A DYNAMIC PROGRAMMING APPROACH TO SINGLE ATTRIBUTE PROCESS CONTROL

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INTRODUCTION

Process control is becoming an integral part of an overall quality program as more managers realize the need to build rather than inspect quality into their product. Management realizes that it is less expensive to monitor and control the process than to incur the costs of producing an undesirable level of quality. Traditionally, purely statistical methods have been used to reduce the risks both to the producer and to the consumer. While the traditional approach does give a desired level of protection against defects, it may not be economically optimal due to the costs of operating the control system. This thesis presents an economic model for use when it is desirable to maintain control of the process fraction defective.

Process Control

A certain amount of variation is inherent in an industrial process. This variation can be due to either random or assignable causes. Random or chance variation is generally relatively small and cannot be controlled. Variation attributed to an assignable cause is larger; the cause can often be isolated and steps taken to eliminate the variation from the process. The purpose of process control is to monitor the system for assignable causes of variation in order to maintain an acceptable level of quality.

Process Control Data

There are two types of data which are used in process control. Variables data are measurable and have values which are continuous within an interval. The actual value measured or a function of this value is used as a basis for process control. For example, the radial torque of a bearing and the outer diameter of a shaft are variables measurements which can be used as the parameter to be controlled. Attributes data, on the other hand, are discrete--usually in a go no-go situation. The article is inspected and determined to be acceptable or unacceptable without considering an actual measurement. For example, either the bearing fits the frame or it doesn't; the insulation is either red or it isn't.

Although variables data tend to provide more information about the process, attributes sampling is well justified in many industrial process control situations. In some cases, such as in the inspection of color codes, there are no variables measurements that can be made. The cost of some of the necessary measurements may prove to be too costly. The training requirements for inspection by attributes are usually less stringent than those that are needed for variables type measurements. Any number of characteristics can be reviewed on an article from the sample instead of requiring separate control and sampling procedures for each characteristic as in variables sampling. An additional advantage of attribute testing is the ease of administration which can be illustrated by the following examples; go no-go gages can be preset so that the item can be easily compared with the tolerances; in order to determine acceptability, some items can be fitted into a mold or compared to a template; a visual inspection is needed to detect cracks, scratches, and burrs. This thesis will deal with process control using attributes testing and sampling procedures.

The p-Chart

The control chart, first introduced by Walter A. Shewhart, is an effective device for statistically monitoring the process. In most cases, a sample of fixed size is taken randomly from the process at frequent intervals. This sample is used to compute an estimate of the process parameter to be controlled. The estimate is then plotted on the control chart and compared with preset control values. If the value plots outside the control limits, assignable causes of variation are assumed to be present and the need for corrective action is indicated. The control chart also provides a method of studying the behavior of the process with respect to time.

The p-chart has been developed to provide a means for recording attributes data by showing variations in the fraction of the product which is defective in the samples. The p-chart, shown in Figure 1-1, consists of a graph of the fraction defective in the sample plotted against a time scale. The central line, or $\overline{p}$, indicates the fraction defective for which the process is considered to be producing an acceptable level of quality. In practice, the central line is often taken to be the historical average of the process, but it may also be the value of the fraction defective to which management is seeking to hold the process. The upper and lower control limits provide an indication of the cause

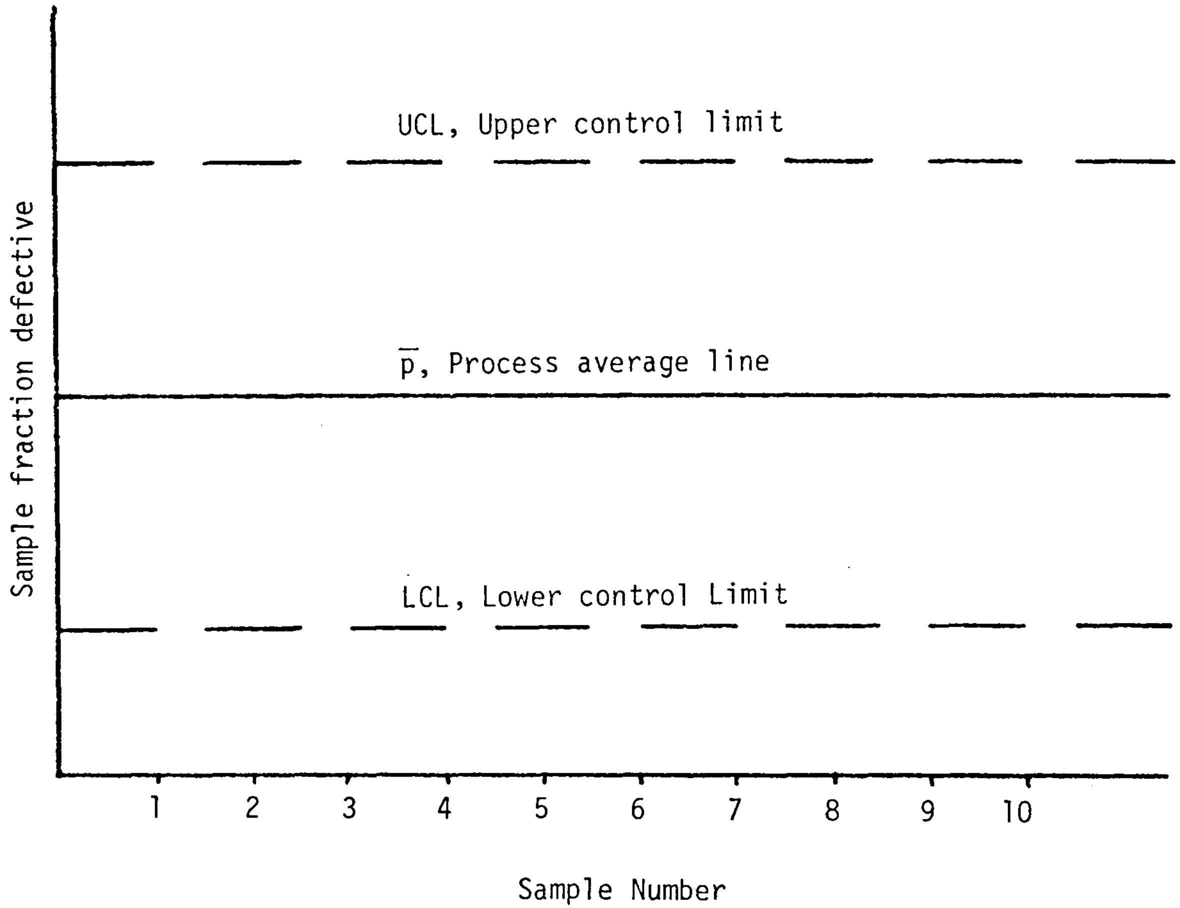

Figure 1-1 P-chart for Control of the Process Fraction Defective

and extent of the variation in the process. If a sample statistic falls within the control limits, the variation in the process is assumed to be due to chance. A point falling outside the control limits indicates a possible out-of-control situation and signals the need to search for assignable causes. The control limits have traditionally been set at three standard deviations, 3σ, above and below the process average. In the case of the p-chart, the 3σ lines fall at $\bar{p} \pm 3\sqrt{\bar{p}(1-\bar{p})/n}$, where $\bar{p}$ is the process average and n is the sample size to be taken. In many cases, when the p-chart is being used, only the upper control limit calls for corrective action. If the process is operating at a fraction defective less than the lower limit, management may want to investigate the process and re-adjust the control limits to the better operating value.

The traditional 3σ limits have not always proved to be the most economical for the process. If the cost of producing a defective item is quite high, perhaps it would be better to have tighter control limits. In some cases, it may be less expensive to run the process at a higher fraction defective rather than incur the expense of correction action or repair. The 3σ limits have worked well in many situations; however, some other multiple of σ may provide a more economical guideline for process control. To minimize the long run cost of operating the process, it is necessary to determine whether it is more economical to continue operating the process at an undesirable level of quality or to suffer the expense of adjusting the process. The control limits can then be adjusted accordingly.

In this thesis, a model is developed for determining the most economical control scheme for the p-chart. The control limit, the optimal sample size, and the frequency of sampling are determined based on the costs associated with operating the process control system. These results can then be used to provide a means for monitoring the process at less expense than if the traditional, purely statistical methods had been used.

Survey of Previous Literature

The traditional, purely statistical techniques for process control have been well developed in earlier works. Walter Shewhart [23] first introduced the control chart as a statistical tool for process control. Most of the later works on statistical process control employ the techniques presented by Shewhart. For additional discussion on the use of statistical sampling schemes and charting procedures, see Duncan [10] or Grant [13].

Many of the authors of recent works in process control are tending away from the traditional methods. Numerous papers have been published on cost-based process control. Most of the basic groundwork has been developed for the case in which the process is monitored by means of an $\overline{X}$ control chart. Duncan [9] developed an economic model for the $\overline{X}$ chart; his assumptions and methods yielded an approximation to the optimal control scheme. Duncan's cost model was optimized with respect to the sample size, the interval between samples, and the multiple of the standard deviation to be used in determining the control limits. Goel et al. [12] extended Duncan's model and developed a solution procedure which gives an exact minimum for the model.

Several authors have developed control procedures on the assumption that the process possesses the Markov property. This was basic to the model used by Duncan and Goel et al. Other research in this area has included work by Girschick and Rubin [11], who applied a Bayesian decision approach to variables sampling under the assumption of four process states--two productive and two repair states. Lave [20] also used the Markov assumption to present alternate control plans for a two state process and guidelines for selection of the optimal plan. Knappenberger and Grandage [16] assumed the Markov property for a process monitored by an $\overline{X}$ chart. This model had one in-control state and allowed for a shift to any of a fixed number of out-of-control states. Baker [2] developed an economic model based on the geometric distribution and used this model to challenge the widespread use of the Markov property. He warned that the process should be carefully studied in order to justify the Markov assumption.

The use of the Markov assumption has also been extended to multi-characteristic process control. Montgomery and Klatt [21] developed an expected cost model for the Hotellings T^2 control chart. Latimer et al. [19] have developed a model in which the design provides for simultaneous control of several independent characteristics and also determines the disposition of the lot.

In most of the work in cost-based process control, the model has been developed using a strictly Bayesian decision approach. Some recent authors have used dynamic programming procedures to find the optimal solution. Bather [4] considered a process which runs for an infinite period of time interrupted only to overhaul the process when necessary.

Dynamic programming was used to determine the optimal stopping region based on the costs of running and of overhauling the process. Pruzan [22] used a dynamic programming approach to determine the points of inspection for a system which provides for either 100% inspection or no inspection at all. Carter [7] extended the work of Bather to include the determination of the optimal sample size and the optimal time between samples by using dynamic programming in a Bayesian framework.

Several authors have developed economic models for the process fraction defective. Hald [14] formulated a model for the prior distribution of the proportion defective in lots. He used the binomial, Polya, and hypergeometric as examples of the prior distribution. Breakwell [6] adapted the minimax criterion to develop an economic model for the process fraction defective based on risk. Barnard [3] assumed and provided some justification for a mixed binomial prior and his paper summarizes some of the research which has been done in the field of cost-based process control.

Cost models have been developed for use with control charts for the fraction defective. Westbrook [25] developed a minimum cost sampling plan for the np control chart. For this model, he assumed that the proportion defective among lots represents a Beta random variable and that the proportion defective in a sample is binomially distributed. Ladany [17] developed a cost model which is an extension of Duncan's model for use with a p-chart.

Purpose

Cost-based attributes sampling plans can be used as a basis for control in many industrial processes. The type of situation in which this research may be applied can best be illustrated by example.

Consider a manufacturing process in which thin glass wafers are produced. Quality Engineering has noticed that many of these parts are defective after an etching process. To produce an acceptable wafer, the etching process is dependent on the wafer being free from cracks and chips. While Engineering realizes that some defects are inevitable, they would like to keep these defects to a minimum.

An operator is assigned to the operation in which the wafers are cut, and a system for controlling the process is established. A p-chart is designed and posted at the operation point. The operator is instructed to sample from the process at specified intervals based on the number of items produced between samples, and to inspect this sample visually for chips or cracks in the wafer surface. If the fraction defective of this sample exceeds the upper control limit, the operator is to perform an adjustment on the cutting tool to eliminate some of the defects.

It has been determined from past experience with this operation that the time the process remains in the in-control state before going out-of-control is an exponentially distributed random variable. Thus the process fraction defective possesses the Markov property. Although the process fraction defective is a continuous random variable, in most cases, the density function can be adequately represented by a discrete mass function. The prior distribution is thus assumed to be a mixed binomial; this is intuitively appealing and has been shown to work well in practice [3].

Since it is impossible to consider an infinite number of values of the process fraction defective, the values or intervals of values to be considered must be specified. Appropriate values to be used can usually be discerned from data taken on the process. The process is considered to be in-control for only one specified value of the process fraction defective, although the process may take on a finite number of out-of-control values. Also, it is assumed that the process, once out-of-control must be corrected by the operator in order to return to the in-control state.

Management has set the following objectives for the control system. The samples should indicate the need for outside action only when the process is actually out-of-control, but they should also be sensitive enough to signal an out-of-control state when it does occur. It is also necessary to minimize the costs of running the process and the control system. The cost of producing defectives, the cost of detecting an out-of-control situation, and the cost of repairing or adjusting the process must all be taken into consideration. This may imply running the process at a higher level of defectives if this cost is less than that of adjusting the process.

The main purpose of this thesis is to design an economically optimal process control system based on the process fraction defective. The model will be developed for the purpose of determining the optimal sample size, number of items produced between samples, and the upper control limit. It is assumed that the lower control limit does not signal the need for operator action and therefore does not play an important role in the optimal process control system.

Two approaches will be taken in the development of the economic model. In the first approach the steady state model with fixed values of the control parameters will be developed. In order to optimize this model, a suitable search procedure, such as the pattern search, will be employed. This economic model is to be designed for direct use with a p-chart. A model will also be developed which is solved by dynamic programming techniques. This approach will consider the future costs to the system at any point in time. There is no control chart for use with this model as the need for adjustment is signaled only by the economic impact of this action upon the future of the process. In this model, therefore, the upper control limit will not be a design variable.

The dynamic programming approach is necessarily more complex. At each point in time, the sampling scheme for the next time period must be computed based on the future costs to the system. This model could best be employed if a computer were used to monitor the process. The new control parameters could then be computed at each test point with little additional effort.

Outline of Succeeding Chapters

The fixed parameter model is developed in Chapter 2. The model is first developed for the case in which only two states are considered--one in-control state and one out-of-control state. The mathematical formulation of the cost model is presented. The transition matrix of the Markov chain describing the behavior of the process fraction defective is defined and all appropriate probability vectors are developed. The model is then extended to the multistate case in which there is only one in-control state but any number of out-of-control states.

The dynamic programming model is developed in Chapter 3. The cost model is presented and a recursive relationship developed which considers the current cost of a decision as well as the economic impact of this decision on all future decisions.

Optimization of the dynamic programming model is discussed in Chapter 4. The steady state solution to the dynamic programming model is proved to exist and the procedure for determining the steady state solution is presented.

An analysis is conducted in Chapter 5 for a hypothetical process. Examples are discussed and the results of optimizing the two models are compared for these examples.

The models are compared in Chapter 6. The advantages and disadvantages of each model are summarized and the limitations of the models are presented. Suggestions for future research are then presented.

Chapter 2

DEVELOPMENT OF THE FIXED PARAMETER MODEL

Introduction

In this chapter, the steady state model with fixed parameters will be developed. The assumptions of the model are stated, parameters defined, and the cost factors to be considered are discussed. The transition matrix of the Markov chain describing the behavior of the process fraction defective is also defined. The economic cost model is first formulated for the two state case, in which there is only one in-control state and one out-of-control state. The model is then extended to the multistate case, in which only one in-control state is considered, but where any number of out-of-control states may exist.

Model Description

The economic model to be developed describes a process control situation for which a p-chart is the main guideline for action. Samples of size n are to be taken at regular intervals corresponding to every k items that are produced. Each item is inspected on the required attribute, the fraction defective of the sample is computed, and then plotted on the p-chart. The process is adjusted only when it is assumed to be out-of-control; that is, when the sample fraction defective falls above the upper control limit on the p-chart. If the plotted value lies below the upper control limit, the process is assumed to be in-control and no corrective action is taken. It is also assumed that the process, once out-of-control, cannot return to the in-control state unless it is manually adjusted.

The model will be used to determine the economically optimal values for the number of items produced between samples, the sample size to be taken, and the upper control limit for the p-chart. It should be noted that although the model pertains to attribute inspection, the development parallels that of Knappenberger and Grandage [16] and Montgomery and Klatt [21].

Notation

The following notation will be used in the development of the model:

a_1 = the fixed cost of sampling;

a_2 = the variable sampling cost;

a_3 = the cost of adjusting the process;

a_4 = the cost of producing a defective item;

a_5 = the cost of a false indication of the state of the process;

n = the sample size;

x = the number of defective items in the sample;

k = the number of items to be produced between sampling;

f_i = the fraction defective for the ith process state;

R = the rate of production;

λ = the mean time between shifts in the process fraction defective;

α_i = the probability that the process is in state f_i at the time of sampling;

P_i = the probability of remaining in state f_i while k items are being produced;

q_i = the probability that the test procedure indicates that the process is out-of-control given that the process is in state f_i;

γ_i = the conditional probability that the process is in state f_i at any point in time;

Δ = the average fraction of time that elapses before a shift occurs, given that the process fraction defective does shift during the interval.

Cost Considerations

The expected total cost per unit of product depends on four cost components. They are: the cost of sampling, C_1, the cost of adjusting the process when it is determined to be out-of-control, C_2, the cost of producing a defective item, C_3, and the cost of falsely identifying the process as being out-of-control, C_4.

The cost of sampling involves two components. The first is a fixed cost, incurred at every inspection point, including set-up costs for the test. The second component is dependent on the number of items being sampled and includes such costs as the time involved in the test, the use of the test equipment, and the cost of the sampled unit, if destructive testing is being used.

When the process is determined to be out-of-control, an adjustment cost is incurred. This cost component involves the expense of searching for the cause of the variation, the time lost if the process must be halted, and the cost of repairing or adjusting the process.

A separate cost term is associated with falsely identifying the process as out-of-control. In many cases, the cost of a false identification is identical to the cost of correctly identifying the process state, and includes the adjustment costs as well as the cost of searching for the cause of variation. This is espcially true when only a minor adjustment, such as the twist of a dial, is needed to correct the process. However, if a part must be replaced to correct the process, there may be a penalty cost associated with discarding a part prematurely. Also in many situations, time can be wasted searching for a cause of variation which doesn't exist. Therefore, the two cost components, corresponding to correctly and incorrectly identifying the process as out-of-control, are included in the model as separate terms.

The final cost component is incurred whenever a defective item is produced. This includes the repair cost for the defective part or the cost of scrapping an item that cannot be repaired. Also included in this term is the loss of customer goodwill if the defective unit should get shipped.

The expected total cost can now be formed as the sum of these four cost components. Therefore,

$$E(C_T) = E(C_1) + E(C_2) + E(C_3) + E(C_4) ,$$

where C_T is the total cost of the control system.

The Two State Case

The model development which follows is for the case in which there are only two possible states in which the process can operate, the in-control state, f_0, and the out-of-control state, f_1.

The Cost Model

The cost of sampling, C_1, is composed of two terms--a_1, the fixed cost of sampling, and a_2, a variable cost which is dependent on the number of items in the sample. This cost component is then divided by k, the total number of items produced between samples, in order to determine the expected cost per item. Thus, the expected cost of sampling is given by

$$E(C_1) = \frac{a_1 + a_2 n}{k} .$$

The cost of adjusting the process, C_2, is incurred only when the process is assessed to be out-of-control. Let α_1 be the probability that the process is in the out-of-control state, f_1, and let q_1 be the conditional probability that the test procedure indicates that the process is out-of-control given that it is indeed out-of-control. The expected cost of adjusting the process, per item produced, is then,

$$E(C_2) = \frac{a_3}{k} (q_1 \alpha_1) ,$$

where a_3 is the cost of an adjustment to the process.

Let a_4 be the cost incurred whenever a defective item is produced. The fraction defective produced when the process is in a given state, i, is f_i. Also, let the probability that the process is in state f_i at any point in time be denoted by γ_i. Then the expected probability of producing a defective if the process is in state f_0 is given by $f_0\gamma_0$, and if the process is in state f_1, by $f_1\gamma_1$. Thus, the expected cost of producing a defective is

$$E(C_3) = a_4(f_0\gamma_0+f_1\gamma_1) .$$

The cost of false indication is incurred whenever the process is in-control but an out-of-control state is indicated. This occurs if the sample fraction defective exceeds the upper control limit when the true state of the process is f_0. Let a_5 be the cost of a false indication, α_0 be the probability that the process is in state f_0 at the time of sampling, and q_0 be the conditional probability that the test procedure indicates that the process is out-of-control when it is really in-control. The expected cost of falsely identifying the process as out-of-control is then given by

$$E(C_4) = \frac{a_5}{k} (q_0\alpha_0) .$$

Note that if the cost of false indication is the same as that of adjusting the process, then these two terms can be combined so that the cost of adjusting the process is given by

$$E(C_2) = \frac{a_3}{k} (q_0\alpha_0+q_1\alpha_1) .$$

The expected total cost model for the two state case is thus given by

$$E(C_T) = \frac{a_1 + a_2n}{k} + \frac{a_3}{k} (q_1\alpha_1) + a_4(f_0\gamma_0+f_1\gamma_1) + \frac{a_5}{k} q_0\alpha_0 .$$

The probability vectors $\underset{\sim}{q}$, $\underset{\sim}{\alpha}$, and $\underset{\sim}{\gamma}$ will be developed in the following sections. Also, the relationship of these vectors to the test parameters will be determined.

