ESTIMATION OF INDIVIDUAL VARIATIONS IN AN UNREPLICATED TWO-WAY CLASSIFICATION

by

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I. INTRODUCTION

This dissertation is concerned with the estimation of error variances in an unreplicated two-way classification. The results are of interest in a wide class of applications. In general, the procedures may be used in checking the assumption of homogeneous error variances in randomized block designs under certain conditions and the estimators derived may be used in the measurement and comparison of the consistencies of judges in subjective experimentation and of the precisions of analytical methods in quantitative experimentation.

There are many experiments in which it is necessary for a group of individuals to give subjective scores to test items such as in judging the quality of food, the appearance of a uniform, or the ease of operation of equipment. The scores given by the judges are usually based on a discrete numerical scale where the numbers on the scale refer to different descriptive definitions such that the lower end of the scale indicates the qualities which are not desired and the upper end of the scale indicates desired qualities. The judge examines the item to be graded and then assigns the grade or score which in his judgment corresponds to the appropriate descriptive definition.

In some of these experiments, it is possible to test a judge's ability to give "good" subjective scores by some quantitative process. For example, in the judging of the wear-resistance of a uniform, standard-wear uniforms are used in the training and testing of the judges. Another such method of testing the reliability of a judge in certain taste testing panels is to have him taste different solutions in which know amounts of a given reagent has been added to see if he can discriminate accurately between the different dilutions.

In other experiments no standards exist such as noted above but it is possible to obtain an estimate of a judge's ability by replication of the experiment. In order that an experiment can be repeated statistically, successive observations or grades must be independent. Although the above two classes of experiments give different types of estimates, these estimates still provide a criterion for determining if a judge can give good subjective scores.

There is yet a third type of experiment in which neither of the above methods is possible for obtaining estimates of the abilities of the judges. For example, no standard may be satisfactory for judging the "ease of operation" of a vehicle and it may be impossible to repeat such an experiment, since a judge may remember the grade previously given an item.

Therefore, a method is needed which assesses the ability of an individual to give "good" subjective scores to items for this third type of experiment; this research proposes such a method.

The criterion used in the proposed method to determine the rating ability of a panel member is that of whether he "expresses the opinion of all the members". This criterion is necessitated by the restrictions already given that the experiment cannot be repeated and there exists no practical quantitative measurement with which to compare the scores. The estimators proposed here measure the abilities of the members of a panel to give subjective scores which express the composite opinion of all the members of the panel. These estimators were obtained by the use of

i. the principle of maximum likelihood

and

ii. a general quadratic form.

Although estimators were obtained by the first method only for a three-member panel grading any number of items, these estimators are shown to be identical to those obtained by the second method for which there are no restrictions on the number of members of the panel or on the number of items graded.

In addition to the derivations of the estimators, two different tests of significance are given. One test provides a method to determine if a particular judge is significantly worse (or significantly better) than the average when it is assumed that all tho other judges are equally good. This test is applicable for three or more judges as well as for two or more items. When there are only three judges and two or more items, the other test provides a method to determine if the three judges are equally good.

Although the above discussion has mentioned only the testing of the ability of a panel member to give good subjective scores, the method is applicable to any quantitative process which fits the model. For example, suppose that it is desired to estimate the variability of several light meters by the use of flashbulbs. In an example such as this, it is impossible to repeat the experiment. Another example is that of estimating the variabilities of several chemical processes where it is too expensive to obtain more than one sample from each process at one time.

For the benefit of those readers who are interested in applications only, it is suggested that they read Chapter II on the mathematical model in addition to Chapter VI wherein numerical examples are discussed.

The discussion of available literature has been delayed to Chapter VII, Review of Literature, in order that the results of this research can be compared with those in the literature.

II. MODEL

Statistical analysis of data assumes that there is some underlying mathematical model which expresses in mathematical symbols as nearly as possible the behavior of the physical process being observed. Thus, for this problem, it is necessary to state the assumed composition of the observation x_{ij} in a two-way classification. In line with the introductory comments, x_{ij} may represent the grade given the i^{th} item by the j^{th} judge, the amount of a given chemical in the i^{th} compound as determined by the j^{th} analytical method, or some similar specification in a host of possible applications. We shall discuss the problem in terms of the grading of items remembering that many other applications are possible.

Firstly, we postulate that each item being graded has a true mean, say μ_i, where $i = 1, 2, \ldots, n$ and identifies the item being graded. Secondly, we recognize that the judge may be biased; that is, one judge may be easily pleased and thus give consistently high scores while another individual may be more conservative and give lower scores. This bias is represented symbolically by β_j, where $j = 1, 2, \ldots, r$ and identifies each of the r judges. Lastly, we recognize the fact that the judges are not perfect; thus the grades of each may have some error which is attributed to the individual judge. This error is represented by ε_{ij}, where i and j are as defined before. Now since the error term is attributed to the individual judge, variability in this error is taken as a measure of the consistency of the judge. We then take the variance of ϵ_{ij} to be σ_j^2. We can write down the mathematical model

2.1 $$x_{ij} = \mu_i + \beta_j + \epsilon_{ij}, \qquad \begin{array}{l} i = 1,2,\ldots,n \\ j = 1,2,\ldots,r, \end{array}$$

where x_{ij} = grade given the i^{th} item by the j^{th} panel member

μ_i = true mean of i^{th} item

β_j = bias of the j^{th} judge

ϵ_{ij} = random error, distributed normally with mean zero and variance σ_j^2.

Now

2.2 $$E(x_{ij}) = \mu_i + \beta_j$$

and

2.3 $$V(\epsilon_{ij}) = \sigma_j^2.$$

The model given by 2.1 embodies the concepts discussed above and represents them in mathematical form. We assume that, when we discuss the use of a scoring scale, the resultant scores are consistent with the assumptions of the model. Less difficulty will be experienced when x_{ij} represents a quantitative measurement.

The magnitude of the error variance, σ_j^2, is a measure of the ability of the j^{th} panel member to express the opinion of the panel and this is the quantity which is to be estimated.

Since it is necessary to use matrix notation later on in deriving the estimators of σ_j^2, let us now consider the mathematical representation of the experimental situation in matrix notation. Let X be an nr by one matrix* of the observed scores; thus for X' we have

2.4a $$X' = \left\| X_1' \quad X_2' \quad \cdots \quad X_j' \quad \cdots \quad X_r' \right\|,$$

where X_j' is itself an one by n row matrix given by

2.4b $$X_j' = \left\| x_{1j} \; x_{2j} \; \cdots \; x_{ij} \; \cdots \; x_{nj} \right\|.$$

If

2.5 $$B' = \left\| \mu_1 \; \mu_2 \cdots \mu_i \cdots \mu_n \quad \beta_1 \; \beta_2 \cdots \beta_j \cdots \beta_r \right\|,$$

a one by (n+r) matrix, then it is possible to write

2.6 $$E(X) = A\,B,$$

where

2.7a $$A' = \left\| A_1' \quad A_2' \quad \cdots \quad A_j' \quad \cdots \quad A_r' \right\|$$

is an (n+r) by nr matrix with matrix elements,

---

* The dimension of a matrix is denoted by the number of rows followed by the number of columns and primes are used to denote the transpose of a matrix.

2.7b $$A_j = \begin{Vmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{Vmatrix},$$

an n by n+r matrix with the unit elements occurring in the $(n+j)^{th}$ column.

If we let e_{ij} be the error by the j^{th} judge on the i^{th} item, and ξ' be a row vector of nr such elements corresponding to the row vector X', then the column vector of observations may be expressed in terms of the model 2.1 by equating

2.8 $$X = AB + \xi.$$

III. MAXIMUM LIKELIHOOD ESTIMATORS

3.1 Maximum Likelihood Equations Using the Joint Density Function of the Original Observations.

The main purpose of this research is to obtain estimators of the variances, σ_j^2 defined in the discussion of the model in Chapter II. When suitable estimators have been obtained, we shall consider the inferences that can be based on them and related to specified tests of hypotheses. The method of maximum likelihood is a general method that often yields good estimators [5] and we shall first consider this approach to the problem. In this chapter we shall see that we can use this method successfully to obtain estimators for three judges only, $r = 3$, and when $n \geq 2$.

The method of obtaining maximum likelihood estimators may usually be applied most easily by taking the logarithm of the joint density function of the observations and then equating the partial derivatives of this logarithm with respect to each of the parameters to be estimated to zero. Thus, conditions are provided for maximizing the function with respect to the set of parameters to be estimated.

The joint density function of the observations required is

3.1 $$f(x_{11},x_{21},\ldots,x_{n1},x_{12},x_{22},\ldots,x_{n2},\ldots,x_{1r},x_{2r},\ldots,x_{nr})=$$

$$(2\pi)^{-\frac{nr}{2}}\left(\prod_{j=1}^{r}\sigma_j^2\right)^{-\frac{n}{2}}\exp.-\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{r}\frac{(x_{ij}-\mu_i-\beta_j)^2}{\sigma_j^2}$$

in view of the statement of the model in Chapter II. The set of equations, with some simplification, obtained when the above steps are followed is

3.2a $$\sum_{j=1}^{r} \frac{(x_{ij}-\hat{\mu}_i-\hat{\beta}_j)}{\hat{\sigma}_j^2} = 0, \qquad i = 1,2,\ldots,n,$$

3.2b $$\sum_{i=1}^{n} (x_{ij}-\hat{\mu}_i-\hat{\beta}_j) = 0, \qquad j = 1,2,\ldots,r,$$

and

3.2c $$\sum_{i}^{n} \frac{(x_{ij}-\hat{\mu}_i-\hat{\beta}_j)^2}{\hat{\sigma}_j^2} = \hat{\sigma}_j^2, \qquad j = 1,2,\ldots,r.$$

The estimates obtained for the variance, say $\hat{\sigma}_t^2$, is

3.3 $$\hat{\sigma}_t^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_{it} - x_{.t} - \frac{\sum_{j=1}^{r}\frac{x_{ij}}{\hat{\sigma}_j^2} - \sum_{j=1}^{r}\frac{x_{.j}}{\hat{\sigma}_j^2}}{\sum_{j=1}^{r}\frac{1}{\hat{\sigma}_j^2}}\right)^2, t=1,2,\ldots,r,$$

where

$$x_{.j} = \frac{1}{n}\sum_{i=1}^{n} x_{ij} \qquad j = 1,2,\ldots,t,\ldots,r.$$

Although no explicit function was obtained for $\hat{\sigma}_t^2$, an iteration process was used in an attempt to evaluate $\hat{\sigma}_t^2$ in several numerical examples. The initial estimate of each variance for this iteration process was obtained by setting

3.4 $$\hat{\sigma}_t^2(1) = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_{it}-x_{i.}-x_{.t}+x_{..})^2, \quad t = 1,2,\ldots,r,$$

where

$$x_{i.} = \frac{1}{n} \sum_{j=1}^{r} x_{ij},$$

$$x_{..} = \frac{1}{nr} \sum_{i=1}^{n} \sum_{j=1}^{r} x_{ij},$$

and $x_{.j}$ is as defined following 3.3. These first approximations to $\hat{\sigma}_t^2$ were substituted in the right-hand member of 3.3 and a new second estimate was obtained for each $\hat{\sigma}_t^2$. This process was then repeated. After several iterations, the smallest initial variance estimate, say $\hat{\sigma}_p^2$ of all the estimates, $\hat{\sigma}_t^2$, $t = 1,2,\ldots,r$ approached zero. The effect of this is that

3.5 $$\lim_{\hat{\sigma}_p^2 \to 0} \frac{\sum_{j=1}^{r} \frac{x_{ij}}{\hat{\sigma}_j^2} - \sum_{j=1}^{r} \frac{x_{.j}}{\hat{\sigma}_j^2}}{\sum_{j=1}^{r} \frac{1}{\hat{\sigma}_j^2}} = x_{ip} - x_{.p}, \qquad 1 \le p \le r,$$

thus yielding, as estimates

3.6 $$\hat{\sigma}_t^2 = \frac{1}{n} \sum_{i=1}^{n} (x_{it} - x_{.t} - x_{ip} + x_{.p})^2, \qquad t = 1,2,\ldots,r.$$

But if we re-examine 3.2a, 3.2b and 3.2c, we find that, if any $\hat{\sigma}_t^2$ is taken to be zero, the other variance estimates have the form 3.6 and the equations are satisfied. The method of maximum likelihood employed here cannot then be considered successful.

3.2 Maximum Likelihood Equations Using a Linear Function of the Original Observations.

In this section we still use the principle of maximum likelihood estimation but a density function for certain linear functions of the observations is used instead of the joint density function of all of the observations. The purpose of selecting linear functions of the observations is to obtain a new set of variables, which are independent of the parameters μ_i and β_j. A set of (n-1)(r-1) linear contrasts may be selected in many ways so that the resultant linear functions are linearly independent. We now turn our attention to obtaining such a set of contrasts.

Let us define $\dot{C}'$ as a row vector of nr elements such that

$$E(\dot{C}'X) = \dot{C}'E(X) = \dot{C}'AB = 0 \tag{3.7}$$

where X, A, and B are defined in 2.4a, 2.5 and 2.7a respectively. $\dot{C}'AB = 0$ must be an identity in the B matrix, thus we require that

$$\dot{C}'A = 0. \tag{3.8}$$

Equation 3.8 is a homogeneous equation in nr unknown c's and the rank of A is n+r-1, thus, equation 3.8 has $nr-(n+r-1) = (n-1)(r-1)$ linearly independent solutions. Let us call these solutions C_1', C_2', ..., $C_{(n-1)(r-1)}'$ and define

3.9 $$C = \begin{Vmatrix} C'_1 \\ C'_2 \\ \cdot \\ \cdot \\ \cdot \\ C'_{(n-1)(r-1)} \end{Vmatrix}$$

We now define Z, the set of linearly independent linear functions of the observation such that

3.10 $$Z = CX$$

and

3.11 $$E(Z) = 0.$$

Of the many possible matrices C with which we can define contrasts like those of Z, we choose

3.12a $\quad C =$

$$
\left\| \begin{matrix}
-1 & 1 & 0 & \cdots & 0 & 1 & -1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 & 1 & 0 & -1 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
-1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & \cdots & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
-1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
-1 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
-1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & -1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
-1 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & 0 & \cdots & -1
\end{matrix} \right\|
$$

an $(n-1)(r-1)$ by nr matrix. This may be rewritten as

3.12b

$$
C = \left\| \begin{matrix}
-D & D & 0 & 0 & \cdots & 0 \\
-D & 0 & D & 0 & \cdots & 0 \\
-D & 0 & 0 & D & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
-D & 0 & 0 & 0 & \cdots & D
\end{matrix} \right\|
$$

an $(r-1)$ by r matrix, where

3.13
$$D = \begin{Vmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & -1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ 1 & 0 & 0 & 0 & 0 & \cdots & -1 \end{Vmatrix}$$

an n-1 by n matrix.

It is easily proved that C satisfies the imposed conditions (i) $CA = 0$ and (ii) the functions formed from CX are linearly independent. The first condition is equivalent to

$$-D(A_j - A_h) = 0 \qquad j \neq h,\ j,h = 1,2,\ldots,r,$$

where

$$A_j - A_h = \begin{Vmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \end{Vmatrix}$$

where the positive ones are in the $n+j^{th}$ column and the negative ones are in the $n+h^{th}$ column. Each element of the product $-D(A_j - A_h)$ is clearly zero. The second condition implies that the rank of C must be $(n-1)(r-1)$ which can be seen by inspection of C and D.

