STABILITY OF STATIC INPUT-OUTPUT SYSTEMS WITH OPTIMIZED SUBSYSTEMS: QUANTITY AND PRICE MODELS

by

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Chapter I

INTRODUCTION

Optimization problems occur frequently in many branches of pure and applied mathematics. In economic applications, such problems are especially natural. For example, industries try to maximize profits or minimize costs; and, consumers wish to spend their income in such a way as to maximize their satisfaction.

This thesis is concerned with special classes of optimization models which are employed to analyze the stability of complex systems when one or more of its subsystems are operationally optimized. In the context of this study, the word "stability," which embraces a rather comprehensive concept in industry-consumer relationships, is primarily due to the work conducted by Moon [M2]. In his dissertation, Moon investigated the pattern of the structural change in a system with optimized subsystems, and formulated a quantity model for stability analyses. As a complementary study to Moon's, the main objective of this thesis is to present several price models and to draw some economic interpretations. A clarification of the terminology, "operational optimization," as originated by Moon and Ghare [M2, M5], is made in section 1.1

For this presentation, the well known static input-output model of Leontief [L2, L3] is chosen to describe a system containing multiple subsystems. Hence, the term "system" is used in this study to refer to a multisector economy, while its subsystems may be referred to as "sectors." The terms, "interindustry system" and "industry" are used interchangeably with a multisector economy and a sector, respectively. In a broader sense, however, the system could be any functional representation, as long as the interactions among its subsystems can be described mathematically, along with some consistent properties that are of necessity for optimizing subsystems for the given problem structure.

In optimizing individual industries, a linear programming model is used. This choice of an optimization method may require some explanations. Firstly, it is due to the selection of an input-output model to describe the economic system which is linear in its form. As will be seen later, the properties of linearity will help in maintaining a consistency in the stability analyses. Secondly, the post-optimality theory of linear programming has been well established, as compared with that of other mathematical programming. The post-optimality analysis is used to a considerable extent in the developments of stability models. Thirdly, primal-dual relationships can easily be observed through a linear programming model. In the last chapter of this thesis, duality theory will provide a basis for understanding some of the interesting phenomena that occur between quantity and price of a linear economic system.

§1.1 Nature of the Problem

Within the framework of an explicitely formulated input-output system, economic change can be explained by several factors such as technological change, changes in product mix, changes in relative price of inputs, or changes in the level of gross outputs, which all fall into a category of structural change of a system. The "structural change" in this sense may be referred to as the changes in the input-output coefficients which describe interrelationships of industries.

The intersectoral input requirements of the economy are uniquely characterized by the production technology used in each sector. If there is a new production technology developed in a sector, the most efficient production process arising from the new technology can be selected through an explicit optimization process. Even in the absence of an explicit optimization, a (non) substitution theorem [S1] justifies that essentially one single "preferred" process suffices to ensure all the efficient input-output configurations of the economy. In either case, a new production process is picked from among processes available under the new technology, whose input coefficients are fixed but may differ from those of the sector's old process. The immediate effect of the aforementioned technological change in a sector is a structural change in the input-output system for the entire economy.

One may now consider another realistic possibility, an "operational optimization" of one or more subsystems, which may cause the structural change of the entire input-output system. Given an existing production technology in a sector, a set of alternative production processes may be available, each of which is characterized by the level of its various activities of production. The input requirements associated with each process are fixed but may differ from one process to the other. By operationally optimizing a sector, a process is selected in such a way that input cost requirement from each of the sectors in the economy does not exceed the current level, at the same time minimizing the total sum of inefficiency factors for production. If the production process thus selected is not the same as the current preferred process, a substitution should take place between the two processes, in order for the sector to be operationally optimal.

A study of this kind of structural change must answer two questions: "How will the structure of a particular sector be actually changed?" and "How will this affect the outputs of the separate industries, the prices of different goods; in short, how will it affect the magnitudes of the dependent variables of the given economic system?" Thus, the nature of the problem is to specify a precise methodology in selecting operationally optimal processes and to investigate the characteristics of the overall economy that is followed by operational optimization.

§1.2 Purpose and Approach

The purpose of this thesis is to investigate the pattern of structural change in an input-output system, when a series of operational optimizations are performed in a set of industries. Specifically, changes in the relative price of each commodity in an economy together with its implications for the associated quantities, will be analyzed. An iterative process of stability analysis, will be developed in an effort to trace the interdependence between the status of the optimized sectoral operation and the extent of the price changes (or changes in values added) in the input-output system.

In very general terms, the system may be said to be stable if in the course of iterative process the sectoral optimization does not cause any further structural changes in the system. A specific solution, which we may call an "equilibrium" solution, should exist as the structural perturbation ceases; in addition, the operation of those optimized sectors must remain optimal. This study concludes that under all normal circumstances such a state of economy can be reached.

The formulation of production will be based on the concept of production technology sets, as opposed to the concept of classical production functions. In developing price models, this concept will again be adjusted to give an interpretation of cost-revenue relationship, rather than the usual quantitative interpretations. The optimal status of the sectoral operation will then be investigated via this adjusted technology set concept, as well as with sensitivity analysis for linear programs.

Depending on the set of assumptions given, a stable state of economy may be achieved through different algorithmic approaches.

These approaches, which may be either sequential or simultaneous, will be described in detail. Matrix theory and the product form of the inverse can then be adopted as analytical tools in examining any change in the input-output structure. Throughout the stability study, a framework of linearity will be maintained.

§1.3 Plan and Scope

The primary purpose of Chapter II is to build some mathematical machinery, and to provide background information about the basic Leontief systems with a brief description of a general linear programming problem and duality theory. A short literature survey on the studies of the structural change of the input-output system is followed by a definition of the production technology set and a discussion of its properties.

A mathematical description of the generalized stability problem is introduced in Chapter III. A few terminologies are defined also. The quantity and price problems are then described in detail with the descriptions of iterative procedures to solve these problems. Finally, the criteria for stability is offered under different assumptions.

For the purpose of completeness, Chapter IV is constituted by Moon's quantity formulation of stability analyses. The model will be first described diagrammatically and a concise development will be furnished. A rather interesting conclusion will then be made observing the phenomena that exist in the quantitative side of input-output system.

The main body of the price models for stability analyses is contained in Chapters V, VI, and VII. The acquisition model, the simplest of all, is presented in Chapter V while consolidation model and appreciation model are described in Chapter VI and VII, respectively. Economic reasons behind these price models are stated briefly at the beginning of each chapter, followed by a unique set of assumptions which characterize the specific price model. In Chapters V, VI and VII, the case when only one sector of the system is optimized is first developed and then the case is extended to accommodate multiple-sector optimization problems. Concluding remarks are also made regarding each of the price models.

The entire study is summarized, with some major conclusions drawn, in Chapter VIII. Economic interpretations of the price models with regard to quantities associated are contained also. A specific set of observations that preserves duality is made along with some comments. Some of the possible applications are discussed and areas for further research are recommended at the end of the chapter.

Chapter II

BACKGROUND INFORMATION

The primary purpose of this chapter is to furnish the mathematical as well as economical background for all the subsequent chapters. This is done by first presenting a brief exposition of a few concepts pertaining to convex sets in Euclidean space, which are of systematic and frequent use in what follows. Then both quantity and price formulations of the Leontief input-output model are described along with a general model of linear programming and its duality theory. A production technology set is defined, and some of its properties are stated. Finally, a short literature survey on the studies regarding the structural change of an input-output system is given.

§2.1 Elementary Mathematical Basis

Modern economic analyses are based primarily upon the concepts and tools of the theory of linear systems. To a certain extent, it seems reasonable to say that the properties of linear equality and inequality play the key role in analyzing the behavior of general economic systems.

Suppose the following system of n linear equations in n variables written in a matrix notation, is under our consideration:

$$AX = b \tag{2.1}$$

where A = an nxn matrix of coefficients a_{ij},

X = an n-component column vector of unknowns, and

b = an n-component column vector of constants.

Also assume $b \neq 0$, in which case (2.1) is said to be a nonhomogeneous system of equations. If A is a nonsingular matrix then A^{-1} exists and the unique solution vector, X, can be determined by

$$X = A^{-1}b \tag{2.1a}$$

Now consider the system of linear inequalities of the form,

$$AX \leq b \tag{2.2}$$

If the same assumptions on A and b hold true, then the solution set $\bar{X}$ for (2.2) can be written as

$$\bar{X} = \{X | X = A^{-1}b - A^{-1}S\} \tag{2.2a}$$

where S is an n-component vector for non-negative slack variables that represents n degrees of freedom in the set of inequalities.

Among the properties of linear systems, the one that is most important in production theory and hence that will be very useful in this study, is the concept of convex set. We shall now examine a few properties concerning this concept.

Definition 2.1 (A Convex Set): Let S be a subset of R^n, an n-dimensional vector space, and X^r, X^s any points of S. Then S is said to be a convex set if and only if for $\lambda \varepsilon [0, 1]$, $\lambda X^r + (1-\lambda)X^s \varepsilon S$.

Definition 2.2 (A Cone): A cone Ω is a set of points with the following property: If $X\varepsilon\Omega$, so is λX for all $\lambda \geq 0$. The cone generated by a set of points $\overline{X}=\{X\}$ is the set

$$\Omega = \{Y|Y = \lambda X, \text{ all } \lambda \geq 0, \text{ and all } X\varepsilon\overline{X}\}$$

If R^2 and R^3, a cone as a set of points is often identical with the usual geometrical concept of a cone. The negative Ω^- of a cone $\Omega=\{u\}$ is the set of points $\Omega=\{-u\}$. Naturally, Ω^- is a cone if Ω is. If $\Omega=\{u\}$ is a cone, then Ω^+, the cone polar to Ω, is the collection of points $\{v\}$ such that $v'u \geq 0$ for each v in the set, and all $u\varepsilon\Omega$. It is easy to see that Ω^+ is a cone. Intuitively, a polar cone is the collection of all vectors which form a nonobtuse angle with all the vectors in Ω.

Definition 2.3 (A Convex Cone): A cone is a convex cone if it is a convex set.

Definition 2.4 (A Convex Polyhedral Cone): Given a single point $a \neq 0$, L, a half-line or ray, be defined as the set $L=\{Y|Y=\lambda a, \text{ all } \lambda \geq 0\}$.
A convex polyhedral cone Ω is the sum of a finite number of half lines.

If $A=(a_1, a_2, \ldots, a_n)$ is an nxn matrix, then the set of points $Y=AX=\sum_{j=1}^{n} x_j a_j$, is a convex polyhedral cone in R^n. The columns a_j of A generate the half-lines whose sum yields the polyhedral cone. Thus we see that there is a non-negative solution to (2.1) if and only if b is an element of the convex polyhedral cone generated by the columns of A.

We shall now develop some additional notations and define some terms, which will be useful particularly for our subsequent discussions on the solvability of systems of the type (2.1) or (2.2). The definitions given below are adopted as usually stated in the literature of mathematical economics [F2, K2, N1].

For any two vectors $U=\{u_i\}$ and $V=\{v_i\}$, $U, V \varepsilon R^n$,

(a) $U \geqq V$, if $u_i \geqq v_i$ for all i

(b) $U \geq V$, if $u_i \geq v_i$ and $U \neq V$,

(c) $U > V$, if $u_i > v_i$ for all i,

(d) $U \otimes V = \{u_i v_i\}$, and

(e) $U \oslash V = \{u_i / v_i\}$.

Adopting similar notations for two nxn square matrices M and N whose elements are m_{ij}, $n_{ij} \varepsilon R^1$, all i and j,

(f) $M \geqq N$, if $m_{ij} \geqq n_{ij}$ for all i and j,

(g) $M \geq N$, if $m_{ij} \geq n_{ij}$ and $M \neq N$,

(h) $M > N$, if $m_{ij} > n_{ij}$ for all i and j,

(i) $M \otimes N = \{m_{ij} n_{ij}\}$, and

(j) $M \oslash N = \{m_{ij} / n_{ij}\}$.

Moreover, the following operations on a matrix and a vector should be understood:

(k) $M \otimes U = \{m_{ij} u_i\}$, and

(l) $M \oslash U = \{m_{ij} / u_i\}$.

Some matrices of the special type are now defined; viz, an indecomposable (or decomposable) matrix, a class Z matrix, and a class K matrix.

<u>Definition 2.5</u> (Indecomposable Matrices): An nxn matrix $M=\{m_{ij}\}$ is said to be decomposable if there is a nonempty subset J of [1, 2, ... , n] such that $m_{ij}=0$ $(i\varepsilon J,\ j\notin J)$; $M=\{m_{ij}\}$ is said to be indecomposable, if it is not decomposable and is not the null matrix with n=1.

<u>Definition 2.6</u> (Class Z Matrices): The set of all real square matrices whose off-diagonal elements are all non-positive forms class Z.

<u>Definition 2.7</u> (Class K Matrices): An nxn matrix M is a class K matrix if $M\varepsilon Z$ and $M^{-1}\geq 0$ where Z is the set of all class Z matrices.

From definitions 2.6 and 2.7 we can extract some relationships between a class Z matrix and class K matrix. Let $M\varepsilon K$, $N\varepsilon Z$, and $N\geq M$. Then

(a) $N\varepsilon K$,

(b) $\det(N) \geq \det(M) > 0$, and

(c) $M^{-1} \geqq N^{-1} \geqq 0$.

§2.2 Quantity Formulation for a Leontief Static Input-Output Model

The static input-output model was first introduced by Leontief [L2, L3, L4] in the 1930's to describe production and flow of goods in a system. This model is based on the notion of economic interdependence in that a significant proportion of the effort of an economy is devoted to the production of intermediate goods, and the production of these goods is linearly related to the gross outputs of the end products. If there is any change in the gross output level of a final product, there would be consequent changes in the output levels of intermediate products used to produce this end product and also similar changes in the products used to produce the intermediates.

Consider a system in which final consumption is regarded as being exogeneously determined to include consumer demand and government use, as well as foreign trade. Then the total, final consumption would have approximately the same meaning as the gross national product. This type of system is often called an open Leontief system. The essence of the open Leontief input-output analysis is the determination of the gross output levels of various industries, which are consistent with the specified pattern of external consumptions, as well as the intermediate demands.

The basic assumptions of the Leontief static input-output model for an n-sector economy may be summarized as follows:

(1) Each kind of n different goods, which may be labeled by $i=1, 2, \ldots, n$, is supplied by a single sector.

(2) Each sector produces only one kind of good, and uses a single production process to make this good, allowing no joint production.

(3) Sector j ($j=1, \ldots, n$) uses $a_{ij} \geq 0$ units of good i ($i=1, \ldots, n$) to produce one unit of its good j; the required input of any factor is directly proportional to the production. Hence if sector j produces G_j units of its good j, $a_{ij}G_j$ units of good i are needed as inputs. The constants a_{ij} are called input coefficients, or technological coefficients.

A sector's gross output is, thus, partially consumed as intermediate inputs within the economy to support production activities of other sectors as well as its own production activity. When exogeneous final demands are known, the total amount G_i which sector i must produce can be determined from the algebraic equation written as follows:

$$G_i - \sum_{j=1}^{n} a_{ij}G_j = C_i, \quad i = 1, \ldots, n \tag{2.3}$$

where G_i = the gross output level of sector i,

$\sum_{j=1}^{n} a_{ij}G_j$ = the total amount of good i consumed as intermediate inputs within the economy, and

C_i = the exogeneous final consumption of good i.

Rewriting the above balance equation as

$$\sum_{j=1}^{n} (\delta_{ij} - a_{ij})\, G_j = C_i, \quad i = 1, \ldots, n,$$

where Kronecker's deltas $\delta_{ij}=0$ for $i \neq j$, 1 for $i=j$, we can readily convert the system of equations to a simple matrix form. That is:

$$(I - A)\, G = C \tag{2.3a}$$

where $A = \{a_{ij}\}$: an nxn input coefficient matrix,

$G = \{G_i\}$: an n-component column vector for gross outputs,

$C = \{C_i\}$: an n-component column vector for final consumptions, and,

$I = \{\delta_{ij}\}$: an identity matrix of order n.

The input coefficients a_{ij} are often computed by g_{ij}/G_j, of which g_{ij} represents the quantity of good i used to produce the total production quantity G_j of sector j. The pertinent data may be obtained from the sectoral production records. The dimensions of the a_{ij} depends on the dimensions in which the g_{ij}, G_j are measured; they may be either physical units or monetary units. In this study, monetary units are used when the quantity side of a stability of an input-output analysis is presented while physical units for a_{ij} are adopted when the price formulation is of interest.

The matrix (I-A) of (2.3a) is known as the Leontief matrix, and its inverse $(I-A)^{-1}$ is termed a matrix multiplier. If $(I-A)^{-1}$ exists, then (2.3a) may be solved, and the unique solution is given by

$$G = (I - A)^{-1} C \tag{2.3b}$$

The unique feature of this system of equations lies in that for any meaningful economic system, the solution $G \geq 0$ for given $C \geq 0$ and $A \geq 0$. It is also true that, under all normal circumstances, $(I-A)^{-1}$ always exists.

The net demand C is often referred to as a "bill of goods." If the columns of A are denoted by a_j, then there exists a $G \geq 0$ satisfying (2.3a) provided that C lies in the convex polyhedral cone spanned by the vectors $e_j - a_j$, $j=1, \ldots, n$, where e_j denotes the jth column of an nth order identity matrix. A bill of goods C will be called producible if there exists a $G \geq 0$ satisfying (2.3a). Thus, in order that there exists at least one producible bill of goods for an economy, it must be true that the intersection of the convex polyhedral cone Ω generated by the $e_j - a_j$ and the non-negative orthant of R^n contains at least one point $C^o \varepsilon \Omega$ such that $C^o \geq 0$, $C^o \neq 0$. Then C^o will be a producible bill of goods, as will be any point $C = \lambda C^o$, $\lambda > 0$. The set of all producible bills of goods is the intersection of Ω and the non-negative orthant of R^n.

Often, it is assumed that labor is the only primary factor used in an economy described by a Leontief model, i.e., the only productive factor that is not produced by the economy. Suppose that $\ell_j G_j$ units of labor are needed to produce G_j units of good j; ℓ_j is a constant. Given a set of G_j, the total labor requirement is then $\sum_{j=1}^{n} \ell_j G_j = \ell G$, $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$. Since the gross production level required to meet any final bill of good C is given by (2.3b), the labor necessary to produce C is $[\ell(I-A)^{-1}]C$.

Several conditions have been developed and reported in the literature [H2, M7, N1] regarding the solvability of system (2.3a). Some of these conditions are given below:

Hawkins-Simon Condition: A necessary and sufficient condition that the G satisfying (2.3a) be all positive for any $C \geq 0$ is that all principal minors of (I-A) be positive;

Brauer-Solow Condition: A sufficient condition that there exists a $G \geq 0$ for any given $C > 0$ in (2.3a) is $\sum_{j=1}^{n} a_{ij} > 1$, all j;

Invertibility-Producibility: The Leontief matrix (I-A) is nonnegatively invertible if and only if (I-A) is productive.

§2.3 Linear Programming and Duality Theory

Linear programming is another method for analyzing linear economic problems. The first mathematical technique was developed by Danzig [D1, D2] in the late 1940's. Among other useful computational algorithms, Danzig's simplex method is the most typical and best known.

Whereas the Leontief model considers a single mode of operation for any sector of the economy, the linear programming model takes into consideration alternative modes of operation, which may then lead to many possible solutions. Thus, a criterion for selecting the "optimal" solution needs to be established.

The canonical form of a linear programming problem can be stated as:

$$\begin{aligned} &\text{Maximize} && WX \\ &\text{subject to} && DX \leq d \\ & && X \geq 0 \end{aligned} \tag{2.4}$$

where X = an m-component column vector for activities,

W = an m-component row vector for coefficients of valuation,

D = an nxm matrix for constraint coefficients, and

d = an n-component column vector for resources.

One of the most important facts about linear programming is that to every linear programming problem there corresponds a dual problem. If (2.4) is called the primal problem, then the corresponding dual problem can be stated as:

$$\begin{aligned} &\text{Minimize} \quad d'Y \\ &\text{subject to} \quad D'Y \geq W' \\ &\qquad\qquad\quad Y \geq 0 \end{aligned} \tag{2.5}$$

where Y is an n-component column vector for activities, and the prime (′) denotes the transpose of a vector or matrix.

Similarities and differences between primal problem and dual problem should be evident. We will now exhibit some of the fundamental relationships between (2.4) and (2.5) without proofs. Rigorous proofs can be found in [C2, D2, H1].

(1) If X is any feasible solution to the primal problem and Y is any feasible solution to the dual problem, then $WX \leq d'Y$.

(2) If X^* is a feasible solution to the primal problem and Y^* is a feasible solution to the dual problem such that $WX^* = d'Y^*$, then both X^* and Y^* are optimal solutions to their respective problems.

(3) A feasible solution X* to the primal problem is optimal if and only if there exists a feasible solution Y* to the dual problem such that $WX^*=d'Y^*$.

§2.4 Price Formulation for a Leontief Static Input-Output Model

The Leontief model presented in section 2.2 was purely a quantity formulation, meaning no mention has been made about prices in our hypothetical economy. If prices are directly incorporated to describe an equilibrium state of an economy, another interesting set of balance equations can be exhibited, as it will be described in the following paragraph.

Let

P_j = the unit price of good j,

a_{ij} = units of good i required to produce one unit of good j, measured in physical units, so that $a_{ij}P_i$ represents the cost of the a_{ij} units of good i required to make one unit of good j,

$\sum_{i=1}^{n} a_{ij}P_i$ = the cost of goods 1, . . . , n needed to make one unit of good j, and

V_j = the difference between the price of good j and the total cost of inputs needed to produce one unit of good j; this difference is termed as the value added for sector j.

Then by definition,

$$P_j - \sum_{i=1}^{n} a_{ij}P_i = V_j, \qquad j = 1, \ldots, n \qquad (2.6)$$

Thus, from producer j's point of view it would be his high interest to increase as much V_j as possible for it may include profit as well as labor costs.

In a matrix notation, a set of equations (2.6) can be written as:

$$(I - A')P' = V' \qquad (2.6a)$$

or

$$(I - A)'P' = V' \qquad (2.6b)$$

where $A = \{a_{ij}\}$ = an nxn input coefficient matrix measured in physical units,

$P = \{P_j\}$ = an n-component row vector describing unit prices of goods 1, . . ., n, and

$V = \{V_j\}$ = an n-component row vector for values added.

If (I-A) is nonsingular, so is $(I-A)'$ and $[(I-A)']^{-1}$ exists. Hence if the value added for each sector is known, it is possible to obtain **unique** values for the prices in the economy as they can be determined by

$$P' = [(I - A)']^{-1}V'. \qquad (2.6c)$$

Now consider an economy that is operated under pure competition and in a state of long-run equilibrium, i.e., an economy in which labor is the only primary factor. Then $V' = w\ell'$ where w is the single wage rate, and from (2.6c),

$$P' = w[\ell(I - A)^{-1}]'. \tag{2.6d}$$

Referring back to section 2.2, if g_{ij}, and G_j are measured in monetary units, they include prices. Denoting by a_{ij}^{p}, a_{ij}^{m} for the input coefficients in their physical and monetary dimensions, respectively, we see that

$$a_{ij}^{p} = \left(\frac{P_j}{P_i}\right) a_{ij}^{m}. \tag{2.7}$$

Since we expect $\sum_{i=1}^{n} a_{ij}^{p} P_i < P_j$ because there must be labor costs in the production of good j, it follows that

$$\begin{aligned} &\text{(a)}\quad a_{ij}^{m} \geq 0, \text{ all i and j} \\ &\text{(b)}\quad \sum_{i=1}^{n} a_{ij}^{m} < 1, \text{ all j.} \end{aligned} \tag{2.8}$$

From the properties (2.8) for a_{ij}^{m}, it should be recognized that

$$0 \leq a_{ij}^{m} < 1, \text{ all i and j.} \tag{2.8a}$$

The duality of prices and quantities in a Leontief system is apparent. To examine this aspect in more detail, consider the linear programming problem,

$$\text{Minimize } z^P = (w\ell')'G = VG$$
$$\text{subject to } (I - A)\ G \geq C$$
$$C \geq 0. \qquad (2.9)$$

This problem seeks to minimize labor costs while producing at least the bill of goods C. The unique optimal solution is given by (2.3b) with the optimum labor cost being $(w\ell')'(I-A)^{-1}C$ or $V(I-A)^{-1}C$. Thus the Leontief model can be a very simple linear programming problem. Now the dual of (2.9) is:

$$\text{Maximize } z^c = C'P'$$
$$\text{subject to } (I - A)'P' \leq V'$$
$$P' \geq 0. \qquad (2.10)$$

The unique optimal solution to this problem can be obtained by (2.6c), or equivalently by (2.6d). Hence the problem of determining prices in a Leontief system under pure competition can also be considered to be a linear programming problem. It is the dual of the problem which determines the quantities to be produced. Notice that the optimal objective function value for (2.9), z^P, has the same meaning as the gross national product while z^c can be interpreted as the gross national consumption. From duality theory, we expect $z^P=z^c$, i.e., $VG=C'P'$.

Duality theory also shows that if the bill of goods C is producible there exists a set of non-negative prices for the economy. Since these prices are independent of C, it also follows from duality theory that the same prices will apply for all producible bill of goods.

The duality of quantities and prices can also be interpreted in terms of convex cones. Since the gross national product, VG>0, $C'P'>0$ provided C is producible. If C is producible, it lies in the convex polyhedral cone Ω spanned by the columns of (I-A). By the relationship, PC=VG>0, P is an element of Ω^+, the cone polar to Ω. According to the discussion of section 2.1, C<0 cannot be contained in Ω, and thus $P' \geq 0$ is expected to be contained in Ω^+. Therefore, from the properties of cones and from duality theory, we see that there should be a set of non-negative prices for the economy.

§2.5 Production Technology Sets

One of the most basic features in the working of a national economy is production. In economics, production means the transformation of inputs to outputs. Each particular realization of production in a sector can be represented by a pair (x,y) of an input vector x and an output vector y, which is referred to as a production process or simply a process. If a particular process produces a single output, the output vector y may be treated as a scaler rather than a vector.

Generally there are many processes under a given technology. From the economic point of view, therefore, a production technolgoy is completely determined by enumerating all the processes that are possible under it. Thus a production technolgoy set T can be defined as the set of all these possible production processes.

Since it is only common sense that an average industry will aim to acquire as great a return as possible at as little sacrifice as possible, the following definitions suggest themselves [N1]:

Definition 2.8 (A More Efficient Process): A production process (x, y) is said to be more efficient than other process (v, w) if

$$\begin{bmatrix} -x \\ y \end{bmatrix} \geq \begin{bmatrix} -v \\ w \end{bmatrix};$$

Definition 2.9 (The Most Efficient Process): A process $(x,y) \varepsilon T$ is said to be the most efficient if (x,y) is more efficient than any other processes in T.

Note that if T is a closed convex set, the most efficient process $(x,y) \varepsilon T$ lies on a boundary point of T.

The concept of efficiency does not premise the profit evaluation under a price system. Its definitions are usually based solely on the comparison of physical quantities of inputs and outputs. But profit (or value added) evaluation is not altogether irrelevant to the efficiency of processes. If we let y and w denote two different (or the same) prices of a given product with corresponding price vectors P' and q', then the same efficiency concepts can apply for the processes $(P' \otimes x, y)$ and $(q' \otimes v, w)$. In this case, $P' \otimes x$ (or $q' \otimes v$) would represent the input costs while y (or w) denotes the output price of a given product.

There are several structural and topological properties of technology set T, which are useful for many economic analyses. Only those properties which we will make use of later are given below as presented by Nikaido [N1]. Let $T \varepsilon R^n$.

(1) (The law of constant returns to scale): If T is a cone then $\alpha T \propto T$ for any scaler $\alpha \geq 0$.

(2) (The absence of external diseconomies): If T is convex and (x,y), $(v,w) \varepsilon T$, then for $\alpha \geq 0$, $\beta \geq 0$, $\alpha+\beta=1$, there is an output vector r such that $(\alpha x+\beta v,\ r) \varepsilon T$, $\alpha y+\beta w \leq r$.

(3) (The closedness of the technology set): T is closed.

In what follows, using the definitions given in the preceding paragraphs, we will now describe the technology of the Leontief static input-output system. Recall that each sector uses only one production process to make one kind of good.

Let a_j be the jth column of the input coefficient matrix A. If G_j represents the amount of the jth good produced in the jth sector, the corresponding technology set is

$$T_j{}^q = \{(G_j a_j,\ G_j) | G_j \geqq 0\},\ j = 1, \ldots, n. \tag{2.11}$$

Obviously, $T_j{}^q$ can be thought of as a ray in R^{n+1} spanned by $\begin{bmatrix} a_j \\ I \end{bmatrix}$, a special polyhedral convex cone.

Defining the national economy-wide technology set by $T^q = \sum_{j=1}^{n} T_j{}^q$, we obtain

$$T^q = \{(AG,\ G) | G \geqq 0\} \tag{2.12}$$

which is also a polyhedral convex cone. Sometimes (2.12) is written, in what is called as the flow version, as

$$T^q = \{(I - A)G \mid G \geqq 0\} \qquad (2.12a)$$

which evidences the net outputs of the economy.

Now if we consider production as the transformation of input costs to output revenue (i.e., price), a different form of T_j for sector j may be defined as

$$T_j^{\,p} = \{(a_j \otimes P', P_j) \mid P' \geqq 0\},\ j = 1, \ldots, n, \qquad (2.13)$$

where $P=(P_1, \ldots, P_j, \ldots, P_n)$ is the price vector and P_j is the unit price of good j. In this case, a technology set T of the entire economy would be

$$T^p = \{(A'P', P') \mid P' \geqq 0\}, \qquad (2.14)$$

or in the flow version,

$$T^p = \{(I - A)'P' \mid P' \geqq 0\}. \qquad (2.14a)$$

Although different symbols have been introduced for technology sets of quantity and monetary transformations in the foregoing paragraphs, without loss of generality, we will drop the superscripts q and p and use T and T_j for the entire economy and sector j, respectively, from this point and on. The similarities and differences should clearly be understood however.

As was discussed in section 2.2, the Leontief system is based on the seemingly very stringent premise that essentially one single process is available in each sector. In view of T_j in (2.11) or (2.13), this premise implies the constancy of a_j. The variability of a_j implies the substitutability among alternative processes. Thus their constancy means the absence of alternative processes, viz., nonsubstitutability. There are several kinds of rationale, justifying the nonsubstitutability, which are referred to as nonsubstitution theorems. Here we will briefly discuss a nonsubstitution theorem advanced by Samuelson [S1], Koopmans [K4], Arrow [A2], and others.