The Probability Vector $\underset{\sim}{q}$

The conditional probability that the test procedure will indicate that the process is out-of-control when it is in state f_i is given by the ith component of the vector $\underset{\sim}{q}$. However, q_i is simply the probability that the sample fraction defective plots above the upper control limit when the true process fraction defective is f_i. Let n be the sample size taken and let x be the number of defectives found in this sample. Since the process can be described by the binomial distribution, the probability that the test procedure indicates an out-of-control situation is given by

$$q_0 = \sum_{x \geq [n(UCL)]} \binom{n}{x} f_0^{\,x} (1-f_0)^{n-x} ,$$

if the process is in state f_0, and by

$$q_1 = \sum_{x \geq [n(UCL)]} \binom{n}{x} f_1^{\,x} (1-f_1)^{n-x} ,$$

if it is in state f_1. In the above equations $[n(UCL)]$ indicates the smallest integer larger than $n(UCL)$.

The Transition Matrix

The time between shifts is assumed to be exponentially distributed. Thus, the process is Markovian with discrete state space and discrete time space. For such a process, it is necessary to define the transition matrix, whose elements give the probability that a process in state f_i on the dth sample will shift to the state f_j before the (d+1)st sample. An explanation of the elements of this matrix, shown in Figure 2-1, is needed.

If the process is in-control on the dth sample, the probability that it will be in-control on the next sample is P_0. The probability that the process will shift to the out-of-control state before the next sample is given by P_1.

If the process is out-of-control, it can be manually adjusted to state f_0 and be in-control on the next sample. The probability that this event will occur is q_1P_0, the probability that the process is detected as out-of-control, adjusted, and remains in adjustment until the next sample. If the process is out-of-control on the present sample, there are two ways that the process can be in the out-of-control state on the next sample. They are: (1) the process remains out-of-control from one sample to the next, i.e., it is not detected as out-of-control, and (2) the process was determined to be out-of-control, was corrected, and then a shift occurred to the out-of-control state before the next sample. Thus, this element is given by $[q_1P_1 + (1-q_1)]$.

Since the matrix is composed of all positive elements, it is regular and hence defines the transition matrix for an irreducible, aperiodic, positive recurrent Markov chain [5]. Thus, letting G denote the transition matrix defined above, there exists an unique vector, $\underset{\sim}{\alpha}$, satisfying the system of equations $\underset{\sim}{\alpha}G = \underset{\sim}{\alpha}$, where $\alpha_0 + \alpha_1 = 1$. Solving this system of equations, one obtains the steady state solution

$$G = \begin{bmatrix} P_0 & P_1 \\ \\ q_1 P_0 & q_1 P_1 + (1-q_1) \end{bmatrix}$$

Figure 2-1: The two state transition matrix.

$$\alpha_0 = \frac{q_1 P_0}{P_1 + q_1 P_0} \quad \text{and} \quad \alpha_1 = \frac{P_1}{P_1 + q_1 P_0} \ .$$

In order to determine the probabilities P_0 and P_1, the probability that the process is in state f_0 and f_1, respectively, recall that the time between shifts is an exponentially distributed random variable with mean λ^{-1} hours. Then the probability of continuing to produce in state f_0 for a period of h hours is given by

$$P_0 = 1 - \int_0^h \lambda \ e^{-\lambda z} dz = e^{-\lambda h}$$

Let R be the production rate of the process, i.e., the number of units produced per hour, and let k be the number of units produced. Then $h = k/R$ and

$$P_0 = e^{-\lambda k/R} \ .$$

Thus, the probability that the process is in state f_1 is given by

$$P_1 = 1 - P_0 = 1 - e^{-\lambda k/R} \ .$$

Estimation of the Transition Probabilities

In order to estimate the process transition probabilities, the probabilities of shifts from one state to another, it is necessary to collect and analyze data from the process which is being studied. There are many methods available; one such method, that of maximum likelihood estimation, will be described here. This method is easily used to test for the existence of the Markov property in the process. For a detailed discussion of this method and other methods of parameter estimation, the analyst should consult a text on mathematical statistics or statistical inference, such as [15].

The first step in analyzing the process is to collect data by sampling from the process at frequent intervals and estimating the process fraction defective. From this data, it is often possible to reduce the entire range to the desired number of intervals. Guidelines for this interval selection can often be inferred by studying frequency plots of the process data.

The maximum likelihood estimates can be used to establish the transition probabilities. Let g_{ij} be the probability of a transition from state i to state j, and let $\hat{g}_{ij}$ be the estimate of this value. Then,

$$\hat{g}_{ij} = n_{ij}/n_i^* ,$$

where n_{ij} is the number of transitions from a state i to a state j, and n_i^* is the total number of transitions from a state i.

The estimates obtained in this manner can be used to test for the existence of the Markov property. This test is discussed in Appendix A.

The Probability Vector $\underset{\sim}{\gamma}$

Finally, it is necessary to determine the vector $\underset{\sim}{\gamma}$, the conditional probability that the process is producing in a particular state at a particular point in time. Duncan [9] has shown that the average fraction of time, Δ, that elapses in an interval before a shift occurs, given that a shift does occur, is

$$\Delta = \frac{1 - (1+\frac{\lambda k}{R})\ \exp(-\lambda k/R)}{[1 - \exp(\frac{-\lambda k}{R})]\ \lambda k/R} .$$

To determine γ_0 note that either of two events can cause the process to be in state f_0. The system could start in state f_0 and remain in-control between samples, or the process can start in-control and shift out-of-control sometime during the interval. Thus, γ_0 is given by

$$\gamma_0 = \alpha_0 P_0 + \Delta\alpha_0 P_1 .$$

In the same manner, to determine γ_1, the process can either be out-of-control when the first sample was taken and not be detected, or the process could be in-control at the start of the sampling interval and shift during the interval, thereby producing 100(1-Δ) percent of the time in state f_1. Hence,

$$\gamma_1 = \alpha_1 + (1-\Delta)\alpha_0 P_1 .$$

The Multistate Cost Model

The preceding model will now be extended to the case in which there are several possible out-of-control states. The fraction defective f_0 is considered to be the only in-control state; f_i, where i can take on any of the values, i = 1, 2, ..., m are the process fraction defectives representing the out-of-control states, and f_m is the worst level of quality which will be considered. It is further assumed that $f_0 < f_1 < \ldots < f_m$. The model does not consider the case in which the process fraction defective is lower than f_0. If the process is operating in a state which is lower than the in-control state, management may want to determine the cause and adjust the control scheme to the better operating value.

The Expected Cost Model

As in the two state case the cost per item of the sampling procedure is given by

$$E(C_1) = \frac{a_1 + a_2 n}{k} .$$

where a_1 is the fixed inspection cost, and a_2 is the variable cost of inspection.

The cost of adjusting the process is incurred whenever the process is detected to be out-of-control. Letting $\underset{\sim}{q}$ be the vector of conditional probabilities that the test procedure indicates that the process is out-of-control, and $\underset{\sim}{\alpha}$ the vector of probabilities that the process is in a particular state at the time of sampling, the expected cost per item of adjusting the process is given by

$$E(C_2) = \frac{a_3}{k} \sum_{i=1}^{m} q_i \alpha_i .$$

The probability of producing a defective item is dependent on the number of defectives produced when the process is in state i, f_i and the probability of the process being in that state, γ_i. Hence, the expected cost of producing defectives is the sum of these probabilities and,

$$E(C_3) = a_4 \sum_{i=0}^{m} f_i \gamma_i .$$

The cost of false indication is only incurred when the process indicates an out-of-control situation, but is actually in-control. Thus, the expected cost of false indication is given by

$$E(C_4) = \frac{a_5}{k} \alpha_0 q_0 .$$

The expected total cost of the fixed parameter, multistate process control system is therefore given by

$$E(C_T) = \frac{a_1 + a_2 n}{k} + \frac{a_3}{k} \sum_{i=1}^{m} q_i \alpha_i + a_4 \sum_{i=0}^{m} f_i \gamma_i + \frac{a_5}{k} \alpha_0 q_0 .$$

The Probability Vector $\underset{\sim}{q}$

As in the previous discussion, let $\underset{\sim}{q} = (q_0, q_1, \ldots, q_m)$ be the probability that the test procedure indicates that the process is out-of-control given that the process is in state f_i. Thus, for all i,

$$q_i = \sum_{x \geq [n(UCL)]} \binom{n}{x} f_i^{\,x} (1-f_i)^{n-x}$$

where [n(UCL)] indicates the smallest integer greater than n(UCL).

The Transition Probabilities

Let P_{ij} denote the probability of a shift from state i to the state j. Then P_{00}, the probability that the process remains in the state f_0 is, as in the two state case,

$$P_{00} = e^{-\lambda k/R} .$$

Thus, the probability of a shift from state f_0 is given by

$$1 - e^{-\lambda k/R} .$$

Since the process can shift from the in-control state to any of the m out-of-control states, this probability will be assigned among the m states using the procedure of Knappenberger and Grandage [16] and Latimer et al. [19].

Consider the binomial mass function, a single parameter distribution which represents the probability of j successes in m trials given by

$$Q(j) = \binom{m}{j} S^j (1-S)^{m-j} \ .$$

There is a fraction a, where $0 < a < 1$, such that

$$\sum_{j=1}^{m} P_{0j} = a \sum_{j=1}^{m} \binom{m}{j} S^j (1-S)^{m-j} \ .$$

It is also true that

$$\sum_{j=0}^{m} \binom{m}{j} S^j (1-S)^{m-j} = 1 \ .$$

Thus,

$$\begin{aligned} \sum_{j=1}^{m} \binom{m}{j} S^j (1-S)^{m-j} &= 1 - Q(0) \\ &= 1 - (1-S)^m \ . \end{aligned}$$

Solving the above equation to determine the value of a gives

$$a = \frac{1 - e^{-\lambda k/R}}{1 - (1-S)^m} \ .$$

Therefore, the probability of a shift from the in-control state, f_0, to a state f_j is given by

$$P_{0j} = \frac{(1 - e^{-\lambda k/R})}{1 - (1-S)^m} \binom{m}{j} S^j (1-S)^{m-j} \ .$$

The probability of a shift, P_{ij}, from state f_i to state f_j, where $i \neq 0$ and $i \leq j$, will be determined as a fraction of the probability of a shift from the in-control state to the jth state. Therefore,

$$P_{ij} = \frac{P_{0j}}{\sum_{j=1}^{m} P_{0j}} = \frac{P_{0j}}{1 - P_{00}} .$$

Then, assigning the remaining probability,

$$P_{ii} = 1 - \sum_{j=i}^{m} P_{ij}$$

$$= \sum_{j=1}^{i} P_{ij}$$

$$= \frac{\sum_{j=1}^{i} P_{0j}}{1-P_{00}} .$$

The value of s to be used in the binomial mass function can often be discerned from the data taken on the process and provides an indication of the skewness of the distribution of the probabilities. If $s < \frac{1}{2}$, the skewness is positive and the transition probabilities are seen to decrease as the magnitude of the shift increases. If $s = \frac{1}{2}$, the distribution is symmetric. This represents the situation in which the transition probabilities increase as the magnitude of the shift increases up to a certain point. Then, as the magnitude continues to increase, the transition probabilities begin to decrease. Finally, if $s > \frac{1}{2}$, there is negative skewness and, as the magnitude of the shift increases, so do the transition probabilities.

By carefully selecting the values of s and m from the data of the process, it is possible to accurately describe many practical situations.

The Transition Matrix

Using the transition probabilities defined above, the elements of the transition matrix, G, can be determined.

The probability of a shift, g_{0j}, from the in-control state to an out-of-control state is just

$$g_{0j} = P_{0j}.$$

If $i \leq j$, where $i \neq 0$, then g_{ij} has two components--the probability that the process shifts to state j, and the probability that the process was determined to be out-of-control, was corrected to the in-control state, and then shifted immediately to the state f_j. Thus,

$$g_{ij} = (1-q_i)P_{ij} + q_i P_{0j}.$$

If $i > j$, the only way that a shift can occur is for the process to have been detected as out-of-control, corrected back to the in-control state, and then the process immediately shifted to the jth state. Thus,

$$g_{ij} = q_i P_{0j} .$$

The Probability Vector $\underset{\sim}{\alpha}$

The transition matrix described above is shown in Figure 2-2. In Appendix B, it is shown that the matrix is regular, and thus describes a transition matrix of an irreducible, aperiodic, positive recurrent Markov chain. Thus, it is possible to solve for the steady state

$$
\begin{array}{c|lllll}
 & f_0 & f_1 & \cdots\ f_j & & \cdots\ f_m \\
\hline
f_0 & P_{00} & P_{01} & \cdots\ P_{0j} & & \cdots\ P_{0m} \\
f_1 & P_{00}q_1 & P_{01}q_1+P_{11}(1-q_1) & \cdots\ q_1P_{01}+(1-q_1)P_{1j} & & \cdots\ q_1P_{0m}+(1-q_1)P_{1m} \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
f_i & P_{00}q_i & q_iP_{01} & \cdots\ q_iP_{0j}+(1-q_i)P_{ij} & & \cdots\ q_iP_{0m}+(1-q_i)P_{im} \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
f_m & P_{00}q_m & q_mP_{01} & \cdots\ q_mP_{0j} & & \cdots\ P_{0m}q_m+(1-q_m)P_{mm}
\end{array}
$$

Figure 2-2 The Multistate Transition Matrix vector $\underset{\sim}{\alpha}$, the probability that the process is in a particular state at the time of sampling.

Latimer [18] derives the solution vector using the system $\underset{\sim}{\alpha}G = \underset{\sim}{\alpha}$, subject to the constraint that $\sum_{i=0}^{m} \alpha_i = 1$. For our situation the elements of $\underset{\sim}{\alpha}$ are given by

$$\alpha_j = \frac{\alpha_0}{P_{00}[1-(1-q_j)P_{jj}]} \{P_{0j} + \sum_{i=1}^{\alpha-1} \frac{P_{0i}(1-q_i)P_{ij}}{[1-P_{ii}(1-q_i)]}$$

$$+ \sum_{i=1}^{j-2} \sum_{k=i+1}^{j-1} \frac{P_{0i}P_{ik}P_{kj}(1-q_i)(1-q_k)}{[1-P_{ii}(1-q_i)][1-P_{kk}(1-q_k)]}$$

$$+ \sum_{i=1}^{j-3} \sum_{k=i+1}^{j-2} \sum_{\ell=k+1}^{j-1} \frac{P_{0i}P_{ik}P_{k\ell}P_{\ell j}(1-q_i)(1-q_k)(1-q_\ell)}{[1-P_{ii}(1-q_i)][1-P_{kk}(1-q_k)][1-P_{\ell\ell}(1-q_\ell)]}$$

$$+ \ldots + \sum_{i=1}^{j-(j-1)} \sum_{k=i+1}^{j-(j-2)} \cdots \sum_{s=j-2}^{j-2} \sum_{t=s+1}^{j-1}$$

$$\frac{P_{0i}P_{ik}\ldots P_{st}P_{tj}(1-q_i)\ldots(1-q_s)(1-q_t)}{[1-P_{ii}(1-q_i)]\ldots[1-P_{ss}(1-q_s)][1-P_{tt}(1-q_t)]}\} .$$

Furthermore, α_0 can be written as

$$\alpha_0 = \{1 + \sum_{j=1}^{k} \frac{F(j)}{P_{00}[1-P_{jj}(1-q_j)]}\}^{-1}$$

where F(j) represents the term in braces in the α_j expression given above.

The Probability Vector $\underset{\sim}{\gamma}$

In order to solve for the vector $\underset{\sim}{\gamma}$, the conditional probability of the process being in a particular state at a given time, let Δ be defined as before. Then

$$\gamma_0 = \alpha_0 P_{00} + \Delta\alpha_0(1-P_{00}) \ .$$

To determine γ_i, where $i \neq 0$, four cases in which the system could be in state f_i must be considered. They are:

1) the process is in state f_i and remains in state f_i;
2) the process is in state f_0 and shifts to state f_i;
3) the process is in state f_j and shifts to state f_i, where $i > j$;
4) the process is in state f_i and shifts to state f_j, where $i < j$.

Thus γ_i depends on these four cases, and, as developed by Knappenberger and Grandage [16], is given by

$$\gamma_i = \frac{\alpha_i \sum_{j=1}^{i} P_{0j}}{1 - P_{00}} + \alpha_0(1-\Delta)P_{0i} + \sum_{j=1}^{i-1} \alpha_j \frac{P_{0i}}{1 - P_{00}}(1-\Delta)$$

$$+ \frac{\alpha_i}{1 - P_{00}}\Delta \sum_{j=i+1}^{m} P_{0j} \ .$$

Chapter 3

THE DYNAMIC PROGRAMMING MODEL

Introduction

The model in the previous chapter was used to find the values of the sample size and interval between sampling which were optimal in the steady state. The assumption that the process is operating in a state of equilibrium is not always satisfactory. Dynamic programming techniques are used in the model of this chapter and this approach avoids the assumption of steady state conditions. The optimal values of the parameters are selected based on the expected process state as determined from sample results. Using dynamic programming, at each decision point, the cost of the alternate actions, overhaul the process or allow it to continue to run at the present level, are balanced with the economic impact of these actions on all future decisions. A recursive relationship is developed which guides in the selection of the action that minimizes the resulting cost for the case in which there are only two possible process states. The model is then extended to allow for a finite number of states.

Dynamic Programming

Dynamic programming uses a recursive formulation to balance present and future costs to the system. Consider the decision tree in Figure 3-1. The circular nodes represent the points at which the operator must make a decision about the process. In order to determine the appropriate action,

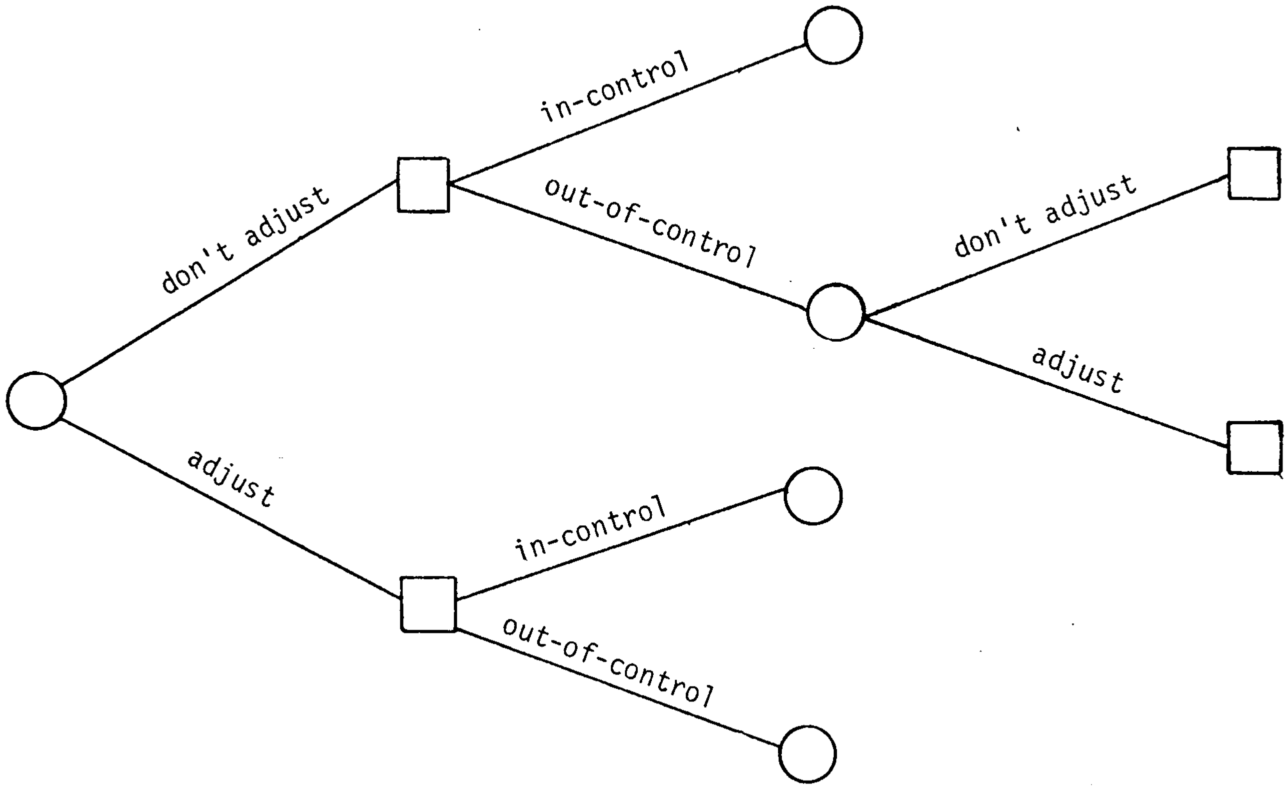

Figure 3-1 The Decision Tree for the Dynamic Programming Model

a prior distribution is computed before sampling. After sampling, the posterior distribution can be computed using the prior distribution and the sampling statistics. Based on this posterior distribution, one can determine the expected costs of adjusting and of not adjusting the process, and can also predict the economic impact of these decisions on all future decisions. The most economical alternative, either to adjust the process or to continue to run at the present level, is then chosen.

Notation

The following notation will be used in the development of the dynamic programming cost model:

a_1 = the fixed cost of sampling;

a_2 = the variable sampling cost;

a_3 = the cost of an adjustment to the process;

a_4 = the cost of producing a defective item;

t = the time index;

f_i = the process fraction defective for the ith state;

$\alpha'_{i,t}$ = the prior probability that the process is in state f_i at time t;

$\alpha''_{i,t}$ = the posterior probability that the process is in state f_i at time t;

x = the number of defective items found in the sample;

$h(x/f_i)$ = the conditional probability of obtaining the sample fraction defective, given that the true state of the process is f_i;

Δ = the average amount of time that elapses before a shift in the process fraction defective occurs, if a shift does occur;

$\gamma''_{i,t}$ = the posterior probability at time t, of producing f_i defectives at any point in time;

P_i = the probability of remaining in state f_i while k items are being produced;

n_t = the sample size at time t;

k_t = the number of items to be produced before the next sample; in other words, the length of the sampling interval at time t;

λ = the mean time between shifts in the process fraction defective;

R = the rate of production.

