A SHORT CUT METHOD FOR LINEAR REGRESSION

by

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I. INTRODUCTION

The method of Least Squares is usually used to estimate the parameters in a linear regression situation. It is however sometimes desirable to use some short-cut methods of estimation, which require less computational effort.

Bose [2] developed three alternative methods for evaluating the regression coefficient for relationship between variables x and y, where the independent variable x is observed at equally spaced intervals. The three methods are (1) method of Successive Differences, (2) method of Difference at Half Range, and (3) method of Range. Nair and Shrivastava [6] developed the method of Group Averages for the same case, and pointed out that Bose's methods could be seen as special cases of the method of Group Averages.

It is the purpose of this paper to review several versions of the method of Group Averages, and to compare their efficiencies relative to the method of Least Squares. The use of this method is also discussed for the case when the independent variable is measured with error, in which case the Least Squares method is known to produce biased results.

II. LINEAR REGRESSION

2.1 Assumptions and Notation

2.1.1 In General

Consider two sets of variables

$$x_1 \quad x_2 \cdots\cdots\cdots \quad x_n$$

$$y_1 \quad y_2 \cdots\cdots\cdots \quad y_n$$

where x_i are observed sample values of the independent variable, assumed to be accurately known, and the y_i are sample values of the dependent variates. In practice this means that x is really known with an error which is so small that it can be neglected. The expected value of y_i, denoted by Y_i, is defined

$$\mathcal{E}(y_i) = Y_i \quad ,$$

and a single linear relationship is assumed to hold between the value x_i and Y_i. That is to say

$$Y_i = \alpha + \beta x_i \quad ,$$

or $$y_i = \alpha + \beta x_i + e_i \quad ,$$

where $$\mathcal{E}(e_i \, e_j) = 0 \quad \text{for all } i \neq j$$

and $$e_i \frown N(0, \sigma^2) \quad i = 1, 2, \ldots n \quad .$$

The normality of e_i is only necessary for tests of hypotheses.

The problem to be solved is: given a set of paired observations

$$\begin{cases} x_1, x_2 \cdots x_n \\ y_1, y_2 \cdots y_n \end{cases} ,$$

to estimate, and to make tests of hypotheses concerning the parameters α, β, and σ^2.

2.1.2 Equally Spaced Data

Suppose n values of x_i be equally spaced, which, for the sake of simplicity, are assumed to be 1, 2, ... n . Therefore the problem now is to fit a straight line of the form:

$$Y_i = \alpha + \beta x_i \quad ,$$

where $\quad x_i = 1, 2, \ldots n$.

2.2 Method of Least Squares

Given a set of paired observations

$$\begin{cases} x_1, x_2 \cdots x_n \\ y_1, y_2 \cdots y_n \end{cases} .$$

If the following linear relationship holds, i.e.,

$$y_i = \alpha + \beta x_i + e_i \quad ,$$

it is desired to minimize, with respect to α and β

$$\sum_1^n e_i^2 = \sum_1^n (y_i - \alpha - \beta x_i)^2 \quad ,$$

resulting in the Least Squares estimators of α and β:

$$a = \bar{y} - b\bar{x}$$

$$b = \frac{\sum_1^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_1^n (x_i - \bar{x})^2} = \frac{\sum_1^n (x_i - \bar{x})\, y_i}{\sum_1^n (x_i - \bar{x})^2} \quad ,$$

and the variances

$$\mathrm{Var}\ (b) = \frac{\sigma^2}{[\sum_1^n (x_i - \bar{x})^2]}$$

and $$\mathrm{Var}\ (a) = \frac{\sigma^2 \sum_1^n x_i^2}{n \sum_1^n (x_i - \bar{x})^2} \quad .$$

We can use Student's t as a test criterion to test the null hypothesis $\beta = \beta_o$, $\alpha = \alpha_o$:

$$t = \frac{b - \beta_o}{\sqrt{\dfrac{S^2}{\sum_1^n (x_i - \bar{x})^2}}} \quad , \qquad \text{d.f. } (n - 2) \ ,$$

and

$$t = \frac{\alpha - \alpha_0}{\sqrt{\dfrac{S^2 \sum_1^n x_i^2}{n \sum_1^n (x_i - \bar{x})^2}}} \quad , \qquad \text{d.f. } (n - 2) ,$$

where $\qquad S^2 = \sum_1^n (y_i - \hat{Y}_i)^2/(n-2)$,

and $\qquad \hat{Y}_i = a + bx_i$.

Now we assume x_i to be equally spaced. Then the Least Squares estimators of α, β, and their variances will be:

$$a = \bar{y} - b\,(\frac{n+1}{2}) , \qquad (2.2.1)$$

$$\frac{6}{n-1}\,5 + \frac{12}{n(n^2-1)} \sum_{i=1}^n i y_i = b = \frac{12}{n(n^2-1)} \sum_1^n (i - \frac{n+1}{2})\, y_i , \qquad (2.2.2)$$

$$\text{Var}\ (a) = \frac{2(2n+1)\sigma^2}{n(n-1)} , \qquad (2.2.3)$$

and

$$\text{Var}\ (b) = \frac{12\ \sigma^2}{n(n^2-1)} . \qquad (2.2.4)$$

2.3 Method of Group Averages

Nair and Shrivastava [6] gave the Group Averages Method as follows. Under the same conditions as in previous sections, the following residual equations are obtained:

$$\begin{cases} y_1 - \alpha - \beta\, 1 = e_1 \\ \vdots \\ y_i - \alpha - \beta\, 2 = e_i \\ \vdots \\ y_n - \alpha - \beta\, n = e_n \end{cases} .$$

We divide the n residual equations into three groups, say, where the first p equations form the first group, the last q equations form the third group, and the middle $n - p - q = r$ equations form the second group. By equating to zero the sum of the residuals in the first group we get a single equation, and by equating to zero the last group we get another equation. Ignoring the middle group, we have two simultaneous equations in the two unknowns α and β:

$$p\,\alpha + (1 + 2 + \dots + p)\,\beta = \sum_1^p y_i$$

$$q\,\alpha + [(n - q + 1) + \dots + n]\,\beta = \sum_{n-q+1}^{n} y_i \quad .$$

Solving for α and β , we obtain estimators a' and b'

$$b' = \frac{2[\frac{1}{q} \sum_{n-q+1}^{n} y_i - \frac{1}{p} \sum_1^p y_i]}{2n - p - q}$$

$$= \frac{2[\bar{y}_3 - \bar{y}_1]}{2n - p - q} \quad , \qquad (2.3.1)$$

where $\bar{y}_i$ = mean of ith group,

and

$$a' = \frac{\frac{2n+1-q}{p} \sum_{1}^{p} y_i - \frac{p+1}{q} \sum_{n-q+1}^{n} y_i}{2n-p-q}$$

$$= \frac{(2n+1-q)\ \bar{y}_1 - (p+1)\ \bar{y}_3}{2n-p-q} \quad . \qquad (2.3.2)$$

