THE ROLE OF MANIPULATIVES IN LEARNING TO MULTIPLY AND FACTOR POLYNOMIALS

by

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CHAPTER 1

INTRODUCTION

Some mistakes in algebra seem to be made by pupils of all ages and from all countries. These mistakes have almost the standing of "international math heresies" (Sawyer, p. 165). For example, teachers are concerned about conveying to their pupils the principle that if $(x + y)$ is to be doubled, the result is $2x + 2y$. But many pupils meet the problem of squaring $(x + y)$ by adding the square of x to the square of y. Also, students factor $3x + 3$ to get $3(x)$ or factor $x^2 + x$ to get $x(x + 0)$. In short, they are manipulating symbols without meaning. There seems to be no corresponding transformation of images in their minds when they multiply or factor polynomials (Dienes, 1971).

Learning should be more than symbol manipulation and rote memorization. It should involve understanding. According to Skemp (p. 46) to understand something means to assimilate it into an appropriate mental structure. Thus, there is a need for some kind of structure to be formed in the mind of each student. The general psychological term for a mental structure is a schema.

Schemas consist of concepts joined together by various interrelationships. Concepts can be interrelated through an awareness that something is in common between pairs of objects. These connecting ideas are examples of relations (Skemp, p. 37). There is another source of cross-linkages with the first structure called transformations. This source arises from the ability to "turn one idea into another" by doing something to it (Skemp, p. 38). For example, $x^2 + 6x + 5 \rightarrow (x + 5)(x + 1)$ and $(x + 2)(x + 6) \rightarrow x^2 + 8x + 12$ are transformations.

According to Piaget's theory (Fennema, p. 636), schemas are formed by a continual process of adaptation to the individual's environment. This adaptation is possible because of the actions performed by the individual upon his environment. Schemas are abstracted from two possible sources. The first is that, when an individual acts upon an object, his knowledge is derived from the object itself. The second possibility indicates that when this individual is acting upon an object, he can also take into account the action itself since the transformation can be carried out mentally. That is, the abstraction is drawn not from the object that is acted upon, but from the action itself.

Thus, as students use manipulatives to learn mathematical transformations, these transformations are abstracted from the actions performed upon the concrete materials. Images are put in the mind as action. The student is then able to take these abstractions and to extend their applicability beyond the set of problems done with the manipulatives (Dienes, 1961). He generalizes from these experiences to a larger class of problems.

Students in algebra must be able to generalize from the concepts they learned in arithmetic. For example, when learning to multiply 4(x + 6), the students must be able to generalize from problems such as 4(5 + 6), that he learned to do in arithmetic. This must be true in order for the manipulatives to aid the learning of algebra. Otherwise, the manipulatives become instruments for rote learning.

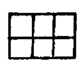
The age interval from 12 to 15 years encompasses most of the initial instruction in algebra. According to Piaget (Richmond, p. 56), we can expect students to develop from the concrete operational stage to the formal operational stage at about the age of 11 or 12 and to essentially complete their basic intellectual development by the age of 15. Of course, there is much room for individual variation within these age limits.

Thus, since we can assume that the algebra students are at least at Piaget's concrete operational stage of development, it is reasonable that they can acquire an understanding of how to multiply and factor polynomials if this understanding is gained in a concrete and action-oriented context. The student must be allowed to manipulate objects himself and to see the operations as actions. Imagery will be internalized and later this imagery will become operational in the sense that the actions can be performed in thought as well as through manipulatives. According to

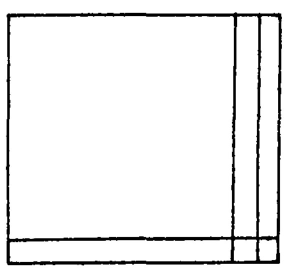
Piaget (Copeland, p. 36), an operation is not only a part of a larger structure, but also is an action that is reversible; that is, it can take place in one direction or the opposite, supposing some conservation or invariance of matter. Thus, in the psychological sense, multiplication and factorization are seen as a single, reversible operation and will be considered as such in the remainder of the dissertation.

The operation of multiplication and its reverse form, factorization, transforms polynomials. There are concrete embodiments that can be used to perform these transformations. Action upon the embodiments may provide corresponding transformations of images in the mind.

One such embodiment depicts multiplication in terms of area. For example, $2 \cdot 3$ is represented by a rectangular array of six square units with width and length dimensions of two and three linear units, respectively.



Products of larger numbers, such as 11 and 12, can be represented by figures involving unit squares, strips representing 10 square units, and larger squares representing 100 square units.



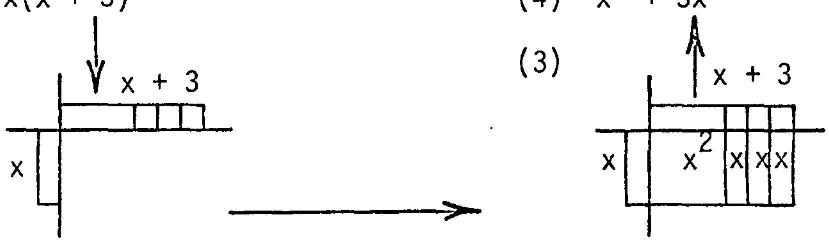
The arithmetic embodiment can be generalized to an algebraic embodiment by arbitrarily assigning a linear measure to the variable x. Then x and x^2 are represented by an area of x and x^2 square units, respectively.

The unit square, of course, is represented by: □

Thus, the multiplication of two polynmials, such as $x(x + 3)$, is embodied by forming a rectangle where x and $(x + 3)$ are the dimensions. Then the product of $x(x + 3)$ is the area of the rectangle or $x^2 + 3x$. The following are the steps a student goes through to find the product of $x(x + 3)$:

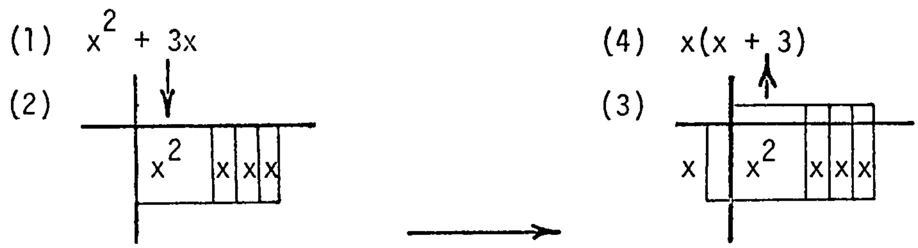
(1) $x(x + 3)$ (4) $x^2 + 3x$

(2) (3)



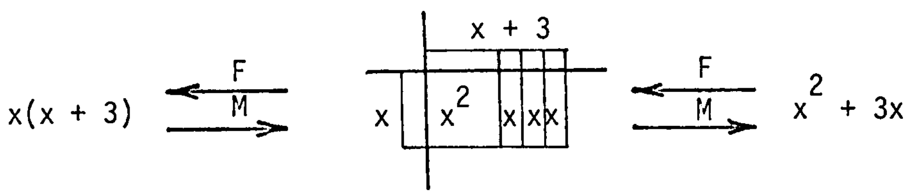
Hence, the multiplication of polynomials, using the embodiment, entails finding the area of a rectangle, given its dimensions.

The factorization transformation is embodied by forming a rectangle which has the given area, such as $x^2 + 3x$. Then the factors are the dimensions of the newly-formed rectangle or $x(x + 3)$. The following are the steps a student goes through to factor the polynomial, $x^2 + 3x$:



This action upon the strips and squares not only conserves area, but also reverses the action of multiplication. Hence, the factorization of polynomials, using the embodiment, entails finding the dimensions of a rectangle, given its area.

Therefore the multiplication and factorization transformations are inverse transformations;



In short, the goal for the learner is to internalize the operation of multiplication and factorization of polynomials. A coherent system of schemas must be developed which is not only adaptable to the learner's present cognitive structure but also exhibits the capability of correctly multiplying and factoring various algebraic expressions.

There is much theoretical support for the use of manipulatives to teach multiplication and factorization of polynomials in algebra. However, there is a lack of empirical evidence to indicate whether the use of manipulatives aids the learning process. Since the use of manipulatives involves teacher training and the spending of extra money for the materials, educators must find out if they make learning more meaningful.

Problem

Does the manipulation of concrete materials by students aid the learning of the mathematical transformation of multiplication and factorization of polynomials?

Purpose

The purpose of this study is to investigate the hypothesis that the manipulation of concrete materials can contribute substantively to the learning of the operation of multiplication and factorization of polynomials in children who are in eighth grade pre-algebra mathematics classes. The study involved a comparison of the achievement of students who used manipulatives to learn how to multiply and factor polynomials with the achievement of those who did not use manipulatives to learn to operate on the polynomials. The nonmanipulative groups solved problems using symbols only. The manipulative groups used the area embodiment to solve problems.

Limitations of the Study

This study was limited to the eighth grade level. However, the same methods and procedures would be applicable to subjects at grade seven or grade nine.

The study was limited to two schools within one school system. Thus, the results of the study must be generalized with caution to students from a different kind of school setting and with different ability levels among students.

A quasi-experimental design was chosen for this study since the treatment groups were intact pre-algebra classes and true randomization could not be met for assignment to manipulative and nonmanipulative groups. However, there was random assignment of treatments within each of the two schools chosen for the experiment.

The treatment time was ten to fourteen days since each instructional booklet contained seven sections and each section was designed to take no more than two days of instruction. Thus, depending on the individual school's schedule, the instruction was able to be completed within this time period.

Definition of Terms

Definitions of the following terms are provided to facilitate understanding of the role of manipulatives in learning to multiply and factor polynomials.

1. Abstraction is defined as the process of drawing from a number of different situations something which is common to them all (Dienes, 1961).
2. A class is constructed out of elements that share common attributes.
3. Concrete Model (concrete aid, manipulative aid, concrete embodiment) represents a mathematical idea by means of three-dimensional objects.
4. Generalization is essentially the realization that a certain type of situation (class of situations) could be considered to belong to a wider class than had been thought previously (Dienes, 1961).
5. A polynomial in x is either a term or a sum of terms, and each term is either a number or a product of a number by a positive integral power of x.
6. Semi-concrete Model represents a mathematical idea by means of pictures or visuals of three-dimensional objects.
7. Symbolic Model (symbols) represents a mathematical idea by means of commonly accepted numberals and signs that denote mathematical operations or relationships.

Summary

Symbol manipulation in mathematics is all too often meaningless because there is no corresponding transformation of images (Dienes, 1964). This is particularly so when students are learning algebraic concepts. It is true that the concepts the students encounter in learning algebra are more abstract than most primary concepts; but there are concrete embodiments of many such concepts. Such embodiments aid the learning process by providing experiences that lead to actions being internalized as operations.

There are embodiments that are used to depict the operation of multiplication and factorization of polynomials. One such embodiment depicts multiplication and factorization in terms of area. For example, the multiplication of $(x + 1)(x + 2)$ is embodied by forming a rectangle where $(x + 1)$ and $(x + 2)$ are the dimensions. Then the product of $(x + 1)(x + 2)$ is the area of the rectangle or $x^2 + 3x + 2$.

The factorization transformation is embodied by forming a rectangle which has the given area, such as $x^2 + 3x + 2$. Then the factors are the dimensions of the newly-formed rectangle or $(x + 1)(x + 2)$. This action upon the area embodiment not only conserves area, but also reverses the action of multiplication. Hence, the factorization of polynomials entails finding the dimensions of a rectangle given its area.

Therefore, the role of the area embodiment appears to be that of providing area-preserving and reversible transformations which can be internalized as an operation. The operation remains an action and is reduced neither to figures nor symbols.

This study was an effort to show that students who manipulate the area embodiment when learning to multiply and factor polynomials have a higher level of achievement, transfer, and retention than do those students who work only with symbols. They have abstracted the actions upon the embodiment and are not carrying on symbol manipulation without meaning.

Organization of the Remainder of the Study

This introductory chapter consisted of an explanation of the need for the study, an explanation of the concrete embodiment used to multiply and factor polynomials, statement of the problem and purpose, limitations of the study, and defined terminology. The second chapter presents an explanation of the learning theory that gives support to the study and a review of literature related to the study. A detailed explanation of the design of the study is presented in Chapter 3. Chapter 4 consists of the findings that resulted from this study. Finally, Chapter 5 contains the summary, the conclusions drawn from the data, and the implications and recommendations for future studies.

CHAPTER 2

REVIEW OF LITERATURE

Much literature was reviewed in developing a theory of learning mathematics. This theory includes the role of concrete materials in mathematics learning and instruction. Thus, the learning theory is included in the review of literature to give a basis for the present study.

The remainder of the chapter includes reviews of experimental studies designed to evaluate the effectiveness of concrete aids in the learning of mathematics as well as reviews of literature that encourages the use of the area embodiment to teach multiplication and factorization of polynomials.

Several studies are reviewed that involve the use of an area embodiment such as the one used in this study. There is also some literature that encourages the use of such an embodiment in various contexts even though there has been little or no associated research.

Many studies on the role of concrete materials in learning mathematics have been conducted with elementary children. Some of these studies are reviewed here if the concrete aids on the mathematical topics used in the research are similar to those used in this particular study.

Reviews of research are also done on the studies that involve the use of concrete aids at the middle school or junior high school level. Particular attention is paid to those studies that involve eighth grade students.

Some studies evaluating particular aids, such as Cuisenaire Rods or Dienes Blocks, are included; but the main emphasis is on concrete approaches versus symbolic approaches and on the similarity of these aids to those used in this particular study.

The use of semi-concrete aids (pictures, visuals) is included in many of the studies reviewed. Several researchers are interested in the effect of semi-concrete aids on learning, in addition to the effect of concrete aids and symbolic aids.

Finally, the section of the review entitled "Other Related Research Studies" includes those that are indirectly related to the study at hand. One such study involves the use of a geometric interpretation of factoring in algebra, even though it does not involve manipulative devices. The multiplication of whole numbers is an important prerequisite operation to this study. Hence, some studies involving the use of manipulatives in learning multiplication are included in the review.

Of course, there is some overlap of the categories for the review of literature. For example, one study involves the use of the area embodiment to teach the distributive principle and also involves eighth grade students. Thus, the "overlapping" studies are included in the category considered to be the most appropriate by this writer.

Theory of Learning Mathematics

Skemp (p. 133) indicates that a learning theory is required which takes account of the systematic development of an organized body of knowledge, which not only integrates what has been learned, but is a major factor in new learning. He feels that a knowledge of arithmetic makes possible the learning of algebra, and this knowledge of algebra is subsequently used for the understanding of analytical geometry. Since much of Skemp's theory of learning is based upon that of Piaget, the learning theory proposed in this chapter is a synthesis of both Piaget's and Skemp's learning theories.

The changes in an individual's cognitive system are stage-like, reflecting qualitative differences in structure and performance (Beilin, p. 23). These stages one goes through in order to attain operational structures can be identified, according to Piaget, as sensori-motor, intuitive preoperational, concrete operational, and formal operational (Steffe, p. 10). It is suggested here that what Piaget perceived as a developmental cycle in the large also occurs in the formation of every abstract concept. For example, the stages one goes through to acquire the operation of multiplication and factorization of polynomials may be similar in comparison to Piaget's levels of cognitive development from actions to operations. Assuming this to be true, the following paragraphs summarize the stages a person goes through to obtain some kind of structure in his mind:

1. Development of Sensori-Motor Thinking

During this period of development the individual gradually builds up internal action models of the objects around him by virtue of the actions he has performed with them; he recognizes objects by means of these. An internal model allows him to perform mental experiments upon the objects he is manipulating physically. The result of performing actions with this model is sensori-motor thinking (Richmond, p. 13).

The individual performs an action directly on an object at this stage of development. He manipulates a model, as well as, takes into account the action upon the model itself. In terms of the area embodiment, he moves the strips and squares by putting pieces together to form rectangles or by separating them to form a series of rectangles. At the same time, he puts into his mind images of this action upon the manipulatives.

2. The Emergence and Development of Intuitive Preoperational Thinking

By the end of the sensori-motor period the individual can recreate the internal imitations of external activity and so produce a mental image. These mental images are the symbols which allow the further development of thought. In symbolic thought, the images arise first, and the external activity follows them. This distinguishes it from sensori-motor thought (Richmond, p. 16). The symbolic activity that is now emerging is sometimes called preconceptual.

Thinking at this stage of development is still characterized by the need for concrete objects. Although a person can perform mental manipulations, he only manipulates what he can see "concretely." His mental manipulations merely represent what he would actually be doing physically with objects (Richmond, p. 16). Thus, the strips and squares and the activity accompanying them are put in the head as mental images. But these images only represent the actions that can be performed physically with the area embodiment.

3. The Emergence and Development of Concrete Operational Thinking

According to Piaget (Copeland, p. 36), an operation is the essence of knowledge. He defines an operation in terms of four fundamental characteristics:

i. An operation is an action that can be internalized, that is, carried out in thought as well as physically.

ii. An operation is an action that is reversible--that can take place in one direction or the opposite.

iii. An operation supposes some conservation or invariance, even though it is a transformation or action.

iv. An operation does not exist alone but is a part of a larger structure or a system of operations that include many mathematical structures.

An individual at the concrete operational stage of structure formation can deal with reversibility and realize that certain aspects of a changing condition are invariant, despite those changes. Thus, conservation is conceived as the resultant of operational reversibility (Richmond, p. 45).

Persons at this stage are capable of learning with symbols if those symbols represent actions the learners have done previously (Fennema, p. 638). Thus, the learning of most mathematical ideas need to be facilitated through concrete representation both because the developmental level of the individual indicates that this is the appropriate learning style and because his experiential background is meager (Fennema, p. 638). In working with strips and squares, symbols such as $2(x + 2)$ and $2x + 4$ are understood as being equivalent as long as they are associated with the action performed by the student on the embodiment.

Although this use of concrete models may enhance the relationship between symbols and reality, the individual should at some point learn to operate efficiently and effectively with symbols that do represent the abstract world of mathematics. But, unless knowledge of these abstract symbols is based on meaningful, concrete experiences, children are unable to use the symbols except in a rote way.

4. The Emergence and Development of Formal Operational Thinking

At this stage of development, the student is no longer tied to his environment. He still performs actions, but not directly with the world. He performs actions with statements about the world. The student knows that $2(x + 2)$ is equivalent to $2x + 4$ without having to depend upon the strips and squares or upon pictures that he has drawn of the embodiment.

During this formal operational stage of development the individual synthesizes operations into propositions which become a part of his cognitive structure. His thinking is no longer tied to past experience but rather to the reorganization within his own mental framework (Schminke, p. 28). The content of the problem is subordinate to its form.

It cannot be assumed that the pre-algebra students involved in this particular study have as yet reached Piaget's formal operational stage of development, in the large. Thus, this study is not concerned with determining whether or not students have reached this stage of concept formation.

Now, how are these mental structures (schemas) acquired? In order to get at this, one must look at how concepts are formed and at how they fit together to form schemas.

The processes of abstraction and generalization are very important to understanding how a concept is formed. In fact, Skemp (p. 22) describes a concept to be an abstraction which enables an individual to recognize new experiences as having the similarities of an already formed class.

The process of abstraction is defined by Dienes (1961) as the drawing from a number of different situations something which is common to them all. It is a search for an attribute which would describe certain elements felt somehow to belong together. A class is constructed out of some elements which will then be said to belong to the class (Dienes, 1961).

For example, as the student manipulates the area embodiment to multiply and factor numerical expressions, he is abstracting the operation by building up a stock of imagery in his head. Through this variety of experiences, he is able to abstract what is common to each experience at the operational level.

The process of generalization is essentially the realization that a certain type of situation (class of situations) could be considered to belong to a wider class than had been thought previously (Dienes, 1961). Thus, instead of leading from elements to classes, the process of generalization leads from classes to classes. For generalization to take place it is necessary to vary the same kind of thing in a situation in order to point to as large as possible a class in which the situation is applicable.

In other words, after first multiplying and factoring numerical expressions, the operation can be generalized to polynomials. The actions on the strips and large squares whose length is ten units are generalized to actions on the strips and large squares whose length is "x."

In mathematics learning the processes of abstraction and generalization go on side by side, even though generalizations take longer to complete. A generalization which seems to take a long time is the one from the initial class of small familiar numbers to any number. Before these later and more difficult generalizations become possible, a certain amount of abstraction work needs to be done on the elements of these classes.

It can safely be assumed that generalizations, just as abstractions, take place as a result of experience (Dienes, 1961). By experience, we must understand either purely mental experience or experience consisting of contacts with the outside world. The area embodiment provides for students experience with the outside world. It provides opportunity for students to abstract the operation of multiplication and factorization of whole numbers as well as an opportunity to generalize the operation to polynomials.

Every experience a person has, is taken into the mind and made to fit into the experiences which already exist there. This process is called assimilation. The new experience will be changed in some degree in order for it to fit. Some experiences do not fit and are, thus, rejected. The intellect assimilates new experiences into itself by transforming them to fit the structure which has been built (Richmond, p. 68).

Hence, with each new experience, the structures which have already been built up will need to modify themselves to accept that new experience. This process by which the intellect continually adjusts its model of the world to fit in each new experience is called accommodation (Richmond, p. 68).

These two processes working together (equilibration) produce the adaptation of the intellect to the environment, at any given time in the developmental process (Richmond, p. 69). An appropriate mixture of assimilation and accommodation results in successful adaptation (Sawada, p. 296).

Language contributes to the development of mental actions; but it alone is not sufficient. Words, as well as symbols, are an important part of mathematics. However, the particular blend should be determined by the specific activity and characteristics of the individual (Steffe, p. 10). Van Engen stated in 1949 (p. 397) that the "meaning of words cannot be thrown back on the meaning of other words. When the child has seen the action and performed the act himself, he is ready for the symbol for the act."

Studies and Literature that Involve the Use of an Area Embodiment

An exploratory study conducted by Mick and Altizer (1976) involved the use of the area embodiment to teach the distributive principle to pre-algebra eighth graders. Both groups of students involved in the study used the embodiment; but they were tested differently. One group was allowed to use the embodiment during the posttest and the other group was not allowed to do so. The second group could draw pictures of the embodiment on their test papers if they felt the need to do so. Also a few students were interviewed two months later in order to gain insight with regard to retention as well as internalization of the distributive principle. The results include the following:

(1) The group permitted to use strips and squares on the posttest achieved significantly better ($p < .01$) than the group not permitted to use strips. This result suggests that the Distributive Principle was not internalized as an operation.

(2) Seventy-one percent of the second group did multiply and factor through figures drawn on their examination papers. This is evidence of a partial representation of the actions as symbolic or visual representations.

(3) The subjects were able to factor better than multiply: 48% of all subjects answered the multiplication problems correctly as compared to the 62% who answered the factoring problems correctly.

(4) Subjects multiplied and factored trinomials with integer roots (52%) better than trinomials with rational roots (36%).

(5) Forty-five percent of all subjects answered the transfer problems correctly: 50% for the group using the embodiment on the test and 40% for the group not using the embodiment on the posttest.

(6) The student interviews indicated that there was little retention of the distributive principle where the students had to give their answers without using the embodiment. One student was able to draw pictures of the embodiment and form the correct rectangles, but could not give the answers.

There are no other reported research studies where either the learning of the distributive principle or the learning of the multiplication and factorization of polynomials had been done using an area embodiment. Many mathematics educators have encouraged the use of such an embodiment to multiply and factor polynomials, however. Bidwell (1972) indicates that an embodiment of "strips and squares" be used to factor quadratic polynomials. He feels that this approach to factoring has the advantage of easily leading the learner from the concrete stage of manipulation to the abstract stage of symbols. The algebra student will always have recourse to a physical model that reinforces the area model of factors and products. Bidwell also indicates that the actual manipulation with pieces and the recording of results can be started in the intermediate grades. He believes that in the eighth grade the manipulation of these pieces is excellent practice for factoring quadratics of the form $ax^2 + bx + c$, where a, b, c are whole numbers. Finally, he recommends that a large L-shaped piece of tagboard be used as a frame in which to place the pieces. This helps to eliminate "dimension" problems that occur because of the width of a rod being "one" unit.

Gibb (1974) emphasizes that during the initial stages of factoring that attention should be on the area embodiment exclusively. As the structure develops, the student should move to a mixture of concrete and symbolic experiences and finally to the purely symbolic stage. Gibb also suggests that the model can be extended over the set of real numbers.

Bruner (1966), Sawyer (1964), and Dienes (1971) indicate that strips and squares may be used to teach both multiplication and factorization of polynomials. They also note that an embodiment of this sort can be used in teaching other mathematical concepts such as operations with fractions. They indicate that richness of concrete experiences with such an embodiment can aid all children in gleaning the essential abstractions.

There are several publications that have been developed for use by teachers and students that encourage active manipulation of materials to learn various algebraic concepts. Laycock and Schadler (1973) show in their book entitled Algebra in the Concrete how manipulatives can be used in constructing some of the fundamental concepts of algebra, such as: Sequences, Building Algebraic Expressions (multiplication and factorization of polynomials), Linear Equations, Simultaneous Linear Equations. They use the area embodiment to multiply and factor polynomials.

Rasmussen (1976), in his book entitled The Mathtiles Book, shows how the area embodiment can be used to teach multiplication and factorization. Miller (1974) gives some activities using "binomial strips" (area embodiment) in his book Laboratory Activities in Algebra. These strips furnish a physical model for developing concepts in multiplying binomials and factoring quadratic expressions. The author emphasizes that these activities should be completed before the student has learned how to multiply binomials and factor quadratics. Symbolization should be introduced as the student progresses through the material.

Cuisenaire rods and Dienes blocks are also area embodiments. They may be used to teach many concepts in arithmetic as well as algebra. The teaching materials accompanying the rods and blocks indicate how these materials may be used to develop understanding of various mathematical concepts. Several experimental studies have been done to determine if the use of these materials aids learning. Although the studies have been done with elementary students, there are indications of how effective an area embodiment is in teaching various concepts in arithmetic.

Crowder's (1966) study was concerned with determining the effectiveness of two methods of teaching arithmetic in the first grade. The primary dimension of this problem was to determine and compare the arithmetic achievement of an experimental group of pupils using the Cuisenaire program with the achievement of a control group of pupils using a conventional program for the purpose of determining which was the more efficient. The results indicate that the experimental group's achievement was significantly greater than the control group's achievement not only on an entire group basis but also on a subgroup basis. The above average and the average pupils profited the most from the Cuisenaire program.

A similar study by Hollis (1965) was conducted to compare the effects of using the Cuisenaire materials and the Cuisenaire-Gattegno approach to teaching first grade mathematics with a traditional approach and to compare the effects of the Cuisenaire-Gattegno approach with a traditional approach at the end of the second grade. The results indicated that, at the end of grade 1, the Cuisenaire-Gattegno method has taught the traditional subject matter as well as the traditional method, when measured by an achievement test and a traditional test. Also, pupils taught by the Cuisenaire-Gattegno method acquired additional concepts and skills to the ones taught in the traditional program. At the end of the second grade, the Cuisenaire-Gattegno method has taught the traditional subject matter better than the traditional method, when measured by an achievement test and a traditional test. The pupils taught by the Cuisenaire-Gattegno method acquired additional concepts and skills that were not presented in the traditional program.

The results of a survey conducted by Howard (1957) indicated general agreement among British teachers that the Cuisenaire-Gattegno method is a desirable technique particularly suited to advanced pupils and is better than any other currently available approach for the development of certain mathematical concepts. Also, the results of a study conducted by Karatzinas and Renshaw (1958) indicated that teachers felt that the development of more advanced topics could easily be accomplished through use of the Cuisenaire-Gattegno method.

An experimental study was conducted by Rich (1972) to investigate the use of Multibase Arithmetic Blocks and Cuisenaire Rods in teaching selected fractional concepts to seventh grade students attending inner-city schools. The results indicated that instruction centering around these embodiments did not negatively effect the inner-city seventh grade student's mathematical achievement, his ability to compute in arithmetic, nor his understanding of arithmetic concepts.

Lucow's (1963) experimental study involved students in Grade 3. One group had used Cuisenaire materials during their three years in school. The other group had been exposed to only traditional methods of instruction. The criterion test was on mutliplication and division since these were new topics at the Grade 3 level. The results of the study indicated that the Cuisenaire method is an effective one in the teaching of multiplication and division in grade 3. But there is some doubt of its superiority over traditional methods of instruction. The Cuisenaire method seemed to operate better in rural settings than in urban settings and with bright and average children in a rural setting. Urban children thrived as well under any method at all levels of intelligence.

The purpose of another study involving Cuisenaire rods, conducted by Haynes (1963), was to compare the effectiveness of the Cuisenaire method of instruction with that of a selected conventional method in teaching multiplication to third-grade children. The results indicate that the Cuisenaire method was no more effective in teaching multiplication to third grade pupils than was the selected conventional method within the framework in which the study was made.

Fennema's study (1969) had the purpose of determining the relative effectiveness of a meaningful concrete and a meaningful symbolic model in learning multiplication by second graders. Cuisenaire rods served as the concrete model. The traditional symbolic approach to instruction served as the symbolic model. The findings indicated that the groups that learned with the symbolic model did somewhat better, although not significantly so in overall learning of multiplication. The groups which had learned with the symbolic model performed somewhat better, but not significantly so on the test of direct recall. The groups which had learned with the symbolic model performed better on the two tests of symbolic transfer. There were no significant differences in performance on the test of concrete transfer between groups which had learned with the symbolic model, concrete model, or had received no instruction on multiplication. Hence, this study indicates that there were no significant differences in the overall learning of a mathematical principle when learning was facilitated by a meaningful concrete or a meaningful symbolic model. But children who had learned with a symbolic model could transfer this learning to solving untaught symbolic instances of the principle significantly better than could children who had learned with a concrete model.

There seems to be only one study where the data indicates that children (third graders), utilizing Cuisenaire materials, achieved significantly less at the 5% level of significance on the arithmetic subtests of the Stanford Achievement Test, Elementary Battery, than either of the two samples that were used for purposes of comparison. This study by Passy (1963) was an evaluation of a program of elementary school mathematics that had been in operation for four years.

Other adaptations of the area embodiment are advocated by various mathematics educators. Grossman (1974) indicates that multi-base blocks-units, longs, flats can be used to teach counting and place value. Schminke and Arnold (1971) encourage the use of the "area" concept as the basis for teaching elementary students to multiply. Factorization of whole numbers can also be taught by using the area concept, as has been indicated by Pereira-Mendoza (1974) in an article entitled "Rectangles, Trees and Factoring."

A couple of research studies have been conducted to determine if area embodiments aid in the learning of addition, subtraction, place value, and multiplication of fractions. In Green's (1969) study two approaches were used to teach multiplication of fractional numbers in grade 5, one based on area of a rectangular region, labeled "area", and one on finding a fractional part of a region or set labeled "of." Two kinds of instructional materials were used with each: (1) diagrams and (2) cardboard strips, labeled "materials". The four treatments were: (1) Area - Diagram, (2) Area - Materials, (3) Of - Diagram, and (4) Of - Materials. Measures of the effects of the treatments included computation, concepts, applications, and attitudes. The findings indicate that the Area approach was substantially more effective than the Of approach in learning multiplication of fractional numbers. It was also concluded that diagrams and materials were equally effective in learning multiplication of fractional numbers. Finally, it was concluded that the Area - Diagram approach was the best of the four treatments for learning the multiplication of fractional numbers; the Of - Materials approach appeared to rank second.