Cost Considerations

The following three cost components are considered in the development of the cost model: the cost of sampling, C_1, the cost of process adjustment, C_2, and the cost of producing a defective item, C_3.

The sampling cost is composed of two components--a fixed inspection cost, and a variable cost of sampling which is based on the number of items in the sample. The sampling cost includes such factors as the time and equipment needed to sample the items from the process and to run the test. The cost of the sampled unit is also included in the inspection cost if destructive testing is being used.

The cost of adjusting the process is incurred only when the results of the sample indicate that the process is to be adjusted. Among the factors included in this term are the cost of searching for the cause of variation in the process and the cost of adjusting, repairing, or overhauling the process.

Whenever a defective item is produced, a cost is incurred which includes repair, rework, or scrap costs, and the cost associated with the loss of customer goodwill if a defective part should get shipped to the customer.

The cost of a false indication that the process is out-of-control, as discussed in Chapter 2, is no longer a factor in the cost model. The process is only considered to require adjustment when adjustment is the most economical decision with respect to the future of the system.

These cost components will be discussed for the two possible actions, adjust the process or continue to run at the present level. If the process is adjusted, the cost model for the present decision will be

$$E_t(C_T) = E_t(C_1) + E_t(C_2) + E_t(C_3) ,$$

where $E_t(C_T)$ is the expected cost of the decision. If the process is not adjusted, the expected total cost model for the present decision is given by

$$E_t(C_T) = E_t(C_1) + E_t(C_3) .$$

Note that the cost of process adjustment is not incurred if the process is allowed to continue running at the current level.

Development of the Two State Case

The dynamic programming model will first be developed for the case in which there can be only two possible states, the in-control state, f_0, and the out-of-control state, f_1. The model will be considered for the two actions, adjust the process or continue running without adjustment.

The Expected Cost Model

The cost of sampling, C_1, is composed of a fixed cost, a_1, and a variable cost, a_2, divided among the k_t items produced in the interval between samples. Thus,

$$E_t(C_1) = \frac{a_1}{k_t} + \frac{n_t a_2}{k_t} .$$

Note that the number of items in the sample and the length of the sampling interval are functions of time.

The cost of adjustment, C_2, is only incurred when the process is adjusted. This cost is omitted if no adjustment is made. Let a_3 be the cost of an adjustment to the process; then, if the process is adjusted,

$$E_t(C_2) = \frac{a_3}{k_t} ,$$

where the cost is divided by the number of items produced in the sampling interval.

The expected cost, at time t, of producing a defective is dependent on the probability that a defective unit is produced during the interval. For each of the possible process states, the fraction defective, f_i, produced in the state is multiplied by the probability of being in that state at the time of sampling, $\gamma''_{i,t}$. Letting a_4 be the cost associated with producing a defective item, the expected cost per item is given by

$$E_t(C_3) = a_4(f_0\gamma''_{0,t}+f_1\gamma''_{1,t}) .$$

After a sample is taken at time t, the expected cost model for the process, if it is to be adjusted, is

$$E_t(C_T) = E_t(C_1) + E_t(C_2) + E_t(C_3) .$$

If the process is not adjusted, the expected cost will be

$$E_t(C_T) = E_t(C_1) + E_t(C_3) .$$

Therefore, after each sample, the policy chosen will be:

$$\min \begin{cases} E_t(C_2) + E_t(C_3) + V_{t-1}(\alpha'_{0,t-1},\alpha'_{1,t-1}) , \\ E_t(C_3) + V_{t-1}(\alpha'_{0,t-1},\alpha'_{1,t-1}) , \end{cases}$$

where $V_{t-1}(\alpha'_{0,t-1},\alpha'_{1,t-1})$ represents the future cost of the decision to the system and is dependent on the action taken.

Letting $\varepsilon = 1$ if the process is to be adjusted, and $\varepsilon = 0$ otherwise, the expected total cost of the system can therefore be described by the recursive equation:

$$\begin{aligned} E_t(C_T) &= \min_{n_t} E_{x|\alpha'_{0,t},\alpha'_{1,t}} \{\min[\frac{a_3}{k_t}\varepsilon + a_4(f_0\gamma''_{0,t}+f_1\gamma''_{1,t}) \\ &\quad + V_{t-1}(\alpha'_{0,t-1},\alpha'_{1,t-1})] + \frac{a_1+a_2n_t}{k_t}\} \\ &= V_t(\alpha'_{0,t},\alpha'_{1,t}) \end{aligned}$$

where $\gamma''_{0,t}$ and $V_{t-1}(\alpha'_{0,t-1},\alpha'_{1,t-1})$ are dependent on the posterior distribution based on the decision being considered.

In the following sections, the probability vectors $\underset{\sim}{\gamma}''_t$ and $\underset{\sim}{\alpha}_t$ will be developed and the relationship of these vectors to the test parameters will be discussed.

The Transition Matrix

As in the fixed parameter model, the process is assumed to possess the Markov property. Therefore, the time between shifts in the process fraction defective is an exponentially distributed random variable with mean λ^{-1} hours. The probability, P_0, of continuing to produce in state f for a period of h hours is given by

$$P_0 = 1 - \int_0^h \lambda\, e^{-\lambda z} dz = e^{-\lambda h} .$$

Letting R be the production rate and k_t the number of units produced, P_0 reduces to

$$P_0 = e^{-\lambda k_t/R} .$$

The probability that the process is in state f is, thus,

$$P_1 = 1 - P_0 = 1 - e^{-\lambda k_t/R} .$$

Using the transition probabilities, P_0 and P_1, the transition matrix, whose elements give the probability of a shift from state f_i on the dth sample to any state f_j on the (d-1)st sample, can be defined. If the process is in-control on the dth sample, on the next sample it can only be out-of-control on the next sample, since one of the basic assumptions of the model is that the process must be manually adjusted to return to the in-control state. The transition matrix for the process, when it is not adjusted, is thus given by

$$G = \begin{bmatrix} P_0 & P_1 \\ 0 & 1 \end{bmatrix}$$

Now consider the case in which the process is adjusted at the dth sample. The process is manually returned to the in-control state. The probability that the process remains in-control is P_0, and the probability that the process shifts out-of-control is P_1. In the next section, it will be shown that the transition matrix for the decision not to adjust, G, can be used for either decision.

The Probability Vector, $\underset{\sim}{\alpha}_t$

Let α_0 be the probability that the process is in state f_0 at the time of sampling, and let α_1 be the probability that the process is in state f_1. Since the dynamic programming model is time dependent, $\underset{\sim}{\alpha}$ and the other time dependent vectors will be subscripted with the time parameter, t. When $\alpha_{i,t}$ is used to indicate a prior probability, it will be superscripted with a prime, $\alpha'_{i,t}$; if it indicates a posterior probability, the probability will be denoted by a superscripted double prime, $\alpha''_{i,t}$. Figure 3-2 shows the relationship of these vectors with respect to time.

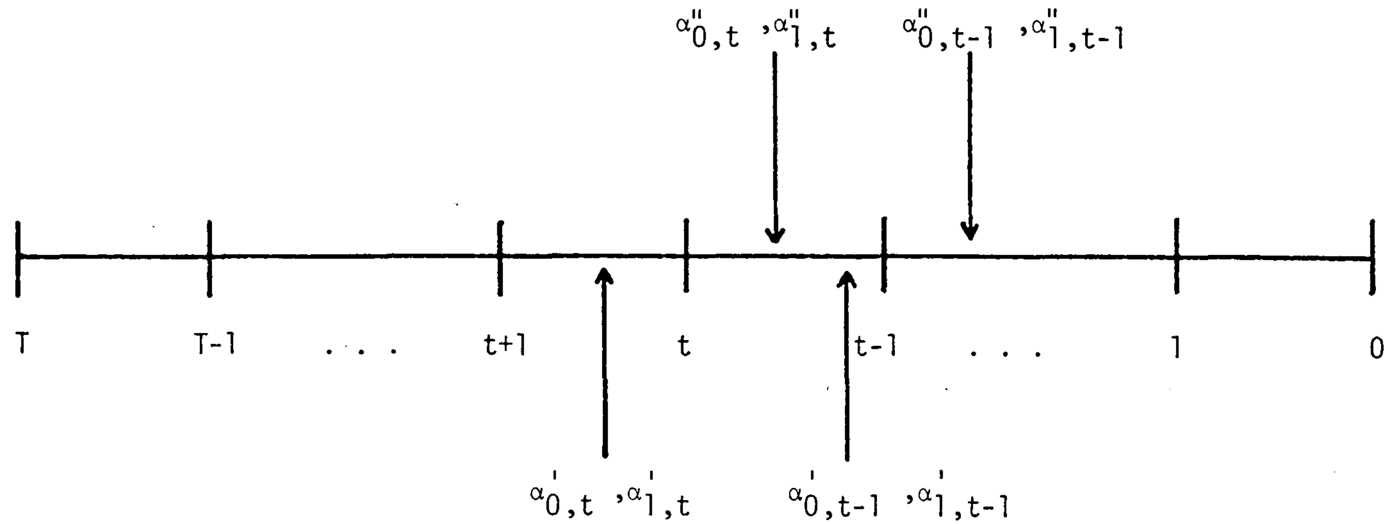

Figure 3-2 Relationship of the Prior and Posterior Distributions of $\underline{\alpha}_t$ with Respect to Time

If the process is adjusted at time t, then $\alpha''_{0,t} = 1$ and $\alpha''_{1,t} = 0$. In other words, the adjustment made will be the correct one, back to the in-control state, f_0. If the process is not adjusted at time t, the posterior probabilities, based on the prior distribution and the sample fraction defective can be found using Bayes formula. Thus,

$$\alpha''_{0,t} = \frac{h(x|f_0)\alpha'_{0,t}}{h(x|f_0)\alpha'_{0,t} + h(x|f_1)\alpha'_{1,t}} ,$$

and

$$\alpha''_{1,t} = \frac{h(x|f_1)\alpha'_{1,t}}{h(x|f_0)\alpha'_{0,t} + h(x|f_1)\alpha'_{1,t}} .$$

The prior distribution for the next time period can be determined from the relation $\underset{\sim}{\alpha}''_t G = \underset{\sim}{\alpha}'_{t-1}$ where G is the transition matrix whose elements give the probability of a shift occuring between the dth and the (d-1)st sample.

If the process is not adjusted, the following equations are derived from $\underset{\sim}{\alpha}''_t G = \underset{\sim}{\alpha}'_{t-1}$:

$$\alpha''_{0,t} P_0 + \alpha''_{1,t}(0) = \alpha'_{0,t} ,$$

and

$$\alpha''_{0,t} P_1 + \alpha''_{1,t}(1) = \alpha'_{1,t}$$

This implies that $\alpha'_{0,t-1} = \alpha''_{0,t}P_0$ and $\alpha'_{1,t-1} = \alpha''_{0,t}P_1 + \alpha''_{1,t}$.

Since $\alpha''_{0,t} = 1$ and $\alpha''_{1,t} = 0$, if the process is adjusted, these prior probabilities are given by $\alpha'_{0,t-1} = P_0$ and $\alpha'_{1,t-1} = P_1$. Also, by using the transition matrix for not adjusting, G, if the process is adjusted $\underset{\sim}{\alpha}''_t G = \underset{\sim}{\alpha}'_{t-1}$. Since $\alpha''_t = (1,0)$, these equations become

$\alpha'_{0,t-1} = P_0$ and $\alpha'_{1,t-1} = P_1$, which is the same result that was obtained from the use of the transition vector for adjustment. Thus, only the transition matrix, G, need be considered.

The Probability Vector $\underset{\sim}{\gamma}''_t$

Let Δ be the average amount of time that elapses before a shift in the process fraction defective occurs, given that a shift does indeed occur. Duncan [9] has shown that

$$\Delta = \frac{1 - (1+\lambda k_t/R)e^{-\lambda k_t/R}}{\lambda k_t/R\ (1-e^{-\lambda k_t/R})}$$

The probability vector, $\underset{\sim}{\gamma}''_t$, the conditional probability of producing f_i defectives at any point in time, is a function of $\underset{\sim}{\alpha}''_t$. As in the fixed parameter case, the elements of this vector are

$$\gamma''_{0,t} = \alpha''_{0,t}P_0 + \Delta\alpha''_{0,t}P_1 ,$$

and

$$\gamma''_{1,t} = \alpha''_{1,t} + \alpha''_{0,t}(1-\Delta)P_1 .$$

If the process is adjusted at time t, $\alpha''_{0,t} = 1$ and $\alpha''_{1,t} = 0$. Thus, γ''_t reduces to $\gamma''_{0,t} = P_0 + \Delta P_1$ and $\gamma''_{1,t} = (1-\Delta)P_1$.

The Multistate Case

The preceding model allowed for only one in-control state and one out-of-control state. It will now be extended to the case in which there can be any number of out-of-control states.

The Expected Cost Model

As in the two state model, the cost per item of the sampling procedure is given by

$$E_t(C_1) = \frac{a_1 + a_2 n_t}{k_t} .$$

A cost, a_3, is incurred whenever the process is adjusted. If the process is allowed to continue running, this cost term is not included. The expected cost, per item, of process adjustment is thus,

$$E_t(C_2) = \frac{a_3}{k_t} .$$

The expected cost of producing a defective item is dependent upon the number of defectives produced when the process is in a state i, multiplied by the probability of the process being in that state at the time of sampling. This cost is summed for all states; thus,

$$E_t(C_3) = a_4 \underset{\sim}{f} \underset{\sim}{\gamma}''_t = a_4 \sum_{i=0}^{m} f_i \gamma''_{i,t} .$$

At time t, after a sample is taken, the expected cost model for the process is given by

$$E_t(C_T) = E_t(C_1) + E_t(C_2) + E_t(C_3) ,$$

if the process is adjusted, and by

$$E_t(C_T) = E_t(C_1) + E_t(C_3) ,$$

if the process is not adjusted.

Minimizing the above relationships with respect to this decision and the future cost of this decision, the expected total cost of the system can be described by the following model

$$E_t(C_T) = \min_{n_t} E_{x|\underset{\sim}{\alpha}'_t} \{\min[\frac{a_3}{k_t}\varepsilon + a_4 \underset{\sim}{f}\,\underset{\sim}{\gamma}''_t + V_{t-1}(\underset{\sim}{\alpha}'_{t-1})]$$

$$+ \frac{a_1+a_2n_t}{k_t}\}$$

$$= V_t(\underset{\sim}{\alpha}'_t)$$

where $V_t(\underset{\sim}{\alpha}'_{t-1})$ is the future cost of the decision and $\underset{\sim}{\gamma}''_t$ and $\underset{\sim}{\alpha}''_t$ are the posterior probabilities for that decision.

The Transition Matrix

Let P_{ij} denote the probability of a shift from state i to a state j. Then P_{00}, the probability that the process remains in state f_0 is

$$P_{00} = e^{-\lambda k_t/R} .$$

The probability of a shift out of state f_0 is

$$1 - e^{-\lambda k_t/R},$$

and thus,

$$\sum_{j=1}^{m} P_{0j} = 1 - e^{-\lambda k_t/R} .$$

The binomial mass function is used to assign the probability among the m states. The probabilities are developed in detail in Chapter 2, and so only the results will be presented here. The probability of a shift from state f_0 to any other state f_j is given by

$$P_{0j} = \frac{(1-e^{-\lambda k_t/R})}{1 - (1-S)^m} \binom{m}{j} S^j (1-S)^{m-j} .$$

The probability, P_{ij}, of a shift from state f_i to state f_j, where $i \neq 0$ and $i \leq j$, is a fraction of the probability of a shift from the in-control state to the jth state. Hence,

$$P_{ij} = \frac{P_{0j}}{1 - P_{00}} ,$$

and

$$P_{ii} = \sum_{j=1}^{i} \frac{P_{0j}}{1 - P_{00}} .$$

The transition matrix can now be defined. Let g_{ij} be the probability of a transition from state i to state j. If the process is in-control on the dth sample, then the probability that the process shifts to state j on the next sample is just the probability that the process is in state j. Thus, $g_{0j} = P_{0j}$.

On the dth sample, if the process is adjusted, the process is in state f_0 at the beginning of the next sampling interval. Therefore, on the (d-1)st sample, the probability that the process is in state j is P_{0j}. The probability of a shift, if it is adjusted, is given by the vector $(P_{00}, P_{01}, \ldots, P_{0m})$.

If the process is not adjusted, then $g_{ij} = 0$ if $i > j$. In other words, the process cannot improve itself without manual adjustment. If $i \leq j$, then $g_{ij} = P_{ij}$, the probability of a shift, without manual adjustment, from state i to state j. The transition matrix, G, for the process, if it is not adjusted, is given by

$$G = \begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0m} \\ 0 & P_{11} & \cdots & P_{1m} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & P_{mm} \end{bmatrix}.$$

<u>The Probability Vector $\underset{\sim}{\alpha}_t$</u>

Let $\alpha'_{i,t}$ be the prior probability that the process is in state f_i at the time of sampling, and let $\alpha''_{i,t}$ be the posterior probability. If the process is adjusted at time t, the adjustment is made back to the in-control state. In other words, $\alpha''_{i,t} = 0$ for $i = 1, 2, \ldots, m$, and $\alpha''_{0,t} = 1$. If the process is not adjusted at time t, the posterior probabilities can be found using Bayes formula. The posterior distribution for all i is therefore given by

$$\alpha''_{i,t} = \frac{h(x|f_i)\alpha'_{i,t}}{\sum_{j=0}^{m} h(x|f_j)\alpha'_{j,t}} .$$

The prior distribution for the next time period can be determined using the relationship $\underset{\sim}{\alpha}''_t G = \underset{\sim}{\alpha}'_{t-1}$. If the process is adjusted, the prior probability that the process is in state f_i at the time of sampling is $\alpha'_{i,t-1} = P_{0i}$. If the process is not adjusted, the prior probability for state f_i is given by $\alpha'_{i,t-1} = \sum_{j=0}^{i} \alpha''_{j,t}P_{ji}$.

The Probability Vector $\underline{\gamma}''_t$

Let Δ be defined as in the two state model. Then $\gamma''_{0,t}$, the conditional probability of the process being in-control at a given time is

$$\gamma''_{0,t} = \alpha''_{0,t} P_{00} + \Delta\alpha''_{0,t}(1-P_{00}).$$

The probability that the process is in state f_i is dependent on the probabilities of the following four events:

1) The process is in state f_i and remains in state f_i;
2) The process is in state f_0 and shifts to state f_i;
3) The process is in state f_j and shifts to state f_i where $i > j$;
4) The process is in state f_i and shifts to state f_j where $i < j$.

Taking the expectation over these four cases, as developed by Knappenberger and Grandage [16], $\gamma''_{i,t}$ is given by

$$\gamma''_{i,t} = \frac{\alpha''_{i,t}\left(\sum_{j=1}^{i} P_{0j}\right)}{1 - P_{00}} + \alpha''_{0,t}(1-\Delta)P_{0i}$$

$$+ \sum_{j=1}^{i-1} \alpha''_{j,t}\left(\frac{P_{0i}}{1-P_{00}}\right)(1-\Delta) + \frac{\alpha''_{i,t}\Delta}{1 - P_{00}} \sum_{j=i+1}^{m} P_{0j} .$$

Chapter 4

OPTIMIZATION OF THE DYNAMIC PROGRAMMING MODEL

Introduction

The fixed parameter model, developed in Chapter 2, can be optimized by means of a pattern search. However, optimization of the dynamic programming model of Chapter 3 is more complex. The procedure for optimization of the model will be discussed in this chapter for both the bounded and the infinite time horizon cases. The infinite horizon problem will be shown to have a steady state solution. The concept of gain as an optimality criterion will also be discussed.

The Finite Horizon Case

Situations do exist where the policy determined by the fixed parameter model of Chapter 2 is not optimal. Consider the case where only a limited number of items are produced by a process run in a job shop and there is not enough time for the process to reach steady state. Even in a large shop, near the end of a production run it may be more economical to complete the last few items without an expensive adjustment to the process. The process has a finite or bounded horizon in many such situations, and therefore can be solved by classical dynamic programming techniques.

A process with a finite horizon runs for a fixed and known length of time. Let h be the length of one sampling interval, and let T be the total time to run, then T/h is the total number of time periods which must be considered. If N items are to be produced in a run and k is the number of units produced in one sampling interval, then N/k represents the number of intervals to consider for a job shop situation.

The optimal policy can be found using the recursive cost function developed in Chapter 3, and optimizing for the required number of periods. The optimal values of n, the sample size, and k, the length of the sampling interval can be found by iterating over possible values of these parameters, since they must be integer valued. The optimal policy will consist of values for n and k, and will be directly dependent on the number of time periods left to run in the process.

A detailed discussion of optimization of a dynamic model under a bounded horizon can be found in Wagner [24].

The Infinite Horizon Case

The length of the planning horizon is indefinite or sufficiently long in most cases so that evaluating over all time periods is impractical; this situation is referred to as the infinite horizon problem. The solution procedure to be developed for this case must necessarily directly incorporate the assumption of an infinite horizon, since the optimal policy for a finite value of T, the total time under consideration, may not be optimal for larger values of T.

Convergence to the Steady State

As the time parameter increases and tends to infinity, the optimal policy should become independent of time and a steady state policy should develop. Note, however, that this policy differs from the steady state policy for the fixed parameter model of Chapter 2. The dynamic programming policy specifies an optimal value of n and k for each value of the sample fraction defective. The values for the next period are chosen based on the results of the sample. The same values of n and k are always used regardless of the sample results for the model of Chapter 2.