Using standard formulae and the conditions of the model,

$$\text{Var}\ (b') = \frac{4}{(2n-p-q)^2}\ \text{Var}\ (\bar{y}_3 - \bar{y}_1)$$

$$= \frac{4}{(2n-p-q)^2}\ [\ \frac{\sigma^2}{q} + \frac{\sigma^2}{p}\ ]$$

$$= \frac{4(p+q)\ \sigma^2}{pq(2n-p-q)^2} \quad , \qquad (2.3.3)$$

and

$$\text{Var}\ (a') = \frac{(2n+1-q)^2\ \text{Var}\ (\bar{y}_1) + (p+1)^2\ \text{Var}\ (\bar{y}_3)}{(2n-p-q)^2}$$

$$= \sigma^2\ [\ \frac{(2n+1-q)^2}{p(2n-p-q)^2} + \frac{(p+1)^2}{q(2n-p-q)^2}\ ]$$

$$= \sigma^2\ [\ \frac{1}{p}(1+\frac{p+1}{2n-p-q})^2 + \frac{1}{q}(\frac{p+1}{2n-p-q})^2]\ . \qquad (2.3.4)$$

We can use Student's t as a test criterion to test null hypotheses $\beta = \beta_o$ and $\alpha = \alpha_o$:

$$t_{\alpha/2} = \frac{b' - \beta_o}{\dfrac{4(p+q)S^2}{pq(2n-p-q)^2}} \quad , \qquad d.f. = (n-2)$$

and

$$t_{\alpha/2} = \frac{a' - \alpha_o}{S^2[\frac{1}{p}(1+\frac{p+1}{2n-p-q})^2+\frac{1}{q}(\frac{p+1}{2n-p-q})^2]} \quad , \ d.f. = (n-2)$$

where $$S^2 = \frac{\sum_1^n (y_i - \hat{Y}_i)^2}{n-2} \quad ,$$

and

$$\hat{Y}_i = a' + b' x_i \quad .$$

2.3.1 Optimum Division for the Residual Equation

It is of interest to determine best values for p and q, so as to maximize the efficiency of b', with respect to b where b is the estimator by Least Squares method, which is known to be the most efficient estimator of β .

Defining the efficiencies as the inverse ratio of variances, we have.

$$E(b') = \frac{Var\ (b)}{Var\ (b')} = \frac{3pq(2n-p-q)^2}{n(n^2-1)(p+q)} \qquad (2.3.1.1)$$

$$E(a') = \frac{Var\ (a)}{Var\ (a')} = \frac{2(2n+1)/n(n+1)}{\frac{1}{p}(1+\frac{p+1}{2n-p-q})^2+\frac{1}{q}(\frac{p+1}{2n-p-q})^2} \qquad (2.3.1.2)$$

Differentiating the logarithmic function of E(b') with respect to p and q we obtain

$$\begin{cases} \frac{1}{p} - \frac{2}{2n-p-q} - \frac{1}{p+q} = 0 \\ \frac{1}{q} - \frac{2}{2n-p-q} - \frac{1}{p+q} = 0 \end{cases} .$$

Solving for p and q we have

$$p = q = \frac{n}{3} .$$

This result was also given by Nair and Shrivastava [6].

Thus the most efficient method of estimating β by the methods of Group Averages is to divide the whole set of residual equations into three equal parts thus ignoring the middle set. Substituting $p = q = \frac{n}{3}$ into the efficiency equations (2.3.1.1), and (2.3.1.2), we have

$$E(b') = \frac{\text{Var (b)}}{\text{Var (b')}} = \frac{8}{9} \cdot \frac{n^2}{n^2-1} > \frac{8}{9} , \qquad (2.3.1.3)$$

$$E(a') = \frac{\text{Var (a)}}{\text{Var (a')}} = \frac{16n^2(2n+1)}{3(n-1)(13n^2+18n+9)}$$

$$= \frac{32}{39} \cdot \frac{n^2(n+\frac{1}{2})}{(n^2+\frac{18}{13}n+\frac{9}{13})(n-1)}$$

$$= \frac{32}{39} \cdot \frac{n^3+\frac{1}{2}n^2}{n^3+\frac{5}{13}n^2-\frac{9}{13}n-\frac{9}{13}} > \frac{32}{39} . \qquad (2.3.1.4)$$

If n is not a multiple of 3, two cases are possible, namely $n = 3m \pm 1$; in either case the optimum value of p and q is m .

2.3.2 Efficiency of b' as A Function of The Number of Omitted Residual Equations

The efficiency of b' has been derived as:

$$E(b') = \frac{Var\ (b)}{Var\ (b')} = \frac{3pq(2n-p-q)^2}{n(n^2-1)(p+q)} \quad .$$

In general, if we take an equal, but arbitrary, number of residual equations, i.e. $p = q$, we have

$$E(b') = \frac{6p(n-p)^2}{n(n^2-1)} \quad .$$

In this case the number of omitted residual equations, $r = n - p - q = n - 2p$, and we have

$$E(b') = \frac{3(n^3+n^2r-nr^2-r^3)}{4n(n^2-1)} \quad . \qquad (2.3.2.1)$$

Setting $\frac{d\ E(b')}{d\ r}$ equal to zero, we have the previously obtained result:

