pi - \frac{n}{2}\sum_{i=1}^{K}\log|\hat{\hat{U}}_i| - \tfrac{1}{2}\operatorname{tr}\sum_{i=1}^{K}\hat{\hat{U}}_i^{-1}\,\hat{\hat{P}}_i\,\hat{\hat{P}}_i' \quad .$$

Hence

(4.2.18) $$-2\log\lambda = n\sum_{i=1}^{K}\log|\hat{\hat{U}}_i| - n\sum_{i=1}^{K}\log|\hat{U}_i| + \operatorname{tr}\sum_{i=1}^{K}\hat{\hat{U}}_i^{-1}\,\hat{\hat{P}}_i\,\hat{\hat{P}}_i' - \operatorname{tr}\sum_{i=1}^{K}\hat{U}_i^{-1}\,\hat{P}_i\,\hat{P}_i' \quad .$$

V. VARIABLE DESIGNS

5.1 Description of the Designs

Certain designs of the variables will result in somewhat simplified computations of $\phi(\Sigma)$ and $\phi^{[k\ell]}(\Sigma)$. There are several basic designs which we might consider. The first that we shall investigate are balanced designs. These are analogous to balanced incomplete blocks in a univariate analysis with variables playing the role of treatments, and groups that of blocks. For example, if seven variables are observed in sets of three, the seven groups (124), (235), (346), (457), (156), (267) and (137) comprise a balanced design in which each variable is measured in three groups and each pair of variables occurs once. In general, these "balanced" designs have the advantage that the same number of observations is available for each variable, and an equal number of observations (of course, different from the former) is available for each pair of variables. Thus the information on variances is the same for each variable, and the information on covariances is the same for each pair.

Another design which we shall consider is that formed by cyclic arrays of p digits in sets of size u in which every variable occurs an equal number of times. For example, in the design (123), (234), (345), (456), (567), (671), (712), each variable is measured in three groups. This design is formed by assigning the first u digits to the first group and starting each successive group with the second digit from the preceding group. The process is continued until every digit occurs an equal number of times. This arrangement will be referred to as a circular link design.

If the last two groups in the above design are omitted, we obtain a new design which will be called an incomplete circular link. It is constructed in the same way as the circular link design, except that the formation of groups stops when the last variable is reached.

The arrangement (123), (345), (567) will be called a minimum linkage design. As the name suggests, this design is formed by repeating only the last digit in a group as the first digit in the next group and continuing in the natural order until the last digit occurs.

5.2 The Submatrices of Σ.

The matrices U_i are submatrices of Σ formed from the rows and columns corresponding to the variables contained in the i'th group. For example, if $p=3$ and variables one and two are measured in group one, then

$$(5.2.1) \qquad U_1 = M_1'\Sigma M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

We could equally well use the product

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_{22} & \sigma_{12} \\ \sigma_{12} & \sigma_{11} \end{bmatrix}$$

However, for convenience, we shall assume that the M_i matrices are chosen as in (5.2.1) in such a way as to preserve the natural order of the variable numbers within the U_i matrices.

If for the case p=3, variables one and two are contained in group one and variables two and three in group two, then the (22) element of U_1 will be the same as the (11) element of U_2. This will not be true of the corresponding matrices U_i^{-1}. Thus if

$$U_1^{-1} = \begin{bmatrix} \sigma^{11}_{(1)} & \sigma^{12}_{(1)} \\ \sigma^{12}_{(1)} & \sigma^{22}_{(1)} \end{bmatrix} \qquad U_2^{-1} = \begin{bmatrix} \sigma^{22}_{(2)} & \sigma^{23}_{(2)} \\ \sigma^{23}_{(2)} & \sigma^{33}_{(2)} \end{bmatrix}$$

then $\sigma^{22}_{(1)} \neq \sigma^{22}_{(2)}$.

The groups will be referred to by the numbers of the variables observed in the group. The u row and column subscripts of the U_i matrices will then be designated 1,2,...,u. Thus, in groups (12) and (23) the U_i matrices will be written

$$U_{(12)} = \begin{bmatrix} \sigma_{11}^{(12)} & \sigma_{12}^{(12)} \\ \sigma_{12}^{(12)} & \sigma_{22}^{(12)} \end{bmatrix} \qquad U_{(23)} = \begin{bmatrix} \sigma_{11}^{(23)} & \sigma_{12}^{(23)} \\ \sigma_{12}^{(23)} & \sigma_{22}^{(23)} \end{bmatrix}$$

where superscripts in parentheses denote the variables in the respective group. They are replaced by subscripts in inverse matrices. So, for example, $\sigma_{11}^{(23)}$ denotes the variance of the first variable in the group which contains variables two and three, that is, the variance of variable two. Similarly $\sigma_{12}^{(23)}$ denotes the covariance of the first and second variables in group (23), that is, the covariance of variables two and three. Hence $\sigma_{11}^{(23)}$ is actually σ_{22} and $\sigma_{12}^{(23)}$ is actually σ_{23}. The U_i^{-1} matrices will be denoted by

$$U_{(12)}^{-1} = \begin{bmatrix} \sigma_{(12)}^{11} & \sigma_{(12)}^{12} \\ \sigma_{(12)}^{12} & \sigma_{(12)}^{22} \end{bmatrix} \qquad U_{(23)}^{-1} = \begin{bmatrix} \sigma_{(23)}^{11} & \sigma_{(23)}^{12} \\ \sigma_{(23)}^{12} & \sigma_{(23)}^{22} \end{bmatrix}$$

As before, we note that $\sigma_{(12)}^{22} \neq \sigma_{(23)}^{11}$.

The matrices $M_i U_i^{-1}$ for the two groups discussed above are

$$M_{(12)} U_{(12)}^{-1} = \begin{bmatrix} \sigma_{(12)}^{11} & \sigma_{(12)}^{12} \\ \sigma_{(12)}^{12} & \sigma_{(12)}^{22} \\ 0 & 0 \end{bmatrix} \qquad M_{(23)} U_{(23)}^{-1} = \begin{bmatrix} 0 & 0 \\ \sigma_{(23)}^{11} & \sigma_{(23)}^{12} \\ \sigma_{(23)}^{12} & \sigma_{(23)}^{22} \end{bmatrix}$$

In general, $M_iU_i^{-1}$ has zeros in the rows corresponding to the variables not contained in the i'th group. The submatrix U_i^{-1} occupies the rows corresponding to the variables contained in the group.

The matrices $M_iU_i^{-1}M_i'$ for groups (12) and (23) are

$$M_{(12)}U_{(12)}^{-1}M_{(12)}' = \begin{bmatrix} \sigma_{(12)}^{11} & \sigma_{(12)}^{12} & 0 \\ \sigma_{(12)}^{12} & \sigma_{(12)}^{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{(23)}U_{(23)}^{-1}M_{(23)}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{(23)}^{11} & \sigma_{(23)}^{12} \\ 0 & \sigma_{(23)}^{12} & \sigma_{(23)}^{22} \end{bmatrix}$$

In general, these matrices have zeros in the rows and columns corresponding to the variables which are absent in the i'th group. The submatrix U_i^{-1} occupies the row and column positions corresponding to the variables present in the i'th group.

In the matrix $U_i^{[k\ell]} = M_i'\Sigma^{[k\ell]}M_i$, the pre-factor and post-factor matrices select the u rows and columns of $\Sigma^{[k\ell]}$ corresponding to the variables contained in the i'th group. If $k=\ell$ and the i'th group does not contain variable k or if $k\neq\ell$ and the i'th group does not contain both variables k and ℓ, then $U_i^{[k\ell]}$ is a null matrix. If $k=\ell$ and the i'th group contains variable k, then $U_i^{[k\ell]}$ has a one in the diagonal position corresponding to the rank of variable k within the i'th group and zeros elsewhere. If $k\neq\ell$ and the i'th group contains the variable pair $(k\ell)$ then $U_i^{[k\ell]}$ has ones in the rs and sr positions, where r is the rank of variable k and s is the rank of variable ℓ in the i'th group.

Thus for the group discussed above, containing variables one and two

$$U_{(12)}^{[11]} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad U_{(12)}^{[12]} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad U_{(12)}^{[13]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U_{(12)}^{[22]} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad U_{(12)}^{[23]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad U_{(12)}^{[33]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

These matrices may be written

$$U_{(12)}^{[k\ell]} = \begin{bmatrix} \delta_{1k}\delta_{1\ell} & \delta_{1k}\delta_{2\ell} \\ \delta_{1k}\delta_{2\ell} & \delta_{2k}\delta_{2\ell} \end{bmatrix} \qquad k \leq \ell$$

For the group containing variables two and three we have

$$U_{(23)}^{[k\ell]} = \begin{bmatrix} \delta_{2k}\delta_{2\ell} & \delta_{2k}\delta_{3\ell} \\ \delta_{2k}\delta_{3\ell} & \delta_{3k}\delta_{3\ell} \end{bmatrix} \qquad k \leq \ell$$

5.3 Components of the Maximum Likelihood Equation

Aside from the matrices discussed in the preceding section, the only factor of the maximum likelihood equation that is affected by the design of the variables is the matrix V. Since this matrix is computed only once in each iteration (in the process of obtaining $\phi(\Sigma)$), while some factors of the $\phi^{[k\ell]}(\Sigma)$ matrix are computed as many times as there are variables and pairs of variables in the design, simplification of this matrix by design of the variables might not appear to be so important at first glance. However the matrix V is used to pre-multiply and post-multiply the matrices $V^{[k\ell]}$ in the course of computing $\phi^{[k\ell]}(\Sigma)$. Furthermore, in practice, after convergence is approached in the Newton iterative method, the same matrix $\nabla\phi$ of derivatives is used repeatedly for successive iterations. This is equivalent to the use of an unchanged $f'(x)$ in the denominator in the standard Newton iterative method for one variable, when the numerator of $f(x)/f'(x)$ becomes small, and thus the slight change of the denominator from one iteration to the next produces no appreciable difference in the correction term. Hence after a certain stage only the matrix $\phi(\Sigma)$, and not the $\phi^{[k\ell]}(\Sigma)$ matrices, needs to be computed.

The 7 x 7 matrix V for the balanced design with p=7, u=3 described in section 5.1 has the form

$$\begin{bmatrix} \sigma^{11}_{(124)}+\sigma^{11}_{(156)}+\sigma^{11}_{(137)} & \sigma^{12}_{(124)} & \cdots & \sigma^{13}_{(137)} \\ \sigma^{12}_{(124)} & \sigma^{22}_{(124)}+\sigma^{11}_{(235)}+\sigma^{11}_{(267)} & \cdots & \sigma^{13}_{(267)} \\ \sigma^{12}_{(137)} & \sigma^{12}_{(235)} & \cdots & \sigma^{23}_{(137)} \\ \sigma^{13}_{(124)} & \sigma^{23}_{(124)} & \cdots & \sigma^{13}_{(457)} \\ \sigma^{12}_{(156)} & \sigma^{13}_{(235)} & \cdots & \sigma^{23}_{(457)} \\ \sigma^{13}_{(156)} & \sigma^{12}_{(267)} & \cdots & \sigma^{23}_{(267)} \\ \sigma^{13}_{(137)} & \sigma^{13}_{(267)} & \cdots & \sigma^{33}_{(457)}+\sigma^{33}_{(267)}+\sigma^{33}_{(137)} \end{bmatrix}$$

The matrix U_i^{-1} occupies the u^2 positions in the rows and columns of V corresponding to the variables in the i'th group. Thus the matrix $U_{(124)}^{-1}$ is obtained by inverting the square submatrix of Σ formed from rows and columns one, two, and four. This inverse matrix occupies rows and columns one, two, and four of V. The U_i^{-1} matrices overlap so that in certain positions of V there are sums of elements from the U_i^{-1} for several groups. If variable m occurs r times in the design, there will be r terms in the m'th diagonal position. If the pair mn occurs λ times, there will be λ terms in the (mn)'th position. If the pair mn does not occur in the same group, there will be a zero in the (mn)'th position.

5.4 Components of the Derivative Equation

Of the matrices affected by the design of the variables, $V^{[k\ell]}$ is the most frequently used in the calculation of $\phi^{[k\ell]}(\Sigma) = 0$, the derivative equation. $V^{[k\ell]}$ is the first term of $\phi^{[k\ell]}(\Sigma)$ and is used twice in the computation of $R_i^{\{k\ell\}}$, a factor of the second term of $\phi^{[k\ell]}(\Sigma)$. (See formulas (3.1.12), (3.1.22), and (3.2.9)).

We shall show in detail a typical $V^{[kk]}$ and $V^{[k\ell]}$ $(k\neq\ell)$ matrix for the balanced design with p=7, u=3 described in section 5.1. To construct $V^{[11]}$ for this design, we first select the groups which include variable one, namely (124), (156), and (137). (The factor $U_i^{[11]}$ in $V^{[11]}$ is a null matrix for any group which does not include variable one.) The inverses of the submatrices of Σ

$$U_{(124)} = \begin{bmatrix} \sigma_{11}\sigma_{12}\sigma_{14} \\ \sigma_{12}\sigma_{22}\sigma_{24} \\ \sigma_{14}\sigma_{24}\sigma_{44} \end{bmatrix} \qquad U_{(156)} = \begin{bmatrix} \sigma_{11}\sigma_{15}\sigma_{16} \\ \sigma_{15}\sigma_{55}\sigma_{56} \\ \sigma_{16}\sigma_{56}\sigma_{66} \end{bmatrix} \qquad U_{(137)} = \begin{bmatrix} \sigma_{11}\sigma_{13}\sigma_{17} \\ \sigma_{13}\sigma_{33}\sigma_{37} \\ \sigma_{17}\sigma_{37}\sigma_{77} \end{bmatrix}$$

will be denoted respectively by

$$U^{-1}_{(124)} = \begin{bmatrix} \sigma^{11}_{(124)} & \sigma^{12}_{(124)} & \sigma^{13}_{(124)} \\ \sigma^{12}_{(124)} & \sigma^{22}_{(124)} & \sigma^{23}_{(124)} \\ \sigma^{13}_{(124)} & \sigma^{23}_{(124)} & \sigma^{33}_{(124)} \end{bmatrix},$$

$$U^{-1}_{(156)} \quad \begin{bmatrix} \sigma^{11}_{(156)} & \sigma^{12}_{(156)} & \sigma^{13}_{(156)} \\ \sigma^{12}_{(156)} & \sigma^{22}_{(156)} & \sigma^{23}_{(156)} \\ \sigma^{13}_{(156)} & \sigma^{23}_{(156)} & \sigma^{33}_{(156)} \end{bmatrix},$$

$$U^{-1}_{(137)} \quad \begin{bmatrix} \sigma^{11}_{(137)} & \sigma^{12}_{(137)} & \sigma^{13}_{(137)} \\ \sigma^{12}_{(137)} & \sigma^{22}_{(137)} & \sigma^{23}_{(137)} \\ \sigma^{13}_{(137)} & \sigma^{23}_{(137)} & \sigma^{33}_{(137)} \end{bmatrix},$$

Then

$$V^{[11]} = -\sum_{i=1}^{7} (M_i U_i^{-1}) U_i^{[11]} (U_i^{-1} M_i) =$$

$$-\begin{bmatrix} \sigma^{11}_{(124)} & \sigma^{12}_{(124)} & \sigma^{13}_{(124)} \\ \sigma^{12}_{(124)} & \sigma^{22}_{(124)} & \sigma^{23}_{(124)} \\ 0 & 0 & 0 \\ \sigma^{13}_{(124)} & \sigma^{23}_{(124)} & \sigma^{33}_{(124)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma^{11}_{(124)} & \sigma^{12}_{(124)} & 0 & \sigma^{13}_{(124)} & 0 & 0 & 0 \\ \sigma^{12}_{(124)} & \sigma^{22}_{(124)} & 0 & \sigma^{23}_{(124)} & 0 & 0 & 0 \\ \sigma^{13}_{(124)} & \sigma^{23}_{(124)} & 0 & \sigma^{33}_{(124)} & 0 & 0 & 0 \end{bmatrix}$$

$$
- \begin{bmatrix} \sigma^{11}_{(156)} & \sigma^{12}_{(156)} & \sigma^{13}_{(156)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{12}_{(156)} & \sigma^{22}_{(156)} & \sigma^{23}_{(156)} \\ \sigma^{13}_{(156)} & \sigma^{23}_{(156)} & \sigma^{33}_{(156)} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma^{11}_{(156)} & 0 & 0 & 0 & \sigma^{12}_{(156)} & \sigma^{13}_{(156)} & 0 \\ \sigma^{12}_{(156)} & 0 & 0 & 0 & \sigma^{22}_{(156)} & \sigma^{23}_{(156)} & 0 \\ \sigma^{13}_{(156)} & 0 & 0 & 0 & \sigma^{23}_{(156)} & \sigma^{33}_{(156)} & 0 \end{bmatrix}
$$

$$
- \begin{bmatrix} \sigma^{11}_{(137)} & \sigma^{12}_{(137)} & \sigma^{13}_{(137)} \\ 0 & 0 & 0 \\ \sigma^{12}_{(137)} & \sigma^{22}_{(137)} & \sigma^{23}_{(137)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{13}_{(137)} & \sigma^{23}_{(137)} & \sigma^{33}_{(137)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma^{11}_{(137)} & 0 & \sigma^{12}_{(137)} & 0 & 0 & 0 & \sigma^{13}_{(137)} \\ \sigma^{12}_{(137)} & 0 & \sigma^{22}_{(137)} & 0 & 0 & 0 & \sigma^{23}_{(137)} \\ \sigma^{13}_{(137)} & 0 & \sigma^{23}_{(137)} & 0 & 0 & 0 & \sigma^{33}_{(137)} \end{bmatrix}
$$

since $U_1^{[11]} = U_{(124)}^{[11]}$, $U_2^{[11]} = U_{(156)}^{[11]}$, etc.

Hence

(5.4.1)
$$
V^{[11]} = \begin{array}{c}
\begin{array}{ccccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad & \quad 6 \quad & \quad 7 \quad \end{array} \\
\left[\begin{array}{ccccccc}
[\sigma^{11}_{(124)}]^2+[\sigma^{11}_{(156)}]^2+[\sigma^{11}_{(137)}]^2 & \sigma^{11}_{(124)}\sigma^{12}_{(124)} & \sigma^{11}_{(137)}\sigma^{12}_{(137)} & \sigma^{11}_{(124)}\sigma^{13}_{(124)} & \sigma^{11}_{(156)}\sigma^{12}_{(156)} & \sigma^{11}_{(156)}\sigma^{13}_{(156)} & \sigma^{11}_{(137)}\sigma^{13}_{(137)} \\
 & [\sigma^{12}_{(124)}]^2 & 0 & \sigma^{12}_{(124)}\sigma^{13}_{(124)} & 0 & 0 & 0 \\
 & & [\sigma^{12}_{(137)}]^2 & 0 & 0 & 0 & \sigma^{12}_{(137)}\sigma^{13}_{(137)} \\
 & & & [\sigma^{13}_{(124)}]^2 & 0 & 0 & 0 \\
 & & & & [\sigma^{12}_{(156)}]^2 & \sigma^{12}_{(156)}\sigma^{13}_{(156)} & 0 \\
 & & & & & [\sigma^{13}_{(156)}]^2 & 0 \\
 & & & & & & [\sigma^{13}_{(137)}]^2
\end{array}\right]^{*}
\end{array}
$$

To construct $V^{[12]}$ select the single group (124) which contains the pair of variables (12). Then

---

*Large matrices are presented in two parts. The headings 1, 2, 3,...7 indicate the column numbers. Omission of the elements below the principal diagonal indicates that the matrix is symmetrical.

(5.4.2)

$$V^{[12]} = -\begin{bmatrix} \sigma^{11}_{(124)} & \sigma^{12}_{(124)} & \sigma^{13}_{(124)} \\ \sigma^{12}_{(124)} & \sigma^{22}_{(124)} & \sigma^{23}_{(124)} \\ 0 & 0 & 0 \\ \sigma^{13}_{(124)} & \sigma^{23}_{(124)} & \sigma^{33}_{(124)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma^{11}_{(124)} & \sigma^{12}_{(124)} & 0 & \sigma^{13}_{(124)} & 0 & 0 & 0 \\ \sigma^{12}_{(124)} & \sigma^{22}_{(124)} & 0 & \sigma^{23}_{(124)} & 0 & 0 & 0 \\ \sigma^{13}_{(124)} & \sigma^{23}_{(124)} & 0 & \sigma^{33}_{(124)} & 0 & 0 & 0 \end{bmatrix}$$

$$= -\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 2\sigma^{12}_{(124)}\sigma^{11}_{(124)} & (\sigma^{12}_{(124)})^2+\sigma^{11}_{(124)}\sigma^{22}_{(124)} & 0 & \sigma^{12}_{(124)}\sigma^{13}_{(124)}+\sigma^{11}_{(124)}\sigma^{23}_{(124)} & 0 & 0 & 0 \\ & 2\sigma^{22}_{(124)}\sigma^{12}_{(124)} & 0 & \sigma^{22}_{(124)}\sigma^{13}_{(124)}+\sigma^{12}_{(124)}\sigma^{23}_{(124)} & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 \\ & & & 2\sigma^{23}_{(124)}\sigma^{13}_{(124)} & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix} \end{array}$$

A general method for the construction of the matrices $V^{[kk]}$ and $V^{[k\ell]}$ $(k\neq\ell)$ may now be stated. To construct $V^{[55]}$, for example, first select the groups which contain variable five and arrange the variable numbers within each group in the natural order: (235), (457), (156). The symbols $\sigma_{(235)}\sigma_{(235)}$, $\sigma_{(457)}\sigma_{(457)}$, and $\sigma_{(156)}\sigma_{(156)}$ (without superscripts) will be referred to as sigma products associated with groups (235), (457), and (156) respectively. In the m'th diagonal position fill in the sigma products or sum of products associated with the group or groups which contain variable m. If none of the selected groups contains variable m, fill in a zero in the m'th diagonal position. In the non-diagonal position (m,n) fill in the sigma product or sum of products associated with the group or groups which contain the pair of variables m and n. If none of the three groups (235), (457), (156) contains the variable pair mn, fill in a zero in the (m,n)'th position. Thus as a first step in the construction of $V^{[55]}$ we have the array

$$
\begin{array}{c}
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}\\
\left[\begin{array}{ccccccc}
\sigma_{(156)}\sigma_{(156)} & 0 & 0 & 0 & \sigma_{(156)}\sigma_{(156)} & \sigma_{(156)}\sigma_{(156)} & 0 \\
0 & \sigma_{(235)}\sigma_{(235)} & \sigma_{(235)}\sigma_{(235)} & 0 & \sigma_{(235)}\sigma_{(235)} & 0 & 0 \\
0 & \sigma_{(235)}\sigma_{(235)} & \sigma_{(235)}\sigma_{(235)} & 0 & \sigma_{(235)}\sigma_{(235)} & 0 & 0 \\
0 & 0 & 0 & \sigma_{(457)}\sigma_{(457)} & \sigma_{(457)}\sigma_{(457)} & 0 & \sigma_{(457)}\sigma_{(457)} \\
\sigma_{(156)}\sigma_{(156)} & \sigma_{(235)}\sigma_{(235)} & \sigma_{(235)}\sigma_{(235)} & \sigma_{(457)}\sigma_{(457)} & \sigma_{(235)}\sigma_{(235)}+\sigma_{(457)}\sigma_{(457)}+\sigma_{(156)}\sigma_{(156)} & \sigma_{(156)}\sigma_{(156)} & \sigma_{(457)}\sigma_{(457)} \\
\sigma_{(156)}\sigma_{(156)} & 0 & 0 & 0 & \sigma_{(156)}\sigma_{(156)} & \sigma_{(156)}\sigma_{(156)} & 0 \\
0 & 0 & 0 & \sigma_{(457)}\sigma_{(457)} & \sigma_{(457)}\sigma_{(457)} & 0 & \sigma_{(457)}\sigma_{(457)}
\end{array}\right]
\end{array}
$$

In group (235), variable 5 has rank order <u>three</u>. The factors of the $\sigma_{(235)}\sigma_{(235)}$ products are elements from the <u>third</u> row (or column) of the symmetric matrix $U^{-1}_{(235)}$. The first factors in these products in <u>rows</u> 2, 3, and 5 are $\sigma^{13}_{(235)}$, $\sigma^{23}_{(235)}$, and $\sigma^{33}_{(235)}$ respectively. The second factors in these products in <u>columns</u> 2, 3, and 5 are $\sigma^{13}_{(235)}$, $\sigma^{23}_{(235)}$, and $\sigma^{33}_{(235)}$ respectively. In group (457), variable 5 has rank order <u>two</u>. The factors of the $\sigma_{(457)}\sigma_{(457)}$ products are elements from the <u>second</u> row (or column) of $U^{-1}_{(457)}$. The first factors in these products in <u>rows</u> 4, 5, and 7 are $\sigma^{12}_{(457)}$, $\sigma^{22}_{(457)}$, and $\sigma^{23}_{(457)}$ respectively. The second factors in these products in <u>columns</u> 4, 5, and 7 are $\sigma^{12}_{(457)}$, $\sigma^{22}_{(457)}$, and $\sigma^{23}_{(457)}$. In the same way we find that the first factors of the $\sigma_{(156)}\sigma_{(156)}$ products in <u>rows</u> 1, 5, and 6 are $\sigma^{12}_{(156)}$, $\sigma^{22}_{(156)}$, and $\sigma^{23}_{(156)}$ respectively and the second factors in these products in <u>columns</u> 1, 5, and 6 are $\sigma^{12}_{(156)}$, $\sigma^{22}_{(156)}$, and $\sigma^{23}_{(156)}$ respectively. Thus we may write

$$V^{[55]} = -\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\left[\begin{array}{ccccccc}
\sigma^{12}_{(156)}\sigma^{12}_{(156)} & 0 & 0 & 0 & \sigma^{12}_{(156)}\sigma^{22}_{(156)} & \sigma^{12}_{(156)}\sigma^{23}_{(156)} & 0 \\
0 & \sigma^{13}_{(235)}\sigma^{13}_{(235)} & \sigma^{13}_{(235)}\sigma^{23}_{(235)} & 0 & \sigma^{13}_{(235)}\sigma^{33}_{(235)} & 0 & 0 \\
0 & \sigma^{23}_{(235)}\sigma^{13}_{(235)} & \sigma^{23}_{(235)}\sigma^{23}_{(235)} & 0 & \sigma^{23}_{(235)}\sigma^{33}_{(235)} & 0 & 0 \\
0 & 0 & 0 & \sigma^{12}_{(457)}\sigma^{12}_{(457)} & \sigma^{12}_{(457)}\sigma^{22}_{(457)} & 0 & \sigma^{12}_{(457)}\sigma^{23}_{(457)} \\
\sigma^{22}_{(156)}\sigma^{12}_{(156)} & \sigma^{33}_{(235)}\sigma^{13}_{(235)} & \sigma^{33}_{(235)}\sigma^{23}_{(235)} & \sigma^{22}_{(457)}\sigma^{12}_{(457)} & \sigma^{33}_{(235)}\sigma^{33}_{(235)}+\sigma^{22}_{(457)}\sigma^{22}_{(457)}+\sigma^{22}_{(156)}\sigma^{22}_{(156)} & \sigma^{22}_{(156)}\sigma^{23}_{(156)} & \sigma^{22}_{(457)}\sigma^{23}_{(457)} \\
\sigma^{23}_{(156)}\sigma^{12}_{(156)} & 0 & 0 & 0 & \sigma^{23}_{(156)}\sigma^{22}_{(156)} & \sigma^{23}_{(156)}\sigma^{23}_{(156)} & 0 \\
0 & 0 & 0 & \sigma^{23}_{(457)}\sigma^{12}_{(457)} & \sigma^{23}_{(457)}\sigma^{22}_{(457)} & 0 & \sigma^{23}_{(457)}\sigma^{23}_{(457)}
\end{array}\right]
\end{array}$$

In general, to obtain $V^{[kk]}$, select the groups which contain variable k and arrange the variable numbers within each group from small to large. Obtain the array of sigma products and zeros following the procedure given above. If variable k has rank order r in a particular group, the first factors of the group sigma products in the u rows corresponding to the variables contained in the group are the u successive elements of the r'th row (or column) of U_i^{-1}. The second factors of the group sigma products in the u columns corresponding to the variables contained in the i'th group are also the u successive elements in the r'th row (or column) of U_i^{-1}. The elements in the diagonal positions of $V^{[kk]}$ are always squares.

To obtain $V^{[k\ell]}$ $(k \neq \ell)$ select the groups which contain the pair of variables $(k\ell)$. Obtain the array of sigma products in the same way as for $V^{[kk]}$. Suppose that the rank orders of variables k and ℓ in a particular group are r and s respectively. The group sigma products occupy the u rows and columns in the array corresponding to the u variables contained in the particular group. Fill in superscripts for the group sigma products in such a way that the first factors in these u rows are the u successive elements of the r'th row (or column) of U_i^{-1} and the second factors in the u columns are the u successive elements of the s'th row (or column) of U_i^{-1}. If this procedure is followed for all groups which contain both variables k and ℓ and the resulting matrix is then added to its transpose, $V^{[k\ell]}$ is obtained.

For example, to obtain $V^{[47]}$ for the balanced design given in section 5.1, select the single group (457) which contains the variable pair (47). Variables 4 and 7 have ranks $\underline{r=1}$ and $\underline{s=3}$ respectively. The $\sigma_{(457)}\sigma_{(457)}$ products occupy rows and columns 4, 5, and 7 in the array of sigma products. Fill in superscripts for the $\sigma_{(457)}\sigma_{(457)}$ products in such a way that the first factors in <u>rows</u> 4, 5, and 7 respectively are the three successive elements of the <u>first</u> row (or column) of $U^{-1}_{(457)}$ and the second factors in <u>columns</u> 4, 5, and 7 respectively are the three successive elements of the <u>third</u> row (or column) of $U^{-1}_{(457)}$. The following matrix is obtained.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^{11}_{(457)}\sigma^{13}_{(457)} & \sigma^{11}_{(457)}\sigma^{23}_{(457)} & 0 & \sigma^{11}_{(457)}\sigma^{33}_{(457)} \\ 0 & 0 & 0 & \sigma^{12}_{(457)}\sigma^{13}_{(457)} & \sigma^{12}_{(457)}\sigma^{23}_{(457)} & 0 & \sigma^{12}_{(457)}\sigma^{33}_{(457)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^{13}_{(457)}\sigma^{13}_{(457)} & \sigma^{13}_{(457)}\sigma^{13}_{(457)} & 0 & \sigma^{13}_{(457)}\sigma^{33}_{(457)} \end{bmatrix}$$

If this matrix is added to its transpose and the result is multiplied by -1, we obtain

$$V^{[47]} = -\begin{array}{c} \begin{array}{ccccccc} & 1 & & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\sigma^{11}_{(457)}\sigma^{13}_{(457)} & \sigma^{11}_{(457)}\sigma^{23}_{(457)}+\sigma^{12}_{(457)}\sigma^{13}_{(457)} & 0 & \sigma^{11}_{(457)}\sigma^{33}_{(457)}+(\sigma^{13}_{(457)})^2 \\
0 & 0 & 0 & \sigma^{11}_{(457)}\sigma^{23}_{(457)}+\sigma^{12}_{(457)}\sigma^{13}_{(457)} & 2\sigma^{12}_{(457)}\sigma^{23}_{(457)} & 0 & \sigma^{12}_{(457)}\sigma^{33}_{(457)}+\sigma^{13}_{(457)}\sigma^{23}_{(457)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma^{11}_{(457)}\sigma^{33}_{(457)}+(\sigma^{13}_{(457)})^2 & \sigma^{12}_{(457)}\sigma^{33}_{(457)}+\sigma^{13}_{(457)}\sigma^{23}_{(457)} & 0 & 2\sigma^{13}_{(457)}\sigma^{33}_{(457)}
\end{array}\right] \end{array}$$

Another term in $\phi^{[k\ell]}(\Sigma)$ which is affected by the design of the variables is $F^{[k\ell]}$. The first factors in the i'th term of the summation for $F^{[k\ell]}$ are the same as those for $V^{[k\ell]}$ and the mechanical process for constructing $F^{[k\ell]}$ is similar. Let

$$Z_i^{\{k\ell\}} = M_i U_i^{-1} U_i^{[k\ell]} U_i^{-1}$$

Then

$$F^{[k\ell]} = -\sum_{i=1}^{K} Z_i^{\{k\ell\}} X_i' \tag{5.4.3}$$

where X_i is the matrix of standard least squares estimates of the parameters in the i'th group.

In general the p x u matrix $Z_i^{\{k\ell\}}$ may be obtained by selecting those elements associated with the i'th group from the u columns of $V^{[k\ell]}$ associated with the i'th group and filling in zeros in the remaining positions. Thus in the balanced design introduced in section 5.1, $Z_{(124)}^{\{11\}}$ is obtained by selecting those elements associated with group (124) from columns 1, 2, and 4 of $V^{[11]}$, $Z_{(156)}^{\{11\}}$ is obtained by selecting appropriate elements from columns 1, 5, and 6 of $V^{[11]}$, and $Z_{(137)}^{\{11\}}$ is obtained by selecting appropriate elements from columns 1, 3, and 7 of $V^{[11]}$. Thus from (5.4.1) and (5.4.3) we have

$$F^{[11]} = -\begin{bmatrix} (\sigma^{11}_{(124)})^2 & \sigma^{11}_{(124)}\sigma^{12}_{(124)} & \sigma^{11}_{(124)}\sigma^{13}_{(124)} \\ \sigma^{11}_{(124)}\sigma^{12}_{(124)} & (\sigma^{12}_{(124)})^2 & \sigma^{12}_{(124)}\sigma^{13}_{(124)} \\ 0 & 0 & 0 \\ \sigma^{11}_{(124)}\sigma^{13}_{(124)} & \sigma^{12}_{(124)}\sigma^{13}_{(124)} & (\sigma^{13}_{(124)})^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X'_{(124)}$$

$$-\begin{bmatrix} (\sigma^{11}_{(156)})^2 & \sigma^{11}_{(156)}\sigma^{12}_{(156)} & \sigma^{11}_{(156)}\sigma^{13}_{(156)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{11}_{(156)}\sigma^{12}_{(156)} & (\sigma^{12}_{(156)})^2 & \sigma^{12}_{(156)}\sigma^{13}_{(156)} \\ \sigma^{11}_{(156)}\sigma^{13}_{(156)} & \sigma^{12}_{(156)}\sigma^{13}_{(156)} & (\sigma^{13}_{(156)})^2 \\ 0 & 0 & 0 \end{bmatrix} X'_{(156)}$$

$$-\begin{bmatrix} (\sigma^{11}_{(137)})^2 & \sigma^{11}_{(137)}\sigma^{12}_{(137)} & \sigma^{11}_{(137)}\sigma^{13}_{(137)} \\ 0 & 0 & 0 \\ \sigma^{11}_{(137)}\sigma^{12}_{(137)} & (\sigma^{12}_{(137)})^2 & \sigma^{12}_{(137)}\sigma^{13}_{(137)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{11}_{(137)}\sigma^{13}_{(137)} & \sigma^{12}_{(137)}\sigma^{13}_{(137)} & (\sigma^{13}_{(137)})^2 \end{bmatrix} X'_{(137)}$$

Similarly $Z^{\{12\}}_{(124)}$ is obtained by selecting columns 1, 2, and 4 from $V^{[12]}$ in (5.4.2). Hence

$$F^{[12]} = -\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\left[\begin{array}{ccc}
2\sigma^{12}_{(124)}\sigma^{11}_{(124)} & (\sigma^{12}_{(124)})^2+\sigma^{11}_{(124)}\sigma^{22}_{(124)} & \sigma^{12}_{(124)}\sigma^{13}_{(124)}+\sigma^{11}_{(124)}\sigma^{23}_{(124)} \\
(\sigma^{12}_{(124)})^2+\sigma^{11}_{(124)}\sigma^{22}_{(124)} & 2\sigma^{22}_{(124)}\sigma^{12}_{(124)} & \sigma^{22}_{(124)}\sigma^{13}_{(124)}+\sigma^{12}_{(124)}\sigma^{23}_{(124)} \\
0 & 0 & 0 \\
\sigma^{12}_{(124)}\sigma^{13}_{(124)}+\sigma^{11}_{(124)}\sigma^{23}_{(124)} & \sigma^{22}_{(124)}\sigma^{13}_{(124)}+\sigma^{12}_{(124)}\sigma^{23}_{(124)} & 2\sigma^{23}_{(124)}\sigma^{13}_{(124)} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
\end{array} X'_{(124)}$$

A third factor of $\phi^{k\ell}(\Sigma)$ which should be examined for possible simplification is

$$T_i^{\{k\ell\}} = U_i^{[k\ell]} U_i^{-1} P_i P_i'$$

If we let $p_{mn}^{(i)}$ stand for the (m,n)'th element of $P_i P_i'$, then some typical $T_i^{\{k\ell\}}$ matrices for the balanced design introduced in section 5.1 are

$$T_{(124)}^{\{11\}} = \begin{bmatrix} \sigma_{(124)}^{11} & \sigma_{(124)}^{12} & \sigma_{(124)}^{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11}^{(124)} & p_{12}^{(124)} & p_{13}^{(124)} \\ p_{12}^{(124)} & p_{22}^{(124)} & p_{23}^{(124)} \\ p_{13}^{(124)} & p_{23}^{(124)} & p_{33}^{(124)} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{3} \sigma_{(124)}^{1j} p_{1j}^{(124)} & \sum_{j=1}^{3} \sigma_{(124)}^{1j} p_{2j}^{(124)} & \sum_{j=1}^{3} \sigma_{(124)}^{1j} p_{3j}^{(124)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T_{(124)}^{\{22\}} = \begin{bmatrix} 0 & 0 & 0 \\ \sum_{j=1}^{3} \sigma_{(124)}^{2j} p_{1j}^{(124)} & \sum_{j=1}^{3} \sigma_{(124)}^{2j} p_{2j}^{(124)} & \sum_{j=1}^{3} \sigma_{(124)}^{2j} p_{3j}^{(124)} \\ 0 & 0 & 0 \end{bmatrix}$$

$$T_{(124)}^{\{12\}} = \begin{bmatrix} \sum_{j=1}^{3} \sigma_{(124)}^{2j} p_{1j}^{(124)} & \sum_{j=1}^{3} \sigma_{(124)}^{2j} p_{2j}^{(124)} & \sum_{j=1}^{3} \sigma_{(124)}^{2j} p_{3j}^{(124)} \\ \sum_{j=1}^{3} \sigma_{(124)}^{1j} p_{1j}^{(124)} & \sum_{j=1}^{3} \sigma_{(124)}^{1j} p_{2j}^{(124)} & \sum_{j=1}^{3} \sigma_{(124)}^{1j} p_{3j}^{(124)} \\ 0 & 0 & 0 \end{bmatrix}$$

Note that the first row of $T_{(124)}^{\{12\}}$ is the same as the second row of $T_{(124)}^{\{22\}}$ and the second row of $T_{(124)}^{\{12\}}$ is the same as the first row of $T_{124}^{\{11\}}$. In general $T_i^{\{kk\}}$ has only one non-zero row. If the rank of variable k in the i'th group is r, the non-zero elements will be in the r'th row of $T_i^{\{kk\}}$. If the rank of variable ℓ in the i'th group is s, then $T_i^{\{\ell\ell\}}$ will have a single row (the s'th row) of non-zero elements. The r'th row of $T_i^{\{k\ell\}}$ $(k\neq\ell)$ is the same as the s'th row of $T_i^{\{\ell\ell\}}$, while the s'th row of $T_i^{\{k\ell\}}$ is the same as the r'th row of $T_i^{\{kk\}}$. Thus only u distinct non-null $T_i^{\{kk\}}$ matrices need to be calculated for each group. The remaining $T_i^{\{k\ell\}}$ $(k\neq\ell)$ may be obtained from the non-zero rows in $T_i^{\{kk\}}$ and $T_i^{\{\ell\ell\}}$.

5.5 Balanced Designs

5.5a Three Variables in Three Groups of Two.

Group	
(1)	12
(2)	23
(3)	13

The submatrices of Σ for this design are

$$U_1 = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \quad U_2 = \begin{bmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{bmatrix} \quad U_3 = \begin{bmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{bmatrix}$$

These submatrices are also denoted by

$$U_{(12)} = \begin{bmatrix} \sigma_{11}^{(12)} & \sigma_{12}^{(12)} \\ \sigma_{12}^{(12)} & \sigma_{22}^{(12)} \end{bmatrix} \quad U_{(23)} = \begin{bmatrix} \sigma_{11}^{(23)} & \sigma_{12}^{(23)} \\ \sigma_{12}^{(23)} & \sigma_{22}^{(23)} \end{bmatrix} \quad U_{(13)} = \begin{bmatrix} \sigma_{11}^{(13)} & \sigma_{12}^{(13)} \\ \sigma_{12}^{(13)} & \sigma_{22}^{(13)} \end{bmatrix}$$

The inverses of these matrices are

$$U^{-1}_{(12)} = \begin{bmatrix} \sigma^{11}_{(12)} & \sigma^{12}_{(12)} \\ \sigma^{12}_{(12)} & \sigma^{22}_{(12)} \end{bmatrix} \quad U^{-1}_{(23)} = \begin{bmatrix} \sigma^{11}_{(23)} & \sigma^{12}_{(23)} \\ \sigma^{12}_{(23)} & \sigma^{22}_{(23)} \end{bmatrix} \quad U^{-1}_{(13)} = \begin{bmatrix} \sigma^{11}_{(13)} & \sigma^{12}_{(13)} \\ \sigma^{12}_{(13)} & \sigma^{22}_{(13)} \end{bmatrix}$$

The matrices $M_i U_i^{-1} M_i'$ are

$$M_{(12)} U^{-1}_{(12)} M'_{(12)} = \begin{bmatrix} \sigma^{11}_{(12)} & \sigma^{12}_{(12)} & 0 \\ \sigma^{12}_{(12)} & \sigma^{22}_{(12)} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{(23)} U^{-1}_{(23)} M'_{(23)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma^{11}_{(23)} & \sigma^{12}_{(23)} \\ 0 & \sigma^{12}_{(23)} & \sigma^{22}_{(23)} \end{bmatrix}$$

$$M_{(13)} U^{-1}_{(13)} M'_{(13)} = \begin{bmatrix} \sigma^{11}_{(13)} & 0 & \sigma^{12}_{(13)} \\ 0 & 0 & 0 \\ \sigma^{12}_{(13)} & 0 & \sigma^{22}_{(13)} \end{bmatrix}$$

The sum of these three matrices is

$$V = \begin{bmatrix} \sigma^{11}_{(12)}+\sigma^{11}_{(13)} & \sigma^{12}_{(12)} & \sigma^{12}_{(13)} \\ \sigma^{12}_{(12)} & \sigma^{22}_{(12)}+\sigma^{11}_{(23)} & \sigma^{12}_{(23)} \\ \sigma^{12}_{(13)} & \sigma^{12}_{(23)} & \sigma^{22}_{(23)}+\sigma^{22}_{(13)} \end{bmatrix}$$

The matrices $U_i^{[k\ell]} = M_i'\Sigma^{[k\ell]}M_i$ are as follows.

$k=1,\ \ell=1$

$$U_{(12)}^{[11]} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad U_{(23)}^{[11]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad U_{(13)}^{[11]} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$k=1,\ \ell=2$

$$U_{(12)}^{[12]} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad U_{(23)}^{[12]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad U_{(13)}^{[12]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$k=1,\ \ell=3$

$$U_{(12)}^{[13]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad U_{(23)}^{[13]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad U_{(13)}^{[13]} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$k=2,\ \ell=2$

$$U_{(12)}^{[22]} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad U_{(23)}^{[22]} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad U_{(13)}^{[22]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$k=2,\ \ell=3$

$$U_{(12)}^{[23]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad U_{(23)}^{[23]} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad U_{(13)}^{[23]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$k=3,\ \ell=3$

$$U_{(12)}^{[33]} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad U_{(23)}^{[33]} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad U_{(13)}^{[33]} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The $V^{[k\ell]}$ matrices for this design are

$$V^{[11]} = -\begin{bmatrix} [\sigma^{11}_{(12)}]^2+[\sigma^{11}_{(13)}]^2 & \sigma^{11}_{(12)}\sigma^{12}_{(12)} & \sigma^{11}_{(13)}\sigma^{12}_{(13)} \\ \sigma^{11}_{(12)}\sigma^{12}_{(12)} & [\sigma^{12}_{(12)}]^2 & 0 \\ \sigma^{11}_{(13)}\sigma^{12}_{(13)} & 0 & [\sigma^{12}_{(13)}]^2 \end{bmatrix}$$

$$V^{[12]} = -\begin{bmatrix} 2\sigma^{11}_{(12)}\sigma^{12}_{(12)} & \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 & 0 \\ \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 & 2\sigma^{12}_{(12)}\sigma^{22}_{(12)} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V^{[13]} = -\begin{bmatrix} 2\sigma^{11}_{(13)}\sigma^{12}_{(13)} & 0 & \sigma^{11}_{(13)}\sigma^{22}_{(13)}+[\sigma^{12}_{(13)}]^2 \\ 0 & 0 & 0 \\ \sigma^{11}_{(13)}\sigma^{22}_{(13)}+[\sigma^{12}_{(13)}]^2 & 0 & 2\sigma^{12}_{(13)}\sigma^{22}_{(13)} \end{bmatrix}$$

$$V^{[22]} = -\begin{bmatrix} [\sigma^{12}_{(12)}]^2 & \sigma^{12}_{(12)}\sigma^{22}_{(12)} & 0 \\ \sigma^{12}_{(12)}\sigma^{22}_{(12)} & [\sigma^{22}_{(12)}]^2+[\sigma^{11}_{(23)}]^2 & \sigma^{11}_{(23)}\sigma^{12}_{(23)} \\ 0 & \sigma^{11}_{(23)}\sigma^{12}_{(23)} & [\sigma^{12}_{(23)}]^2 \end{bmatrix}$$

$$V^{[23]} = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\sigma^{11}_{(23)}\sigma^{12}_{(23)} & \sigma^{11}_{(23)}\sigma^{22}_{(23)}+[\sigma^{12}_{(23)}]^2 \\ 0 & \sigma^{11}_{(23)}\sigma^{22}_{(23)}+[\sigma^{12}_{(23)}]^2 & 2\sigma^{12}_{(23)}\sigma^{22}_{(23)} \end{bmatrix}$$

$$V^{[33]} = \begin{bmatrix} [\sigma^{12}_{(13)}]^2 & 0 & \sigma^{12}_{(13)}\sigma^{22}_{(13)} \\ 0 & [\sigma^{12}_{(23)}]^2 & \sigma^{12}_{(23)}\sigma^{22}_{(23)} \\ \sigma^{12}_{(13)}\sigma^{22}_{(13)} & \sigma^{12}_{(23)}\sigma^{22}_{(23)} & [\sigma^{22}_{(13)}]^2+[\sigma^{22}_{(23)}]^2 \end{bmatrix}$$

The terms of the $F^{[k\ell]}$ matrices are as follows.

$$F^{[11]} = \begin{bmatrix} [\sigma^{11}_{(12)}]^2 & \sigma^{11}_{(12)}\sigma^{12}_{(12)} \\ \sigma^{11}_{(12)}\sigma^{12}_{(12)} & [\sigma^{12}_{(12)}]^2 \\ 0 & 0 \end{bmatrix} X'_{(12)}$$

$$- \begin{bmatrix} [\sigma^{11}_{(13)}]^2 & \sigma^{11}_{(13)}\sigma^{12}_{(13)} \\ 0 & 0 \\ \sigma^{11}_{(13)}\sigma^{12}_{(13)} & [\sigma^{12}_{(13)}]^2 \end{bmatrix} X'_{(13)}$$

$$F^{[12]} = \begin{bmatrix} 2\sigma^{11}_{(12)}\sigma^{12}_{(12)} & \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 \\ \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 & 2\sigma^{12}_{(12)}\sigma^{22}_{(12)} \\ 0 & 0 \end{bmatrix} X'_{(12)}$$

$$F^{[13]} = \begin{bmatrix} 2\sigma^{11}_{(13)}\sigma^{12}_{(13)} & \sigma^{11}_{(13)}\sigma^{22}_{(13)}+[\sigma^{12}_{(13)}]^2 \\ 0 & 0 \\ \sigma^{11}_{(13)}\sigma^{22}_{(13)}+[\sigma^{12}_{(13)}]^2 & 2\sigma^{12}_{(13)}\sigma^{22}_{(13)} \end{bmatrix} X'_{(13)}$$

$$F^{[22]} = -\begin{bmatrix} [\sigma^{12}_{(12)}]^2 & \sigma^{12}_{(12)}\sigma^{22}_{(12)} \\ \sigma^{12}_{(12)}\sigma^{22}_{(12)} & [\sigma^{22}_{(12)}]^2 \\ 0 & 0 \end{bmatrix} X'_{(12)}$$

$$-\begin{bmatrix} 0 & 0 \\ [\sigma^{11}_{(23)}]^2 & \sigma^{11}_{(23)}\sigma^{12}_{(23)} \\ \sigma^{11}_{(23)}\sigma^{12}_{(23)} & [\sigma^{12}_{(23)}]^2 \end{bmatrix} X'_{(23)}$$

$$F^{[23]} = -\begin{bmatrix} 0 & 0 \\ 2\sigma^{11}_{(23)}\sigma^{12}_{(23)} & \sigma^{11}_{(23)}\sigma^{22}_{(23)}+[\sigma^{12}_{(23)}]^2 \\ \sigma^{11}_{(23)}\sigma^{12}_{(23)}+[\sigma^{12}_{(23)}]^2 & 2\sigma^{12}_{(23)}\sigma^{22}_{(23)} \end{bmatrix} X'_{(23)}$$

$$F^{[33]} = -\begin{bmatrix} 0 & 0 \\ [\sigma^{12}_{(23)}]^2 & \sigma^{12}_{(23)}\sigma^{22}_{(23)} \\ \sigma^{12}_{(23)}\sigma^{22}_{(23)} & [\sigma^{22}_{(23)}]^2 \end{bmatrix} X'_{(23)}$$

$$-\begin{bmatrix} [\sigma^{12}_{(13)}]^2 & \sigma^{12}_{(13)}\sigma^{22}_{(13)} \\ 0 & 0 \\ \sigma^{12}_{(13)}\sigma^{22}_{(13)} & [\sigma^{22}_{(13)}]^2 \end{bmatrix} X'_{(13)}$$

where $X_{(12)}$, $X_{(23)}$, and $X_{(13)}$ are the matrices of standard least squares estimates of the parameters in groups (12), (23), and (13) respectively.

The $T_i^{\{k\ell\}}$ matrices are listed below according to groups. The symbol p_{mn} (with superscripts indicating the variables in the i'th group) is used to denote the (m,n) term of the product P_iP_i'. There is no simple way to obtain the elements of P_iP_i'. However, the matrix P_iP_i' is small (of order u x u). The formula for P_iP_i' is given in step 18 of the computational procedure outlined in section 3.4. The equation for $T_i^{\{k\ell\}}$ is given in step 23.

