COHERENT AND PARTIALLY COHERENT CW LASER BEAM PROPAGATION IN LENS-LIKE MEDIA; A DIFFRACTIVE RAY-THEORETIC APPROACH

by

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I. INTRODUCTION

Since the advent of the laser as an important source of coherent light, with the potential for various applications (e.g., light wave communications,* laser-induced plasma fusion,[4] remote sensing of biological substances,[5] etc.), a great amount of interest has been directed recently in the propagation of optical beams in various media such as the atmosphere, the ocean, the ionosphere, fiber-optical waveguides, laboratory plasmas, biological particle concentrations, etc.

The output of a good, single-mode laser** is a pure Gaussian beam. The goal, usually, is to study the evolution of such a beam (after it has been adequately focused by an optical system, e.g., a lens) in the presence of appropriate electromagnetic parameters characterizing the particular channel under consideration. Material properties which ordinarily play a significant role in this respect are inhomogeneities, absorption, dispersion, nonlinearities, discontinuities, etc. In more realistic situations, however, one has to account for statistical fluctuations which enter into the medium parameters and the boundary conditions.

Diffraction effects are completely ignored within the framework of the first-order geometrical optics approximation. As a consequence,

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* An overview of this subject can be found in the May 1976 special issue of Physics Today.

** In contrast to an inexpensive multi-mode semiconductor laser, for example an LED, whose output light is incoherent.

beam spreading of an electromagnetic signal, a physical phenomenon which occurs even in the absence of inhomogeneities and becomes significant for long ranges, cannot be accounted for.

A technique, referred to as the wave-kinetic method, has been developed recently by Tappert and Besieris (cf. Refs. 1-3) for extending conventional ray-tracing methods to incorporate diffractive, refractive, as well as stochastic effects associated with the propagation of beamed signals through inhomogeneous media. At each point in space, rays are provided with a distribution of directions (wave numbers). This is achieved via the Wigner "phase-space" distribution density function. The latter may, in turn, be used to obtain physical observables (e.g., the slope of rays, the spread of ray angles, etc.) by taking appropriate phase-space moments.

It is our intent in this thesis to illustrate the wave-kinetic method by examining the propagation of coherent and partially coherent cw Gaussian focused laser beams in lens-like media (e.g., selfoc fibers, laser-induced plasmas, etc.), i.e., media having a quadratic variation in the index of refraction. The discussion is limited to media which are lossless, linear and unbounded. Although statistical imperfections in the focusing lens are taken into consideration, the medium itself is considered to be purely deterministic.

The main results presented here (cf., also, Ref. 6) extend those for vacuum propagation determined by Papoulis[7] and Tappert,[8] as well as the ones reported by McCoy and Beran.[9]

II. TIME-HARMONIC MAXWELL'S EQUATIONS FOR LENS-LIKE MEDIA

For harmonic time variation, with an assumed time dependence of the form $\exp(-i\omega t)$, where ω is the angular frequency, and in the absence of external sources, Maxwell's equations become

$$\nabla \times \underline{E}(\underline{r},\omega) = i\omega \underline{B}(\underline{r},\omega), \tag{2.1a}$$

$$\nabla \times \underline{H}(\underline{r},\omega) = -i\omega \underline{D}(\underline{r},\omega), \tag{2.1b}$$

$$\nabla \cdot \underline{D}(\underline{r},\omega) = 0, \tag{2.1c}$$

$$\nabla \cdot \underline{B}(\underline{r},\omega) = 0, \tag{2.1d}$$

where $\underline{E}$, $\underline{H}$ signify, respectively, the electric and magnetic field intensities, and $\underline{D}$, $\underline{B}$ the electric and magnetic displacements, all referred to the mks system of units.

For a linear, inhomogeneous medium, we specify the constitutive relations

$$\underline{D}(\underline{r},\omega) = \varepsilon(\underline{r},\omega)\ \underline{E}(\underline{r},\omega), \tag{2.2a}$$

$$\underline{B}(\underline{r},\omega) = \mu \underline{H}(\underline{r},\omega). \tag{2.2b}$$

The magnetic permeability is assumed to be a real, constant quantity. On the other hand, the permittivity depends on the spatial coordinate $\underline{r}$ (inhomogeneity), on the frequency ω(dispersion), and, furthermore, it is allowed to be complex (accounting for the presence of absorption or gain).

Introducing (2.2) into (2.1), Maxwell's equations assume the following "definite" form:

$$\nabla \times \underline{E}(\underline{r},\omega) = i\omega\mu \, \underline{H}(\underline{r},\omega), \tag{2.3a}$$

$$\nabla \times \underline{H}(\underline{r},\omega) = -i\omega\varepsilon(\underline{r},\omega)\underline{E}(\underline{r},\omega), \tag{2.3b}$$

$$\nabla \cdot [\varepsilon(\underline{r},\omega)\underline{E}(\underline{r},\omega)] = 0, \tag{2.3c}$$

$$\nabla \cdot \underline{H}(\underline{r},\omega) = 0. \tag{2.3d}$$

In general, these equations are rendered closed mathematically by specifying appropriate boundary conditions.

Eliminating the magnetic field intensity from (2.3), results in the following equation for $\underline{E}$:

$$\nabla^2 \underline{E} + \omega^2 \mu\varepsilon \underline{E} + \nabla\left(\frac{\nabla\varepsilon}{\varepsilon} \cdot \underline{E}\right) = 0 \; . \tag{2.4}$$

The last term on the left-hand side of this equation shows that the three components of $\underline{E}$ are coupled because of the inhomogeneity. This, also, applies for the components of the magnetic field intensity by virtue of (2.3a). In such a case, we say that the electromagnetic fields are depolarized.