This second study by Knaupp (1970) had as its main objective to measure the attitudes of second grade students toward specific activities used during arithmetic instruction. Two modes of instruction and two manipulative models for the numeration system were used in presenting addition and subtraction algorithms and the ideas of base and place value to four second grade classes. Instruction was teacher-centered utilizing either a teacher-demonstration mode or a student-activity mode. The two manipulative models were made of blocks of wood or ice cream sticks. The results indicated that all four classes showed achievement gains significant at the .01 level. Therefore, all four treatments were considered effective at increasing understanding and skill proficiency. (The attitude results are not included because they are not pertinent to the literature review for this particular study.)

Studies that Involve Students Who Are at the Middle School Age Level

Anderson's (1957) experimental study was an attempt to measure the efficacy of a kit of sixteen visual-tactual devices used in an eighth grade arithmetic unit involving areas, volumes, and the Pythagorean relation. He did find some evidence that visual-tactual devices aid in learning the section of arithmetic involved in this study. Differences were not significant at the 5% level, but on both the criterion and retention tests the experimental groups consistently scored higher than the control group.

Bledsoe (1974) conducted a study to compare the mathematics achievement and retention of seventh grade pupils taught by a method involving use of manipulative activities measuring objects with a steel scale and micrometer with a control group which worked similar problems without the use of manipulative activities. He used learning packages during the instruction. The results indicated that the manipulative activities produced greater gain on both posttest and retention test.

A study was done by Boersig (1973) for the purpose of testing the effects of the enactive mode of representation on the penetration of multivariable verbal problems and the derivation of algebraic equations used to solve them. Students in the control treatment group received instruction in coin, mixture, and uniform motion problems through programmed materials using only the iconic and symbolic modes of representation. Students in the experimental treatment group received instruction in the same problems through programmed materials and video tape. The video tape instruction employed the enactive mode of representation by simulating the problems with concrete manipulatives in addition to using the iconic and symbolic modes of representation. The results indicated there were no significant differences between the treatment groups in deriving algebraic equations. Students receiving instruction in the enactive mode of representation did significantly better at the .10 level of significance in penetration of problems similar to those taught in the formal training period and those problems involving non-specific transfer.

The study conducted by Jamison (1964) sought to isolate statistically significant differences in pupils' achievement attributable to the use of abaci as visual aids or as visual-tactile aids. He used seventh grade students for the three treatment groups: A - instruction in counting in other numeration systems which utilized a large variable base abacus as a teaching aid; B - instruction in counting in other numeration systems which utilized a large variable base abacus and, in addition, a small variable base abacus for each pupil; C - instruction in other numeration systems which utilized only the blackboard and chalk. The results indicated that there was no difference in the mean gains of the three groups. But these mean gains were not arithmetically equal. An examination of the means shows that the mean gain of Treatment B was smaller than any other mean gain.

Olley (1973) uses four sequences of instruction to teach one operation on a permutation group to seventh graders. The four sequences include the following:

(1) Classical - the concrete; semi-concrete; abstract sequence.

(2) Experimental - the concrete; concrete to abstract; abstract to concrete; abstract sequence.

(3) Pictorial - the iconic; iconic to abstract; abstract sequence.

(4) Abstract - the abstract sequence.

With respect to retention of mathematical constructs, the results indicated that the use of model devices in the learning sequence is not a significant variable. But where the objective is to promote an ability to transfer mathematical concepts at the seventh grade level, a model device sequence involving "hands-on" manipulation is preferable to a no-device sequence or a sequence involving only iconic representation of a model device.

Purser's (1973) study had as its major objective to determine if certain manipulative activities using measuring instruments are significantly associated with student gains in achievement and retention scores in mathematics at the seventh-grade level. The learning packages that were developed for use in the study were composed of two areas of mathematics - fractions and decimals and two areas of measuring - using a rule and using a micrometer. The results indicated that students of all ability levels in the experimental treatment achieved significantly higher scores on the post-test and the retention test than students of all ability levels in the control treatment.

Other Related Research Studies

Bunch (1972) investigated the effects of the addition of geometry and geometric illustrations as a supplement to the teaching of factoring of second-degree polynomials. It was hypothesized that exposure to a geometrical interpretation along with the instruction in algebra would: (1) increase the problem-solving ability of students, as well as (2) increase the retention of this ability, and (3) some increase could be evidenced midway through the instructional program. In this experiment a group of eighth-grade algebra students were taught the factoring of second degree polynomials through a computer-assisted program. The independent variable was the insertion into the computer program of geometrical interpretations of the algebraic operations being taught. Six treatment groups were generated by varying these insertions. The results of the analysis of data were interpreted to mean that the insertion of geometrical interpretations into the algebraic instruction offered in this computer program did not make a significant difference in the problem-solving ability of these students. Also, problem-solving ability did not necessarily increase with an increase in the number of opportunities to practice.

Carmody's (1970) research study investigated, both theoretically and experimentally the assumption that the use of concrete and semi-concrete materials can contribute significantly to the learning of mathematics at the elementary school level. A significant difference was found only between the Symbolic Group and the Semi-Concrete Group on the Post-Test on Numeration ($p = .05$). The difference favored the Semi-Concrete Group. Differences at the 1 per cent level were found between the Semi-Concrete and the Symbolic Group on two transference tests and between the Concrete and the Symbolic Group on one transference test. The differences favored the Semi-Concrete and Concrete Groups. No significant differences were found between the Concrete and Semi-Concrete Groups.

The purpose of a study by Moody (1971) was to examine the efficacy of activity-oriented instruction in the learning of multiplication in the third grade. Activity-oriented instruction in multiplication consisted of the manipulation of concrete materials by the learners prior to the introduction of the process in symbolic form. The four treatments were as follows:

(1) The activity-oriented treatment (A) consisted of multiplication instruction starting each day with a class activity in which all subjects manipulated the instructional materials.

(2) The rote treatment (R) consisted of instruction in the multiplication unit of the American Book Company's <u>Developing Mathematics</u> (1963) without activity suggestions and word problem instruction. Emphasis was placed upon memorization of basic mutliplication facts and algorithms.

(3) The rote-word problem (RW) consisted of the same multiplication instruction as R with the addition of practice in solving multiplication word problems.

(4) The control treatment (C) received instruction in addition.

The results indicated that original learning occurred as a function of instruction.

The addition of instruction in the solving of word problems did not significantly affect computational performance involving basic facts. Activity-oriented instruction did not result in superior original learning when compared to treatments R and RW. The transfer of learned multiplication facts did not occur for the instructed subjects. Activity-oriented instruction did not result in superior transfer as compared to the other instructional methods. No significant difference existed between activity-oriented instruction and treatments R and RW in retention of multiplication facts and word problem-solving performance.

Nichols' (1971) study compared two methods for instruction of multiplication and division at the third grade level. Treatment A utilized instruction by means of manipulative materials and pupil discovery. Treatment B utilized instruction by means of abstract and semi-concrete materials combined with teacher explanation and exposition. The findings were such that the null hypotheses that there would be no significant differences between treatment groups were rejected as significant differences were found for all hypotheses stated. All differences favored subjects in Treatment A over subjects in Treatment B. The use of manipulative materials and pupil discovery was found to be the more effective of the two methods of instruction.

A study by Punn (1973) was conducted to determine whether teaching multiplication facts to a group of third graders using all

3 modes of representation (Treatment 3) - enactive (manipulative materials), ikonic (pictorial devices), and symbolic (mathematics symbols) - enabled them to solve missing factor multiplication problems and multiplication word problems at a higher achievement level than two other groups taught multiplication facts using two modes of representation each: enactive plus symbolic (Treatment 1), and ikonic plus symbolic (Treatment 2). The results indicated that pupils in treatment 1 and 3 performed at a statistically higher level of achievement on the Multiplication Usage Test than the pupils in treatment 2. The difference between treatment 1 and 3 were statistically non-significant.

The problem of a research study by St. Martin (1974) centered about an investigation into the effects upon initial achievement and retention of selected fifth grade mathematics operations attributed to the sequence of introductory experiences and the Piagetian Developmental Level of the students. Two types of introductory experiences were compared, namely, a concrete - semi-concrete - abstract sequence and a semi-concrete - abstract sequence. All students were classified as presently in either Level III or Level IV of Piagetian Developmental Stages. The two topics selected were multiplication and division of fractions. The findings indicate that Level IV students secure significantly higher achievement and retention scores and when mathematics achievement was introduced as a covariate, retention differences remained significant. In the area of comprehension, a teaching sequence utilizing concrete aids results in higher retention results. In the area of computation, interaction occurs between a teaching procedure utilizing concrete aids and Level III students.

Toney's (1968) research problem was to compare the achievement in basic mathematical understandings when students individually manipulated the instructional materials and when they saw only a teacher demonstration of the same materials. Although no statistically significant difference was found between the class means on the test for basic mathematical understanding, the data indicated a trend toward greater achievement by the group using the individually manipulated materials. The use of individually manipulated materials seemed to be a somewhat more effective means for building understanding than does a teacher demonstration. A teacher demonstration of instructional materials seems to promote general mathematical achievement as efficiently as does individual manipulation of the materials by the students.

In Trask's (1972) research study one class was taught by a symbolic method (Method S) in which the only teaching aids were the text, blackboard and multiplication flash cards. The second class was given a manipulative approach (Method M). This method covered the same assignments and used the same text as Method S. In addition, the students manipulated concrete objects to introduce or augment the textbook topics. None of the statistical tests were significant. Thus it was concluded that Method M was not superior to Method S.

The two treatment groups in Weber's (1969) study were (1) reinforcement of mathematical concepts through the use of paper and pencil follow-up activities and (2) reinforcement of mathematical concepts through the use of manipulative and concrete materials for follow-up activities. The results indicated that there was not significant difference between methods as measured on the Metropolitan Readiness Test, although a definite trend favored manipulative materials. Children from the manipulative materials groups scored significantly higher on the Oral Test of Understanding, both in number of correct responses and in levels of understanding.

Summary

The review of literature revealed increased interest and attention given to the use of manipulative aids in mathematics instruction. The interest of educational psychologists concerned with learning and concept development may be responsible for this growing interest. A number of psychologists and mathematics educators have recommended increased use of manipulative aids to facilitate the acquisition and retention of concepts and principles. They have encouraged the use of manipulative aids in the early stages of the development of new concepts.

There seems to be much literature that encourages the use of the area embodiment to teach multiplication and factorization of polynomials. Dienes (1971), Bidwell (1972), and Gibb (1974) indicate that the richness of these concrete experiences can aid students in gleaning the essential abstractions. But, with the exception of the exploratory study conducted prior to this research study, none of this literature involves experimental studies.

Although many research studies have been conducted using concrete embodiments to teach the basic operations with whole numbers and fractions, the results are far from conclusive. The number of studies where the use of manipulative materials does aid learning is slightly greater than the number of studies where they seem to make no difference.

Many of these studies were pilot studies, small in scale, and perhaps far too lacking in control and in potential generalizability to be considered good research. Many studies have been done with elementary students, while only a few have been done with middle school or junior high school students, high school students, and college students. Nevertheless, they represent first steps toward answering the question, "What is the effect of manipulative materials on the learning of mathematics?"

Probably the best conclusion that can be made from all of the literature reviewed is that in only one case (Passy, 1963) did the results of the studies indicate that the concrete treatments and/or the semi-concrete treatments had detrimental effects on learning. The results of the other studies were such that either the concrete aids improved learning or they made no difference in learning. There was almost no indication that students learned less after having used concrete aids than they learned by using symbols alone.

CHAPTER 3

METHODOLOGY

Introduction

The purpose of this study was to investigate the hypothesis that the manipulation of concrete materials can contribute substantively to the learning of the operation of multiplication and factorization of polynomials. The achievement of students who used manipulatives to learn how to multiply and factor polynomials was compared with the achievement of students who did not use manipulatives to operate on the polynomials. A *Polynomial Test*, developed especially for the study, was used as both an immediate posttest and a retention test. The *Orleans Hanna Algebra Prognosis Test* served as the pretest and the scores from it were used as the covariate in the analyses of the data. The study involved two experiments, Experiment II being a replication of Experiment I.

Subjects

The subjects were eighth graders enrolled in pre-algebra mathematics classes at Dublin Middle School and Pulaski Middle School. Both of these schools are in the Pulaski County School System, Pulaski, Virginia. Eight intact classes were used in the study. Experiment I involved four classes, two at each school. Experiment II involved four classes at Dublin Middle School. The regular classroom teachers at each school were involved in the study. During Experiment I each classroom teacher taught both a manipulative and a nonmanipulative group. Due to class scheduling at Dublin Middle School, Experiment II involved one classroom teacher teaching three groups and the other teacher teaching one group. Treatments were randomly assigned to the classes within each school. A total of 173 students were involved in the study with 81 of these students being in the manipulative groups and 92 of these students being in the nonmanipulative groups.

Development of the Materials

The instructional booklets (Appendix A), for use by both the teachers and the students, were written especially for this study in accordance with the theory of learning mathematics developed in Chapter 2. The content as well as the instructional sequence of the booklets were based on this theory. Both the manipulative and the nonmanipulative groups learned to multiply and factor whole number expressions prior to learning how to multiply and factor polynomials in one variable. That is, the students had to generalize from the operation of multiplication and factorization of whole number expressions to the operation of multiplication and factorization of polynomials in one variable.

An advisory committee composed of two area mathematics supervisors and the four teachers participating in the study approved the booklets. In addition to the theory of learning mathematics, the problems and procedures used in the booklet came from the following references:

1. Laboratory Activities in Algebra (Walch)
2. Using Algebra (Laidlaw)
3. Algebra I (Holt
4. Modern School Mathematics: Algebra I (Houghton-Mifflin)
5. Introductory Algebra (Harcourt-Brace)
6. Algebra (Addison-Wesley)
7. Mathtiles (Key Curriculum Project)
8. Instructional booklet from exploratory study (Mick and Altizer)

The manipulative groups used an area embodiment that consisted of strips and squares cut out of masonite and painted various colors. The embodiment used for the whole numbers were 3/4" by 3/4" unit squares, 3/4" by 7 1/2" strips, and 7 1/2" by 7 1/2" large squares. For the operation on polynomials, the same unit squares were used; but the strips measured 3/4" by 5 1/3" and the large squares were 5 1/3" by 5 1/3." This change was made so that there would be a much smaller chance of a student finding a solution to a problem that was unique to that particular strip length.

A physical layout was needed to accentuate the difference between dimensions and areas of rectangles. So a permanent border, consisting of two soldered rectangular brass tubing of length 12 inches was used. This layout is illustrated in Figure 1.

As the result of an exploratory study that was conducted prior to this study, many revisions were made to alleviate some of the problems that had been encountered. The physical layout of the manipulatives was improved through the addition of the border. The manipulatives were made on a larger scale and were made out of more durable materials. This exploratory study is discussed in the Literature Review of Chapter 2.

Selection of the Instruments

The scores from the Orleans-Hanna Algebra Prognosis Test (OHAPT) was used as the covariate in the statistical analyses for this study. The primary use of the OHAPT is for identifying, before instruction is begun, those students who may be expected to achieve success and those who may be expected to encounter difficulties in an algebra course (Orleans, 1968). The OHAPT may also be used by teachers as an instructional aid for planning lessons and assignments that will meet the needs of students of different abilities.

The test was designed to be used with grades 7-11; but norms are only available for grades 7 and 8. There is only one form of the test. Forty minutes is allotted to answer the 58 Test Questions and fifteen to twenty minutes to answer the five Questionnaire Items.

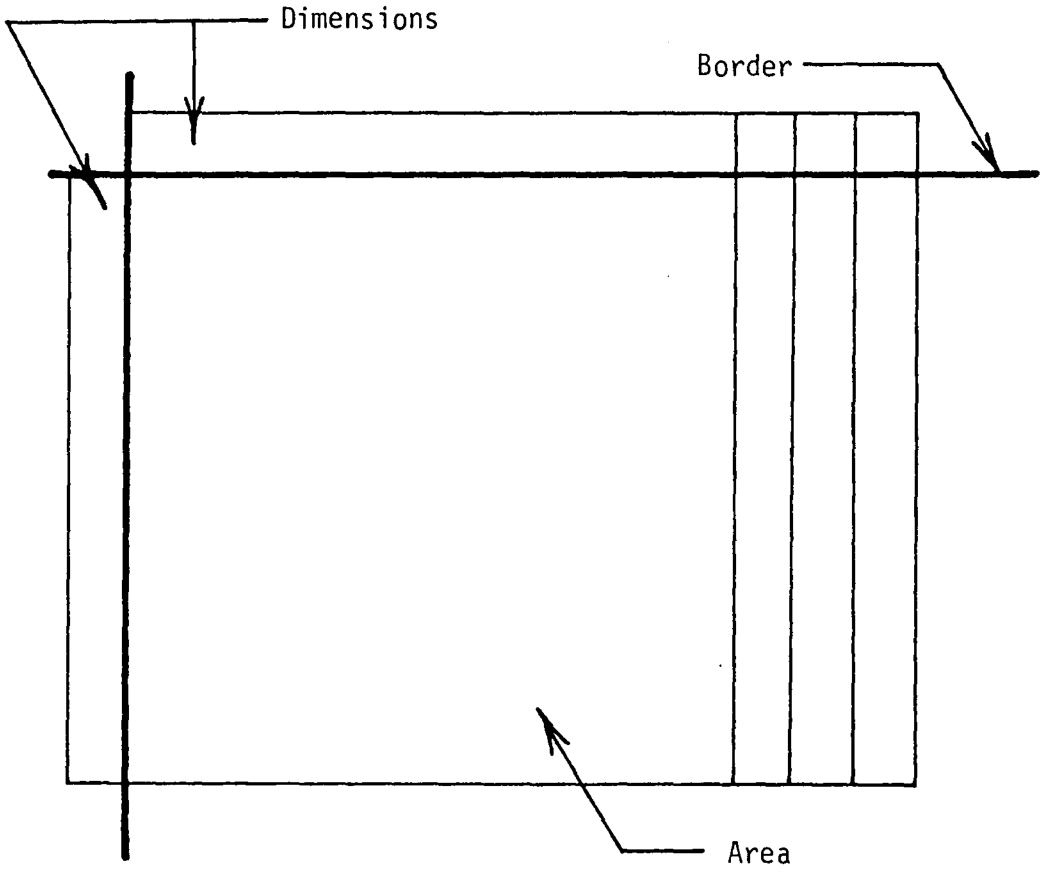


Figure 1. Physical Layout of Area Embodiment

The test contains items in multiple-choice form, divided into ten subtests of five or six items each. Each subtest is preceded by "lesson" material explaining and illustrating the principles required for answering the test items.

The Five Questionnaire Items consist of self-report questions concerning previous grades in mathematics, science, English, and social studies and an estimate of the expected algebra grade. The items are included in the total test score.

Emphases of algebra textbooks, articles published in professional journals, and suggestions of experienced teachers provided the authors with information for determining valid content for the test. A variety of investigations were undertaken to obtain information on the predictive validity of the test and to establish guidelines for interpreting test results. Predictive validities, based on correlation with a Mid-Year Algebra Test, Mid-Year Algebra Grade, Lankton First-Year Algebra Test, and Final Algebra Grade, ranged from .39 to .82, median .71.

Reliability data were based on test-retest estimation procedures with a two-week time interval. Coefficients range from .91 to .96 for total scores, .89 to .95 for questionnaire items, and .87 to .94 for test items.

The Polynomial Test (Appendix B) served as both an immediate posttest and a retention test. The retention test was given six weeks after the immediate posttest was given. The Polynomial Test was of the short-answer form and consisted of twenty-two problems that the students either had to multiply or factor. The test items included four problems that the students had done previously (recall problems), twelve problems that were of the same form as had been done previously (application problems), and six transfer problems. Transfer problems were defined as those not discussed or practiced during the duration of the study. These transfer problems included the following: $2x^2 + 5x$, $x(x - 2)$, $(2x + 3)(2x + 5)$, $x(2x + 3)$, $5x - 15$, and $2x^2 + 7x + 6$. These problems differed from those practiced during the study to the extent that the operation of subtraction was included in some problems while other problems had an "x^2" term with a coefficient other than one. Problems of the form $ax^2 + bx + c$ where a, b, or c are negative integers were not included as transfer problems because of the students' limited experience with operating on negative integers. The scores on the Polynomial Test were indicated as the number of problems the students answered correctly.

This test was developed from a similar Polynomial Test of seventeen items that was used as the immediate posttest and the retention test in an exploratory research study conducted by this writer. The reliability of the first Polynomial Test was computed by the Kuder-Richardson formula (KR-20) and the reliability of the test was estimated as .82. The length of the second Polynomial Test was increased to twenty-two items and the reliability again was computed by the Kuder-Richardson formula. The reliability estimate of this second test was .89.

The content validity of the Polynomial Test was reviewed and recommended by the previously mentioned advisory committee. They indicated that the Polynomial Test was appropriate, considering the instructional materials and the types of problems completed by the students during the period of time prior to the administering of the test. Also, the textbooks and supplementary materials that served as sources for the writing of the instructional booklets, served as sources for the problems on the Polynomial Test.

Procedures

The research study involved two experiments in the Pulaski County School System. This writer met with the teachers prior to the beginning of the study to explain the procedures and to explain the teacher's role in administering the study. Workshops were conducted to show the teachers how to use the manipulatives as well as to explain the rationale of the instruction to be given to both kinds of groups.

Experiment I began December 3, 1976, in both Dublin Middle School and Pulaski Middle School. One teacher at each school was involved in the study. Each teacher taught both a manipulative group and a nonmanipulative group. Although instruction took place on approximately 12 days, these days were not necessarily on consecutive weekdays due to Christmas vacation and inclement weather. However, reviews were conducted each time the students came back to class. The last day of the study was on February 2, 1977.

Experiment II began February 3, 1977, in Dublin Middle School. Again, two teachers and four classes were involved. However, one teacher had three pre-algebra mathematics classes and the second teacher had only one. Through random assignment of treatments to groups, two of the first teacher's classes were nonmanipulative and the third was manipulative. The second teacher's one class was a manipulative group. Again instruction took place on approximately twelve days, but this time instruction took place on consecutive weekdays. The study ended by February 23, 1977.

The number of days of instruction varied among teachers and according to whether the group was manipulative or nonmanipulative. Some groups finished instruction in eight days where others took as long as thirteen days. (See Chapter 4 for the amount of time spent by each group.)

The Orleans-Hanna Algebra Prognosis Test was given on the first day of each experiment. Classroom instruction followed, where both the teachers and students used the booklets prepared for the study. Students learned how to multiply and factor polynomials in one variable by generalizing from the multiplication and factorization of whole numbers.

Both the overhead projector and the chalkboard were used by the teachers to illustrate how to solve problems. The students in the manipulative groups practiced working problems using their packets of strips and squares, sometimes following the teacher's movements, but mostly working by themselves at their desks.

Assistance was available from both the prepared booklet and from the classroom teachers. The students in the nonmanipulative groups practiced working problems using only pencil and paper. Assistance was also available to these groups from the prepared booklet and from the classroom teachers.

The classroom teachers and students were observed periodically by this writer to determine:

1. That the classroom teachers were following the prepared instructional materials.

2. What kinds of problems the students were having as they were learning and practicing with the material.

3. If periodic changes in school schedules effected the amount of time available to spend on classwork.

4. That the students were not assigned homework. All practice was to be done during the regular class periods.

The observations of teachers and students were written down in the form of comments by this investigator. Then the comments were compiled and became a part of the results.

After the instruction took place, all students took the immediate posttest. The manipulative groups were not allowed to use the area embodiment. However, during the last two to three days of instruction, the teachers showed the manipulative groups how they could draw pictures of the embodiment if it were necessary in working the problems. It was conjectured that many of the subjects would have internalized the operation by the time of the immediate posttest, but if not, then images would have been internalized.

Six weeks after each group completed the immediate posttest, a retention test was given to all subjects. It was the same test as the immediate posttest. Again, the manipulative groups could draw pictures, but they could not use the area embodiment.

Student interviews were conducted after the retention tests were given (see Interview Instruments in Appendix C.). The students interviewed were chosen because of their unusually high or unusually low performance on the two tests. The purpose of the interviews was to see what trouble the students were having and to determine what kinds of things they had learned and how they had learned them.

Time logs were kept by each teacher involved in both experiments. The purpose of the logs was to see whether or not the manipulative groups needed more time than the nonmanipulative groups to complete their work in the instructional booklets.

Each teacher was interviewed at the end of the study to get reactions to the instructional materials and methods used. These comments became a part of the results and are available in Chapter 4.

Treatment of Data Used in Study

The Nonequivalent Control Group Design was selected for this study. The groups were intact classes as similar as availability permitted but yet not so similar that one could dispense with some kind of pretest. There was, however, random assignment of treatments to the intact classes within each of the two schools.

This study involved the initial introduction of new subject matter. A pretest that was of an equivalent form to the posttest would have given no information concerning the groups. Thus, the OHAPT seemed to be an appropriate pretest since it measured general achievement in algebra-related topics.

Since the manipulative and nonmanipulative groups came from eighth grade pre-algebra classes and the students were selected to be in these classes according to county-wide criteria, they were similar in many respects. However, since the subjects could not be matched or assigned at random to groups, there was a need to confirm their similarity by comparing their mean scores on the pretest. The statistical procedure of analysis of variance was used for this purpose. If it were found that there were no significant differences in mean scores among the groups on the pretest, then the statistical procedure of analysis of covariance was used to analyze the differences between the nonmanipulative and the manipulative groups on both the immediate posttest and the retention test after taking into account initial differences in performance on the OHAPT. The computer programs ONEWAY (Nie, 1975) and MANOVA (Clyde, 1969) were used for the analyses.

If the analysis of variance on the mean pretest scores indicated no significant difference in achievement among the four groups participating in Experiment I, then the immediate posttest and retention test mean scores were analyzed using three oneway analyses of covariance. The paradigms for these analyses are illustrated in Figure 2.

The data from Experiment II was analyzed using three oneway analyses of covariance, provided that there were no significant differences among groups on the mean scores of the OHAPT. The paradigms for these analyses are illustrated in Figure 3.

Statement of Null Hypotheses - Experiment I

Hypothesis 1. There will be no difference in mean achievement scores between the manipulative and the nonmanipulative groups.

Hypothesis 2. There will be no difference in mean achievement scores between the manipulative and the nonmanipulative groups taught by Teacher A.

Hypothesis 3. There will be no difference in mean achievement scores between the manipulative and the nonmanipulative groups taught by Teacher B.

Hypothesis 4. There will be no difference in mean retention scores between the manipulative and the nonmanipulative groups.

Total Population for Exp. I	
Nonmanip.	Manip.

Teacher A's Groups for Exp. I	
Nonmanip.	Manip.

Teacher B's Groups for Exp. I	
Nonmanip.	Manip.

Figure 2. Paradigms for Analysis of Data from Experiment I.

Total Population for Exp. II	
Nonmanip.	Manip.

Teacher D's Groups for Exp. II*	
Nonmanip. 1	Manip.

Teacher D's Groups for Exp. II*	
Nonmanip. 2	Manip.

Figure 3. Paradigms for Analysis of Data from Experiment II.

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*Teacher D taught two nonmanipulative groups and one manipulative group while Teacher C taught only one manipulative group. Since an attempt was made throughout the study to control for the teacher variable, it was decided not to use the data from Teacher C's class in the statistical analyses.

Hypothesis 5. There will be no difference in mean retention scores between the manipulative and the nonmanipulative groups taught by Teacher A.

Hypothesis 6. There will be no difference in mean retention scores between the manipulative and the nonmanipulative groups taught by Teacher B.

Statement of Null Hypotheses - Experiment II

Hypothesis 1. There will be no difference in mean achievement scores between the manipulative and the nonmanipulative groups.

Hypothesis 2. There will be no difference in mean achievement scores between the manipulative group and nonmanipulative group 1 taught by Teacher D.

Hypothesis 3. There will be no difference in mean achievement scores between the manipulative group and nonmanipulative group 2 taught by Teacher D.

Hypothesis 4. There will be no difference in mean retention scores between the manipulative and the nonmanipulative groups.

Hypothesis 5. There will be no difference in mean retention scores between the manipulative group and nonmanipulative group 1 taught by Teacher D.

Hypothesis 6. There will be no difference in mean retention scores between the manipulative group and nonmanipulative group 2 taught by Teacher D.

Informal analyses of the items on the posttest and the retention test were performed using chi-square tests of independence. The individual items were tested to determine whether a systematic relationship exists between those items solved by the manipulative groups and those items solved by the nonmanipulative groups.

CHAPTER 4

FINDINGS OF THE STUDY

Introduction

The purpose of this chapter is to present data gathered during the investigation of the problem under consideration: Does the manipulation of concrete materials by students aid the learning of the mathematical transformation of multiplication of polynomials and of its inverse, factorization? The chapter is divided into six sections: immediate posttest and retention test data; item analyses; time log; classroom observations; teacher interviews; and student interviews.

Immediate Posttest and Retention Test Data

The data reported in this section were derived from raw scores made on the *Polynomial Test* that was given immediately following the treatments and from raw scores on the same test given again six weeks later to measure retention. Since the groups were intact classes and the students could not be assigned randomly to treatments nor could they be matched on some pertinent characteristic, it was necessary to use the Nonequivalent Control Group Design for this study. Hence, the *Orleans-Hanna Algebra Prognosis Test* (*OHAPT*) was given as a pretest and the scores from this test were used as a covariate in the analysis of the data, provided the data met the assumptions of ANCOVA and similarity of groups. A one-factor ANOVA was conducted on the OHAPT data to determine whether the treatment groups differed significantly prior to the treatments. The means and standard deviations of the pretest scores from both Experiment I and Experiment II as well as the final summaries of the one-factor ANOVA are located in Appendix D.

Since there were no statistical differences ($p = .172$) in mean pretest scores among the classes participating in Experiment I, the statistical procedure of analysis of covariance was used to analyze the differences between the manipulative and the nonmanipulative groups on both the immediate posttest and the retention test. The computer program MANOVA (Clyde, 1969), multivariate analysis of variance, was used for the analyses.

Since there were statistical differences ($p < .002$) in mean pretest scores among the classes participating in Experiment II, it was decided that the statistical procedure of analysis of covariance was not an appropriate statistical test to analyze the differences in achievement between the nonmanipulative and the manipulative groups. However, in order to see if at least one of the two nonmanipulative classes (Class 7 or Class 8) taught by Teacher D and the manipulative class (Class 6) taught by Teacher D could be used in the analysis of the data, two more one factor ANOVA's were conducted, using only two classes at a time. The final summaries of the analyses on the pretest scores are located in Appendix D. Class 5, which was taught by Teacher C, was not used in the statistical analyses since its use would have confounded the teacher variable. (Teacher C did not teach a nonmanipulative group.)

Since there were no statistical differences ($p = .523$) in mean scores on the pretest between classes 6 and 8, the statistical procedure of analysis of covariance was used to analyze the differences in achievement between the nonmanipulative and the manipulative groups on both the immediate posttest and the retention test. The computer program MANOVA was again used for the analyses.

The tests for homogenity of regression of scores on both the immediate posttests and retention tests from Experiments I and II are presented in Appendix E. The pooled regression coefficients are also given in this same appendix.

The posttest results from Experiment I were derived from raw scores made on the Polynomial Test given immediately following the treatments. Three analyses of covariance were performed on the unadjusted posttest scores. The first analysis involved the total population of Experiment I. The second analysis involved only those classes taught by Teacher A; and the third analysis involved only those classes taught by Teacher B. The last two analyses were done in order to get an indication of whether the teacher variable or possibly the school variable was still playing a role in the results, even though each teacher taught both a manipulative and a nonmanipulative group in their respective schools.