In order for us to express the joint density function of these new variables, the elements of Z, we must obtain the variances and covariances of these variables. The elements of Z are linear functions of the original observations which, for the model used in this research, were assumed to be independently and normally distributed with the variance of x_{ij} equal to σ_j^2. Thus, by the theorem, if W' is a vector of n random variables with variance-covariance matrix Σ_w and if $Y = KW$, then the variance-covariance matrix Σ_y is given by $\Sigma_y = K'\Sigma_w K$,

3.14 $$\Sigma_z = C'\Sigma_x C$$

where

3.15 $$\Sigma_x = \left\| \begin{array}{ccccccccccccc} \sigma_1^2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_1^2 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_1^2 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \sigma_2^2 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \sigma_2^2 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \sigma_2^2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \sigma_r^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \sigma_r^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & \sigma_r^2 \end{array} \right\|$$

an nr square matrix, and C is as defined in 3.12a. Then we obtain

3.16 $\Sigma_z =$

$$
\begin{Vmatrix}
2(\sigma_1^2+\sigma_2^2) & \sigma_1^2+\sigma_2^2 & \cdots & \sigma_1^2+\sigma_2^2 & 2\sigma_1^2 & \sigma_1^2 & \cdots & \sigma_1^2 & \cdots & 2\sigma_1^2 & \sigma_1^2 & \cdots & \sigma_1^2 \\
\sigma_1^2+\sigma_2^2 & 2(\sigma_1^2+\sigma_2^2) & \cdots & \sigma_1^2+\sigma_2^2 & \sigma_1^2 & 2\sigma_1^2 & \cdots & \sigma_1^2 & \cdots & \sigma_1^2 & 2\sigma_1^2 & \cdots & \sigma_1^2 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\sigma_1^2+\sigma_2^2 & \sigma_1^2+\sigma_2^2 & \cdots & 2(\sigma_1^2+\sigma_2^2) & \sigma_1^2 & \sigma_1^2 & \cdots & 2\sigma_1^2 & \cdots & \sigma_1^2 & \sigma_1^2 & \cdots & 2\sigma_1^2 \\
2\sigma_1^2 & \sigma_1^2 & \cdots & \sigma_1^2 & 2(\sigma_1^2+\sigma_3^2) & \sigma_1^2+\sigma_3^2 & \cdots & \sigma_1^2+\sigma_3^2 & \cdots & 2\sigma_1^2 & \sigma_1^2 & \cdots & \sigma_1^2 \\
\sigma_1^2 & 2\sigma_1^2 & \cdots & \sigma_1^2 & \sigma_1^2+\sigma_3^2 & 2(\sigma_1^2+\sigma_3^2) & \cdots & \sigma_1^2+\sigma_3^2 & \cdots & \sigma_1^2 & 2\sigma_1^2 & \cdots & \sigma_1^2 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\sigma_1^2 & \sigma_1^2 & \cdots & 2\sigma_1^2 & \sigma_1^2+\sigma_3^2 & \sigma_1^2+\sigma_3^2 & \cdots & 2(\sigma_1^2+\sigma_3^2) & \cdots & \sigma_1^2 & \sigma_1^2 & \cdots & 2\sigma_1^2 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
2\sigma_1^2 & \sigma_1^2 & \cdots & \sigma_1^2 & 2\sigma_1^2 & \sigma_1^2 & \cdots & \sigma_1^2 & \cdots & 2(\sigma_1^2+\sigma_r^2) & \sigma_1^2+\sigma_r^2 & \cdots & \sigma_1^2+\sigma_r^2 \\
\sigma_1^2 & 2\sigma_1^2 & \cdots & \sigma_1^2 & \sigma_1^2 & 2\sigma_1^2 & \cdots & \sigma_1^2 & \cdots & \sigma_1^2+\sigma_r^2 & 2(\sigma_1^2+\sigma_r^2) & \cdots & \sigma_1^2+\sigma_r^2 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\sigma_1^2 & \sigma_1^2 & \cdots & 2\sigma_1^2 & \sigma_1^2 & \sigma_1^2 & \cdots & 2\sigma_1^2 & \cdots & \sigma_1^2+\sigma_r^2 & \sigma_1^2+\sigma_r^2 & \cdots & 2(\sigma_1^2+\sigma_r^2)
\end{Vmatrix},
$$

an $(n-1)(r-1)$ square matrix.

We can simplify 3.16 to

3.17 $$\Sigma_z = DD' \cdot H,$$

where D is as defined in 3.13; thus, DD' is

3.18 $$DD' = \begin{Vmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{Vmatrix} ,$$

an (n-1) square matrix, and

3.19 $$H = \begin{Vmatrix} \sigma_1^2+\sigma_2^2 & \sigma_1^2 & \cdots & \sigma_1^2 \\ \sigma_1^2 & \sigma_1^2+\sigma_3^2 & \cdots & \sigma_1^2 \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_1^2 & \sigma_1^2 & \cdots & \sigma_1^2+\sigma_r^2 \end{Vmatrix}$$

an (r-1) square matrix. The notation used in 3.17 implies that each element in H acts as a scalar multiplying DD' and yields a matrix which is a scalar times DD' to replace each element of H. We shall define this operation as the "dot product" of two matrices . Thus, the joint density function of the linear functions of the observations which weshall use to obtain the maximum likelihood estimators is

3.20 $$f\ z_1, z_2, \ldots, z_{(n-1)(r-1)} = (2\pi)^{-\frac{1}{2}(n-1)(r-1)} \left|\Sigma_z^{-1}\right|^{\frac{1}{2}} \exp -\tfrac{1}{2}\, Z'\Sigma_z^{-1}Z$$

where Z is defined in 3.10, Σ_z^{-1} is the inverse of Σ_z and Σ_z^{-1} is the determinant of Σ_z^{-1}.

The inverse of Σ_z defined in 3.17 is

$$\Sigma_z^{-1} = (DD')^{-1} \cdot H^{-1}$$

where

3.21 $$(DD')^{-1} = \frac{1}{n} \begin{Vmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{Vmatrix},$$

and

$$\begin{Vmatrix} \frac{(c\sigma_2^2-\sigma_1^2)}{\sigma_2^4} & \frac{-\sigma_1^2}{\sigma_2^2\sigma_3^2} & \cdots & \frac{-\sigma_1^2}{\sigma_2^2\sigma_r^2} \\ \frac{-\sigma_1^2}{\sigma_2^2\sigma_3^2} & \frac{(c\sigma_3^2-\sigma_1^2)}{\sigma_3^4} & \cdots & \frac{-\sigma_1^2}{\sigma_3^2\sigma_r^2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{-\sigma_1^2}{\sigma_2^2\sigma_r^2} & \frac{-\sigma_1^2}{\sigma_3^2\sigma_r^2} & \cdots & \frac{(c\sigma_r^2-\sigma_1^2)}{\sigma_r^4} \end{Vmatrix}$$

where

3.23 $$c = \sigma_1^2 \sum_{j=1}^{r} \frac{1}{\sigma_j^2}.$$

$(DD')^{-1}$ follows simply from the definition of (DD') and is easily verified. H^{-1} requires more algebraic derivation but that 3.22 is correct is again easily verified by showing

$HH^{-1} = I$. We shall need the result that

3.24 $$|H| = c \prod_{j=2}^{r} \sigma_j^2.$$

The logarithm of the function in 3.20 is

3.25 $$L = k + \tfrac{1}{2} \ln \left|\Sigma_z^{-1}\right| - \tfrac{1}{2} Z'\Sigma_z^{-1} Z$$

where k is a constant independent of the elements of Σ_z^{-1}. Following the method outlined at the beginning of this chapter, the right-hand member of 3.25 is partially differentiated with respect to the r different variances σ_j^2 (which are the only parameters in 3.25). Actually, instead of partially differentiating 3.25 with respect to σ_j^2, we partially differentiated with respect to ω_j where

3.26 $$\omega_j = \frac{\sigma_j^2}{|H|}, \qquad j = 1,2,\ldots,r,$$

and $|H|$ is the determinant of H in 3.24. This is permissible by the theorem, if $\hat{\Theta}$ is the vector of maximum-likelihood estimators for the elements of Θ, and if $u(\Theta)$ is any vector of the function of Θ, such that a unique solution for Θ may be obtained from $u(\Theta) = c$, c a vector of constants, then $u(\hat{\Theta})$ is the maximum-likelihood estimator for $u(\Theta)$.

Thus

3.27 $$\frac{L}{\omega_j} = \frac{1}{2\left|\Sigma_z^{-1}\right|} \frac{\partial \Sigma_z^{-1}}{\partial \omega_j} - \tfrac{1}{2} Z' \left\| \frac{\partial \Sigma_z^{-1}}{\partial \omega_j} \right\| Z .$$

Let

3.28 $$\left\| \frac{\partial \Sigma_z^{-1}}{\partial \omega_j} \right\| = \left\| f_{jsu} \right\| , \qquad s,u = 1,2,\ldots,(n-1)(r-1),$$

where

3.29 $$f_{jsu} = \frac{\partial \sigma^{su}}{\partial \omega_j}$$

and

σ^{su} = the element in the s^{th} row and u^{th} column of Σ_z^{-1}.

Thus, the last expression on the right of 3.27 is simplified to

3.30 $$\tfrac{1}{2} Z' \left\| \frac{\partial \Sigma_z^{-1}}{\partial \omega_j} \right\| Z = \tfrac{1}{2} Z' \left\| f_{jsu} \right\| Z = \tfrac{1}{2} \sum_s \sum_u z_s z_u f_{jsu} \ .$$

To complete the reduction of the right-hand member of 3.27, we need only note, using the usual rule for differentiating a determinant, that

3.31 $$\frac{\partial \left| \Sigma_z^{-1} \right|}{\partial \omega_j} = \sum_s \sum_u \left(\frac{\partial \sigma^{su}}{\partial \omega_j} \right) (\text{cofactor of } \sigma^{su})$$

and

3.32 $$\frac{1}{\left| \Sigma_z^{-1} \right|} \frac{\partial \left| \Sigma_z^{-1} \right|}{\partial \omega_j} = \sum_s \sum_u f_{jsu} \, \sigma_{su}$$

since $\left(\frac{1}{\left| \Sigma_z^{-1} \right|} \right) (\text{cofactor of } \sigma^{su}) = \sigma_{su}$ and $f_{jsu} = \frac{\partial \sigma^{su}}{\partial \omega_j}$ by definition. The results of 3.30 and 3.32 are substituted in 3.27 and then these results equated to zero to obtain

3.33a $$\sum_s \sum_u f_{jsu} \left(\sigma_{su} - z_s z_u \right) = 0,$$

or

3.33b $$\mathrm{tr}\left[F_j(\Sigma_z - ZZ')\right] = 0,$$

where tr indicates the trace which is defined as the sum of the principal diagonal terms,

and $F_j = \left\|f_{jsu}\right\|$.

It is to be noted that the solution of 3.33a or 3.33b yields the maximum-likelihood estimators of the σ_j^2 and they will be designated by $\hat{\sigma}_j^2$.

Since Σ_z^{-1} is a dot product of $(DD')^{-1}$, a matrix which does not contain the parameters, and H^{-1}, a matrix involving σ_j^2, then we need only to differentiate H^{-1} with respect to ω_j; then multiple by $(DD')^{-1}$ to obtain F_j.

Although 3.33a and 3.33b yield maximum-likelihood estimators for $n \geq 2$, $r \geq 3$, we have been unable to solve these equations for $\hat{\sigma}_j^2$ except for $r = 3$. The inherent difficulty in the problem is illustrated by examination of the equations for $r = 4$ and $n = 3$ now written:

3.34a $$6(\hat{\sigma}_2^2\hat{\sigma}_3^2+\hat{\sigma}_2^2\hat{\sigma}_4^2+\hat{\sigma}_3^2\hat{\sigma}_4^2) = (k_2+k_3+k_4)\hat{\sigma}_2^2 + (k_1+k_3+k_5)\hat{\sigma}_3^2 + (k_1+k_2+k_6)\hat{\sigma}_4^2$$

3.34b $$6(\hat{\sigma}_1^2\hat{\sigma}_3^2+\hat{\sigma}_1^2\hat{\sigma}_4^2+\hat{\sigma}_3^2\hat{\sigma}_4^2) = (k_2+k_3+k_4)\hat{\sigma}_1^2 + k_3\hat{\sigma}_3^2 + k_2\hat{\sigma}_4^2$$

3.34c $$6(\hat{\sigma}_1^2\hat{\sigma}_2^2+\hat{\sigma}_1^2\hat{\sigma}_4^2+\hat{\sigma}_2^2\hat{\sigma}_4^2) = (k_1+k_3+k_5)\hat{\sigma}_1^2 + k_3\hat{\sigma}_2^2 + k_1\hat{\sigma}_4^2$$

3.34d $$6(\hat{\sigma}_1^2\hat{\sigma}_2^2+\hat{\sigma}_1^2\hat{\sigma}_3^2+\hat{\sigma}_2^2\hat{\sigma}_3^2) = (k_1+k_2+k_6)\hat{\sigma}_1^2 + k_2\hat{\sigma}_2^2 + k_1\hat{\sigma}_3^2$$

where

$$k_1 = z_1^2 - z_1z_2 + z_2^2$$

$$k_2 = z_3^2 - z_3z_4 + z_4^2$$

$$k_3 = z_5^2 - z_5z_6 + z_6^2$$

$$k_4 = -2z_3z_5 + z_3z_6 + z_4z_5 - 2z_4z_6$$

$$k_5 = -2z_1z_5 + z_1z_6 + z_2z_5 - 2z_2z_6$$

$$k_6 = -2z_1z_3 + z_1z_4 + z_2z_3 - 2z_2z_4$$

Inspection of these equations revelas that one trivial solution is $\hat{\sigma}_j^2 = 0$ for all j, j = 1,2,3,4, but this is not helpful. In general, the degree of the terms on the left in 3.34a-3.34d is (r-2) in $\hat{\sigma}_j^2$ and on the right, (r-3). We now turn to the special case wherein solution is possible.

3.3 Maximum-Likelihood Estimators for $n \geq 2$ and $r = 3$.

It was observed in the previous section that Σ_z^{-1} could be factored into a dot product of $(DD')^{-1}$ and H^{-1} and that $(DD')^{-1}$ does not involve any of the parameters, σ_j^2. Thus, in order to obtain the maximum-likelihood estimators of σ_j^2 for the three judges and with n items, $n \geq 2$, we need only differentiate H^{-1} with respect to ω_j where ω_j is defined in 3.26 and then multiply by $(DD')^{-1}$ to obtain F_j which is defined in 3.33b.

For three judges, r = 3,

3.35 $$H^{-1} = \frac{1}{H}\begin{Vmatrix} \sigma_1^2+\sigma_3^2 & -\sigma_1^2 \\ -\sigma_1^2 & \sigma_1^2+\sigma_2^2 \end{Vmatrix},$$

3.36 $$\frac{\partial H^{-1}}{\partial \frac{\sigma_1^2}{H}} = \begin{Vmatrix} 1 & -1 \\ -1 & 1 \end{Vmatrix},$$

3.37 $$\frac{\partial H^{-1}}{\partial \frac{\sigma_2^2}{H}} = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix},$$

and

3.38 $$\frac{\partial H^{-1}}{\partial \frac{\sigma_3^2}{H}} = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}.$$

F_1, F_2 and F_3 are obtained by dot multiplying 3.36, 3.37 and 3.38 by $(DD')^{-1}$ and the required equations are obtained through evaluation of 3.33b which results in

3.39 $$n(n-1)(\hat{\sigma}_2^2+\hat{\sigma}_3^2) = Z'F_1Z,$$

3.40 $$n(n-1)(\hat{\sigma}_1^2+\hat{\sigma}_3^2) = Z'F_2Z,$$

and

3.41 $$n(n-1)(\hat{\sigma}_1^2+\hat{\sigma}_2^2) = Z'F_3Z.$$

These are solve simultaneously to give

3.42 $$2n(n-1)\hat{\sigma}_1^2 = Z'(F_2+F_3-F_1)Z,$$

3.43 $$2n(n-1)\hat{\sigma}_2^2 = Z'(F_1+F_3-F_2)Z,$$

and

3.44 $$2n(n-1)\hat{\sigma}_3^2 = Z'(F_1+F_2-F_3)Z,$$

where

3.45 $$F_3+F_2-F_1 = \begin{Vmatrix} 0 & (DD')^{-1} \\ (DD')^{-1} & 0 \end{Vmatrix},$$

3.46 $$F_1+F_3-F_2 = \begin{Vmatrix} 2(DD')^{-1} & -(DD')^{-1} \\ -(DD')^{-1} & 0 \end{Vmatrix},$$

and

3.47 $$F_1+F_2-F_3 = \begin{Vmatrix} 0 & -(DD')^{-1} \\ -(DD')^{-1} & 2(DD')^{-1} \end{Vmatrix}$$

We shall not solve these equations, 3.38, 3.39 and 3.40, explicitly for σ_j^2, j=1,2,3 in terms of the original observations until later, since we shall obtain estimators by another method in Chapter IV with which we shall compare the estimators obtained in this section.

IV. QUADRATIC FORM ESTIMATORS

4.1 Derivation of the Quadratic Form Estimators

In the previous chapter we obtained, with limited success, estimators by the principle of maximum likelihood. In this chapter we shall use a general quadratic form on which certain desirable conditions are imposed. The selection of the restrictions follows somewhat naturally from the properties desired for good estimators.

We shall first define Q to be a quadratic form in the original observations and then impose our restrictions which will permit determination of the coefficients of the quadratic form. In line with this method of procedure we can write the general quadratic

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{r} \sum_{k=1}^{n} \sum_{h=1}^{r} m_{ijkh}\, x_{ij}\, x_{kh} \tag{4.1a}$$

where the x's are the observations and the m's are the constants to be determined by the conditions to be imposed.

There are three reasonable and obvious conditions that the quadratic form, Q, must satisfy in order to yield a good estimator. These condtions are:

i. The quadratic form must be invariant when the items are interchanged.

ii. The estimator obtained from the quadratic form must be free of the nuisance parameters μ_i and β_j.

iii. The quadratic form must provide an unbiased estimate of of σ_t^2, $t=1,2,\ldots,r$, when σ_t^2 is the paramter being estimated, that is $E(Q_t) = \sigma_t^2$.