In the following we shall assume that

$$L\left|\frac{\nabla\varepsilon}{\varepsilon}\right| \ll 1 \; , \tag{2.5}$$

where L is a characteristic length. This, of course, is tantamount to neglecting depolarization. In this case, each component of $\underline{E}$ and $\underline{H}$ obeys a Helmholtz equation of the form

$$\nabla^2 \, u(\underline{r},\omega) + \omega^2 \mu\varepsilon(\underline{r},\omega) u(\underline{r},\omega) = 0 \; . \tag{2.6}$$

For isotropic, lens-like media, the permittivity has the following general form:

$$\varepsilon(\underline{r},\omega) = (\varepsilon_r - G_r^2 x^2) - i(\varepsilon_i - G_i^2 x^2); \qquad (2.7a)$$

$$\underline{r} = (\underline{x},z), \; x = |\underline{x}|. \qquad (2.7b)$$

Here, ε_r, ε_i, G_r^2 and G_i^2 are real quantities which may, in general, depend on frequency. Depending upon their relative values, the medium is lossless, absorbing, or is characterized by gain.[10-13]

In the following, we shall restrict the discussion to media which have neither loss nor gain, viz., $\varepsilon_i = 0$, $G_i^2 = 0$. In this simple case, the permittivity simplifies to

$$\varepsilon(\underline{r},\omega) = \varepsilon(o,\omega)(1 - g^2x^2); \qquad (2.8a)$$

$$g^2 = \frac{G_r^2}{\varepsilon_r}. \qquad (2.8b)$$

Introducing this expression to (2.6), we obtain

$$\nabla^2 u(\underline{r},\omega) + k^2(1 - g^2x^2)u(\underline{r},\omega) = 0; \qquad (2.9a)$$

$$k^2 = \omega^2\mu\varepsilon(o,\omega). \qquad (2.9b)$$

Our goal is to investigate the solution of the Helmholtz equation (2.9a) subject to the boundary condition

$$u(\underline{x},o,\omega) = u_o(\underline{x},\omega).^{*} \qquad (2.10)$$

Since our work will be restricted to cw solutions, the parametric dependence of u on the frequency ω will be suppressed in the sequel.

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* Strictly speaking, the problem is not closed mathematically unless a radiation condition is imposed in addition to (2.10).

III. THE PARABOLIC APPROXIMATION

For beam propagation in the z direction, it is convenient to resort to the transformation

$$u(\underline{r}) = \psi(\underline{x},z)\exp(ikz). \tag{3.1}$$

Let

$$\left|\frac{1}{k}\frac{\partial}{\partial z}\psi(\underline{x},z)\right| >> \left|\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\psi(\underline{x},z)\right| . \tag{3.2}$$

This condition, known as the <u>small-angle approximation</u>[14] is satisfied rather well in the quasi-optical regime. It follows, then, that the slowly varying amplitude function $\psi(\underline{x},z)$ is governed by the parabolic equation

$$\frac{i}{k}\frac{\partial}{\partial z}\psi(\underline{x},z) = -\frac{1}{2k^2}\frac{\partial^2}{\partial \underline{x}^2}\psi(\underline{x},z) + \frac{1}{2}g^2x^2\psi(\underline{x},z), \tag{3.3a}$$

where $(\frac{\partial^2}{\partial \underline{x}^2})$ denotes the transverse (with respect to z) Laplacian operator. Corresponding to the boundary condition (2.10), we have now

$$\psi(\underline{x},o) \equiv \psi_o(\underline{x}) = u_o(\underline{x}). \tag{3.3b}$$

It is our goal in this exposition to consider the boundary-value problem in an unbounded (with respect to $\underline{x}$) domain. It should be pointed out, however, that this idealized problem provides a good approximation to the forward propagation of low-order modes in a fiber light guide having a parabolically graded refractive index.[15,16]

We consider next a conservation law associated with the parabolic equation (3.3a). Let the quantities

$$i(\underline{x},z) = \psi^*(\underline{x},z)\psi(\underline{x},z) \tag{3.4}$$

and

$$\underline{j}(\underline{x},z) = \frac{i}{2k}\,[\psi(\underline{x},z)\frac{\partial}{\partial \underline{x}}\,\psi^*(\underline{x},z) - \psi^*(\underline{x},z)\frac{\partial}{\partial \underline{x}}\,\psi(\underline{x},z)] \tag{3.5}$$

be referred to as the <u>intensity</u> and <u>intensity flux densities</u>, respectively. They are related by the following conservation law:

$$\frac{\partial}{\partial z}\,i(\underline{x},z) + \frac{\partial}{\partial \underline{x}} \cdot \underline{j}(\underline{x},z) = 0. \tag{3.6}$$

As a consequence, the <u>total intensity</u>, defined by

$$I(z) = \int_{R^2} d\underline{x}\; i(\underline{x},z), \tag{3.7}$$

is conserved, i.e.,

$$\frac{d}{dz}\,I(z) = 0. \tag{3.8}$$

It follows, then, that (3.9)

$$I(z) = I(o) = \int_{R^2} d\underline{x}\;\psi_o^*(\underline{x})\;\psi_o(\underline{x}). \tag{3.9}$$

Henceforward, we shall assume that this quantity is normalized to unity.

Let us define next a vector field $\underline{s}(\underline{x},z)$ by

$$\underline{s}(\underline{x},z) = i(\underline{x},z)\hat{z} + \underline{j}(\underline{x},z), \tag{3.10}$$

where $\hat{z}$ is a unit vector in the z direction. The conservation law (3.6) can be rewritten in terms of $\underline{s}(\underline{x},z)$ as follows:

$$\nabla\cdot\underline{s}(\underline{x},z) = 0. \tag{3.11}$$

From physical considerations, $\underline{s}(\underline{x},z)$ can be interpreted as a <u>power flux density</u>. On the strength of the divergence theorem,

$$\iiint_V \nabla\cdot\underline{s}(\underline{x},z) = \oiint_S \underline{s}(\underline{x},z) \cdot d\underline{A} = 0, \tag{3.12}$$

where S is a regular surface bounding a volume V. From a more practical point of view, the power intercepted by a detector can be written as

$$\iint_S \underline{s}(\underline{x},z) \cdot d\vec{A} \ ,$$

where S is an open surface (cf. Fig. 3-1).