A steady state solution for the infinite horizon problem will be shown to exist by proving that the recursive cost equation developed in Chapter 3 converges to a limit as time tends to infinity. To do this, it is first necessary to add a discount factor, a, to prevent costs from growing to infinity as time increases. The recursive cost equation then becomes

$$V_t(\underset{\sim}{\alpha}'_t) = \min_{n_t} \{E_{x|\underset{\sim}{\alpha}'_t} [\frac{a_3\varepsilon}{k_t} + a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''_t + aV_{t-1}(\underset{\sim}{\alpha}'_{t-1})]$$

$$+ \frac{a_1+a_2n_t}{k_t}\}$$

where $\underset{\sim}{\gamma}''_t$ and $\underset{\sim}{\alpha}'_t$ are dependent on the decision whether or not to adjust the process and

$$\varepsilon = \begin{cases} 1 & \text{if the process is adjusted} \\ 0 & \text{otherwise.} \end{cases}$$

To show that the sequence $\{V_t(\underset{\sim}{\alpha}'_t)\}$ does converge, it is necessary to show that it is uniformly bounded and monotonic nondecreasing with respect to time. This then implies that the sequence must be uniformly convergent. The following theorems which parallel those of Carter [8] lead to this result.

Theorem I:

Assume that the costs of adjusting the process and of producing a defective item are continuous and bounded. Then $V_t(\underset{\sim}{\alpha}'_t)$ is uniformly bounded for all values of $\underset{\sim}{\alpha}'_t$ and t.

Proof:

Since a_1, a_2, a_3, and a_4 are fixed costs, k and n are continuous and k > 0, then a_3/k_t and $\frac{a_1+a_2n_t}{k_t}$ are finite. Let K_0 be an upper bound on the cost of adjustment and let K_1 be an upper bound on the cost of sampling. Then

$$\frac{a_3}{k_t} < K_0 \quad \text{and} \quad \frac{a_1+a_2n_t}{k_t} < K_1 \; .$$

Also, $a_4 \underset{\sim}{f}\, \gamma''_t \leq a_4$. Thus the cost terms are continuous and bounded.

The decision maker can decide not to sample and to adjust the process at any point in time. The cost of this decision at time t = 1 is given by

$$V_1(\underset{\sim}{\alpha}'_1) \leq K_0 + K_1 + a_4 \; .$$

Using the same decision at time t = 2 yields

$$V_2(\underset{\sim}{\alpha}'_2) \leq K_0 + K_1 + a_4 + aV_1(\underset{\sim}{\alpha}'_1) = [K_0+K_1+a_4](1-a) \; .$$

By induction, it can be shown that

$$V_t(\underset{\sim}{\alpha}'_t) \leq [K_0+K_1+a_4][1+a+a^2+\ldots+a^{t-1}] \; ,$$

and so

$$V_t(\underset{\sim}{\alpha}'_t) \leq \frac{K_0 + K_1 + a_4}{1 - a} \; .$$

Thus $V_t(\underset{\sim}{\alpha}'_t)$ is uniformly bounded. Q. E. D.

The values of f_i, α_i, and γ_i are restricted to a finite region. In particular,

$$0 \leq f_i \leq 1 \ ,$$

$$0 \leq \alpha_i \leq 1 \ ,$$

and

$$0 \leq \gamma_i \leq 1 \ .$$

The values of n_t and k_t are not restricted to a finite range. Although an infinite sample is impractical, theoretically, n_t can approach infinity. The variable cost of sampling tends to infinity as n_t increases since it is a linear function of n_t. However, the policy of choosing the minimum cost decision ensures that n_t will remain finite.

Theorem II:

If the sampling cost tends to infinity as n increases, the sequence $\{V_t(\underset{\sim}{\alpha}'_t)\}$ is uniformly convergent as $t \to \infty$ and the limiting function $V(\underset{\sim}{\alpha}')$ is continuous.

Proof:

Let $\bar{n}_t$ denote the value of n_t which minimizes

$$E_{x|\underset{\sim}{\alpha}'_t} \ \min \begin{bmatrix} \dfrac{a_3}{k_t} + a_4 \underset{\sim}{f} \, \underset{\sim}{\gamma}''_{t,0} + aV_{t-1}(\underset{\sim}{\alpha}'_{t-1,0}) \\ \\ a_4 \underset{\sim}{f} \, \underset{\sim}{\gamma}''_t + aV_{t-1}(\underset{\sim}{\alpha}'_{t-1}) \end{bmatrix}$$

$$+ \frac{a_1 + a_2\bar{n}_t}{k_t} \quad ,$$

where the subscripted vectors, $\underset{\sim}{\gamma}''_{t,0}$ and $\underset{\sim}{\alpha}'_{t,0}$, indicate that the process has been adjusted. Define $G_t(\underset{\sim}{\alpha}'_t)$ as

$$G_t(\underset{\sim}{\alpha}'_t) = \min \begin{bmatrix} \frac{a_3}{k_t} + a_4 \underset{\sim}{f} \underset{\sim}{\gamma}''_{t,0} + aV_{t-1}(\underset{\sim}{\alpha}'_{t-1,0}) \\ \\ a_4 \underset{\sim}{f} \underset{\sim}{\gamma}''_t + aV_{t-1}(\underset{\sim}{\alpha}'_{t-1}) \end{bmatrix} .$$

Then the recursive equation becomes

$$V_t(\underset{\sim}{\alpha}'_t) = \min_{n_t} \{E_{x|\underset{\sim}{\alpha}'_t}[G_t(\underset{\sim}{\alpha}'_t)] + \frac{a_1+a_2\bar{n}_t}{k_t}\} .$$

Note that $\underset{\sim}{\alpha}'_t$ is also a function of $\bar{n}_t$.

Evaluating the cost function at both time t and time t+1 and $\bar{n}_{t+1}$ yields

$$V_{t+1}(\underset{\sim}{\alpha}'_{t+1}) = E_{x|\underset{\sim}{\alpha}'_{t+1}}[G_{t+1}(\underset{\sim}{\alpha}'_{t+1})] + \frac{a_1+a_2\bar{n}_{t+1}}{k_t}$$

and

$$V_t(\underset{\sim}{\alpha}'_t) = E_{x|\underset{\sim}{\alpha}_t}[G_t(\underset{\sim}{\alpha}'_t)] + \frac{a_1+a_2\bar{n}_{t+1}}{k_t} .$$

It then follows that

$$\begin{aligned} V_t(\underset{\sim}{\alpha}'_t) &= E_{x|\underset{\sim}{\alpha}'_t}[G_t(\underset{\sim}{\alpha}'_t)] + \frac{a_1+a_2\bar{n}_t}{k_t} \\ &\leq E_{x|\underset{\sim}{\alpha}'_{t*}}[G_t(\underset{\sim}{\alpha}'_{t*})] + \frac{a_1+a_2\bar{n}_{t+1}}{k_t} \end{aligned}$$

where $\underset{\sim}{\alpha}'_{t*}$ is a function of $\bar{n}_{t+1}$ evaluated at time t. Subtracting these two expressions yields

$$V_{t+1}(\underset{\sim}{\alpha}'_t) - V_t(\underset{\sim}{\alpha}'_t) \geq E_{x|\underset{\sim}{\alpha}'}[G_{t+1}(\underset{\sim}{\alpha}'_t) - G_t(\underset{\sim}{\alpha}'_t)] .$$

Since the parameters range over the same set of values, for clarity the time index will be dropped in the rest of the proof. It will now be shown by induction that

$$G_{t+1}(\underset{\sim}{\alpha}') - G_t(\underset{\sim}{\alpha}') \geq 0 ,$$

for all values of $\underset{\sim}{\alpha}'$ and n.

The term, $G_1(\underset{\sim}{\alpha}')$, at time t = 1 is given by

$$G_1(\underset{\sim}{\alpha}') = \min \begin{bmatrix} \frac{a_3}{k} + a_4 \underset{\sim}{f} \underset{\sim}{\gamma}''_0 \\ \\ a_4 \underset{\sim}{f} \gamma'' \end{bmatrix} ,$$

and at time t = 2 by

$$G_2(\underset{\sim}{\alpha}') = \min \begin{bmatrix} \frac{a_3}{k} + a_4 \underset{\sim}{f} \underset{\sim}{\gamma}''_0 + aV_1(\underset{\sim}{\alpha}'_0) \\ \\ a_4 \underset{\sim}{f} \underset{\sim}{\gamma}'' + aV_1(\underset{\sim}{\alpha}') \end{bmatrix}$$

The difference, $G_2(\underset{\sim}{\alpha}') - G_1(\underset{\sim}{\alpha}')$ will be considered for the following four combinations of decisions for the two periods:

1) Adjust the process in both periods;
2) Adjust the process in the first period but let it run in the second period;
3) Let the process run in the first period but adjust it in the second;
4) Do not adjust the process in either period.

Case 1:

The cost function evaluated at time t = 1 and time = 2 is given by

$$G_1(\underset{\sim}{\alpha}') = \frac{a_3}{k} + a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''_0 ,$$

and

$$G_2(\underset{\sim}{\alpha}') = \frac{a_3}{k} + a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''_0 + aV_1(\underset{\sim}{\alpha}'_0) .$$

The difference is thus

$$G_2(\underset{\sim}{\alpha}') - G_1(\underset{\sim}{\alpha}') = aV_1(\underset{\sim}{\alpha}') \geq 0 ,$$

since all the cost terms are positive and thus $V_t(\underset{\sim}{\alpha}'_0)$ is positive.

Case 2:

For this case, the function for the first periods is given by

$$G_1(\underset{\sim}{\alpha}') = \frac{a_3}{k} + a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''_0$$

and

$$G_2(\underset{\sim}{\alpha}') = a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}'' + aV_1(\underset{\sim}{\alpha}') ,$$

and the difference is given by

$$G_2(\underset{\sim}{\alpha}') - G_1(\underset{\sim}{\alpha}') = a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}'' - \frac{a_3}{k} - a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''_0 + aV_1(\underset{\sim}{\alpha}') .$$

The least cost decision at time t = 1 was to adjust the process. So

$$\frac{a_3}{k} + a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''_0 \leq a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''$$

and

$$a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}'' - \frac{a_3}{k} - a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''_0 = \delta \geq 0 .$$

Case 3:

The cost functions for the two periods are given by

$$G_1(\underset{\sim}{\alpha}') = a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}'' ,$$

and

$$G_2(\underset{\sim}{\alpha}') = \frac{a_3}{k} + a_4 \underset{\sim}{f} \underset{\sim}{\gamma}''_0 + a_1 V_1(\underset{\sim}{\alpha}'_0) \ .$$

The difference between these two functions is

$$G_2(\underset{\sim}{\alpha}') - G_1(\underset{\sim}{\alpha}') = \frac{a_3}{k} + a_4 \underset{\sim}{f} \underset{\sim}{\gamma}''_0 + aV_1(\underset{\sim}{\alpha}'_0) - a_4 \underset{\sim}{f} \underset{\sim}{\gamma}'' \ .$$

Recall that at time t = 1, run without adjustment was the least cost decision and

$$\frac{a_3}{k} + a_4 \underset{\sim}{f} \underset{\sim}{\gamma}''_0 - a_4 \underset{\sim}{f} \underset{\sim}{\gamma}'' = \delta \geq 0 \ .$$

Hence,

$$G_2(\underset{\sim}{\alpha}') - G_1(\underset{\sim}{\alpha}') = \Delta + aV_1(\underset{\sim}{\alpha}') \geq 0 \ .$$

Case 4:

The cost functions for the decision in both periods not to adjust the process are given by

$$G_1(\underset{\sim}{\alpha}') = a_4 \underset{\sim}{f} \underset{\sim}{\gamma}'' \ ,$$

and

$$G_2(\underset{\sim}{\alpha}') = a_4 \underset{\sim}{f} \underset{\sim}{\gamma}'' + aV_1(\underset{\sim}{\alpha}') \ .$$

Hence,

$$G_2(\underset{\sim}{\alpha}') - G_1(\underset{\sim}{\alpha}') = aV_1(\underset{\sim}{\alpha}') \geq 0 \ ,$$

since all costs are positive and thus $V_1(\underset{\sim}{\alpha}')$ is positive.

Thus, it has been shown that

$$V_{t+1}(\underset{\sim}{\alpha}') - V_t(\underset{\sim}{\alpha}') \geq 0$$

for t = 1. In order to show that the inequality holds for all t,

$[V_t(\underset{\sim}{\alpha}') - V_{t-1}(\underset{\sim}{\alpha}')]$ is assumed to be greater than or equal to zero at time t. It will now be shown that it also holds at time t+1. To show this, the same four cases discussed above will be considered.

Case 1:

The cost function evaluated at time t and at time t+1 are given by

$$G_t(\underset{\sim}{\alpha}') = \frac{a_3}{k} + a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''_0 + aV_{t-1}(\underset{\sim}{\alpha}'_0)$$

and

$$G_{t+1}(\underset{\sim}{\alpha}') = \frac{a_3}{k} + a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''_0 + aV_t(\underset{\sim}{\alpha}'_0) ,$$

and their difference is

$$G_{t+1}(\underset{\sim}{\alpha}') - G_t(\underset{\sim}{\alpha}') = aV_t(\underset{\sim}{\alpha}') - aV_{t-1}(\underset{\sim}{\alpha}') .$$

Since $V_t(\underset{\sim}{\alpha}')$ is assumed to be greater than $V_{t-1}(\underset{\sim}{\alpha}')$,

$$a[V_t(\underset{\sim}{\alpha}') - V_{t-1}(\underset{\sim}{\alpha}')] \geq 0 .$$

Case 2:

The functions for the times t and t+1 are given by

$$G_t(\underset{\sim}{\alpha}') = \frac{a_3}{k} + a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''_0 + aV_{t-1}(\underset{\sim}{\alpha}') ,$$

and

$$G_{t+1}(\underset{\sim}{\alpha}') = a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}'' + aV_t(\underset{\sim}{\alpha}') .$$

The difference between these two functions is

$$G_{t+1}(\underset{\sim}{\alpha}') - G_t(\underset{\sim}{\alpha}') = a_4 \underset{\sim}{f}\,\underset{\sim}{\gamma}'' - \frac{a_3}{k} - a_4 \underset{\sim}{f}\,\underset{\sim}{\gamma}''_0 + aV_t(\underset{\sim}{\alpha}') - aV_{t-1}(\underset{\sim}{\alpha}'_0)$$

$$= a_4 \underset{\sim}{f}\,\underset{\sim}{\gamma}'' - \frac{a_3}{k} - a_4 \underset{\sim}{f}\,\underset{\sim}{\gamma}''_0 + aV_t(\underset{\sim}{\alpha}') - aV_{t-1}(\underset{\sim}{\alpha}'_0)$$

$$+ [aV_{t-1}(\underset{\sim}{\alpha}') - aV_{t-1}(\underset{\sim}{\alpha}')]$$

$$= [a_4 \underset{\sim}{f}\,\underset{\sim}{\gamma}'' + aV_{t-1}(\underset{\sim}{\alpha}')] - [\frac{a_3}{k} + a_4 \underset{\sim}{f}\,\underset{\sim}{\gamma}''_0$$

$$+ aV_{t-1}(\underset{\sim}{\alpha}'_0)] + a[V_t(\underset{\sim}{\alpha}') - V_{t-1}(\underset{\sim}{\alpha}')] .$$

Since the decision to adjust the process was chosen at time t, it follows that

$$a_4 \underset{\sim}{f}\,\underset{\sim}{\gamma}'' + aV_{t-1}(\underset{\sim}{\alpha}') \geq \frac{a_3}{k} + a_4 \underset{\sim}{f}\,\underset{\sim}{\gamma}''_0 + aV_{t-1}(\underset{\sim}{\alpha}'_0) ,$$

and by assumption,

$$V_t(\underset{\sim}{\alpha}') - V_{t-1}(\underset{\sim}{\alpha}') \geq 0 .$$

Thus,

$$G_{t+1}(\underset{\sim}{\alpha}') - G_t(\underset{\sim}{\alpha}') \geq 0 .$$

Case 3:

The functions for this case are given by

$$G_t(\underset{\sim}{\alpha}') = a_4 \underset{\sim}{f}\,\underset{\sim}{\gamma}'' + aV_{t-1}(\underset{\sim}{\alpha}') ,$$

and

$$G_{t+1}(\underset{\sim}{\alpha}') = \frac{a_3}{k} + a_4 \underset{\sim}{f}\,\underset{\sim}{\gamma}''_0 + aV_t(\underset{\sim}{\alpha}'_0) .$$

and their difference by

$$G_{t+1}(\underset{\sim}{\alpha}') - G_t(\underset{\sim}{\alpha}') = \frac{a_3}{k} + a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''_0 + aV_t(\underset{\sim}{\alpha}'_0) - a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}'' - aV_{t-1}(\underset{\sim}{\alpha}')$$

$$= [\frac{a_3}{k} + a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}''_0 + aV_{t-1}(\underset{\sim}{\alpha}'_0)]$$

$$- [a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}'' + aV_{t-1}(\underset{\sim}{\alpha}')]$$

$$+ a[V_t(\underset{\sim}{\alpha}'_0) - V_{t-1}(\underset{\sim}{\alpha}'_0)] .$$

Since the decision to allow the process to continue running was chosen at time t, the term in the first bracket is greater than the term in the second bracket. Also, by assumption,

$$V_t(\underset{\sim}{\alpha}'_0) - V_{t-1}(\underset{\sim}{\alpha}'_0) \geq 0 .$$

Thus,

$$G_{t+1}(\underset{\sim}{\alpha}') - G_t(\underset{\sim}{\alpha}') \geq 0 .$$

Case 4:

The cost functions at time t and time t+1 are given by

$$G_t(\underset{\sim}{\alpha}') = a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}'' + aV_{t-1}(\underset{\sim}{\alpha}')$$

and

$$G_{t+1}(\underset{\sim}{\alpha}') = a_4 \underset{\sim}{f}\, \underset{\sim}{\gamma}'' + aV_t(\underset{\sim}{\alpha}') ,$$

and the difference between the two functions is given by

$$G_{t+1}(\underset{\sim}{\alpha}') - G_t(\underset{\sim}{\alpha}') = a[V_t(\underset{\sim}{\alpha}') - V_{t-1}(\underset{\sim}{\alpha}')] \geq 0$$

which follows by assumption.

Thus, the inequality

$$G_{t+1}(\underset{\sim}{\alpha}') - G_t(\underset{\sim}{\alpha}') \geq 0$$

has been shown by induction to hold for all values of t.

Since,

$$V_{t+1}(\underset{\sim}{\alpha}') - V_t(\underset{\sim}{\alpha}') \geq E_{x|\underset{\sim}{\chi}'}[G_{t+1}(\underset{\sim}{\alpha}') - G_t(\underset{\sim}{\alpha}')] ,$$

for all values of $\underset{\sim}{\alpha}'$, n, and t,

$$V_{t+1}(\underset{\sim}{\alpha}') - V_t(\underset{\sim}{\alpha}') \geq 0 .$$

Hence, $V_t(\underset{\sim}{\alpha}')$ is monotonically nondecreasing in t, and is uniformly bounded from Theorem I. Thus, $V_t(\underset{\sim}{\alpha}')$ is uniformly convergent. Also, since all costs are assumed to be continuous, $V_t(\underset{\sim}{\alpha}')$ is continuous.
Q. E. D.

Optimization of the Model

The model has been shown to possess a steady state solution and therefore is to be independent of time. Thus the above analysis can be applied to a particular process, after estimation from actual data of the various costs, the process parameters, and the underlying distributions of the process. The following procedure should be followed at the end of each sampling interval:

1) The new prior variables are computed using the transition matrix;
2) Using this prior distribution, the optimal sample size is determined and the required number of items is taken from the process and tested;
3) The results for this test are used to determine the posterior distribution;

4) Using the posterior variables, the optimal operating decision, whether to adjust the process or to allow it to continue running, is then selected.

Note that if the optimal sample size had been zero, the optimal operating decision would have been selected immediately, based on the prior distribution. Since the steady state solution procedure is being used, the operator does not have to take note of the time before conducting the analysis. The above procedure will be used in determining the steady state policy with all results based on the expected outcome.

Let

$$u_t = \min_n \; E_{x|\underset{\sim}{\alpha}} \; \min \begin{bmatrix} \frac{a_3}{k} + a_4 \underset{\sim}{f} \, \underset{\sim}{\gamma}''_0 + aV_{t-1}(\underset{\sim}{\alpha}'_0) \\ a_4 \underset{\sim}{f} \, \underset{\sim}{\gamma}'' + aV_{t-1}(\underset{\sim}{\alpha}') \end{bmatrix} + \frac{a_1 + a_2 n}{k} \; .$$

The criterion of optimality to be used will be the gain of the process which is defined as

$$\nu = \lim_{t \to \infty} [u_t(\underset{\sim}{\alpha}') - u_{t-1}(\underset{\sim}{\alpha}')]$$

with a = 1. The gain can be described as the time average of the total expected future cost viewed from any position and will be independent of $\underset{\sim}{\alpha}$ and n. The optimal policy will yield minimum gain and so the model will be optimized by minimizing the gain. Bather [4] gives a more complete discussion of the concept of gain. The flow chart of Figure 4-1 outlines the general procedure to be used in optimizing the model and the use of the process gain as a test for optimality.