Let us consider the first of these conditions by interchanging the α^{th} item with the β^{th} item. It is possible to expand 4.1a to give

4.1b
$$Q = \sum_{\substack{i=1 \\ i,k \neq \alpha,\beta}}^{n} \sum_{j=1}^{r} \sum_{k=1}^{n} \sum_{h=1}^{r} m_{ijkh}\, x_{ij}\, x_{kh} +$$

$$\sum_{j=1}^{r} \sum_{h=1}^{r} m_{\alpha j\alpha h}\, x_{\alpha j}\, x_{\alpha h} + \sum_{j=1}^{r} \sum_{h=1}^{r} m_{\beta j\beta h}\, x_{\beta j}\, x_{\beta h} +$$

$$\sum_{j=1}^{r} \sum_{h=1}^{r} m_{\alpha j\beta h}\, x_{\alpha j}\, x_{\beta h} + \sum_{j=1}^{r} \sum_{h=1}^{r} m_{\beta j\alpha h}\, x_{\beta j}\, x_{\alpha h} .$$

Let Q' be the quadratic form when the β^{th} item is interchanged with the α^{th} item, that is,

4.2
$$Q' = \sum_{\substack{i=1 \\ i,k \neq \alpha,\beta}}^{n} \sum_{j=1}^{r} \sum_{k=1}^{n} \sum_{h=1}^{r} m_{ijkh}\, x_{ij}\, x_{kh} +$$

$$\sum_{j=1}^{r} \sum_{h=1}^{r} m_{\beta j\beta h}\, x_{\alpha j}\, x_{\alpha h} + \sum_{j=1}^{r} \sum_{h=1}^{r} m_{\alpha j\alpha h}\, x_{\beta j}\, x_{\beta h} +$$

$$\sum_{j=1}^{r} \sum_{h=1}^{r} m_{\alpha j\beta h}\, x_{\beta j}\, x_{\alpha h} + \sum_{j=1}^{r} \sum_{h=1}^{r} m_{\beta j\alpha h}\, x_{\alpha j}\, x_{\beta h} .$$

For the quadratic form to be invariant with respect to an interchange of the items, then

4.3
$$Q = Q' .$$

When 4.1b is equated to 4.2, the resultant equation is an identity in the x's, and the corresponding coefficients must be equal. Thus

$$4.4 \qquad m_{\alpha j\alpha h} = m_{\beta j\beta h} \qquad \begin{matrix} \alpha,\beta = 1,2,\ldots,n \\ j,h = 1,2,\ldots,r \end{matrix}$$

and

$$4.5 \qquad m_{\alpha j\beta h} = m_{\beta j\alpha h} \qquad \begin{matrix} \alpha,\beta = 1,2,\ldots,n \\ j,h = 1,2,\ldots,r\ . \end{matrix}$$

In order to understand what this means, consider M a a symmetric nr by nr matrix such that

$$4.6 \qquad M = \begin{Vmatrix} M_{11} & M_{12} & \cdots & M_{1h} & \cdots & M_{1r} \\ M_{21} & M_{22} & \cdots & M_{2h} & \cdots & M_{2r} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ M_{j1} & M_{j2} & \cdots & M_{jh} & \cdots & M_{jr} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ M_{r1} & M_{r2} & \cdots & M_{rh} & \cdots & M_{rr} \end{Vmatrix} ,$$

and r square matrix in terms of n square matrices M_{jh}, where

$$4.7 \qquad M_{jh} = \begin{Vmatrix} m_{1j1h} & m_{1j2h} & \cdots & m_{1jkh} & \cdots & m_{1jnh} \\ m_{2j1h} & m_{2j2h} & \cdots & m_{2jkh} & \cdots & m_{2jnh} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ m_{ij1h} & m_{ij2h} & \cdots & m_{ijkh} & \cdots & m_{ijnh} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ m_{nj1h} & m_{nj2h} & \cdots & m_{njkh} & \cdots & m_{njnh} \end{Vmatrix} , \qquad \begin{matrix} j,h=1,2,\ldots,r, \\ i,k=1,2,\ldots,n. \end{matrix}$$

Equation 4.4 states that the terms along the principal diagonal of any M_{jh} are equal and 4.5 states that M_{jh} is symmetric.

To impose the condition that the estimator obtained from the quadratic form is independent of μ_i and β_j, we need only to make use of the transformation we use in Chapter III where we obtained certain linear functions of the original observations. The set of linear functions, Z, defined in 3.10 is independent of μ_i and β_j, thus, one way of proceeding is to write our quadratic form as

$$Q'' = Z'PZ, \tag{4.8}$$

where P is an $(n-1)(r-1)$ square matrix and Q'' is now a quadratic function which is independent of μ_i and β_j.

Equation 3.10 is substituted in the right-hand member of 4.8 to obtain

$$Q'' = X'C'PCX. \tag{4.9}$$

In order to see what effect this has on Q as defined in 4.1a, which is matrix notation is

$$Q = X'MX, \tag{4.10}$$

where M is as defined in 4.6, the right-hand members of 4.9 and 4.10 are equated to obtain

4.11 $$X'MX = X'C'PCX.$$

The identity of 4.11 is equivalent to the matrix equation,

4.12 $$M = C'PC.$$

If 4.12 is post multiplied by A, defined in 2.7a,

4.13 $$MA = 0$$

since CA = 0. We can expand 4.13 in terms of submatrices to give

4.14 $$\sum_{j=1}^{r} M_{hj}A_j = \sum_{j=1}^{r} M_{jh}\, A_j = 0,$$

where M_{jh} is defined in 4.7 and 0 is an nr by (n+r) matrix whose elements are zero. Equation 4.14 may be written in terms of its elements as

4.15 $$A_j = \begin{Vmatrix} a_{j11} & a_{j12} & \cdots & a_{j1,n+j} & \cdots & a_{j1,n+r} \\ a_{j21} & a_{j22} & \cdots & a_{j2,n+j} & \cdots & a_{j2,n+r} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{jn1} & a_{jn2} & \cdots & a_{jn,n+j} & \cdots & a_{jn,n+r} \end{Vmatrix}$$

where

4.16 $$a_{jip} = 1 \text{ when } p = i \text{ or } n+j, \qquad p = 1,2,\ldots,n+r,$$
$$= 0 \text{ otherwise.}$$

Upon expanding 4.14 in terms of products of the elements of

M_{jh} and A_j, we find that the elements of the matrix of 4.14 are

4.17 $$\sum_{i=1}^{n} \sum_{j=1}^{r} m_{khij}\, a_{jip} = 0, \qquad \begin{array}{l} k = 1,2,\ldots,n, \\ h = 1,2,\ldots,r, \\ p = 1,2,\ldots,n+r. \end{array}$$

When we let $i = p$ in 4.17, we obtain

4.18 $$\sum_{j=1}^{r} m_{khpj} = 0, \qquad \begin{array}{l} h = 1,2,\ldots,r, \\ k,p = 1,2,\ldots,n, \end{array}$$

since $a_{jpp} = 1$ for these terms and zero for all the others by 4.16. If we let $j = p-n$, $p > n$, in 4.17, we obtain

4.19 $$\sum_{i=1}^{n} m_{khi,p-n} = 0, \qquad k = 1,2,\ldots,n$$

or

$$\sum_{i=1}^{n} m_{khij} = 0, \qquad \begin{array}{l} k = 1,2,\ldots,n, \\ h,j = 1,2,\ldots,r. \end{array}$$

Equation 4.18 states that the sum of the corresponding elements from each M_{jh} (or M_{hj}) in any column or row of M is zero and 4.19 states that the sum of any row (or column) of M_{hj} is zero.

We now consider the unbiasedness of the quadratic form to be obtained for each of the estimators that are required with the form of Q. The expected value of Q before any of the conditions are applied is

4.20 $$E(Q) = \sum_{i=1}^{n} \sum_{j=1}^{r} m_{ijij}\, \sigma_j^2 + \sum_{i=1}^{n} \sum_{j=1}^{r} \sum_{k=1}^{n} \sum_{h=1}^{r} m_{ijkh} \cdot (\mu_i\mu_k + \mu_i\beta_h + \mu_k\beta_j + \beta_j\beta_h) .$$

When equations 4.18 and 4.19 are substituted in 4.20 above,

4.21 $$E(Q) = \sum_{i=1}^{n} \sum_{j=1}^{r} m_{ijij}\, \sigma_j^2$$

is obtained. If the quadratic form is to be unbiased, then, when σ_t^2, $t = 1,2,\ldots,r$, is being estimated, we should have

4.22 $$E(Q_t) = \sigma_t^2$$

Thus

4.23 $$\sum_{i=1}^{n} m_{iqiq} = 1 \text{ when } q = t, \quad = 0 \text{ otherwise}, \qquad q = 1,2,\ldots,r.$$

To recapitulate, then

(i) for invariance under interchange of items,

4.24a $$m_{ihij} = m_{khkj}, \qquad i,k = 1,2,\ldots,n, \quad h,j = 1,2,\ldots,r,$$

and

4.24b $$m_{ihkj} = m_{khij};$$

(ii) for independence of μ_i and β_j,

4.25a $$\sum_{j=1}^{r} m_{khij} = 0, \qquad i,k = 1,2,\ldots,n, \quad j,h = 1,2,\ldots,r,$$

and

4.25b $$\sum_{i=1}^{n} m_{khij} = 0;$$

and

(iii) for unbiasedness when estimating σ_t^2, 4.26a $$\sum_{i=1}^{n} m_{itit} = 1, \qquad t = 1,2,\ldots,r,$$

and

4.26b $$\sum_{i=1}^{n} m_{iqiq} = 0, \qquad q \neq t, q = 1,2,\ldots,r.$$

Let us consider the quadratic form for estimating σ_t^2. If we substitute equation 4.24a in 4.26a, we obtain

4.27a $$m_{itit} = 1/n \qquad i = 1,2,\ldots,n.$$

When 4.24a is substituted in 4.26b,

4.27b $$m_{iqiq} = 0, \qquad q \neq t, q = 1,2,\ldots,r.$$

Equation 4.27a states that the elements of the principal diagonal of M_{tt} are each 1/n and 4.27b states that the elements on the principal diagonal of M_{qq}, $p \neq t$, are each zero.

If we substitute 4.27a in 4.25a wherein we take k =i, h = t, and q = j,

4.28 $$\sum_{\substack{q=1 \\ q\neq t}}^{r} M_{itiq} = -1/n \qquad i = 1,2,\ldots,n.$$

This states that the sum of the corresponding elements of the principal diagonal of each M_{tq}, $q \neq t$, is - 1/n.

In a similar manner, after substituting 4.27a in 4.25b,

4.29 $$\sum_{\substack{v=1 \\ v\neq i}}^{n} m_{itvt} = - 1/n \qquad i = 1,2,\ldots,n,$$

is obtained. This states that the sum of the elements off the principal diagonal of any row (or any column, since M_{tt} is symmetric) of M_{tt} is $-1/n$.

If the additional assumption that certain elements of M are equal is made, then the determination of M is unique. To elaborate on this statement, consider M as composed of elements which determine weights given to the products of the observations; that is, m_{ijkh} is the weight given $x_{ij}x_{kh}$. We are estimating the abilities of the judges to give good subjective scores and, a priori, we do not know the nature of any possible differences in the judges' abilities. Thus, rather than use unknown weights which should in fact depend on the σ_j^2 themselves, we assume that it is reasonable to give equal weights to all the other judges when σ_t^2 is being estimated. Thus, we should have certain of the elements of M equal.

When we estimate σ_t^2, we set

i. $M_{pt} = M_{qt}$, $p,q \neq t$, $p,q = 1,2,\ldots,r$.

ii. $M_{pq} = M_{uv}$, $p \neq q$, $u \neq v$, $p,q,u,v \neq t$, $p,q,u,v = 1,2,\ldots,r$.

iii. $M_{pp} = M_{qq}$, $p,q \neq t$, $p,q = 1,2,\ldots,r$.

iv. $m_{ijkh} = m_{fjgh}$ $i \neq k$, $f \neq g$, $f,g,i,k = 1,2,\ldots,n$, $j,h = 1,2,\ldots,r$.

These restrictions define the equal weighting system discussed.

Let us consider the determination of M_{tt} first. The elements along the principal diagonal have already been obtained (4.27a). When we make the assumptions that certain sets of elements of M are equal, then, using 4.29

4.30 $$m_{itkt} = -1/n(n-1) \qquad i \neq k,\ i,k=1,2,\ldots,n.$$

This completes all the elements in M_{tt} to give

4.31 $$M_{tt} = \begin{Vmatrix} 1/n & -1/n(n-1) & \cdots & -1/n(n-1) \\ -1/n(n-1) & 1/n & \cdots & -1/n(n-1) \\ \vdots & \vdots & \cdots & \vdots \\ -1/n(n-1) & -1/n(n-1) & \cdots & 1/n \end{Vmatrix}$$

Next we consider M_{qq}, $q=1,2,\ldots,r$, $q \neq t$.

The elements along the principal diagonal of M_{qq} are zero by 4.27b. When 4.26b and the assumption that certain elements of M are equal are used, then

4.32 $$m_{iqkq} = 0 \qquad i \neq k,\ i,k=1,2,\ldots,n.$$

Thus for M_{qq}, we have

4.33 $$M_{qq} = \begin{Vmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{Vmatrix}, \qquad q=1,2,\ldots,r,\ q \neq t.$$

When we consider M_{tq}, $q \neq t$, we again use the assumption of equality of certain elements of M and 4.28. From these we obtain

4.34 $$m_{iqit} = -1/n(r-1), \qquad q \neq t,\ q=1,2,\ldots,r,\ i=1,2,\ldots,n.$$

Then we use 4.25b, 4.34, and our last assumtion of equality of certain elements to obtain

4.35 $$m_{iqkt} = 1/n(n-1)(r-1), \qquad q \neq t,\ q=1,2,\ldots,r,\ i \neq k;\ i,k=1,2,\ldots,n.$$

This completes M_{tq} to give

4.36 $$M_{tq} = \begin{Vmatrix} -1/n(r-1) & 1/n(n-1)(r-1) & \ldots & 1/n(n-1)(r-1) \\ 1/n(n-1)(r-1) & -1/n(r-1) & \ldots & 1/n(n-1)(r-1) \\ \vdots & \vdots & \ldots & \vdots \\ 1/n(n-1)(r-1) & 1/n(n-1)(r-1) & \ldots & -1/n(r-1) \end{Vmatrix}, \qquad q=1,2,\ldots,r,\ q \neq t.$$

The last matrix in M to be obtained is M_{pq}, $p \neq q$; p, $q \neq t$. Equation 4.34 is substituted in 4.25a and then applying 4.24a as well as equality of certain elements of M, we have

4.37 $$m_{iqip} = 1/n(r-1)(r-2), \qquad i=1,2,\ldots,n,\ p,q=1,2,\ldots,r,\ p \neq q.$$

Again we use the equality of certain elements of M, 4.25b, and 4.37 to obtain

4.38 $$m_{iqkp} \; -1/n(n-1)(r-1)(r-2) \qquad \begin{array}{l} i,k=1,2,\ldots,n, \\ i\neq k \\ p,q=1,2,\ldots,r, \\ p,q\neq t, \\ q\neq p. \end{array}$$

Thus

4.39 $$M_{pq}=\left\| \begin{array}{cccc} \frac{1}{n(r-1)(r-2)} & -\frac{1}{n(n-1)(r-1)(r-2)} & \cdots & -\frac{1}{n(n-1)(r-1)(r-2)} \\ -\frac{1}{n(n-1)(r-1)(r-2)} & \frac{1}{n(r-1)(r-2)} & \cdots & -\frac{1}{n(n-1)(r-1)(r-2)} \\ \vdots & \vdots & \cdots & \vdots \\ -\frac{1}{n(n-1)(r-1)(r-2)} & -\frac{1}{n(n-1)(r-1)(r-2)} & \cdots & \frac{1}{n(n-1)(r-1)(r-2)} \end{array} \right\|$$

$$\begin{array}{l} p,q=1,2,\ldots,r, \\ q\neq p, \\ p,q\neq t. \end{array}$$

To simplify the notation, let us obtain the quadratic form for $(n)(n-1)(r-1)(r-2)\;Q_t$ and let the coefficients of that quadratic form be denoted by $M^{(t)}$. Thus

4.40 $$n(n-1)(r-1)(r-2)Q_t = X'M^{(t)}X$$

where

4.41a $$M^{(t)} = \left\| \begin{array}{cccc} M^{(t)}_{11} & M^{(t)}_{12} & \cdots & M^{(t)}_{1r} \\ M^{(t)}_{21} & M^{(t)}_{22} & \cdots & M^{(t)}_{2r} \\ \vdots & \vdots & \cdots & \vdots \\ M^{(t)}_{r1} & M^{(t)}_{r2} & \cdots & M^{(t)}_{rr} \end{array} \right\|, \quad t=1,2,\ldots,r,$$

with

4.41b $$M^{(t)}_{tt}= \left\| \begin{array}{cccc} (n-1)(r-1)(r-2) & -(r-1)(r-2) & \cdots & -(r-1)(r-2) \\ -(r-1)(r-2) & (n-1)(r-1)(r-2) & \cdots & -(r-1)(r-2) \\ \vdots & \vdots & \cdots & \vdots \\ -(r-1)(r-2) & -(r-1)(r-2) & \cdots & (n-1)(r-1)(r-2) \end{array} \right\|,$$

4.41c

$$M_{tp}^{(t)} = M_{pt}^{(t)} = \begin{Vmatrix} -(n-1)(r-2) & (r-2) & \cdots & (r-2) \\ (r-2) & -(n-1)(r-2) & \cdots & (r-2) \\ \vdots & \vdots & \cdots & \vdots \\ (r-2) & (r-2) & \cdots & -(n-1)(r-2) \end{Vmatrix}, \quad p \neq t,$$

4.41d

$$M_{pp}^{(t)} = \begin{Vmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{Vmatrix}, \quad p \neq t,$$

4.41e

$$M_{pq}^{(t)} = M_{qp}^{(t)} = \begin{Vmatrix} (n-1) & -1 & \cdots & -1 \\ -1 & (n-1) & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & \cdots & (n-1) \end{Vmatrix}, \quad \begin{matrix} p \neq q \\ p, q \neq t, \end{matrix}$$

and where X is defined in 2.4a.

This completes the actual derivation of the quadratic form estimators. We shall leave the estimators in the form as expressed in 4.40 since this is a form with which it is convenient to compare those estimators obtained in Chapter III.