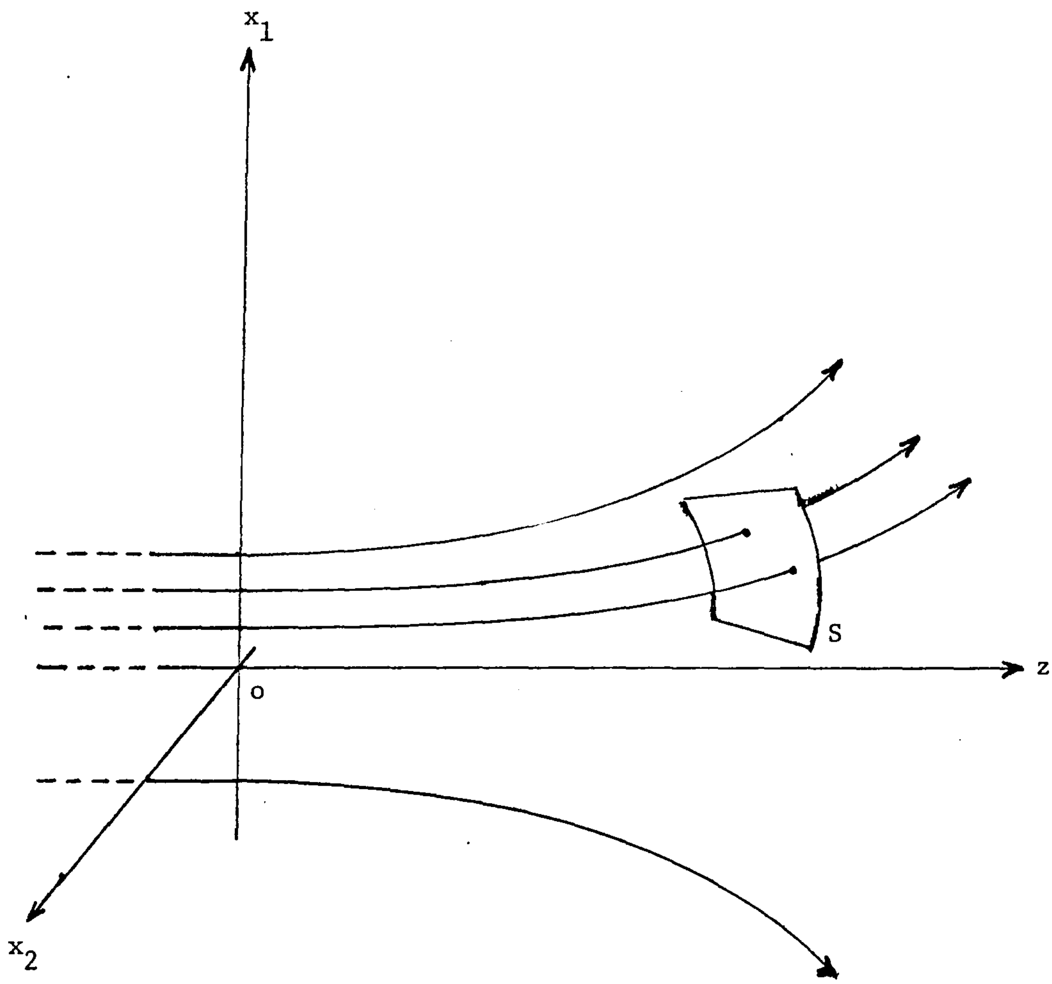

Figure 3-1

IV. THE COHERENT WIGNER DISTRIBUTION DENSITY FUNCTION

A two-(transverse) point field density function is introduced next as follows in terms of the wave function $\psi(\underline{x},z)$:

$$\rho(\underline{x}_2,\underline{x}_1,z) = \psi^*(\underline{x}_2,z)\psi(\underline{x}_1,z) \ . \tag{4.1}$$

The "phase-space" analog of the field density function is provided by the Wigner distribution density function which is defined as follows:[2,17]

$$f(\underline{x},\underline{\theta},z) = \left(\frac{k}{2\pi}\right)^2 \int_{R^2} d\underline{y}\ e^{ik\underline{\theta}\cdot\underline{y}}\ \rho(\underline{x} + \tfrac{1}{2}\,\underline{y},\underline{x} - \tfrac{1}{2}\,\underline{y},z). \tag{4.2}$$

The magnitude of the two-dimensional vector $\underline{\theta}$ can be interpreted physically as the angle with respect to the z-axis in the small angle limit (cf. Fig. 4-1).

The Wigner distribution function is real, but not necessarily positive everywhere. It is shown in Appendix A that, in general, $|f(\underline{x},\underline{\theta},z)| \leq \left(\frac{k}{\pi}\right)^2$. Provided that $f(\underline{x},\underline{\theta},z)$ is normalized (to unity), this means that the Wigner distribution function is different from zero in a region of which the volume in phase-space is at least equal to $\left(\frac{\pi}{k}\right)^2$. Hence, $f(\underline{x},\underline{\theta},z)$ can never be sharply localized in $\underline{x}$ and $\underline{\theta}$. This situation is a reflection of the ambiguity or uncertainty principle, and is analogous to the quantum mechanical Heisenberg uncertainty principle, the ambiguity arising in Fourier optics,[7] and the radar ambiguity discussed originally by Woodward.[18]

Using the definition of $f(\underline{x},\underline{\theta},z)$ in conjunction with (3.3a) and (4.1), it is shown in Appendix B that the Wigner distribution function