Group One

$$T_{(12)}^{\{11\}} = \begin{bmatrix} \sigma_{(12)}^{11}p_{11}^{(12)}+\sigma_{(12)}^{12}p_{12}^{(12)} & \sigma_{(12)}^{11}p_{12}^{(12)}+\sigma_{(12)}^{12}p_{22}^{(12)} \\ 0 & 0 \end{bmatrix}$$

$$T_{(12)}^{\{12\}} = \begin{bmatrix} \sigma_{(12)}^{12}p_{11}^{(12)}+\sigma_{(12)}^{22}p_{12}^{(12)} & \sigma_{(12)}^{12}p_{12}^{(12)}+\sigma_{(12)}^{22}p_{22}^{(12)} \\ \sigma_{(12)}^{11}p_{11}^{(12)}+\sigma_{(12)}^{12}p_{12}^{(12)} & \sigma_{(12)}^{11}p_{12}^{(12)}+\sigma_{(12)}^{12}p_{22}^{(12)} \end{bmatrix}$$

$$T_{(12)}^{\{22\}} = \begin{bmatrix} 0 & 0 \\ \sigma_{(12)}^{12}p_{11}^{(12)}+\sigma_{(12)}^{22}p_{12}^{(12)} & \sigma_{(12)}^{12}p_{12}^{(12)}+\sigma_{(12)}^{22}p_{22}^{(12)} \end{bmatrix}$$

$T_{(12)}^{\{13\}}$, $T_{(12)}^{\{23\}}$, and $T_{(12)}^{\{33\}}$ are null matrices.

Group Two

$$T_{(23)}^{\{22\}} = \begin{bmatrix} \sigma_{(23)}^{11}p_{11}^{(23)}+\sigma_{(23)}^{12}p_{12}^{(23)} & \sigma_{(23)}^{11}p_{12}^{(23)}+\sigma_{(23)}^{12}p_{22}^{(23)} \\ 0 & 0 \end{bmatrix}$$

$$T_{(23)}^{\{23\}} \quad \begin{bmatrix} \sigma_{(23)}^{12}P_{11}^{(23)}+\sigma_{(23)}^{22}P_{12}^{(23)} & \sigma_{(23)}^{12}P_{12}^{(23)}+\sigma_{(23)}^{22}P_{22}^{(23)} \\ \sigma_{(23)}^{11}P_{11}^{(23)}+\sigma_{(23)}^{12}P_{12}^{(23)} & \sigma_{(23)}^{11}P_{12}^{(23)}+\sigma_{(23)}^{12}P_{22}^{(23)} \end{bmatrix}$$

$$T_{(23)}^{\{33\}} = \begin{bmatrix} 0 & 0 \\ \sigma_{(23)}^{12}P_{11}^{(23)}+\sigma_{(23)}^{22}P_{12}^{(23)} & \sigma_{(23)}^{12}P_{12}^{(23)}+\sigma_{(23)}^{22}P_{22}^{(23)} \end{bmatrix}$$

$T_{(23)}^{\{11\}}$, $T_{(23)}^{\{12\}}$, and $T_{(23)}^{\{13\}}$ are null.

Group Three

$$T_{(13)}^{\{11\}} = \begin{bmatrix} \sigma_{(13)}^{11}P_{11}^{(13)}+\sigma_{(13)}^{12}P_{12}^{(13)} & \sigma_{(13)}^{11}P_{12}^{(13)}+\sigma_{(13)}^{12}P_{22}^{(13)} \\ 0 & 0 \end{bmatrix}$$

$$T_{(13)}^{\{13\}} = \begin{bmatrix} \sigma_{(13)}^{12}P_{11}^{(13)}+\sigma_{(13)}^{22}P_{12}^{(13)} & \sigma_{(13)}^{12}P_{12}^{(13)}+\sigma_{(13)}^{22}P_{22}^{(13)} \\ \sigma_{(13)}^{11}P_{11}^{(13)}+\sigma_{(13)}^{12}P_{12}^{(13)} & \sigma_{(13)}^{11}P_{12}^{(13)}+\sigma_{(13)}^{12}P_{22}^{(13)} \end{bmatrix}$$

$$T_{(13)}^{\{33\}} = \begin{bmatrix} 0 & 0 \\ \sigma_{(13)}^{12}P_{11}^{(13)}+\sigma_{(13)}^{22}P_{12}^{(13)} & \sigma_{(13)}^{12}P_{12}^{(13)}+\sigma_{(13)}^{22}P_{22}^{(13)} \end{bmatrix}$$

$T_{(13)}^{\{12\}}$, $T_{(13)}^{\{22\}}$, and $T_{(13)}^{\{23\}}$ are null.

5.5b Four Variables in Six Groups of Two.

Group

(1)	1	2	(4)	2	3
(2)	1	3	(5)	2	4
(3)	1	4	(6)	3	4

The submatrices of Σ for this design are

$$U_1 = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \quad U_2 = \begin{bmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{bmatrix} \quad U_3 = \begin{bmatrix} \sigma_{11} & \sigma_{14} \\ \sigma_{14} & \sigma_{44} \end{bmatrix}$$

$$U_4 = \begin{bmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{bmatrix} \quad U_5 = \begin{bmatrix} \sigma_{22} & \sigma_{24} \\ \sigma_{24} & \sigma_{44} \end{bmatrix} \quad U_6 = \begin{bmatrix} \sigma_{33} & \sigma_{34} \\ \sigma_{34} & \sigma_{44} \end{bmatrix}$$

These submatrices are also written as

$$U_{(12)} = \begin{bmatrix} \sigma_{11}^{(12)} & \sigma_{12}^{(12)} \\ \sigma_{12}^{(12)} & \sigma_{22}^{(12)} \end{bmatrix} \quad U_{(13)} = \begin{bmatrix} \sigma_{11}^{(13)} & \sigma_{12}^{(13)} \\ \sigma_{12}^{(13)} & \sigma_{22}^{(13)} \end{bmatrix} \quad U_{(14)} = \begin{bmatrix} \sigma_{11}^{(14)} & \sigma_{12}^{(14)} \\ \sigma_{12}^{(14)} & \sigma_{22}^{(14)} \end{bmatrix}$$

$$U_{(23)} = \begin{bmatrix} \sigma_{11}^{(23)} & \sigma_{12}^{(23)} \\ \sigma_{12}^{(23)} & \sigma_{22}^{(23)} \end{bmatrix} \quad U_{(24)} = \begin{bmatrix} \sigma_{11}^{(24)} & \sigma_{12}^{(24)} \\ \sigma_{12}^{(24)} & \sigma_{22}^{(24)} \end{bmatrix} \quad U_{(34)} = \begin{bmatrix} \sigma_{11}^{(34)} & \sigma_{12}^{(34)} \\ \sigma_{12}^{(34)} & \sigma_{22}^{(34)} \end{bmatrix}$$

The matrices $M_i U_i^{-1} M_i'$ are

$$M_{(12)} U_{(12)}^{-1} M'_{(12)} = \begin{bmatrix} \sigma_{(12)}^{11} & \sigma_{(12)}^{12} & 0 & 0 \\ \sigma_{(12)}^{12} & \sigma_{(12)}^{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{(13)} U_{(13)}^{-1} M'_{(13)} = \begin{bmatrix} \sigma_{(13)}^{11} & 0 & \sigma_{(13)}^{12} & 0 \\ 0 & 0 & 0 & 0 \\ \sigma_{(13)}^{12} & 0 & \sigma_{(13)}^{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{(14)}U^{-1}_{(14)}M'_{(14)} = \begin{bmatrix} \sigma^{11}_{(14)} & 0 & 0 & \sigma^{12}_{(14)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma^{12}_{(14)} & 0 & 0 & \sigma^{22}_{(14)} \end{bmatrix}$$

$$M_{(23)}U^{-1}_{(23)}M'_{(23)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma^{11}_{(23)} & \sigma^{12}_{(23)} & 0 \\ 0 & \sigma^{12}_{(23)} & \sigma^{22}_{(23)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{(24)}U^{-1}_{(24)}M'_{(24)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma^{11}_{(24)} & 0 & \sigma^{12}_{(24)} \\ 0 & 0 & 0 & 0 \\ 0 & \sigma^{12}_{(24)} & 0 & \sigma^{22}_{(24)} \end{bmatrix}$$

$$M_{(34)}U^{-1}_{(34)}M'_{(34)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^{11}_{(34)} & \sigma^{12}_{(34)} \\ 0 & 0 & \sigma^{12}_{(34)} & \sigma^{22}_{(34)} \end{bmatrix}$$

These matrices add to

$$
V = \begin{array}{c}
\begin{array}{cccc} \quad\quad 1 \quad\quad & \quad\quad 2 \quad\quad & \quad\quad 3 \quad\quad & \quad\quad 4 \quad\quad \end{array} \\
\begin{bmatrix}
\sigma^{11}_{(12)}+\sigma^{11}_{(13)}+\sigma^{11}_{(14)} & \sigma^{12}_{(12)} & \sigma^{12}_{(13)} & \sigma^{12}_{(14)} \\
\sigma^{12}_{(12)} & \sigma^{22}_{(12)}+\sigma^{11}_{(23)}+\sigma^{11}_{(24)} & \sigma^{12}_{(23)} & \sigma^{12}_{(24)} \\
\sigma^{12}_{(13)} & \sigma^{12}_{(23)} & \sigma^{22}_{(13)}+\sigma^{22}_{(23)}+\sigma^{11}_{(34)} & \sigma^{12}_{(34)} \\
\sigma^{12}_{(14)} & \sigma^{12}_{(24)} & \sigma^{12}_{(34)} & \sigma^{22}_{(14)}+\sigma^{22}_{(24)}+\sigma^{22}_{(34)}
\end{bmatrix}
\end{array}
$$

The matrices $U^{[k\ell]}_i = M'_i \Sigma^{[k\ell]} M_i$ are as follows.

$k=1,\ \ell=1$

$$
U^{[11]}_{(12)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad U^{[11]}_{(13)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad U^{[11]}_{(14)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
$$

$U^{[11]}_{(23)}$, $U^{[11]}_{(24)}$, and $U^{[11]}_{(34)}$ are null.

$k=1,\ \ell=2$

$$
U^{[12]}_{(12)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
$$

$U^{[12]}_{(13)}$, $U^{[12]}_{(14)}$, $U^{[12]}_{(23)}$, $U^{[12]}_{(24)}$, and $U^{[12]}_{(34)}$ are null.

$k=1,\ \ell=3$

$$U^{[13]}_{(13)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$U^{[13]}_{(12)}$, $U^{[13]}_{(14)}$, $U^{[13]}_{(23)}$, $U^{[13]}_{(24)}$, and $U^{[13]}_{(34)}$ are null matrices.

$k=1,\ \ell=4$

$$U^{[14]}_{(14)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$U^{[14]}_{(12)}$, $U^{[14]}_{(13)}$, $U^{[14]}_{(23)}$, $U^{[14]}_{(24)}$, and $U^{[14]}_{(34)}$ are null.

$k=2,\ \ell=2$

$$U^{[22]}_{(12)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad U^{[22]}_{(23)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad U^{[22]}_{(24)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$U^{[22]}_{(13)}$, $U^{[22]}_{(14)}$, and $U^{[22]}_{(34)}$ are null matrices.

$k=2,\ \ell=3$

$$U^{[23]}_{(23)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$U^{[23]}_{(12)}$, $U^{[23]}_{(13)}$, $U^{[23]}_{(14)}$, $U^{[23]}_{(24)}$, and $U^{[23]}_{(34)}$ are null.

$k=2,\ \ell=4$

$$U^{[24]}_{(24)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$U^{[24]}_{(12)}$, $U^{[24]}_{(13)}$, $U^{[24]}_{(14)}$, $U^{[24]}_{(23)}$, and $U^{[24]}_{(34)}$ are null.

$k=3,\ \ell=3$

$$U^{[33]}_{(13)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad U^{[33]}_{(23)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad U^{[33]}_{(34)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$U^{[33]}_{(12)}$, $U^{[33]}_{(14)}$, and $U^{[33]}_{(24)}$ are null.

$k=3,\ \ell=4$

$$U^{[34]}_{(34)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$U^{[34]}_{(12)}$, $U^{[34]}_{(13)}$, $U^{[34]}_{(14)}$, $U^{[34]}_{(23)}$, and $U^{[34]}_{(24)}$ are null.

$k=4,\ \ell=4$

$$U^{[44]}_{(14)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad U^{[44]}_{(24)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad U^{[44]}_{(34)} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$U^{[44]}_{(12)}$, $U^{[44]}_{(13)}$, and $U^{[44]}_{(23)}$ are null.

The $V^{[k\ell]}$ matrices for this design are

$$V^{[11]} = \begin{array}{c} \begin{array}{cc} \qquad\qquad 1 \qquad\qquad & \qquad 2 \end{array} \\ \left[\begin{array}{cc} [\sigma^{11}_{(12)}]^2+[\sigma^{11}_{(13)}]^2+[\sigma^{11}_{(14)}]^2 & \sigma^{11}_{(12)}\sigma^{12}_{(12)} \\ \sigma^{11}_{(12)}\sigma^{12}_{(12)} & [\sigma^{12}_{(12)}]^2 \\ \sigma^{11}_{(13)}\sigma^{12}_{(13)} & 0 \\ \sigma^{11}_{(14)}\sigma^{12}_{(14)} & 0 \end{array} \right. \end{array}$$

$$
\begin{array}{cc}
3 & 4 \\
\left.\begin{matrix}
\sigma^{11}_{(13)}\sigma^{12}_{(13)} & \sigma^{11}_{(14)}\sigma^{12}_{(14)} \\
0 & 0 \\
[\sigma^{12}_{(13)}]^2 & 0 \\
0 & [\sigma^{12}_{(14)}]^2
\end{matrix}\right]
\end{array}
$$

$$
V^{[12]} = -\begin{bmatrix}
2\sigma^{11}_{(12)}\sigma^{12}_{(12)} & \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 & 0 & 0 \\
\sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 & 2\sigma^{12}_{(12)}\sigma^{22}_{(12)} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
V^{[13]} = -\begin{bmatrix}
2\sigma^{11}_{(13)}\sigma^{12}_{(13)} & 0 & \sigma^{11}_{(13)}\sigma^{22}_{(13)}+[\sigma^{12}_{(13)}]^2 & 0 \\
0 & 0 & 0 & 0 \\
\sigma^{11}_{(13)}\sigma^{22}_{(13)}+[\sigma^{12}_{(13)}]^2 & 0 & 2\sigma^{12}_{(13)}\sigma^{22}_{(13)} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
V^{[14]} = -\begin{bmatrix}
2\sigma^{11}_{(14)}\sigma^{12}_{(14)} & 0 & 0 & \sigma^{11}_{(14)}\sigma^{22}_{(14)}+[\sigma^{12}_{(14)}]^2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\sigma^{11}_{(14)}\sigma^{12}_{(14)}+[\sigma^{12}_{(14)}]^2 & 0 & 0 & 2\sigma^{12}_{(14)}\sigma^{22}_{(14)}
\end{bmatrix}
$$

$$V^{[22]} = -\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} [\sigma^{11}_{(12)}]^2 & \sigma^{11}_{(12)}\sigma^{12}_{(12)} & 0 & 0 \\ \sigma^{11}_{(12)}\sigma^{12}_{(12)} & [\sigma^{12}_{(12)}]^2+[\sigma^{11}_{(23)}]^2+[\sigma^{11}_{(24)}]^2 & \sigma^{11}_{(23)}\sigma^{12}_{(23)} & \sigma^{11}_{(24)}\sigma^{12}_{(24)} \\ 0 & \sigma^{11}_{(23)}\sigma^{12}_{(23)} & [\sigma^{12}_{(23)}]^2 & 0 \\ 0 & \sigma^{11}_{(24)}\sigma^{12}_{(24)} & 0 & [\sigma^{12}_{(24)}]^2 \end{bmatrix} \end{array}$$

$$V^{[23]} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\sigma^{11}_{(23)}\sigma^{12}_{(23)} & \sigma^{11}_{(23)}\sigma^{22}_{(23)}+[\sigma^{12}_{(23)}]^2 & 0 \\ 0 & \sigma^{11}_{(23)}\sigma^{22}_{(23)}+[\sigma^{12}_{(23)}]^2 & 2\sigma^{12}_{(23)}\sigma^{22}_{(23)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V^{[24]} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\sigma^{11}_{(24)}\sigma^{12}_{(24)} & 0 & \sigma^{11}_{(24)}\sigma^{22}_{(24)}+[\sigma^{12}_{(24)}]^2 \\ 0 & 0 & 0 & 0 \\ 0 & \sigma^{11}_{(24)}\sigma^{22}_{(24)}+[\sigma^{12}_{(24)}]^2 & 0 & 2\sigma^{12}_{(24)}\sigma^{22}_{(24)} \end{bmatrix}$$

$$V^{[33]} = -\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} [\sigma^{11}_{(13)}]^2 & 0 & \sigma^{11}_{(13)}\sigma^{12}_{(13)} & 0 \\ 0 & [\sigma^{11}_{(23)}]^2 & \sigma^{11}_{(23)}\sigma^{12}_{(23)} & 0 \\ \sigma^{11}_{(13)}\sigma^{12}_{(13)} & \sigma^{11}_{(23)}\sigma^{12}_{(23)} & [\sigma^{12}_{(13)}]^2+[\sigma^{12}_{(23)}]^2+[\sigma^{11}_{(34)}]^2 & \sigma^{11}_{(34)}\sigma^{12}_{(34)} \\ 0 & 0 & \sigma^{11}_{(34)}\sigma^{12}_{(34)} & [\sigma^{12}_{(34)}]^2 \end{bmatrix} \end{array}$$

$$V^{[34]} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sigma^{11}_{(34)}\sigma^{12}_{(34)} & \sigma^{11}_{(34)}\sigma^{22}_{(34)}+[\sigma^{12}_{(34)}]^2 \\ 0 & 0 & \sigma^{11}_{(34)}\sigma^{22}_{(34)}+[\sigma^{12}_{(34)}]^2 & 2\sigma^{12}_{(34)}\sigma^{22}_{(34)} \end{bmatrix}$$

$$V^{[44]} = -\begin{bmatrix} [\sigma^{11}_{(14)}]^2 & 0 & 0 & \sigma^{11}_{(14)}\sigma^{12}_{(14)} \\ 0 & [\sigma^{11}_{(24)}]^2 & 0 & \sigma^{11}_{(24)}\sigma^{12}_{(24)} \\ 0 & 0 & [\sigma^{11}_{(34)}]^2 & \sigma^{11}_{(34)}\sigma^{12}_{(34)} \\ \sigma^{11}_{(14)}\sigma^{12}_{(14)} & \sigma^{11}_{(24)}\sigma^{12}_{(24)} & \sigma^{11}_{(34)}\sigma^{12}_{(34)} & [\sigma^{12}_{(14)}]^2+[\sigma^{12}_{(24)}]^2+[\sigma^{12}_{(34)}]^2 \end{bmatrix}$$

The terms of the $F^{[k\ell]}$ matrices are as follows.

$$F^{[11]} = -\begin{bmatrix} [\sigma^{11}_{(12)}]^2 & \sigma^{11}_{(12)}\sigma^{12}_{(12)} \\ \sigma^{11}_{(12)}\sigma^{12}_{(12)} & [\sigma^{12}_{(12)}]^2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} X'_{(12)}$$

$$-\begin{bmatrix} [\sigma^{11}_{(13)}]^2 & \sigma^{11}_{(13)}\sigma^{12}_{(13)} \\ 0 & 0 \\ \sigma^{11}_{(13)}\sigma^{12}_{(13)} & [\sigma^{12}_{(13)}]^2 \\ 0 & 0 \end{bmatrix} X'_{(13)}$$

$$-\begin{bmatrix} [\sigma^{11}_{(14)}]^2 & \sigma^{11}_{(14)}\sigma^{12}_{(14)} \\ 0 & 0 \\ 0 & 0 \\ \sigma^{11}_{(14)}\sigma^{12}_{(14)} & [\sigma^{12}_{(14)}]^2 \end{bmatrix} X'_{(14)}$$

$$F^{[12]} = -\begin{bmatrix} 2\sigma^{11}_{(12)}\sigma^{12}_{(12)} & \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 \\ \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 & 2\sigma^{11}_{(12)}\sigma^{12}_{(12)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} X'_{(12)}$$

$$F^{[13]} = -\begin{bmatrix} 2\sigma^{11}_{(13)}\sigma^{12}_{(13)} & \sigma^{11}_{(13)}\sigma^{22}_{(13)}+[\sigma^{12}_{(13)}]^2 \\ 0 & 0 \\ \sigma^{11}_{(13)}\sigma^{22}_{(13)}+[\sigma^{12}_{(13)}]^2 & 2\sigma^{12}_{(13)}\sigma^{22}_{(13)} \\ 0 & 0 \end{bmatrix} X'_{(13)}$$

$$F^{[14]} = -\begin{bmatrix} 2\sigma^{11}_{(14)}\sigma^{12}_{(14)} & \sigma^{11}_{(14)}\sigma^{22}_{(14)}+[\sigma^{12}_{(14)}]^2 \\ 0 & 0 \\ 0 & 0 \\ \sigma^{11}_{(14)}\sigma^{12}_{(14)}+[\sigma^{12}_{(14)}]^2 & 2\sigma^{12}_{(14)}\sigma^{22}_{(14)} \end{bmatrix} X'_{(14)}$$

$$F^{[22]} = -\begin{bmatrix} [\sigma^{11}_{(12)}]^2 & \sigma^{11}_{(12)}\sigma^{12}_{(12)} \\ \sigma^{11}_{(12)}\sigma^{12}_{(12)} & [\sigma^{12}_{(12)}]^2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} X'_{(12)}$$

$$-\begin{bmatrix} 0 & 0 \\ [\sigma^{11}_{(23)}]^2 & \sigma^{11}_{(23)}\sigma^{12}_{(23)} \\ \sigma^{11}_{(23)}\sigma^{12}_{(23)} & [\sigma^{12}_{(23)}]^2 \\ 0 & 0 \end{bmatrix} X'_{(23)}$$

$$-\begin{bmatrix} 0 & 0 \\ [\sigma^{11}_{(24)}]^2 & \sigma^{11}_{(24)}\sigma^{12}_{(24)} \\ 0 & 0 \\ \sigma^{11}_{(24)}\sigma^{12}_{(24)} & [\sigma^{12}_{(24)}]^2 \end{bmatrix} X'_{(24)}$$

$$F^{[23]} = -\begin{bmatrix} 0 & 0 \\ 2\sigma^{11}_{(23)}\sigma^{12}_{(23)} & \sigma^{11}_{(23)}\sigma^{22}_{(23)}+[\sigma^{12}_{(23)}]^2 \\ \sigma^{11}_{(23)}\sigma^{22}_{(23)}+[\sigma^{12}_{(23)}]^2 & 2\sigma^{12}_{(23)}\sigma^{22}_{(23)} \\ 0 & 0 \end{bmatrix} X'_{(23)}$$

$$F^{[24]} = -\begin{bmatrix} 0 & 0 \\ 2\sigma^{11}_{(24)}\sigma^{12}_{(24)} & \sigma^{11}_{(24)}\sigma^{22}_{(24)}+[\sigma^{12}_{(24)}]^2 \\ 0 & 0 \\ \sigma^{11}_{(24)}\sigma^{22}_{(24)}+[\sigma^{12}_{(24)}]^2 & 2\sigma^{12}_{(24)}\sigma^{22}_{(24)} \end{bmatrix} X'_{(24)}$$

$$F^{[33]} = -\begin{bmatrix} [\sigma^{11}_{(13)}]^2 & \sigma^{11}_{(13)}\sigma^{12}_{(13)} \\ 0 & 0 \\ \sigma^{11}_{(13)}\sigma^{12}_{(13)} & [\sigma^{12}_{(13)}]^2 \\ 0 & 0 \end{bmatrix} X'_{(13)}$$

$$-\begin{bmatrix} 0 & 0 \\ [\sigma^{11}_{(23)}]^2 & \sigma^{11}_{(23)}\sigma^{12}_{(23)} \\ \sigma^{11}_{(23)}\sigma^{12}_{(23)} & [\sigma^{12}_{(23)}]^2 \\ 0 & 0 \end{bmatrix} X'_{(23)}$$

$$-\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ [\sigma^{11}_{(34)}]^2 & \sigma^{11}_{(34)}\sigma^{12}_{(34)} \\ \sigma^{11}_{(34)}\sigma^{12}_{(34)} & [\sigma^{12}_{(34)}]^2 \end{bmatrix} X'_{(34)}$$

$$F^{[34]} = -\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2\sigma^{11}_{(34)}\sigma^{12}_{(34)} & \sigma^{11}_{(34)}\sigma^{22}_{(34)}+[\sigma^{12}_{(34)}]^2 \\ \sigma^{11}_{(34)}\sigma^{22}_{(34)}+[\sigma^{12}_{(34)}]^2 & 2\sigma^{11}_{(34)}\sigma^{22}_{(34)} \end{bmatrix} X'_{(34)}$$

$$F^{[44]} = -\begin{bmatrix} [\sigma^{11}_{(14)}]^2 & \sigma^{11}_{(14)}\sigma^{12}_{(14)} \\ 0 & 0 \\ 0 & 0 \\ \sigma^{11}_{(14)}\sigma^{12}_{(14)} & [\sigma^{12}_{(14)}]^2 \end{bmatrix} X'_{(14)}$$

$$-\begin{bmatrix} 0 & 0 \\ [\sigma^{11}_{(24)}]^2 & \sigma^{11}_{(24)}\sigma^{12}_{(24)} \\ 0 & 0 \\ \sigma^{11}_{(24)}\sigma^{12}_{(24)} & [\sigma^{12}_{(24)}]^2 \end{bmatrix} X'_{(24)}$$

$$-\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ [\sigma^{11}_{(34)}]^2 & \sigma^{11}_{(34)}\sigma^{12}_{(34)} \\ \sigma^{11}_{(34)}\sigma^{12}_{(34)} & [\sigma^{12}_{(34)}]^2 \end{bmatrix} X'_{(34)}$$

where $X_{(12)}$, $X_{(13)}$, $X_{(14)}$, $X_{(23)}$, $X_{(24)}$, and $X_{(34)}$ are the matrices of standard least squares estimates of the parameters in groups 1, 2,...,6 respectively.

The $T_i^{\{k\ell\}}$ matrices for the six groups are listed below. The $p_{mn}^{(vw)}$ are elements of P_iP_i' for the i'th group containing variables v and w. The equation for P_iP_i' is given in step 18 of the computational procedure in section 3.4. The computation of $T_i^{\{k\ell\}}$ is discussed in step 23.

Group One

$$T_{(12)}^{\{11\}} = \begin{bmatrix} \sigma_{(12)}^{11}p_{11}^{(12)}+\sigma_{(12)}^{12}p_{12}^{(12)} & \sigma_{(12)}^{11}p_{12}^{(12)}+\sigma_{(12)}^{12}p_{22}^{(12)} \\ 0 & 0 \end{bmatrix}$$

$$T_{(12)}^{\{12\}} = \begin{bmatrix} \sigma_{(12)}^{12}p_{11}^{(12)}+\sigma_{(12)}^{22}p_{12}^{(12)} & \sigma_{(12)}^{12}p_{12}^{(12)}+\sigma_{(12)}^{22}p_{22}^{(12)} \\ \sigma_{(12)}^{11}p_{11}^{(12)}+\sigma_{(12)}^{12}p_{12}^{(12)} & \sigma_{(12)}^{11}p_{12}^{(12)}+\sigma_{(12)}^{12}p_{22}^{(12)} \end{bmatrix}$$

$$T_{(12)}^{\{22\}} = \begin{bmatrix} 0 & 0 \\ \sigma_{(12)}^{12}p_{11}^{(12)}+\sigma_{(12)}^{22}p_{12}^{(12)} & \sigma_{(12)}^{12}p_{12}^{(12)}+\sigma_{(12)}^{22}p_{22}^{(12)} \end{bmatrix}$$

$T_{(12)}^{\{13\}}$, $T_{(12)}^{\{14\}}$, $T_{(12)}^{\{23\}}$, $T_{(12)}^{\{24\}}$, $T_{(12)}^{\{33\}}$, $T_{(12)}^{\{34\}}$, and $T_{(12)}^{\{44\}}$ are null.

Group Two

$$T_{(13)}^{\{11\}} = \begin{bmatrix} \sigma_{(13)}^{11}p_{11}^{(13)}+\sigma_{(13)}^{12}p_{12}^{(13)} & \sigma_{(13)}^{11}p_{12}^{(13)}+\sigma_{(13)}^{12}p_{22}^{(13)} \\ 0 & 0 \end{bmatrix}$$

$$T_{(13)}^{\{13\}} = \begin{bmatrix} \sigma_{(13)}^{12}p_{11}^{(13)}+\sigma_{(13)}^{22}p_{12}^{(13)} & \sigma_{(13)}^{12}p_{12}^{(13)}+\sigma_{(13)}^{22}p_{22}^{(13)} \\ \sigma_{(13)}^{11}p_{11}^{(13)}+\sigma_{(13)}^{12}p_{12}^{(13)} & \sigma_{(13)}^{11}p_{12}^{(13)}+\sigma_{(13)}^{12}p_{22}^{(13)} \end{bmatrix}$$

$$T^{\{33\}}_{(13)} = \begin{bmatrix} 0 & 0 \\ \sigma^{12}_{(13)}P^{(13)}_{11}+\sigma^{22}_{(13)}P^{(13)}_{12} & \sigma^{12}_{(13)}P^{(13)}_{12}+\sigma^{22}_{(13)}P^{(13)}_{22} \end{bmatrix}$$

$T^{\{12\}}_{(13)}$, $T^{\{14\}}_{(13)}$, $T^{\{22\}}_{(13)}$, $T^{\{23\}}_{(13)}$, $T^{\{24\}}_{(13)}$, $T^{\{34\}}_{(13)}$, and $T^{\{44\}}_{(13)}$ are null.

Group Three

$$T^{\{11\}}_{(14)} = \begin{bmatrix} \sigma^{11}_{(14)}P^{(14)}_{11}+\sigma^{12}_{(14)}P^{(14)}_{12} & \sigma^{11}_{(14)}P^{(14)}_{12}+\sigma^{12}_{(14)}P^{(14)}_{22} \\ 0 & 0 \end{bmatrix}$$

$$T^{\{14\}}_{(14)} = \begin{bmatrix} \sigma^{12}_{(14)}P^{(14)}_{11}+\sigma^{22}_{(14)}P^{(14)}_{12} & \sigma^{12}_{(14)}P^{(14)}_{12}+\sigma^{22}_{(14)}P^{(14)}_{22} \\ \sigma^{11}_{(14)}P^{(14)}_{11}+\sigma^{12}_{(14)}P^{(14)}_{12} & \sigma^{11}_{(14)}P^{(14)}_{12}+\sigma^{12}_{(14)}P^{(14)}_{22} \end{bmatrix}$$

$$T^{\{44\}}_{(14)} = \begin{bmatrix} 0 & 0 \\ \sigma^{12}_{(14)}P^{(14)}_{11}+\sigma^{22}_{(14)}P^{(14)}_{12} & \sigma^{12}_{(14)}P^{(14)}_{12}+\sigma^{22}_{(14)}P^{(14)}_{22} \end{bmatrix}$$

$T^{\{12\}}_{(14)}$, $T^{\{13\}}_{(14)}$, $T^{\{22\}}_{(14)}$, $T^{\{23\}}_{(14)}$, $T^{\{24\}}_{(14)}$, $T^{\{33\}}_{(14)}$, and $T^{\{34\}}_{(14)}$ are null.

Group Four

$$T^{\{22\}}_{(23)} = \begin{bmatrix} \sigma^{11}_{(23)}P^{(23)}_{11}+\sigma^{12}_{(23)}P^{(23)}_{12} & \sigma^{11}_{(23)}P^{(23)}_{12}+\sigma^{12}_{(23)}P^{(23)}_{22} \\ 0 & 0 \end{bmatrix}$$

$$T^{\{23\}}_{(23)} = \begin{bmatrix} \sigma^{12}_{(23)}P^{(23)}_{11}+\sigma^{22}_{(23)}P^{(23)}_{12} & \sigma^{12}_{(23)}P^{(23)}_{12}+\sigma^{22}_{(23)}P^{(23)}_{22} \\ \sigma^{11}_{(23)}P^{(23)}_{11}+\sigma^{12}_{(23)}P^{(23)}_{12} & \sigma^{11}_{(23)}P^{(23)}_{12}+\sigma^{12}_{(23)}P^{(23)}_{22} \end{bmatrix}$$

$$T^{\{33\}}_{(23)} = \begin{bmatrix} 0 & 0 \\ \sigma^{12}_{(23)}P^{(23)}_{11}+\sigma^{22}_{(23)}P^{(23)}_{12} & \sigma^{12}_{(23)}P^{(23)}_{12}+\sigma^{22}_{(23)}P^{(23)}_{22} \end{bmatrix}$$

$T_{(23)}^{\{11\}}$, $T_{(23)}^{\{12\}}$, $T_{(23)}^{\{13\}}$, $T_{(23)}^{\{14\}}$, $T_{(23)}^{\{24\}}$, $T_{(23)}^{\{34\}}$, and $T_{(23)}^{\{44\}}$ are null.

Group Five

$$T_{(24)}^{\{22\}} = \begin{bmatrix} \sigma_{(24)}^{11}P_{11}^{(24)}+\sigma_{(24)}^{12}P_{12}^{(24)} & \sigma_{(24)}^{11}P_{12}^{(24)}+\sigma_{(24)}^{12}P_{22}^{(24)} \\ 0 & 0 \end{bmatrix}$$

$$T_{(24)}^{\{24\}} = \begin{bmatrix} \sigma_{(24)}^{12}P_{11}^{(24)}+\sigma_{(24)}^{22}P_{12}^{(24)} & \sigma_{(24)}^{12}P_{12}^{(24)}+\sigma_{(24)}^{22}P_{22}^{(24)} \\ \sigma_{(24)}^{11}P_{11}^{(24)}+\sigma_{(24)}^{12}P_{12}^{(24)} & \sigma_{(24)}^{11}P_{12}^{(24)}+\sigma_{(24)}^{12}P_{22}^{(24)} \end{bmatrix}$$

$$T_{(24)}^{\{44\}} = \begin{bmatrix} 0 & 0 \\ \sigma_{(24)}^{12}P_{11}^{(24)}+\sigma_{(24)}^{22}P_{12}^{(24)} & \sigma_{(24)}^{12}P_{12}^{(24)}+\sigma_{(24)}^{22}P_{22}^{(24)} \end{bmatrix}$$

$T_{(24)}^{\{11\}}$, $T_{(24)}^{\{12\}}$, $T_{(24)}^{\{13\}}$, $T_{(24)}^{\{14\}}$, $T_{(24)}^{\{23\}}$, $T_{(24)}^{\{33\}}$, and $T_{(24)}^{\{34\}}$ are null.

Group Six

$$T_{(34)}^{\{33\}} = \begin{bmatrix} \sigma_{(34)}^{11}P_{11}^{(34)}+\sigma_{(34)}^{12}P_{12}^{(34)} & \sigma_{(34)}^{11}P_{12}^{(34)}+\sigma_{(34)}^{12}P_{22}^{(34)} \\ 0 & 0 \end{bmatrix}$$

$$T_{(34)}^{\{34\}} = \begin{bmatrix} \sigma_{(34)}^{12}P_{11}^{(34)}+\sigma_{(34)}^{22}P_{12}^{(34)} & \sigma_{(34)}^{12}P_{12}^{(34)}+\sigma_{(34)}^{22}P_{22}^{(34)} \\ \sigma_{(34)}^{11}P_{11}^{(34)}+\sigma_{(34)}^{12}P_{12}^{(34)} & \sigma_{(34)}^{11}P_{12}^{(34)}+\sigma_{(34)}^{12}P_{22}^{(34)} \end{bmatrix}$$

$$T_{(34)}^{\{44\}} = \begin{bmatrix} 0 & 0 \\ \sigma_{(34)}^{12}P_{11}^{(34)}+\sigma_{(34)}^{22}P_{12}^{(34)} & \sigma_{(34)}^{12}P_{12}^{(34)}+\sigma_{(34)}^{22}P_{22}^{(34)} \end{bmatrix}$$

$T_{(34)}^{\{11\}}$, $T_{(34)}^{\{12\}}$, $T_{(34)}^{\{13\}}$, $T_{(34)}^{\{14\}}$, $T_{(34)}^{\{22\}}$, $T_{(34)}^{\{23\}}$, and $T_{(34)}^{\{24\}}$ are null.

5.5c Case of p Variables in K Groups of Two.

If the variables in the i'th group are denoted by v and w, then the U_i matrices have the form

$$U_i = \begin{bmatrix} \sigma_{vv} & \sigma_{vw} \\ \sigma_{vw} & \sigma_{ww} \end{bmatrix}$$

These matrices are also denoted by

$$U_{(vw)} = \begin{bmatrix} \sigma_{11}^{(vw)} & \sigma_{12}^{(vw)} \\ \sigma_{12}^{(vw)} & \sigma_{22}^{(vw)} \end{bmatrix}$$

where $\sigma_{11}^{(vw)}$ is the variance of the first variable in group (vw), $\sigma_{12}^{(vw)}$ is the covariance between the first and second variables in group (vw), and $\sigma_{22}^{(vw)}$ is the variance of the second variable in group (vw).

The matrix U_{vw}^{-1} has the form

$$U_{(vw)}^{-1} = \begin{bmatrix} \sigma_{(vw)}^{11} & \sigma_{(vw)}^{12} \\ \sigma_{(vw)}^{12} & \sigma_{(vw)}^{22} \end{bmatrix}$$

$M_{(vw)} U_{(vw)}^{-1} M'_{(vw)}$ is a square matrix of order p with $U_{(vw)}^{-1}$ as a submatrix in rows and columns u and v and zeros in the remaining positions. The symmetric matrix $V = \sum_{i=1}^{K} M_i U_i^{-1} M_i'$ has a single element $\sigma_{(vw)}^{12}$ in each non-diagonal position vw. The m'th diagonal element of V is the sum of p-1 terms, each associated with a group containing variable m.

The matrix $U_i^{[k\ell]} = M_i' \Sigma^{[k\ell]} M_i$ can be written

$$U_{(vw)}^{[k\ell]} = \begin{bmatrix} \delta_{vk}\delta_{v\ell} & \delta_{vk}\delta_{w\ell} \\ \delta_{vk}\delta_{w\ell} & \delta_{wk}\delta_{w\ell} \end{bmatrix} \qquad k \leq \ell$$

for the group which contains variables v and w.

The $V^{[kk]}$ matrix has zeros in all positions (uv) such that both $u \neq k$ and $v \neq k$. The diagonal and the k'th row and column of $V^{[kk]}$ consist of non-zero elements. All of the remaining elements of $V^{[kk]}$ are zero. The element in the k'th diagonal position is the sum of p-1 squares, each associated with a group containing variable k. The m'th diagonal position ($m \neq k$) contains the square of a single element from the U_i^{-1} matrix for the group which contains variables k and m.

The $V^{[k\ell]}$ ($k \neq \ell$) matrix has a 2 x 2 submatrix of non-zero elements in rows and columns k and ℓ and zeros in the remaining positions. The non-zero diagonal elements are $2\sigma_{(k\ell)}^{11}\sigma_{(k\ell)}^{12}$ and $2\sigma_{(k\ell)}^{12}\sigma_{(k\ell)}^{22}$. The non-zero element in two symmetric off diagonal positions is $\sigma_{(k\ell)}^{11}\sigma_{(k\ell)}^{22} + [\sigma_{(k\ell)}^{12}]^2$.

The $F^{[kk]}$ matrices are sums of (p-1) matrix products each associated with a group which contains variable k. Each matrix product has (p-2) rows of zeros. However, all elements of the sum, $F^{[kk]}$, are non-zero. $F^{[k\ell]}$ ($k \neq \ell$) is a single product of a matrix containing (p-2) rows of zeros and the matrix of least squares estimates of the parameters for the group which contains variables k and ℓ.

Associated with each group there are three non-null $T_i^{\{k\ell\}}$ matrices. The three non-null matrices for the group containing variables v and w are

$$T_{(vw)}^{\{vv\}} = \begin{bmatrix} \sigma_{(vw)}^{11} p_{11}^{(vw)} + \sigma_{(vw)}^{12} p_{12}^{(vw)} & \sigma_{(vw)}^{11} p_{12}^{(vw)} + \sigma_{(vw)}^{12} p_{22}^{(vw)} \\ 0 & 0 \end{bmatrix}$$

$$T_{(vw)}^{\{vw\}} = \begin{bmatrix} \sigma_{(vw)}^{12} p_{11}^{(vw)} + \sigma_{(vw)}^{22} p_{12}^{(vw)} & \sigma_{(vw)}^{12} p_{12}^{(vw)} + \sigma_{(vw)}^{22} p_{22}^{(vw)} \\ \sigma_{(vw)}^{11} p_{11}^{(vw)} + \sigma_{(vw)}^{12} p_{12}^{(vw)} & \sigma_{(vw)}^{11} p_{12}^{(vw)} + \sigma_{(vw)}^{12} p_{22}^{(vw)} \end{bmatrix}$$

$$T_{(vw)}^{\{ww\}} = \begin{bmatrix} 0 & 0 \\ \sigma_{(vw)}^{12} p_{11}^{(vw)} + \sigma_{(vw)}^{22} p_{12}^{(vw)} & \sigma_{(vw)}^{12} p_{12}^{(vw)} + \sigma_{(vw)}^{22} p_{22}^{(vw)} \end{bmatrix}$$

where $p_{mn}^{(vw)}$ denotes the element in the m'th row and n'th column of the matrix product $P_i P_i'$ for the group which contains variables v and w. (See step 18 of section 3.4). The first row of $T_{(vw)}^{\{vw\}}$ is the same as the non-null row of $T_{(vw)}^{\{ww\}}$, while the second row is the same as the non-null row of $T_{(vw)}^{\{vv\}}$. Thus only the single non-null matrix $T_{(vw)}^{\{vw\}}$ needs to be computed in each group (vw).

5.5d Five Variables in Ten Groups of Three

Group

(1)	1 2 3	(6)	1 2 4
(2)	1 2 5	(7)	1 3 4
(3)	1 4 5	(8)	1 3 5
(4)	2 3 4	(9)	2 3 5
(5)	3 4 5	(10)	2 4 5

The submatrices of Σ for this design are

$$U_1 = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \qquad U_2 = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{15} \\ \sigma_{12} & \sigma_{22} & \sigma_{25} \\ \sigma_{15} & \sigma_{25} & \sigma_{55} \end{bmatrix}$$

$$U_3 = \begin{bmatrix} \sigma_{11} & \sigma_{14} & \sigma_{15} \\ \sigma_{14} & \sigma_{44} & \sigma_{45} \\ \sigma_{15} & \sigma_{45} & \sigma_{55} \end{bmatrix} \qquad U_4 = \begin{bmatrix} \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{23} & \sigma_{33} & \sigma_{34} \\ \sigma_{24} & \sigma_{34} & \sigma_{44} \end{bmatrix}$$

$$U_5 = \begin{bmatrix} \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{34} & \sigma_{44} & \sigma_{45} \\ \sigma_{35} & \sigma_{45} & \sigma_{55} \end{bmatrix} \qquad U_6 = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{14} \\ \sigma_{12} & \sigma_{22} & \sigma_{24} \\ \sigma_{14} & \sigma_{24} & \sigma_{24} \end{bmatrix}$$

$$U_7 = \begin{bmatrix} \sigma_{11} & \sigma_{13} & \sigma_{14} \\ \sigma_{13} & \sigma_{33} & \sigma_{34} \\ \sigma_{14} & \sigma_{34} & \sigma_{44} \end{bmatrix} \qquad U_8 = \begin{bmatrix} \sigma_{11} & \sigma_{13} & \sigma_{15} \\ \sigma_{13} & \sigma_{33} & \sigma_{35} \\ \sigma_{15} & \sigma_{35} & \sigma_{55} \end{bmatrix}$$

$$U_9 = \begin{bmatrix} \sigma_{22} & \sigma_{23} & \sigma_{25} \\ \sigma_{23} & \sigma_{33} & \sigma_{35} \\ \sigma_{25} & \sigma_{35} & \sigma_{55} \end{bmatrix} \qquad U_{10} = \begin{bmatrix} \sigma_{22} & \sigma_{24} & \sigma_{25} \\ \sigma_{24} & \sigma_{44} & \sigma_{45} \\ \sigma_{25} & \sigma_{45} & \sigma_{55} \end{bmatrix}$$

The inverses of these matrices are denoted by *

$$U^{-1}_{(123)} = \begin{bmatrix} \sigma^{11}_{(123)} & \sigma^{12}_{(123)} & \sigma^{13}_{(123)} \\ \sigma^{12}_{(123)} & \sigma^{22}_{(123)} & \sigma^{23}_{(123)} \\ \sigma^{13}_{(123)} & \sigma^{23}_{(123)} & \sigma^{33}_{(123)} \end{bmatrix} \qquad U^{-1}_{(125)} = \begin{bmatrix} \sigma^{11}_{(125)} & \sigma^{12}_{(125)} & \sigma^{13}_{(125)} \\ \sigma^{12}_{(125)} & \sigma^{22}_{(125)} & \sigma^{23}_{(125)} \\ \sigma^{13}_{(125)} & \sigma^{23}_{(125)} & \sigma^{33}_{(125)} \end{bmatrix}$$

$$U^{-1}_{(145)} = \begin{bmatrix} \sigma^{11}_{(145)} & \sigma^{12}_{(145)} & \sigma^{13}_{(145)} \\ \sigma^{12}_{(145)} & \sigma^{22}_{(145)} & \sigma^{23}_{(145)} \\ \sigma^{13}_{(145)} & \sigma^{23}_{(145)} & \sigma^{33}_{(145)} \end{bmatrix} \qquad U^{-1}_{(234)} = \begin{bmatrix} \sigma^{11}_{(234)} & \sigma^{12}_{(234)} & \sigma^{13}_{(234)} \\ \sigma^{12}_{(234)} & \sigma^{22}_{(234)} & \sigma^{23}_{(234)} \\ \sigma^{13}_{(234)} & \sigma^{23}_{(234)} & \sigma^{33}_{(234)} \end{bmatrix}$$

$$U^{-1}_{(345)} = \begin{bmatrix} \sigma^{11}_{(345)} & \sigma^{12}_{(345)} & \sigma^{13}_{(345)} \\ \sigma^{12}_{(345)} & \sigma^{22}_{(345)} & \sigma^{23}_{(345)} \\ \sigma^{13}_{(345)} & \sigma^{23}_{(345)} & \sigma^{33}_{(345)} \end{bmatrix} \qquad U^{-1}_{(124)} = \begin{bmatrix} \sigma^{11}_{(124)} & \sigma^{12}_{(124)} & \sigma^{13}_{(124)} \\ \sigma^{12}_{(124)} & \sigma^{22}_{(124)} & \sigma^{23}_{(124)} \\ \sigma^{13}_{(124)} & \sigma^{23}_{(124)} & \sigma^{33}_{(124)} \end{bmatrix}$$

$$U^{-1}_{(134)} = \begin{bmatrix} \sigma^{11}_{(134)} & \sigma^{12}_{(134)} & \sigma^{13}_{(134)} \\ \sigma^{12}_{(134)} & \sigma^{22}_{(134)} & \sigma^{23}_{(134)} \\ \sigma^{13}_{(134)} & \sigma^{23}_{(134)} & \sigma^{33}_{(134)} \end{bmatrix} \qquad U^{-1}_{(135)} = \begin{bmatrix} \sigma^{11}_{(135)} & \sigma^{12}_{(135)} & \sigma^{13}_{(135)} \\ \sigma^{12}_{(135)} & \sigma^{22}_{(135)} & \sigma^{23}_{(135)} \\ \sigma^{13}_{(135)} & \sigma^{23}_{(135)} & \sigma^{33}_{(135)} \end{bmatrix}$$

---

* $U^{-1}_{(123)}$ is the inverse of U_1, etc., in accordance with notation defined on page 87.

$$U^{-1}_{(235)} = \begin{bmatrix} \sigma^{11}_{(235)} & \sigma^{12}_{(235)} & \sigma^{13}_{(235)} \\ \sigma^{12}_{(235)} & \sigma^{22}_{(235)} & \sigma^{23}_{(235)} \\ \sigma^{13}_{(235)} & \sigma^{23}_{(235)} & \sigma^{33}_{(235)} \end{bmatrix} \quad U^{-1}_{(245)} = \begin{bmatrix} \sigma^{11}_{(245)} & \sigma^{12}_{(245)} & \sigma^{13}_{(245)} \\ \sigma^{12}_{(245)} & \sigma^{22}_{(245)} & \sigma^{23}_{(245)} \\ \sigma^{13}_{(245)} & \sigma^{23}_{(245)} & \sigma^{33}_{(245)} \end{bmatrix}$$

The 5 x 5 matrix $V = \sum_{i=1}^{10} M_i U_i^{-1} M_i'$ for this design is

$$V=\begin{bmatrix}
\sigma^{11}_{(123)}+\sigma^{11}_{(125)}+\sigma^{11}_{(145)}+\sigma^{11}_{(134)}+\sigma^{11}_{(124)}+\sigma^{11}_{(135)} & \sigma^{12}_{(123)}+\sigma^{12}_{(125)}+\sigma^{12}_{(124)} & \sigma^{13}_{(123)}+\sigma^{12}_{(134)}+\sigma^{12}_{(135)} & \sigma^{13}_{(124)}+\sigma^{12}_{(145)}+\sigma^{13}_{(134)} & \sigma^{13}_{(125)}+\sigma^{13}_{(145)}+\sigma^{13}_{(135)} \\
 & \sigma^{22}_{(123)}+\sigma^{22}_{(125)}+\sigma^{11}_{(234)}+\sigma^{22}_{(124)}+\sigma^{11}_{(235)}+\sigma^{11}_{(245)} & \sigma^{23}_{(123)}+\sigma^{12}_{(234)}+\sigma^{12}_{(235)} & \sigma^{23}_{(124)}+\sigma^{13}_{(234)}+\sigma^{12}_{(245)} & \sigma^{23}_{(125)}+\sigma^{13}_{(235)}+\sigma^{13}_{(245)} \\
 & & \sigma^{33}_{(123)}+\sigma^{22}_{(234)}+\sigma^{11}_{(345)}+\sigma^{22}_{(134)}+\sigma^{22}_{(135)}+\sigma^{22}_{(235)} & \sigma^{23}_{(134)}+\sigma^{23}_{(234)}+\sigma^{12}_{(345)} & \sigma^{23}_{(135)}+\sigma^{23}_{(235)}+\sigma^{13}_{(345)} \\
 & & & \sigma^{22}_{(145)}+\sigma^{33}_{(234)}+\sigma^{22}_{(345)}+\sigma^{33}_{(124)}+\sigma^{33}_{(134)}+\sigma^{22}_{(245)} & \sigma^{23}_{(145)}+\sigma^{23}_{(245)}+\sigma^{23}_{(345)} \\
 & & & & \sigma^{33}_{(125)}+\sigma^{33}_{(145)}+\sigma^{33}_{(345)}+\sigma^{33}_{(135)}+\sigma^{33}_{(235)}+\sigma^{33}_{(245)}
\end{bmatrix}$$

The matrices $U_i^{[k\ell]} = M_i'\Sigma^{[k\ell]}M_i$ are as follows.

k=1, ℓ=1

$$U^{[11]}_{(123)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[11]}_{(125)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[11]}_{(145)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$U^{[11]}_{(124)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[11]}_{(134)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[11]}_{(135)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$U^{[11]}_{(234)}$, $U^{[11]}_{(345)}$, $U^{[11]}_{(235)}$, and $U^{[11]}_{(245)}$ are null.

k=1, ℓ=2

$$U^{[12]}_{(123)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[12]}_{(125)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[12]}_{(124)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$U^{[12]}_{(145)}$, $U^{[12]}_{(234)}$, $U^{[12]}_{(345)}$, $U^{[12]}_{(134)}$, $U^{[12]}_{(135)}$, $U^{[12]}_{(235)}$, and $U^{[12]}_{(245)}$ are null.

k=1, ℓ=3

$$U^{[13]}_{(123)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad U^{[13]}_{(134)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[13]}_{(135)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$U^{[13]}_{(125)}$, $U^{[13]}_{(145)}$, $U^{[13]}_{(234)}$, $U^{[13]}_{(345)}$, $U^{[13]}_{(124)}$, $U^{[13]}_{(235)}$, and $U^{[13]}_{(245)}$ are null.

