AN ANALYTIC SOLUTION FOR THE STRESS DISTRIBUTION IN A SEMI-INFINITE STRIP LOADED ON THE TRANSVERSE EDGE

by

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NORMENCLATURE

x, y : rectangular coodinates.

σ_x : normal stress in x direction.

σ_y : normal stress in y direction.

τ_{xy} : shearing stress.

a : one half of width of strip, regarded as a unit in measuring length.

q : uniform load per unit width of strip.

P : concentrated normal load.

Q : concentrated tangential load.

χ : stress funtion.

A_n : parametric coefficient.

ϕ_m : parametric function.

$I_{n,k}$: an integral.

$J_{n,k}$: an integral.

$S_n(x)$: an integral.

$C_n(x)$: an integral.

$S_n^*(x)$: an integral.

$C_n^*(x)$: an integral.

INTRODUCTION

The problem of determining the stresses and displacements in a rectangular strip of isotropic elastic material under the action of any system of edge tractions was attacked by Filon [1] as early as 1903. Filon gave a solution which is almost complete if the breadth of the strip is small compared with its length. The cognate problem of finding the stresses and displacements produced by forces acting within the strip instead of on its edges was given by Howland [2] who presented a satisfactory solution for the problem of a force, either longitudinal or transverse, acting at any point within the strip. The stresses due to any distribution of forces within the strip may be deduced by integration. Howland's solution, however, is for the case in which the strip is of infinite length. The problem of finding the stresses in a rectangular plate under a pair of symmetrical concentrated loads acting inside the plate was solved by Ling [3] who attacked the problem by using a method of images and obtained a solution applicable to a plate of any length-width ratio.

Solution of a finite plate compressed by two symmetrical concentrated loads is given by Timoshenko [4], Coker and Filon [5], Goodier [6], Pickett [7] and others. This in fact is the limiting case of Ling's solution. Horvay [8] derived a solution for the stresses in a rectangular plate when one of its narrow edges is under a self-equilibrating system of tractions. This solution is in the form of a series of products of orthogonal polynomials. The series is obtained by expanding the boundary values of the stress function into a set of orthogonal polynomials and the other functions are determined from the Euler-Lagrange equation of the associated variational problem.

The problem of a semi-infinite strip was considered by several investigators. Horvay and Born [9] gave a solution for the semi-infinite strip, when the longitudinal edges are traction-free and the transverse edge is subjected to (i) a shear displacement and zero normal stress, and (ii) a normal displacement and zero shear stress. The solution was based on an explicit series expansion of polynomial functions. The resulting boundary-value problem for the whole strip was solved by using a Fourier integral and the solution was expanded into an infinite series with the aid of the theorem of residues. Another paper by Horvay [10] dealt with the problem of a semi-infinite strip which is traction free along the longitudinal edge but under a self-equilibrating normal and shear tractions along the transverse edge. The solution is obtained by using biharmonic eigenfunctions. Benthem [11] used a Laplace transform method for the solution of semi-infinite and finite strip problems. The method can be applied to strips loaded in, or normal to, the plane of the strip and also applied to finite strips. Johnson and Little [12] solved the same problem of a semi-infinite strip traction free along the longitudinal edges but under specified boundary conditions of stress and/or displacement along the transverse edge. This solution was obtained by solving a system of four first order differential equations, namely: two equilibrium equations, one compatibility equation and an equation consisting of a newly introduced function. This method differs from the usual procedure of solving a single fourth order differential equation for Airy's stress function.

Theocaris [13] obtained a solution for the stress distribution in a semi-infinite strip subjected to a concentrated load acting normal to the transverse edge. The solution is based on his method of isostatics [14]. It consists of determining a complex function representing the field of isostatics in a basic equation of elasticity. Conformal mapping of this field into the field of isostatics of the semi-infinite strip, however, introduced normal stresses to the longitudinal edges, which are eliminated by applying the minimum strain-energy principle over a part of the strip. The normal stresses along these edges are, however, only imperfectly eliminated in this way. Another paper by Theocaris [15] dealt with a similar strip but subjected to a segment of uniform load over the transverse edge. The normal stresses along the longitudinal edges are eliminated by using a method of multiple

Fourier series over a part of the strip. Ling [16] solved the same problem of semi-infinite strip subjected to an axial concentrated load by the method of images. In his solution, Ling presented an interesting analysis which satisfies all the boundary conditions. It is the aim of this thesis to extend Ling's work to the same strip but under different types of loadings over the transverse edge. The solutions are also constructed by using the method of images. The procedures used in the solutions are essentially those of Ling's. However, in solving the resulting system of equations, further attempts are made to eliminate one of them so that only one system of linear equations is left, which can be solved by method of successive approximations.

Three different cases are considered, when the transverse edge is under the following symmetrical loadings: (i) a segment of uniform load; (ii) two concentrated normal loads and (iii) two concentrated tangential loads. The solution of the first case is given in some detail and those of the remaining two cases are given in a somewhat abbreviated manner. In each case, the expressions for normal stresses along the longitudinal axis are derived and calculated at several locations.

METHOD OF SOLUTION

Case (i) Semi-infinite Strip Subjected to a Segment of Uniform Load on the Transverse Edge

Consider a semi-infinite plane strip of isotropic elastic material and of uniform thickness. Let the strip be bounded in the right half-plane by the lines $x = 0$ and $y = \pm 1$ as shown in Fig. 1; the strip being measured by a typical length a or one-half of the width of the strip. The transverse edge of the strip is under a segment of uniformly distributed load of q per unit width, symmetrical with respect to the x-axis from $y = -c$ to $y = c$.

The loading on the transverse edge is assumed parallel to the plane of the strip and distributed uniformly over the thickness. The strip is regarded as in a state of generalized plane stress. The solution of the problem consists of finding a function χ which satisfies the biharmonic equation $\nabla^4 \chi = 0$. The function χ is generally known as Airy stress function. The stress components in rectangular coordinates are derived from χ in the following form:

$$\sigma_x = \frac{\partial^2 \chi}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 \chi}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}.$$

The parametric coefficients involved in the solution are determined by adjusting the boundary conditions along the edges of the strip. As will be described in the subsequence, the solution is constructed by using the method of images.

Suppose that the given strip is under successive reflections about the longitudinal edges. Then the strip and the resulting images occupy the entire right half-plane $x \geqslant 0$ as shown in Fig. 2. The given load and its images become a series of segments of uniformly distributed loads acting along the edge $x = 0$ from $y = 2s - c$ to $y = 2s + c$, respectively, where s takes all positive and negative integral values including zero.

The stress function for such a semi-infinite plate can be constructed in the following way:

It is well known that the stress function for the semi-infinite plate loaded as shown in Fig. 3 is given by the following biharmonic function [4]

$$\chi = \frac{q a^2}{2\pi}\left(r^2\theta - r'^2\theta'\right) \qquad (1)$$

where (r, θ) and (r', θ') are two sets of polar coordinates with poles at the points $(0, \pm c)$, respectively.

Thus the stress function for the semi-infinite plate under a series of uniform loads as shown in Fig. 2 is obtained by superposition as follows:

$$\chi_0 = \frac{q a^2}{2\pi}\left[r^2\theta - r'^2\theta' + \sum_{s=1}^{\infty}\left\{r_s^2\theta_s - r_s'^2\theta_s' + r_{-s}^2\theta_{-s} - r_{-s}'^2\theta_{-s}'\right\}\right] \qquad (2)$$

where (r_s, θ_s) and (r_s', θ_s') are two sets of polar coordinates with poles at the points $(0, 2s + c)$ and $(0, 2s - c)$ respectively. In particular, (r, θ) and (r', θ') are written in place of (r_0, θ_0) and $r_0', \theta_0')$ as shown in Fig. 4.

The polar coordinates (r_s, θ_s) are connected with (x, y) by

$$
\begin{aligned}
x + i(y - 2s - c) &= z - (2s + c)i = r_s \exp(i\theta_s) \\
x + i(y - 2s + c) &= z - (2s - c)i = r_s' \exp(i\theta_s')
\end{aligned}
\tag{3}
$$

The stress function χ_0 in (2) can be expressed as the real part of a complex function as follows:

$$
\begin{aligned}
\chi_0 = \frac{qa^2}{2\pi} \mathcal{R} \Big[i\{x^2 + (y+c)^2\} \log(z + ci) - i\{x^2 + (y-c)^2\} \log(z - ci) \Big] \\
+ \frac{qa^2}{2\pi} \mathcal{R} \sum_{s=1}^{\infty} i \Big[\{x^2 + (y - 2s + c)^2\} \log\left(1 + \frac{iz - c}{2s}\right) - \{x^2 + (y - 2s - c)^2\} \log\left(1 + \frac{iz + c}{2s}\right) \\
+ \{x^2 + (y + 2s + c)^2\} \log\left(1 - \frac{iz - c}{2s}\right) - \{x^2 + (y + 2s - c)^2\} \log\left(1 - \frac{iz + c}{2s}\right) \Big]
\end{aligned}
\tag{4}
$$

The stress function given in this form is divergent. However, the divergence can be removed if a linear term $\mathcal{R}[4ci(ix - 3y)]$ is added to each term of the series. For the proof of this, refer to Appendix 1. This is permissible since such a modification does not affect the stresses derived from the function. Then the stress function χ_0 appears in the following form:

$$
\chi_0 = \frac{qa^2}{2\pi} \mathcal{R} \Big[i\{x^2 + (y+c)^2\} \log(z + ci) - i\{x^2 + (y-c)^2\} \log(z - ic) \Big] +
$$

$$+ \frac{qa^2}{2\pi} R \sum_{s=1}^{\infty} i\left[\left\{x^2+(y-2s+c)^2\right\}\log\left(1+\frac{iz-c}{2s}\right)-\left\{x^2+(y-2s-c)^2\right\}\log\left(1+\frac{iz+c}{2s}\right)\right.$$

$$\left.+\left\{x^2+(y+2s+c)^2\right\}\log\left(1-\frac{iz-c}{2s}\right)-\left\{x^2+(y+2s-c)\right\}\log\left(1-\frac{iz+c}{2s}\right)+4c(ix-3y)\right] \quad (5)$$

The preceding stress function gives no normal and tangential stresses along the edge x = 0 besides the segments of uniform loads. Furthermore, by symmetry, it gives no tangential stress along the lines y = ±1. The normal stresses given by the stress function are as follows:

$$\sigma_x = \frac{1}{a^2}\frac{\partial^2 \chi_0}{\partial y^2}$$

$$= \frac{q}{\pi} R\left[i\log(z+ci) - i\log(z-ci) - \frac{ix}{z+ci} + \frac{ix}{z-ci}\right]$$

$$+ \frac{q}{\pi} R \sum_{s=1}^{\infty} i\left[\log\left(1+\frac{(z+ci)^2}{4s^2}\right) - \log\left(1+\frac{(z-ci)^2}{4s^2}\right)\right.$$

$$\left.+\frac{x}{z-2si-ci}+\frac{x}{z+2si-ci}-\frac{x}{z-2si+ci}-\frac{x}{z+2si+ci}\right] \quad (6)$$

$$\sigma_y = \frac{1}{a^2}\frac{\partial^2 \chi_0}{\partial x^2}$$

$$= \frac{q}{\pi} R\left[i\log(z+ci) - i\log(z-ci) + \frac{ix}{z+ci} - \frac{ix}{z-ci}\right]$$

$$+ \frac{q}{\pi} R \sum_{s=1}^{\infty} i\left[\log\left(1+\frac{(z+ci)^2}{4s^2}\right) - \log\left(1+\frac{(z-ci)^2}{4s^2}\right)\right.$$

$$\left.-\frac{x}{z-2si-ci}-\frac{x}{z+2si-ci}+\frac{x}{z-2si+ci}+\frac{x}{z+2si+ci}\right] \quad (7)$$

To sum up the series in eqs. (6) and (7), we observe that the series below can be written as infinite products:

$$\log(z+ci)+\sum_{s=1}^{\infty}\log\left(1+\frac{(z+ci)^2}{4s^2}\right)-\log(z-ci)-\sum_{s=1}^{\infty}\log\left(1+\frac{(z-ci)^2}{4s^2}\right)$$

$$=\log\left[(z+ci)\prod_{s=1}^{\infty}\left(1+\frac{(z+ci)^2}{4s^2}\right)\right]-\log\left[(z-ci)\prod_{s=1}^{\infty}\left(1+\frac{(z-ci)^2}{4s^2}\right)\right] \tag{8}$$

Comparing the infinite product with the well known formula [17]

$$\theta\prod_{s=1}^{\infty}\left(1-\frac{\theta^2}{\pi^2 s^2}\right)=\sin\theta \tag{9}$$

we find

$$\log(z+ci)+\sum_{s=1}^{\infty}\log\left(1+\frac{(z+ci)^2}{4s^2}\right)-\log(z-ci)-\sum_{s=1}^{\infty}\log\left(1+\frac{(z-ci)^2}{4s^2}\right)$$

$$=\log\sinh\frac{\pi(z+ci)}{2}-\log\sinh\frac{\pi(z-ci)}{2} \tag{10}$$

Next we compare the series

$$\left[\frac{1}{z-ci}+\sum_{s=1}^{\infty}\left(\frac{1}{(z-ci)-2si}+\frac{1}{(z-ci)+2si}\right)\right]-\left[\frac{1}{z+ci}+\sum_{s=1}^{\infty}\left(\frac{1}{(z+ci)-2si}+\frac{1}{(z+ci)+2si}\right)\right]$$

with the known formula [17]

$$\frac{1}{\theta}+\sum_{s=-\infty}^{\infty}{}'\left(\frac{1}{\theta+s\pi}-\frac{1}{s\pi}\right)=\cot\theta \tag{11}$$

where the prime on the summation sign means that the term corresponding to s = 0 is excluded. We find

$$\left[\frac{1}{z-ci}+\sum_{s=1}^{\infty}\left(\frac{1}{(z-ci)-2si}+\frac{1}{(z-ci)+2si}\right)\right]-\left[\frac{1}{z+ci}+\sum_{s=1}^{\infty}\left(\frac{1}{(z+ci)-2si}+\frac{1}{(z+ci)+2si}\right)\right]$$

$$=\frac{\pi}{2}\coth\frac{\pi(z-ci)}{2}-\frac{\pi}{2}\coth\frac{\pi(z+ci)}{2} \qquad (12)$$

Substitution into (6) and (7) gives

$$\sigma_x=\frac{q}{\pi}\mathcal{R}\left[i\log\sinh\frac{\pi(z+ci)}{2}-i\log\sinh\frac{\pi(z-ci)}{2}\right.$$

$$\left.-\frac{\pi i x}{2}\coth\frac{\pi(z+ci)}{2}+\frac{\pi i x}{2}\coth\frac{\pi(z-ci)}{2}\right] \qquad (13)$$

$$\sigma_y=\frac{q}{\pi}\mathcal{R}\left[i\log\sinh\frac{\pi(y+ci)}{2}-i\log\sinh\frac{\pi(z-ci)}{2}\right.$$

$$\left.+\frac{\pi i x}{2}\coth\frac{\pi(z+ci)}{2}-\frac{\pi i x}{2}\coth\frac{\pi(z-ci)}{2}\right] \qquad (14)$$

The above expressions can further be simplified if we observe that

$$\coth\frac{\pi(z-ci)}{2}=\coth\frac{\pi}{2}\left[x+i(y-c)\right]=\frac{\sinh\pi x-i\sin\pi(y-c)}{\cosh\pi x-\cos\pi(y-c)} \qquad (15)$$

and

$$\coth\frac{\pi(z+ci)}{2}=\coth\frac{\pi}{2}\left[x+i(y+c)\right]=\frac{\sinh\pi x-i\sin\pi(y+c)}{\cosh\pi x-\cos\pi(y+c)} \qquad (16)$$

also

$$\log \sinh \frac{\pi(z-ci)}{2} - \log \sinh \frac{\pi(z+ci)}{2} = \log \frac{\sinh \frac{\pi}{2}[x+i(y-c)]}{\sinh \frac{\pi}{2}[x+i(y+c)]}$$

$$= \log \left[\frac{\cosh \pi x \cos \pi c - \cos \pi y}{\cosh \pi x - \cos \pi(y+c)} - i \frac{\sinh \pi x \sin \pi c}{\cosh \pi x - \cos \pi(y+c)} \right]$$

$$= \frac{1}{2} \log \frac{(\cosh \pi x \cos \pi c - \cos \pi y)^2 + (\sinh \pi x \sin \pi c)^2}{[\cosh \pi x - \cos \pi(y+c)]^2} - i \tan^{-1} \frac{\sinh \pi x \sin \pi c}{\cosh \pi x \cos \pi c - \cos \pi y} \tag{17}$$

Thus we have

$$\sigma_x = -\frac{q}{\pi} \tan^{-1} \frac{\sin \pi c \sinh \pi x}{\cos \pi c \cosh \pi x - \cos \pi y}$$

$$- \frac{qx}{2} \left(\frac{\sin \pi(y+c)}{\cosh \pi x - \cos \pi(y+c)} - \frac{\sin \pi(y+c)}{\cosh \pi x - \cos \pi(y-c)} \right) \tag{18}$$

$$\sigma_y = -\frac{q}{\pi} \tan^{-1} \frac{\sin \pi c \sinh \pi x}{\cos \pi c \cosh \pi x - \cos \pi y}$$

$$+ \frac{qx}{2} \left(\frac{\sin \pi(y+c)}{\cosh \pi x - \cos \pi(y+c)} - \frac{\sin \pi(y-c)}{\cosh \pi x - \cos \pi(y-c)} \right) \tag{19}$$

At x infinity, σ_x tends to $-qc$ as required by the static condition of the plate. Along the lines $y = \pm 1$ the normal stress is:

$$[\sigma_y]_{y=\pm 1} = -\frac{q}{\pi} \tan^{-1} \frac{\sin \pi c \sinh \pi x}{1 + \cos \pi c \cosh \pi x} - \frac{qx \sin \pi c}{\cosh \pi x + \cos \pi c} \tag{20}$$

Now, consider the strip in question. Suppose that a second biharmonic function χ_1 is added to χ_0 so that the required stress function χ for the given strip is

$$\chi = \chi_0 + \chi_1 \tag{21}$$

The function χ_1 may be constructed in the following way.