Table 1 gives the means, standard deviations, and adjusted means of pretest and posttest scores from the total population involved in Experiment I, divided according to treatment. The inclement weather and Christmas vacation interrupted the instructional sequence of this experiment. Although the students were reviewed on the material each time they returned to school, it appears that their mean scores were lower, on the whole, than those of the students involved in Experiment II, where there were no interruptions during the instruction.

Table 1 shows that the adjusted posttest mean of the nonmanipulative groups was higher than the adjusted posttest mean of the manipulative groups. But when taking the standard deviations into account, it would be expected that there would be no difference in the mean scores of the two treatment groups. Table 2 presents the final summary of the analysis of covariance on the adjusted posttest scores of the total population of Experiment I.

The F ratio presented in Table 2 indicates that there was no statistical difference in mean scores on the posttest between the manipulative and nonmanipulative groups ($p = 0.385$). Although this observation seems to be contrary to the theory developed in Chapter 2, Hypothesis 1, in its null form, cannot be rejected.

Table 1

Means, Standard Deviations, and Adjusted Means of Pretest and Posttest Scores from the Total Population of Experiment I, Divided According to Treatment

Group	N	Pretest $\bar{x}$ and S. D.	Posttest $\bar{x}$ and S. D.	Posttest Adj. $\bar{x}$
Manip. (Classes 1 & 3)	35	58.657 11.805	6.371 6.417	6.365
Nonmanip. (Classes 2 & 4)	45	58.578 10.168	7.489 5.442	7.494

Table 2

Final Summary of Analysis of
Covariance on Adjusted Posttest
Scores of the Total Population
of Experiment 1

Source	DF	SS	MS	F	P Less Than
Within Cells	77	2530.081	32.858		
Regression	1	173.334	173.334	5.275	0.024
T	1	25.063	25.063	0.763	0.385

Table 3 gives the means, standard deviations, and adjusted means of pretest and posttest scores from the manipulative and the nonmanipulative groups taught by Teacher A. There is indication from Table 3 that the adjusted posttest mean of the manipulative group was higher than the adjusted posttest mean of the nonmanipulative group. But since the adjusted mean score difference was less than one point and the standard deviations were approximately the same, it is doubtful that the difference would be significant. Table 4 presents the final summary of the analysis of covariance on the adjusted posttest scores of the classes taught by Teacher A.

The F ratio presented in Table 4 indicates that there was no statistical difference in mean scores on the posttest between the manipulative and nonmanipulative groups ($p = 0.609$). Thus, Hypothesis 2, in its null form, cannot be rejected.

Table 5 gives the means, standard deviations, and adjusted means of pretest and posttest scores from the manipulative groups taught by Teacher B. The table shows that the adjusted posttest mean of the nonmanipulative group was higher than the adjusted posttest mean of the manipulative group. Table 6 presents the final summary of the analysis of covariance on the adjusted posttest scores of the classes taught by Teacher B.

The F ratio presented in Table 6 indicates a marginal difference ($p = 0.071$) in mean scores on the posttest between the

Table 3

Means, Standard Deviations, and Adjusted Means of Pretest and Posttest Scores from the Classes Taught by Teacher A during Experiment I

Group	N	Pretest $\bar{x}$ and S. D.	Posttest $\bar{x}$ and S. D.	Posttest Adj. $\bar{x}$
Manip. (Class 1)	18	59.667 10.901	9.278 5.245	8.903
Nonmanip. (Class 2)	23	55.043 10.222	7.696 5.927	7.989

Table 4

Final Summary of Analysis
of Covariance on Adjusted Posttest
Scores of the Classes Taught
by Teacher A during Experiment I

Source	DF	SS	MS	F	P Less Than
Within Cells	38	1150.332	30.272		
Regression	1	90.147	90.147	2.978	0.093
T	1	8.037	8.037	0.266	0.609

Table 5

Means, Standard Deviations, and Adjusted Means of Pretest and Posttest Scores from the Classes Taught by Teacher B during Experiment I

Group	N	Pretest $\bar{x}$ and S. D.	Posttest $\bar{x}$ and S. D.	Posttest Adj. $\bar{x}$
Manip. (Class 3)	17	57.588 12.942	3.294 6.223	3.641
Nonmanip. (Class 4)	22	62.273 8.897	7.273 5.016	7.004

Table 6

Final Summary of Analysis
of Covariance on Adjusted
Posttest Scores of the Classes
Taught by Teacher B during
Experiment I

Source	DF	SS	MS	F	P Less Than
Within Cells	36	1072.964	29.805		
Regression	1	74.930	74.930	2.514	0.122
T	1	103.461	103.461	3.471	0.071

manipulative and nonmanipulative groups, favoring the nonmanipulative group. However, Hypothesis 3, in its null form, cannot be rejected.

These latter two analyses of covariance may indicate that some variance in student achievement resulted from the differences between teachers. Thus, the teacher variable was not entirely removed by having each teacher teach both a manipulative and a nonmanipulative group. Although it was not the intent of this study to compare teachers to see what these differences are, it should be noted that there was possibly some interaction.

The retention test results from Experiment I were derived from raw scores made on the Polynomial Test given six weeks after the treatments. The same three analyses of covariance were performed on the unadjusted retention test scores, as were performed on the unadjusted posttest scores.

Table 7 gives the means, standard deviations, and adjusted means of pretest and retention test scores from the total population involved in Experiment I, divided according to treatment. The table shows that the adjusted retention test mean of the manipulative groups was higher than the adjusted retention test mean of the nonmanipulative groups. Table 8 presents the final summary of the analysis of covariance on the adjusted retention test scores of the total population of Experiment I.

The F ratio presented in Table 8 indicates that there was a statistical difference in mean scores on the retention test between

Table 7

Means, Standard Deviations, and Adjusted Means of Pretest and Retention Test Scores from the Total Population of Experiment I, Divided According to Treatment

Group	N	Pretest $\bar{x}$ and S. D.	Retention Test $\bar{x}$ and S. D.	Retention Test Adj. $\bar{x}$
Manip. (Classes 1 & 3)	35	58.657 11.805	5.571 6.878	5.565
Nonmanip. (Classes 2 & 4)	45	58.578 10.168	2.333 3.357	2.338

Table 8

Final Summary of Analysis
of Covariance on Adjusted
Retention Test Scores of the
Total Population of Experiment I

Source	DF	SS	MS	F	P Less Than
Within Cells	77	1902.115	24.703		
Regression	1	202.455	202.455	8.196	0.005
T	1	204.934	204.934	8.296	0.005

the manipulative groups and the nonmanipulative groups ($p < 0.005$), favoring the manipulative groups. Thus, Hypothesis 4, in its null form, can be rejected. This observation seems to be compatible with the theory of learning developed in Chapter 2.

Table 9 gives the means, standard deviations, and adjusted means of pretest and retention test scores from the manipulative and the nonmanipulative groups taught by Teacher A. There is indication from the table that the adjusted retention test mean of the manipulative group was higher than the adjusted retention test mean of the nonmanipulative group. Table 10 presents the final summary of the analysis of covariance on the adjusted retention test scores of the classes taught by Teacher A.

The F ratio presented in Table 10 indicates that there was a statistical difference in mean scores on the retention test between the manipulative and nonmanipulative groups ($p < 0.009$), favoring the manipulative group. Thus, Hypothesis 5, in its null form, can be rejected. Again, the results indicated here are compatible with the theory of learning developed in Chapter 2.

Table 11 gives the means, standard deviations, and adjusted means of pretest and retention test scores from the manipulative and the nonmanipulative groups taught by Teacher B. The table shows that the adjusted retention test mean of the manipulative group was higher than the adjusted retention test mean of the nonmanipulative group.

Table 9

Means, Standard Deviations, and Adjusted Means of Pretest and Retention Test Scores from the Classes Taught by Teacher A during Experiment I

Group	N	Pretest $\bar{x}$ and S. D.	Retention Test $\bar{x}$ and S. D.	Retention Test Adj. $\bar{x}$
Manip. (Class 1)	18	59.667 10.901	8.722 7.019	8.211
Nonmanip. (Class 2)	23	55.043 10.222	3.174 4.075	3.574

Table 10

Final Summary of Analysis
of Covariance on Adjusted
Retention Test Scores of the Classes
Taught by Teacher A during
Experiment I

Source	DF	SS	MS	F	P Less Than
Within Cells	38	1035.173	27.241		
Regression	1	167.743	167.743	6.158	0.018
T	1	206.799	206.799	7.591	0.009

Table 11

Means, Standard Deviations, and Adjusted Means of Pretest and Retention Test Scores from the Classes Taught by Teacher B during Experiment I

Group	N	Pretest $\bar{x}$ and S. D.	Retention Test $\bar{x}$ and S. D.	Retention Test Adj. $\bar{x}$
Manip. (Class 3)	17	57.588 12.942	2.235 5.019	2.565
Nonmanip. (Class 4)	22	62.273 8.897	1.455 2.154	1.200

Table 12 presents the final summary of the analysis of covariance on the adjusted retention test scores of the classes taught by Teacher B.

The F ratio presented in Table 12 indicates that there was no statistical difference in mean scores on the retention test between the manipulative and nonmanipulative groups ($p = 0.241$). Although this conclusion is very tenuous due to the extremely low mean scores from both groups, it should be noted that the mean score of the nonmanipulative group fell considerably during the six weeks period of time (5.804 points). However, Hypothesis 6, in its null form, cannot be rejected.

The posttest results from Experiment II were also based on raw scores made on the Polynomial Test, given immediately following the treatments. It was intended that three analyses of covariance be performed on the unadjusted posttest scores. However, after the preliminary one factor ANOVA indicated differences in mean pretest scores among classes involved in Experiment II and differences in mean pretest scores between the manipulative group (Class 6) and nonmanipulative group 1 (Class 7) taught by Teacher D, only one analysis of covariance was performed. This analysis involved only the manipulative group (Class 6) and nonmanipulative group 2 (Class 8) taught by Teacher D. Thus, Hypotheses 1, 2, 4, 5 of Experiment II could not be tested for statistical significance.

Table 13 gives the means, standard deviations, and adjusted means of pretest and posttest scores from the manipulative group and

Table 12

Final Summary of
Analysis of Covariance on
Adjusted Retention Test
Scores of the Classes
Taught by Teacher B during
Experiment I

Source	DF	SS	MS	F	P Less Than
Within Cells	36	432.884	12.025		
Regression	1	67.629	67.629	5.624	0.023
T	1	17.051	17.051	1.418	0.241

Table 13

Means, Standard Deviations,
and Adjusted Means of Pretest
and Posttest Scores from
Classes Taught by Teacher D
during Experiment II

Group	N	Pretest $\bar{x}$ and S. D.	Posttest $\bar{x}$ and S. D.	Posttest Adj. $\bar{x}$
Manip. (Class 6)	24	47.000 9.682	12.708 5.767	12.914
Nonmanip. 2 (Class 8)	23	48.826 12.231	12.652 5.967	12.437

nonmanipulative group 2 taught by Teacher D. The table indicates that there was no difference between the adjusted posttest means of the two groups since the difference in scores was less than 0.5 and the difference in standard deviations was 0.2. Table 14 presents the final summary of the analysis of covariance on the adjusted posttest scores of the manipulative group and nonmanipulative group 2 taught by Teacher D.

The F ratio presented in Table 14 indicates that there was no statistical difference in mean scores on the posttest between the manipulative group and nonmanipulative group 2 ($p = 0.762$). This result is contrary to the theory presented in Chapter 2 and Hypothesis 3, in its null form, cannot be rejected.

The retention test results from Experiment II were also derived from raw scores made on the Polynomial Test given six weeks after the treatments. The same analysis of covariance was performed on the unadjusted retention test scores as was performed on the unadjusted posttest scores.

Table 15 gives the means, standard deviations, and adjusted means of pretest and retention test scores from the manipulative group and nonmanipulative group 2 taught by Teacher D. The table indicates that the adjusted retention test mean of the manipulative group was higher than the adjusted retention test of the nonmanipulative groups. But when taking the standard deviations into account, it might be expected that there would be no difference in the mean

Table 14

Final Summary of Analysis of Covariance on Adjusted Posttest Scores of the Manipulative Group and Nonmanipulative Group 2 Taught by Teacher D during Experiment II

Source	DF	SS	MS	F	P Less Than
Within Cells	44	1258.296	28.598		
Regression	1	289.879	289.879	10.136	0.003
T	1	2.657	2.657	0.093	0.762

Table 15

Means, Standard Deviations, and Adjusted Means of Pretest and Retention Test Scores from Classes Taught by Teacher D during Experiment II

Group	N	Pretest $\bar{x}$ and S. D.	Retention Test $\bar{x}$ and S. D.	Retention Test Adj. $\bar{x}$
Manip. (Class 6)	24	47.000 9.682	11.583 5.021	11.810
Nonmanip. (Class 8)	23	48.826 12.231	9.696 6.898	9.459

scores of the two treatment groups. Table 16 presents the final summary of the analysis of covariance on the adjusted retention test scores of the manipulative group and nonmanipulative group 2 taught by Teacher D.

The F ratio presented in Table 16 indicates that there was no statistical difference in mean scores on the retention test between the manipulative group and nonmanipulative group 2 ($p = 0.143$). Thus, Hypothesis 6, in its null form, cannot be rejected. It should be noted that the mean score of the nonmanipulative group fell approximately three points during the six weeks period of time whereas the mean score of the manipulative group fell approximately one point during this time.

Item Analyses

Informal analyses of the items on the immediate posttest and the retention test were performed using chi-square (x^2) tests of statistical significance. The individual items were tested to determine whether a systematic relationship exists between those items solved by the manipulative groups and those items solved by the nonmanipulative groups. The data are reported in Summary Tables 17 and 18.

Table 16

Final Summary of Analysis of Covariance on Adjusted Retention Test Scores of the Manipulative Group and Nonmanipulative Group 2 Taught by Teacher D during Experiment II

Source	DF	SS	MS	F	P Less Than
Within Cells	44	1276.196	29.004		
Regression	1	350.507	350.507	12.085	0.001
T	1	64.446	64.446	2.222	0.143

Table 17

x^2 Summary Table
of Immediate Posttest Items

ITEM (Type of)	Manipulative Students n = 59		Nonmanipulative Students n = 68		x^2	df	P Less Than
	Number Correct	Percent	Number Correct	Percent			
1. $6x + 10$ (Application)	10	16.9	18	26.5	1.15843	1	0.2818
2. $2x^2 + 5x$ (Transfer)	19	32.2	14	20.6	1.65323	1	0.1985
3. $x(x - 2)$ (Transfer)	8	13.6	28	41.2	10.54182	1	0.0012
4. $x^2 + 9x$ (Application)	28	47.5	20	29.4	3.64183	1	0.0563
5. $2(2x + 1)$ (Application)	26	44.1	32	47.1	0.02525	1	0.8737
6. $(x + 7)(x + 2)$ (Application)	29	49.2	41	60.3	1.16681	1	0.2801
7. $x^2 + 5x + 4$ (Recall)	30	50.8	33	48.5	0.00683	1	0.9341
8. $4x + 12$ (Recall)	28	47.5	28	41.2	0.28289	1	0.5948
9. $x^2 + 11x + 18$ (Application)	22	37.3	31	45.6	0.58621	1	0.4439
10. $(2x + 3)(2x + 5)$ (Transfer)	17	28.8	5	7.4	8.71549	1	0.0032
11. $x^2 + 10x + 25$ (Application)	37	62.7	31	45.6	3.06729	1	0.0799
12. $x(2x + 3)$ (Transfer)	29	49.2	34	50.0	0.00683	1	0.9341

Table 17 (Continued)

ITEM (Type of)	Manipulative Students n = 59		Nonmanipulative Students n = 68		x^2	df	P Less Than
	Number Correct	Percent	Number Correct	Percent			
13. $5x - 15$ (Transfer)	11	18.6	13	19.1	0.02536	1	0.8735
14. $2x^2 + 7x + 6$ (Transfer)	11	18.6	0	0.0	11.62355	1	0.0007
15. $(x + 6)(x + 6)$ (Application)	29	49.2	35	51.5	0.00683	1	0.9341
16. $7(x + 6)$ (Recall)	28	47.5	36	52.9	0.19229	1	0.6610
17. $(x + 5)(x + 3)$ (Recall)	31	52.5	41	60.3	0.48966	1	0.4841
18. $x^2 + 10x + 16$ (Application)	28	47.5	31	45.6	0.00104	1	0.9742
19. $(x + 8)x$ (Application)	24	40.7	40	58.8	3.46667	1	0.0626
20. $x^2 + 6x + 7$ (Application)	32	54.2	38	55.9	0.00005	1	0.9944
21. $x(x + 3)$ (Application)	22	37.3	39	57.4	4.32305	1	0.0376
22. $(x + 4)(x + 6)$ (Application)	29	49.2	38	55.9	0.33578	1	0.5623

Table 18

x^2 Summary Table
of Retention Test Items

ITEM (Type of)	Manipulative Students n = 59		Nonmanipulative Students n = 68		x^2	df	P Less Than
	Number Correct	Percent	Number Correct	Percent			
1. $6x + 10$ (Application)	15	25.4	10	14.7	1.66744	1	0.1966
2. $2x^2 + 5x$ (Transfer)	17	28.8	11	16.2	2.24614	1	0.1339
3. $x(x - 2)$ (Transfer)	2	3.4	21	30.9	14.29982	1	0.0002
4. $x^2 + 9x$ (Application)	23	39.0	18	26.5	1.72623	1	0.1889
5. $2(2x + 1)$ (Application)	30	50.8	23	33.8	3.09754	1	0.0784
6. $(x + 7)(x + 2)$ (Application)	22	37.3	21	30.9	0.32814	1	0.5668
7. $x^2 + 5x + 4$ (Recall)	22	37.3	12	17.6	5.25481	1	0.0219
8. $4x + 12$ (Recall)	24	40.7	15	22.1	4.30896	1	0.0379
9. $x^2 + 11x + 18$ (Application)	21	35.6	10	14.7	6.38046	1	0.0115
10. $(2x + 3)(2x + 5)$ (Transfer)	22	37.3	4	5.9	17.25731	1	0.0000
11. $x^2 + 10x + 25$ (Application)	31	52.5	9	13.2	20.83664	1	0.0000
12. $x(2x + 3)$ (Transfer)	23	39.0	20	29.4	0.90022	1	0.3427

Table 18 (Continued)

ITEM (Type of)	Manipulative Students n = 59		Nonmanipulative Students n = 68				
	Number Correct	Percent	Number Correct	Percent	x^2	df	P Less Than
13. 5x - 15 (Transfer)	3	5.1	4	5.9	0.03739	1	0.8467
14. $2x^2 + 7x + 6$ (Transfer)	13	22.0	0	0.0	14.37975	1	0.0001
15. (x + 6) (x + 6) (Application)	29	49.2	14	20.6	10.26953	1	0.0014
16. 7(x + 6) (Recall)	30	50.8	29	42.6	0.55617	1	0.4558
17. (x + 5) (x + 3) (Recall)	28	47.5	18	26.5	5.14893	1	0.0233
18. $x^2 + 10x + 16$ (Application)	20	33.9	12	17.6	3.60630	1	0.0576
19. (x + 8)x (Application)	23	39.0	25	36.8	0.00543	1	0.9413
20. $x^2 + 6x + 7$ (Application)	29	49.2	7	10.3	21.61081	1	0.0000
21. x(x + 3) (Application)	23	39.0	26	38.2	0.00929	1	0.9232
22. (x + 4) (x + 6) (Application)	31	52.5	19	27.9	7.01222	1	0.0081

Classroom Observations

The following classroom observations of both treatment groups were made by this writer as instruction was taking place in the individual classes at both Dublin Middle School and Pulaski Middle School:

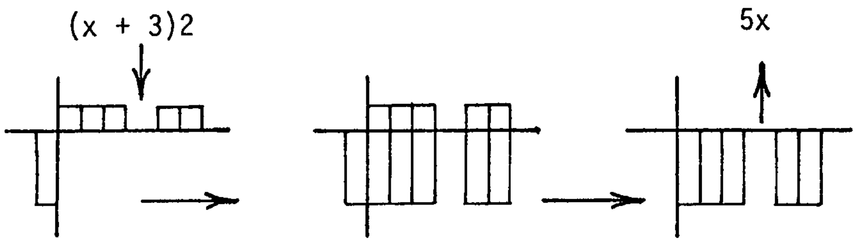
1. Manipulative Groups. The teachers followed the instructional booklets very closely. They used only examples from the booklet. They allowed the students to use the manipulatives to solve all the problems, if necessary. Teacher B was not explicit about the students writing the answers to the problems in their booklets. Thus, many students worked the problems using the manipulatives but did not write the answers in symbols in their booklets.

   As the students began to generalize and used the "x" strip instead of the "10" strip to solve the problems, some had trouble deciding which kind of strip to use. They had trouble comprehending that the "x" strip was not some specific whole number length.

   Some students had trouble forming the dimensions in order to multiply $4x \cdot x$. They did not seem to know how to make the dimension using the "4x" term.

   Many of the manipulative students, particularly at the beginning of instruction, had trouble knowing how to read their answers. They could find the area given the dimensions or find the dimensions given the area, but they had trouble interpreting the manipulative configurations into symbolic form. This seemed to be somewhat of a problem throughout the entire study.

When multiplying (x + 3)2 a few students did the following:

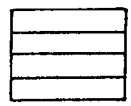


Another mistake was illustrated when the students were asked to multiply x(x + 4). Some students got a "7x" instead of an "x^2" as part of their answer because seven "x's" formed a rectangle that was very close in size to a square whose area was "x^2." Therefore, these students got "11x" as their final answer to the problem.

Very few students, if any, seemed to discover a pattern to multiply and factor polynomials, without having to depend upon the manipulatives. Even with using the strips and squares to multiply a problem such as (x + 3) (x + 7), many students wanted to attach an "x" or an "x^2" to the "21" when giving their answers in symbolic form.

For the most part, the students were very receptive to working with the strips and squares. Some of Teacher C's students indicated that the activity was childish. However, this attitude seemed to change as the study progressed and the teacher was able to get their cooperation.

At the beginning of each experiment many students were reluctant to try various arrangements of the strips and squares. They wanted to always stack them in the following arrangement:



A few students did not understand the "x^2" symbol for $x \cdot x$. They seemed to understand that a square with dimensions $x \cdot x$ had to have an area, but the notation "x^2" was not well understood even though they seemed to comprehend that $4 \cdot 4$ was 4^2.

2. Nonmanipulative Groups. The teachers followed the instructional booklet quite closely. They worked only those problems given in the book. Those teachers who taught both manipulative and nonmanipulative groups kept the treatments separated. They did not perform manipulative instruction in nonmanipulative classes or vice versa.

Many students felt that using repeated addition and expanded notation to multiply numerical expressions was a waste of time. The teachers explained that they would build on these notions to solve problems later in the study.

One of the major difficulties encountered in learning to solve multiplication and factorization problems was the combining of unlike terms:

$5x + 10 = 15x$

$x^2 + 5x = 5x^3$

$(x + 7)\ (x + 3) = (7x)\ (3x) = 10x$ or $21x^2$

Other students seemed to have trouble correctly adding like terms. They would get such answers as $11x^2$ when adding $5x + 6x$.

Many students confused $x + x$ with $x \cdot x$. In both cases they would get x^2 as their answer. Still others had trouble understanding why $x \cdot x$ was written as x^2.

Some other specific mistakes included the following:

a. $x(5 + x) = 5x + x$

b. $(x + 7)\ (x + 3) = x^2 + 21x + 21$

c. The factors of $10x$ are $2x$ and $5x$.

d. When factoring, $x^2 + x = x(x)$.

e. When asked to factor $5 + x^2$, many gave the answer as $x(5 + x)$.

A few students saw the pattern for factoring such problems as $x^2 + 4x + 3$ right away. However, there were still many mistakes as far as the "x" term was concerned. For example, many students factored $x^2 + 7x + 6$ to get $(x + 2)\ (x + 3)$.

<u>Time Log</u>

Each teacher kept a log of how much time he/she spent on instruction within each class. That information is summarized in Table 19.

Table 19

Time Log Summary

		Number of Class Periods of Instruction Per Group (Class Period - 50 min)	
		Manipulative Classes	Nonmanipulative Classes
Teachers	Teacher A*	8 1/2	8
	Teacher B*	13 1/2	13
	Teacher C+	13 1/2	--
	Teacher D+	11 1/2	9 1/2 (Class I) 9 1/2 (Class II)

*Due to the inclement weather and Christmas vacation during Experiment I it was necessary for the students to be reviewed on the previously taught material upon their return to school.

+Teacher C taught only one class, a manipulative group. Teacher D taught two nonmanipulative groups and one manipulative group.

Teacher Interviews

Each teacher was interviewed by this writer to get reactions and feedback from the teachers involved in the study. Each one verbalized his/her opinions, but also wrote reactions in a short paper. The following paragraphs indicate each teachers reaction to the study.

Teacher A. The nonmanipulative students seemed to enjoy the experiment. They enjoyed learning something new and somewhat difficult. Some may have grasped a little of the concept being taught but most just did the problems by copying the example problems. This group had fewer interruptions due to changes in the school's schedule than did the manipulative group. The manipulative students seemed somewhat skeptical about the entire experiment. Most seemed to lack confidence in their ability to come up with correct answers using the manipulatives. Very few learned how to do the problems without the manipulatives or pictures of the manipulatives. Many seemed to have trouble in knowing how to start a problem and then how to write their answer. This class was interrupted on several occasions due to changes in the school's schedule.

Teacher B. The manipulative group started out slower than the nonmanipulative group at the beginning of the experiment but began to move more quickly toward the end of the experiment because the students were visualizing the problems as they attempted to solve them. However, in some cases, the students got too caught up in the actual manipulating and little learning took place. The manipulatives seemed to help some of the slower students to see the problems and solutions much better than the slower students in the nonmanipulative group. The slower students in the non-manipulative group seemed to fall behind as they progressed to "multiple step" problems. It is important that students correlate the manipulative actions they are making and the problems at hand if they are not to have the manipulatives for testing purposes. Even if they are able to draw pictures on the test to work a problem, they might get their drawn pictures confused and thereby do poorly on the test. The manipulatives work very well, provided they can be used the entire time, including on the test.

Teacher C (She taught only a manipulative group.). The experiment went okay. Many students, however, did not accept the blocks very well. They felt like they were too old to be playing with blocks, that the blocks were childish. However, the ones who did have a positive attitude toward the experiment learned something. There were also those who tried but were just not capable of grasping algebra concepts yet and, thus, only did a fair job on the work. It was frustrating toward the end of the experiment because the manipulatives were so noisy and hard to keep up with while trying to teach at the same time. Many students did not like doing every example and every assigned problem with the manipulatives. They wanted to go faster than they were really able.

Teacher D (She taught two nonmanipulative groups and one manipulative group.). The nonmanipulative groups probably did better than the manipulative group. The students seemed to grasp the basic concepts as readily as the manipulative students without being slowed down by the manipulatives or without becoming dependent on them. I felt more comfortable with the nonmanipulative approach since it was closer to the approach I would normally use with students. As for student reaction, most in the nonmanipulative groups seemed to enjoy the experiment. Some of the manipulative group enjoyed the study but were slowed down by the manipulatives to the point of dragging. Perhaps more weaning away from the strips and squares would have improved their performance on the posttest. Both booklets were set up and organized well. They were easy for students to follow and understand.

Student Interviews

Students were selected to be interviewed because of their extremely good or extremely poor scores on the posttest and retention test. Students selected to be interviewed from the manipulative groups either drew no pictures or drew incorrect pictures on their test papers. The interview instruments are located in Appendix C.

Student 1 (nonmanipulative treatment). This student performed very well on both the posttest and the retention test. He also performed quite well in solving the problems he was asked to do during the interview. He only missed factoring problems of the form $ax^2 + bx + c$ where a, b, and c are positive integers. He indicated that he had forgotten the pattern. Once he was shown how the factoring was done he could do similar problems on his own but with not too much assurance that the problems were correct. The student did not use the notion of repeated addition to solve any of the problems at the interview. He seemed to be past the stage of having to depend on that notion. Also, this student had no trouble solving the transfer problem that involved multiplying $2x \cdot 2x$ to get $4x^2$. He did hesitate once when he needed to add $x + x$ to get $2x$. He got the correct answer, however. The only transfer problem that he could not do was to factor $2x^2 + 17x + 21$. Finally, this student did not seem to make the connection that factoring is the inverse of multiplying, even though he could perform both operations.

Student 2 (nonmanipulative treatment). This student did very poorly on both the posttest and the retention test. She also did not do well working other problems during the interview. The only problem she did correctly was $(x + 3)(x + 4)$. This she did using vertical multiplication. She attempted to do a similar problem, using vertical multiplication but she could not complete it because she could not multiply $2x \cdot 2x$ to get $4x^2$. This student did not seem to distinguish between addition and multiplication. She combined terms in the following way:

$x(2x + 3)$	$x(x + 3)$	$12 + 8x = 20x$
$= x + 2x + 3$	$= x + x + 3$	
$= 2x^2 + 3$	$= x^2 + 3$	
$= 5x^2$	$= 3x^2$	

She also seemed to confuse $x \cdot x$ with $x + x$. In both instances she got x^2 when she performed the operation. Finally, she could not begin to do the transfer problems that involved the operation of subtraction. She also did not attempt to factor any of the trinomials, whether they were transfer problems or not. In summary, Student 2 seemed to grasp only the pattern (or rule) for multiplying binomials using the vertical multiplication algorithm. The interchanging of the operations of addition and multiplication indicated that she has confused the rules for combining the terms of a polynomial.

Student 3 (manipulative treatment). This student did extremely well on both the posttest and the retention test, missing only the two transfer problems that involved the operation of subtraction. She did equally as well in solving the problems she was asked to do during the interview. She did not have to draw any pictures of the manipulatives in order to solve the problems on any of the three occasions. Although Student 3 demonstrated during the interview that she could use the manipulatives with great proficiency, she felt that she did not need to use them at all. When using the manipulatives she knew exactly how to arrange her "x" strips so that she would be able to fill in the small squares in such a way to get a rectangle whose dimensions were the factors of the "c" term in a polynomial of the form $ax^2 + bx + c$. Since she did not have to use the strips and squares or pictures of them to multiply or factor polynomials, Student 3 was asked how she did the problems. She responded that she visualized the solutions in her head. She could see the action of multiplication and factorization, without having to draw anything on paper. When asked if she had figured out any set of rules for operating on the polynomials she replied that she had not. She said that the reason she could not do the problems that involved the operation of subtraction was because she could not visualize the solutions in terms of strips and squares.

Student 4 (manipulative treatment). This student had not done well on either the posttest or the retention test. He had drawn pictures in order to multiply or factor, but most of the pictures were wrong. Also, he could not do the interview problems by drawing pictures. He indicated that he got confused when he drew pictures. However, when he was given the strips and squares at the interview he could work most of the problems correctly. The two problems that he missed were the transfer problems that involved the operation of subtraction and the manipulatives could not be used to get solutions. This student did not seem to have any trouble knowing when to multiply and when to factor. He knew how to write his answers in symbolic form. When factoring problems like $x^2 + 5x + 6$, he knew exactly how to arrange his "x" strips so that the arrangement of small squares would be a rectangle whose dimensions were 2 and 3, the factors of 6. He was also able to do this when factoring transfer problems like $2x^2 + 17x + 21$.