$k=1,\ \ell=4$

$$U^{[14]}_{(124)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad U^{[14]}_{(134)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad U^{[14]}_{(145)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$U^{[14]}_{(123)}$, $U^{[14]}_{(125)}$, $U^{[14]}_{(234)}$, $U^{[14]}_{(345)}$, $U^{[14]}_{(135)}$, $U^{[14]}_{(235)}$, and $U^{[14]}_{(245)}$ are null.

$k=1,\ \ell=5$

$$U^{[15]}_{(125)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad U^{[15]}_{(145)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad U^{[15]}_{(135)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$U^{[15]}_{(123)}$, $U^{[15]}_{(234)}$, $U^{[15]}_{(345)}$, $U^{[15]}_{(124)}$, $U^{[15]}_{(134)}$, $U^{[15]}_{(235)}$, and $U^{[15]}_{(245)}$ are null.

$k=2,\ \ell=2$

$$U^{[22]}_{(123)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[22]}_{(125)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[22]}_{(234)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$U^{[22]}_{(124)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[22]}_{(235)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[22]}_{(245)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$U^{[22]}_{(145)}$, $U^{[22]}_{(345)}$, $U^{[22]}_{(134)}$, and $U^{[22]}_{(135)}$ are null.

$k=2,\ \ell=3$

$$U^{[23]}_{(123)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad U^{[23]}_{(234)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[23]}_{(235)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$U^{[23]}_{(125)}$, $U^{[23]}_{(145)}$, $U^{[23]}_{(345)}$, $U^{[23]}_{(124)}$, $U^{[23]}_{(134)}$, $U^{[23]}_{(135)}$ and $U^{[23]}_{(245)}$ are null.

$k=2,\ \ell=4$

$$U^{[24]}_{(234)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad U^{[24]}_{(124)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad U^{[24]}_{(245)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$U^{[24]}_{(123)}$, $U^{[24]}_{(125)}$, $U^{[24]}_{(145)}$, $U^{[24]}_{(345)}$, $U^{[24]}_{(134)}$, $U^{[24]}_{(135)}$ and $U^{[24]}_{(235)}$ are null.

$k=2,\ \ell=5$

$$U^{[25]}_{(125)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad U^{[25]}_{(235)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad U^{[25]}_{(245)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$U^{[25]}_{(123)}$, $U^{[25]}_{(145)}$, $U^{[25]}_{(234)}$, $U^{[25]}_{(345)}$, $U^{[25]}_{(124)}$, $U^{[25]}_{(134)}$ and $U^{[25]}_{(135)}$ are null.

$k=3,\ \ell=3$

$$U^{[33]}_{(123)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad U^{[33]}_{(234)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[33]}_{(345)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$U^{[33]}_{(134)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[33]}_{(135)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[33]}_{(235)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$U^{[33]}_{(125)}$, $U^{[33]}_{(145)}$, $U^{[33]}_{(124)}$, and $U^{[33]}_{(245)}$ are null.

$k=3$, $\ell=4$

$$U^{[34]}_{(234)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad U^{[34]}_{(345)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[34]}_{(134)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$U^{[34]}_{(123)}$, $U^{[34]}_{(125)}$, $U^{[34]}_{(145)}$, $U^{[34]}_{(124)}$, $U^{[34]}_{(135)}$, $U^{[34]}_{(235)}$, and $U^{[34]}_{(245)}$ are null

$k=3$, $\ell=5$

$$U^{[35]}_{(345)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad U^{[35]}_{(135)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad U^{[35]}_{(235)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$U^{[35]}_{(123)}$, $U^{[35]}_{(125)}$, $U^{[35]}_{(145)}$, $U^{[35]}_{(234)}$, $U^{[35]}_{(124)}$, $U^{[35]}_{(134)}$, and $U^{[35]}_{(245)}$ are null.

$k=4$, $\ell=4$

$$U^{[44]}_{(145)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[44]}_{(234)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad U^{[44]}_{(124)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U^{[44]}_{(345)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad U^{[44]}_{(134)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad U^{[44]}_{(245)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$U^{[44]}_{(123)}$, $U^{[44]}_{(125)}$, $U^{[44]}_{(135)}$, and $U^{[44]}_{(235)}$ are null.

$k=4,\ \ell=5$

$$U^{[45]}_{(145)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad U^{[45]}_{(345)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad U^{[45]}_{(245)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$U^{[45]}_{(123)}$, $U^{[45]}_{(125)}$, $U^{[45]}_{(234)}$, $U^{[45]}_{(124)}$, $U^{[45]}_{(134)}$, $U^{[45]}_{(135)}$, and $U^{[45]}_{(235)}$ are null.

$k=5,\ \ell=5$

$$U^{[55]}_{(125)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad U^{[55]}_{(145)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad U^{[55]}_{(345)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U^{[55]}_{(135)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad U^{[55]}_{(235)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad U^{[55]}_{(245)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$U^{[55]}_{(123)}$, $U^{[55]}_{(234)}$, $U^{[55]}_{(124)}$, and $U^{[55]}_{(134)}$ are null.

The $V^{[k\ell]}$ matrices for this design are

$$
V^{[11]} = \begin{array}{c}
\begin{array}{ccc} \qquad\qquad\qquad 1 \qquad\qquad\qquad & \qquad\qquad 2 \qquad\qquad & \qquad\qquad 3 \qquad\qquad \end{array} \\
\left[\begin{array}{ccc}
\begin{array}{l}[\sigma^{11}_{(123)}]^2+[\sigma^{11}_{(125)}]^2 \\ +[\sigma^{11}_{(145)}]^2+[\sigma^{11}_{(124)}]^2 \\ +[\sigma^{11}_{(134)}]^2+[\sigma^{11}_{(135)}]^2\end{array} &
\begin{array}{l}\sigma^{11}_{(123)}\sigma^{12}_{(123)} \\ +\sigma^{11}_{(125)}\sigma^{12}_{(125)} \\ +\sigma^{11}_{(124)}\sigma^{12}_{(124)}\end{array} &
\begin{array}{l}\sigma^{11}_{(123)}\sigma^{13}_{(123)} \\ +\sigma^{11}_{(134)}\sigma^{12}_{(134)} \\ +\sigma^{11}_{(135)}\sigma^{12}_{(135)}\end{array} \\
 & \begin{array}{l}[\sigma^{12}_{(123)}]^2 \\ +[\sigma^{12}_{(125)}]^2 \\ +[\sigma^{12}_{(124)}]^2\end{array} & \sigma^{12}_{(123)}\sigma^{13}_{(123)} \\
 & & \begin{array}{l}[\sigma^{13}_{(123)}]^2 \\ +[\sigma^{12}_{(134)}]^2 \\ +[\sigma^{12}_{(135)}]^2\end{array} \\
 & & \\
 & &
\end{array} \right.
\end{array}
$$

$$
\begin{array}{cc}
4 & 5 \\
\end{array}
$$

$$
\left.
\begin{array}{cc}
\sigma^{11}_{(145)}\sigma^{12}_{(145)} & \sigma^{11}_{(125)}\sigma^{13}_{(125)} \\
+\sigma^{11}_{(124)}\sigma^{13}_{(124)} & +\sigma^{11}_{(145)}\sigma^{13}_{(145)} \\
+\sigma^{11}_{(134)}\sigma^{13}_{(134)} & +\sigma^{11}_{(135)}\sigma^{13}_{(135)} \\
 & \\
\sigma^{12}_{(124)}\sigma^{13}_{(124)} & \sigma^{12}_{(125)}\sigma^{13}_{(125)} \\
 & \\
\sigma^{12}_{(134)}\sigma^{13}_{(134)} & \sigma^{12}_{(135)}\sigma^{13}_{(135)} \\
 & \\
[\sigma^{12}_{(145)}]^2 & \sigma^{12}_{(145)}\sigma^{13}_{(145)} \\
+[\sigma^{13}_{(124)}]^2 & \\
+[\sigma^{13}_{(134)}]^2 & \\
 & \\
 & [\sigma^{13}_{(125)}]^2 \\
 & +[\sigma^{13}_{(145)}]^2 \\
 & +[\sigma^{13}_{(135)}]^2
\end{array}
\right]
$$

$$
V^{[12]} = -\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
\begin{gathered} 2\sigma^{11}_{(123)}\sigma^{12}_{(123)} \\ +2\sigma^{11}_{(125)}\sigma^{12}_{(125)} \\ +2\sigma^{11}_{(124)}\sigma^{12}_{(124)} \end{gathered} &
\begin{gathered} \sigma^{11}_{(123)}\sigma^{22}_{(123)}+[\sigma^{12}_{(123)}]^2 \\ +\sigma^{11}_{(125)}\sigma^{22}_{(125)}+[\sigma^{12}_{(125)}]^2 \\ +\sigma^{11}_{(124)}\sigma^{22}_{(124)}+[\sigma^{12}_{(124)}]^2 \end{gathered} &
\begin{gathered} \sigma^{11}_{(123)}\sigma^{23}_{(123)} \\ +\sigma^{13}_{(123)}\sigma^{12}_{(123)} \end{gathered} &
\begin{gathered} \sigma^{11}_{(124)}\sigma^{23}_{(124)} \\ +\sigma^{13}_{(124)}\sigma^{12}_{(124)} \end{gathered} &
\begin{gathered} \sigma^{11}_{(125)}\sigma^{23}_{(125)} \\ +\sigma^{13}_{(125)}\sigma^{12}_{(125)} \end{gathered} \\
 &
\begin{gathered} 2\sigma^{12}_{(123)}\sigma^{22}_{(123)} \\ +2\sigma^{12}_{(125)}\sigma^{22}_{(125)} \\ +2\sigma^{12}_{(124)}\sigma^{22}_{(124)} \end{gathered} &
\begin{gathered} \sigma^{12}_{(123)}\sigma^{23}_{(123)} \\ +\sigma^{13}_{(123)}\sigma^{22}_{(123)} \end{gathered} &
\begin{gathered} \sigma^{12}_{(124)}\sigma^{23}_{(124)} \\ +\sigma^{13}_{(124)}\sigma^{22}_{(124)} \end{gathered} &
\begin{gathered} \sigma^{12}_{(125)}\sigma^{23}_{(125)} \\ +\sigma^{13}_{(125)}\sigma^{22}_{(125)} \end{gathered} \\
 & & 2\sigma^{13}_{(123)}\sigma^{23}_{(123)} & 0 & 0 \\
 & & & 2\sigma^{13}_{(124)}\sigma^{23}_{(124)} & 0 \\
 & & & & 2\sigma^{13}_{(125)}\sigma^{23}_{(125)}
\end{bmatrix}
\end{array}
$$

$$
V^{[13]} = -
\begin{array}{c}
\begin{array}{ccccc}
\quad\quad 1 \quad\quad & \quad\quad 2 \quad\quad & \quad\quad 3 \quad\quad & \quad\quad 4 \quad\quad & \quad\quad 5 \quad\quad
\end{array}
\\
\left[\begin{array}{ccccc}
\begin{matrix} 2\sigma^{11}_{(123)}\sigma^{13}_{(123)} \\ +2\sigma^{11}_{(134)}\sigma^{12}_{(134)} \\ +2\sigma^{11}_{(135)}\sigma^{12}_{(135)} \end{matrix}
&
\begin{matrix} \sigma^{11}_{(123)}\sigma^{23}_{(123)} \\ +\sigma^{12}_{(123)}\sigma^{13}_{(123)} \end{matrix}
&
\begin{matrix} \sigma^{11}_{(123)}\sigma^{33}_{(123)}+[\sigma^{13}_{(123)}]^2 \\ +\sigma^{11}_{(134)}\sigma^{22}_{(134)}+[\sigma^{12}_{(134)}]^2 \\ +\sigma^{11}_{(135)}\sigma^{22}_{(135)}+[\sigma^{12}_{(135)}]^2 \end{matrix}
&
\begin{matrix} \sigma^{11}_{(134)}\sigma^{23}_{(134)} \\ +\sigma^{13}_{(134)}\sigma^{12}_{(134)} \end{matrix}
&
\begin{matrix} \sigma^{11}_{(135)}\sigma^{23}_{(135)} \\ +\sigma^{13}_{(135)}\sigma^{12}_{(135)} \end{matrix}
\\[2ex]
&
2\sigma^{12}_{(123)}\sigma^{23}_{(123)}
&
\sigma^{12}_{(123)}\sigma^{33}_{(123)}+\sigma^{13}_{(123)}\sigma^{23}_{(123)}
&
0
&
0
\\[2ex]
& &
\begin{matrix} 2\sigma^{13}_{(123)}\sigma^{33}_{(123)} \\ +2\sigma^{12}_{(134)}\sigma^{22}_{(134)} \\ +2\sigma^{12}_{(135)}\sigma^{22}_{(135)} \end{matrix}
&
\begin{matrix} \sigma^{12}_{(134)}\sigma^{23}_{(134)} \\ +\sigma^{13}_{(134)}\sigma^{22}_{(134)} \end{matrix}
&
\begin{matrix} \sigma^{12}_{(135)}\sigma^{23}_{(135)} \\ +\sigma^{13}_{(135)}\sigma^{22}_{(135)} \end{matrix}
\\[2ex]
& & &
2\sigma^{13}_{(134)}\sigma^{23}_{(134)}
&
0
\\[2ex]
& & & &
2\sigma^{13}_{(135)}\sigma^{23}_{(135)}
\end{array}\right]
\end{array}
$$

$$V^{[14]} = - \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \begin{array}{l} 2\sigma^{11}_{(145)}\sigma^{12}_{(145)} \\ +2\sigma^{11}_{(124)}\sigma^{13}_{(124)} \\ +2\sigma^{11}_{(134)}\sigma^{13}_{(134)} \end{array} & \begin{array}{l} \sigma^{11}_{(124)}\sigma^{23}_{(124)} \\ +\sigma^{12}_{(124)}\sigma^{13}_{(124)} \end{array} & \begin{array}{l} \sigma^{11}_{(134)}\sigma^{23}_{(134)} \\ +\sigma^{12}_{(134)}\sigma^{13}_{(134)} \end{array} & \begin{array}{l} \sigma^{11}_{(145)}\sigma^{22}_{(145)}+[\sigma^{12}_{(145)}]^2 \\ +\sigma^{11}_{(124)}\sigma^{33}_{(124)}+[\sigma^{13}_{(124)}]^2 \\ +\sigma^{11}_{(134)}\sigma^{33}_{(134)}+[\sigma^{13}_{(134)}]^2 \end{array} & \begin{array}{l} \sigma^{11}_{(145)}\sigma^{23}_{(145)} \\ +\sigma^{13}_{(145)}\sigma^{12}_{(145)} \end{array} \\ & 2\sigma^{12}_{(124)}\sigma^{23}_{(124)} & 0 & \sigma^{12}_{(124)}\sigma^{33}_{(124)}+\sigma^{13}_{(124)}\sigma^{23}_{(124)} & 0 \\ & & 2\sigma^{12}_{(134)}\sigma^{23}_{(134)} & \sigma^{12}_{(134)}\sigma^{33}_{(134)}+\sigma^{13}_{(134)}\sigma^{23}_{(134)} & 0 \\ & & & \begin{array}{l} 2\sigma^{12}_{(145)}\sigma^{22}_{(145)}+2\sigma^{13}_{(124)}\sigma^{33}_{(124)} \\ +2\sigma^{13}_{(134)}\sigma^{33}_{(134)} \end{array} & \begin{array}{l} \sigma^{12}_{(145)}\sigma^{23}_{(145)} \\ +\sigma^{13}_{(145)}\sigma^{22}_{(145)} \end{array} \\ & & & & 2\sigma^{13}_{(145)}\sigma^{23}_{(145)} \end{bmatrix} \end{array}$$

$$V^{[15]} = -\begin{array}{c} \begin{array}{ccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad \end{array} \\ \begin{bmatrix} \begin{array}{l} 2\sigma^{11}_{(125)}\sigma^{13}_{(125)} \\ +2\sigma^{11}_{(145)}\sigma^{13}_{(145)} \\ +2\sigma^{11}_{(135)}\sigma^{13}_{(135)} \end{array} & \begin{array}{l} \sigma^{11}_{(125)}\sigma^{23}_{(125)} \\ +\sigma^{12}_{(125)}\sigma^{13}_{(125)} \end{array} & \begin{array}{l} \sigma^{11}_{(135)}\sigma^{23}_{(135)} \\ +\sigma^{12}_{(135)}\sigma^{13}_{(135)} \end{array} & \begin{array}{l} \sigma^{11}_{(145)}\sigma^{23}_{(145)} \\ +\sigma^{12}_{(145)}\sigma^{13}_{(145)} \end{array} & \begin{array}{l} \sigma^{11}_{(125)}\sigma^{33}_{(125)}+[\sigma^{13}_{(125)}]^2 \\ +\sigma^{11}_{(145)}\sigma^{33}_{(145)}+[\sigma^{13}_{(145)}]^2 \\ +\sigma^{11}_{(135)}\sigma^{33}_{(135)}+[\sigma^{13}_{(135)}]^2 \end{array} \\ & 2\sigma^{12}_{(125)}\sigma^{23}_{(125)} & 0 & 0 & \sigma^{12}_{(125)}\sigma^{33}_{(125)}+\sigma^{13}_{(126)}\sigma^{33}_{(125)} \\ & & 2\sigma^{12}_{(135)}\sigma^{23}_{(135)} & 0 & \sigma^{12}_{(135)}\sigma^{33}_{(135)}+\sigma^{13}_{(135)}\sigma^{23}_{(135)} \\ & & & 2\sigma^{12}_{(145)}\sigma^{23}_{(145)} & \sigma^{12}_{(145)}\sigma^{33}_{(145)}+\sigma^{13}_{(145)}\sigma^{23}_{(145)} \\ & & & & \begin{array}{l} 2\sigma^{13}_{(125)}\sigma^{33}_{(125)} \\ +2\sigma^{13}_{(145)}\sigma^{33}_{(145)} \\ +2\sigma^{13}_{(135)}\sigma^{33}_{(135)} \end{array} \end{bmatrix} \end{array}$$

$$
V^{[22]} = \begin{array}{c}
\begin{array}{ccc} \qquad\qquad 1 \qquad\qquad & \qquad\qquad 2 \qquad\qquad & \qquad\qquad 3 \qquad\qquad \end{array} \\
\left[\begin{array}{ccc}
\begin{array}{l} [\sigma^{12}_{(123)}]^2 \\ +[\sigma^{12}_{(125)}]^2 \\ +[\sigma^{12}_{(124)}]^2 \end{array} &
\begin{array}{l} \sigma^{12}_{(123)}\sigma^{22}_{(123)} \\ +\sigma^{12}_{(125)}\sigma^{22}_{(125)} \\ +\sigma^{12}_{(124)}\sigma^{22}_{(124)} \end{array} &
\sigma^{12}_{(123)}\sigma^{23}_{(123)} \\
 &
\begin{array}{l} [\sigma^{22}_{(123)}]^2+[\sigma^{22}_{(125)}]^2 \\ +[\sigma^{22}_{(124)}]^2+[\sigma^{11}_{(234)}]^2 \\ +[\sigma^{11}_{(235)}]^2+[\sigma^{11}_{(245)}]^2 \end{array} &
\begin{array}{l} \sigma^{22}_{(123)}\sigma^{23}_{(123)} \\ +\sigma^{11}_{(234)}\sigma^{12}_{(234)} \\ +\sigma^{11}_{(235)}\sigma^{12}_{(235)} \end{array} \\
 & &
\begin{array}{l} [\sigma^{23}_{(123)}]^2 \\ +[\sigma^{12}_{(234)}]^2 \\ +[\sigma^{12}_{(235)}]^2 \end{array}
\end{array}\right.
\end{array}
$$

$$
\left.\begin{array}{cc}
4 & 5 \\
\sigma^{12}_{(124)}\sigma^{23}_{(124)} & \sigma^{12}_{(125)}\sigma^{23}_{(125)} \\[2ex]
\sigma^{11}_{(234)}\sigma^{13}_{(234)} & \sigma^{22}_{(125)}\sigma^{23}_{(125)} \\
+\sigma^{22}_{(124)}\sigma^{23}_{(124)} & +\sigma^{11}_{(235)}\sigma^{13}_{(235)} \\
+\sigma^{11}_{(245)}\sigma^{12}_{(245)} & +\sigma^{11}_{(245)}\sigma^{13}_{(245)} \\[2ex]
\sigma^{12}_{(234)}\sigma^{13}_{(234)} & \sigma^{12}_{(235)}\sigma^{13}_{(235)} \\[2ex]
[\sigma^{13}_{(234)}]^2 & \sigma^{12}_{(245)}\sigma^{13}_{(245)} \\
+[\sigma^{23}_{(124)}]^2 & \\
+[\sigma^{12}_{(245)}]^2 & \\[2ex]
 & [\sigma^{23}_{(125)}]^2 \\
 & +[\sigma^{13}_{(235)}]^2 \\
 & +[\sigma^{13}_{(245)}]^2
\end{array}\right]
$$

$$
V^{[23]} = -\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
2\sigma^{12}_{(123)}\sigma^{13}_{(123)} &
\begin{matrix} \sigma^{12}_{(123)}\sigma^{23}_{(123)} \\ +\sigma^{22}_{(123)}\sigma^{13}_{(123)} \end{matrix} &
\begin{matrix} \sigma^{12}_{(123)}\sigma^{33}_{(123)} \\ +\sigma^{23}_{(123)}\sigma^{13}_{(123)} \end{matrix} &
0 & 0 \\
 &
\begin{matrix} 2\sigma^{22}_{(123)}\sigma^{23}_{(123)} \\ +2\sigma^{11}_{(234)}\sigma^{12}_{(234)} \\ +2\sigma^{11}_{(235)}\sigma^{12}_{(235)} \end{matrix} &
\begin{matrix} \sigma^{22}_{(123)}\sigma^{33}_{(123)}+[\sigma^{23}_{(123)}]^2 \\ +\sigma^{11}_{(234)}\sigma^{22}_{(234)}+[\sigma^{12}_{(234)}]^2 \\ +\sigma^{11}_{(235)}\sigma^{22}_{(235)}+[\sigma^{12}_{(235)}]^2 \end{matrix} &
\begin{matrix} \sigma^{11}_{(234)}\sigma^{23}_{(234)} \\ +\sigma^{13}_{(234)}\sigma^{12}_{(234)} \end{matrix} &
\begin{matrix} \sigma^{11}_{(235)}\sigma^{23}_{(235)} \\ +\sigma^{13}_{(235)}\sigma^{12}_{(235)} \end{matrix} \\
 & &
\begin{matrix} 2\sigma^{23}_{(123)}\sigma^{33}_{(123)} \\ +2\sigma^{12}_{(234)}\sigma^{22}_{(234)} \\ +2\sigma^{12}_{(235)}\sigma^{22}_{(235)} \end{matrix} &
\begin{matrix} \sigma^{12}_{(234)}\sigma^{23}_{(234)} \\ +\sigma^{13}_{(234)}\sigma^{22}_{(234)} \end{matrix} &
\begin{matrix} \sigma^{12}_{(235)}\sigma^{23}_{(235)} \\ +\sigma^{13}_{(235)}\sigma^{22}_{(235)} \end{matrix} \\
 & & & 2\sigma^{13}_{(234)}\sigma^{23}_{(234)} & 0 \\
 & & & & 2\sigma^{13}_{(235)}\sigma^{23}_{(235)}
\end{array}\right]
\end{array}
$$

$$
V^{[24]} = -\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
2\sigma^{12}_{(124)}\sigma^{13}_{(124)} & \begin{array}{l}\sigma^{12}_{(124)}\sigma^{23}_{(124)} \\ +\sigma^{22}_{(124)}\sigma^{13}_{(124)}\end{array} & 0 & \begin{array}{l}\sigma^{12}_{(124)}\sigma^{33}_{(124)} \\ +\sigma^{23}_{(124)}\sigma^{13}_{(124)}\end{array} & 0 \\
 & \begin{array}{l}2\sigma^{22}_{(124)}\sigma^{23}_{(124)} \\ +2\sigma^{11}_{(234)}\sigma^{13}_{(234)} \\ +2\sigma^{11}_{(245)}\sigma^{12}_{(245)}\end{array} & \begin{array}{l}\sigma^{11}_{(234)}\sigma^{23}_{(234)} \\ +\sigma^{12}_{(234)}\sigma^{13}_{(234)}\end{array} & \begin{array}{l}\sigma^{22}_{(124)}\sigma^{33}_{(124)}+[\sigma^{23}_{(124)}]^2 \\ +\sigma^{11}_{(234)}\sigma^{33}_{(234)}+[\sigma^{13}_{(234)}]^2 \\ +\sigma^{11}_{(245)}\sigma^{22}_{(245)}+[\sigma^{12}_{(245)}]^2\end{array} & \begin{array}{l}\sigma^{11}_{(245)}\sigma^{23}_{(245)} \\ +\sigma^{13}_{(245)}\sigma^{12}_{(245)}\end{array} \\
 & & 2\sigma^{12}_{(234)}\sigma^{23}_{(234)} & \begin{array}{l}\sigma^{12}_{(234)}\sigma^{33}_{(234)} \\ +\sigma^{13}_{(234)}\sigma^{23}_{(234)}\end{array} & 0 \\
 & & & \begin{array}{l}2\sigma^{23}_{(124)}\sigma^{33}_{(124)} \\ +2\sigma^{13}_{(234)}\sigma^{33}_{(234)} \\ +2\sigma^{12}_{(245)}\sigma^{22}_{(245)}\end{array} & \begin{array}{l}\sigma^{12}_{(245)}\sigma^{23}_{(245)} \\ +\sigma^{13}_{(245)}\sigma^{22}_{(245)}\end{array} \\
 & & & & 2\sigma^{13}_{(245)}\sigma^{33}_{(245)}
\end{array}\right]
\end{array}
$$

$$
V^{[25]} = -\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array}
$$

$$
\left[\begin{array}{ccccc}
2\sigma^{12}_{(125)}\sigma^{13}_{(125)} & \begin{array}{l}\sigma^{12}_{(125)}\sigma^{23}_{(125)}\\ +\sigma^{22}_{(125)}\sigma^{13}_{(125)}\end{array} & 0 & 0 & \begin{array}{l}\sigma^{12}_{(125)}\sigma^{33}_{(125)}\\ +\sigma^{23}_{(125)}\sigma^{13}_{(125)}\end{array} \\
 & \begin{array}{l}2\sigma^{22}_{(125)}\sigma^{23}_{(125)}\\ +2\sigma^{11}_{(235)}\sigma^{13}_{(235)}\\ +2\sigma^{11}_{(245)}\sigma^{13}_{(245)}\end{array} & \begin{array}{l}\sigma^{11}_{(235)}\sigma^{23}_{(235)}\\ +\sigma^{12}_{(235)}\sigma^{13}_{(235)}\end{array} & \begin{array}{l}\sigma^{11}_{(245)}\sigma^{23}_{(245)}\\ +\sigma^{12}_{(245)}\sigma^{13}_{(245)}\end{array} & \begin{array}{l}\sigma^{22}_{(125)}\sigma^{33}_{(125)}+[\sigma^{23}_{(125)}]^2\\ +\sigma^{11}_{(235)}\sigma^{33}_{(235)}+[\sigma^{13}_{(235)}]^2\\ +\sigma^{11}_{(245)}\sigma^{33}_{(245)}+[\sigma^{13}_{(245)}]^2\end{array} \\
 & & 2\sigma^{12}_{(235)}\sigma^{23}_{(235)} & 0 & \sigma^{12}_{(235)}\sigma^{33}_{(235)}+\sigma^{13}_{(235)}\sigma^{23}_{(235)} \\
 & & & 2\sigma^{12}_{(245)}\sigma^{23}_{(245)} & \sigma^{12}_{(245)}\sigma^{33}_{(245)}+\sigma^{13}_{(245)}\sigma^{23}_{(245)} \\
 & & & & \begin{array}{l}2\sigma^{23}_{(125)}\sigma^{33}_{(125)}\\ +2\sigma^{13}_{(235)}\sigma^{33}_{(235)}\\ +2\sigma^{13}_{(245)}\sigma^{33}_{(245)}\end{array}
\end{array}\right]
$$

$$V^{[33]} = -\begin{array}{c} \begin{array}{ccc} \qquad 1 \qquad & \qquad 2 \qquad & \qquad 3 \qquad \end{array} \\ \left[\begin{array}{ccc} \begin{array}{l} [\sigma^{13}_{(123)}]^2 \\ +[\sigma^{12}_{(134)}]^2 \\ +[\sigma^{12}_{(135)}]^2 \end{array} & \sigma^{13}_{(123)}\sigma^{23}_{(123)} & \begin{array}{l} \sigma^{13}_{(123)}\sigma^{33}_{(123)} \\ +\sigma^{12}_{(134)}\sigma^{22}_{(134)} \\ +\sigma^{12}_{(135)}\sigma^{22}_{(135)} \end{array} \\ & \begin{array}{l} [\sigma^{23}_{(123)}]^2 \\ +[\sigma^{12}_{(234)}]^2 \\ +[\sigma^{12}_{(235)}]^2 \end{array} & \begin{array}{l} \sigma^{23}_{(123)}\sigma^{33}_{(123)} \\ +\sigma^{12}_{(234)}\sigma^{22}_{(234)} \\ +\sigma^{12}_{(235)}\sigma^{22}_{(235)} \end{array} \\ & & \begin{array}{l} [\sigma^{33}_{(123)}]^2+[\sigma^{22}_{(234)}]^2 \\ +[\sigma^{11}_{(345)}]^2+[\sigma^{22}_{(134)}]^2 \\ +[\sigma^{22}_{(135)}]^2+[\sigma^{22}_{(235)}]^2 \end{array} \end{array}\right. \end{array}$$

$$
\begin{array}{cc}
4 & 5 \\
\left.\begin{matrix}
\sigma^{12}_{(134)}\sigma^{23}_{(134)} & \sigma^{12}_{(135)}\sigma^{23}_{(135)} \\[2ex]
\sigma^{12}_{(234)}\sigma^{23}_{(234)} & \sigma^{12}_{(235)}\sigma^{23}_{(235)} \\[2ex]
\begin{gathered}\sigma^{22}_{(234)}\sigma^{23}_{(234)} \\ +\sigma^{11}_{(345)}\sigma^{12}_{(345)} \\ +\sigma^{22}_{(134)}\sigma^{23}_{(134)}\end{gathered} & \begin{gathered}\sigma^{11}_{(345)}\sigma^{13}_{(345)} \\ +\sigma^{22}_{(135)}\sigma^{23}_{(135)} \\ +\sigma^{22}_{(235)}\sigma^{23}_{(235)}\end{gathered} \\[2ex]
\begin{gathered}[\sigma^{23}_{(234)}]^2 \\ +[\sigma^{12}_{(345)}]^2 \\ +[\sigma^{23}_{(134)}]^2\end{gathered} & \sigma^{12}_{(345)}\sigma^{13}_{(345)} \\[2ex]
 & \begin{gathered}[\sigma^{13}_{(345)}]^2 \\ +[\sigma^{23}_{(135)}]^2 \\ +[\sigma^{23}_{(235)}]^2\end{gathered}
\end{matrix}\right]
\end{array}
$$

$$V^{[34]} = -\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} 2\sigma^{12}_{(134)}\sigma^{13}_{(134)} & 0 & \begin{array}{c}\sigma^{12}_{(134)}\sigma^{23}_{(134)}\\ +\sigma^{22}_{(134)}\sigma^{13}_{(134)}\end{array} & \begin{array}{c}\sigma^{12}_{(134)}\sigma^{33}_{(134)}\\ +\sigma^{23}_{(134)}\sigma^{13}_{(134)}\end{array} & 0 \\ & 2\sigma^{12}_{(234)}\sigma^{13}_{(234)} & \begin{array}{c}\sigma^{12}_{(234)}\sigma^{23}_{(234)}\\ +\sigma^{22}_{(234)}\sigma^{13}_{(234)}\end{array} & \begin{array}{c}\sigma^{12}_{(234)}\sigma^{33}_{(234)}\\ +\sigma^{23}_{(234)}\sigma^{13}_{(234)}\end{array} & 0 \\ & & \begin{array}{c}2\sigma^{22}_{(234)}\sigma^{23}_{(234)}\\ +2\sigma^{11}_{(345)}\sigma^{12}_{(345)}\\ +2\sigma^{22}_{(134)}\sigma^{23}_{(134)}\end{array} & \begin{array}{c}\sigma^{22}_{(234)}\sigma^{33}_{(234)}+[\sigma^{23}_{(234)}]^2\\ +\sigma^{11}_{(345)}\sigma^{22}_{(345)}+[\sigma^{12}_{(345)}]^2\\ +\sigma^{22}_{(134)}\sigma^{33}_{(134)}+[\sigma^{23}_{(134)}]^2\end{array} & \begin{array}{c}\sigma^{11}_{(345)}\sigma^{23}_{(345)}\\ +\sigma^{13}_{(345)}\sigma^{12}_{(345)}\end{array} \\ & & & \begin{array}{c}2\sigma^{23}_{(234)}\sigma^{33}_{(234)}\\ +2\sigma^{12}_{(345)}\sigma^{22}_{(345)}\\ +2\sigma^{23}_{(134)}\sigma^{33}_{(134)}\end{array} & \begin{array}{c}\sigma^{12}_{(345)}\sigma^{23}_{(345)}\\ +\sigma^{13}_{(345)}\sigma^{22}_{(345)}\end{array} \\ & & & & 2\sigma^{13}_{(345)}\sigma^{23}_{(345)} \end{array}\right] \end{array}$$

$$
V^{[35]} = -\begin{array}{c}
\begin{array}{ccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad \end{array} \\
\left[\begin{array}{ccccc}
2\sigma^{12}_{(135)}\sigma^{13}_{(135)} & 0 & \begin{array}{l}\sigma^{12}_{(135)}\sigma^{23}_{(135)} \\ +\sigma^{22}_{(135)}\sigma^{13}_{(135)}\end{array} & 0 & \begin{array}{l}\sigma^{12}_{(135)}\sigma^{33}_{(135)} \\ +\sigma^{23}_{(135)}\sigma^{13}_{(135)}\end{array} \\
 & 2\sigma^{12}_{(235)}\sigma^{13}_{(235)} & \begin{array}{l}\sigma^{12}_{(235)}\sigma^{23}_{(235)} \\ +\sigma^{22}_{(235)}\sigma^{13}_{(235)}\end{array} & 0 & \begin{array}{l}\sigma^{12}_{(235)}\sigma^{33}_{(235)} \\ +\sigma^{23}_{(235)}\sigma^{13}_{(235)}\end{array} \\
 & & \begin{array}{l}2\sigma^{11}_{(345)}\sigma^{13}_{(345)} \\ +2\sigma^{22}_{(135)}\sigma^{23}_{(135)} \\ +2\sigma^{22}_{(235)}\sigma^{23}_{(235)}\end{array} & \begin{array}{l}\sigma^{11}_{(345)}\sigma^{23}_{(345)} \\ +\sigma^{12}_{(345)}\sigma^{13}_{(345)}\end{array} & \begin{array}{l}\sigma^{11}_{(345)}\sigma^{33}_{(345)}+[\sigma^{13}_{(345)}]^2 \\ +\sigma^{22}_{(135)}\sigma^{33}_{(135)}+[\sigma^{23}_{(135)}]^2 \\ +\sigma^{22}_{(235)}\sigma^{33}_{(235)}+[\sigma^{23}_{(235)}]^2\end{array} \\
 & & & 2\sigma^{12}_{(345)}\sigma^{23}_{(345)} & \begin{array}{l}\sigma^{12}_{(345)}\sigma^{33}_{(345)} \\ +\sigma^{13}_{(345)}\sigma^{23}_{(345)}\end{array} \\
 & & & & \begin{array}{l}2\sigma^{13}_{(345)}\sigma^{33}_{(345)} \\ +2\sigma^{23}_{(135)}\sigma^{33}_{(135)} \\ +2\sigma^{23}_{(235)}\sigma^{33}_{(235)}\end{array}
\end{array}\right]
\end{array}
$$

$$
V^{[44]} = - \begin{array}{c}
\begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\
\left[\begin{array}{ccc}
\begin{array}{l} [\sigma^{12}_{(145)}]^2 \\ +[\sigma^{13}_{(124)}]^2 \\ +[\sigma^{13}_{(134)}]^2 \end{array} & \sigma^{13}_{(124)}\sigma^{23}_{(124)} & \sigma^{13}_{(134)}\sigma^{23}_{(134)} \\
 & \begin{array}{l} [\sigma^{23}_{(124)}]^2 \\ +[\sigma^{13}_{(234)}]^2 \\ +[\sigma^{12}_{(245)}]^2 \end{array} & \sigma^{13}_{(234)}\sigma^{23}_{(234)} \\
 & & \begin{array}{l} [\sigma^{23}_{(234)}]^2 \\ +[\sigma^{12}_{(345)}]^2 \\ +[\sigma^{23}_{(134)}]^2 \end{array}
\end{array} \right.
\end{array}
$$

$$
\left.\begin{array}{cc}
4 & 5 \\
\begin{matrix} \sigma^{12}_{(145)}\sigma^{22}_{(145)} \\ +\sigma^{13}_{(124)}\sigma^{33}_{(124)} \\ +\sigma^{13}_{(134)}\sigma^{33}_{(134)} \end{matrix} & \sigma^{12}_{(145)}\sigma^{23}_{(145)} \\[2ex]
\begin{matrix} \sigma^{23}_{(124)}\sigma^{33}_{(124)} \\ +\sigma^{13}_{(234)}\sigma^{33}_{(234)} \\ +\sigma^{12}_{(245)}\sigma^{22}_{(245)} \end{matrix} & \sigma^{12}_{(245)}\sigma^{23}_{(245)} \\[2ex]
\begin{matrix} \sigma^{23}_{(234)}\sigma^{33}_{(234)} \\ +\sigma^{12}_{(345)}\sigma^{22}_{(345)} \\ +\sigma^{23}_{(134)}\sigma^{33}_{(134)} \end{matrix} & \sigma^{12}_{(345)}\sigma^{23}_{(345)} \\[2ex]
\begin{matrix} [\sigma^{22}_{(145)}]^2+[\sigma^{33}_{(234)}]^2 \\ +[\sigma^{22}_{(345)}]^2+[\sigma^{33}_{(124)}]^2 \\ +[\sigma^{33}_{(134)}]^2+[\sigma^{22}_{(245)}]^2 \end{matrix} & \begin{matrix} \sigma^{22}_{(145)}\sigma^{23}_{(145)} \\ +\sigma^{22}_{(345)}\sigma^{23}_{(345)} \\ +\sigma^{22}_{(245)}\sigma^{23}_{(245)} \end{matrix} \\[2ex]
 & \begin{matrix} [\sigma^{23}_{(145)}]^2 \\ +[\sigma^{23}_{(345)}]^2 \\ +[\sigma^{23}_{(245)}]^2 \end{matrix}
\end{array}\right]
$$

$$
V^{[45]} = -\begin{array}{c}
\begin{array}{ccccc}
\quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad
\end{array} \\
\left[\begin{array}{ccccc}
2\sigma^{12}_{(145)}\sigma^{13}_{(145)} & 0 & 0 & \begin{array}{c}\sigma^{12}_{(145)}\sigma^{23}_{(145)} \\ +\sigma^{22}_{(145)}\sigma^{13}_{(145)}\end{array} & \begin{array}{c}\sigma^{12}_{(145)}\sigma^{33}_{(145)} \\ +\sigma^{23}_{(145)}\sigma^{13}_{(145)}\end{array} \\
 & 2\sigma^{12}_{(245)}\sigma^{13}_{(245)} & 0 & \begin{array}{c}\sigma^{12}_{(245)}\sigma^{23}_{(245)} \\ +\sigma^{22}_{(245)}\sigma^{13}_{(245)}\end{array} & \begin{array}{c}\sigma^{12}_{(245)}\sigma^{33}_{(245)} \\ +\sigma^{23}_{(245)}\sigma^{13}_{(245)}\end{array} \\
 & & 2\sigma^{12}_{(345)}\sigma^{13}_{(345)} & \begin{array}{c}\sigma^{12}_{(345)}\sigma^{23}_{(345)} \\ +\sigma^{22}_{(345)}\sigma^{13}_{(345)}\end{array} & \begin{array}{c}\sigma^{12}_{(345)}\sigma^{33}_{(345)} \\ +\sigma^{23}_{(345)}\sigma^{13}_{(345)}\end{array} \\
 & & & \begin{array}{c}2\sigma^{22}_{(145)}\sigma^{23}_{(145)} \\ +2\sigma^{22}_{(345)}\sigma^{23}_{(345)} \\ +2\sigma^{22}_{(245)}\sigma^{23}_{(245)}\end{array} & \begin{array}{c}\sigma^{22}_{(145)}\sigma^{33}_{(145)}+[\sigma^{23}_{(145)}]^2 \\ +\sigma^{22}_{(345)}\sigma^{33}_{(345)}+[\sigma^{23}_{(345)}]^2 \\ +\sigma^{22}_{(245)}\sigma^{33}_{(245)}+[\sigma^{23}_{(245)}]^2\end{array} \\
 & & & & \begin{array}{c}2\sigma^{23}_{(145)}\sigma^{33}_{(145)} \\ +2\sigma^{23}_{(345)}\sigma^{33}_{(345)} \\ +2\sigma^{23}_{(245)}\sigma^{33}_{(245)}\end{array}
\end{array}\right]
\end{array}
$$

$$
V^{[55]} = -\left[\begin{array}{ccc}
1 & 2 & 3 \\
[\sigma^{13}_{(125)}]^2 + [\sigma^{13}_{(145)}]^2 + [\sigma^{13}_{(135)}]^2 & \sigma^{13}_{(125)}\sigma^{23}_{(125)} & \sigma^{13}_{(135)}\sigma^{23}_{(135)} \\
 & [\sigma^{23}_{(125)}]^2 + [\sigma^{13}_{(235)}]^2 + [\sigma^{13}_{(245)}]^2 & \sigma^{13}_{(235)}\sigma^{23}_{(235)} \\
 & & [\sigma^{13}_{(345)}]^2 + [\sigma^{23}_{(135)}]^2 + [\sigma^{23}_{(235)}]^2
\end{array}\right.
$$

4	5
$\sigma^{13}_{(145)}\sigma^{23}_{(145)}$	$\sigma^{13}_{(125)}\sigma^{33}_{(125)} + \sigma^{13}_{(145)}\sigma^{33}_{(145)} + \sigma^{13}_{(135)}\sigma^{33}_{(135)}$
$\sigma^{13}_{(245)}\sigma^{23}_{(245)}$	$\sigma^{23}_{(125)}\sigma^{33}_{(125)} + \sigma^{13}_{(235)}\sigma^{33}_{(235)} + \sigma^{13}_{(245)}\sigma^{33}_{(245)}$
$\sigma^{13}_{(345)}\sigma^{23}_{(345)}$	$\sigma^{13}_{(345)}\sigma^{33}_{(345)} + \sigma^{23}_{(135)}\sigma^{33}_{(135)} + \sigma^{23}_{(235)}\sigma^{33}_{(235)}$
$[\sigma^{23}_{(145)}]^2 + [\sigma^{23}_{(345)}]^2 + [\sigma^{23}_{(245)}]^2$	$\sigma^{23}_{(145)}\sigma^{33}_{(145)} + \sigma^{23}_{(345)}\sigma^{33}_{(345)} + \sigma^{23}_{(245)}\sigma^{33}_{(245)}$
	$[\sigma^{33}_{(125)}]^2 + [\sigma^{33}_{(145)}]^2 + [\sigma^{33}_{(345)}]^2 + [\sigma^{33}_{(135)}]^2 + [\sigma^{33}_{(235)}]^2 + [\sigma^{33}_{(245)}]^2$

The terms of the $F^{[k\ell]}$ matrices for this design are as follows. (The X_i matrices contain the standard least squares estimates of the parameters in the i'th group.)