First consider an infinite strip with the same width as the semi-infinite strip in our problem. Suppose a system of forces symmetrical with respect to both the x and y axes acting on the edges of the strip. A stress function for this infinite strip was found by Howland [2] in the following form:

$$\chi_s = \int_0^\infty \frac{\phi_m \sinh m}{m^2(\sinh 2m + 2m)} \left\{ my \sinh my - (1 + m\coth m) \cosh my \right\} \cos mx \, dm \tag{22}$$

where ϕ_m is an arbitrary function of m depending on the load distribution on the longitudinal edges.

This stress function gives no shearing stress along the edges $y = \pm 1$ and since it is even both in x and y, it gives only a normal stress symmetrical with both the x and y axes.

Next let us consider a semi-infinite plate with the x and y axes perpendicular to and along the straight edge of the plate, respectively. Suppose the plate is subjected to a series of loads which is symmetrical with respect to x-axis and let the dimension of the plate be also measured by a typical length a. Consider, in this case, a biharmonic function

$$\chi_p = (Ax + B)\,e^{-mx}\cos my + (Cx + D)\,e^{mx}\cos my \qquad (23)$$

which is even in y. The condition that the normal stress σ_y approaches zero at x infinity requires the exclusion of the terms e^{mx} from the above expression and thus χ_p is reduced to the form:

$$\chi_p = (Ax + B)\,e^{-mx}\cos my \qquad (24)$$

The shear stresses given by this function at lines $y = \pm 1$ are zero if $m = n\pi$ where n is an integer. It satisfies the condition of zero shear on the edge $x = 0$ if we take

$$A = n\pi B \qquad (25)$$

Thus

$$\chi_p = B(1 + n\pi x)\,e^{-n\pi x}\cos n\pi y \qquad (26)$$

This holds for all integer values of n so that, provided the series converges, a more general solution is

$$\chi_p = \sum_{n=1}^{\infty} \frac{A_n}{n^2\pi^2}(1 + n\pi x)\,e^{-n\pi x}\cos n\pi y \qquad (27)$$

where $A_n/n^2\pi^2$ is written in place of B for convenience.

Now the stress function χ_1 can be constructed by combining χ_s and χ_p in the following form:

$$\chi_1 = -2a^2\int_0^\infty \frac{\phi_m \sinh m}{m^2(\sinh 2m + 2m)}\left\{my\sinh my - (1+m\coth m)\cosh my\right\}\cos my\, dm$$

$$+ \frac{a^2}{2}A_0 x^2 + a^2\sum_{n=1}^\infty \frac{A_n}{n^2\pi^2}(1+n\pi x)e^{-n\pi x}\cos n\pi y \tag{28}$$

This stress function gives no tangential stress along all three edges of the strip but the following normal stresses:

$$[\sigma_x]_{x=0} = \frac{1}{a^2}\left[\frac{\partial^2 \chi_1}{\partial y^2}\right]_{x=0}$$

$$= -\sum_{n=1}^\infty A_n \cos n\pi y - 2\int_0^\infty \frac{\phi_m \sinh m}{\sinh 2m + 2m}$$

$$\times\left\{(1-m\coth m)\cosh my + my\sinh my\right\}dm \tag{29}$$

$$[\sigma_y]_{y=\pm 1} = \frac{1}{a^2}\left[\frac{\partial^2 \chi}{\partial x^2}\right]_{y=\pm 1}$$

$$= A_0 - \int_0^\infty \phi_m \cos mx\, dm$$

$$-\sum_{n=1}^\infty (-1)^n A_n(1-n\pi x)e^{-n\pi x} \tag{30}$$

Hence, the normal stress σ_x along the edge x = 0 becomes zero provided that

$$\sum_{n=1}^{\infty} A_n \cos n\pi y = -2\int_0^{\infty} \frac{\phi_m \sinh m}{\sinh 2m + 2m}\left\{(1-m\coth m)\cosh my + my\sinh my\right\} dm \tag{31}$$

By Fourier series expansion we have

$$A_n = -2\int_0^{\infty}\int_{-1}^{1} \frac{\phi_m \sinh m}{\sinh 2m + 2m}\left[(1-m\coth)\cosh my + my\sinh my\right]\cos n\pi y\, dy\, dm \tag{32}$$

Again, the normal stress σ_y given by χ along the edges $y = \pm 1$ becomes zero provided that $A_o = qc$ and

$$\int_0^{\infty} \phi_m \cos mx\, dm = -\frac{q}{\pi}\left[\tan^{-1}\frac{\sinh \pi x \sin \pi c}{1+\cos \pi c \cosh \pi x} - \pi c\right]$$

$$- \frac{qx\sin \pi c}{\cosh \pi x + \cos \pi c} - \sum_{n=1}^{\infty} (-1)^n A_n (1-n\pi x) e^{-n\pi x} \tag{33}$$

By Fourier cosine transform we have

$$\phi_m = -\frac{2q}{\pi^2}\int_0^\infty \left[\tan^{-1}\frac{\sin\pi c\,\sinh\pi x}{1+\cos\pi c\,\cosh\pi x} - \pi c\right]\cos mx\,dx$$

$$-\frac{2q\sin\pi c}{\pi}\int_0^\infty \frac{x\cos mx\,dx}{\cosh\pi x+\cos\pi c}$$

$$-\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n A_n\int_0^\infty (1-n\pi x)e^{-n\pi x}\cos mx\,dx \tag{34}$$

On performing the integration, the following results are obtained.

$$\int_{-1}^{1}\left[(1-m\coth m)\cosh my + my\sinh my\right]\cos n\pi y\,dy$$

$$=\frac{4(-1)^n n^2\pi^2 m\sinh m}{(m^2+n^2\pi^2)^2} \qquad \text{(see Appendix 2(i))}$$

$$\frac{2q\sin\pi c}{\pi}\int_0^\infty \frac{x\cos mx\,dx}{\cosh\pi x+\cos\pi c}$$

$$= 4qc\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\pi c\,(m^2-n^2\pi^2)}{\pi c\,(m^2+n^2\pi^2)^2} \qquad \text{(see Appendix 2(ii))}$$

$$\frac{2q}{\pi^2}\int_0^\infty\left[\tan^{-1}\frac{\sin\pi c\,\sinh\pi x}{1+\cos\pi c\,\cosh\pi x}-\pi c\right]\cos mx\,dx$$

$$= 4qc\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\pi c}{\pi c\,(m^2+n^2\pi^2)} \qquad \text{(see Appendix 2(iii))}$$

$$\int_0^\infty (1-n\pi x)e^{-n\pi x}\cos mx\,dx$$

$$=\frac{2n\pi m^2}{(m^2+n^2\pi^2)^2} \qquad \text{(see Appendix 2(iv))}$$

Thus eqs. (32) and (34) are finally reduced to the following forms, respectively,

$$\phi_m = -4m^2 \sum_{n=1}^{\infty} \frac{(-1)^n n}{(m^2+n^2\pi^2)^2} \left[\frac{2q \sin n\pi c}{n\pi} + A_n \right] \tag{35}$$

$$A_n = -8(-1)^n n^2 \pi^2 \int_0^{\infty} \frac{\phi_m \, m \sinh^2 m \, dm}{(m^2+n^2\pi^2)^2(\sinh 2m + 2m)} \, , \quad (n \geq 1) \tag{36}$$

which consists partly of a system of algebraic equations and partly a system of integral equations. These systems of equations may further be simplified as follows:

Substituting (35) into (36) and interchanging the integration and summation signs we obtain:

$$A_n = 32(-1)^n n^2 \pi^2 \sum_{k=1}^{\infty} (-1)^k k \left(\frac{2q \sin k\pi c}{k\pi} + A_k \right) \int_0^{\infty} \frac{m^2 \sinh^2 m \, dm}{(m^2+n^2\pi^2)^2 (m^2+k^2\pi^2)^2 (\sinh 2m + 2m)} \tag{37}$$

Further, if we denote

$$A_n^* = \frac{(-1)^n}{\sqrt{n}} A_n \tag{38}$$

$$I_{n,k} = 32\pi^2 (nk)^{3/2} \int_0^{\infty} \frac{m^3 \sinh^2 m \, dm}{(m^2+n^2\pi^2)^2 (m^2+k^2\pi^2)^2 (\sinh 2m + 2m)} \tag{39}$$

Eq. (37) becomes

$$A_n^* = \sum_{k=1}^{\infty} \left(\frac{2q(-1)^k \sin k\pi c}{\pi k^{3/2}} + A_k^* \right) I_{n,k} \tag{40}$$

By successive approximation with

$$A_n^{*(0)} = \frac{2q}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k \sin k\pi c}{k^{3/2}} I_{n,k}$$

$$A_n^{*(p)} = \sum_{k=1}^{\infty} I_{n,k} A_k^{*(p-1)} \quad , \qquad (p \geq 1) \tag{41}$$

we have

$$A_n^* = \sum_{p=0}^{\infty} A_n^{*(p)} \tag{42}$$

This is true since

$$\sum_{p=0}^{\infty} A_n^{*(p)} = \sum_{k=1}^{\infty} \left[\frac{2(-1)^k \sin k\pi c}{\pi c k^{3/2}} + \sum_{p=0}^{\infty} A_k^{*(p)} \right] I_{n,k}$$

$$= \sum_{k=1}^{\infty} \left[\frac{2(-1)^k \sin k\pi c}{\pi c k^{3/2}} + A_k^* \right] I_{n,k}$$

$$= A_n^*$$

With A_n^* calculated in this way, A_n may be found immediately from

$$A_n = (-1)^n \sqrt{n}\, A_n^* \tag{43}$$

The integral $I_{n,k}$ can be evaluated by Cauchy's contour integration as shown in Appendix 2(v). Table 1 shows the values of $I_{n,k}^*$ for n and k ranging from 1 to 20, where $I_{n,k}^*$ are related to $I_{n,k}$ by

$$I_{n,k}^* = \frac{3}{4}\pi^2 (nk)^{\frac{1}{2}} I_{n,k}$$

The values of A_n are shown in Table 3. Note that $A_o = qc$ has been previously obtained. The stresses at any point within the strip can now be derived from the stress function χ. In particular, the normal stresses σ_x and σ_y along the longitudinal axis y = 0 are:

$$\begin{aligned}\left[\sigma_x\right]_{y=0} = &-\frac{q}{\pi}\tan^{-1}\frac{\sin\pi c \sinh\pi x}{\cos\pi c\cosh\pi x - 1} - \frac{qx\sin\pi c}{\cosh\pi x - \cos\pi c}\\ &-\sum_{n=1}^{\infty} A_n(1+n\pi x)e^{-n\pi x} + 8\sum_{n=1}^{\infty}(-1)^n\left[\frac{2q}{\pi}\sin n\pi c + nA_n\right]\left\{S_n(x) - C_n(x)\right\}\end{aligned} \tag{44}$$

$$\begin{aligned}\left[\sigma_y\right]_{y=0} = &-\frac{q}{\pi}\tan^{-1}\frac{\sin\pi c\sinh\pi x}{\cos\pi c\cosh\pi x - 1} - \frac{qx\sin\pi c}{\cosh\pi x - \cos\pi c}\\ &+qc - \sum_{n=1}^{\infty} A_n(1-n\pi x)e^{-n\pi x} + 8\sum_{n=1}^{\infty}(-1)^n\left[\frac{2q}{\pi}\sin n\pi c + nA_n\right]\left\{S_n(x) + C_n(x)\right\}\end{aligned} \tag{45}$$

where $S_n(x)$ and $C_n(x)$ represent the following two integrals:

$$S_n(x) = \int_0^{\infty} \frac{m^2 \sinh m \cos mx\, dx}{(m^2 + n^2\pi^2)^2 (\sinh 2m + 2m)}$$

$$C_n(x) = \int_0^{\infty} \frac{m^3 \cosh m \cos mx\, dm}{(m^2 + n^2\pi^2)^2 (\sinh 2m + 2m)} \tag{46}$$

The above two integrals can also be evaluated by Cauchy's contour integration as shown in Appendix 2(vi). The values are tabulated in Table 4. The normal stress σ_x along y = 0 for c = 1/2 has been calculated for several values of x. The results are tabulated in Table 6 and shown graphically in Fig. 7.

Case (ii) Semi-infinite Strip Subjected to Two Concentrated Normal Loads

Consider the same semi-infinite strip as in case (i) but subjected to two normal concentrated compressive loads each of P per unit thickness acting symmetrically at $y = \pm c$ on the edge x = 0 as shown in Fig. 5. By successive reflections about the longitudinal edges, the given strip and the resulting images occupy the entire right half-plane $x \geqslant 0$, and the given loads and their images become a series of normal concentrated loads acting along the edge x = 0 at the points $(0, 2s \pm c)$, where s takes all positive and negative integral values including zero.

The stress function for a semi-infinite plate subjected to a concentrated normal force P at a point of the straight boundary is given by [4]

$$\chi' = -\frac{P}{\pi} r\theta \sin\theta \tag{47}$$

Now the stress function for the semi-infinite plate constructed by images and acted by a series of concentrated load may be obtained by superposing the above stress function in the following form:

$$\chi_0 = -\frac{Pa}{\pi}\Big\{ r\theta\sin\theta + r'\theta'\sin\theta$$

$$+\sum_{s=1}^{\infty}\left(r_s\theta_s\sin\theta_s + r_s'\theta_s'\sin\theta_s' + r_{-s}\theta_{-s}\sin\theta_{-s} + r_{-s}'\theta_{-s}'\sin\theta_{-s}'\right)\Big\} \tag{48}$$

where (r_s, θ_s) and (r_s', θ_s') are two sets of polar coordinates with poles at the points $(0, 2s \pm c)$, respectively. In particular, (r, θ) and (r', θ') are written in place of (r_0, θ_0) or (r_0', θ_0'), respectively. These sets of polar coordinates are connected with (x, y) in the same way as (3) in case (i).