$$\frac{d\ E(b')}{d\ r} = n^2 - 2nr - 3r^2 = 0$$

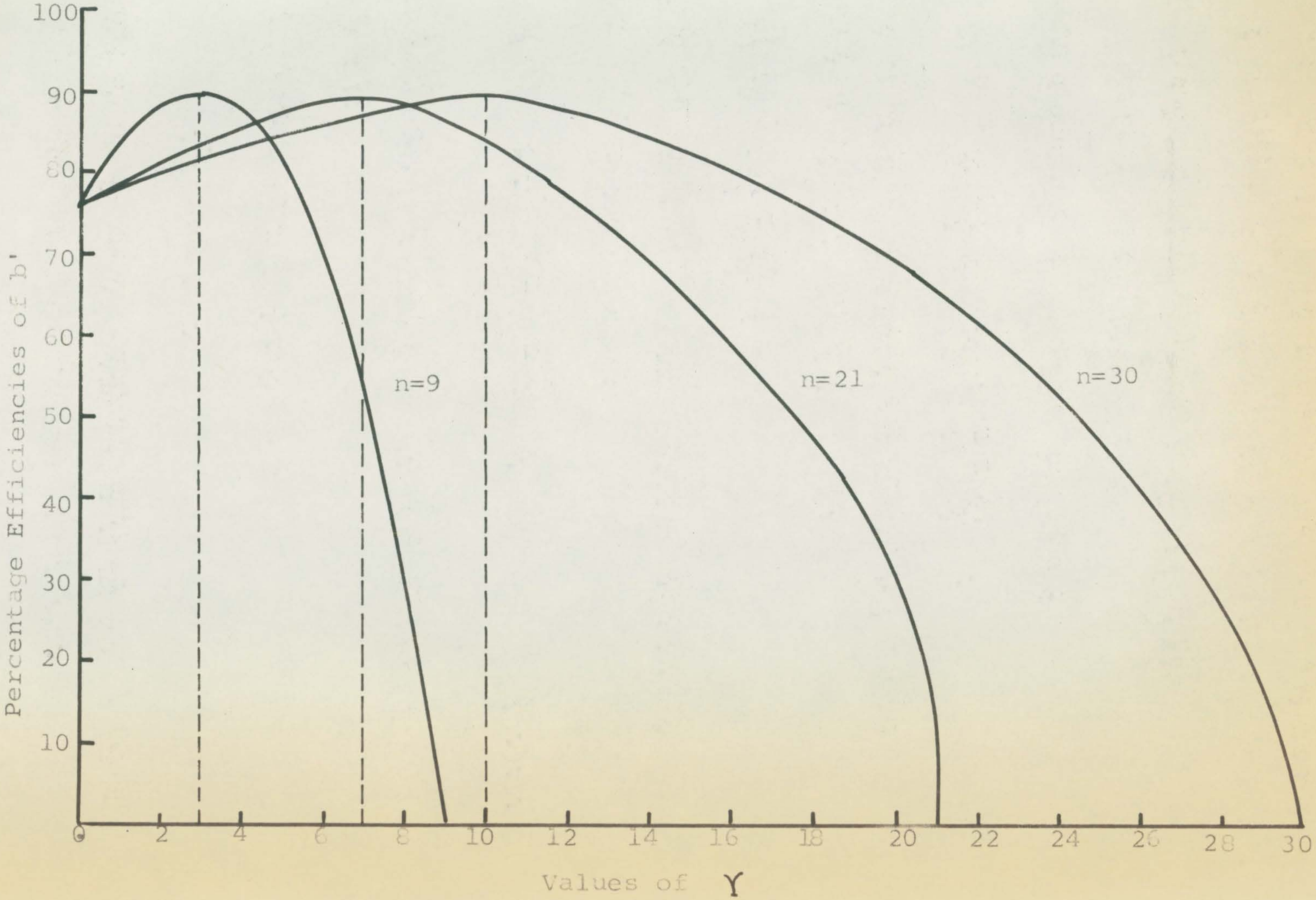
$$\therefore \quad r = \frac{n}{3} \quad .$$

The efficiencies of the Group Averages estimator as a function of r for selected values of n are in Chart 1.

Chart 1

Percentage Efficiencies of b' As A Function of

γ For n=9, 21, 30



2.4 A Numerical Example

Snedecor [8] gave the following numerical example. The heights of four soybean plants were measured a week after emergence, showing a mean height of 5 centimeters. Four other plants were similarly measured two weeks after emergence, and so on to the seventh week; see Table 1.

Table 1

Data for Example

Age in Weeks	1	2	3	4	5	6	7
Height in centimeters	5	13	16	23	33	38	40

Using the Least Squares method for estimating parameters of a regression of height on age, we have:

$$\bar{x} = 4 \quad ,$$

$$\bar{y} = 24 \quad ,$$

$$\sum_{1}^{7} (x_i - \bar{x})^2 = 28 \quad ,$$

$$\sum_{1}^{7} (y_i - \bar{y}) = 1{,}080 \quad ,$$

$$\sum_{1}^{7} (y_i - \bar{y})(x_i - \bar{x}) = 172 \quad ,$$

$$b = \frac{172}{28} = 6.143 \quad ,$$

$$a = \bar{y} - b\bar{x} = -0.572 \quad .$$

The estimated regression line equation:

$$Y_i - 6.143x_i - 0.572 \quad ,$$

and

$$S^2 = \frac{\sum_{1}^{7} (y_i - Y_i)^2}{5}$$

$$= \frac{1}{5} \left[\sum_{1}^{7} (y_i - \bar{y})^2 - \frac{\left(\sum_{1}^{7} (x_i - \bar{x})(y_i - \bar{y})\right)^2}{\Sigma (x_i - \bar{x})^2} \right]$$

$$= \frac{23.92}{5} = 4.784 \quad .$$

The test for H_o: $\beta = 0$, H_A: $\beta > 0$,

$$t_{.05}(5) = \frac{6.143}{\sqrt{\frac{4.784}{28}}} \qquad \text{d.f.} = 5$$

$$= 14.84^* \quad ,$$

and we reject the null hypothesis.

Using the method of Group Averages, we obtain the estimates:

$$b' = \frac{2(\bar{y}_3 - \bar{y}_1)}{\frac{n}{3}} = 6.42 \quad ,$$

$$a' = \frac{(5n + 3)\bar{y}_1 - (n + 3)\bar{y}_3}{4n} = -1.71 \quad ,$$

and the estimated regression line equation is

$$Y_i = 6.42x_i - 1.71 \quad .$$

The test for H_o: $\beta = 0$, H_A: $\beta > 0$,

$$t_{.05}(5) = \frac{6.42}{\sqrt{\frac{18}{n^2} S^2}}$$

$$= \frac{6.42}{1.05} = 6.11^* \quad ,$$

and we reject the null hypothesis. Note that the estimates and tests are quite comparable for both methods.

III. QUADRATIC POLYNOMIAL REGRESSION

As before we assume x_i to be equally spaced, and for the sake of simplicity, the x_i's will be assumed to be 1, 2, ... n . We now desire to estimate a quadratic regression of the form:

$$Y_i = \alpha + \beta x_i + \gamma x_i^2 \quad .$$

The estimators of α, β, and γ will be obtained by the method of Least Squares and the method of Group Averages. The optimum division of residual equations for the Group Averages method will also be obtained.