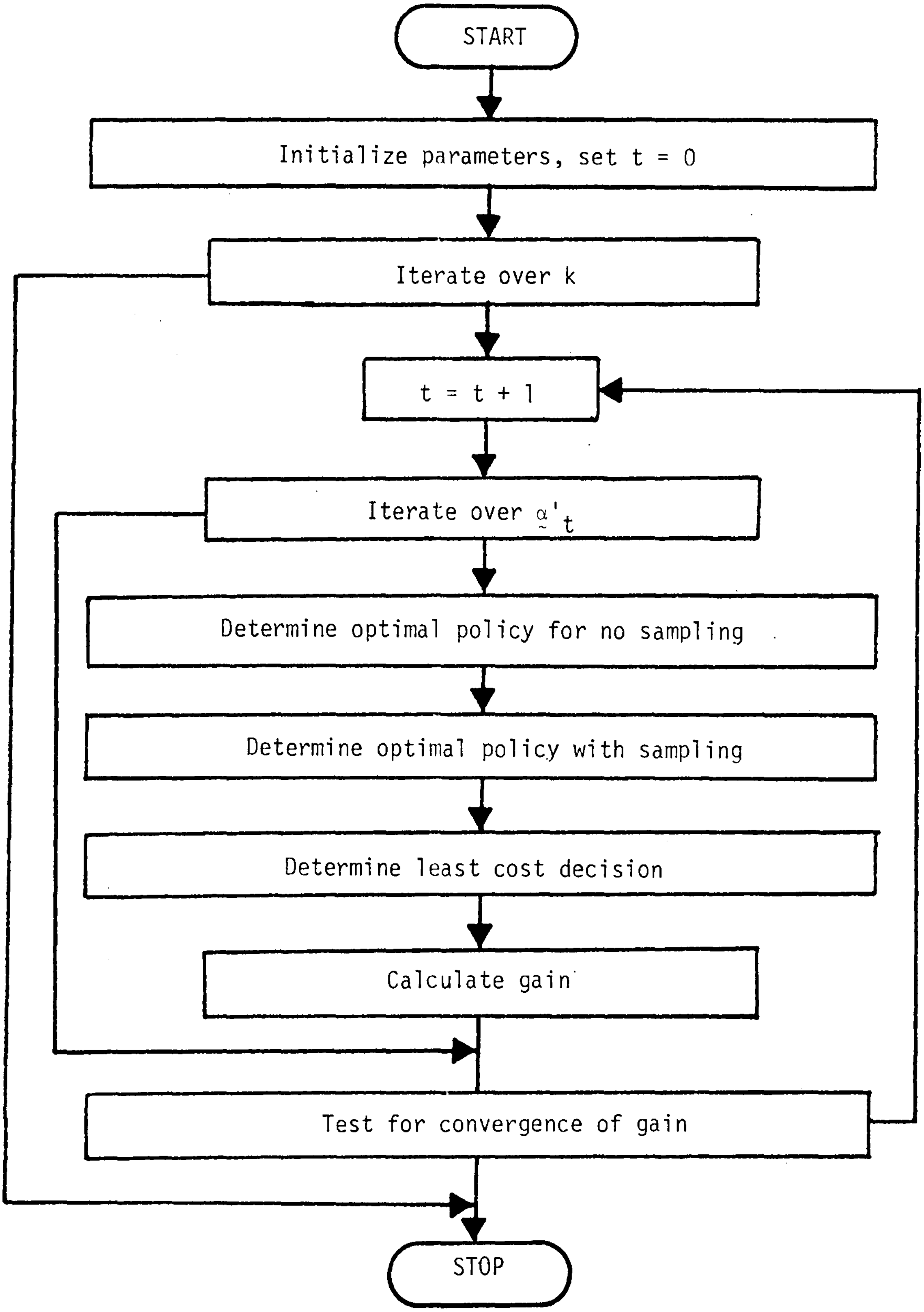

Figure 4-1 Flow Chart for Optimization of the Dynamic Programming Model

Chapter 5

NUMERICAL ANALYSIS OF A HYPOTHETICAL PROCESS

The results of an analysis carried out to find the optimal policies for a hypothetical process which possesses the Markov property are presented in this chapter. The results illustrate the considerable improvement which can be obtained using the dynamic programming cost model of Chapter 3 rather than the fixed parameter model of Chapter 2.

Discussion of the Numerical Values Chosen

Arbitrary values of the process parameters and the costs are chosen to carry out the calculations for the examples of this chapter. Although these values are intended to be reasonable and representative, they are not from an actual process.

Process Parameters

The values chosen for the process parameters will remain fixed throughout the analysis. This will give some indication of the behavior of the process when costs are changed. The actual values used in the analysis were:

$\lambda = 1.00$

$R = 500$

$S = .5$

In addition, the analysis will be conducted for the case in which there is only one in-control state f_0, and one out-of control state, f_1. The values of the process fraction defective for these two states were taken to be

$$f_0 = .04$$

$$f_1 = .15 .$$

Cost Components

The fixed cost of sampling is incurred each time the process is examined, even when the optimal sample size is zero, an event that can occur in the dynamic programming model. Therefore, this cost does not play a major role in the optimization of the sample size. For this reason, the fixed cost of sampling was kept constant in the analysis at 25.00.

The other cost terms are varied in the analysis. Table 5-1 lists the cost terms used in the four examples. These terms are not intended to provide a sensitivity analysis of the models with respect to cost, but do allow for a comparison of the models. Note that the cost of a false indication in the fixed parameter model has been set equal to the cost of process adjustment in order to facilitate comparison of the two models.

Optimization Procedure

The Fixed Parameter Model

The steady state model was optimized using a pattern search on the three variables, sample size, length of the sampling interval and upper control limit. The search began by varying one independent variable at a time, measuring the objective function at each perturbation, until all

Table 5-1

Cost Terms Used in the Analysis

	Example 1	Example 2	Example 3	Example 4
Fixed Cost of Sampling a_1	25.00	25.00	25.00	25.00
Variable Cost of Sampling a_2	1.00	10.00	1.00	1.00
Cost of Process Adjustment a_3	100.00	60.00	60.00	60.00
Cost of Producing a Defective Item a_4	10.00	5.00	10.00	20.00

variables have been perturbed. These results establish a direction or "pattern." The search then steps to another point in the cost surface in the direction established, and the perturbation of the variables is repeated. If the same pattern is repeated for two consecutive iterations, the step size is doubled. If a step in a direction fails to produce a better solution the step sizes are reduced. This procedure is repeated until no improvement can be found. The search is then terminated. For a complete discussion of the pattern search, see Wilde and Beightler [26].

The Dynamic Programming Model

Optimization of the fixed parameter model was performed over all of the decision variables simultaneously. The dynamic programming procedure required different approaches to find the optimal sample size and the optimal sampling interval. The criterion of optimality in both cases is the process gain, ν, defined in Chapter 4 as

$$\nu = \lim_{t\to\infty} [u_t(\underset{\sim}{\alpha}') - u_{t-1}(\underset{\sim}{\alpha}')] \ .$$

The optimization procedures for n, the sample size, and k, the length of the sampling interval, employed the concept of gain in slightly different manners.

An interval of possible values of k was selected for the optimization procedure and the function was evaluated for each of the desired sampling intervals. The optimal length of the sampling interval was then chosen after all values had been enumerated. Since the cost function is a non-linear function of k, it was not known with certainty how the costs would vary with the sampling interval. The graphs shown in Figures 5-1, 2, 3 and 4 illustrate how k and the expected unit-cost are related in the four examples of this chapter. As can be seen from these graphs, it is difficult to predict how the expected unit cost will vary as k is incremented. This is because the cost model is a non-linear function of k and involves k in an exponential term. It was noted, however, that for the four examples considered, the curves tended to be smooth and to flatten out in the region of the optimum. This can be seen by comparing the values in Table 5-2 which correspond to the graph in Figure 5-1. The optimal sampling interval selected is k = 450 with a cost per unit of 1.0525. However, if k is decreased to 400 units, the cost is 1.0553, a difference of only .0028. Although no general inference can be made from these observations, it does lead to an area of further study. It may be that only a rough estimate of the sampling interval is needed to insure a near-optimal operating doctrine, which would result in a savings in the design of the optimal policy.

Optimal sampling plans must be found for each value of the sampling interval under consideration and values of $\underset{\sim}{\alpha}_t'$, the prior position, must be selected prior to the optimization procedure. The following eleven vectors were used in the example problems:

$$(1,0),\ (.9,.1),\ (.8,.2)\ \ldots\ (.1,.9),\ (0,\ 1)\ .$$

A finer grid of prior values may be desirable in an actual process although the results should not differ greatly. The selected grid should logically discretize the continuous range of possible prior probabilities and the selection should be based on the process under consideration.

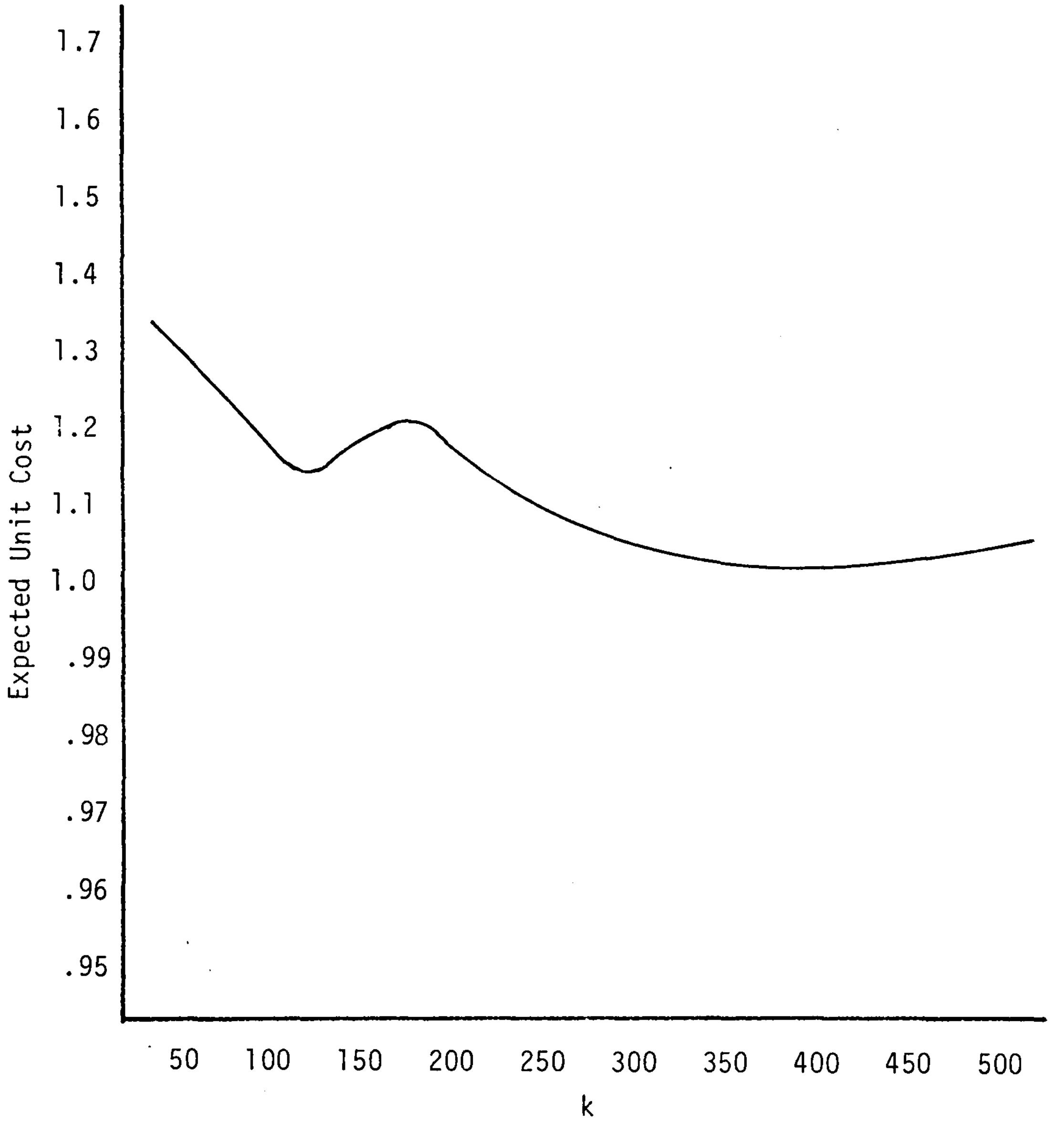

Figure 5-1 Relationship of k to Expected Unit Cost for Example 1

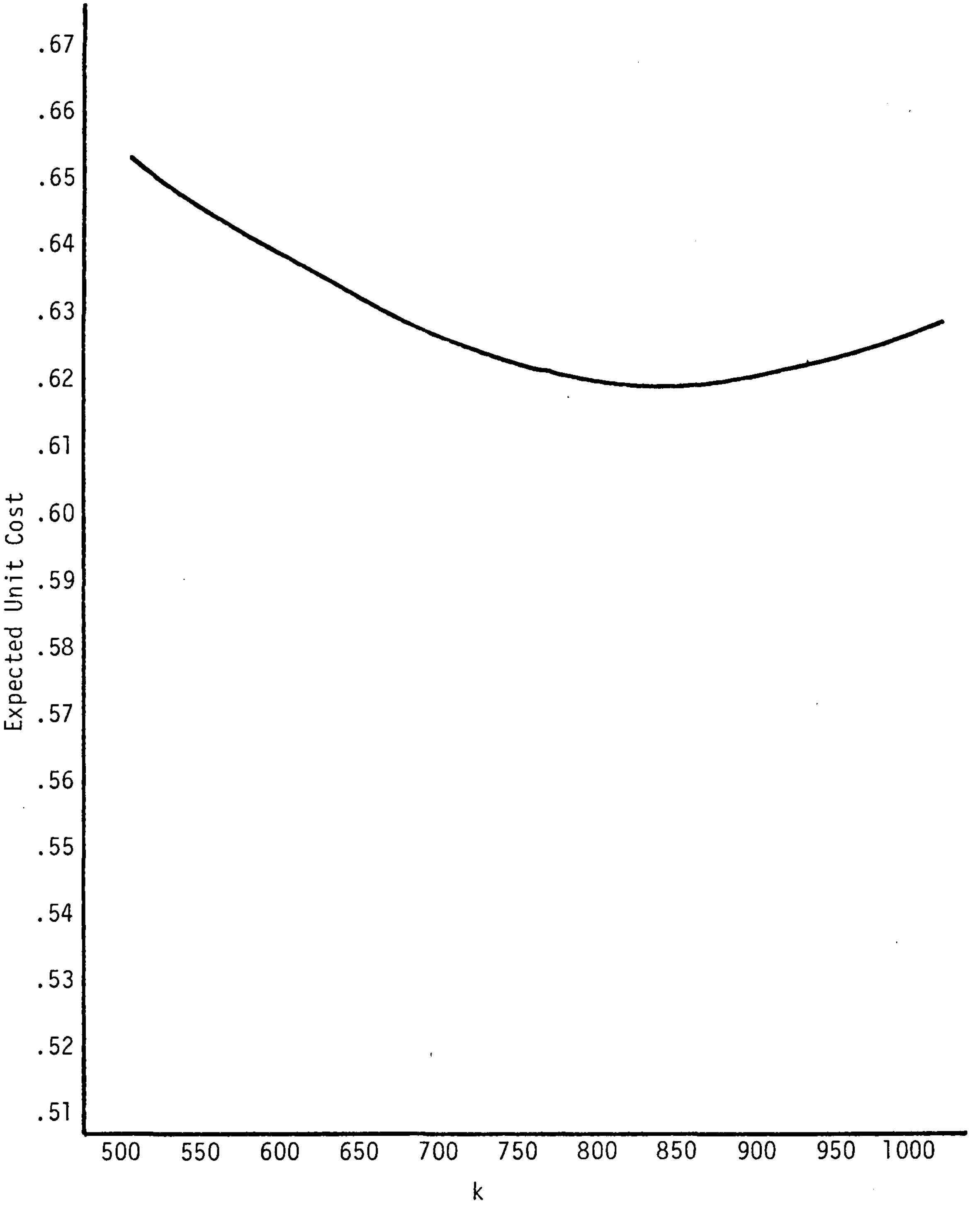

Figure 5-2 Relationship of k to Expected Unit Cost for Example 2

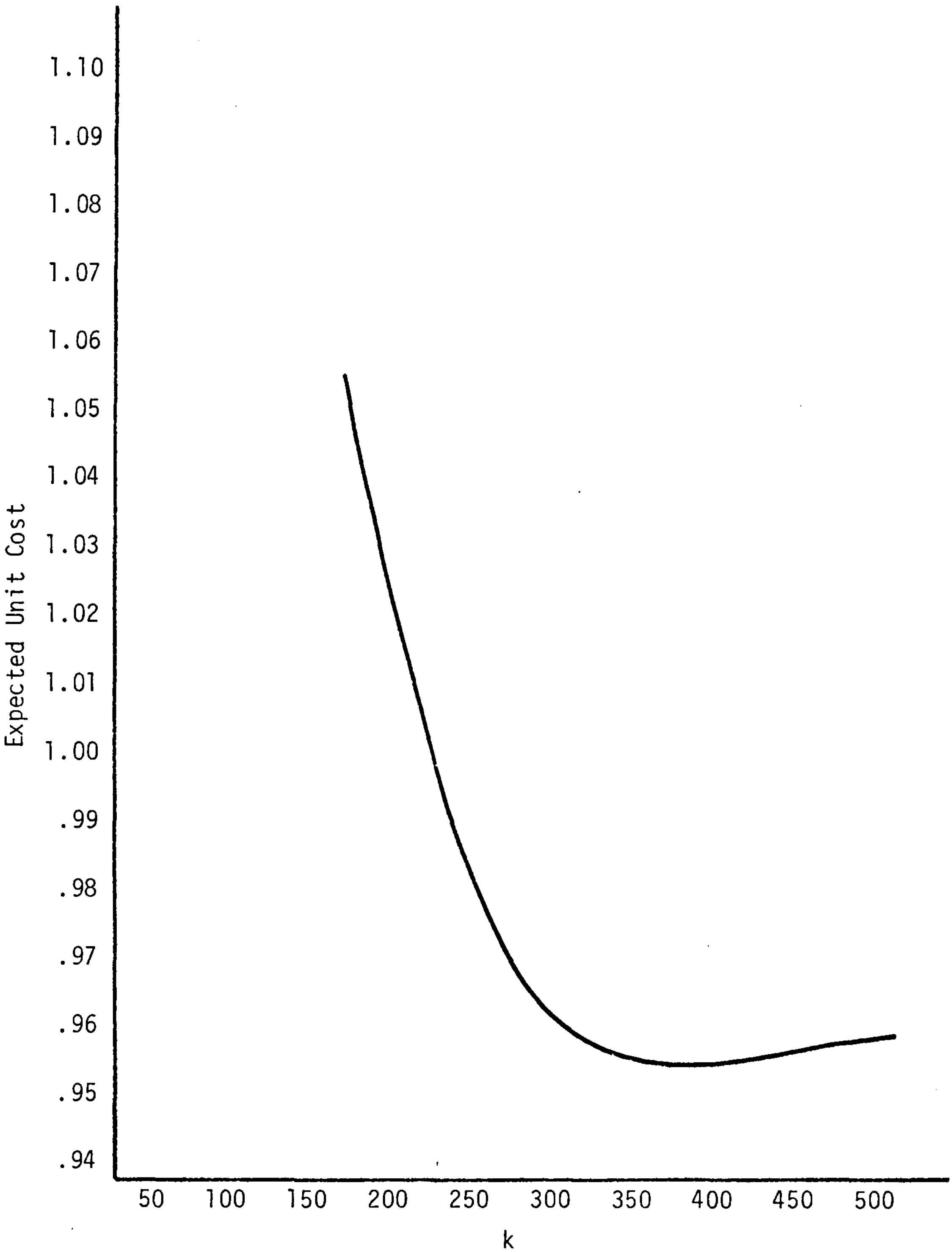

Figure 5-3 Relationship of k to Expected Unit Cost for Example 3

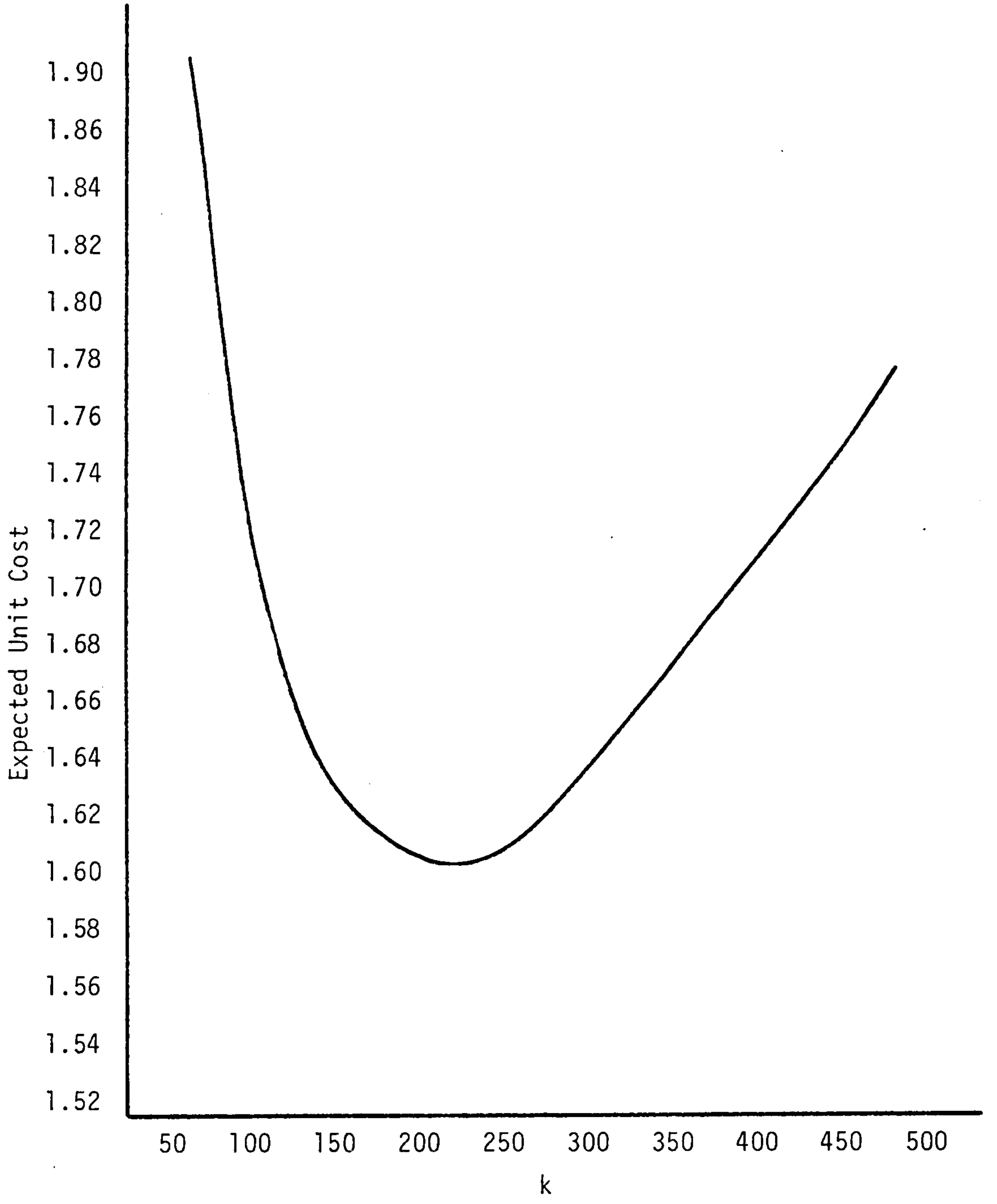

Figure 5-4 Relationship of k to Expected Unit Cost for Example 4

Table 5-2

Values of k and the Expected Unit Cost
for Example 1

k	Unit Cost
50	1.3980
100	1.2888
150	1.1561
200	1.2184
250	1.3440
300	1.0646
350	1.0661
400	1.0553
450	1.0525
500	1.0547

The optimization was performed for each of the prior vectors by enumerating over consecutive integer values of n, the sample size. In the example problems, the sample size was allowed to take on any of the integer values up to fifteen. The sample size was also allowed to assume the value of zero implying that an operating decision is to be made without inspection, based on the prior probabilities. The cost of sampling was chosen sufficiently high in the example problems to insure that sample sizes larger than 15 would be impractical.