CHAPTER 5

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Summary

This study was designed to answer the following question: Does the manipulation of concrete materials by students aid the learning of the mathematical transformation of multiplication of polynomials and its inverse, factorization. In order to answer this question, a theory of learning mathematics was synthesized and used as a basis for designing the instructional materials. It was theorized that as students use manipulatives to learn mathematical concepts, the actions performed upon the concrete materials would be abstracted or internalized in the mind as operations. The student would then be able to take these operations or abstractions and extend their applicability through thought processes beyond the set of problems done with the manipulatives (Dienes, 1961). From these experiences the learner would generalize to a larger class of problems in a manner similar to how a student generalizes from arithmetic to algebra.

An area embodiment was used in the study to provide area preserving and reversible transformations which would be internalized as the multiplication operation on polynomials. There is literature that encourages the use of the area embodiment to teach multiplication and factorization of polynomials. Dienes (1971), Bidwell (1972), and Gibb (1974) all indicate that these concrete experiences help students to abstract essential concepts. However, with the exception of the exploratory study conducted prior to this study, none of the related literature contains references to any experimental research on this particular topic and accompanying embodiment.

Many research studies have been conducted where other concrete embodiments were used for teaching various topics, but the results were far from conclusive. It seems that the number of studies where the use of manipulative materials did aid the learning of some mathematical concept is slightly greater than the number of studies where they seemed to make no difference. However, most of these studies were conducted with elementary school students rather than with middle school students.

This study involved a comparison of the achievement of eighth grade pre-algebra students in the Pulaski County School System who used manipulatives to multiply and factor polynomials with the achievement of those who did not use manipulatives to operate on polynomials. Two experiments were conducted during the study, Experiment II being essentially a replication of Experiment I. Instructional materials were developed by this writer for use with both groups of students. The instructional booklet used by the nonmanipulative groups differed from the booklet used by the manipulative groups only to the extent that the manipulative groups used the area embodiment to solve problems. A Polynomial Test, developed especially for the study, was used as both an immediate posttest and a retention test six weeks later. The Orleans-Hanna Algebra Prognosis Test (OHAPT) served as the pretest and the scores from this test were used as the covariate in several analyses of covariance to determine whether differences existed between the two treatment groups on the immediate posttest and the retention test mean scores. Chi-square tests of statistical significance were performed on the individual posttest and retention test items. Time logs were kept by the participating teachers. Classroom observations were made by this writer; and student and teacher interviews were conducted at the conclusion of the study. All of this was done to better assess the role of manipulatives in learning to operate on polynomials.

The F ratios from the analyses of covariance conducted on the immediate posttest scores from Experiment I indicated that (1) using the total population of students, there was no statistical difference in mean scores between the manipulative and nonmanipulative groups ($p = .385$); (2) using only Teacher A's students, there was no statistical difference in mean scores between the manipulative and nonmanipulative groups ($p = .609$); and (3) using only Teacher B's students, there was a marginal difference in mean scores between the manipulative and nonmanipulative groups ($p = .071$), favoring the nonmanipulative group.

The F ratios from analyses of covariance conducted on the retention test scores from Experiment I indicated that (1) using the total population of students, there was a statistical difference in mean scores between the manipulative and nonmanipulative groups ($p < .005$), favoring the manipulative groups; (2) using only Teacher A's students, there was a statistical difference in mean scores between the manipulative and nonmanipulative groups ($p < .009$), favoring the manipulative group; and (3) using only Teacher B's students there was no statistical difference in mean scores between the manipulative and the nonmanipulative groups ($p = .241$).

The F ratios from an analysis of covariance conducted on the immediate posttest scores of the manipulative group and non-manipulative group 2 taught by Teacher D during Experiment II indicated that there was no statistical difference in mean scores between the two treatment groups ($p = 0.762$). The F ratios from an analysis of covariance conducted on the retention test scores of these same groups indicated that there was no statistical difference in mean scores between the manipulative and nonmanipulative groups ($p = 0.143$).

A summary of the results of the chi-square tests on both the immediate posttest and the retention test items indicates that there were statistical differences in performance on various test items between the manipulative groups and the nonmanipulative groups.

There were statistical differences in performance on the immediate posttest items favoring the manipulative treatment groups on items 4, 10, 11, 14 and favoring the nonmanipulative treatment groups on items 3, 19, 21. There were statistical differences in performance on the retention test items favoring the manipulative treatment groups on items 5, 7, 8, 9, 10, 11, 14, 15, 17, 18, 20, 22, and favoring the nonmanipulative treatment groups on item 3.

A summary of classroom observations indicates that the manipulative groups had some trouble throughout the study knowing how to read their answers to the problems and very few students could solve the problems without at least having to draw pictures of the manipulatives. The nonmanipulative group had trouble combining like terms and they did not seem to know when to add and when to multiply. They interchanged the operations at will.

The time log summary indicates that in only one case did the manipulative treatment group take longer to complete the instruction than did the nonmanipulative treatment group, taught by the same teacher. Teacher D felt, however, that this was due to disciplinary problems that were normally present in the class that served as a manipulative group in Experiment II.

Teacher reaction to the study indicated that some of the students in the manipulative groups seemed to lack confidence in their ability to come up with the right answers. They had trouble writing their answers in symbolic form. One teacher felt that the students got too caught up in the actual manipulating and little learning took place. Also, it was felt that many students could do the problems with the manipulatives but could not do them when the manipulatives were taken away, even though they were allowed to draw pictures. Teacher reaction to the performance of the nonmanipulative groups indicated that the students did the multiplication and factorization problems without learning concepts; they did the problems by copying the examples. One teacher indicated that she felt more comfortable with the nonmanipulative treatment than she did with the manipulative treatment.

Student interviews with those participating in the nonmanipulative treatment groups indicated that the students had memorized rules. The student who had performed well during testing had forgotten the rule for factoring trinomials. The poorer student only knew the rule for multiplying binomials and also interchanged the operations of addition and multiplication when attempting other problems. The students interviewed from the manipulative groups had not figured out symbolic patterns for multiplying and factoring polynomials. The student who performed well on all the tests indicated that she had to see the action with the strips and squares in her mind before she could work the problems. The student who did not test well could do all the problems as soon as he was given the strips and squares. He could not draw accurate pictures. He was totally dependent on the concrete materials.

Conclusions

The results of no statistical differences from the analysis of covariance on the mean scores of the immediate posttest between groups could be interpreted, at first glance, to mean that the theory of learning developed in Chapter 2 is not supported by the data. However, this may not be true for a number of reasons. Initially, it seemed that the nonmanipulative groups were learning more and perhaps at a slightly faster pace. However, they were using a familiar mode of instruction and they were asked to take the tests in the symbolic mode, with which they were already comfortable. The manipulative groups seemed slow and unsure of themselves. They had not been exposed to this mode of instruction previously. Also, they were asked to take the tests without using the strips and squares. This testing procedure was necessary in order to determine whether the students had internalized the actions associated with the manipulatives. Hence, for short-termed learning, the nonmanipulative treatment seems to be as effective, if not more so, than the manipulative treatment. However, six weeks later, the manipulative treatment groups were scoring higher than the nonmanipulative treatment groups on the retention test. Although Teacher B's and Teacher D's manipulative groups were not scoring statistically higher than their nonmanipulative groups, the mean retention scores of the nonmanipulative groups had fallen below the mean retention scores of the manipulative groups, while the manipulative retention scores remained in the same range as the immediate posttest scores.

Thus, it must be assumed that the manipulative groups acquired some understanding and retained what they had acquired, be it ever so little. Skemp (p. 46) indicates that to understand something means to assimilate it into an appropriate mental structure or schema. The actions performed on the concrete materials seemed to aid in building up a stock of imagery in the student's mind. Thus, he was able to abstract what was common to each experience and thereby assimilate these experiences into an appropriate schema or perhaps build a new and separate schema.

An attempt was made to have the nonmanipulative treatment be schematic also. This was done to the extent that the instruction in both treatment groups differed only in terms of the use of the concrete materials by the manipulative groups. However, due to the lack of retention over the six weeks period of time, it is likely that rote memorization of rules took place. Hence, it appears that the lack of experience with manipulating the area embodiment seemed to have fostered manipulation of symbols without meaning for these particular students. They did not build cognitive structures or schemas in their minds.

Student performance on individual items from both the immediate posttest and the retention test also seems to give support to the contention that the manipulation of concrete materials does aid the learning of the transformation of multiplication of polynomials and its inverse, factorization. Students in the manipulative treatment groups performed statistically better on four posttest items and on twelve retention test items whereas non-manipulative students performed statistically better on three posttest items and one retention test item.

The results of this study also tend to support the part of the theory of learning that indicates that a person goes through various stages in order to attain operational structures in his mind and that these stages are similar in comparison to Piaget's levels of cognitive development from actions to operations. It seems quite likely that many students did not internalize the associated actions at an operational level during the instructional period. There was evidence through student interviews and classroom observation that some students could multiply and factor using the manipulatives themselves, but could neither draw pictures nor visualize solutions to the problems. Thus, the actions with the concrete materials had not been internalized at either a representative or an operational level. Many of these same students did not seem to see factoring as the inverse of multiplication. They could do two separate operations, but never seemed to make the connection between the two actions.

There were other manipulative students who internalized the operation of multiplication of polynomials and its inverse, factorization, in that they could draw pictures of the strips and squares to solve the problems. Other students did not draw pictures but were able to multiply and factor. Student interviews as well as classroom observation indicated that these students visualized the strips and squares in their minds in order to solve the problems.

The kinds of transfer problems that the manipulative students were able to solve also indicated that these students were not generalizing to symbolic patterns to multiply and factor polynomials. They could solve transfer problems like $(2x + 3)(2x + 5)$ and $2x^2 + 7x + 6$ because they could visualize the actions with the strips and squares. However, they could not factor or multiply transfer problems that contained the operation of subtraction.

The nonmanipulative students seemed to be memorizing rules rather than internalizing operations and building cognitive structures. Very few students could solve a transfer problem like $(2x + 3)(2x + 5)$ and none of the students could solve a transfer problem like $2x^2 + 7x + 6$. The multiplication and factorization of these problems do not follow exactly the rules learned during instruction. The nonmanipulative students were able to multiply the transfer problem $x(x - 2)$. But this multiplication is very similar to that of $x(x + 3)$. The nonmanipulative students did not seem to be able to factor as well as they could multiply. This was probably due, in part, to their not seeing factorization as the inverse of multiplication and, in part, because factoring is more a trial and error process than is multiplication. Hence, most of the nonmanipulative students did not appear to have developed any schematic structure at the end of the instructional period. In fact, they appeared to be manipulating symbols in rotely learned patterns. This was further evidenced by their consistency in combining unlike terms incorrectly and in interchanging the operations of addition and multiplication point that inclement weather and Christmas vacation seemed to play a role in the amount of learning that took place during Experiment I. Even though there were reviews each time the students returned to school, the mean scores for all groups were below 50% of the problems done correctly. However, these kinds of interruptions in school schedules are very typical, so the data perhaps reflects what happens in quite a few learning environments when the daily sequence of instruction is broken.

Another factor that could possibly have influenced the results of the study, is the teacher variable. As has been indicated previously, there was an attempt to factor out the teacher variable by assigning each teacher both a manipulative and a nonmanipulative group. However, student achievement did vary somewhat from teacher to teacher. Although it was not the purpose of this study to determine those differences, it should be kept in mind that many teachers were not familiar with a manipulative approach to teaching concepts. When they were required to shift their strategy and the pupils were required to manipulate concrete aids illustrating their understandings, teachers might not have been as skillful in providing assistance. Classroom observations by this writer did indicate that one teacher was not helpful as would have been desired in weaning the students away from the manipulatives. It should also be noted that the nonmanipulative treatment was unfamiliar to the teachers because it was not the usual "textbook" approach to teaching multiplication and factorization of polynomials. Finally, a teacher's effectiveness could possibly have been related to the preference developed by that teacher as he/she used and compared both treatments. Teacher D did indicate in her reactions to the study that she preferred the nonmanipulative treatment. However, this preference was not detectable as this writer observed her classroom instruction.

Recommendations for Further Research

Since this study was an exploratory one, another study should be conducted using the same treatments, but involving more teachers, students, and geographical locations. Also, if at all possible, both students and teachers should be selected and assigned randomly to two treatment groups. Teacher workshops should be expanded to include more demonstration of the instruction that should take place in both treatment groups.

It is highly recommended that either in conjunction with a study similar to this one or as a separate study, that extensive use be made of student interviews. This would aid in gaining a better understanding of the role of concrete materials in the learning processes associated with acquiring the operation of multiplication of polynomials and its inverse, factorization. The interviews would give indication of how conceptual structures are formed and whether students actually do go through discernible stages in order to acquire these structures.

It is also recommended to add a third treatment using materials already developed with an emphasis on the visual modality of the area embodiment. In this way, the researcher may determine whether it is necessary for each student to manipulate concrete materials to learn an operation or whether it is only necessary that the students experience the actions through teacher demonstrations of the concrete materials and through visual manipulations. Again, extensive use of student interviews would give information as to how learning is taking place.

A study concerning the various measures of teacher characteristics related to pupil achievement would be appropriate. The purpose of the research would be to identify those characteristics which might possibly cause a manipulative treatment, a visual treatment, or a nonmanipulative treatment to be more effective for one teacher than for another in facilitating retention of mathematical concepts.

REFERENCES

Anderson, George R. "Visual-Tactual Devices: Their Efficacy in Teaching Area, Volume, and the Pythagorean Relationship to Eighth Grade Children." (Pennsylvania State University, 1957) DAI 18:160; Jan., 1958.

Beilin, Harry. "Linguistic, Logical and Cognitive Models for Learning Mathematical Concepts." A paper presented to the Research Workshop on Models for Learning Mathematics, University of Georgia Center for the Study of Learning and Teaching Mathematics, May, 1975.

Bidwell, James K. "A Physical Model for Factoring Quadratic Polynomials," Mathematics Teacher, Mar., 1972, 201-205.

Bledsoe, Joseph C., Jerry D. Purser, and Frantz Nevin, Jr. "Effects of Manipulative Activities on Arithmetic Achievement and Retention." Psychological Reports, Vol. 35 (Aug., 1974), 247-252.

Boersig, Teresa Marie. "The Effects of Instruction in the Enactive Mode of Representation on Multi-Variable Verbal Problems Encountered in Elementary Algebra." (Purdue University, 1973) DAI 34A:3230; Dec., 1973.

Bruner, Jerome S. Towards a Theory of Instruction. New York: W. W. Norton & Co., 1966.

Bunch, Martha A. "A Study of the Effects on Retention and on the Problem-Solving Ability of Students When Geometry Is Used as an Aid in Teaching Factoring of Second-Degree Polynomials." (University of Missouri - Kansas City, 1972) DAI 34A:1057-1058; Sept., 1973.

Carmody, Lenone M. "A Theoretical and Experimental Investigation Into the Role of Concrete and Semi-Concrete Materials in the Teaching of Elementary School Mathematics." (Ohio State University, 1970) DAI 31A:3407; Jan., 1971.

Copeland, Richard W. How Children Learn Mathematics. New York: Macmillan Publishing Co., Inc., 1974.

Crowder, Alex B. "A Comparative Study of Two Methods of Teaching Arithmetic in the First Grade." (N. Texas State University, 1965). DAI 26:3778; Jan., 1966.

Dienes, Z. P. Building Up Mathematics. London: Hutchinson Educational Ltd., 1971.

Dienes, Z. P. "On Abstraction and Generalization," Harvard Educational Review, 31 (Summer, 1961), 281-301.

Fennema, Elizabeth H. "A Study of the Relative Effectiveness of a Meaningful Concrete and a Meaningful Symbolic Model in Learning a Selected Mathematical Principle." (University of Wisconsin, 1969) DAI 30A:5338-5339; June, 1970.

Fennema, Elizabeth H. "Models and Mathematics," Arithmetic Teacher, 18 (Dec., 1972), 635-640.

Gibb, Allan A. "More on Physical Models For Factoring Polynomials," Mathematics Teacher, Feb., 1974, 133-138.

Green, Geraldine A. "A Comparison of Two Approaches, Area and Finding a Part of, and Two Instructional Materials, Diagrams and Manipulative Aids, on Multiplication of Fractional Numbers in Grade Five." (University of Michigan, 1969) DAI 31A: 676-677; Aug., 1970.

Grossman, Rose. "Lorena and Molly and Maths and Me." Mathematics Teaching, 68 (Sept., 1974), 4-6.

Haynes, Jerry O. "Cuisenaire Rods and the Teaching of Multiplication to Third Grade Children." (Florida State University, 1963). DAI 24:4545; May, 1964.

Hollis, Loye Y. "A Study to Compare the Effects of Teaching First and Second Grade Mathematics by the Cuisenaire-Gattegno Method with a Traditional Method," School Science and Mathematics, Vol. 65 (Nov., 1965), 683-687.

Howard, Charles F. "British Teacher Reactions to the Cuisenaire-Gattegno Materials," Arithmetic Teacher, Vol. 4 (Nov., 1957), 191-195.

Jamison, King W., Jr. "An Experiment With a Variable Base Abacus," Arithmetic Teacher, Vol. 11 (Feb., 1964), 81-84.

Karatzinas, D. and T. Renshaw. "Teachers' Views of the Cuisenaire Method," The Scottish Educational Journal, Sept. 19 and 26, Oct. 3, 1958.

Knaupp, Jonathan E. "A Study of Achievement and Attitude of Second Grade Students Using Two Modes of Instruction and Two Manipulative Models for the Numeration System." (University of Illinois at Urbana - Champaign, 1970) DAI 31A:6471; June, 1971.

Laycock, Mary and Reuben A. Schadler. Algebra in the Concrete. Hayward, Calif.: Activities Resources Co., Inc., 1973.

Lucow, William H. "Testing the Cuisenaire Method," Arithmetic Teacher, Vol. 10 (Nov., 1963), 435-438.

MANOVA Program. Miami, Fla.: Clyde Computing Service, 1969.

Mick, Harold W. and Carol J. Altizer. "The Concrete and Symbolic Modes of Representation in Learning the Distributive Principle in Elementary Algebra: An Exploratory Study." A research project conducted at VPI & SU, Blacksburg, Va.

Miller, William A. Laboratory Activities in Algebra. Portland, Maine: J. Weston Walch, 1974.

Moody, William B., Roberta Abell, and R. Barker Bausell. "The Effect of Activity-Oriented Instruction Upon Original Learning, Transfer, and Retention," Journal for Research in Mathematics Education, Vol. 2, No. 3 (May, 1971), 207-212.

Nichols, Edith June. "A Comparison of Two Methods of Instruction in Multiplication and Division for Third-Grade Pupils." (University of California, Los Angeles, 1971) DAI 32A:6011; May, 1972.

Nie, Norman H. et al. Statistical Package for Social Sciences. New York: McGraw Hill Book Co., 1975.

Olley, Peter George. "The Relative Efficacy of Four Experimental Protocols in the Use of Model Devices to Teach Selected Mathematical Constructs." (Washington State University, 1973) DAI 34A:4993; Feb., 1974.

Orleans, Joseph B. and Gerald S. Hanna. Orleans-Hanna Algebra Prognosis Test Manual. New York: Harcourt, Brace, Jovanovich, Inc., 1968.

Passy, Robert A. "The Effect of Cuisenaire Materials on Reasoning and Computation." Arithmetic Teacher, Vol. 10 (Nov., 1963), 439-440.

Pereira-Mendoza, L. "Rectangle, Trees and Factoring." School Science and Mathematics, 74 (Dec., 1974), 708-710.

Punn, Autar Kaur. "The Effects of Using Three Modes of Representation in Teaching Multiplication Facts on the Achievement and Attitudes of Third Grade Pupils." (University of Denver, 1973) DAI 34A:6954-6955; May, 1974.

Purser, Jerry D. "The Relation of Manipulative Activities, Achievement and Retention, In a Seventh-Grade Mathematics Class: An Exploratory Study." (University of Georgia, 1973) DAI 34A: 3255-3256; Dec., 1973.

Rasmussen, Peter. The Mathtiles Book. Berkley: Key Curriculum Project, 1976.

Rich, Littleton W. "The Effects of a Manipulative Instructional Mode in Teaching Mathematics To Selected Seventh Grade Inner-City Students." (Temple University, 1972) DAI 33B:330; July, 1972.

Richmond, P. G. An Introduction To Piaget. New York: Basic Books, Inc., 1970.

St. Martin, Allen H. "An Analysis of the Relationship Between Two Alternate Procedures for the Utilization of Teaching Aids and Piaget's Developmental Theory During the Initial Introduction of Selected Fifth Grade Mathematical Topics." (University of Houston, 1974) DAI 35A:7037-7038; May, 1975.

Sawada, Daiyo. "Piaget and Pedegogy: Fundamental Relationships." Arithmetic Teacher, April, 1972, 293-298.

Sawyer, W. W. Vision in Elementary Mathematics. Baltimore: Penguin Books, 1964.

Schminke, C. W. and William R. Arnold. Mathematics Is a Verb. Hinsdale, Illinois: The Dryden Press, Inc., 1971.

Skemp, Richard R. The Psychology of Learning Mathematics. Baltimore: Penguin Books, Inc., 1971.

Steffe, Leslie and Charles D. Smock. "On a Model for Learning and Teaching Mathematics." In L. P. Steffe (ed.), Research on Mathematical Thinking of Young Children: Six Empirical Studies. Washington, D. C.: National Council of Teachers of Mathematics.

Toney, Jo Anne. "The Effectiveness of Individual Manipulation of Instructional Materials as Compared to a Teacher Demonstration in Developing Understanding in Mathematics." (Indiana University, 1968) DAI 29A:1831-1832; Dec., 1968.

Trask, Marvin W. "A Study on Interaction Between Aptitudes and Concrete vs. Symbolic Teaching Methods as Presented to Third-Grade Students in Multiplication and Division." (University of Oklahoma, 1972) DAI 33A:4253-4254; Feb., 1973.

Van Engen, H. "An Analysis of Meaning in Arithmetic." Elementary School Journal, Vol. 49 (February-March, 1949), 397.

Weber, Audra W. "Introducing Mathematics to First Grade Children: Manipulatives vs. Paper and Pencil." (University of California, 1969) DAI 30A:3372-3373; Feb., 1970.

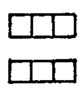
APPENDIX A

Instructional Booklets

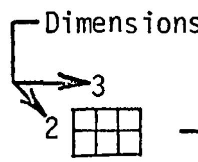
MULTIPLICATION OF POLYNOMIALS

I. Let's look at the operation of multiplication of whole numbers. 2 x 3 or 2 · 3 can be thought of as repeated addition. Thus, 2 x 3 means 2 three's added together. This repeated addition can be pictured through the use of unit squares, such as □ , and by putting these units together to form a rectangle whose dimensions are 2 and 3 and whose area is the product of 2 and 3 or the number of unit squares in the rectangle.

Two "3's" →



→

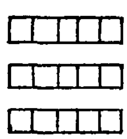


→ Area = 6

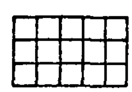
∴ 2 x 3 = 6

Let's look at another example:

Three "5's" →



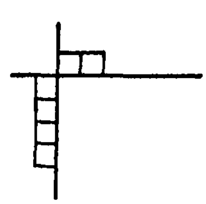
→



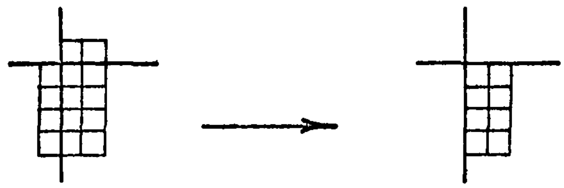
→ Area = 15

∴ 3 x 5 = 15

Now let's use the embodiment the teacher has given you to multiply 4 x 2. Since 4 and 2 from the dimensions of a rectangle, let's place 4 unit squares along the outside of the left-hand side of the dividers and 2 unit squares above the dividers, as illustrated below:

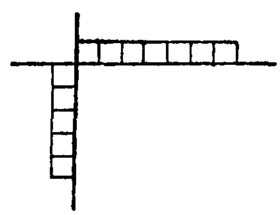


Now fill in the rectangle with four "2's":

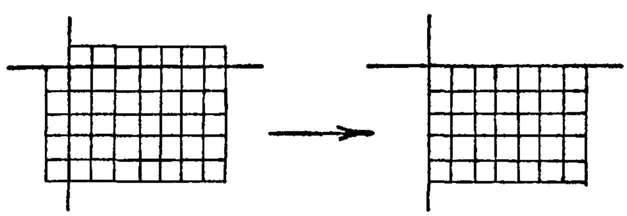


Thus, it took 8 unit squares to fill in the rectangle and the area of the rectangle is 8 unit squares. Therefore, 4 x 2 = 8.

Let's look at a second example: Multiply 5 x 7. First place 5 unit squares along the left-hand side of the dividers and 7 unit squares above the dividers:

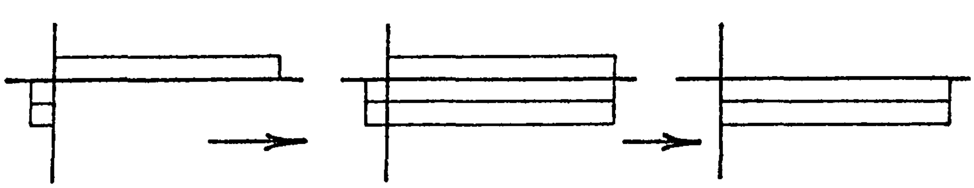


Next fill in the rectangle having these dimensions:



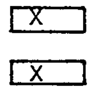
Thus, we filled in the rectangle with five "7's" or 35 unit squares. The area of the rectangle is 35 unit squares and, therefore, 5 x 7 = 35.

Now, when multiplying 2 x 10, using the embodiment we will replace ten "1's" with a "10" strip. Thus, two "10's" look like:

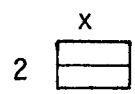


Therefore, 2 · 10 = 20.

Hence, by letting x stand for any whole number, 2 · x can be thought of in terms of repeated addition also: two "x's" added together, x + x or 2x. Using the embodiment, the strip, can be used to stand for any whole number x:

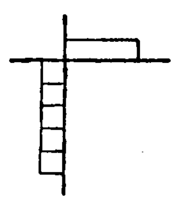
Two "x's": 

These "x's" can be put together in such a manner that the dimensions of the newly formed rectangle are 2 and x:

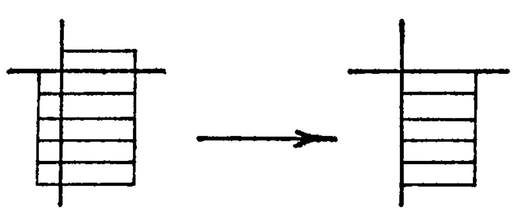


It took two "x" strips to form the rectangle. Hence, the area of the rectangle is 2x and 2 · x = 2x.

Now let's use the dividers, strips, and squares to multiply 5 · x: First place 5 unit squares on the left-hand side of the dividers and the "x" strip along the top of the dividers:

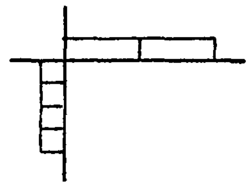


Next fill in the rectangle having these dimensions:

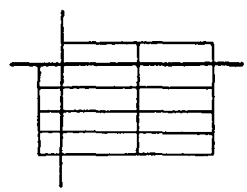


It took 5 "x" strips. Thus, the area is 5x and 5 · x = 5x.

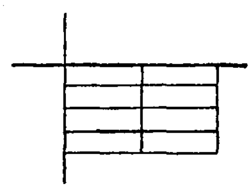
Let's look at another example: Multiply 4 · 2x. First place 4 unit squares on the left-hand side of the dividers to form the first dimension of the rectangle and then form the second dimension by placing two "x" strips along the top of the dividers:



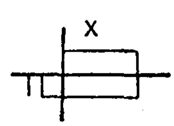
Next fill in the rectangle having these dimensions:



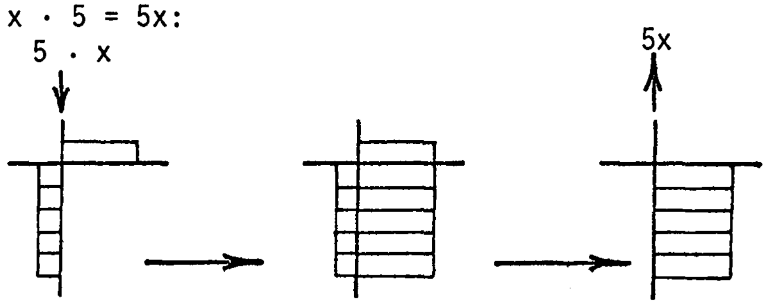
Thus, the area of the rectangle is 8x and $4 \cdot 2x = 8x$:



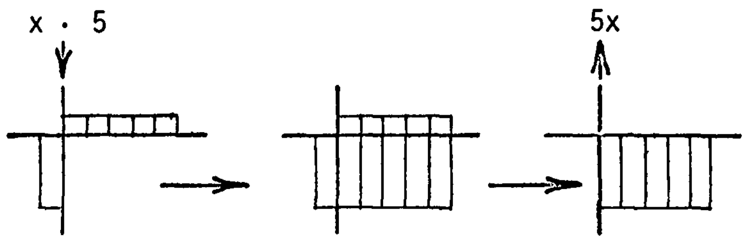
Note that 1x can be written as "x" since 1x is the product of $1 \cdot x$ and $1 \cdot x$ indicates that we have 1 copy of x:



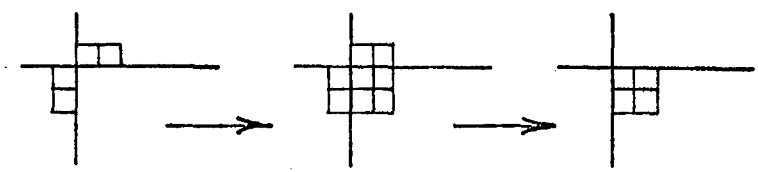
We can also illustrate, with strips and squares, that $5 \cdot x = x \cdot 5 = 5x$:



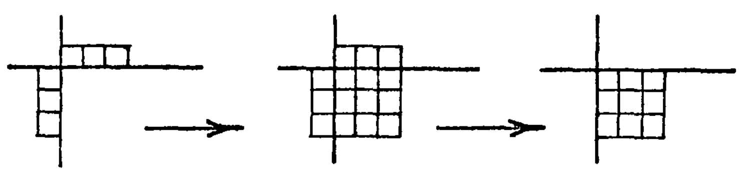
<u>or</u>



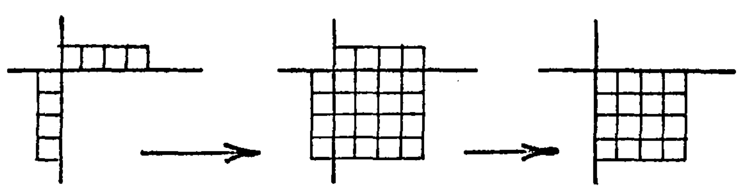
Now let's look at the following multiplications: 2 x 2 can be written as 2^2 where the exponent indicates how many two's are multiplied together. 2^2 is the product of 2 x 2 and is illustrated as follows, with unit squares:



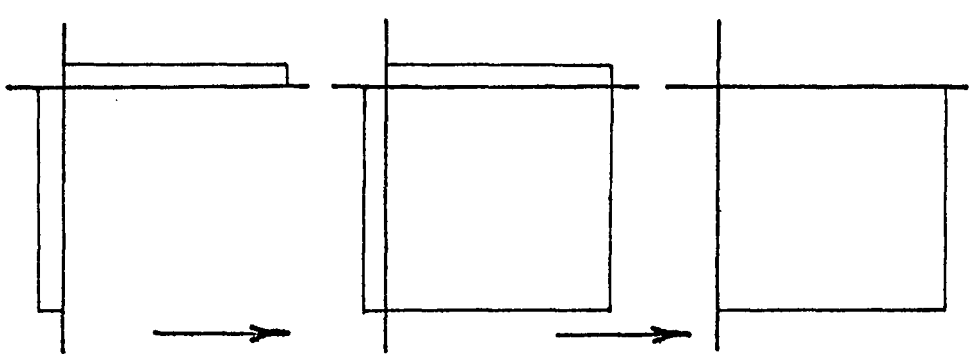
Also, $3 \cdot 3 = 3^2$ or



$4 \cdot 4 = 4^2$ or

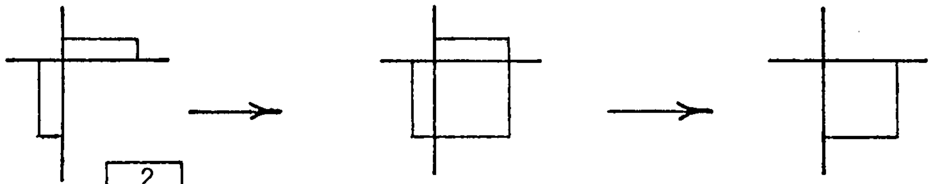


$10 \cdot 10 = 10^2$ or



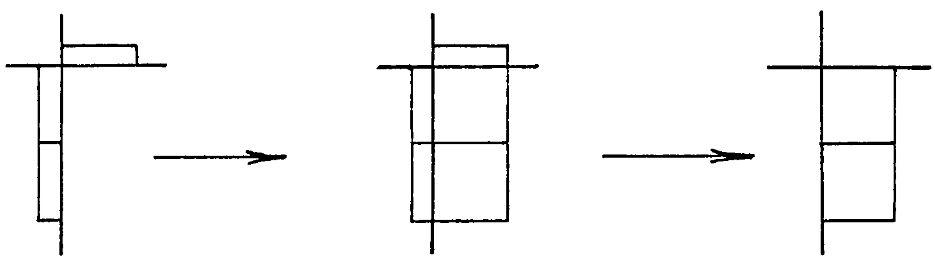
where the "10" square replaces ten "10" strips.