The stress function given in this form appears to be divergent. However, the divergence can be removed if it is expressed as the real part of a complex function and modified as follows:

$$\chi_0 = \frac{Pa}{\pi} R\left[i(y-c)\log(z+ci) + i(y+c)\log(z-ic)\right]$$

$$+\frac{Pa}{\pi} R\sum_{s=1}^{\infty} i\left[(y-2s-c)\log\left(1+\frac{iz+c}{2s}\right) + (y-2s+c)\log\left(1+\frac{iz-c}{2s}\right)\right.$$

$$\left.+(y+2s-c)\log\left(1-\frac{iz+c}{2s}\right) + (y+2s+c)\log\left(1-\frac{iz-c}{2s}\right) + 4(ix-y)\right] \tag{49}$$

The foregoing stress function gives no normal and tangential stresses along the edge x = 0 besides the series of concentrated loads. Furthermore, by symmetry, it gives no tangential stress along the lines $y = \pm 1$. The normal stresses given by the stress function are as follows:

$$\sigma_x = \frac{1}{a^2}\frac{\partial^2 \chi_0}{\partial y^2}$$

$$= -\frac{P}{\pi a} R\left[\frac{2x+i(y-c)}{[x+i(y-c)]^2} + \frac{2x+i(y+c)}{[x+i(y+c)]^2}\right.$$

$$+\sum_{s=1}^{\infty}\left\{\frac{2x+i(y-2s-c)}{[x+i(y-2s-c)]^2} - \frac{2x+i(y-2s+c)}{[x+i(y-2s+c)]^2}\right.$$

$$\left.\left.+\frac{2x+i(y+2s-c)}{[x+i(y+2s-c)]^2} + \frac{2x+i(y+2s+c)}{[x+i(y+2s+c)]^2}\right\}\right] \tag{50}$$

$$\sigma_y = \frac{1}{a^2}\,\frac{\partial^2 \chi_0}{\partial x^2}$$

$$= -\frac{P}{\pi a}\,\mathcal{R}\Bigg[\frac{i(y-c)}{[x+i(y-c)]^2} + \frac{i(y+c)}{[x+i(y+c)]^2}$$

$$+\sum_{s=1}^{\infty}\Bigg\{\frac{i(y-2s-c)}{[x+i(y-2s-c)]^2} + \frac{i(y-2s+c)}{[x+i(y-2s+c)]^2}$$

$$+\frac{i(y+2s-c)}{[x+i(y+2s-c)]^2} + \frac{i(y+2s+c)}{[x+i(y+2s+c)]^2}\Bigg\}\Bigg] \tag{51}$$

These series are summable and the sums are

$$\sigma_x = -\frac{P}{2a}\,\mathcal{R}\Bigg[\coth\frac{\pi[x+i(y-c)]}{2} + \frac{\pi x}{2}\operatorname{csch}^2\frac{\pi[x+i(y-c)]}{2}$$

$$+\coth\frac{\pi[x+i(y+c)]}{2} + \frac{\pi x}{2}\operatorname{csch}^2\frac{\pi[x+i(y+c)]}{2}\Bigg]$$

$$= -\frac{P}{2a}\Bigg\{\frac{\sinh\pi x}{\cosh\pi x - \cos\pi(y-c)} + \frac{\pi x[\cosh\pi x\cos\pi(y-c) - 1]}{[\cosh\pi x - \cos\pi(y-c)]^2}$$

$$+\frac{\sinh\pi x}{\coth\pi x - \cos\pi(y+c)} - \frac{\pi x[\cosh\pi x\cos(y+c) - 1]}{[\cosh\pi x - \cos\pi(y+c)]^2}\Bigg\} \tag{52}$$

$$\sigma_y = -\frac{P}{2a}\mathcal{R}\left[\coth\frac{\pi[x+i(y-c)]}{2} - \frac{\pi x}{2}\operatorname{csch}^2\frac{\pi[x+i(y-c)]}{2}\right.$$
$$\left. + \coth\frac{\pi[x+i(y+c)]}{2} - \frac{\pi x}{2}\operatorname{csch}^2\frac{\pi[x+i(y+c)]}{2}\right]$$
$$= -\frac{P}{2a}\left\{\frac{\sinh\pi x}{\cosh\pi x - \cos\pi(y-c)} - \frac{\pi x[\cosh\pi x\cos\pi(y-c) - 1]}{[\cosh\pi x - \cos\pi(y-c)]^2}\right.$$
$$\left. + \frac{\sinh\pi x}{\cosh\pi x - \cos\pi(y+c)} - \frac{\pi x[\cosh\pi x\cos\pi(y+c) - 1]}{[\cosh\pi x - \cos\pi(y+c)]^2}\right\} \qquad (53)$$

At x infinity, σx tends to -P/a. Along the lines $y = \pm 1$, the normal stress is

$$\left[\sigma_y\right]_{y=\pm 1} = -\frac{P}{a}\left[\frac{\sinh\pi x}{\cosh\pi x + \cos\pi c} + \frac{\pi x(\cosh\pi x\cos\pi c + 1)}{(\cosh\pi x + \cos\pi c)^2}\right] \qquad (54)$$

Now, consider the strip in question. Suppose that a second biharmonic function χ_1 is added to χ_o so that the required stress function for the given strip is

$$\chi = \chi_o + \chi_1 \qquad (55)$$

The function χ_1 may be constructed in the same way as in case (i) and the expression for χ_1 in (28) of case (i) can be used here directly without modification. Thus the normal stresses $[\sigma x]_{x=0}$ and $[\sigma y]_{y=\pm 1}$ derived from χ_1 are in the same form as (29) and (30) of case (i).

Hence, the normal stress σx along the edge x = 0 becomes zero provided that by Fourier expansion from -1 to 1, for $n \geqslant 1$

$$A_n = -2\int_0^\infty \int_{-1}^{1} \frac{\phi_m \sinh m}{\sinh 2m + 2m} \Big\{(1 - m\coth m)\cosh my$$

$$+ my\sinh my\Big\} \cos n\pi y \, dy \, dm \tag{56}$$

Again, the normal stress σ_y given by χ along the edges $y = \pm 1$ becomes zero provided that $A_o = P/a$ and by Fourier cosine transform

$$\phi_m = -\frac{2P}{\pi a}\int_0^\infty \left(\frac{\sinh \pi x}{\cosh \pi x + \cos \pi c} - 1\right) \cos mx \, dx$$

$$- \frac{2P}{a}\int_0^\infty \frac{x(\cosh \pi x \cos \pi c + 1)}{(\cosh \pi x + \cos \pi c)^2} \cos mx \, dx$$

$$- \frac{2}{\pi}\sum_{n=1}^{\infty} (-1)^n A_n (1 - n\pi x) e^{-n\pi x} \cos mx \, dx \tag{57}$$

On performing the integration, the following system of equations are obtained:

$$\phi_m = -4m^2 \sum_{n=1}^{\infty} \frac{(-1)^n n}{(m^2 + n^2\pi^2)^2}\left(\frac{2P\cos n\pi c}{a} + A_n\right) \tag{58}$$

$$A_n = -8(-1)^n n^2\pi^2 \int_0^\infty \frac{\phi_m\, m \sinh^2 m\, dm}{(m^2+n^2\pi^2)^2(\sinh 2m + 2m)}$$

$$= 32(-1)^n n^2\pi^2 \sum_{k=1}^{\infty} (-1)^k k\left(\frac{2P\cos k\pi c}{a} + A_k\right)\int_0^\infty \frac{m^3 \sinh^2 m\, dm}{(m^2+n^2\pi^2)^2(m^2+k^2\pi^2)^2(\sinh 2m+2m)}$$

$$(n \geqslant 1) \qquad (59)$$

By (38) and (39) we reach

$$A_n^* = \sum_{k=1}^{\infty}\left(\frac{2(-1)^k P\cos k\pi c}{a k^{1/2}} + A_k^*\right) I_{n,k} \qquad (n \geqslant 1) \qquad (60)$$

which can be solved by the method of successive approximation. The values of An thus found are tabulated in Table 3. Note that Ao = P/a has been determined previously. The stresses at any point within the strip can now be derived from the stress function χ. In particular, the normal stresses σx and σy along the longitudinal axis y = 0 are:

$$\left[\sigma_x\right]_{y=0} = -\frac{P}{a}\left[\frac{\sinh \pi x}{\cosh \pi x - \cos \pi c} + \frac{\pi x(\cosh \pi x \cos \pi c - 1)}{(\cosh \pi x - \cos \pi c)^2}\right]$$

$$- \sum_{n=1}^{\infty} A_n (1 + n\pi x) e^{-n\pi x}$$

$$+ 8\sum_{n=1}^{\infty} (-1)^n n\left(\frac{2P}{a}\cos n\pi c + A_n\right)\left\{S_n(x) - C_n(x)\right\}$$

$$(61)$$

$$[\sigma_y]_{y=0} = -\frac{P}{a}\left[\frac{\sinh \pi x}{\cosh \pi x - \cos \pi c} - \frac{\pi x(\cosh \pi x \cos \pi c - 1)}{(\cosh \pi x - \cos \pi c)^2}\right]$$

$$+ \frac{P}{a} - \sum_{n=1}^{\infty} A_n (1 - n\pi x) e^{-n\pi x}$$

$$+ 8\sum_{n=1}^{\infty} (-1)^n n \left(\frac{2P}{a} \cos n\pi c + A_n\right)\left\{S_n(x) + C_n(x)\right\} \tag{62}$$

where $Sn(x)$ and $Cn(x)$ stand for the same integrals as in (46) of case (i). The normal stress σ_x along $y = 0$ for $c = \frac{1}{2}$ has been calculated corresponding to several values of x; the results are shown numerically in Table 6 and graphically in Fig. 7.

Case (iii) Semi-infinite Strip Subjected to Two Concentrated Tangential Loads

Consider the same infinite strip as in case (i) or case (ii) but subjected to two concentrated tangential loads each of V per unit thickness acting symmetrically at $y = \pm c$ on the edge $x = 0$ as shown in Fig. 6. By successive reflections about the longitudinal edges, the given strip and the resulting images occupy the entire right half-plane $x \geqslant 0$, and the given loads and their images become a series of concentrated tangential loads acting along the edge $x = 0$ at the points $(0, 2s \pm c)$, where s takes all positive and negative integral values including zero.

The stresses in such a semi-infinite plate may be obtained by superposing the function [4]

$$\chi' = \frac{V}{\pi} r\theta \cos\theta \tag{63}$$

which is the stress function for a semi-infinite plate under a concentrated tangential load V at a point on the straight edge. We thus obtain

$$\chi_0 = \frac{Va}{\pi}\Big\{ r\theta\cos\theta - r'\theta'\cos\theta' + \sum_{s=1}^{\infty}\Big[r_s\theta_s\cos\theta_s - r_s'\theta_s'\cos\theta_s' + r_{-s}\theta_{-s}\cos\theta_{-s} - r_{-s}'\theta_{-s}'\cos\theta_{-s}'\Big]\Big\} \tag{64}$$

where (r_s, θ_s) and (r_s', θ_s') are two sets of polar coordinates with poles at the points (0, 2s $\pm$c), respectively. In particular, (r, θ) and (r', θ') is written in place of (r_0, θ_0) and (r_0', θ_0'), respectively. These sets of polar coordinates are connected with (x, y) as shown in (3) of case (i).

The stress function can be expressed as the real part of a complex function as follows:

$$\chi_o = \frac{Va}{\pi} \mathcal{R}\left[ix \log(z+ci) - ix \log(z-ic)\right]$$

$$+ \frac{Va}{\pi} \mathcal{R} \sum_{s=1}^{\infty} ix \left[\log\left(1 + \frac{iz-c}{2s}\right) - \log\left(1 + \frac{iz+c}{2s}\right)\right.$$

$$\left. + \log\left(1 - \frac{iz-c}{2s}\right) - \log\left(1 - \frac{iz+c}{2s}\right)\right] \tag{65}$$

This stress function gives no normal and tangential stresses along the edge $x = 0$ besides the series of concentrated tangential loads. Furthermore, by symmetry, it gives no tangential stress along the lines $y = \pm 1$. The normal stresses given by this stress function are as follows:

$$\sigma_x = \frac{1}{a^2} \frac{\partial^2 \chi_o}{\partial y^2}$$

$$= \frac{V}{\pi a} \mathcal{R}\left[\frac{ix}{[x+i(y+c)]^2} - \frac{ix}{[x+i(y-c)]^2}\right.$$

$$+ \sum_{s=1}^{\infty} \left\{ \frac{ix}{[x+i(y-2s+c)]^2} - \frac{ix}{[x+i(y-2s-c)]^2} \right.$$

$$\left.\left. + \frac{ix}{[x+i(y+2s+c)]^2} - \frac{ix}{[x+i(y+2s-c)]^2} \right\}\right] \tag{66}$$

$$\sigma_y = \frac{1}{a^2}\frac{\partial^2 \chi_0}{\partial x^2}$$

$$= \frac{V}{\pi a}\mathcal{R}\Big[\frac{2i}{x+i(y+c)} - \frac{xi}{[x+i(y+c)]^2} - \frac{2i}{x+i(y-c)} + \frac{xi}{[x+i(y-c)]^2}$$

$$+\sum_{s=1}^{\infty}\Big\{\frac{2i}{x+i(y-2s+c)} - \frac{xi}{[x+i(y-2s+c)]^2} - \frac{2i}{x+i(y-2s-c)} + \frac{xi}{[x+i(y-2s-c)]^2}$$

$$+\frac{2i}{x+i(y+2s+c)} - \frac{xi}{[x+i(y+2s+c)]^2} - \frac{2i}{x+i(y+2s-c)} + \frac{xi}{[x+i(y+2s-c)]^2}\Big\}\Big] \quad (67)$$

These series are summable and the sums are

$$\sigma_x = \frac{V\pi}{4a}\mathcal{R}\left[ix\operatorname{csch}^2\frac{\pi[x+i(y-c)]}{2} - ix\operatorname{csch}^2\frac{\pi[x+i(y+c)]}{2}\right]$$

$$= \frac{V\pi}{2a}\left\{\frac{x\sinh\pi x\sin\pi(y-c)}{[\cosh\pi x - \cos\pi(y-c)]^2} - \frac{x\sinh\pi x\sin\pi(y+c)}{[\cosh\pi x - \cos\pi(y+c)]^2}\right\} \quad (68)$$

$$\sigma_y = \frac{V}{a}\mathcal{R}\Big[i\coth\frac{\pi[x+i(y+c)]}{2} - i\coth\frac{\pi[x+i(y-c)]}{2}$$

$$+\frac{i\pi x}{4}\operatorname{csch}^2\frac{\pi[x+i(y+c)]}{2} + \frac{i\pi x}{4}\operatorname{csch}^2\frac{\pi[x+i(y-c)]}{2}\Big]$$

$$= \frac{V}{a}\Big\{\frac{\sin\pi(y+c)}{\cosh\pi x - \cos\pi(y+c)} - \frac{\sin\pi(y-c)}{\cosh\pi x - \cos\pi(y-c)}$$

$$+\frac{\pi}{2}\frac{x\sinh\pi x\sin\pi(y-c)}{[\cosh\pi x - \cos\pi(y-c)]^2} - \frac{\pi}{2}\frac{x\sinh\pi x\sin\pi(y+c)}{[\cosh\pi x - \cos\pi(y+c)]^2}\Big\} \quad (69)$$

At x infinity, σ_x tends to zero. Along the lines $y = \pm 1$, the normal stress is

$$\left[\sigma_y\right]_{y=\pm 1} = -\frac{V}{a}\left[\frac{2\sin \pi c}{\cosh \pi x + \cos \pi c} - \frac{\pi x \sinh \pi x \sin \pi c}{(\cosh \pi x + \cos \pi c)^2}\right] \tag{70}$$

Now, consider the strip in question. Suppose that a second biharmonic function χ_1 is added to χ_o so that the required stress function for the given strip is

$$\chi = \chi_o + \chi_1 \tag{71}$$

The function χ_1 in (28) of case (i) may be used here directly and the normal stresses $[\sigma_x]_{x=0}$ and $[\sigma_y]_{y=\pm 1}$ derived from χ_1 are also in the same form as (29) and (30) of case (i).