The sample plan for one value of the prior position is given by the optimal sample size and the value of the maximum number of defectives in the sample for which the least cost decision is to continue to run. The optimal plans are determined for each of the possible prior positions. The gain is tested for convergence when the function has been evaluated at all prior positions. If the gain has not converged, the process is repeated for the next time period. In all examples it was found that the gain converges in four or five sampling intervals regardless of the length of the sampling interval and is independent of the prior position.

Policy Implementation

The fixed parameter model provides an optimal policy consisting of the sample size, the frequency of sampling and the upper control limit. These values are utilized in the same manner as those of a traditional statistical control chart. Consider the results of Example 1, displayed in Table 5-3 along with the optimal policies for the other examples. If this control chart were posted at the work station, the operator would sample three units every 125 units of production, plot the sample

Table 5-3

Optimal Policies for the Steady State Model Examples

	Example 1	Example 2	Example 3	Example 4
k*	125	350	125	75
n*	3	1	4	7
UCL*	.169	.098	.098	.11

fraction defective on the chart and adjust the process if the value exceeded .169, the upper control limit; otherwise, the process would be allowed to continue running.

The use of the optimal policy as determined by the dynamic approach is much more complex and involves extra computations. The operator determines the prior position either by multiplying the posterior probability vector from the previous period by the process transition matrix or by referring to a table in which these calculations have been made previously. The appropriate sample size to be taken would be selected by referring to Table 5-4, in the case of Example 1. The optimal sample size may be zero in some cases such as for the prior positions (1,0) and (.2,.8). When this occurs the operator immediately reads the correct operating decision from the table.

Consider Table 5-4 and assume that the operators prior probability of being in-control is 0.7 and of being out-of-control is 0.3. The chart would then instruct the operator to inspect 9 units and to let the process continue running if no defective units were found. If one or more defectives were found then the process would require adjustment.

The optimal sampling plans for Examples 2, 3, and 4 are given in Tables 5-5, 5-6, and 5-7 respectively.

Results of the Optimization

The optimal policy determined by the dynamic programming model yielded substantial savings in each of the four example problems over the fixed parameter model. For these cases, the savings were between

Table 5-4

Optimal Policy for the Dynamic Programming Model - Example 1

$k^* = 450$				
Prior Position		Optimal Sample Size	Break-even Point	Optimal Operating Decision
α_0'	α_1'			
1.0	0.0	0	-	run
.9	.1	0	-	run
.8	.2	4	0	-
.7	.3	9	0	-
.6	.4	12	0	-
.5	.5	15	0	-
.4	.6	0	-	adjust
.3	.7	0	-	adjust
.2	.8	0	-	adjust
.1	.9	0	-	adjust
0.0	1.0	0	-	adjust

Table 5-5

Optimal Policy for the Dynamic Programming Model - Example 2

k* = 750				
Prior Position		Optimal Sample Size	Break-even Point	Optimal Operating Decision
α_0'	α_1'			
1.0	0.0	0	-	run
.9	.1	0	-	run
.8	.2	0	-	run
.7	.3	0	-	run
.6	.4	0	-	adjust
.5	.5	0	-	adjust
.4	.6	0	-	adjust
.3	.7	0	-	adjust
.2	.8	0	-	adjust
.1	.9	0	-	adjust
0.0	1.0	0	-	adjust

Table 5-6

Optimal Policy for the Dynamic Programming Model - Example 3

k* = 350				
Prior Position		Optimal Sample Size	Break-even Point	Optimal Operating Decision
α_0'	α_1'			
1.0	0.0	0	-	run
.9	.1	1	0	-
.8	.2	8	0	-
.7	.3	12	0	-
.6	.4	0	-	adjust
.5	.5	0	-	adjust
.4	.6	0	-	adjust
.3	.7	0	-	adjust
.2	.8	0	-	adjust
.1	.9	0	-	adjust
0.0	1.0	0	-	adjust

Table 5-7

Optimal Policy for the Dynamic Programming Model - Example 4

k* = 250				
Prior Position		Optimal Sample Size	Break-even Point	Optimal Operating Decision
α_0'	α_1'			
1.0	0.0	0	-	run
.9	.1	5	0	-
.8	.2	10	0	-
.7	.3	0	-	adjust
.6	.4	0	-	adjust
.5	.5	0	-	adjust
.4	.6	0	-	adjust
.3	.7	0	-	adjust
.2	.8	0	-	adjust
.1	.9	0	-	adjust
0.0	1.0	0	-	adjust

20% and 30% of the cost incurred by the fixed parameter model. The expected unit cost for each of the four example cases is shown in Table 5-8.

The optimal policy for each set of cost coefficients is also substantially different. The fixed parameter model recommends sampling one unit after 350 units are produced for Example 2. The dynamic programming model recommends that sampling never be carried out for that particular process; the operating decision is made based on the assessed prior position. Similarly the results for the other three examples indicate that not only do the optimal sample sizes vary between the models, but that there is considerable difference in the frequency of the sampling to be carried out.

Table 5-8

EXPECTED COSTS PER UNIT OF BOTH MODELS AND RESULTING SAVINGS

Example	Fixed Parameter Model	Dynamic Model	Savings Per Unit	Percent Savings
1	.8932	.6318	.2614	29%
2	1.3249	1.0525	.2727	20%
3	1.2365	.9518	.2847	23%
4	2.0172	1.6087	.4085	20%

Chapter 6

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER RESEARCH

The purpose of this chapter is to summarize the results of this thesis. The two models developed are compared and their limitations discussed. Finally, several possible areas of future research are discussed.

Comparison of the Models

The traditional purely statistical approach to quality control employs a control chart to aid in monitoring the process for assignable causes of variation. The optimal policy for the control chart specifies a fixed sample size at fixed intervals. However, the size of these samples and the length of the sampling interval are not economically optimal. The operating decision is determined from either heuristic rules or the control chart designer's intuitive feel for the process. The costs involved in the process are never explicitly taken into account.

The model developed in Chapter 2 is a logical extension of the statistical control chart. The operating policy is designed for a control chart but is based on costs rather than statistical considerations. Hence this model offers an economic improvement over the traditional control chart. It is limited in that it requires a fixed sample size and a fixed sampling interval.

The dynamic programming model of Chapter 3 moves away from the control chart and hence is not limited to a fixed sampling size, although the length of the sampling interval remains fixed. Again, the optimal operating doctrine is based on costs. In determining the optimal sample size, the operator's subjective feelings are formally incorporated into the policy. The sample size is selected based on the assessed value of the process fraction defective. Also, the decision to adjust or not adjust is not based on how far the process is out of control but on which decision is the most economical. Since the operating doctrine is freed from the limitations of the control chart, the dynamic programming policy offers considerable economic improvement over the cost model based on a control chart.

It was shown in Chapter 5 that the dynamic programming model provides substantial savings over the fixed parameter model. Although the improvement has been shown to be considerable, the analyst should be aware of several disadvantages which will be discussed in the remainder of this section. Most of these involve the increased cost and complexity of designing the cost model. The resulting savings should far outweigh the increased design costs, if the policies developed are to be used for an extended production run.

There are disadvantages to both the steady state cost model and the dynamic programming model. The process must be studied before determining the process control operating doctrine, even for traditional control chart development. Once it has been determined that the process is operating in-control, only the in-control state of the process must be found. This can be done with relatively few observations before the control chart is put into effect.

Many more observations are needed to set up an economic model. One of the assumptions of both the steady state model and the dynamic programming model is that the process possesses the Markov property. The process must be thoroughly studied to ensure that this assumption does indeed hold. In addition some feel for the transition probabilities must be developed in order to select a value of s, as discussed in Chapter 2, to be used in determining the transition probabilities. Also the average number of shifts in the process fraction defective must be determined from actual process data.

The process states to be considered must also be determined. Obviously the more states used in the model, the more accurately the process will be represented.

In addition, the various costs must be determined. Some costs such as that of sampling and of an adjustment to the process are relatively easy to obtain. However, the cost of producing a defective may be difficult to estimate due to such subjective factors as the loss of customer goodwill. These costs must be determined as accurately as possible to ensure an optimal operating policy.

There is another disadvantage of the cost models over the statistical control chart operating policy. The cost of designing the operating doctrine is much higher for the economic models. Any text on industrial quality control [10] offers simple formulas to determine the control limits and rules of thumb for selecting sample size and the sampling interval. Thus, there is little additional cost after the initial estimate of the process mean is determined.

A pattern search was employed to find the optimal policy for the steady state model. In most instances this technqiue will require the use of the computer. Even before the model is optimized, time must be put into designing the cost model to ensure that it accurately reflects the system under consideration.

Although the dynamic programming model offers substantial savings over the other two approaches, it is by far the most expensive method and a computer is absolutely essential. Considerable amounts of computer time were needed to find the optimal solution for even a simple two state case.

A final disadvantage of the dynamic programming model over the control chart approach is that more work is required by the operator. When a control chart is used, the operator always takes a fixed number of items, plots the results on the chart and reads his action. Using the dynamic programming model, the operator must first make an assessment of the process state and determine the prior distribution. Using this prior, the optimal sample size must be selected, the items tested and a decision made on the results of the sample. Although graphs and charts can be developed which display the optimal decision based on his sample size and results of the sample, this model has added several calculations. This approach, however, is ideal when a computer is being used to control the process and the additional computations can be made with little effort.

Model Limitations

The major limitations of the models developed in Chapters 2 and 3 are the restrictions which must apply to the actual process. One of the basic assumptions is that the process must possess the Markov property. By this assumption, the time between shifts in the process fraction defective is exponentially distributed. If this assumption is not valid for the process, the solution derived will not be optimal. Baker [2] discusses the effects of violating the Markov assumption.

The prior density function is assumed to be a mixed binomial. This assumption is valid in many situations. However, it would be wise to re-derive the model for the particular density function, if the process under consideration does not follow a mixed binomial distribution.

Finally, the costs are assumed to be linear. This type of cost is a good approximation for many processes; however, a quadratic or other form may suit the process better and the model should be adapted to these costs.

Suggestions for Further Research

One of the first steps to be taken should be to perform sensitivity analysis on the model. Both the effects of the cost values and the process states chosen should be analyzed in order to determine the robustness of the optimal policy.

The cost values are interrelated. If the cost of producing a defective is relatively high compared to the cost of sampling, the optimal policy will be to sample quite frequently. It will be more economical to take large samples to be certain that the process is indeed out-of-control if the cost of an adjustment to the process is high. An interesting field of research would be the investigation of how the various costs do affect the optimal policy.

Obviously, when many process states are considered, the model will describe the process more accurately than if only two states were used.

However, as more states are added to the model, the complexity increases as the time required to obtain a solution and the cost of determining the solution. Hence, a study could be made to determine the number of process states to consider. Also, since the process states must be discretized it would be interesting to study the robustness of the model with respect to the process states chosen for consideration.

Baker [2] emphasized the importance of ensuring that the process does indeed possess the Markov property. The models developed in this thesis could be examined for robustness with respect to the Markov assumption.

Carter [8] discussed various forms of the cost terms. His model was optimized for a quadratic defective cost. One possible area of future research would be to extend the model to allow for non-linear cost terms.

Another possible extension of the model would be to the multi-attribute situation. Two or more attributes, such as color and the presence of burrs, of the same item could be used simultaneously as a basis for quality control.

A third possible area of research would be to develop a dual purpose attribute model. In this case, the model would be used to determine an optimal policy which has two goals: a) to control the disposition of the lot and b) to monitor the process for assignable causes of variation.

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Appendix A

A TEST FOR THE EXISTENCE OF THE MARKOV PROPERTY

A likelihood ratio test can be used on actual process data to test the process to determine if it possesses the Markov property. The null hypothesis of this test is that the observations are statistically independent. This hypothesis is tested against the alternative that the observations do come from a Markov process.

Let P_{ij} be the probability of a transition from state i to state j. It is assumed that the probabilities are independent of time and are estimated by the maximum likelihood method. The maximum likelihood estimates are given by

$$P_{ij} = \frac{n_{ij}}{n_i^*} ,$$

where n_{ij} is the number of transitions from state i to state j in T observations, and

$$n_i^* = \sum_j n_{ij} .$$

The null hypothesis of this test is that the transition probabilities for a state j are independent of the previous state i, where i = 1, 2, ..., m. In other words,

$$H_0 : P_{1j} = P_{2j} = \dots = P_{ij} = \dots = P_{mj} = P_j .$$

The alternate hypothesis is that the probabilities are dependent on the previous state i.

The likelihood ratio used for testing this hypothesis is given by

$$\lambda_j = \prod_{i=1}^{m} (\hat{P}_j/\hat{P}_{ij})^{n_{ij}}$$

where $\hat{P}_j = \sum_i \frac{n_{ij}}{T}$.

Anderson and Goodman [1] show that the random variable $X = -2 \ln \lambda_j$ has an asymptotic chi square distribution with $(m-1)^2$ degrees of freedom where m is the number of process states. Let $x_0^{\,2}$ be the critical value of the test statistic as determined by the degrees of freedom and the desired probability of a type 1 error. The null hypothesis is rejected if the test statistic determined from the actual data is greater than the actual value for this statistic. This leads to the conclusion that the process under consideration does possess the Markov property.

The same likelihood ratio test can be used to test if the process is a uth order Markov process against the alternative hypothesis that it is an rth but not uth process, where the order of the Markov process is determined by the number of past observations upon which the process is dependent.

For further discussion on statistical inference about Markov chains, see Anderson and Goodman [1].

Appendix B

PROOF THAT THE TRANSITION MATRIX OF CHAPTER 2 IS ERGODIC

The matrix of transition probabilities is said to be stochastic if it can be shown that each of the elements is non-negative and the sum of the elements in each row is unity. The elements of this matrix are developed in Chapter 2. All of the elements are non-negative since each is the convex combination of two probabilities. The following argument will show that the elements in each row of the matrix is equal to unity.

For i = 0,

$$\sum_{j=0}^{m} g_{0j} = \sum_{j=0}^{m} P_{0j}$$

$$= \sum_{j=1}^{m} P_{0j} + P_{00}$$

$$= (1-e^{-\lambda k/R}) + e^{-\lambda k/R}$$

$$= 1 .$$

For i = 1, 2, ..., m,

$$\sum_{j=0}^{m} g_{ij} = \sum_{j=0}^{i-1} q_i P_{0j} + \sum_{j=i}^{m} [(1-q_i)P_{ij} + q_i P_{0j}]$$

$$= q_i \sum_{j=0}^{m} P_{0j} + (1-q_i) \sum_{j=1}^{m} P_{ij} .$$

Using the relationship $\sum_{j=0}^{m} P_{0j} = 1$ from above, the result becomes

$$\sum_{j=0}^{m} g_{ij} = q_i + (1-q_i) \sum_{j=1}^{m} P_{ij} .$$

In Chapter 2, it was shown that

$$P_{ii} = \frac{\sum_{j=1}^{i} P_{0j}}{1 - P_{00}} \qquad \text{and} \qquad P_{ij} = \frac{P_{0j}}{1 - P_{00}}$$

where $i < j$. Thus

$$\sum_{j=1}^{m} P_{ij} = \frac{1}{1 - P_{00}} \sum_{j=1}^{i} P_{0j} + \frac{1}{1 - P_{00}} \sum_{j=i+1}^{m} P_{0j}$$

$$= \frac{1}{1 - P_{00}} \sum_{j=1}^{m} P_{0j} \ .$$

However, since $\sum_{j=1}^{m} P_{0j} = 1 - P_{00}$, this becomes

$$\sum_{j=1}^{m} P_{ij} = \frac{1 - P_{00}}{1 - P_{00}}$$

$$= 1 \ .$$

Hence,

$$\sum_{j=0}^{m} g_{ij} = q_i + (1-q_i)$$

$$= 1 \ .$$

Hence, the matrix is ergodic since it is stochastic and all elements are positive.

Appendix C

DOCUMENTATION OF THE COMPUTER PROGRAM USED TO OPTIMIZE THE STEADY STATE MODEL OF CHAPTER 2

A. TITLE

A program to optimize the expected total cost function for the steady state model developed in Chapter 2.

1. Programmer - Nancy L. Orndorff
2. Machine - IBM 370/155
3. Language - FORTRAN IV
4. Date completed - May 25, 1974
5. Approximate compile time - .64 seconds
6. Execution time - Variable, depending on the number of iterations required in the pattern search. Approximate time for a two state example: .9 seconds.
7. Lines of output - Approximately 700 lines
8. Storage required - 3330 bytes.

B. PURPOSE

This program finds the optimal solution for the decision variables sample size, sampling interval and upper control chart limit. The mathematical cost function is optimized using pattern search techniques.

C. RESTRICTIONS

This program can be used for up to ten process states. The dimension statements can be changed for problems in which more than ten states are considered.

D. DEFINITIONS

The subscripted and unsubscripted variables are described in the listing.

E. INPUT

In the following card input information RJND indicates that the number punched in the card must be right justified in the columns given and punched without a decimal point. Those numbers which can be punched in any of the columns given provided a decimal point is used are indicated by IRPD.

CARD 1

Cols. 2-6 RJND--The number of separate functions to be optimized.

CARD 2

Cols. 2-6 RJND--Output option: 0 indicates output data is written for only those solutions which are better than the previous best; 1 indicates output is written for each point evaluated.

CARD 3

Cols. 1-3 RJND--Number of out-of-control states.

Cols. 4-13 IRPD--Expected number of shifts in the lot mean per unit time.

Cols. 14-23 IRPD--The value of S in the equation which determines the transition probabilities.

Cols. 24-33 IRPD--The production rate.

CARD 4

Cols. 1-10 IRPD--The fixed cost of sampling.

Cols. 11-20 IRPD--The variable cost of sampling.

Cols. 21-30 IRPD--The cost of a process adjustment.

Cols. 31-40 IRPD--The cost of producing a defective item.

Cols. 41-50 IRPD--The cost of a false indication of the process state.

CARD 5

Cols. 1-7 IRPD--The value of the in-control state of the process fraction defective. The states should be numbered lowest to highest.

Cols. 8-14 IRPD--The value of the first out-of-control state.

Cols. 15-21

. . . .

. . . .

. . . .

Cols. 64-70 IRPD--The value of the last out-of-control state.

CARD 6

Cols. 2-9 RJND--The number of decision variables.

Cols. 10-17 RJND--The maximum number of "steps" in the pattern search procedure that the program will make.

Note: Card 7 is repeated for each decision variable.

CARD 7

Cols. 2-9 IRPD--Value of the decision variable at which search originates.

Cols. 10-17 IRPD--Minimum value to consider in the search.

Cols. 18-26 IRPD--Maximum value to consider in the search.

Cols. 27-35 IRPD--Starting perturbation increment.

Cols. 36-44 IRPD--Minimum perturbation increment.

Cols. 45-53 IRPD--Maximum perturbation increment.

F. COMPUTER SYSTEM LANGUAGE LIMITATIONS

The input and output statements were written with I/O unit numbers 5 and 6 respectively. Other systems may require different unit designations.

The on-line subroutine MINV was used for matrix inversion and the on-line subroutine GMPRD was used to find the product of two matrices. Both of these subroutines are a part of the IBM System/360 Scientific Subroutine Package, Version III.

G. OUTPUT

For each point evaluated on the cost surface the following information is available.

1. The matrix of transition probabilities.
2. The values of the decision variables.
   a. N, the sample size.
   b. K, the length of the sampling interval.
   c. UCL, the upper control limit.
3. The expected cost per unit.
   a. The cost of sampling.
   b. The cost of adjustment.
   c. The cost of producing a defective.
   d. The cost of a false indication.
   e. The total cost.