If x represents any whole number, then $x \cdot x = x^2$ or

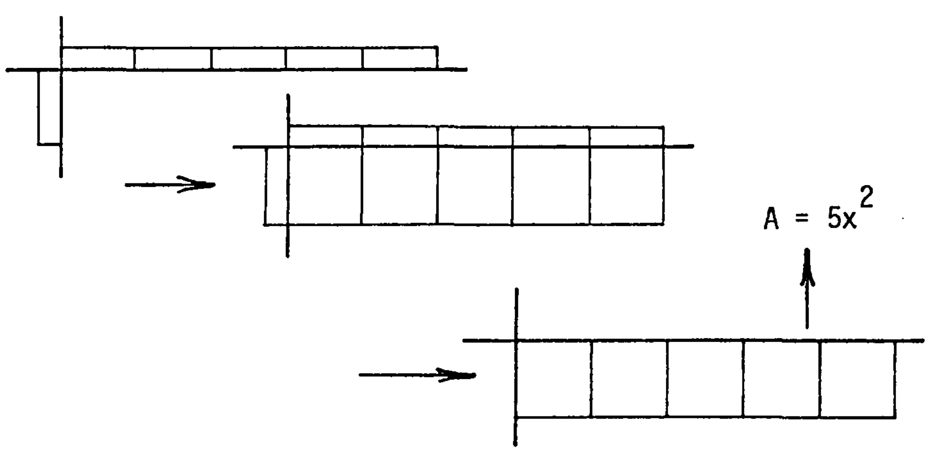


where $\boxed{x^2}$ replaces "x" number of "x" strips.

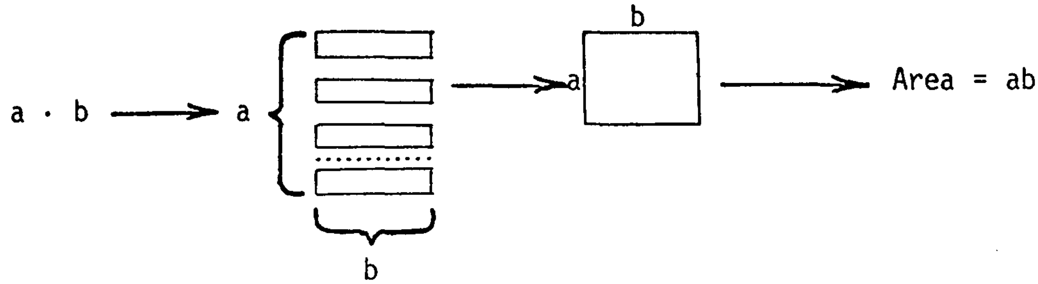
Also, $2x \cdot x$ is a rectangle whose dimensions are 2x and x and whose area is $2x^2$:



Finally, $x \cdot 5x$ is $5x^2$:



Thus, when we multiply we find "a" copies of "b" and this can be illustrated through the use of the strips and squares embodiment, where both "a" and "b" are dimensions of a rectangle and their product "ab" is the area of the rectangle:



<u>Problems</u>

Multiply the following:

1. $2 \cdot 7$

2. $5 \cdot 3$

3. $5 \cdot y$

4. $x \cdot 10$

5. $3 \cdot 3x$

6. $4x \cdot 9$

7. $4x \cdot x$

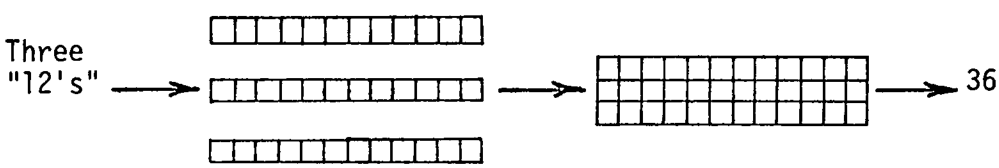
8. $2 \cdot x^2$

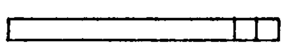
9. $0 \cdot x$

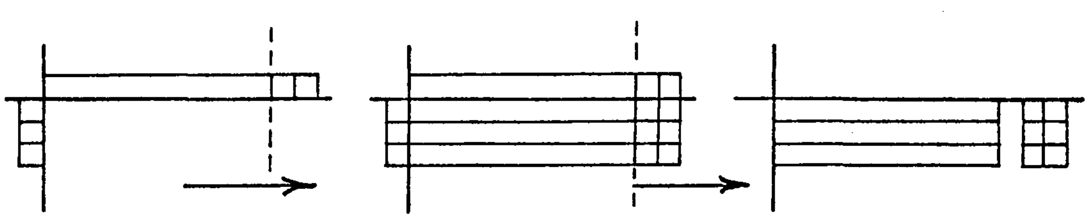
10. $1 \cdot x$

11. $x \cdot 3x$

II. When we begin to multiply numbers with more than one digit, multiplication is still thought of in terms of repeated addition. For example, $3 \cdot 12$ means 3 twelve's added together. This also means that $3 \cdot 12$ can be represented by using unit squares and forming a rectangle with these squares whose dimensions are 3 and 12:



But in order to make the addition easier, we sometimes use the notion of place value and write 12 as (10 + 2) and represent it as  . Thus, we have 3(10 + 2) or 3 "10 + 2's". Using our embodiment we have:



or 3 ten's (3(10)) and 3 two's (3(2)) which gives us

$3(10) + 3(2)$

$= 30 + 6$

$= 36$

Let's look at another example: Multiply 4 · 13.

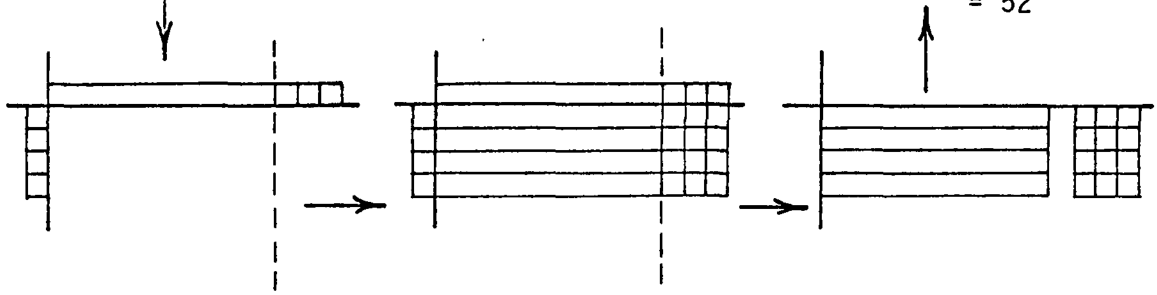
4(13)

= 4(10 + 3)

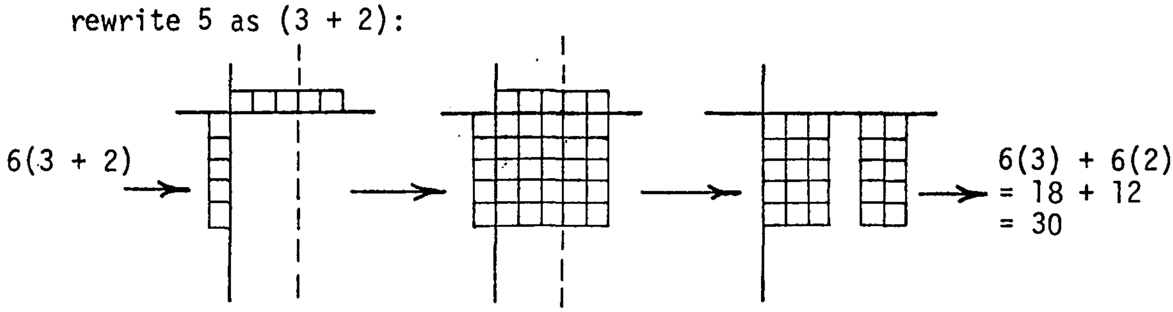
4(10) + 4(3)

= 40 + 12

= 52

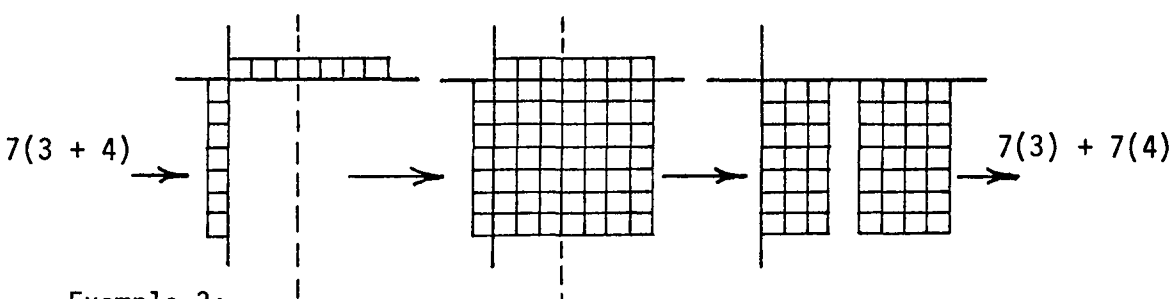


This procedure can also be used with 6 x 5 where we can rewrite 5 as (3 + 2):



Example 2:

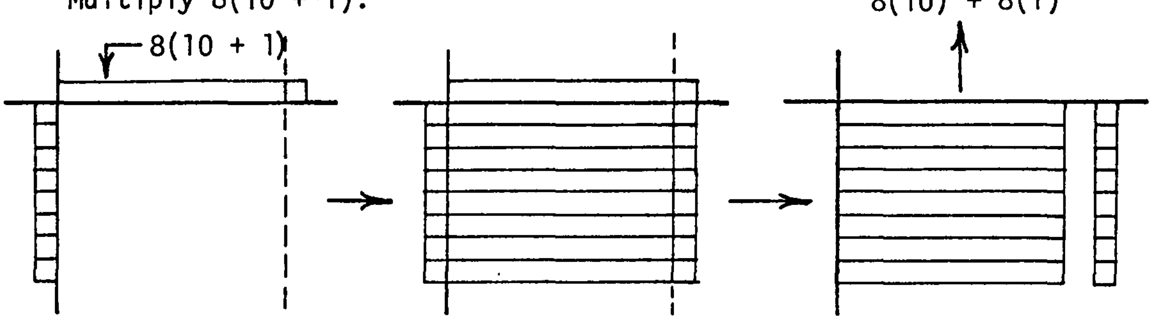
Multiply 7(3 + 4).



Example 3:

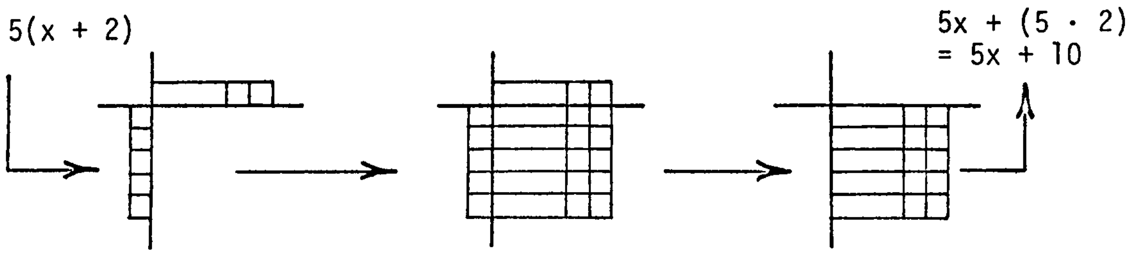
Multiply 8(10 + 1).

8(10) + 8(1)



Example 4:

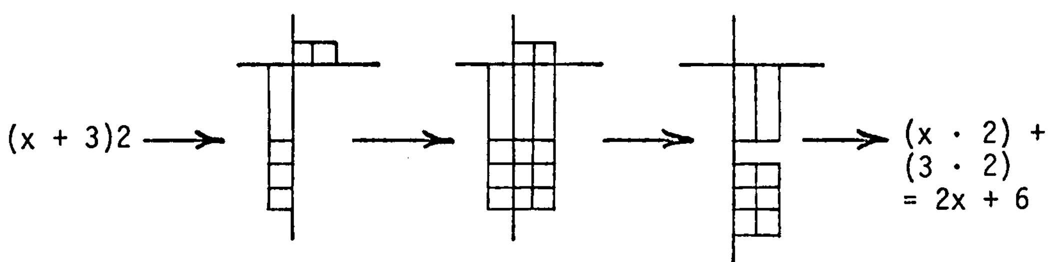
Now let's generalize to 5(x + 2), where x is any whole number:



That is, we have 5 "x + 2's" or (x + 2) + (x + 2) + (x + 2) + (x + 2) + (x + 2). Hence, we have five "x's" or 5 · x and five "2's" or 5 · 2. Since 5 · x = 5x and 5 · 2 = 10, we have 5x + 10.

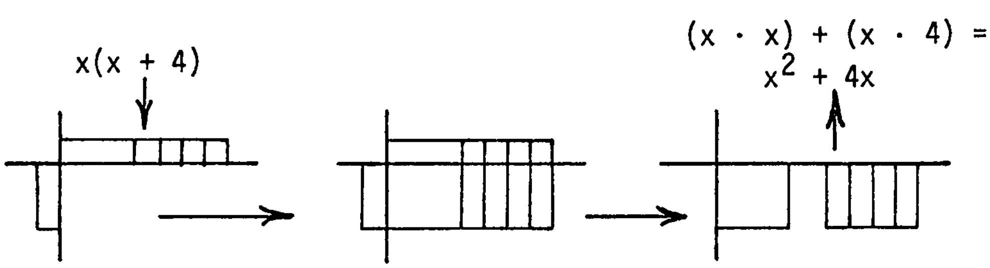
Example 5:

Multiply (x + 3)2.



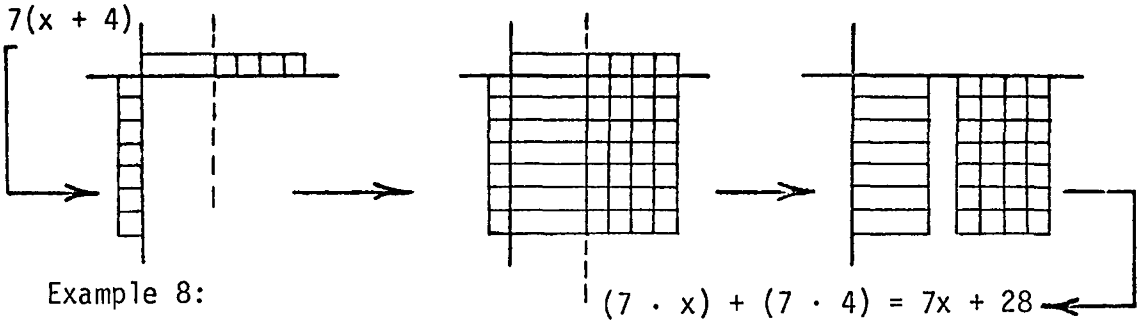
Example 6:

Multiply x(x + 4).



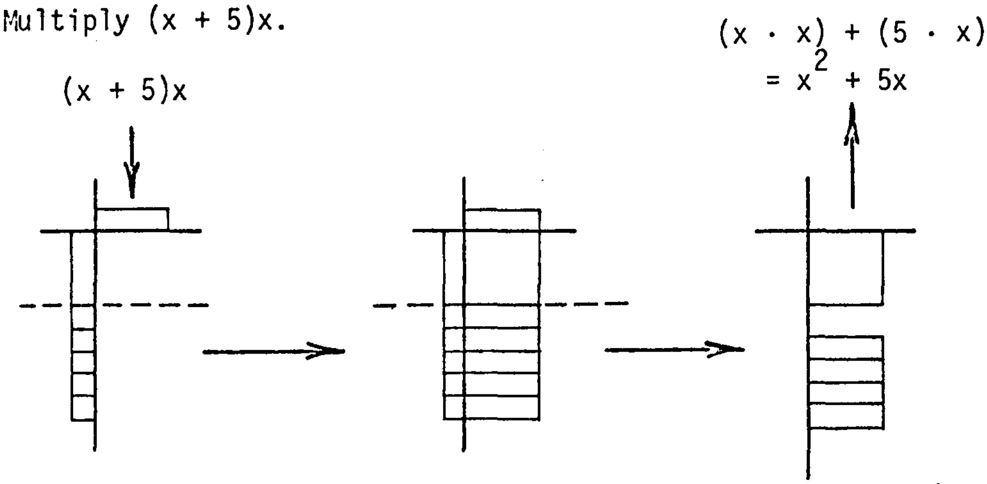
Example 7:

Multiply 7(x + 4).



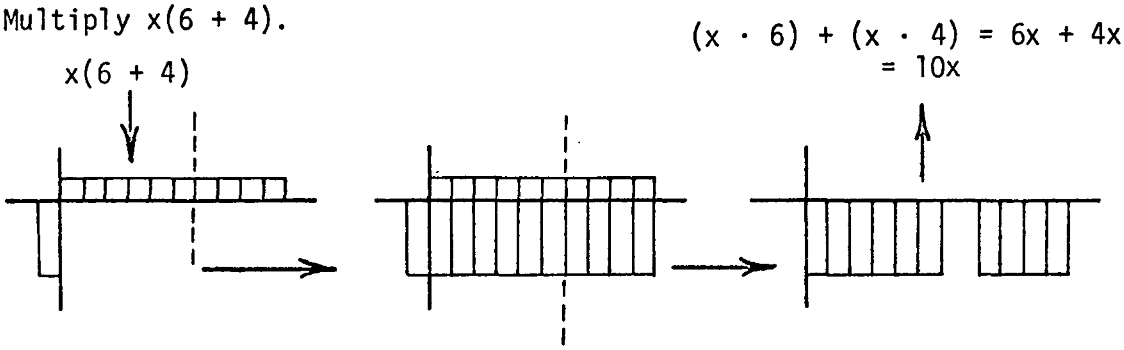
Example 8:

Multiply (x + 5)x.



Example 9:

Multiply x(6 + 4).



Problems

Multiply the following:

1. 4(3 + 5)

2. $x(x + 4)$

3. $5(10 + 2)$

4. $7(x + 6)$

5. $x(5 + x)$

6. $(10 + 6)9$

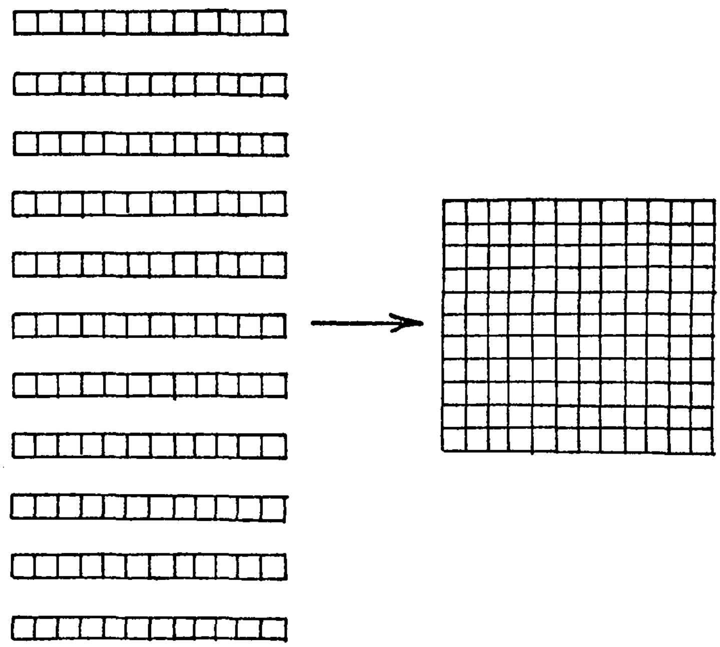
7. $(x + 8)6$

8. $x(5 + 7)$

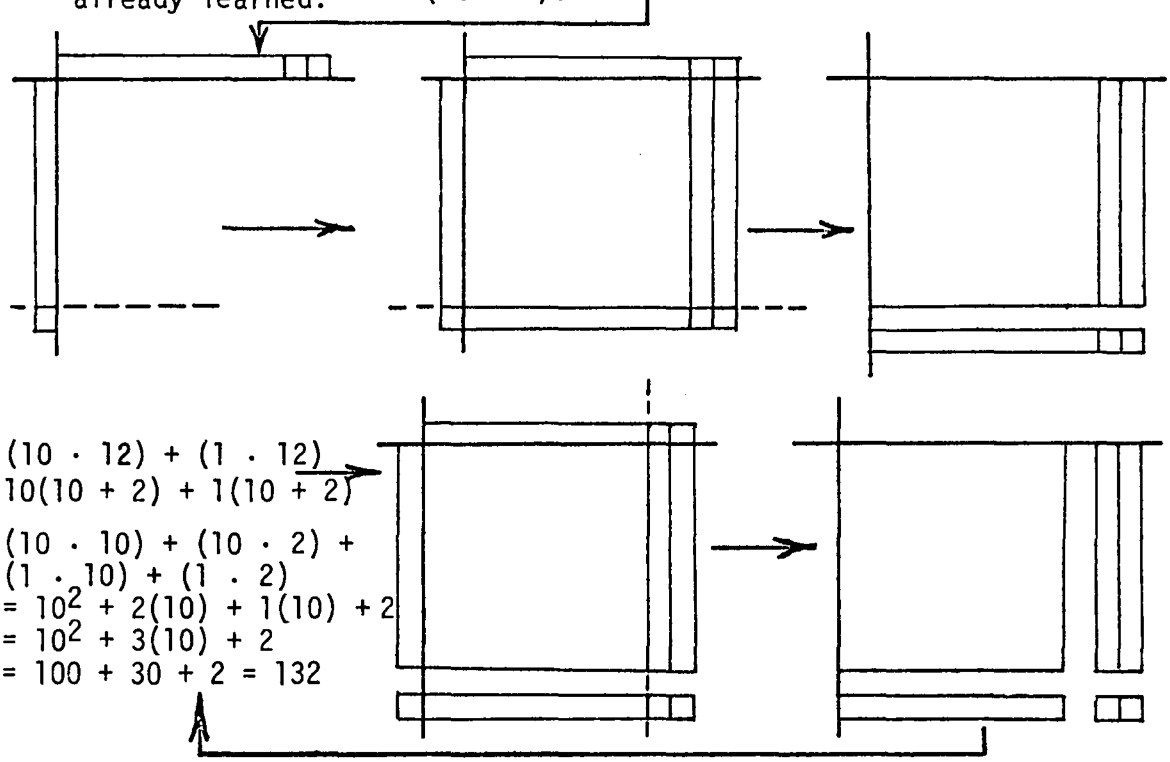
9. $(x + 11)x$

10. $12(x + 7)$

III. Finally, let's look at a two-digit number multiplied by a two-digit number: 11 x 12 can be thought of as eleven "12's" added together:



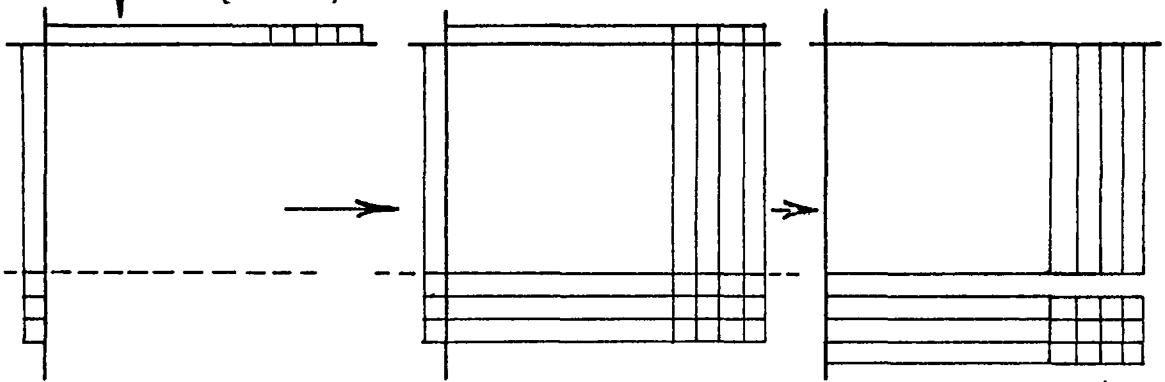
But when multiplying we can think of 11 as (10 + 1) and 12 as (10 + 2) and apply the multiplication procedures we have already learned: (10 + 1)12



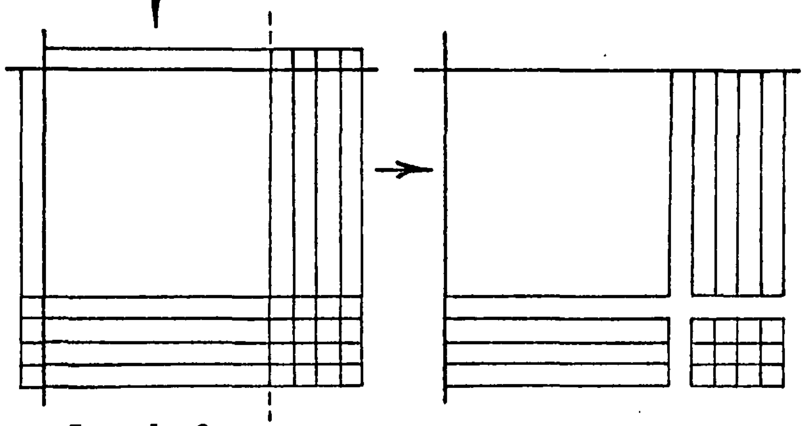
$$(10 \cdot 12) + (1 \cdot 12)$$
$$10(10 + 2) + 1(10 + 2)$$
$$(10 \cdot 10) + (10 \cdot 2) + (1 \cdot 10) + (1 \cdot 2)$$
$$= 10^2 + 2(10) + 1(10) + 2$$
$$= 10^2 + 3(10) + 2$$
$$= 100 + 30 + 2 = 132$$

Let's look at another such problem: Multiply 13 x 14.

$$13 \times 14 = (10 + 3)14$$



$$(10 \cdot 14) + (3 \cdot 14) = 10(10 + 4) + 3(10 + 4)$$



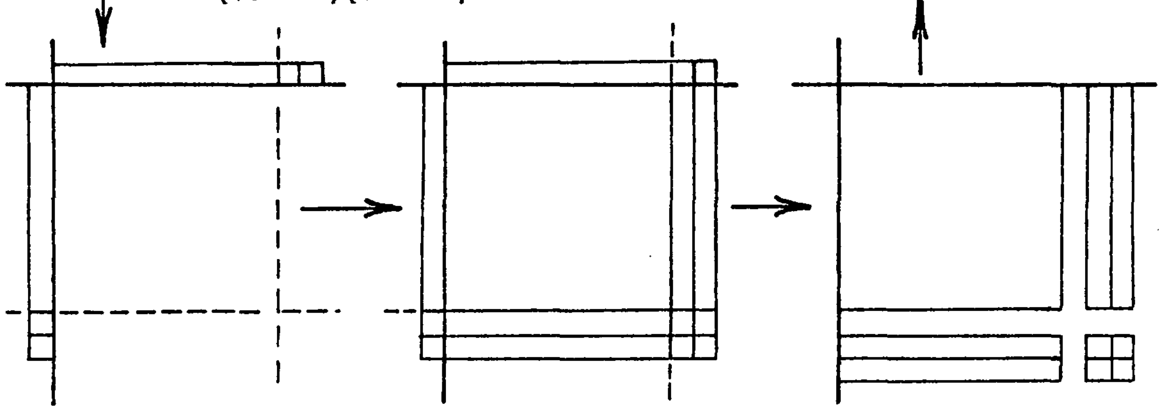
$$(10 \cdot 10) + (4 \cdot 10) + (3 \cdot 10) + (3 \cdot 4) = 10^2 + 7(10) + 12 = 100 + 70 + 12 = 182$$

Example 3:

Multiply 12 x 12.

$$12 \times 12 = (10 + 2)(10 + 2)$$

$$10(10) + 2(10) + 2(10) + 2(2) = 10^2 + 4(10) + 4 = 100 + 40 + 4 = 144$$

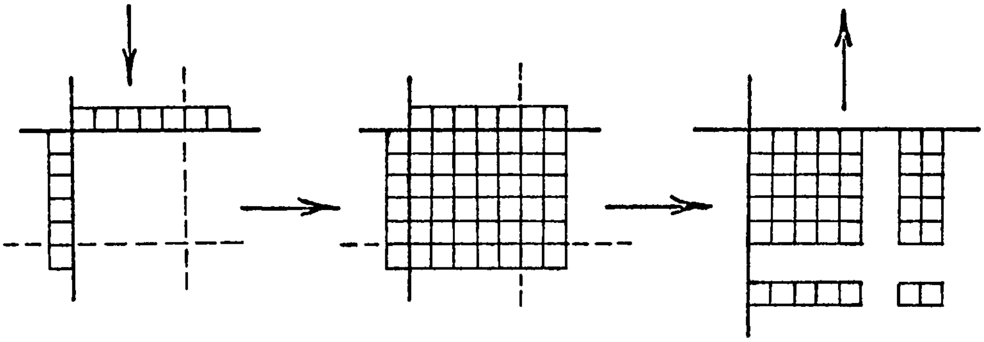


Example 4:

Multiply 6 x 7.

$$6 \times 7 = (5 + 1)(5 + 2)$$

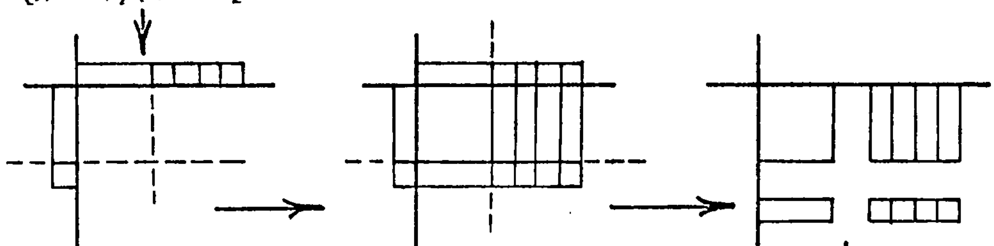
$$5(5) + 2(5) + 1(5) + 1(2) = 5^2 + 3(5) + 2 = 25 + 15 + 2 = 42$$



Example 5:

Multiply (x + 1) (x + 4) where x stands for any whole number.