4.2. Identity of the Quadratic Form Estimators and the Maximum Likelihood Estimators for $n \geq 2$ and $r = 3$.

In Section 3.3 we obtained maximum likelihood estimators of the variances for $n \geq 2$ and $r = 3$ by using linear functions of the original observations and we defined these estimators as $\hat{\sigma}_t^2$, t=1,2,3. In Section 4.1 we derived, by the use of a quadratic form, estimators Q_t, $t=1,2,\ldots,r$, for $n \geq 2$ and $r \geq 3$. We shall show in this section that the two estimators, $\hat{\sigma}_t^2$ and Q_t, are identical for $n \geq 2$ and $r = 3$.

We shall show only that the two estimators are identical for t=1 and the proofs for t=2 and 3 follow in the same way. If we substitute r=3 and t=1 in 4.40, 4.41b, 4.41c and 4.41e, we can write

4.42 $$2n(n-1)Q_1 = X'M^{(1)}X$$

where X is defined in 2.4a and

4.43 $$M^{(1)} = \begin{Vmatrix} 2\dot{M} & -\dot{M} & -\dot{M} \\ -\dot{M} & 0 & \dot{M} \\ -\dot{M} & \dot{M} & 0 \end{Vmatrix}$$

with

4.44 $$\dot{M} = \begin{Vmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & \cdots & n-1 \end{Vmatrix}$$

an n-square matrix. The form of $M^{(1)}$ as now defined is slightly different by algebraically equivalent to that defined in 4.41a.

In order for σ_1^2 and Q_1 to be equal, the right-hand members of 3.42 and 4.42 must be equal, that is

4.45 $$Z'(F_2+F_3-F_1)Z = X'M^{(1)}X;$$

or, if we substitute 3.10 in the left-hand member of 4.45,

4.46 $$X'C'(F_2+F_3-F_1)CX = X'M^{(1)}X$$

is obtained. The identity 4.46 is equivalent to the requirement that

4.47 $$C'(F_2+F_3-F_1)C = M^{(1)}.$$

When the multiplication in the left-hand member of 4.47 is effected, we obtain

4.48 $$C'(F_2+F_3-F_1)C =$$
$$\begin{Vmatrix} 2nD'(DD')^{-1}D & -nD'(DD')^{-1}D & -nD'(DD')^{-1}D \\ -nD'(DD')^{-1}D & 0 & nD'(DD')^{-1}D \\ -nD'(DD')^{-1}D & nD'(DD')^{-1}D & 0 \end{Vmatrix}$$

where D is defined in 3.13 and $(DD')^{-1}$ is shown in 3.21.

Inspection of 4.48 and 4.43 reveals that in order for 4.47 to be true, we need only show

4.49 $$nD'(DD')^{-1}D = \dot{M}.$$

A form equivalent to 4.49 may be obtained by post-multiplying both sides of the equation by D'. This allows us to write

4.50 $$(\dot{M}-nI)D' = 0,$$

where I is an n by n identity matrix and this is the relationship to be proved. But we can see by referring to 4.44 that

4.51 $$(\dot{M}-nI) = \begin{matrix} -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \cdots & \vdots \\ -1 & -1 & \cdots & -1 \end{matrix}$$

an n-square matrix, and this when post-multiplied by D' obviously yields a null matrix. This completes the proof that $\hat{\sigma}_1^2 = Q_1$ for $4 = 3$, $n \geq 2$.

The proof of the identity of the other pairs of estimators is similarly obtained. Thus, for $n \geq 2$ and $4 = 3$, the estimators obtained by the two different methods are identical.

4.3 Reduction of the Estimator Q_t.

We have derived estimators for the variances by two methods; however, we have shown that the corresponding estimators are identical for $r = 3$ and $n \geq 2$. Since the estimators obtained in Section 4.1 are applicable for $n \geq 2$ and $r \geq 3$, and hence more generally useful than the maximum likelihood estimators, we shall in this section reduce their matrix expressions to more simple quadratic forms dependent on the observations. The resultant forms will then be reduced by means of a linear transformation of the observations to a function of two functionally independent sum of squares.

It is observed by inspection of 4.41b, 4.41c, 4.41d and 4.41e that

4.52
$$\dot{M} = \begin{Vmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \cdots\cdot & \vdots \\ -1 & -1 & \cdots & n-1 \end{Vmatrix}$$

an n square matrix, is a common factor of each submatrix $M^{(t)}_{hj}$, $h,j=1,2,\ldots,r$, of $M^{(t)}$. Therefore,

4.53
$$M^{(t)} = \begin{Vmatrix} 0 & \dot{M} & \cdots & \dot{M} & -(r-2)\dot{M} & \dot{M} & \cdots & \dot{M} \\ \dot{M} & 0 & \cdots & \dot{M} & -(r-2)\dot{M} & \dot{M} & \cdots & \dot{M} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \dot{M} & \dot{M} & \cdots & 0 & -(r-2)\dot{M} & \dot{M} & \cdots & \dot{M} \\ -(r-2)\dot{M} & -(r-2)\dot{M} & \cdots & -(r-2)\dot{M} & (r-1)(r-2)\dot{M} & -(r-2)\dot{M} & \cdots & -(r-2)\dot{M} \\ \dot{M} & \dot{M} & \cdots & \dot{M} & -(r-2)\dot{M} & 0 & \cdots & \dot{M} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \dot{M} & \dot{M} & \cdots & \dot{M} & -(r-2)\dot{M} & \dot{M} & \cdots & 0 \end{Vmatrix}$$

an r square matrix of n square submatrices. It is observed that the submatrices in the t^{th} row and t^{th} column of $M^{(t)}$ contain matrix elements $-(r-2)\dot{M}$ except in the diagonal position where the matrix is $(r-1)(r-2)\dot{M}$.

Let us suppose that $M^{(t)}$ is made up of the sealar elements m_{khij} where $i,k=1,\ldots,n$, $h,j=1,\ldots,r$. We shall require a specification of these elements in any column of $M^{(t)}$. Their values are apparent from 4.53 but the following listing will be helpful. We take i and j to be fixed and to identify a column <u>when $j\neq t$</u>, we obtain

$$m_{itij} = -(r-2)(n-1),$$

$$m_{ktij} = (r-2), \qquad k\neq i,\ k=1,\ldots,n,$$

$$m_{ihij} = (n-1), \qquad h\neq j,t;\ h=1,\ldots,r,$$

$$m_{khij} = -1, \qquad h\neq j,t;\ h=1,\ldots,r,\ k\neq i;i=1,\ldots,n,$$

and

$$m_{kjij} = 0 \qquad k=1,\ldots,n.$$

<u>When $j=t$</u>, we have

$$m_{itit} = (r-1)(r-2)(n-1),$$

$$m_{ktit} = -(r-1)(r-2), \qquad k\neq i;\ k=1,\ldots,n,$$

$$m_{ihit} = -(r-2)(n-1), \qquad h\neq t;\ h=1,\ldots,r,$$

and

$$m_{khit} = (r-2), \qquad h\neq t;h=1,\ldots,r,k\neq i,k=1,\ldots,n.$$

Now from 4.40, $n(n-1)(r-1)(r-2)Q_t = X'M^{(t)}X$ and reduction of Q_t to a quadratic form is to be effected. We first consider $X'M_{ij}$ where M_{ij} is the $(i,j)^{th}$ column vector of $M^{(t)}$ with elements m_{khij} as defined in 4.7. From matrix multiplication,

4.53

$$X'M_{ij} = -(r-2)(n-1)x_{it}+(r-2)\sum_{\substack{k=1\\k\neq i}}^{n} x_{kt}+(n-1)\sum_{\substack{h=1\\h\neq j,t}}^{r} x_{ih}-\sum_{\substack{k=1\\k\neq i}}^{n}\sum_{\substack{h=1\\h\neq j,t}}^{r} x_{kh}$$

$$= -(r-2)(n-1)x_{it}+[n(r-2)x_{.t}-(r-2)x_{it}]+[(n-1)rx_{i.}-(n-1)x_{ij}$$
$$-(n-1)x_{it}]-[nrx_{..}-rx_{i.}-nx_{.t}-xn_{.j}+x_{it}+x_{ij}]$$

$$= rn(x_{i.}-x_{..}) - n(x_{ij}-x_{.j}) - n(r-1)(x_{it}-x_{.t}), \qquad j\neq t,$$

where, as usual,

$$x_{i.} = \frac{1}{r}\sum_{j=1}^{r} x_{ij},$$

$$x_{.j} = \frac{1}{n}\sum_{i=1}^{n} x_{ij},$$

$$x_{..} = \frac{1}{nr}\sum_{i=1}^{n}\sum_{j=1}^{r} x_{ij}.$$

Similarly, when $j=t$, we obtain

4.54 $$X'M_{it} = nr(r-2)(x_{it}-x_{.t}) - nr(r-2)(x_{i.}-x_{..}).$$

But 4.53 and 4.54 define the elements in the $(i,j)^{th}$ column of $X'M^{(t)}$ and consequently it follows at once that

4.55

$$X'M^{(t)}X = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq t}}^{r} [rn(x_{i.}-x_{..})-n(x_{ij}-x_{.j})-n(r-1)(x_{it}-x_{.t})]x_{ij}$$

$$+ \sum_{i=1}^{n} [nr(r-2)(x_{it}-x_{.t})-nr(r-2()x_{i.}-x_{..})]x_{it}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{r} [rn(x_{i.}-x_{..})-n(x_{ij}-x_{.j})-n(r-1)(x_{it}-x_{.t})](x_{ij}-x_{.j})$$

$$+ \sum_{i=1}^{n} [nr(r-1)(x_{it}-x_{.t})-nr(r-1)(x_{i.}-x_{..})](x_{it}-x_{.t}).$$

The additional terms in the second factor of the two products in 4.55 have simply been added for convenience and they do not contribute to the sums. Multiplication of the products yields

4.56 $$X'M^{(t)}X = -n \sum_{i=1}^{n} \sum_{j=1}^{r} (x_{ij}-x_{.j})^2+r^2n \sum_{i=1}^{n}(x_{i.}-x_{..})^2$$

$$+ nr(r-1)\sum_{i=1}^{n}(x_{it}-x_{.t})^2-2nr(r-1)\sum_{i=1}^{n}(x_{it}-x_{.t})(x_{i.}-x_{..})$$

and final reduction lets us write

4.57

$$(n-1)(r-1)(r-2)Q_t = \frac{1}{n} X'M^{(t)}X = r(r-1)\sum_{i=1}^{n}(x_{it}-x_{i.}-x_{.t}+x_{..})^2$$

$$-\sum_{i}\sum_{j}(x_{ij}-x_{i.}-x_{.j}+x_{..})^2,$$

the desired simplified form for Q_t.

We shall now show that

5.58 $$(n-1)(r-1)(r-2)Q_t = \overset{(n-1)(r-1)}{\underset{u=1}{-\Sigma}} y_u^2$$

$$+ r(r-1)\sum_{p=1}^{n-1}\left(\frac{t-1}{t}\, y_{(t-2)(n-1)+p} \overset{r-2}{\underset{q=t-1}{-\Sigma}} \frac{y_{q(n-1)+p}}{\sqrt{(q+1)(q+2)}}\right)^2 ,$$

$$t=1,2,\ldots,r,$$

where, when $t-1 > r-2$, the last term in the parentheses has no meaning and is defined to be zero. We shall see that $y_1,\ldots,r_{(n-1)(r-1)}$ are linear functions of $x_{11},\ldots,x_{nr}$.

Let us consider an orthogonal transformation F on the observations such that

4.59 $$Y = FX$$

where Y is an nr by one column vector, X is an nr by one column vector of the observations as defined in 2.4a, and F is an nr-square orthogonal Helmert-type transformation discussed in more detail below. In terms of the usual theory or analysis of variance the new variables will be arranged so that the first (n-1)(r-1) of them are error contrasts, the next (r-1) measure the effects of judges (or blocks), the next (n-1) measure the effects of items (or treatments), and the remaining new variable estimates the grand mean. To reduce the form of Q_t in 4.57, we shall need the inverse transformation,

4.60 $$X = F'Y.$$

While it is cumbersome to attempt to show the complete matrix F', we can write down the $(i,j)^{th}$ row of F'. The elements in this row are

4.61

$$f_{i,j;q(n-1)+p} = 0, \quad i=1,\ldots,n;\ j=1,\ldots,(r-1);\ q=(j-1),\ldots,(r-2);\ p=1,\ldots,(i-2),$$

$$f_{i,j;q(n-1)+p} = [p(p+1)(q+1)(q+2)]^{-\frac{1}{2}}, i=1,\ldots,n; j=1,\ldots,(r-1);\ q=(j-1),\ldots,(r-2); p=i,\ldots,(n-1),$$

$$f_{i,j;(j-1)(n-1)+(i-1)} = -(i-1)[i(i-1)j(j+1)]^{-\frac{1}{2}},\ i=1,\ldots,n;\ j=1,\ldots,(r-1),$$

$$f_{i,j;(j-2)(n-1)+p} = 0; \quad i=1,\ldots,n;\ j=2,\ldots,r;\ p=1,\ldots,(i-2),$$

$$f_{i,j;(j-2)(n-1)+p} = -(j-1)[(j-1)jp(p+1)]^{-\frac{1}{2}}, \quad i=1,\ldots,n;\ j=2,\ldots,r;\ p=1,\ldots,(n-1),$$

$$f_{ij;(j-2)(n-1)+(i-1)} = (i-1)(j-1)[i(i-1)j(j-1)]^{-\frac{1}{2}},\ i=2,\ldots,n;\ j=2,\ldots,r,$$

$$f_{i,j;(r-1)(n-1)+q} = 0,\ i=1,\ldots,n;\ j=3,\ldots,r;\ q=1,\ldots,(j-2),$$

$$f_{i,j;(r-1)(n-1)+q} = [q(q+1)n]^{-\frac{1}{2}},\ i=1,\ldots,n;\ j=1,\ldots,(r-1);\ q=j,\ldots,(r-1),$$

$$f_{i,j;(r-1)(n-1)+(j-1)} = -(j-1)[j(j-1)n]^{-\frac{1}{2}}, \quad i=1,\ldots,n;\ j=2,\ldots,r;$$

$$f_{i,j;(r-1)n+p} = 0, \quad i=3,\ldots,n;\ j=1,\ldots,r;\ p=1,\ldots,(i-2),$$

$f_{i,j;(r-1)n+p} = [p(p+1)r]^{-\frac{1}{2}}$ $\qquad i=1,\ldots,(n-1);\ j=1,\ldots,r;$
$\qquad p=1,\ldots,(n-1),$

$f_{i,j;(r-1)n+(i-1)} = -(i-1)[i(i-1)r]^{-\frac{1}{2}}$ $\qquad i=2,\ldots,n,$

$f_{i,j;q(n-1)+p} = 0,$ $\qquad i=1,\ldots,n;\ j=3,\ldots,r;$
$\qquad q=0,\ldots,(j-3);\ p=1,\ldots,(n-1),$

and

$f_{i,j;rn} = (nr)^{-\frac{1}{2}},$ $\qquad i=1,\ldots,n;\ j=1,\ldots,r.$

In the definition of the elements of F' we have used a double subscript notation, i,j, to define the rows and a single (third) subscript to define the column. The rows run in the same way as the elements of X, that is, $i=1,\ldots,n$ with $j=1$, down to $i=1,\ldots,n$ with $j=r$.

To further illustrate the definitions of the last two paragraphs, F' for $n=4$ and $r=3$ is given in Table 4.1. The numbers in the last row of the table are the divisors for the corresponding elements in the rows above.