3.1 Method of Least Squares

Estimates of α, β, and γ are obtained by minimizing the following function:

$$\sum_1^n e_i^2 = \sum_1^n (y_i - \alpha - \beta i - \gamma i^2)^2 \quad ,$$

which results in the normal equations:

$$\begin{cases} n\,\alpha + \beta \sum_1^n i + \gamma \sum_1^n i^2 = \sum_1^n y_i \\ \alpha \sum_1^n i + \beta \sum_1^n i^2 + \gamma \sum_1^n i^3 = \sum_1 i\, y_i \\ \alpha \sum_1^n i^2 + \beta \sum_1^n i^3 + \gamma \sum_1^n i^4 = \sum_1^n i^2 y_i \end{cases} \quad .$$

Using the well known relationship

$$\sum_1^n i = \frac{1}{2} n(n+1)$$

$$\sum_1^n i^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_1^n i^3 = \{\frac{1}{2} n(n+1)\}^2$$

$$\sum_1^n i^4 = \frac{1}{30} n(n+1)(6n^3+9n^2+n-1) \quad ,$$

we solve the simultaneous equations for α, β, and γ, and obtain the Least Squares estimators a, b, c:

$$a = \frac{3[H_1(3n^2+3n+2)-6H_2(2n+1)+10H_3]}{n(n-1)(n-2)} \quad ,$$

$$b = \frac{6[3H_1(n+1)(n+2)(-2n-1)+2H_2(16n^2+30n+11)-30(n+1)H_3]}{n(n^2-1)(n^2-4)} \quad ,$$

and $$c = \frac{30[H_1(n+1)(n+2)-6H_2(n+1)+6H_3]}{n(n^2-1)(n^2-4)} \quad ,$$

where $H_1 = \sum_1^n y_i$, $H_2 = \sum_1^n i\ y_i$, $H_3 = \sum_1^n i^2\ y_i$.

Further, we have

$$\text{Var (a)} = \frac{3(3n^2+3n+2)\sigma^2}{n(n-1)(n-2)} \qquad (3.1.1)$$

$$\text{Var (b)} = \frac{12(16n^2+30n+11)\sigma^2}{n(n^2-1)(n^2-4)} \tag{3.1.2}$$

and
$$\text{Var (c)} = \frac{180\,\sigma^2}{n(n^2-1)(n^2-4)} \quad . \tag{3.1.3}$$

3.2 Method of Group Averages

Nair and Shrivastava [6] give the method of Group Averages for a quadratic polynomial as follows. Using the model

$$y_i = \alpha + \beta i + \gamma i^2 + e_i \quad ,$$

we have the following residual equations:

$$\begin{cases} y_1 - \alpha - \beta - \gamma = e_1 \\ y_2 - \alpha - 2\beta - 2^2\gamma = e_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ y_n - \alpha - n\beta - n^2\gamma = e_n \end{cases} .$$

Here we require three simultaneous equations to solve for a', b', c', the estimators of α, β, γ. From our experience of fitting a straight line, we will assume that these three equations are based on equal numbers of residual equations. We take the first p equations, middle p equations, last p equations to form the first, second and last group. Again using our experience from the linear case, we will assume that some equations between these groups will not be used; thus w and u equations, say, will be omitted between the first and second, and second and third sets of p, respectively. Thus

$$p + w + p + u + p = n \quad .$$

From the first, second and last groups, we get three simultaneous equations:

$$\begin{cases} \ell_1 \alpha + k_1 \beta + m_1 \gamma = S_1 \\ \ell_2 \alpha + k_2 \beta + m_2 \gamma = S_2 \\ \ell_3 \alpha + k_3 \beta + m_3 \gamma = S_3 \end{cases}$$

where, under assumption of equally spaced x's,

$$\ell_1 = \ell_2 = \ell_3 = p$$

$$k_1 = \sum_1^p i = \frac{1}{2}\, p(p+1)$$

$$k_2 = \sum_{p+w+1}^{2p+w} i = \frac{1}{2}\, p(3p+2w+1)$$

$$k_3 = \sum_{n-p+1}^{n} i = \frac{1}{2}\, p(2n-p+1)$$

$$m_1 = \sum_1^p i^2 = \frac{1}{6}\, p(p+1)(2p+1)$$

$$m_2 = \sum_{p+w+1}^{2p+w} i^2 = \frac{1}{6}\, p\left[\begin{matrix} (3p+2w+1)(4p+2w+1) \\ +2(p+w)(p+w+1) \end{matrix} \right]$$

$$m_3 = \sum_{n-p+1}^{n} i^2 = \frac{1}{6}\, p[(2n-p+1)(2n+1)+2(n-p)(n-p+1)]$$

$$S_1 = \sum_{1}^{p} y_i$$

$$S_2 = \sum_{p+w+1}^{2p+w} y_i$$

$$S_3 = \sum_{n-p+1}^{n} y_i \quad .$$

Note that u does not appear in these equations, since, given n, p, and w uniquely determine u.

Solving the simultaneous equations for a', b', c' the estimators of α, β, γ. We obtain:

$$a' = \frac{S_1V_1+S_2V_2+S_3V_3}{12p(n-2p-w)(n-p)(p+w)}$$

$$b' = \frac{S_1Z_1+S_2Z_2+S_3Z_3}{p(n-2p-w)(n-p)(p+w)}$$

$$c' = \frac{S_1(n-2p-w)+S_2(n-p)+S_3(p+w)}{p(n-2p-w)(n-p)(p+w)}$$

where

$$\begin{aligned} V_1 = {} & 18n^2p - 46np^2 + 12n^2w - 48pwn + 22p^2w \\ & + 6n^2 - 6np - 12p^2 + 4n - 4w - 18wp \\ & + 20p^3 - 12nw^2 - 8p + 6pw^2 - 6w^2 \end{aligned}$$

$$V_2 = 10np^2 + 6np - 4n - 4p^3 + 4p - 6n^2 - 6n^2p$$

$$V_3 = 8p^3 + 12p^2 + 4p + 18pw + 4w + 6w^2 + 14p^2w + 6pw^2$$

$$Z_1 = 2p^2 + 2p + 3nw + w + w^2 - n^2 + np - n$$

$$Z_2 = n^2 - np - p + n = (n+1)(n-p)$$

$$Z_3 = -2p^2 - p - 3pw - w - w^2 = -(p+w)(2p+w+1) \quad .$$

3.3 Optimum Division of the Residual Equations

It is of interest to find the values for p, w, u, such that the efficiencies of c' and b' will be maximized relative to the Least Squares estimators. First we obtain:

$$\text{Var }(c') = \frac{[(n-2p-w)^2+(n-p)^2+(p+w)^2]\sigma^2}{p(n-2p-w)^2(n-p)^2(p+w)^2} \quad ,$$

and

$$\text{Var }(b') = \frac{\sigma^2[Z_1{}^2+Z_2{}^2+Z_3{}^2]}{p(n-2p-w)^2(n-p)^2(p+w)^2} \quad .$$