Hence, the normal stress σ_x along the edge $x = 0$ becomes zero provided that by Fourier expansion from -1 to 1, for $n \geqslant 1$

$$A_n = -2\int_0^\infty \int_{-1}^{1} \frac{\phi_m \sinh m}{\sinh 2m + 2m}\Big\{(1 - m\coth m)\cosh my$$

$$+ my \sinh my\Big\} \cos n\pi y \, dy \, dm \tag{72}$$

Again, the normal stress σ_y given by χ along the edges $y = \pm 1$ becomes zero provided that $A_o = 0$ and by Fourier cosine transform

$$\phi_m = -\frac{4}{\pi}\int_0^\infty \frac{\sin \pi c}{\cosh \pi x + \cos \pi c} \cos mx\,dx$$

$$+ 2\int_0^\infty \frac{x \sinh \pi x \sin \pi c}{(\cosh \pi x + \cos \pi c)^2} \cos mx\,dx$$

$$-\frac{2}{\pi}\sum_{n=1}^\infty (-1)^n A_n \int_0^\infty (1-n\pi x)e^{-n\pi x} \cos mx\,dx \tag{73}$$

On performing the integration, the following system of equations is obtained:

$$\phi_m = 4m^2 \sum_{n=1}^\infty \frac{(-1)^n n}{(m^2+n^2\pi^2)^2}\left[\frac{V}{a}\left(3+\frac{n^2\pi^2}{m^2}\right)\sin n\pi c - A_n\right] \tag{74}$$

$$A_n = -8(-1)^n n^2\pi^2 \int_0^\infty \frac{\phi_m\, m \sinh^2 m\,dm}{(m^2+n^2\pi^2)^2(\sinh 2m + 2m)}$$

$$= -32(-1)^n n^2\pi^2 \sum_{k=1}^\infty (-1)^k k \left[\left(\frac{3V}{a}\sin k\pi c - A_k\right)\int_0^\infty \frac{m^3 \sinh^2 m\,dm}{(m^2+n^2\pi^2)^2(m^2+k^2\pi^2)^2(\sinh 2m+2m)}\right.$$

$$\left. + \frac{V\pi^2k^2}{a}\sin k\pi c \int_0^\infty \frac{m \sinh^2 m\,dm}{(m^2+n^2\pi^2)^2(m^2+k^2\pi^2)^2(\sinh 2m+2m)}\right] \tag{75}$$

Now if we denote

$$A_n^* = \frac{(-1)^n}{\sqrt{n}} A_n \tag{76}$$

$$I_{n,k} = 32\pi^2 (nk)^{3/2} \int_0^\infty \frac{m^3 \sinh^2 m \, dm}{(m^2+n^2\pi^2)^2(m^2+k^2\pi^2)^2(\sinh 2m+2m)} \tag{77}$$

$$J_{n,k} = \frac{32\pi^4}{3} (nk)^{5/2} \int_0^\infty \frac{m \sinh^2 m \, dm}{(m^2+n^2\pi^2)^2(m^2+k^2\pi^2)^2(\sinh 2m+2m)} \tag{78}$$

Equ. (75) then becomes

$$A_n^* = \sum_{k=1}^{\infty} \left[\frac{-3(-1)^k V \sin k\pi c}{a k^{1/2}} \left(I_{n,k} + \frac{k}{n} J_{n,k} \right) + A_k^* I_{n,k} \right] \tag{79}$$

Here $I_{n,k}$ is the same as in case (i) but $J_{n,k}$ has not appeared before. The evaluation of $J_{n,k}$ is given in Appendix 6. Table 2 shows the values of $J_{n,k}$* which are related to $J_{n,k}$ by

$$J_{n,k}^* = \frac{9}{8}\pi^2 (nk)^{1/2} J_{n,k}$$

Eq. (39) may be solved by successive approximation with

$$A_n^{*(0)} = -\frac{3V}{a} \sum_{k=1}^{\infty} \frac{(-1)^k \sin k\pi c}{k^{1/2}} \left(I_{n,k} + \frac{k}{n} J_{n,k} \right) \tag{80}$$

$$A_n^{*(p)} = \sum_{k=1}^{\infty} I_{n,k} A_k^{*(p-1)} , \qquad (p \geqslant 1)$$

We have

$$A_n^* = \sum_{p=0}^{\infty} A_n^{*(p)} \tag{81}$$

From (80) and (76) the values of An can be calculated and the results are tabulated in Table 3. Note that Ao = 0 has been determined previously. The stresses at any location within the strip can now be derived by differentiation from the stress function χ. In particular, the normal stresses σx and σy along the longitudinal axis y = 0 are:

$$\begin{aligned}\left[\sigma_x\right]_{y=0} = &-\frac{V\pi}{a}\frac{x\sinh\pi x\sin\pi c}{(\cosh\pi x-\cos\pi c)^2}-\sum_{n=1}^{\infty}A_n(1+n\pi x)e^{-n\pi x}\\&-8\sum_{n=1}^{\infty}(-1)^n n\left(\frac{3V}{a}\sin n\pi c-A_n\right)\left\{S_n(x)-C_n(x)\right\}\\&-\frac{8\pi^2V}{a}\sum_{n=1}^{\infty}(-1)^n n^3\sin n\pi c\left\{S_n^*(x)-C_n^*(x)\right\}\end{aligned} \tag{82}$$

$$\begin{aligned}\left[\sigma_y\right]_{y=0} = &\frac{V\sin\pi c}{a}\left[\frac{2}{\cosh\pi x-\cos\pi c}-\frac{\pi x\sinh\pi x}{(\cosh\pi x-\cos\pi c)^2}\right]\\&-\sum_{n=1}^{\infty}A_n(1-n\pi x)e^{-n\pi x}-8\sum_{n=1}^{\infty}(-1)^n n\left(\frac{3V}{a}\sin n\pi c-A_n\right)\left\{S_n(x)+C_n(x)\right\}\\&-\frac{8\pi^2V}{a}\sum_{n=1}^{\infty}(-1)^n n^3\sin n\pi c\left\{S_n^*(x)+C_n^*(x)\right\}\end{aligned} \tag{83}$$

where $Sn(x)$ and $Cn(x)$ stand for the same integrals as in (46) of case (i), but $S_n{}^*(x)$ and $C_n{}^*(x)$ are new and represent:

$$S_n{}^*(x) = \int_0^\infty \frac{\sinh m \cos mx \, dm}{(m^2+n^2\pi^2)^2(\sinh 2m+2m)}$$

$$C_n{}^*(x) = \int_0^\infty \frac{m \cosh m \cos mx \, dm}{(m^2+n^2\pi^2)^2(\sinh 2m+2m)} \tag{84}$$

the evaluation of which is shown in Appendix 2(vi) and the numerical values of which are tabulated in Table 4. The normal stress σ_y along $y = 0$ for $c = 1/2$ has been calculated corresponding to several values of x. The results are shown numerically in Table 6 and graphically in Fig. 8.

CONCLUSION

An analytic solution of the stress distribution in a semi-infinite strip loaded on the transverse edge is presented by using a method of images. Three cases of loading on the transverse edge are considered in detail. It is found that in each case, when the boundary conditions are adjusted, a system of algebraic equations for the parametric coefficients involved in the stress function is obtained. This system is similar in form in all the cases and can be solved by method of successive approximations. The values of the coefficients in all the three cases are alternately positive and negative. The convergence of the coefficients is rather slow. Nevertheless, in computing the stresses these values are to be multiplied by an exponential function so that the resulting terms converge very rapidly. Accurate values of the stresses can therefore be obtained without difficulty.

In each case the normal stress σ_x or σ_y along the longitudinal axis is computed. When the values of σ_x in the case (i) are compared with those obtained by Theocaris [15], it is found that the values by Theocaris check well with the present results.

From Figure 1, it is seen that in the case (i) the greatest value of σ_x along the longitudinal axis occurs at the point where the uniform load is applied. This stress drops rapidly with x and becomes substantially uniform at x equal to the width of the strip. In the case (ii), this stress increases rapidly from zero to a uniform stress at x equal to one-half of the width of the strip. From Figure 3, it is seen that the largest value of σ_y occurs at the origin and drops very rapidly to zero at x equal to about 0.9 of the half-width and then further drops to a small negative value until finally it rises asymptotically to zero again at x equal to about one and a half times the width of the strip.

FIGURES

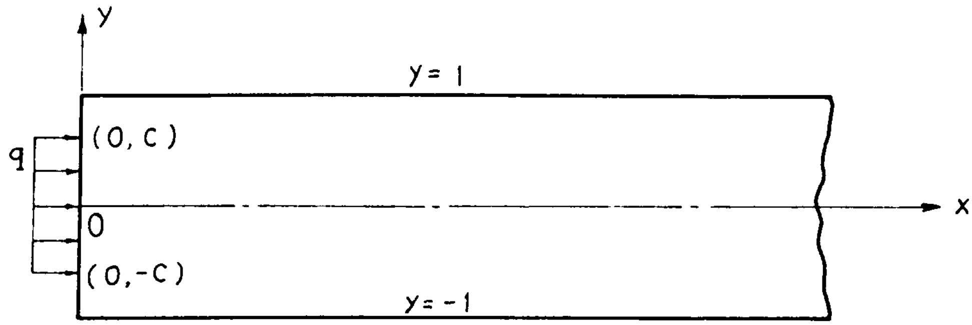

Fig. 1. The Semi-infinite Strip under Segment of Symmetrically placed Uniform Load; Length Measured in Unit a. (Longitudinal edges are free from traction.)

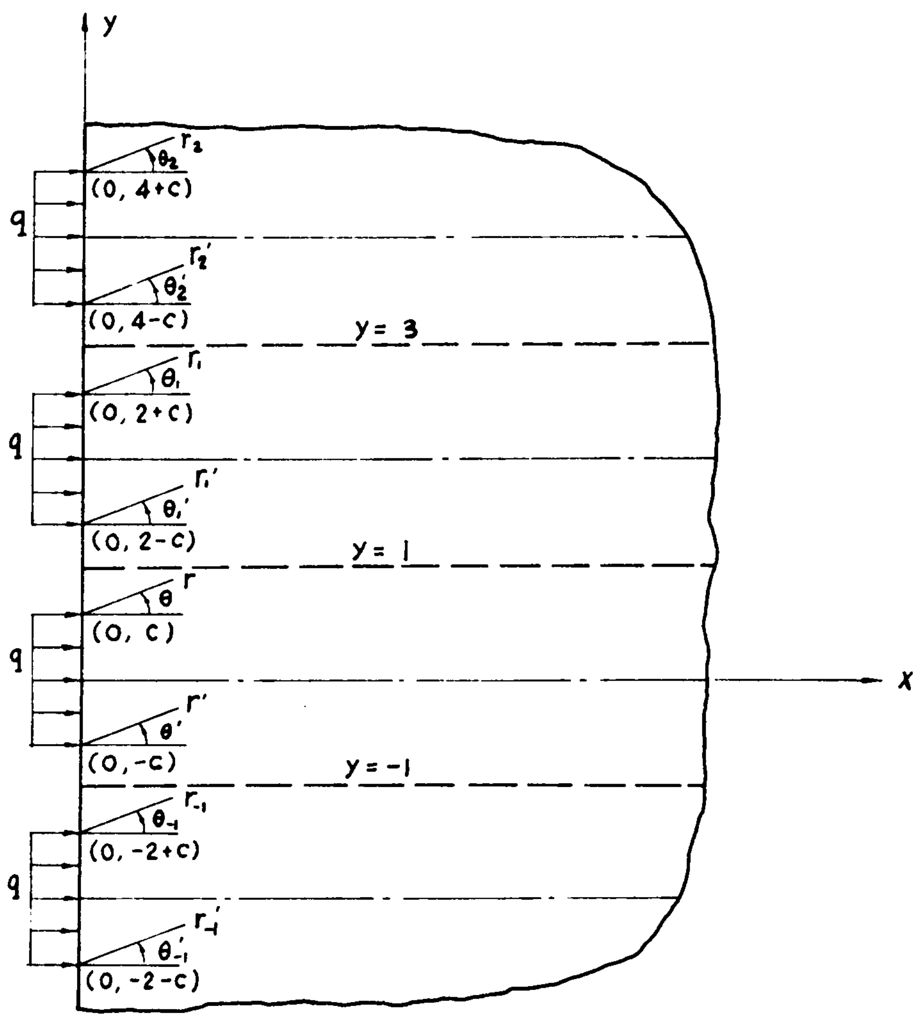

Fig. 2. The Semi-infinite Plate under a Series of Uniform Load

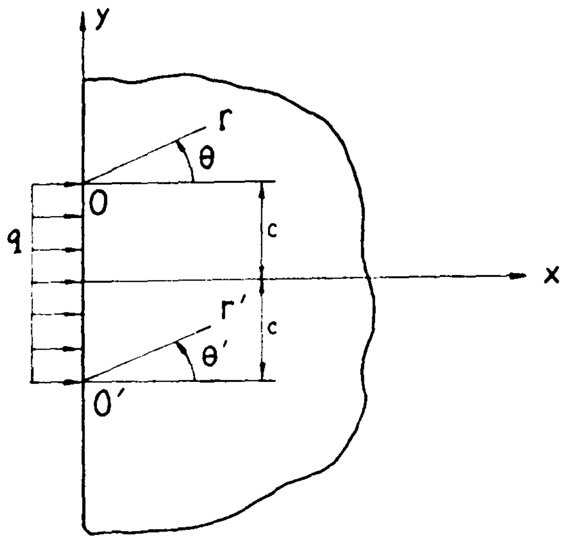

Fig.3. A Semi-infinite Plate Under a Segment of Uniform Load

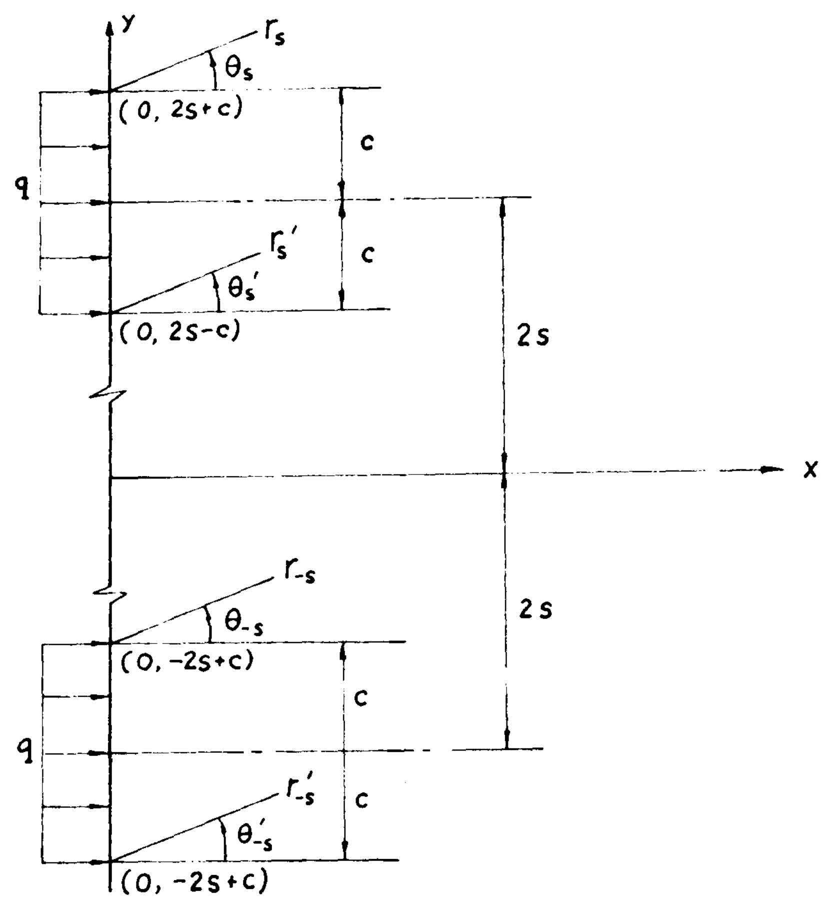

Fig. 4. The Relation between Polar Coordinates and (x, y) Coordinate

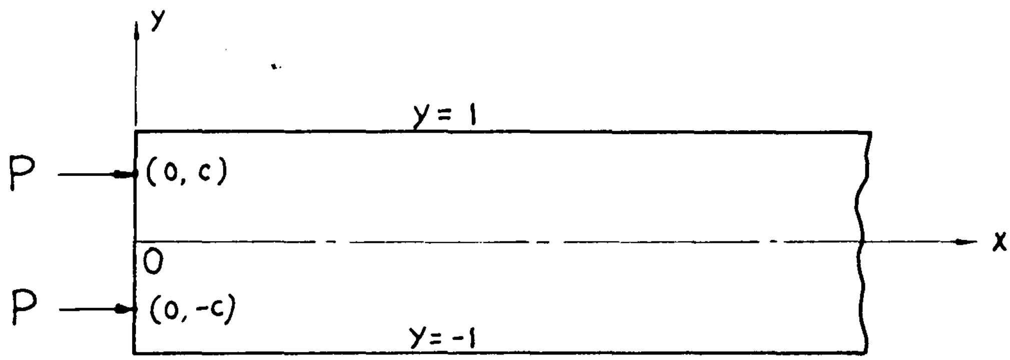

Fig. 5. The Semi-infinite Strip under Two Symmetrical Concentrated Loads; Length Being Measured in Unit a. (Longitudinal edges are traction free.)