$$F^{[11]} = -\begin{bmatrix} [\sigma^{11}_{(123)}]^2 & \sigma^{11}_{(123)}\sigma^{12}_{(123)} & \sigma^{11}_{(123)}\sigma^{13}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{12}_{(123)} & [\sigma^{12}_{(123)}]^2 & \sigma^{12}_{(123)}\sigma^{13}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{13}_{(123)} & \sigma^{12}_{(123)}\sigma^{13}_{(123)} & [\sigma^{13}_{(123)}]^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X'_{(123)}$$

$$-\begin{bmatrix} [\sigma^{11}_{(125)}]^2 & \sigma^{11}_{(125)}\sigma^{12}_{(125)} & \sigma^{11}_{(125)}\sigma^{13}_{(125)} \\ \sigma^{11}_{(125)}\sigma^{12}_{(125)} & [\sigma^{12}_{(125)}]^2 & \sigma^{12}_{(125)}\sigma^{13}_{(125)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{11}_{(125)}\sigma^{13}_{(125)} & \sigma^{12}_{(125)}\sigma^{13}_{(125)} & [\sigma^{13}_{(125)}]^2 \end{bmatrix} X'_{(125)}$$

$$-\begin{bmatrix} [\sigma^{11}_{(145)}]^2 & \sigma^{11}_{(145)}\sigma^{12}_{(145)} & \sigma^{11}_{(145)}\sigma^{13}_{(145)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{11}_{(145)}\sigma^{12}_{(145)} & [\sigma^{12}_{(145)}]^2 & \sigma^{12}_{(145)}\sigma^{13}_{(145)} \\ \sigma^{11}_{(145)}\sigma^{13}_{(145)} & \sigma^{12}_{(145)}\sigma^{13}_{(145)} & [\sigma^{13}_{(145)}]^2 \end{bmatrix} X'_{(145)}$$

$$- \begin{bmatrix} [\sigma^{11}_{(124)}]^2 & \sigma^{11}_{(124)}\sigma^{12}_{(124)} & \sigma^{11}_{(124)}\sigma^{13}_{(124)} \\ \sigma^{11}_{(124)}\sigma^{12}_{(124)} & [\sigma^{12}_{(124)}]^2 & \sigma^{12}_{(124)}\sigma^{13}_{(124)} \\ 0 & 0 & 0 \\ \sigma^{11}_{(124)}\sigma^{13}_{(124)} & \sigma^{12}_{(124)}\sigma^{13}_{(124)} & [\sigma^{13}_{(124)}]^2 \\ 0 & 0 & 0 \end{bmatrix} X'_{(124)}$$

$$- \begin{bmatrix} [\sigma^{11}_{(134)}]^2 & \sigma^{11}_{(134)}\sigma^{12}_{(134)} & \sigma^{11}_{(134)}\sigma^{13}_{(134)} \\ 0 & 0 & 0 \\ \sigma^{11}_{(134)}\sigma^{12}_{(134)} & [\sigma^{12}_{(134)}]^2 & \sigma^{12}_{(134)}\sigma^{13}_{(134)} \\ \sigma^{11}_{(134)}\sigma^{13}_{(134)} & \sigma^{12}_{(134)}\sigma^{13}_{(134)} & [\sigma^{13}_{(134)}]^2 \\ 0 & 0 & 0 \end{bmatrix} X'_{(134)}$$

$$- \begin{bmatrix} [\sigma^{11}_{(135)}]^2 & \sigma^{11}_{(135)}\sigma^{12}_{(135)} & \sigma^{11}_{(135)}\sigma^{13}_{(135)} \\ 0 & 0 & 0 \\ \sigma^{11}_{(135)}\sigma^{12}_{(135)} & [\sigma^{12}_{(135)}]^2 & \sigma^{12}_{(135)}\sigma^{13}_{(135)} \\ 0 & 0 & 0 \\ \sigma^{11}_{(135)}\sigma^{13}_{(135)} & \sigma^{12}_{(135)}\sigma^{13}_{(135)} & [\sigma^{13}_{(135)}]^2 \end{bmatrix} X'_{(135)}$$

$$F^{[12]} = -\begin{array}{c} \begin{array}{ccc} \qquad\qquad 1 \qquad\qquad & \qquad\qquad 2 \qquad\qquad & \qquad\qquad 3 \qquad\qquad \end{array} \\ \left[\begin{array}{ccc} 2\sigma^{11}_{(123)}\sigma^{12}_{(123)} & \sigma^{11}_{(123)}\sigma^{22}_{(123)}+[\sigma^{12}_{(123)}]^2 & \sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{12}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{22}_{(123)}+[\sigma^{12}_{(123)}]^2 & 2\sigma^{12}_{(123)}\sigma^{22}_{(123)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{22}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{12}_{(123)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{22}_{(123)} & 2\sigma^{13}_{(123)}\sigma^{23}_{(123)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} X'_{(123)}$$

$$-\begin{array}{c} \begin{array}{cc} \qquad\qquad 1 \qquad\qquad & \qquad\qquad 2 \qquad\qquad \end{array} \\ \left[\begin{array}{cc} 2\sigma^{11}_{(125)}\sigma^{12}_{(125)} & \sigma^{11}_{(125)}\sigma^{22}_{(125)}+[\sigma^{12}_{(125)}]^2 \\ \sigma^{11}_{(125)}\sigma^{22}_{(125)}+[\sigma^{12}_{(125)}]^2 & 2\sigma^{12}_{(125)}\sigma^{22}_{(125)} \\ 0 & 0 \\ 0 & 0 \\ \sigma^{11}_{(125)}\sigma^{23}_{(125)}+\sigma^{13}_{(125)}\sigma^{12}_{(125)} & \sigma^{12}_{(125)}\sigma^{23}_{(125)}+\sigma^{13}_{(125)}\sigma^{22}_{(125)} \end{array}\right. \end{array}$$

$$\begin{array}{c} 3 \\ \left.\begin{matrix} \sigma^{11}_{(125)}\sigma^{23}_{(125)}+\sigma^{13}_{(125)}\sigma^{12}_{(125)} \\ \sigma^{12}_{(125)}\sigma^{23}_{(125)}+\sigma^{13}_{(125)}\sigma^{22}_{(125)} \\ 0 \\ 0 \\ 2\sigma^{13}_{(125)}\sigma^{23}_{(125)} \end{matrix}\right] \end{array} X'_{(125)}$$

$$-\begin{array}{ccc} 1 & 2 & 3 \\ \multicolumn{3}{c}{\left[\begin{array}{ccc} 2\sigma^{11}_{(124)}\sigma^{12}_{(124)} & \sigma^{11}_{(124)}\sigma^{22}_{(124)}+[\sigma^{12}_{(124)}]^2 & \sigma^{11}_{(124)}\sigma^{23}_{(124)}+\sigma^{13}_{(124)}\sigma^{12}_{(124)} \\ \sigma^{11}_{(124)}\sigma^{22}_{(124)}+[\sigma^{12}_{(124)}]^2 & 2\sigma^{12}_{(124)}\sigma^{22}_{(124)} & \sigma^{12}_{(124)}\sigma^{23}_{(124)}+\sigma^{13}_{(124)}\sigma^{22}_{(124)} \\ 0 & 0 & 0 \\ \sigma^{11}_{(124)}\sigma^{23}_{(124)}+\sigma^{13}_{(124)}\sigma^{12}_{(124)} & \sigma^{12}_{(124)}\sigma^{23}_{(124)}+\sigma^{13}_{(124)}\sigma^{22}_{(124)} & 2\sigma^{13}_{(124)}\sigma^{23}_{(124)} \\ 0 & 0 & 0 \end{array}\right]} \end{array} X'_{(124)}$$

$$
F^{[13]} = -\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc}
2\sigma^{11}_{(123)}\sigma^{13}_{(123)} & \sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{12}_{(123)}\sigma^{13}_{(123)} & \sigma^{11}_{(123)}\sigma^{33}_{(123)}+[\sigma^{12}_{(123)}]^2 \\
\sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{12}_{(123)}\sigma^{13}_{(123)} & 2\sigma^{12}_{(123)}\sigma^{23}_{(123)} & \sigma^{12}_{(123)}\sigma^{33}_{(123)}+\sigma^{13}_{(123)}\sigma^{23}_{(123)} \\
\sigma^{11}_{(123)}\sigma^{33}_{(123)}+[\sigma^{13}_{(123)}]^2 & \sigma^{12}_{(123)}\sigma^{33}_{(123)}+\sigma^{13}_{(123)}\sigma^{23}_{(123)} & 2\sigma^{13}_{(123)}\sigma^{33}_{(123)} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] \end{array} X'_{(123)}
$$

$$
-\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc}
2\sigma^{11}_{(134)}\sigma^{12}_{(134)} & \sigma^{11}_{(134)}\sigma^{22}_{(134)}+[\sigma^{12}_{(134)}]^2 \\
0 & 0 \\
\sigma^{11}_{(134)}\sigma^{22}_{(134)}+[\sigma^{12}_{(134)}]^2 & 2\sigma^{12}_{(134)}\sigma^{22}_{(134)} \\
\sigma^{11}_{(134)}\sigma^{23}_{(134)}+\sigma^{13}_{(134)}\sigma^{12}_{(134)} & \sigma^{12}_{(134)}\sigma^{23}_{(134)}+\sigma^{13}_{(134)}\sigma^{22}_{(134)} \\
0 & 0
\end{array}\right. \end{array}
$$

$$
\begin{array}{c}
3 \\
\left.\begin{array}{c}
\sigma^{11}_{(134)}\sigma^{23}_{(134)}+\sigma^{13}_{(134)}\sigma^{12}_{(134)} \\
0 \\
\sigma^{12}_{(134)}\sigma^{23}_{(134)}+\sigma^{13}_{(134)}\sigma^{22}_{(134)} \\
2\sigma^{13}_{(134)}\sigma^{23}_{(134)} \\
0
\end{array}\right]
\end{array}
X'_{(134)}
$$

$$
-\begin{array}{cc}
1 & 2 \\
\left[\begin{array}{cc}
2\sigma^{11}_{(135)}\sigma^{12}_{(135)} & \sigma^{11}_{(135)}\sigma^{22}_{(135)}+[\sigma^{12}_{(135)}]^2 \\
0 & 0 \\
\sigma^{11}_{(135)}\sigma^{22}_{(135)}+[\sigma^{12}_{(135)}]^2 & 2\sigma^{12}_{(135)}\sigma^{22}_{(135)} \\
0 & 0 \\
\sigma^{11}_{(135)}\sigma^{23}_{(135)}+\sigma^{13}_{(135)}\sigma^{12}_{(135)} & \sigma^{12}_{(135)}\sigma^{23}_{(135)}+\sigma^{13}_{(135)}\sigma^{22}_{(135)}
\end{array}\right.
\end{array}
$$

$$
\begin{array}{c}
3 \\
\left.\begin{array}{c}
\sigma^{11}_{(135)}\sigma^{23}_{(135)}+\sigma^{13}_{(135)}\sigma^{12}_{(135)} \\
0 \\
\sigma^{12}_{(135)}\sigma^{23}_{(135)}+\sigma^{13}_{(135)}\sigma^{22}_{(135)} \\
0 \\
2\sigma^{13}_{(135)}\sigma^{23}_{(135)}
\end{array}\right]
\end{array}
X'_{(135)}
$$

$$
F^{[14]} = -\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\left[\begin{array}{ccc}
2\sigma^{11}_{(145)}\sigma^{12}_{(145)} & \sigma^{11}_{(145)}\sigma^{22}_{(145)}+[\sigma^{12}_{(145)}]^2 & \sigma^{11}_{(145)}\sigma^{23}_{(145)}+\sigma^{13}_{(145)}\sigma^{12}_{(145)} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\sigma^{11}_{(145)}\sigma^{22}_{(145)}+[\sigma^{12}_{(145)}]^2 & 2\sigma^{12}_{(145)}\sigma^{22}_{(145)} & \sigma^{12}_{(145)}\sigma^{23}_{(145)}+\sigma^{13}_{(145)}\sigma^{22}_{(145)} \\
\sigma^{11}_{(145)}\sigma^{23}_{(145)}+\sigma^{13}_{(145)}\sigma^{12}_{(145)} & \sigma^{12}_{(145)}\sigma^{23}_{(145)}+\sigma^{13}_{(145)}\sigma^{22}_{(145)} & 2\sigma^{13}_{(145)}\sigma^{23}_{(145)}
\end{array}\right]
\end{array} X'_{(145)}
$$

$$
- \begin{array}{c}
\begin{array}{cc} 1 & 2 \end{array} \\
\left[\begin{array}{cc}
2\sigma^{11}_{(124)}\sigma^{13}_{(124)} & \sigma^{11}_{(124)}\sigma^{23}_{(124)}+\sigma^{12}_{(124)}\sigma^{13}_{(124)} \\
\sigma^{11}_{(124)}\sigma^{23}_{(124)}+\sigma^{12}_{(124)}\sigma^{13}_{(124)} & 2\sigma^{12}_{(124)}\sigma^{23}_{(124)} \\
0 & 0 \\
\sigma^{11}_{(124)}\sigma^{33}_{(124)}+[\sigma^{13}_{(124)}]^2 & \sigma^{12}_{(124)}\sigma^{33}_{(124)}+\sigma^{13}_{(124)}\sigma^{23}_{(124)} \\
0 & 0
\end{array}\right.
\end{array}
$$

$$\begin{array}{c} 3 \\ \left.\begin{matrix} \sigma^{11}_{(124)}\sigma^{33}_{(124)}+[\sigma^{13}_{(124)}]^{2} \\ \sigma^{12}_{(124)}\sigma^{33}_{(124)}+\sigma^{13}_{(124)}\sigma^{23}_{(124)} \\ 0 \\ 2\sigma^{13}_{(124)}\sigma^{33}_{(124)} \\ 0 \end{matrix}\right] \end{array} X'_{(124)}$$

$$- \begin{array}{cc} 1 & 2 \\ \multicolumn{2}{l}{\left[\begin{matrix} 2\sigma^{11}_{(134)}\sigma^{13}_{(134)} & \sigma^{11}_{(134)}\sigma^{23}_{(134)}+\sigma^{12}_{(134)}\sigma^{13}_{(134)} \\ 0 & 0 \\ \sigma^{11}_{(134)}\sigma^{23}_{(134)}+\sigma^{12}_{(134)}\sigma^{13}_{(134)} & 2\sigma^{12}_{(134)}\sigma^{23}_{(134)} \\ \sigma^{11}_{(134)}\sigma^{33}_{(134)}+[\sigma^{13}_{(134)}]^{2} & \sigma^{12}_{(134)}\sigma^{33}_{(134)}+\sigma^{13}_{(134)}\sigma^{23}_{(134)} \\ 0 & 0 \end{matrix}\right.} \end{array}$$

$$\begin{array}{c} 3 \\ \left.\begin{matrix} \sigma^{11}_{(134)}\sigma^{33}_{(134)}+[\sigma^{13}_{(134)}]^{2} \\ 0 \\ \sigma^{12}_{(134)}\sigma^{33}_{(134)}+\sigma^{13}_{(134)}\sigma^{23}_{(134)} \\ 2\sigma^{13}_{(134)}\sigma^{33}_{(134)} \\ 0 \end{matrix}\right] \end{array} X'_{(134)}$$

$$
P^{[15]} = -\begin{array}{c}
\begin{array}{ccc} \quad 1 & \quad 2 & \quad 3 \end{array} \\
\left[\begin{array}{ccc}
2\sigma^{11}_{(125)}\sigma^{13}_{(125)} & \sigma^{11}_{(125)}\sigma^{23}_{(125)}+\sigma^{12}_{(125)}\sigma^{13}_{(125)} & \sigma^{11}_{(125)}\sigma^{33}_{(125)}+[\sigma^{13}_{(125)}]^2 \\
\sigma^{11}_{(125)}\sigma^{23}_{(125)}+\sigma^{12}_{(125)}\sigma^{13}_{(125)} & 2\sigma^{12}_{(125)}\sigma^{23}_{(125)} & \sigma^{12}_{(125)}\sigma^{33}_{(125)}+\sigma^{13}_{(125)}\sigma^{23}_{(125)} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\sigma^{11}_{(125)}\sigma^{33}_{(125)}+[\sigma^{13}_{(125)}]^2 & \sigma^{12}_{(125)}\sigma^{33}_{(125)}+\sigma^{13}_{(125)}\sigma^{23}_{(125)} & 2\sigma^{13}_{(125)}\sigma^{23}_{(125)}
\end{array}\right]
\end{array} x'_{(125)}
$$

$$
- \begin{array}{c}
\begin{array}{cc} \quad 1 & \quad 2 \end{array} \\
\left[\begin{array}{cc}
2\sigma^{11}_{(145)}\sigma^{13}_{(145)} & \sigma^{11}_{(145)}\sigma^{23}_{(145)}+\sigma^{12}_{(145)}\sigma^{13}_{(145)} \\
0 & 0 \\
0 & 0 \\
\sigma^{11}_{(145)}\sigma^{23}_{(145)}+\sigma^{12}_{(145)}\sigma^{13}_{(145)} & 2\sigma^{12}_{(145)}\sigma^{23}_{(145)} \\
\sigma^{11}_{(145)}\sigma^{33}_{(145)}+[\sigma^{13}_{(145)}]^2 & \sigma^{12}_{(145)}\sigma^{33}_{(145)}+\sigma^{13}_{(145)}\sigma^{23}_{(145)}
\end{array}\right.
\end{array}
$$

$$
\begin{array}{c}
3 \\
\left.\begin{array}{c}
\sigma^{11}_{(145)}\sigma^{33}_{(145)}+[\sigma^{13}_{(145)}]^2 \\
0 \\
0 \\
\sigma^{12}_{(145)}\sigma^{33}_{(145)}+\sigma^{13}_{(145)}\sigma^{23}_{(145)} \\
2\sigma^{13}_{(145)}\sigma^{33}_{(145)}
\end{array}\right]
\end{array}
X'_{(145)}
$$

$$
-\begin{array}{cc}
1 & 2 \\
\left[\begin{array}{c}
2\sigma^{11}_{(135)}\sigma^{13}_{(135)} \\
0 \\
\sigma^{11}_{(135)}\sigma^{23}_{(135)}+\sigma^{12}_{(135)}\sigma^{13}_{(135)} \\
0 \\
\sigma^{11}_{(135)}\sigma^{33}_{(135)}+[\sigma^{13}_{(135)}]^2
\end{array}\right.
&
\begin{array}{c}
\sigma^{11}_{(135)}\sigma^{23}_{(135)}+\sigma^{12}_{(135)}\sigma^{13}_{(135)} \\
0 \\
2\sigma^{12}_{(135)}\sigma^{23}_{(135)} \\
0 \\
\sigma^{12}_{(135)}\sigma^{33}_{(135)}+\sigma^{13}_{(135)}\sigma^{33}_{(135)}
\end{array}
\end{array}
$$

$$
\begin{array}{c}
3 \\
\left.\begin{array}{c}
\sigma^{11}_{(135)}\sigma^{33}_{(135)}+[\sigma^{13}_{(135)}]^2 \\
0 \\
\sigma^{12}_{(135)}\sigma^{33}_{(135)}+\sigma^{13}_{(135)}\sigma^{23}_{(135)} \\
0 \\
2\sigma^{13}_{(135)}\sigma^{33}_{(135)}
\end{array}\right]
\end{array}
X'_{(135)}
$$

$$
F^{[22]} = \begin{bmatrix}
[\sigma^{12}_{(123)}]^2 & \sigma^{12}_{(123)}\sigma^{22}_{(123)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)} \\
\sigma^{12}_{(123)}\sigma^{22}_{(123)} & [\sigma^{22}_{(123)}]^2 & \sigma^{22}_{(123)}\sigma^{23}_{(123)} \\
\sigma^{12}_{(123)}\sigma^{23}_{(123)} & \sigma^{22}_{(123)}\sigma^{23}_{(123)} & [\sigma^{23}_{(123)}]^2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} X'_{(123)}
- \begin{bmatrix}
[\sigma^{12}_{(125)}]^2 & \sigma^{12}_{(125)}\sigma^{22}_{(125)} & \sigma^{12}_{(125)}\sigma^{23}_{(125)} \\
\sigma^{12}_{(125)}\sigma^{22}_{(125)} & [\sigma^{22}_{(125)}]^2 & \sigma^{22}_{(125)}\sigma^{23}_{(125)} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\sigma^{12}_{(125)}\sigma^{23}_{(125)} & \sigma^{22}_{(125)}\sigma^{23}_{(125)} & [\sigma^{23}_{(125)}]^2
\end{bmatrix} X'_{(125)}
- \begin{bmatrix}
0 & 0 & 0 \\
[\sigma^{11}_{(234)}]^2 & \sigma^{11}_{(234)}\sigma^{12}_{(234)} & \sigma^{11}_{(234)}\sigma^{13}_{(234)} \\
\sigma^{11}_{(234)}\sigma^{12}_{(234)} & [\sigma^{12}_{(234)}]^2 & \sigma^{12}_{(234)}\sigma^{13}_{(234)} \\
\sigma^{11}_{(234)}\sigma^{13}_{(234)} & \sigma^{12}_{(234)}\sigma^{13}_{(234)} & [\sigma^{13}_{(234)}]^2 \\
0 & 0 & 0
\end{bmatrix} X'_{(234)}
$$

$$
- \begin{bmatrix}
[\sigma^{12}_{(124)}]^2 & \sigma^{12}_{(124)}\sigma^{22}_{(124)} & \sigma^{12}_{(124)}\sigma^{23}_{(124)} \\
\sigma^{12}_{(124)}\sigma^{22}_{(124)} & [\sigma^{22}_{(124)}]^2 & \sigma^{22}_{(124)}\sigma^{23}_{(124)} \\
0 & 0 & 0 \\
\sigma^{12}_{(124)}\sigma^{23}_{(124)} & \sigma^{22}_{(124)}\sigma^{23}_{(124)} & [\sigma^{23}_{(124)}]^2 \\
0 & 0 & 0
\end{bmatrix} x'_{(124)}
$$

$$
- \begin{bmatrix}
0 & 0 & 0 \\
[\sigma^{11}_{(235)}]^2 & \sigma^{11}_{(235)}\sigma^{12}_{(235)} & \sigma^{11}_{(235)}\sigma^{13}_{(235)} \\
\sigma^{11}_{(235)}\sigma^{12}_{(235)} & [\sigma^{12}_{(235)}]^2 & \sigma^{12}_{(235)}\sigma^{13}_{(235)} \\
0 & 0 & 0 \\
\sigma^{11}_{(235)}\sigma^{13}_{(235)} & \sigma^{12}_{(235)}\sigma^{13}_{(235)} & [\sigma^{13}_{(235)}]^2
\end{bmatrix} x'_{(235)}
$$

$$
- \begin{bmatrix}
0 & 0 & 0 \\
[\sigma^{11}_{(245)}]^2 & \sigma^{11}_{(245)}\sigma^{12}_{(245)} & \sigma^{11}_{(245)}\sigma^{13}_{(245)} \\
0 & 0 & 0 \\
\sigma^{11}_{(245)}\sigma^{12}_{(245)} & [\sigma^{12}_{(245)}]^2 & \sigma^{12}_{(245)}\sigma^{13}_{(245)} \\
\sigma^{11}_{(245)}\sigma^{13}_{(245)} & \sigma^{12}_{(245)}\sigma^{13}_{(245)} & [\sigma^{13}_{(245)}]^2
\end{bmatrix} x'_{(245)}
$$

$$F^{[23]} = - \begin{array}{ccc} 1 & 2 & 3 \end{array}$$
$$\begin{bmatrix} 2\sigma^{12}_{(123)}\sigma^{13}_{(123)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{22}_{(123)}\sigma^{13}_{(123)} & \sigma^{12}_{(123)}\sigma^{33}_{(123)}+\sigma^{23}_{(123)}\sigma^{13}_{(123)} \\ \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{22}_{(123)}\sigma^{13}_{(123)} & 2\sigma^{22}_{(123)}\sigma^{23}_{(123)} & \sigma^{22}_{(123)}\sigma^{33}_{(123)}+[\sigma^{23}_{(123)}]^2 \\ \sigma^{12}_{(123)}\sigma^{33}_{(123)}+\sigma^{23}_{(123)}\sigma^{13}_{(123)} & \sigma^{22}_{(123)}\sigma^{33}_{(123)}+[\sigma^{23}_{(123)}]^2 & 2\sigma^{23}_{(123)}\sigma^{33}_{(123)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X'_{(123)}$$

$$\begin{array}{cc} 1 & 2 \end{array}$$
$$- \left[\begin{matrix} 0 & 0 \\ 2\sigma^{11}_{(234)}\sigma^{12}_{(234)} & \sigma^{11}_{(234)}\sigma^{22}_{(234)}+[\sigma^{12}_{(234)}]^2 \\ \sigma^{11}_{(234)}\sigma^{22}_{(234)}+[\sigma^{12}_{(234)}]^2 & 2\sigma^{12}_{(234)}\sigma^{22}_{(234)} \\ \sigma^{11}_{(234)}\sigma^{23}_{(234)}+\sigma^{13}_{(234)}\sigma^{12}_{(234)} & \sigma^{12}_{(234)}\sigma^{23}_{(234)}+\sigma^{13}_{(234)}\sigma^{22}_{(234)} \\ 0 & 0 \end{matrix} \right.$$

$$
\left.\begin{array}{c}
3 \\
\begin{array}{c}
0 \\
\sigma^{11}_{(234)}\sigma^{23}_{(234)}+\sigma^{13}_{(234)}\sigma^{12}_{(234)} \\
\sigma^{12}_{(234)}\sigma^{23}_{(234)}+\sigma^{13}_{(234)}\sigma^{22}_{(234)} \\
2\sigma^{13}_{(234)}\sigma^{23}_{(234)} \\
0
\end{array}
\end{array}\right] X'_{(234)}
$$

$$
- \begin{array}{ccc}
1 & 2 & 3 \\
\end{array}
$$

$$
-\left[\begin{array}{ccc}
0 & 0 & 0 \\
2\sigma^{11}_{(235)}\sigma^{12}_{(235)} & \sigma^{11}_{(235)}\sigma^{22}_{(235)}+[\sigma^{12}_{(235)}]^2 & \sigma^{11}_{(235)}\sigma^{23}_{(235)}+\sigma^{13}_{(235)}\sigma^{12}_{(235)} \\
\sigma^{11}_{(235)}\sigma^{22}_{(235)}+[\sigma^{12}_{(235)}]^2 & 2\sigma^{12}_{(235)}\sigma^{22}_{(235)} & \sigma^{12}_{(235)}\sigma^{23}_{(235)}+\sigma^{13}_{(235)}\sigma^{22}_{(235)} \\
0 & 0 & 0 \\
\sigma^{11}_{(235)}\sigma^{23}_{(235)}+\sigma^{13}_{(235)}\sigma^{12}_{(235)} & \sigma^{12}_{(235)}\sigma^{23}_{(236)}+\sigma^{13}_{(235)}\sigma^{22}_{(235)} & 2\sigma^{13}_{(235)}\sigma^{23}_{(235)}
\end{array}\right] X'_{(235)}
$$

$$
F^{[24]} = -\begin{array}{c}
\begin{array}{ccc} \quad\quad\quad\quad\quad 1 & \quad\quad\quad\quad\quad\quad\quad\quad 2 & \quad\quad\quad\quad\quad\quad\quad\quad 3 \end{array} \\
\left[\begin{array}{ccc}
0 & 0 & 0 \\
2\sigma^{11}_{(234)}\sigma^{13}_{(234)} & \sigma^{11}_{(234)}\sigma^{23}_{(234)}+\sigma^{12}_{(234)}\sigma^{13}_{(234)} & \sigma^{11}_{(234)}\sigma^{33}_{(234)}+[\sigma^{13}_{(234)}]^2 \\
\sigma^{11}_{(234)}\sigma^{23}_{(234)}+\sigma^{12}_{(234)}\sigma^{13}_{(234)} & 2\sigma^{12}_{(234)}\sigma^{23}_{(234)} & \sigma^{12}_{(234)}\sigma^{33}_{(234)}+\sigma^{13}_{(234)}\sigma^{23}_{(234)} \\
\sigma^{11}_{(234)}\sigma^{33}_{(234)}+[\sigma^{13}_{(234)}]^2 & \sigma^{12}_{(234)}\sigma^{33}_{(234)}+\sigma^{13}_{(234)}\sigma^{23}_{(234)} & 2\sigma^{13}_{(234)}\sigma^{33}_{(234)} \\
0 & 0 & 0
\end{array}\right]
\end{array} X'_{(234)}
$$

$$
-\begin{array}{c}
\begin{array}{cc} \quad\quad\quad\quad\quad 1 & \quad\quad\quad\quad\quad\quad\quad\quad 2 \end{array} \\
\left[\begin{array}{cc}
2\sigma^{12}_{(124)}\sigma^{13}_{(124)} & \sigma^{12}_{(124)}\sigma^{23}_{(124)}+\sigma^{22}_{(124)}\sigma^{13}_{(124)} \\
\sigma^{12}_{(124)}\sigma^{23}_{(124)}+\sigma^{22}_{(124)}\sigma^{13}_{(124)} & 2\sigma^{22}_{(124)}\sigma^{23}_{(124)} \\
0 & 0 \\
\sigma^{12}_{(124)}\sigma^{33}_{(124)}+\sigma^{23}_{(124)}\sigma^{13}_{(124)} & \sigma^{22}_{(124)}\sigma^{33}_{(124)}+[\sigma^{23}_{(124)}]^2 \\
0 & 0
\end{array}\right.
\end{array}
$$

$$
\begin{array}{c}
3 \\
\left.\begin{array}{c}
\sigma^{12}_{(124)}\sigma^{33}_{(124)}+\sigma^{23}_{(124)}\sigma^{13}_{(124)} \\
\sigma^{22}_{(124)}\sigma^{33}_{(124)}+[\sigma^{23}_{(124)}]^{2} \\
0 \\
2\sigma^{23}_{(124)}\sigma^{33}_{(124)} \\
0
\end{array}\right]
\end{array}
X'_{(124)}
$$

$$
-\begin{array}{cc}
1 & 2 \\
\left[\begin{array}{c}
0 \\
2\sigma^{11}_{(245)}\sigma^{12}_{(245)} \\
0 \\
\sigma^{11}_{(245)}\sigma^{22}_{(245)}+[\sigma^{12}_{(245)}]^{2} \\
\sigma^{11}_{(245)}\sigma^{23}_{(245)}+\sigma^{13}_{(245)}\sigma^{12}_{(245)}
\end{array}\right. &
\begin{array}{c}
0 \\
\sigma^{11}_{(245)}\sigma^{22}_{(245)}+[\sigma^{12}_{(245)}]^{2} \\
0 \\
2\sigma^{12}_{(245)}\sigma^{22}_{(245)} \\
\sigma^{12}_{(245)}\sigma^{23}_{(245)}+\sigma^{13}_{(245)}\sigma^{22}_{(245)}
\end{array}
\end{array}
$$

$$
\begin{array}{c}
3 \\
\left.\begin{array}{c}
0 \\
\sigma^{11}_{(245)}\sigma^{23}_{(245)}+\sigma^{13}_{(245)}\sigma^{12}_{(245)} \\
0 \\
\sigma^{12}_{(245)}\sigma^{23}_{(245)}+\sigma^{13}_{(245)}\sigma^{22}_{(245)} \\
2\sigma^{13}_{(245)}\sigma^{23}_{(245)}
\end{array}\right]
\end{array}
X'_{(245)}
$$

$$F^{[25]} = - \begin{array}{c} \begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \left[\begin{array}{ccc} 2\sigma^{12}_{(125)}\sigma^{13}_{(125)} & \sigma^{12}_{(125)}\sigma^{23}_{(125)}+\sigma^{22}_{(125)}\sigma^{13}_{(125)} & \sigma^{12}_{(125)}\sigma^{33}_{(125)}+\sigma^{23}_{(125)}\sigma^{13}_{(125)} \\ \sigma^{12}_{(125)}\sigma^{23}_{(125)}+\sigma^{22}_{(125)}\sigma^{13}_{(125)} & 2\sigma^{22}_{(125)}\sigma^{23}_{(125)} & \sigma^{22}_{(125)}\sigma^{33}_{(125)}+[\sigma^{23}_{(125)}]^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{12}_{(125)}\sigma^{33}_{(125)}+\sigma^{23}_{(125)}\sigma^{13}_{(125)} & \sigma^{22}_{(125)}\sigma^{33}_{(125)}+[\sigma^{23}_{(125)}]^2 & 2\sigma^{23}_{(125)}\sigma^{33}_{(125)} \end{array} \right] \end{array} X'_{(125)}$$

$$- \begin{array}{c} \begin{array}{cc} \quad 1 \quad & \quad 2 \quad \end{array} \\ \left[\begin{array}{cc} 0 & 0 \\ 2\sigma^{11}_{(235)}\sigma^{13}_{(235)} & \sigma^{11}_{(235)}\sigma^{23}_{(235)}+\sigma^{12}_{(235)}\sigma^{13}_{(235)} \\ \sigma^{11}_{(235)}\sigma^{33}_{(235)}+\sigma^{12}_{(235)}\sigma^{13}_{(235)} & 2\sigma^{12}_{(235)}\sigma^{23}_{(235)} \\ 0 & 0 \\ \sigma^{11}_{(235)}\sigma^{33}_{(235)}+[\sigma^{13}_{(235)}]^2 & \sigma^{12}_{(235)}\sigma^{33}_{(235)}+\sigma^{13}_{(235)}\sigma^{23}_{(235)} \end{array} \right. \end{array}$$

$$
\begin{array}{c}
3 \\
\left.\begin{array}{c}
0 \\
\sigma^{11}_{(235)}\sigma^{33}_{(235)}+[\sigma^{13}_{(235)}]^2 \\
\sigma^{12}_{(235)}\sigma^{33}_{(235)}+\sigma^{13}_{(235)}\sigma^{23}_{(235)} \\
0 \\
2\sigma^{13}_{(235)}\sigma^{33}_{(235)}
\end{array}\right]
\end{array}
X'_{(235)}
$$

$$
-\;
\begin{array}{cc}
1 & 2 \\
\multicolumn{2}{c}{\left[\begin{array}{cc}
0 & 0 \\
2\sigma^{11}_{(245)}\sigma^{13}_{(245)} & \sigma^{11}_{(245)}\sigma^{23}_{(245)}+\sigma^{12}_{(245)}\sigma^{13}_{(245)} \\
0 & 0 \\
\sigma^{11}_{(245)}\sigma^{23}_{(245)}+\sigma^{12}_{(245)}\sigma^{13}_{(245)} & 2\sigma^{12}_{(245)}\sigma^{23}_{(245)} \\
\sigma^{11}_{(245)}\sigma^{33}_{(245)}+[\sigma^{13}_{(245)}]^2 & \sigma^{12}_{(245)}\sigma^{33}_{(245)}+\sigma^{13}_{(245)}\sigma^{33}_{(245)}
\end{array}\right.}
\end{array}
$$

$$
\begin{array}{c}
3 \\
\left.\begin{array}{c}
0 \\
\sigma^{11}_{(245)}\sigma^{33}_{(245)}+[\sigma^{13}_{(245)}]^2 \\
0 \\
\sigma^{12}_{(245)}\sigma^{33}_{(245)}+\sigma^{13}_{(245)}\sigma^{23}_{(245)} \\
2\sigma^{13}_{(245)}\sigma^{33}_{(245)}
\end{array}\right]
\end{array}
X'_{(245)}
$$

$$
F^{[33]} = -\begin{bmatrix}
[\sigma^{13}_{(123)}]^2 & \sigma^{13}_{(123)}\sigma^{23}_{(123)} & \sigma^{13}_{(123)}\sigma^{33}_{(123)} \\
\sigma^{13}_{(123)}\sigma^{23}_{(123)} & [\sigma^{23}_{(123)}]^2 & \sigma^{23}_{(123)}\sigma^{33}_{(123)} \\
\sigma^{13}_{(123)}\sigma^{33}_{(123)} & \sigma^{23}_{(123)}\sigma^{33}_{(123)} & [\sigma^{33}_{(123)}]^2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} X'_{(123)}
$$

$$
-\begin{bmatrix}
0 & 0 & 0 \\
[\sigma^{12}_{(234)}]^2 & \sigma^{12}_{(234)}\sigma^{22}_{(234)} & \sigma^{12}_{(234)}\sigma^{23}_{(234)} \\
\sigma^{12}_{(234)}\sigma^{22}_{(234)} & [\sigma^{22}_{(234)}]^2 & \sigma^{22}_{(234)}\sigma^{23}_{(234)} \\
\sigma^{12}_{(234)}\sigma^{23}_{(234)} & \sigma^{22}_{(234)}\sigma^{23}_{(234)} & [\sigma^{23}_{(234)}]^2 \\
0 & 0 & 0
\end{bmatrix} X'_{(234)}
$$

$$
-\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
[\sigma^{11}_{(345)}]^2 & \sigma^{11}_{(345)}\sigma^{12}_{(345)} & \sigma^{11}_{(345)}\sigma^{13}_{(345)} \\
\sigma^{11}_{(345)}\sigma^{12}_{(345)} & [\sigma^{12}_{(345)}]^2 & \sigma^{12}_{(345)}\sigma^{13}_{(345)} \\
\sigma^{11}_{(345)}\sigma^{13}_{(345)} & \sigma^{12}_{(345)}\sigma^{13}_{(345)} & [\sigma^{13}_{(345)}]^2
\end{bmatrix} X'_{(345)}
$$

$$
- \begin{bmatrix}
[\sigma^{12}_{(134)}]^2 & \sigma^{12}_{(134)}\sigma^{22}_{(134)} & \sigma^{12}_{(134)}\sigma^{23}_{(134)} \\
0 & 0 & 0 \\
\sigma^{12}_{(134)}\sigma^{22}_{(134)} & [\sigma^{22}_{(134)}]^2 & \sigma^{22}_{(134)}\sigma^{23}_{(134)} \\
\sigma^{12}_{(134)}\sigma^{23}_{(134)} & \sigma^{22}_{(134)}\sigma^{23}_{(134)} & [\sigma^{23}_{(134)}]^2 \\
0 & 0 & 0
\end{bmatrix} X'_{(134)}
$$

$$
- \begin{bmatrix}
[\sigma^{12}_{(135)}]^2 & \sigma^{12}_{(135)}\sigma^{22}_{(135)} & \sigma^{12}_{(135)}\sigma^{23}_{(135)} \\
0 & 0 & \\
\sigma^{12}_{(135)}\sigma^{22}_{(135)} & [\sigma^{22}_{(135)}]^2 & \sigma^{22}_{(135)}\sigma^{23}_{(135)} \\
0 & 0 & \\
\sigma^{12}_{(135)}\sigma^{23}_{(135)} & \sigma^{22}_{(135)}\sigma^{23}_{(135)} & [\sigma^{23}_{(135)}]^2
\end{bmatrix} X'_{(135)}
$$

$$
- \begin{bmatrix}
0 & 0 & 0 \\
[\sigma^{12}_{(235)}]^2 & \sigma^{12}_{(235)}\sigma^{22}_{(235)} & \sigma^{12}_{(235)}\sigma^{23}_{(235)} \\
\sigma^{12}_{(235)}\sigma^{22}_{(235)} & [\sigma^{22}_{(235)}]^2 & \sigma^{22}_{(235)}\sigma^{23}_{(235)} \\
0 & 0 & 0 \\
\sigma^{12}_{(235)}\sigma^{23}_{(235)} & \sigma^{22}_{(235)}\sigma^{23}_{(235)} & [\sigma^{23}_{(235)}]^2
\end{bmatrix} X'_{(235)}
$$

$$
F^{[34]} = - \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc}
0 & 0 & 0 \\
2\sigma^{12}_{(234)}\sigma^{13}_{(234)} & \sigma^{12}_{(234)}\sigma^{23}_{(234)}+\sigma^{22}_{(234)}\sigma^{13}_{(234)} & \sigma^{12}_{(234)}\sigma^{33}_{(234)}+\sigma^{23}_{(234)}\sigma^{13}_{(234)} \\
\sigma^{12}_{(234)}\sigma^{23}_{(234)}+\sigma^{22}_{(234)}\sigma^{13}_{(234)} & 2\sigma^{22}_{(234)}\sigma^{23}_{(234)} & \sigma^{22}_{(234)}\sigma^{33}_{(234)}+[\sigma^{23}_{(234)}]^2 \\
\sigma^{12}_{(234)}\sigma^{33}_{(234)}+\sigma^{23}_{(234)}\sigma^{13}_{(234)} & \sigma^{22}_{(234)}\sigma^{33}_{(234)}+[\sigma^{23}_{(234)}]^2 & 2\sigma^{23}_{(234)}\sigma^{33}_{(234)} \\
0 & 0 & 0
\end{array}\right] \end{array} X'_{(234)}
$$

$$
- \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc}
0 & 0 \\
0 & 0 \\
2\sigma^{11}_{(345)}\sigma^{12}_{(345)} & \sigma^{11}_{(345)}\sigma^{22}_{(345)}+[\sigma^{12}_{(345)}]^2 \\
\sigma^{11}_{(345)}\sigma^{22}_{(345)}+[\sigma^{12}_{(345)}]^2 & 2\sigma^{12}_{(345)}\sigma^{22}_{(345)} \\
\sigma^{11}_{(345)}\sigma^{23}_{(345)}+\sigma^{13}_{(345)}\sigma^{12}_{(345)} & \sigma^{12}_{(345)}\sigma^{23}_{(345)}+\sigma^{13}_{(345)}\sigma^{22}_{(345)}
\end{array}\right. \end{array}
$$

$$
\begin{array}{c}
3 \\
\left.\begin{matrix}
0 \\
0 \\
\sigma^{11}_{(345)}\sigma^{23}_{(345)}+\sigma^{13}_{(345)}\sigma^{12}_{(345)} \\
\sigma^{12}_{(345)}\sigma^{23}_{(345)}+\sigma^{13}_{(345)}\sigma^{22}_{(345)} \\
2\sigma^{13}_{(345)}\sigma^{23}_{(345)}
\end{matrix}\right]
\end{array}
X'_{(345)}
$$

$$
-\begin{array}{ccc}
1 & 2 & 3 \\
\multicolumn{3}{c}{\left[\begin{matrix}
2\sigma^{12}_{(134)}\sigma^{13}_{(134)} & \sigma^{12}_{(134)}\sigma^{23}_{(134)}+\sigma^{22}_{(134)}\sigma^{13}_{(134)} & \sigma^{12}_{(134)}\sigma^{33}_{(134)}+\sigma^{23}_{(134)}\sigma^{13}_{(134)} \\
0 & 0 & 0 \\
\sigma^{12}_{(134)}\sigma^{23}_{(134)}+\sigma^{22}_{(134)}\sigma^{13}_{(134)} & 2\sigma^{22}_{(134)}\sigma^{23}_{(134)} & \sigma^{22}_{(134)}\sigma^{33}_{(134)}+[\sigma^{23}_{(134)}]^2 \\
\sigma^{12}_{(134)}\sigma^{33}_{(134)}+\sigma^{23}_{(134)}\sigma^{13}_{(134)} & \sigma^{22}_{(134)}\sigma^{33}_{(134)}+[\sigma^{23}_{(134)}]^2 & 2\sigma^{23}_{(134)}\sigma^{33}_{(134)} \\
0 & 0 & 0
\end{matrix}\right]}
\end{array}
X'_{(134)}
$$

$$
F^{[35]} = -\begin{array}{c}
\begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\
\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
2\sigma^{11}_{(345)}\sigma^{13}_{(345)} & \sigma^{11}_{(345)}\sigma^{23}_{(345)}+\sigma^{12}_{(345)}\sigma^{13}_{(345)} & \sigma^{11}_{(345)}\sigma^{33}_{(345)}+[\sigma^{13}_{(345)}]^{2} \\
\sigma^{11}_{(345)}\sigma^{23}_{(345)}+\sigma^{12}_{(345)}\sigma^{13}_{(345)} & 2\sigma^{12}_{(345)}\sigma^{23}_{(345)} & \sigma^{12}_{(345)}\sigma^{33}_{(345)}+\sigma^{13}_{(345)}\sigma^{23}_{(345)} \\
\sigma^{11}_{(345)}\sigma^{33}_{(345)}+[\sigma^{13}_{(345)}]^{2} & \sigma^{12}_{(345)}\sigma^{33}_{(345)}+\sigma^{13}_{(345)}\sigma^{23}_{(345)} & 2\sigma^{13}_{(345)}\sigma^{33}_{(345)}
\end{array}\right]
\end{array} X'_{(345)}
$$

$$
-\begin{array}{c}
\begin{array}{cc} \quad 1 \quad & \quad 2 \quad \end{array} \\
\left[\begin{array}{cc}
2\sigma^{12}_{(135)}\sigma^{13}_{(135)} & \sigma^{12}_{(135)}\sigma^{23}_{(135)}+\sigma^{22}_{(135)}\sigma^{13}_{(135)} \\
0 & 0 \\
\sigma^{12}_{(135)}\sigma^{23}_{(135)}+\sigma^{22}_{(135)}\sigma^{13}_{(135)} & 2\sigma^{22}_{(135)}\sigma^{23}_{(135)} \\
0 & 0 \\
\sigma^{12}_{(135)}\sigma^{33}_{(135)}+\sigma^{23}_{(135)}\sigma^{13}_{(135)} & \sigma^{22}_{(135)}\sigma^{33}_{(135)}+[\sigma^{23}_{(135)}]^{2}
\end{array}\right.
\end{array}
$$

$$
\begin{array}{c}
3 \\
\left.\begin{array}{c}
\sigma^{12}_{(135)}\sigma^{33}_{(135)}+\sigma^{23}_{(135)}\sigma^{13}_{(135)} \\
0 \\
\sigma^{22}_{(135)}\sigma^{33}_{(135)}+[\sigma^{23}_{(135)}]^{2} \\
0 \\
2\sigma^{23}_{(135)}\sigma^{33}_{(135)}
\end{array}\right]
\end{array}
X'_{(135)}
$$

$$
-\begin{array}{cc}
1 & 2 \\
\multicolumn{2}{c}{\left[\begin{array}{cc}
0 & 0 \\
2\sigma^{12}_{(235)}\sigma^{13}_{(235)} & \sigma^{12}_{(235)}\sigma^{23}_{(235)}+\sigma^{22}_{(235)}\sigma^{13}_{(235)} \\
\sigma^{12}_{(235)}\sigma^{23}_{(235)}+\sigma^{22}_{(235)}\sigma^{13}_{(235)} & 2\sigma^{22}_{(235)}\sigma^{23}_{(235)} \\
0 & 0 \\
\sigma^{12}_{(235)}\sigma^{33}_{(235)}+\sigma^{23}_{(235)}\sigma^{13}_{(235)} & \sigma^{22}_{(235)}\sigma^{33}_{(235)}+[\sigma^{23}_{(235)}]^{2}
\end{array}\right.}
\end{array}
$$

$$
\begin{array}{c}
3 \\
\left.\begin{array}{c}
0 \\
\sigma^{12}_{(235)}\sigma^{33}_{(235)}+\sigma^{23}_{(235)}\sigma^{13}_{(235)} \\
\sigma^{22}_{(235)}\sigma^{33}_{(235)}+[\sigma^{23}_{(235)}]^{2} \\
0 \\
2\sigma^{23}_{(235)}\sigma^{33}_{(235)}
\end{array}\right]
\end{array}
X'_{(235)}
$$

$$F^{[44]} = - \begin{bmatrix} [\sigma^{12}_{(145)}]^2 & \sigma^{12}_{(145)}\sigma^{22}_{(145)} & \sigma^{12}_{(145)}\sigma^{23}_{(145)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{12}_{(145)}\sigma^{22}_{(145)} & [\sigma^{22}_{(145)}]^2 & \sigma^{22}_{(145)}\sigma^{23}_{(145)} \\ \sigma^{12}_{(145)}\sigma^{23}_{(145)} & \sigma^{22}_{(145)}\sigma^{23}_{(145)} & [\sigma^{23}_{(145)}]^2 \end{bmatrix} X'_{(145)}$$

$$- \begin{bmatrix} 0 & 0 & 0 \\ [\sigma^{13}_{(234)}]^2 & \sigma^{13}_{(234)}\sigma^{23}_{(234)} & \sigma^{13}_{(234)}\sigma^{33}_{(234)} \\ \sigma^{13}_{(234)}\sigma^{23}_{(234)} & [\sigma^{23}_{(234)}]^2 & \sigma^{23}_{(234)}\sigma^{33}_{(234)} \\ \sigma^{13}_{(234)}\sigma^{33}_{(234)} & \sigma^{23}_{(234)}\sigma^{33}_{(234)} & [\sigma^{33}_{(234)}]^2 \\ 0 & 0 & 0 \end{bmatrix} X'_{(234)}$$

$$- \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ [\sigma^{12}_{(345)}]^2 & \sigma^{12}_{(345)}\sigma^{22}_{(345)} & \sigma^{12}_{(345)}\sigma^{23}_{(345)} \\ \sigma^{12}_{(345)}\sigma^{22}_{(345)} & [\sigma^{22}_{(345)}]^2 & \sigma^{22}_{(345)}\sigma^{23}_{(345)} \\ \sigma^{12}_{(345)}\sigma^{23}_{(345)} & \sigma^{22}_{(345)}\sigma^{23}_{(345)} & [\sigma^{23}_{(345)}]^2 \end{bmatrix} X'_{(345)}$$

$$
- \begin{bmatrix} [\sigma^{13}_{(124)}]^2 & \sigma^{13}_{(124)}\sigma^{23}_{(124)} & \sigma^{13}_{(124)}\sigma^{33}_{(124)} \\ \sigma^{13}_{(124)}\sigma^{23}_{(124)} & [\sigma^{23}_{(124)}]^2 & \sigma^{23}_{(124)}\sigma^{33}_{(124)} \\ 0 & 0 & 0 \\ \sigma^{13}_{(124)}\sigma^{33}_{(124)} & \sigma^{23}_{(124)}\sigma^{33}_{(124)} & [\sigma^{33}_{(124)}]^2 \\ 0 & 0 & 0 \end{bmatrix} X'_{(124)}
$$

$$
- \begin{bmatrix} [\sigma^{13}_{(134)}]^2 & \sigma^{13}_{(134)}\sigma^{23}_{(134)} & \sigma^{13}_{(134)}\sigma^{33}_{(134)} \\ 0 & 0 & 0 \\ \sigma^{13}_{(134)}\sigma^{23}_{(134)} & [\sigma^{23}_{(134)}]^2 & \sigma^{23}_{(134)}\sigma^{33}_{(134)} \\ \sigma^{13}_{(134)}\sigma^{33}_{(134)} & \sigma^{23}_{(134)}\sigma^{33}_{(134)} & [\sigma^{33}_{(134)}]^2 \\ 0 & 0 & 0 \end{bmatrix} X'_{(134)}
$$

$$
- \begin{bmatrix} 0 & 0 & 0 \\ [\sigma^{12}_{(245)}]^2 & \sigma^{12}_{(245)}\sigma^{22}_{(245)} & \sigma^{12}_{(245)}\sigma^{23}_{(245)} \\ 0 & 0 & 0 \\ \sigma^{12}_{(245)}\sigma^{22}_{(245)} & [\sigma^{22}_{(245)}]^2 & \sigma^{22}_{(245)}\sigma^{23}_{(245)} \\ \sigma^{12}_{(245)}\sigma^{23}_{(245)} & \sigma^{22}_{(245)}\sigma^{23}_{(245)} & [\sigma^{23}_{(245)}]^2 \end{bmatrix} X'_{(245)}
$$

$$F^{[45]} = - \begin{array}{c} \begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \left[\begin{array}{ccc} 2\sigma^{12}_{(145)}\sigma^{13}_{(145)} & \sigma^{12}_{(145)}\sigma^{23}_{(145)}+\sigma^{22}_{(145)}\sigma^{13}_{(145)} & \sigma^{12}_{(145)}\sigma^{33}_{(145)}+\sigma^{23}_{(145)}\sigma^{13}_{(145)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{12}_{(145)}\sigma^{23}_{(145)}+\sigma^{22}_{(145)}\sigma^{13}_{(145)} & 2\sigma^{22}_{(145)}\sigma^{23}_{(145)} & \sigma^{22}_{(145)}\sigma^{33}_{(145)}+[\sigma^{23}_{(145)}]^2 \\ \sigma^{12}_{(145)}\sigma^{33}_{(145)}+\sigma^{23}_{(145)}\sigma^{13}_{(145)} & \sigma^{22}_{(145)}\sigma^{33}_{(145)}+[\sigma^{23}_{(145)}]^2 & 2\sigma^{23}_{(145)}\sigma^{33}_{(145)} \end{array} \right] \end{array} X'_{(145)}$$

$$- \begin{array}{c} \begin{array}{cc} \quad 1 \quad & \quad 2 \quad \end{array} \\ \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 2\sigma^{12}_{(345)}\sigma^{13}_{(345)} & \sigma^{12}_{(345)}\sigma^{23}_{(345)}+\sigma^{22}_{(345)}\sigma^{13}_{(345)} \\ \sigma^{12}_{(345)}\sigma^{23}_{(345)}+\sigma^{22}_{(345)}\sigma^{13}_{(345)} & 2\sigma^{22}_{(345)}\sigma^{23}_{(345)} \\ \sigma^{12}_{(345)}\sigma^{33}_{(345)}+\sigma^{23}_{(345)}\sigma^{13}_{(345)} & \sigma^{22}_{(345)}\sigma^{33}_{(345)}+[\sigma^{23}_{(345)}]^2 \end{array} \right. \end{array}$$

$$
\begin{array}{c}
3 \\
\left.\begin{matrix}
0 \\[2ex]
0 \\[2ex]
\sigma^{12}_{(345)}\sigma^{33}_{(345)}+\sigma^{23}_{(345)}\sigma^{13}_{(345)} \\[2ex]
\sigma^{22}_{(345)}\sigma^{33}_{(345)}+[\sigma^{23}_{(345)}]^{2} \\[2ex]
2\sigma^{23}_{(345)}\sigma^{33}_{(345)}
\end{matrix}\right]
\end{array}
X'_{(345)}
$$

$$
-\begin{array}{c}
\begin{matrix} \qquad\qquad 1 \qquad\qquad & \qquad\qquad 2 \qquad\qquad & \qquad\qquad 3 \qquad\qquad \end{matrix} \\
\left[\begin{matrix}
0 & 0 & 0 \\[2ex]
2\sigma^{12}_{(245)}\sigma^{13}_{(245)} & \sigma^{12}_{(245)}\sigma^{23}_{(245)}+\sigma^{22}_{(245)}\sigma^{13}_{(245)} & \sigma^{12}_{(245)}\sigma^{33}_{(245)}+\sigma^{23}_{(245)}\sigma^{13}_{(245)} \\[2ex]
0 & 0 & 0 \\[2ex]
\sigma^{12}_{(245)}\sigma^{23}_{(245)}+\sigma^{22}_{(245)}\sigma^{13}_{(245)} & 2\sigma^{22}_{(245)}\sigma^{33}_{(245)} & \sigma^{22}_{(245)}\sigma^{33}_{(245)}+[\sigma^{23}_{(245)}]^{2} \\[2ex]
\sigma^{12}_{(245)}\sigma^{33}_{(245)}+\sigma^{23}_{(245)}\sigma^{13}_{(245)} & \sigma^{22}_{(245)}\sigma^{33}_{(245)}+[\sigma^{23}_{(245)}]^{2} & 2\sigma^{23}_{(245)}\sigma^{33}_{(245)}
\end{matrix}\right]
\end{array}
X'_{(245)}
$$

$$F^{[55]} = -\begin{bmatrix} [\sigma^{13}_{(125)}]^2 & \sigma^{13}_{(125)}\sigma^{23}_{(125)} & \sigma^{13}_{(125)}\sigma^{33}_{(125)} \\ \sigma^{13}_{(125)}\sigma^{23}_{(125)} & [\sigma^{23}_{(125)}]^2 & \sigma^{23}_{(125)}\sigma^{33}_{(125)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{13}_{(125)}\sigma^{33}_{(125)} & \sigma^{23}_{(125)}\sigma^{33}_{(125)} & [\sigma^{33}_{(125)}]^2 \end{bmatrix} X'_{(125)}$$

$$-\begin{bmatrix} [\sigma^{13}_{(145)}]^2 & \sigma^{13}_{(145)}\sigma^{23}_{(145)} & \sigma^{13}_{(145)}\sigma^{33}_{(145)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{13}_{(145)}\sigma^{23}_{(145)} & [\sigma^{23}_{(145)}]^2 & \sigma^{23}_{(145)}\sigma^{33}_{(145)} \\ \sigma^{13}_{(145)}\sigma^{33}_{(145)} & \sigma^{23}_{(145)}\sigma^{33}_{(145)} & [\sigma^{33}_{(145)}]^2 \end{bmatrix} X'_{(145)}$$

$$-\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ [\sigma^{13}_{(345)}]^2 & \sigma^{13}_{(345)}\sigma^{23}_{(345)} & \sigma^{13}_{(345)}\sigma^{33}_{(345)} \\ \sigma^{13}_{(345)}\sigma^{23}_{(345)} & [\sigma^{23}_{(345)}]^2 & \sigma^{23}_{(345)}\sigma^{33}_{(345)} \\ \sigma^{13}_{(345)}\sigma^{33}_{(345)} & \sigma^{23}_{(345)}\sigma^{33}_{(345)} & [\sigma^{33}_{(345)}]^2 \end{bmatrix} X'_{(345)}$$

$$
- \begin{bmatrix}
[\sigma^{13}_{(135)}]^2 & \sigma^{13}_{(135)}\sigma^{23}_{(135)} & \sigma^{13}_{(135)}\sigma^{33}_{(135)} \\
0 & 0 & 0 \\
\sigma^{13}_{(135)}\sigma^{23}_{(135)} & [\sigma^{23}_{(135)}]^2 & \sigma^{23}_{(135)}\sigma^{33}_{(135)} \\
0 & 0 & 0 \\
\sigma^{13}_{(135)}\sigma^{33}_{(135)} & \sigma^{23}_{(135)}\sigma^{33}_{(135)} & [\sigma^{33}_{(135)}]^2
\end{bmatrix} X'_{(135)}
$$

$$
- \begin{bmatrix}
0 & 0 & 0 \\
[\sigma^{13}_{(235)}]^2 & \sigma^{13}_{(235)}\sigma^{23}_{(235)} & \sigma^{13}_{(235)}\sigma^{33}_{(235)} \\
\sigma^{13}_{(235)}\sigma^{23}_{(235)} & [\sigma^{23}_{(235)}]^2 & \sigma^{23}_{(235)}\sigma^{33}_{(235)} \\
0 & 0 & 0 \\
\sigma^{13}_{(235)}\sigma^{33}_{(235)} & \sigma^{23}_{(235)}\sigma^{33}_{(235)} & [\sigma^{33}_{(235)}]^2
\end{bmatrix} X'_{(235)}
$$

$$
- \begin{bmatrix}
0 & 0 & 0 \\
[\sigma^{13}_{(245)}]^2 & \sigma^{13}_{(245)}\sigma^{23}_{(245)} & \sigma^{13}_{(245)}\sigma^{33}_{(245)} \\
0 & 0 & 0 \\
\sigma^{13}_{(245)}\sigma^{23}_{(245)} & [\sigma^{23}_{(245)}]^2 & \sigma^{23}_{(245)}\sigma^{33}_{(245)} \\
\sigma^{13}_{(245)}\sigma^{33}_{(245)} & \sigma^{23}_{(245)}\sigma^{33}_{(245)} & [\sigma^{33}_{(245)}]^2
\end{bmatrix} X'_{(245)}
$$

The $T_i^{\{k\ell\}}$ matrices for the first group are listed below. The symbol $p_{mn}^{(i)}$ is used to denote the element in the m'th row and n'th column of the matrix product P_iP_i' whose equation is given in step 18 of section 3.4.

$$T_{(123)}^{\{11\}} = \begin{bmatrix} \sum_{j=1}^{3} \sigma_{(123)}^{1j} p_{1j}^{(123)} & \sum_{j=1}^{3} \sigma_{(123)}^{1j} p_{2j}^{(123)} & \sum_{j=1}^{3} \sigma_{(123)}^{1j} p_{3j}^{(123)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T_{(123)}^{\{12\}} = \begin{bmatrix} \sum_{j=1}^{3} \sigma_{(123)}^{2j} p_{1j}^{(123)} & \sum_{j=1}^{3} \sigma_{(123)}^{2j} p_{2j}^{(123)} & \sum_{j=1}^{3} \sigma_{(123)}^{2j} p_{3j}^{(123)} \\ \sum_{j=1}^{3} \sigma_{(123)}^{1j} p_{1j}^{(123)} & \sum_{j=1}^{3} \sigma_{(123)}^{1j} p_{2j}^{(123)} & \sum_{j=1}^{3} \sigma_{(123)}^{1j} p_{3j}^{(123)} \\ 0 & 0 & 0 \end{bmatrix}$$

$$T_{(123)}^{\{13\}} = \begin{bmatrix} \sum_{j=1}^{3} \sigma_{(123)}^{3j} p_{1j}^{(123)} & \sum_{j=1}^{3} \sigma_{(123)}^{3j} p_{2j}^{(123)} & \sum_{j=1}^{3} \sigma_{(123)}^{3j} p_{3j}^{(123)} \\ 0 & 0 & 0 \\ \sum_{j=1}^{3} \sigma_{(123)}^{1j} p_{1j}^{(123)} & \sum_{j=1}^{3} \sigma_{(123)}^{1j} p_{2j}^{(123)} & \sum_{j=1}^{3} \sigma_{(123)}^{1j} p_{3j}^{(123)} \end{bmatrix}$$

$$T_{(123)}^{\{22\}} = \begin{bmatrix} 0 & 0 & 0 \\ \sum_{j=1}^{3} \sigma_{(123)}^{2j} p_{1j}^{(123)} & \sum_{j=1}^{3} \sigma_{(123)}^{2j} p_{2j}^{(123)} & \sum_{j=1}^{3} \sigma_{(123)}^{2j} p_{3j}^{(123)} \\ 0 & 0 & 0 \end{bmatrix}$$

$$T^{(23)}_{(123)} = \begin{bmatrix} 0 & 0 & 0 \\ \sum_{j=1}^{3}\sigma^{3j}_{(123)}p^{(123)}_{1j} & \sum_{j=1}^{3}\sigma^{3j}_{(123)}p^{(123)}_{2j} & \sum_{j=1}^{3}\sigma^{3j}_{(123)}p^{(123)}_{3j} \\ \sum_{j=1}^{3}\sigma^{2j}_{(123)}p^{(123)}_{1j} & \sum_{j=1}^{3}\sigma^{2j}_{(123)}p^{(123)}_{2j} & \sum_{j=1}^{3}\sigma^{2j}_{(123)}p^{(123)}_{3j} \end{bmatrix}$$

$$T^{(33)}_{(123)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sum_{j=1}^{3}\sigma^{3j}_{(123)}p^{(123)}_{1j} & \sum_{j=1}^{3}\sigma^{3j}_{(123)}p^{(123)}_{2j} & \sum_{j=1}^{3}\sigma^{3j}_{(123)}p^{(123)}_{3j} \end{bmatrix}$$

$T^{(14)}_{(123)}$, $T^{(15)}_{(123)}$, $T^{(24)}_{(123)}$, $T^{(25)}_{(123)}$, $T^{(34)}_{(123)}$, $T^{(35)}_{(123)}$, $T^{(44)}_{(123)}$, $T^{(45)}_{(123)}$, and $T^{(55)}_{(123)}$ are null.