Table 4.1

F' for n=4 and r=3

	Columns											
Rows	1	2	3	4	5	6	7	8	9	10	11	12
1,1	1	1	1	1	1	1	1	1	1	1	1	1
2,1	-1	1	1	-1	1	1	-1	1	-1	1	1	1
3,1	0	-2	1	0	-2	1	1	1	0	-2	1	1
4,1	0	0	-3	0	0	-3	1	1	0	0	-3	1
1,2	-1	-1	-1	1	1	1	-1	1	1	1	1	1
2,2	1	-1	-1	-1	1	1	-1	1	-1	1	1	1
3,2	0	2	-1	0	-2	1	-1	1	0	-2	1	1
4,2	0	0	3	0	0	-3	-1	1	0	0	-3	1
1,3	0	0	0	-2	-2	-2	0	-2	1	1	1	1
2,3	0	0	0	2	-2	-2	0	-2	-1	1	1	1
3,3	0	0	0	0	4	-2	0	-2	0	-2	1	1
3,4	0	0	0	0	0	6	0	-2	0	0	-3	1
Divisor	2	$2\sqrt{3}$	$2\sqrt{6}$	$2\sqrt{3}$	6	$6\sqrt{2}$	$2\sqrt{3}$	$2\sqrt{6}$	2	$2\sqrt{3}$	6	$2\sqrt{6}$

To express Q_t in terms of the elements of Y, we first expand the second member on the right of 4.57 to obtain

4.62

$$\sum_{i=1}^{n}\sum_{j=1}^{r}(x_{ij}-x_{i.}-x_{.j}+x_{..})^2=\sum_{i=1}^{n}\sum_{j=1}^{r}x_{ij}^2-r\sum_{i=1}^{n}x_{i.}^2-n\sum_{j=1}^{r}x_{.j}^2+nrx_{..}^2 .$$

Since F is orthogonal,

4.63 $$\sum_{i=1}^{n}\sum_{j=1}^{r}x_{ij}^2=\sum_{u=1}^{nr}y_u^2 .$$

Other relations required follow from 4.61 and they may be easily verified by inspection of Table 4.1 when n=4 and r=3. These relations are:

4.64 $$x_{..} = \frac{y_{rn}}{\sqrt{rn}}, \qquad x^2_{..} = y^2_{rn}/rn,$$

4.65 $$\frac{1}{\sqrt{r}} \cdot \left\| \begin{matrix} y_{n(r-1)+1} \\ y_{n(r-1)+2} \\ \vdots \\ y_{nr-1} \\ y_{nr} \end{matrix} \right\| = \left\| \begin{matrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{-(n-1)}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{matrix} \right\| \cdot \left\| \begin{matrix} x_{1.} \\ x_{2.} \\ \vdots \\ x_{n-1.} \\ x_{n.} \end{matrix} \right\|,$$

from whence, owing to the orthogonality of the relationship 4.65,

4.66 $$\sum_{j=1}^{r} x^2_{i.} = \frac{1}{r}\left(\sum_{p=1}^{r-1} y_{n(r-1)+p} + y^2_{rn}\right).$$

In addition, a form similar to 4.65 yields

4.67 $$\sum_{j=1}^{r} x^2_{i.} = \frac{1}{n}\left(\sum_{p=1}^{r-1} y_{(n-1)(r-1)+q} + y^2_{rn}\right).$$

Substitution from 4.63, 4.64, 4.66, and 4.67 in 4.62 allows us to write

4.68
$$\sum_{i=1}^{n}\sum_{j=1}^{r}(x_{ij}-x_{i.}-x_{.j}+x_{..})^2 = \sum_{n=1}^{nr} y^2_n - \sum_{p=1}^{n-1} y^2_{n(r-1)+p}$$
$$- \sum_{q=1}^{r-1} y^2_{(n-1)(r-1)+2} - y^2_{rn} = \sum_{u=1}^{(n-1)(r-1)} y^2_u .$$

Reduction of the first sum of squares in the right-hand member of 4.57 is more difficult but may be accomplished with considerable algebraic reduction. We first require

4.69

$$(x_{it}-x_{i.}-x_{.t}+x_{..}) = \frac{-(t-1)}{\sqrt{t(t-1)}}\left[\frac{-(i-1)}{\sqrt{i(i-1)}}\, y_{(t-2)(n-1)+(i-1)} + \sum_{p=i}^{n-1} \frac{y_{(t-2)(n-1)+p}}{\sqrt{p(p+1}}\right] + \sum_{q=t-1}^{r-2} \frac{1}{\sqrt{(q+1)(q+2)}}$$

$$\left[\frac{-(i-1)}{\sqrt{i(i-1)}}\, y_{q(n-1)+(i-1)} + \sum_{p=i}^{n-1} \frac{y_{q(n-1)+p}}{\sqrt{p(p+1}}\right]$$

where the second term in both square brackets is taken to be zero when i=n and the second member of the right-hand side of 4.69 is zero when t=r. This result depends on 4.61 and can readily be verified in the special case illustrated by Table 4.1. Note that 4.69 depends at most on the first (n-1)(r-1) of the y's.

We must evaluate $\sum_{i=1}^{n}(x_{it}-x_{i.}-x_{.t}+x_{..})^2$ and this can most easily be done through use of Table 4.2 which essentially lists the results of 4.69. We now refer to Table 4.2 where we have only the y's involved in $(x_{it}-x_{i.}-x_{.t}+x_{..})$, i=1,...,n. The heading of each main column shows the y's involved for q=t-2, t-1, ..., r-1. The values of p=1,2,..., n-1, are shown in the second heading of each main column. The coefficients of the y's are obtained from the main body

Table 4.2

Coefficients of y's in $(x_{it}-x_{i.}-x_{.t}+x_{..})$

$y_{(t-2)(n-1)+p}$					$y_{(t-1)(n-1)+p}$						$y_{(r-2)(n-1)+p}$				
1	2	3	...	n-1	1	2	3	...	n-1	...	1	2	3	...	n-1
-(t-1)	-(t-1)	-(t-1)	...	-(t-1)	11	1	1	...	1	...	1	1	1	...	1
(t-1)	-(t-1)	-(t-1)	...	-(t-1)	-1	1	1	...	1	...	1	1	1	...	1
0	2(t-1)	-(t-1)	...	-(t-1)	0	-2	1	...	1	...	0	-2	1	...	1
0	0	3(t-1)	...	-(t-1)	0	0	-3	...	1	...	0	0	-3	...	1
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
0	0	0	...	(n-1)(t-1)	0	0	0	...	-(n-1)	...	0	0	0	...	-(n-1)
Column Divisors															
$\sqrt{2t(t-1)}$	$\sqrt{6t(t-1)}$	$\sqrt{12t(t-1)}$		$\sqrt{nt(n-1)(t-1)}$	$\sqrt{2t(t+1)}$	$\sqrt{6t(t+1)}$	$\sqrt{12t(t+1)}$		$\sqrt{nt(n-1)(t+1)}$		$\sqrt{2r(r-1)}$	$\sqrt{6r(r-1)}$	$\sqrt{12r(r-1)}$		$\sqrt{nr(n-1)(r-1)}$

of the table for $i=1,2,\ldots,n$. The number at the bottom of a column is the divisor for each y in that column.

It should be noted that if $t=1$, then the first main set of columns of Table 4.2 has no meaning, this column would be omitted and the first set of columns would be $y_{(t-1)(n-1)+p}$.

$\sum_{i=1}^{n}(x_{it}-x_{i.}-x_{.t}+x_{..})^2$ is the sum of squares of the linear functions defined in Table 4.2. In the addition, the sum of the coefficients for the cross-products of any two y's in the same main column of Table 4.2 vanish as does the sum of the coefficients for the cross-products of any two y's in different main columns of Table 4.2 except for those cases where both values of y have a common value of p. The following results can be verified:

(i) The terms in the sum involving squares of the y's in the first main column of Table 4.2 are

$$\sum_{p=1}^{n-1} \frac{(t-1)}{t} y^2_{(t-2)(n-1)+p} .$$

(ii) The corresponding terms for the remaining y's are

$$\sum_{p=1}^{n-1} \sum_{q=t-1}^{r-2} \frac{y^2_{q(n-1)+p}}{(q+1)(q+2)} .$$

(iii) The cross-product terms involving the y's in the first main column are

$$-2 \sum_{p=1}^{n-1} \sqrt{\frac{(t-1)}{t}}\, y_{(t-2)(n-1)+p} \sum_{q=t-1}^{r-2} \frac{y_{q(n-1)+p}}{\sqrt{(q+1)(q+2)}}$$

(iv) Finally, the remaining cross-product terms may be written as

$$2 \sum_{p=1}^{n-1} \sum_{q=t-1}^{r-2} \sum_{\substack{q'=t \\ q < q'}}^{r-2} \frac{y_{q(n-1)+p}\, y_{q'(n-1)+p}}{\sqrt{(q+1)(q+2)(q'+1)(q'+2)}} \,.$$

It now follows that

4.70 $$\sum_{i=1}^{n} (x_{it}-x_{i.}-x_{.t}+x_{..})^2 = \sum_{p=1}^{n-1} \frac{(t-1)}{t}\, y^2_{(t-2)(n-1)+p}$$

$$- 2 \sum_{p=1}^{n-1} \sqrt{\frac{(t-1)}{t}}\, y_{(t-2)(n-1)+p} \sum_{q=t-1}^{n-2} \frac{y_{q(n-1)+p}}{\sqrt{(q+1)(q+2)}}$$

$$+ \sum_{p=1}^{n-1} \left[\sum_{q=t-1}^{r-2} \frac{y^2_{q(n-1)+p}}{(q+1)(q+2)} + 2 \sum_{\substack{q=t-1 \\ q}}^{r-2} \sum_{\substack{q'=t \\ q'}}^{r-2} \frac{y_{q(n-1)+p}\, y_{q'(n-1)+p}}{\sqrt{(q+1)(q+2)(q'+1)(q'+2)}} \right].$$

The term in square brackets in the right-hand member of 4.70 reduces to

$$\left[\sum_{q=t-1}^{r-2} \frac{y_{q(n-1)+p}}{\sqrt{(q+1)(q+2)}} \right]^2$$

and finally we have

4.71 $$\sum_{i=1}^{n} (x_{it}-x_{i.}-x_{.t}+x_{..})^2 = \sum_{p=1}^{n-1} \left[\sqrt{\frac{t-1}{t}}\, y_{(t-2)(n-1)+p} - \sum_{q=t-1}^{r-2} \frac{y_{q(n-1)+p}}{\sqrt{(q+1)(q+2)}} \right]^2 .$$

Equations 4.68 and 4.71 are now used to substitute in 4.57 and thus we obtain

4.72 $$(n-1)(r-1)(r-2)Q_t =$$

$$r(r-1)\sum_{p=1}^{n-1}\left(\sqrt{\frac{t-1}{t}}\, y_{(t-2)(n-1)+p} - \sum_{q=t-1}^{r-2}\frac{y_{q(n-1)+p}}{\sqrt{(q+1)(q+2)}}\right)^2 - \sum_{u=1}^{(n-1)(r-1)} y_u^2,$$

where t=1,2,...,r and when t=r, the second term in the parentheses is defined to be zero.

When t=r,

4.73
$$(n-1)(r-1)(r-2)Q_r = r(r-2)\sum_{p=1}^{n-1} y^2_{(r-2)(n-1)+p} - \sum_{u=1}^{(n-1)(r-2)} y_u^2,$$

which is a linear difference of two sums of squares.

V. DISTRIBUTION THEORY AND TESTS OF SIGNIFICANCE

5.1 Distribution of $\frac{Q_t}{2}$ Assuming Homogeneous Variances.

We have just shown that the estimator Q_t that we have proposed is a linear function of two sums of squares of y's. The y's in 4.73 are, for the model used in this research, normally distributed with means zero and variance-covariance matrix given by the first $(n-1)(r-1)$ square principal minor of Σy where

$$\Sigma y = F' \,{}_x F \tag{5.1}$$

by the theorem of Section 3.2 and in view of the transformation 4.59. The transformation F is such that y_u, $u=1,2,\ldots,(n-1)(r-1)$, are independent and have variances σ^2 if all the judges have equal ability, that is, if $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_r^2 = \sigma^2$. Therefore, when both sides of 4.73 are divided by σ^2, we have

$$(n-1)(r-1)(r-2)\,\frac{Q_r}{\sigma^2} = r(r-2)\,\chi^2_{(n-1)} - \chi^2_{(n-1)(r-2)} \tag{5.2}$$

under the assumption that $\sigma_j^2 = \sigma^2$, $j=1,\ldots,r$ and where $\chi^2_{(n-1)}$ and $\chi^2_{(n-1)(r-2)}$ are independent central chi square variates with $(n-1)$ and $(n-1)(r-2)$ degrees of freedom respectively. While 5.2 indicates the form of the distribution of Q_r/σ^2, it is clear from the symmetry of the problem that Q_r may be replaced by Q_t, $t=1,\ldots,r$, in 5.2 and the result still holds.

The difference of two independent chi square variates has been studied both by Pachares [6] and Gurland [3]. Pachares has shown that an indefinite quadratic form such as that in 5.23 has a density function that can be expressed in terms of Bessel functions. Gurland expressed the density function of an indefinite quadratic form as finite series of LaGuerre polynomials. The computation for compiling the necessary tables for the distribution of Q_t/σ^2 based on either of the above results seems impractical. Thus a new approach was sought.

5.2 Tests of Significance for a Particular Variance Assuming the Other Variances are Equal.

In Section 4.3 we showed that an estimator, Q_t, could be expressed as a weighted difference of two sums of squares, while the result was demonstrated for Q_r, a similar result is possible for any Q_t by redefining F in 4.59. We now repeat that

5.3
$$(n-1)(r-1)(r-2)Q_t = r(r-2)\sum_{p=1}^{n-1} y^2_{(r-2)(n-1)+p} - \sum_{u=1}^{(n-1)(r-2)} y_u^2, \qquad t=1,\ldots,r,$$

for use in this section.

We shall, in this section, obtain a statistic based upon 5.3 with which we can test the null hypothesis,

5.6
$$H_o: \sigma_t^2 = \sigma^2$$

against any one of the alternative hypotheses,

$$H_{a_1}: \sigma_t^2 < \sigma^2,$$

5.7 $$H_{a_2}: \sigma_t^2 > \sigma^2,$$

$$H_{a_3}: \sigma_t^2 \neq \sigma^2,$$

under the assumption that $\sigma_j^2=\sigma^2$, $j\neq t$, $j=1,\ldots,r$.

We may rewrite 5.3 as

5.8

$$(n-1)(r-1)(r-2)Q_t = (r-1)^2 \sum_{p=1}^{n-1} y^2_{(r-2)(n-1)+p} - \sum_{v=1}^{(n-1)(r-1)} y_v^2.$$

Now instead of using the statistic Q_t/σ^2 for the test procedures, we use a studentized form by replacing σ^2 by an estimator of σ^2. Under H_o and given the assumptions of the test situations, we are in fact assuming the usual model of analysis of variance in the sense that we are postulating homogeneity of variances. Then the error mean square of the analysis of variance yields an estimator of σ^2. We actually work with the error sum of squares which we shall designate by E and it follows that

5.9 $$E = \sum_{v=1}^{(n-1)(r-1)} y_v^2 = \sum_{p=1}^{n-1} y^2_{(r-2)(n-1)+p} - \sum_{u=1}^{(n-1)(r-2)} y_u^2.$$

Similar algebra based on 5.8 and 5.9 yields

5.10

$$(n-1)(r-1)(r-2)\frac{Q_t}{E} = (r-1)^2 \left[1 + \frac{\sum_{u=1}^{(n-1)(r-2)} y_u^2}{\sum_{p=1}^{n-1} y^2_{(r-2)(n-1)+p}} \right]^{-1} - 1.$$

Let us consider the ratio $\dfrac{\sum_{u=1}^{(n-1)(r-2)} y_u^2}{\sum_{p=1}^{n-1} y^2_{(r-2)(n-1)+p}}$. If we assume that $\sigma_j^2=\sigma^2$, $j=1,\ldots,r$, then by the transformation used in 4.59, the y's in the numerator are independent of those in the denominator of this ratio and all of the y's are normally distributed with variances σ^2. Therefore, if we divide both the numerator and denominator of this fraction by σ^2, a ratio of two independent chi square variates is obtained. If we multiply and divide these two chi square variates by their respective degrees of freedom, we obtain a F-variate. Thus, we can write

5.11 $$(n-1)(r-1)(r-2)\frac{Q_t}{E} = \frac{(r-1)^2}{1+(r-2)F_{[(n-1)(r-2),(n-1)]}} - 1$$

where $(n-1)(r-2)$ are the degrees of freedom for the numerator of the F-variate and $(n-1)$ are the degrees of the denominator. Under the stated conditions and H_o, $F_{[(n-1)(r-2),n-1]}$ has the usual central F-distribution of the analysis of variance.

Equation 5.11 shows that $\frac{Q_t}{E}$, $t=1,\ldots,r$, is a monotone function of F. Therefore, we can use the F-variate to test H_o: $\sigma_t^2=\sigma^2$ against any one of the alternative in 5.7, assuming $\sigma_j^2=\sigma^2$, $j\neq t$, $j=1,\ldots,r$.

If we solve 5.11 for F, we obtain

5.12 $$F_{[(n-1)(r-2),(n-1)]} = \frac{rE - (n-1)(r-1)Q_t}{E + (n-1)(r-1)(r-2)Q_t}$$

since F decreases as Q_t increases, to test

$$H_o: \sigma_t^2 = \sigma^2$$

against

$$H_a: \sigma_t^2 < \sigma^2$$

assuming $\sigma_j^2=\sigma^2$; $j=1,\ldots,r$; $j\neq t$, we reject H_o in favor of H_a if

5.13

$$F_{[(n-1)(r-2),(n-1)]} = \frac{rE-n(n-1)(r-1)Q_t}{e+(n-1)(r-1)(r-2)Q_t} \geq F_{\alpha,[(n-1)(r-2)(n-1)]}$$

where α is the significance level of the one-sided test and $F_{\alpha,[(n-1)(r-2),(r-1)]}$ is the appropriate tabular value of the F-distribution.

To test the stated null hypothesis H_o against

$$H_a: \sigma_t^2 > \sigma^2$$

with the usual assumptions of this section, we must use the lower tail of the F-distribution. To obtain the lower tail of the F-distribution, one obtains the tabulated value of F at significance level α with the degrees of freedom reversed and then takes the reciprocal of the tabulated value. The critical region for this second one-sided test is given by

5.14

$$F_{[(n-1)(r-2),(n-1)]} = \frac{rE-(n-1)(r-1)Q_t}{E+(n-1)(r-1)(r-2)Q_t} \leq 1/F_{\alpha[(n-1),(n-1)(r-2)]}.$$

Then if F in the left-hand member of 5.14 is exceeded by $1/F_\alpha$ in the right-hand member, we reject H_o and accept H_a with α as the risk of a Type I error.

We use both 5.13 and 5.14 to test the stated null hypothesis against the two-sided alternative by replacing α with $\frac{\alpha}{2}$ in those inequalities in order to obtain a two-sided, equal tails test at significance level α.

The procedures as outlined provide valid tests of significance for a particular variance, say σ_t^2 if:

(i) The particular variance that is to be tested is selected before an examination is made of the data

and

(ii) It is assumed that all of the other variance σ_j^2, $j=1,\ldots,r$; $j\neq t$ are equal.

The first statement above is not so limiting as it may appear, since quite often it is desired to determine if the variability of a particular judge or process is different from that of the other judges or processes. Also in the next section, a test of significance is proposed, based upon large sample theory, which will provide a method to determine if three variances are homogeneous.