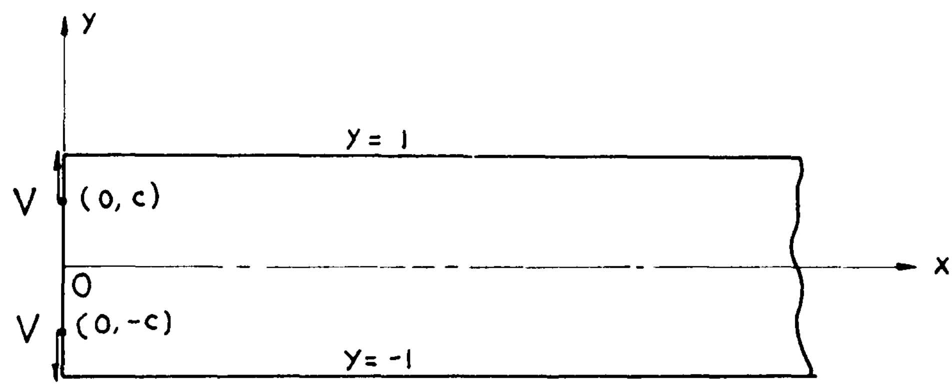

Fig. 6. The Semi-infinite Strip under Two Symmetrical Concentrated Tangential Loads; Length Being Measured in Unit a. (Longitudinal edges are traction free.)

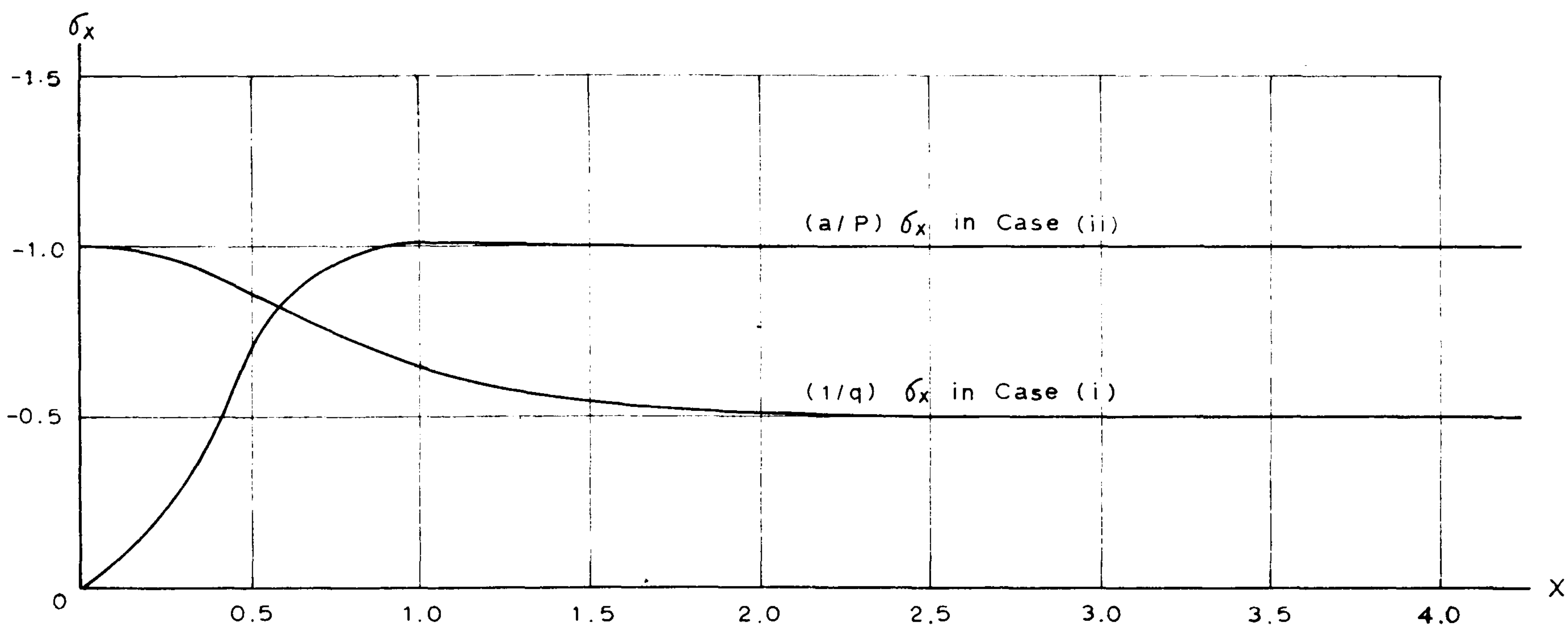

Fig. 7. Normal Stress σ_x along x-axis for Case (i) and Casc (ii)

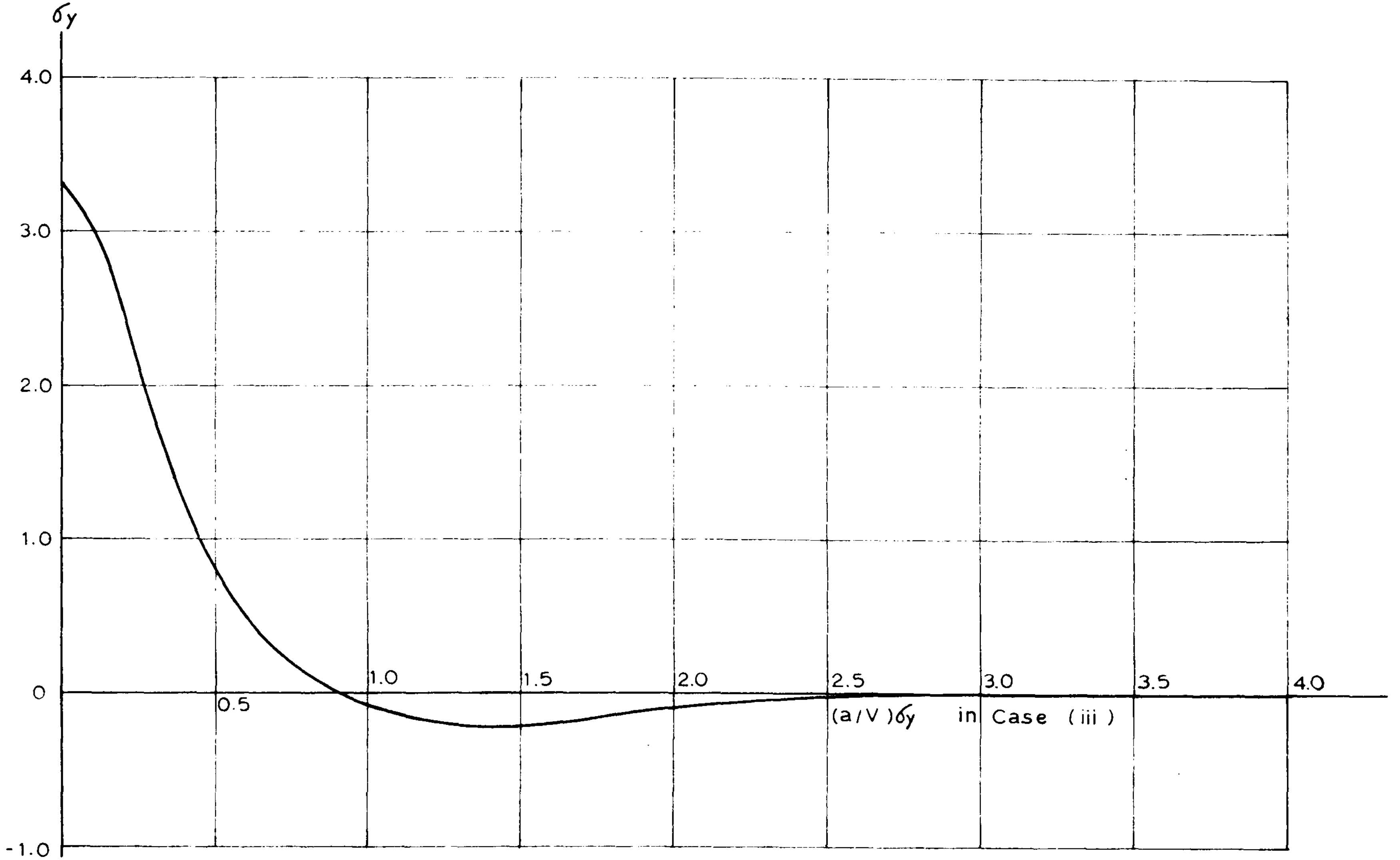

Fig. 8. The Normal Stress σ_y along x-axis for Case (iii)

K \ N	1	2	3	4	5	6	7	8	9	10
1	0.908997	0.793520	0.612861	0.477389	0.380755	0.310749	0.258712	0.219043	0.188109	0.163505
2	0.793520	0.985858	0.929558	0.823718	0.719255	0.627915	0.550796	0.486242	0.432135	0.386545
3	0.612861	0.929558	0.996320	0.965345	0.900361	0.826956	0.755279	0.689081	0.629422	0.576234
4	0.477389	0.823718	0.965345	0.998692	0.979453	0.936125	0.883101	0.827626	0.773300	0.721825
5	0.380755	0.719255	0.900361	0.979453	0.999431	0.986407	0.955614	0.915871	0.872293	0.827825
6	0.310749	0.627915	0.826956	0.936125	0.986407	0.999716	0.990333	0.967371	0.936605	0.901696
7	0.258712	0.550796	0.755279	0.883101	0.955614	0.990333	0.999843	0.992778	0.975014	0.950538
8	0.219043	0.486242	0.689081	0.827626	0.915871	0.967371	0.992778	0.999906	0.994393	0.980252
9	0.188109	0.432135	0.629422	0.773300	0.872293	0.936605	0.975014	0.994393	0.999939	0.995517
10	0.163505	0.386545	0.576234	0.721825	0.827825	0.901696	0.950538	0.980252	0.995517	0.999958
11	0.143598	0.347868	0.529009	0.673906	0.784139	0.864974	0.922045	0.960341	0.984002	0.996344
12	0.127250	0.314818	0.487105	0.629727	0.742152	0.827899	0.891358	0.936691	0.967498	0.986779
13	0.113647	0.286373	0.449873	0.589208	0.702330	0.791382	0.859709	0.910738	0.947573	0.972873
14	0.102198	0.261724	0.416715	0.552140	0.664872	0.755977	0.827931	0.883517	0.925384	0.955882
15	0.092464	0.240225	0.387100	0.518255	0.629816	0.722000	0.796568	0.855747	0.901774	0.936707
16	0.084112	0.221359	0.360566	0.487277	0.597106	0.689618	0.765980	0.827943	0.877380	0.916076
17	0.076889	0.204712	0.336715	0.458929	0.566638	0.658900	0.736394	0.800464	0.852659	0.894520
18	0.070595	0.189945	0.315205	0.432955	0.538281	0.629847	0.707939	0.773551	0.827948	0.872451
19	0.065074	0.176782	0.295744	0.409120	0.511891	0.602427	0.680686	0.747370	0.803497	0.850185
20	0.060204	0.164996	0.278084	0.387210	0.487326	0.576578	0.654661	0.722026	0.779476	0.827950

K \ N	11	12	13	14	15	16	17	18	19	20
1	0.143598	0.127250	0.113647	0.102198	0.092464	0.084112	0.076889	0.070595	0.065074	0.060204
2	0.347868	0.314818	0.286373	0.261724	0.240225	0.221359	0.204712	0.189945	0.176782	0.164996
3	0.529009	0.487105	0.449873	0.416715	0.387100	0.360566	0.336715	0.315205	0.295744	0.278084
4	0.673906	0.629727	0.589208	0.552140	0.518255	0.487277	0.458929	0.432955	0.409120	0.387210
5	0.784139	0.742152	0.702330	0.664872	0.629816	0.597106	0.566638	0.538281	0.511891	0.487326
6	0.864974	0.827899	0.791382	0.755977	0.722000	0.689618	0.658900	0.629847	0.602427	0.576578
7	0.922045	0.891358	0.859709	0.827931	0.796568	0.765980	0.736394	0.707939	0.680686	0.654661
8	0.960341	0.936691	0.910738	0.883517	0.855747	0.827943	0.800464	0.773551	0.747370	0.722026
9	0.984002	0.967498	0.947573	0.925384	0.901774	0.877380	0.852659	0.827948	0.803497	0.779476
10	0.996344	0.986779	0.972873	0.955882	0.936707	0.916076	0.894520	0.872451	0.850185	0.827950
11	0.999968	0.996953	0.988874	0.977031	0.962360	0.945642	0.927475	0.908320	0.888537	0.868409
12	0.996953	0.999973	0.997462	0.990544	0.980300	0.967524	0.952824	0.936715	0.919593	0.901779
13	0.988874	0.997462	0.999976	0.997781	0.991823	0.982904	0.971673	0.958659	0.944288	0.928908
14	0.977031	0.990544	0.997781	0.999976	0.998072	0.992871	0.985027	0.975081	0.963478	0.950583
15	0.962360	0.980300	0.991823	0.998072	0.999973	0.998306	0.993726	0.986775	0.977906	0.967499
16	0.945642	0.967524	0.982904	0.992871	0.998306	0.999970	0.998497	0.994433	0.988230	0.980274
17	0.927475	0.952824	0.971673	0.985027	0.993726	0.998497	0.999964	0.998654	0.995023	0.989454
18	0.908320	0.936715	0.958659	0.975081	0.986775	0.994433	0.998654	0.999957	0.998783	0.995519
19	0.888537	0.919593	0.944288	0.963478	0.977906	0.988230	0.995023	0.998783	0.999948	0.998891
20	0.868409	0.901779	0.928908	0.950583	0.967499	0.980274	0.989454	0.995519	0.998891	0.999938