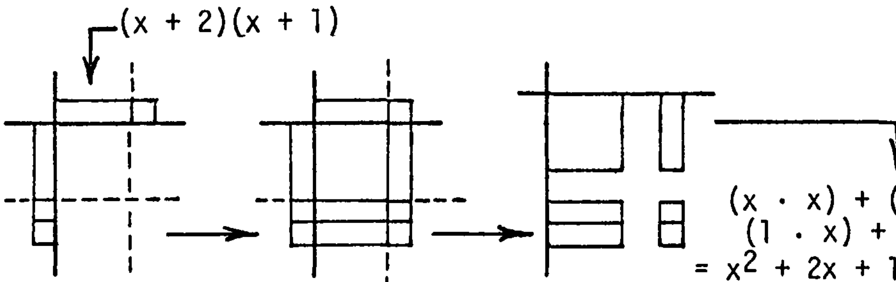
$(x + 1)(x + 4)$



$$(x \cdot x) + (4 \cdot x) + (1 \cdot x) + (1 \cdot 4) = x^2 + 4x + 1x + 4 = x^2 + 5x + 4$$

Example 6:

Multiply (x + 2) (x + 1).

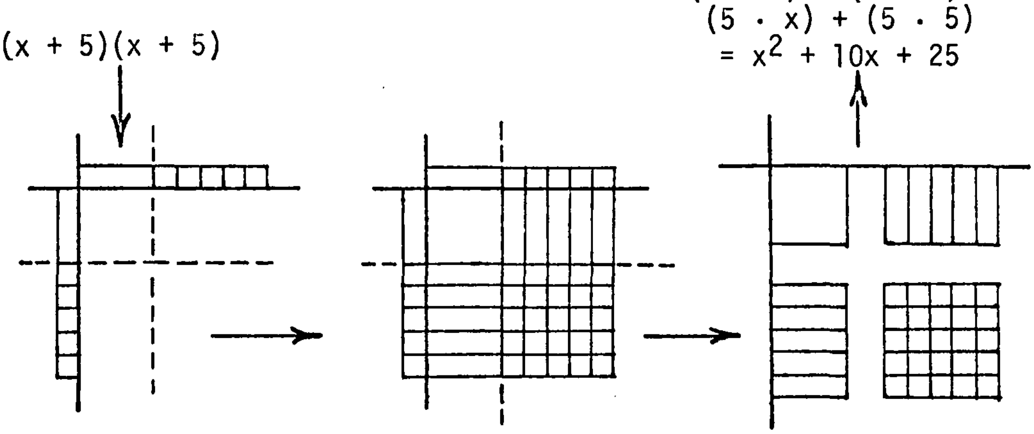


$(x \cdot x) + (2 \cdot x) +$
$(1 \cdot x) + (1 \cdot 2)$
$= x^2 + 2x + 1x + 2$
$= x^2 + 3x + 2$

Example 7:

Multiply (x + 5) (x + 5).

$(x \cdot x) + (5 \cdot x) +$
$(5 \cdot x) + (5 \cdot 5)$
$= x^2 + 10x + 25$



<u>Problems</u>

Multiply the following:

1. 12 x 15

2. (10 + 3) (10 + 2)

3. (x + 3) (x + 1)

4. (x + 2) (x + 2)

5. $(x + 10)(x + 8)$

6. $(x + 7)(x + 3)$

7. $(x + 4)(x + 3)$

8. $(x + 6)(x + 1)$

9. 14×14

10. $(x + 8)(x + 3)$

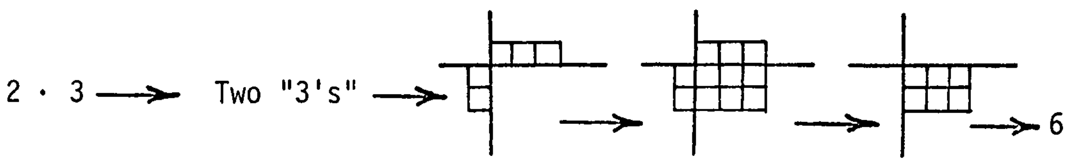
11. $(x + 2)(x + 11)$

12. $(x + 5)(x + 3)$

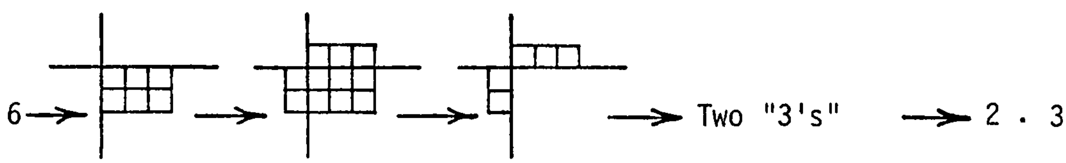
FACTORIZATION OF POLYNOMIALS

IV. Factoring a number is the "reverse" of multiplying two numbers to get a product. When we factor we are given the product and are asked to find the factors. In other words, we want to know how many copies of what number does it take to obtain the given number. In terms of strips and squares, we are given an area and are asked to find the dimensions of a rectangle having that given area. Let's look at both multiplication and factorization in the example below:

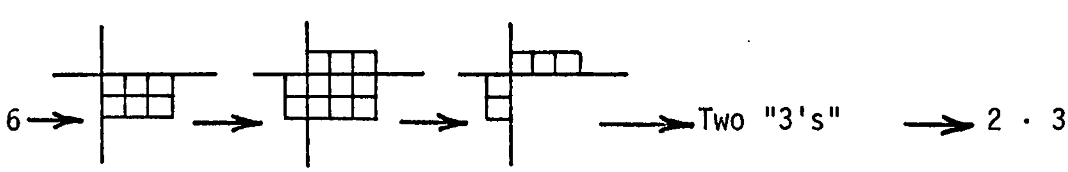
Multiply 2 · 3:



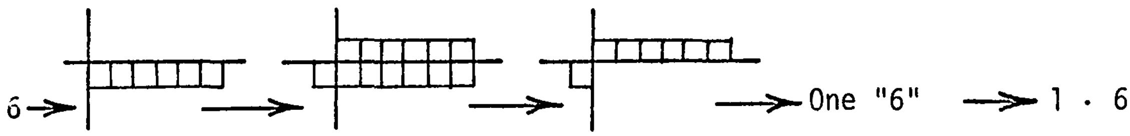
Factor 6:



Hence, factorization is the "reverse" operation of multiplication. But by factoring a number we do not always get unique answers like we do when we multiply. For example, when we factor the number 6, we want to know how many copies of a number gives us 6. Well, two copies of three give us six or one copy of six gives us six:



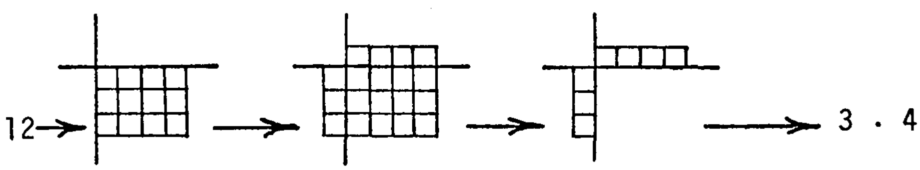
or

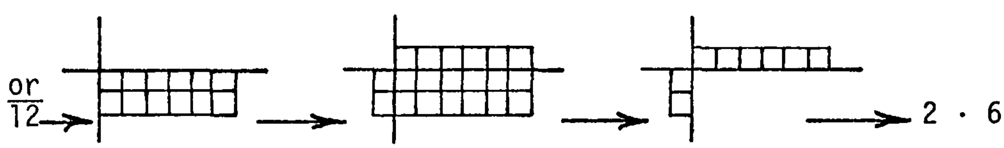


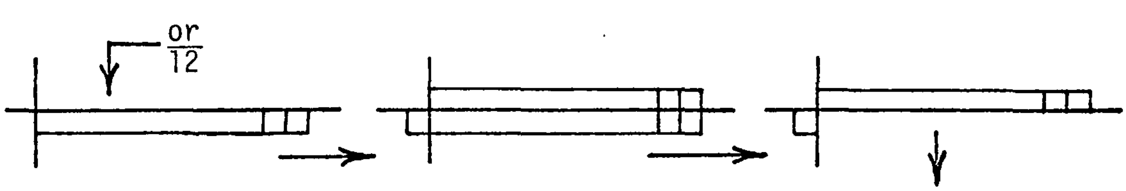
Because of the commutative property of multiplication, the order of the factors does not make any difference:

$$2 \cdot 3 = 3 \cdot 2 \text{ and } 1 \cdot 6 = 6 \cdot 1$$

Let's look at another example: Factor 12.



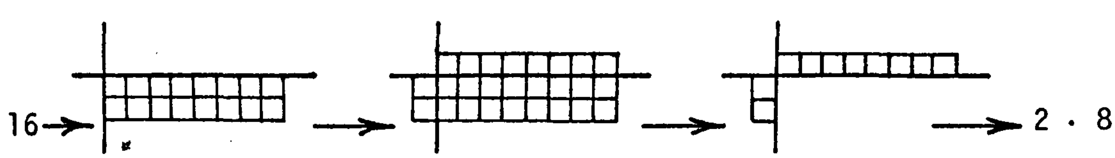


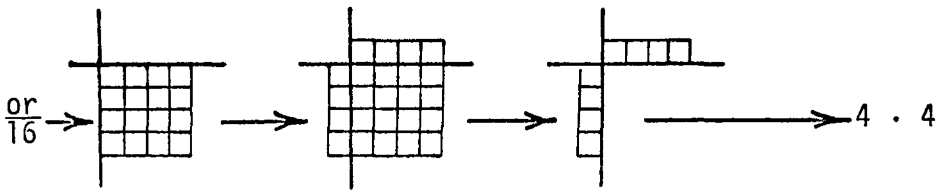


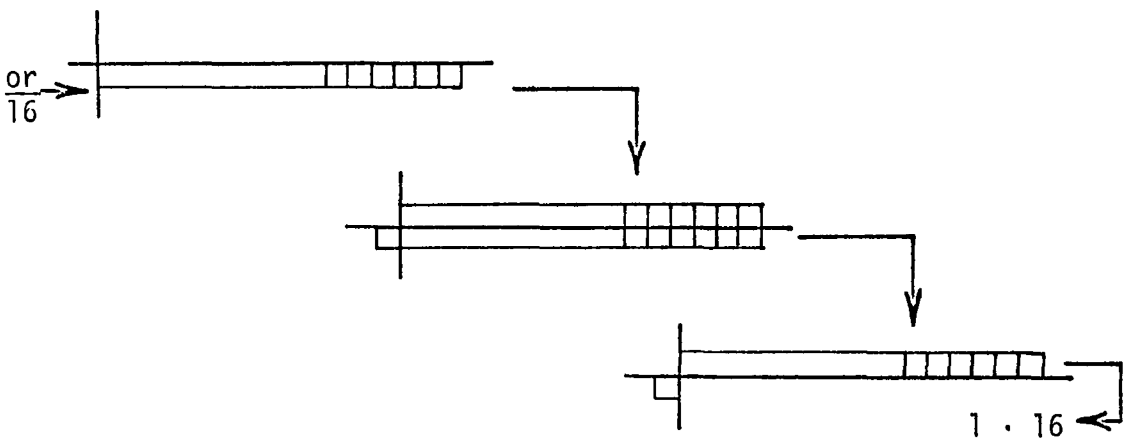
$1 \cdot 12$ or
$1 \cdot (10 + 2)$

Example 3:

Factor 16.

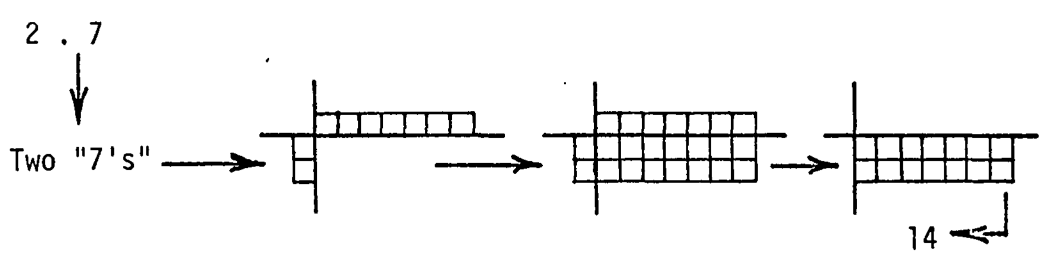




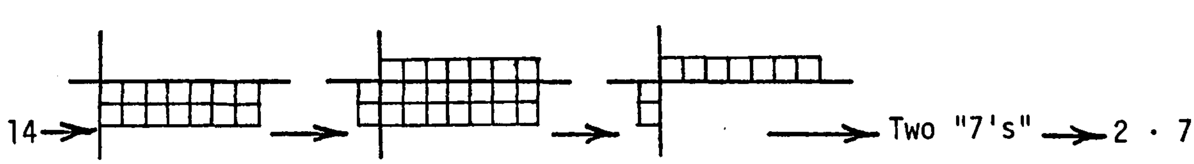


Therefore, multiplication and factorization are "reverse" operations. Let's again look at the operations in terms of strips and squares:

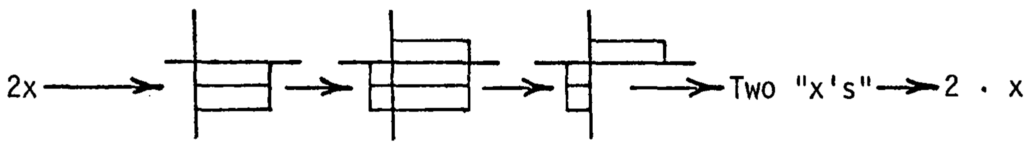
Multiplication:



Factorization:

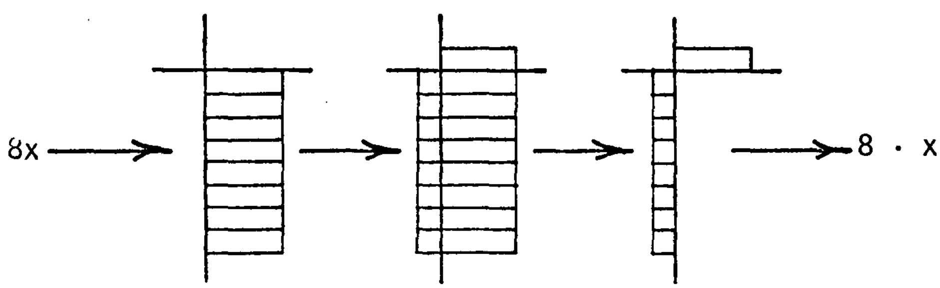


Now let's look at the factors of the product 2x. What number times what number gives the product 2x? Let's look at the strips and squares:

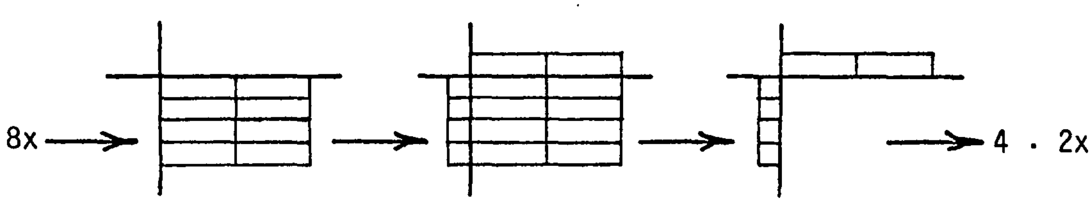


Example 2:

Factor 8x.

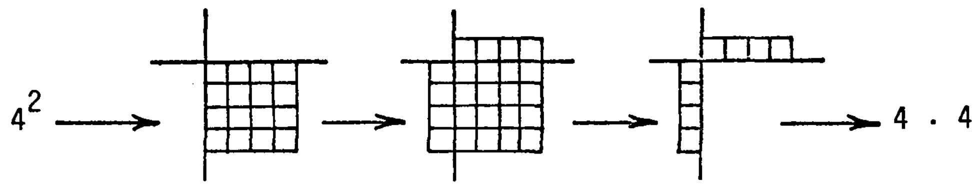


or

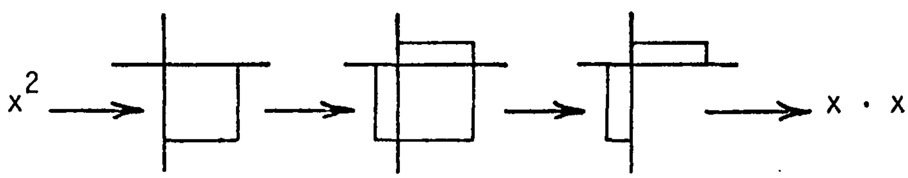


Example 3:

Factor 4^2:

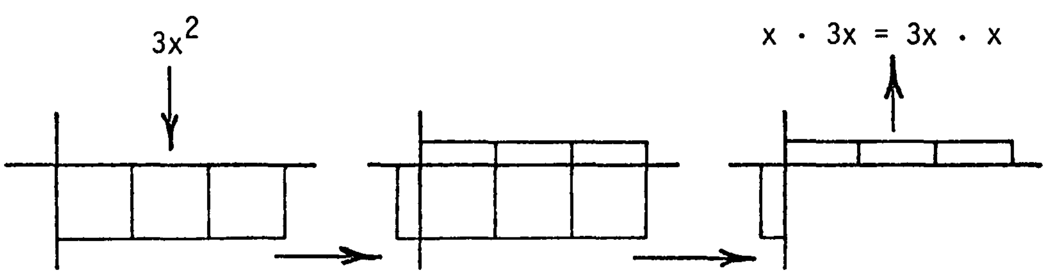


Generalized, if we factor x^2, we get $x \cdot x$ as the factors:



Example 4:

Factor $3x^2$.



<u>Problems</u>

Factor the following:

1. 8

2. $5x$

3. 7^2

4. 10

5. $9x$

6. $4x^2$

7. 18

8. $10x$

9. 15

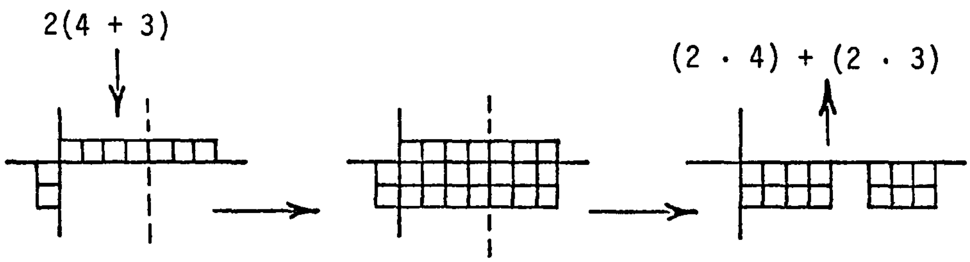
10. $2x^2$

11. 24

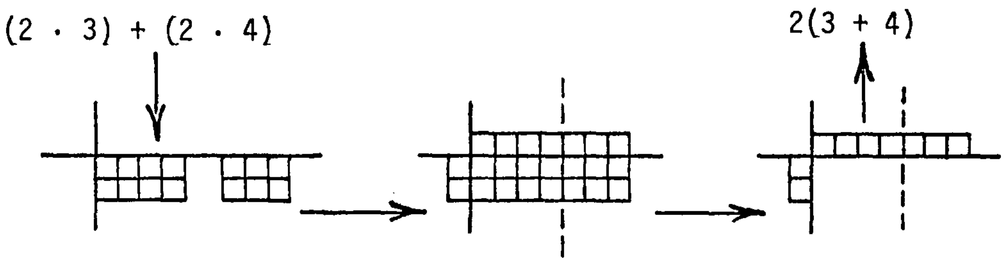
12. 11

V. Now let's look at sums of products to see if we can find their factors. But first a reminder of how we multiply to get a sum of a product. Then we will reverse the process in order to factor:

Multiplication:



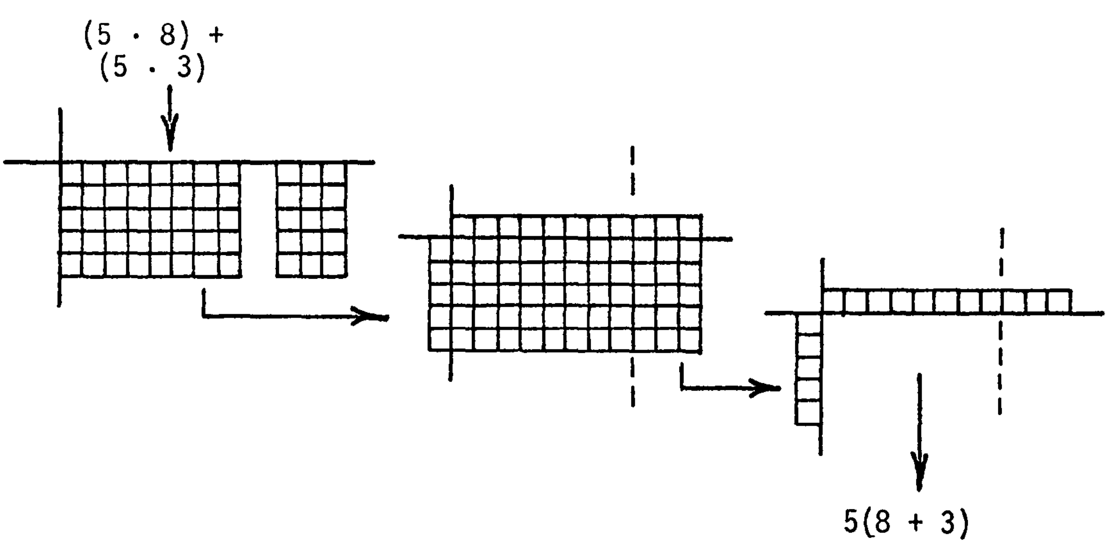
Factorization:



Thus, when factoring we want to know how many copies of a sum gives us the sum of the products. In terms of strips and squares, we are given the area of a rectangle and are asked to find the dimensions.

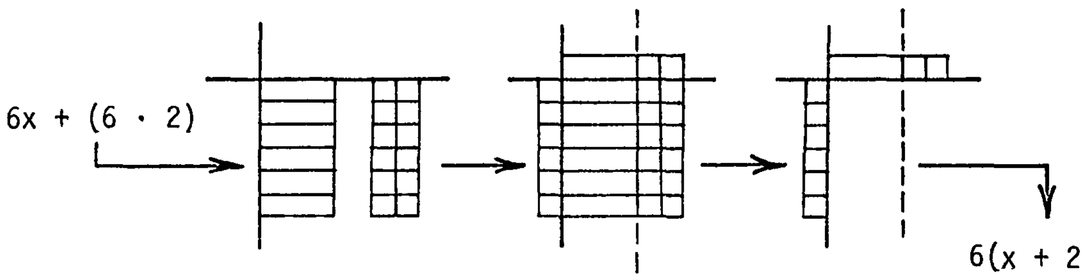
Example 2:

Factor (5 · 8) + (5 · 3)



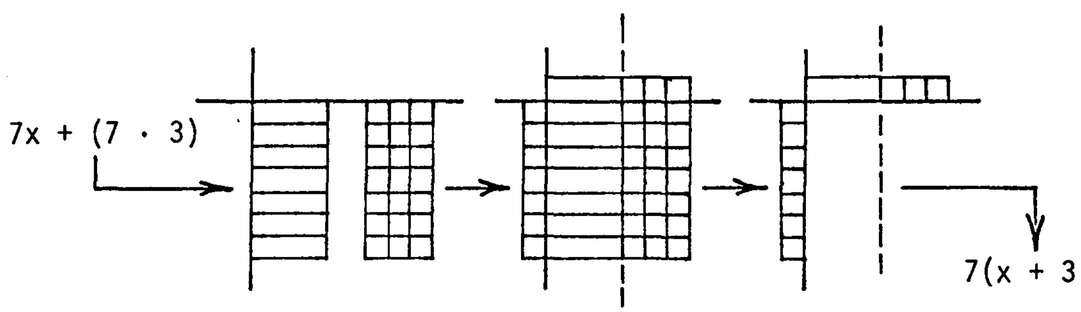
Example 3:

Let's look at the generalized form: Factor $6x + (6 \cdot 2)$.



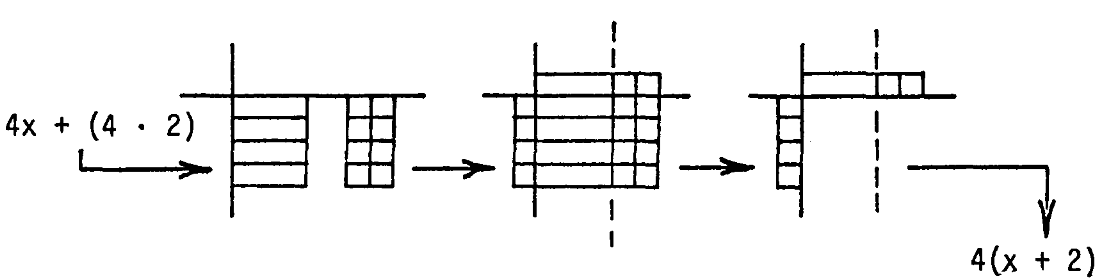
Example 4:

Factor $7x + (7 \cdot 3)$.



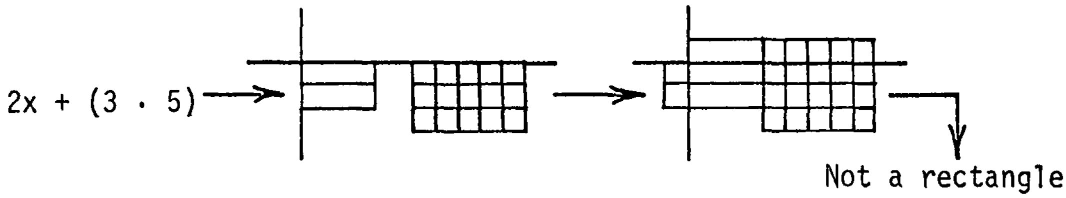
Example 5:

Factor $4x + (4 \cdot 2)$.

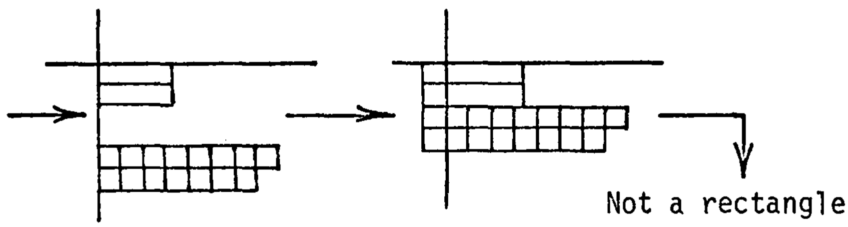


Example 6:

Factor 2x + (3 · 5).

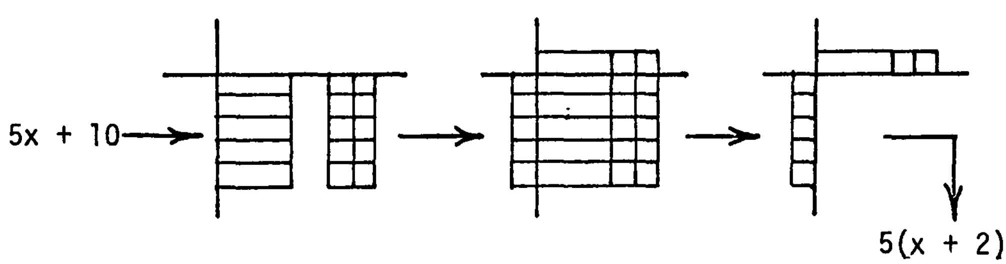


or



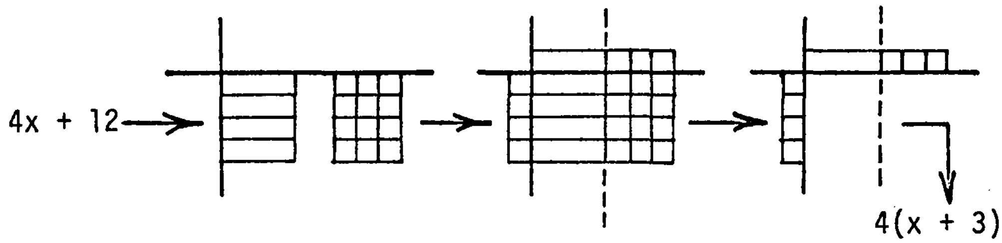
Since there is no way to make a rectangle, 2x + 3 · 5 cannot be factored.

Now suppose we were asked to factor the 5x + 10. By thinking of 5x + 10 as the area of a rectangle, we want to find its dimensions in order to find the factors. Thus, let's make a rectangle with area 5x + 10 and find its dimensions:



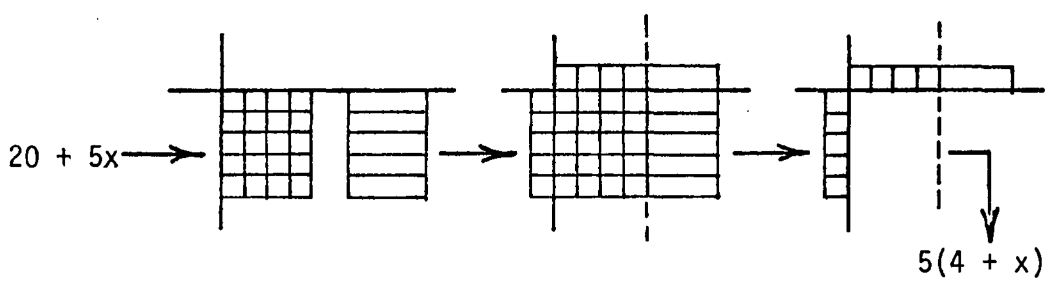
Example 2:

Factor $4x + 12$.