Hence the efficiencies of c', b' as compared to Least Squares estimators c and b are:

$$E(c') = \frac{\text{Var }(c)}{\text{Var }(c')} \tag{3.3.1}$$

$$= \frac{180}{n(n^2-1)(n^2-4)} \cdot \frac{p(n-2p-w)^2(n-p)^2(p+w)^2}{(n-2p-w)^2+(n-p)^2+(p+w)^2} \quad ,$$

and

$$E(b') = \frac{\text{Var }(b)}{\text{Var }(b')} \tag{3.3.2}$$

$$= \frac{12(16n^2+30n+11)}{n(n^2-1)(n^2-4)} \frac{p(n-2p-w)^2(n-p)^2(p+w)^2}{(Z_1{}^2+Z_2{}^2+Z_3{}^2)} ,$$

where $Z_1{}^2+Z_2{}^2+Z_3{}^2$

$$= (2p^2+2p+3nw+w-n^2+np-n)^2 + (n+1)^2(n-p)^2 + (p+w)^2(2p+w+1)^2 .$$

Since we are fitting a quadratic regression curve, we are more interested in $E(c')$ than in $E(b')$. Since it is not likely that both efficiencies will be maximized by one division pattern, we will obtain those values for p, w, u, such that $E(c')$ will be maximized. If we maximize the numerator of (3.3.1) with respect to w we have:

$$\frac{1}{p+w} - \frac{1}{n-2p-w} = 0$$

$$w = \frac{1}{2}(n-3p) . \tag{3.3.3}$$

Fortunately (3.3.3) simultaneously minimizes the denominator with respect to w. Therefore (3.3.3) is the condition to maximize $E(c')$. Thus

$$w = u = \frac{1}{2}(n-3p) ,$$

since $n = 3p + w + u$. With this value of w, (3.3.1) reduces to

$$E(c') = \frac{15}{2} \cdot \frac{p(n-p)^4}{n(n^2-1)(n^2-4)}$$

which attains maximum value when p = n/5. Thus for maximum efficiency of c' we should choose our groups in such a way that

$$p = w = u = \frac{n}{5} \quad .$$

In other words, we should divide the set of residual equations into five equal parts and then use the first, third, and fifth parts for the residual equations, ignoring the second and fourth sets. This result is also given by Nair and Shrivastava. [6]

This division of the residual equations does not maximize the efficiency of b'. The efficiencies of a' and b', using this division are, however, quite satisfactory, as will be shown. Let $w = p = \frac{n}{5}$, then a', b', c', and their efficiencies are:

$$a' = \frac{5}{48n^3} [S_1(67n^2+105n+50)-S_2(26n^2+150n+100)+S_3(7n^2+45+50)] ,$$

$$b' = \frac{25[(10n+10)S_2 - (3n+5)S_3 - (7n+5)S_1]}{8n^3} ,$$

and

$$c' = \frac{125[S_3 + S_1 + 2S_2]}{8n^3} ;$$

$$E(a') = \frac{\text{Var }(a)}{\text{Var }(a')} = \frac{6912n^4(3n^2+3n+2)}{5(n-1)(n-2)\,\phi_1} \quad ,$$

where $\phi_1 = 5214n^4+22500n^3+48150n^2+45000n+15000$,

$$E(b') = \frac{\text{Var }(b)}{\text{Var }(b')} = \frac{384}{125}\,\frac{n^4(16n^3+30n+11)}{(n^2-1)(n^2-4)\,\phi_2},$$

where $\phi_2 = 79n^2+150n+75$,

and

$$E(c') = \frac{\text{Var }(c)}{\text{Var }(c')} = \frac{384}{625}\,\frac{n^4}{(n^2-1)(n^2-4)} \quad .$$

The efficiencies of a', b', c', are presented in Table 2 for a selected set of values of n.

Table 2

Percentage Efficiencies of a', b', c',
For Different Sample Sizes

n	E(a')	E(b')	E(c')
5	88.14	76.33	76.19
10	79.76	65.19	64.65
15	79.00	63.47	62.83
20	78.90	62.90	62.22
25	78.92	62.64	61.93

IV. FUNCTIONAL RELATIONSHIP SITUATION

4.1 Assumptions and Notation

We will now consider the problem of estimating a linear relationship between two variables, X and Y, when both are subject to errors of observation. The problem is no longer to estimate a regression coefficient in the usual sense; it is rather to estimate the constants in a functional relation:

$$Y_i = \alpha + \beta X_i ;$$

however, Y_i and X_i cannot be directly observed because of errors of observation. Instead of Y_i and X_i, values denoted y_i and x_i are observed where

$$y_i = Y_i + e_i \quad ,$$

$$x_i = X_i + d_i \quad ,$$

$$e_i \frown \text{NID } (0, \sigma_e^2) \; , \quad i = 1, \dots n \; ,$$

$$d_j \frown \text{NID } (0, \sigma_d^2) \; , \quad j = 1, \dots n \; ,$$

and $$\mathcal{E}(e_i \, d_j) = 0 \quad \text{for all } i, j \; .$$

We shall use the Maximum Likelihood method and the Group Averages method to estimate α, β, σ_e^2, σ_d^2, and compare, under some restrictions, the estimators obtained by these two methods.

4.2 Maximum Likelihood Method, $\lambda = \sigma_d^2/\sigma_e^2$ Known

Lindley [5] gave the maximum likelihood estimators for the functional relationship situation as follows under the assumption $\lambda = \frac{\sigma_d^2}{\sigma_e^2}$ is known:

$$\hat{\beta} = b_M = \pm[U^2 + \tfrac{1}{\lambda}]^{\frac{1}{2}} + U \tag{4.2.1}$$

$$\text{where} \quad U = \frac{\sum_1^n (y_i-\bar{y})^2 - \frac{1}{\lambda}\sum_1^n (x_i-\bar{x})^2}{2\sum_1^n (x_i-\bar{x})(y_i-\bar{y})} ,$$

$$\hat{\alpha} = \bar{y} - \hat{\beta}\,\bar{x} , \tag{4.2.2}$$

and

$$\sigma_e^2 = \frac{1}{n-2}[\sum_1^n (y_i-\bar{y})^2 - \beta\sum_1^n (x_i-\bar{x})(y_i-\bar{y})] , \tag{4.2.3}$$

where the sign used for the first term in $\hat{\beta}$ is that which will maximize the likelihood function.

Lindley pointed out that it is not feasible to obtain reasonable maximum likelihood estimators without additional information; a knowledge of the ratio $\lambda = \sigma_d^2/\sigma_e^2$ seemed to be the most convenient form of additional information to use.