A nonsubstitution theorem states that even if a set of alternative processes is available in each sector of the Leontief system, only a single fixed process would be selected, and that the combined operation of those n fixed processes, one for each sector, suffices to ensure all the efficiency resulting from using all the processes available. According to the above references, this practice prevails as long as labor is the only primary factor used in production, and joint production is not allowed.

A mathematical representation of this theorem can be provided by using technology sets. Adding the amount of labor ℓ_j, used in sector j to T_j in (2.13), we would have

$$T_j = \{(\ell_j, a_j \otimes P', P_j) | P' \geqq 0\}, \quad j = 1, \ldots, n, \qquad (2.15)$$

where $\ell_j \geq 0$ is a scaler. We assume that T_j is a closed convex cone and that $\ell_j=0$ implies $P_j=0$, that is, labor is indispensable.

Likewise, from (2.14),

$$T = \{(\ell, A'P', P') | P' \geq 0\} \tag{2.16}$$

where $\ell = \sum_{j=1}^{n} \ell_j$. Also, from (2.14a), define

$$T_+ = \{(\ell, P' - A'P') | P' \geq A'P'\}, \tag{2.16a}$$

which is a rearranged representation of the productive portion of T. Now if we define T_+^* as the set of most efficient processes $(\ell, P'-A'P')$ in T_+, the formal statement for a nonsubstitution theorem can be in order.

<u>A Nonsubstitution Theorem</u>: There are special processes $(1, (a_j\ P')^*, P_j^*) \varepsilon T_j$, for all j, fulfilling the following conditions:

(1) T_+^* is completely described by $\sum_{j=1}^{n} \theta_j (1, P_j^* - (a_j \otimes P')^*)$, when n parameters $\theta_j (j=1, \ldots, n)$ range all non-negative scalers.

(2) For any vector describing values added $V \geq 0$, there exists an $\ell \geq 0$ such that $(\ell, V) \varepsilon T_+^*$.

A similar type of theorem can be developed [M2] if we describe T_j, T, T_+ and T_+^* for the case of quantity transformations as

$$T_j = \{(\ell_j, G_j a_j, G_j) | G_j \geq 0\}, \quad j = 1, \ldots, n, \tag{2.17}$$

$$T = \{(\ell, AG, G) | G \geq 0\}, \tag{2.18}$$

$$T_+ = \{(\ell, G - AG) | G \geq AG\}, \tag{2.18a}$$

from (2.11), (2.12), and (2.12a), respectively, and T_+^* be the set of most efficient processes $(\ell, G-AG)$ in T_+ of (2.18a). However, without restating the theorem and with the similarities and differences already understood, the term, a nonsubstitution theorem will be used to apply for both cases, whichever is appropriate at the point of discussion.

§2.6 A Brief Literature Survey

Since the Leontief model first appeared in the 1930's, an enormous amount of studies have been conducted by many mathematicians as well as by a large number of economists.

Amongst many topics governing these studies, this section devotes its brief discussion on the structural change of an input-output system, which is responsible for the developments of price-models in later chapters; no comprehensive survey is offered here however.

In general, the structural change of a system can be affected by changes in its input-output coefficients. For several possible causes, each of which precipitates changes in the intermediate inputs of the sector under consideration, Forssell [F4] classifies them as follows:

(1) Technological changes; this change may appear through changes in quality of inputs, in technical renovation of production equipment, or in learning a better production design, etc.

(2) Changes in relative price of inputs; this change may lead to substitution among inputs.

(3) Changes in product mix; this is due to a change in the combination of production methods as the relative importance among similar products changes within industry.

(4) Changes in the level of gross outputs; this may also cause a structural change because of the existence of increasing or decreasing returns to scale.

These categories are not mutually exclusive but indicate the range of phenomena falling under the general heading of structural change.

Leontief [L2] himself pioneered both empirical and theoretical analysis of structural changes of an economy. Based on the empirical data for the American economic system between 1919 and 1939, he shows how the outputs of various industries and the prices of their products would have reacted to different types of primary changes. His study of price and output reactions are empirical insofar as they are computed from actual statistical data; they are also theoretical insofar as the computation formulae are deduced from abstract theoretical assumptions.

A study on the effects of technological change in a linear model of production functions is offered by Simmon [S4]. By partitioning activity functions of an industry by three levels, viz., for final products, for intermediate products, and for primary factors, his major concern was to find the unique set of activities that will give a greater output for some combinations of inputs than formerly.

Once the unique activity levels are found imposing certain optimizing conditions, his indication was that input-output coefficients for the industry may experience a change due to this optimization process.

Morishima [M6] discusses the effects of technological improvement of a sector concerning the equilibrium state of economy. If a system makes a technological improvement so as to save the ith good required to produce a unit of the jth good, then he concludes, the induced relative change in the equilibrium price P_i of the input directly involved falls short of the relative change in the output G_j directly produced. His observation on the price-quantity relationships are known to be a significant contribution to input-output analysis.

Carter [C1] offers a study that is descriptive of changing intermediate input structure. Presenting a broad survey of structural change between 1939 and 1961, she observes that the total intermediate output required to produce a fixed final demand remains fairly stable and even increases slightly. As to the reason for stability, her interpretation was that somewhat greater specialization has come with technological change.

Stability of input coefficients has been studied by several individuals. Among these individuals is Savaldson [S3]. He gives a definition of stability with statistical concepts as follows: Let $u=\overline{G}-G$ and $v=\overline{A}-A$, where G is an estimated gross output vector and A is an estimated input coefficients matrix. Those quantities corresponding to observed data are denoted by $\overline{G}$ and $\overline{A}$. Then the requirement for stability in a_{ij} is that the distribution of the v must not be such as to cause a distribution of u which leads to a rejection of the model for relevant values of consumption vector C.

Other statistical stability studies of input coefficients include Carter [C1], Forssell [F4], Stäglin and Wessels [S5]. Discussions on technological forecasting have appeared in Aujac [A3], Miernyk [M1], and others.

Whereas most of these studies on structural change leading to a system stability have been treated as statistical analyses of changing coefficients based on empirical data, the current work done by Moon [M2] views the problem in a slightly different perspective. Employing linear programming as the optimization process for a set of optimized sectors within an economy, he proposes a specific methodology for obtaining a operationally optimal set of input coefficients for each of the optimized sectors. He then relates those sectoral changes to the entire system through iterative processes, and gives a definition of stability as "when a given set of optimized sectors remain feasible and optimal under a given set of technologies so as not to yield a further perturbation in input coefficients in their equilibrium, the stability of system may be said to be achieved."

As indicated before, this study relies heavily upon the conceptual basis of Moon's definition of regulatory optimization leading to a system stability. In Chapter III, a precise description of the price problem will be given which is comparable with Moon's quantity formulation.

Chapter III

STABILITY OF AN INPUT-OUTPUT SYSTEM WITH OPTIMIZED SUBSYSTEMS

The precise descriptions of the quantity and price problems are presented in this chapter. Section 3.1 describes the overall concept of a stability study for the general problem. Rather vaguely defined terms of Chapter I, "regulatory optimization," "equilibrium state of system," and that of "stability" are clarified mathematically. The quantity problem of Moon constitutes section 3.2. Assumptions and the procedure of stability study are stated. Section 3.3 deals with the price problem and its characteristics. Depending upon different assumptions to be made, and under distinct economic reasonings behind these assumptions, three individual kinds of the price models are introduced. Criteria for stability are established for all the quantity and price problems.

§3.1 A General Problem

To develop an extremely general model for the problem of stability, we first begin with an assumption that there are p different types of interaction in a productive system with n interacting subsystems. These interactions are described by p functions $t = (t_1, t_2, \ldots, t_p)$, one t_i for each interaction type i, $i = 1, \ldots, p$. Given t, the problem can be stated as a system of equations of the general dimension:

$$t_i(D) = b_i, \quad i = 1, \ldots, p \tag{3.1}$$

or

$$t(D) = b \tag{3.1a}$$

where $b = \{b_i\}$: a set of demand vectors, $i = 1, \ldots, p$,

$D = \{D_k\}$: a production vector, $k = 1, \ldots, n$

$b_i = \{b_{k_i}\}$: $k = 1, \ldots, n$ for each i, and

$t_i = \{t_{k_i}\}$: $k = 1, \ldots, n$ for each i.

In Moon's general model [M3], each b_i is assumed to be fixed. However, a further generalization will be made in this study such that b, as well as D, can assume variability. Then the problem becomes one of solving for D and b given t.

Before investigating the problem in detail, we now define the term "equilibrium" that will be essential in understanding the concept of stability.

<u>Definition 2.10</u> (A System in Equilibrium): A system is said to be in a state of equilibrium if, given t, there exist specific, nontrivial solutions D and b for (3.1a).

Now consider a particular subsystem j ($j = 1, \ldots, n$) whose "activities" X are involved in its interactions with other subsystems within the system. Suppose that subsystem j interacts with q, $1 \leq q \leq n$, subsystems for interaction type i. Consider known functions, h_i, which may be defined by

$$h_i = (h_{1_i}, h_{2_i}, \ldots, h_{q_i}),\ i = 1, \ldots, p. \tag{3.2}$$

Here h_{k_i} denotes a function which describes a level of the i^{th} interaction between subsystem j and the k^{th} interacting subsystems. Also define g(X) which will represent the production level of subsystem j.

Then current operation of the j^{th} subsystem can be described by

$$\begin{cases} g(X) = D_j \\ h_i(X) = t_i(D_j), \quad i = 1, \ldots, p, \end{cases} \tag{3.3}$$

or

$$\begin{cases} g(X) = D_j \\ h(X) = t(D_j) \end{cases} \tag{3.3a}$$

where D_j is the j^{th} component of D. If, for some reason, each t_i should be described as a function of D, rather than D_j, (3.3a) can be rewritten as

$$\begin{cases} g(X) = D_j \\ h(X) = t(D). \end{cases} \tag{3.3b}$$

To generalize these two different function types of t, we will simply use t(D) in which case D_k, $k \neq j$, $k = 1, \ldots, n$ would be constants in (3.3a).

It is clear that if production level D_j of subsystem j is changed below or above some percent of its current level, the state of equilibrium for system (3.1a) or (3.3b) may be affected. Denoting these "some percents" by $\alpha > 0$, $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$ where $\beta_i = \{\beta_{k_i}\}$, $\beta_{k_i} > 0$, $k = 1, \ldots, q$ for each $i = 1, \ldots, p$, the above statements may be written as

$$\begin{cases} g(X) \gtreqless \alpha D_j & \text{:production} \\ h(X) \gtreqless \beta t(D) & \text{:level of interactions.} \end{cases} \tag{3.4}$$

With presence of inequalities, many possible solutions can exist. Thus if we develop an appropriate objective function f(X), the formal "regulatory optimization" problem for subsystem j can be stated as follows [M2, M3]:

$$\begin{aligned} &\max(\min) && f(X) \\ &\text{subject to} && g(X) \gtreqless \alpha D_j \\ & && h(X) \gtreqless \beta t(D) \\ & && X \geq 0 \end{aligned} \tag{3.5}$$

which may or may not yield a "regulatory optimal" program X^*. Many forms of mathematical programming, i.e., a linear or nonlinear, integer, dynamic, stochastic or goal programming, could be applied to solve (3.5). The choice of a programming technique would be based on the nature of t, i.e., the nature of the interactions among subsystems.

The obvious effect of the above optimization for the j^{th} subsystem is the perturbation of the function t. If X^* were obtained in equalities in (3.5) while α=1 and β_{k_i}=1 for all k and i, the level of current production D_j would also be operationally optimal. Otherwise t may be modified as a function of X^* to seek another regulatory optimal production level D_k, k=1,...,n, k$\neq$j, as well as new D_j. Some predetermined rule for the perturbation of t is assumed to be known. It is for this reason that the term "operational optimization" has been originated [M2].

With the foregoing paragraphs understood, we may proceed to introduce the concept of stability. Consider a composite, diagrammatic model of (3.1a) and (3.5) as the following:

$$
\begin{tikzcd}
\max(\min) & f(X) & \\
\text{subject to} \arrow[d] & \begin{array}{l} g(X)\gtreqless\alpha D_j \\ h(X)\gtreqless\beta t(D) \\ X \geq 0 \end{array} & \\
t=\phi_1(X^*) \arrow[r] & t(D)=b \arrow[r] & d=\begin{bmatrix}\alpha D_j \\ \beta t(D)\end{bmatrix}=\phi_2(D) \arrow[ul]
\end{tikzcd}
\tag{3.6}
$$

which implies that an optimization of sector j should be conducted in accordance with an equilibrium state of the overall system.

The major assumptions for a stability analysis, in regard to the above model may be summarized as:

(1) The system is initially in equilibrium, i.e., $t(D)=b$;

(2) Functions t, g, and h, are fixed and known; the direction of (in)equalities remains unchanged as is first determined;

(3) For the sake of meaningful stability analysis, the first optimal program $X^{*(1)}$ is assumed to exist;

(4) The function ϕ_1 is fixed and known; the function ϕ_2 is also fixed as determined by α and β.

To achieve an equilibrium state, it has been stated that the problem becomes one of solving for D and b given a predetermined function t. However, if every component of D and b are unknown, as the theory of linear equality tells us, the solvability of (3.1) may not exist. Thus, to ensure solvability for D and b, an important set of assumptions is made as follows:

$$\begin{array}{ll} (1) & D_k \text{ is a variable if } b_k \text{ is known, } k=1,\ldots,n, \\ (2) & D_k \text{ is fixed if } b_k \text{ can assume its variability, } k=1,\ldots,n, \\ (3) & \text{such that the total number of unknowns among the com-} \\ & \text{ponents of D and b is to be exactly n.} \end{array} \qquad (3.7)$$

In applying (3.6) given a system in its equilibrium state, an iterative process begins as t is perturbed to become $t^{(1)}=\phi_1(X^*)$, where ϕ_1 describes the mode of perturbation. With $t^{(1)}$, the system may not produce a set of equilibrium solutions. Even if it does, the corresponding equilibrium solutions denoted by $D^{(1)}$ and $b^{(1)}$ may most likely not be the same as the initial solutions D and b, although at least n components of $D^{(1)}$ and $b^{(1)}$ should be exactly the same as those components of D and b. Then the optimality status of X^* is in question as $\phi_2(D^{(1)})\neq\phi_2(D)$. If X^* does not remain with $d^{(1)}$, another regulatory optimization of the j^{th} subsystem may be conducted to obtain $X^{*(1)}$, yielding $t^{(2)}$ as a function of $X^{*(1)}$, etc.

The precise definition of stability can now be stated.

<u>Definition 2.10</u> (A Stable System): A system is said to be stable if a series of perturbations in t terminates, while the corresponding equilibrium solutions D and b converge to finite vectors.

In view of the above definition, note that a stable system obtained after i iterations simultaneously satisfies the following criteria:

(1) The existence of a regulatorily optimal program, $X^{*(i)}$, for the optimized subsystems;

(2) The existence of equilibrium solution vectors, $D^{(i)}$ and $b^{(i)}$, to the system;

(3) The convergence of $D^{(i)}$ and $b^{(i)}$; the existence of a finite number N such that $D \to D^{(N)} = D^{(i)}$ and $b \to b^{(N)} = b^{(i)}$.

§3.2 The Quantity Problem

The primary purpose of this section is to define in detail the quantity problem of a stability analysis, considering production as the transformation of inputs to outputs in the sense of quantity. In doing so, the quantity formulation of Leontief input-output model introduced in section 2.2 is chosen to describe the (national economic) system. We note that for this choice of a system, there is only one type of interaction among subsystems, i.e., the intermediate use of goods. Furthermore, a linear programming is considered as the optimization technique for any subsystem (sector).

If the operation of a particular sector, say, the kth sector of an n-sector economy, is to be optimized, then a diagrammatic model corresponding to (3.6) can be stated as follows:

For sector k: Minimize WX :inefficiency

Subject to $MX = G_k$:output

$LX \leq G_k a_k$:input

$X \geq 0$

$a_k = \phi_1(X^*)$ $\qquad$ $d = \phi_2(G_p)$

For economy: $(I-A)G=C$

where X = an m-component column vector of activities,

W = an m-component row vector for coefficients of primary factors,

M = an m-component row vector of output constraint coefficients,

L = an nxm matrix of input constraint coefficients,

G_k = the k^{th} component of G, for output level of sector k, and

a_k = the k^{th} column of $A=\{a_{ij}\}$ where each a_{ij} is measured in monetary units.

As optimization implies more efficient use of resources, sector k will be using a smaller amount of the outputs from other sectors. This will result in a new input-output matrix A and consequently a new final production level G with a possible change in the already determined operationally optimal program X^*. Every optimization iteration would perturb the input-output system. This system will be considered stable if the series of perturbations terminates after a finite number of iterations with the overall economy in equilibrium and the operation of sector k feasible and optimal.

We will now list two sets of assumptions that are necessary in defining the problem in detail.

For the original system $(I-A)G=C$, assume that:

(1) All a_{ij} are in monetary units, i.e., $a_{ij}=a^m_{ij}$.

(2) $C>0$ for a meaningful analysis, and C remains unchanged.

(3) $G>0$; the system is indecomposable, productive, and the Hawkins-Simon Condition is satisfied.

For the regulatory optimization of the k^{th} sector, it is assumed that:

(1) W, M and L are fixed and for obvious economic reasons, $W \geq 0$, $M \geq 0$ and $L \geq 0$.

(2) For the output constraint, an equality holds; for input constraints, inequalities ($\leq$) hold.

(3) $\alpha=1$ and $\beta_{k1}=1$, $k=1,\ldots,n$, for the single type of interaction so that

$$d^{(1)} = \begin{bmatrix} G_k \\ G_k a_k \end{bmatrix}_{(n+1)x1}.$$

(4) An operationally optimal program $X^{*(1)}$.

As the k^{th} sector is optimized, an underlying assumption is that there may be at least one realizable process under those less-than-or-equal-to constraints in which the strict equalities hold for the current process. Thus a different set of input coefficients may be generated to obtain a new feasible program.

Let $a_k^{(1)}$ be a set of input coefficients vectors corresponding to a set of all realizable processes, when the k^{th} sector is operationally optimized:

$$a_k^{(1)} = \{a_k^{(i1)} \mid i \geq 1\} \tag{3.8}$$

where $a_k^{(1)}$ is assumed to be nonempty. The corresponding technology set T_k is then defined by, from (2.11),

$$T_k = \{(G_k a_k^{(i1)}, G_k) \mid a_k^{(i1)} \varepsilon a_k^{(1)}, G_k > 0\}, k = 1, \ldots, n, \tag{3.9}$$

which is a polyhedral convex cone.

Adopting the Leontief's assumption of a linear technology, each $a_k^{(i1)}$ can be computed by

$$a_k^{(i1)} = \frac{L^{(i1)}X^{(i1)}}{M^{(i1)}X^{(i1)}} = \frac{L^{(i1)}X^{(i1)}}{G_k}, \quad i \geq 1 \tag{3.10}$$

where $X^{(i1)}$ is the i^{th} vector of a set of feasible programs $X^{(1)}$, $X^{(i1)} \varepsilon X^{(1)}$; and $L^{(i1)}$ and $M^{(i1)}$ are the corresponding basic input constraint matrix and output constraint vector, respectively.

Since $L^{(i1)}X^{(1)} \leq G_k a_k$ for all i, note that

$$a_k^{(i1)} \leq a_k, \; a_k^{(i1)} \varepsilon a_k^{(1)}, \quad i \geq 1 \tag{3.11}$$

which then implies that any feasible production process $(G_k a_k^{(i1)}, G_k)$ is at least as efficient as the current process $(G_k a_k, G_k)$. If there is more than one such process in T_k, we may select the operationally optimal process $(G_k a_k^{*(1)}, G_k)$ T_k such that

$$\begin{aligned} &(1) \quad a_k^{*(1)} = \frac{L^* X^{*(1)}}{G_k}, \\ &(2) \quad a_k^{*(1)} \leqq a_k, \; a_k^{*(1)} \varepsilon a_k^{(1)}, \\ &(3) \quad WX^{*(1)} \text{ is the minimum, } X^{*(1)} \varepsilon X^{(1)}, \end{aligned} \tag{3.12}$$

where $X^{*(1)}$ is the operationally optimal program, and L^* is the corresponding input constraints matrix. From the definitions 2.8 and 2.9 given in section 2.5, the following relationship can then be observed:

$$\begin{bmatrix} -G_k a_k^{*(1)} \\ G_k \end{bmatrix} \geqq \begin{bmatrix} -G_k a_k^{(i1)} \\ G_k \end{bmatrix} \geqq \begin{bmatrix} -G_k a_k \\ G_i \end{bmatrix} \tag{3.13}$$

If $a_i^{*(1)} = a_k$, the current process is also said to be operationally optimal and no perturbation of the current matrix A is necessary. However, if $a_k^{*(1)} \neq a_k$, a new perturbed matrix $A^{(1)}$ whose k^{th} column is now $a_k^{*(1)}$, produces a new set of gross outputs $G^{(1)} \neq G$ of which $G_k^{(1)} \neq G_k$. Thus a modification for the process of the k^{th} sector is made to $(G_k^{(1)} a_k^{*(1)}, G_k^{(1)})$. As a natural inclination, we ask ourselves the following set of questions:

(1) Does this modified process still belong to the technology set T_k?

(2) Would it remain optimal under T_k?

If "Yes" is the answer for both of the above questions, the iterative process need not be continued further. Otherwise, the process may continue to reoptimize the operation of sector k. The cycle of perturbations is illustrated in Figure 1 [M2].

As the final part of discussion in describing the quantity problem of a stability analysis, a formal set of criteria for the problem will now be developed. These criteria are provided below: at the i^{th} iteration, i = 1, 2, . . . ,

(1) Operational optimality of $X^{*(i)}$ with

$$d^{(i)} = \begin{bmatrix} G_k^{(i-1)} \\ G_k^{(i-1)} a_k^{*(i-1)} \end{bmatrix}$$

for the k^{th} sector which undergoes an optimization;

(2) The equilibrium status of $G^{(i)} = (I-A^{(i)})^{-1}C$ for the input-output system;

(3) Convergence of $\{G^{(i)}\}$.

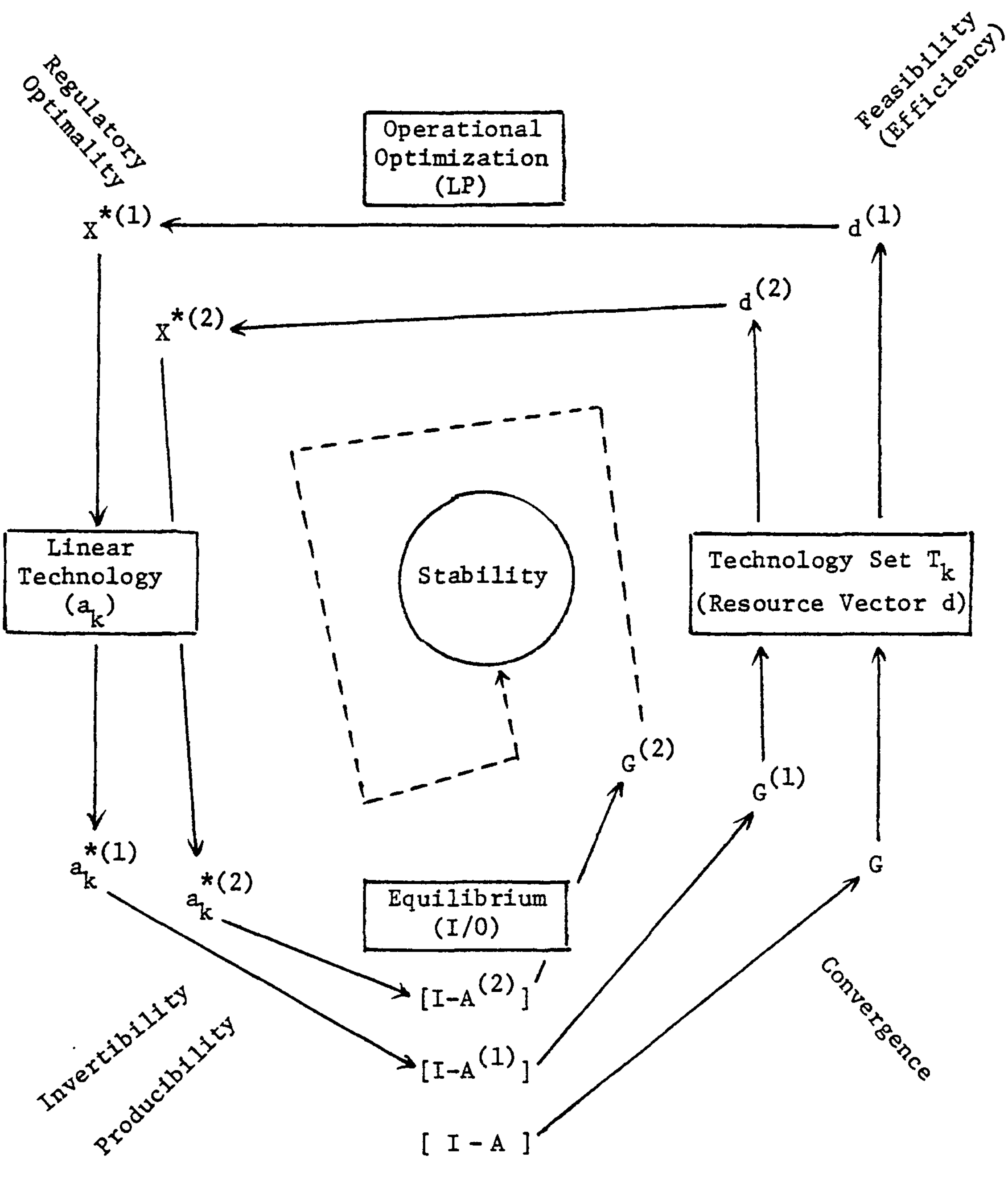

Figure 1: Iterative procedure of a quantitative static stability analysis.

Concerning criterion (1), post optimality analysis tells us that the optimality condition is unaffected by a change in the resource vector. Therefore, if $X^{*(i)}$ is feasible, it is also optimal. Criterion (2) for achieving an equilibrium state may be divided into the invertibility of $(I-A^{(i)})$ and the producibility of C. These are complementary of each other as discussed in section 2.2. For criterion (3), since convergence means $G^{(i)} = G^{(i-1)}$, it may be obtained if $a_k^{*(i)} = a_k^{*(i-1)}$.

Based on these arguments, an alternative set of more specific criteria can be developed. That is:

(1) The feasibility of $X^{*(i)}$;

(2) The invertibility of $(I-A^{(i)})$;

(3) The producibility of a given C;

(4) The equality of $a_k^{*(i)} = a_k^{*(i-1)}$.

Here we will consider the original input coefficient for sector k, a_k, as being equal to $a_k^{*(o)}$. Four conditions described above will be extensively used to examine the stability of the quantity problem in Chapter IV.

§3.3 The Price Problem

In this section, several versions of stability analysis in terms of cost-revenue relationships are developed. Unlike the quantity problem, those which we are about to take up will have more the flavor of the problem structures which occur in ordinary economic situations. From a producing sector point of view, we consider the cost of production as the amount in dollar values for buying some quantities of goods with their associated prices from the economy, while revenue is regarded as the amount (also in dollar value) that comes from selling the sector's products with a designated price to the economy. The interpretation of production as the transformation of input costs to output revenue (or, price of output) in which the difference yields the value added is, thus, essential.

The General Model

To state the problem in a well-posed form, we first choose the price formulation of a Leontief input-output model introduced in section 2.4, as the proper description of the system under consideration. Observe that the use of intermediate goods is the only type of interaction for the system. In addition, assume that linear programming model satisfies our needs for the optimization of any subsystem.

Suppose that the operation of the k^{th} sector is to be optimized. Then a diagrammatic model corresponding to (3.6) can be stated as follows:

$$
\begin{array}{llll}
\text{For sector } k: & \text{minimize} & bY & :\text{inefficiency} \\
 & \text{subject to} & SY = P_k & :\text{output revenue} \\
 & & HY \leq a_k \otimes P' & :\text{input costs} \\
 & & Y \geq 0 & \\
 & a_k = \psi_1(Y^*) & \qquad d = \psi_2(P) & \\
\text{For the economy:} & & (I-A')P' = V' &
\end{array}
$$

where Y = an m-component column vector,

b = an m-component row vector for coefficients of inefficiency,

S = an m-component row vector of output constraint coefficients,

H = an nxm matrix of input constraint coefficients,

P_k = the k^{th} component of P; the unit price of good k, and

a_k = the k^{th} column of $A=\{a_{ij}\}$ where each a_{ij} is measured in natural (physical) dimension .

During the course of optimization, sector k will constantly seek to obtain the most efficient way of using its resources until a certain set of stopping conditions are met. The type of resources, here, would mean the budget requirement for purchasing goods from the economy, and the stopping conditions would naturally be referred to as criteria for a stability. Consequently, this constant change of a_k will bring a new price vector P and a new vector of values added V such that another equilibrium state can be achieved, which in turn may affect the feasibility and optimality status of current optimal program Y^*. Thus, note that every optimization iteration would perturb the input-output system completely by means of changing P and V as well as the matrix A. As before, we will say that the system is stable if a series of perturbations terminates after a finite number of iterations with the overall economy in equilibrium and the operation of sector k feasible and optimal.

General Assumptions

Before describing specific models for the price problem we state a set of general assumptions and draw some conclusions from the assumptions made.

For the original system, $(I-A')P'=V'$, assume that:

(1) All $a_{ij} \geq 0$ are in physical units, having no upper bounds for the magnitudes of a_{ij}, except $a_{ii} < 1$ for all i.

(2) $P' > 0$, $V' > 0$; the system is indecomposable, in equilibrium, and the Hawkins-Simon Condition is satisfied. (3.15)

(3) The number of unknowns of the set, $\{u | u \varepsilon V,\ u \varepsilon P\}$ whose total number of elements comprises 2n, is n.

To specify the structure of ψ_2, choose:

(1) $\alpha=1$ and $\beta_{k1}=1$, $k=1,\ldots,n$ for interaction type 1,

(2) so that the limitation vector $d^{(1)}$ in the first iteration (3.16)

$$d^{(1)} = \begin{bmatrix} P_k \\ a_k \otimes P \end{bmatrix}_{(n+1)x1}.$$

For the operational optimization of the k^{th} sector, further assumptions are made as:

(1) $b' \geq 0$, $S' \geq 0$ and $H \geq 0$ are fixed.

(2) For the output constraint, an equality holds; for input constraints, inequalities ($\leq$) hold; (3.17)

(3) For an obvious economic reason, an operationally optimal program $Y^{*(1)}$ exists.