H. LISTING OF PROGRAM

<u>Name of Subroutine</u>

MAIN PROGRAM

SUBSYS

STSTPR

COST

PMTX

BINOM

NDTR

```
C     MAIN PROGRAM
C     PURPOSE--PROVIDES THE PATTERN SEARCH MECHANISM FOR OPTIMIZING THE
C     EXPECTED TOTAL COST FUNCTION
C
C     DESCRIPTION OF FREQUENTLY USED VARIABLES
C       NPROB--NUMBER OF INDEPENDENT FUNCTIONS TO BE SEARCHED
C       MX--NUMBER OF VARIABLES TO BE SEARCHED
C       NS--MAXIMUM NUMBER OF 'STEPS' IN THE PATTERN SEARCH PROCEDURE
C       BASE--VECTOR DEFINING THE STARTING POINT OF THE SEARCH
C       XMIN--MINIMUM VALUE TO CONSIDER FOR EACH DECISION VARIABLE
C       XMAX--MAXIMUM VALUE TO CONSIDER FOR EACH DECISION VARIABLE
C       DELT--VECTOR OF INITIAL STEP SIZES
C       DELTM--VECTOR OF MINIMUM STEP SIZES
C       DELTU--VECTOR OF MAXIMUM STEP SIZES
C       RF--VECTOR OF CURRENT POINT OF SEARCH
C       AOPT--VECTOR OF DECISION VARIABLES HAVING BEST SOLUTION
C       N--SAMPLE SIZE
C       K--NUMBER OF ITEMS PRODUCED BETWEEN SAMPLES
C       M--NUMBER OF OUT-OF-CONTROL STATES
C       MM--TOTAL NUMBER OF PROCESS STATES
C       UCL--UPPER CONTROL LIMIT
C       ALAM--AVERAGE NUMBER OF SHIFTS IN THE PROCESS FRACTION DEFECTIVE
C       PER UNIT TIME
C       S--PARAMETER WHICH DETERMINES THE TRANSITION PROBABILITIES
C       R--RATE OF PRODUCTION
C       ALPHA--VECTOR OF STEADY STATE PROBABILITIES
C       F--VECTOR OF VALUES OF THE PROCESS STATE
C       Q--VECTOR OF CONDITIONAL PROBABILITIES THAT THE SAMPLE INDICATES
C       THAT THE PROCESS IS OUT-OF-CONTROL
C       G--ELEMENTS OF THE TRANSITION MATRIX
C       P--THE TRANSITION PROBABILITIES
C       DELTA--AVERAGE FRACTION OF THE TIME BEFORE A SHIFT OCCURS GIVEN
```

```
C         THAT A SHIFT DOES OCCUR
C         SHIFT DOES OCCUR
C         AGAM--THE VECTOR OF PROBABILITIES OF BEING IN A GIVEN STATE AT
C         ANY POINT IN TIME
C         A(1)--FIXED COST OF SAMPLING
C         A(2)--VARIABLE COST OF SAMPLING
C         A(3)--COST OF AN ADJUSTMENT
C         A(4)--COST OF PRODUCING A DEFECTIVE ITEM
C         A(5)--COST OF A FALSE INDICATION
C         C1--EXPECTED COST OF SAMPLING PER ITEM
C         C2--EXPECTED COST OF ADJUSTMENT PER ITEM
C         C3--EXPECTED COST OF PRODUCING A DEFECTIVE
C         C4--EXPECTED COST OF FALSE INDICATION
C         ECTOT--EXPECTED TOTAL COST PER ITEM
C
      DIMENSION BASE(40), XMIN(40), XMAX(40), DELT(40), RF(40), TB(40)
      DIMENSION DELTM(40), DELTU(40), PB(40), BEST(40)
      COMMON /BLOKL/ RF
      READ (5,27) NPROB
      WRITE (6,27) NPROB
      READ (5,27) IPRINT
      WRITE (6,27) IPRINT
      DO 21 IJ=1,NPROB
      IND=0
      CALL SUBSYS (IND,YY)
      IND=1
      READ (5,22) MX,NS
      WRITE (6,22) MX,NS
      NITER=0
      DO 1 I=1,MX
      READ (5,23) BASE(I),XMIN(I),XMAX(I),DELT(I),DELTM(I),DELTU(I)
      WRITE (6,23) BASE(I),XMIN(I),XMAX(I),DELT(I),DELTM(I),DELTU(I)
```

```
      TB(I)=BASE(I)
      RF(I)=BASE(I)
      PB(I)=BASE(I)
    1 CONTINUE
      WRITE (6,29)
      IC=1
      YMAX=-10.**20
      JJ=1
    2 II=1
      DO 3 I=1,MX
    3 RF(I)=TB(I)
      CALL SUBSYS (IND,YY)
      YY=-YY
      IF (IPRINT.LE.0) GO TO 4
    4 IF (YY.LE.YMAX) GO TO 7
      IF (IPRINT.GT.0) GO TO 6
      DO 5 I=1,MX
    5 BEST(I)=RF(I)
    6 NITER=0
      YMAX=YY
      IF (JJ.NE.1) II=2
    7 DO 12 I=1,MX
      RF(I)=TB(I)+DELT(I)
      IF (RF(I).GT.XMAX(I)) RF(I)=XMAX(I)
      CALL SUBSYS (IND,YY)
      YY=-YY
      IF (IPRINT.LE.0) GO TO 8
    8 IF (YY.GT.YMAX) GO TO 10
      RF(I)=TB(I)-DELT(I)
      IF (RF(I).LT.XMIN(I)) RF(I)=XMIN(I)
      CALL SUBSYS (IND,YY)
      YY=-YY
```

```
      IF (IPRINT.LE.0) GO TO 9
9     IF (YY.GT.YMAX) GO TO 10
      RF(I)=TB(I)
      GO TO 12
10    YMAX=YY
      BEST(I)=RF(I)
      IF (IPRINT.GT.0) GO TO 11
11    NITER=0
      II=2
12    CONTINUE
      GO TO (15,13), II
13    JJ=2
      DO 14 I=1,MX
      PB(I)=BASE(I)
      BASE(I)=RF(I)
      TB(I)=2.*BASE(I)-PB(I)
      IF (TB(I).LT.XMIN(I)) TB(I)=XMIN(I)
      IF (TB(I).GT.XMAX(I)) TB(I)=XMAX(I)
14    CONTINUE
      GO TO 2
15    IF (IC.GT.NS) GO TO 19
      IC=IC+1
      DO 16 I=1,MX
      DELT(I)=DELT(I)/2.
      IF (DELT(I).LT.DELTM(I)) GO TO 17
      TB(I)=BASE(I)
      PB(I)=BASE(I)
      RF(I)=BASE(I)
16    CONTINUE
      GO TO 7
17    NITER=NITER+1
      IF (NITER.GT.1) GO TO 19
```

```
      DO 18 I=1,MX
      DELT(I)=DELTU(I)
      BASE(I)=TB(I)
   18 RF(I)=BASE(I)
      GO TO 2
   19 WRITE (6,25)
      IC=IC-1
      WRITE (6,26) YMAX
      WRITE (6,24) (BEST(I),I=1,MX)
      WRITE (6,28) IC
      DO 20 I=1,MX
   20 RF(I)=BEST(I)
      NND=1
   21 CONTINUE
      RETURN
C
   22 FORMAT (1X,8I5)
   23 FORMAT (1X,8F6.0)
   24 FORMAT (1X,7(3X,F14.7))
   25 FORMAT (1H0,14HFINAL SOLUTION,//)
   26 FORMAT (1X,21HOBJECTIVE FUNCTION = ,E14.7,/,1X,19HDECISION VARIABL
     1ES ,//)
   27 FORMAT (1X,I5)
   28 FORMAT (1X,13HITERATIONS = ,I5)
   29 FORMAT (1H0,//)
      END
```

```
      SUBROUTINE SUBSYS (IND,YY)
C
C     PURPOSE--TO PROVIDE DIRECTION AND CONTROL OF THE PROGRAM BY
C     PERFORMING THE FOLLOWING FUNCTIONS: (1) READ INPUT, (2) INITIALIZE
C     THE PROGRAM VARIABLES AND ARRAYS, (3) CALLS OTHER ROUTINES
C
      INTEGER N,K,M,MM
      REAL UCL,ALAM,S,R
      REAL YY,ECTOT,YMIN,A(5)
      REAL F(10),ALPHA(10),RF(40),AOPT(40)
      INTEGER IND
      COMMON /BLOKL/ RF
      COMMON /PARAM/ N,K,UCL,M,MM,ALAM,R,S
      COMMON /COEFF/ A
      COMMON /STATES/ F
      COMMON /VECTOR/ ALPHA
      IF (IND.EQ.1) GO TO 1
      YMIN=10.**8
C
C     READ PROGRAM INPUT
C
      READ (5,6) M,ALAM,S,R
      WRITE (6,9) M,ALAM,S,R
      READ (5,7) (A(I),I=1,5)
      WRITE (6,10) (A(I),I=1,5)
      MM=M+1
      READ (5,8) (F(I),I=1,MM)
      WRITE (6,11) (F(I),I=1,MM)
      IF (IND.EQ.0) RETURN
C
C     INITIALIZE THE PROGRAM PARAMETERS
C
```

```
1     DO 2 J=1,MM
2     ALPHA(J)=0.0
C
C     DEFINE THE CURRENT POINT OF THE SEARCH
C
      N=RF(1)
      K=RF(2)
      UCL=RF(3)
C
C     DETERMINE IF UCL IS LOWER THAN F(1)
C
      IF (UCL.LT.F(1)) GO TO 5
C
C     DETERMINE IF THE SAMPLE SIZE IS GREATER THAN THE NUMBER OF ITEMS
C     PRODUCED
C
      IF (N.GT.K) GO TO 5
C
C     DETERMINE IF SAMPLE SIZE EQUALS ZERO
C
      IF (N.EQ.0) UCL=1.0
C
C     EVALUATE THE FUNCTION AT THE CURRENT POINT OF SEARCH
C
      CALL PMTX
      CALL STSTPR
      CALL COST (ECTOT)
      YY=ECTOT
      IF (YY.LT.YMIN) GO TO 3
      RETURN
3     YMIN=YY
      DO 4 J=1,3
```

```
4      AOPT(J)=RF(J)
       RETURN
5      YY=999999.
       RETURN
C
6      FORMAT (I3,3F10.5)
7      FORMAT (5(F10.4))
8      FORMAT (10(F7.5))
9      FORMAT (1H ,2X,2HM=,I3,5X,5HALAM=,F10.8,5X,2HS=,F10.8,5X,2HR=,F10.
     15)
10     FORMAT (1H ,2X,21HCOST COEFFICIENTS ARE,2X,5(F10.4,5X))
11     FORMAT (1H ,2X,22HTHE PROCESS STATES ARE,2X,10(F10.8,5X))
       END
```

```
      SUBROUTINE STSTPR
C
C     PURPOSE--CALCULATES THE STEADY STATE PROBABILITIES USING THE
C     RELATION:
C          ALPHA*AA=(0,1)
C     WHERE (0,1) IS A VECTOR OF ZEROES AUGMENTED BY A ONE IN THE LAST
C     COLUMN
C
      REAL G(10,10),F(10),Q(10),AA(100),B(10),N1(10),M1(10),R1(10),P(10,
     110),ALPHA(10)
      REAL X,SUM
      REAL UCL,ALAM,S,R
      INTEGER N,K,M,MM
      INTEGER IX,LC,LE
      REAL LL
      COMMON /PARAM/ N,K,UCL,M,MM,ALAM,R,S
      COMMON /STATES/ F
      COMMON /VECTOR/ ALPHA
      COMMON /TRANS/ P
      COMMON /PROB/ Q
C
C     INITIALIZING THE MM*MM TRANSITION MATRIX
C
      DO 1 I=1,MM
      DO 1 J=1,MM
1     G(I,J)=0.0
C
C     DETERMINE THE VALUES OF Q(I) USING THE NORMAL APPROXIMATION TO THE
C     BINOMIAL
C     THE UPPER CONTROL LIMIT FOR A P-CHART IS
C                    F*(1-F)/N
C     WHERE F REPRESENTS THE IN-CONTROL STATE OF THE PROCESS
```

```
C
      LL=SQRT(F(1)*(1-F(1))/N)*UCL
      DO 2 I=1,MM
      D=F(I)-F(1)
      DENOM=SQRT(F(I)*(1-F(I))/N)
      X=(LL-D)/DENOM
      CALL NDTR (X,Z,D)
C
C     DETERMINE THE PROBABILITY OF A SHIFT FROM THE IN-CONTROL STATE
C     TO EACH OUT-OF-CONTROL STATE
C
2     Q(I)=1-Z
      DO 3 J=1,MM
3     G(1,J)=P(1,J)
C
C     DETERMINE THE VALUES OF G(I,J) WHERE I<J
C
      DO 4 J=2,MM
      DO 4 I=2,J
4     G(I,J)=(1-Q(I))*P(I,J)+Q(I)*P(1,J)
C
C     DETERMINE THE VALUES OF G(I,J) WHERE I>J
C
      DO 5 I=2,MM
      DO 5 J=1,I
      IF (I.LE.J) GO TO 5
      G(I,J)=Q(I)*P(1,J)
5     CONTINUE
C
C     PUT THE ELEMENTS OF THE MATRIX INTO VECTOR AA
C
      IX=0
```

```
      DO 6 I=1,MM
      DO 6 J=1,MM
      IX=IX+1
      AA(IX)=G(J,I)
 6    IF (I.EQ.J) AA(IX)=AA(IX)-1
      LC=MM**2-MM+1
      LE=MM**2
C
C     FILL OUT ONE ROW OF THE MATRIX EQUAL TO 1
C
      DO 7 LX=LC,LE
 7    AA(LX)=1.0
      IVPI=MM*MM
      DO 8 I=1,M
 8    B(I)=0.0
      B(MM)=1.0
C
C     INVERT THE MATRIX AA AND MULTIPLY B BY AA INVERSE
C
      CALL MINV (AA,MM,D,N1,M1,IVPI)
      CALL GMPRD (B,AA,R1,1,MM,MM,MM,IVPI,MM)
C
C     ALPHA=B * AI WHERE AI IS THE INVERSE OF THE MATRIX AA
C
      DO 9 I=1,MM
 9    ALPHA(I)=R1(I)
      RETURN
      END
```

```
      SUBROUTINE COST (ECTOT)
C
C     PURPOSE--CALCULATES THE VECTOR AGAM AND COMPUTES THE EXPECTED
C     TOTAL COST PER UNIT
C
      REAL AGAM(10),ALPHA(10),F(10),Q(10)
      REAL P(10,10)
      REAL DELTA,D,SUM,C1,C2,C3,C4,A(5),ECTOT
      INTEGER N,K,M,MM
      REAL UCL,ALAM,S,R
      COMMON /PARAM/ N,K,UCL,M,MM,ALAM,R,S
      COMMON /PROB/ Q
      COMMON /TRANS/ P
      COMMON /STATES/ F
      COMMON /COEFF/ A
      COMMON /VECTOR/ ALPHA
C
C     DETERMINE THE VALUE OF DELTA
C
      D=(ALAM*K)/R
      DELTA=(1-(1+D)*EXP(-D))/((1-EXP(-D))*D)
C
C     DETERMINE AGAM(1)
C
      AGAM(1)=ALPHA(1)*P(1,1)+DELTA*ALPHA(1)*(1-P(1,1))
C
C     DETERMINE AGAM(I)
C
      DO 5 I=2,MM
      SUM=0.0
      DO 1 J=2,I
1     SUM=SUM+P(1,J)
```

```
      AGAM(I)=(ALPHA(I)*SUM)/(1-P(1,1))
      AGAM(I)=AGAM(I)+ALPHA(1)*(1-DELTA)*P(1,I)
      II=I-1
      IF (II.LT.2) GO TO 3
      SUM=0.0
      DO 2 J=2,II
2     SUM=SUM+ALPHA(J)
      AGAM(I)=AGAM(I)+SUM*P(1,I)*(1-DELTA)/(1-P(1,1))
3     II=I+1
      IF (II.GT.MM) GO TO 6
      SUM=0.0
      DO 4 J=II,MM
4     SUM=SUM+P(1,J)
      AGAM(I)=AGAM(I)+ALPHA(I)*DELTA*SUM/(1-P(1,1))
5     CONTINUE
6     CONTINUE
C
C     DETERMINE THE EXPECTED COST OF SAMPLING
C
      C1=(A(1)+A(2)*N)/K
C
C     DETERMINE THE EXPECTED COST OF ADJUSTING THE PROCESS
C
      SUM=0.0
      DO 7 I=2,MM
7     SUM=SUM+Q(I)*ALPHA(I)
      C2=A(3)*SUM/K
C
C     DETERMINE THE EXPECTED COST OF PRODUCING A DEFECTIVE ITEM
C
      SUM=0.0
      DO 8 I=1,MM
```

```
8      SUM=SUM+F(I)*AGAM(I)
       C3=A(4)*SUM
C
C      DETERMINE THE COST OF A FALSE INDICATION
C
       C4=A(5)*ALPHA(1)*Q(1)/K
       ECTOT=C1+C2+C3+C4
       WRITE (6,9) N,K,UCL,C1,C2,C3,C4,ECTOT
       RETURN
C
9      FORMAT (1H ,2X,2HN=,I3,2X,2HK=,I3,2X,4HUCL=,F10.8,/,2X,17HCOST OF
      1SAMPLING=,F10.8,/,2X,19HCOST OF ADJUSTMENT=,F10.8,/,2X,20HCOST OF
      2A DEFECTIVE=,F10.4,/,2X,25HCOST OF FALSE INDICATION=,F10.8,/,2X,20
      3HEXPECTED TOTAL COST=,F10.8,//)
       END
```

```
      SUBROUTINE PMTX
C
C     PURPOSE--CALCULATES THE TRANSITION PROBABILITIES
C
      REAL BI,SUM
      REAL P(10,10)
      INTEGER N,K,M,MM
      REAL UCL,ALAM,S,R
      COMMON /PARAM/ N,K,UCL,M,MM,ALAM,R,S
      COMMON /TRANS/ P
C
C     INITIALIZE THE TRANSITION MATRIX TO ZERO
C
      DO 1 I=1,MM
      DO 1 J=1,MM
1     P(I,J)=0.0
C
C     FIND P(1,1)
C
      P(1,1)=EXP(-ALAM*K/R)
C
C     FIND P(1,J)
C
      DO 2 J=2,MM
      CALL BINOM (M,J-1,S,BI)
2     P(1,J)=((1-P(1,1))/(1-(1-S)**M))*BI
      IF (MM.LT.3) GO TO 4
C
C     FIND P(I,J)
C
      DO 3 J=3,MM
      DO 3 I=2,J
```

```
      IF (I.GE.J) GO TO 3
      P(I,J)=P(1,J)/(1-P(1,1))
3     CONTINUE
4     CONTINUE
C
C     FIND VALUE OF P(I,I)
C
      DO 6 I=2,MM
      SUM=0.0
      DO 5 J=2,I
5     SUM=SUM+P(1,J)
6     P(I,I)=SUM/(1-P(1,1))
      WRITE (6,9)
      DO 7 I=1,MM
      WRITE (6,8) (P(I,J),J=1,MM)
7     CONTINUE
      RETURN
C
8     FORMAT (1H ,2X,10F10.8)
9     FORMAT (1H ,2X,28HTHE TRANSITION PROBABILITIES)
      END
```

```
      SUBROUTINE BINOM (N,J,S,X)
C
C     PURPOSE--FINDS THE BINOMIAL PROBABILITY
C
C       N--NUMBER IN SAMPLE
C       J--NUMBER CHOSEN IN THE SAMPLE
C       S--THE STATE PROBABILITY
C       X--VALUE RETURNED
C
      INTEGER N,J
      REAL S,X,NN,NJ,JJ
      NN=N+1
      NJ=N-J+1
      JJ=J+1
      A=GAMMA(NN)/((GAMMA(NJ)*GAMMA(JJ)))
      X=A*(S**J)*((1-S)**(N-J))
      RETURN
      END
```

```
      SUBROUTINE NDTR (X,P,D)
C
C     PURPOSE--CALCULATES THE NORMAL PROBABILITY
C
C       X--INPUT SCALAR FOR WHICH P(X) IS COMPUTED
C       P--OUTPUT PROBABILITY
C       D--OUTPUT DENSITY
C
      AX=ABS(X)
      T=1.0/(1.0+.2316419*AX)
      D=0.3989423*EXP(-X*X/2.0)
      P=1.0-D*T*((((1.330274*T-1.821256)*T+1.781478)*T-0.3565638)*T+0.31
     193815)
      IF (X) 1,2,2
1     P=1.0-P
2     RETURN
      END
```

```
      SUBROUTINE PMTX
C
C     PURPOSE--CALCULATES THE TRANSITION PROBABILITIES
C
      REAL P(10,10)
      COMMON /PARAM/ N,K,M,MM,ALAM,R,S
      COMMON /TRANS/ P
C
C     INITIALIZE THE MATRIX TO ZERO
C
      DO 1 I=1,MM
      DO 1 J=1,MM
1     P(I,J)=0.0
C
C     DETERMINE THE PROBABILITY OF REMAINING IN-CONTROL FOR ONE PERIOD
C
      P(1,1)=EXP(-ALAM*K/R)
C
C     DETERMINE THE PROBABILITY OF A TRANSITION FROM THE IN-CONTROL
C     TO ANY OUT-OF-CONTROL STATE
C
      DO 2 J=2,MM
      CALL BINOM (M,J-1,S,BI)
2     P(1,J)=((1-P(1,1))/(1-(1-S)**M))*BI
      IF (MM.LT.3) GO TO 4
C
C     DETERMINE THE PROBABILITY OF A TRANSITION FROM STATE I TO
C     TO STATE J WHERE I<J
C
      DO 3 J=3,MM
      DO 3 I=2,J
      IF (I.GE.J) GO TO 3
```

```
      P(I,J)=P(1,J)/(1-P(1,1))
3     CONTINUE
4     CONTINUE
C
C     DETERMINE THE PROBABILITY OF REMAINING IN STATE I FOR ONE INTERVAL
C
      DO 6 I=2,MM
      SUM=0.0
      DO 5 J=2,I
5     SUM=SUM+P(1,J)
6     P(I,I)=SUM/(1-P(1,1))
      WRITE (6,9)
      DO 7 I=1,MM
      WRITE (6,8) (P(I,J),J=1,MM)
7     CONTINUE
      RETURN
C
8     FORMAT (1H ,2X,10(F10.8,5X))
9     FORMAT (1H ,2X,28HTHE TRANSITION PROBABILITIES)
      END
```

Appendix D

DOCUMENTATION OF THE COMPUTER PROGRAM USED TO OPTIMIZE THE DYNAMIC MODEL OF CHAPTER 3

A. TITLE

A program to optimize the expected total cost function for the dynamic programming model developed in Chapter 3.