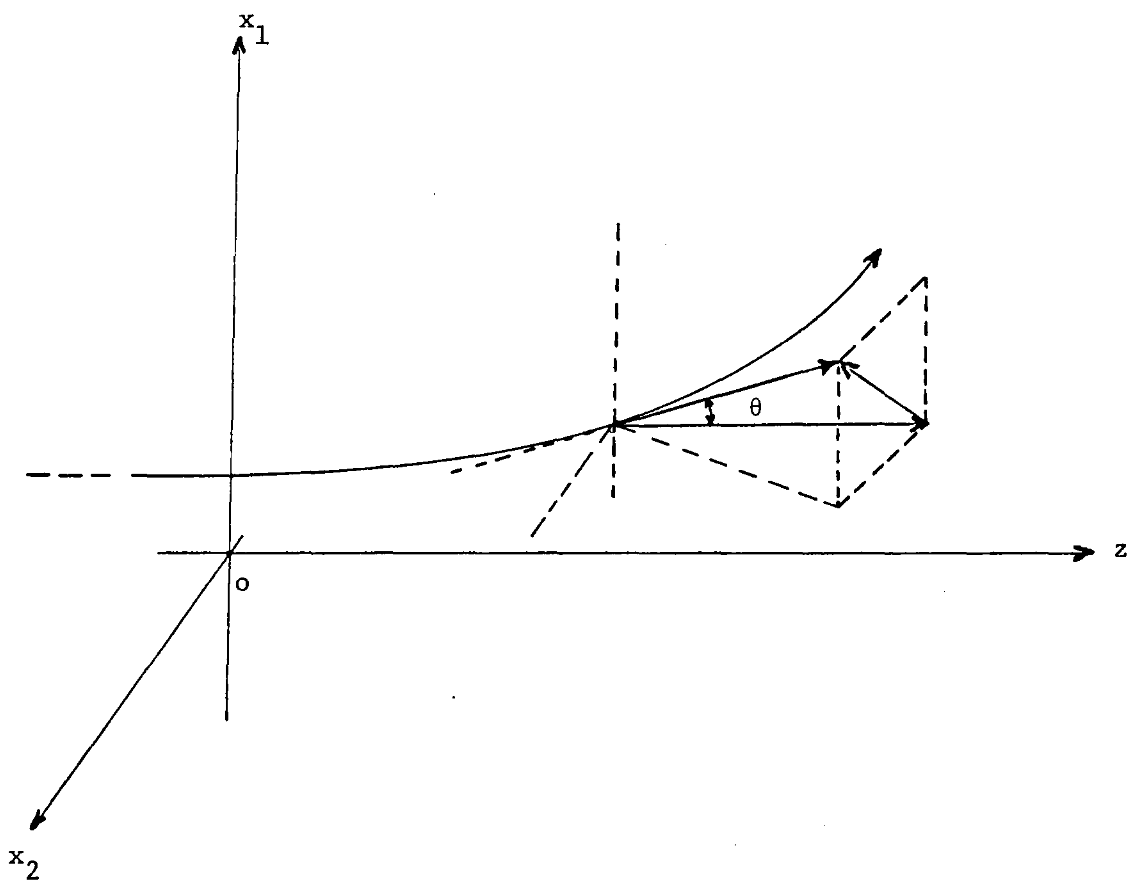

Figure 4-1

"evolves" according to the equation:

$$\frac{\partial}{\partial z} f(\underline{x},\underline{\theta},z) + \underline{\theta} \cdot \frac{\partial}{\partial \underline{x}} f(\underline{x},\underline{\theta},z) - g^2\underline{x} \cdot \frac{\partial}{\partial \underline{\theta}} f(\underline{x},\underline{\theta},z) = 0, \quad z > 0, \qquad (4.3a)$$

with the boundary condition

$$f(\underline{x},\underline{\theta},o) \equiv f_o(\underline{x},\underline{\theta}) = \left(\frac{k}{2\pi}\right)^2 \int_{R^2} d\underline{y} e^{ik\underline{\theta}\cdot\underline{y}} \psi_o(\underline{x} + \tfrac{1}{2}\underline{y})\psi_o(\underline{x} - \tfrac{1}{2}\underline{y}). \qquad (4.3b)$$

The collisionless, Liouville-type, kinetic equation (4.3) generalizes the classical first-order geometrical optics approximation to include coherent wave processes such as diffractive effects.[3] As a consequence, beam spreading of an electromagnetic signal, a physical phenomenon which occurs even in the absence of inhomogeneities and becomes significant at long ranges, can be accounted for. Within the framework of the kinetic formulation, this is achieved via the Wigner distribution function which allows, at each point in space, the specification of "rays" with a distribution of directions (wave numbers).

The initial value problem (4.3) can be integrated easily (cf. Appendix C) using the method of characterisitcs. The final solution is given as follows:

$$f(\underline{x},\underline{\theta},z) = f_o(\underline{x} \cos gz - \frac{1}{g}\underline{\theta} \sin gz,\ g\underline{x} \sin gz + \underline{\theta} \cos gz). \qquad (4.4)$$

Having established an expression for the Wigner distribution function [cf. Eq. (4.4)], the following physically meaningful quantities can be obtained by straight-forward integration:

(i) Intensity Density:

$$i(\underline{x},z) = \int_{R^2} d\underline{\theta} f(\underline{x},\underline{\theta},z); \tag{4.5a}$$

(ii) Total Intensity:*

$$I(z) = \int_{R^2} d\underline{x} \int_{R^2} d\underline{\theta} f(\underline{x},\underline{\theta},z); \tag{4.5b}$$

(iii) Intensity Flux Density:

$$\underline{j}(\underline{x},z) = \int_{R^2} d\underline{\theta}\,\underline{\theta} f(\underline{x},\underline{\theta},z); \tag{4.5c}$$

(iv) Power Flux Density:

$$\underline{s}(\underline{x},z) = [\int_{R^2} d\underline{\theta} f(\underline{x},\underline{\theta},z)]\hat{z} + \int_{R^2} d\underline{\theta}\,\underline{\theta} f(\underline{x},\underline{\theta},z); \tag{4.5d}$$

(v) Centroid of a Beam:

$$\underline{x}(z) = \int_{R^2} d\underline{x} \int_{R^2} d\underline{\theta}\,\underline{x} f(\underline{x},\underline{\theta},z); \tag{4.5e}$$

(vi) The Square of Standard Deviation (Spread) of a Beam:

$$\sigma_x^2(z) = \int_{R^2} d\underline{x} \int_{R^2} d\underline{\theta} [\underline{x} - \underline{x}(z)]^2 f(\underline{x},\underline{\theta},z); \tag{4.5f}$$

(vii) Mean Angle:

$$\underline{\theta}(\underline{x},z) = [\int_{R^2} d\underline{\theta}\,\underline{\theta} f(\underline{x},\underline{\theta} z)]/[\int_{R^2} d\underline{\theta} f(\underline{x},\underline{\theta},z)]; \tag{4.5g}$$

(viii) Variance of the Angle:

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* On the basis of the discussion in Sec. 3, this quantity is normalized (to unity).

$$\sigma_{\theta}{}^{2}(z) = \int_{R^2} d\underline{x} \int_{R^2} d\underline{\theta}[\underline{\theta} - \underline{\theta}(\underline{x},z)]^2 \; f(\underline{x},\underline{\theta},z). \tag{4.5h}$$

V. COHERENT GAUSSIAN BEAM

Consider the parabolic equation (3.3a), with the boundary condition

$$\psi_o(\underline{x}) = \frac{1}{\sigma_o\sqrt{\pi}}\, e^{-x^2/2\sigma_o^2}\, e^{-ikx^2/2R_o} \,. \tag{5.1}$$

The latter represents the field of a Gaussian laser beam immediately after passing through a lens (located at z=0) whose radius of curvature is equal to R_o (R_o > o for focusing). Furthermore, σ_o is the Gaussian aperture radius, and $\psi_o(\underline{x})$ is normalized to unity.