Observe that as the k^{th} sector is operationally optimized, there may be a nonempty set of input coefficient vectors, $a_k^{(1)}$, that corresponds to a set of all realizable processes under a given technology, where

$$a_k^{(1)} = \{a_k^{(i1)} | i \geq 1\}. \qquad (3.18)$$

From (2.13), the corresponding technology set T_k is, then, defined by

$$T_k = \{(a_k^{(i1)} \otimes P', P_k) | P' \geq 0\}, \quad k=1, \ldots, n \tag{3.19}$$

which is a polyhedral cone.

Based on the assumption of a linear technology, an $a_k^{(i1)} \varepsilon a_k^{(1)}$ is computed by

$$a_k^{(i1)} = (H^{(i1)} Y^{(i1)}) \oslash P' \tag{3.20}$$

where each element of P still remains as a constant, and

$$P_k = S^{(i1)} Y^{(i1)}. \tag{3.21}$$

$Y^{(i1)}$ is the i^{th} feasible program of a set of feasible programs $Y^{(1)}$, $Y^{(i1)} \varepsilon Y^{(1)}$, and the $H^{(i1)}$ and $S^{(i1)}$ are corresponding basic constraint matrices.

From the relationship, $H^{(i1)} Y^{(1)} \leq a_k \otimes P$, and (3.20), note that

$$a_k^{(i1)} \leq a_k, \; a_k^{(i1)} \varepsilon a_k^{(1)}, \quad i \geq 1, \tag{3.22}$$

which then implies, in terms of efficiency of production processes,

$$\begin{bmatrix} -a_k^{(i)} \otimes P' \\ P_k \end{bmatrix} \geqq \begin{bmatrix} -a_k \otimes P' \\ P_k \end{bmatrix} \quad \text{for all } i \tag{3.23}$$

where $P' > 0$. The precise interpretation of (3.23) is that any feasible production $(a_k^{(i1)} \otimes P', P_k)$ is at least as efficient as the current process $(a_k \otimes P', P_k)$. To find the operationally optimal process $(a_k^{*(1)} \otimes P', P_k) \varepsilon T_k$ and the optimal program $Y^{*(1)}$, we use the following set of conditions:

$$\begin{aligned} &(1) \quad a_k^{*(1)} = H^{*}Y^{*(1)} \oslash P' \\ &(2) \quad bY^{*(1)} \leqq bY^{(i1)} \text{ for all } i \end{aligned} \qquad (3.24)$$

where $a_k^{*(1)} \leqq a_k$, $a_k^{*(1)} \varepsilon a_k^{(1)}$, $Y^{*(1)} \varepsilon Y^{(1)}$, and H^{*} is the corresponding input constraints matrix. With the possible equality relationship in $a_k^{*(1)} \leqq a_k$, note that it is conceivable to have two equally efficient processes which may contain different activities.

Cases

We now turn our attention to seriously analyze the set of assumptions (3.15) for the original system, $(I-A')P'=V'$. Particularly, of utmost interest will be assumption (3) which states that the total number of unknowns, as the optimization progresses among the elements of P and V is to be exactly n. The reasons behind this assumption are, in large part, economical as well as mathematical.

Mathematically, if we are to solve a system of n equations and to obtain the unique solution, the theory of linear equality tells us that there must be exactly n number of unknowns in the system. Thus, given $(I-A')$, whether each of these n unknowns is an element of a vector V or P, as long as another n elements of P and V are fixed constants such that P_j is fixed when V_j is a variable, and vice versa, it is possible to find the solution for the system.

Depending upon the configurations of the system in terms of n unknown variables, let us categorize the system into three following cases: j denotes the sector that is to be operationally optimized.

(1) P_j fixed as well as P_k, $k \neq j$; all V_k assume their variabilities.

(2) P_j changes with fixed V_j; V_k change with fixed P_k, $k \neq j$, $k = 1, \ldots, n$.

(3) P_j changes as well as P_k, $k \neq j$, $k = 1, \ldots, n$; all V_k are fixed.

Economic reasons behind each of the above suppositions are given below.

Case 1 (Acquisition Model):

Consider a particular sector j which has just been encountered with the most efficient method of using its resources in producing its goods. As an immediate consequence, sector j will experience an increase in its profits, that is, if it chooses to sell its products at the same price as before. What the management of this sector will do with this increased profit is not our concern, although there are some implications, especially regarding the wage rates for the employees of this sector. Hence, if sector j is optimized and the price remains as a constant there would be positive acquisition (in dollars) to the sector. Since there is no reason to believe that this new acquisition of sector j will yield the changes of other goods' prices, the model remains within the framework of constancy of prices.

Case 2 (Consolidation Model):

Now suppose that sector j with its optimized production process decides to lower the price of its good j, maintaining the previous amount of profits. This assumption may seem rather unrealistic at first glance. However, (i) to attract the market for the products, (ii) to observe the reactions of other sectors in the economy, (iii) to enhance the sector's reputation, etc., sector j may well choose to do so, at least for a short period of time. Undoubtedly this will affect the sectors which require some amount of good j for their productions, and consequently, overall economy. If the prices of other goods remain the same, then, this compounding effect will be consolidated to obtain a new equilibrium state for the system.

Case 3 (Appreciation Model):

One of the most useful capabilities of this model may be the forecasting power for prices in the national economy. If we know how much each sector expects from its production, it is possible to forecast equilibrium prices for all the sectors in the economy. In this sense, the quantity formulation given in the previous section exhibits a close relationship with this model. Stated differently, if it is known how much to produce so that how much to expect from the production can be determined, what the appreciation of each sector's good must be should also be calculable. Most of the economic literature dealing with the price side of input-output system assumes these types of assumptions. However, the appreciation model in this study is concerned with the operational optimization within the economy rather than the economy itself.

These three different cases based on different economic reasons are illustrated in Figure 2. We will explore case 1 in more detail in Chapter V while case 2 and case 3 are expounded in Chapters VI and VII, respectively.

The Iterative Procedure

The same type of iterative procedure as in the quantity model can be assumed to prevail for all three of the price models. That is, if $a_k^{*(1)}=a_k$, the current process used in sector k is also operationally optimal and perturbation of matrix A is not necessary; if $a_k^{*(1)}\neq a_k$, $a_k^{*(1)}$ will replace a_k in A resulting in a new perturbed matrix $A^{(1)}$, which in turn will generate new vectors $P^{(1)}$ and $V^{(1)}$. Note that a simultaneous change of P and V only occur in the consolidation model since P is fixed in the acquisition model and V is fixed for the appreciation model, throughout the iterative procedure. With the modified process $(a_k^{*(1)}\otimes P^{(1)},\ P_k^{(1)})$, the iterative procedure should determine the answers to the following two questions, consecutively:

(1) Does $(a_k^{*(1)}\otimes P^{(1)},\ P_k)$ belong to the technology set T_k described by (3.19)?

(2) Would $(a_k^{*(1)}\otimes P^{(1)},\ P_k^{(1)})$ remain as the operationally optimal process?

If both of the above two questions are answered "Yes," the iterative process need not be continued further. Otherwise, the process will continue to reoptimize the operation of sector k.

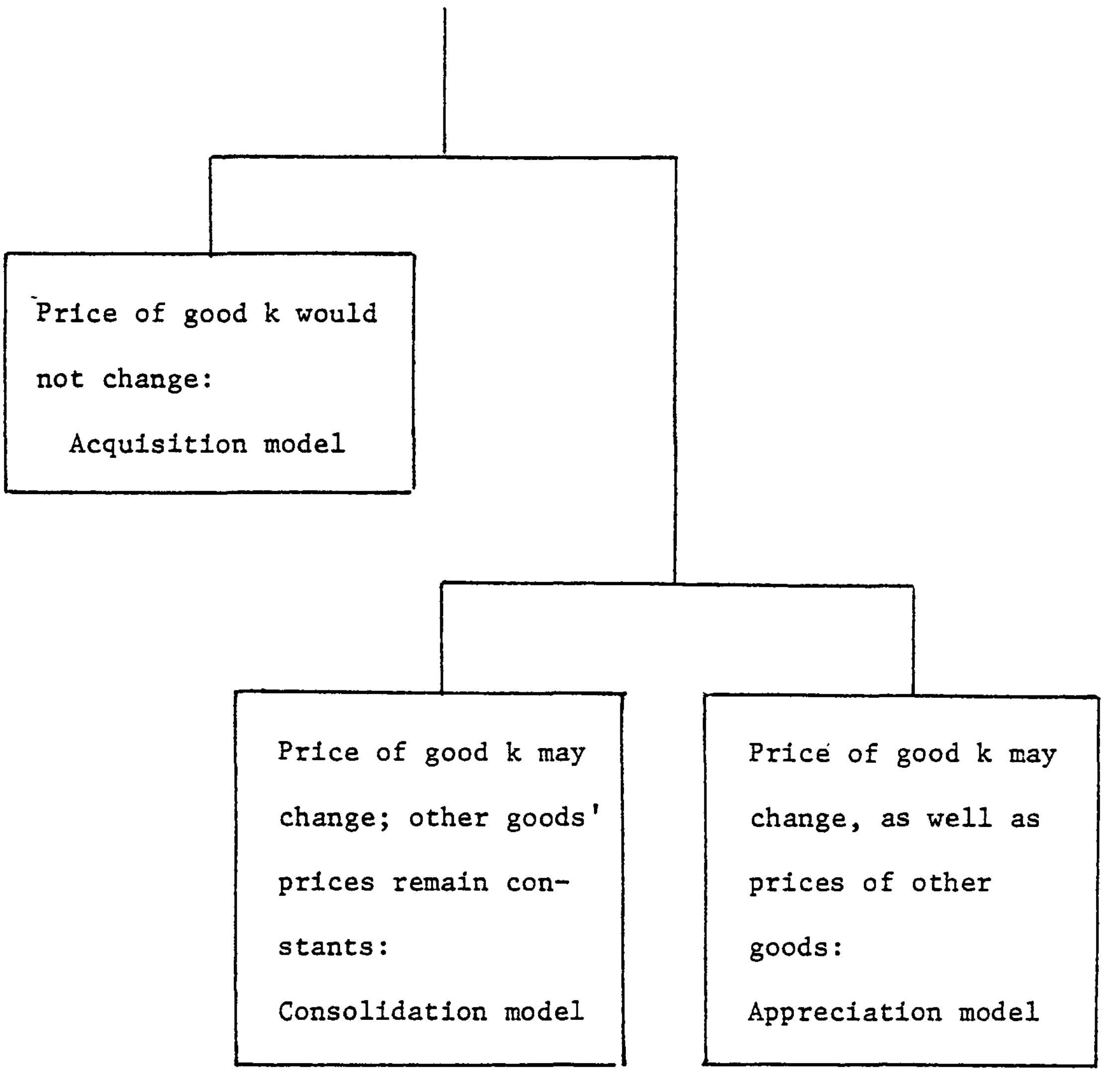

Figure 2. Three different models for the price formulation of static input-output analysis.

Criteria for Stability

A formal set of stability criteria may now be developed. We will first state the criteria in rather general terms as follows: at the i^{th} iteration, $i = 1, \ldots,$

(1) Operational optimality of $Y^{*(i)}$ with

$$d^{(i)} = \begin{bmatrix} P_k^{(i-1)} \\ a_k^{*(i-1)} \otimes P^{(i-1)} \end{bmatrix}.$$

(2) Equilibrium status of $(I-A^{(i)'})P^{(i)'} = V^{(i)'}$ for the input-output system.

(3) Convergence of $\{P^{(i)}\}$ and $\{V^{(i)}\}$.

Observe that the criterion (1) for the optimality of $Y^{*(i)}$ has the same interpretation as the feasibility of $Y^{*(i)}$ by the post optimality theorem; and the convergence of $P^{(i)}$ in criterion (3) implies the equality of $a_k^{*(i)} = a_k^{*(i-1)}$. Also note that depending on a specific case the criterion (2) may require an invertibility of $(I-A^{(i)'})$ at each i^{th} iteration to obtain an equilibrium state of the system. Furthermore, positivity of $P^{(i)}$ should be satisfied.

A specific set of criteria for each of the price models is tabulated in Table 1 in which the criterion with an asterisk (*) for the given price model is a necessary condition for a stable system. Those criteria without asterisk signs are, as will be seen later, conditions that are automatically satisfied or not required by the nature of the corresponding price model.

Table 1

Stability Criteria for Price Models

Criterion	Price Model: Acquisition Model	Consolidation Model	Appreciation Model
Feasibility of $Y^{*(i)}$	*	*	*
Invertibility of $(I-A^{(i)'})$			*
Positivity of P		*	*
Convergence of $P^{(i)}$, or equality of $a_k^{*(i)}=a_k^{*(i-1)}$		*	*
Convergence of $V^{(i)}$	*	*	

Chapter IV

AN INPUT-OUTPUT SYSTEM WITH OPTIMIZED SUBSYSTEMS: THE QUANTITY MODEL

The chapter offers a stability study of an input-output system in the sense of quantity, as developed by Moon. The model is described in section 4.1 along with assumptions to be made. A development for the case of single sectoral optimization is contained in section 4.2, based on the product form of the inverse. Mathematical relationships satisfying the conditions for the system to be stable are shown in terms of scalar coefficients of the product form. A rather interesting conclusion that under all normal circumstances the current system can reach a stable state in one iteration is then made. Section 4.3 generalizes the single sectoral problem to accommodate the situations when more than one sector undergoes an operational optimization. The overall conclusion on the quantitative stability study can be found in section 4.4.

§4.1 The Model

Consider an economy in which q, $1 \leq q \leq n$, number of sectors are to be optimized. If we designate these sectors by $k_1, k_2, \ldots, k_q$, $1 \leq k_p \leq n$, $p = 1, \ldots, q$, the diagrammatic model in its general form corresponding to the model introduced in section 3.2 can be shown as follows:

$$
\begin{array}{lll}
\text{For sector } k_p: & \text{minimize} & W_{k_p} X_{k_p} \\
 & \text{subject to} & M_{k_p} X_{k_p} = G_{k_p} \\
 & & L_{k_p} X_{k_p} \leqq G_{k_p} a_{k_p} \\
 & & X_{k_p} \geq 0 \\
 & a_{k_p} = \phi_1(X^*_{k_p}) & \qquad d = \phi_2(G_{k_p}) \\
\text{For economy:} & (I-A)^{-1}C = G &
\end{array}
\qquad (4.1)
$$

where variable vectors X_{k_p}, a_{k_p}, G_{k_p}, and constant vectors W_{k_p}, M_{k_p}, L_{k_p} are associated with a particular sector k_p, $p = 1, \ldots, q$.

This model implies that optimization of multiple-sector problem optimizes each sector consecutively until all the q sectors are operationally optimized while overall economy is in equilibrium. All the assumptions made in section 3.2 are retained for this analysis. Specifically, we recall that $C>0$, $G>0$, $0<\sum_j a_{ij}<1$ for all j as all a_{ij} are measured in monetary units and $X^{*(1)}_{k_p}$ exists for each $p = 1, \ldots, q$, given a system in its equilibrium.

§4.2 Development for the Single-Sector Optimization Problem

Suppose that stability of the system with only one optimized sector, say sector k_1, is to be investigated. For a notational convenience, we will adopt the simplified version of model (4.1) for this single-sector case by dropping subscripts from X_{k_p}, W_{k_p}, L_{k_p} and M_{k_p} so that those variables without subscripts can be assumed to have an association with that particular sector k_1, or simply sector k, i.e., $k = k_1$. If the gross outputs for all the sectors, G, are found by solving the set of equations of the Leontief input-output system, the iterative algorithm is ready to function. The resource vector $d^{(1)}$ in the first iteration, then, would be

$$d^{(1)} = \begin{bmatrix} G_k \\ G_k a_k \end{bmatrix} \tag{4.2}$$

where a_k is the original input coefficient of sector k, and scalar G_k is the equilibrium gross output for the sector.

By an assumption, $X^{*(1)}$, an m-component column vector describing the optimal program in the first iteration exists. To develop a very useful relationship let $X^{*(1)}$ be redesignated by $X_R^{*(1)}$ for real variables and write $X^{*(1)}$ which now constitutes not only real variables $X_R^{*(1)}$, but also slack and surplus variables $X_S^{*(1)}$, as:

$$X^{*(1)} = \begin{bmatrix} X_R^{*(1)} \\ X_S^{*(1)} \end{bmatrix}, \tag{4.3}$$

with an assumption that $X^{*(1)}$ can be partitioned into $X_R^{*(1)}$ and $X_S^{*(1)}$ as shown above. Note that $X^{*(1)}$ is an (n+1)-component column vector and $X_S^{*(1)}$ is an (n+1-m)-component column vector. Then from the theory of linear programming the following are obvious:

$$X^{*(1)} = B^{-1}d^{(1)}, \text{ and } d^{(1)} = BX^{*(1)} \tag{4.4}$$

where B represents the optimal basis. Expressing B in terms of the elements of M and the column of L, we assume that B can be structured as:

$$B = \begin{bmatrix} \hat{M} \\ \hat{L} \end{bmatrix} \tag{4.5}$$

which is an (n+1)x(n+1) matrix. If the columns of B, corresponding to $X_S^{*(1)}$, any slack or surplus activity of $X^{*(1)}$ are filled by (n+1) component null vectors, this modified optimal basis, which will be denoted by B^*, can have the form:

$$B^* = \begin{bmatrix} M^* \\ L^* \end{bmatrix}. \tag{4.5a}$$

By the characteristics of B^* just defined we see that

$$\begin{bmatrix} X_R^{*(1)} \\ 0 \end{bmatrix} = B^{*-1} d^{(1)} \tag{4.6}$$

and also,

$$d^{(1)} = B^* \begin{bmatrix} X_R^{*(1)} \\ 0 \end{bmatrix} = B^* \begin{bmatrix} X_R^{*(1)} \\ X_S^{*(1)} \end{bmatrix} = B^* X^{*(1)} \tag{4.6a}$$

Based on the assumption of a linear technology prevailing in each sector, the new optimal input coefficients $a_k^{*(1)} \varepsilon a_k^{(1)}$ is computed by:

$$a_k^{*(1)} = \frac{L^* X^{*(1)}}{M^* X^{*(1)}} = \frac{L^* X^{*(1)}}{G_k} \tag{4.7}$$

where $a_k^{*(1)} \leqq a_k$ from the arguments given in section 3.2. Note that

$$0 \leq \sum_{j=1}^{n} a_{jk}^{*(1)} \leqq \sum_{j=1}^{n} a_{ij} < 1, \text{ and } a_{jk}^{*(1)} \geq 0 \text{ for all } j. \tag{4.8}$$

If $a_k^{*(1)} = a_k$, then no perturbation would occur in A. In this case, the current process $(G_k a_k, G_k)$ is also an operationally optimal process and the system is considered stable.

On the other hand, if $a_k^{*(1)} \leq a_k$, the gross outputs of the entire economy may be affected and need to be recomputed for a new equilibrium state. The new input coefficients matrix of the economy $A^{(1)}$, resultant from the perturbation of A, will have its new k^{th} column $a_k^{*(1)}$ in place of a_k; all other columns remaining the same. To take advantage of this unique feature of the once perturbed matrix $A^{(1)}$, let us adopt the product-form approach to find the expression for $(I-A^{(1)})^{-1}$. If the inverse of $(I-A^{(1)})$ exists, the new equilibrium solution $G^{(1)}$ can be obtained by

$$G^{(1)} = (I-A^{(1)})^{-1}C. \tag{4.9}$$

Since (I-A) constitutes a basis for R^n we can express the k^{th} column of $(I-A^{(1)})$ as a linear combination of the columns of (I-A). Hence,

$$e_k - a_k^{*(1)} = \sum_{j=1}^{n} \gamma_j (e_j - a_j) = (I-A)\gamma \tag{4.10}$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)'$ is a vector of scalar coefficients. Only if

$$\gamma_k \neq 0, \tag{4.11}$$

will $(I-A^{(1)})$ be non-singular and hence have an inverse. Therefore, rearranging (4.10),

$$e_k - a_k = \frac{1}{\gamma_k}(e_k - a_k^{*(1)}) - \sum_{\substack{i=1\\ j\neq k}}^{n} \frac{\gamma_j}{\gamma_k}(e_j - a_j) \tag{4.12}$$

which can be rewritten more conveniently in matrix notation as

$$e_k - a_k = (I-A^{(1)})\eta_k \tag{4.13}$$

where

$$\eta_k = (\eta_{1k}, \eta_{2k}, \ldots, \eta_{k-1,R}, \eta_{kk}, \eta_{k+1,k}, \ldots, \eta_{nk})$$

$$= (-\frac{\gamma_1}{\gamma_k}, -\frac{\gamma_2}{\gamma_k}, \ldots, -\frac{\gamma_{k-1}}{\gamma_k}, \frac{1}{\gamma_k}, -\frac{\gamma_{k+1}}{\gamma_k}, \ldots, -\frac{\gamma_n}{\gamma_k})' . \tag{4.14}$$

From equations (4.13) and (4.14) we see that the relationship between (I-A) and $(I-A^{(1)})$ as

$$(I-A) = (I-A^{(1)})E_k, \tag{4.15}$$

or $$(I-A^{(1)})^{-1} = E_k(I-A)^{-1} \tag{4.15a}$$

where $$E_k = (e_1, e_2, \ldots, e_{k-1}, \eta_k, e_{k+1}, \ldots, e_n) \tag{4.16}$$

which is an nxn operational matrix. Equation (4.15a) gives $(I-A^{(1)})^{-1}$ in terms of $(I-A)^{-1}$ which has already been computed while obtaining the initial equilibrium output levels G. Since $G = (I-A)^{-1}C$, from (4.9) and (4.15a),

$$G^{(1)} = E_k G. \tag{4.17}$$

We also develop an expression which will be useful for our later analysis;

$$\eta_k = (I-A^{(1)})^{-1}(e_k - a_k). \tag{4.18}$$

Now observe that (4.17) is valid only if (4.11) is true. However, since (4.8) suffices to ensure the solvability of (4.9) according to the Brauer-Solow Condition introduced in section 2.2, (4.11) is satisfied and hence (4.17) can be a valid statement. Furthermore, by the invertibility-producibility relationship, (4.8) also assures the producibility of given C, i.e., $G^{(1)} \geq 0$.

From (4.17) note that

$$G_j^{(1)} = G_j + \eta_{jk}G_k, \; j = 1, \ldots, n; \; j \neq k, \text{ and}$$
$$G_k^{(1)} = \eta_{kk}G_k, \tag{4.19}$$

where $\eta_{jk} = \frac{-\gamma_j}{\gamma_k}$ for $j \neq k$ and $\eta_{kk} = \frac{1}{\gamma_k}$. Since it has already been shown $G^{(1)} \geq 0$, (4.19) is equivalent to

$$\gamma_j \leq \gamma_k \cdot \frac{G_j}{G_k}, \text{ all } j \neq k, \text{ and}$$
$$\gamma_k > 0. \tag{4.20}$$

Let us proceed with further investigation on the magnitude of the components of vector γ which will be useful for describing the variability of $G^{(1)}$ from its original value, G. Since $a_k^{*(1)} \leqq a_k$, it is obvious that

$$e_k - a_k^{*(1)} \geqq e_k - a_k, \text{ and} \tag{4.21}$$

$$(I-A^{(1)}) \geqq (I-A). \tag{4.21a}$$

In addition, both $(I-A^{(1)})$ and $(I-A)$ are class K matrices. Hence it follows that

$$(I-A^{(1)})^{-1} \leqq (I-A)^{-1}. \tag{4.21b}$$

From (4.10) and (4.18), and inequalities (4.21) and (4.21b), we have

$$\gamma \geqq \eta_k \tag{4.22}$$

which supplies the information that

$$\gamma_j \geqq -\frac{\gamma_j}{\gamma_k} \text{ for all } j \neq k, \text{ and } \gamma_k \geqq \frac{1}{\gamma_k}. \tag{4.22a}$$

Therefore

$$\gamma_j \geq 0 \text{ for all } j \neq k, \text{ and } \gamma_k \geq 1.0. \tag{4.23}$$

Referring back to (4.19) with the knowledge obtained in (4.23), we may state that the new gross output level of each sector, including that of sector k, could decrease as sector k switches its production process from its original process to its operationally optimal process, i.e.,

$$G_j^{(1)} = G_j - \frac{\gamma_j}{\gamma_k} G_k \leq G_j, \text{ for all } j \neq k \tag{4.24}$$

$$G_k^{(1)} = \frac{1}{\gamma_k} G_k \leq G_k. \tag{4.25}$$

We now concentrate our discussion on the more efficient process, $(G_k^{(1)} a_k^{*(1)}, G_k^{(1)})$, which has arrived as an immediate consequence of the optimization of sector k with respect to the new equilibrium state,

$(I-A^{(1)})G^{(1)} = C$. Specifically, our concerns here are the feasibility of this new process for a given technology set T_k and the optimality status.

Recalling $d^{(1)} = B^*X^{*(1)}$ from (4.6a), we compute the net change in the resource vector as

$$\Delta d^{(2)} = d^{(2)} - d^{(1)}$$

$$= \begin{bmatrix} G_k^{(1)} \\ G_k^{*(1)} - a_k^{*(1)} \end{bmatrix} - B^*X^{*(1)}.$$

Using equations (4.6) and (4.24), and also noting that $G_k = M^*X^{*(1)}$ by the very nature of the equality constraint for the output in sector k,

$$\Delta d^{(2)} = \frac{1}{\gamma_k}\begin{bmatrix} M^*X^{*(1)} \\ G_k \cdot \frac{L^*X^{*(11)}}{G_k} \end{bmatrix} - B^*X^{*(1)} \tag{4.26}$$

$$= (\frac{1}{\gamma_k} - 1)(B^*X^{*(1)}).$$

With $\Delta d^{(2)}$, the new optimal program $X^{*(2)}$ for the optimization problem can be found as

$$X^{*(2)} = X^{*(1)} + B^{*-1}(\Delta d^{(2)})$$

$$= X^{*(1)} + (\frac{1}{\gamma_k} - 1)\ B^{*-1} \cdot B^*X^{*(1)} \tag{4.27}$$

$$= \frac{1}{\gamma_k}\ X^{*(1)}.$$

Since $\gamma_k \geq 1.0$, (4.27) reveals an obvious feasibility of $X^{*(2)}$. This, then implies that the new production process certainly belongs to the technology set given, i.e.,

$$(G_k^{(1)} a_k^{*(1)},\ G_k^{(1)}) \varepsilon T_k. \tag{4.28}$$

According to the post-optimality analysis, the optimality of $X^{*(2)}$ is guaranteed by its feasibility. Consequently, there would be no need to search for another process, $(G_k^{(1)} a_k^{*(2)},\ G_k^{(1)}) \varepsilon T_k$, which may be more efficient than $(G_k^{(1)} a_k^{*(1)},\ G_k^{(1)}) \varepsilon T_k$ in the sense that

$$\begin{bmatrix} -G_k^{(1)} a_k^{*(2)} \\ G_k^{(1)} \end{bmatrix} \geq \begin{bmatrix} -G_k^{(1)} a_k^{*(1)} \\ G_k^{(1)} \end{bmatrix}$$

which implies $a_k^{*(2)} \leq a_k^{*(1)}$. This assurance can be further evidenced when an attempt to compute $a_k^{*(2)}$ is made:

$$a_k^{*(2)} = \frac{L^* X^{*(2)}}{M^* X^{*(2)}} = \frac{L^* (\frac{1}{\gamma_k}) X^{*(1)}}{(\frac{1}{\gamma_k}) G_k} = a_k^{*(1)}. \tag{4.29}$$

By having $a_k^{*(2)} = a_k^{*(1)}$, the last condition for stability is satisfied. The Leontief matrix $(I-A^{(1)})$ need not be perturbed further and therefore the system, with its unaffected gross outputs $G^{(1)}$, can be said to have reached a stable state.

4.3 Development for the Multiple-Sector Optimization Problem

We shall now study the behavior of the economy, when two or more sectors are optimized. First, the case of two optimized sectors will be considered, as we proceed with optimizing one more sector, say $k_2 (k=k_1 \neq k_2)$, in addition to sector k_1 which has already been optimized in the previous section. The study will then be extended to the stability of the economy, of which q of its n sectors are optimized.

As was concluded at the end of section 4.2, the optimization procedure for sector k_1 was terminated when the once perturbed input coefficient matrix, $A^{(1)}$, produced a new equilibrium solution, $G^{(1)}$, to the system.

Two-Sector Optimization

For our current problem for sector k_2, the gross output $G_{k_2}^{(1)}$ and the original input coefficients a_{k_2} will play a role as the initial resource vector

$$d_{k_2}^{(1)} = \begin{bmatrix} G_{k_2}^{(1)} \\ G_{k_2}^{(1)} a_{k_2} \end{bmatrix}. \tag{4.30}$$

The initial problem for sector k_2 is given below:

$$\begin{array}{lll} \text{For sector } k_2\text{:} & \text{minimize} & W_{k_2} X_{k_2} \\ & \text{subject to} & M_{k_2} X_{k_2} = G_{k_2}^{(1)} \\ & & L_{k_2} X_{k_2} \leq G_{k_2}^{(1)} a_{k_2} \\ & & X_{k_2} \geq 0 \\ & a_{k_2}^{*} = \phi_1(X_{k_2}^{*}) & \qquad\qquad d_{k_2} = \phi_2(G_{k_2}^{(1)}) \\ \text{For economy:} & & (I-A^{(1)})^{-1} C = G^{(1)} \end{array} \tag{4.31}$$

with an assumption that $G_{k_2}^{(1)} > 0$ for a meaningful analysis.

If there is any solution to the sectoral problem, then

$$\begin{bmatrix} X_{R,k_2}^{*(1)} \\ 0 \end{bmatrix} = B_{k_2}^{*-1} d_{k_2}^{(1)} \quad \text{and} \quad d_{k_2}^{(1)} = B_{k_2}^{*} X_{k_2}^{*(1)} \tag{4.32}$$

where

$$B^*_{k_2} = \begin{bmatrix} M^*_{k_2} \\ L^*_{k_2} \end{bmatrix} \qquad (4.33)$$

is a modified, (n+1)-dimensional optimal basis, and

$$X^{*(1)}_{k_2} = \begin{bmatrix} X^{*(1)}_{R,k_2} \\ X^{*(1)}_{S,k_2} \end{bmatrix} \qquad (4.34)$$

is an (n+1)-component activity vector of which $X^{*(1)}_{R,k_2}$ and $X^{*(1)}_{S,k_2}$ correspond to the real and slack variables, respectively. From (4.32) and (4.33), we can obtain a new set of optimal input coefficients as:

$$a^{*(1)}_{k_2} = \frac{L^*_{k_2} X^{*(1)}_{k_2}}{G^{(1)}_{k_2}} \qquad (4.35)$$

which satisfies the relationship

$$0 \leq a^{*(1)}_{k_2} \leqq a_{k_2} < 1. \qquad (4.36)$$

Observe that (4.36) reveals the satisfaction of the Brauer-Solow Condition and hence $(I-A^{(2)})$, whose k_2th column is now $e_{k_2} - a^{*(1)}_{k_2}$, will have its inverse.