1. Programmer - Nancy L. Orndorff
2. Machine - IBM 370/155
3. Language - FORTRAN IV
4. Date completed - May 25, 1974
5. Approximate compile time - .52 seconds
6. Execution time - Variable, depending on the number of iterations required. Approximate time for a two state example: 8 minutes.
7. Lines of output - Approximately 2000 lines
8. Storage required - 5000 bytes

B. PURPOSE

This program finds the optimal solution for the decision variables sample size and sampling interval. The mathematical cost function is optimized using dynamic programming techniques. A steady state solution is found by testing for the convergence of the gain.

C. RESTRICTIONS

This program can be used for up to ten process states. The dimension statements can be changed for problems in which more than ten states are considered. The number of prior probability vectors cannot exceed 200. Again, the dimension statements can be changed to increase this capacity.

D. DEFINITIONS

The subscripted and unsubscripted variables are described in the listing.

E. INPUT

In the following card input information RJND indicates that the number punched in the card must be right justified in the columns given and punched without a decimal point. Those numbers which can be punched in any of the columns given provided a decimal point is used are indicated by IRPD.

CARD 1

Cols. 1-3 RJND--The number of out-of-control states.

Cols. 4-13 IRPD--The expected number of shifts in the lot mean per unit time.

Cols. 14-23 IRPD--The value of s in the equation which determines the transition probabilities.

Cols. 24-33 IRPD--The production rate.

CARD 2

Cols. 1-10 IRPD--The fixed cost of sampling.

Cols. 11-20 IRPD--The variable cost of sampling.

Cols. 21-30 IRPD--The cost of a process adjustment.

Cols. 31-40 IRPD--The cost of producing a defective item.

CARD 3

Cols. 1-7 IRPD--The value of the in-control state of the process fraction defective. The states should be numbered lowest to highest.

Cols. 8-14 IRPD--The value of the first out-of-control state.

Cols. 15-21

. . . .

. . . .

. . . .

Cols. 64-70 IRPD--The value of the last out-of-control state.

CARD 4

Cols. 1-4 RJND--Starting value for the length of the sampling interval.

Cols. 5-8 RJND--Maximum length of the sampling interval.

Cols. 9-12 RJND--Step size to be used in iterating over values of the sampling interval.

CARD 5

Cols. 1-4 RJND--Starting sample size.

Cols. 5-8 RJND--Maximum sample size.

Cols. 9-12 RJND--Step size to be used in iterating over values of the sample size.

CARD 6

Cols. 1-10 IRPD--The increment of perturbation for the values of the prior distribution.

CARD 7

Cols. 1-4 RJND--The maximum number of time periods to be considered.

F. COMPUTER SYSTEM LANGUAGE LIMITATIONS

The input and output statements were written with I/O unit numbers 5 and 6 respectively. Other systems may require different unit designations.

G. OUTPUT

The program iterates over values of the sampling interval. The following procedure is performed for each value of the sampling interval. First, the time parameter is incremented. The function is then evaluated at each of the desired prior vectors. For each set of values of the prior probabilities, the following information is printed in tabular form:

1. The values of the prior vector being considered.
2. The optimal sample size for that vector.
3. The expected cost of the system, including the future costs.
4. The process gain.
5. The breakeven point. If the value observed is less than or equal to the break-even point, then run is the least cost decision. If this value is greater than the break-even point, then the process should be adjusted.

H. LISTING OF PROGRAM

<u>Name of Subroutine</u>

MAIN PROGRAM

NOSAMP

SAMP

FUTURE

ATPRM

COST

BAYES

BINOM

PMTX

```
C      MAIN PROGRAM
C
C      PURPOSE--TO PROVIDE DIRECTION AND CONTROL OF THE PROGRAM BY
C      PERFORMING THE FOLLOWING FUNCTIONS: (1) READ INPUT, (2) INITIALIZE
C      THE PROGRAM VARIABLES AND ARRAYS, (3) CALLS OTHER ROUTINES
C
C      DESCRIPTION OF FREQUENTLY USED VARIABLES
C        IBEP--MAXIMUM VALUE OF X FOR WHICH RUN IS THE LEAST COST
C        DECISION
C        X--NUMBER OF DEFECTIVES IN THE SAMPLE
C        BASE--STARTING VALUE OF THE DECISION VARIABLE
C        BMAX--MAXIMUM VALUE OF THE DECISION VARIABLE
C        DELT--STEP SIZE
C        ALPRI--PRIOR PROBABILITY VECTOR
C        ALPOS--POSTERIOR PROBABILITY VECTOR
C        ALPT--THE PRIOR PROBABILITY VECTOR AT TIME (T-1)
C        ADELT--THE STEP SIZE TO USE IN ITERATING OVER THE PRIOR
C        PROBABILITIES
C        F--THE VALUES OF THE PROCESS STATES
C        P--THE TRANSITION PROBABILITIES
C        ALST--THE STARTING VALUE OF THE PRIOR PROBABILITY VECTOR
C        V--THE FUTURE COST ASSOCIATED WITH A PARTICULAR PRIOR VECTOR
C        U1--THE FUTURE COST DETERMINED AT TIME T-1
C        U2--THE FUTURE COST DETERMINED IN THE CURRENT TIME PERIOD T
C        GAIN--THE PROCESS GAIN
C        GANE--THE PROCESS GAIN FROM THE PREVIOUS PERIOD
C        T--TIME INDEX
C        N--SAMPLE SIZE
C        K--NUMBER OF UNITS PRODUCED BETWEEN SAMPLES
C        TMAX--MAXIMUM NUMBER OF TIME PERIODS TO BE CONSIDERED FOR EACH
C        SET OF VALUES OF THE PRIOR VECTOR
C        M--NUMBER OF OUT-OF-CONTROL STATES
```

```
C        MM--TOTAL NUMBER OF STATES (MM=M+1)
C        ALAM--AVERAGE NUMBER OF SHIFTS IN THE PROCESS FRACTION
C        DEFECTIVE PER UNIT TIME
C        R--PRODUCTION RATE
C        S--PARAMETER WHICH DEFINES THE TRANSITION PROBABILITIES
C        CRIT--STOPPING CRITERION USED TO TEST FOR CONVERGENCE OF GAIN
C        KV--POINTER TO VALUE OF FUTURE COST TO BE ADDED TO EXPECTED
C        TOTAL COST
C        EC1--EXPECTED TOTAL COST IF NO ADJUSTMENT IS MADE
C        EC2--EXPECTED TOTAL COST IF AN ADJUSTMENT IS MADE
C        CSTN--EXPECTED TOTAL COST OF BEST DECISION
C        AGAM--THE VECTOR OF PROBABI3ITIES OF BEING IN A GIVEN STATE AT
C        ANY POINT IN TIME
C        A(1)--FIXED COST OF SAMPLING
C        A(2)--VARIABLE COST OF SAMPLING
C        A(3)--COST OF AN ADJUSTMENT
C        A(4)--COST OF PRODUCING A DEFECTIVE ITEM
C        C1--EXPECTED COST OF SAMPLING PER ITEM
C        C2--EXPECTED COST OF ADJUSTMENT PER ITEM
C        C3--EXPECTED COST OF PRODUCING A DEFECTIVE
C        ECTOT--EXPECTED TOTAL COST PER ITEM
C
      INTEGER IBEP(25)
      INTEGER BMAX(2),BASE(2),DELT(2)
      REAL A(4),F(10),P(10,10)
      REAL ALST(10),ALPRI(10),V(200)
      REAL ALPOS(10),ALPT(10),U1(200),GAIN(200),GANE(200)
      REAL U2(200)
      INTEGER T,TMAX
      COMMON /PARAM/ N,K,M,MM,ALAM,R,S
      COMMON /TRANS/ P
      COMMON /STATES/ F
```

```
      COMMON /COEFF/ A
      COMMON /FUT/ V
      COMMON /XYZ/ ADELT
      COMMON /VECTOR/ ALPOS,ALPRI,ALPT
C
C     READ PROGRAM INPUT
C
      READ (5,15) M,ALAM,S,R,CRIT
      WRITE (6,20) M,ALAM,S,R,CRIT
      READ (5,16) (A(I),I=1,4)
      WRITE (6,21) (A(I),I=1,4)
      MM=M+1
      READ (5,17) (F(I),I=1,MM)
      WRITE (6,22) (F(I),I=1,MM)
C
C     DEFINE GRID VALUES FOR K AND N
C
      DO 1 I=1,2
      READ (5,18) BASE(I),BMAX(I),DELT(I)
1     WRITE (6,23) BASE(I),BMAX(I),DELT(I)
C
C     READ STEP SIZE FOR ALPHA GRID
C
      READ (5,24) ADELT,(ALST(I),I=1,MM)
      WRITE (6,25) ADELT,(ALST(I),I=1,MM)
      READ (5,19) TMAX
      WRITE (6,26) TMAX
C
C     INITIALIZE PROGRAM PARAMETERS
C
      DO 2 I=1,MM
      ALPRI(I)=0.0
```

```
      ALPOS(I)=0.0
    2 ALPT(I)=0.0
      IV=(1./ADELT)+.5
      K1=BASE(1)
      K2=BMAX(1)
      K3=DELT(1)
      N1=BASE(2)
      N2=BMAX(2)
      N3=DELT(2)
C
C     ITERATE OVER VALUES OF K
C
      DO 14 K=K1,K2,K3
      IF (N1.GT.K) GO TO 14
      DO 3 I=1,200
      V(I)=0.0
      U1(I)=0.0
      U2(I)=0.0
      GANE(I)=0.0
      GAIN(I)=0.0
    3 CONTINUE
      GSUM=0.0
      T=0
      WRITE (6,29) K
      CALL PMTX
    4 T=T+1
      IF (T.GT.TMAX) GO TO 14
      WRITE (6,28) T
      DO 5 I=1,MM
    5 ALPRI(I)=ALST(I)
      I=1
    6 CMIN=999999.
```

```
C
C      DETERMINE THE EXPECTED COST IF NO SAMPLING
C
       CALL NOSAMP (CSTN,JBEP)
       CMIN=CSTN
       NOPT=N
C
C      DETERMINE THE OPTIMAL VALUE OF N AND THE ASSOCIATED EXPECTED COST
C
       DO 7 N=N1,N2,N3
       IF (N.GT.K) GO TO 8
       CALL SAMP (CSTN,IBEP(N))
       IF (CSTN.GT.CMIN) GO TO 7
       CMIN=CSTN
       NOPT=N
7      CONTINUE
8      U2(I)=CMIN
       IF (NOPT.EQ.0) GO TO 9
       KBEP=IBEP(NOPT)
       GO TO 10
9      KBEP=JBEP
C
C      CALCULATE THE GAIN
C
10     GAIN(I)=U2(I)-U1(I)
       WRITE (6,27) ALPRI(1),NOPT,U2(I),GAIN(I),KBEP
C
C      REDEFINE PROGRAM PARAMETERS FOR NEXT PERIOD
C
       V(I)=U2(I)
       U1(I)=U2(I)
       I=I+1
```

```
      ALPRI(1)=ALPRI(1)-ADELT
      DO 11 JJ=2,MM
11    ALPRI(JJ)=ALPRI(JJ)+(ADELT/M)
      IF (ALPRI(1).LE.0.0) GO TO 12
      GO TO 6
12    CONTINUE
C
C     TEST FOR CONVERGENCE OF GAIN
C
      DO 13 KK=1,IV
      G=ABS(GAIN(KK)-GANE(KK))
      GANE(KK)=GAIN(KK)
      IF (G.GT.CRIT) GO TO 4
13    CONTINUE
      GO TO 14
14    CONTINUE
      RETURN
C
15    FORMAT (I3,4F10.5)
16    FORMAT (5(F10.4))
17    FORMAT (10(F10.8))
18    FORMAT (3I4)
19    FORMAT (I4)
20    FORMAT (1H ,2X,2HM=,I3,5X,5HALAM=,F10.8,5X,2HS=,F10.8,5X,2HR=,F10.
     15,5X,5HCRIT=,F10.5)
21    FORMAT (1H ,2X,21HCOST COEFFICIENTS ARE,2X,10(F10.4,5X))
22    FORMAT (1H ,2X,22HTHE PROCESS STATES ARE,2X,10(F10.8,5X))
23    FORMAT (1H ,2X,4HBASE,I3,2X,4HBMAX,I3,2X,4HDELT,I3,2X)
24    FORMAT (11F10.8)
25    FORMAT (1H ,2X,5HADELT,F10.8,/,2X,4HALST,10(F10.8,2X))
26    FORMAT (1H ,2X,5HTMAX=,I4)
27    FORMAT (1H ,2X,F10.8,T25,I4,T40,F10.4,T70,F10.4,T90,I4)
```

```
28    FORMAT (1H ,///,2X,2HT=,2X,I4,//,5X,8HALPRI(1),T28,1HN,T46,1HV,T75
     1,4HGAIN,T91,3HBEP,/)
29    FORMAT (1H ,///,2X,2HK=,I4)
      END
```

```
      SUBROUTINE NOSAMP (CSTN,IBEP)
C     PURPOSE--DETERMINE MINIMUM COST DECISION IF NO SAMPLING
C
      REAL ALPOS(10),ALPRI(10),ALPT(10),V(200)
      INTEGER ADJ
      COMMON /PARAM/ N,K,M,MM,ALAM,R,S
      COMMON /VECTOR/ ALPOS,ALPRI,ALPT
      COMMON /FUT/ V
C
C     IF NO ADJUSTMENT
C
      N=0
      DO 1 I=1,MM
1     ALPOS(I)=ALPRI(I)
      CALL ATPRM
      CALL FUTURE (KV)
      ADJ=0
      CALL COST (EC1,ADJ)
      EC1=EC1+V(KV)
C
C     IF ADJUSTMENT
C
      DO 2 I=2,MM
2     ALPOS(I)=0.0
      ALPOS(1)=1.0
      CALL ATPRM
      CALL FUTURE (KV)
      ADJ=1
      CALL COST (EC2,ADJ)
      EC2=EC2+V(KV)
C
C     DETERMINE MINIMUM COST DECISION
```

```
C
      IF (EC1.LE.EC2) GO TO 3
      CSTN=EC2
      IBEP=99
      RETURN
3     CSTN=EC1
      IBEP=55
      RETURN
      END
```

```
      SUBROUTINE SAMP (CSTN,IBEP)
C     PURPOSE--DETERMINES COST AND BREAK EVEN POINT FOR A PARTICULAR
C     VALUE OF N
C
      REAL ALPOS(10),ALPRI(10),ALPT(10),V(200)
      REAL PR(25),F(10)
      INTEGER XX
      INTEGER ADJ,X
      COMMON /PARAM/ N,K,M,MM,ALAM,R,S
      COMMON /VECTOR/ ALPOS,ALPRI,ALPT
      COMMON /FUT/ V
      COMMON /STATES/ F
      IBEP=-1
C
C     IF AN ADJUSTMENT IS MADE
C
      DO 1 I=2,MM
1     ALPOS(I)=0.0
      ALPOS(1)=1.0
      CALL ATPRM
      CALL FUTURE (KV)
      ADJ=1
      CALL COST (EC2,ADJ)
      EC2=EC2+V(KV)
C
C     IF NO ADJUSTMENT
      NN=N+1
      CTOT=0
      ADJ=0
      DO 7 X=1,NN
      XX=X-1
      DO 2 I=1,MM
```

```
2     CALL BAYES (XX,I,ALPRI,ALPOS(I))
      CALL COST (EC1,ADJ)
      CALL ATPRM
      CALL FUTURE (KV)
      EC1=EC1+V(KV)
C
C     TAKE THE EXPECTATION OF THE TOTAL COST WITH RESPECT TO X
C
      SUM=0.0
      DO 3 I=1,MM
      CALL BINOM (N,XX,F(I),Z)
3     SUM=SUM+Z*ALPRI(I)
      IF (EC1-EC2) 4,4,5
4     SK=EC1*SUM
      IBEP=XX
      GO TO 6
5     SK=EC2*SUM
6     CTOT=CTOT+SK
7     CONTINUE
      CSTN=CTOT
      RETURN
      END
```

```
      SUBROUTINE FUTURE (KV)
C     PURPOSE--DETERMINES THE POINTER TO THE FUTURE COST ASSOCIATED
C     WITH THE PRIOR VECTOR AT TIME T-1
C
      REAL ALPOS(10),ALPRI(10),ALPT(10)
      COMMON /VECTOR/ ALPOS,ALPRI,ALPT
      COMMON /XYZ/ ADELT
      KV=0
      ALF1=1.0
      ALF2=ALF1-ADELT/2
1     KV=KV+1
      IF (ALPT(1).LE.ALF1.AND.ALPT(1).GT.ALF2) RETURN
      IF (ALF1.LE.0.0) RETURN
      ALF1=ALF2
      ALF2=ALF1-ADELT
      GO TO 1
      END
```

```
      SUBROUTINE ATPRM
C     COMPUTES THE VALUES OF ALPHA PRIME AT TIME T-1, USING THE
C     TRANSITION MATRIX
C
      REAL P(10,10),ALPOS(10),ALPRI(10),ALPT(10)
      COMMON /PARAM/ N,K,M,MM,ALAM,R,S
      COMMON /VECTOR/ ALPOS,ALPRI,ALPT
      COMMON /TRANS/ P
      DO 2 I=1,MM
      SUM=0.0
      DO 1 J=1,I
1     SUM=SUM+ALPOS(J)*P(J,I)
      ALPT(I)=SUM
2     CONTINUE
      RETURN
      END
```

```
      SUBROUTINE COST (ECTOT,IADJ)
C     PURPOSE--CALCULATES THE VECTOR AGAM AND COMPUTES THE EXPECTED
C     TOTAL COST PER ITEM
C
      INTEGER ADJ
      REAL F(10),A(4),AGAM(10),ALPOS(10),ALPRI(10),ALPT(10),P(10,10)
      COMMON /PARAM/ N,K,M,MM,ALAM,R,S
      COMMON /TRANS/ P
      COMMON /STATES/ F
      COMMON /COEFF/ A
      COMMON /VECTOR/ ALPOS,ALPRI,ALPT
C
C     DETERMINE THE VALUE OF DELTA
C
      D=(ALAM*K)/R
      DELTA=(1-(1+D)*EXP(-D))/((1-EXP(-D))*D)
C
C     DETERMINE AGAM(1)
C
      AGAM(1)=ALPOS(1)*P(1,1)+DELTA*ALPOS(1)*(1-P(1,1))
C
C     DETERMINE AGAM(I)
      DO 5 I=2,MM
      SUM=0.0
      DO 1 J=2,I
1     SUM=SUM+P(1,J)
      AGAM(I)=(ALPOS(I)*SUM)/(1-P(1,1))
      AGAM(I)=AGAM(I)+ALPOS(1)*(1-DELTA)*P(1,I)
      II=I-1
      IF (II.LT.2) GO TO 3
      SUM=0.0
      DO 2 J=2,II
```

```
2     SUM=SUM+ALPOS(J)
      AGAM(I)=AGAM(I)+SUM*P(1,I)*(1-DELTA)/(1-P(1,1))
3     II=I+1
      IF (II.GT.MM) GO TO 6
      SUM=0.0
      DO 4 J=II,MM
4     SUM=SUM+P(1,J)
      AGAM(I)=AGAM(I)+ALPOS(I)*DELTA*SUM/(1-P(1,1))
5     CONTINUE
6     CONTINUE
C
C     DETERMINE THE EXPECTED COST OF SAMPLING
C
      C1=(A(1)+A(2)*N)/K
C
C     DETERMINE THE EXPECTED COST OF ADJUSTING THE PROCESS
C
      IF (IADJ.EQ.0) GO TO 7
      C2=A(3)/K
      GO TO 8
7     C2=0.0
8     CONTINUE
C
C     DETERMINE THE COST OF PRODUCING A DEFECTIVE ITEM
C
      SUM=0.0
      DO 9 I=1,MM
9     SUM=SUM+F(I)*AGAM(I)
      C3=A(4)*SUM
      ECTOT=C1+C2+C3
      RETURN
      END
```

```
      SUBROUTINE BAYES (X,J,A,B)
C     PURPOSE--FINDS THE PROBABILITY USING BAYES FORMULA
C
      REAL A(10),F(10),H(20)
      INTEGER X
      COMMON /STATES/ F
      COMMON /PARAM/ N,K,M,MM,ALAM,R,S
      DO 1 I=1,MM
      CALL BINOM (N,X,F(I),Y)
1     H(I)=Y
      SUM=0.0
      DO 2 I=1,MM
2     SUM=SUM+H(I)*A(I)
      B=H(J)*A(J)/SUM
      RETURN
      END
```

```
      SUBROUTINE BINOM (N,J,S,X)
C     PURPOSE--FINDS THE BINOMIAL PROBABILITY
C
      INTEGER N,J
      REAL S,X,NN,NJ,JJ
      NN=N+1
      NJ=N-J+1
      JJ=J+1
      A=GAMMA(NN)/((GAMMA(NJ)*GAMMA(JJ)))
      X=A*(S**J)*((1-S)**(N-J))
      RETURN
      END
```

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A DYNAMIC PROGRAMMING APPROACH

TO SINGLE ATTRIBUTE PROCESS CONTROL

by

Nancy L. Orndorff

(ABSTRACT)

This thesis focuses on the economic design of process control procedures for attributes sampling. The process is modeled as a continuous time, discrete space stochastic process which possesses the Markov property, and hence a Markov chain is used to describe its behavior.

Two models are developed. The first model has fixed values of the decision variables and is optimized using the pattern search procedure. The second model is a dynamic formulation. The optimal decision policies developed using this model vary with the expected state of the process.

Several cost components are considered in the mathematical development of each model. They are: the cost of sampling, the cost of process adjustment, and the cost of producing a defective unit. The cost of a false indication of the process state is also included in the fixed parameter model.

Computer programs, written in Fortran IV are developed and used to find the optimal system designs. Example problems are presented to illustrate both of the models. The dynamic programming model is shown to offer considerable economic improvement over the steady state model in all of the examples.