By virtue of (4.3b), the Wigner distribution function at the boundary z=0 is given by

$$f_o(\underline{x},\underline{\theta}) = \frac{k^2}{\pi^2}\, e^{-x^2/\sigma_o^2}\, e^{-k^2\sigma_o^2[\underline{\theta} + (\frac{1}{R_o})\underline{x}]^2}. \tag{5.2}$$

Introducing this expression into (4.5), we find the following initial properties of the beam:

(i) $$i(\underline{x},o) = \frac{1}{\sigma_o^2\pi}\, e^{-x^2/\sigma_o^2}; \tag{5.3a}$$

(ii) $$I(o) = 1; \tag{5.3b}$$

(iii) $$\underline{j}(\underline{x},o) = -\,\underline{x}/R_o; \tag{5.3c}$$

(iv) $$\underline{s}(\underline{x},o) = \frac{1}{\sigma_o^2\pi}\, e^{-x^2/\sigma_o^2}\,\hat{z} - \frac{1}{R_o}\,\underline{x}; \tag{5.3d}$$

(v) $$\underline{x}(o) = 0; \tag{5.3e}$$

(vi) $$\sigma_x^2(o) = \sigma_o^2; \tag{5.3f}$$

(vii) $$\underline{\theta}(\underline{x},o) = -\,\frac{\sigma_o^2\pi}{R_o}\, e^{x^2/\sigma_o^2}\,\underline{x}; \tag{5.3g}$$

(viii) $\sigma_\theta^{\,2}(o) = \dfrac{1}{k^2\sigma_o^{\,2}}$. (5.3h)

The Wigner distribution function for z > 0 can be determined by substituting the boundary condition (5.2) into (4.4). The final expression is given by

$$f(\underline{x},\underline{\theta},z) = \left(\frac{k}{\pi}\right)^2 e^{-x^2/\sigma_1^{\,2}(z)}\, e^{-k^2\sigma_2^{\,2}(z)[\underline{\theta} + \underline{x}/\sigma_3(z)]^2}\,, \tag{5.4}$$

where

$$\sigma_1^{\,2}(z) = \sigma_o^{\,2}\Big(\cos^2 gz + \frac{1}{k^2\sigma_o^{\,4}g^2}\sin^2 z + \frac{1}{R_o^{\,2}g^2}\sin^2 gz - \frac{1}{R_o g}\sin 2\,gz\Big), \tag{5.5a}$$

$$\sigma_2^{\,2}(z) = \sigma_1^{\,2}(z), \tag{5.5b}$$

$$\sigma_3^{\,-1}(z) = -\frac{1}{2}\left(\frac{\sin 2\,gz(1 - k^2\sigma_o^{\,4}g^2) + \dfrac{k^2\sigma_o^{\,4}}{R_o^{\,2}}\sin 2\,gz - \dfrac{k^2\sigma_o^{\,4}}{R_o}g\cos 2\,gz}{\sin^2 gz + k^2\sigma_o^{\,4}g^2\cos^2 gz + \dfrac{k^2\sigma_o^{\,4}}{R_o^{\,2}}\sin^2 gz - \dfrac{k^2\sigma_o^{\,4}}{R_o}g\sin 2\,gz}\right). \tag{5.5c}$$

Having established an expression for the Wigner distribution function, the physical observables discussed in Sec. IV can be determined from the definitions incorporated in (4.5):

(i) $i(\underline{x},z) = \dfrac{1}{\pi}\,\dfrac{1}{\sigma_1^{\,2}(z)}\, e^{-x^2/\sigma_1^{\,2}(z)}$; (5.6a)

$$\text{(ii)} \quad I(z) = 1; \tag{5.6b}$$

$$\text{(iii)} \quad \underline{j}(\underline{x},z) = \underline{x}/\sigma_3(z); \tag{5.6c}$$

$$\text{(iv)} \quad \underline{s}(\underline{x},z) = \frac{1}{\pi}\frac{1}{\sigma_1^{\,2}(z)}\, e^{-x^2/\sigma_1^{\,2}(z)}\,\hat{z} + \frac{1}{\sigma_3(z)}\,\underline{x}; \tag{5.6d}$$

$$\text{(v)} \quad \underline{x}(z) = 0; \tag{5.6e}$$

$$\text{(vi)} \quad \sigma_x^{\,2}(z) = \sigma_1^{\,2}(z); \tag{5.6f}$$

$$\text{(vii)} \quad \underline{\theta}(\underline{x},z) = \frac{\pi\sigma_1^{\,2}(z)}{\sigma_3(z)}\, e^{x^2/\sigma_1^{\,2}(z)}\,\underline{x}; \tag{5.6g}$$

$$\text{(viii)} \quad \sigma_\theta^{\,2}(z) = \frac{1}{k^2}\frac{1}{\sigma_1^{\,2}(z)} \quad . \tag{5.6h}$$

In the absense of a lens-like medium ($g \to o$), these results specialize to those reported recently by Tappert (cf. Ref. 8).

A detailed examination of the square of the standard deviation $\sigma_x^{\,2}(z)$ [cf. Eq. (5.6f) will be undertaken in Sec. IX.

VI. THE STOCHASTIC WIGNER DISTRIBUTION DENSITY FUNCTION

The propagation of a partially coherent beam in a deterministic lens-like medium is governed by the stochastic parabolic equation

$$\frac{i}{k}\frac{\partial}{\partial z}\psi(\underline{x},z;\alpha) = -\frac{1}{2k^2}\frac{\partial^2}{\partial \underline{x}^2}\psi(\underline{x},z;\alpha) + \frac{1}{2}g^2x^2\,\psi(\underline{x},z;\alpha), \qquad z > 0, \tag{6.1a}$$

$$\psi(\underline{x},o;\alpha) = \psi_o(\underline{x};\alpha). \tag{6.1b}$$

Because of the random boundary condition at z=0, $\psi(\underline{x},z;\alpha)$ is a complex scalar random field depending on a parameter $\alpha \in A$, A being an appropriate probability measure space.