To find $(I-A^{(2)})^{-1}$ the product-form approach is again employed as follows:

$$(I-A^{(2)})^{-1} = E_{k_2}(I-A^{(1)})^{-1} \qquad (4.37)$$

where

$$E_{k_2} = (e_1, e_2, \ldots, e_{k_2-1}, \eta_{k_2}, e_{k_2+1}, \ldots, e_n), \tag{4.38}$$

$$\eta_{k_2} = (-\frac{\gamma_{1k_2}}{\gamma_{k_2k_2}}, -\frac{\gamma_{2k_2}}{\gamma_{k_2k_2}}, \ldots, -\frac{\gamma_{k_2-1,k_2}}{\gamma_{k_2k_2}}, \frac{1}{\gamma_{k_2k_2}}, -\frac{\gamma_{k_2+1,k_2}}{\gamma_{k_2k_2}}, \ldots, -\frac{\gamma_{nk_2}}{\gamma_{k_2k_2}})' \tag{4.38a}$$

From (4.9), (4.15a) and (4.37), the new equilibrium solution can be found by

$$\begin{aligned} G^{(2)} &= (I-A^{(2)})^{-1}C \\ &= E_{k_2}G^{(1)} \end{aligned} \tag{4.39}$$

which then, in component-wise, gives the information that:

$$\begin{aligned} G_j^{(2)} &= G_j^{(1)} + \eta_{jk_2}G_{k_2}^{(1)}, \quad \eta_{jk_2} = -\frac{\gamma_{jk_2}}{\gamma_{k_2k_2}}; \; j \neq k_2 \\ G_{k_2}^{(2)} &= \eta_{k_2k_2}G_{k_2}^{(1)} = \frac{1}{\gamma_{k_2k_2}}\, G_{k_2}^{(1)}. \end{aligned} \tag{4.40}$$

Note that $G^{(2)}$ can also be expressed in terms of the original gross output vector G, if we utilize the expression (4.17) as

$$G_{k_2}^{(2)} = E_{k_2}E_{k_1}G. \tag{4.41}$$

As $(I-A^{(2)})$ is a productive class-K matrix the following relationships similar to (4.23) can be developed as

$$\gamma_{jk_2} \geq \text{ for all } j \neq k_2 \text{ and } \gamma_{k_2k_2} \geq 1 \tag{4.42}$$

and also

$$0 \leq \frac{\gamma_{jk_2}}{\gamma_{k_2k_2}} \leq \frac{G_j^{(1)}}{G_{k_2}^{(1)}}, \; j \neq k_2 \tag{4.43}$$

where for sector k_1 the corresponding inequality to (4.43) can be written

$$0 \leq \frac{\gamma_{jk_1}}{\gamma_{k_1k_1}} \leq \frac{G_j}{G_{k_2}}, \; j \neq k_1. \tag{4.43a}$$

From the information obtained in (4.40), (4.42) and (4.43a), we observe that gross outputs for all sectors, including those optimized sectors, are constantly decreasing unless equalities hold.

The next step for our stability study is to see whether or not the modified production process of sector k_2, $(G_{k_2}^{(2)} a_{k_2}^{*(1)}, G_{k_2}^{(2)})$, is still an element of T_{k_2} where

$$T_{k_2} = \{(G_{k_2}^{(1)} a_{k_2}^{(i1)}, G_{k_2}^{(1)}) \,|\, a_{k_2}^{(i1)} \varepsilon a_{k_2}^{(1)}, G_{k_2}^{(1)} > 0\} \tag{4.44}$$

A usual post-optimality analysis can be performed. From (4.35) and noting that $G_{k_2}^{(1)} = M_{k_2}^* X_{k_2}^{*(1)}$,

$$\begin{aligned}
\Delta d_{k_2}^{(2)} &= d_{k_2}^{(2)} - d_{k_2}^{(1)} \\
&= \begin{bmatrix} G_{k_2}^{(2)} \\ \\ G_{k_2}^{(2)} a_{k_2}^{*(1)} \end{bmatrix} - B_{k_2}^* X_{k_2}^{*(1)} \\
&= \left(\frac{1}{\gamma_{k_2k_2}} - 1\right) B_{k_2}^* X_{k_2}^{*(1)}.
\end{aligned} \tag{4.45}$$

With this change in the resource vector, and since $\gamma_{k_2k_2} \geq 1$ by (4.42) the new feasible program which is also optimal, can be found as

$$
\begin{aligned}
X_{k_2}^{*(2)} &= X_{k_2}^{*(1)} + B_{k_2}^{*-1}(\Delta d_{k_2}^{(2)}) \\
&= \left(\frac{1}{\gamma_{k_2k_2}}\right) X_{k_2}^{*(1)} \\
&\leq X_{k_2}^{*(1)}
\end{aligned}
\tag{4.46}
$$

This optimality status that is guaranteed by the feasibility of $X_{k_2}^{*(2)}$ as indicated in (4.46) can be further supported by

$$
a_{k_2}^{*(2)} = \frac{L_{k_2}^{*} X_{k_2}^{*(2)}}{G_{k_2}^{(2)}} = \frac{L_{k_2}^{*} \left(\frac{1}{\gamma_{k_2k_2}}\right) X_{k_2}^{*(1)}}{\left(\frac{1}{\gamma_{k_2k_2}}\right) G_{k_2}^{(1)}} = a_{k_2}^{*(1)} \tag{4.47}
$$

Hence, the operationally optimal process $(G_{k_2}^{(2)} a_{k_2}^{*(1)}, G_{k_2}^{(2)})$ remains optimal and the system is stable.

Now let us examine the impact of the optimization of sector k_2 on the already optimized sector k_1 whose current production process is now $(G_{k1}^{(2)} a_{k1}^{*(1)}, G_{k1}^{(2)})$ due to the new equilibrium state, $(I-A^{(2)})G^{(2)}=C$. From (4.40), the value of $G_{k_1}^{(2)}$ may be found as

$$
G_{k_1}^{(2)} = G_{k_1}^{(1)} - \frac{\gamma_{k_1k_2}}{\gamma_{k_2k_2}} G_{k_2}^{(1)} . \tag{4.48}
$$

To digress for a moment, define a vector $\alpha^{(p)}$, which can be called as a vector of the decrease factors, by

$$\alpha^{(p)} = (\alpha_1^{(p)}, \alpha_2^{(p)}, \ldots, \alpha_j^{(p)}, \ldots, \alpha_n^{(p)})' \tag{4.49}$$

$$= G^{(p)} \oslash G^{(p-1)}$$

where p denotes the iteration number so that k_p is the index for the sector being optimized. Using (4.49), we may be able to find those factors for k_1, $\alpha_{k_1}^{(1)}$ and $\alpha_{k_1}^{(2)}$, corresponding to iterations 1 and 2 respectively, by

$$\alpha_{k_1}^{(1)} = \frac{G_{k_1}^{(1)}}{G_{k_1}} = \frac{(\frac{1}{\gamma_{k_1k_1}})G_{k_1}}{G_{k_1}} = \frac{1}{\gamma_{k_1k_1}} \tag{4.50}$$

$$\alpha_{k_1}^{(2)} = \frac{G_{k_1}^{(2)}}{G_{k_1}^{(1)}} = \frac{G_{k_1}^{(1)} - \frac{\gamma_{k_1k_2}}{\gamma_{k_2k_2}} G_{k_1}^{(1)}}{G_{k_1}^{(1)}} = 1 - \frac{\gamma_{k_2k_2}}{\gamma_{k_2k_2}}(\frac{G_{k_2}^{(1)}}{G_{k_1}^{(1)}}). \tag{4.50a}$$

From the preceding analysis, recall that

$$0 \leq G_{k_1}^{(2)} \leq G_{k_1}^{(1)} \leq G_{k_1}. \tag{4.51}$$

This, along with (4.50) and (4.50a), implies that

$$0 \leq \alpha_{k_1}^{(1)} \leq 1 \quad \text{and} \quad 0 \leq \alpha_{k_1}^{(2)} \leq 1 \tag{4.52}$$

where

$$G_{k_1}^{(p)} = \alpha_{k_1}^{(p)} G_{k_1}^{(p-1)}, \; p = 1, 2; \; G_{k_1}^{(o)} = G_{k_1}. \tag{4.53}$$

Also from (4.27) and (4.50) we would have

$$X_{k_1}^{*(2)} = \alpha_{k_1}^{(1)} X_{k_1}^{*(1)} \leq X_{k_1}^{*(1)}. \tag{4.54}$$

As the stability analysis proceeds to check the post optimality for the new resource vector $d_{k_1}^{(3)}$ in sector k_1 due to the optimization of sector k_2, we may hope that a relationship between $X_{k_1}^{(2)}$ and $X_{k_2}^{(3)}$ can be found. To be specific, if $X_{k_1}^{*(3)} = \alpha_{k_1}^{(2)} X_{k_1}^{*(2)}$ then we can conclude that $X_{k_1}^{*(3)}$ is feasible and therefore optimal.

We now compute the net change in the resource vector of sector k_1 after the optimization of sector k_2 by

$$\begin{aligned} \Delta d_{k_1}^{(3)} &= d_{k_1}^{(3)} - d_{k_1}^{(2)} \\ &= \begin{bmatrix} G_{k_1}^{(2)} \\ G_{k_1}^{(2)} a_{k_1}^{*(1)} \end{bmatrix} - \begin{bmatrix} G_{k_1}^{(1)} \\ G_{k_1}^{(1)} a_{k_1}^{*(1)} \end{bmatrix} \end{aligned} \tag{4.55}$$

Substituting (4.5a), (4.7) and (4.48), and noting that $G_{k_1}^{(1)} = M_{k_1}^{*} X_{k_1}^{*(2)}$, we have

$$\Delta d_{k_1}^{(3)} = - \frac{\gamma_{k_1 k_2}}{\gamma_{k_2 k_2}} \frac{G_{k_2}^{(1)}}{G_{k_1}^{(1)}} B^{*} \cdot X_{k_1}^{(3)}. \tag{4.56}$$

The new program of sector k_1, $X_{k_1}^{*(3)}$, corresponding to the modified production process, $(G_{k_1}^{(2)} a_{k_1}^{*(1)}, G_{k_1}^{(2)})$, may now be computed by

$$\begin{aligned} X_{k_1}^{*(3)} &= X_{k_1}^{*(2)} + B^{*-1}(\Delta d_{k_1}^{(3)}) \\ &= X_{k_1}^{*(2)} + B^{*-1} \cdot B^{*} \left(- \frac{\gamma_{k_1 k_2}}{\gamma_{k_2 k_2}}\right)\left(\frac{G_{k_2}^{(1)}}{G_{k_1}^{(1)}}\right) \cdot X_{k_1}^{*(2)} \\ &= \alpha_{k_1}^{(2)} X_{k_1}^{*(2)} \end{aligned} \tag{4.57}$$

As a result, the optimization of sector k_2 did not affect sector k_1 at all, in that the first selection of sector k_1 of the operationally optimal production process still remains optimal. Again, since $X_{k_1}^{*(3)}$ and $G_{k_1}^{(2)}$ were changed by the same decrease factor, $\alpha_{k_1}^{(2)}$, from $X_{k_1}^{*(2)}$ and $G_{k_1}^{(1)}$, respectively, we have

$$a_{k_1}^{*(1)} = a_{k_1}^{*(2)} = a_{k_1}^{*(3)}. \tag{4.58}$$

General Case

Finally, we shall proceed to development of a general model in which each of q ($\leq$n) sectors of the n-sector economy is seeking an operationally optimal production process under a given technology. Without loss of generality, we may optimize q sectors sequentially, one at a time, starting from sector k_1 and completing the optimization of the sector k_q at last, no matter what each k_p, p=1, ..., q, actually represents. Therefore q optimization processes will be needed through q iterations, as shown in diagrammatic model (4.1). At iteration p (p=1, ..., q), then, sector k_p can be assumed being optimized.

If sector k_p is to be optimized at the p^{th} iteration, its resource vector would be

$$d_{k_p}^{(1)} = \begin{bmatrix} G_{k_p}^{(p-1)} \\ G_{k_p}^{(p-1)} a_{k_p} \end{bmatrix}, \quad 1 \leq p \leq q, \tag{4.59}$$

which represents its current production process $(G_{k_p}^{(p-1)} a_{k_p}, G_{k_p}^{(p-1)})$ under the given technology set T_{k_p}, where

$$T_{k_p} = \{(G_{k_p}^{(p-1)} a_{k_p}^{(\ell 1)},\ G_{k_p}^{(p-1)}) \,|\, a_{k_p}^{(\ell 1)} \varepsilon a_{k_p}^{(1)},\ G_{k_p}^{(p-1)} > 0\},\ 1 \leq p \leq q. \tag{4.60}$$

As before, we find an optimal program for sector k_p at iteration p:

$$X_{k_p}^{*(1)} = B_{k_p}^{*-1} d_{k_p}^{(1)} = \begin{bmatrix} M_{k_p}^{*} \\ L_{k_p}^{*} \end{bmatrix}^{-1} \begin{bmatrix} G_{k_p}^{(p-1)} \\ G_{k_p}^{(p-1)} a_{k_p} \end{bmatrix} \tag{4.61}$$

which exists by an assumption; $B_{k_p}^{*}$ denotes the modified optimal basis. The new optimal input coefficients for sector k_p is then found by

$$a_{k_p}^{*(1)} = \frac{L_{k_p}^{*} X_{k_p}^{*(1)}}{G_{k_p}^{(p-1)}} \leq a_{k_p} \tag{4.62}$$

where $G_{k_p}^{(p-1)} = M_{k_p}^{*} X_{k_p}^{*(1)}$ from the equality constraint.

Then, $a_{k_p}^{*(1)}$ replaces the current k_pth column, a_{k_p} in $A^{(p-1)}$ resulting in a new matrix $A^{(p)}$. A new equilibrium solution $G^{(p)}$ to this p^{th} perturbed system is obtained by

$$\begin{aligned} G^{(p)} &= (I-A^{(p)})^{-1} C \\ &= E_{k_p} (I-A^{(p-1)})^{-1} C \end{aligned} \tag{4.63}$$

where

$$E_{k_p} = (e_1,\ e_2,\ \ldots,\ e_{k_p-1},\ \eta_{k_p},\ e_{k_p+1},\ \ldots,\ e_n), \tag{4.64}$$

$$\eta_{k_p} = \left(-\frac{\gamma_{1k_p}}{\gamma_{k_p k_p}},\ \ldots,\ -\frac{\gamma_{k_p-1,k_p}}{\gamma_{k_p k_p}},\ \frac{1}{\gamma_{k_p k_p}},\ -\frac{\gamma_{k_p+1,k_p}}{\gamma_{k_p k_p}},\ \ldots,\ -\frac{\gamma_{nk_p}}{\gamma_{k_p k_p}}\right)' \tag{4.64a}$$

and

$$\gamma_{k_p} = (\gamma_{1k_p}, \gamma_{2k_p}, \ldots, \gamma_{nk_p})'$$
$$= (I-A^{(p-1)})^{-1}(e_{k_p} - a_{k_p}^{(1)}). \tag{4.65}$$

As was discussed in two-sector optimization model, we have the assertion for

$$\gamma_{k_p k_p} \geq 1, \; 1 \leq p \leq q \tag{4.66}$$

and

$$0 \leq \gamma_{jk_p} < \gamma_{k_p k_p} \frac{G_j^{(p-1)}}{G_{k_p}^{(p-1)}}, \; 1 \leq p \leq q, \; 1 \leq j \leq n, \; j \neq k_p, \tag{4.66a}$$

from the facts that $(I-A^{(p)})$ is a class K matrix and the p^{th} perturbed system is productive based on an implication of (4.62). From (4.66) and (4.66a) we note that

$$0 \leq G^{(p)} \leqq G^{(p-1)} \leqq \ldots \leqq G^{(1)} \leqq G. \tag{4.67}$$

due to the relationships:

$$G_{k_p}^{(p)} = \eta_{k_p k_p} G_{k_p}^{(p-1)}$$
$$G_j^{(p)} = G_j^{(p-1)} + \eta_{jk_p} G_{k_p}^{(p-1)}, \; j \neq k_p, \; p = 1, \ldots, q. \tag{4.68}$$

The vector of decrease factors, $\alpha^{(p)}$, as defined by (4.49) is computed by components as

$$\alpha_j^{(p)} = \begin{cases} \eta_{jk_p}, \; j = k_p, \; 1 \leq p \leq q \\ 1 + \eta_{jk_p} \cdot \dfrac{G_{k_p}^{(p-1)}}{G_j^{(p-1)}} \text{ for all } j \neq k_p. \end{cases} \tag{4.69}$$

Equations in (4.68) can now be written as

$$G_j^{(p)} = \alpha_j^{(p)} G_j^{(p-1)} \text{ for all } j. \tag{4.70}$$

where

$$0 \leq \alpha_j^{(p)} \leqq 1 \tag{4.71}$$

We now proceed to post-optimality analysis to determine whether or not the modified production process $(G_{k_p}^{(p)} a_{k_p}^{*(1)}, G_{k_p}^{(p)}) \epsilon T_{k_p}$. The resulting program will be feasible and therefore optimal by (4.71), and it can be written as

$$X_{k_p}^{*(2)} = \alpha_{k_p}^{(p)} X_{k_p}^{*(1)}, \; 1 \leq p \leq q. \tag{4.72}$$

Also, from (4.70) we have

$$G_{k_p}^{(p)} = \alpha_{k_p}^{(p)} G_{k_p}^{(p-1)}, \; 1 \leq p \leq q. \tag{4.73}$$

As was discussed in two-sector optimization model, since both $X_{k_p}^{*(2)}$ and $G_{k_p}^{(p)}$ employ the same decrease factor $\alpha_{k_p}^{(p)}$, there will be no other production process under T_{k_p} which may yield $a_{k_p}^{*(2)} \leq a_{k_p}^{*(1)}$. Therefore, $a_{k_p}^{*(2)} = a_{k_p}^{*(1)}$, and this completes the optimization process of sector k_p.

As the final touch to this well established stability analysis, let us consider the optimality status of industries $k_1, k_2, \ldots, k_{p-1}$, which have already optimized their production processes, prior to optimization of sector k_p. At iteration p ($1 \leq p \leq q$), the new gross output for sector j, $k_1 \leq j \leq k_{p-1}$, can be found by (4.70). It should also be recalled that the optimal input coefficients for sector j are still $a_j^{*(1)}$. Therefore, the post-optimality test at iteration p yields a new program

$$X_j^{*(p-j+2)} = \alpha_j^{(p-j+1)} X_{k_p}^{(p-j+1)}, \quad k_1 \leq j \leq k_{p-1}, \quad 1 \leq p \leq q \qquad (4.74)$$

which is feasible and hence also optimal.

§4.4 Conclusions

In conclusion, the static input-output system reaches a stable state as any subset of the sectors undergoes the operational optimization. As a sector is optimized, an operationally optimal production process is selected under a given technology; a new equilibrium state of the system does always exist under the normal assumptions, and it does not affect the optimality of the process thus selected.

As an additional sector is optimized, the same argument given above holds for the newly optimized sector. Thus optimality status of the previously optimized production process is unaffected by this additional sectoral optimization.

For each of any $q \leq n$ sectoral optimizations, only one iteration is required to select its operationally optimal process. Thus, at the p^{th} iteration, $p=1, \ldots, q$, if the operationally optimal process of $(G_{k_p}^{(p-1)} a_{k_p}^{*(1)}, G_{k_p}^{(p-1)})$ is not the same as the sector's old process of $(G_{k_p}^{(p-1)} a_{k_p}, G_{k_p}^{(p-1)})$, then a modified process of $(G_{k_p}^{(p)} a_{k_p}^{*(1)}, G_{k_p}^{(p)})$ becomes the operationally optimal process for sector k_p and it remains optimal. This then requires every sector of the economy to reduce its gross output level from $G^{(p-1)}$ to $G^{(p)}$ in order for the system to maintain an equilibrium state; the system remains stable, as all the q sectors are optimized, under all normal circumstances.

Chapter V

AN INPUT-OUTPUT SYSTEM WITH OPTIMIZED SUBSYSTEMS: THE ACQUISITION MODEL

This chapter presents the acquisition model for a stability study of an input-output system when any subset of its subsystems undergoes an operational optimization. In section 5.1, the model is diagrammatically described, and a specific set of assumptions which characterize the acquisition-model-type price system of an economy are stated. Based on the concept of production technology set which is described in cost-to-revenue structure, a development of single-sectoral optimization problem is offered in section 5.2. A generalization of the single-sectoral problem is then established using the theory of linear equality in section 5.3. Finally, some concluding remarks concerning the acquisition model are made in section 5.4.

§5.1 The Diagrammatic Model

Consider a system in which the acquisition model can represent the proper problem structure of stability analysis for the optimizations of its subsystems. Suppose there are q sectors, k_1, k_2, ..., k_q, $1 \leq q \leq n$, each of which seeks for an optimal production process. Then, a diagrammatic model corresponding to (3.14) can be shown as follows:

For sector k_p: Minimize $b_{k_p} Y_{k_p}$

$$\text{subject to } S_{k_p} Y_{k_p} = P_{k_p} \tag{5.1}$$

$$H_{k_p} Y_{k_p} \leqq a_{k_p} \otimes P'$$

$$Y_{k_p} \geq 0$$

$$a_{k_p} = \psi_1 (Y^*_{k_p}) \qquad d_{k_p} = \psi_2 (a_{k_p})$$

For economy: $V' = (I - A')P'$

where

Y_{k_p} = an (n+1) component column vector of activities of which the first m components, Y_{R,k_p}, are for real variables, and the next (n+1-m) components, Y_{S,k_p}, are for slack or surplus variables, i.e.,

$$Y_{k_p} = \begin{bmatrix} Y_{R,k_p} \\ \hline Y_{S,k_p} \end{bmatrix} \tag{5.2}$$

b_{k_p} = an (n+1) component row vector for inefficiency coefficients with the last (n+1-m) components having the value 0,

S_{k_p} = an (n+1) component row vector of output constraint coefficients,

H_{k_p} = an nx(n+1) matrix for input constraints coefficients.

Note that we have included (n+l-m) elements (or columns) in S_{k_p} and H_{k_p} to accommodate slack or surplus variables as was done for Y_{k_p} and b_{k_p}.

By the very specific assumption of the acquisition model, the price vector P will remain constant throughout the iterative process whereas value added for each sector can assume its variability whenever necessary. Also, recall that each element of input coefficients matrix A is measured in physical dimension. All other general assumptions for price models made in section 3.3 are retained for this analysis. Specifically, we recognize that $V'>0$, $P'>0$, and $Y^{*(1)}_{k_p}$ exists for each p for a meaning analysis.

§5.2 Development for the Single-Sector Optimization Problem

For the optimization of only one sector, sector k_1 or simply sector k, we will adopt a similar approach to the quantity formulation by dropping the subscript k wherever possible.

Suppose given matrix A and the price vector P, a set of values added are obtained by a simple matrix-vector multiplication, i.e.,

$$V' = (I - A')P'. \tag{5.3}$$

Then the cost-revenue vector, which we will call "limitation vector," in the first iteration, would be formed as:

$$d^{(1)} = \begin{bmatrix} P_k \\ a_k \otimes P' \end{bmatrix} \tag{5.4}$$

where a_k is the original input coefficients of sector k, measured in physical units, and P is the given unit price vector.

By an assumption, $Y^{*(1)}$, an (n+1) component column vector describing the optimal program exists. If the optimal basis for $Y^{*(1)}$ is denoted by

$$B = \begin{bmatrix} \hat{S} \\ \hat{H} \end{bmatrix}_{(n+1) \times (n+1)}, \tag{5.5}$$

then we would have

$$Y^{*(1)} = B^{-1}d^{(1)} \text{ and } d^{(1)} = BY^{*(1)} \tag{5.6}$$

Substituting with null vectors for the columns that correspond to slack or surplus variables of $Y^{*(1)}$ in B, a modified optimal basis can be obtained as

$$B^* = \begin{bmatrix} S^* \\ H^* \end{bmatrix} \tag{5.7}$$

whose interests are only on those real variables, $Y_R^{*(1)}$. With (5.7), a similar set of equations as (5.6) can be obtained as

$$\begin{aligned} \begin{bmatrix} Y_R{}^{*(1)} \\ 0 \end{bmatrix} &= B^{*-1}d^{(1)}; \\ d^{(1)} &= B^* \begin{bmatrix} Y_R{}^{*(1)} \\ 0 \end{bmatrix} = \begin{bmatrix} Y_R{}^{*(1)} \\ Y_S{}^{*(1)} \end{bmatrix} = B^*Y^{*(1)} \end{aligned} \tag{5.8}$$

Based on the Leontief assumption of a linear technology, the new optimal input coefficients $a_k{}^{*(1)} \varepsilon a_k{}^{(1)}$ is computed by

$$a_k^{*(1)} = (H^*Y^{*(1)}) \oslash P' \tag{5.9}$$

The reason behind $a_k^{*(1)}$ being optimal is obvious. As H* contains null vectors for columns that correspond to slack or surplus variables, by having $a_k^{*(1)}$ as in (5.9) it eliminates unnecessary production costs that will not decrease objective function value. From (5.9) it should also be noted that

$$\hat{H} \begin{bmatrix} Y_R^{*(1)} \\ 0 \end{bmatrix} = a_k^{*(1)} \otimes P' \tag{5.10}$$

and also,

$$\hat{S} \begin{bmatrix} Y_R^{*(1)} \\ 0 \end{bmatrix} = S^* \begin{bmatrix} Y_R^{*(1)} \\ Y_S^{*(1)} \end{bmatrix} = \hat{S} \begin{bmatrix} Y_R^{*(1)} \\ Y_S^{*(1)} \end{bmatrix} = P_k \tag{5.10a}$$

In (5.9), if we substitute (5.10a) for P_k,

$$a_{kk}^{*(1)} = \frac{H_k^{*}Y^{*(1)}}{S^{*}Y^{*(1)}} = \frac{H_k^{*}Y^{*(1)}}{\hat{S}\ Y^{*(1)}} \tag{5.11}$$

where $a_{kk}^{*(1)}$ is the kth element of $a_k^{*(1)}$, and H_k^* denotes the kth row of H^*, or equivalently (k+1) st row of B^*. Since the original input coefficients a_k can be described by

$$a_k = (\hat{H}Y^{*(1)}) \oslash P' \tag{5.12}$$

and

$$H^*Y^{*(1)} \leqq \hat{H}Y^{*(1)}, \tag{5.13}$$

we see that

$$a_k^{*(1)} \leqq a_k. \tag{5.14}$$

If $a_k^{*(1)}=a_k$ (which indicates that $Y^{*(1)}$ is obtained by $H^*Y=a_k\ P$) in (5.14), then no perturbation would occur in A. If that happens, the current production process ($a_k \otimes P'$, P_k) is also an operationally optimal process and the system is stable.

However if $a_k^{*(1)} \leq a_k$, the vector of value added has to be changed to reach a new equilibrium state as the prices of all goods are fixed. To compute new set of values added $V^{(1)}$, we apply a simple matrix-vector multiplication rule,

$$V^{(1)'} = (I - A^{(1)'})P' \tag{5.15}$$

where $A^{(1)}$ is formed by substituting the kth column of A by $a_k^{*(1)}$. Clearly, since only the kth column has been changed in A, only the value added for the kth sector will experience a change; values added for all other sectors remaining the same. Hence, $V^{(1)}$ can be obtained by

$$V_j^{(1)} = V_j,\ j \neq k \tag{5.15a}$$

$$V_k^{(1)} = (-a_{1k}^{*(1)}, -a_{zk}^{*(1)}, \ldots, 1 - a_{kk}^{*(1)}, \ldots, -a_{nk}^{*(1)})P'$$

$$= P_k - \sum_{i=1}^{n} a_{ik}^{*(1)} P_i. \tag{5.15b}$$

Originally, V_k, using its original input coefficients, was:

$$V_k = P_k - \sum_{i=1}^{n} a_{ik} P_i. \tag{5.16}$$

From (5.15b) and (5.16), we can then compute the amount of change that occurred in the value added of sector k as

$$\Delta V_k^{(1)} = V_k^{(1)} - V_k$$

$$= (P_k - \sum_{i=1}^{n} a_{ik}^{*(1)} P_i) - (P_k - \sum_{i=1}^{n} a_{ik} P_i)$$

$$= \sum_{i=1}^{n} a_{ik} P_i - \sum_{i=1}^{n} a_{ik}^{*(1)} P_i$$

$$= \sum_{i=1}^{n} (a_{ik} - a_{ik}^{*(1)}) P_i \qquad (5.17)$$

from which we can determine, by the fact $a_k^{*(1)} \leq a_k$, that

$$\Delta V_k^{(1)} > 0. \qquad (5.17a)$$

The interpretation of (5.17) and (5.17a) is straightforward. As sector k undergoes an optimization, its only effects are the positive acquisition, namely $\Delta V_k^{(1)}$, to the sector and the change of input coefficients for obtaining a new production process. All other values would remain the same as the price of each kind of good, including the good for the sector that is operated under operationally optimal production process.

The new limitation vector $d^{(2)}$ can be used to compute $Y^{*(2)}$ where

$$d^{(2)} = \begin{bmatrix} P_k \\ a_k^{*(1)} \otimes P' \end{bmatrix} \qquad (5.18)$$

by

$$Y^{*}(2) = B^{-1}d^{(2)} \quad (5.19)$$

However since we can describe $d^{(2)}$ by, from (5.10) and (5.10a)

$$d^{(2)} = \begin{bmatrix} \hat{S} \\ \hat{H} \end{bmatrix} \begin{bmatrix} Y_R^{*(1)} \\ 0 \end{bmatrix}, \quad (5.20)$$

substitution of (5.20) in (5.19) will yield a very obvious result:

$$Y^{*}(2) = \begin{bmatrix} Y_R^{*(1)} \\ 0 \end{bmatrix} \quad (5.21)$$

A satisfaction of feasibility for $Y^{*(2)}$ is clear, and hence by the post-optimality theory, $Y^{*(2)}$ is also optimal, i.e.,

$$bY^{*(1)} = bY^{*(2)} \quad (5.22)$$

If we attempt to compute $a_k^{*(2)}$, a new optimal input coefficients that may result in a better production process, optimality status of $Y^{*(2)}$ or equivalently, that of the operationally optimal process of $(a_k^{*(1)} \otimes P', P_k)$ can be further supported:

$$\begin{aligned} a_k^{*(2)} &= (\hat{H}Y^{*(2)}) \oslash P' \\ &= (\hat{H}\begin{bmatrix} Y_R^{*(1)} \\ 0 \end{bmatrix}) \oslash P' \\ &= (H^{*}\begin{bmatrix} Y_R^{*(1)} \\ Y_S^{*(1)} \end{bmatrix}) \oslash P' \\ &= (H^{*}Y^{*(1)}) \oslash P' \\ &= a_k^{*(1)} \end{aligned} \quad (5.23)$$

Having satisfied all the criteria for our stability analysis, we can now conclude that optimization of single-sector problems can be solved after only one iteration. The new optimal input coefficients vector $a_k^{*(1)}$ is computed by a simple matrix manupulation and the increased value added for the sector is determined from it. The optimal program, once obtained, retains its values except for any slack or surplus variables.