In each of the ten groups there are nine null $T_i^{(k\ell)}$ matrices corresponding to the variables and pairs of variables not contained in the i'th group. The six non-null $T_i^{(k\ell)}$ matrices in each group are exactly the same as those in the first group, except that the subscript for σ and the superscript for p must be replaced by a triplet denoting the variables contained in the i'th group. For example, in group (125) the matrix $T^{(55)}_{(125)}$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sum_{j=1}^{3} \sigma^{3j}_{(125)} P^{(125)}_{1j} & \sum_{j=1}^{3} \sigma^{3j}_{(125)} P^{(125)}_{2j} & \sum_{j=1}^{3} \sigma^{3j}_{(125)} P^{(125)}_{3j} \end{bmatrix}$$

5.5e Case of p Variables in K Groups of Three

If the variables in the i'th group are denoted by v, w, and x then the U_i matrices will have the form

$$U_i = \begin{bmatrix} \sigma_{vv} & \sigma_{vw} & \sigma_{vx} \\ \sigma_{vw} & \sigma_{ww} & \sigma_{wx} \\ \sigma_{vx} & \sigma_{wx} & \sigma_{xx} \end{bmatrix} .$$

These matrices are also denoted by

$$U_{(vwx)} = \begin{bmatrix} \sigma^{(vwx)}_{11} & \sigma^{(vwx)}_{12} & \sigma^{(vwx)}_{13} \\ \sigma^{(vwx)}_{12} & \sigma^{(vwx)}_{22} & \sigma^{(vwx)}_{23} \\ \sigma^{(vwx)}_{13} & \sigma^{(vwx)}_{23} & \sigma^{(vwx)}_{33} \end{bmatrix} .$$

$\sigma^{(vwx)}_{11}$ denotes the variance of the first variable in group (vwx); $\sigma^{(vwx)}_{12}$, the covariance between the first and second variables; $\sigma^{(vwx)}_{13}$, the covariance between the first and third variables, etc.

The matrix U_i^{-1} for group (vwx) has the form

$$U^{-1}_{(vwx)} = \begin{bmatrix} \sigma^{11}_{(vwx)} & \sigma^{12}_{(vwx)} & \sigma^{13}_{(vwx)} \\ \sigma^{12}_{(vwx)} & \sigma^{22}_{(vwx)} & \sigma^{23}_{(vwx)} \\ \sigma^{13}_{(vwx)} & \sigma^{23}_{(vwx)} & \sigma^{33}_{(vwx)} \end{bmatrix} .$$

$M_i U_i^{-1} M_i'$ for group (vwx) is a square matrix of order p with $U^{-1}_{(vwx)}$ as a submatrix in rows and columns v, w, and x and zeros in the remaining positions. The matrix $U_i^{[k\ell]} = M_i' \Sigma^{[k\ell]} M_i$ can be written

$$U^{[k\ell]}_{(vwx)} = \begin{bmatrix} \delta_{vk}\delta_{v\ell} & \delta_{vk}\delta_{w\ell} & \delta_{vk}\delta_{x\ell} \\ \delta_{vk}\delta_{w\ell} & \delta_{wk}\delta_{w\ell} & \delta_{wk}\delta_{x\ell} \\ \delta_{vk}\delta_{x\ell} & \delta_{wk}\delta_{x\ell} & \delta_{xk}\delta_{x\ell} \end{bmatrix} \qquad k \leq \ell$$

for the group which contains variables v, w, and x.

The k'th diagonal element of $V^{[kk]}$ is a sum of r terms associated with the r groups which contain variable k. The m'th diagonal element, where $m \neq k$, is a sum of λ terms associated with groups which contain both variables k and m. For example in section 5.5d, the fourth diagonal element of $V^{[44]}$ is

$$[\sigma^{22}_{(145)}]^2 + [\sigma^{33}_{(234)}]^2 + [\sigma^{22}_{(345)}]^2 + [\sigma^{33}_{(124)}]^2 + [\sigma^{33}_{(134)}]^2 + [\sigma^{22}_{(245)}]^2,$$

a sum of six terms corresponding to the six groups which contain variable four. The fifth diagonal element is

$$[\sigma^{23}_{(145)}]^2+[\sigma^{23}_{(345)}]^2+[\sigma^{23}_{(245)}]^2 \quad ,$$

a sum of three terms corresponding to the three groups which contain both variables four and five. The terms in the (k,ℓ) non-diagonal position of $V^{[kk]}$ are associated with groups which contain variables k and ℓ. For instance, the terms in the (4,5) position of $V^{[44]}$ are

$$\sigma^{22}_{(145)}\sigma^{23}_{(145)}+\sigma^{22}_{(345)}\sigma^{23}_{(345)}+\sigma^{22}_{(245)}\sigma^{23}_{(245)} \quad ,$$

elements associated with groups which contain variables four and five. The single element in the (ℓ,m) non-diagonal position of $V^{[kk]}$ is a term corresponding to the group which contains variables k, ℓ, and m. See for example the (3,5) term of $V^{[44]}$, i.e. $\sigma^{12}_{(345)}\sigma^{23}_{(345)}$.

The k'th and ℓ'th diagonal elements of $V^{[k\ell]}$ $(k\neq\ell)$ are sums of terms associated with groups which contain both variables k and ℓ. For instance, in section 5.5d, the fourth and fifth diagonal elements of $V^{[45]}$ are

$$2\sigma^{22}_{(145)}\sigma^{23}_{(145)}+2\sigma^{22}_{(345)}\sigma^{23}_{(345)}+2\sigma^{22}_{(245)}\sigma^{23}_{(245)}$$

and

$$2\sigma^{23}_{(145)}\sigma^{33}_{(145)}+2\sigma^{23}_{(345)}\sigma^{33}_{(345)}+2\sigma^{23}_{(245)}\sigma^{33}_{(245)}$$

respectively. These are terms coming from the U_i^{-1} matrices for groups which contain both variables four and five. The m'th diagonal position of $V^{[k\ell]}$ (where m is different from both k and ℓ) contains a single term associated with the group which contains variables k, ℓ, and m. For example, the third diagonal element of $V^{[45]}$ is $2\sigma^{12}_{(345)}\sigma^{13}_{(345)}$. The (m,k) and (m,$\ell$) non-diagonal elements of $V^{[k\ell]}$ (where m is different from both k and ℓ) are sums of two terms associated with the group which contains variables k, ℓ, and m. Note for example that the (3,4) and (3,5) elements of $V^{[45]}$ are

$$\sigma^{12}_{(345)}\sigma^{23}_{(345)}+\sigma^{22}_{(345)}\sigma^{13}_{(345)}$$

and

$$\sigma^{12}_{(345)}\sigma^{33}_{(345)}+\sigma^{23}_{(345)}\sigma^{13}_{(345)}$$

respectively. If m and n are different from both k and ℓ, then $V^{[k\ell]}$ has a zero in the (m,n) non-diagonal position. The (k,ℓ) position of $V^{[k\ell]}$ is a sum of six terms, two each from the groups containing both k and ℓ.

The $F^{[kk]}$ matrices are sums of r matrix products each associated with a group which contains variable k. The $F^{[k\ell]}$ matrices are sums of λ matrix products each associated with a group which contains the variable pair (kℓ). Each matrix product in $F^{[kk]}$ and $F^{[k\ell]}$ has p-3 rows of zeros.

Associated with each group there are six non-null $T_i^{\{k\ell\}}$ matrices. The six non-null matrices for the group containing variables v, w, and x are

$$T^{\{vv\}}_{(vwx)} = \begin{bmatrix} \sum_{j=1}^{3}\sigma^{1j}_{(vwx)}P^{(vwx)}_{1j} & \sum_{j=1}^{3}\sigma^{1j}_{(vwx)}P^{(vwx)}_{2j} & \sum_{j=1}^{3}\sigma^{1j}_{(vwx)}P^{(vwx)}_{3j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T^{\{vw\}}_{(vwx)} = \begin{bmatrix} \sum_{j=1}^{3}\sigma^{2j}_{(vwx)}P^{(vwx)}_{1j} & \sum_{j=1}^{3}\sigma^{2j}_{(vwx)}P^{(vwx)}_{2j} & \sum_{j=1}^{3}\sigma^{2j}_{(vwx)}P^{(vwx)}_{3j} \\ \sum_{j=1}^{3}\sigma^{1j}_{(vwx)}P^{(vwx)}_{1j} & \sum_{j=1}^{3}\sigma^{1j}_{(vwx)}P^{(vwx)}_{2j} & \sum_{j=1}^{3}\sigma^{1j}_{(vwx)}P^{(vwx)}_{3j} \\ 0 & 0 & 0 \end{bmatrix}$$

$$T^{\{vx\}}_{(vwx)} = \begin{bmatrix} \sum_{j=1}^{3}\sigma^{3j}_{(vwx)}P^{(vwx)}_{1j} & \sum_{j=1}^{3}\sigma^{3j}_{(vwx)}P^{(vwx)}_{2j} & \sum_{j=1}^{3}\sigma^{3j}_{(vwx)}P^{(vwx)}_{3j} \\ 0 & 0 & 0 \\ \sum_{j=1}^{3}\sigma^{1j}_{(vwx)}P^{(vwx)}_{1j} & \sum_{j=1}^{3}\sigma^{1j}_{(vwx)}P^{(vwx)}_{2j} & \sum_{j=1}^{3}\sigma^{1j}_{(vwx)}P^{(vwx)}_{3j} \end{bmatrix}$$

$$T^{\{ww\}}_{(vwx)} \quad \begin{bmatrix} 0 & 0 & 0 \\ \sum_{j=1}^{3}\sigma^{2j}_{(vwx)}P^{(vwx)}_{1j} & \sum_{j=1}^{3}\sigma^{2j}_{(vwx)}P^{(vwx)}_{2j} & \sum_{j=1}^{3}\sigma^{2j}_{(vwx)}P^{(vwx)}_{3j} \\ 0 & 0 & 0 \end{bmatrix}$$

$$T^{\{wx\}}_{(vwx)} = \begin{bmatrix} 0 & 0 & 0 \\ \sum_{j=1}^{3} \sigma^{3j}_{(vwx)} p^{(vwx)}_{1j} & \sum_{j=1}^{3} \sigma^{3j}_{(vwx)} p^{(vwx)}_{2j} & \sum_{j=1}^{3} \sigma^{3j}_{(vwx)} p^{(vwx)}_{3j} \\ \sum_{j=1}^{3} \sigma^{2j}_{(vwx)} p^{(vwx)}_{1j} & \sum_{j=1}^{3} \sigma^{2j}_{(vwx)} p^{(vwx)}_{2j} & \sum_{j=1}^{3} \sigma^{2j}_{(vwx)} p^{(vwx)}_{3j} \end{bmatrix}$$

$$T^{\{xx\}}_{(vwx)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sum_{j=1}^{3} \sigma^{3j}_{(vwx)} p^{(vwx)}_{1j} & \sum_{j=1}^{3} \sigma^{3j}_{(vwx)} p^{(vwx)}_{2j} & \sum_{j=1}^{3} \sigma^{3j}_{(vwx)} p^{(vwx)}_{3j} \end{bmatrix}$$

where $p^{(vwx)}_{mn}$ denotes the element in the m'th row and n'th column of the matrix product $P_i P_i'$ for the group which contains variables v, w, and x. A formula for $P_i P_i'$ is given in step 18 of section 3.4. The non-null rows of $T^{\{k\ell\}}_{(vwx)}$ $(k \neq \ell)$ can be obtained from the non-null rows of $T^{\{kk\}}_{(vwx)}$ and $T^{\{\ell\ell\}}_{(vwx)}$.

5.5f Seven Variables in Seven Groups of Four

Group			
(1)	3 5 6 7	(5)	2 3 4 7
(2)	1 4 6 7	(6)	1 3 4 5
(3)	1 2 5 7	(7)	2 4 5 6
(4)	1 2 3 6		

The submatrices of Σ for this design are

$$U_1 = \begin{bmatrix} \sigma_{33} & \sigma_{35} & \sigma_{36} & \sigma_{37} \\ \sigma_{35} & \sigma_{55} & \sigma_{56} & \sigma_{57} \\ \sigma_{36} & \sigma_{56} & \sigma_{66} & \sigma_{67} \\ \sigma_{37} & \sigma_{57} & \sigma_{67} & \sigma_{77} \end{bmatrix} \qquad U_2 = \begin{bmatrix} \sigma_{11} & \sigma_{14} & \sigma_{16} & \sigma_{17} \\ \sigma_{14} & \sigma_{44} & \sigma_{46} & \sigma_{47} \\ \sigma_{16} & \sigma_{46} & \sigma_{66} & \sigma_{67} \\ \sigma_{17} & \sigma_{47} & \sigma_{67} & \sigma_{77} \end{bmatrix}$$

$$U_3 = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{15} & \sigma_{17} \\ \sigma_{12} & \sigma_{22} & \sigma_{25} & \sigma_{27} \\ \sigma_{15} & \sigma_{25} & \sigma_{55} & \sigma_{57} \\ \sigma_{17} & \sigma_{27} & \sigma_{57} & \sigma_{77} \end{bmatrix} \qquad U_4 = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{16} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} & \sigma_{26} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} & \sigma_{36} \\ \sigma_{16} & \sigma_{26} & \sigma_{36} & \sigma_{66} \end{bmatrix}$$

$$U_5 = \begin{bmatrix} \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{27} \\ \sigma_{23} & \sigma_{33} & \sigma_{34} & \sigma_{37} \\ \sigma_{24} & \sigma_{34} & \sigma_{44} & \sigma_{47} \\ \sigma_{27} & \sigma_{37} & \sigma_{47} & \sigma_{77} \end{bmatrix} \qquad U_6 = \begin{bmatrix} \sigma_{11} & \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{13} & \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{14} & \sigma_{34} & \sigma_{44} & \sigma_{45} \\ \sigma_{15} & \sigma_{35} & \sigma_{45} & \sigma_{55} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{22} & \sigma_{24} & \sigma_{25} & \sigma_{26} \\ \sigma_{24} & \sigma_{44} & \sigma_{45} & \sigma_{46} \\ \sigma_{25} & \sigma_{45} & \sigma_{55} & \sigma_{56} \\ \sigma_{26} & \sigma_{46} & \sigma_{56} & \sigma_{66} \end{bmatrix} .$$

The inverses of these matrices are denoted by

$$U_1^{-1} = \begin{bmatrix} \sigma_{(1)}^{11} & \sigma_{(1)}^{12} & \sigma_{(1)}^{13} & \sigma_{(1)}^{14} \\ \sigma_{(1)}^{12} & \sigma_{(1)}^{22} & \sigma_{(1)}^{23} & \sigma_{(1)}^{24} \\ \sigma_{(1)}^{13} & \sigma_{(1)}^{23} & \sigma_{(1)}^{33} & \sigma_{(1)}^{34} \\ \sigma_{(1)}^{14} & \sigma_{(1)}^{24} & \sigma_{(1)}^{34} & \sigma_{(1)}^{44} \end{bmatrix} \qquad U_2^{-1} = \begin{bmatrix} \sigma_{(2)}^{11} & \sigma_{(2)}^{12} & \sigma_{(2)}^{13} & \sigma_{(2)}^{14} \\ \sigma_{(2)}^{12} & \sigma_{(2)}^{22} & \sigma_{(2)}^{23} & \sigma_{(2)}^{24} \\ \sigma_{(2)}^{13} & \sigma_{(2)}^{23} & \sigma_{(2)}^{33} & \sigma_{(2)}^{34} \\ \sigma_{(2)}^{14} & \sigma_{(2)}^{24} & \sigma_{(2)}^{34} & \sigma_{(2)}^{44} \end{bmatrix}$$

$$U_3^{-1} = \begin{bmatrix} \sigma_{(3)}^{11} & \sigma_{(3)}^{12} & \sigma_{(3)}^{13} & \sigma_{(3)}^{14} \\ \sigma_{(3)}^{12} & \sigma_{(3)}^{22} & \sigma_{(3)}^{23} & \sigma_{(3)}^{24} \\ \sigma_{(3)}^{13} & \sigma_{(3)}^{23} & \sigma_{(3)}^{33} & \sigma_{(3)}^{34} \\ \sigma_{(3)}^{14} & \sigma_{(3)}^{24} & \sigma_{(3)}^{34} & \sigma_{(3)}^{44} \end{bmatrix} \qquad U_4^{-1} = \begin{bmatrix} \sigma_{(4)}^{11} & \sigma_{(4)}^{12} & \sigma_{(4)}^{13} & \sigma_{(4)}^{14} \\ \sigma_{(4)}^{12} & \sigma_{(4)}^{22} & \sigma_{(4)}^{23} & \sigma_{(4)}^{24} \\ \sigma_{(4)}^{13} & \sigma_{(4)}^{23} & \sigma_{(4)}^{33} & \sigma_{(4)}^{34} \\ \sigma_{(4)}^{14} & \sigma_{(4)}^{24} & \sigma_{(4)}^{34} & \sigma_{(4)}^{44} \end{bmatrix}$$

$$U_5^{-1} = \begin{bmatrix} \sigma_{(5)}^{11} & \sigma_{(5)}^{12} & \sigma_{(5)}^{13} & \sigma_{(5)}^{14} \\ \sigma_{(5)}^{12} & \sigma_{(5)}^{22} & \sigma_{(5)}^{23} & \sigma_{(5)}^{24} \\ \sigma_{(5)}^{13} & \sigma_{(5)}^{23} & \sigma_{(5)}^{33} & \sigma_{(5)}^{34} \\ \sigma_{(5)}^{14} & \sigma_{(5)}^{24} & \sigma_{(5)}^{34} & \sigma_{(5)}^{44} \end{bmatrix} \qquad U_6^{-1} = \begin{bmatrix} \sigma_{(6)}^{11} & \sigma_{(6)}^{12} & \sigma_{(6)}^{13} & \sigma_{(6)}^{14} \\ \sigma_{(6)}^{12} & \sigma_{(6)}^{22} & \sigma_{(6)}^{23} & \sigma_{(6)}^{24} \\ \sigma_{(6)}^{13} & \sigma_{(6)}^{23} & \sigma_{(6)}^{33} & \sigma_{(6)}^{34} \\ \sigma_{(6)}^{14} & \sigma_{(6)}^{24} & \sigma_{(6)}^{34} & \sigma_{(6)}^{44} \end{bmatrix}$$

$$U_7^{-1} = \begin{bmatrix} \sigma_{(7)}^{11} & \sigma_{(7)}^{12} & \sigma_{(7)}^{13} & \sigma_{(7)}^{14} \\ \sigma_{(7)}^{12} & \sigma_{(7)}^{22} & \sigma_{(7)}^{23} & \sigma_{(7)}^{24} \\ \sigma_{(7)}^{13} & \sigma_{(7)}^{23} & \sigma_{(7)}^{33} & \sigma_{(7)}^{34} \\ \sigma_{(7)}^{14} & \sigma_{(7)}^{24} & \sigma_{(7)}^{34} & \sigma_{(7)}^{44} \end{bmatrix} .$$

The 7 x 7 matrix $V = \sum_{i=1}^{7} M_i U_i^{-1} M_i'$ for this design is

$$V = -\begin{array}{c} \\ \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \sigma^{11}_{(2)}+\sigma^{11}_{(3)}+\sigma^{11}_{(4)}+\sigma^{11}_{(6)} & \sigma^{12}_{(3)}+\sigma^{12}_{(4)} & \sigma^{13}_{(4)}+\sigma^{12}_{(6)} & \sigma^{12}_{(2)}+\sigma^{13}_{(6)} \\ & \sigma^{22}_{(3)}+\sigma^{22}_{(4)}+\sigma^{11}_{(5)}+\sigma^{11}_{(7)} & \sigma^{23}_{(4)}+\sigma^{12}_{(5)} & \sigma^{13}_{(5)}+\sigma^{12}_{(7)} \\ & & \sigma^{11}_{(1)}+\sigma^{33}_{(4)}+\sigma^{22}_{(5)}+\sigma^{22}_{(6)} & \sigma^{23}_{(5)}+\sigma^{23}_{(6)} \\ & & & \sigma^{22}_{(2)}+\sigma^{33}_{(5)}+\sigma^{33}_{(6)}+\sigma^{22}_{(7)} \\ & & & \\ & & & \\ & & & \end{array}\right.\end{array}$$

$$
\begin{array}{c}
\begin{array}{ccc} \qquad 5 \qquad & \qquad 6 \qquad & \qquad 7 \qquad \end{array} \\
\left.\begin{array}{ccc}
\sigma^{13}_{(3)}+\sigma^{14}_{(6)} & \sigma^{13}_{(2)}+\sigma^{14}_{(4)} & \sigma^{14}_{(2)}+\sigma^{14}_{(3)} \\
\sigma^{23}_{(3)}+\sigma^{13}_{(7)} & \sigma^{24}_{(4)}+\sigma^{14}_{(7)} & \sigma^{24}_{(3)}+\sigma^{14}_{(5)} \\
\sigma^{12}_{(1)}+\sigma^{24}_{(6)} & \sigma^{13}_{(1)}+\sigma^{34}_{(4)} & \sigma^{14}_{(1)}+\sigma^{24}_{(5)} \\
\sigma^{34}_{(6)}+\sigma^{23}_{(7)} & \sigma^{23}_{(2)}+\sigma^{24}_{(7)} & \sigma^{24}_{(2)}+\sigma^{34}_{(5)} \\
\begin{array}{c}\sigma^{22}_{(1)}+\sigma^{33}_{(3)} \\ +\sigma^{44}_{(6)}+\sigma^{33}_{(7)}\end{array} & \sigma^{23}_{(1)}+\sigma^{34}_{(7)} & \sigma^{24}_{(1)}+\sigma^{34}_{(3)} \\
 & \begin{array}{c}\sigma^{33}_{(1)}+\sigma^{33}_{(2)} \\ +\sigma^{44}_{(4)}+\sigma^{44}_{(7)}\end{array} & \sigma^{34}_{(1)}+\sigma^{34}_{(2)} \\
 & & \begin{array}{c}\sigma^{44}_{(1)}+\sigma^{44}_{(2)} \\ +\sigma^{44}_{(3)}+\sigma^{44}_{(5)}\end{array}
\end{array}\right]
\end{array}
$$

Two typical $V^{[k\ell]}$ matrices for this design are

$$V^{[11]} = - \begin{array}{l} \begin{array}{cccc} \quad 1 & \quad 2 & \quad 3 & \quad 4 \end{array} \\ \left[\begin{array}{cccc} \begin{array}{l} [\sigma^{11}_{(2)}]^2 \\ +[\sigma^{11}_{(3)}]^2 \\ +[\sigma^{11}_{(4)}]^2 \\ +[\sigma^{11}_{(5)}]^2 \end{array} & \begin{array}{l} \sigma^{11}_{(3)}\sigma^{12}_{(3)} \\ +\sigma^{11}_{(4)}\sigma^{12}_{(4)} \end{array} & \begin{array}{l} \sigma^{11}_{(4)}\sigma^{13}_{(4)} \\ +\sigma^{11}_{(6)}\sigma^{12}_{(6)} \end{array} & \begin{array}{l} \sigma^{11}_{(2)}\sigma^{12}_{(2)} \\ +\sigma^{11}_{(6)}\sigma^{13}_{(6)} \end{array} \\ & \begin{array}{l} [\sigma^{12}_{(3)}]^2 \\ +[\sigma^{12}_{(4)}]^2 \end{array} & \sigma^{12}_{(4)}\sigma^{13}_{(4)} & 0 \\ & & \begin{array}{l} [\sigma^{13}_{(4)}]^2 \\ +[\sigma^{12}_{(6)}]^2 \end{array} & \sigma^{12}_{(6)}\sigma^{13}_{(6)} \\ & & & \begin{array}{l} [\sigma^{12}_{(2)}]^2 \\ +[\sigma^{13}_{(6)}]^2 \end{array} \end{array} \right. \end{array}$$

$$
\begin{array}{ccc}
5 & 6 & 7
\end{array}
$$

$$
\left.
\begin{array}{ccc}
\sigma^{11}_{(3)}\sigma^{13}_{(3)} + \sigma^{11}_{(6)}\sigma^{14}_{(6)} & \sigma^{11}_{(2)}\sigma^{13}_{(2)} + \sigma^{11}_{(4)}\sigma^{14}_{(4)} & \sigma^{11}_{(2)}\sigma^{14}_{(2)} + \sigma^{11}_{(3)}\sigma^{14}_{(3)} \\
\sigma^{12}_{(3)}\sigma^{13}_{(3)} & \sigma^{12}_{(4)}\sigma^{14}_{(4)} & \sigma^{12}_{(3)}\sigma^{14}_{(3)} \\
\sigma^{12}_{(6)}\sigma^{14}_{(6)} & \sigma^{13}_{(4)}\sigma^{14}_{(4)} & 0 \\
\sigma^{13}_{(6)}\sigma^{14}_{(6)} & \sigma^{12}_{(2)}\sigma^{13}_{(2)} & \sigma^{12}_{(2)}\sigma^{14}_{(2)} \\
[\sigma^{13}_{(3)}]^2 + [\sigma^{14}_{(6)}]^2 & 0 & \sigma^{13}_{(3)}\sigma^{14}_{(3)} \\
 & [\sigma^{13}_{(2)}]^2 + [\sigma^{14}_{(4)}]^2 & \sigma^{13}_{(2)}\sigma^{14}_{(2)} \\
 & & [\sigma^{14}_{(2)}]^2 + [\sigma^{14}_{(3)}]^2
\end{array}
\right]
$$

$$
V^{[12]} = -\begin{array}{c}
\begin{array}{ccccccc}
\quad 1 \quad & \quad 2 \quad & \quad 3 \quad & 4 & \quad 5 \quad & \quad 6 \quad & \quad 7 \quad
\end{array} \\
\left[\begin{array}{ccccccc}
\begin{array}{c} 2\sigma^{11}_{(3)}\sigma^{12}_{(3)} \\ +2\sigma^{11}_{(4)}\sigma^{12}_{(4)} \end{array} &
\begin{array}{c} \sigma^{11}_{(3)}\sigma^{22}_{(3)} \\ [\sigma^{12}_{(3)}]^2 \\ +\sigma^{11}_{(4)}\sigma^{22}_{(4)} \\ [\sigma^{12}_{(4)}]^2 \end{array} &
\begin{array}{c} \sigma^{11}_{(4)}\sigma^{23}_{(4)} \\ +\sigma^{13}_{(4)}\sigma^{12}_{(4)} \end{array} &
0 &
\begin{array}{c} \sigma^{11}_{(3)}\sigma^{23}_{(3)} \\ +\sigma^{13}_{(3)}\sigma^{12}_{(3)} \end{array} &
\begin{array}{c} \sigma^{11}_{(4)}\sigma^{24}_{(4)} \\ +\sigma^{14}_{(4)}\sigma^{12}_{(4)} \end{array} &
\begin{array}{c} \sigma^{11}_{(3)}\sigma^{24}_{(3)} \\ +\sigma^{14}_{(3)}\sigma^{12}_{(3)} \end{array} \\
 &
\begin{array}{c} 2\sigma^{12}_{(3)}\sigma^{22}_{(3)} \\ +2\sigma^{12}_{(4)}\sigma^{22}_{(4)} \end{array} &
\begin{array}{c} \sigma^{12}_{(4)}\sigma^{23}_{(4)} \\ +\sigma^{13}_{(4)}\sigma^{22}_{(4)} \end{array} &
0 &
\begin{array}{c} \sigma^{12}_{(3)}\sigma^{23}_{(3)} \\ +\sigma^{13}_{(3)}\sigma^{22}_{(3)} \end{array} &
\begin{array}{c} \sigma^{12}_{(4)}\sigma^{24}_{(4)} \\ +\sigma^{14}_{(4)}\sigma^{22}_{(4)} \end{array} &
\begin{array}{c} \sigma^{12}_{(3)}\sigma^{24}_{(3)} \\ +\sigma^{14}_{(3)}\sigma^{22}_{(3)} \end{array} \\
 & & 2\sigma^{13}_{(4)}\sigma^{23}_{(4)} & 0 & 0 &
\begin{array}{c} \sigma^{13}_{(4)}\sigma^{24}_{(4)} \\ +\sigma^{14}_{(4)}\sigma^{23}_{(4)} \end{array} & 0 \\
 & & & 0 & 0 & 0 & 0 \\
 & & & & 2\sigma^{13}_{(3)}\sigma^{23}_{(3)} & 0 &
\begin{array}{c} \sigma^{13}_{(3)}\sigma^{24}_{(3)} \\ +\sigma^{14}_{(3)}\sigma^{23}_{(3)} \end{array} \\
 & & & & & 2\sigma^{14}_{(4)}\sigma^{24}_{(4)} & 0 \\
 & & & & & & 2\sigma^{14}_{(3)}\sigma^{24}_{(3)}
\end{array}\right]
\end{array}
$$

.

The corresponding $F^{[k\ell]}$ matrices are as follows.

$$F^{[11]} = -\begin{bmatrix} [\sigma^{11}_{(2)}]^2 & \sigma^{11}_{(2)}\sigma^{12}_{(2)} & \sigma^{11}_{(2)}\sigma^{13}_{(2)} & \sigma^{11}_{(2)}\sigma^{14}_{(2)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma^{11}_{(2)}\sigma^{12}_{(2)} & [\sigma^{12}_{(2)}]^2 & \sigma^{12}_{(2)}\sigma^{13}_{(2)} & \sigma^{12}_{(2)}\sigma^{14}_{(2)} \\ 0 & 0 & 0 & 0 \\ \sigma^{11}_{(2)}\sigma^{13}_{(2)} & \sigma^{12}_{(2)}\sigma^{13}_{(2)} & \sigma^{13}_{(2)}{}^{2} & \sigma^{13}_{(2)}\sigma^{14}_{(2)} \\ \sigma^{11}_{(2)}\sigma^{14}_{(2)} & \sigma^{12}_{(2)}\sigma^{14}_{(2)} & \sigma^{13}_{(2)}\sigma^{14}_{(2)} & [\sigma^{14}_{(2)}]^2 \end{bmatrix} X'_{(2)}$$

$$-\begin{bmatrix} [\sigma^{11}_{(3)}]^2 & \sigma^{11}_{(3)}\sigma^{12}_{(3)} & \sigma^{11}_{(3)}\sigma^{13}_{(3)} & \sigma^{11}_{(3)}\sigma^{14}_{(3)} \\ \sigma^{11}_{(3)}\sigma^{12}_{(3)} & [\sigma^{12}_{(3)}]^2 & \sigma^{12}_{(3)}\sigma^{13}_{(3)} & \sigma^{12}_{(3)}\sigma^{14}_{(3)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma^{11}_{(3)}\sigma^{13}_{(3)} & \sigma^{12}_{(3)}\sigma^{13}_{(3)} & [\sigma^{13}_{(3)}]^2 & \sigma^{13}_{(3)}\sigma^{14}_{(3)} \\ 0 & 0 & 0 & 0 \\ \sigma^{11}_{(3)}\sigma^{14}_{(3)} & \sigma^{12}_{(3)}\sigma^{14}_{(3)} & \sigma^{13}_{(3)}\sigma^{14}_{(3)} & [\sigma^{14}_{(3)}]^2 \end{bmatrix} X'_{(3)}$$

$$
- \begin{bmatrix}
[\sigma^{11}_{(4)}]^2 & \sigma^{11}_{(4)}\sigma^{12}_{(4)} & \sigma^{11}_{(4)}\sigma^{13}_{(4)} & \sigma^{11}_{(4)}\sigma^{14}_{(4)} \\
\sigma^{11}_{(4)}\sigma^{12}_{(4)} & [\sigma^{12}_{(4)}]^2 & \sigma^{12}_{(4)}\sigma^{13}_{(4)} & \sigma^{12}_{(4)}\sigma^{14}_{(4)} \\
\sigma^{11}_{(4)}\sigma^{13}_{(4)} & \sigma^{12}_{(4)}\sigma^{13}_{(4)} & [\sigma^{13}_{(4)}]^2 & \sigma^{13}_{(4)}\sigma^{14}_{(4)} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\sigma^{11}_{(4)}\sigma^{14}_{(4)} & \sigma^{12}_{(4)}\sigma^{14}_{(4)} & \sigma^{13}_{(4)}\sigma^{14}_{(4)} & [\sigma^{14}_{(4)}]^2 \\
0 & 0 & 0 & 0
\end{bmatrix} X'_{(4)}
$$

$$
- \begin{bmatrix}
[\sigma^{11}_{(6)}]^2 & \sigma^{11}_{(6)}\sigma^{12}_{(6)} & \sigma^{11}_{(6)}\sigma^{13}_{(6)} & \sigma^{11}_{(6)}\sigma^{14}_{(6)} \\
0 & 0 & 0 & 0 \\
\sigma^{11}_{(6)}\sigma^{12}_{(6)} & [\sigma^{12}_{(6)}]^2 & \sigma^{12}_{(6)}\sigma^{13}_{(6)} & \sigma^{12}_{(6)}\sigma^{14}_{(6)} \\
\sigma^{11}_{(6)}\sigma^{13}_{(6)} & \sigma^{12}_{(6)}\sigma^{13}_{(6)} & [\sigma^{13}_{(6)}]^2 & \sigma^{13}_{(6)}\sigma^{14}_{(6)} \\
\sigma^{11}_{(6)}\sigma^{14}_{(6)} & \sigma^{12}_{(6)}\sigma^{14}_{(6)} & \sigma^{13}_{(6)}\sigma^{14}_{(6)} & [\sigma^{14}_{(6)}]^2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} X'_{(6)}
$$

where $X_{(2)}$, $X_{(3)}$, $X_{(4)}$, and $X_{(6)}$ are the matrices of standard least squares estimates of the parameters for the groups that contain variable one.

$$F^{[12]} = -\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{matrix} 2\sigma^{11}_{(3)}\sigma^{12}_{(3)} & \sigma^{11}_{(3)}\sigma^{22}_{(3)}+[\sigma^{12}_{(3)}]^2 & \sigma^{11}_{(3)}\sigma^{23}_{(3)}+\sigma^{13}_{(3)}\sigma^{12}_{(3)} & \sigma^{11}_{(3)}\sigma^{24}_{(3)}+\sigma^{14}_{(3)}\sigma^{12}_{(3)} \\ \sigma^{11}_{(3)}\sigma^{22}_{(3)}+[\sigma^{12}_{(3)}]^2 & 2\sigma^{12}_{(3)}\sigma^{22}_{(3)} & \sigma^{12}_{(3)}\sigma^{23}_{(3)}+\sigma^{13}_{(3)}\sigma^{22}_{(3)} & \sigma^{12}_{(3)}\sigma^{24}_{(3)}+\sigma^{14}_{(3)}\sigma^{22}_{(3)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma^{11}_{(3)}\sigma^{23}_{(3)}+\sigma^{13}_{(3)}\sigma^{12}_{(3)} & \sigma^{12}_{(3)}\sigma^{23}_{(3)}+\sigma^{13}_{(3)}\sigma^{22}_{(3)} & 2\sigma^{13}_{(3)}\sigma^{23}_{(3)} & \sigma^{13}_{(3)}\sigma^{24}_{(3)}+\sigma^{14}_{(3)}\sigma^{23}_{(3)} \\ 0 & 0 & 0 & 0 \\ \sigma^{11}_{(3)}\sigma^{24}_{(3)}+\sigma^{14}_{(3)}\sigma^{12}_{(3)} & \sigma^{12}_{(3)}\sigma^{24}_{(3)}+\sigma^{14}_{(3)}\sigma^{22}_{(3)} & \sigma^{13}_{(3)}\sigma^{24}_{(3)}+\sigma^{14}_{(3)}\sigma^{23}_{(3)} & 2\sigma^{14}_{(3)}\sigma^{24}_{(3)} \end{matrix}\right] \end{array} X'_{(3)}$$

$$
- \begin{array}{c}
\begin{array}{cc} \qquad 1 \qquad\qquad\qquad & \qquad\qquad 2 \qquad \end{array} \\
\left[\begin{array}{cc}
2\sigma^{11}_{(4)}\sigma^{12}_{(4)} & \sigma^{11}_{(4)}\sigma^{22}_{(4)}+[\sigma^{12}_{(4)}]^2 \\
\sigma^{11}_{(4)}\sigma^{22}_{(4)}+[\sigma^{12}_{(4)}]^2 & 2\sigma^{12}_{(4)}\sigma^{22}_{(4)} \\
\sigma^{11}_{(4)}\sigma^{23}_{(4)}+\sigma^{13}_{(4)}\sigma^{12}_{(4)} & \sigma^{12}_{(4)}\sigma^{23}_{(4)}+\sigma^{13}_{(4)}\sigma^{22}_{(4)} \\
0 & 0 \\
0 & 0 \\
\sigma^{11}_{(4)}\sigma^{24}_{(4)}+\sigma^{14}_{(4)}\sigma^{12}_{(4)} & \sigma^{12}_{(4)}\sigma^{24}_{(4)}+\sigma^{14}_{(4)}\sigma^{22}_{(4)} \\
0 & 0
\end{array}\right.
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cc} \qquad 3 \qquad\qquad\qquad & \qquad\qquad 4 \qquad \end{array} \\
\left.\begin{array}{cc}
\sigma^{11}_{(4)}\sigma^{23}_{(4)}+\sigma^{13}_{(4)}\sigma^{12}_{(4)} & \sigma^{11}_{(4)}\sigma^{24}_{(4)}+\sigma^{14}_{(4)}\sigma^{12}_{(4)} \\
\sigma^{12}_{(4)}\sigma^{23}_{(4)}+\sigma^{13}_{(4)}\sigma^{22}_{(4)} & \sigma^{12}_{(4)}\sigma^{24}_{(4)}+\sigma^{14}_{(4)}\sigma^{22}_{(4)} \\
2\sigma^{13}_{(4)}\sigma^{23}_{(4)} & \sigma^{13}_{(4)}\sigma^{24}_{(4)}+\sigma^{14}_{(4)}\sigma^{23}_{(4)} \\
0 & 0 \\
0 & 0 \\
\sigma^{13}_{(4)}\sigma^{24}_{(4)}+\sigma^{14}_{(4)}\sigma^{23}_{(4)} & 2\sigma^{14}_{(4)}\sigma^{24}_{(4)} \\
0 & 0
\end{array}\right]
\end{array} X'_{(4)}
$$

where $X_{(3)}$ and $X_{(4)}$ are matrices of standard least squares estimates of the parameters for the groups that contain both variables one and two.

The $T_i^{\{k\ell\}}$ matrices for the first group are listed below. As before, $p_{mn}^{(i)} = (P_i P_i')_{mn}$, the (m,n) term of the matrix product whose equation is given in step 18 of section 3.4.

$$T_{(1)}^{\{33\}} = \begin{bmatrix} \sum_{j=1}^{4} \sigma_{(1)}^{1j} p_{1j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{1j} p_{2j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{1j} p_{3j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{1j} p_{4j}^{(1)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{(1)}^{\{35\}} = \begin{bmatrix} \sum_{j=1}^{4} \sigma_{(1)}^{2j} p_{1j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{2j} p_{2j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{2j} p_{3j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{2j} p_{4j}^{(1)} \\ \sum_{j=1}^{4} \sigma_{(1)}^{1j} p_{1j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{1j} p_{2j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{1j} p_{3j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{1j} p_{4j}^{(1)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{(1)}^{\{36\}} = \begin{bmatrix} \sum_{j=1}^{4} \sigma_{(1)}^{3j} p_{1j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{3j} p_{2j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{3j} p_{3j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{3j} p_{4j}^{(1)} \\ 0 & 0 & 0 & 0 \\ \sum_{j=1}^{4} \sigma_{(1)}^{1j} p_{1j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{1j} p_{2j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{1j} p_{3j}^{(1)} & \sum_{j=1}^{4} \sigma_{(1)}^{1j} p_{4j}^{(1)} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{(1)}^{\{37\}}=\begin{bmatrix} \sum_{j=1}^{4}\sigma_{(1)}^{4j}P_{1j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{4j}P_{2j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{4j}P_{3j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{4j}P_{4j}^{(1)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sum_{j=1}^{4}\sigma_{(1)}^{1j}P_{1j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{1j}P_{2j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{1j}P_{3j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{1j}P_{4j}^{(1)} \end{bmatrix}$$

$$T_{(1)}^{\{55\}}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ \sum_{j=1}^{4}\sigma_{(1)}^{2j}P_{1j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{2j}P_{2j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{2j}P_{3j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{2j}P_{4j}^{(1)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{(1)}^{\{56\}}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ \sum_{j=1}^{4}\sigma_{(1)}^{3j}P_{1j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{3j}P_{2j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{3j}P_{3j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{3j}P_{4j}^{(1)} \\ \sum_{j=1}^{4}\sigma_{(1)}^{2j}P_{1j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{2j}P_{2j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{2j}P_{3j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{2j}P_{4j}^{(1)} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{(1)}^{\{57\}}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ \sum_{j=1}^{4}\sigma_{(1)}^{3j}P_{1j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{3j}P_{2j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{3j}P_{3j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{3j}P_{4j}^{(1)} \\ 0 & 0 & 0 & 0 \\ \sum_{j=1}^{4}\sigma_{(1)}^{2j}P_{1j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{2j}P_{2j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{2j}P_{3j}^{(1)} & \sum_{j=1}^{4}\sigma_{(1)}^{2j}P_{4j}^{(1)} \end{bmatrix}$$

$$T^{\{66\}}_{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sum_{j=1}^{4} \sigma^{3j}_{(1)} P^{(1)}_{1j} & \sum_{j=1}^{4} \sigma^{3j}_{(1)} P^{(1)}_{2j} & \sum_{j=1}^{4} \sigma^{3j}_{(1)} P^{(1)}_{3j} & \sum_{j=1}^{4} \sigma^{3j}_{(1)} P^{(1)}_{4j} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T^{\{67\}}_{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sum_{j=1}^{4} \sigma^{4j}_{(1)} P^{(1)}_{1j} & \sum_{j=1}^{4} \sigma^{4j}_{(1)} P^{(1)}_{2j} & \sum_{j=1}^{4} \sigma^{4j}_{(1)} P^{(1)}_{3j} & \sum_{j=1}^{4} \sigma^{4j}_{(1)} P^{(1)}_{4j} \\ \sum_{j=1}^{4} \sigma^{3j}_{(1)} P^{(1)}_{1j} & \sum_{j=1}^{4} \sigma^{3j}_{(1)} P^{(1)}_{2j} & \sum_{j=1}^{4} \sigma^{3j}_{(1)} P^{(1)}_{3j} & \sum_{j=1}^{4} \sigma^{3j}_{(1)} P^{(1)}_{4j} \end{bmatrix}$$

$$T^{\{77\}}_{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sum_{j=1}^{4} \sigma^{4j}_{(1)} P^{(1)}_{1j} & \sum_{j=1}^{4} \sigma^{4j}_{(1)} P^{(1)}_{2j} & \sum_{j=1}^{4} \sigma^{4j}_{(1)} P^{(1)}_{3j} & \sum_{j=1}^{4} \sigma^{4j}_{(1)} P^{(1)}_{4j} \end{bmatrix}$$

The remaining $T^{\{k\ell\}}_{(1)}$ are null.

5.5g Case of p Variables in K Groups of Size u

In general, if variable m occurs r times in the design, the symmetric matrix V has r terms in the m'th diagonal position, each associated with a group containing variable m. If the variable pair mn occurs λ times, V has λ terms in the m'th row and n'th column, each associated with a group containing both variables m and n. The elements of the matrix U_i^{-1} occur in a square array formed from the rows and columns corresponding to the variables contained in the i'th group.

If variable k does not occur in the same group with variables m and n, then $V^{[kk]}$ has a zero in the m'th row and n'th column and in the n'th row and m'th column. For example, in the balanced design discussed in section 5.5f, variable one does not occur in the same group with the pairs (24), (37), and (56). Hence $V^{[11]}$ has zeros in positions (24), (37), (56) and the corresponding symmetric positions.