4.3 Method of Group Averages

Wald [9] suggested that when the number of observations, n, is even one can order the x_i's such that $x_i \leq x_{i+1}$, divide the observations into two equal groups, and estimate the parameters of the functional relation

$$Y_i = \alpha + \beta X_i$$

as follows:

$$\hat{\beta} = b_w = \frac{\bar{y}_2 - \bar{y}_1}{\bar{x}_2 - \bar{x}_1} \tag{4.3.1}$$

$$\hat{\alpha} = a_w = \bar{y} - b_w \bar{x} \tag{4.3.2}$$

$$\hat{\sigma}_d^2 = (S_x^2 - \frac{S_{xy}}{b_w}) \frac{n}{n-2} \tag{4.3.3}$$

$$\hat{\sigma}_e^2 = (S_y^2 - b_w S_{xy}) \frac{n}{n-2} \tag{4.3.4}$$

where

$$\bar{y} = \sum_1^n y_i/n \;, \quad \bar{y}_1 = \sum_1^{\frac{n}{2}} y_i/\tfrac{n}{2} \;, \quad \bar{y}_2 = \sum_{\frac{n}{2}+1}^{n} y_i/\tfrac{n}{2} \;,$$

$$\bar{x} = \sum_1^n x_i/n \;, \quad \bar{x}_1 = \sum_1^{\frac{n}{2}} x_i/\tfrac{n}{2} \;, \quad \bar{x}_2 = \sum_{\frac{n}{2}+1}^{n} x_i/\tfrac{n}{2} \;,$$

$$S_x^2 = \frac{1}{n}\left[\sum_1^{\frac{n}{2}}(x_i - \bar{x}_1)^2 + \sum_{\frac{n}{2}+1}^{n}(x_i - \bar{x}_2)^2\right]$$

$$S_y^2 = \frac{1}{n}\left[\sum_1^{\frac{n}{2}}(y_i - \bar{y}_1)^2 + \sum_{\frac{n}{2}+1}^{n}(y_i - \bar{y}_2)^2\right]$$

and

$$S_{xy} = \frac{\sum_{1}^{\frac{n}{2}}(x_i - \bar{x}_1)(y_i - \bar{y}_1) + \sum_{\frac{n}{2}+1}^{n}(x_i - \bar{x}_2)(y_i - \bar{y}_2)}{n} .$$

All these estimators are consistent provided

$$\Pr \{ |d_i| \geq \tfrac{1}{2} c \} \text{ is negligible}$$

$$\text{where } c = \min. |X_i - X_{i+1}| .$$

That is to say the probabilities that the d_i (errors of observations on X) have values exceeding $\pm \frac{1}{2}$ c may be neglected. This assumption simply insures that the ordering of the x_i is identical with the ordering according to the X_i .

Bartlett [1] provided a modification of Wald's method by using three groups instead of two groups to estimate α, β, σ_e^2, and σ_d^2 . The resulting estimators are:

$$\hat{\beta} = b_B = \frac{\bar{y}_3 - \bar{y}_1}{\bar{x}_3 - \bar{x}_1} \tag{4.3.5}$$

$$\hat{\alpha} = a_B = \bar{y} - b_B \bar{x} \tag{4.3.6}$$

$$\hat{\sigma}_e^2 = (S_y^2 - b_B S_{xy}) \frac{n}{n-3} \tag{4.3.7}$$

$$\hat{\sigma}_d^2 = (S_x^2 - S_{xy}/b_B) \frac{n}{n-3} \tag{4.3.8}$$

where

$$\bar{y}_3 = \sum_{\frac{2}{3}n+1}^{n} y_i/\frac{n}{3} \quad ,$$

$$\bar{y}_2 = \sum_{\frac{n}{3}+1}^{\frac{2n}{3}} y_i/\frac{n}{3} \quad ,$$

$$\bar{y}_1 = \sum_{1}^{\frac{n}{3}} y_i/\frac{n}{3} \quad ,$$

$$\bar{x}_3 = \sum_{\frac{2n}{3}+1}^{n} x_i/\frac{n}{3} \quad ,$$

$\bar{x}_2$ and $\bar{x}_1$ are defined as they are for y ,

$$S_x^2 = \frac{1}{n} \left[\sum_{1}^{\frac{n}{3}} (x_i - \bar{x}_1)^2 + \sum_{\frac{n}{3}+1}^{\frac{2n}{3}} (x_i - \bar{x}_2)^2 + \sum_{\frac{2n}{3}+1}^{n} (x_i - \bar{x}_3)^2 \right] \quad ,$$

S_y^2 and S_{xy} are defined analogously.

Nair and Banerjee [7] used a sampling experiment to show that Bartlett's method gave a more efficient estimator of β. They assumed the functional relationship to be

$$Y_i = 1 + X_i \quad ,$$

i.e. $\alpha = \beta = 1$.

The X_i were given integral values from 1 to 30, thus the corresponding values of Y_i were 2 to 31. However the observed variables x_i and y_i were of the form

$$x_i = X_i + d_i \quad ,$$

$$y_i = Y_i + e_i \quad , \text{ where}$$

$$d_i \frown \text{NID}\ (0,\ 0.01)$$

$$e_i \frown \text{NID}\ (0,\ 0.01) \quad .$$

Using random normal deviates, they obtained the required samples of size 30, with x_i, y_i both subject to uncorrelated errors. Since the variances were quite small the ordering according to x_i was, for all practical purposes, identical to the ordering according to the X_i.

One hundred samples of 30 paired observations were obtained, and estimates of α and β were obtained by using both Wald's and Bartlett's methods. The results of the samples are presented in Table 3.

Table 3

Results of Sampling Experiment

Method of	a		b	
Estimating	Mean	S. D.	Mean	S. D.
Wald's	0.998583	0.0104	1.010063	0.0991
Bartlett's	0.999793	0.0035	1.003634	0.0767

The above table indicates that while both the methods give consistent estimates for α and β, Bartlett's method gives the more efficient estimates. Therefore Bartlett's method will be used in succeeding sections.

4.4 Comparison of the Relative Efficiency of Two Estimators of β

For the comparison of the efficiency of the two estimators, b_B (group average) and b_M (maximum likelihood), we need to find their variances.

4.4.1 Variance of b_B

Dorff and Gurland [3] gave the asymptotic variance of b_B as follows, using the notation of Section 4.1:

$$\text{Var}\ (b_B) = \frac{\frac{2n}{3}\ (\sigma_e^2 + \beta^2\ \sigma_d^2)}{\left(\sum\limits_{\frac{2n}{3}+1}^{n} X_i - \sum\limits_{1}^{\frac{n}{3}} X_i\right)^2} \quad .$$

Since we shall compare the Var (b_B) with the Var (b_M) under the assumption $\lambda\sigma_e^2 = \sigma_d^2$, the variance formula to be used is:

$$\text{Var}\ (b_B) = \frac{\frac{2n}{3}\ \sigma_e^2(1 + \lambda\beta^2)}{\left(\sum\limits_{\frac{2}{3}n+1}^{n} X_i - \sum\limits_{1}^{\frac{n}{3}} X_i\right)^2} \quad . \qquad (4.4.1.1)$$

4.4.2 Variance of b_M

The general procedure for obtaining an asymptotic variance of a maximum likelihood estimator is as follows:

1. Take the partial derivatives of the logarithms of the likelihood function with respect to all parameters, i.e., $\frac{\partial \log L}{\partial \theta_i}$, $i = 1 \ldots k$, where k is the number of parameters.