§5.3 The Multiple-Sector Optimization Problem

To generalize the development of stability analysis in the single-sector problem we now assume that there are q, $1 \leq q \leq n$, sectors to be optimized. In the usual manner, we will denote these sectors by $k_1, k_2, \ldots, k_q$.

First recall that optimization of any one sector does not affect other sectors in any means due to the assumption of constant prices. This, greatly simplifies our stability study even in the multi-sectoral problem. As (5.1) diagrammatically illustrates, a sequential approach can be taken to optimize sectors k_1 through sector k_1, one by one. As is already evident, once a sector is optimized, it will not be affected no matter what, or how many sectors are optimized therafter.

Suppose given original input coefficients matrix A and the fixed price vector P, a set of values added are computed to have the initial equilibrium state for the system. Then for each sector k_p, P=1, ..., q, limitation vector in the first iteration would be

$$d_{k_p}{}^{(1)} = \begin{bmatrix} P_{k_p} \\ a_{k_p} \otimes P' \end{bmatrix} \qquad (5.24)$$

which corresponds to the current production process $(a_{k_p} \otimes P', P_{k_p})$ under given technology set,

$$T_{k_p} = \{(a_{k_p}{}^{(i1)} \otimes P', P_{k_p}) | P' > 0\}, \; P = 1, \ldots, n, \qquad (5.25)$$

a polyhedral cone.

Assume $Y_{k_p}{}^{*(1)}$ exists as the operationally optimal program. If the optimal basis B_{k_p} can be structured as

$$B_{k_p} = \begin{bmatrix} \hat{S}_{k_p} \\ \hat{H}_{k_p} \end{bmatrix}, \quad p = 1, \ldots, q, \qquad (5.26)$$

we have

$$Y_{k_p}^{*(1)} = B_{k_p}^{-1} d_{k_p}^{(1)} \text{ and } d_{k_p}^{(1)} = B_{k_p} Y_{k_p}^{*(1)} . \qquad (5.27)$$

For optimal basis of each sector k_p, B_{k_p}, if the columns which associate with slack or surplus activity of $Y_{k_p}^{*(1)}$ are filled with null vectors, a modified optimal basis can be obtained as

$$B_{k_p}{}^{*} = \begin{bmatrix} S_{k_p}{}^{*} \\ H_{k_p}{}^{*} \end{bmatrix} \qquad (5.28)$$

so that

$$B_{k_p}^{*-1} d^{(1)} = \begin{bmatrix} Y_{R,k_p}{}^{*(1)} \\ 0 \end{bmatrix} \qquad (5.29)$$

where $Y_{R,k_p}^{*(1)}$ is an m-component column vector of real variables for sector k_p.

Using (5.28) and the presence of equality in the input cost constraints a new input coefficients $a_{k_p}^{*(1)}$ is computed by

$$a_{k_p}^{*(1)} = (H_{k_p}^{*} Y_{k_p}^{*(1)}) \oslash P', \; p = 1, \ldots, q, \tag{5.30}$$

which is optimal, in the sense that it eliminates production costs which would not decrease the factors of inefficiency for the objective function value. For the constraints of operational optimization problem, we recognize that

$$P_{k_p} = \hat{S}_{k_p} Y_{k_p}^{*(1)} = S_{k_p}^{*} Y_{k_p}^{*(1)} = \hat{S}_{k_p} \begin{bmatrix} Y_{R,k_p}^{*(1)} \\ 0 \end{bmatrix}$$

and (5.31)

$$a_{k_p}^{*(1)} \otimes P' = \hat{H}_{k_p} \begin{bmatrix} Y_{R,k_p}^{*(1)} \\ 0 \end{bmatrix}$$

where

$$a_{k_p}^{*(1)} \leqq a_{k_p}.$$

When the new optimal input coefficients are found by (5.30) for all of the q sectors, we may proceed to investigate a new equilibrium state, resultant from q number of optimizations. Since all the prices are fixed, the problem becomes to solve $V^{(1)}$ in terms of $A^{(1)}$ where columns k_1, k_2, ..., k_q of A which were originally a_k, a_{k2}, ...,

a_{kq}, are now $a_{k_1}^{*(1)}$, $a_{k2}^{*(2)}$, ..., $a_{k_q}^{*(1)}$, in $A^{(1)}$. Note that if for some sectors $a_{k_p}^{*(1)}=a_{k_p}$, meaning the current production process is also regulatorily optimal, substitution of $a_{k_p}^{*(1)}$ in place of a_{k_p} is not necessary to form $A^{(1)}$.

The problem can be described as the following:

$$\begin{bmatrix} V_{k_1}^{(1)} \\ V_{k_2}^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ V_{k_q}^{(1)} \end{bmatrix} = \begin{bmatrix} -a_{1k_1}^{*(1)}, & -a_{2k_1}^{*(1)}, & \ldots, & 1 - a_{k_1k_1}^{*(1)}, & \ldots, & a_{nk_1}^{*(1)} \\ -a_{1k_2}^{*(1)}, & -a_{2k_2}^{*(1)}, & \ldots, & 1 - a_{k_2k_2}^{*(1)}, & \ldots, & -a_{nk_1}^{*(1)} \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & \cdot & & \\ -a_{1k_q}^{*(1)}, & -a_{2k_q}^{*(1)}, & \ldots, & 1 - a_{k_qk_q}^{*(1)}, & \ldots, & -a_{nk_q}^{*(1)} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix} \qquad (5.32)$$

which is only a part of equation (5.15) as there's no effects on the values added of sector j, $j \notin \{k_1, k_2, \ldots, k_q\}$, i.e.,

$$V_j^{(1)} = V_j, \; j \notin \{k_1, k_2, \ldots, k_q\} \qquad (5.33)$$

In view of (5.32), each $V_{k_p}^{(1)}$ can be written as

$$V_{k_p}^{(1)} = P_{k_p} - \sum_{i=1}^{n} a_{ik_p}^{*(1)} P_i, \quad p = 1, \ldots, q. \qquad (5.34)$$

Since from (5.3), original V_{k_p} has been computed by

$$V_{k_p} = P_{k_p} - \sum_{i=1}^{n} a_{ik_p} P_i, \quad p = 1, \ldots, q, \qquad (5.35)$$

we can observe that each sector k_p will experience an increase in the amount of value added (or, profit) due to the optimization, i.e.,

$$\begin{aligned}\Delta V_{k_p}^{(1)} &= V_{k_p}^{(1)} - V_{k_p}\\ &= \sum_{i=1}^{n} (a_{ik_p} - a_{ik_p}^{*(1)})P_i\\ &\geq 0. \end{aligned} \tag{5.36}$$

For the sake of completeness, let us proceed to test feasibility and optimality of the new production process $(a_{k_p}^{*(1)} \otimes P', P_{k_p})$, or equivalently, feasibility of $Y_{k_p}^{*(2)}$, computed by

$$Y_{k_p}^{*(2)} = B_{k_p}^{-1} d_{k_p}^{(2)}, \quad p = 1, \ldots, q \tag{5.37}$$

where

$$d_{k_p}^{(2)} = \begin{bmatrix} P_k \\ a_{k_p}^{*(1)} \otimes P' \end{bmatrix} \tag{5.38}$$

From (5.31), $d_{k_p}^{(2)}$ can be rewritten in terms of B_{k_p} and $Y_{k_p}^{*(1)}$ as

$$d_{k_p}^{(2)} = \begin{bmatrix} \hat{S}_{k_p} \\ H_{k_p} \end{bmatrix} \begin{bmatrix} Y_{R,k_p}^{*(1)} \\ 0 \end{bmatrix} = B_{k_p} \begin{bmatrix} Y_{R,k_p}^{*(1)} \\ 0 \end{bmatrix} \tag{5.38a}$$

Substituting (5.38a) into (5.37), we have

$$Y_{k_p}^{*(2)} = B_{k_p}^{-1} \cdot B_{k_p} \begin{bmatrix} Y_{R,k_p}^{*(1)} \\ 0 \end{bmatrix} = \begin{bmatrix} Y_{R,k_p}^{*(1)} \\ 0 \end{bmatrix} \leqq Y_{k_p}^{*(1)}. \tag{5.39}$$

An obvious feasibility can be observed in (5.39). Hence, according to a post-optimality theorem, $Y_{k_p}^{*(2)}$ is also optimal.

Finally, we now want to show the equality of $a_{k_p}^{*(1)}$ and $a_{k_p}^{*(2)}$ holds true when an attempt is made to find another set of input coefficients, $a_{k_p}^{*(2)}$, that may be better than $a_{k_p}^{*(1)}$. Note that by having shown

$$(a_{k_p}^{*(1)} \otimes P', P_{k_p}) \in T_{k_p}, \tag{5.40}$$

we can expect that $a_{k_p}^{*(1)}$ should remain as the optimal input coefficients for sector k_p. Undoubtly, the following relationship can be easily obtained:

$$\begin{aligned} a_{k_p}^{*(2)} &= (\hat{H}_{k_p} Y_{k_p}^{*(2)}) \oslash P' \\ &= (H_{k_p}^{*} Y_{k_p}^{*(1)}) \oslash P' \\ &= a_k^{*(1)} \end{aligned} \tag{5.41}$$

§5.4 Conclusion

In this chapter, the development of a price model for a stability analysis under the assumptions of constancy of prices and variability of values added is given.

As one sector of the static input-output system is optimized an operationally otpimal production process is selected under a given technology which yields an increase in value added for the optimized sector without altering any other values in the system. The amount of increase in the value added is shown to be exactly the same as the amount of cost reduction in the sector's production. Thus, a new stable state of system can immediately be achieved when a substitution of the optimal input coefficients is taken in place of the old input coefficients. Because of constancy in prices, once an operationally optimal process is selected, it remains optimal even under the new equilibrium system.

For multi(q)-sectoral optimization problems, the same arguments given above hold, i.e., each optimization only affects the corresponding input coefficients and value added, making no changes for the values of other sectors. This implies that the new production process $(a^{*(1)}_{k_p} \otimes P', P_{k_p})$ becomes the optimal process for any optimized sector k_p, p=1, ..., q, and the system reaches a stable state as q number of substitutions are made for the input-output coefficients matrix.

Chapter VI

AN INPUT-OUTPUT SYSTEM WITH OPTIMIZED SUBSYSTEMS: THE CONSOLIDATION MODEL

This chapter offers a stability study of a static system under the assumption that the price vector as well as the vector of values added can assume its variability. The conceptual basis of the model is described in section 6.1 along with assumptions to be made. In section 6.2, development for the single-sectoral problem is presented. The development is then extended to the multi-sectoral problems in section 6.3 using mathematical induction. The computational scheme illustrating multi-sectoral problems with the consoldiation model is also shown diagrammatically. Finally in section 6.4, some concluding remarks are given explaining the general nature of the overall developments.

§6.1 Ingredients of the Model

As briefly described in section 3.3, the conceptual basis for the consolidation model lies in that prices of certain products may be able to decrease as the products' producers encounter with better production processes due to sectoral optimization. Production technology is assumed to be fixed for each sector in economy. Hence, a better production process would not imply any technological improvements, but rather, its implications correspond to a better way of using limited budget with respect to the price economy can accept. To refresh our memory, recall the following assumptions characterize the consolidation model.

Suppose there are q sectors, $1 \le q \le n$, in an n-sector economy, whose primary interests are finding the optimal way of buying certain products that are required to produce their own products, given technology. If those sectors are designated by $k_1, k_2, \ldots, k_q$, $1 \le k_p \le n$, $p=1, \ldots, q$, the specific set of assumptions are:

(1) P_{k_p} is a variable and V_{k_p} is a constant for each $p=1, \ldots, q$;

(2) P_j is fixed and V_j can assume its variability for each $j \notin \{k_1, k_2, \ldots, k_q\}$, $j=1, \ldots, n$, (6.1)

where P_j is the unit price of good j and V_j is the value added for sector j.

The diagrammatic model shown in (5.1) is considered again. However, as will be clear in the later discussion, a concomitantial method will be adopted for optimizing multi-sectoral problems instead of the usual simultaneous method. The general assumptions made in section 3.3 are retained for this price model. One specific assumption that should be recalled is the satisfaction of Hawkins-Simon condition, i.e., all principal minors of (I-A) is positive.

§6.2 The Single-Sectoral Problem

In this section, development for stability analysis will be presented when only one sector is regulatorily optimized. Denoting this sector with subscript k, the following diagrammatic model can be adopted:

For sector k: Minimize bY

$$\begin{tikzcd}
\text{subject to } SY = P_k \arrow[ddd] & & \\
& HY \le a_k \otimes P' & \\
& Y \ge 0 & \\
a_k = \psi_1(Y^*) \arrow[dr] & & d = \psi_2(P) \arrow[uul] \\
& (I - A')P' = V \arrow[ur] &
\end{tikzcd} \tag{6.2}$$

where definition of each variable may be found in section 5.1.

To start the iterative process, first assume that the current system is in equilibrium, and the operationally optimal program $Y^{*(1)}$ exists with the limitation vector $d^{(1)}$ as

$$d^{(1)} = \begin{bmatrix} P_k \\ a_k \otimes P' \end{bmatrix}. \tag{6.3}$$

If B denotes the optimal basis, by the theory of linear programming, we have

$$Y^{*(1)} = B^{-1}d^{(1)} \text{ and } d^{(1)} = BY^{*(1)}. \tag{6.4}$$

The columns of B, which correspond to any slack or surplus activity of $Y^{*(1)}$, may be substituted with null vectors to yield a modified optimal basis,

$$B^* = \begin{bmatrix} S^* \\ H^* \end{bmatrix} \tag{6.5}$$

so that the followings can be obvious:

$$Y_R^{*(1)} = B^{*-1}d^{(1)}; \tag{6.6}$$

$$0$$

$$d^{(1)} = B^* \begin{bmatrix} Y_R^{*(1)} \\ 0 \end{bmatrix} = B^* \begin{bmatrix} Y_R^{*(1)} \\ Y_S^{*(1)} \end{bmatrix} = B^*Y^{*(1)}. \tag{6.6a}$$

Assume that the original optimal basis B can be structured as

$$B = \begin{bmatrix} \hat{S} \\ \hat{H} \end{bmatrix} \tag{6.7}$$

whose components are associated with constraints coefficients S and H.

Having defined B^* to be partitioned into S^* and H^*, the new optimal input coefficients $a_k^{*(1)}$ can be computed by

$$a_k^{*(1)} = (H^*Y^{*(1)}) \oslash P' \tag{6.8}$$

based on the Leontief basic assumption of a linear technology prevailing in sector k. From (6.8) note a very useful set of equations:

$$P_k = \hat{S} \begin{bmatrix} Y_R^{*(1)} \\ Y_S^{*(1)} \end{bmatrix} = S^*Y^{*(1)} = \hat{S} \begin{bmatrix} Y_R^{*(1)} \\ 0 \end{bmatrix}; \tag{6.9}$$

$$a_k^{*(1)} \otimes P' = \hat{H} \begin{bmatrix} Y_R^{*(1)} \\ 0 \end{bmatrix}. \tag{6.9a}$$

Substituting (6.9) in (6.8) for the ith element of $a_k^{*(1)}$,

$$a_{kk}^{*(1)} = \frac{H_k{}^*Y^{*(1)}}{P_k} = \frac{H_k{}^*Y^{*(1)}}{S^*Y^{*(1)}}$$

where $H_k{}^*$ is the kth row of H^*. Also, by the characteristics of $a_k^{*(1)}$, we should have

$$a_k^{*(1)} \leqq a_k. \tag{6.10}$$

If $a_k^{*(1)} = a_k$ in (6.11), the current production process ($a_k \otimes P'$, P_k) is also operationally optimal process. Hence, no perturbation would occur in A and the current system is considered stable.

On the other hand if $a_k^{*(1)} \leq a_k$, a new equilibrium state for the economy has to be recomputed by finding $P^{(1)}$ and $V^{(1)}$ such that

$$(I - A^{(1)'})P^{(1)'} = V^{(1)'} \qquad (6.12)$$

where $A^{(1)}$ is formed by substituting $a_k^{*(1)}$ in place of a_k in A. Since V_k is a fixed constant along with P_j, $j \neq k$, by the consolidation model's basic assumption described in (6.1), we would have

$$P_j^{(1)} = P_j,\ j \neq k, \text{ and } V_k^{(1)} = V_k. \qquad (6.13)$$

Hence the problem (6.12) becomes to find $P_k^{(1)}$ and $V_j^{(1)}$ ($j \neq k$) which simultaneously satisfy the set of equations:

$$\begin{bmatrix} 1-a_{11} & -a_{21} & \cdots & -a_{k1} & \cdots & -a_{n1} \\ -a_{12} & 1-a_{22} & \cdots & -a_{k2} & \cdots & -a_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ -a_{1k}^{*(1)} & -a_{2k}^{*(1)} & \cdots & 1-a_{kk}^{*(1)} & \cdots & -a_{nk}^{*(1)} \\ \vdots & \vdots & & \vdots & & \vdots \\ -a_{1n} & -a_{2n} & \cdots & -a_{kn} & \cdots & 1-a_{nn} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_k^{(1)} \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} V_1^{(1)} \\ V_2^{(1)} \\ \vdots \\ V_k \\ \vdots \\ V_n^{(1)} \end{bmatrix} . \qquad (6.14)$$

From the kth equation in (6.14)

$$\sum_{\substack{i=1\\i\neq k}}^{n} (-a_{ik}^{*(1)})P_i + (1-a_{kk}^{*(1)})P_k^{(1)} = V_k \tag{6.14a}$$

which then implies that

$$P_k^{(1)} = \frac{V_k + \sum\limits_{\substack{i=1\\i\neq k}}^{n} a_{ik}^{*(1)}P_i}{1-a_{kk}^{*(1)}} \geq 0 \tag{6.14b}$$

Once all the components of $P^{(1)}$ are thus determined by (6.13) and (6.14b), $V^{(1)}=(V_1^{(1)}, V_2^{(1)}, \ldots, V_k^{(1)}, \ldots, V_n^{(1)})$ can be computed by

$$V_j^{(1)} = \sum_{\substack{i=1;\\i\neq j}}^{n} (-a_{ij}P_i^{(1)}) + (1-a_{jj})P_j^{(1)}, \quad j \neq k \tag{6.15}$$

where $V_k^{(1)}=V_k$. Note that as long as $P_k^{(1)}\neq P_k$, $V_j^{(1)}\neq V_j$, $j\neq k$, and $P_k^{(1)}$ will always be different from P_k if $a_k^{*(1)}\leq a_k$.

To compute the change of unit price for the good k, let us consider the original system from which we can extract the information

$$P_k = \frac{V_k + \sum\limits_{\substack{i=1\\i\neq k}}^{n} a_{ik}P_i}{1-a_{kk}} . \tag{6.16}$$

Thus,

$$\Delta P_k^{(1)} = P_k^{(1)} - P_k$$

$$= \frac{V_k + \sum\limits_{\substack{i=1\\i\neq k}}^{n} a_{ik}^{*(1)} P_i}{1-a_{kk}^{*(1)}} - \frac{V_k + \sum\limits_{\substack{i=1\\i\neq k}}^{n} a_{ik} P_i}{1-a_{kk}} . \tag{6.17}$$

Recognizing the facts from $a_k^{*(1)} \leq a_k$,

$$\sum_{\substack{i=1\\i\neq k}}^{n} a_{ik}^{*(1)} P_i \leq \sum_{\substack{i=1\\i\neq k}}^{n} a_{ik} P_i \text{ and } 1-a_{kk}^{*(1)} \geq 1-a_{kk}, \tag{6.18}$$

it is easy to observe from (6.17) that

$$\Delta P_k^{(1)} < 0. \tag{6.19}$$

The inequality in (6.19) is exactly what has been expected from the beginning. As sector k undergoes an optimization the price of sector k will be decreased if the value added for the sector remains as the same. The decrease amount is described as a function of the previous input coefficients and the newly found optimal input coefficients as well as the unit prices of other goods which are assumed to be fixed.

To observe the effects on the other sectors due to optimization of sector k we now want to investigate $V_j^{(1)}$, $j\neq k$, $j=1, \ldots, n$. For the original system,

$$V_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} (-a_{ij}P_i) + (1-a_{jj})P_j$$

$$= \sum_{\substack{i=1 \\ i \neq j,k}}^{n} (-a_{ij}P_i) + (1-a_{jj})P_j - a_{kj}P_k, \qquad j \neq k,\ j=1,\ \ldots,\ n. \tag{6.20}$$

Since (6.15) can be written, with the assumptions (6.13), as

$$V_j^{(1)} = \sum_{\substack{i=1 \\ i \neq j,k}}^{n} (-a_{ij}P_i) + (1-a_{jj})P_i - a_{kj}P_k^{(1)}, \qquad j \neq k,\ j=1,\ \ldots,\ n, \tag{6.21}$$

the change in the value added for each of the sectors other than that of sector k can be computed by

$$\Delta V_j^{(1)} = V_j^{(1)} - V_j$$

$$= -a_{kj}(P_k^{(1)} - P_k)$$

$$= -a_{kj}(\Delta P_k^{(1)}). \qquad j \neq k,\ j=1,\ \ldots,\ n \tag{6.22}$$

At this point recall that all $a_{ij} \geq 0$ for a meaning analysis and $\Delta P_k^{(1)}$ has the negative sign by (6.19) if $a_k^{*(1)} \leq a_k$. Hence we can conclude that

$$\Delta V_j^{(1)} > 0. \qquad j \neq k,\ j=1,\ \ldots,\ n. \tag{6.23}$$

Of course $\Delta V_k^{(1)} = 0$ from the assumption (6.13).

The equation (6.22) reveals an obvious economic reason. That is, if a particular sector k lowers its price without affecting other prices in the economy, other sectors will gain in their production costs as much as the amount of change in the price of good k multiplied by how many of good k are required. For example, if two units of good k are needed to make one unit of good r, production cost of sector r will be decreased by $2(\Delta P_k^{(1)})$ whereas any sector which does not require any amount of good k would not be affected at all. For this reason, we observe that the amount of increase in the value added for the sectors other than sector k are totally dependent upon the kth row of the original input-output matrix A and $\Delta P_k^{(1)}$.

We now proceed our stability study to analyze the new production process $(a_k^{*(1)} \otimes P^{(1)'}, P_k^{(1)})$, or $(a_k^{*(1)} \otimes P^{(1)'}, P_k)$. Two questions are to be answered:

(1) feasibility of the new production process in the given technology set, i.e.,

$$(a_k^{*(1)} \otimes P^{(1)'}, P_k^{(1)}) \in T_k \tag{6.24}$$

where

$$T_k = \{(a_k^{(i1)} \otimes P', P_k) | P' > 0\} \tag{6.25}$$

(2) optimality of the new production process, i.e.,

$$\begin{bmatrix} -a_k^{*(1)} \otimes P^{(1)'} \\ P_k^{(1)} \end{bmatrix} \geqq \begin{bmatrix} -a_k^{(i1)} \otimes P^{(1)'} \\ P_k \end{bmatrix} \text{ for all } i \tag{6.26}$$

where each $a_k^{(i1)}$ is an element of the set of all realizable processes $a_k^{(1)}$.

To answer question (1), we begin with computation of the net change in the limitation vector due to optimization,

$$\Delta d^{(2)} = d^{(2)} - d^{(1)}$$
$$= \begin{bmatrix} P_k^{(1)} \\ a_k^{*(1)} \otimes P^{(1)\prime} \end{bmatrix} - \begin{bmatrix} P_k \\ a_k \otimes P^{\prime} \end{bmatrix}. \tag{6.27}$$

By (6.13), each component of $\Delta d^{(2)}=(\Delta d_0^{(2)}, \Delta d_1^{(2)}, \ldots, \Delta d_n^{(2)})^{\prime}$ is then,

$$\Delta d^{(2)} = \begin{bmatrix} P_k^{(1)} - P_k \\ a_{1k}^{*(1)}P_1 - a_{1k}P_1 \\ a_{2k}^{*(1)}P_2 - a_{2k}P_2 \\ \vdots \\ a_{kk}^{*(1)}P_k^{(1)} a_{kk}P_k \\ \vdots \\ a_{nk}^{*(1)}P_n - a_{nk}P_n \end{bmatrix} = \begin{bmatrix} \Delta P_k^{(1)} \\ (a_{1k}^{*(1)} - a_{1k})P_1 \\ (a_{2k}^{*(1)} - a_{2k})P_2 \\ \vdots \\ a_{kk}^{*(1)}P_k^{(1)} - a_{kk}P_k \\ \vdots \\ (a_{nk}^{*(1)} - a_{nk})P_n \end{bmatrix}. \tag{6.27a}$$

Let us analyze each component of (6.27a) with respect to feasibility of the new process. First, note that the new operationally optimal program $Y^{*(2)}$ may be found as

$$Y^{*(2)} = Y^{*(1)} + B^{-1}(\Delta d^{(2)}). \tag{6.28}$$

For there are maximum amounts of decrease from $d^{(1)}$, that can be allowed to maintain the feasibility of $Y^{*(2)}$, the following three conditions have to be simultaneously satisfied:

(a) For the output constraint,

$$d_0^{(2)} = \hat{S}Y^{*(2)} \tag{6.29}$$

(b) For the kth input constraint,

$$-\Delta d_k^{(2)} \geq (a_{kk} - a_{kk}^{*(1)})P_k \tag{6.29a}$$

(c) For the input constraints other than the kth's,

$$-\Delta d_i^{(2)} \geq (a_{ik} - a_{ik}^{*(1)})P_i, \; i\neq k, \; i=1, \ldots, n. \tag{6.29a}$$

where (a) is the strict equality requirement and each $(a_{ik}-a_{ik}^{*(1)})P_i$, $i=1, \ldots, n$, denotes the maximum decrease possible from $d_i^{(1)}$ to ensure the feasibility.

Consider the condition in (6.29). Since $P_k^{(1)}=d_0^{(2)}$, and $P_k=\hat{S}Y^{*(1)}$, by (6.9) the following relationship can be developed:

$$\begin{aligned} P_k^{(1)} &= P_k + \Delta P_k^{(1)} \\ &= P_k + (1, 0, \ldots, 0)\;(\Delta d^{(2)}) \\ &= \hat{S}Y^{*(1)} + \hat{S}\;B^{-1}(\Delta d^{(2)}) \\ &= \hat{S}[Y^{*(1)} + B^{-1}(\Delta d^{(2)})]. \end{aligned} \tag{6.30}$$

Substitution of (6.28) into (6.30) yields

$$P_k^{(1)} = d_0^{(2)} = \hat{S}Y^{*(2)} \tag{6.31}$$

which implies the satisfaction of condition (6.29).

Considering condition imbedded in (6.29a), first note from (6.27a) that

$$-\Delta d_k^{(2)} = a_{kk}P_k - a_{kk}^{*(1)}P_k^{(1)}.$$

Hence, knowing $P_k^{(1)} \leq P_k$,

$$\begin{aligned} -\Delta d_k^{(2)} &\geq a_{kk}P_k - a_{kk}^{*(1)}P_k \\ &= (a_{kk} - a_{kk}^{*(1)})P_k. \end{aligned} \tag{6.32}$$

Conditions (6.29b) are automatically satisfied in equalities as

$$-\Delta d_i^{(2)} = (a_{ik} - a_{ik}^{*(1)})P_i, \; i \neq k, \; i=1, \ldots, n. \tag{6.33}$$

Having shown validities of all the three conditions (6.29), (6.29a) and (6.29b), we may now state that the new optimal program $Y^{*(2)}$ as determined by (6.28) would be indeed feasible, or equivalently, the new optimal production process belongs to the technology set given, i.e.,

$$(a_k^{*(1)} \otimes P^{(1)'}, P_k^{(1)}) \in T_k.$$

According to a post-optimality theorem, the change in the limitation vector would not affect the optimality. Hence, optimality of $Y^{*(2)}$ is maintained as it is feasible. In terms of production processes, this implies that

$$\begin{bmatrix} -a_k^{*(1)} \otimes P^{(1)'} \\ P_k^{(1)} \end{bmatrix} \geqq \begin{bmatrix} -a_k^{(i1)} \otimes P^{(1)'} \\ P_k \end{bmatrix} \text{for all } i \quad .$$

Optimality criterion of the new production process can further be evidenced if

$$a_k^{*(2)} = a_k^{*(1)} \tag{6.34}$$

where $a_k^{*(2)}$ is the optimal input coefficients determined with $Y^{*(2)}$ as

$$a_k^{*(2)} = (H^* Y^{*(2)}) \oslash P^{(1)}. \tag{6.35}$$

In trying to acquire (6.34) let us expand the right hand side of (6.35) by substituting (6.28),

$$\begin{aligned} a_k^{*(2)} &= [H^*(Y^{*(1)} + B^{-1}(\Delta d^{(2)}))] \oslash P^{(1)} \\ &= \hat{H}Y^{*(1)} \oslash P^{(1)} + \hat{H}B^{-1}(\Delta d^{(2)}) \oslash P^{(1)} \\ &= \hat{H}Y^{*(1)} \oslash P^{(1)} + (0, e_1, e_2, \ldots, e_n)(\Delta d^{(2)}) \oslash P^{(1)}. \end{aligned}$$

If we let $\hat{H}_i$ be the ith row of $\hat{H}$, then from (6.27a),

$$a_k^{*(2)} = \begin{bmatrix} \hat{H}_1 Y^{*(1)}/P_1^{(1)} \\ \hat{H}_2 Y^{*(1)}/P_2^{(1)} \\ \vdots \\ \hat{H}_k Y^{*(1)}/P_k^{(1)} \\ \vdots \\ \hat{H}_n Y^{*(1)}/P_n^{(1)} \end{bmatrix} + \begin{bmatrix} a_{1k}^{*(1)} - a_{1k})P_1/P_1^{(1)} \\ a_{2k}^{*(1)} - a_{2k})P_2/P_2^{(1)} \\ \vdots \\ (a_{kk}^{*(1)}P_k^{(1)} - a_{kk}P_k)/P_k^{(1)} \\ \vdots \\ (a_{nk}^{*(1)} - a_{nk})P_n/P_n^{(1)} \end{bmatrix}.$$

Since $a_{ik}=\hat{H}_i Y^{*(1)}/P_i$ and $P_j^{(1)}=P_j$ for all $j\neq k$,

$$a_k^{*(2)} = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{k-1,k} \\ \hat{H}Y^{*(1)}/P_k^{(1)} \\ a_{k+1,k} \\ \vdots \\ a_{nk} \end{bmatrix} + \begin{bmatrix} a_{1k}^{*(1)} - a_{1k} \\ a_{2k}^{*(1)} - a_{2k} \\ \vdots \\ a_{k-1,k}^{*(1)} - a_{k-1,k} \\ a_{kk}^{*(1)} - a_{kk}P_k/P_k^{(1)} \\ a_{k+1,k}^{*(1)} - a_{k+1,k} \\ \vdots \\ a_{nk}^{*(1)} - a_{nk} \end{bmatrix} = \begin{bmatrix} a_{1k}^{*(1)} \\ a_{2k}^{*(1)} \\ \vdots \\ a_{k-1,k}^{*(1)} \\ \pi \\ a_{k+1,k}^{*(1)} \\ \vdots \\ a_{nk}^{*(1)} \end{bmatrix}.$$

In view of the above equation, each component of $a_k^{*(2)}$ is seen to be equal to the corresponding component of the right-hand side vector except a possible question on the kith component. Enumerating π we have

$$\begin{aligned}\pi &= \hat{H}_k Y^{*(1)}/P_k^{(1)} + a_{kk}^{*(1)} - a_{kk}P_k/P_k^{(1)} \\ &= a_{kk}^{*(1)} + \frac{\hat{H}_k Y^{*(1)} - a_{kk}P_k}{P_k^{(1)}} \quad .\end{aligned}$$

Substituting a_{kk} by $\hat{H}_k Y^{*(1)}/P_k$,

$$\begin{aligned}\pi &= a_{kk}^{*(1)} + \frac{\hat{H}_k Y^{*(1)} - (\hat{H}_k Y^{*(1)}/P_k)P_k}{P_k^{(1)}} \\ &= a_{kk}^{*(1)}\end{aligned}$$

Therefore,

$$a_k^{*(2)} = a_k^{*(1)}. \tag{6.36}$$

By having $a_k^{*(2)}=a_k^{*(1)}$ we have now shown all the criteria that are necessary to achieve a stable state. Referring back to Table 1 in section 3.3 and to be specific, positivity of the new price vector is shown by (6.14b); convergence or the existence of positive value added is seen in (6.21) together with (6.23); feasibility of the new production process is indicated by (6.31), (6.32) and (6.33), which also leads to the conclusion (6.36) for its optimality.