K \ N	1	2	3	4	5	6	7	8	9	10
1	0.681975	0.674845	0.575458	0.487695	0.418928	0.365425	0.323196	0.289256	0.261495	0.238426
2	0.674845	0.876495	0.864820	0.801682	0.731194	0.665499	0.607395	0.556841	0.512994	0.474875
3	0.575458	0.864820	0.937810	0.927759	0.886950	0.836322	0.784509	0.735031	0.689188	0.647300
4	0.487695	0.801682	0.927759	0.963198	0.955398	0.927259	0.890165	0.849914	0.809472	0.770333
5	0.418928	0.731194	0.886950	0.955398	0.975847	0.969822	0.949347	0.921270	0.889555	0.856518
6	0.365425	0.665499	0.836322	0.927259	0.969822	0.982987	0.978261	0.962731	0.940828	0.915356
7	0.323196	0.607395	0.784509	0.890165	0.949347	0.978261	0.987391	0.983604	0.971437	0.953914
8	0.289256	0.556841	0.735031	0.849914	0.921270	0.962731	0.983604	0.990291	0.987201	0.977416
9	0.261495	0.512994	0.689188	0.809472	0.889555	0.940828	0.971437	0.987201	0.992298	0.989731
10	0.238426	0.474875	0.647300	0.770333	0.856518	0.915356	0.953914	0.977416	0.989731	0.993743
11	0.218984	0.441580	0.609256	0.733204	0.823509	0.888099	0.933076	0.963096	0.981697	0.991586
12	0.202398	0.412338	0.574772	0.698373	0.791303	0.860196	0.910309	0.945767	0.969785	0.984868
13	0.188093	0.386508	0.543506	0.665895	0.760328	0.832371	0.886553	0.926515	0.955164	0.974810
14	0.175637	0.363561	0.515116	0.635709	0.730802	0.805078	0.862446	0.906115	0.938699	0.962316
15	0.164700	0.343064	0.489281	0.607691	0.702812	0.778598	0.838423	0.885118	0.921030	0.948089
16	0.155023	0.324663	0.465711	0.581694	0.676369	0.753090	0.814775	0.863919	0.902627	0.932657
17	0.146403	0.308063	0.444150	0.557559	0.651438	0.728638	0.791691	0.842798	0.883842	0.916425
18	0.138679	0.293021	0.424372	0.535133	0.627957	0.705278	0.769295	0.821951	0.864936	0.899702
19	0.131718	0.279335	0.406180	0.514269	0.605849	0.683007	0.747657	0.801515	0.846097	0.882722
20	0.125414	0.266834	0.389403	0.494831	0.585033	0.661805	0.726815	0.781583	0.827468	0.865666

K \ N	11	12	13	14	15	16	17	18	19	20
1	0.218984	0.202398	0.188093	0.175637	0.164700	0.155023	0.146403	0.138679	0.131718	0.125414
2	0.441580	0.412338	0.386508	0.363561	0.343064	0.324663	0.308063	0.293021	0.279335	0.266834
3	0.609256	0.574772	0.543506	0.515116	0.489281	0.465711	0.444150	0.424372	0.406180	0.389403
4	0.733204	0.698373	0.665895	0.635709	0.607691	0.581694	0.557559	0.535133	0.514269	0.494831
5	0.823509	0.791303	0.760328	0.730802	0.702812	0.676369	0.651438	0.627957	0.605849	0.585033
6	0.888099	0.860196	0.832371	0.805078	0.778598	0.753090	0.728638	0.705278	0.683007	0.661805
7	0.933076	0.910309	0.886553	0.862446	0.838423	0.814775	0.791691	0.769295	0.747657	0.726815
8	0.963096	0.945767	0.926515	0.906115	0.885118	0.863919	0.842798	0.821951	0.801515	0.781583
9	0.981697	0.969785	0.955164	0.938699	0.921030	0.902627	0.883842	0.864936	0.846097	0.827468
10	0.991586	0.984868	0.974810	0.962316	0.948089	0.932657	0.916425	0.899702	0.882722	0.865666
11	0.994818	0.992969	0.987290	0.978674	0.967886	0.955478	0.941897	0.927491	0.912528	0.897223
12	0.992969	0.995639	0.994020	0.989153	0.981709	0.972295	0.961390	0.949356	0.936498	0.923048
13	0.987290	0.994020	0.996279	0.994896	0.990653	0.984150	0.975870	0.966206	0.955473	0.943930
14	0.978674	0.989153	0.994896	0.996788	0.995573	0.991857	0.986128	0.978790	0.970171	0.960544
15	0.967886	0.981709	0.990653	0.995573	0.997200	0.996125	0.992842	0.987758	0.981210	0.973478
16	0.955478	0.972295	0.984150	0.991857	0.996125	0.997537	0.996579	0.993659	0.989117	0.983239
17	0.941897	0.961390	0.975870	0.986128	0.992842	0.996579	0.997817	0.996958	0.994343	0.990261
18	0.927491	0.949356	0.966206	0.978790	0.987758	0.993659	0.996958	0.998052	0.997277	0.994923
19	0.912528	0.936498	0.955473	0.970171	0.981210	0.989117	0.994343	0.997277	0.998251	0.997549
20	0.897223	0.923048	0.943930	0.960544	0.973478	0.983239	0.990261	0.994923	0.997549	0.998421

Table 3. Values of the Coefficients A_n

N	Case (i)	Case (ii)	Case (iii)
	$(1/q)\,A_n$	$(a/P)\,A_n$	$(a/V)\,A_n$
1	0.09319	0.11232	-0.49531
2	-0.08362	-0.12929	0.36570
3	0.06666	0.11669	-0.26402
4	-0.05372	-0.10076	0.19985
5	0.04437	0.08673	-0.15815
6	-0.03749	-0.07521	0.12965
7	0.03229	0.06589	-0.10924
8	-0.02825	-0.05832	0.09403
9	0.02504	0.05209	-0.08231
10	-0.02243	-0.04692	0.07303
11	0.02027	0.04258	-0.06552
12	-0.01846	-0.03868	0.05932
13	0.01692	0.03571	-0.05412
14	-0.01559	-0.03296	0.04969
15	0.01444	0.03056	-0.04588
16	-0.01343	-0.02845	0.04257
17	0.01254	0.02657	-0.03967
18	-0.01175	-0.02491	0.03710
19	0.01104	0.02341	-0.03481
20	-0.01040	-0.02206	0.03276

Table 4. Values of $S_n(x)$, $S_n^*(x)$, $C_n(x)$, $C_n^*(x)$

x	N	$S_n(X)$	$S_n^*(x)$	$C_n(X)$	$C_n^*(X)$
0	1	0.60316E-02†	0.38233E-02	0.13825E-01	0.54089E-02
	2	0.71700E-03	0.29764E-03	0.19743E-02	0.47221E-03
	3	0.17410E-03	0.62623E-04	0.51685E-03	0.10362E-03
	4	0.60337E-04	0.20336E-04	0.18601E-03	0.34307E-04
	5	0.25919E-04	0.64387E-05	0.81618E-04	0.14384E-04
	6	0.12851E-04	0.40997E-05	0.40999E-04	0.70296E-05
	7	0.70595E-05	0.22231E-05	0.22713E-04	0.38263E-05
	8	0.41870E-05	0.13071E-05	0.13550E-04	0.22547E-05
	9	0.26355E-05	0.81773E-06	0.85642E-05	0.14130E-05
	10	0.17395E-05	0.53731E-06	0.56698E-05	0.92921E-06
	11	0.11934E-05	0.36741E-06	0.38988E-05	0.63597E-06
	12	0.84552E-06	0.25963E-06	0.27672E-05	0.44914E-06
	13	0.61552E-06	0.18863E-06	0.20173E-05	0.32640E-06
	14	0.45860E-06	0.14031E-06	0.15047E-05	0.24249E-06
	15	0.34861E-06	0.10652E-06	0.11448E-05	0.18413E-06
0.25	1	0.49964E-02	0.36404E-02	0.10464E-01	0.49953E-02
	2	0.53634E-03	0.27623E-03	0.12706E-02	0.41448E-03
	3	0.12313E-03	0.57466E-04	0.30147E-03	0.88704E-04
	4	0.41323E-04	0.18557E-04	0.10213E-03	0.28980E-04
	5	0.17408E-04	0.76768E-05	0.43120E-04	0.12057E-04
	6	0.85250E-05	0.37226E-05	0.21119E-04	0.58654E-05
	7	0.46441E-05	0.20162E-05	0.11498E-04	0.31828E-05
	8	0.27385E-05	0.11845E-05	0.67757E-05	0.18722E-05
	9	0.17166E-05	0.74059E-06	0.42441E-05	0.11716E-05
	10	0.11295E-05	0.48643E-06	0.27910E-05	0.77001E-06
	11	0.77308E-06	0.33251E-06	0.19093E-05	0.52659E-06
	12	0.54672E-06	0.23492E-06	0.13498E-05	0.37217E-06
	13	0.39742E-06	0.17064E-06	0.98083E-06	0.27041E-06
	14	0.29576E-06	0.12691E-06	0.72972E-06	0.20116E-06
	15	0.22461E-06	0.96336E-07	0.55404E-06	0.15272E-06
0.50	1	0.25906E-02	0.31525E-02	0.33624E-02	0.39478E-02
	2	0.16347E-03	0.22232E-03	0.88103E-06	0.28033E-03
	3	0.25102E-04	0.44864E-04	-0.50273E-04	0.55645E-04
	4	0.62803E-05	0.14280E-04	-0.26497E-04	0.17492E-04
	5	0.21394E-05	0.58622E-05	-0.13546E-04	0.71233E-05
	6	0.89782E-06	0.28299E-05	-0.73575E-05	0.34209E-05
	7	0.43658E-06	0.15283E-05	-0.42655E-05	0.18411E-05
	8	0.23648E-06	0.89615E-06	-0.26188E-05	0.10769E-05
	9	0.13896E-06	0.55957E-06	-0.16876E-05	0.67130E-06
	10	0.86984E-07	0.36718E-06	-0.11327E-05	0.43994E-06
	11	0.57250E-07	0.25081E-06	-0.78671E-06	0.30022E-06
	12	0.39243E-07	0.17710E-06	-0.56260E-06	0.21183E-06
	13	0.27817E-07	0.12859E-06	-0.41254E-06	0.15371E-06
	14	0.20279E-07	0.95603E-07	-0.30913E-06	0.11423E-06
	15	0.15141E-07	0.72549E-07	-0.23608E-06	0.86651E-07

† E-02 STAND FOR $\times 10^{-2}$

Table 4. Values of $S_n(x)$, $S_n^*(x)$, $C_n(x)$, $C_n^*(x)$. (cont'd)

X	N	Sn(X)	Sn* (X)	Cn (X)	Cn* (X)
1.0	1	-0.14530E-02	0.18388E-02	-0.51147E-02	0.15572E-02
	2	-0.24824E-03	0.10032E-03	-0.75203E-03	0.46229E-04
	3	-0.60232E-04	0.18486E-04	-0.17376E-03	0.50547E-05
	4	-0.20392E-04	0.56665E-05	-0.57172E-04	0.10659E-05
	5	-0.85995E-05	0.22837E-05	-0.23702E-04	0.33064E-06
	6	-0.42097E-05	0.10912E-05	-0.11480E-04	0.13125E-06
	7	-0.22921E-05	0.58562E-06	-0.62078E-05	0.61611E-07
	8	-0.13509E-05	0.34199E-06	-0.36418E-05	0.32592E-07
	9	-0.84650E-06	0.21295E-06	-0.22744E-05	0.18837E-07
	10	-0.55683E-06	0.13945E-06	-0.14925E-05	0.11650E-07
	11	-0.38105E-06	0.95113E-07	-0.10195E-05	0.75986E-08
	12	-0.26943E-06	0.67085E-07	-0.71986E-06	0.51726E-08
	13	-0.19583E-06	0.48665E-07	-0.52264E-06	0.36467E-08
	14	-0.14572E-06	0.36156E-07	-0.38857E-06	0.26470E-08
	15	-0.11066E-06	0.27422E-07	-0.29486E-06	0.19693E-08
2.0	1	-0.12879E-02	0.27814E-03	-0.13635E-02	-0.13776E-03
	2	-0.71764E-04	0.50103E-05	-0.19638E-04	-0.21368E-04
	3	-0.12740E-04	0.58419E-06	0.50996E-06	-0.43486E-05
	4	-0.38475E-05	0.14442E-06	0.62096E-06	-0.13761E-05
	5	-0.15402E-05	0.51865E-07	0.33645E-06	-0.56272E-06
	6	-0.73330E-06	0.23152E-07	0.18301E-06	-0.27102E-06
	7	-0.39273E-06	0.11900E-07	0.10540E-06	-0.14615E-06
	8	-0.22904E-06	0.67511E-08	0.64262E-07	-0.85616E-07
	9	-0.14248E-06	0.41191E-08	0.41171E-07	-0.53424E-07
	10	-0.93246E-07	0.26579E-08	0.27503E-07	-0.35039E-07
	11	-0.63569E-07	0.17929E-08	0.19031E-07	-0.23926E-07
	12	-0.44820E-07	0.12539E-08	0.13569E-07	-0.16890E-07
	13	-0.32504E-07	0.90354E-09	0.99261E-08	-0.12260E-07
	14	-0.24144E-07	0.66775E-09	0.74235E-08	-0.91139E-08
	15	-0.18308E-07	0.50426E-09	0.56600E-08	-0.69152E-08
3.0	1	-0.17967E-03	0.86427E-05	0.11065E-03	-0.69918E-04
	2	-0.22665E-05	-0.10571E-05	0.16553E-04	-0.31282E-05
	3	-0.17171E-06	-0.22252E-06	0.32676E-05	-0.52571E-06
	4	-0.27790E-07	-0.71045E-07	0.10239E-05	-0.15611E-06
	5	-0.66533E-08	-0.29166E-07	0.41692E-06	-0.62024E-07
	6	-0.20082E-08	-0.14076E-07	0.20034E-06	-0.29412E-07
	7	-0.70039E-09	-0.76000E-08	0.10790E-06	-0.15714E-07
	8	-0.26627E-09	-0.44556E-08	0.63151E-07	-0.91497E-08
	9	-0.10495E-09	-0.27818E-08	0.39383E-07	-0.56859E-08
	10	-0.40250E-10	-0.18252E-08	0.25819E-07	-0.37182E-08
	11	-0.13091E-10	-0.12466E-08	0.17625E-07	-0.25334E-08
	12	-0.15331E-11	-0.88022E-09	0.12439E-07	-0.17854E-08
	13	0.32337E-11	-0.63907E-09	0.90278E-08	-0.12944E-08
	14	0.49637E-11	-0.47513E-09	0.67100E-08	-0.96120E-09
	15	0.53314E-11	-0.36054E-09	0.50907E-08	-0.72872E-09

Table 4. Values of $Sn(x)$, $Sn^*(x)$, $Cn(x)$, $Cn^*(x)$. (cont'd)