2. Take the second derivatives for all equations in (1.) with respect to all parameters again, i.e.,

$$\frac{\partial^2 \log L}{\partial\theta_i \, \partial\theta_j} \quad \text{for } i, j = 1, 2, \ldots k .$$

3. Take the expected value of each second partial derivative, i.e.

$$\mathcal{E}\left(\frac{\partial^2 \log L}{\partial\theta_i \, \partial\theta_j}\right) .$$

4. Form the "information" matrix whose i, jth element is

$$-\mathcal{E}\left(\frac{\partial^2 \log L}{\partial\theta_i \, \partial\theta_j}\right) \quad i, j = 1 \ldots k .$$

5. The diagonal elements of the inverse of this matrix will be the asymptotic variances, and off-diagonal elements will be the asymptotic covariances of maximum likelihood estimators of $\theta_1 \, \theta_2 \ldots \theta_k$.

We shall follow this procedure to obtain the asymptotic variance of b_M. The likelihood function is:

$$L(e_1, e_2, \ldots, e_n, d_1, d_2, \ldots, d_n)$$

$$= \frac{1}{(\sigma_e\ \sigma_d\ 2\pi)^n} \exp\{ -\frac{1}{2} \sum_1^n (\frac{e_i^2}{\sigma_e^2} + \frac{d_i^2}{\sigma_d^2}) \} \quad ,$$

and since $\quad e_i = y_i - Y_i \quad ,$

and $\quad d_i = x_i - X_i \quad ,$

$$L = \frac{1}{(\sigma_e\ \sigma_d\ 2\pi)^n} \exp\{ -\frac{1}{2} \sum_1^n [\frac{(y_i - Y_i)^2}{\sigma_e^2} + \frac{(x_i - X_i)^2}{\sigma_d^2}]\} \quad .$$

Using the relationship:

$$\lambda = \sigma_d^2/\sigma_e^2 \quad ,$$

$$Y_i = \alpha + \beta\ X_i \quad ,$$

and taking the logarithm, we have

$$\log L = -\frac{n}{2} \log \sigma_e^2 - \frac{n}{2} \log \lambda\ \sigma_e^2 - n \log 2\pi$$

$$- \frac{1}{2} [\frac{\sum_1^n (y_i - \alpha - \beta\ X_i)^2}{\sigma_e^2} + \frac{\sum_1^n (x_i - x_i)^2}{\lambda\ \sigma_e^2}] \quad .$$

The first derivatives have been derived in Graybill [4], and are as follows:

$$\frac{\partial \log L}{\partial X_i} = \frac{(y_t - \alpha - \beta\ X_i)\beta}{\sigma_e^2} + \frac{x_i - X_i}{\lambda\ \sigma_e^2} \quad , \quad i = 1, 2, \ldots n,$$

$$\frac{\partial \log L}{\partial \sigma_e^2} = -\frac{n}{\sigma_e^2} + \frac{1}{2\sigma_e^4}\left[\sum_1^n (y_i - \alpha - \beta X_i)^2 + \frac{\sum_1^n (x_i - X_i)^2}{\lambda}\right],$$

$$\frac{\partial \log L}{\partial \alpha} = \frac{\sum_1^n (y_i - \alpha - \beta X_i)X_i}{\sigma_e^2} .$$

Using these, we take second partials and their expected values. Reversing the sign and inserting into the information matrix denoted by A:

$$A = \begin{array}{cc}
\begin{array}{ccccccc} X_1 & X_2 & \cdots & X_n & \sigma_e^2 & \beta & \alpha \end{array} & \\
\left(\begin{array}{ccccc|cc}
\frac{1+\gamma\beta^2}{\lambda\,\sigma_e^2} & 0 & \cdots & 0 & 0 & \frac{\beta X_1}{\sigma_e^2} & \frac{\beta}{\sigma_e^2} \\
0 & \frac{1+\lambda\beta^2}{\lambda\,\sigma_e^2} & & \vdots & \vdots & \frac{\beta X_2}{\sigma_e^2} & \frac{\beta}{\sigma_e^2} \\
\vdots & & \ddots & & & \vdots & \vdots \\
0 & \cdots & & \frac{1+\lambda\beta^2}{\lambda\,\sigma_e^2} & 0 & \frac{\beta X_n}{\sigma_e^2} & \frac{\beta}{\sigma_e^2} \\
0 & \cdots & & & \frac{n}{\sigma_e^4} & 0 & 0 \\
\hline
\frac{\beta X_1}{\sigma_e^2} & \frac{\beta X_2}{\sigma_e^2} & \cdots & \frac{\beta X_n}{\sigma_e^2} & 0 & \sum_1^n X_i^2/\sigma_e^2 & \sum_1^n X_i/\sigma_e^2 \\
\frac{\beta}{\sigma_e^2} & \frac{\beta}{\sigma_e^2} & \cdots & \frac{\beta}{\sigma_e^2} & 0 & \frac{\sum_1^n X_i}{\sigma_e^2} & \frac{n}{\sigma_e^2}
\end{array}\right) &
\begin{array}{c} X_1 \\ X_2 \\ \vdots \\ X_n \\ \sigma_e^2 \\ \beta \\ \alpha \end{array}
\end{array} .$$

We partition the matrix A:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}' & A_{22} \end{bmatrix} ,$$

and denote the inverse of A as

$$A^{-1} = \begin{bmatrix} A^{11} & A^{12} \\ (A^{12})' & A^{22} \end{bmatrix} .$$

Since we are only interested in the variance of b_M, which is the first diagonal element of A^{22}, we need to obtain only A^{22}. Using the relationship

$$A^{22} = [A_{22} - A_{12}' A_{11}^{-1} A_{12}]^{-1}$$

we have

$$[A^{22}]^{-1} = \frac{1}{\sigma_e^2} \begin{bmatrix} \sum_1^n X_i^2 & \sum_1^n X_i \\ \sum_1^n X_i & n \end{bmatrix} - \frac{\beta}{\sigma_e^2} \begin{bmatrix} X_1 & \cdots & X_n & 0 \\ 1 & \cdots & 1 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} \frac{\lambda\ \sigma_e^2}{1+\lambda\beta^2} & 0 & \cdots & 0 & 0 \\ 0 & \cdot & & \vdots & \vdots \\ \vdots & & \cdot & \frac{\lambda\ \sigma_e^2}{1+\lambda\beta^2} & \vdots \\ 0 & \cdots & & & \\ 0 & \cdots & & 0 & \frac{\sigma_e^4}{n} \end{bmatrix} \begin{bmatrix} X_1 & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ X_n & 1 \\ 0 & 0 \end{bmatrix} \frac{\beta}{\sigma_e^2}$$

$$[A^{22}]^{-1} = \frac{1}{\sigma_e^2(1+\lambda\beta^2)} \begin{bmatrix} \sum_1^n X_i^2 & \sum_1^n X_i \\ \sum_1^n X_i & n \end{bmatrix}$$

$$A^{22} = \frac{\sigma_e^2(1+\lambda\beta^2)}{n\sum_1^n X_i^2 - (\sum_1^n X_i)^2} \begin{bmatrix} n & -\sum_1^n X_i \\ -\sum_1^n X_i & \sum_1^n X_i^2 \end{bmatrix} .$$