In conclusion, a consolidation model for a static input-output system reaches a stable state in one iteration as one subsystem (kth) of the system undergoes an operational optimization. If the operationally optimal process $(a_k^{*(1)} \otimes P', P_k)$ is not the same as the sector's old process $(a_k \otimes P', P_k)$, a modified process $(a_k^{*(1)} \otimes P^{(1)'}, P_k^{(1)})$ becomes the operationally optimal process and it remains optimal. In this case, the price of good k will be decreased and each sector of the economy other than sector k will experience an increase in the value added that is proportional to its input coefficient associated with sector k. The stable state of economy thus obtained can be expressed by $(I-A^{(1)'})P^{(1)'}=V^{(1)'}$.

§6.3 The Multiple-Sector Optimization Problem

A generalized version of the single sectoral problem in section 6.2 is considered in this section to accommodate q, $1 \leq q \leq n$, number of sectors simultaneously into optimization process. Observing the development in the previous section, it should be clear that a sequential method cannot apply in this case because of the fact that optimization of one sector will most likely try to vary value added of another sector which also may want to be optimized, and this effect can violate a rule of the consolidation model given in (6.1). Hence, unlike the methods employed in the quantity model or in the acquisition model, we will attempt to establish a new stable state in a concomitant fashion.

Designating those q to-be-optimized sectors by subscripts k_p, p=1, ..., q, a list of assertions from section 6.2 is in order for the later developments. Assuming $Y_{k_p}^{*(1)}$, an operational optimal process for the original sectoral problem, exists given a system in its initial equilibrium, $(I-A')P'=V'$,

(a) $Y^{*(1)}_{k_p} = B^{-1}_{k_p} d^{(1)}_{k_p}$ and $d^{(1)}_{k_p} = B_{k_p} Y^{*(1)}_{k_p}$, $p=1, \ldots, q$,

where optimal basis B can be assumed to be structured as

$$B_{k_p} = \begin{bmatrix} \hat{S}_{k_p} \\ \hat{H}_{k_p} \end{bmatrix}$$

and initial limitation vector $d^{(1)}_{k_p}$ being

$$d^{(1)}_{k_p} = \begin{bmatrix} P_{k_p} \\ a_{k_p} \otimes P' \end{bmatrix};$$

(b) a modified optimal basis $B^{*}_{k_p}$ can be described as

$$B^{*}_{k_p} = \begin{bmatrix} S^{*}_{k_p} \\ H^{*}_{k_p} \end{bmatrix}, \quad p=1, \ldots, q,$$

so that

$$\begin{bmatrix} Y^{*(1)}_{R,k_p} \\ 0 \end{bmatrix} = B^{*-1}_{k_p} d^{(1)}_{k_p}$$

where $Y^{*(1)}_{R,k_p}$ is the optimal solution vector corresponding to real activities of $Y^{*(1)}_{k_p}$;

(c) Using the definition of $B^{*}_{k_p}$, the new optimal input coefficients for each sector k_p, $p=1, \ldots, q$, can be computed by

$$a_{k_p}^{*(1)} = (H_{k_p}^{*} Y_{k_p}^{*(1)}) \oslash P'$$

where

$$a_{k_p}^{*(1)} \leq a_{k_p}; \text{ and}$$

(d) $T_{k_p} = \{(a_{k_p}^{(i1)} \otimes P', P_{k_p}) | P'>0\}$ describes the given technology set for each sector k_p, p=1, ..., q. Note that

$$(a_{k_p}^{*(1)} \otimes P', P_{k_p}) \in T_{k_p}.$$

Having calculated $a_{k_p}^{*(1)}$ for all p=1, ..., q, we may now proceed to perceive a new equilibrium state of economy which is hoped to exist. To do this, a set of vectors $a_{k_1}^{*(1)}$, $a_{k_2}^{*(1)}$, ..., $a_{k_q}^{*(1)}$ will first replace a_{k_1}, a_{k_2}, ..., a_{k_q} in A, respectively, and the resulting input matrix would be denoted as $A^{(1)}$ in the usual manner. If an equilibrium state exists for $A^{(1)}$ there should be new price vector $P^{(1)}$ and the vector of values added $V^{(1)}$ such that

$$(I - A^{(1)'})P^{(1)'} = V^{(1)'}. \tag{6.37}$$

Since V_{k_p} is fixed for each p=1, ..., q, and P_j is fixed for $j \notin Q=\{k_1, k_2, ..., k_q\}$, (6.37) becomes to solve $P_{k_p}^{(1)}$, p=1, ..., q, and $V_j^{(1)}$, $j \notin Q$, j=1, ..., n, satisfying the following n equations simultaneously:

$$\begin{bmatrix} 1-a_{11} & -a_{21} & \cdots & -a_{k_1 1} & \cdots & -a_{k_q 1} & \cdots & -a_{n1} \\ -a_{12} & -a_{22} & \cdots & -a_{k_1 2} & \cdots & -a_{k_q 2} & \cdots & -a_{n2} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ -a^{*(1)}_{1k_1} & -a^{*(1)}_{2k_1} & \cdots & 1-a^{*(1)}_{k_1k_1} & \cdots & -a^{*(1)}_{k_qk_1} & \cdots & -a^{*(1)}_{nk_1} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ -a^{*(1)}_{1k_q} & -a^{*(1)}_{2k_q} & \cdots & -a^{*(1)}_{k_1k_q} & \cdots & 1-a^{*(1)}_{k_qk_q} & \cdots & -a^{*(1)}_{nk_q} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ -a_{1n} & -a_{2n} & \cdots & -a_{k_1 n} & \cdots & -a_{k_q n} & \cdots & 1-a_{nn} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P^{*(1)}_{k_1} \\ \vdots \\ P^{*(1)}_{k_q} \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} V^{(1)}_1 \\ V^{(1)}_2 \\ \vdots \\ V_{k_1} \\ \vdots \\ V_{k_q} \\ \vdots \\ V^{(1)}_n \end{bmatrix} \qquad (6.37a)$$

From (6.37a), we have q equations of the form

$$\sum_{\substack{i=1 \\ i\notin Q}}^{n} (-a^{*(1)}_{ik_p})P_i + \sum_{\substack{i=1 \\ i\varepsilon Q, i\notin k_p}}^{n} (-a^{*(1)}_{ik_p})P^{(1)}_i + (1-a^{*(1)}_{k_pk_p})P^{(1)}_{k_p} = V_{k_p}, \quad p=1, \ldots, q. \qquad (6.38)$$

Thus, each $P^{(1)}_{k_p}$, p=1, ..., q, can be expressed by

$$P^{(1)}_{k_p} = (V_{k_p} + \sum_{\substack{i=1 \\ i\notin Q}}^{n} a^{*(1)}_{ik_p} P_i + \sum_{\substack{i=1 \\ i\varepsilon Q, i\neq k_p}}^{n} a^{*(1)}_{ik_p} P^{(1)}_i)/(1-a^{*(1)}_{k_pk_p}) \qquad (6.39)$$

which is observed to be non-negative as all the terms in the expression are non-negative. Also we have (n-q) equations as

$$V_j^{(1)} = P_j + \sum_{\substack{i=1 \\ i \notin Q}}^{n} (-a_{ij})P_i + \sum_{\substack{i=1 \\ i \in Q}}^{n} (-a_{ij})P_i^{(1)}. \tag{6.40}$$

The evaluations of each $P_{k_p}^{(1)}$, $k_p \varepsilon Q$, and $V_j^{(1)}$, $j \notin Q$, are difficult in the sense that the law of substitution should be applied consecutively until all the unknowns are expressed in terms of currently known elements. To avoid this computational difficulty, let us first try to solve (6.37a) for two-sector optimization problems, i.e., q=2.

Without loss of generality, let k_1=1 and k_2=2 such that Q={1, 2}. Then (6.37a) may be rewritten as:

$$\begin{bmatrix} 1-a_{11}^{*(1)} & -a_{21}^{*(1)} & -a_{31}^{*(1)} & \cdots & -a_{n1}^{*(1)} \\ -a_{12}^{*(1)} & 1-a_{22}^{*(1)} & -a_{32}^{*(1)} & \cdots & -a_{n2}^{*(1)} \\ -a_{13} & -a_{23} & 1-a_{33} & \cdots & -a_{n3} \\ \vdots & \vdots & \vdots & & \vdots \\ -a_{1n} & -a_{2n} & -a_{3n} & \cdots & 1-a_{nn} \end{bmatrix} \begin{bmatrix} P_1^{(1)} \\ P_2^{(1)} \\ P_3 \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3^{(1)} \\ \vdots \\ V_n^{(1)} \end{bmatrix} \tag{6.41}$$

From the first equation of (6.41), we have

$$(1-a_{11}^{*(1)})P_1^{(1)} - a_{21}^{*(1)}P_2^{(1)} - \sum_{i=3}^{n} a_{i1}^{*(1)}P_i = V_1 \tag{6.42}$$

so that $P_1^{(1)}$ can be expressed as

$$P_1^{(1)} = \frac{V_1 + a_{21}^{*(1)} P_2^{(1)} + \sum_{i=3}^{n} a_{i1}^{*(1)} P_i}{1-a_{11}^{*(1)}} . \tag{6.42a}$$

In the same manner, the second equation of (6.41) yields

$$-a_{12}^{*(1)} P_1^{(1)} + (1-a_{22}^{*(1)}) P_2^{(1)} - \sum_{i=3}^{n} a_{i2}^{*(1)} P_i = V_2 \tag{6.43}$$

when then implies that

$$P_2^{(1)} = \frac{V_2 + a_{12}^{*(1)} P_1^{(1)} + \sum_{i=3}^{n} a_{i2}^{*(1)} P_i}{(1-a_{22}^{*(1)})} . \tag{6.43a}$$

Substituting (6.42a) in (6.43a),

$$P_2^{(1)} = \frac{V_2 + \frac{a_{12}^{*(1)}}{(1-a_{11}^{*(1)})} (V_1 + a_{21}^{*(1)} P_2^{(1)} + \sum_{i=3}^{n} a_{i1}^{*(1)} P_i) + \sum_{i=3}^{n} a_{i2}^{*(1)} P_i}{(1-a_{22}^{*(1)})} . \tag{6.44}$$

Rearranging (6.44),

$$P_2^{(1)} = \frac{V_2 + \frac{a_{12}^{*(1)}}{(1-a_{11}^{*(1)})} (V_1 + \sum_{i=3}^{n} a_{i1}^{*(1)} P_i) + \sum_{i=3}^{n} a_{i2}^{*(1)} P_i}{(1-a_{22}^{*(1)}) - \frac{a_{12}^{*(1)} \; a_{21}^{*(1)}}{(1-a_{11}^{*(1)})}} \tag{6.45}$$

whose sign depends totally on the sign of its denominator as its numerator part is positive. However since $(I-A^{(1)})$ satisfies the Hawkins-Simon Condition and the terms in the denominator reveals the second order principal minor of $(I-A^{(1)})$, a conclusion can be made as to the sign of $P_2^{(1)}$ being positive.

If $P_2^{(1)}>0$, then in view of (6.42a) we can also see that

$$P_1^{(1)} > 0. \tag{6.46}$$

Now let us relate the magnitude of $P_1^{(1)}$ in the single-sectoral problem of the previous section to the current two-sectoral problems. Although a mathematical proof is unavailable, if $\Delta P_k^{(1)}$ of (6.19) is observed to be negative, this decreasing characteristic of price for the optimized sector should not be altered by the other sectors' behaviors. Therefore, at this point we will make a rather strong statement for the later analyses that

$$\Delta P_1^{(1)} = P_1^{(1)} - P_1 < 0. \tag{6.47}$$

Originally, P_2 could have been expressed as

$$P_2 = \frac{V_2 + a_{12}P_1 + \sum_{i=3}^{n} a_{i2}P_i}{(1 - a_{22})}. \tag{6.48}$$

Comparing (6.48) with (6.43a) using (6.46) and (6.47), it is obvious that

$$\Delta P_2^{(1)} = P_2^{(1)} - P_2 < 0. \tag{6.49}$$

The relationships (6.41) through (6.49) only correspond to the two-sectoral optimization problems. If we go back to the original problem of (6.37) for multi-sectoral problems, the same types of arguments can be applied inductively so that each $P_{k_p}^{(1)}$, p=1, ..., q, of (6.39), can have the properties of

$$\text{(a) } P_{k_p}^{(1)} > 0, \quad p=1, \ldots, q;$$

$$\text{(b) } P_{k_p}^{(1)} < P_{k_p}, \quad p=1, \ldots, q, \tag{6.50}$$

under the assumption that $a_{k_p}^{*(1)} \leq a_{k_p}$, p=1, ..., q. Using (6.50), each $V_j^{(1)}$, $j \notin Q=\{k_1, k_2, \ldots, k_q\}$, described in (6.40) then yields the information as:

$$\text{(a) } V_j^{(1)} > 0, \quad j \notin Q;$$

$$\text{(b) } V_j^{(1)} > V_j, \quad j \in Q. \tag{6.51}$$

Therefore, the system reaches a new equilibrium state,

$$(I - A^{(1)\prime})P^{(1)\prime} = V^{(1)\prime} \tag{6.52}$$

where

$$P^{(1)\prime} < P^{\prime} \text{ and } V^{(1)\prime} > V^{\prime}. \tag{6.53}$$

We now proceed our stability study to analyze the new production process of each of q optimized sectors, $(a_{k_p}^{*(1)} \otimes P^{(1)\prime}, P_{k_p}^{(1)})$, p=1, ..., q. Two questions to be answered are:

$$\text{(1) } (a_{k_p}^{*(1)} \otimes P^{(1)\prime}, P_{k_p}^{(1)}) \varepsilon\ T_{k_p} = \{(a_{k_p}^{(i1)} \otimes P^{(1)\prime}, P_{k_p}^{(1)}) \mid P^{(1)} > 0\}; \tag{6.54}$$

$$(2)\begin{bmatrix} -a^{*(1)}_{k_p} \otimes P^{(1)'} \\ P^{(1)}_{k_p} \end{bmatrix} \geqq \begin{bmatrix} -a^{(i1)}_{k_p} \otimes P^{(1)'} \\ P^{(1)}_{k_p} \end{bmatrix} \quad \text{for all } i$$

in which $a^{(i1)}_{k_p}$ denotes an element of the set of all realizable processes $a^{(1)}_{k_p}$. Note that questions (1) and (2) of (6.54) are merely mathematical representations of the feasibility and optimability of the operationally optimal process that must be satisfied for a stable state of system.

In order to answer question (1), we begin with computation of the net change in the limitation vector of sector k_p due to the sector's optimization with respect to the new equilibrium state of system.

$$\Delta d^{(2)}_{k_p} = (\Delta d^{(2)}_{k_p,0}, \Delta d^{(2)}_{k_p,1}, \ldots, \Delta d^{(2)}_{k_p,n})$$

$$= d^{(2)} - d^{(1)}$$

$$= \begin{bmatrix} P^{(1)}_{k_p} \\ a^{*(1)}_{k_p} \otimes P^{(1)'} \end{bmatrix} - \begin{bmatrix} P_{k_p} \\ a_{k_p} \otimes P' \end{bmatrix}. \tag{6.55}$$

Using $\Delta d^{(2)}_{k_p}$, the new optimal program, $Y^{*(2)}_{k_p}$, may be obtained by

$$Y^{*(2)}_{k_p} = Y^{*(1)}_{k_p} + B^{-1}_{k_p}(\Delta d^{(2)}_{k_p}), \quad p=1, \ldots, q. \tag{6.56}$$

The major concern is then, whether or not each element of $\Delta d_{k_p}^{(2)}$ lies within the boundary to ensure the feasibility of $Y_{k_p}^{*(2)}$. As there are three types of constraints, the following conditions can be developed:

(a) For the output constraint,

$$d_{k_p,0}^{(2)} = \hat{S}_{k_p} Y_{k_p}^{*(2)}; \tag{6.57}$$

(b) For each of the pth constraint, p=1, ..., q,

$$-\Delta d_{k_p,k_p}^{(2)} \geqq (a_{k_p k_p} - a_{k_p k_p}^{*(1)}) P_{k_p}; \tag{6.57a}$$

(c) For each of the ith input constraints, $i \notin Q$,

$$-\Delta d_{k_p,i}^{(2)} \geqq (a_{i_k p} - a_{i_k p}^{*(1)}) P_i. \tag{6.57b}$$

Of course, all of the above three conditions have to be satisfied for the feasibility of $Y_{k_p}^{*(2)}$.

Consider the condition in (6.57) first. Since $d_{k_p,0}^{(2)} = P_{k_p}^{(1)}$ and $P_{k_p} = \hat{S}_{k_p} Y_{k_p}^{*(1)}$, the following relationship can be developed for each p=1, ..., q:

$$\begin{aligned} P_{k_p}^{(1)} &= P_{k_p} + \Delta P_{k_p}^{(1)} \\ &= P_{k_p} + (1, 0, \ldots, 0)(\Delta d_{k_p}^{(2)}) \\ &= \hat{S}_{k_p} Y_{k_p}^{*(1)} + \hat{S}_{k_p} B_{k_p}^{-1} (\Delta d_{k_p}^{(2)}) \\ &= \hat{S}_{k_p} [Y_{k_p}^{*(1)} + B_{k_p}^{-1} (\Delta d_{k_p}^{(2)})]. \end{aligned} \tag{6.58}$$

Substituting (6.56) in (6.58), we have

$$P_{k_p}^{(1)} = d_{k_p,0}^{(2)} = \hat{S}_{k_p} Y_{k_p}^{*(2)}. \tag{6.59}$$

Now to show (6.57a), first note that

$$-\Delta d_{k_p,k_p}^{(2)} = a_{k_p k_p} P_{k_p} - a_{k_p k_p}^{*(1)} P_{k_p}^{(1)}. \tag{6.60}$$

As $P_{k_p}^{(1)} < P_{k_p}$ from (6.50), (6.60) can be rewritten as

$$-\Delta d_{k_p,k_p}^{(2)} \geq (a_{k_p k_p} - a_{k_p k_p}^{*(1)}) P_{k_p}. \tag{6.61}$$

Also, it can be seen that the condition in (6.57a) is automatically satisfied by an assumption of the consolidation model, i.e.,

$$\begin{aligned} -\Delta d_{k_p,i}^{(2)} &= a_{ik_p} P_i - a_{ik_p}^{*(1)} P_i^{(1)}, \quad i \notin Q \\ &= (a_{ik_p} - a_{ik_p}^{*(1)}) P_i, \quad i \notin Q \end{aligned} \tag{6.62}$$

which shows the satisfaction of condition (6.57a) in equalities.

Having (6.59), (6.61) and (6.62), $Y_{k_p}^{*(2)}$ can now be said to be a feasible program and this then also implies that its optimality is unaffected. Therefore the two questions of (6.54) can be answered "yes."

This completes the stability study of multi-sectoral problems with the consolidation model. The development fully described in this section can be illustrated as shown in Figure 3. Although the method

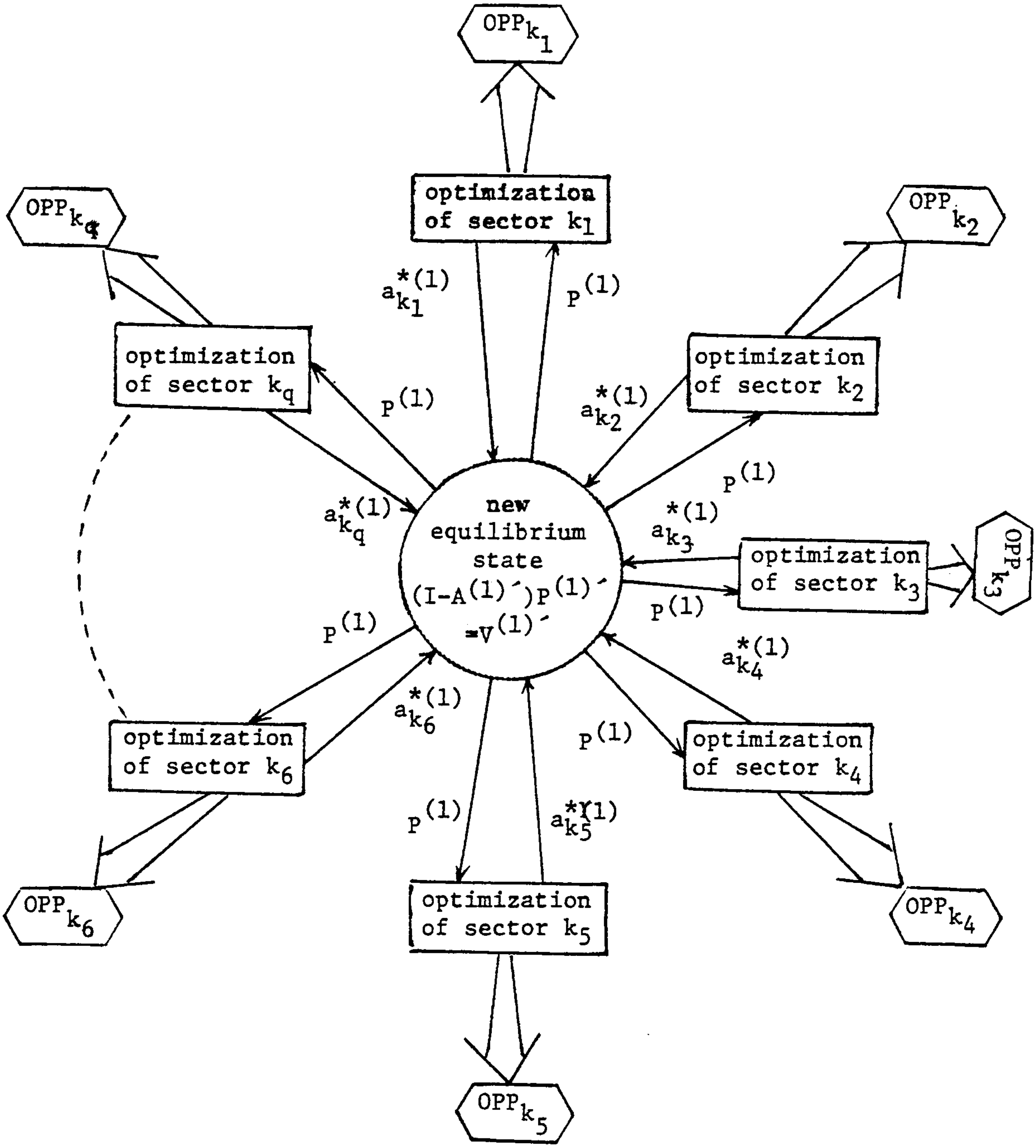

Figure 3: Computational scheme illustrating multi(q)-sectoral problem with a consolidation model (OPP$_{k_p}$ denotes for optimal production process for sector k_p).

of finding the new optimal input coefficients for the optimized sectors are no different from those of the acquisition model, it should be noted that a stable state can only be achieved when operationally optimal input coefficients for all the optimized sectors are gathered simultaneously to form a new system in equilibrium.

§6.4 Concluding Remarks

In this chapter, we dealt with a stability study under the assumptions that the prices of optimized sectors change due to regulatory optimizations and the prices of other sectors remain as constants. The effects of these assumptions were shown to be consolidated to reach a new equilibrium state of system.

For the single sectoral problem, an operationally optimal production process is selected in the same manner as in the acquisition model case. However, when the new input coefficients matrix for the entire system is formed incorporating a change for the optimized sector, a simple substitution rule for the mathematical equation corresponding to the optimized sector's behavior was taken to obtain the new set of values added for the system. An increase in vector V and a decrease in vector P were observed. The new equilibrium state thus determined was seen to be stable according to a post-optimality analysis.

As the single sectoral problem is generalized to accommodate $q(\leq n)$ sectoral problems, a simultaneous method is performed to determine a new equilibrium solution. The reasons for a sequential approach not being suitable, was simply because of the characteristics of economy which does not permit changes in the values added for the optimized sectors. Hence, operationally optimal processes for all the q sectors are gathered to form a new input coefficients matrix, and a new equilibrium state of economy was achieved accordingly. Describing more precisely, the original production process for sector k_p, p=1, ..., q, $(a_{k_p} \otimes P', P_{k_p})$ was changed to $(a^{*(1)}_{k_p} \otimes P', P_{k_p})$ following the sector's optimization, which in turn created a modification in itself to become $(a^{*(1)}_{k_p} \otimes P^{(1)\prime}, P^{(1)}_{k_p})$ in order to maintain a new equilibrium.

Conducting post-optimality analyses, the modified production process for each sector k_p, is shown to be not only a realizable process under a given technology, but also optimal. Therefore, a stable state of system was said to be reached.

Chapter VII

AN INPUT-OUTPUT SYSTEM WITH OPTIMIZED SUBSYSTEMS: THE APPRECIATION MODEL

The most important price model is presented. As a set of sectors are optimized, the development given in this chapter is highly useful in the sense that the model will be able to forecast the price changes in an economy with a reasonable easiness. In section 7.1, the model is described with the major assumption that characterizes the appreciation model. The development for the single-sectoral problem is contained in section 7.2 in terms of the scalar coefficients of the product form. The properties of class K matrices are used to determine the nature of price changes. Then, it is shown in section 7.3 that the development for the multi-sectoral problems can be easily obtained using the same types of arguments. Finally, some concluding remarks are given as to the overall developments with the appreciation model.

§7.1 Description of the Model

As we have already experienced in our everyday life, prices of certain products do change over time as our socio-economic judgment of a value fluctuates from time to time, and hence supply and demand become unsteady. It is natural to foresee that price of a particular product, say product k, will be increased if a market for the product expands given constant supply whereas if demands are fixed and the supply grows, the price of good k will most likely be decreased. In a highly competitive economy, it is assumed that a set of equilibrium prices always exist taking into consideration all the dynamic effects of the economy.

Consider sector k at a particular point in time, which has just found an optimal way of purchasing goods from the economy to produce its own goods. If the value added for sector k is fixed, then the price of good k will certainly be changed, and this will affect the entire economy. For the sake of simplicity, we ignore considering any mathematically unrepresentables such as consumers' value of judgment or inflationery factors, and choose the effects of price change of good k on the overall economy would be only on the prices of other goods, for value added of each sector being fixed. Quantities associated with price changes in economy are not considered. These, then, characterize the appreciation model for operational optimization of single sector k. In a symbolic description, the major assumption of the appreciation model can be written as

$$P_j \text{ is a variable given fixed } V_j \text{ for all } j = 1, \ldots, n. \quad (7.1)$$

Therefore, an appreciation model for the operational optimization is constituted by determining a new appreciation value of each product in an economy that is followed by optimization of sectoral problems.

Other types of assumptions made in section 3.3 are retained. For this reason, note that the only difference between the appreciation model and other price models lies in a computational scheme used to obtain a new equilibrium solution vector P.

§7.2 The Single-Sector Optimization Problem

Since precisely the same development for computing a new set of optimal input coefficients can be applied to the appreciation model of optimizing the single sector k, we will simply claim that equations (6.3) through (6.10) and the inequality (6.11) also hold in this section. For the initial stage of development, note that original equilibrium state given input matrix A and a vector of values added V was obtained by computing price vector P from

$$P' = (I-A')^{-1}V' \tag{7.2}$$

with the assumption that $(I-A')^{-1}$ existed. Let a new input matrix $A^{(1)}$ be formed by substituting the k^{th} column of A, a_k, by $a_k^{*(1)}$. Our objective here is mainly to investigate the existence and non-negativity of a new equilibrium price vector $P^{(1)}=(P_1^{(1)}, P_2^{(1)}, \ldots, P_n^{(1)})$ due to optimization of sector k such that

$$(I-A^{(1)'})P^{(1)'} = V' \tag{7.3}$$

where V is a vector of constant values added for an n-sector economy.