X	N	$Sn(x)$	$Sn^*(x)$	$Cn(x)$	$Cn^*(x)$
4.0	1	-0.26161E-05	-0.29256E-05	0.49028E-04	-0.76134E-05
	2	0.84406E-06	-0.18559E-06	0.20091E-05	-0.95797E-08
	3	0.17125E-06	-0.31977E-07	0.32742E-06	0.95691E-08
	4	0.54107E-07	-0.95835E-08	0.96105E-07	0.40772E-08
	5	0.22112E-07	-0.38242E-08	0.37971E-07	0.18535E-08
	6	0.10646E-07	-0.18178E-08	0.17950E-07	0.94002E-09
	7	0.57404E-08	-0.97260E-09	0.95723E-08	0.52207E-09
	8	0.33623E-08	-0.56686E-09	0.55668E-08	0.31152E-09
	9	0.20979E-08	-0.35249E-09	0.34565E-08	0.19681E-09
	10	0.13759E-08	-0.23061E-09	0.22589E-08	0.13021E-09
	11	0.93946E-09	-0.15718E-09	0.15384E-08	0.89484E-10
	12	0.66317E-09	-0.11080E-09	0.10838E-08	0.63474E-10
	13	0.48139E-09	-0.80346E-10	0.78553E-09	0.46248E-10
	14	0.35784E-09	-0.59675E-10	0.58322E-09	0.34481E-10
	15	0.27151E-09	-0.45247E-10	0.44208E-09	0.26224E-10
5.0	1	0.22643E-05	-0.42239E-06	0.46325E-05	0.97861E-07
	2	0.12210E-06	-0.38045E-08	-0.34971E-07	0.45342E-07
	3	0.20495E-07	-0.57329E-10	-0.14084E-07	0.87897E-08
	4	0.60838E-08	0.47288E-10	-0.50896E-08	0.27396E-08
	5	0.24167E-08	0.30950E-10	-0.21937E-08	0.11130E-08
	6	0.11459E-08	0.17855E-10	-0.10851E-08	0.53417E-09
	7	0.61220E-09	0.10572E-10	-0.59431E-09	0.28747E-09
	8	0.35645E-09	0.65483E-11	-0.35159E-09	0.16818E-09
	9	0.22150E-09	0.42371E-11	-0.22085E-09	0.10485E-09
	10	0.14485E-09	0.28495E-11	-0.14553E-09	0.68722E-10
	11	0.98690E-10	0.19812E-11	-0.99719E-10	0.46904E-10
	12	0.69551E-10	0.14176E-11	-0.70578E-10	0.33098E-10
	13	0.50422E-10	0.10397E-11	-0.51337E-10	0.24020E-10
	14	0.37444E-10	0.77923E-12	-0.38224E-10	0.17851E-10
	15	0.28387E-10	0.59510E-12	-0.29040E-10	0.13542E-10

Table 5. Roots of $\sinh z + z = 0$ in the first quadrant

n	x_n	y_n
0	0	0
1	2.25072 861160	4.21239 223049
2	3.10314 874583	10.71253 739728
3	3.55108 734702	17.07336 485315
4	3.85880 899311	23.39835 522565
5	4.09370 492477	29.70811 982528
6	4.28378 158778	36.00986 601637
7	4.44344 583032	42.30682 671764
8	4.58110 457345	48.60068 412409
9	4.70209 646036	54.89240 578807
10	4.81002 513746	61.18259 019683
11	4.90743 841652	67.47162 863498
12	4.99620 440987	73.75978 834683
13	5.07773 373224	80.04725 843589
14	5.15311 770139	86.33417 669040
15	5.22321 798925	92.62064 601433
16	5.28872 685706	98.90674 489377
17	5.35020 884863	105.19253 428953
18	5.40813 039638	111.47806 230791
19	5.46288 131611	117.76336 744566
20	5.51479 071942	124.04848 089410
21	5.56413 899816	130.33342 820720
22	5.61116 698985	136.61823 053015
23	5.65608 308428	142.90290 551843
24	5.69906 880246	149.18746 803487
25	5.74028 322579	155.47193 068522
26	5.77986 654860	161.75630 423437
27	5.81794 295439	168.04059 793320
28	5.85462 296454	174.32481 977787
29	5.89000 537156	180.60897 671725
30	5.92417 884209	186.89307 482034
31	5.95722 325503	193.17711 941234
32	5.98921 082568	199.46111 518617
33	6.02020 705574	205.74506 629433
34	6.05027 154048	212.02897 642512
35	6.07945 865814	218.31284 886632
36	6.10781 816165	224.59668 655849
37	6.13539 568867	230.88049 214004
38	6.16223 320333	237.16426 798534
39	6.18836 938015	243.44801 623726
40	6.21383 993910	249.73173 883489
41	6.23867 793915	256.01543 753741
42	6.26291 403608	262.29911 394465
43	6.28657 670998	268.58276 951489
44	6.30969 246633	274.86640 558027
45	6.33228 601441	281.15002 336028
46	6.35438 042604	287.43362 397350
47	6.37599 727713	293.71720 844791
48	6.39715 677422	300.00077 772996
49	6.41787 786803	306.28433 269255
50	6.43817 835534	312.56787 414210
51	6.45807 497086	318.85140 282482
52	6.47758 347008	325.13491 943225
53	6.49671 870419	331.41842 460624
54	6.51549 468798	337.70191 894333
55	6.53392 466153	343.98540 299878
56	6.55202 114621	350.26887 729003
57	6.56979 599585	356.55234 229995
58	6.58726 044328	362.83579 847966
59	6.60442 514302	369.11924 625115
60	6.62130 021030	375.40268 600960
61	6.63789 525683	381.68611 812549
62	6.65421 942367	387.96954 294651
63	6.67028 141150	394.25296 079931
64	6.68608 950843	400.53637 199109
65	6.70165 161568	406.81977 681101
66	6.71697 527130	413.10317 553152
67	6.73206 767211	419.38656 840957
68	6.74693 569392	425.66995 568768
69	6.76158 591042	431.95333 759497
70	6.77602 461060	438.23671 434808
71	6.79025 781491	444.52008 615199
72	6.80429 129041	450.80345 320083
73	6.81813 056474	457.08681 567857
74	6.83178 093919	463.37017 375969
75	6.84524 750094	469.65352 760981
76	6.85853 513437	475.93687 738621
77	6.87164 853175	482.22022 323837
78	6.88459 220318	488.50356 530849
79	6.89737 048585	494.78690 373184
80	6.90998 755284	501.07023 863728
81	6.92244 742124	507.35357 014758
82	6.93475 395984	513.63689 837978
83	6.94691 089636	519.92022 344556
84	6.95892 182422	526.20354 545149
85	6.97079 020890	532.48686 449938
86	6.98251 939394	538.77018 068649
87	6.99411 260664	545.05349 410585
88	7.00557 296331	551.33680 484640
89	7.01690 347439	557.62011 299329
90	7.02810 704910	563.90341 862807
91	7.03918 650000	570.18672 182883
92	7.05014 454720	576.47002 267043
93	7.06098 382236	582.75332 122467
94	7.07170 687248	589.03661 756041
95	7.08231 616351	595.31991 174375
96	7.09281 408375	601.60320 383814
97	7.10320 294707	607.88649 390456
98	7.11348 499598	614.16978 200159
99	7.12366 240450	620.45306 818555
100	7.13373 728097	626.73635 251061

Table 6. The Normal Stress along the x-axis

x	Case (i)	Case (ii)	Case (iii)
	$(1/q)\,\sigma_x$	$(a/p)\,\sigma_x$	$(a/V)\,\sigma_y$
0	−1.000	0	3.317
0.25	−0.974	−0.228	2.098
0.50	−0.863	−0.713	0.781
1.00	−0.651	−1.011	−0.084
2.00	−0.510	−1.000	−0.079
3.00	−0.499	−0.999	−0.007
4.00	−0.500	−1.000	0.000
5.00	−0.500	−1.000	0.000
∞	−0.500	−1.000	0

APPENDIX 1

Modification of Stress Function χ_o for Convergence for Case (i)

The logarithmic term inside the summation sign of the following stress function

$$\chi_o = \frac{q a^2}{2\pi} R\left[i\left\{x^2+(y+c)^2\right\}\log(z+ci) - i\left\{x^2+(y-c)^2\right\}\log(z-ci)\right]$$

$$+ \frac{q a^2}{2\pi} R \sum_{s=1}^{\infty} i\left[\left\{x^2+(y-2s+c)^2\right\}\log\left(1+\frac{iz-c}{2s}\right) - \left\{x^2+(y-2s+c)^2\right\}\log\left(1+\frac{iz+c}{2s}\right)\right.$$

$$\left. + \left\{x^2+(y+2s+c)^2\right\}\log\left(1-\frac{iz-c}{2s}\right) - \left\{x^2+(y+2s-c)^2\right\}\log\left(1-\frac{iz+c}{2s}\right)\right] \quad \text{(A.1)}$$

can be expanded into infinite series. For example,

$$\log\left(1+\frac{iz-c}{2s}\right) = \left(\frac{iz-c}{2s}\right) - \frac{1}{2}\left(\frac{iz-c}{2s}\right)^2 + \frac{1}{3}\left(\frac{iz-c}{2s}\right)^3 - \cdots\cdots \quad \text{(A.2)}$$

In doing this, the series in the above stress function becomes

$$\sum_{s=1}^{\infty} i\left[2(x^2+y^2+c^2-cy)\left(\frac{1}{2}S^2+\frac{1}{4}S^4+\cdots\cdots\right) - 2(x^2+y^2+c^2+cy)\left(\frac{1}{2}S'^2+\frac{1}{4}S'^4+\cdots\cdots\right)\right.$$

$$+ 8s^2\left(\frac{1}{4}S^4+\frac{1}{6}S^6+\cdots\cdots\right) - 8s^2\left(\frac{1}{4}S'^4+\frac{1}{6}S'^6+\cdots\cdots\right) - 8s(y-c)\left(\frac{1}{3}S^3+\frac{1}{5}S^5+\cdots\cdots\right)$$

$$\left. + 8s(y+c)\left(\frac{1}{3}S'^3+\frac{1}{5}S'^5+\cdots\cdots\right) - 4c(ix-3y)\right] \quad \text{(A.3)}$$

where S and S' denote $(iz + c)/2s$ and $(iz - c)/2s$, respectively.

From (A.3) we conclude that if the linear term $-4c(ix-3y)$ is removed from each term of the series, the stress function becomes convergent, since the series

$$\sum_{S=1}^{\infty} \frac{1}{S^n}$$

is convergent for $n \geq 2$.

APPENDIX 2

(i) Evaluation of the Integral I_1

$$I_1 = \int_{-1}^{1} \left[(1 - m \coth m) \cosh my + my \sinh my \right] \cos n\pi y \, dy \tag{A.4}$$

From the table of integrals , we find

$$\int e^{my} \cos n\pi y \, dy = \frac{e^{my} (m \cos n\pi y + n\pi \sin n\pi y)}{m^2 + n^2 \pi^2} \tag{A.5}$$

Imposing the limit we have:

$$\begin{aligned} \int_{-1}^{1} e^{my} \cos n\pi y \, dy &= \frac{(-1)^n \, 2m \sinh m}{m^2 + n^2 \pi^2} \\ \int_{-1}^{1} e^{-my} \cos n\pi y \, dy &= \frac{(-1)^n \, 2m \sinh m}{m^2 + n^2 \pi^2} \end{aligned} \tag{A.6}$$

The combination of the above two equations yields

$$\int_{-1}^{1} \cosh my \cos n\pi y \, dy = \frac{2(-1)^n m \sinh m}{m^2 + n^2 \pi^2} \tag{A.7}$$

Differentiating both sides of (A.7) partially with respect to m we obtain:

$$\int_{-1}^{1} y \sinh my \cos n\pi y \, dy$$

$$= \frac{2(-1)^n \left[(m^2 + n^2\pi^2)(m \cosh m + \sinh m) - 2m^2 \sinh m \right]}{(m^2 + n^2 \pi^2)^2} \tag{A.3}$$

Substituting (A.7) and (A.8) into (A.4) we have

$$I_1 = \frac{4(-1)^n n^2 \pi^2 m \sinh m}{(m^2 + n^2\pi^2)^2} \qquad (A.9)$$

APPENDIX 2 (cont'd)

(ii) Evaluation of the Integral I_2

$$I_2 = \frac{2q \sin \pi c}{\pi} \int_0^\infty \frac{x \cos mx\, dx}{\cosh \pi x + \cos \pi c} \qquad \text{(A.10)}$$

The fraction $F = 1/(\cosh \pi x + \cos \pi c)$ may be expanded into a series in the following way:

$$F = \frac{2}{e^{\pi x} + e^{-\pi x} + e^{i\pi c} + e^{-i\pi c}}$$

$$= \frac{2e^{-\pi x}}{1 + (e^{i\pi c} + e^{-i\pi c})e^{-\pi x} + e^{-2\pi x}}$$

$$= \frac{2e^{-\pi x}}{(1 + e^{i\pi c}e^{-\pi x})(1 + e^{-i\pi c}e^{-\pi x})} \qquad \text{(A.11)}$$

Denoting $e^{-\pi x}$ by z and resolving into partial fractions we obtain:

$$F = \frac{2z}{(1 + e^{i\pi c}z)(1 + e^{-i\pi c}z)}$$

$$= \frac{-1}{i \sin \pi c}\left[\frac{1}{1 + e^{i\pi c}z} - \frac{1}{1 + e^{-i\pi c}z}\right] \qquad \text{(A.12)}$$

Now using the binomial theorem we have

$$\frac{1}{1+e^{i\pi c}z} = (1+e^{i\pi c}z)^{-1} = 1 - e^{i\pi c}z + e^{2i\pi c}z^2 - e^{3i\pi c}z^3 + \cdots\cdots \tag{A.13}$$

and

$$\frac{1}{1+e^{-i\pi c}z} = (1+e^{-i\pi c}z)^{-1} = 1 - e^{-i\pi c}z + e^{-2i\pi c}z^2 - e^{-3i\pi c}z^3 + \cdots\cdots \tag{A.14}$$

Substituting (A.13) and (A.14) into (A.12) we obtain

$$\frac{1}{\cosh \pi x + \cos \pi c} = \frac{-2}{\sin \pi c} \sum_{n=1}^{\infty} (-1)^n \sin n\pi c \, e^{-n\pi x} \tag{A.15}$$

Substituting (A.15) into (A.10) we obtain

$$I_2 = -\frac{4q}{\pi} \sum_{n=1}^{\infty} (-1)^n \sin n\pi c \int_0^{\infty} x e^{-n\pi x} \cos mx dx \tag{A.16}$$

The integral involved in (A.16) may be evaluated as follows:

From integration table we find

$$\int e^{-n\pi x} \cos mx dx = \frac{e^{-n\pi x}(-n\pi \cos mx + m \sin mx)}{m^2 + n^2\pi^2} \tag{A.17}$$

Hence

$$\int_0^{\infty} e^{-n\pi x} \cos mx dx = \frac{n\pi}{m^2 + n^2\pi^2} \tag{A.18}$$

Differentiating both sides partially with respect to n we obtain:

$$\int_0^\infty x e^{-n\pi x} \cos mx \, dx = \frac{-(m^2 - n^2\pi^2)}{(m^2 + n^2\pi^2)^2} \tag{A.19}$$

Substituting (A.19) into (A.16) we have finally:

$$I_2 = 4qc \sum_{n=1}^{\infty} \frac{(-1)^n \sin n\pi c \, (m^2 - n^2\pi^2)}{\pi c \, (m^2 + n^2\pi^2)^2} \tag{A.20}$$

APPENDIX 2 (cont'd)

(iii) Evaluation of the Integral I_3

$$I_3 = \frac{2q}{\pi^2}\int_0^\infty \left[\tan^{-1}\frac{\sin \pi c \sinh \pi x}{1 + \cos \pi c \cosh \pi x} - \pi c\right]\cos mx\, dx \tag{A.21}$$

Using integration by parts we let

$$u = \tan^{-1}\frac{\sin \pi c \sinh \pi x}{1 + \cos \pi c \cosh \pi x} - \pi c \tag{A.22}$$

$$dv = \cos mx\, dx$$

Then

$$du = \frac{\pi \sin \pi c}{\cosh \pi x + \cos \pi c}\, dx \quad , \qquad v = \frac{\sin mx}{m} \tag{A.23}$$

From Appendix 3 eq. (A.15) the fraction $1/(\cosh \pi x + \cos \pi c)$ may be expanded into an infinite series thus

$$du = -2\pi \sum_{n=1}^{\infty} (-1)^n \sin n\pi c\, e^{-n\pi x}\, dx \tag{A.24}$$

Hence

$$I_3 = \frac{2q}{\pi^2}\left[\frac{\sin mx}{m}\left\{\tan^{-1}\frac{\sin \pi c \sinh \pi x}{\cos \pi c \cosh \pi x + 1} - \pi c\right\}\right]_0^\infty$$

$$+ \frac{4}{\pi c m}\sum_{n=1}^{\infty} (-1)^n \sin n\pi c \int_0^\infty e^{-n\pi x} \sin mx\, dx \tag{A.25}$$