The variable pair (12) does not occur in the same group with variable four. It will be noted that $V^{[12]}$ has zeros in the fourth row and column. In general, if variable m does not occur in the same group with the pair $(k\ell)$, $V^{[k\ell]}$ $(k\neq\ell)$ has zeros in the m'th row and column. $V^{[12]}$ also has zeros in the non-diagonal positions (34), (35), (37), (45), (46), (56), and (67). These variable pairs never occur in the same group with the pair (12). In general, if the variable pair mn does not occur in the same group with the pair $k\ell$, $V^{[k\ell]}$ has a zero in the mn position. For the cases u=2 and u=3, all non-diagonal elements except those in the k'th and ℓ'th rows and columns are zero.

Each matrix product in $F^{[kk]}$ and $F^{[k\ell]}$ has (p-u) rows of zeros. In the example in section 5.5f, all elements of $F^{[11]}$ are non-zero, while the fourth row of $F^{[12]}$ consists entirely of zeros. In general, if variable m does not occur in the same group with the pair $(k\ell)$, $F^{[k\ell]}$ has only zero elements in the m'th row.

$T_i^{\{kk\}}$ is a square matrix of order u with a single row of non-zero elements. If k is the r'th smallest variable number in the i'th group, the rq element of $T_i^{\{kk\}}$ is

$$\sum_{j=1}^{u} \sigma_{(i)}^{rj} p_{qj}^{(i)}$$

where $p_{qj}^{(i)}$ denotes the element in the q'th row and j'th column of the matrix product $P_i P_i'$. If variables k and ℓ have group ranks r and s respectively, then the r'th row of $T_i^{\{k\ell\}}$ is the same as the s'th row of $T_i^{\{\ell\ell\}}$ and the s'th row of $T_i^{\{k\ell\}}$ is the same as the r'th row of $T_i^{\{kk\}}$. There are u(u+1)/2 non-null $T_i^{\{k\ell\}}$ matrices for each group.

5.6 Circular Link Designs

5.6a Four Variables in Four Groups of Two

Group			
(1)	1 2	(3)	3 4
(2)	2 3	(4)	1 4

The V matrix for this design has the form

$$V = \begin{bmatrix} \sigma^{11}_{(12)}+\sigma^{11}_{(14)} & \sigma^{12}_{(12)} & 0 & \sigma^{12}_{(14)} \\ \sigma^{12}_{(12)} & \sigma^{22}_{(12)}+\sigma^{11}_{(23)} & \sigma^{12}_{(23)} & 0 \\ 0 & \sigma^{12}_{(23)} & \sigma^{22}_{(23)}+\sigma^{11}_{(34)} & \sigma^{12}_{(34)} \\ \sigma^{12}_{(14)} & 0 & \sigma^{12}_{(34)} & \sigma^{22}_{(34)}+\sigma^{22}_{(14)} \end{bmatrix}$$

Since the pairs (13) and (24) do not occur together in the same group, $V^{[13]}$ and $V^{[24]}$ are null matrices. Two typical non-null $V^{[k\ell]}$ matrices are

$$V^{[11]} = - \begin{bmatrix} [\sigma^{11}_{(12)}]^2+[\sigma^{11}_{(14)}]^2 & \sigma^{11}_{(12)}\sigma^{12}_{(12)} & 0 & \sigma^{11}_{(14)}\sigma^{12}_{(14)} \\ \sigma^{11}_{(12)}\sigma^{12}_{(12)} & [\sigma^{12}_{(12)}]^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma^{11}_{(14)}\sigma^{12}_{(14)} & 0 & 0 & [\sigma^{12}_{(14)}]^2 \end{bmatrix}$$

$$V^{[12]} = - \begin{bmatrix} 2\sigma^{11}_{(12)}\sigma^{12}_{(12)} & \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 & 0 & 0 \\ \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 & 2\sigma^{12}_{(12)}\sigma^{22}_{(12)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

$F^{[13]}$ and $F^{[24]}$ are null matrices. Two typical non-null

$F^{[k\ell]}$ are

$$F^{[11]} = -\begin{bmatrix} [\sigma^{11}_{(12)}]^2 & \sigma^{11}_{(12)}\sigma^{12}_{(12)} \\ \sigma^{11}_{(12)}\sigma^{12}_{(12)} & [\sigma^{12}_{(12)}]^2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} X'_{(12)}$$

$$-\begin{bmatrix} [\sigma^{11}_{(14)}]^2 & \sigma^{11}_{(14)}\sigma^{12}_{(14)} \\ 0 & 0 \\ 0 & 0 \\ \sigma^{11}_{(14)}\sigma^{12}_{(14)} & [\sigma^{12}_{(14)}]^2 \end{bmatrix} X'_{(14)}$$

$$F^{[12]} = -\begin{bmatrix} 2\sigma^{11}_{(12)}\sigma^{12}_{(12)} & \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 \\ \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 & 2\sigma^{12}_{(12)}\sigma^{22}_{(12)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} X'_{(12)}$$

where $X_{(12)}$ and $X_{(14)}$ are matrices of the parameter least squares estimates in groups one and four respectively.

The $T_i^{\{k\ell\}}$ matrices for group one are

$$T^{\{11\}}_{(1)} = \begin{bmatrix} \sum_{j=1}^{2} \sigma^{1j}_{(1)} p^{(1)}_{1j} & \sum_{j=1}^{2} \sigma^{1j}_{(1)} p^{(1)}_{2j} \\ 0 & 0 \end{bmatrix}$$

$$T^{\{12\}}_{(1)} = \begin{bmatrix} \sum_{j=1}^{2} \sigma^{2j}_{(1)} p^{(1)}_{1j} & \sum_{j=1}^{2} \sigma^{2j}_{(1)} p^{(1)}_{2j} \\ \sum_{j=1}^{2} \sigma^{1j}_{(1)} p^{(1)}_{1j} & \sum_{j=1}^{2} \sigma^{1j}_{(1)} p^{(1)}_{2j} \end{bmatrix}$$

$$T^{\{22\}}_{(1)} \quad \begin{bmatrix} 0 & 0 \\ \sum_{j=1}^{2} \sigma^{2j}_{(1)} p^{(1)}_{1j} & \sum_{j=1}^{2} \sigma^{2j}_{(1)} p^{(1)}_{2j} \end{bmatrix}$$

where $p^{(i)}_{mn}$ is the (m,n) element of the matrix $P_i P_i'$ whose formula appears in step 18 of section 3.4.

With slight modifications, the remarks which were made in section 5.5c concerning balanced designs will hold also for circular link designs with p variables in groups of two. In addition, certain simplifications result when the circular link designs are used.

Since in the balanced designs, all pairs occur together in the same group λ times, the matrix V has all non-zero elements. In the circular link designs, the matrix V has zeros in all non-diagonal positions corresponding to the variable pairs which are missing in the design.

In the example in the preceding section, variable three never occurs in the same group with variable one. The third row of $V^{[11]}$ consists entirely of zeros. In general if variable m is not measured in the same group with variable k, $V^{[kk]}$ (and also $F^{[kk]}$) will have only zero elements in the m'th row. $V^{[kk]}$ also will have zeros in the non-diagonal positions corresponding to those variable pairs which do not occur in the same group along with k. If the variable pair $k\ell$ does not occur in the design, $V^{[k\ell]}$ and $F^{[k\ell]}$ are null.

5.6b Five Variables in Five Groups of Three

Group			
(1)	1 2 3	(4)	4 5 1
(2)	2 3 4	(5)	5 1 2
(3)	3 4 5		

The V matrix for this design is

$$
V = \begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array}
$$

$$
V = \left[\begin{array}{ccccc}
\sigma^{11}_{(123)}+\sigma^{11}_{(145)}+\sigma^{11}_{(125)} & \sigma^{12}_{(123)}+\sigma^{12}_{(125)} & \sigma^{13}_{(123)} & \sigma^{12}_{(145)} & \sigma^{13}_{(125)}+\sigma^{13}_{(145)} \\
 & \sigma^{22}_{(123)}+\sigma^{22}_{(125)}+\sigma^{11}_{(234)} & \sigma^{23}_{(123)}+\sigma^{12}_{(234)} & \sigma^{13}_{(234)} & \sigma^{23}_{(125)} \\
 & & \sigma^{33}_{(123)}+\sigma^{22}_{(234)}+\sigma^{11}_{(345)} & \sigma^{23}_{(234)}+\sigma^{12}_{(345)} & \sigma^{13}_{(345)} \\
 & & & \sigma^{22}_{(145)}+\sigma^{33}_{(234)}+\sigma^{22}_{(345)} & \sigma^{23}_{(145)}+\sigma^{23}_{(345)} \\
 & & & & \sigma^{33}_{(125)}+\sigma^{33}_{(145)}+\sigma^{33}_{(345)}
\end{array}\right] .
$$

Three typical $V^{[k\ell]}$ matrices are shown below. $V^{[11]}$ is typical of all $V^{[kk]}$. $V^{[12]}$ is typical of $V^{[k\ell]}$ $(k\neq\ell)$ when the pair $(k\ell)$ occurs twice in the design. $V^{[13]}$ is typical of $V^{[k\ell]}$ $(k\neq\ell)$ when the pair $(k\ell)$ occurs only once.

$$
V^{[11]} = -\begin{array}{c}
\begin{array}{ccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad \end{array} \\
\left[\begin{array}{ccccc}
[\sigma^{11}_{(123)}]^2+[\sigma^{11}_{(125)}]^2+[\sigma^{11}_{(145)}]^2 & \sigma^{11}_{(123)}\sigma^{12}_{(123)}+\sigma^{11}_{(125)}\sigma^{12}_{(125)} & \sigma^{11}_{(123)}\sigma^{13}_{(123)} & \sigma^{11}_{(145)}\sigma^{12}_{(145)} & \sigma^{11}_{(125)}\sigma^{13}_{(125)}+\sigma^{11}_{(145)}\sigma^{13}_{(145)} \\
 & [\sigma^{12}_{(123)}]^2+[\sigma^{12}_{(125)}]^2 & \sigma^{12}_{(123)}\sigma^{13}_{(123)} & 0 & \sigma^{12}_{(125)}\sigma^{13}_{(125)} \\
 & & [\sigma^{13}_{(123)}]^2 & 0 & 0 \\
 & & & [\sigma^{12}_{(145)}]^2 & \sigma^{12}_{(145)}\sigma^{13}_{(145)} \\
 & & & & [\sigma^{13}_{(125)}]^2+[\sigma^{13}_{(145)}]^2
\end{array}\right]
\end{array}
$$

$$
V^{[12]} = -\begin{array}{c}
\begin{array}{cc} \quad 1 \quad & \quad 2 \quad \end{array} \\
\left[\begin{array}{cc}
2\sigma^{11}_{(123)}\sigma^{12}_{(123)}+2\sigma^{11}_{(125)}\sigma^{12}_{(125)} & \sigma^{11}_{(123)}\sigma^{22}_{(123)}+[\sigma^{12}_{(123)}]^2+\sigma^{11}_{(125)}\sigma^{22}_{(125)}+[\sigma^{12}_{(125)}]^2 \\
 & 2\sigma^{12}_{(123)}\sigma^{22}_{(123)}+2\sigma^{12}_{(125)}\sigma^{22}_{(125)}
\end{array}\right.
\end{array}
$$

$$
\begin{array}{ccc}
3 & 4 & 5 \\
\left.\begin{matrix}
\sigma^{11}_{(123)}\sigma^{23}_{(123)} \\ +\sigma^{13}_{(123)}\sigma^{12}_{(123)} \\
\sigma^{12}_{(123)}\sigma^{23}_{(123)} \\ +\sigma^{13}_{(123)}\sigma^{22}_{(123)} \\
2\sigma^{13}_{(123)}\sigma^{23}_{(123)} \\
\\
\\
\end{matrix}\right.
&
\begin{matrix}
0 \\ \\ 0 \\ \\ 0 \\ 0 \\ \\
\end{matrix}
&
\left.\begin{matrix}
\sigma^{11}_{(125)}\sigma^{23}_{(125)} \\ +\sigma^{13}_{(125)}\sigma^{12}_{(125)} \\
\sigma^{12}_{(125)}\sigma^{23}_{(125)} \\ +\sigma^{13}_{(125)}\sigma^{22}_{(125)} \\
0 \\
0 \\
2\sigma^{13}_{(125)}\sigma^{23}_{(125)}
\end{matrix}\right]
\end{array}
$$

$$
V^{[13]} = -\begin{array}{cc}
1 & 2 \\
\left[\begin{matrix}
2\sigma^{11}_{(123)}\sigma^{13}_{(123)} \\
\\
\end{matrix}\right.
&
\begin{matrix}
\sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{12}_{(123)}\sigma^{13}_{(123)} \\
2\sigma^{12}_{(123)}\sigma^{23}_{(123)}
\end{matrix}
\end{array}
$$

$$
\begin{array}{ccc}
3 & 4 & 5 \\
\begin{matrix}
\sigma^{11}_{(123)}\sigma^{33}_{(123)}+[\sigma^{13}_{(123)}]^2 \\
\sigma^{12}_{(123)}\sigma^{33}_{(123)}+\sigma^{13}_{(123)}\sigma^{23}_{(123)} \\
2\sigma^{13}_{(123)}\sigma^{33}_{(123)} \\
\\
\\
\end{matrix}
&
\begin{matrix}
0 \\ 0 \\ 0 \\ 0 \\ \\
\end{matrix}
&
\left.\begin{matrix}
0 \\ 0 \\ 0 \\ 0 \\ 0
\end{matrix}\right]
\end{array}
$$

The corresponding $F^{[k\ell]}$ matrices are

$$F^{[11]} = -\begin{bmatrix} [\sigma^{11}_{(123)}]^2 & \sigma^{11}_{(123)}\sigma^{12}_{(123)} & \sigma^{11}_{(123)}\sigma^{13}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{12}_{(123)} & [\sigma^{12}_{(123)}]^2 & \sigma^{12}_{(123)}\sigma^{13}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{13}_{(123)} & \sigma^{12}_{(123)}\sigma^{13}_{(123)} & [\sigma^{13}_{(123)}]^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X'_{(123)}$$

$$-\begin{bmatrix} [\sigma^{11}_{(125)}]^2 & \sigma^{11}_{(125)}\sigma^{12}_{(125)} & \sigma^{11}_{(125)}\sigma^{13}_{(125)} \\ \sigma^{11}_{(125)}\sigma^{12}_{(125)} & [\sigma^{12}_{(125)}]^2 & \sigma^{12}_{(125)}\sigma^{13}_{(125)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{11}_{(125)}\sigma^{13}_{(125)} & \sigma^{12}_{(125)}\sigma^{13}_{(125)} & [\sigma^{13}_{(125)}]^2 \end{bmatrix} X'_{(125)}$$

$$-\begin{bmatrix} [\sigma^{11}_{(145)}]^2 & \sigma^{11}_{(145)}\sigma^{12}_{(145)} & \sigma^{11}_{(145)}\sigma^{13}_{(145)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^{11}_{(145)}\sigma^{12}_{(145)} & [\sigma^{12}_{(145)}]^2 & \sigma^{12}_{(145)}\sigma^{13}_{(145)} \\ \sigma^{11}_{(145)}\sigma^{13}_{(145)} & \sigma^{12}_{(145)}\sigma^{13}_{(145)} & [\sigma^{13}_{(145)}]^2 \end{bmatrix} X'_{(145)}$$

$$F^{[12]} = - \begin{array}{c} \begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \left[\begin{array}{ccc} 2\sigma^{11}_{(123)}\sigma^{12}_{(123)} & \sigma^{11}_{(123)}\sigma^{22}_{(123)}+[\sigma^{12}_{(123)}]^2 & \sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{12}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{22}_{(123)}+[\sigma^{12}_{(123)}]^2 & 2\sigma^{12}_{(123)}\sigma^{22}_{(123)}+2\sigma^{12}_{(125)}\sigma^{22}_{(125)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{22}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{12}_{(123)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{22}_{(123)} & 2\sigma^{13}_{(123)}\sigma^{23}_{(123)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} X'_{(123)}$$

$$- \begin{array}{c} \begin{array}{cc} \quad 1 \quad & \quad 2 \quad \end{array} \\ \left[\begin{array}{cc} 2\sigma^{11}_{(125)}\sigma^{12}_{(125)} & \sigma^{11}_{(125)}\sigma^{22}_{(125)}+[\sigma^{12}_{(125)}]^2 \\ \sigma^{11}_{(125)}\sigma^{22}_{(125)}+[\sigma^{12}_{(125)}]^2 & 2\sigma^{12}_{(125)}\sigma^{22}_{(125)} \\ 0 & 0 \\ 0 & 0 \\ \sigma^{11}_{(125)}\sigma^{23}_{(125)}+\sigma^{13}_{(125)}\sigma^{12}_{(125)} & \sigma^{12}_{(125)}\sigma^{23}_{(125)}+\sigma^{13}_{(125)}\sigma^{22}_{(125)} \end{array}\right. \end{array}$$

$$
\begin{array}{c}
3 \\
\left.\begin{matrix}
\sigma^{11}_{(123)}\sigma^{23}_{(123)}+[\sigma^{12}_{(123)}]^2 \\
\sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{22}_{(123)} \\
0 \\
0 \\
2\sigma^{13}_{(123)}\sigma^{23}_{(123)}
\end{matrix}\right]
\end{array}
X'_{(123)}
$$

$$
F^{[13]} = \begin{array}{cc}
1 & 2 \\
\left[\begin{matrix}
2\sigma^{11}_{(123)}\sigma^{13}_{(123)} \\
\sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{12}_{(123)}\sigma^{13}_{(123)} \\
\sigma^{11}_{(123)}\sigma^{33}_{(123)}+[\sigma^{13}_{(123)}]^2 \\
0 \\
0
\end{matrix}\right.
&
\begin{matrix}
\sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{12}_{(123)}\sigma^{13}_{(123)} \\
2\sigma^{12}_{(123)}\sigma^{23}_{(123)} \\
\sigma^{12}_{(123)}\sigma^{33}_{(123)}+\sigma^{13}_{(123)}\sigma^{23}_{(123)} \\
0 \\
0
\end{matrix}
\end{array}
$$

$$
\begin{array}{c}
3 \\
\left.\begin{matrix}
\sigma^{11}_{(123)}\sigma^{33}_{(123)}+[\sigma^{13}_{(123)}]^2 \\
\sigma^{12}_{(123)}\sigma^{33}_{(123)}+\sigma^{13}_{(123)}\sigma^{23}_{(123)} \\
2\sigma^{13}_{(123)}\sigma^{33}_{(123)} \\
0 \\
0
\end{matrix}\right]
\end{array}
X'_{(123)}
$$

The $T_i^{\{k\ell\}}$ matrices for the first group are the same as those given in section 5.5d for the balanced design with p=5, u=3.

In the particular design given in this section, all pairs occur at least once and the matrix V has all non-zero elements. However, for the case p=6, the pairs (14), (25) and (36) are absent in the circular link designs with u=3. Similarly, when p=7, the pairs (14), (15), (25), (26), (36), (37), and (47) do not occur in the same group. For these designs, V has zeros in the positions corresponding to the missing pairs.

For $p \geq 6$, $V^{[kk]}$ and $F^{[kk]}$ have one or more rows of zeros corresponding to the variables which do not appear in the same group with variable k. For $p \geq 4$, $V^{[k\ell]}$ and $F^{[k\ell]}$ have one or more rows of zeros corresponding to the variables which are not observed in the same group with the pair $(k\ell)$.

If the pair (k,ℓ) occurs only once in the design, $V^{[k\ell]}$ will have non-zero elements only at the intersection of rows (α,β,γ) with columns (α,β,γ) where (α,β,γ) denote the variables in that group in which the pair (k,ℓ) occurs; thus, either $\alpha=k$, $\beta=\ell$, $\alpha=k$, $\gamma=\ell$, or $\beta=k$, $\gamma=\ell$. There will be nine such non-zero terms. If the pair (k,ℓ) occurs twice, in groups (k,ℓ,α) and (k,ℓ,β), in the u=3 circular link design, $V^{[k\ell]}$ will have non-zero elements at the intersection of rows (k,ℓ,α) with columns (k,ℓ,α) and at the intersection of rows (k,ℓ,β) with columns (k,ℓ,β). If the pair (k,ℓ) does not occur in the design, $V^{[k\ell]}$ and $F^{[k\ell]}$ will be null.

5.7 Incomplete Circular Link Designs

5.7a Four Variables in Three Groups of Two

Group

(1)	1 2	(3)	3 4
(2)	2 3		

The V matrix for this design is

$$V = \begin{bmatrix} \sigma^{11}_{(12)} & \sigma^{12}_{(12)} & 0 & 0 \\ \sigma^{12}_{(12)} & \sigma^{22}_{(12)}+\sigma^{11}_{(23)} & \sigma^{12}_{(23)} & 0 \\ 0 & \sigma^{12}_{(23)} & \sigma^{22}_{(23)}+\sigma^{11}_{(34)} & \sigma^{12}_{(34)} \\ 0 & 0 & \sigma^{12}_{(34)} & \sigma^{22}_{(34)} \end{bmatrix} .$$

Since the pairs (13), (14), and (24) do not appear in in the same group, $V^{[13]}$, $V^{[14]}$, and $V^{[24]}$ are null. Three typical $V^{[k\ell]}$ matrices are shown below. $V^{[11]}$ is typical of the two $V^{[kk]}$ for which variable k occurs in only one group;

$V^{[22]}$, of the $V^{[kk]}$ for which variable k occurs in two groups. $V^{[12]}$ is typical of any pair which occurs in a group in a u=2 incomplete circular link design.

$$V^{[11]} = -\begin{bmatrix} [\sigma^{11}_{(12)}]^2 & \sigma^{11}_{(12)}\sigma^{12}_{(12)} & 0 & 0 \\ \sigma^{12}_{(12)}\sigma^{12}_{(12)} & [\sigma^{12}_{(12)}]^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V^{[12]} = -\begin{bmatrix} 2\sigma^{11}_{(12)}\sigma^{12}_{(12)} & \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 & 0 & 0 \\ \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 & 2\sigma^{12}_{(12)}\sigma^{22}_{(12)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V^{[22]} = -\begin{bmatrix} [\sigma^{12}_{(12)}]^2 & \sigma^{12}_{(12)}\sigma^{22}_{(12)} & 0 & 0 \\ \sigma^{12}_{(12)}\sigma^{22}_{(12)} & [\sigma^{22}_{(12)}]^2+[\sigma^{12}_{(23)}]^2 & \sigma^{12}_{(23)}\sigma^{22}_{(23)} & 0 \\ 0 & \sigma^{12}_{(23)}\sigma^{22}_{(23)} & [\sigma^{22}_{(23)}]^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding $F^{[k\ell]}$ are

$$F^{[11]} = -\begin{bmatrix} [\sigma^{11}_{(12)}]^2 & \sigma^{11}_{(12)}\sigma^{12}_{(12)} \\ \sigma^{12}_{(12)}\sigma^{12}_{(12)} & [\sigma^{12}_{(12)}]^2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} X'_{(12)}$$

$$F^{[12]} = -\begin{bmatrix} 2\sigma^{11}_{(12)}\sigma^{12}_{(12)} & \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 \\ \sigma^{11}_{(12)}\sigma^{22}_{(12)}+[\sigma^{12}_{(12)}]^2 & 2\sigma^{12}_{(12)}\sigma^{22}_{(12)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} X'_{(12)}$$

$$F^{[22]} = -\begin{bmatrix} [\sigma^{12}_{(12)}]^2 & \sigma^{12}_{(12)}\sigma^{22}_{(12)} \\ \sigma^{12}_{(12)}\sigma^{22}_{(12)} & [\sigma^{22}_{(12)}]^2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} X'_{(12)}$$

$$-\begin{bmatrix} 0 & 0 \\ [\sigma^{12}_{(23)}]^2 & \sigma^{12}_{(23)}\sigma^{22}_{(23)} \\ \sigma^{12}_{(23)}\sigma^{22}_{(23)} & [\sigma^{22}_{(23)}]^2 \\ 0 & 0 \end{bmatrix} X'_{(23)}$$

The $T_i^{\{k\ell\}}$ for groups one, two, three, and four are the same as those given in section 5.5b for groups one, four, six, and three respectively.

As in the circular link designs, if pairs of variables are missing, V will have zeros in the positions corresponding to these missing pairs. Moreover the $V^{[k\ell]}$ corresponding to these (k,ℓ) pairs are null.

The incomplete circular link designs differ from the balanced and complete circular link designs in that all variables do not occur the same number of times. If variable k occurs only once in the u=2 incomplete circular link design, $V^{[kk]}$ will have non-zero elements at the intersection of rows (α,β) with columns (α,β) where (α,β) denote the variables in that group in which variable k occurs. If variable k occurs twice, in groups $(k\alpha)$ and $(k\beta)$, then $V^{[kk]}$ will have non-zero elements at the intersection of rows (k,α) with columns (k,α) and at the intersection of rows (k,β) with columns (k,β).

5.7b Five Variables in Three Groups of Three

Group

(1)	1 2 3	(3)	3 4 5
(2)	2 3 4		

The V matrix for this design has the form

$$V = \begin{array}{c} \begin{matrix} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad \end{matrix} \\ \begin{bmatrix} \sigma^{11}_{(123)} & \sigma^{12}_{(123)} & \sigma^{13}_{(123)} & 0 & 0 \\ & \sigma^{22}_{(123)}+\sigma^{11}_{(234)} & \sigma^{23}_{(123)}+\sigma^{12}_{(234)} & \sigma^{13}_{(234)} & 0 \\ & & \sigma^{33}_{(123)}+\sigma^{22}_{(234)}+\sigma^{11}_{(345)} & \sigma^{23}_{(234)}+\sigma^{12}_{(345)} & \sigma^{13}_{(345)} \\ & & & \sigma^{33}_{(234)}+\sigma^{22}_{(345)} & \sigma^{23}_{(345)} \\ & & & & \sigma^{33}_{(345)} \end{bmatrix} \end{array} .$$

Since the pairs (14), (15), and (25) do not occur in the same group, $V^{[14]}$, $V^{[15]}$, and $V^{[25]}$ are null. Five typical $V^{[k\ell]}$ matrices are shown below. $V^{[11]}$ is typical of the $V^{[kk]}$ for which variable k occurs once; $V^{[22]}$, of those for which variable k occurs twice; $V^{[33]}$, of those for which variable k occurs three times. $V^{[12]}$ is typical of the $V^{[k\ell]}$ for which the pair $(k\ell)$ occurs once; $V^{[23]}$, of those for which the pair $(k\ell)$ occurs twice.

$$V^{[11]} = -\begin{bmatrix} [\sigma^{11}_{(123)}]^2 & \sigma^{11}_{(123)}\sigma^{12}_{(123)} & \sigma^{11}_{(123)}\sigma^{13}_{(123)} & 0 & 0 \\ & [\sigma^{12}_{(123)}]^2 & \sigma^{12}_{(123)}\sigma^{13}_{(123)} & 0 & 0 \\ & & [\sigma^{13}_{(123)}]^2 & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$$

$$V^{[12]} = -\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} 2\sigma^{11}_{(123)}\sigma^{12}_{(123)} & \sigma^{11}_{(123)}\sigma^{22}_{(123)}+[\sigma^{12}_{(123)}]^2 \\ & 2\sigma^{12}_{(123)}\sigma^{22}_{(123)} \end{array}\right. \end{array}$$

$$\begin{array}{c} \begin{array}{ccc} 3 & 4 & 5 \end{array} \\ \left.\begin{array}{ccc} \sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{12}_{(123)} & 0 & 0 \\ \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{22}_{(123)} & 0 & 0 \\ 2\sigma^{13}_{(123)}\sigma^{23}_{(123)} & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{array}\right] \end{array}$$

$$V^{[22]} = -\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} [\sigma^{12}_{(123)}]^2 & \sigma^{12}_{(123)}\sigma^{22}_{(123)} \\ & [\sigma^{22}_{(123)}]^2+[\sigma^{11}_{(234)}]^2 \end{array}\right. \end{array}$$

$$
\begin{array}{ccc}
3 & 4 & 5 \\
\sigma^{12}_{(123)}\sigma^{23}_{(123)} & 0 & 0 \\
\sigma^{22}_{(123)}\sigma^{23}_{(123)}+\sigma^{11}_{(234)}\sigma^{12}_{(234)} & \sigma^{11}_{(234)}\sigma^{13}_{(234)} & 0 \\
[\sigma^{23}_{(123)}]^2+[\sigma^{12}_{(234)}]^2 & \sigma^{12}_{(234)}\sigma^{13}_{(234)} & 0 \\
 & [\sigma^{13}_{(234)}]^2 & 0 \\
 & & 0
\end{array}
\Bigg]
$$

$$
V^{[23]} = - \left[
\begin{array}{ccc}
1 & 2 & 3 \\
2\sigma^{12}_{(123)}\sigma^{13}_{(123)} & \begin{array}{c}\sigma^{12}_{(123)}\sigma^{23}_{(123)} \\ +\sigma^{22}_{(123)}\sigma^{13}_{(123)}\end{array} & \begin{array}{c}\sigma^{12}_{(123)}\sigma^{33}_{(123)} \\ +\sigma^{23}_{(123)}\sigma^{13}_{(123)}\end{array} \\
 & \begin{array}{c}2\sigma^{22}_{(123)}\sigma^{23}_{(123)} \\ +2\sigma^{11}_{(234)}\sigma^{12}_{(234)}\end{array} & \begin{array}{c}\sigma^{22}_{(123)}\sigma^{33}_{(123)}+[\sigma^{23}_{(123)}]^2 \\ +\sigma^{11}_{(234)}\sigma^{22}_{(234)}+[\sigma^{12}_{(234)}]^2\end{array} \\
 & & \begin{array}{c}2\sigma^{23}_{(123)}\sigma^{33}_{(123)} \\ +2\sigma^{12}_{(234)}\sigma^{22}_{(234)}\end{array}
\end{array}
\right.
$$

$$
\left.
\begin{array}{cc}
4 & 5 \\
0 & 0 \\
\begin{array}{c}\sigma^{11}_{(234)}\sigma^{23}_{(234)} \\ +\sigma^{13}_{(234)}\sigma^{12}_{(234)}\end{array} & 0 \\
\begin{array}{c}\sigma^{12}_{(234)}\sigma^{23}_{(234)} \\ +\sigma^{13}_{(234)}\sigma^{22}_{(234)}\end{array} & 0 \\
2\sigma^{13}_{(234)}\sigma^{23}_{(234)} & 0 \\
 & 0
\end{array}
\right]
$$

$$V^{[33]} = - \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} [\sigma^{13}_{(123)}]^2 & \sigma^{13}_{(123)}\sigma^{23}_{(123)} & \sigma^{13}_{(123)}\sigma^{33}_{(123)} & 0 & 0 \\ & [\sigma^{23}_{(123)}]^2+[\sigma^{12}_{(234)}]^2 & \sigma^{23}_{(123)}\sigma^{33}_{(123)}+\sigma^{12}_{(234)}\sigma^{22}_{(234)} & \sigma^{12}_{(234)}\sigma^{23}_{(234)} & 0 \\ & & [\sigma^{33}_{(123)}]^2+[\sigma^{22}_{(234)}]^2+[\sigma^{11}_{(345)}]^2 & \sigma^{22}_{(234)}\sigma^{23}_{(234)}+\sigma^{11}_{(345)}\sigma^{12}_{(345)} & \sigma^{11}_{(345)}\sigma^{13}_{(345)} \\ & & & [\sigma^{23}_{(234)}]^2+[\sigma^{12}_{(345)}]^2 & \sigma^{12}_{(345)}\sigma^{13}_{(345)} \\ & & & & [\sigma^{13}_{(345)}]^2 \end{array} \right] \end{array}$$

The corresponding $F^{[k\ell]}$ are

$$F^{[11]} = - \begin{bmatrix} [\sigma^{11}_{(123)}]^2 & \sigma^{11}_{(123)}\sigma^{12}_{(123)} & \sigma^{11}_{(123)}\sigma^{13}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{12}_{(123)} & [\sigma^{12}_{(123)}]^2 & \sigma^{12}_{(123)}\sigma^{13}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{13}_{(123)} & \sigma^{12}_{(123)}\sigma^{13}_{(123)} & [\sigma^{13}_{(123)}]^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad X'_{(123)}$$

$$F^{[12]} = - \begin{array}{c} \begin{array}{ccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \left[\begin{array}{ccc} 2\sigma^{11}_{(123)}\sigma^{12}_{(123)} & \sigma^{11}_{(123)}\sigma^{22}_{(123)}+[\sigma^{12}_{(123)}]^2 & \sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{12}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{22}_{(123)}+[\sigma^{12}_{(123)}]^2 & 2\sigma^{12}_{(123)}\sigma^{22}_{(123)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{22}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{12}_{(123)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{22}_{(123)} & 2\sigma^{13}_{(123)}\sigma^{23}_{(123)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \end{array} x'_{(123)}$$

$$F^{[22]} = - \left[\begin{array}{ccc} [\sigma^{12}_{(123)}]^2 & \sigma^{12}_{(123)}\sigma^{22}_{(123)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)} \\ \sigma^{12}_{(123)}\sigma^{22}_{(123)} & [\sigma^{22}_{(123)}]^2 & \sigma^{22}_{(123)}\sigma^{23}_{(123)} \\ \sigma^{12}_{(123)}\sigma^{23}_{(123)} & \sigma^{22}_{(123)}\sigma^{23}_{(123)} & [\sigma^{23}_{(123)}]^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] x'_{(123)}$$

$$
- \begin{bmatrix}
0 & 0 & 0 \\
[\sigma^{11}_{(234)}]^2 & \sigma^{11}_{(234)}\sigma^{12}_{(234)} & \sigma^{11}_{(234)}\sigma^{13}_{(234)} \\
\sigma^{11}_{(234)}\sigma^{12}_{(234)} & [\sigma^{12}_{(234)}]^2 & \sigma^{12}_{(234)}\sigma^{13}_{(234)} \\
\sigma^{11}_{(234)}\sigma^{13}_{(234)} & \sigma^{12}_{(234)}\sigma^{13}_{(234)} & [\sigma^{13}_{(234)}]^2 \\
0 & 0 & 0
\end{bmatrix} X'_{(234)}
$$

$$
F^{[23]} = - \begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\left[\begin{array}{ccc}
2\sigma^{12}_{(123)}\sigma^{13}_{(123)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{22}_{(123)}\sigma^{13}_{(123)} & \sigma^{12}_{(123)}\sigma^{33}_{(123)}+\sigma^{23}_{(123)}\sigma^{13}_{(123)} \\
\sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{22}_{(123)}\sigma^{13}_{(123)} & 2\sigma^{22}_{(123)}\sigma^{23}_{(123)} & \sigma^{22}_{(123)}\sigma^{33}_{(123)}+[\sigma^{23}_{(123)}]^2 \\
\sigma^{12}_{(123)}\sigma^{33}_{(123)}+\sigma^{23}_{(123)}\sigma^{13}_{(123)} & \sigma^{22}_{(123)}\sigma^{33}_{(123)}+[\sigma^{23}_{(123)}]^2 & 2\sigma^{23}_{(123)}\sigma^{33}_{(123)} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
\end{array} X'_{(123)}
$$

$$
- \begin{array}{c}
\begin{matrix} \qquad\qquad 1 \qquad\qquad & \qquad\qquad 2 \qquad\qquad & \qquad\qquad 3 \qquad\qquad \end{matrix} \\
\begin{bmatrix}
0 & 0 & 0 \\
2\sigma^{11}_{(234)}\sigma^{12}_{(234)} & \sigma^{11}_{(234)}\sigma^{22}_{(234)}+[\sigma^{12}_{(234)}]^2 & \sigma^{11}_{(234)}\sigma^{23}_{(234)}+\sigma^{13}_{(234)}\sigma^{12}_{(234)} \\
\sigma^{11}_{(234)}\sigma^{22}_{(234)}+[\sigma^{12}_{(234)}]^2 & 2\sigma^{11}_{(234)}\sigma^{12}_{(234)} & \sigma^{11}_{(234)}\sigma^{22}_{(234)}+[\sigma^{12}_{(234)}]^2 \\
\sigma^{11}_{(234)}\sigma^{23}_{(234)}+\sigma^{13}_{(234)}\sigma^{12}_{(234)} & \sigma^{11}_{(234)}\sigma^{22}_{(234)}+[\sigma^{12}_{(234)}]^2 & 2\sigma^{13}_{(234)}\sigma^{23}_{(234)} \\
0 & 0 & 0
\end{bmatrix}
\end{array}
X'_{(234)}
$$

$$
F^{[33]} = - \begin{bmatrix}
[\sigma^{13}_{(123)}]^2 & \sigma^{13}_{(123)}\sigma^{23}_{(123)} & \sigma^{13}_{(123)}\sigma^{33}_{(123)} \\
\sigma^{13}_{(123)}\sigma^{23}_{(123)} & [\sigma^{23}_{(123)}]^2 & \sigma^{23}_{(123)}\sigma^{33}_{(123)} \\
\sigma^{13}_{(123)}\sigma^{33}_{(123)} & \sigma^{23}_{(123)}\sigma^{33}_{(123)} & [\sigma^{33}_{(123)}]^2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
X'_{(123)}
$$

$$
- \begin{bmatrix}
0 & 0 & 0 \\
[\sigma^{12}_{(234)}]^2 & \sigma^{12}_{(234)}\sigma^{22}_{(234)} & \sigma^{12}_{(234)}\sigma^{23}_{(234)} \\
\sigma^{12}_{(234)}\sigma^{22}_{(234)} & [\sigma^{22}_{(234)}]^2 & \sigma^{22}_{(234)}\sigma^{23}_{(234)} \\
\sigma^{12}_{(234)}\sigma^{23}_{(234)} & \sigma^{22}_{(234)}\sigma^{23}_{(234)} & [\sigma^{23}_{(234)}]^2 \\
0 & 0 & 0
\end{bmatrix} X'_{(234)}
$$

$$
- \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
[\sigma^{11}_{(345)}]^2 & \sigma^{11}_{(345)}\sigma^{12}_{(345)} & \sigma^{11}_{(345)}\sigma^{13}_{(345)} \\
\sigma^{11}_{(345)}\sigma^{12}_{(345)} & [\sigma^{12}_{(345)}]^2 & \sigma^{12}_{(345)}\sigma^{13}_{(345)} \\
\sigma^{11}_{(345)}\sigma^{13}_{(345)} & \sigma^{12}_{(345)}\sigma^{13}_{(345)} & [\sigma^{13}_{(345)}]^2
\end{bmatrix} X'_{(345)} .
$$

The $T_i^{\{k\ell\}}$ matrices for the first group are the same as those listed in section 5.5d for the balanced design with $p=5$, $u=3$.

With a few slight changes, the observations which were made for incomplete circular link designs for which $u=2$ apply also to those for which $u=3$. In the designs with $u=3$, two variables occur once, two occur twice, and the remaining $p-4$ variables occur three times. If variable k occurs once, $V^{[kk]}$ will have non-zero elements at the intersection of rows

(α,β,γ) with columns (α,β,γ) where (α,β,γ) denote the variables in that group in which variable k occurs. If variable k occurs twice, in groups $(k\ \alpha\ \beta)$ and $(k\ \gamma\ \delta)$, then $V^{[kk]}$ will have non-zero elements at the intersection of rows (k,α,β) with columns (k,α,β) and at the intersection of rows (k,γ,δ) with columns (k,γ,δ).

5.8 Minimum Linkage Designs

5.8a Seven Variables in Three Groups of Three

Group

(1)	1 2 3	(3)	5 6 7
(2)	3 4 5		

The V matrix for this design is

$$V = \begin{array}{c} \begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} \\ \left[\begin{array}{cccc} \sigma^{11}_{(123)} & \sigma^{12}_{(123)} & \sigma^{13}_{(123)} & 0 \\ & \sigma^{22}_{(123)} & \sigma^{23}_{(123)} & 0 \\ & & \sigma^{33}_{(123)}+\sigma^{11}_{(345)} & \sigma^{12}_{(345)} \\ & & & \sigma^{22}_{(345)} \end{array} \right. \end{array}$$

$$
\begin{array}{c}
\begin{array}{ccc} \quad 5 \quad & \quad 6 \quad & \quad 7 \quad \end{array} \\
\left.\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
\sigma^{13}_{(345)} & 0 & 0 \\
\sigma^{23}_{(345)} & 0 & 0 \\
\sigma^{33}_{(345)}+\sigma^{11}_{(567)} & \sigma^{12}_{(567)} & \sigma^{13}_{(567)} \\
 & \sigma^{22}_{(567)} & \sigma^{23}_{(567)} \\
 & & \sigma^{33}_{(567)}
\end{array}\right]
\end{array} .
$$

In this design, twelve pairs of variables do not occur together in the same group. The $V^{[k\ell]}$ corresponding to these $(k\ell)$ pairs are zero. The missing pairs can be read from the row and column positions of the zero elements in V.

Three typical $V^{[k\ell]}$ matrices are shown below. $V^{[11]}$ is typical of those $V^{[kk]}$ for which variable k occurs once in the design; $V^{[33]}$, of those for which variable k occurs twice.

$$V^{[11]} = \begin{bmatrix} [\sigma^{11}_{(123)}]^2 & \sigma^{11}_{(123)}\sigma^{12}_{(123)} & \sigma^{11}_{(123)}\sigma^{13}_{(123)} & 0 & 0 & 0 & 0 \\ & [\sigma^{12}_{(123)}]^2 & \sigma^{12}_{(123)}\sigma^{13}_{(123)} & 0 & 0 & 0 & 0 \\ & & [\sigma^{13}_{(123)}]^2 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix}$$

$$V^{[12]} = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 2\sigma^{11}_{(123)}\sigma^{12}_{(123)} & \sigma^{11}_{(123)}\sigma^{22}_{(123)}+[\sigma^{12}_{(123)}]^2 & \sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{12}_{(123)} & 0 & 0 & 0 & 0 \\ & 2\sigma^{12}_{(123)}\sigma^{22}_{(123)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{22}_{(123)} & 0 & 0 & 0 & 0 \\ & & 2\sigma^{13}_{(123)}\sigma^{23}_{(123)} & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix} \end{array}$$

$$V^{[33]} = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \end{array}$$

$$V^{[33]} = \left[\begin{array}{ccccccc}
[\sigma^{13}_{(123)}]^2 & \sigma^{13}_{(123)}\sigma^{23}_{(123)} & \sigma^{13}_{(123)}\sigma^{33}_{(123)} & 0 & 0 & 0 & 0 \\
 & [\sigma^{23}_{(123)}]^2 & \sigma^{23}_{(123)}\sigma^{33}_{(123)} & 0 & 0 & 0 & 0 \\
 & & [\sigma^{33}_{(123)}]^2+[\sigma^{11}_{(345)}]^2 & \sigma^{11}_{(345)}\sigma^{12}_{(345)} & \sigma^{11}_{(345)}\sigma^{13}_{(345)} & 0 & 0 \\
 & & & [\sigma^{12}_{(345)}]^2 & \sigma^{12}_{(345)}\sigma^{13}_{(345)} & 0 & 0 \\
 & & & & [\sigma^{13}_{(345)}]^2 & 0 & 0 \\
 & & & & & 0 & 0 \\
 & & & & & & 0
\end{array}\right].$$

The corresponding $F^{[k\ell]}$ are

$$F^{[11]} = \left[\begin{array}{ccc}
[\sigma^{11}_{(123)}]^2 & \sigma^{11}_{(123)}\sigma^{12}_{(123)} & \sigma^{11}_{(123)}\sigma^{13}_{(123)} \\
\sigma^{11}_{(123)}\sigma^{12}_{(123)} & [\sigma^{12}_{(123)}]^2 & \sigma^{12}_{(123)}\sigma^{13}_{(123)} \\
\sigma^{11}_{(123)}\sigma^{13}_{(123)} & \sigma^{12}_{(123)}\sigma^{13}_{(123)} & [\sigma^{13}_{(123)}]^2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] X'_{(123)}$$

$$F^{[12]} = - \begin{array}{c} \begin{array}{ccc} \quad\quad\quad 1 \quad\quad\quad & \quad\quad\quad 2 \quad\quad\quad & \quad\quad\quad 3 \quad\quad\quad \end{array} \\ \begin{bmatrix} 2\sigma^{11}_{(123)}\sigma^{12}_{(123)} & \sigma^{11}_{(123)}\sigma^{22}_{(123)}+[\sigma^{12}_{(123)}]^2 & \sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{12}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{22}_{(123)}+[\sigma^{12}_{(123)}]^2 & 2\sigma^{12}_{(123)}\sigma^{22}_{(123)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{22}_{(123)} \\ \sigma^{11}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{12}_{(123)} & \sigma^{12}_{(123)}\sigma^{23}_{(123)}+\sigma^{13}_{(123)}\sigma^{22}_{(123)} & 2\sigma^{13}_{(123)}\sigma^{23}_{(123)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} x'_{(123)}$$

$$F^{[33]} = -\begin{bmatrix} [\sigma^{13}_{(123)}]^2 & \sigma^{13}_{(123)}\sigma^{23}_{(123)} & \sigma^{13}_{(123)}\sigma^{33}_{(123)} \\ \sigma^{13}_{(123)}\sigma^{23}_{(123)} & [\sigma^{23}_{(123)}]^2 & \sigma^{23}_{(123)}\sigma^{33}_{(123)} \\ \sigma^{13}_{(123)}\sigma^{33}_{(123)} & \sigma^{23}_{(123)}\sigma^{33}_{(123)} & [\sigma^{33}_{(123)}]^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X'_{(123)}$$

$$-\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ [\sigma^{11}_{(345)}]^2 & \sigma^{11}_{(345)}\sigma^{12}_{(345)} & \sigma^{11}_{(345)}\sigma^{13}_{(345)} \\ \sigma^{11}_{(345)}\sigma^{12}_{(345)} & [\sigma^{12}_{(345)}]^2 & \sigma^{12}_{(345)}\sigma^{13}_{(345)} \\ \sigma^{11}_{(345)}\sigma^{13}_{(345)} & \sigma^{12}_{(345)}\sigma^{13}_{(345)} & [\sigma^{13}_{(345)}]^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X'_{(345)}$$

The $T_i^{\{k\ell\}}$ matrices for the first group are the same as those given in section 5.5d for the balanced design with $p=5$, $u=3$.

Minimum linkage designs are not given for the case $u=2$ since these are the same as the incomplete circular link designs discussed in the preceding section. In the minimum linkage designs, all pairs occur either once or not at all. Every non-null $V^{[k\ell]}$ has a submatrix of order u containing non-zero elements and p-u rows and columns of zeros.

If there are K groups in the design, K-1 variables will occur twice. The remaining p-K+1 variables will occur once. If variable k appears in only one group, $V^{[kk]}$ will have a non-null submatrix of order u and p-u rows and columns of zeros elsewhere. If variable k is observed in two groups, $V^{[kk]}$ will have a submatrix of order 2u-1 containing $2(u-1)^2$ zeros and p-2u+1 rows and columns of zeros.

5.9 General Remarks

In a univariate analysis, balance of treatments in blocks almost invariably leads to computational simplicity. The same situation does not hold in the multivariate situation even for the simplest cases when balance of variables in groups is attempted.

If all pairs are present in the design, the chief advantage of balance is that the terms of V, $V^{[kk]}$, and $V^{[k\ell]}$ can more readily be checked after these matrices have been constructed. Thus if each variable occurs r times and each pair of variables occurs λ times in the design, the diagonal elements of V should each consist of r terms, while the non-diagonal elements should consist of λ terms. The k'th diagonal element of $V^{[kk]}$ should have r terms and the m'th diagonal element $(m \neq k)$ should have λ terms. The k'th and ℓ'th diagonal elements of $V^{[k\ell]}$ $(k \neq \ell)$ should also have λ terms. The number of elements of various types in each $V^{[kk]}$ and $V^{[k\ell]}$ matrix is the same.

The balanced, circular link, incomplete circular link, and minimum linkage designs represent decreasing linkage among the groups and a corresponding increasing simplicity in the terms of $\phi(\Sigma)$ and $\phi^{[k\ell]}(\Sigma)$ which are affected by the design of the variables. They represent also an increase in the number of pairs of variables which do not occur together in the same group, and hence, in the number of covariances which cannot be estimated. The experimenter must choose between balance and simplicity on the basis of their relative importance in his particular study and the computational equipment at his disposal.

VI. DEMONSTRATION STUDY

6.1 Trivariate Analysis

To illustrate the iterative technique for the solution of equation (2.5.30), and to compare approximate methods of estimation with the maximum likelihood method, we construct an artificial example. We obtain a random sample of size 45 from the following model

(6.1.1)
$$\xi = \begin{bmatrix} 10 & 15 & 20 \\ 3 & 2 & 1 \end{bmatrix}$$

(6.1.2)
$$\Sigma = \begin{bmatrix} 14 & 6 & -4 \\ 6 & 3 & 0 \\ -4 & 0 & 32 \end{bmatrix}$$

In order to produce such a sample, we take three sets of 45 numbers u, v, w from a table of random normal numbers. Then we take the following linear combinations

$$\delta_1 = u + 2v + 3w$$
$$\delta_2 = u + v + w$$
$$\delta_3 = 4u - 4v$$

in order to produce normally distributed errors which have the variances and covariances given in the theoretical error covariance matrix (6.1.2).

Artificial regression relations

$$y_1 = 10 + 3t + \ell_1$$

$$y_2 = 15 + 2t + \ell_2$$

$$y_3 = 20 + t + \ell_3$$

are used to construct the three groups in Table 1 each consisting of 15 observation vectors $[y_1, y_2, y_3]$. The expected values and regression coefficients in the above equations are row vectors of the parameter matrix (6.1.1). The 45 observations on three variables constitute the X' (45x3) matrix discussed in Chapter II. With this information we can perform a standard trivariate analysis.

Table 1

Group One			Group Two			Group Three		
Y_1	Y_2	Y_3	Y_1	Y_2	Y_3	Y_1	Y_2	Y_3
11.560	15.597	25.748	5.952	13.197	28.576	9.190	14.531	22.208
19.263	19.862	19.472	17.257	18.841	15.852	14.678	16.463	15.808
20.650	20.905	21.688	16.692	19.723	25.620	7.942	16.480	23.344
16.644	19.813	13.564	11.588	18.050	33.924	27.978	25.052	27.424
15.084	19.878	22.796	22.699	23.565	18.228	20.063	22.243	34.576
25.806	25.701	24.780	24.619	24.180	30.724	25.731	24.832	26.304
31.762	28.263	27.176	26.951	27.436	35.820	25.054	25.141	33.368
34.890	30.601	17.384	32.403	29.328	26.132	28.897	28.568	35.736
33.704	31.263	29.672	38.307	32.415	28.332	27.581	28.660	29.648
37.817	32.105	26.424	41.237	34.580	24.384	39.185	34.251	20.712
40.322	34.947	24.844	37.857	33.953	26.204	40.287	35.176	33.428
40.902	35.921	22.056	47.986	37.699	24.064	44.125	37.511	34.600
46.134	39.667	34.720	45.550	38.856	35.216	42.353	37.692	40.272
47.515	39.488	30.408	46.496	40.250	42.496	50.979	40.915	32.276
51.723	42.821	36.288	52.839	42.346	34.896	51.181	43.211	48.364

The model for the three groups can be written

$$E[y_{j1}^{(1)}\ y_{j2}^{(1)}\ y_{j3}^{(1)}] = [\mu_1+\beta_1(t_j-\bar{t}),\ \mu_2+\beta_2(t_j-\bar{t}),\ \mu_3+\beta_3(t_j-\bar{t})]$$

$$E[y_{j1}^{(2)}\ y_{j2}^{(2)}\ y_{j3}^{(2)}] = [\mu_1+\beta_1(t_j-\bar{t}),\ \mu_2+\beta_2(t_j-\bar{t}),\ \mu_3+\beta_3(t_j-\bar{t})]$$

$$E[y_{j1}^{(3)}\ y_{j2}^{(3)}\ y_{j3}^{(3)}] = [\mu_1+\beta_1(t_j-\bar{t}),\ \mu_2+\beta_2(t_j-\bar{t}),\ \mu_3+\beta_3(t_j-\bar{t})]$$

$$j = 1,2,\ldots,15$$

where the concomitant variable t assumes the values 0,1,2,...14 in each group.