If the inverse of $(I-A^{(1)'})$ exists, the new price vector $P^{(1)}$ can be determined by

$$\begin{aligned} P^{(1)'} &= (I-A^{(1)'})^{-1}V' \\ &= [(I-A^{(1)})']^{-1}V'. \end{aligned} \tag{7.3a}$$

Since inverse of transpose (of a square matrix) is the same as transpose of inverse, (7.3a) may also be written as

$$P^{(1)'} = [(I-A^{(1)})^{-1}]'V' \tag{7.3b}$$

We now want to prove validity of (7.3) by showing existence of $(I-A^{(1)})^{-1}$. Consider (7.2). Given fixed V, the existence of non-negative vector P lies within polyhedral cone of vectors $\{e_j-a_j\}$ and also in the non-negative orthant R^n_+. When the k^{th} vector of A is perturbed to $a_k+\Delta a_k$, then, the intersection of the polyhedral cone spanned by $\{e_k-a_k-\Delta a_k\}$ and $\{e_j-a_j|j\neq k\}$, and non-negative orthant should not be empty to assure the equation (7.3b). However from (7.2) since it is already known that there exists at least one $(e_\ell-a_\ell)$, $1\leq\ell\leq n$, in R^n_+, a non-negative vector $P^{(1)}$ can be assumed to exist with perturbed input matrix $A+\Delta A=A^{(1)}$ if a non-negative vector P was obtainable with A.

Knowing $(I-A^{(1)})$ is non-singular, the product form of the inverse approach is again employed to find $(I-A^{(1)})^{-1}$. Expressing the k^{th} column of $(I-A^{(1)})$ as a linear combination of the columns of (I-A) to have

$$e_k - a_k^{*(1)} = \sum_{j=1}^{n} \gamma_j(e_j - a_j) = (I-A)\gamma \tag{7.4}$$

where each γ_j is a scalar, we would have a unique set of γ_j, j=1,..., n, as

$$\begin{aligned}\gamma &= (\gamma_1, \gamma_2, \ldots, \gamma_n)\\ &= (I-A)^{-1}(e_k - a_k^{*(1)}).\end{aligned} \tag{7.5}$$

A non-singularity of $(I-A^{(1)})$ implies that $\gamma_k\neq 0$. Hence the new price vector can be expressed by

$$P^{(1)\prime} = [(I-A^{(1)})^{-1}]'V'$$
$$= [E_k(I-A)^{-1}]'V'$$
$$= [(I-A)^{-1}]'E_k'(I-A)'P' \qquad (7.6)$$

where

$$E_k = (e_1, e_2, \ldots, e_{k-1}, \eta_k, e_{k+1}, \ldots, e_n) \qquad (7.7)$$

of which

$$\eta_k = (\eta_{1k}, \eta_{2k}, \ldots, \eta_{k-1,k}, \eta_{kk}, \eta_{k+1,k}, \ldots, \eta_{nk})'$$
$$= (-\frac{\gamma_1}{\gamma_k}, -\frac{\gamma_2}{\gamma_k}, \ldots, -\frac{\gamma_{k-1}}{\gamma_k}, \frac{1}{\gamma_k}, -\frac{\gamma_{k+1}}{\gamma_k}, \ldots, -\frac{\gamma_n}{\gamma_k})'. \qquad (7.8)$$

To evaluate (7.6), let $[(I-A)^{-1}]' = \{\alpha_{ij}\}$ and $(I-A)' = \{\beta_{ij}\}$. Then by the properties of E_k defined by (7.7) the following relationships can be obtained after some algebraic manipulations:

$$P_j^{(1)} = P_j + \alpha_{jk} \sum_{i=1}^{n} [(\sum_{\ell=1}^{n} \eta_{\ell k}\beta_{\ell i}) - \beta_{ki}]P_i, \; j = 1, \ldots, n. \qquad (7.9)$$

which can be rewritten as

$$P_j^{(1)} = P_j + \alpha_{jk} \cdot r_o \qquad (7.9a)$$

where

$$r_o = \sum_{i=1}^{n} [(\sum_{\ell=1}^{n} \eta_{\ell k}\beta_{\ell i}) - \beta_{ki}]P_i. \qquad (7.9b)$$

Hence, the difference between the original price and the new price for good j can be described by

$$\Delta P_j^{(1)} = P_j^{(1)} - P_j$$
$$= \alpha_{jk} \cdot r_o. \quad j = 1, \ldots, n. \qquad (7.10)$$

In trying to determine the sign of $\Delta P_j^{(1)}$, the knowledge on the signs of α_{jk} and r_o is inevitable. However, properties of the class K matrices can be used instead of enumerating those quantities directly. Since (I-A) and $(I-A^{(1)})$ both belong to the class K and $A^{(1)} \leq A$, observe that

$$\begin{aligned} \Delta P^{(1)} &= P^{(1)} - P \\ &= [(I-A^{(1)})^{-1}]'V' - [(I-A)^{-1}]'V' \\ &\leq 0. \end{aligned} \tag{7.11}$$

Also by the properties of class K, every element of $(I-A)^{-1}$ can be shown to be non-negative, i.e.,

$$\alpha_{ij} \geq 0 \text{ for all i and j} \tag{7.12}$$

Observation of (7.10) with (7.11) and (7.12) would then yield

$$r_o \leq 0; \quad -\alpha_{jk} \cdot r_o \leq P_j \tag{7.13}$$

Let us now turn our attention to the effects of the new equilibrium state $(I-A^{(1)'})P^{(1)'}=V'$ upon the already optimized sector k. With the new price vector $P^{(1)}$ the current production process $(a_k^{*(1)} \otimes P', P_k)$, which is changed from the original process $(a_k \otimes P', P_k)$ due to optimization, will again be modified to $(a_k^{*(1)} \otimes P^{(1)'}, P_k^{(1)})$ to maintain economic equilibrium. Showing the feasibility and optimality given the technology set for sector k

$$T_k = \{(a_k^{(i1)} \otimes P', P_k) | P' > 0\} \tag{7.14}$$

for the modified optimal production process becomes our objective.

The new limitation vector $d^{(2)}$ is another representation of $(a_k^{*(1)} \otimes P^{(1)\prime}, P_k^{(1)})$. Consider the net change from $d^{(1)}$ to $d^{(2)}$:

$$
\begin{aligned}
d^{(2)} &= d^{(2)} - d^{(1)} \\
&= \begin{bmatrix} P_k^{(1)} \\ a_k^{*(1)} \otimes P^{(1)\prime} \end{bmatrix} - \begin{bmatrix} P_k \\ a_k \otimes P' \end{bmatrix} \\
&= \begin{bmatrix} \Delta P_k^{(1)} \\ (a_k^{*(1)} \otimes P^{(1)\prime}) - (a_k \otimes P') \end{bmatrix}
\end{aligned} \tag{7.15}
$$

where $\Delta P_k^{(1)} \leq 0$.

Using $\Delta d^{(2)}$ in (7.15), a new solution $Y^{*(2)}$ can be obtained as

$$Y^{*(2)} = Y^{*(1)} + B^{-1}(\Delta d^{(2)}). \tag{7.16}$$

The feasibility as well as the optimality of $Y^{*(2)}$ is unclear, however, at this point. If we let $d^{(2)} = (d_o^{(2)}, d_1^{(2)}, d_2^{(2)}, \ldots, d_n^{(2)})'$ the following equality and inequalities have to be simultaneously satisfied to insure the feasibility of $Y^{*(2)}$:

(a) For the output price constraint,

$$\hat{S}Y^{*(2)} = d_o^{(2)} \tag{7.17}$$

(b) For the input cost constraints,

$$(a_{ik} - a_{ik}^{*(1)})\, P_i \leq -\Delta d_i^{(2)},\ i = 1, \ldots, n \tag{7.17a}$$

as $(a_{ik} - a_{ik}^{*(1)})P_i$ denotes the maximum amount decrease to maintain the feasibility of the (i+1) at constraint, where $\Delta d^{(2)} = (\Delta d_o^{(2)}, \Delta d_1^{(2)}, \Delta d_2^{(2)}, \ldots, \Delta d_n^{(2)})'$.

First, consider the equation (7.17). Since $d_o^{(2)} = P_k^{(1)}$ and $P_k = \hat{S}Y^{*(1)}$, we see that

$$\begin{aligned} P_k^{(1)} &= P_k + \Delta P_k^{(1)} \\ &= P_k + (1, 0, \ldots, 0)(\Delta d^{(2)}) \\ &= \hat{S}Y^{*(1)} + (\hat{S}B^{-1})(\Delta d^{(2)}) \\ &= \hat{S}[Y^{*(1)} + B^{-1}(\Delta d^{(2)})] \end{aligned} \tag{7.18}$$

Substituting (7.16) in (7.18), we have

$$P_k^{(1)} = \hat{S}Y^{*(2)} = d_o^{(2)},$$

which implies the satisfaction of the output price constraint with $Y^{*(2)}$ as given in (7.17).

From (7.15), each $-\Delta d_i^{(2)}$ can be expressed by

$$-\Delta d_i^{(2)} = a_{ik}P_i - a_{ik}^{*(1)}P_i^{(1)}. \quad i = 1, \ldots, n. \tag{7.19}$$

For there is a relationship, $P_i^{(1)} \leq P_i$ from (7.12),

$$\begin{aligned} -\Delta d_i^{(2)} &\geq a_{ik}P_i - a_{ik}^{*(1)}P_i \\ &= (a_{ik} - a_{ik}^{*(1)})P_i, \quad i = 1, \ldots, n. \end{aligned} \tag{7.20}$$

Thus we see that the conditions in (7.17a) are also satisfied. This means that the feasibility of a new program $Y^{*(2)}$ is assured and hence by post-optimality theory, it is also optimal, i.e.,

$$\text{(a)} \quad (a_k^{*(1)} \otimes P^{(1)'}, P_k^{(1)}) \epsilon T_k \qquad \text{[feasibility]} \tag{7.21}$$

$$\text{(b)} \quad \begin{bmatrix} -a_k^{*(1)} \otimes P^{(1)'} \\ P_k^{(1)} \end{bmatrix} \geq \begin{bmatrix} -a_k^{(i1)} \otimes P^{(1)'} \\ P_i^{(1)} \end{bmatrix} \quad \begin{array}{l}\text{for all } i \\ \text{[optimality]}\end{array} \tag{7.21a}$$

as feasibility and optimality of $Y^{*(2)}$ are synonymous with those of the new production process $(a_k^{*(1)} \otimes P^{(1)'},\ P_k^{(1)})$.

To clarify the already proven optimality criterion (7.23), we now proceed to compute a new set of optimal input coefficients for sector k, $a_k^{*(2)}$, under the new price system of an economy. Our speculation is, of course, the equality of $a_k^{*(2)}$ and $a_k^{*(1)}$.

Since the same optimal basis B and the modified optimal basis B^* are retained, $a_k^{*(2)}$ can be computed by

$$a_k^{*(2)} = (H^* Y^{*(2)}) \oslash P^{(1)'} . \tag{7.22}$$

One should recognize that $\hat{H}Y^{*(2)} = H^* Y^{*(2)}$ in this case due to substitution of $a_k^{*(1)}$ in place a_k in forming the limitation vector $d^{(2)}$ to get $Y^{*(2)}$. Hence from (7.16), (7.22) can be rewritten as

$$\begin{aligned} a_k^{*(2)} &= \hat{H}[Y^{*(1)} + B^{-1} (\Delta d^{(2)})] \oslash P^{(1)'} \\ &= \hat{H}Y^{*(1)} \oslash P^{(1)'} + \hat{H}B^{-1}(\Delta d^{(2)}) \oslash P^{(1)'} . \end{aligned} \tag{7.22a}$$

Concerning the second term of right hand side of equation in (7.22a), we observe

$$\hat{H}B^{-1} = (0,\ e_1,\ e_2,\ \ldots,\ e_n). \tag{7.23}$$

When (7.23) is post multiplied by $\Delta d^{(2)}$ described in (7.15),

$$\hat{H}B^{-1}(\Delta d^{(2)}) = (0,\ e_1,\ e_2,\ .\ .\ .\ ,\ e_n)(\Delta d^{(2)})$$
$$=(a_k^{*(1)} \otimes P^{(1)'}) - (a_k \otimes P') \qquad (7.24)$$

so that from (7.22a),

$$a_k^{*(2)} = \hat{H}Y^{*(1)} \oslash P^{(1)'} + [(a_k^{*(1)} \otimes P^{(1)'}) - (a_k \otimes P')] \oslash P^{(1)'}$$
$$= \hat{H}Y^{*(1)} \oslash P^{(1)'} + a_k^{*(1)} - a_k \otimes P' \oslash P^{(1)'}$$
$$= a_k^{*(1)} + (\hat{H}Y^{*(1)} - a_k \otimes P') \oslash P^{(1)'} . \qquad (7.25)$$

Since the original input coefficients a_k can be written as

$$a_k = \hat{H}Y^{*(1)} \oslash P' \qquad (7.26)$$

substitution of (7.26) into (7.25) yields

$$a_k^{*(2)} = a_k^{*(1)} + [\hat{H}Y^{*(1)} - (\hat{H}Y^{*(1)} \oslash P') \otimes P'] \oslash P^{(1)'}$$
$$= a_k^{*(1)} + (\hat{H}Y^{*(1)} - \hat{H}Y^{*(1)}) \oslash P^{(1)'}$$
$$= a_k^{*(1)} \qquad (7.27)$$

which corresponds to our previous speculation, and reassures the optimality of the new production process described in (7.21a).

Having satisfied all the criteria that are necessary to achieve a stable system, a few remarks are in order. As a single sector undergoes an operational optimization, an input-output system reaches a stable state in one iteration. If the operationally optimal set of input coefficients are not the same as the original ones, the appreciation model leads the system such that the prices of all goods are decreased. The amounts of decrease for sector i, i=1,...,n, is proportional to the

i^{th} component of the k^{th} column of $[(I-A)^{-1}]'$ where k denotes the index of the sector just optimized. When the new price vector is represented by $P^{(1)}$, the system thus being reached a stable state by the appreciation model is described by $[(I-A^{(1)})^{-1}]P^{(1)'}=V'$ where $A^{(1)}$ is the once perturbed input matrix.

§7.3 The Multiple-Sector Optimization Problem

We shall now study the behavior of the economy when $q(\leq n)$ number of sectors are optimized under their existing technology sets. To analyze the interaction between one optimized sector and the already optimized sectors, the case of two optimized sectors is first considered as we proceed one more sector, say sector k_2 $(k_2 \neq k_1 = k)$, in addition to sector k, which has already been optimized in section 7.2. A generalized version of the appreciation model for the stability of economy will then be presented to accommodate the q-sector optimization problem. To avoid notational complications, subscripts k_p, $p=1,\ldots,q$, will be attached to variables and their components whenever making a distinction between sectors is of necessity.

Two-Sector Optimization

For the current optimization problem for sector k_2, the new price vector $P^{(1)}$ and the original input coefficients a_{k_2} play a role in forming the initial limitation vector.

$$d_{k_2}^{(1)} = \begin{bmatrix} P_{k_2}^{(1)} \\ a_{k_2} \otimes P^{(1)'} \end{bmatrix} \tag{7.28}$$

for the problem diagrammatically described as:

$$
\begin{array}{lll}
\text{For sector } k_2: & \text{minimize} & b_{k_2} Y_{k_2} \\
 & \text{subject to} & S_{k_2} Y_{k_2} = P_{k_2}^{(1)} \\
 & & H_{k_2} Y_{k_2} \leq a_{k_2} \otimes P^{(1)'} \\
 & & Y_{k_2} \geq 0 \\
 & a_{k_2}^{*(1)} & \qquad\qquad (P^{(1)'}, a_{k_2}) \\
\text{For economy:} & & \longrightarrow P^{(1)'} = (I-A^{(1)'})V' \longrightarrow
\end{array}
\tag{7.29}
$$

Assuming a solution exists to this sectoral problem with the optimal basis B_{k_2}, we have

$$
Y_{k_2}^{*(1)} = B_{k_2}^{-1} d_{k_2}^{(1)} = \begin{bmatrix} \hat{S}_{k_2} \\ \hat{H}_{k_2} \end{bmatrix} d_{k_2}^{(1)}, \tag{7.30}
$$

and with a modified optimal basis $B_{k_2}^{*}$

$$
\begin{bmatrix} Y_{R,k_2}^{*(1)} \\ 0 \end{bmatrix} = B_{k_2}^{*-1} d_{k_2}^{(1)} = \begin{bmatrix} S_{k_2}^{*} \\ H_{k_2}^{*} \end{bmatrix} d_{k_2}^{(1)} \tag{7.31}
$$

where $Y_{R,k_2}^{*(1)}$ is an m-component column vector of the optimal real activities.

Using the definition of $B_{k_2}^{*(1)}$, a new set of optimal input coefficients is found by

$$
a_{k_2}^{*(1)} = (H_{k_2}^{*} Y_{k_2}^{*(1)}) \oslash P^{(1)'} \tag{7.32}
$$

each of whose components is observed to be less than or equal to the corresponding component of a_{k_2}.

To find new equilibrium solution vector $P^{(2)}$ at this second iteration, the current Leontief matrix $(I-A^{(1)'})$ is once more perturbed by replacing its k_2^{th} row by $(e_{k_2}-a_{k_2}^{*1)})'$, and the resulting matrix will be $(I-A^{(2)'})$. Since by an assumption all the values added are fixed, the new price vector can be determined by

$$P^{(2)'} = [(I-A^{(2)'})]^{-1}V' \tag{7.33}$$

or

$$P^{(2)'} = [(I-A^{(2)})^{-1}]'V'. \tag{7.33a}$$

If $(I-A^{(2)})$ is non-singular, the product form approach can be applied to find its inverse as $(I-A^{(1)})^{-1}$ is already known and only difference between $(I-A^{(2)})$ and $(I-A^{(1)})$is for the k_2^{th} column. The new price vector $P^{(2)}$ then can be written as

$$P^{(2)'} = [(I-A^{(1)})^{-1}]'E_{k_2}'(I-A^{(1)})'P^{(1)'} \tag{7.34}$$

where

$$E_{k_2} = (e_1, e_2, \ldots, e_{k_2-1}, \eta_{k_2}, e_{n_2+1}, \ldots, e_n) \tag{7.35}$$

and

$$\eta_{k_2} = (\eta_{1k_2}, \eta_{2k_2}, \ldots, \eta_{k_2-1,k_2}, \eta_{k_2k_2}, \eta_{k_2+1,k_2}, \ldots, \eta_{nk_2})'$$

$$= (-\frac{\gamma_{1k_2}}{\gamma_{k_2k_2}}, -\frac{\gamma_{2k_2}}{\gamma_{k_2k_2}}, \ldots, -\frac{\gamma_{k_2-1,k_2}}{\gamma_{k_2k_2}}, \frac{1}{\gamma_{k_2k_2}},$$

$$-\frac{\gamma_{k_2+1,k_2}}{\gamma_{k_2k_2}}, \ldots, -\frac{\gamma_{nk_2}}{\gamma_{k_2k_2}})'. \tag{7.36}$$

Letting $[(I-A^{(1)})^{-1}]' = \{\alpha_{ij}^{(2)}\}$ and $(I-A^{(1)})' = \{\beta_{ij}^{(2)}\}$, each component of $P^{(2)}$ may be represented by, after some algebraic manipulations,

$$P_j^{(2)} = P_j^{(1)} + \alpha_{jk_2}^{(2)} \sum_{i=1}^{n} [(\sum_{\ell=1}^{n} \eta_{\ell k_2} \beta_{\ell i}^{(2)}) + \beta_{k_2 i}^{(2)}] P_i^{(1)}, \; j=1,\ldots,n. \tag{7.37}$$

which can be rewritten as

$$P_j^{(2)} = P_j^{(1)} + \alpha_{jk_2}^{(2)} \cdot r_o^{(2)}, \quad j=1, \ldots, n. \tag{7.37a}$$

where

$$r_o^{(2)} = \sum_{i=1}^{n} [(\sum_{\ell=1}^{n} \eta_{\ell k_2} \beta_{\ell i}^{(2)}) + \beta_{k_2 i}^{(2)}] P_i^{(1)} \tag{7.38}$$

The existence of non-negative price vector $P^{(2)}$ can be argued in terms of polyhedral convex cones generated by $\{e_j - a_j^{(1)}\}$ and the non-negative orthant R_+^n. Furthermore, a set of relationships that the price changes due to optimization of sector k are non-positive for all goods, may be drawn from the characteristics of class K matrices, i.e.,

$$\begin{aligned} &\text{(a)} \quad \Delta P_j^{(2)} = P_j^{(2)} - P_j^{(1)} \leq 0, \quad j=1, \ldots, n \\ &\text{(b)} \quad P_j^{(2)} \geq 0, \quad j=1, \ldots, n. \end{aligned} \tag{7.39}$$

Since each element of $\{\alpha_{ij}^{(2)}\}$ is non-negative, from (7.38a) and (7.39),

$$\Delta P_j^{(2)} = \alpha_{jk}^{(2)} \cdot r_o^{(2)} \leq 0 \tag{7.40}$$

where $r_o^{(2)} \leq 0$. For $r_o^{(2)}$ being a constant, this then implies that price change of each good is proportional to the magnitude of corresponding component of $\alpha_{k_2}^{(2)} = [(I-A^{(1)})^{-1}]'_{k_2}$.

The next step for the stability study concerns with the feasibility and optimality of the new optimal production process which results from the operational optimization of sector k_2 as well as new equilibrium state of economy. Specifically, we would like to test the validities of:

$$\text{(a)} \quad (a_{k_2}^{*(1)} \otimes P^{(2)'},\ P_{k_2}^{(2)}) \varepsilon T_{k_2} \tag{7.41}$$

where

$$T_{k_2} = \{(a_{k_2}^{(1)} \otimes P^{(2)'},\ P_{k_2}^{(2)}) \mid P^{(2)'} \geq 0\};$$

$$\text{(b)} \quad \begin{bmatrix} -a_{k_2}^{*(1)} \otimes P^{(2)'} \\ P_k^{(2)} \end{bmatrix} \geqq \begin{bmatrix} -a_{k_2}^{(i1)} \otimes P^{(2)'} \\ P_k^{(2)} \end{bmatrix} \text{ for all i.} \tag{7.42}$$

As the new production process is the synonymous description of $d_{k_2}^{(2)}$, the new limitation vector for sector k_2, we choose to adopt a usual post-optimality analysis for testing (7.41) and (7.42). Consider the new change from $d_{k_2}^{(1)}$ to $d_{k_2}^{(2)}$:

$$\begin{aligned} \Delta d_{k_2}^{(2)} &= d_{k_2}^{(2)} - d_{k_2}^{(1)} \\ &= \begin{bmatrix} P_{k_2}^{(2)} \\ a_{k_2}^{*(1)} \otimes P^{(2)'} \end{bmatrix} - \begin{bmatrix} P_{k_2}^{(1)} \\ a_{k_2} \otimes P^{(1)'} \end{bmatrix} \\ &= \begin{bmatrix} \Delta P_{k_2}^{(2)} \\ (a_{k_2}^{*(1)} \otimes P^{(2)'}) - (a_{k_2} \otimes P^{(1)'}) \end{bmatrix} \end{aligned} \tag{7.43}$$

where

$$d_{k_2}^{(i)} = (d_{k_2,0}^{(i)}, d_{k_2,1}^{(i)}, \ldots, d_{k_2,n}^{(i)})', \ i = 1, 2, \text{ and}$$

$$\Delta d_{k_2}^{(2)} = (\Delta d_{k_2,0}^{(2)}, \Delta d_{k_2,1}^{(2)}, \ldots, \Delta d_{k_2,n}^{(2)})'.$$

Now feasibility condition (7.41) will be satisfied if we can show the feasibility of new optimal program that may be determined by

$$Y_{k_2}^{*(2)} = Y_{k_2}^{*(1)} + B_{k_2}^{-1} (\Delta d_{k_2}^{(2)}). \tag{7.44}$$

Two conditions must be satisfied concurrently to support the feasibility of $Y_{k_2}^{*(2)}$:

(1) For the output price constraint,

$$\hat{S}_{k_2} Y_{k_2}^{*(2)} = d_{k_2,0}^{(2)} ; \tag{7.45}$$

(2) For the input cost constraints,

$$(a_{ik_2} - a_{ik_2}^{*(1)}) P_{k_2}^{(1)} \leq - \Delta d_{ik_2}^{(2)}, \quad i = 1, \ldots, n. \tag{7.45a}$$

Consider the equation (7.45) first. Since $d_{k_2,0}^{(2)} = P_{k_2}^{(2)}$ and $P_{k_2}^{(1)} = \hat{S}_{k_2} Y_{k_2}^{*(1)}$,

$$\begin{aligned} P_{k_2}^{(2)} &= P_{k_2}^{(1)} + \Delta P_{k_2}^{(2)} \\ &= P_{k_2}^{(1)} + (1, 0, \ldots, 0) (\Delta d_{k_2}^{(2)}) \\ &= \hat{S}_{k_2} Y_{k_2}^{*(1)} + (\hat{S}_{k_2} B_{k_2}^{-1})(\Delta d_{k_2}^{(2)}) \\ &= \hat{S}_{k_2} [Y_{k_2}^{*(1)} + B_{k_2}^{-1}(\Delta d_{k_2}^{(2)})]. \end{aligned} \tag{7.46}$$

Substitution of (7.44) into (7.46) yields,

$$d^{(2)}_{k_2,0} = P^{(2)}_{k_2} = \hat{S}_{k_2} Y^{*(2)}_{k_2} . \tag{7.47}$$

Next, consider $-\Delta d^{(2)}_{ik_2}$ which may be expressed by

$$-\Delta d^{(2)}_{ik_2} = a_{ik_2} P^{(1)}_i - a^{*(1)}_{ik_2} P^{(2)}_i , \quad i = 1, \ldots, n \tag{7.48}$$

Since each $P^{(2)}_i \leq P^{(1)}_i$, clearly then,

$$-\Delta d^{(2)}_{ik_2} \geq a_{ik_2} P^{(1)}_i - a^{*(1)}_{ik_2} P^{(1)}_i$$

$$= (a_{ik_2} - a^{*(1)}_{ik_2}) P^{(1)}_i \tag{7.49}$$

Hence we can now conclude that a new program $Y^{*(2)}_{k_2}$ is feasible. By the post-optimality theory this then also implies the optimality of $Y^{*(2)}_{k_2}$, or equivalently, satisfaction of the condition (7.42).

Interactions Between Two Optimized Sectors

We now want to make an observation on the optimal production process previously selected by sector k_1. Note that the optimal input coefficients of sector k_1, $a^{*(1)}_{k_1}$, remain intact in the course of optimizing sector k_2. In order to maintain a new equilibrium state as sector k_2 selects its operationally optimal production process, the change has occurred on the price vector $P^{(1)}$ to become $P^{(2)}$ such that

$$P^{(2)} \leqq P^{(1)} \leqq P . \tag{7.50}$$

Therefore, a scaled-down production process $(a^{*(1)}_{k_1} \otimes P^{(2)\prime}, P^{(2)}_{k_1})$ is under an examination. Our objective here is to see whether or not this modified process is still capable of realization under the given technology set T_{k_1}.

Computing the net change from $d_{k_1}^{(2)}$ to $d_{k_1}^{(3)}$,

$$\Delta d_{k_1}^{(3)} = d_{k_1}^{(3)} - d_{k_1}^{(2)}$$
$$= \begin{bmatrix} P_{k_1}^{(2)} \\ a_{k_1}^{*(1)} \otimes P^{(2)'} \end{bmatrix} - \begin{bmatrix} P_{k_1}^{(1)} \\ a_{k_1}^{*(1)} \otimes P^{(1)'} \end{bmatrix}. \tag{7.51}$$

Then a new optimal program can be determined by

$$Y_{k_1}^{*(3)} = Y_{k_1}^{*(2)} + B_{k_1}^{-1}(\Delta d_{k_1}^{(3)}). \tag{7.52}$$

In the usual manner, we will show the feasibility and optimality of $Y_{k_1}^{*(3)}$ to show a realizable capability of the new production process.

Letting $d_{k_1}^{(3)} = (d_{k_1,0}^{(3)}, d_{k_1,1}^{(3)}, \ldots, d_{k_1,n}^{(3)}$ and $\Delta d_{k_1}^{(3)} = (\Delta d_{k_1,0}^{(3)}, \Delta d_{k_1,1}^{(3)}, \ldots, \Delta d_{k_1,n}^{(3)})$, the feasibility of $Y_{k_1}^{*(3)}$ can be satisfied if all the following conditions are met.

(a) For the output constraint,

$$\hat{S}_{k_1} Y_{k_1}^{*(3)} = d_{k_1 0}^{(3)} ; \tag{7.53}$$

(b) For the input constraints,

$$\hat{H}_{k_1 i}(Y_{k_1}^{*(2)} - Y_{k_1}^{*(3)}) \leq -\Delta d_{k_1 i}^{(3)}, \quad i = 1, \ldots, n. \tag{7.54}$$

Since $d_{k_1 0}^{(3)} = P_{k_1}^{(2)}$, from (7.18), (7.51), and (7.52),

$$\begin{aligned} P_{k_1}^{(2)} &= P_{k_1}^{(1)} + (P_{k_1}^{(2)} - P_{k_1}^{(1)}) \\ &= \hat{S}_{k_1} Y_{k_1}^{*(2)} + (1, 0, \ldots, 0)(\Delta d_{k_1}^{(3)}) \\ &= \hat{S}_{k_1} Y_{k_1}^{*(2)} + (\hat{S}_{k_1} B_{k_1}^{-1})(\Delta d_{k_1}^{(3)}) \\ &= \hat{S}_{k_1} [Y_{k_1}^{*(2)} + B_{k_1}^{-1}(\Delta d_{k_1}^{(3)})] \\ &= \hat{S}_{k_1} Y_{k_1}^{*(3)}, \end{aligned} \tag{7.55}$$

which implies the satisfaction of condition (a) imbedded in (7.53).

Now consider (7.54). From (7.51) we observe

$$\begin{aligned} -\Delta d_{k_1 i}^{(3)} &= a_{ik_1}^{*(1)} P_i^{(1)} - a_{ik_1}^{*(1)} P_i^{(2)} \\ &= a_{ik_1}^{*(1)} (P_i^{(1)} - P_i^{(2)}), \quad i = 1, \ldots, n. \end{aligned} \tag{7.56}$$

Also, from (7.51) and (7.52),

$$\begin{aligned} \hat{H}_{k_1}(Y_{k_1}^{*(2)} - Y_{k_1}^{*(3)}) &= \hat{H}_{k_1}[B_{k_1}^{-1}(-\Delta d_{k_1}^{(3)})] \\ &= (0, e_1, \ldots, e_n)(-\Delta d_{k_1}^{(3)}) \\ &= a_{k_1}^{*(1)}(P^{(1)\prime} - P^{(2)\prime}). \end{aligned} \tag{7.57}$$

Comparing each element of (7.57) with (7.56), we see that

$$\hat{H}_{k_1}(Y_{k_1}^{*(2)} - Y_{k_1}^{*(3)}) = -\Delta d_{k_1}^{*(3)} \tag{7.58}$$

Therefore condition (b) of (7.54) is also seen to be satisfied in equality. This means that the new optimal program $Y_{k_1}^{*(3)}$ is feasible, and hence by the post-optimality theory, it is also optimal.