A stochastic Wigner distribution density function can be defined by analogy to the coherent one introduced in Sec. IV. Specifically,

$$f(\underline{x},\underline{\theta},z;\alpha) = \left(\frac{k}{2\pi}\right)^2 \int_{R^2} d\underline{y}\; e^{ik\underline{\theta}\cdot\underline{y}}\,\psi^*(\underline{x}+\tfrac{1}{2}\underline{y},z;\alpha)\,\psi(\underline{x}-\tfrac{1}{2}\underline{y},z;\alpha). \tag{6.2}$$

It evolves according to the stochastic kinetic equation

$$\frac{\partial}{\partial z}f(\underline{x},\underline{\theta},z;\alpha) + \underline{\theta}\cdot\frac{\partial}{\partial \underline{x}}f(\underline{x},\underline{\theta},z;\alpha) - g^2\underline{x}\cdot\frac{\partial}{\partial \underline{\theta}}f(\underline{x},\underline{\theta},z;\alpha) = 0, \qquad z > 0, \tag{6.3a}$$

$$f(\underline{x},\underline{\theta},o;\alpha) = f_o(\underline{x},\underline{\theta};\alpha) \tag{6.3b}$$

Taking the ensemble average of both sides of (6.3), one has the following equation for the mean Wigner distribution function:

$$\frac{\partial}{\partial z}\langle f(\underline{x},\underline{\theta},z;\alpha)\rangle + \underline{\theta}\cdot\frac{\partial}{\partial \underline{x}}\langle f(\underline{x},\underline{\theta},z;\alpha)\rangle - g^2\underline{x}\cdot\frac{\partial}{\partial \underline{\theta}}\langle f(\underline{x},\underline{\theta},z;\alpha)\rangle$$

$$= 0, \qquad z > 0, \tag{6.4a}$$

$$\langle f(\underline{x},\underline{\theta},o;\alpha)\rangle = \langle f_o(\underline{x},\underline{\theta};\alpha)\rangle \tag{6.4b}$$

This averaged (or transport) equation can be formally integrated as follows:

$$\langle f(\underline{x},\underline{\theta},z;\alpha)\rangle = \langle f_o(\underline{x}\cos gz - \frac{1}{g}\underline{\theta}\sin gz,\ \underline{x}g\sin gz + \underline{\theta}\cos gz;\alpha)\rangle . \tag{6.5}$$

Once an expression has been established for the mean Wigner distribution function, the following physically meaningful averaged quantities can be obtained by direct integration:

(i) Mutual Coherence Function:

$$\langle \psi^*(\underline{x} + \frac{1}{2}\underline{y},z;\alpha)\psi(\underline{x} - \frac{1}{2}\underline{y},z;\alpha)\rangle = \int_{R^2} d\underline{\theta}\, e^{-ik\underline{\theta}\cdot\underline{y}} \langle f(\underline{x},\underline{\theta},z;\alpha)\rangle ; \tag{6.6a}$$

(ii) Mean Intensity Density:

$$\langle i(\underline{x},z;\alpha)\rangle = \int_{R^2} d\underline{\theta}\ \langle f(\underline{x},\underline{\theta},z;\alpha)\rangle ; \tag{6.6b}$$

(iii) Total Mean Intensity:*

$$\langle I(z;\alpha)\rangle = \int_{R^2} d\underline{x} \int_{R^2} d\underline{\theta}\ \langle f(\underline{x},\underline{\theta},z;\alpha)\rangle ; \tag{6.6c}$$

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* This quantity is conserved, i.e., $\langle I(z;\alpha)\rangle = \langle I(o;\alpha)\rangle$. Furthermore, it is assumed to be normalized to unity.

(iv) Mean Intensity Flux Density:

$$\langle \underline{j}(\underline{x},z;\alpha)\rangle = \int_{R^2} d\underline{\theta}\ \underline{\theta}\ \langle f(\underline{x},\underline{\theta},z;\alpha)\rangle \ ; \tag{6.6d}$$

(v) Mean Power Flux Density:

$$\langle \underline{s}(\underline{x},z;\alpha)\rangle = [\int_{R^2} d\underline{\theta}\ \langle f(\underline{x},\underline{\theta},z;\alpha)\rangle\ ]\hat{z} + \int_{R^2} d\underline{\theta}\ \underline{\theta}\langle f(\underline{x},\underline{\theta},z;\alpha)\rangle \ ; \tag{6.6e}$$

(vi) Mean Centroid of a Beam:

$$\langle \underline{x}(z;\alpha)\rangle = \int_{R^2} d\underline{x} \int_{R^2} d\underline{\theta}\ \underline{x}\ \langle f(\underline{x},\underline{\theta},z;\alpha)\rangle \ ; \tag{6.6f}$$

(vii) The Square of Standard Deviation (Spread) of a Beam:

$$\sigma_x^{\,2}(z) = \int_{R^2} d\underline{x} \int_{R^2} d\underline{\theta}[\underline{x} - \langle \underline{x}(z;\alpha)\rangle\ ]^2 \langle f(\underline{x},\underline{\theta},z;\alpha)\rangle \ ; \tag{6.6g}$$

(viii) Mean Angle:

$$\langle \underline{\theta}(\underline{x},z;\alpha)\rangle = [\int_{R^2} d\underline{\theta}\ \underline{\theta}\langle f(\underline{x},\underline{\theta},z;\alpha)\rangle\ ]/[\int_{R^2} d\underline{\theta}\ \langle f(\underline{x},\underline{\theta},z;\alpha)\rangle\ ]; \tag{6.6h}$$

(ix) Variance of the Angle:

$$\sigma_\theta^{\,2}(z) = \int_{R^2} d\underline{x} \int_{R^2} d\underline{\theta}[\underline{\theta} - \langle \underline{\theta}(\underline{x},z;\alpha)\rangle\ ]^2 \langle f(\underline{x},\underline{\theta},z;\alpha)\rangle \ . \tag{6.6i}$$

VII. PARTIALLY COHERENT GAUSSIAN BEAM

In actual experiments, it is seldom possible to focus a laser beam to the diffraction-limited focal spots (cf. Sec. IX). To a certain extent, this is due to the various imperfections in the optical system (focusing lens). In this section, we shall modify the results in Sec. V in order to account for the degradation of focusing capability of a laser beam caused by imperfections in the focusing lens. Our work in this section is a generalization of that reported recently by Tappert (cf. Ref. 8) in the sense that propagation takes place in the presence of a parabolic profile of the index of refraction.

Consider the stochastic parabolic equation (6.1a), with the boundary condition

$$\psi_o(\underline{x};\alpha) = \frac{1}{\sigma_o\sqrt{\pi}} e^{-x^2/2\sigma_o^2} e^{-ikx^2/2R_o} e^{i\phi(\underline{x};\alpha)}. \tag{7.1}$$

This represents the wave function of a Gaussian laser beam immediately after passing through a lens (located at z=0), with a radius of curvature R_o ($R_o > 0$ for focusing).