The regression sums of squares and products are given by

$$r_{11} = \frac{[\sum_{i=1}^{3}\sum_{j=1}^{15}(t_j-\bar{t})y_{j1}^{(i)}]^2}{s_{tt}} = \frac{(2498.867)^2}{840} = 7433.7337$$

$$r_{22} = \frac{[\sum_{i=1}^{3}\sum_{j=1}^{15}(t_j-\bar{t})y_{j2}^{(i)}]^2}{s_{tt}} = \frac{(1655.806)^2}{840} = 3263.9208$$

$$r_{33} = \frac{[\sum_{i=1}^{3}\sum_{j=1}^{15}(t_j-\bar{t})y_{j3}^{(i)}]^2}{s_{tt}} = \frac{(843.752)^2}{840} = 847.52085$$

$$r_{12} = \frac{[\sum_{i=1}^{3}\sum_{j=1}^{15}(t_j-\bar{t})y_{j1}^{(i)}][\sum_{i=1}^{3}\sum_{j=1}^{15}(t_j-\bar{t})y_{j2}^{(i)}]}{s_{tt}} = 4925.7607$$

$$r_{13} = \frac{[\sum_{i=1}^{3}\sum_{j=1}^{15}(t_j-\bar{t})y_{j1}^{(i)}][\sum_{i=1}^{3}\sum_{j=1}^{15}(t_j-\bar{t})y_{j3}^{(i)}]}{s_{tt}} = 2510.0286$$

$$r_{23} = \frac{[\sum_{i=1}^{3}\sum_{j=1}^{15}(t_j-\bar{t})y_{j2}^{(i)}][\sum_{i=1}^{3}\sum_{j=1}^{15}(t_j-\bar{t})y_{j3}^{(i)}]}{s_{tt}} = 1663.2019$$

In the above formulas $s_{tt} = 3\sum_{j=1}^{15}(t_j-\bar{t})^2 = 840.$

The error sums of squares and products are

$$\ell_{11} = \sum_{i=1}^{3}\sum_{j=1}^{15}(y_{j1}^{(i)})^2 - \frac{(\sum_{i=1}^{3}\sum_{j=1}^{15} y_{j1}^{(i)})^2}{45} - r_{11}$$

$= 51,381.5738 - 43,395.9775 - 7,433.7337$

$= 551.8626$

$$\ell_{22} = \sum_{i=1}^{3}\sum_{j=1}^{15}(y_{j2}^{(i)})^2 - \frac{(\sum_{i=1}^{3}\sum_{j=1}^{15} y_{j2}^{(i)})^2}{45} - r_{22}$$

$= 41,026.5379 - 37,669.869 - 3,263.9208$

$= 92.7481$

$$\ell_{33} = \sum_{i=1}^{3}\sum_{j=1}^{15}(y_{j3}^{(i)})^2 - \frac{(\sum_{i=1}^{3}\sum_{j=1}^{15} y_{j3}^{(i)})^2}{45} - r_{33}$$

$= 37,987.5177 - 35.591.8220 - 847.5208$

$= 1548.1749$

$$\ell_{12} = \sum_{i=1}^{3}\sum_{j=1}^{15} y_{j1}^{(i)} y_{j2}^{(i)} - \frac{(\sum_{i=1}^{3}\sum_{j=1}^{15} y_{j1}^{(i)})(\sum_{i=1}^{3}\sum_{j=1}^{15} y_{j2}^{(i)})}{45} - r_{12}$$

$= 45,569.2893 - 40,431.680 - 4,925.7606$

$= 211.8480$

$$\ell_{13} = \sum_{i=1}^{3}\sum_{j=1}^{15} y_{j1}^{(i)} y_{j3}^{(i)} - \frac{(\sum_{i=1}^{3}\sum_{j=1}^{15} y_{j1}^{(i)})(\sum_{i=1}^{3}\sum_{j=1}^{15} y_{j3}^{(i)})}{45} - r_{13}$$

$= 41,491.3598 - 39,300.660 - 2510.0286$

$= -319.3292$

$$s_{23} = \sum_{i=1}^{3} \sum_{j=1}^{15} y_{j2}^{(i)} y_{j3}^{(i)} - \frac{\left(\sum_{i=1}^{3} \sum_{j=1}^{15} y_{j2}^{(i)}\right)\left(\sum_{i=1}^{3} \sum_{j=1}^{15} y_{j3}^{(i)}\right)}{45} - r_{23}$$

$$= 38{,}196.8766 - 36{,}616.106 - 1663.2019$$

$$= -82.4321$$

6.2 Test for the Absence of Regression

The regression estimates for the three variables are

$$b_1 = \frac{\sum_{i=1}^{3} \sum_{j=1}^{15} (t_j - \bar{t}) y_{j1}^{(i)}}{s_{tt}} = 2.97484$$

$$b_2 = \frac{\sum_{i=1}^{3} \sum_{j=1}^{15} (t_j - \bar{t}) y_{j2}^{(i)}}{s_{tt}} = 1.97120$$

$$b_3 = \frac{\sum_{i=1}^{3} \sum_{j=1}^{15} (t_j - \bar{t}) y_{j3}^{(i)}}{s_{tt}} = 1.00447$$

To test the hypothesis that the regression coefficients are zero, we use Wilks' lambda criterion, setting up the determinant ratio

$$\Lambda = \frac{\begin{vmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{12} & \ell_{22} & \ell_{23} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{vmatrix}}{\begin{vmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & t_{23} \\ t_{13} & t_{23} & t_{33} \end{vmatrix}}$$

where the elements t_{ij} in the determinant in the denominator are corrected total sums of squares and products. This is essentially the likelihood ratio statistic for testing the hypothesis $C\xi=0$ which was discussed in section 4.1. In this case C is the row vector (o 1) and ξ is the 2x3 matrix of means and regression coefficients.

The statistic lambda for testing the significance of regression is distributed as $B(\frac{n_e-p+1}{2}, \frac{p}{2})$ where n_e is the number of error degrees of freedom and p is the number of variables under study.

In this case the number of error degrees of freedom is 43 and the number of variables is 3 so that here the lambda ratio is distributed as $B(\frac{41}{2}, \frac{3}{2})$. The lambda ratio is

(6.2.1)

$$\Lambda = \frac{\begin{vmatrix} 551.8626 & 211.8480 & -319.3292 \\ 211.8480 & 92.7481 & -82.4321 \\ -319.3292 & -82.4321 & 1548.1749 \end{vmatrix}}{\begin{vmatrix} 7985.5963 & 5137.6087 & 2190.6994 \\ 5137.6087 & 3356.6689 & 1580.7698 \\ 2190.6994 & 1580.7698 & 2395.6957 \end{vmatrix}} = 0.02$$

Since this is less than $B_{.05}(\frac{41}{2}, \frac{3}{2}) = 0.83$, we reject the hypothesis.

6.3 Confidence Bounds for the Regression Coefficients

A confidence ellipsoid for the three regression coefficients is obtained from the relation

(6.3.1)

$$F_{1-\alpha}(p, n_e-p+1) = \left(\frac{n_e-p+1}{p}\right) s_{tt}[b_1-\beta_1, b_2-\beta_2, b_3-\beta_3]E^{-1}\begin{bmatrix} b_1-\beta_1 \\ b_2-\beta_2 \\ b_3-\beta_3 \end{bmatrix}$$

where $F_{1-\alpha}$ denotes the upper tail α value of the F distribution with (p, n_e-p+1) degrees of freedom, E is the matrix of error sums of squares and products, p is the number of variables, n_e is the degrees of freedom due to error, and

$s_{tt} = \sum_{j=1}^{N}(t_j-\bar{t})^2$. Let

(6.3.2)
$$x = b_1 - \beta_1$$
$$y = b_2 - \beta_2$$
$$z = b_3 - \beta_3$$

The 95% confidence ellipsoid for the regression coefficients which were estimated in the preceding section can then be written

(6.3.3)
$$F_{.95}(3,41) = \frac{41}{3}(840)[x\ y\ z]E^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The equation of the elliptical contour $b_3 = \beta_3$ can be found by putting $z = 0$ in the transformed equation 6.3.3. The equation of the elliptical trace in the xy plane is then

(6.3.4)
$$[x\quad y]\begin{bmatrix} \ell^{11} & \ell^{12} \\ \ell^{12} & \ell^{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0.000247$$

To put the equation of this ellipse into standard form, we first write the equation as

$$Ax^2 + Bxy + Cy^2 = K$$

with

$$A = \ell^{11},\ B = 2\ell^{12},\ C = \ell^{22},\ K = 0.000247$$

Then, using the rotation transformation

$$x = x' \cos\theta - y' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta$$

we can reduce the equation to canonical form. The angle of rotation which will eliminate the xy term can of course be determined from the relation

$$\tan 2\theta = \frac{B}{A-C}$$

The equation (6.3.4) is most easily simplified by substituting the equations of rotation in matrix form, i.e.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

in the quadratic form on the left hand side. The simplified equation in matrix form is then

$$(6.3.5) \qquad [x', y'] \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \ell^{11} & \ell^{12} \\ \ell^{12} & \ell^{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 0.000247$$

The product of the three matrices is the array of coefficients in the simplified form of the equation of the ellipse. If this equation is written in scalar form and divided by the constant K = 0.000247, the standard form of the equation of the ellipse

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$$

is obtained. In this equation, a is the semi-axis along the x' axis and b, the semi-axis along the y' axis.

The inverse of E,* which can be obtained by inverting the matrix of the determinant in the numerator of (6.2.1) is

(6.3.6)
$$E^{-1} = \begin{bmatrix} 0.017752524 & -0.039147221 & 0.0015772987 \\ -0.039147221 & 0.097643482 & -0.0028756138 \\ 0.0015772987 & -0.0028756138 & 0.00081814752 \end{bmatrix}$$

The angle of rotation, $\theta = 22^\circ\ 13'$ is determined from the relation

$$\tan 2\theta = \frac{2(-0.039147221)}{0.017752524 - 0.097643482} = 0.98001631$$

The trigonometric functions

(6.3.7) $\quad \sin\theta = 0.3780 \qquad \cos\theta = 0.9258$

which are substituted in the rotation equations, are found by using the half angle formulas with $\cos 2\theta = 0.7142070$. The product of the three inner matrices in (6.3.5) is

$$\begin{bmatrix} 0.9258 & 0.3780 \\ -0.3780 & 0.9258 \end{bmatrix} \begin{bmatrix} 0.017752524 & -0.039147221 \\ -0.039147221 & 0.097643482 \end{bmatrix}$$

$$\begin{bmatrix} 0.9258 & -0.3780 \\ 0.3780 & 0.9258 \end{bmatrix} = \begin{bmatrix} 0.00176815 & -0.00000176 \\ -0.00000176 & 0.11362666 \end{bmatrix}$$

Hence the equation of the contour of the ellipsoid corresponding to $\beta_3 = b_3$ is

$$0.00177\, x'^2 + 0.11363\, y'^2 = 0.000247$$

---

* with elements $\ell^{11}, \ell^{12}, \ldots, \ell^{33}$.

or in standard form

$$\frac{x'^2}{(0.37)^2} + \frac{y'^2}{(0.05)^2} = 1$$

The standard equation of any contour $\beta_3 = k$ is found by substituting $z = z_o = b_3 - k$ in 6.3.3 and making the transformation

$$(6.3.9) \qquad \begin{bmatrix} x \\ y \\ z_o \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z_o \end{bmatrix}$$

The equation (6.3.3) then becomes

$$(6.3.10) \qquad [x', y', z_o] \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ell^{11} & \ell^{12} & \ell^{13} \\ \ell^{12} & \ell^{22} & \ell^{23} \\ \ell^{13} & \ell^{23} & \ell^{33} \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0.000247$$

With the values given in (6.3.6) and (6.3.7), this equation is

$$[x', y', z_o] \begin{bmatrix} 0.00176815 & -0.00000176 & 0.00037328 \\ -0.00000176 & 0.11362666 & -0.00325846 \\ 0.00037328 & -0.00325846 & 0.00081815 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z_o \end{bmatrix}$$

$$= 0.000247$$

In scalar form this equation is

$$0.00176815\, x'^2 + 0.11362666\, y'^2 + 2(0.00037328) z_o x' - 2(0.00325846) z_o y' = 0.000247 - 0.00081815\, z_o^2 \tag{6.3.11}$$

Completing squares we obtain

$$0.00176815\left(x' + \frac{0.00037328}{0.00176815} z_o\right)^2 + 0.11362666\left(y' - \frac{0.00325846}{0.11362666} z_o\right)^2 = 0.000247 - 0.00081815\, z_o^2 + \frac{(0.00037328\, z_o)^2}{0.00176815} + \frac{(0.00325846\, z_o)^2}{0.11362666}$$

which can easily be put into standard form.

The standard equation of the contour $\beta_3 = 0.5$, found by substituting $z_o = 0.50447$ in (6.3.11) and completing squares is

$$\frac{(x' + 0.11)^2}{(0.22)^2} + \frac{(y' - 0.01)^2}{(0.03)^2} = 1$$

The center of the ellipse has coordinates (-0.11, 0.01) in the (x', y') system and the semi-axes in the x' and y' directions are 0.22 and 0.03 respectively. The centers and semi-axes for several contours $\beta_3 = k$ in the $\beta_1\beta_2$ plane are given below. (The origin of the (x,y) and (x',y') systems correspond to $\beta_1 = b_1$, $\beta_2 = b_2$ and the angle of rotation is approximately 22°.)

Table 2

Centers and Semi-Axes for Contours of Ellipsoid

Based on Complete Sets and Exact Distribution

Contours		Centers		Semi-Axes	
$\beta_3=k$	$z=z_o$	x'	y'	a	b
0.4	0.60447	-0.13	0.02	0.08	0.01
0.5	0.50447	-0.11	0.01	0.22	0.03
b_3	0	0	0	0.37	0.05
1.5	-0.49553	0.10	-0.01	0.22	0.03
1.6	-0.59553	0.13	-0.02	0.10	0.01

The x' and y' coordinates of the points where the elliptical sections become imaginary are (-0.13, 0.02) and (0.13, -0.02) which correspond to $\beta_3 = 0.39$ and $\beta_3 = 1.62$ respectively.

In the next chapter we shall obtain the equation of a confidence ellipsoid based on incomplete sets of data and the approximate asymptotic distribution of the quadratic form

$$s_{tt}[b_1-\beta_1, b_2-\beta_2, b_3-\beta_3]\hat{\Sigma}_o^{-1} \begin{bmatrix} b_1-\beta_1 \\ b_2-\beta_2 \\ b_3-\beta_3 \end{bmatrix} \tag{6.3.12}$$

as chi-square with three degrees of freedom. The matrix $\hat{\Sigma}_o$ in (6.3.12) is an approximation for the maximum likelihood estimate of the dispersion matrix.

This ellipsoid will be compared with that constructed from complete sets of data in this chapter. In order to make a fair comparison, we shall replace $F_{1-\alpha}(p, N-p-1)$ in (6.3.1) by $F_{1-\alpha}(p, \infty)$ or, what is the same thing, by $\chi^2_{1-\alpha}(p)/\text{d.f.}$ In this way we shall obtain the equation of a confidence ellipsoid based on complete sets of data and the asymptotic distribution of the quadratic form

$$(6.3.13) \qquad (n-p-1)s_{tt}[b_1-\beta_1, b_2-\beta_2, b_3-\beta_3]E^{-1} \begin{bmatrix} b_1-\beta_1 \\ b_2-\beta_2 \\ b_3-\beta_3 \end{bmatrix}$$

as chi-square with p degrees of freedom.

6.4 Confidence Ellipsoid Based on Asymptotic Distribution

Equations of the contours of the confidence ellipsoid based on the asymptotic distribution of (6.3.13) are obtained by replacing

$$\frac{3F_{.95}(3,41)}{41(840)} = 0.000247$$

by

$$\frac{3F_{.95}(3,\infty)}{41(840)} = 0.0002265$$

The following table shows the centers and semi-axes for the elliptical contours in the $\beta_1\beta_2$ plane corresponding to various values of β_3.

Table 3

Centers and Semi-Axes for Contours of Ellipsoid Based on Complete Sets and Asymptotic Distribution

Contours		Centers		Semi-Axes	
$\beta_3=k$	$z=z_o$	x'	y'	a	b
0.45	0.55447	-0.12	0.02	0.13	0.02
0.50	0.50447	-0.11	0.01	0.19	0.02
b_3	0	0	0	0.36	0.04
1.50	-0.49553	0.10	-0.01	0.20	0.02
1.55	-0.54553	0.12	-0.02	0.14	0.02

The x' and y' coordinates of the points where the elliptical sections become imaginary are (-0.13; 0.02) and (0.13; -0.02) which correspond to $\beta_3 = 0.41$ and $\beta_3 = 1.6$ respectively. As before, the angle of rotation for the x' and y' axes is 22^o and the origin of the xy and $x'y'$ systems corresponds to $\beta_1=b_1$, $\beta_2=b_2$. The contours for the two ellipsoids are shown in figures 1 and 2.

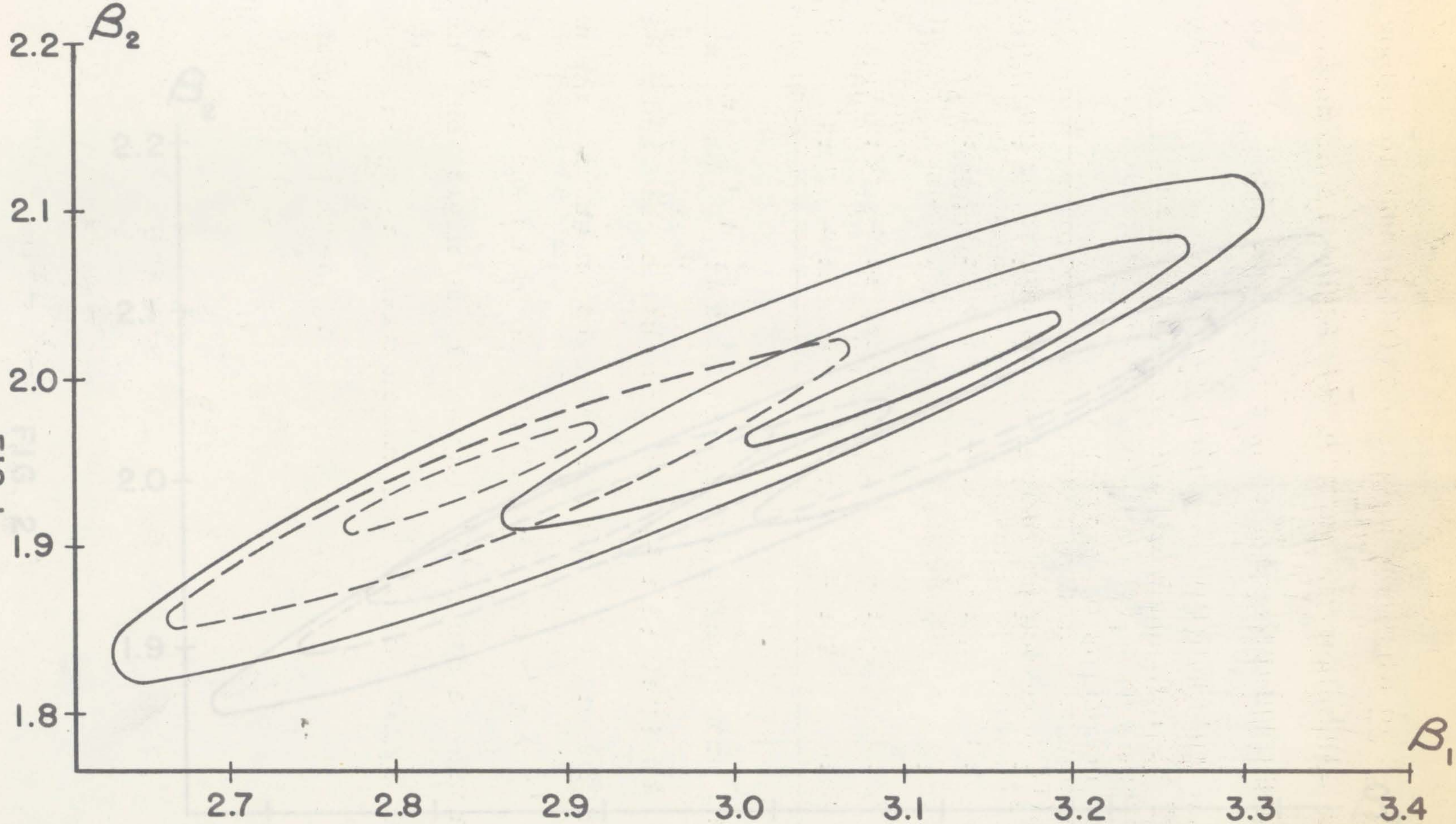
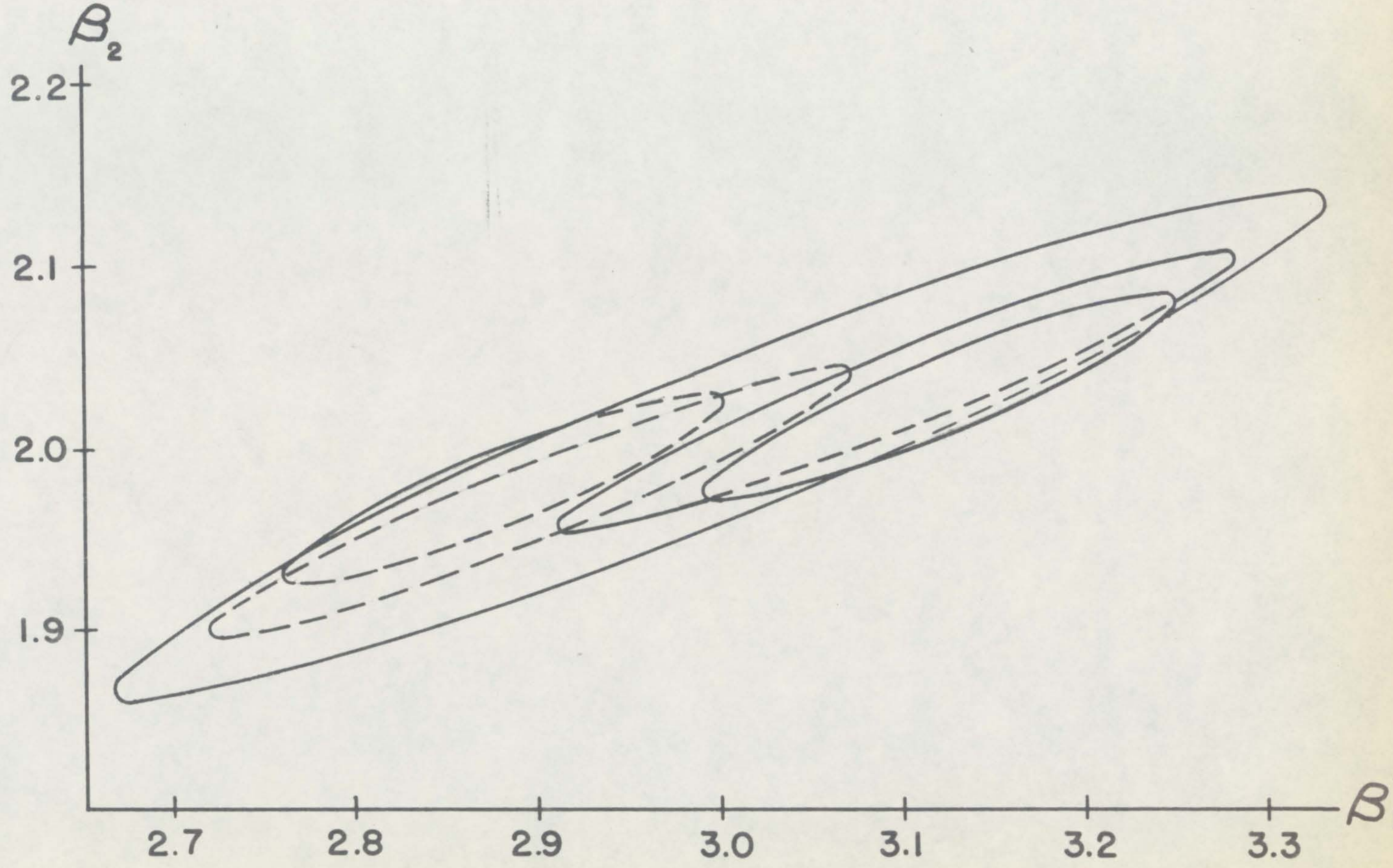


FIG. I

CONTOURS OF ELLIPSOID BASED ON
COMPLETE SETS AND EXACT DISTRIBUTION



CONTOURS OF ELLIPSOID BASED ON
COMPLETE SETS AND ASYMPTOTIC DISTRIBUTION

FIG. 2

VII. ANALYSIS OF DEMONSTRATION STUDY

7.1 Three Independent Bivariate Analyses

Now we shall consider the case where measurements on a different variable are assumed to be omitted in each group so that the observed variables in groups one, two, and three are (12), (23), and (13). Following the general procedure used in the last chapter, we shall make three distinct bivariate analyses based on the two variables in each group.

The model for the three groups can be written

$$E[y_{j1}^{(1)} y_{j2}^{(1)}] = [\mu_1+\beta_1(t_j-\bar{t}),\ \mu_2+\beta_2(t_j-\bar{t})]$$

$$E[y_{j2}^{(2)} y_{j3}^{(2)}] = [\mu_2+\beta_2(t_j-\bar{t}),\ \mu_3+\beta_3(t_j-\bar{t})]$$

$$E[y_{j1}^{(3)} y_{j3}^{(3)}] = [\mu_1+\beta_1(t_j-\bar{t}),\ \mu_3+\beta_3(t_j-\bar{t})]$$

$$j = 1,2,\ldots,15$$

where the observations on variable t are equally spaced.

In group one, the regression sums of squares and products for variables one and two are

$$r_{11}^{(1)} = \frac{[\sum_{j=1}^{15}(t_j-\bar{t})y_{j1}^{(1)}]^2}{s_{tt}^{(1)}} = \frac{(776.783)^2}{280} = 2154.9708$$

$$r_{22}^{(1)} = \frac{[\sum_{j=1}^{15}(t_j-\bar{t})y_{j2}^{(1)}]^2}{s_{tt}^{(1)}} = \frac{(527.581)^2}{280} = 994.0775$$

$$r_{12}^{(1)} = \frac{[\sum_{j=1}^{15}(t_j-\bar{t})y_{j1}^{(1)}][\sum_{j=1}^{15}(t_j-\bar{t})y_{j2}^{(1)}]}{s_{tt}^{(1)}} = 1463.6284$$

where $s_{tt}^{(i)} = \sum_{j=1}^{15}(t_j-\bar{t})^2$, $(i=1,2,3)$; they are identical for all three groups.

The error sums of squares and products are given by

$$\ell_{11}^{(1)} = \sum_{j=1}^{15}(y_{j1}^{(1)})^2 - \frac{(\sum_{j=1}^{15} y_{j1}^{(1)})^2}{15} - r_{11}^{(1)}$$

$$= 17,253.9677 - 14,964.2465 - 2154.9708$$

$$= 134.7504$$

$$\ell_{22}^{(1)} = \sum_{j=1}^{15}(y_{j2}^{(1)})^2 - \frac{(\sum_{j=1}^{15} y_{j2}^{(1)})^2}{15} - r_{22}^{(1)}$$

$$= 13,744.3624 - 12,721.4797 - 994.0775$$

$$= 28.8052$$

$$\ell_{12}^{(1)} = \sum_{j=1}^{15} y_{j1}^{(1)}\ y_{j2}^{(1)} - \frac{(\sum_{j=1}^{15} y_{j1}^{(1)})(\sum_{j=1}^{15} y_{j2}^{(1)})}{15} - r_{12}^{(1)}$$

$$= 15,320.0738 - 13,797.3678 - 1,463.6284$$

$$= 59.0776$$

In group two, the regression sums of squares and products for variables two and three are

$$r_{22}^{(2)} = \frac{(563.701)^2}{280} = 1134.8529$$

$$r_{33}^{(2)} = \frac{(216.404)^2}{280} = 167.2525$$

$$r_{23}^{(2)} = 435.6684$$

The error sums of squares and products are

$$\ell_{22}^{(2)} = 13,740.3550 - 12,581.3245 - 1134.8529$$
$$= 24.1776$$

$$\ell_{33}^{(2)} = 13,044.0887 - 12,353.5133 - 167.2525$$
$$= 523.3229$$

$$\ell_{23}^{(2)} = 12,830.0190 - 12,466.8985 - 435.6684$$
$$= -72.5479$$

In group three, the regression sums of squares and products are

$$r_{11}^{(3)} = \frac{(838.493)^2}{280} = 2510.9661$$

$$r_{33}^{(3)} = \frac{(376.896)^2}{280} = 507.3236$$

$$r_{13}^{(3)} = 1128.6595$$

The error sums of squares and products are

$$\ell_{11}^{(3)} = 16,551.4551 - 13,815.2593 - 2,510.9661$$
$$= 225.2297$$

$$\ell_{33}^{(3)} = 14,943.2242 - 13,988.4195 - 507.3236$$
$$= 447.4811$$

$$s_{13}^{(3)} = 14,957.0958 - 13,901.5698 - 1,128.6595$$
$$= -73.1335$$

The regression estimates in group one for variables one and two are

$$b_1^{(1)} = \frac{\sum_{j=1}^{15}(t_j-\bar{t})y_{j1}^{(1)}}{s_{tt}^{(1)}} = \frac{776.783}{280} = 2.7742$$

$$b_2^{(1)} = \frac{\sum_{j=1}^{15}(t_j-\bar{t})y_{j2}^{(1)}}{s_{tt}^{(1)}} = \frac{527.581}{280} = 1.8842$$

The regression estimates in group two for variables two and three are

$$b_2^{(2)} = \frac{563.701}{280} = 2.0132$$

$$b_3^{(2)} = \frac{216.404}{280} = 0.7729$$

The regression estimates in group three for variables one and three are

$$b_1^{(3)} = \frac{838.493}{280} = 2.9946$$

$$b_3^{(3)} = \frac{376.896}{280} = 1.3461$$

To test for the absence of regression in group one, we set up the determinant ratio

$$\Lambda = \frac{\begin{vmatrix} \ell_{11}^{(1)} & \ell_{12}^{(1)} \\ \ell_{12}^{(1)} & \ell_{22}^{(1)} \end{vmatrix}}{\begin{vmatrix} t_{11}^{(1)} & t_{12}^{(1)} \\ t_{12}^{(1)} & t_{22}^{(1)} \end{vmatrix}}$$

where the elements in the determinant in the denominator are corrected total sums of squares and products for group one. In this case n_e is 13 and p is 2 so that the lambda statistic is distributed as B(6, 1).

In group one,

(7.1.1)
$$\Lambda = \frac{\begin{vmatrix} 134.7504 & 59.0776 \\ 59.0776 & 28.8052 \end{vmatrix}}{\begin{vmatrix} 2289.7212 & 1522.7060 \\ 1522.7060 & 1022.8827 \end{vmatrix}} = 0.02$$

Since $0.02 < B_{.05}(6,1) = 0.61$, we reject the hypothesis that $\beta_1 = 0$, $\beta_2 = 0$.

In group two,

(7.1.2)
$$\Lambda = \frac{\begin{vmatrix} 24.1776 & -72.5479 \\ -72.5479 & 523.3229 \end{vmatrix}}{\begin{vmatrix} 1159.0305 & 363.1205 \\ 363.1205 & 690.5754 \end{vmatrix}} = 0.01$$

Since $0.01 < B_{.05}(6,1)$, we reject the hypothesis that $\beta_1=0, \beta_3=0$.

In group three,

(7.1.3)
$$\Lambda = \frac{\begin{vmatrix} 225.2297 & -73.1335 \\ -73.1335 & 447.4811 \end{vmatrix}}{\begin{vmatrix} 2736.1958 & 1055.5260 \\ 1055.5260 & 954.8047 \end{vmatrix}} = 0.06$$

Since $0.06 < B_{.05}(6,1)$, we reject the hypothesis that $\beta_1=0, \beta_3=0$.

A 95% confidence ellipse for two regression coefficients can be constructed for group one, using the relation

(7.1.4)
$$F_{.95}(p, n_e-p+1) = \frac{(n_e-p+1)}{p} s_{tt}^{(1)} [b_1^{(1)}-\beta_1, b_2^{(1)}-\beta_2] E_{(1)}^{-1} \begin{bmatrix} b_1^{(1)}-\beta_1 \\ b_2^{(1)}-\beta_2 \end{bmatrix}$$

with $n_e=13$, $p=2$, $s_{tt}^{(1)}=280$, $b_1^{(1)}=2.7742$, $b_2^{(1)}=1.8842$. Let

$$[x \quad y] = [b_1^{(1)}-\beta_1 \quad b_2^{(1)}-\beta_2]$$

and perform a rotation through an angle θ chosen so as to eliminate the x y term. Equation (7.1.4) then becomes

(7.1.5)
$$[x', y'] \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \ell_{(1)}^{11} & \ell_{(1)}^{12} \\ \ell_{(1)}^{12} & \ell_{(1)}^{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 0.002312696$$

The matrix $E_{(1)}^{-1}$ obtained by inverting the matrix of the determinant in the numerator of (7.1.1) is

$$E_{(1)}^{-1} = \begin{bmatrix} 0.073604814 & -0.150958708 \\ -0.150958708 & 0.34432249 \end{bmatrix}$$

The angle θ must satisfy

$$\tan 2\theta = \frac{2\ell_{(1)}^{12}}{\ell_{(1)}^{11}-\ell_{(1)}^{22}} = 1.11524825$$

From this we find that $\sin\theta = 0.40768$, $\cos\theta = 0.91312$, and $\theta = 24^\circ\ 4'$. Substituting the particular values of $\sin\theta$, $\cos\theta$, and $E^{-1}_{(1)}$ in (7.1.5) we obtain

$$[x',y']\begin{bmatrix} 0.00620624 & -0.00000022 \\ -0.00000022 & 0.41171735 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= 0.002312696$$

Hence the equation of the ellipse in standard form is

$$\text{(7.1.6)} \qquad \frac{x'^2}{(0.61)^2} + \frac{y'^2}{(0.07)^2} = 1$$

For group two

$$E^{-1}_{(2)} = \begin{bmatrix} 0.070819846 & 0.009817706 \\ 0.009817706 & 0.003271887 \end{bmatrix}$$

$$\tan 2\theta = \frac{2\ell^{23}_{(2)}}{\ell^{22}_{(2)} - \ell^{33}_{(2)}} = 0.29068845$$

$$\sin\theta = 0.14097$$

$$\cos\theta = 0.99001$$

$$\theta = 8^\circ\ 6'$$

The equation of the confidence ellipse in a form corresponding to (7.1.5) is

$$[x',y']\begin{bmatrix} 0.07221731 & 0.00000032 \\ 0.00000032 & 0.00187386 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= 0.002312696$$

Hence the canonical equation of the 95% confidence ellipse for group two is

$$\text{(7.1.7)} \qquad \frac{x'^2}{(0.18)^2} + \frac{y'^2}{(1.11)^2} = 1$$

For group three,

$$E^{-1}_{(3)} = \begin{matrix} 0.004688733 & 0.000766297 \\ 0.000766297 & 0.002359970 \end{matrix}$$

$$\tan 2\theta = \frac{2\ell^{13}_{(3)}}{\ell^{11}_{(3)} - \ell^{33}_{(3)}} = 0.65811505$$

$$\sin \theta = 0.28694$$

$$\cos \theta = 0.95795$$

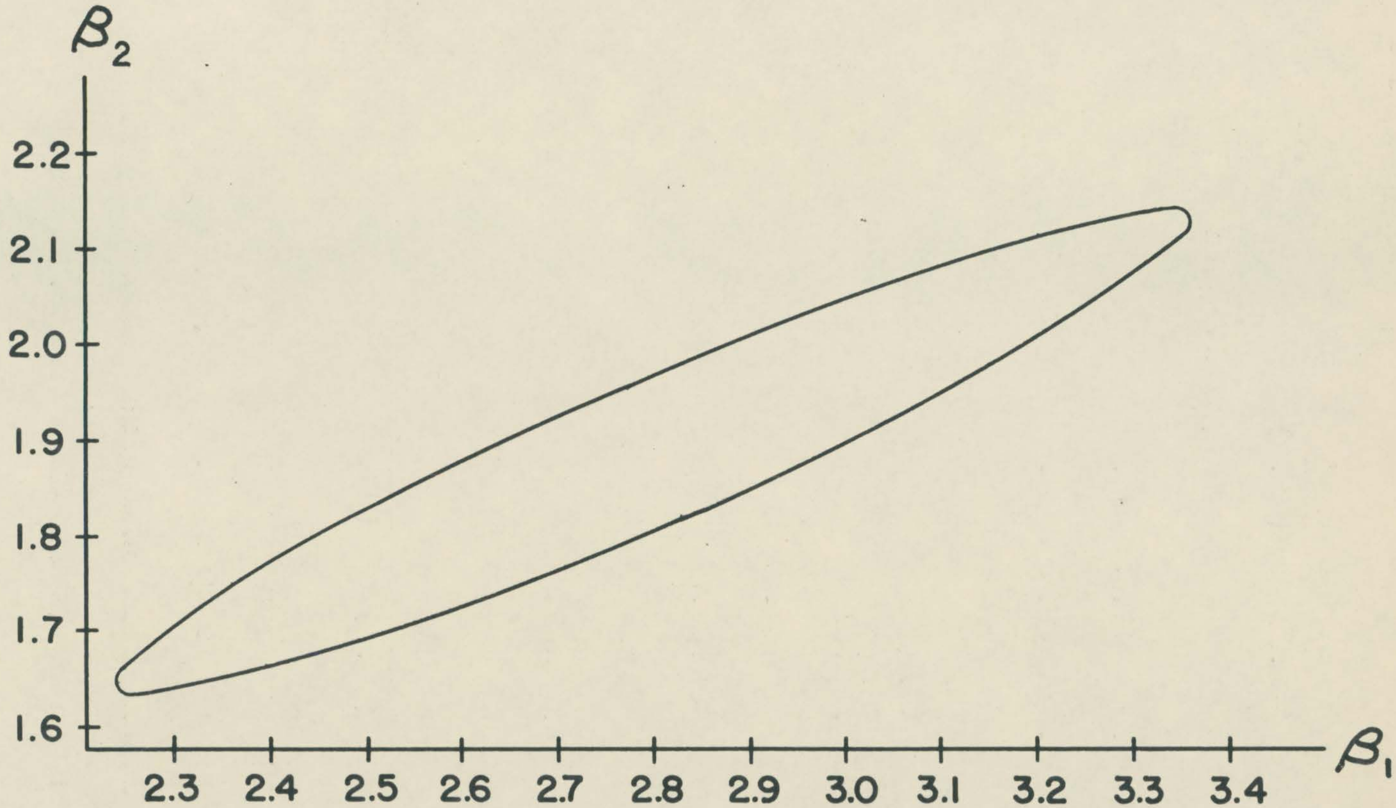
$$\theta = 16^{\circ}\ 29'$$

The standard form of the equation of the 95% confidence ellipse for group three is

$$\text{(7.1.8)} \qquad \frac{x'^2}{(0.69)^2} + \frac{y'^2}{(1.04)^2} = 1$$

The confidence ellipses for the three groups are shown in figures 3, 4, and 5. If we compare these ellipses with the (x y z) coordinate plane sections of the ellipsoid whose contours are shown in figure 2, we see that, as we would expect, the elliptical traces of the confidence ellipsoid for the trivariate data are considerably smaller than the ellipses

FIG. 3



CONFIDENCE ELLIPSE FOR GROUP ONE

CONFIDENCE ELLIPSE FOR GROUP TWO

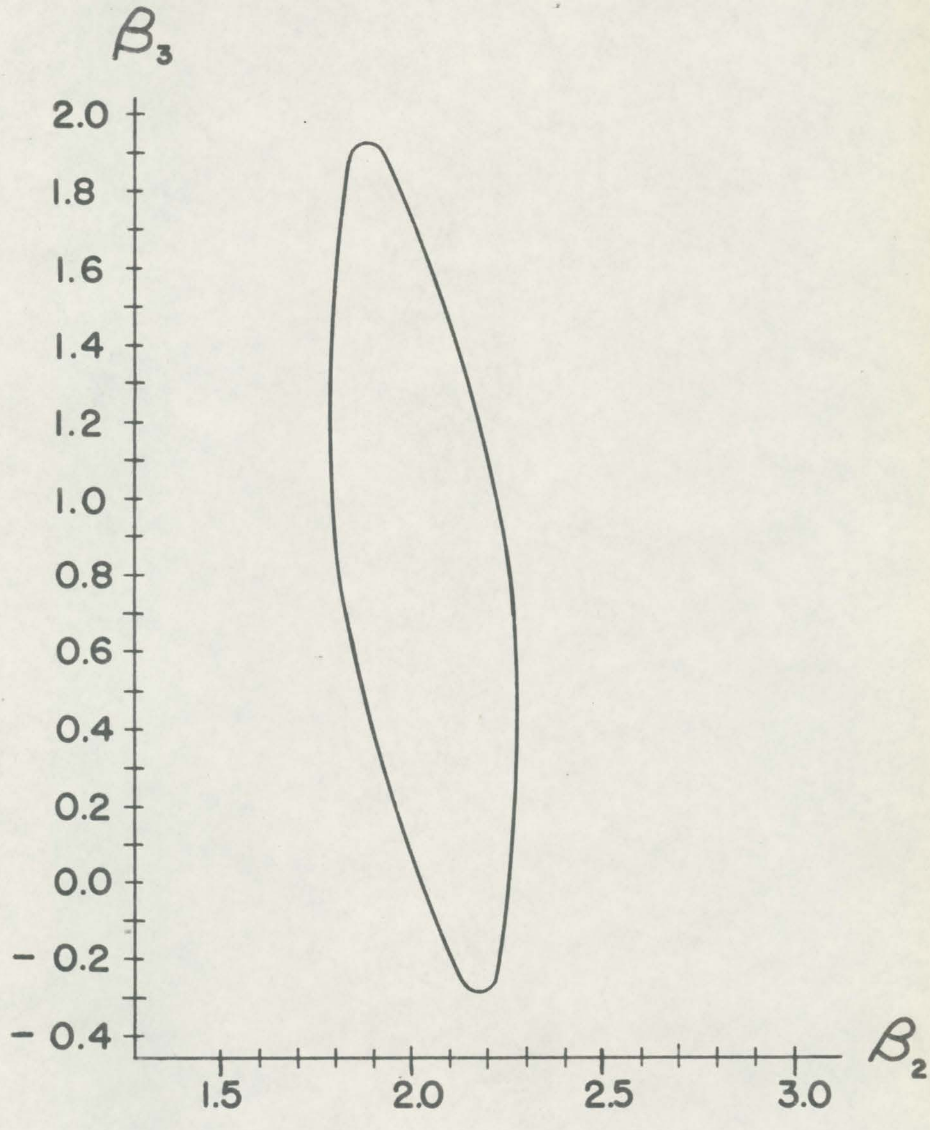


FIG. 4

CONFIDENCE ELLIPSE FOR GROUP THREE

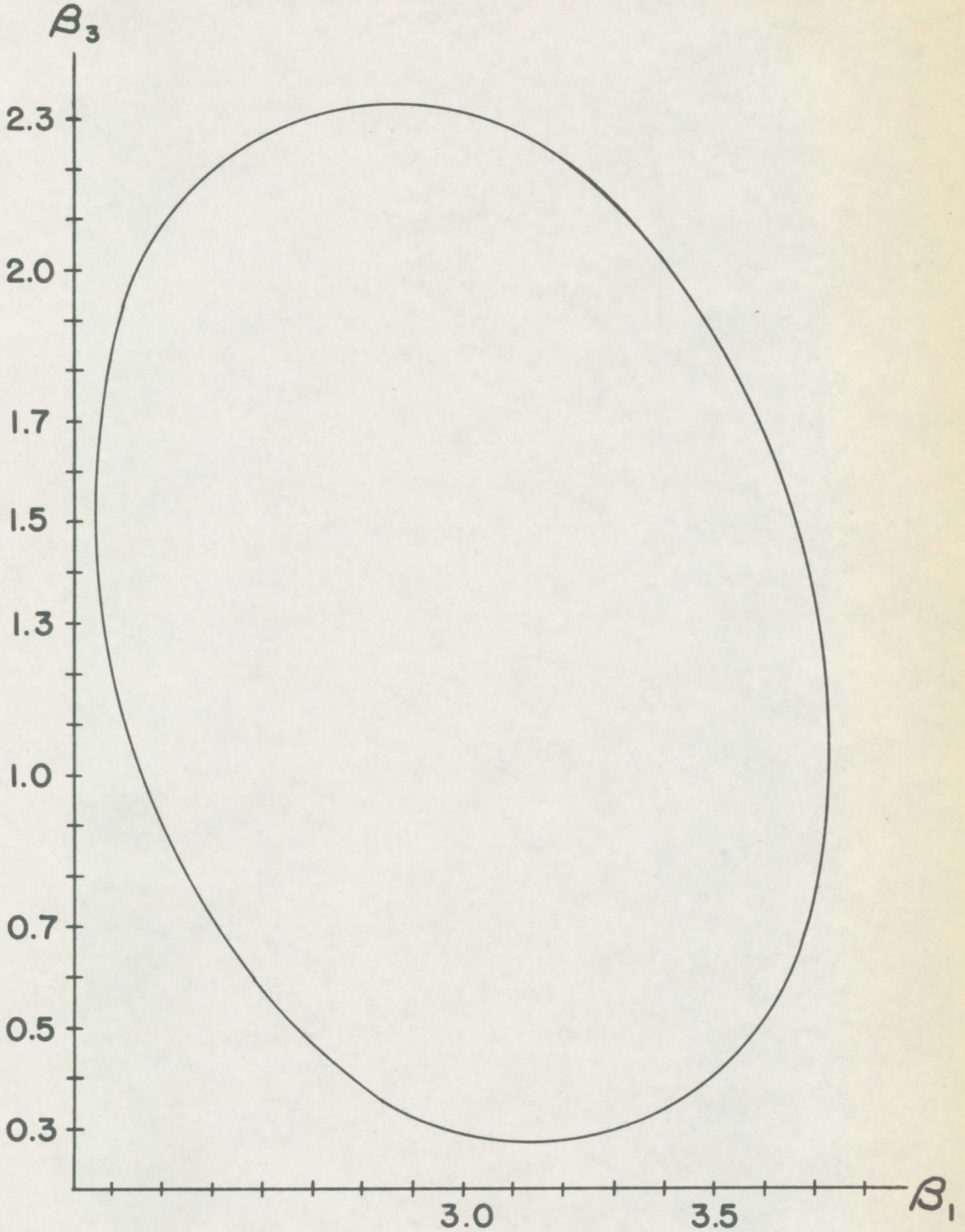


FIG. 5

constructed for the bivariate sets. The semi-axes for the trace in the x y plane, which correspond to variables one and two respectively, are 0.37 and 0.05, while the analogous semi-axes of the ellipse for group one are 0.61 and 0.07. Similar results hold for the other ellipses. Due to the fact that each set is now based on only 13 error degrees of freedom (instead of 43 before), the axes for the confidence ellipses obtained when measurements on one variable in each group are omitted are on the average almost twice as large as the central sections of the confidence ellipsoid based on complete trivariate sets. Moreover the estimates $b_1=2.97$, $b_2=1.97$, $b_3=1.00$ based on the complete data are somewhat closer on the average to the theoretical values 3, 2, and 1 than are the pairs of group estimates $b_1^{(1)}=2.77$ and $b_1^{(3)}=2.99$, $b_2^{(2)}=1.88$ and $b_2^{(3)}=2.01$, $b_3^{(2)}=0.77$ and $b_3^{(3)}=1.35$.

7.2 Trivariate Analysis Based Upon Initial Estimate of Σ

The most difficult task in obtaining the maximum likelihood estimate of the parameter matrix $\hat{\xi}$ is the iterative solution of equation (2.5.30) for the maximum likelihood estimate $\hat{\Sigma}$ of the covariance matrix. In Chapter 3, a detailed procedure was outlined for obtaining an initial estimate $\hat{\Sigma}_o$ and substituting this estimate in equation (2.5.30).

The final solution $\hat{\Sigma}$ of equation (2.5.30) is then substituted in the expression (2.5.12) for $\hat{\xi}$ to obtain the maximum likelihood estimate of the parameter matrix.

In this section, we shall use the initial estimate $\hat{\Sigma}_o$ in place of the maximum likelihood estimate $\hat{\Sigma}$ to obtain an approximate estimate

$$(7.2.1) \qquad \hat{\xi}_o \doteq \sum_{i=1}^{3}(A'A)^{-1}A'Y_i'(M_i'\hat{\Sigma}_o^{-1}M_i)M_i'[\sum_{i=1}^{3}M_i(M_i'\hat{\Sigma}_o^{-1}M_i)M_i']^{-1}$$

of the parameter matrix for the incomplete sets of data in the demonstration study discussed in section 7.1. A confidence ellipsoid will then be constructed from the approximate relation

$$(7.2.2) \qquad \chi^2_{1-\alpha}(3) \doteq s_{tt}[\tilde{\beta}_1-\beta_1,\tilde{\beta}_2-\beta_2,\tilde{\beta}_3-\beta_3]\hat{\Sigma}_o^{-1}\begin{bmatrix}\tilde{\beta}_1-\beta_1\\ \tilde{\beta}_2-\beta_2\\ \tilde{\beta}_3-\beta_3\end{bmatrix}$$

where

$$[\tilde{\beta}_1 \quad \tilde{\beta}_2 \quad \tilde{\beta}_3]$$

is the second row vector of $\hat{\xi}_o$. We shall then compare the confidence region with that obtained in section 6.4 for the complete sets of data.