Hence

$$\text{Var}\ (b_M) = \frac{n\,\sigma_e^2\ (1+\lambda\beta^2)}{n\ (\sum_1^n X_i^2 - (\sum_1^n X_i)^2/n)}$$

$$= \frac{\sigma_e^2\ (1+\lambda\beta^2)}{\sum_1^n (X_i - \bar{X})^2} \qquad (4.4.2.1)$$

4.4.3 The Relative Efficiency of b_B and b_M

The efficiency of Group Averages estimator b_B compared to the maximum likelihood estimator b_M is:

$$E(b_B) = \frac{\text{Var}\ (b_M)}{\text{Var}\ (b_B)}$$

$$= \frac{(\sum_{\frac{2}{3}n+1}^{n} X_i - \sum_1^{\frac{n}{3}} X_i)^2}{\frac{2n}{3}\ [\sum_1^n (X_i - \bar{X})^2]} . \qquad (4.4.3.1)$$

If we assume that X_i are equally spaced, and for the sake of simplicity, let X_i be 1, 2, ... n ,

$$\frac{2n}{3} \sum_{1}^{n} (X_i - \bar{X})^2 = \frac{n^2}{18} (n^2 - 1)$$

$$\left(\sum_{\frac{2}{3}n+1}^{n} X_i - \sum_{1}^{\frac{n}{3}} X_i\right)^2 = \left(\sum_{\frac{2}{3}n+1}^{n} i - \sum_{1}^{\frac{n}{3}} i\right)^2$$

$$= \left[\frac{n}{18} (5n + 3) - \frac{n}{18} (n + 3)\right]^2$$

$$= \frac{4n^4}{81} .$$

Hence

$$E(b_B) = \frac{\mathrm{Var}\ (b_M)}{\mathrm{Var}\ (b_B)}$$

$$= \frac{\frac{4n^4}{81}}{\frac{n^2}{18} (n^2 - 1)}$$

$$= \frac{8n^2}{9(n^2 - 1)} \geq \frac{8}{9} \quad . \qquad (4.4.3.2)$$

Thus we conclude that the relative efficiency of Group Averages estimator compared with the maximum likelihood estimator is always greater than $\frac{8}{9}$. Efficiencies of b_B when the X_i are not equally spaced are not as straightforward, but estimates can be made based on formula (4.4.3.1).

4.5 A Numerical Example

Graybill [4] gave the following example in which he assumed that all values of x_i and y_i below are recorded under the assumptions of Section (4.1) and $\lambda = 1$.

Table 4

Data for Numerical Example

y_i	6.3	7.9	8.2	9.4	10.8	11.5	12.9
x_i	11.1	9.9	8.2	7.3	6.1	6.0	4.2

Using the maximum likelihood estimators, we have

$$\sum_{1}^{7} (x_i - \bar{x})^2 = 34.337 \ , \qquad \sum_{1}^{7} (y_i - \bar{y})^2 = 31.714 \ ,$$

$$\sum_{1}^{7} (y_i - \bar{y})(x_i - \bar{x}) = -32.311 \ , \qquad n = 7 \ ,$$

$$\bar{x} = 7.543 \ , \qquad \bar{y} = 9.571 \ , \qquad U = .040589 \ ,$$

and hence

$$b_M = -0.960 \ , \qquad a_M = 16.81 \ , \quad \text{and} \quad \hat{\sigma}_e^2 = 0.139 \ .$$

Using the method of Group Averages we get the estimates

$$b_B = \frac{\bar{y}_3 - \bar{y}_1}{\bar{x}_3 - \bar{x}_1} = \frac{7.1 - 12.2}{10.5 - 5.1} = -.944$$

$$a_B = \bar{y} - b_B\,\bar{x} = 16.684$$

$$\hat{\sigma}_e^2 = (S_y^2 - b_B\,S_{xy})\,\frac{n}{n-3}$$

$$= [0.813 - (-0.944)(-0.711)]\,\frac{7}{4}$$

$$= 0.252\quad .$$

The estimates are quite comparable. However, the Group Averages method is much easier to compute.

V. CONCLUSION

The use of the Group Averages method for the case of X_i equally spaced in linear regression and quadratic polynomial regression can readily be recommended, since it involves less computational work and has quite satisfactory efficiencies when compared to the Least Squares method. The method will be particularly useful to those who have no automatic calculators, and for those uses where answers are needed quickly.

More caution is needed to use the Group Averages method in the functional relationship situation, since the following condition must be satisfied:

$$\Pr \{ |d_i| > \tfrac{1}{2} c \} \text{ is negligible}$$

$$\text{where } c = \text{Min. } |X_{i+1} - X_i| \ ,$$

as this insures b_B to be a consistent estimator of β. However, the efficiency of the Group Averages estimator of β is quite satisfactory compared with the maximum likelihood estimator, at least when the X_i are equally spaced. Another advantage to Group Averages method is that no prior knowledge is required about the relative magnitudes of σ_e^2 and σ_d^2, while such information is essential for the maximum likelihood method.

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ABSTRACT

This thesis reviews and discusses the so-called "Group Averages method" in the linear regression, the quadratic regression, and the functional relation situations.

In the linear and quadratic regression situations, under the assumption of X_i equally spaced, the efficiency of the Group Averages estimator is quite satisfactory as compared with Least Squares estimators.

In the functional relation situation we used the Group Averages method and the Maximum Likelihood method for estimation of parameters. To compare their efficiencies we used the variance of the Group Averages estimator which was given by Dorff and Gurland [3], and developed the variance of Maximum Likelihood estimators. Under the assumption of X_i equally spaced, we found the efficiency of the Group Averages estimator to be quite satisfactory. However, caution is needed for using the Group Averages method in functional relationships, since it requires the following condition to be satisfied:

$$\Pr \{ |d_i| \geq \frac{1}{2} c \} \text{ negligible}$$

$$\text{where } c = \text{Min.} |X_{i+1} - X_i| \ .$$