Imperfections in the lens are accounted for by means of the phase $\phi(\underline{x};\alpha)$ which, for the sake of simplicity, is assumed to be a zero-mean, homogeneous, isotropic, real, Gaussian random process. It is, therefore, fully characterized by the correlation function:

$$\Gamma(|\underline{x} - \underline{x}'|) = \langle\phi(\underline{x};\alpha)\phi(\underline{x}';\alpha)\rangle / \langle\phi^2(\underline{x};\alpha)\rangle . \tag{7.2}$$

Let $y = |\underline{x} - \underline{x}'|$. Expanding $\Gamma(y)$ in a Taylor series, one obtains

$$\Gamma(y) \simeq 1 - \frac{1}{2}\frac{1}{L^2} y^2 \ . \tag{7.3}$$

Use has been made here of the properties $\Gamma(o) = 1$, $\Gamma(-y) = \Gamma(y)$. Furthermore, a scale length of the phase front fluctuations has been defined by the relation $L^{-2} = -\Gamma''(o)$.

Within the framework of the assumptions made so far, the mean Wigner distribution function at the boundary z=0, is given by

$$\left\langle f_o(\underline{x},\underline{\theta};\alpha)\right\rangle = \frac{k^2D^2}{\pi^2} e^{-x^2/\sigma_o^2} e^{-k^2D^2\sigma_o^2[\underline{\theta} + (\frac{1}{R_o})\underline{x}]^2}, \tag{7.4}$$

where

$$D = [1 + 2\ \sigma_o^2 \frac{1}{L^2} \left\langle\phi^2(\underline{x};\alpha)\right\rangle]^{-\frac{1}{2}} \ . \tag{7.5}$$

The Wigner distribution function for z > 0 can be found by substituting the boundary condition (7.4) into (6.5). The final expression has the form [6]

$$\left\langle f(\underline{x},\underline{\theta},z;\alpha)\right\rangle = \frac{k^2D^2}{\pi^2} e^{-x^2/\Sigma_1{}^2(z)} e^{-k^2D^2\Sigma_2{}^2(z)[\underline{\theta} + \Sigma_3{}^{-1}(z)\underline{x}]^2}, \tag{7.6}$$

where

$$\Sigma_1^2(z) = \sigma_o{}^2(\cos^2 gz + \frac{1}{k^2D^2\sigma_o{}^4 g^2} \sin^2 gz + \frac{1}{R_o^2 g^2} \sin^2 gz$$

$$- \frac{1}{R_o g} \sin 2\ gz), \tag{7.7a}$$

$$\Sigma_2^2(z) = \Sigma_1^2(z), \tag{7.7b}$$

$$\Sigma_3^{-1}(z) = -\frac{1}{2}\left\{\frac{\sin 2\,gz(1 - k^2D^2\sigma_o^4g^2) + \dfrac{k^2D^2\sigma_o^4}{R_o}\sin 2\,gz - \dfrac{k^2D^2\sigma_o^4}{R_o}\,g\cos 2\,gz}{\sin^2 gz + k^2D^2\sigma_o^4g^2\cos^2 gz + \dfrac{k^2D^2\sigma_o^4}{R_o^2}\sin^2 gz - \dfrac{k^2D^2\sigma_o^4}{R_o}\,g\sin 2\,gz}\right\}. \tag{7.7c}$$

It should be noted that Eqs. (7.7a) - (7.7c) reduce to Eqs. (5.5a) - (5.5c) provided that kD in the former is replaced by k. Thus, the presence of aberrations in the focusing length is accounted for by means of an effective wave number $k_{eff} = kD$.

The averaged physical observables listed in (6.6a) - (6.6i) can be determined by direct integration of the mean Wigner distribution function given in (7.6). The final results are given as follows:

(i) $\langle\psi^*(\underline{x} + \frac{1}{2}\,\underline{y}, z;\alpha)\psi(\underline{x} - \frac{1}{2}\,\underline{y}, z;\alpha)\rangle$

$$= \frac{1}{\pi}\frac{1}{\Sigma_1^2(z)}\,e^{-x^2/\Sigma_1^2(z)}\,e^{-y^2/4D^2\Sigma_1^2(z)}\,e^{ik\underline{x}\cdot\underline{y}/\Sigma_3(z)}; \tag{7.8a}$$

(ii) $$\langle i(\underline{x}, z;\alpha)\rangle = \frac{1}{\pi}\frac{1}{\Sigma_1^2(z)}\,e^{-x^2/\Sigma_1^2(z)}; \tag{7.8b}$$

(iii) $$\langle I(z;\alpha)\rangle = 1\ ; \tag{7.8c}$$

(iv) $$\langle \underline{j}(\underline{x}, z;\alpha)\rangle = \underline{x}/\Sigma_3(z)\ ; \tag{7.8d}$$

(v) $$\langle \underline{s}(\underline{x}, z;\alpha)\rangle = \frac{1}{\pi}\frac{1}{\Sigma_1^2(z)}\,e^{-x^2/\Sigma_1^2(z)}\,\hat{z} + \frac{1}{\Sigma_3(z)}\,\underline{x}\ ; \tag{7.8e}$$

$$\text{(vi)} \quad \langle \underline{x}(z;\alpha) \rangle = 0 \; ; \tag{7.8f}$$

$$\text{(vii)} \quad \sigma_x{}^2(z) = \Sigma_1^2(z) \; ; \tag{7.8g}$$

$$\text{(viii)} \quad \langle \underline{\theta}(\underline{x},z;\alpha) \rangle = \pi \frac{\Sigma_1^2(z)}{\Sigma_3(z)} e^{x^2/\Sigma_1^2(z)} \, \underline{x} \; ; \tag{7.8h}$$

$$\text{(ix)} \quad \sigma_\theta{}^2(z) = \frac{1}{k^2 D^2} \frac{1}{\Sigma_1^2(z)} \quad . \tag{7.8i}$$

Detailed numerical results pertaining to the standard deviation $\sigma_x{}^2(z)$ will be presented and discussed in Sec. IX.