To conclude this section, we note that to apply the tests proposed we must compute $F_{[(n-1)(r-2),(n-1)]}$ and this can be most easily done from results in Section 4.3. We require basically two quantities,

5.15 $$E = \sum_{i=1}^{n} \sum_{j=1}^{r} (x_{ij}-x_{i.}-x_{.j}+x_{..})^2,$$

the error sum of squares from the analysis of variance of the two-way classification (which will usually be available), and

5.16 $$J = \sum_{i=1}^{n} (x_{it}-x_{i.}-x_{.t}+x_{..})^2$$

which is more difficult to compute but is obtained most easily through direct calculation of the residuals $(x_{it}-x_{i.}-x_{.t}+x_{..})$. (It is becoming more common for statisticians to examine residuals from analyses of variance and, if this has been done, computation of 5.16 follows easily.) Now 4.57 becomes

5.17 $$(n-1)(r-1)(r-2)Q_t = r(r-1)J - E$$

and 5.12 becomes

5.18 $$F_{[(n-1)(r-2),(n-1)]} = \frac{(r-1)E - rJ}{r(r-2)J} .$$

5.3 Likelihood Ratio Test of Homogeneity of Variances for $n \geq 2$ and $r = 3$.

We have shown that the estimators obtained by a method of maximum likelihood and through the use of a quadratic form are the same when $r=3$. Therefore, in this special case a test of homogeneity of variances is possible using the likelihood-ratio criterion [5].

Briefly, the likelihood-ratio test consists of obtaining

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$$

where $L(\hat{\Omega})$ is the maximum of the likelihood function defined in 3.20 obtained with respect to σ_1^2, σ_2^2 and σ_3^2 while $L(\hat{\omega})$ is a similar maximum obtained with respect to σ^2 on the assumption that $\sigma_j^2 = \sigma^2$, $j=1,2,3$.

Under certain general conditions [5], $-2 \ln \lambda$ is approximately distributed as a chi square variate with k-v degrees of freedom for large samples when a null hypothesis is true and where k is the number of parameters estimated under the alternative hypothesis and v is the number of parameters estimated under the null hypothesis.

Using the procedure as just outlined, we can test the null hypothesis,

5.19 $$H_o\colon \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2,$$

against the alternative hypothesis

H_a: $\sigma_j^2 = \sigma^2$ for at least one j, j=1,2,3.

In order to perform this test we must first determine $L(\hat{\omega})$ and $L(\hat{\Omega})$ using the joint density function

5.20

$$f[z_1,z_2,\ldots,z_{(n-1)(r-1)}] = \frac{1}{(2\pi)^{\frac{(n-1)(r-1)}{2}}} \frac{1}{|\Sigma_z|^{\frac{1}{2}}} e^{-\frac{1}{2}Z'\Sigma_z^{-1}Z},$$

(see 3.20).

Given H_0, $\sigma_1^2=\sigma_2^2=\sigma_3^2=\sigma^2$ and then

5.21 $$\Sigma_z = \sigma^2(DD') \begin{Vmatrix} 2 & 1 \\ 1 & 2 \end{Vmatrix}$$

from 3.17, 3.18, and 3.19,

5.22 $$|\Sigma_z| = \sigma^{4(n-1)}|DD'|^2 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}^{n-1} = \sigma^{4(n-1)}n^2 3^{n-1}$$

by the definition of the determinant of a "dot product" [4] and since $|DD'| = n$ while σ^2 in 5.21 is a scalar, and

5.23 $$\Sigma_z^{-1} = \frac{1}{3\sigma^2}(DD') \cdot \begin{Vmatrix} 2 & -1 \\ -1 & 2 \end{Vmatrix}.$$

We alternatively write

5.24 $$\Sigma_z^{-1} = \frac{1}{3\sigma^2} G$$

where

5.25 $$G = (DD')^{-1} \cdot \begin{Vmatrix} 2 & -1 \\ -1 & 2 \end{Vmatrix}.$$

Equations 5.22 and 5.24 are used to substitute in the right-hand member of 5.20 and the logarithm of the resulting function is then differentiated with respect to σ^2. Upon equating that derivative to zero, we obtain the value of σ^2 leading to the maximum $L(\hat{\omega})$ to be

5.26 $$\sigma^2 = \frac{Z'GZ}{6(n-1)}$$

It can be shown that

5.27 $$Z'GZ = 3 \sum_{i=1}^{n} \sum_{j=1}^{r} (x_{ij}-x_{i.}-x_{.j}+x_{..})^2$$

or

5.28 $$z'GZ = 3E$$

where E is defined in 5.15. The procedure for obtaining 5.27 is first to replace Z by CX and Z' by X'C' where X and C are defined in 2.4a and 3.12. We obtain

5.29

$$C'GC = \frac{1}{n}\left\|\begin{array}{cccc|cccc|cccc}
2(n-1) & -2 & \dots & -2 & -(n-1) & 1 & \dots & 1 & -(n-1) & 1 & \dots & 1 \\
-2 & 2(n-1) & \dots & -2 & 1 & -(n-1) & \dots & 1 & 1 & -(n-1) & \dots & 1 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
-2 & -2 & \dots & 2(n-1) & 1 & 1 & \dots & -(n-1) & 1 & 1 & \dots & -(n-1) \\
-(n-1) & 1 & \dots & 1 & 2(n-1) & -2 & \dots & -2 & -(n-1) & 1 & \dots & 1 \\
1 & -(n-1) & \dots & 1 & -2 & 2(n-1) & \dots & -2 & 1 & -(n-1) & \dots & 1 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
1 & 1 & \dots & -(n-1) & -2 & -2 & \dots & 2(n-1) & 1 & 1 & \dots & -(n-1) \\
-(n-1) & 1 & \dots & 1 & -(n-1) & 1 & \dots & 1 & 2(n-1) & -2 & \dots & -2 \\
1 & -(n-1) & \dots & 1 & 1 & -(n-1) & \dots & 1 & -2 & 2(n-1) & \dots & -2 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
1 & 1 & \dots & -(n-1) & 1 & 1 & \dots & -(n-1) & -2 & -2 & \dots & 2(n-1)
\end{array}\right\|,$$

a 3n-square matrix. When C'GC in 5.29 is premultiplied by X', the first n elements of the resulting row vector are

$$n[2(x_{i1}-x_{.1}) - (x_{i2}-x_{.2}) - (x_{i3}-x_{.3})], \quad i=1,2,\ldots,n.$$

The next n elements are

$$n[-(x_{i1}-x_{.1}) + 2(x_{i2}-x_{.2}) - (x_{i3}-x_{.3})], \quad i=1,2,\ldots,n,$$

and the last n elements are

$$n[-(x_{i1}-x_{.1}) - (x_{i2}-x_{.2}) + 2(x_{i3}-x_{.3})], \quad i=1,2,\ldots,n.$$

The row vector of these elements is now post-multiplied by the column vector X to yield

5.30

$$X'C'GCX = \frac{1}{n}\Big[2n\Big(\sum_{i}\sum_{j} x^2_{ij}\Big) - 2n \sum_{i}(x_{i1}x_{i2}+x_{i1}x_{i3}+x_{i2}x_{i3}) + 2n^2(x_{.1}x_{.2}+x_{.1}x_{.3}+x_{.2}x_{.3}) - 2n^2(x^2_{.1}+x^2_{.2}+x^2_{.3})\Big],$$

which can be reduced by adding and subtracting the appropriate terms to

5.31 $$X'C'GCX = 3 \sum_{i}\sum_{j}(x_{ij}-x_{i.}-x_{.j}+x_{..})^2$$

or

5.32 $$X'C'GCX = 3E$$

where E is as defined in 5.15.

This last result is substituted in 5.26 to obtain

5.32 $$\hat{\sigma}^2 = \frac{E}{2(n-1)}$$

where σ^2 is the estimator required under H_o. The maximum of the likelihood function under H_o for r=3 is then

5.33 $$L(\hat{\omega}) = \left(\frac{1}{\pi}\right)^{n-1} \frac{(n-1)^{n-1}}{(E)^{n-1}\ 3^{\frac{n-1}{2}} n} e^{-(n-1)} ,$$

where ^{2} is as defined in 5.32.

The above process is repeated except that now we have three parameters, σ_1^2, σ_2^2, and σ_3^2, to estimate in order to obtain $L(\hat{\Omega})$. We have already obtained the required estimators in Chapter III (or Chapter IV when r=3). Under the alternative hypothesis, we require $|\Sigma_z|$. Σ_z is defined in 3.17 in terms of "dot product" multiplication of matrices (DD') and H. It follows that

5.34 $$|\Sigma_z| = DD'^{\ h} \cdot |H|^d$$

where h is the order of $|H|$ and d is the order of $|DD'|$. We can substitute n for $|DD'|$, 2 for the order of $|H|$ and n-1 for the order of $|DD'|$ to obtain

5.35 $$|\Sigma_z| = n^2\ |H|^{n-1} .$$

When we substitute the values of the estimators of σ_j^2 and the value of Σ_z in 5.35, in 5.20 we obtain

5.36 $$L(\hat{\Omega}) = \frac{1}{2\pi}^{n-1} \frac{e^{-(n-1)}}{n|\hat{H}|^{\frac{n-1}{2}}} .$$

$|H|$ is a function of the $\hat{\sigma}_j^2$ and we take $|\hat{H}|$ to be the same function of the σ_j^2.

The ratio of the right-hand members of 5.33 and 5.36 is now obtained to give

5.37 $$\lambda = \frac{4(n-1)^2\ |\hat{H}|}{3E^2}^{\frac{n-1}{2}}$$

It now follows that

5.38
$$-2\ \ln \lambda = -(n-1)[\ln 4 + 2 \ln (n-1) + \ln |\hat{H}| - 2 \ln E - \ln 3]$$

where we use ln to represent a natural logarithm.

Since $\ln |\hat{H}|$ would have no meaning if $|\hat{H}| \leq 0$, it is necessary to show that $|\hat{H}| > 0$. In order to prove that $|\hat{H}| > 0$ we shall use the forms Q_1, Q_2, and Q_3 as equivalent to $\hat{\sigma}_1^2$, $\hat{\sigma}_2^2$, and $\hat{\sigma}_3^2$ and as defined in 4.72. When $r=3$, we can take

5.39
$$\hat{\sigma}_1^2 = \frac{[2(y_1^2+y_2^2+\dots+y_{n-1}^2) + 2\sqrt{3}(y_1y_n+y_2y_{n+1}+\dots+y_{n-1}y_{2(n-1)})]}{2(n-1)},$$

5.40
$$\hat{\sigma}^2 = \frac{[2(y_1^2+y_2^2+\dots+y_{n-1}^2) - 2\sqrt{3}(y_1y_n+y_2y_{n+1}+\dots+y_{n-1}y_{2(n-1)})]}{2(n-1)},$$

and

5.41

$$\hat{\sigma}^2 = \frac{[-(y_1^2+y_2^2+\dots+y_{n-1}^2) + 3(y_n^2+y_{n+1}^2+\dots+y_{2(n-1)}^2)]}{2(n-1)} .$$

Observe that

5.42 $$|\hat{H}| = (\hat{\sigma}_1^2+\hat{\sigma}_2^2)(\hat{\sigma}_1^2+\hat{\sigma}_3^2) - \hat{\sigma}_1^4 ,$$

when r=3 and this follows from the definition of H in 3.19.

Now from 5.39, 5.40, and 5.41,

5.43 $$\hat{\sigma}_1^2 + \hat{\sigma}_2^2 = \frac{4}{2(n-1)} \sum_{u=1}^{2(n-1)} y_u^2 ,$$

5.44 $$\hat{\sigma}_1^2 + \hat{\sigma}_3^2 = \frac{1}{2(n-1)} \sum_{u=1}^{n-1} (y_u + \sqrt{3}\, y_{u+n-1})^2 ,$$

and

5.45 $$\hat{\sigma}_1^2 = \frac{2}{2(n-1)} \left(\sum_{u=1}^{n-1} y_u^2 + \sqrt{3} \left(\sum_{u=1}^{n-1} y_u y_{u+n-1} \right) \right) .$$

Substitution in 5.42 yields

5.46 $$|\hat{H}| = \frac{3}{(n-1)^2} \left[\sum_{u=1}^{n-1} y_u^2 \sum_{u=1}^{n-1} y_{u+n-1}^2 - \left(\sum_u y_u y_{u+n-1} \right)^2 \right] .$$

Therefore for $|H| > 0$, we require

5.47 $$\sum_{u=1}^{n-1} y_u^2 \sum_{u=1}^{n-1} y_{u+n-1}^2 > \left(\sum_{u=1}^{n-1} y_u y_{u+n-1} \right)^2 ,$$

which is Cauchy's inequality and is true for all values of the variables except when the quotient y_u/y_{u+n-1} for all u is a constant. This latter exception is, of course, an event that occurs with probability zero.

Since we estimate three parameters under the alternative hypothesis and one under the null hypothesis, $-2 \ln \lambda$ as given in 5.38 is approximately distributed as a chi square variate with two degrees of freedom for large values of n. For computational purposes we use the formula

5.48
$$-2 \ln \lambda = -(n-1) [2 \ln(n-1) + \ln(Q_1Q_2+Q_1Q_3+Q_2Q_3) - 2 \ln E + \ln 4/3]$$

where we have substituted for $|\hat{H}|$ in terms of the estimators obtained. A modified form of 5.48 may be preferred by some readers and it is written

5.49
$$-2 \ln \lambda = -(n-1)\left[\ln \frac{3}{2} + \ln \left\{1 - \frac{4(n-1)^2 \sum_{j=1}^{3} Q_j^2}{9E^2}\right\}\right].$$

This form results from 5.48 when it is noted that, when $r=3$,

$$E = \frac{2}{3}(n-1) \sum_{j=1}^{3} Q_j$$

and this is obvious from 4.57. Now 5.49 basically depends on E^2 and $\sum_{j=1}^{3} Q_j^2$ and a statistic based on these quantities is also considered in the next section.

5.4 Discussion of the Distribution of a Proposed Test Statistic for n,r=3.

The test of significance proposed in the last section is only applicable for $r=3$ and $n \geq 2$ and, in fact, is at best only approximate for finite n. Thus, it is desired to obtain an exact, small-sample test for any r and n. That is, we would like to have a test of significance for testing the homogeneity of any number of variance-estimators. In this section, although we obtained no significant results, we shall propose a test statistic which could be used for such a test of homogeneity and examine the possibilities of determining its small sample distribution.

The statistic we propose is

$$T = \frac{\sum_{j=1}^{r}\left(Q_j - \frac{E}{(n-1)(r-1)}\right)^2}{E^2} \tag{5.50}$$

where E is defined in 5.15. It should be noted that if $\sigma_j^2=\sigma^2$, $j=1,\ldots,r$, the expected value of Q_j and $\frac{E}{(n-1)(r-1)}$ are the same, namely σ^2. Also, $\sum_{j=1}^{r} Q_j = \frac{r}{(n-1)(r-1)} E$ as seen by summing the values Q_t in 4.57 and the definition of E in 5.15. Thus, the last term inside the parentheses in 5.50 is the average of the estimators. Note also that T is monotonically related to $\sum_{j=1}^{r} Q_j^2/E^2$ as was the case of $-2 \ln \lambda$ in 5.49 when we had $r=3$.

We shall attempt to find the small-sample distribution of T when $r=3$ and $n=3$ with a view to considering the feasibility of generalization. The test statistic for n,r=3 is

5.51 $$T = \frac{\sum_{j=1}^{3}\left(Q_j - \frac{E}{4}\right)^2}{E^2} = \frac{\sum_{j=1}^{3} Q_j^2 - \frac{3}{16}E^2}{E^2}$$

The estimators for n,r=3, obtained by the use of 4.58, are

5.52 $$Q_1 = \tfrac{1}{2}(y_1^2+y_2^2) + \frac{\sqrt{3}}{2}(y_1y_3+y_2y_4) ,$$

5.53 $$Q_2 = \tfrac{1}{2}(y^2+y^2) - \frac{\sqrt{3}}{2}(y_1y_3+y_2y_4) ,$$

and

5.54 $$Q_3 = -\frac{1}{4}(y_1^2+y_2^2) + \frac{3}{4}(y_3^2+y_4^2)$$

and the joint density function for the y's assuming that all the variances are equal is

5.55 $$f(y_1,y_2,y_3,y_4) = \frac{1}{(2\pi)^2}\,\frac{1}{(\sigma)^4}\,e^{-\frac{1}{2}\sigma^2(y_1^2+y_2^2+y_3^2+y_4^2)}$$

Briefly, the transformations used are

(i) $z_j = \frac{y_j}{\sigma}$, $j=1,2,3,4.$ $0 \le z \le \infty$

(ii) $\rho \sin\Theta = z_1,$

$\rho \cos\Theta = z_2,$ $0 \le \rho,\rho' \le \infty$

$\rho' \sin\phi = z_3,$ $0 \le \Theta,\phi \le 2\pi$

$\rho' \cos\phi = z_4,$

(iii) $p' = \delta \sin\alpha,$ $0 \le \delta \le \infty$

$p = \delta \cos\alpha,$ $0 \le \alpha \le \pi/2$

(iv) $\gamma = \phi - \theta$ $\quad 0 \leq \gamma \leq 2\pi$ $\quad \gamma \leq \eta \leq 4\pi - \gamma$

$\eta = \phi + \theta$ $\quad -2\pi \leq \gamma \leq 0$ $\quad -r \leq \eta \leq 4\pi + \gamma$

(v) $\beta = 2\alpha,$ $\quad 0 \leq \beta \leq \pi,$

to obtain the marginal elements of probability,

5.56 $$e.p.(\beta,\gamma) = \frac{2\pi-\gamma}{2\pi^2} \sin\beta \, d\beta d\gamma, \qquad 0 \leq \beta \leq \pi/2, \quad 0 \leq \gamma \leq 2\pi$$

on which the distribution of T depends in view of 5.57 below. It should be noted due to the symmetry obtained during some of the transformations that we are permitted to have the intervals of β and γ different than those specified in (v) and (iv) respectively. We also obtained, by the use of the transformations just given,

5.57 $$T = 3/8[\,1 - \sin^2\beta \sin^2\gamma\,] \,.$$

Considerable difficulty results when one attempts to work from 5.56 and 5.57 to actually obtain the density function of T. It turns out that it is necessary to subdivide the domain of T, which has the limits, 0 and 3/8 in this case (the limits could be made 0 and 1 with a slight modeficiation of the definition of T in 5.50), and to use different expressions for the density function over the resulting intervals and even then these expressions cannot be given in closed form. It is somewhat easier to consider the distribution function, $P(T \leq t)$ for $0 \leq t \leq 3/8$ and we now note the form to which $P(T \leq t)$ may be reduced:

5.58 $$P(T \leq t) = \frac{5}{4} - \frac{1}{\pi^2} \int_0^1 \frac{\sin^{-1}\sqrt{(1-t)(a+v)}}{(a+v)} \frac{\sqrt{v}}{\sqrt{1-v}}\, dv - \frac{5a}{4\pi} \int_0^1 \frac{dv}{(a+v)\sqrt{v(1-v)}}$$

where

5.59 $$t' = 1 - \frac{8}{3} t$$

and

5.60 $$a = t'/(1-t').$$

Now the integrals in 5.58 could be evaluated by expanding parts of their integrands in series followed by term by term integration or by methods of numerical integration. Either possibility is somewhat formidable and the results would be too limited for practical use for we have not been able to get integrals approaching the simplicity of those in 5.58 for larger values of r and n. Accordingly, we have included this section to indicate the difficulties that have been encountered and to forstall any suggestion that a direct approach to the sampling distribution to T should be straight-forward and obvious. It is hoped that further study may be more successful than the attempt shown here and the author hopes to have an opportunity of attempting such study.