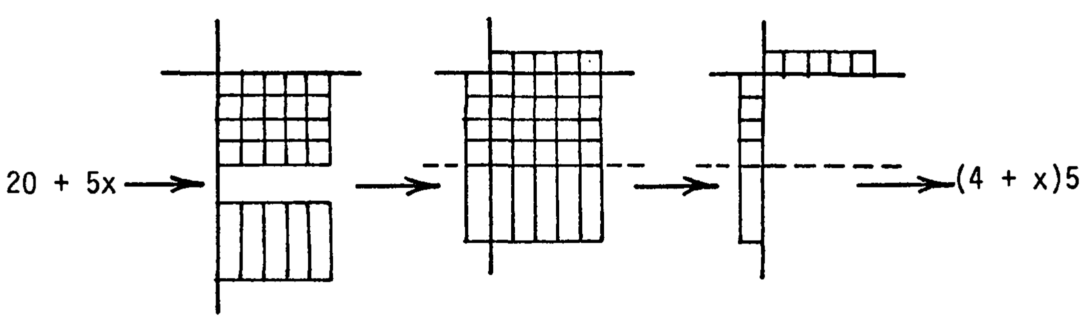


Example 3:

Factor $20 + 5x$.

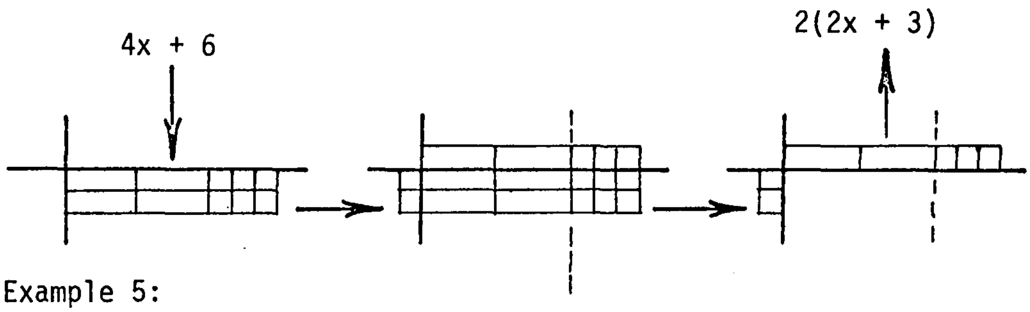


or



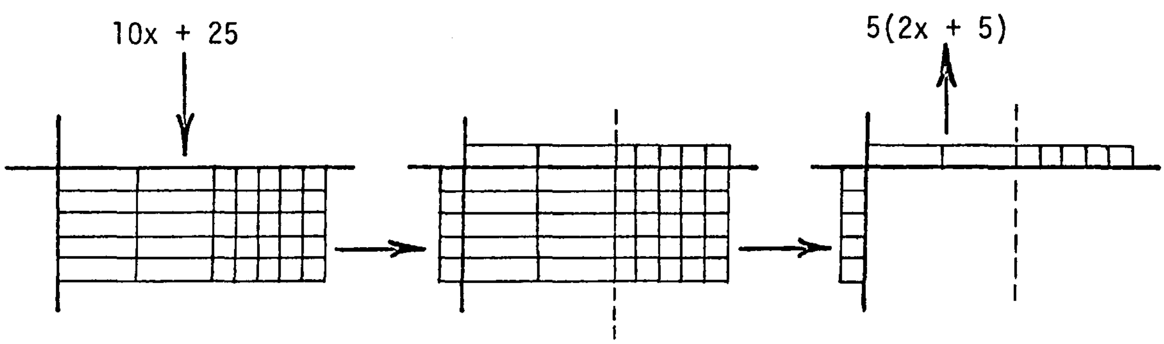
Example 4:

Factor $4x + 6$.



Example 5:

Factor 10x + 25.



Problems

Factor the following:

1. $3x + (3 \cdot 1)$

2. $(2 \cdot 6) + (2 \cdot 4)$

3. $7x + 14$

4. $(5 \cdot 2) + (5 \cdot x)$

5. $10x + 60$

6. $3x + 4$

7. $2x + 2$

8. $16 + 4x$

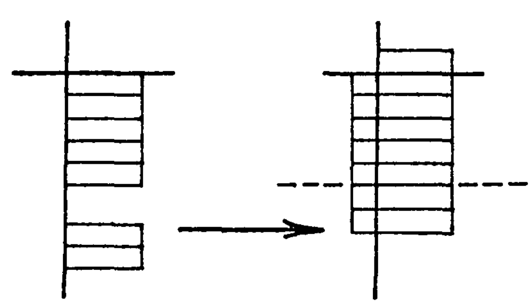
9. $4x + 10$

10. $(8 \cdot 6) + (8 \cdot 3)$

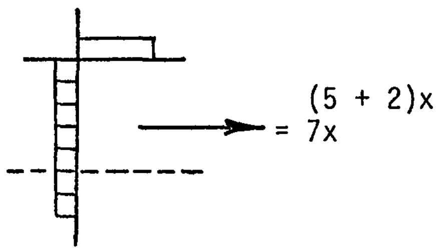
11. $8x + 18$

12. $6x + 54$

VI. Now suppose we were asked to factor $5x + 2x$. Let's look at the rectangle whose area is $5x + 2x$:

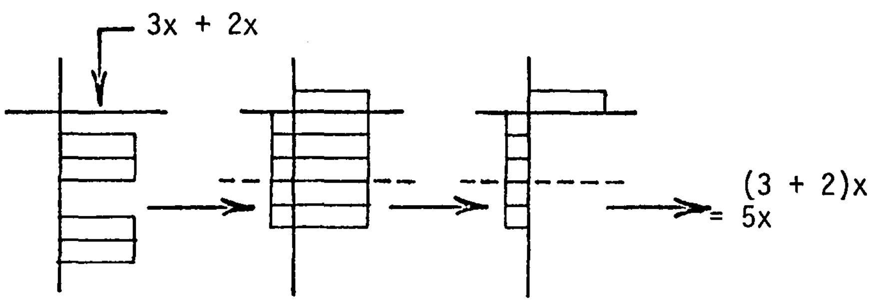


Now let's look at the dimensions:

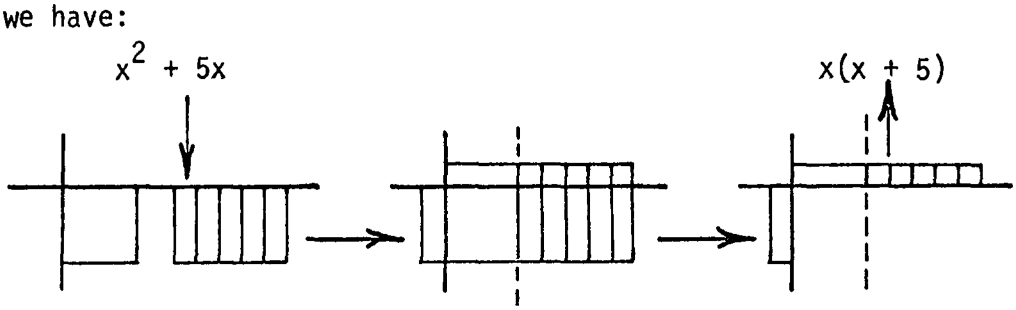


Example 2:

Factor 3x + 2x.

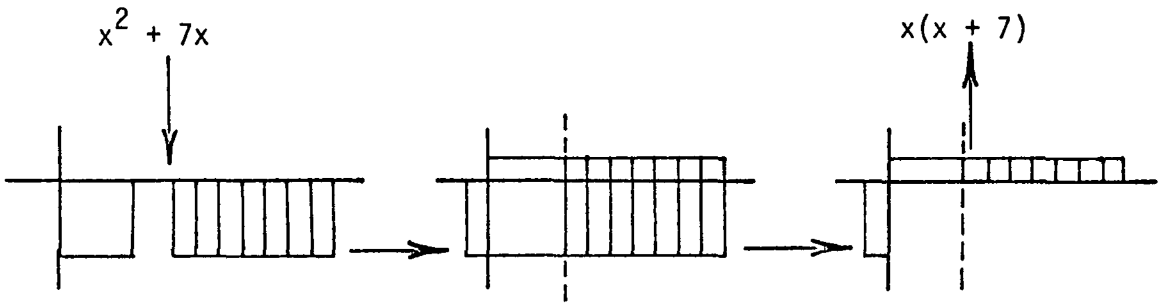


Now let's suppose we were asked to factor $x^2 + 5x$. We know $x \cdot x = x^2$, so generalizing from the previous sections, we have:



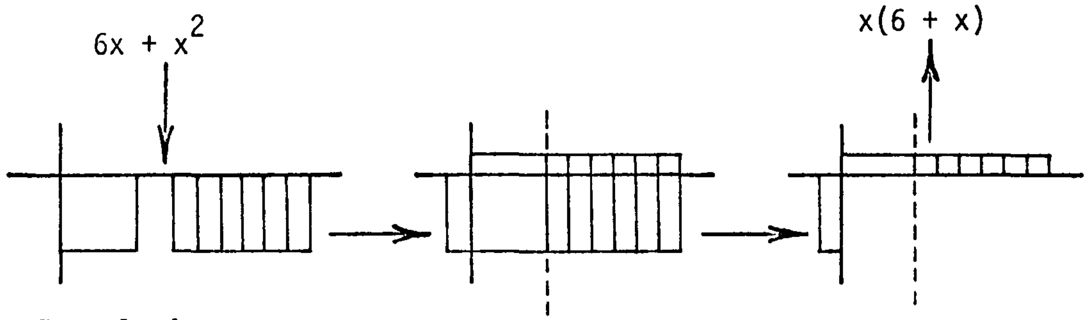
Example 2:

Factor $x^2 + 7x$.



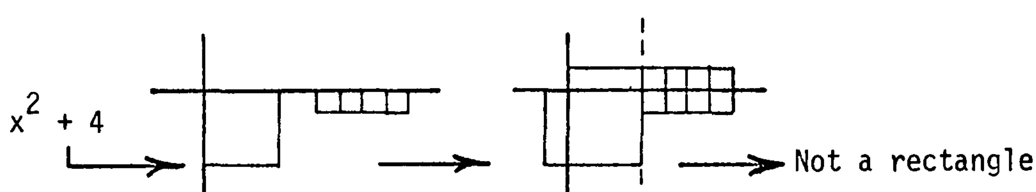
Example 3:

Factor $6x + x^2$.

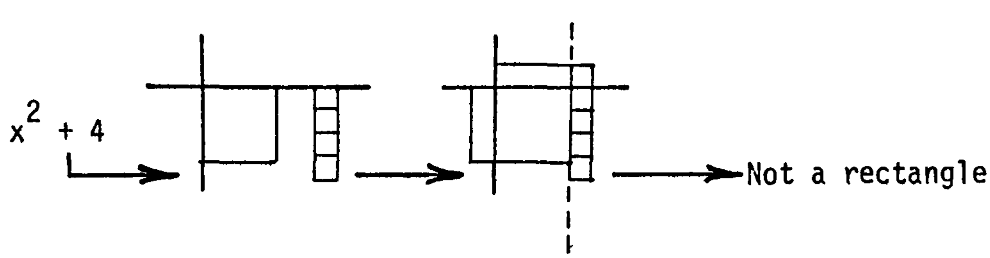


Example 4:

Factor $x^2 + 4$.

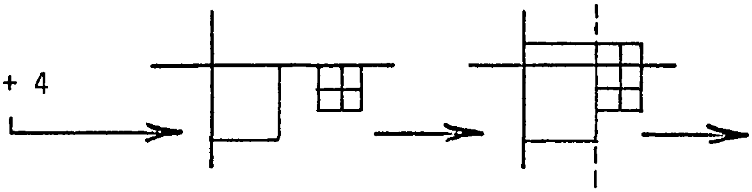


<u>or</u>



<u>or</u>

$x^2 + 4$



Since no rectangle can be formed that has area $x^2 + 4$, then there are no factors, other than one. Therefore, $x^2 + 4 = x^2 + 4$.

Problems

Factor the following:

1. $x^2 + 8x$

2. $x^2 + 9$

3. $7x + x^2$

4. $4x + 6x$

5. $5 + x^2$

6. $x^2 + 3x$

7. $x^2 + x$

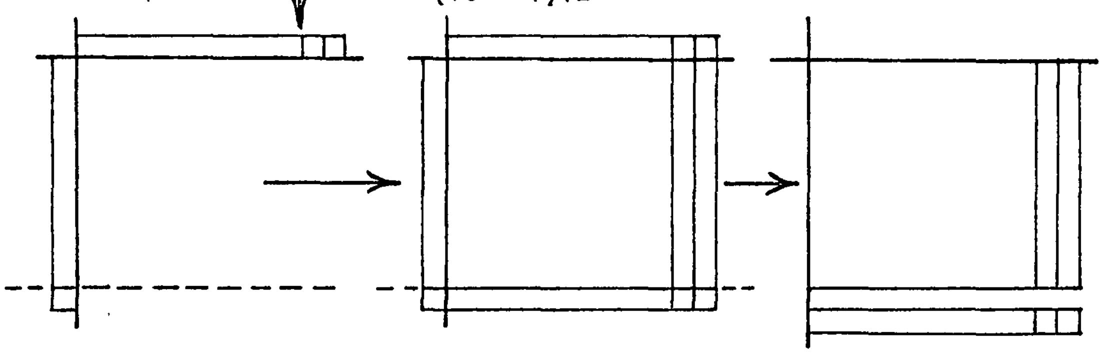
8. $x^2 + 11x$

9. $2x + 8x$

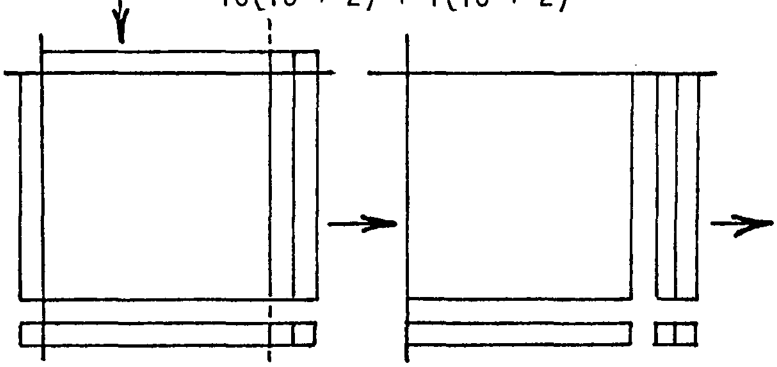
10. $3x + 5x$

VII. In this final section, let's go back and look at how we multiplied:

$$\begin{aligned} &11(12) \\ &= (10 + 1)12 \end{aligned}$$



$$\begin{aligned} &10 \cdot 12 + 1 \cdot 12 \\ &= 10(10 + 2) + 1(10 + 2) \end{aligned}$$

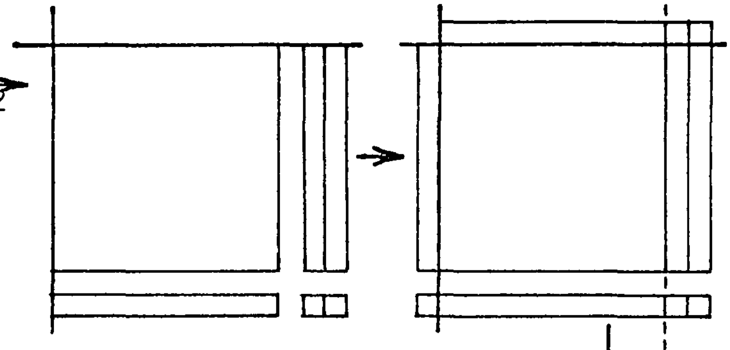


$$\begin{aligned} &(10 \cdot 10) + (10 \cdot 2) \\ &+ (1 \cdot 10) + (1 \cdot 2) \\ &= 10^2 + 2(10) + 1(10) + 2 \\ &= 10^2 + 3(10) + 2 \\ &= 100 + 30 + 2 = 132 \end{aligned}$$

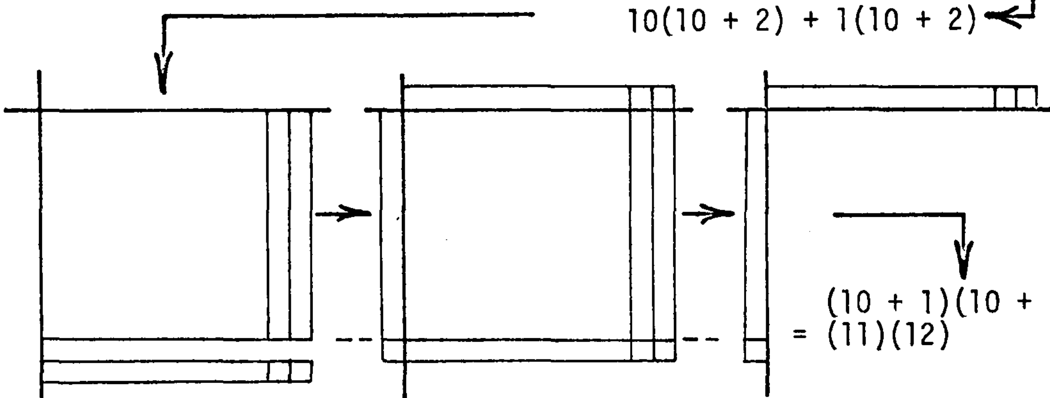
We were given the dimensions and were asked to find the area.

Now, let's see if we can reverse the process and find the dimensions, given the area:

$$
\begin{aligned}
& 132 \\
&= 100 + 30 + 2 \\
&= 10^2 + 3(10) + 2 \\
&= 10^2 + 2(10) + 1(10) + 2 \\
&= 10(10) + 2(10) + 1(10) + 2
\end{aligned}
$$



$10(10 + 2) + 1(10 + 2)$

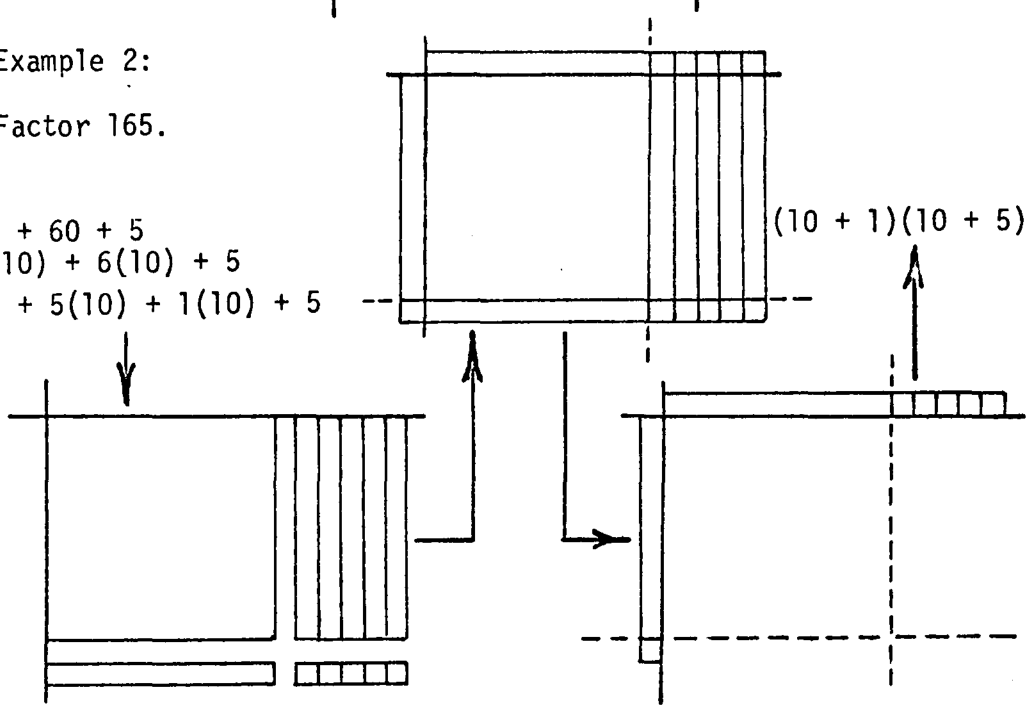


$$
\begin{aligned}
& (10 + 1)(10 + 2) \\
&= (11)(12)
\end{aligned}
$$

Example 2:

Factor 165.

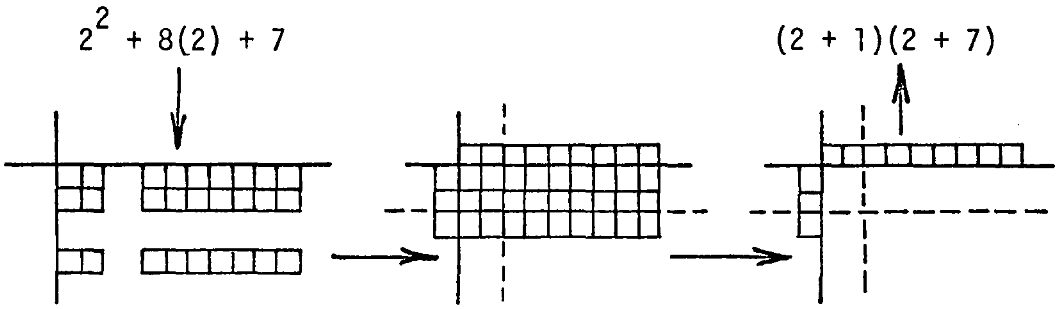
$$
\begin{aligned}
& 165 \\
&= 100 + 60 + 5 \\
&= 10(10) + 6(10) + 5 \\
&= 10^2 + 5(10) + 1(10) + 5
\end{aligned}
$$



$(10 + 1)(10 + 5)$

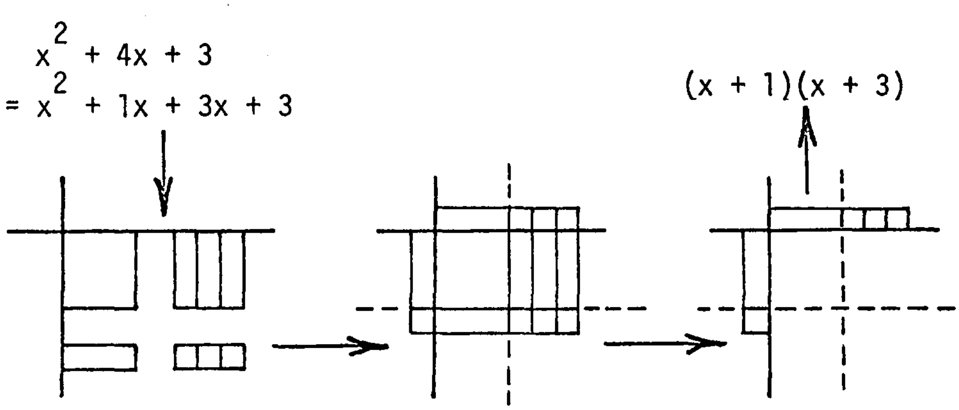
Example 3:

Factor $2^2 + 8(2) + 7$.



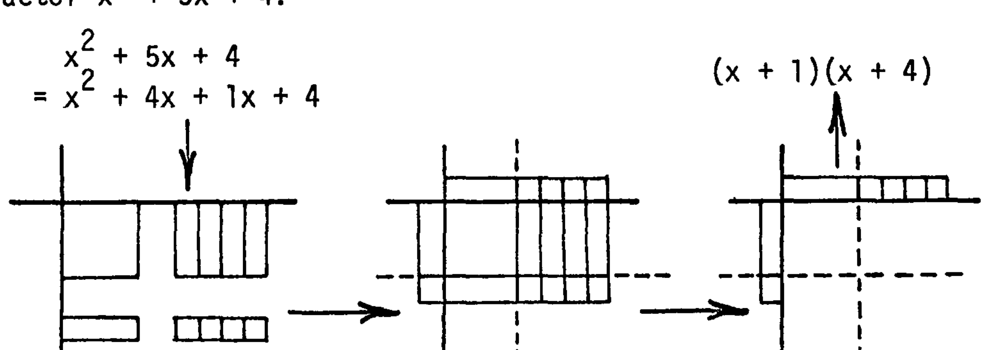
Example 4:

Generalizing, let's factor $x^2 + 4x + 3$:

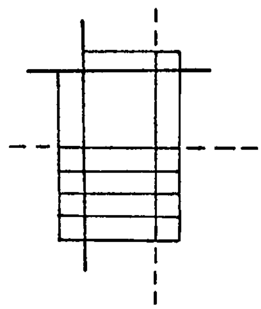


Example 5:

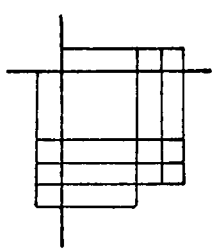
Factor $x^2 + 5x + 4$.



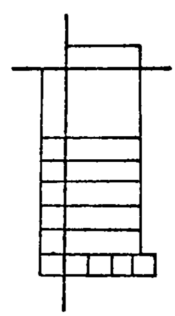
Note: There is no other arrangement for the strips and squares except the following:



If the strips and squares were arranged in any other fashion, a rectangle would not be formed:



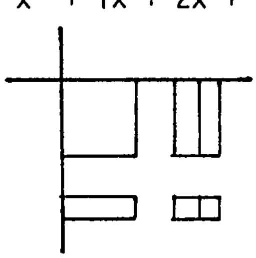
<u>OR</u>

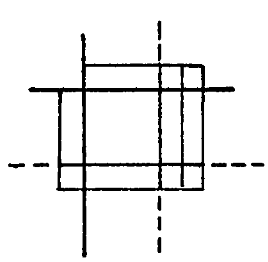


Example 6:

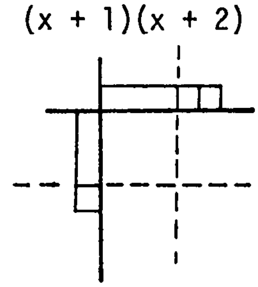
Factor $x^2 + 3x + 2$.

$$x^2 + 3x + 2$$
$$= x^2 + 1x + 2x + 2$$





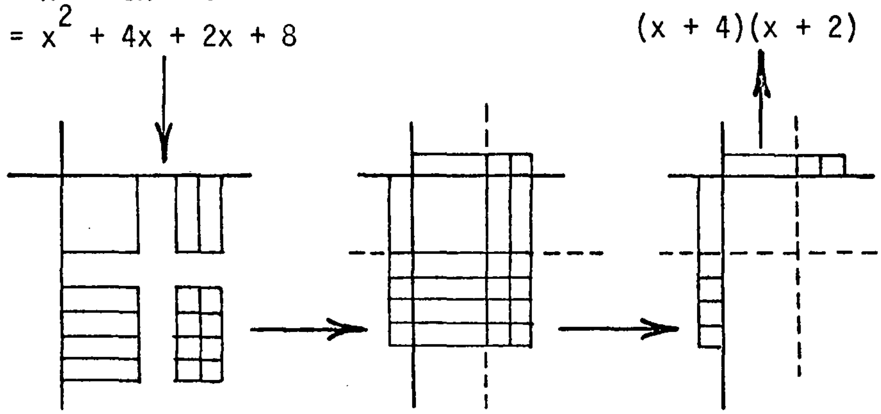
$(x + 1)(x + 2)$



Example 7:

Factor $x^2 + 6x + 8$.

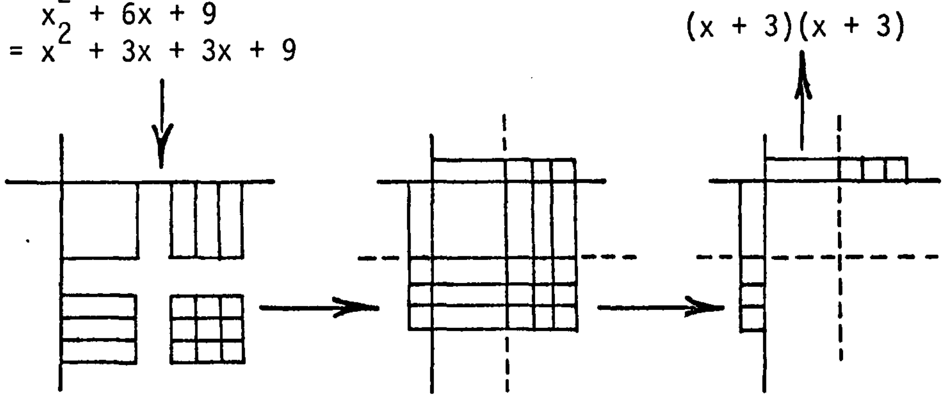
$$x^2 + 6x + 8$$
$$= x^2 + 4x + 2x + 8$$



Example 8:

Factor $x^2 + 6x + 9$.

$$x^2 + 6x + 9$$
$$= x^2 + 3x + 3x + 9$$



Problems

Factor the following:

1. $10^2 + 6(10) + 9$

2. $x^2 + 5x + 6$

3. $x^2 + 2x + 1$

4. $x^2 + 8x + 7$

5. $x^2 + 6x + 5$

6. $x^2 + 7x + 10$

7. $x^2 + 4x + 4$

8. $x^2 + 8x + 16$

9. $x^2 + 12x + 20$

10. $x^2 + 5x + 2$

11. $x^2 + 8x + 5$

12. $x^2 + 13x + 36$

13. $x^2 + 3x + 5$

14. $x^2 + 7x + 12$

15. $x^2 + x + 2$

16. 143

NOTE: The manipulative instructional booklet that was given to students during the study was the same length as the nonmanipulative instructional booklet given to students.

MULTIPLICATION OF POLYNOMIALS

I. Let's look at the operation of multiplication with whole numbers:

2×3 or $2 \cdot 3$ can be thought of as repeated addition. Thus, $2 \cdot 3$ means 2 three's added together, $3 + 3$ or 6.

3×5 or $3 \cdot 5$ can be thought of as 3 five's added together, $5 + 5 + 5$ or 15.

$5 \cdot 7$ can be thought of as 5 seven's added together, $7 + 7 + 7 + 7 + 7$ or 35.

2×10 can be thought of as 2 ten's added together, $10 + 10$ or 20.

Hence, by letting x stand for any whole number, $2(x)$ or $2 \cdot x$ can be thought of in terms of repeated addition: two "x's" added together, $x + x$ or $2x$. Also, $5 \cdot x$ can be thought of as five "x's" added together, $x + x + x + x + x$ or $5x$. (Note that $1x$ can thus be written as "x".)

Also, $4 \cdot 2x$ can be thought of as four "2x's" added together, $2x + 2x + 2x + 2x$ or $8x$.

Finally, because of the commutative property of multiplication, $x \cdot 5 = 5 \cdot x = 5x$. The conventional way of writing $x \cdot 5$ or $5 \cdot x$ is $5x$, not $x5$.

Now let's look at the following multiplications:

2×2 or $2 \cdot 2$ can be written as 2^2 where the exponent indicates how many two's are multiplied together. Hence, 2^2 is the product of 2×2.

Also, $3 \times 3 = 3^2$

$4 \times 4 = 4^2$

$10 \times 10 = 10^2$

If x represents any whole number, then $x \cdot x = x^2$ and $2x \cdot x =$

$$2\,(x \cdot x)$$
$$= 2x^2$$

and
$$\begin{aligned} x(5x) &= (5x)\,(x) \\ &= 5(x \cdot x) \\ &= 5x^2 \end{aligned}$$

Thus, when we multiply we find "a" copies of "b".

Problems

Multiply the following.