To further support the optimality of $Y_{k_1}^{*(3)}$, or equivalently, that of the new production process $(a_{k_1}^{*(1)} \otimes P^{(2)'}, P_{k_1}^{(2)})$, let us compute the new optimal set of input coefficients that can be obtained by

$$a_{k_1}^{*(3)} = (H_{k_1}^{*} Y_{k_1}^{*(3)}) \oslash P^{(2)'} \tag{7.59}$$

Using (7.52), (7.59) may be expanded as:

$$\begin{aligned} a_{k_1}^{*(3)} &= (H_{k_1}^{*} Y_{k_1}^{*(3)}) \oslash P^{(2)'} \\ &= H_{k_1}^{*} [Y_{k_1}^{*(2)} + B_{k_1}^{-1}(\Delta d_{k_1}^{(3)})] \oslash P^{(2)'} \\ &= H_{k_1}^{*} Y_{k_1}^{*(2)} \oslash P^{(2)'} + H_{k_1}^{*} G_{k_1}^{-1}(\Delta d_{k_1}^{(3)}) \oslash P^{(2)'} \\ &= H_{k_1}^{*} Y_{k_1}^{*(2)} \oslash P^{(2)'} + (0, e_1, \ldots, e_n)(\Delta d_{k_1}^{(3)}) \oslash P^{(2)'}. \end{aligned} \tag{7.59a}$$

By (7.55),

$$\begin{aligned} a_{k_1}^{*(3)} &= H_{k_1}^{*} Y_{k_1}^{*(2)} \oslash P^{(2)'} + a_{k_1}^{*(1)} \otimes (P^{(2)'} - P^{(1)'}) \oslash P^{(2)'} \\ &= H_{k_1}^{*} Y_{k_1}^{*(2)} \oslash P^{(2)'} + a_{k_1}^{*(1)} \otimes P^{(2)'} \oslash P^{(2)'} \\ &\quad - a_{k_1}^{*(1)} \otimes P^{(1)'} \oslash P^{(2)'} \\ &= a_{k_1}^{*(1)} + (H_{k_1}^{*} Y_{k_1}^{*(2)} - a_{k_1}^{*(1)} \otimes P^{(1)'}) \oslash P^{(2)'} . \end{aligned} \tag{7.59b}$$

Since $a_{k_1}^{*(1)} = a_{k_1}^{*(2)}$ by (7.27), substituting (7.22) in the above equation yields

$$a_{k_1}^{*(3)} = a_{k_1}^{*(1)} + [H_{k_1}^{*} Y_{k_1}^{*(2)} - (H_{k_1}^{*} Y_{k_1}^{*(2)}) \oslash P^{(1)'} \otimes P^{(1)'}] \oslash P^{(2)'}$$

$$= a_{k_1}^{*(1)} + (0) \oslash P^{(2)'}$$

$$= a_{k_1}^{*(1)} . \tag{7.60}$$

Therefore, the optimal input coefficients obtained in the first iteration still remains optimal for sector k_1 even when an additional sector undergoes an operational optimization after the optimization of sector k_1.

A General Case

Finally, we shall proceed to a general model in which each of $q(\leq n)$ sectors of the n-sector economy is considering its selection of the operationally optimal production process under a given technology. Since precisely the same development of the two-sector optimization model can be generalized, the description of the mathematical derivations is given in less detail. Our objective here is simply to indicate the nature of the problem structure for a given sector k_p, $P = 1,\ldots,q$, and its implications to the overall economy.

As one optimization process is required for each of q sectors, a sequential approach is taken through q iterations, one iteration for each sector. This at iteration p (=1,...,q), sector k_p is optimized with its limitation vector

$$d_{k_p}^{(1)} = \begin{bmatrix} P_{k_p}^{(p-1)} \\ a_{k_p} \otimes P^{(p-1)'} \end{bmatrix} \tag{7.61}$$

which represents its current production process $(a_{k_p} \otimes P^{(p-1)'}, P_{k_p}^{(p-1)})$.

The technology set, T_{k_p}, may be described by

$$T_{k_p} = \{(a_{k_p}^{(i1)} \otimes P^{(p-1)'}, P_{k_p}^{(p-1)}) \mid a_{k_p}^{(i1)} \varepsilon a_{k_p}^{(1)}, P^{(p-1)'} > 0\}, \quad (7.62)$$

a polyhedral convex cone. Note that $a_{k_p}^{(1)} = \{a_{k_p}^{(i1)} | i \geq 1\}$ is a set of input coefficients corresponding to a set of all realizable processes for sector k_p.

To find the optimal input coefficients among the vectors in $a_{k_p}^{(1)}$, the following equation can be used:

$$a_{k_p}^{*(1)} = (H_{k_p}^{*} Y_{k_p}^{*(1)}) \oslash P^{(p-1)'} \leq a_{k_p} \quad (7.63)$$

where $Y_{k_p}^{*(1)}$ is the optimal program for the sectoral problem, and $H_{k_p}^{*}$ is the modified optimal basis corresponding to the input constraints.

If $a_{k_p}^{*(1)} \leq a_{k_p}$, then $a_{k_p}^{*(1)}$ replaces a_{k_p} in the current input coefficient matrix $A^{(p-1)}$ to form $A^{(p)}$. A new equilibrium solution to this perturbed input-output system is obtained by

$$\begin{aligned} P^{(p)'} &= [(I-A^{(p)})^{-1}]'V' \\ &= [(I-A^{(p-1)})^{-1}]'E_{k_p}'(I-A^{(p-1)})'P^{(p-1)'} \quad (7.64) \end{aligned}$$

where

$$E_{k_p} = (e_1, e_2, \ldots, e_{k_p-1}, \eta_{k_p}, e_{k_p+1}, \ldots, e_n), \quad (7.65)$$

$$\eta_{k_p} = (\eta_{1k_p}, \eta_{2k_p}, \ldots, \eta_{nk_p})', \quad (7.65a)$$

$$\eta_{jk_p} = \begin{cases} -\dfrac{\gamma_{jk_p}}{\gamma_{k_p k_p}}, \; j \neq k_p, \; j = 1, \ldots, n \\ \dfrac{1}{\gamma_{k_p k_p}}, \; j = k_p. \end{cases} \tag{7.65b}$$

Letting $[(I-A^{(p-1)})^{-1}]' = \{\alpha_{ij}^{(p)}\}$ and $(I-A^{(p-1)})' = \{\beta_{ij}^{(p)}\}$, each component of $P^{(p)}$ can be expressed as

$$P_j^{(p)} = P_j^{(p-1)} + \alpha_{jk_p}^{(p)} \cdot r_o^{(p)}, \quad j = 1, \ldots, n \tag{7.66}$$

where

$$r_o^{(p)} = \sum_{i=1}^{n} [(\sum_{\ell=1}^{n} \eta_{\ell k_p} \beta_{\ell i}^{(p)}) + \beta_{k_p i}^{(p)}] P_i^{(p-1)} \tag{7.66a}$$

The close examination of (7.64) yields

$$P^{(p)} \leqq P^{(p-1)}, \quad p = 1, \ldots, q \tag{7.67}$$

which then implies that

$$P^{(p)} \leqq P^{(p-1)} \leqq P^{(p-2)} \leqq \ldots \leqq P^{(1)} \leqq P. \tag{7.67a}$$

As the production process for sector k_p now becomes $(a_{k_p}^{*(1)} \otimes P^{(p)'}, P_{k_p}^{(p)})$, we proceed with the post optimality analysis to determine whether or not this modified process is a realizable member of T_{k_p}. In the process of doing so, an indirect approach will again be taken by determining the new optimal program and observing its feasibility.

First, computing the net change in the limitation vector due to the new equilibrium state of the system,

$$\Delta d_{k_p}^{(p)} = d_{k_p}^{(p)} - d_{k_p}^{(p-1)}$$

$$= \begin{bmatrix} P_{k_p}^{(p)} \\ a_{k_p}^{*(1)} \otimes P^{(p)'} \end{bmatrix} - \begin{bmatrix} P_{k_p}^{(p-1)} \\ a_{k_p} \otimes P^{(p-1)'} \end{bmatrix}, \tag{7.68}$$

the new optimal program for the sectoral problem can be obtained by

$$Y_{k_p}^{*(2)} = Y_{k_p}^{*(1)} + B_{k_p}^{-1} (\Delta d_{k_p}^{(p)}). \tag{7.69}$$

When a post-optimal analysis is conducted on the values of $Y_{k_p}^{*(2)}$, it can be seen that it is feasible in the sense that

$$Y_{k_p}^{*(2)} \leq Y_{k_p}^{*(1)} . \tag{7.70}$$

Therefore, it is also optimal and this leads to the conclusion that the modified process $(a_{k_p}^{*(1)} \otimes P^{(p)'},\ P_{k_p}^{(p)})$ is not only a realizable production process, but also it is a better production process than any other processes in T_{k_p}.

The optimality status of the operationally optimal production process can be further supported, when an optimal input coefficient, $a_{k_p}^{*(2)}$, is recomputed under the new equilibrium state of system. The obvious result,

$$a_{k_p}^{*(2)} = a_{k_p}^{*(1)}, \tag{7.71}$$

would then be prevailed.

§7.4 Concluding Comments

In conclusion, the appreciation model leads to a stable state in q iterations as q ($1 \leq q \leq n$) number of sectors are optimized. Once a new set of optimal input coefficients are determined for a sector, it is shown that they remain optimal regardless of how many optimizations for other sectors follow and change the structure of economy thereafter.

Due to the assumption of fixed values added, a decreasing nature of the price changes for all sectors is observed at the end of each iteration. The amount of decrease after i iterations for the price of each product is seen to be proportional to the magnitude of the corresponding component of $[(I-A^{(i-1)})^{-1}]'_{k_i}$. In terms of the production processes, the iterative procedure at the i^{th} iteration began with the original process $(a^{*(1)}_{k_i} \otimes P^{(i-1)'}, P^{(i-1)}_{k_i})$, to create an operationally optimal process $(a^{*(1)}_{k_i} \otimes P^{(i-1)}, P^{(i-1)'}_{k_i})$, and again made a modification as to reach an operationally optimal, stable process $(a^{*(1)}_{k_i} \otimes P^{(i)'}, P^{(i)}_{k_i})$ for sector k_i.

It should be recognized that the solution procedures presented in this chapter have very similar characteristics to those of the quantity model. In section 8.2, a comparison will be made between the two types of changes to come up with the general relationship between the quantity and the price side of an input-output system.

Chapter VIII

CONCLUSIONS AND RECOMMENDATIONS

In section 8.1, a combined analysis between each of the price models and the quantity model is made to observe the quantitative reactions of the economy to the sectoral optimization using a price model. The amount of change in the gross national product will also be explained in terms of the fixed final consumption. The entire study is then summarized with some major conclusions drawn in section 8.2; more detailed conclusions can be found at the end of chapters V, VI and VII. Recommendations for further research and some possible application areas are contained in section 8.3.

§8.1 Economic Interpretations of the Price Models with Quantities Associated

In this section, a combined analysis between each of the price models and Moon's quantity model will be made to observe effects of the operational optimization to both the quantitative and pricing side of an economy. This analysis will be based on the duality theory which states that once prices are determined for an economy in equilibrium, the same input coefficients matrix can also be applied to compute the quantities to produce in order to meet the given bill of goods C.

To use Moon's model (which should have been called a "value model" rather than quantity model, as each input coefficient in his model adopts the monetary unit incorporating the known prices of economy), the relationship given in (2.7) should be recalled in the form of,

$$a_{ij}^{m} = \left(\frac{P_i}{P_j}\right) a_{ij}^{p} \quad \text{for all } i \text{ and } j \tag{8.1}$$

where a_{ij}^{p} is the coefficient measured in physical unit and a_{ij}^{m} is the coefficient measured in monetary unit. Notice that $A^{p}=\{a_{ij}^{p}\}$ was used by all the price models whereas the quantity model assumed each element of A as being a_{ij}^{m}, i.e., $A^{m}=\{a_{ij}^{m}\}$. Also, assume that the fixed final consumption vector C of Chapter IV actually implies the final demand for each product in physical units. To make a distinction between "how many units" and "how much worth," let G_j^{q} and C_i^{q} be the number of units of good j to be produced by sector j and the number of units of good i demanded by economy as the final consumption, respectively. The G and C without superscripts can be assumed to have superscript m. Hence, one can observe that C_i^{q} is fixed for all i although $C_i^{m}(=C_i)$ may vary as it is dependent upon price of good i.

Now suppose that a sectoral optimization has been performed using one of the price models to yield a new equilibrium state of economy.that is also stable. Two questions should arise immediately:

(1) How will this change in A (or, A^{p} in this case) and P affect the quantities to be produced by economy to satisfy a given bill of goods C^{q}?

(2) What would be the amount of change in the Gross National Product (or, Consumption)?

For the sake of simplicity, we will attempt to answer the above questions in the case of single-sectoral (sector k) optimization problem only.

The Acquisition Model

To relate the acquisition model developed in Chapter V with the quantities associated, first recall that the characterization of the model is constituted by the constancy of prices. Since $C=C^q \otimes P$, and C^q and P are fixed each $G_j^{(1)}$ can be obtained by

$$G^{(1)} = (I - A^{m(1)})^{-1}C \tag{8.2}$$

where $G^{(1)}$ corresponds to the amount of goods to be produced by economy in monetary units. Note that each element of $A^{m(1)}$ can be described as

$$\begin{aligned} a_{ij}^{m(1)} &= a_{ij}^{m}, \ i = 1, \ldots, n; \ j \neq k, \ j = 1, \ldots, n \\ a_{ik}^{m(1)} &= a_{ik}^{m*(1)} = \left(\frac{P_i}{P_k}\right) a_{ik}^{p*(1)}, \ i = 1, \ldots, n. \end{aligned} \tag{8.3}$$

Since $a_{ik}^{p*(1)} \leqq a_{ik}^{p}$ by (5.14), it should be obvious that $a_{ik}^{m*(1)} \leqq a_{ik}^{m}$ for all i. If $a_k^{m*(1)} \leqq a_k^m$, according to Moon's derivation of the quantity model, the new gross output level of each product including that of sector k decreases as sector k switches its production process from its original process to its operationally optimal process, i.e.,

$$G_j^{(1)} \leq G_j \text{ for all } j. \tag{8.4}$$

The actual quantities to be produced, then, can be found by dividing each $G_j^{(1)}$ and G_j by P_j so that

$$G_j^{q(1)} = \frac{G_j^{(1)}}{P_j} \leq \frac{G_j}{P_j} = G_j^q \text{ for all } j \tag{8.5}$$

Observing (8.5), one can conclude that an operational optimization of sector k by the acquisition model forces the quantity to be produced by each sector to decrease, at the same time increasing the value added for the sector.

To answer the second question, note that the gross national consumption (GNC) is given by

$$z^c = \sum_{i=1}^{n} C_i = \sum_{i=1}^{n} C_i^q P_i.$$

Since P_i is fixed by the very nature of the acquisition model and C_i^q is fixed by an assumption of the quantity model, GNC would not be affected by an optimization of sector k.

One other interesting observation here is on the variables that are associated with the gross national product, z^p. As z^p should be equal to z^c for an economy in equilibrium, the amount of increase in V_k and the amounts of decrease in G_j^q for all j, should have been made to meet the following requirements:

(1) $z^c = z^p = VG^q$

(2) $z^c = z^p = V^{(1)}G^{q(1)}$

where $V^{(1)} \geq 0$ and $G^{q(1)} \leq G^q$.

The Consolidation Model

In the case of single-sectoral optimization, the consolidation model's major underlying assumption is the variability in the price of the optimized sector's good. If sector k is optimized, a change in P_k should occur as to reach a new equilibrium state of system. Hence, this decreasing characteristic of P_k, i.e., $P_k^{(1)} \leq P_k$, plays the key role in observing implications of the consolidation model on the quantitative side of an economy.

To solve the solution vector $G^{(1)} = (I - A^{m(1)})^{-1}C^{(1)}$, the following set of transformations become necessary tasks to perform:

(a) Change each element of $A^m = \{a_{ij}^m\}$ by

$$a_{ij}^{m(1)} = a_{ij}^m,\ i = 1, \ldots, n,\ i \neq k;\ j = 1, \ldots, n,\ j \neq k, \qquad (8.6)$$

$$a_{kj}^{m(1)} = \left(\frac{P_k^{(1)}}{P_j}\right)a_{kj}^{p(1)},\ j = 1, \ldots, n, \qquad (8.6a)$$

$$a_{ik}^{m(1)} = \left(\frac{P_i}{P_k^{(1)}}\right)a_{ik}^{p(1)},\ i = 1, \ldots, n, \qquad (8.6b)$$

where $A^{p(1)} = \{a_{ij}^{p(1)}\}$ is an input-output matrix corresponding to the new equilibrium state of economy due to optimization of sector k;

(b) Compute the new final consumption vector $C^{(1)}=\{C_i^{(1)}\}$ $=\{C_i^{m(1)}\}$ by

$$C_i^{(1)} = C_i, \ i \neq k, \ i = 1, \ldots, n,$$

$$C_k^{(1)} = \left(\frac{P_k^{(1)}}{P_k}\right) C_k \leq C_k. \tag{8.7}$$

Note that the new set of input coefficients $a_k^{m*(1)}$ associated with the operational optimal input coefficients $a_k^{p*(1)}$ of (6.8) are not necessary to be recomputed since $a_{ik}^{p(1)}=a_{ik}^{p*(1)}$ for all i.

Let us closely examine (8.6a) and (8.6b). With the change in P_k, (8.1) may be rewritten as

$$a_{ik}^{p(1)} = \left(\frac{P_k^{(1)}}{P_i}\right) a_{ik}^m, \ i = 1, \ldots, n. \tag{8.8}$$

Substituting (8.8) in (8.6b),

$$a_{ik}^{m(1)} = \left(\frac{P_i}{P_k^{(1)}}\right)\left(\frac{P_k^{(1)}}{P_i}\right) a_{ik}^m = a_{ik}^m. \tag{8.9}$$

Thus one can see from (8.9) that the transformation as indicated by (8.6b) could have been eliminated. Furthermore, by the relationship $P_k^{(1)} \leq P_k$ and from (8.6a) and (8.1),

$$a_{kj}^{m(1)} = \left(\frac{P_k^{(1)}}{P_j}\right) a_{kj}^{p(1)}$$

$$= \left(\frac{P_k^{(1)}}{P_j}\right) a_{kj}^{p}$$

$$= \left(\frac{P_k^{(1)}}{P_j}\right)\left(\frac{P_j}{P_k}\right) a_{kj}^{m}$$

$$\leq a_{kj}^{m},\ j = 1, \ldots, n. \tag{8.10}$$

As the net results due to transformations described by (8.6a) and (8.7), then, the only changes that occur in the original system of equations $(I-A^m)G=C$ are in the kth row of A^m and in the kth component of C. Moreover, note that the same decrease factor $P_k^{(1)}/P_k$ is multiplied to each of a_{kj}^m, j=1, ..., n, and C_k. Therefore, solving the unique vector $G^{(1)}$ by $G^{(1)}=(I-A^{m(1)})^{-1}C^{(1)}$, one can expect that

$$G^{(1)} = G \tag{8.11}$$

which gives an implication that no change be made in the magnitude of G.

However, computing the physical quantities of $G^{(1)}$ according to the relationship,

$$G_j^{q(1)} = \frac{P_j^{(1)}}{P_j}\ G = G,\ j \neq k,\ j = 1, \ldots, n,$$

$$G_k^{q(1)} = \frac{P_k^{(1)}}{P_k}\ G \geq G, \tag{8.12}$$

conclusions can be made as: an operational optimization of sector k by the consolidation model increases the physical quantity to be produced by sector k while the price of good k being decreased; all other quantities and prices are remained the same; and the amount of gross national product would be decreased as much as $(C_k - C_k^{(1)})$ from its original amount.

The Appreciation Model

In Chapter VII, a substantial amount of developments was made for sectoral problems when a set of sectors undergoes operational optimizations under the assumptions of variability of prices and constancy of values added. In the case of single-sectoral problem, it is seen in section 7.2 that all the prices of an economy are lowered to reach a new equilibrium state. When the new price vector is denoted by $P^{(1)}$, recall that $P^{(1)} \leq P$ and the system may be represented by $(I-A^{p(1)'})P^{(1)'}=V'$, where $A^{p(1)}$ is the once parturbed matrix in physical units.

To study the impacts of so determined equilibrium state of economy on the quantitative side of system, the following rules of transformation are adopted:

(a) For the new input-output coefficients matrix, $A^{m(1)}=\{a_{ij}^{m(1)}\}$,

$$a_{ij}^{m(1)} = \left(\frac{P_i^{(1)}}{P_j^{(1)}}\right)a_{ij}^{(1)} \text{ for all i and j;} \tag{8.13}$$

(b) For the final consumption vector, $C^{(1)}=\{C_i^{m(1)}\}$,

$$C_i^{(1)} = \left(\frac{P_i^{(1)}}{P_i}\right)C_i,\ i = 1, \ldots, n \tag{8.14}$$

Before proceeding to compute the new solution vector $G^{(1)}$ for the set of equations $(I-A^{m(1)})G^{(1)}=C^{(1)}$, let us make an important observation on (8.13). Substituting (8.1) into (8.13), each $a_{ij}^{m(1)}$ can be expressed by

$$a_{ij}^{m(1)} = \left(\frac{P_i^{(1)}}{P_j^{(1)}}\right)a_{ij}^{p}$$

$$= \left(\frac{P_i^{(1)}}{P_j^{(1)}}\right)\left(\frac{P_j}{P_i}\right)a_{ij}^{m}$$

$$= \left(\frac{P_j}{P_j^{(1)}}\right)\left(\frac{P_i^{(1)}}{P_i}\right)a_{ij}^{m}. \tag{8.15}$$

Comparing (8.15) with (8.14), one can see that the scale of decrease in each a_{ij}^{m} is $(P_j/P_j^{(1)})(P_i^{(1)}/P_i)$ while each C_i is decreased by the factor of $(P_i^{(1)}/P_i)$ only. Therefore, if any unique solution $G^{(1)}$ exists, each of its components should be describable as

$$G_j^{(1)} = \left(\frac{P_j^{(1)}}{P_j}\right)G_j,\ j = 1, \ldots, n, \tag{8.16}$$

where the reciprocal of $(P_j^{(1)}/P_j)$ is exactly the same factor multiplied to a_{ij}^{m} to get $a_{ij}^{m(1)}$, which has been missing in computing $C_i^{(1)}$.

As $P_j^{(1)} \leq P_j$ for all j, from (8.16),

$$G_j^{(1)} \leq G_j,\ j = 1, \ldots, n,$$

or,

$$G_j^{q(1)} = \left(\frac{1}{P_j^{(1)}}\right) G_j^{(1)}$$

$$= \left(\frac{1}{P_j^{(1)}} \frac{P_j^{(1)}}{P_j}\right) G_j$$

$$= \left(\frac{1}{P_j}\right) G_j$$

$$= G_j^q \,, \tag{8.17}$$

which reveals an interesting result that under the constancy of values added for the economy, an operational optimization of single-sector k does not affect the quantities to be produced by each sector. However, the gross national product would be decreased as much as $\sum_{i=1}^{n}(C_i^{(1)} - C_i)$, because of the price change of every good in the economy.

§8.2 Summary and Overall Conclusions

In this study the stability of a complex system has been examined when one or more of its subsystems are operationally optimized.

The Leontief static input-output model was considered to describe a (national economic) system; for the operational optimization of a sector, a linear programming model was used so that the property of linearity of the Leontief system also prevailed in each optimized sector. The development was based on the concept of a production technology set, mathematical induction, the product form of the inverse, and post-optimality and duality theory of linear programming.

For a sectoral optimization, a linear programming model with the sectoral production constraints was used in such a way that the current revenue is maintained at the same time minimizing the input costs. The input configurations so determined were called operationally optimal input coefficients and their implications were simply that the optimized sector may choose to buy less than the current level of input from every sector of the economy.

To find a new equilibrium state of economy following the sectoral optimization, three different price models were developed under the three distinct economic reasons. The simplest of all was named the acquisition model as it may yield a positive acquisition to the optimized sector. Because the model assumes the constancy of prices, it has been found that for the single-sectoral problem, the system can reach a stable state by a simple mathematical substitution resulting in an increase in the value added for the sector. For a generalized version of the single-sectoral problem, the same type of arguments could hold. That is, an increase in the value added can be experienced by each of the optimized sectors, under a new stable system.

The second model, that was named the consolidation model, viewed the economy in a slightly different perspective. Assuming that the price for the optimized sector can vary while the other prices should be fixed, the model was able to determine the price change as well as the changes in the values added. A decrease for the price of the optimized sector and an increase in the value added for each of the non-optimized sectors have been observed. For the multi-sectoral problem, a simultaneous analysis was conducted in order to take into consideration all the changes that should occur for the new equilibrium state of economy. After some numerical manipulations and using the mathematical induction, it has been seen that a new stable state can be achieved in which prices for the optimized sectors are decreased while the other sectors gain increases in their values added.

The appreciation model with its highly useful capability of forecasting power for prices in the national economy was the third model presented. In the single-sectoral problem, if the operationally optimal production process is not the same as the sector's old process, it has been observed that every sector of the economy may be required to reduce its unit price in order for the system to maintain a new equilibrium state. Sequential applications of the single-sectoral problem yielded the same type of results at each iteration j in addition, the optimality of a sector's previously chosen operationally optimal process was found to be unaffected by another sector's optimization.

Moon's quantity model has also been presented for the purpose of completeness as well as to analyze the quantitative side of an economy following the performance of each of the price models. It was assumed that Moon's assumption of the fixed demand actually implies the fixed demand in quantitative units rather than in monetary units. With this assumption, a combined analysis between the quantity model and each of the price models was made. When the acquisition model was used to obtain an operationally optimal process given a single sector, the conclusion was that the sectoral optimization forces the quantity to be produced by each sector to decrease; with the consolidation model, the quantity to be produced by the optimized sector would be increased while other quantities remain the same; and the physical quantities to be produced would not be affected with the appreciation model. Summary of the study describing the reactions of operational optimization for the single sectoral case can be seen in Table 2.

In conclusion, the Leontief's input-output system, with the aid of linear programming, can reach a stable state as long as the system's underlying assumptions fall under the three categories of economic behaviors described. Although a direct dual formulation of Moon's quantity model was not possible, the price models presented in this study are certainly capable of calculating the changes for the quantitative side of an economy as well as determining price changes that should occur due to the operational optimizations of sectoral problems.

Table 2

Summarized Results Describing the Reactions of an Input-Output System to Each of the Price Models for the Single-Sectoral (k) Optimization Case

Category	Price Model		
	Acquisition Model	Consolidation Model	Appreciation Model
Price of Good k	unchanged	decreased	decreased
Price of Good $j \neq k$, $j=1, \ldots, n$	unchanged	unchanged	decreased
Value Added for Sector k	increased	unchanged	unchanged
Value Added for Sector $j \neq k$, $j=1, \ldots, n$	unchanged	increased	unchanged
Number of Units of Good k to be Produced by Sector k	decreased	increased	unchanged
Number of Units of Good $j \neq k$ to be Produced by Sector $j \neq k$, $j=1, \ldots, n$	decreased	unchanged	unchanged
Gross National Product	unchanged	decreased	decreased

§8.3 Areas for Further Research and Possible Applications

The developments of a stability analysis for an input-output system have not been exhaustive in this study. Several extensions are of interest and remain to be considered. Among these are the following four areas which appear to be a potentially important body of further research.

(1) A direct dual approach for the quantity model or for each of the price models can be taken so that the price and quantitative side of the system are treated as a single problem. If this is done, a more interesting set of observations could be made directly from the solution of primal problem for the solution of its dual problem.

(2) The linearity assumption prevailing in the current study could be relaxed in either a sectoral problem or the problem of the system, or both.

(3) The production constraint for the price may assume the inequality ($\geq$) such that a regulatorily optimal production process would imply the maximization of revenue as well as the minimization of production costs.

(4) Adding the concept of capital requirements of the economy, development of a price model for the Leontief's dynamic input-output system may also be a highly suggestable area for further research.

Two major categories of application may be of interest: the existing application areas for input-output analyses; and, possible applications to any complex system other than a national economic system.

A broad range of existing applications includes analysis and/or planning of such areas as national economy, multi-regional and multi-national systems, international trade, population and manpower, education, and environment. Moon's quantity model, any of the price models presented in this study, or a combined analysis may be applicable to any of the above application areas whenever it is necessary to perform operational optimizations in some sectors.

As was indicated at the outset of this thesis, any complex system may be analyzed as were the previous economic systems, by the approach taken in developing the stability models. To this end, various types of interactions among its subsystems should be describable mathematically. The system of interest could be any organization such as a firm, academic or medical institution, or others, each of which is involved with various interacting departments. The extent of centralization (or decentralization) of the organization would then be a key factor in defining the necessity for operational optimization.

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STABILITY OF STATIC INPUT-OUTPUT SYSTEMS

WITH OPTIMIZED SUBSYSTEMS:

QUANTITY AND PRICE MODELS

by

John Hearn Lee

(ABSTRACT)

An investigation of the pattern of structural change in a static input-output system is considered, when a series of operational optimizations are performed in one or more subsystems. Specifically, changes in the relative price of each commodity in an economy, along with their implications for the quantities associated, are analyzed.

By operationally optimizing a sector, a production process is selected in such a way that the input requirements from other sectors in the economy do not exceed the current equilibrium level as specified by the input-output economy. When the new optimal process is substituted for the current equilibrium process, the input-output structure may be perturbed and a new equilibrium solution needs to be sought. For the multi-sectoral problems, the system would be considered stable if the necessity for the further perturbation ceases, while an equilibrium solution exists to the current input-output system.

Three different price models are developed in trying to obtain a new stable system. Depending upon three distinct sets of economic

reasons behind, these models are named as the acquisition, the consolidation, and the appreciation model. This study concludes that under all the normal circumstances, a static input-output system does reach a stable state, if the sectoral optimization is conducted through linear programming. This study also investigates the effects of operational optimization of sectors on the quantity side of the economy.