VI. COMPUTATIONAL TECHNIQUES AND NUMERICAL EXAMPLES

6.1 Computational Techniques.

We have derived estimators for the individual variance in an unreplicated two-way classification and have proposed two different tests of significance. In this chapter we show the computation of the estimates and demonstrate the use of the two tests of significance.

The observed data would appear as given algebraically in Table 6.1. The meaning of the symbols as used in the table are

x_{ij} = observation on the i^{th} item by the j^{th} judge (process),

$$X_{i.} = \sum_{j=1}^{r} x_{ij},$$

$$X_{.j} = \sum_{i=1}^{n} x_{ij},$$

and

$$X_{..} = \sum_{i=1}^{n} \sum_{j=1}^{r} x_{ij}.$$

Table 6.1

Algebraic Form of Observed Data

Item	Judges (Processes)						Totals
	1	2	...	j	...	r	
1	x_{11}	x_{12}	...	x_{1j}	...	x_{1r}	$X_{1.}$
2	x_{21}	x_{22}	...	x_{2j}	...	x_{2r}	$X_{2.}$
⋮	⋮	⋮	...	⋮	...	⋮	⋮
i	x_{i1}	x_{i2}	...	x_{ij}	...	x_{ir}	$X_{i.}$
⋮	⋮	⋮	...	⋮	...	⋮	⋮
n	x_{n1}	x_{n2}	...	x_{nj}	...	x_{nr}	$X_{n.}$
Totals	$X_{.1}$	$X_{.2}$	...	$X_{.j}$	...	$X_{.r}$	$X_{..}$

For computational purposes, a new table, such as Table 6.2, is constructed. The values in Table 6.2 are obtained by formulas

$$x'_{ij} = rx_{ij} - X_{i.},$$

$$X'_{.j} = \sum_{i=1}^{n} x'_{ij} ,$$

$$A_j = \sum_{i=1}^{n} x'^{2}_{ij} ,$$

$$B_j = \frac{1}{n}(X'_{ij})^2 ,$$

$$C_j = (r-1)(A_j - B_j),$$

rE = r times the error sum of squares as shown in Table 6.3, Analysis of Variance,

$F_j = C_j - rE,$

$$Q_j = \frac{F_j}{(r)(n-1)(r-1)(r-2)}$$

r = number of judges (processes),

and n - number of items.

Table 6.2 provides a convenient summary for the computations as well as a systematic way of doing the numerical work. The final column simply yields a check on the computation of x'_{ij}'s. A further check depends on the fact that in the last row of the table $\sum_{j=1}^{r} Q_j$ should equal $rE/(n-1)(r-1)$.

Table 6.2

Computation of the Estimates

Item	Judge (Process)						Total
	1	2		j		r	
1	x'_{11}	x'_{12}	...	x'_{1j}	...	x'_{1r}	0
2	x'_{21}	x'_{22}	...	x'_{2j}	...	x'_{2r}	0
⋮	⋮	⋮	...	⋮	...	⋮	⋮
i	x'_{i1}	x'_{i2}	...	x'_{ij}	...	x'_{ir}	0
⋮	⋮	⋮	...	⋮	...	⋮	⋮
n	x'_{n1}	x'_{n2}	...	x'_{nj}	...	x'_{nr}	0
Total	$X'_{.1}$	$X'_{.2}$	...	$X'_{.j}$	...	$X'_{.r}$	0
	A_1	A_2	...	A_j	...	A_r	
	B_1	B_2	...	B_j	...	B_r	
	C_1	C_2	...	C_j	...	C_r	
	rE	rE	...	rE	...	rE	
	F_1	F_2	...	F_j	...	F_r	
	Q_1	Q_2	...	Q_j	...	Q_r	

Table 6.3

Analysis of Variance

Source of Variation	Sum of Squares
Judges	$\frac{1}{n} \sum_{j=1}^{r} X_{\cdot j}^2 - \frac{X_{\cdot\cdot}^2}{nr}$
Items	$\frac{1}{r} \sum_{i=1}^{n} X_{i\cdot}^2 - \frac{X_{\cdot\cdot}^2}{nr}$
Error	Total - (Judges and Items)
Total	$\sum_{i=1}^{n} \sum_{j=1}^{r} x_{ij}^2 - \frac{X_{\cdot\cdot}^2}{nr}$

6.2 First Numerical Example.

We now consider a numerical example of data obtained by Sir Hubert Wilkens of the Quartermaster Research and Development Center, United States Army, Natick, Massachusetts. The data are part of some obtained by having service men grade different tropical combat boots.

In this example, the data are shown for men, t_1, t_2, and t_3, grading a tropical combat boot as to fit and comfort for

1. Length of foot
2. Width of foot
3. Arch of foot
4. Toe area
5. Instep
6. Ankle
7. Heel area
8. Leg area

for the wearer. Each of the men wore a sample boot and then gave it a subjective grade on each characteristic. The grade on each characteristic depended upon the individual's judgment, independently of the grades given by the other men, and was based on the scoring scale in Table 6.4.

Table 6.4

Definition of Scale

Grade	Meaning
1	extremely unsatisfactory
2	very unsatisfactory
3	moderately unsatisfactory
4	slightly unsatisfactory
5	indifferent
6	slightly satisfactory
7	moderately satisfactory
8	very satisfactory
9	extremely satisfactory

It is now desired to test if the three men have the same ability to give good subjective grades. Thus, we want to test the null hypothesis,

$$H_o: \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2$$

against the alternative

$$H_a: \sigma^2 \neq \sigma^2 \text{ for at least one } j,$$
$$j = 1, 2, 3.$$

The pertinent data from the experiment are shown in Table 6.5.

8 Table 6.5

Observed Data for Three Men
(Sir Hubert Wilkens' Data)

Fit and comfort as to	Man 1	2	3	Total
Length of foot	9	9	7	25
Width of foot	9	9	8	26
Arch of foot	9	8	8	25
Toe area	9	9	7	25
Instep	9	6	7	22
Ankle	9	9	8	26
Heel area	9	9	8	26
Leg area	9	9	6	24
Total	72	68	59	199

In order to perform this test, we obtain the necessary values to substitute in

6.1

$$-2 \ln \lambda = -(n-1)[2 \ln (n-1)+\ln(Q_1Q_2+Q_1Q_3+Q_2Q_3)-2 \ln E+0.2877]$$

where 0.2877 is ln 4/3 as required in view of 5.48. The computation of these values is given in Table 6.6 which corresponds to the algebraic form in Table 6.2.

Table 6.6

Calculation of the Estimates for Three Men

Boot Characteristics	Men 1	Men 2	Men 3	Totals
1	2	2	-4	0
2	1	1	-2	0
3	2	1	-1	0
4	2	2	-4	0
5	5	-4	-1	0
6	1	1	-2	0
7	1	1	-2	0
8	3	3	-6	0
Totals	17	5	-22	0
A_j	49	37	82	
B_j	289.125	3.125	60.500	
C_j	25.75	67.75	43.00	
rE	22.75	22.75	22.75	
F_j	3.00	45.00	20.25	
Q_j	0.07	1.07	0.48	1.62

By the use of Table 6.6 we obtain

6.2 $$Q_1Q_2+Q_1Q_3+Q_2Q_3 = (0.07)(1.07) + (0.07)(0.48) + (1.07)(0.48) = 0.62,$$

and

6.3 $$E = \frac{22.75}{3} = 7.58.$$

Analysis of variance as in Table 6.3 was used as a check on the value of E but is not shown here.

Therefore

6.4 $$\ln(Q_1Q_2+Q_1Q_3+Q_2Q_3) = \ln 0.62 = -0.478,$$

6.5 $$\ln E = \ln 7.58 = 2.026,$$

and

6.6 $$\ln(n-1) = \ln(8-1) = \ln 7 = 1.946.$$

When these values are substituted in 6.1, we have

6.7 $$\begin{aligned} -2 \ln \lambda &= -7[2(1.946) - 0.478 - 2(2.026) + 0.288] \\ &= -7[3.892 - 0.478 - 4.052 + 0.288] \\ &= -7(-0.350) \\ &= 2.45. \end{aligned}$$

The value of 2.45 is compared with that of the tabulated chi square variate with two degrees of freedom. The tabulated value of the chi square variate at the 10%

significance level is 4.61. Thus, we would not reject the nuol hypothesis that there is no difference in the variances of the three judges.

6.3 Second Numerical Example.

To illustrate the test of significance developed in Section 5.2, let us use an additional judge from Sir Hubert Wilkens' experiment and consider his ability to give good subjective scores in comparison with the abilities of the original three judges. In order to make this type of test we must assume that the other three men are equally good in giving subjective scores. The null hypothesis for this test is

$$H_o: \sigma_4^2 = \sigma^2$$

against one of the alternatives of Section 5.2,

$$H_a: \sigma_4^2 > \sigma^2$$

under the assumption that $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2$.

The data used for this example are the same as before except that the scores of the fourth man have been included. The data are as given in Table VI and the computed values for the estimators are given in Table 6.6.

Table 6.6

Observed Data for Four Men

Fit and comfort as to	Man 1	2	3	4	Total
Length of foot	9	9	7	7	32
Width of foot	9	9	8	1	27
Arch of foot	9	8	8	1	26
Toe area	9	9	7	7	32
Instep	9	6	7	1	23
Ankle	9	9	8	6	32
Heel area	9	9	8	7	33
Leg area	9	9	6	7	31
Total	72	68	59	37	236

Although it is only necessary to compute Q_4, we have shown the computed values for all four estimators. Also the computational technique is the same for this test as it was for the previous test. It is to be observed that negative estimates are possible and are very likely for a small number of men (judges).

Table 6.7

Computation of the Estimates for Four Men

Item	Men t_1	t_2	t_3	t_4	Total
1	4	4	-4	-4	0
2	9	9	5	-23	0
3	10	6	6	-22	0
4	4	4	-4	-4	0
5	13	1	5	-19	0
6	4	4	0	-8	0
7	3	3	-1	-5	0
8	5	5	-7	-3	0
Total	52	36	0	-88	0
A_j	432	200	168	1504	
B_j	338	162	0	968	
C_j	282	114	504	1608	
rE	209	209	209	209	
F_j	73	-95	295	1399	
Q_j	0.43	-0.57	1.76	8.33	9.95

These values are substituted in the inequality

6.9 $$\frac{E + (n-1)(r-1)(r-2)Q_t}{rE - (n-1)(r-1)Q_t} \geq F_{\alpha}[(n-1),(n-1)(r-2)],$$

a restatement of 5.14, where

E = error sum of squares,

r = number of men on the panel, including the one being tested,

n = number of characteristics (or items) graded,

$F_{\alpha[(n-1),(n-1)(r-2)]}$ = the value found in the F-table at the α^{th} level of probability with $(n-1)$ and $(n-1)(r-2)$ degrees of freedom respectively,

and Q_t = estimate of the ability of t^{th} man to give good subjective scores.

If the computed value of the left-hand side is greater than the tabled value of the F-variate with $(n-1)$ and $(n-1)(r-2)$ degrees of freedom respectively and at the α level of significance, then the man who is being tested is said to have comparatively poor ability to give subjective scores. Thus for our example we compute

$$6.10 \qquad \frac{E + (n-1)(r-1)(r-2)Q_t}{rE - (n-1)(r-1)Q_t} = \frac{52.25 + (7)(3)(2)(8.3274)}{4(52.25) - (7)(3)(8.3274)}$$

$$= \frac{402.0008}{34.1246} = 11.78$$

using the data of Table 6.7 and taking $t=4$. This calculated value is now compared with the tabled value for F at the α level of significance with 7 and 14 degrees of freedom respectively. The tabled value of F at the 5% level with the above degrees of freedom is 2.75 and for F at the 0.05% level, the tabled value is 8.11. Since 11.78 is even greater than the tabled value of F at the 0.05% level, then this fourth man's ability is very poor indeed. It follows that the null hypothesis is rejected and the alternative hypothesis, that the fourth man is a poor judge in comparison with the first three, is accepted.

The procedure for testing the null hypothesis

$$H_o: \sigma_t^2 = \sigma^2$$

against either of the alternatives

$$H_a: \sigma_t^2 < \sigma^2$$

or

$$H_a: \sigma_t^2 \neq \sigma^2$$

is similar and the reader is referred to Section 5.2 where the critical regions are defined. Although we have given only examples for subjective data, it should be remembered that the method suggested in this research is even more likely to be applicable to quantitative data for we have assumed normality of the basic observations.

VII. REVIEW OF LITERATURE

As stated in the introduction, the discussion of the literature was delayed in order that the results of this research could more easily be compared with those in the literature.

Estimators of the individual error variances for non-replicated two-way classifications have been proposed by Grubbs [2] and Ehrenberg [1]. We shall discuss the estimators proposed by Grubbs first.

Grubbs was interested in the problem of estimating two different variances simultaneously using the model

7.1 $$x_{ij} = \mu_i + \epsilon_{ij} \qquad i=1,2,\ldots,n;\ j=1,2,\ldots,r,$$

where x_{ij} = observations on the i^{th} item by the j^{th} instrument,

μ_i = true mean of the i^{th} item,

ϵ_{ij} = error in measurement of the i^{th} item by the j^{th} instrument,

and both μ_i and ϵ_{ij} are assumed normally distributed with means μ and zero and with variance σ_μ^2 and $\sigma_{e_j}^2$ respectively. He also assumed $E(\epsilon_{ij}\epsilon_{kh}) = E(\epsilon_{ij})E(\epsilon_{kh})$, $h \neq j$ and $E(\mu_i\epsilon_{ij}) = E(\mu_i)E(\epsilon_{ij})$. The type of problem he was studying was that of measuring the burning time of a powder train fuse by, say three, timing instruments attached to one rifle. He wanted an estimate of the variability of the powder as well as an estimate of the variability of the instrument measuring the powder variability. Thus, in terms of his model, he wanted an estimate of σ_μ^2, the powder variance, and an estimate of $\sigma_{e_j}^2$, the measuring instrument variability simultaneously. He could, by comparing these two estimates, determine if the measuring instrument had less variability than that of the powder being measured.