1. $2 \cdot 7$

2. $5 \cdot 3$

3. $5 \cdot y$

4. $x \cdot 10$

5. $3 \cdot 3x$

6. $4x \cdot 9$

7. $4x \cdot x$

8. $2 \cdot x^2$

9. $1 \cdot x$

10. $0 \cdot x$

11. $x(3x)$

II. When students begin to multiply numbers with more than one digit, multiplication is still thought of in terms of repeated addition. Hence, 3 x 12 means 3 twelve's added together. But in order to make the addition easier, we sometimes use the notion of place value and write 12 as (10 + 2). Thus, we have 3(10 + 2) or three "(10 + 2)'s", (10 + 2) + (10 + 2) +

(10 + 2). Using the commutative property we get (10 + 10 + 10) + (2 + 2 + 2) or 3(10) + 3(2). $\therefore$ 3(10 + 2) = 3(10) + 3(2) or 30 + 6 or 36.

Let's look at another example:

$$
\begin{aligned}
4(13) = 4(10 + 3) &= (10 + 3) + (10 + 3) + (10 + 3) + (10 + 3) \\
&= 4(10) + 4(3) \\
&= 40 + 12 \\
&= 52
\end{aligned}
$$

This same procedure can also be used with 6 x 5, where we can rewrite 5 as (3 + 2):

$$
\begin{aligned}
6(3 + 2) &= (3 + 2) + (3 + 2) + (3 + 2) + (3 + 2) + (3 + 2) + \\
&\quad (3 + 2) \\
&= 6(3) + 6(2)
\end{aligned}
$$

Hence, let's look at several other examples:

Example 1:

$$
\begin{aligned}
7(3 + 4) &= (3 + 4) + (3 + 4) + (3 + 4) + (3 + 4) + (3 + 4) + \\
&\quad (3 + 4) + (3 + 4) \\
&= 7(3) + 7(4)
\end{aligned}
$$

Example 2:

$$8(10 + 1) = 8 \cdot 10 + 8 \cdot 1$$

Example 3:

Now using "x" in the place of a whole number, we have:

$$5(x + 2) = (x + 2) + (x + 2) + (x + 2) + (x + 2) + (x + 2)$$
$$= x + x + x + x + x + 2 + 2 + 2 + 2 + 2$$
$$= 5(x) + 5(2)$$
$$= 5x + 10$$

Example 4:

$$(x + 3)2 = (x \cdot 2) + (3 \cdot 2)$$
$$= 2x + 6$$

Example 5:

$$7(x + 4) = 7(x) + 7(4)$$
$$= 7x + 28$$

Example 6:

$$x(6 + 4) = (x \cdot 6) + (x \cdot 4)$$
$$= 6x + 4x$$
$$= 10x$$

Example 7:

$$x(x + 4) = (x \cdot x) + (x \cdot 4)$$
$$= x^2 + 4x$$

Example 8:

$$(x + 5)x = (x \cdot x) + (5 \cdot x)$$
$$= x^2 + 5x$$

Problems

Multiply the following:

1. $4(3 + 5)$

2. $x(x + 4)$

3. $5(x + 2)$

4. $7(x + 6)$

5. $y(5 + y)$

6. $(10 + 6)9$

7. $(x + 8)6$

8. $x(5 + 7)$

9. $(x + 11)x$

10. $12(x + 7)$

III. Finally, let's look at a two-digit number multiplied by a two-digit number:

11 x 12 can be thought of as eleven 12's added together.
But when we multiply, we want to rewrite 11 as (10 + 1) and 12 as (10 + 2) and apply the processes that have already been developed:

$$\begin{aligned} 11 \cdot 12 &= (10 + 1)12 \\ &= (10 \cdot 12) + (1 \cdot 12) \\ &= 10(10 + 2) + 1(10 + 2) \\ &= (10 \cdot 10) + (10 \,.\, 2) + (1 \cdot 10) + (1 \cdot 2) \\ &= 10^2 + (2 \cdot 10) + (1 \cdot 10) + 2 \\ &= 10^2 + (3 \cdot 10) + 2 \\ &= 100 + 30 + 2 \\ &= 132 \end{aligned}$$

The use of vertical multiplication helps to organize the terms resulting from the multiplication more precisely. Let's look at 11 x 12 in terms of vertical multiplication:

$$\begin{array}{l} \quad 11 \qquad\quad (10 + 1) \\ \ \underline{\times 12} \qquad \underline{\times(10 + 2)} \\ \qquad\qquad\quad \underline{2(10) + 2(1)} \\ \underline{10(10) + 1(10)} \\ 10(10) + 3(10) + 2(1) \end{array} \begin{aligned} &= 10^2 + 3(10) + 2(1) \\ &= 100 + 30 + 2 \\ &= 132 \end{aligned}$$

Let's look at another such problem:

$$\begin{aligned} 13 \times 14 &= (10 + 3)14 \\ &= (10 \cdot 14) + (3 \cdot 14) \\ &= 10(10 + 4) + 3(10 + 4) \\ &= (10 \cdot 10) + (10 \cdot 4) + (3 \cdot 10) + (3 \cdot 4) \\ &= (10 \cdot 10) + (4 \cdot 10) + (3 \cdot 10) + (3 \cdot 4) \\ &= 10^2 + 7(10) + 12 \\ &= 100 + 70 + 12 \end{aligned}$$

Now let's use the vertical form to multiply:

$$\begin{array}{lll} \quad 13 & \quad (10 + 3) & \\ \underline{\times 14} & \underline{\times (10 + 4)} & \\ & \underline{4(10) + 4(3)} & \\ \multicolumn{2}{l}{\underline{10(10) + 3(10)}} & \\ \multicolumn{2}{l}{10(10) + 7(10) + 4(3)} & = 10^2 + 7(10) + 4(3) \\ & & = 100 + 70 + 12 \\ & & = 182 \end{array}$$

Thus, the vertical multiplication form aids in the organization of terms.

Here is a third example:

$$\begin{aligned} 12 \times 12 &= (10 + 2)12 \\ &= (10 \cdot 12) + (2 \cdot 12) \\ &= 10(10 + 2) + 2(10 + 2) \\ &= (10 \cdot 10) + (10 \cdot 2) + (2 \cdot 10) + (2 \cdot 2) \\ &= (10 \cdot 10) + (2 \cdot 10) + (2 \cdot 10) + (2 \cdot 2) \\ &= 10^2 + (4 \cdot 10) + 4 \\ &= 100 + 40 + 4 \\ &= 144 \end{aligned}$$

The vertical form looks like:

$$\begin{array}{lll} \quad 12 & \quad (10 + 2) & \\ \underline{\times 12} & \underline{\times (10 + 2)} & \\ & \underline{2(10) + 2(2)} & \\ \multicolumn{2}{l}{\underline{10(10) + 2(10)}} & \\ \multicolumn{2}{l}{10(10) + 4(10) + 2(2)} & = 10^2 + 4(10) + 4 \\ & & = 100 + 40 + 4 \\ & & = 144 \end{array}$$

Now let's look at (5 + 1) (5 + 3):

$$\begin{array}{rrr}
 & (5\ + & 1) \\
 & x(5\ + & 3) \\ \hline
 & 3(5)\ + & 3(1) \\
5(5)\ + & 1(5) & \\ \hline
5(5)\ + & 4(5)\ + & 3(1)
\end{array} = 5^2 + 4(5) + 3$$

$$= 25 + 20 + 3$$

$$= 48$$

Finally, let's generalize to where x replaces a whole number:

Example 1:

$$(x + 1)\ (x + 4) = \begin{array}{rrr}
 & (x\ + & 1) \\
 & (x\ + & 4) \\ \hline
 & 4(x)\ + & 4(1) \\
x(x)\ + & 1(x) & \\ \hline
x(x)\ + & 5(x)\ + & 4(1)
\end{array} = x^2 + 5x + 4$$

Example 2:

$$(x + 2)\ (x + 1) = \begin{array}{rrr}
 & (x\ + & 2) \\
 & (x\ + & 1) \\ \hline
 & 1(x)\ + & 1(2) \\
x(x)\ + & 2(x) & \\ \hline
x(x)\ + & 3(x)\ + & 1(2)
\end{array} = x^2 + 3x + 2$$

Example 3:

$$(x + 5)\ (x + 5) = \begin{array}{rrr}
 & (x\ + & 5) \\
 & (x\ + & 5) \\ \hline
 & 5(x)\ + & 5\ (5) \\
x(x)\ + & 5(x) & \\ \hline
x(x)\ + & 10(x)\ + & 5(5)
\end{array} = x^2 + 10x + 25$$

<u>Problems</u>

Multiply the following:

1. 12 x 15

2. $(10 + 3)\ (10 + 2)$

3. $(x + 3)\ (x + 1)$

4. $(x + 2)\ (x + 2)$

5. $(x + 10)\ (x + 8)$

6. $(x + 7)\ (x + 3)$

7. $(x + 4)\ (x + 3)$

8. 14×14

9. $(x + 6)\ (x + 1)$

10. $(x + 8)\ (x + 3)$

11. (x + 2) (x + 11)

12. (x + 5) (x + 3)

Factorization of Polynomials

IV. We want to think of factoring as the "reverse" operation of multiplication. For example, when we multiply 2 · 3, we use the notion of repeated addition and say that we have two "3's" or 3 + 3 or 6. When we factor we want to know how many copies of some number "n" is needed to give us a product of 6. That is, ? x ? must equal 6. Now we know 1 copy of six gives us six, six copies of 1 gives us six, 2 copies of 3 gives us 6, 3 copies of 2 gives us 6. Hence, 1, 2, 3, 6 are factors of 6. We will consider 2 x 3 = 3 x 2 and 1 x 6 = 6 x 1 by the commutative property of multiplication.

If one is asked to factor the number 12, he finds whole numbers that multiply to give him 12:

3 x 4

2 x 6

1 x 12

Suppose you are asked to factor the number 16. Then you find that 2 x 8, 4 x 4, and 16 x 1 multiply to give you 16.

Now let's look at multiplication and factorization together:

$2 \times 7 \rightarrow 14$ Multiplication

$14 \rightarrow 2 \times 7$ Factorization

Notice also that factorization does not always give unique answers. If asked to factor 20, you might give the answer as 2×10 or 4×5.

Let's look at the factors of the product 2x. What number times what number gives the product 2x? Right, $2 \cdot x$.
What factors give the product 8x? $8 \cdot x$ or $4 \cdot 2x$ or $2 \cdot 4x$.

Now let's factor 4^2:
$4^2 = 4 \times 4$. Generalized, if one factors x^2, he gets $x \cdot x$, or, if he factors $3x^2$, he gets $3x \cdot x$.

Problems

Factor the following:

1. 8

2. 10

3. 18

4. 11

5. 15

6. 24

7. 5x

8. 10x

9. $2x^2$

10. 7^2

11. 9x

12. $4x^2$

V. Let's look now at the sums of products to see if we can find their factors. Let's review multiplication first:

$2(4 + 3) = (4 + 3) + (4 + 3)$ (since multiplication is repeated addition)

$= (4 + 4) + (3 + 3)$

$= 2(4) + 2(3)$

Now factorization is the reverse operation:

$2(4) + 2(3)$

$= (4 + 4) + (3 + 3)$

$= (4 + 3) + (4 + 3)$

$= 2(4 + 3)$

Hence, the factors of $2 \cdot 4 + 2 \cdot 3$ are 2 and $(4 + 3)$.

Example 2:

Factor $(5 \cdot 8 + 5 \cdot 3)$:

$5 \cdot 8 + 5 \cdot 3$

$= 8 + 8 + 8 + 8 + 8 + 3 + 3 + 3 + 3 + 3$

$= (8 + 3) + (8 + 3) + (8 + 3) + (8 + 3) + (8 + 3)$

$= 5(8 + 3)$

Example 3:

Let's generalize from the previous examples:

Factor $6x + (6 \cdot 2)$:

$6x + (6 \cdot 2) = x + x + x + x + x + x + 2 + 2 + 2 + 2 + 2 + 2$

$= (x + 2) + (x + 2) + (x + 2) + (x + 2) + (x + 2) + (x + 2)$

$= 6(x + 2)$

Example 4:

Factor $7x + (7 \cdot 3)$

$7x + (7 \cdot 3)$

$= 7(x + 3)$

Example 5:

Factor $4x + (4 \cdot 2)$

$4x + (4 \cdot 2)$

$= 4(x + 2)$

Example 6:

Factor $2x + (3 \cdot 5)$

$2x + (3 \cdot 5)$

$= x + x + 5 + 5 + 5$

$= (x + 5) + (x + 5) + 5$

Since we cannot write the above expression in terms of so many copies of $x + 5$, then $2x + (3 \cdot 5)$ cannot be factored.

Now suppose we were asked to factor the sum $5x + 10$. We know we have 5 "x's" and ten "1's". But the 10 can be factored into $5 \cdot 2$. Then we have 5 "2's", which is the same quantity as ten "1's". Therefore, $5x + 10$

$$= 5x + (5 \cdot 2)$$

$$= 5(x + 2)$$

Example 1:

Factor $4x + 12$

$4x + 12$

$= (4 \cdot x) + (4 \cdot 3)$

$= 4(x + 3)$

Example 2:

Factor $20 + 5x$

$20 + 5x$

$= (5 \cdot 4) + (5 \cdot x)$

$= 5(4 + x)$

Example 3:

Factor $4x + 6$

$4x + 6$

$= (2 \cdot 2x) + (2 \cdot 3)$

$= 2(2x + 3)$

Example 4:

Factor $10x + 25$

$10x + 25$

$= 5(2x) + 5(5)$

$= 5(2x + 5)$

<u>Problems</u>

Factor the following:

1. $3x + (3 \cdot 1)$

2. $(2 \cdot 6) + (2 \cdot 4)$

3. $7x + 14$

4. $(5 \cdot 2) + (5 \cdot x)$

5. $10x + 60$

6. $3x + 4$

7. $2x + 2$

8. $16 + 4x$

9. $4x + 10$

10. $(8 \cdot 6) + (8 \cdot 5)$

11. $8x + 18$

12. $6x + 54$

VI. Let's suppose we were asked to factor $5x + 2x$. Now $5x + 2x = (5 \cdot x) + (2 \cdot x) = (5 + 2)x$.

A similar example involves factoring $x^2 + 5x$. We know $x \cdot x = x^2$, so generalizing from the previous examples we have:

$x^2 + 5x$

$= (x \cdot x) + (5 \cdot x)$

$= (x + 5)x$ or $x(x + 5)$ (using the commutative property of multiplication)

Example 1:

Factor $x^2 + 7x$

$x^2 + 7x$

$= (x \cdot x) + (7 \cdot x)$

$= (x + 7)x$

Example 2:

Factor $6x + x^2$

$6x + x^2$

$= (6 \cdot x) + (x \cdot x)$

$= (6 + x)x$

Example:

Factor $x^2 + 4$

$x^2 + 4$

$= (x \cdot x) + (2 \cdot 2)$

or $(x \cdot x) + (4 \cdot 1)$

But neither "breakdown" of the factors of the terms helps in finding factors of the sum. Therefore, $x^2 + 4 = x^2 + 4$.

Problems

Factor the following:

1. $x^2 + 8x$

2. $x^2 + 3x$

3. $x^2 + 9$

4. $x^2 + x$

5. $7x + x^2$

6. $2x + 8x$

7. $x^2 + 11x$

8. $4x + 6x$

9. $3x + 5x$

10. $5 + x^2$

VII. In this final section, let's first go back and look again at how one multiplies a two-digit number by a two-digit number, using the notion of place-value:

$$
\begin{aligned}
11 \times 12 &= (10 + 1)\ (10 + 2) \\
&= 10(10) + 1(10) + 2(10) + 2 \cdot 1 \\
&= 10^2 + 3(10) + 2 \\
&= 100 + 30 + 2 \\
&= 132
\end{aligned}
$$

Now to factor a number, we want to reverse this process:

$$
\begin{aligned}
&132 \\
&= 100 + 30 + 2 \\
&= 10^2 + 3(10) + 2(1) \\
&= (10 \cdot 10) + (3 \cdot 10) + (2 \cdot 1) \\
&= (10 + 1)\ (10 + 2)
\end{aligned}
$$

Example 1:

Factor 165

$$
\begin{aligned}
165 &= 100 + 60 + 5 \\
&= (10 \cdot 10) + (6 \cdot 10) + (5 \cdot 1)
\end{aligned}
$$

$= (10 \cdot 10) + (5 \cdot 10) + (1 \cdot 10) + (5 \cdot 1)$

$= (10 + 5)\ (10 + 1)$

You can check your answer by multiplying!

Example 2:

Factor $2^2 + 8(2) + 7$

$2^2 + 8(2) + 7$

$= (2 \cdot 2) + (8 \cdot 2) + (7 \cdot 1)$

$= (2 + 7)\ (2 + 1)$

Let's look at multiplication in the next problem. But this time "x" is used in the place of a whole number:

$(x + 1)\ (x + 2)$

$= (x \cdot x) + (1 \cdot x) + (2 \cdot x) + (2 \cdot 1)$

$= x^2 + 3x + 2$

Now let's look at the reverse process of factoring:

$x^2 + 3x + 2$

$= (x \cdot x) + (1 \cdot x) + (2 \cdot x) + (2 \cdot 1)$

$= (x + 1)\ (x + 2)$

Example 1:

Factor $x^2 + 4x + 3$

$x^2 + 4x + 3$

$= (x \cdot x) + (3 \cdot x) + (1 \cdot x) + (3 \cdot 1)$

$= (x + 3)\ (x + 1)$

Example 2:

Factor $x^2 + 5x + 4$

There are two possible ways to factor $x^2 + 5x + 4$ since $4 = 4 \times 1$ and $4 = 2 \times 2$. But, by using 2×2 as the factors of

4, we get $(x \cdot x) + (2 \cdot x) + (2 \cdot x) + (2 \cdot 2)$, which does not yield the correct "x" term. The correct factoring is as follows:

$$x^2 + 5x + 4$$
$$= (x \cdot x) + (4 \cdot x) + (1 \cdot x) + (4 \cdot 1)$$
$$= (x + 4)\ (x + 1)$$

Example 4:

Factor $x^2 + 6x + 8$

Since $8 = 8 \times 1$ and $8 = 4 \times 2$, the possible factors are:

1. $(x \cdot x) + (8 \cdot x) + (1 \cdot x) + (8 \cdot 1) = (x + 8)\ (x + 1)$
2. $(x \cdot x) + (4 \cdot x) + (2 \cdot x) + (4 \cdot 2) = (x + 4)\ (x + 2)$

But we need only to look at the "x" terms to decide that no. 2 is the correct factorization since $(4 \cdot x) + (2 \cdot x) = (6 \cdot x)$.

Example 5:

Factor $x^2 + 6x + 9$

Since $9 = 9 \times 1$ and $9 = 3 \times 3$, the possible factors are:

1. $(x \cdot x) + (9 \cdot x) + (1 \cdot x) + (9 \cdot 1) = (x + 9)\ (x + 1)$
2. $(x \cdot x) + (3 \cdot x) + (3 \cdot x) + (3 \cdot 3) = (x + 3)\ (x + 3)$

Number 2 is again the correct factorization since $(3 \cdot x) + (3 \cdot x) = 6x$.

Example 6:

Factor $x^2 + 8x + 12$

Since $12 = 12 \times 1$, $12 = 3 \times 4$, $12 = 6 \times 2$, the possible factors are:

1. $(x \cdot x) + (12 \cdot x) + (1 \cdot x) + (12 \cdot 1) = (x + 12)\ (x + 1)$

2. $(x \cdot x) + (3 \cdot x) + (4 \cdot x) + (3 \cdot 4) = (x + 3)\ (x + 4)$

3. $(x \cdot x) + (2 \cdot x) + (6 \cdot x) + (2 \cdot 6) = (x + 2)\ (x + 6)$

No. 3 is the correct factorization since $(2 \cdot x) + (6 \cdot x) = (8 \cdot x)$.

Problems

Factor the following:

1. 143

2. $10^2 + 6(10) + 9$

3. $x^2 + 5x + 6$

4. $x^2 + 2x + 1$

5. $x^2 + 8x + 7$

6. $x^2 + 6x + 5$

7. $x^2 + 7x + 10$

8. $x^2 + 4x + 4$

9. $x^2 + 8x + 16$

10. $x^2 + 12x + 20$

11. $x^2 + 5x + 2$

12. $x^2 + 8x + 15$

13. $x^2 + 13x + 36$

14. $x^2 + 3x + 5$

15. $x^2 + 7x + 12$

16. $x^2 + x + 2$

APPENDIX B

Polynomial Test

POSTTEST

RETENTION TEST

Directions:

Multiply or factor the following expressions according to the type of problem given.

1. $6x + 10$

2. $2x^2 + 5x$

3. $x(x - 2)$

4. $x^2 + 9x$

5. $2(2x + 1)$

6. $(x + 7)\ (x + 2)$

7. $x^2 + 5x + 4$

8. $4x + 12$

9. $x^2 + 11x + 18$

10. $(2x + 3)(2x + 5)$

11. $x^2 + 10x + 25$

12. $x(2x + 3)$

13. $5x - 15$

14. $2x^2 + 7x + 6$

15. $(x + 6)(x + 6)$

16. $7(x + 6)$

17. $(x + 5)\ (x + 3)$

18. $x^2 + 10x + 16$

19. $(x + 8)x$

20. $x^2 + 6x + 7$

21. $x(x + 3)$

22. $(x + 4)\ (x + 6)$

APPENDIX C

Interview Instruments

INTERVIEW INSTRUMENT

MANIPULATIVE GROUP

I. Multiply or factor without using the manipulatives:

1. $x(2x + 3)$

2. $8x + 12$

3. $(x + 3)(x + 4)$

4. $2x^2 + 17x + 21$

5. $x(x + 3)$

6. $2x^2 - 14$

7. $x^2 + 4$

II. Multiply or factor with the manipulatives:

1. $x(2x + 3)$

2. $8x + 12$

3. $(x + 3)(x + 4)$

4. $2x^2 + 17x + 21$

5. $x(x + 3)$

6. $2x^2 - 14$

7. $x^2 + 4$

III. Interpret the following expressions in terms of the manipulatives:

1. $x^2 + 5x + 6$

2. $5(x + 2)$

3. $3x^2 + 7x$

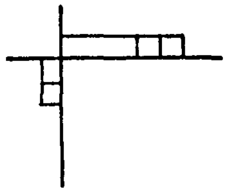
4. $x^2 + x + 1$

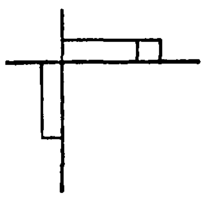
5. $x(x - 2)$

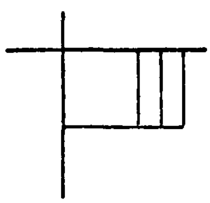
6. $x^2 + 10x$

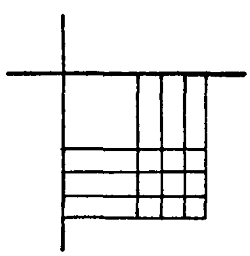
7. $(2x + 1)\ (2x + 5)$

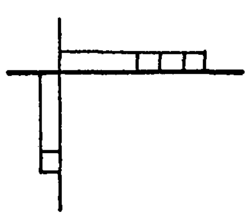
IV. Write symbolic expressions representing the following figures:

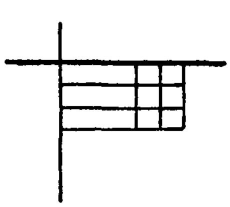
1. 

2. 

3. 

4. 

5. 

6. 

INTERVIEW INSTRUMENT

NONMANIPULATIVE GROUP

I. Multiply or factor:

1. $x(2x + 3)$

2. $8x + 12$

3. $(x + 3)(x + 4)$

4. $2x^2 + 17x + 21$

5. $x(x + 3)$

6. $2x^2 - 14$

7. $x^2 + 4$

II. Multiply or factor, indicating each step one must go through to work the problem:

1. $x^2 + 5x + 6$

2. $5(x + 2)$

3. $3x^2 + 7x$

4. $x^2 + x + 1$

5. $x(x - 2)$

6. $x^2 + 10x$

7. $(2x + 1)\ (2x + 5)$

APPENDIX D

Final Summaries of the Analyses on Pretest (<u>OHAPT</u>) Scores

Table 20

Means, Standard Deviations
of Pretest Scores of the Manipulative
and the Nonmanipulative Classes

	Class	N	Mean ($\overline{x}$)	S. D.
Exp. 1	1 (Manip.)	18	59.667	10.901
	2 (Nonmanip.)	23	55.043	10.222
	3 (Manip.)	17	57.588	12.942
	4 (Nonmanip.)	22	62.897	8.897
	Total	80	58.798	11.129
Exp. 2	5 (Manip.)	22	42.954	8.845
	6 (Manip.)	24	47.000	9.682
	7 (Nonmanip.)	24	54.520	10.477
	8 (Nonmanip.)	23	48.826	12.231
	Total	93	48.325	11.065

Table 21

Final Summary of One Factor ANOVA on Pretest Scores by Classes 1-4 (Experiment I)

Source	DF	SS	MS	F	P Less Than
Between Groups	3	614.1875	204.7292	1.602	0.172
Within Groups	76	9659.5625	127.7445		
Total	79	10273.7500			

Table 22

Final Summary of One Factor
ANOVA on Pretest Scores by
Classes 5-8 (Experiment II)

Source	DF	SS	MS	F	P Less Than
Between Groups	3	1700.6250	566.8750	5.018	0.002
Within Groups	89	10053.4375	112.9599		
Total	96	11754.0625			

Table 23

Final Summary of One Factor
ANOVA on Pretest Scores by
Classes 6 and 7 (Experiment 2)

Source	DF	SS	MS	F	P Less Than
Between Groups	1	867.5625	867.5625	8.218	0.005
Within Groups	47	4961.4375	105.562		
Total	48	5829.0000			

Table 24

Final Summary of One Factor
ANOVA on Pretest Scores by
Classes 6 and 8 (Experiment 2)

Source	DF	SS	MS	F	P Less Than
Between Groups	1	51.0625	51.0625	0.407	0.523
Within Groups	46	5776.1250	125.5679		
Total	47	5827.1875			

APPENDIX E

Final Summaries of the Tests for Homogenity of Regression

Table 25

Test for Homogenity of Regression of
Scores on Posttest (Using Classes 1-4, Exp. I)

Source	DF	SS	MS	F	P Less Than
Within Cells	76	2483.515	32.678		
Regression	1	46.566	46.566	1.425	0.236

(Pooled Regression Coefficient: 0.137)

Table 26

Test for Homogenity of Regression of
Scores on Retention Test (Using Classes 1-4, Exp. I)

Source	DF	SS	MS	F	P Less Than
Within Cells	76	1897.427	24.966		
Regression	1	4.688	4.688	0.188	0.666

(Pooled Regression Coefficient: 0.148)

Table 27

Test for Homogenity of Regression of
Scores on Posttest (Using Classes 1-2, Exp. I)

Source	DF	SS	MS	F	P Less Than
Within Cells	37	1073.637	29.017		
Regression	1	76.695	76.695	2.643	0.112

(Pooled Regression Coefficient: 0.144)

Table 28

Test for Homogenity of Regression of
Scores on Retention Test (Using Classes 1-2, Exp. I)

Source	DF	SS	MS	F	P Less Than
Within Cells	37	1034.996	27.973		
Regression	1	0.176	0.176	0.006	0.937

(Pooled Regression Coefficient: 0.197)

Table 29

Test for Homogenity of Regression of
Scores on Posttest (Using Classes 3-4, Exp. I)

Source	DF	SS	MS	F	P Less Than
Within Cells	35	1059.392	30.268		
Regression	1	13.572	13.572	0.448	0.507

(Pooled Regression Coefficient: 0.131)

Table 30

Test for Homogenity of Regression of
Scores on Retention Test (Using Classes 3-4, Exp. I)

Source	DF	SS	MS	F	P Less Than
Within Cells	35	408.496	11.671		
Regression	1	24.388	24.388	2.090	0.157

(Pooled Regression Coefficient: 0.125)

Table 31

Test for Homogenity of Regression of
Scores on Posttest (Using Classes 6 and 8, Exp. II)

Source	DF	SS	MS	F	P Less Than
Within Cells	43	1251.169	29.097		
Regression	1	7.127	7.127	0.245	0.623

(Pooled Regression Coefficient: 0.231)

Table 32

Test for Homogenity of Regression of
Scores on Retention Test (Using Classes 6 and 8, Exp. II)

Source	DF	SS	MS	F	P Less Than
Within Cells	43	1258.013	29.256		
Regression	1	18.182	18.182	0.621	0.435

(Pooled Regression Coefficient: 0.254)

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THE ROLE OF MANIPULATIVES
IN LEARNING TO MULTIPLY
AND FACTOR POLYNOMIALS

by

Carol Jane Altizer

(ABSTRACT)

The purpose of this exploratory research study was to investigate, both theoretically and experimentally, the hypothesis that the manipulation of concrete materials can contribute substantively to the learning of the operation of multiplication of polynomials and its inverse, factorization, in children who are in eighth grade pre-algebra mathematics classes. The study involved a comparison of the achievement of students who used manipulatives to learn how to multiply and factor polynomials with the achievement of those who did not use manipulatives to learn to operate on the polynomials. The instructional material designed for use by both treatment groups was based on the theory of learning developed by this writer. It was theorized that as students use manipulatives to learn mathematical concepts the actions performed upon the concrete materials would be abstracted or internalized in the mind as operations.

The study involved four teachers and 173 students from two middle schools in the Pulaski County School System, Pulaski, Virginia. The means and standard deviations of the students' scores

on both the immediate posttest and retention test were compared as well as inferences made from the data using several analyses of covariance. The Orleans-Hanna Algebra Prognosis Test served as the pretest for this study.

The F ratios from the analyses of covariance conducted on the immediate posttest scores from Experiment I indicated that (1) using the total population of students, there was no statistical difference in mean scores between the manipulative and nonmanipulative groups ($p = 0.385$); (2) using only Teacher A's students, there was no statistical difference in mean scores between the manipulative and nonmanipulative groups ($p = 0.609$); and (3) using only Teacher B's students, there was a marginal difference in mean scores between the manipulative and nonmanipulative groups ($p = 0.071$), favoring the nonmanipulative group.

The F ratios from analyses of covariance conducted on the retention test scores from Experiment I indicated that (1) using the total population of students, there was a statistical difference in mean scores between the manipulative and nonmanipulative groups ($p < 0.005$), favoring the manipulative groups; (2) using only Teacher A's students, there was a statistical difference in mean scores between the manipulative and nonmanipulative groups ($p < 0.009$), favoring the manipulative group; (3) using only Teacher B's students there was no statistical difference in mean scores between the manipulative and the nonmanipulative groups ($p = 0.241$). However, the mean score of Teacher B's manipulative group was higher than the mean score of his nonmanipulative group.

The study was replicated (Experiment II) immediately following Experiment I in two classes taught by Teacher D.* The F ratio from an analysis of covariance conducted on the immediate posttest scores indicated that there was no statistical difference in mean scores between the manipulative and nonmanipulative groups ($p = 0.762$). The F ratio from an analysis of covariance conducted on the retention test scores indicated that there was no statistical difference in mean scores between the manipulative and nonmanipulative groups ($p = 0.143$). However, the mean score of the manipulative group was higher than the mean score of the nonmanipulative group.

In summary, there are implications from these findings that the manipulation of concrete materials by students does aid the learning of the mathematical transformation of multiplication of polynomials and its inverse, factorization. This was especially evident for retention of the operations. These findings support the theory of learning conceptualized for this study.

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*Teacher C was omitted from the analyses of the data since she taught only a manipulative group.