The limiting value inside the bracket is zero and the integral involved is very easy to evaluate. The result is:

$$\int_0^{\infty} e^{-n\pi x} \sin mx\, dx = \frac{m}{m^2 + n^2 \pi^2} \tag{A.26}$$

Consequently

$$I_3 = 4qc \sum_{n=1}^{\infty} \frac{(-1)^n \sin n\pi c}{\pi c(m^2 + n^2\pi^2)} \tag{A.27}$$

APPENDIX 2 (cont'd)

(iv) Evaluation of the Integral I_4

$$I_4 = \int_0^\infty (1 - n\pi x) e^{-n\pi x} \cos mx\, dx \qquad \text{(A.28)}$$

From Appendix 3 eqs. (A.18) and (A.19) we have:

$$\int_0^\infty e^{-n\pi x} \cos mx\, dx = \frac{n\pi}{m^2 + n^2\pi^2} \qquad \text{(A.29)}$$

$$\int_0^\infty x e^{-n\pi x} \cos mx\, dx = \frac{-(m^2 - n^2\pi^2)}{(m^2 + n^2\pi^2)^2} \qquad \text{(A.30)}$$

Substituting (A.29) and (A.30) into (A.28) we get immediately:

$$I_4 = \frac{2\pi n m^2}{(m^2 + n^2\pi^2)^2} \qquad \text{(A.31)}$$

APPENDIX 2 (cont'd)

(v) Evaluation of Integrals $I_{n,\kappa}$ & $J_{n,\kappa}$

$$I_{n,k} = 32\,\pi^2 (nk)^{3/2} \int_0^\infty \frac{x^3 \sinh^2 x\, dx}{(x^2+n^2\pi^2)^2 (x^2+k^2\pi^2)^2 (\sinh 2x + 2x)} \qquad (A.32)$$

$$J_{n,k} = \frac{32\pi^4}{3} (nk)^{5/2} \int_0^\infty \frac{x \sinh^2 x\, dx}{(x^2+n^2\pi^2)^2 (x^2+k^2\pi^2)^2 (\sinh 2x + 2x)} \qquad (A.33)$$

The Cauchy's theorem of residues states that

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum Res. \qquad (A.34)$$

where $\sum$Res is the sum of residues of f(z) in the upper half-plane. If f(x) is even, i.e. f(x) = f(-x) then

$$\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx = \pi i \sum Res. \qquad (A.35)$$

Write

$$f(z) = \frac{z^3 \sinh^2 z\, dz}{(z^2+n^2\pi^2)^2 (z^2+k^2\pi^2)^2 (\sinh 2z + 2z)} \qquad (A.36)$$

we note that f(z) = f(-z), it is even.

To find $\sum$Res of f(z) first we have to establish poles in the upper half-plane. If n $\neq$ k, poles of f(z) in the upper half-plane are:

(1) In the first quadrant: poles at $z_j = x_j + iy_j$ of order one;

(2) In the second quadrant, poles at $-\bar{z}_j = -x_j + iy_j$ of order one.

where z_j are zeros of the function $\sinh 2z + 2z$.

Thus the residues are:

$$\sum Res[f(z), z_j] = \sum_{j=1}^{\infty} \left[\lim_{z \to z_j} \frac{(z-z_j) z^3 \sinh^2 z}{(z^2+n^2\pi^2)^2 (z^2+k^2\pi^2)^2 (\sinh 2z + 2z)} \right] \tag{A.37}$$

Applying l'Hospital's rule and then taking the limit we find

$$\sum Res[f(z), z_j] = \sum_{j=1}^{\infty} \frac{z_j^3 \tanh^2 z_j}{4(z_j^2+n^2\pi^2)^2 (z_j^2+k^2\pi^2)^2} \tag{A.38}$$

Similarly

$$\sum Res[f(z), -\bar{z}] = \sum_{j=1}^{\infty} \frac{-\bar{z}_j^3 \tanh^2 \bar{z}_j}{4(\bar{z}_j^2+n^2\pi^2)^2 (\bar{z}_j^2+k^2\pi^2)^2} \tag{A.39}$$

Now if $n = k$, $f(z)$ becomes

$$f(z) = \frac{z^3 \sinh^2 z}{(z^2+n^2\pi^2)^4 (\sinh 2z + 2z)} \tag{A.40}$$

In this case, poles of $f(z)$ in the upper half-plane are:

(1) pole at $z_1 - n\pi i$ of order two;

(2) poles at $z_j = x_j + iy_j$ of order one;

(3) poles at $-\bar{z}_j = -x_j + iy_j$ of order one.

where z_j are again the zeros of the function $\sinh 2z + 2z$.

Thus the residues are:

$$\operatorname{Res}[f(z), z_1] = \lim_{z\to n\pi i}\left[\frac{d}{dz}\left\{(z-n\pi i)^2 f(z)\right\}\right]$$

$$= \lim_{z\to n\pi i}\left[\frac{d}{dz}\,\frac{z^3\sinh^2 z}{(z+n\pi i)^4(z-n\pi i)^2(\sinh 2z+2z)}\right] \tag{A.41}$$

Upon performing the differentiation and applying the l'Hospital's rule and taking the limit we finally obtain:

$$\operatorname{Res}[f(z), z_1] = \frac{1}{32n^3\pi^3 i} \tag{A.42}$$

The remaining two residues may be found as follows:

$$\sum \operatorname{Res}[f(z), z_j] = \sum_{j=1}^{\infty}\frac{z_j^3\tanh^3 z_j}{4(z_j^2+n^2\pi^2)^4} \tag{A.43}$$

$$\sum \operatorname{Res}[f(z), z_j] = \sum_{j=1}^{\infty}\frac{-z_j^3\tanh^2\bar{z}_j}{4(\bar{z}_j^2+n^2\pi^2)^4} \tag{A.44}$$

By equation (A.35) we have

$$\int_0^\infty \frac{x^3\sinh^2 x\,dx}{(x^2+n^2\pi^2)^2(x^2+k^2\pi^2)^2(\sinh 2x+2x)} = \pi i\left\{\sum \operatorname{Res}[f(z), z_j] + \sum \operatorname{Res}[f(z), -\bar{z}_j]\right\} \tag{A.45}$$

and

$$\int_0^\infty \frac{x^3\sinh^2 x\,dx}{(x^2+n^2\pi^2)^4(\sinh 2x+2x)} = \pi i\left\{\operatorname{Res}[f(z), z_1] + \sum \operatorname{Res}[f(z), z_j] + \sum \operatorname{Res}[f(z), -\bar{z}_j]\right\} \tag{A.46}$$

Upon substituting the results of (A.38), (A.39), (A.42), (A.43) and (A.44) into (A.45) and (A.46) and combining the results into one expression, we have

$$\int_0^\infty \frac{x^3 \sinh^2 x \, dx}{(x^2+n^2\pi^2)^2(x^2+k^2\pi^2)^2(\sinh 2x+2x)}$$

$$= \frac{1}{32 n^3\pi^2}\,\delta_{nk} + \frac{\pi}{2}\sum_{j=1}^{\infty} \mathcal{R}\left[\frac{i\, z_j^3 \tanh^2 z_j}{(z_j^2+n^2\pi^2)^2(z_j^2+k^2\pi^2)^2}\right] \tag{A.47}$$

or

$$I_{n,k} = \delta_{nk} + 16\pi^3 (nk)^{3/2} \sum_{j=1}^{\infty} \mathcal{R}\left[\frac{i z_j^3 \tanh^2 z_j}{(z_j^2+n^2\pi^2)^2(z_j^2+k^2\pi^2)^2}\right] \tag{A.48}$$

where δ_{nk} is kronecker delta and z_j are the zeros of the function sinh 2z + 2z or one-half of the corresponding zeros of the function sinh z + z. The first one-hundred zeros of the function sinh z + z in the first quadrant have been computed recently to ten decimal places by Dr. C. B. Ling. The one-hundred values of these zeros are shown in Table 5.

As for the integral $J_{n,k}$, one can follow exactly the same procedures shown above to obtain the following result:

$$J_{n,k} = -\delta_{nk} + \frac{16}{3}\pi^5 (nk)^{5/2} \sum_{j=1}^{\infty} \mathcal{R}\left[\frac{i z_j \tanh^2 z_j}{(z_j^2+n^2\pi^2)^2(z_j^2+k^2\pi^2)^2}\right] \tag{A.49}$$

APPENDIX 2 (cont'd)

(vi) Evaluation of the Integrals
$S_n(\alpha)$, $S_n^*(\alpha)$, $C_n(\alpha)$, $C_n^*(\alpha)$

$$S_n(\alpha) = \int_0^\infty \frac{y^2 \sinh y \cos \alpha y \, dy}{(y^2+n^2\pi^2)^2(\sinh 2y + 2y)}$$

$$S_n^*(\alpha) = \int_0^\infty \frac{\sinh y \cos \alpha y \, dy}{(y^2+n^2\pi^2)^2(\sinh 2y + 2y)}$$

$$C_n(\alpha) = \int_0^\infty \frac{y^3 \cosh y \cos \alpha y \, dy}{(y^2+n^2\pi^2)^2(\sinh 2y + 2y)} \tag{A.50}$$

$$C_n^*(\alpha) = \int_0^\infty \frac{y \cosh y \cos \alpha y \, dy}{(y^2+n^2\pi^2)^2(\sinh 2y + 2y)}$$

By contour integration, we have:

$$\int_{-\infty}^{\infty} f(y)\,dy = 2\pi i \sum Res \tag{A.51}$$

Denoting

$$Q(y) = \frac{y^2 \sinh y}{(y^2+n^2\pi^2)^2(\sinh 2y + 2y)} \quad ,$$

we observe here $Q(y) = Q(-y)$ and thus

$$S_n(\alpha) = \int_0^\infty Q(y) \cos \alpha y \, dy = \frac{1}{2}\int_0^\infty \left[Q(y) e^{\alpha i y} + Q(-y) e^{-\alpha i y} \right] dy$$

$$= \frac{1}{2}\int_0^\infty Q(y) e^{\alpha i y} dy + \frac{1}{2}\int_{-\infty}^0 Q(y) e^{\alpha i y} dy$$

$$= \frac{1}{2}\int_{-\infty}^\infty Q(y) e^{\alpha i y} dy$$

By (b) it yields:

$$S_n(\alpha) = \pi i \sum Res. \tag{A.52}$$

where ΣRes is the sum of residues of $Q(z)e^{\alpha iz}$ at its poles in the upper half-plane.

$$Q(z)\,e^{\alpha iz} = \frac{z^2 \sinh z\, e^{\alpha iz}}{(z^2+n^2\pi^2)^2(\sinh 2z + 2z)} \tag{A.53}$$

Poles in the upper half-plane are:

(1) pole at $z_1 = n\pi i$ of order one;

(2) pole at zeros of the function $\sinh 2z + 2z$ in the first quadrant poles at $z_j = x_j + i\,y_j$ of order one; in the second quadrant poles at $-\bar{z}_j = -x_j + iy_j$ of order one.

The residues at these poles are:

$$Res[Q(z)e^{\alpha iz}, z_1] = \lim_{z\to n\pi i} \frac{(z-n\pi i)z^2 \sinh z\, e^{\alpha iz}}{(z^2+n^2\pi^2)^2(\sinh 2z + 2z)}$$

$$= \lim_{z\to n\pi i} \frac{z^2 \sinh z\, e^{\alpha iz}}{(z-n\pi i)(z+n\pi i)^2(\sinh 2z + 2z)}$$

$$= \frac{(-1)^n\, e^{-\alpha n\pi}}{8 n\pi i} \tag{A.54}$$

$$\sum Res[Q(z)e^{\alpha iz}, z_j] = \sum_{j=1}^{\infty} \lim_{z\to z_j} \frac{(z-z_j)z^2 \sinh z\, e^{\alpha iz}}{(z^2+n^2\pi^2)^2(\sinh 2z + 2z)}$$

$$= \sum_{j=1}^{\infty} \frac{z_j^2 \tanh z_j\, e^{\alpha iz_j}}{4(z_j^2+n^2\pi^2)^2 \cosh z_j} \tag{A.55}$$

$$\sum Res[Q(z)e^{\alpha iz}, -\bar{z}_j] = \sum_{j=1}^{\infty} \frac{-\bar{z}_j^2 \tanh \bar{z}_j\, e^{-\alpha i\bar{z}_j}}{4(\bar{z}_j^2+n^2\pi^2)^2 \cosh \bar{z}_j} \tag{A.56}$$

Substituting (A.54), (A.55) and (A.56) into (A.52) we have:

$$S_n(\alpha) = \frac{(-1)^n}{8n} e^{-\alpha n\pi} - \frac{\pi}{4}\sum_{j=1}^{\infty} \mathcal{I}\left[\frac{z_j^2 \tanh z_j\, e^{\alpha i z_j}}{(z_j^2+n^2\pi^2)^2 \cosh z_j} - \frac{\bar{z}_j^2 \tanh \bar{z}_j\, e^{-\alpha i \bar{z}_j}}{(\bar{z}_j^2+n^2\pi^2)^2 \cosh \bar{z}_j}\right]$$

(A.57)

or

$$S_n(\alpha) = \frac{(-1)^n}{8n} e^{-\alpha n\pi} + \frac{\pi}{2}\sum_{j=1}^{\infty} \mathcal{R}\left[\frac{i z_j^2 \tanh z_j\, e^{\alpha i z_j}}{(z_j^2+n^2\pi^2)^2 \cosh z_j}\right] \qquad (A.58)$$

where z_j are zeros of the function $\sinh 2z + 2z$.

The other three integrals may be found in a similar way; the results are:

$$S_n^*(\alpha) = -\frac{(-1)^n}{8n^3\pi^2} e^{-\alpha n\pi} + \frac{\pi}{2}\sum_{j=1}^{\infty} \mathcal{R}\left[\frac{i \tanh z_j\, e^{\alpha i z_j}}{(z_j^2+n^2\pi^2)^2 \cosh z_j}\right] \qquad (A.59)$$

$$C_n(\alpha) = -\frac{(-1)^n \pi\alpha}{8} e^{-\alpha n\pi} + \frac{\pi}{2}\sum_{j=1}^{\infty} \mathcal{R}\left[\frac{i z_j^3\, e^{\alpha i z_j}}{(z_j^2+n^2\pi^2)^2 \cosh z_j}\right] \qquad (A.60)$$

$$C_n^*(\alpha) = \frac{(-1)^n(\alpha n\pi+2)}{8n^3\pi^2} e^{-\alpha n\pi} + \frac{\pi}{2}\sum_{j=1}^{\infty} \mathcal{R}\left[\frac{i z_j\, e^{\alpha i z_j}}{(z_j^2+n^2\pi^2)^2 \cosh z_j}\right]$$

(A.61)

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ABSTRACT

AN ANALYTIC SOLUTION FOR THE STRESS DISTRIBUTION IN A SEMI-INFINITE STRIP LOADED ON THE TRANSVERSE EDGE

This thesis presents an analytic solution for the stress distribution in a semi-infinite strip subjected to symmetrical loads on the transverse edge. Three different types of loading on the transverse edge are considered: (i) a segment of uniform load, (ii) two concentrated normal loads, and (iii) two concentrated tangential loads. The solution is constructed by the method of images. Under successive reflections the given strip and the resulting images become a semi-infinite plate with a series of periodic loads on the edge. The stress function for such a plate is constructed by superposing the known solutions of a simple nature. To satisfy the boundary conditions along the longitudinal edges of the semi-infinite strip, additional stress functions are introduced. When the boundary conditions are adjusted, a system of integral equations and a system of algebraic equations are obtained, which are further reduced to a single system of algebraic equations. The latter system is solved by the method of successive approximations. In each case, the expressions for normal stresses along the longitudinal axis are derived and numerical values for these stresses are given.