One of his methods for obtaining estimators of $\sigma_{e_t}^2$, $t=1,2,\ldots,r$, which corresponds to the variance, σ_j^2, which we estimated in this research, is by the formula

7.2 $$\text{est}(\sigma_{e_t}) = s_{x+e_j}^2 - \frac{2}{r-1} \sum_{\substack{j=1 \\ j \neq t}}^{r} s_{x+e_1,x+e_2}$$

$$+ \frac{2}{(r-1)(r-2)} \sum_{\substack{h=1 \\ h < m \\ h,m \neq t}}^{r} \sum_{m=2}^{r} s_{x+e_h,x+e_m}$$

where, in terms of the original observations,

$$s_{x+e_j}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_{ij}-x_{.j})^2,$$

$$s_{x+e_j,x+e_j} = \frac{1}{n} \sum_{i=1}^{n} (x_{ij}-x_{.j})(x_{ih}-x_{.h}),$$

$$x_{.j} = \frac{1}{n} \sum_{i=1}^{n} x_{ij}.$$

His other method of obtaining the identical estimators, was by forming $\frac{n!}{(n-r)!r!}$ columns of differences of the r instruments taken two at a time and using the formula

7.3 $$\text{est}(\sigma^2_{e_t}) = \frac{1}{r-1}\left\{\sum_{\substack{j=1\\j\neq t}}^{r} s^2_{e_t-e_j} - \frac{1}{(r-2)}\sum_{\substack{j=1\\j<h\\j,h\neq t}}^{r}\sum_{h=2}^{r} s^2_{e_j-e_h}\right\}, \quad r \geq 3$$

where

e_j-e_h = column differences between the j^{th} instrument and the h^{th} instrument for the i^{th} item, $i=1,2,\ldots,n$,

$$s^2_{e_j-e_h} = \frac{1}{n}\sum_{i=1}^{n}\left[(e_j-e_h) - (\overline{e_j-e_h})\right]^2$$

and

$$\overline{e_j-e_h} = \frac{1}{n}\sum_{i=1}^{n}(e_{ij}-e_{ih}).$$

These estimators can be shown to be identical and Grubbs notes that they have variances

7.4 $$\text{Var}(\text{est.}\ \sigma^2_{e_t}) = \frac{2}{n-1}\sigma^4_{e_t} + \frac{1}{(n-1)(r-1)^2}\left\{4\sum_{\substack{j=1\\j\neq t}}^{r}\sigma^2_{e_t}\sigma^2_{e_t} + \frac{4}{(r-2)^2}\sum_{\substack{j=1\\j<h\\j,h\neq t}}^{r}\sum_{h=2}^{r}\beta^2_{e_j}\sigma^2_{e_h}\right\}.$$

When his model is compared with the one we used, it is observed that our model is the same as his if we let $\beta_j = 0$

in our model. Of course, his model still differs from that which we used in that he used what is a form of model II of the analysis of variance while we used model I. However, in so far as estimating σ_j^2 in our notation and σ_e^2 in his notations, the difference in the two models of analysis of variance being used has no effect on the problem and it can be shown by simple but cumbersome algebra that his estimators are identical to those that we derived. (It also follows that his variance formula 7.4 for his estimators apaples to those developed in this dissertation and hence we have not included them previously.)

It would appear that one of the methods that Grubbs used in obtaining his estimators may be rationalized as follows and we limit this discussion for simplicity to the case of three measuring instruments, that is, to r=3. He first obtained the column differences

$$e_{i1} - e_{i2}, \quad e_{i1} - e_{i3}, \quad \text{and} \quad e_{i2} - e_{i3},$$

$$i=1,2,\ldots,n.$$

Since

7.5 $$\sigma^2_{e_j - e_h} = \sigma^2_{e_j} + \sigma^2_{e_h}, \qquad j \neq h, \quad j,h=1,2,\ldots,3,$$

under the assumption that $\sigma_{e_j e_h} = 0$, he set

7.6 $$\text{est}(\sigma^2_{e_1}) = s^2_{e_1 - e_2} - \text{est}(\sigma^2_{e_2})$$

$$7.7 \qquad \text{est}(\sigma^2_{e_1}) = s^2_{e_1-e_3} - \text{est}(\sigma^2_{e_3})$$

and

$$7.8 \qquad \text{est}(\sigma^2_{e_2}) = s^2_{e_2-e_3} - \text{est}(\sigma^2_{e_3})$$

and, when 7.6, 7.7 and 7.8 were solved simultaneously, say for $\text{est}(\sigma^2_{e_1})$,

$$7.9 \qquad \text{est}(\sigma^2_{e_1}) = \tfrac{1}{2}(s^2_{e_1-e_2} + s^2_{e_1-e_3} - s^2_{e_2-e_3}) ,$$

was obtained.

In this form, 7.9, his estimators can easily be seen to be unbiased since

$$7.10 \qquad E[\text{est}(\sigma^2_{e_1})] = \tfrac{1}{2}(\sigma^2_{e_1} + \sigma^2_{e_2} + \sigma^2_{e_1} + \sigma^2_{e_3} - \sigma^2_{e_2} - \sigma^2_{e_3})$$
$$= \sigma^2_{e_1} .$$

It appears also from a statement that Grubbs has in his summary that one may use the statistic,

$$S = \frac{2(n-1)}{5} \cdot \frac{\text{est}(\sigma^2_{e_t})}{\sigma^2} ,$$

with σ^2 as the variance one would obtain if $\sigma^2_{e_j} = \sigma^2$, $j \neq t$, $j=1,2,\ldots,r$, to test if $\sigma^2_{e_t}$ is greater than σ^2. For this test S would be compared with the chi square variate with $\frac{2(n-1)}{5}$ degrees of freedom. Now what grubbs has actually done is to approximate to the distribution that we have obtained in 5.2 for Q_t/σ^2 through the use of a chi square distribution with first and second moments equated to those of the linear function of chi square variates that we obtained. While Grubbs does not give complete details of these procedures, we have presented what we believe to be an accurate interpretation of his comments.

In his conclusion Grubbs suggests other rought test procedures but he did not recognize that the exact F test that we have developed for the situation above was possible. He did, however, recognise that correlations between his estimators (and our estimators) introduce difficulties in formulating test methods.

Ehrenberg used the same model as we used from which he proposed three estimators of the variances. We shall define them as s_j^2, $s_j'^2$ and s_j^{*2}, $j=1,2,\ldots,r$, and discuss them in turn.

His approach to the problem was also similar to that which we used. He first obtained equations similar to those in 3.2a, 3.2b and 3.2c by use of the principle of maximum likelihood and proposed $\frac{1}{2}\sum_{i=1}^{n}(x_{ij}-x_{i.}-x_{.j}+x_{..})^2$ as an approximate solution to these equations. However, he recognized that his estimates were biased and when he corrected for the bias he obtained s_j^2 identical to Q_t. He also gives as the variance of his estimators the result stated by Grubbs and noted in 7.4.

Ehrenberg next considered a general quadratic form and noted the values of the coefficients required to obtain $s_j^2=Q_j$, $j=1,2,\ldots,r$. He did not, however, derive Q_j on the basis of reasonable assumptions from a general quadratic form in the manner discussed in this dissertation.

The second estimator Ehrenberg proposed is

7.11

$$s_j'^2 = \frac{1}{n-1}\left\{\sum_{i=1}^{n}(x_{ij}-x_{.j})^2 - \frac{1}{r(r-1)}\left[r^2 \sum_{i=1}^{p}(x_{i.}-x_{..})^2 - \sum_{i=1}^{n}\sum_{j=1}^{r}(x_{ij}-x_{.j})^2\right]\right\}.$$

Although this estimator is unbiased, its variance is a function of the μ_i. His third estimator $s_j''^2$ is based upon the use of ranges. However, he noted that the efficiencies of the last two estimators are less than the efficiency of s_j^2.

Ehrenberg states in his paper that "for large values of n, it seems reasonable to assume that the distribution of s_t^2 tends to a χ^2 form". It is believed that this statement should be for large r and not large n since

7.12 $$s_t^2 = \frac{r}{(n-1)(r-2)}\left\{\sum_{i=1}^{n}(x_{it}-x_{i.}-x_{.t}+x_{..})^2 - \frac{1}{r(r-1)}\sum_{i=1}^{n}\sum_{j=1}^{r}(x_{ij}-x_{i.}-x_{.j}+x_{..})^2\right\},$$

and he gives

7.13
$$\operatorname{Var}(s_t^2) \underset{r \to \infty}{\longrightarrow} \frac{2}{n-1}\sigma_t^4 .$$

This suggests that $\frac{(n-1)s_t^2}{\sigma_t^2}$ has a chi square distribution with (n-1) degrees of freedom as $r \to \infty$.

The work of Grubbs and Ehrenberg seems to be the only published work on this problem. With this review of those papers it will now be clear where our contributions lie. It is also evident that there is more work to be done, particularly on distribution theory related to our estimators, and that the problems to be solved will be difficult.

VIII. SUMMARY

The problem of obtaining estimators for the individual error variances in an unreplicated two-way classification is studied for the model

$$x_{ij} = \mu_i + \beta_j + \varepsilon_{ij}, \qquad i=1,2,\ldots,n;\ j=1,2,\ldots,r,$$

where x_{ij} = observation on the i^{th} treatment of the j^{th} block,

μ_i = true mean of the i^{th} treatment,

β_j = bias of the j^{th} block,

ε_{ij} = random error, distributed normally with means zero and variance σ_j^2,

and $E(x_{ij}) = \mu_i + \beta_j$.

Such a model is taken to be appropriate both for quantitative data as well as subjective data such as is discussed in this research. Also, by letting β_j be zero a model is obtained where there are no block effects, which is applicable to problems such as those discussed by Grubbs [2].

Estimators of the variances, σ_j^2, are obtained for $r=3$ and $n \geq 2$, by applying the principle of maximum likelihood to the joint density function of a set of new variables obtained from the $(n-1)(r-1)$ contrasts usually ascribed to

error. Also, equations are derived for the maximum likelihood estimators for $n \geq 2$ and $r > 3$ but solutions were not obtained.

Four reasonable and practical assumptions are used to obtain estimators, Q_t, $t=1,2,\ldots,r$, by the use of a general quadratic form. The assumptions are:

(i) The estimator should be invariant when the order of the items is changed.

(ii) The estimator should be free of the nuisance parameters μ_i and β_j.

(iii) The estimator should be unbiased,

and

(iv) Certain coefficients of the quadratic form should be equal.

The first three of these assumptions are obvious assumptions that one would impose to obtain good estimators and the reader is referred to Section 4.1 for the discussion of the last assumption.

The unbiased estimators derived for the variances σ_t^2, $t=1,2,\ldots,r$, are

8.1 $$Q_t = \frac{r}{(n-1)(r-2)} \sum_{i=1}^{n} (x_{it}-x_{i.}-x_{.t}+x_{..})^2$$

$$- \frac{1}{(n-1)(r-1)(r-2)} \sum_{i=1}^{n} \sum_{j=1}^{r} (x_{ij}-x_{i.}-x_{.j}+x_{..})^2,$$

where

$$x_{i.} = \frac{1}{r} \sum_{j=1}^{r} x_{ij},$$

$$x_{.j} = \frac{1}{n} \sum_{i=1}^{n} x_{ij},$$

$$x_{..} = \frac{1}{nr} \sum_{i=1}^{n} \sum_{j=1}^{r} x_{ij},$$

$$n \geq 2,$$

and $$r \geq 3.$$

Q_t is shown to be the same as the estimator obtained by the use of the maximum likelihood procedure when $r=3$ and $n \geq 2$. Also the expected value of Q_t is shown to be σ^2, if $\sigma_j^2=\sigma^2$, $j=1,2,\ldots,r,\ldots,r$. It is also noted that the estimate may be negative and is quite often when r is small.

When an orthogonal Helmert-type transformation is used to transform the original nr variables, it is shown that

8.2 $$Q_t = \frac{r}{(n-1)(r-1)} \sum_{p=1}^{n-1} y^2_{(n-1)(r-2)+p} - \frac{1}{(n-1)(r-1)(r-2)} \sum_{u=1}^{(n-1)(r-2)} y_u^2$$

in terms of the new variables. For the model used and transformation applied, the y's are normally distributed and if $\sigma_j^2=\sigma^2$, $j=1,2,\ldots,r$, then the y's are independently distributed with variances σ^2. Therefore, if both members of 8.2 are divided by σ^2

8.3 $$\frac{Q_t}{\sigma^2} = \frac{r}{(n-1)(r-1)} \chi^2_{(n-1)} - \frac{1}{(n-1)(r-1)(r-2)} \chi^2_{(n-1)(r-2)}$$

is obtained. This is a linear difference of two independent chi square variates with (n-1) and (n-1)(r-2) degrees of freedom. The density function of such an indefinite quadratic form has been discussed by Gurland [3] and Pachares [6].

If both members of 8.1 are divided by E, where

8.4
$$E = \frac{1}{(n-1)(r-1)} \sum_{i=1}^{n} \sum_{j=1}^{r} (x_{ij}-x_{i.}-x_{.j}+x_{..})^2 = \sum_{v=1}^{(n-1)(r-1)} y_u^2$$

it is shown that

8.5
$$\frac{Q_t}{E} = \left[\frac{(r-1)^2}{1+(r-2)F_{[(n-1)(r-2),n-1)]}} - 1\right] \Big/ [(n-1)(r-1)(r-2)]$$

where F is the central F-variate with (n-1)(r-2) degrees of freedom in the numerator and (n-1) degrees of freedom in denominator. Thus, $\frac{Q_t}{E}$, since it is a monotone function of the F-variate, may be used to test.

8.7 $$H_o: \sigma_t^2 = \sigma^2$$

against any one of the three alternatives

$$H_{a_1}: \sigma_t^2 > \sigma^2,$$

8.8 $$H_{a_2}: \sigma_t^2 < \sigma^2,$$

$$H_{a_3}: \sigma_t^2 \neq \sigma^2,$$

assuming that $\sigma_j^2 = \sigma^2$, $j=1,2,\ldots,r$, $j \neq t$.

Another test is proposed using the likelihood ratio criterion for $n \geq 2$ and $r = 3$ to test

8.9 $$H_0: \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2$$

against the alternative

8.10 $$H_a: \sigma_j^2 \neq \sigma^2 \quad \text{for at least one } j, \; j=1,2,3.$$

The test statistic

8.11 $$-2 \ln \lambda = -(n-1)[2 \ln(n-1) + \ln(Q_1Q_2+Q_1Q_3+Q_2Q_3) - 2 \ln E + \ln 4/3]$$

is approximately distributed for large values of n as a central chi square with two degrees of freedom.

Another test statistic for $n \geq 2$ and $r \geq 3$ is proposed and the use of this test statistic is discussed.

Two numerical examples are discussed using subjective data to illustrate the applications of the tests of significances and calculation of the estimates. However, the results obtained in this research are applicable to estimating variances and the testing of homogeneity of these variances of certain data irrespective of whether the data is quantitative or subjective.

In final summary, estimators are derived for estimating the individual error variance in a nonreplicated two-way classification and certain tests of significances are proposed for testing the homogeneity of variances. However, it is felt that some of the distribution theory related to the derived estimators should be investigated further, and the author plans to continue with such investigations.

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Estimation of Individual Variations in an
Unreplicated Two-Way Classification

Abstract

by

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May 1956
Blacksburg, Virginia

Abstract

THOMAS SOLON RUSSELL. Estimation of Individual Variations in an Unreplicated Two-way Classification (under the direction of Ralph Allan Bradley)

Estimators for the individual error variances were derived in a nonreplicated two-way classification by the use of the model

$$x_{ij} = \mu_i + \beta_{ij} + \varepsilon_{ij}, \qquad i=1,2,\ldots,n;\ j=1,2,\ldots,r,$$

where

x_{ij} = observation on the i^{th} treatment of the j^{th} block,

μ_i = true mean of the i^{th} treatment,

β_j = bias of the j^{th} block,

ε_{ij} = random error, distributed normally with means zero and variance σ_j^2,

and $E(x_{ij}) = \mu_i + \beta_j$.

The estimator $\hat{\sigma}_t^2$, for σ_t^2, $t=1,2,3,\ldots,r$, was derived for $n \geq 2$ and $r = 3$, by applying the principle of maximum likelihood to a set of $(n-1)(r-1)$ transformed variables usually ascribed to error. Equations were derived for the maximum likelihood estimators for $n \geq 2$ and $r \geq 3$. A general quadratic form was used and when four reasonable assumptions were applied, estimators of the variances were obtained in for form of

$$Q_t = [r(r-1)\sum_i (x_{it}-x_{i.}-x_{.t}+x_{..})^2-\sum_{ij}\sum (x_{ij}-x_{i.}-x_{.j}+x_{..})^2] \div [(n-1)(r-1)(r-2)]$$

where $x_{i.}$, $x_{.j}$ and $x_{..}$ are the means of i^{th} treatment, j^{th} block and grand mean respectively. $\hat{\sigma}_t^2$ and Q_t were shown to be identical when σ_t^2 was being estimated for the case $n \geq 2$, $r = 3$. It was noted that the derived estimator Q_j is equal to the estimators proposed by Grubbs [J.A.S.A., Vol. 43 (1948)] and Ehrenberg [Biometrika, Vol 37. (1950)]. It was shown that

$$Q_t/\sigma^2 = [(r-1)^2\chi^2_{(n-1)}-\chi^2_{(n-1)(r-2)}] / [(n-1)(r-1)(r-2)],$$ a linear difference of two independent central chi square variates. The statistic Q/E was derived such that

$$Q_t/E = \left[\frac{(r-1)^2}{1+(r-2)F} - 1\right] / [(n-1)(r-1)(r-2)]$$ with F, a central F-statistic with (n-1)(r-2) and (n-1) degrees of freedom in the numerator and denominator respectively and

$E = \sum_{ij}\sum (x_{ij}-x_{i.}-x_{.j}+x_{..})^2$. It was noted that this statistic may be used to test $H_o: \sigma_t^2 = \sigma^2$ against one of $H_{a_1}: \sigma_t^2 > \sigma^2$; $H_{a_2}: \sigma_t^2 < \sigma^2$; and $H_{a_3}: \sigma_t^2 \neq \sigma^2$ assuming $\sigma_j^2 = \sigma^2$, $j \neq t$, $j=1,2,\ldots,r$. A final test was of homogeneity of variances when $r = 3$ and was based on

$$- 2 \ln \lambda = - (n-1)[2 \ln (n-1) + \ln(Q_1Q_2+Q_1Q_3+Q_2Q_3) - 2 \ln E + \ln 4/3],$$

where λ is a likelihood ratio and $- 2 \ln \lambda$ is approximately distributed as χ^2 with 2 degrees of freedom for large n. A more general statistic for testing homogeneity of variances for $r \geq 3$ was proposed and its distribution discussed in a special case.