The elements of the initial estimate Σ_o are found by averaging the bivariate group maximum likelihood estimates from section 7.1. Thus

$$\tilde{\sigma}_{11} = \frac{\ell_{11}^{(1)} + \ell_{11}^{(3)}}{30} = \frac{134.7504 + 225.2297}{30}$$

$$= 11.9993$$

$$\tilde{\sigma}_{22} = \frac{\ell_{22}^{(2)} + \ell_{22}^{(3)}}{30} = \frac{28.8052 + 24.1776}{30}$$

$$= 1.7661$$

$$\tilde{\sigma}_{33} = \frac{\ell_{33}^{(2)} + \ell_{33}^{(3)}}{30} = \frac{523.3230 + 447.4811}{30}$$

$$= 32.3601$$

$$\tilde{\sigma}_{12} = \frac{59.0776}{15} = 3.9385$$

$$\tilde{\sigma}_{23} = -\frac{72.5479}{15} = -4.8365$$

$$\tilde{\sigma}_{13} = -\frac{73.1335}{15} = -4.8756$$

Hence

(7.2.3)
$$\hat{\Sigma}_o = \begin{bmatrix} 11.9993 & 3.9385 & -4.8756 \\ 3.9385 & 1.7661 & -4.8365 \\ -4.8756 & -4.8365 & 32.3601 \end{bmatrix}$$

The approximate maximum likelihood estimate of the parameter matrix can be written

$$\hat{\xi}_o = \begin{bmatrix} \tilde{\mu}_1 & \tilde{\mu}_2 & \tilde{\mu}_3 \\ \tilde{\beta}_1 & \tilde{\beta}_2 & \tilde{\beta}_3 \end{bmatrix} \tag{7.2.4}$$

The model given in section 7.1 can be expressed in matrix form as

$$E(Y_i') = A\xi M_i \qquad i=1,2,3$$

where Y_i' is the 15 x 2 matrix of observations in the i'th group and

$$A = \begin{bmatrix} 1 & -7 \\ 1 & -6 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 6 \\ 1 & 7 \end{bmatrix}$$

The regression estimates (the second row vector in (7.2.4)) are obtained by multiplying the second row vector of $X_i = (A'A)^{-1}A'Y_i'$ in (7.2.1) into the rest of the expression and summing over the groups. The X_i matrices contain the standard least squares estimates of the parameters in the i'th group. The second row vectors of these matrices (which were found in section 7.1 for the bivariate analyses) are

$$(7.2.5) \qquad (X_1)_2 = (b_1^{(1)}, b_2^{(1)}) = (2.7742 \quad 1.8842)$$

$$(X_2)_2 = (b_2^{(2)}, b_3^{(2)}) = (2.0132 \quad 0.7729)$$

$$(X_3)_2 = (b_1^{(3)}, b_3^{(3)}) = (2.9946 \quad 1.3461)$$

Let

$$\hat{U}_{io} = M_i' \hat{\Sigma}_o M_i$$

$$\hat{V}_o = \sum_{i=1}^{3} M_i \hat{U}_{io}^{-1} M_i'$$

Then the second row vector of the expression (7.2.1) can be written

$$(7.2.6) \qquad (\hat{\xi}_o)_2 = \sum_{i=1}^{3} (X_i)_2 \hat{U}_{io}^{-1} M_i' \hat{V}_o^{-1}$$

where

$$(\hat{\xi}_o)_2 = (\tilde{\beta}_1 \quad \tilde{\beta}_2 \quad \tilde{\beta}_3)$$

and the vectors $(X_i)_2$ are given in (7.2.5).

The $\hat{U}_{io}^{-1}$ matrices, obtained by inverting the second order submatrices of $\hat{\Sigma}_o$ are

(7.2.7) $$\hat{U}_{1o}^{-1} = \begin{bmatrix} 0.3109232 & -0.6933757 \\ -0.6933757 & 2.1124853 \end{bmatrix}$$

$$\hat{U}_{2o}^{-1} = \begin{bmatrix} 0.9585497 & 0.1432636 \\ 0.1432636 & 0.0523143 \end{bmatrix}$$

$$\hat{U}_{3o}^{-1} = \begin{bmatrix} 0.0887728 & 0.0133751 \\ 0.0133751 & 0.0329174 \end{bmatrix}$$

Then

(7.2.8) $$\sum_{i=1}^{3} (X_i)_2 \, \hat{U}_{io}^{-1} M_i' =$$

$$(2.7742 \quad 1.8842) \begin{bmatrix} 0.3109232 & -0.6933757 & 0 \\ -0.6933757 & 2.1124853 & 0 \end{bmatrix}$$

$$+(2.0132 \quad 0.7729) \begin{bmatrix} 0 & 0.9585497 & 0.1432636 \\ 0 & 0.1432636 & 0.0523143 \end{bmatrix}$$

$$+(2.9946 \quad 1.3461) \begin{bmatrix} 0.0887728 & 0 & 0.0133751 \\ 0.0133751 & 0 & 0.0329174 \end{bmatrix}$$

$$= (-0.1600557 \quad 4.0972960 \quad 0.4132151)$$

and

$$\hat{V}_o = \begin{bmatrix} 0.3109232 & -0.6933757 & 0 \\ -0.6933757 & 2.1124853 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.9585497 & 0.1432636 \\ 0 & 0.1432636 & 0.0523143 \end{bmatrix}$$

$$+\begin{bmatrix} 0.0887728 & 0 & 0.0133751 \\ 0 & 0 & 0 \\ 0.0133751 & 0 & 0.0329174 \end{bmatrix}$$

$$=\begin{bmatrix} 0.3996960 & -0.6933757 & 0.0133751 \\ -0.6933757 & 3.0710350 & 0.1432636 \\ 0.0133751 & 0.1432636 & 0.0852317 \end{bmatrix}$$

Inverting this matrix we have

$$\hat{V}_o^{-1} = \begin{bmatrix} 4.6182320 & 1.1681028 & -2.6881538 \\ 1.1681028 & 0.6487803 & -1.2738229 \\ -2.6881538 & -1.2738229 & 14.2957053 \end{bmatrix} \tag{7.2.9}$$

Then, multiplying the vector obtained in (7.2.8) by the matrix in (7.2.9), we obtain the vector $(\hat{\xi}_o)_2$ given by (7.2.6). Thus we have

$$(\hat{\xi}_o)_2 = (\tilde{\beta}_1\ \tilde{\beta}_2\ \tilde{\beta}_3) = (2.9361\ 1.9449\ 1.1182) \tag{7.2.10}$$

To test for the absence of regression we shall use an approximation for $-2 \ln \lambda$. If in the maximum for the likelihood function (4.2.15) we replace $\hat{\Sigma}$ by the approximation $\hat{\Sigma}_o$, we obtain

$$\ell(\hat{\Omega}) \doteq -\frac{N_u}{2}\log 2\pi - \frac{n}{2}\log |M_i' \hat{\Sigma}_o M_i| - \tfrac{1}{2}\sum_{i=1}^{K} \operatorname{tr}(M_i' \hat{\Sigma}_o M_i)^{-1} \hat{P}_{io} \hat{P}_{io}' \tag{7.2.11}$$

where

$$\hat{P}_{io}\hat{P}_{io}' = Y_iY_i' - Y_i\, A\, \hat{\xi}_o\, M_i - M_i'\, \hat{\xi}_o'\, A'\, Y_i' + M_i'\, \hat{\xi}_o'\, A'\, A\, \hat{\xi}_o\, M_i \tag{7.2.12}$$

In the expression (7.2.12), $\hat{\xi}_o$ is an approximation for the exact maximum likelihood estimate (2.5.15) obtained by replacing $\hat{U}_i = M_i' \hat{\Sigma} M_i$ by $\hat{U}_{io} = M_i' \hat{\Sigma}_o M_i$. If we denote by $\hat{\hat{\xi}}_o$ the corresponding approximation for the maximum likelihood estimate under the hypothesis $C\xi=0$, we have

$$\ell(\hat{\omega}) \doteq -\frac{N_u}{2}\log 2\pi - \frac{n}{2}\log |M_i' \hat{\Sigma}_o M_i| - \tfrac{1}{2}\sum_{i=1}^{K} \operatorname{tr}(M_i' \hat{\Sigma}_o M_i)^{-1} \hat{\hat{P}}_{io}\hat{\hat{P}}_{io}' \tag{7.2.13}$$

where

$$\hat{\hat{P}}_{io}\hat{\hat{P}}_{io}' = Y_iY_i' - Y_i\, A\, \hat{\hat{\xi}}_o\, M_i - M_i'\, \hat{\hat{\xi}}_o'\, A'\, Y_i' + M_i'\, \hat{\hat{\xi}}_o'\, A'\, A\, \hat{\hat{\xi}}_o\, M_i \tag{7.2.14}$$

Then from (7.2.11) and (7.2.13) we obtain

$$-2\ln\lambda \doteq \sum_{i=1}^{K} \operatorname{tr}\, \hat{U}_{io}^{-1}(\hat{\hat{P}}_{io}\hat{\hat{P}}_{io}' - \hat{P}_{io}\hat{P}_{io}') \tag{7.2.15}$$

Subtracting (7.2.11) from (7.2.13) we obtain

$$\hat{\hat{P}}_{io}\hat{\hat{P}}'_{io}-\hat{P}_{io}\hat{P}'_{io} = M'_i\ \hat{\xi}'_o\ A'\ Y'_i+Y_iA\ \hat{\xi}_oM_i$$

$$-M'_i\ \hat{\xi}'_o\ A'A\ \hat{\xi}_o\ M_i-M'_i\ \hat{\hat{\xi}}'_o\ A'Y'_i$$

$$-Y_iA\ \hat{\hat{\xi}}_o\ M_i+M'_i\ \hat{\hat{\xi}}'_o\ A'A\ \hat{\hat{\xi}}_o\ M_i$$

whence

(7.2.16) $$\hat{\hat{P}}_{io}\hat{\hat{P}}'_{io}-\hat{P}_{io}\hat{P}'_{io} = M'_i(\hat{\xi}'_o-\hat{\hat{\xi}}'_o)A'Y'_i+Y_iA(\hat{\xi}_o-\hat{\hat{\xi}}_o)M_i$$

$$-M'_i(\hat{\xi}'_oA'A\hat{\xi}_o-\hat{\hat{\xi}}'_oA'A\hat{\hat{\xi}}_o)M_i$$

The approximations for $\hat{\xi}$ and $\hat{\hat{\xi}}$ are

(7.2.17) $$\hat{\xi}_o = \sum_{i=1}^{K} X_i\ \hat{U}_{io}^{-1}\ M'_i\ \hat{V}_o^{-1}$$

(7.2.18) $$\hat{\hat{\xi}}_o = \sum_{i=1}^{K} \hat{\hat{X}}_i\ \hat{U}_{io}^{-1}\ M'_i\ \hat{V}_o^{-1}$$

where X_i and $\hat{\hat{X}}_i$ are defined in (4.2.5) and (4.2.10) respectively. Now

(7.2.19) $$\hat{\xi}_o-\hat{\hat{\xi}}_o = \sum_{i=1}^{K}(X_i-\hat{\hat{X}}_i)\hat{U}_{io}^{-1}\ M'_i\ \hat{V}_o^{-1}$$

where

(7.2.20) $$X_i-\hat{\hat{X}}_i = (A'A)^{-1}\ C'[C(A'A)^{-1}C']^{-1}\ C\ X_i$$

The hypothesis under test is $C\xi=0$ where C is the row vector (o 1) and ξ is the (2 x 3) parameter matrix of means and regression coefficients. Using this specific value of C and

$$(\Lambda'\Lambda)^{-1} = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{s_{tt}} \end{bmatrix}$$

in (7.2.20) we obtain

$$X_i - \hat{\hat{X}}_i = \begin{bmatrix} 0 & 0 \\ b_1^{(i)} & b_2^{(i)} \end{bmatrix} \tag{7.2.21}$$

where $b_1^{(i)}$ and $b_2^{(i)}$ are the ordinary least squares estimates of the regression coefficients for the two variables in the i'th group. Hence the matrix $\hat{\xi}_o - \hat{\hat{\xi}}_o$ differs from the matrix $\hat{\xi}_o$ only in that the first row is a vector of zeros. The second row was obtained in (7.2.10). Thus

$$\hat{\xi}_o - \hat{\hat{\xi}}_o = \begin{bmatrix} 0 & 0 & 0 \\ \tilde{\beta}_1 & \tilde{\beta}_2 & \tilde{\beta}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2.9361 & 1.9449 & 1.1182 \end{bmatrix} \tag{7.2.22}$$

Also, we have

$$\hat{\xi}_o'(\Lambda'\Lambda)\hat{\xi}_o - \hat{\hat{\xi}}_o'(\Lambda'\Lambda)\hat{\hat{\xi}}_o = \tag{7.2.23}$$

$$s_{tt} \begin{bmatrix} \tilde{\beta}_1^2 & \tilde{\beta}_1\tilde{\beta}_2 & \tilde{\beta}_1\tilde{\beta}_3 \\ \tilde{\beta}_1\tilde{\beta}_2 & \tilde{\beta}_2^2 & \tilde{\beta}_2\tilde{\beta}_3 \\ \tilde{\beta}_1\tilde{\beta}_3 & \tilde{\beta}_2\tilde{\beta}_3 & \tilde{\beta}_3^2 \end{bmatrix}$$

Substituting (7.2.22) and (7.2.23) in (7.2.16) and letting

$$\tilde{F}_i = \hat{\hat{F}}_{io}\hat{\hat{F}}'_{io} - \hat{F}_{io}\hat{F}'_{io}$$

we obtain

$$\tilde{F}_1 = \begin{bmatrix} 2(776.783)(2.9361) & 776.783(1.9449)+527.581(2.9361) \\ -280(2.9361)^2 & -280(2.9361)(1.9449) \\ & 2(527.581)(1.9449)-280(1.9449)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2,147.633834 & 1,460.8779816 \\ 1,460.8779816 & 993.046491 \end{bmatrix}$$

$$\tilde{F}_2 = \begin{bmatrix} 2(563.701)(1.9449) & 563.701(1.1182)+216.404(1.9449) \\ -280(1.9449)^2 & -280(1.9449)(1.1182) \\ & 2(216.404)(1.1182)-280(1.1182)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1,133.546067 & 442.2741874 \\ 442.2741874 & 133.8619584 \end{bmatrix}$$

$$\tilde{F}_3 \quad \begin{bmatrix} 2(838.493)(2.9361) & 838.493(1.1182)+376.896(2.9361) \\ -280(2.9361)^2 & -280(2.9361)(1.1182) \\ & 2(376.896)(1.1182)-280(1.1182)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2,510.007296 & 1,124.9260526 \\ 1,124.9260526 & 492.7862672 \end{bmatrix}$$

The matrices $\hat{U}_{io}^{-1}$ were obtained in (7.2.7). If we multiply the six rows of these matrices into the corresponding six rows of the matrices $\tilde{F}_i$ and add the products together, we obtain (using (7.2.15))

$$- 2 \ln \lambda \doteq \sum_{i=1}^{K} \operatorname{tr} \hat{U}_{io}^{-1} \tilde{F}_i = 2,229.09$$

Since this exceeds $\chi^2_{.95}(3) = 7.81473$, we reject the hypothesis that the β_i are zero.

The equation of an approximate confidence ellipsoid is then obtained by substituting the regression estimates $\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3$ from (7.2.10) in the expression (7.2.2). Again let

$$(x \quad y \quad z) = (\tilde{\beta}_1 - \beta_1,\ \tilde{\beta}_2 - \beta_2,\ \tilde{\beta}_3 - \beta_3)$$

Then substituting $s_{tt} = 840$, $\chi^2_{.95}(3) = 7.81473$ in (7.2.2) we obtain for the equation of the circumference of the confidence region

$$(x \quad y \quad z)\hat{\Sigma}_o^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.00930325 \qquad (7.2.24)$$

The trace of this surface in the (x y) plane has the equation

$$(7.2.25) \qquad (x \quad y \quad 0)\hat{\Sigma}_o^{-1} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = 0.00930325$$

Inverting (7.2.3) we obtain

$$(7.2.26) \qquad \hat{\Sigma}_o^{-1} = \begin{bmatrix} 0.7199663 & -2.2151574 & -0.2225995 \\ -2.2151574 & 7.7740387 & 0.8281469 \\ -0.2225995 & 0.8281469 & 0.1211376 \end{bmatrix}$$

The equation of the contour $\beta_3=\tilde{\beta}_3$ can then be written

$$(7.2.27) \qquad (x \quad y) \begin{bmatrix} 0.7199663 & -2.2151574 \\ -2.2151574 & 7.7740387 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 0.00930325$$

The angle of rotation θ which will eliminate the product term in (7.2.27) is determined from

$$\tan 2\theta = \frac{2(-2.2151574)}{0.7199663 - 7.7740387}$$

$$= 0.6280506$$

whence $\theta = 16^\circ\ 4'$, $\sin\theta = 0.27673$, $\cos\theta = 0.96095$. Then making the rotation transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.96095 & -0.27673 \\ 0.27673 & 0.96095 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

in (7.2.27), we obtain

$$(x' \ y') \begin{bmatrix} 0.08204104 & -0.00005089 \\ -0.00005089 & 8.41200104 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= 0.00930325$$

Hence the equation of the contour $\beta_3=\tilde{\beta}_3$ in standard form is

$$\text{(7.2.28)} \qquad \frac{x'^2}{(0.34)^2} + \frac{y'^2}{(0.03)^2} = 1$$

The equation of any contour $\beta_3=k$ is found by substituting $z=z_o=\tilde{\beta}_3-k$ in (7.2.24), and making the transformation (6.3.9). Equation (7.2.24) then becomes

(7.2.29)

$$[x' \ y' \ z_o] \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\sigma}^{11} & \tilde{\sigma}^{12} & \tilde{\sigma}^{13} \\ \tilde{\sigma}^{12} & \tilde{\sigma}^{22} & \tilde{\sigma}^{23} \\ \tilde{\sigma}^{13} & \tilde{\sigma}^{23} & \tilde{\sigma}^{33} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z_o \end{bmatrix}$$

$$= 0.00930325$$

where $\tilde{\sigma}^{ij}$ denotes the element in the i'th row and j'th column of $\hat{\Sigma}_o^{-1}$. Substituting $\sin\theta = 0.27673$, $\cos\theta = 0.96095$ in (7.2.16) and multiplying the three inner matrices together, we obtain

$$(7.2.30)\qquad [x' \;\; y' \;\; z_o]\begin{bmatrix} 0.08204104 & -0.00005089 & 0.01526613 \\ -0.00005089 & 8.41200104 & 0.85740770 \\ 0.01526613 & 0.85740770 & 0.12113764 \end{bmatrix}\begin{bmatrix} x' \\ y' \\ z_o \end{bmatrix}$$

$$= 0.00930325$$

In scalar form this equation is

$$(7.2.31)\qquad 0.08204104\, x'^2 + 8.41200104\, y'^2$$

$$+ 2(0.01526613)z_o x' + 2(0.85740770)z_o y'$$

$$= 0.00930325 - 0.12113764\, z_o^2$$

Completing squares we obtain

$$0.08204104\left(x' + \frac{0.01526613z_o}{0.08204104}\right)^2$$

$$+ 8.41200104\left(y' + \frac{0.85740770z_o}{8.41200104}\right)^2$$

$$= 0.00930325 - 0.12113764\, z_o^2$$

$$+ \frac{(0.01526613z_o)^2}{0.08204104} + \frac{(0.85740770z_o)^2}{8.41200104}$$

which can easily be put in standard form.

The standard equations of the contours β_3=k are found by substituting $z_o = \tilde{\beta}_3 - k$ in equation (7.2.18) and completing squares. The centers and semi-axes for several contours β_3=k are given in the table below. (The origin of the x y and x'y' systems corresponds to $\beta_1=\tilde{\beta}_1$, $\beta_2=\tilde{\beta}_2$ and the angle of rotation is approximately 16°.)

Table 4

Contours of Ellipsoid Based on

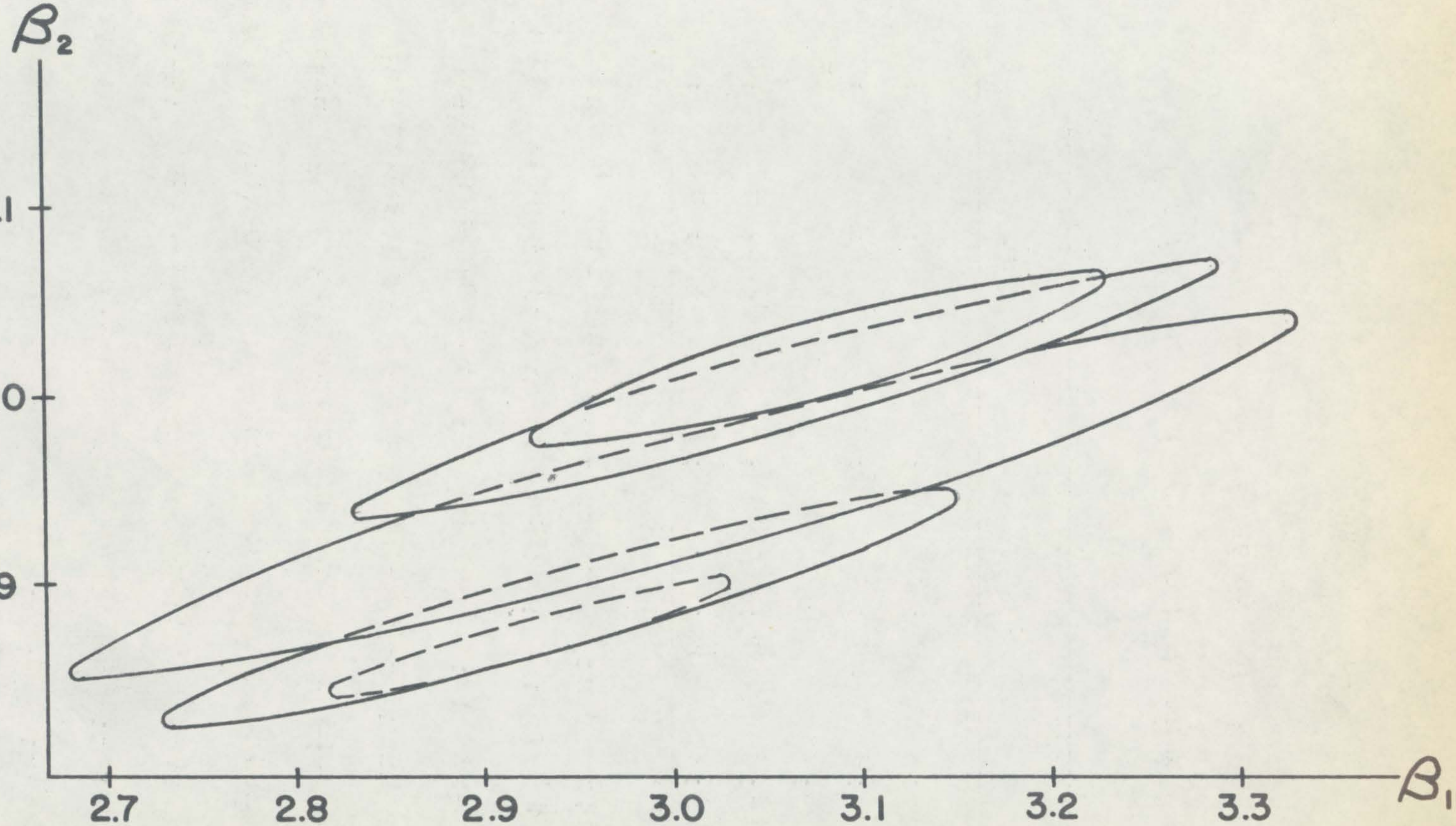
Incomplete Sets and Asymptotic Distribution

Contours		Centers		Semi-Axes	
$\beta_3=k$	$z=z_0$	x'	y'	a	b
0.6	0.5182	-0.10	-0.05	0.11	0.01
0.7	0.4182	-0.08	-0.04	0.22	0.02
$\tilde{\beta}_3$	0	0	0	0.34	0.03
1.5	-0.3818	0.07	0.04	0.24	0.02
1.6	-0.4818	0.09	0.05	0.16	0.02

The x' and y' coordinates of the centers of the sections which are point ellipses are (-0.10, -0.06) and (0.10, 0.06). These correspond to β_3= 0.57 and β_3= 1.67 respectively.

The contours of this ellipsoid are shown in figure 6. The semi-axes of the central sections of the ellipsoid for variables one, two, and three are approximately 0.34, 0.03, and 0.55. The corresponding semi-axes of the ellipsoid discussed in section 7.1 are approximately 0.36, 0.04, and 0.6. The semi-axes of the section $z \doteq 0.50(\beta_3=0.5)$ of the ellipsoid for the complete sets are 0.19 and 0.02 as compared with 0.11 and 0.01 for the section $z \doteq 0.52(\beta_3=0.6)$ of the ellipsoid

FIG. 6



CONTOURS OF ELLIPSOID BASED ON
INCOMPLETE SETS AND ASYMPTOTIC DISTRIBUTION

based on incomplete sets. Similarly the semi-axes of the section $z \doteq -0.50$ of the ellipsoid for the complete sets are 0.20 and 0.02, while those of the section $z \doteq -0.48$ of the ellipsoid for the incomplete data are 0.16 and 0.02. Thus the confidence ellipsoid constructed by the method of this section for the incomplete data is almost the same size as that calculated in section 6.4 for the complete data.

The orientation of the two ellipsoids, however, is different. The centers of the contours of the ellipsoid in figure 2 have signs (+ - -) and (- + +). If one imagines that the negative z (and positive β_3) axis is directed from the plane of the paper toward the reader, then the ellipsoid in figure 2 extends from the octant behind the plane of the paper, above the (x z) plane and to the left of the (y z) plane to the octant in front of the plane of the paper, below the (x z) plane, and to the right of the (y z) plane. The centers of the contours of the ellipsoid in figure 6 have signs (+ + -) and (- - +). This ellipsoid extends from the octant behind the plane of the paper, below the x z plane, and to the left of the y z plane to the octant in front of the plane of the paper, above the x z plane, and to the right of the y z plane.

This difference in orientation is clearly caused by the differences in the elements of the matrices E^{-1} and $\hat{\Sigma}_o^{-1}$. The matrix E/41 in the equation of the ellipsoid for the complete data corresponds to the matrix $\hat{\Sigma}_o$ in the equation for the incomplete data. If we divide the matrix of the determinant in the numerator of (6.2.1) by 41, we obtain

$$E/41 = \begin{bmatrix} 13.46 & 5.17 & -7.79 \\ 5.17 & 2.26 & -2.01 \\ -7.79 & -2.01 & 37.76 \end{bmatrix}$$

The matrix $\hat{\Sigma}_o$ is given in (7.2.3). If we compare these two estimates with the theoretical dispersion matrix in (6.1.2), we see that the elements of $\hat{\Sigma}_o$ are on the whole somewhat closer to the theoretical values. The estimates of σ_{13} and σ_{33} in $\hat{\Sigma}_o$ are considerably better. On the other hand, the estimates in E/41 for two variances and covariances are closer to the true values. In both matrices all elements except the third diagonal element are underestimates of the theoretical values.

The regression estimates $\tilde{\beta}_1 = 2.94$, $\tilde{\beta}_2 = 1.94$, $\tilde{\beta}_3 = 1.12$ are almost as close to the theoretical values as the estimates $b_1 = 2.97$, $b_2 = 1.97$, $b_3 = 1.00$ based on the complete trivariate analysis. As in all the methods of analysis used up to this point, the values found for $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are underestimates while that for $\tilde{\beta}_3$ is an overestimate.

7.3 Maximum-Likelihood Estimation

A set of programs to facilitate the Newton iterative technique for the present case was prepared for the IBM 650 by Prof. R. E. Bargmann. With the aid of these programs he obtained the maximum-likelihood solution for the present example. Due to the serious storage limitation of the IBM 650 for programs of this order of magnitude, these programs are not sufficiently general to permit, at this time, publication and distribution, and considerable special adaptations and a large number of parameter cards are required for each new variable-design. For a variable-design of the types mentioned in this dissertation the analysis can now be performed at the Virginia Polytechnic Institute if the number of variables (p) is less than five or six. More extensive programs and a more general procedure would require the availability of a larger electronic computer.

The application of the iterative technique in the present study will be discussed in detail for the benefit of those who suggest that numerical solutions can always be "obtained readily by an iterative technique". The complexity of the problem of actually developing such a technique is already indicated in the derivation of the ∇ matrix[1] in this dissertation. Other complications arise due to the fact that the Newton method will always converge to the nearest stationary solution. Whereas in most cases, in practice, the heuristic least-squares estimate described in the previous section produces a first guess so close to the maximum-likelihood solution that the Newton method will at once converge to the maximum, the present study may serve as an example of a notable exception. Some adjustment of the initial estimate was required before the iterative method produced the desired stationary value which was proved to be a maximum.

The iteration was begun with $\hat{\Sigma}_o$ as stated in formula (7.2.3). The results of the first and second iterations were

(7.3.1)
$$\hat{\Sigma}_1 = \begin{bmatrix} 17.070 & 3.386 & -6.864 \\ & 1.581 & -4.816 \\ & & 34.612 \end{bmatrix} ,$$

and

---

(1) The expression ∇ is used for the matrix of derivatives in the Newton method. In the present case its columns are $\underline{g}^{[K\ell]}$ as stated in (3.1.22).

(7.3.2)
$$\hat{\Sigma}_2 = \begin{bmatrix} 23.380 & 1.661 & -9.307 \\ & 1.665 & -5.032 \\ & & 35.593 \end{bmatrix} .$$

In spite of the fact that the elements of $\phi(\hat{\Sigma})$ (2.5.30) decreased rapidly indicating that a stationary value was approached (the largest element of this matrix was less than 0.1, compared with a largest element of 0.84 in the first iteration), the drastic changes in Σ seemed to indicate that the iteration was "running away" to another stationary point. This fact was finally proved by evaluation of the likelihood function itself which was, approximately,

(7.3.3)
$$\log_e L_0 = -211$$
$$\log_e L_1 = -217$$
$$\log_e L_2 = -223 \quad ,$$

and thus clearly decreased, so that the solution did not converge to a maximum. On the assumption, and with the hope, that this was due to the fact that the initial estimate was only slightly beyond an inflection point away from the maximum, a slight correction in the direction opposite to that indicated by the iteration was made, and thus a new initial estimate was chosen as

(7.3.4)
$$\hat{\Sigma}_0 = \begin{bmatrix} 11.0 & 4.5 & -4.0 \\ & 2.0 & -4.6 \\ & & 31.0 \end{bmatrix}$$

The results of two iterations of this initial estimate were:

(7.3.5)
$$\hat{\Sigma}_1 = \begin{bmatrix} 10.94 & 4.59 & -4.06 \\ & 2.11 & -5.96 \\ & & 35.27 \end{bmatrix} ,$$

$$\hat{\Sigma}_2 = \begin{bmatrix} 11.12 & 4.72 & -4.17 \\ & 2.19 & -6.21 \\ & & 36.95 \end{bmatrix} .$$

The largest element of the criterion, $\phi(\hat{\Sigma})$ (2.5.30), after this second iteration was .060 (compared with 1.46 in the initial iteration), so that rapid convergence to a stationary value was indicated. Also, the likelihood function itself increased, as follows,

(7.3.6)
$$\log_e L_0 = -207.02$$
$$\log_e L_1 = -206.46$$
$$\log_e L_2 = -206.41 \quad ,$$

so that the stationary value corresponded to a larger likelihood than the initial estimate. Seven iterations were necessary in order to reduce the largest element of $\phi(\hat{\Sigma})$ to a value less than 0.0001, and established the elements of $\hat{\Sigma}$ to three decimal places, i.e., changes from $\hat{\Sigma}_6$ to $\hat{\Sigma}_7$ occurred in the 4'th decimal place only for two elements, and in the 5'th decimal place for the others. The final result was

(7.3.7)
$$\hat{\Sigma}_7 = \begin{bmatrix} 11.136 & 4.738 & -4.183 \\ & 2.206 & -6.244 \\ & & 37.152 \end{bmatrix} .$$

To prove that this solution constitutes, in fact, a maximum, the following procedure was employed:

If we denote by σ_{ij} the distinct element which occurs in the i'th row and j'th column <u>and</u> in the j'th row and i'th column of Σ, the elements of $\phi(\Sigma)$ in (2.5.30) are related to the first derivative of the likelihood function, as

(7.3.8)
$$\varphi_{ij} = -\frac{1}{n}\frac{\partial \log L}{\partial \sigma_{ij}} \quad , \quad i \neq j,$$

$$\varphi_{ii} = -\frac{2}{n}\frac{\partial \log L}{\partial \sigma_{ii}} \quad .$$

Consequently,

$$\phi_{ij}^{[K\ell]} = -\frac{1}{n}\frac{\partial^2 \log L}{\partial \sigma_{ij}\partial \sigma_{K\ell}} \quad , \quad i \neq j,$$

and

$$\phi_{ii}^{[K\ell]} = -\frac{2}{n}\frac{\partial^2 \log L}{\partial \sigma_{ii}\partial \sigma_{K\ell}} \quad .$$

Hence if, in the ∇ matrix, one applies the factor $n/2$ to each row corresponding to a φ_{ii} and n to each row corresponding to a φ_{ij} $(i \neq j)$, one will obtain a symmetric matrix whose elements are the negative second derivatives of log L with respect to each pair of σ_{ij} and $\sigma_{K\ell}$. Thus, if this matrix is positive definite, the obtained solution for Σ must make the likelihood function a maximum.

The ∇ matrix corresponding to the last iteration was

$$(7.3.9) \qquad \begin{bmatrix} 1.018 & -2.241 & .0014 & 5.006 & .0000 & .0001 \\ -4.481 & 10.407 & -.0001 & -24.111 & .0000 & .0000 \\ .0028 & -.0001 & .0033 & -.0001 & .0010 & .0055 \\ 5.004 & -12.052 & -.0003 & 29.523 & .1058 & .0211 \\ .0000 & .0001 & .0001 & .2116 & .0614 & .0159 \\ .0001 & .0000 & .0003 & .0211 & .0080 & .0036 \end{bmatrix}$$

(Four places were recorded for numbers smaller than one and three decimals for numbers greater than one. This reflects the accuracy obtainable by floating point arithmetic in the machine computations. Actually, the accuracy was between 4 and 5 significant digits).

From this ∇ matrix (or rather from the matrix obtained by the machine in floating point form) the matrix of second derivatives of the log-likelihood function was found to be,

$-\partial^2 \log L/\partial\sigma_{ij}\partial\sigma_{K\ell} =$

(7.3.10)
$$\begin{bmatrix} 7.632 & -33.610 & .0213 & 37.538 & -.0002 & .0009 \\ & 156.112 & -.0013 & -18.081 & .0008 & .0000 \\ & & .0491 & -.0044 & .0015 & .0041 \\ & & & 22.142 & .1587 & .1583 \\ & & & & .9208 & .1193 \\ & & & & & .0269 \end{bmatrix}$$

This matrix was proved to be positive definite by the forward Doolittle solution (i.e., if the leading elements of the second to last row--before normalization to unity--are positive in each cycle, the corresponding matrix must be positive definite, for it can then be represented as $\tilde{T}\tilde{T}'$, where $\tilde{T}$ is a triangular matrix with real elements). Thus, (7.3.7) represents the maximum likelihood estimate of Σ, correct to three decimal places.

7.4 Discussion of Results

The matrix (7.3.7) is not very different from the easily obtainable $\hat{\Sigma}_o$ in (7.2.3). The agreement in the estimates of means and regression coefficients is even more striking. They are

(7.3.11) $$\underline{\mu}_0' = [10.404,\ 15.282,\ 21.597]$$

$$\underline{\mu}_7' = [10.530,\ 15.222,\ 21.721] \quad ,$$

$$\underline{\beta}_0' = [2.9362,\ 1.9449,\ 1.1184]$$

$$\underline{\beta}_7' = [2.9199,\ 1.9478,\ 1.1053] \quad .$$

But even in the first set of iterations, where the solution converged in the direction of a saddle point, and where the $\hat{\Sigma}$ matrix had become very different from the original one, the main effects were not seriously affected. They were

(7.3.12) $$\underline{\mu}_2' = [10.469,\ 15.403,\ 21.287]$$

and

$$\underline{\beta}_2' = [2.9379,\ 1.9356,\ 1.1482] \quad .$$

It should be recalled that the theoretical values, on which the sampling experiment was based, were

(6.1.1) $$\underline{\mu}' = [10,\ 15,\ 20]$$

$$\underline{\beta}' = [\ 3,\ 2,\ 1]$$

and

(6.1.2) $$\Sigma = \begin{bmatrix} 14 & 6 & -4 \\ & 3 & 0 \\ & & 32 \end{bmatrix} \quad .$$

It cannot be ascertained by analyses of this type that one type of estimate is consistently superior to another. With 45 observation vectors, the sampling effect is considerably greater than the disagreement between the heuristic approximation $\hat{\Sigma}_o$ and the maximum-likelihood solution, as stated in $\hat{\Sigma}_7$. Here, as in other comparative studies (e.g. bio-assay or factor analysis) the maximum likelihood solution seems to have only theoretical advantages over some heuristic approximations (see, e.g., Brown [1960]). Where, as in factor analysis and in the present problem, the maximum-likelihood estimates can be obtained only with formidable computational work, their theoretical advantage will hardly justify the practical application, because of the closeness of the easily obtainable approximations. However, where measurements on variables can be performed with a high degree of precision <u>and</u> where unusually large samples are involved, an iteration for the purpose of obtaining maximum-likelihood solutions may be worth-while. Especially where correlation matrices are desired (as, for example, in the analysis of parallel test forms in sociological or educational research), the maximum-likelihood solution has the advantage of being the logically consistent one--for correlations and functions of correlations are estimated exclusively by maximum-likelihood, i.e., by the correlation coefficients in the sample.

VIII. SUMMARY

In this investigation methods of estimation and tests of significance have been developed for multivariate experiments in which only a subset of size u of the p variables under study can be measured in each of K groups of subjects. The matrix Y_i' of measurements in the i'th group consists of n rows of observations on u variables. It is obtained by multiplying the matrix of theoretically possible observations in the i'th group by a post-factor matrix M_i consisting of ones and zeros. The matrix of means for the i'th group is $A\xi M_i$ where $A(n \times m)$ is the design matrix for each group and $\xi(m \times p)$, the parameter matrix. The covariance matrix is $U_i = M_i'\Sigma M_i$ where Σ is the dispersion matrix for all p variables.

The logarithm of the likelihood function for the entire sample of $N=Kn$ observations is

$$L(Y') = -\frac{Nu}{2}\log 2\pi - \frac{n}{2}\sum_{i=1}^{K}\log|U_i| - \tfrac{1}{2}\operatorname{tr}\sum_{i=1}^{K}U_i^{-1}P_iP_i'$$

where $P_i = Y_i - M_i'\xi'A'$. Denoting the least squares estimates of observations in the i'th group by X_i and letting

$$\hat{V} = \sum_{i=1}^{K}M_i\,\hat{U}_i^{-1}\,M_i' \text{ and } \hat{F} = \sum_{i=1}^{K}M_i\,\hat{U}_i^{-1}\,X_i',$$

we have derived the maximum likelihood estimate $\hat{\xi} = \hat{F}'\hat{V}^{-1}$ of the parameter matrix. An implicit relation

$$\phi(\hat{\Sigma}) = \hat{V} - \frac{1}{n}\sum_{i=1}^{K} M_i \hat{U}_i^{-1} \hat{P}_i \hat{P}_i' \hat{U}_i^{-1} M_i' = 0$$

has been determined for the maximum likelihood estimate of dispersion.

The Newton method is employed to obtain $\hat{\Sigma}$. An initial estimate $\hat{\Sigma}_o$ is computed by averaging estimates based on the group error sums of squares and products. Successive values for $\hat{\Sigma}$ are found by using the Newton iterative equation $\underline{\sigma}_1 = \underline{\sigma}_o$ $\underline{\sigma}_1 = \underline{\sigma}_o - [\nabla \phi_o]^{-1} \underline{\phi}_o$. In this formula, $\underline{\sigma}_o$ and $\underline{\sigma}_1$ are vectors of estimates of the $p(p+1)/2$ distinct elements of the dispersion matrix, $\underline{\phi}_o$ is a vector of distinct elements of $\phi(\Sigma)$, and $\nabla \phi_o$ is a matrix each of whose columns is a vector of derivatives of distinct elements of $\phi(\Sigma)$ with respect to a particular element $\sigma_{k\ell}$ of the matrix Σ. The elements of the rows and columns of $\nabla \phi_o$ are ordered in the same way as the elements of $\underline{\sigma}$ and $\underline{\phi}$.

The asymptotic distribution of $-2 \ln \lambda$ is used to test the general linear hypothesis $C\xi = 0$, where C is an arbitrary matrix. The maximum likelihood estimates $\hat{\xi}$ and $\hat{\hat{\xi}}$ of the parameter matrices in Ω and ω have essentially the same form, except that in order to obtain $\hat{\hat{\xi}}$, the matrix X_i of group least squares estimates (implicit in the equation $\hat{\xi} = \hat{P}'\hat{V}^{-1}$) is replaced by the corresponding matrix $\hat{\hat{X}}_i$ of least squares estimates under the restricted model in which the null hypothesis is true. The relations $\phi(\hat{\hat{\Sigma}})=0$ and $\phi(\hat{\Sigma})=0$ from which the maximum likelihood estimates of dispersion are obtained, have the same form in ω and Ω.

The effect of certain designs of the variables on some of the matrices used in the calculation of $\phi(\Sigma)$ and $\phi^{[k\ell]}(\Sigma)$ is studied. These arrangements are called balanced, circular link, incomplete circular link, and minimum linkage designs. When pairs of variables are not observed together in the same group in the three latter designs, patterns of zeros occur in the components of the maximum likelihood and derivative equations. Decreasing the linkage among the groups simplifies the form of these matrices.

Two approximate estimates of the parameter and dispersion matrices were made in a demonstration study of three variables arranged in balanced groups of two. The first approximate method is based upon three independent bivariate analyses; the second, upon the substitution of Σ_0 for $\hat{\Sigma}$ in the maximum likelihood expression for $\hat{\xi}$.

The exact maximum likelihood estimate $\hat{\Sigma}$ for the demonstration study was established to three decimal places by seven iterations of the Newton formula. To verify that $\hat{\Sigma}$

is a maximum, the matrix of negative second derivatives of the logarithm of the likelihood function was shown to be positive definite by means of the forward Doolittle procedure.

The matrix $\hat{\Sigma}$ is very similar to the initial estimate $\hat{\Sigma}_o$ used in this demonstration study. The estimates of main effects obtained by the exact and approximate methods are even closer in agreement. However, since only 45 observations were used in the study, the sampling effect is much greater than the difference in results obtained by the exact and approximate methods. The laborious computational work involved in obtaining the maximum likelihood estimate of the dispersion matrix for incomplete variable designs would seem to be justified only in studies involving large samples where observations are taken with a high degree of precision. Particularly in problems involving correlation matrices, (for example, estimation of the reliabilities of standardized tests), maximum likelihood would seem to be the most logically consistent approach to use.

ACKNOWLEDGEMENTS

The author wishes to express her appreciation to the following:

Professor Rolf Bargmann, for suggesting the problem, for his guidance in its solution, and for his contributions, particularly his construction of the program for the IBM 650.

Professor David C. Hurst, , and the members of the Graduate Committee, for helpful suggestions.

of Keuka College, for granting an additional leave of absence after the sabbatical year.

The National Science Foundation, for providing partial support for this project.

, for her careful typing of the manuscript.

Professor Boyd Harshbarger, for his helpfulness and for his generous and encouraging attitude.

BIBLIOGRAPHY

Anderson, R. L. and Bancroft, T. A. (1952). Statistical Theory in Research. New York: McGraw-Hill Book Company, Inc.

Anderson, T. W. (1958). An Introduction to Multivariate Statistical Analysis. New York: John Wiley and Sons, Inc.

Browne, C. T. (1958). Introduction to the Theory of Determinants and Matrices. Chapel Hill: University of North Carolina Press.

Brown, R. H. (1960). A Comparison of the Maximum Determinant Solution in Factor Analysis with Various Approximate Solutions. Unpublished Master's Thesis. Virginia Polytechnic Institute, Blacksburg, Virginia.

Cochran, William G. and Cox, Gertrude M. (1957). Experimental Designs. New York: John Wiley and Sons, Inc.

Crawford, Albert B. and Burnham, Paul S. (1946). Forecasting College Achievement. New Haven: Yale University Press.

Davies, Owen L. (1956). Design and Analysis of Industrial Experiments. New York: Hafner Publishing Company.

Edgett, George L. (1956). Multiple Regression with Missing Observations among the Independent Variables. Journal of the American Statistical Association, 51, 122.

Frazer, R. A., Duncan, W. J., and Collar, A. R. (1947). Elementary Matrices and Some Applications to Dynamics and Differential Equations. New York: The Macmillan Company.

Gulliksen, Harold (1950). Theory of Mental Tests. New York: John Wiley and Sons, Inc.

Heck, D. L. (1958). Some Uses of the Distribution of the Largest Root in Multivariate Analysis. Institute of Statistics, University of North Carolina, Mimeograph Series No. 194.

Kempthorne, Oscar (1952). The Design and Analysis of Experiments. New York: John Wiley and Sons, Inc.

Lord, Frederic M. (1959). Problems in Mental Test Theory Arising from Errors of Measurement. Journal of the American Statistical Association, 54, 472.

Matthai, Abraham (1951). Estimation of Parameters from Incomplete Data with Application to Design of Sample Surveys. Sankhya, 11, 145.

Nielsen, Kaj L. (1957). Methods in Numerical Analysis. New York: The Macmillan Company.

Pearson, E. S. and Hartley, H. O. (1954). Biometrika Tables for Statisticians, Vol. I. Cambridge University Press.

Rao, C. R. (1952). Advanced Statistical Methods in Biometric Research. New York: John Wiley and Sons, Inc.

Rao, C. R. (1955). Analysis of Dispersion for Multiply Classified Data with Unequal Numbers in Cells. Sankhya, 15, 253.

Rao, C. R. (1956). Analysis of Dispersion with Incomplete Observations on One of the Characters. Journal of the Royal Statistical Society, Series B, 18, 259.

Roy, J. (1958). Step-Down Procedure in Multivariate Analysis. Annals of Mathematical Statistics, 29, 1177.

Roy, S. N. (1939). p-Statistics, or Some Generalizations on Analysis of Variance Appropriate to Multivariate Problems. Sankhya, 4, 381.

Roy, S. N. (1942). Analysis of Variance for Multivariate Normal Populations. Sankhya, 6, 35.

Roy, S. N. (1957). Some Aspects of Multivariate Analysis. New York: John Wiley and Sons, Inc.

Wade, Thomas L. (1951). The Algebra of Vectors and Matrices. Cambridge, Mass.: Addison-Wesley Press, Inc.

Wilks, S. S. (1932a). Certain Generalizations in the Analysis of Variance. _Biometrika_, 24, 471.

Wilks, S. S. (1932b). Moments and Distributions of Estimates of Population Parameters from Fragmentary Samples. _Annals of Mathematical Statistics_, 3, 163.

Wilks, S. S. (1935). Test Criteria for Statistical Hypotheses Involving Several Variables. _Journal of the American Statistical Association_, 30, 549.

Wilks, S. S. (1943). _Mathematical Statistics_. Princeton, N. J.: Princeton University Press.

The two page vita has been removed from the scanned document. Page 1 of 2

The two page vita has been removed from the scanned document. Page 2 of 2

ABSTRACT

The maximum likelihood approach is used for the multivariate situation where a different subset of size u of the p variables under study is observed in each of K groups of n sampling units. The matrix of observations in the i'th group is $Y_i' = X_i'M_i$ where M_i is a matrix of ones and zeros which selects from the matrix X_i' (nxp) the n x u array Y_i'. The expectation and dispersion matrices for the i'th group are $A\xi M_i$ and $M_i'\Sigma M_i$ where A is the common design matrix for all groups, ξ, the array of all parameters to be estimated in the study, and Σ, the covariance matrix for all p variables.

An explicit expression for the maximum likelihood estimate of the parameter matrix and an equation involving that of the dispersion matrix are obtained by differentiating the likelihood function for the whole sample of K n observations. The latter equation is solved by the Newton-Raphson method.

Group estimates of dispersion are found by dividing the group error sums of squares and products by n. An initial estimate $\hat{\Sigma}_o$ for the roots of the maximum likelihood equation is calculated by averaging the appropriate group estimates. Successive approximations are obtained by the Newton iterative formula $\underline{\sigma_1} = \underline{\sigma_o} - [\nabla \phi_o]^{-1} \underline{\phi_o}$. In this equation $\underline{\sigma_o}$ is the

vector of the p(p+1)/2 distinct elements of $\hat{\Sigma}_o$ and $\underline{\sigma_1}$ is the corresponding vector of first approximations. Each column of ϕ_o is a vector of the p(p+1)/2 distinct elements of $\phi^{[k\ell]}(\Sigma_o)$ for a specific pair of $(k\ell)$ values, where $\phi^{[k\ell]}(\Sigma)$ is the matrix of derivatives of $\phi(\Sigma)$ with respect to $\sigma_{k\ell}$. $\underline{\phi}_o$ is a vector of the p(p+1)/2 distinct elements of $\phi(\Sigma_o)$.

The general linear hypothesis $C\xi=0$, where C is an arbitrary matrix, is tested by using the asymptotic distribution of $-2 \ln \lambda$. The maximum likelihood estimates $\hat{\xi}$ and $\hat{\hat{\xi}}$ of the parameter matrices in Ω and ω are both obtained in the same way, except that in order to compute $\hat{\hat{\xi}}$, the group least squares estimates (implicit in the expression for $\hat{\xi}$) must be replaced by the corresponding group least squares estimates under the restricted model in which the null hypothesis is true. The condition on the maximum likelihood estimate $\hat{\hat{\Sigma}}$ in ω has the same form as the condition on $\hat{\Sigma}$ in Ω.

The effect of certain arrangements of the variables in groups on components of the maximum likelihood and derivative equations is investigated. These arrangements, called balanced, circular link, incomplete circular link, and minimum linkage designs, represent decreasing linkage among the groups. The absence of pairs of variables in the individual groups in the unbalanced designs results in the formation of patterns of zeros in some of the matrices used in the calculation of $\phi(\Sigma)$ and $\phi^{[k\ell]}(\Sigma)$. Generally speaking, as one would expect, the less linkage there is between the groups, the simpler the form of these matrices.

A demonstration study is introduced to compare approximate methods of estimation with the exact maximum likelihood method for the case p=3, u=2. One such approximate method consists in combining the results of three independent bivariate analyses. Another consists in using the initial estimate $\hat{\Sigma}_o$ in place of the exact maximum likelihood estimate $\hat{\Sigma}$ to obtain an approximate estimate of the parameter matrix. The maximum likelihood estimate of the dispersion matrix obtained in this study is not very different from the more easily obtainable initial estimate $\hat{\Sigma}_o$. The agreement in the exact and approximate parameter estimates is even closer.