VIII. DEGREE OF COHERENCE

Given a wave function $\psi(\underline{x},z;\alpha)$, the degree of coherence, $D(z)$, is defined as follows:

$$D^2(z) = \int_{R^2} d\underline{x}_2 \int_{R^2} d\underline{x}_1 |< \psi^*(\underline{x}_2,z;\alpha)\psi(\underline{x}_1,z;\alpha)>|^2 \ . \tag{8.1}$$

This quantity is intimately linked with the irreversible loss of information (coherence) due to the statistical fluctuations.

The degree of coherence is characterized by the property

$$D^2(z) \leq 1 \ , \tag{8.2}$$

the equality holding for the case of a purely coherent beam. To show this we note that in the absence of random fluctuations (8.1) reduces to

$$\begin{aligned} D^2(z) &= \int_{R^2} d\underline{x}_2 \int_{R^2} d\underline{x}_1 |\psi^*(\underline{x}_2,z)\psi(\underline{x}_1,z)|^2 \\ &= [\int_{R^2} d\underline{x}_2 |\psi(\underline{x}_2,z)|^2][\int_{R^2} d\underline{x}_1 |\psi(\underline{x}_1,z)|^2] = 1 \ , \end{aligned} \tag{8.3}$$

the final equality following because of the conservation of the total intensity.

To prove the inequality $D^2(z) < 1$, which holds for a partially coherent beam, we use the Cauchy-Schwartz inequality [19], viz.,

$$|<\psi^*(\underline{x}_2,z;\alpha)\psi(\underline{x}_1,z;\alpha)>|^2 \leq <|\psi(\underline{x}_2,z;\alpha)|^2><|\psi(\underline{x}_1,z;\alpha)|^2> \ , \tag{8.4}$$

in conjunction with (8.1). We have, then,

$$D^2(z) \leq [\int_{R^2} d\underline{x}_2 \, <|\psi(\underline{x}_2,z;\alpha)|^2>][\int_{R^2} d\underline{x}_1 \, <|\psi(\underline{x}_1,z;\alpha)|^2>] = 1, \tag{8.5}$$

the last equality following from the fact that the total mean intensity is conserved and is normalized to unity (cf. Eq. (7.8c)].

We conclude this section with an important observation: the degree of coherence D(z) is exactly equal to the parameter D defined in (7.5), i.e.,

$$D(z) = [1 + 2\,\sigma_o^2 \frac{1}{L^2} \, <\phi^2(\underline{x};\alpha)>]^{-\frac{1}{2}} . \tag{8.6}$$

This can be shown by recasting (8.1) in terms of the mean Wigner distribution function:

$$D^2(z) = (\frac{2\pi}{k})^2 \int_{R^2} d\underline{x} \int_{R^2} d\underline{\theta} \, <f(\underline{x},\underline{\theta},z;\alpha)>^2 . \tag{8.7}$$

Introducing the solution (7.6) for $<f(\underline{x},\underline{\theta},z;\alpha)>$ into (8.7) and carrying out the integrations, we find that the degree of coherence is, indeed, given by (8.6).

We note that the right-hand side of (8.6) is independent of z. This means that for the particular problem under consideration here the degree of coherence is conserved, i.e, D(z) = D(o). Finally, it is clear from (8.6) that the degree of coherence decreases as the variance of the random phase fluctuations becomes large or the scale length L of the fluctuations becomes small compared to the initial beam aperture σ_o. For a purely incoherent beam the degree of coherence approaches zero.

IX. NUMERICAL RESULTS AND DISCUSSION

To gain some insight into the nature of the effects of focusing lens aberrations on the propagation of a Gaussian beam in a parabolic medium, we shall present in this section numerical results exhibiting the spreading of the beam.

Consider the general expression (7.8g) for the square of the standard deviation:

$$\sigma_x^2(z) = \sigma_o^2\left(\cos^2 gz + \frac{1}{k^2 D^2 \sigma_o^4 g^2}\sin^2 gz + \frac{1}{R_o^2 g^2}\sin^2 gz - \frac{1}{R_o g}\sin 2\,gz\right) . \tag{9.1}$$

In terms of the dimensionless parameters

$$\zeta = \frac{z}{k\sigma_o^2}\ ,\quad \omega = gk\sigma_o^2,\quad m = \frac{k\sigma_o^2}{R_o}\ , \tag{9.2}$$

(9.1) becomes

$$\sigma_x^2(\zeta)/\sigma_o^2 = \cos^2 \omega\zeta + \frac{1}{\omega^2 D^2}\sin^2 \omega\zeta + \frac{m^2}{\omega^2}\sin^2 \omega\zeta - \frac{m}{\omega}\sin 2\,\omega\zeta. \tag{9.3}$$

Considered as a function of the dimensionless quantity ζ, the normalized square of the standard deviation $\sigma_x^2(\zeta)/\sigma_o^2$ depends parametrically on the dimensionless quantities ω, m and D. The first parameter arises solely from the presense of a lens-like medium (ω=0 in the absence of inhomogeneities); the second is due to the presence of a focusing lens at z=0 (m=0 in the absence of a lens); finally, the last parameter accounts for lens aberrations (D=1 for a perfect lens).

The variation of $\sigma_x^2(\zeta)/\sigma_o^2$ with ζ is shown in Figs. (9-1) - (9-6).

The monotonic increase of the standard deviation in the absence of a profile and lens shown in Fig. 9-1 is due to diffraction. A perfect lens with a positive radius of curvature results in the focusing of the beam. (The diffraction-limited spot size depends on the focusing strength of the lens.) Eventually, however, diffraction predominates and the beam spreads monotonically. This is shown in Fig. 9-2. Statistical fluctuations in phase due to aberrations in the lens act as a defocusing mechanism (cf. Fig. 9-3). Figures 9-4a and 9-4b exhibit the oscillatory evolution of $\sigma_x^2(\zeta)/\sigma_o^2$ in the presence of a parabolic profile, without a focusing lens. It should be noted that $\sigma_x^2(\zeta)/\sigma_o^2 \geq 1$ or $\sigma_x^2(\zeta)/\sigma_o^2 \leq 1$ according as $\omega \leq 1$ or $\omega \geq 1$. These relations do not hold if a focusing lens is present. For example, if $\omega = 0.5$, $\sigma_x^2(\zeta)/\sigma_o^2$ evolves in an oscillatory manner; however it can assume values less than unity (cf. Fig. 9-5a). The degradation of the focusing effect due to imperfections in the lens, with the allowance for a parabolic profile, is shown in Figs. 9-6a and 9-6b.

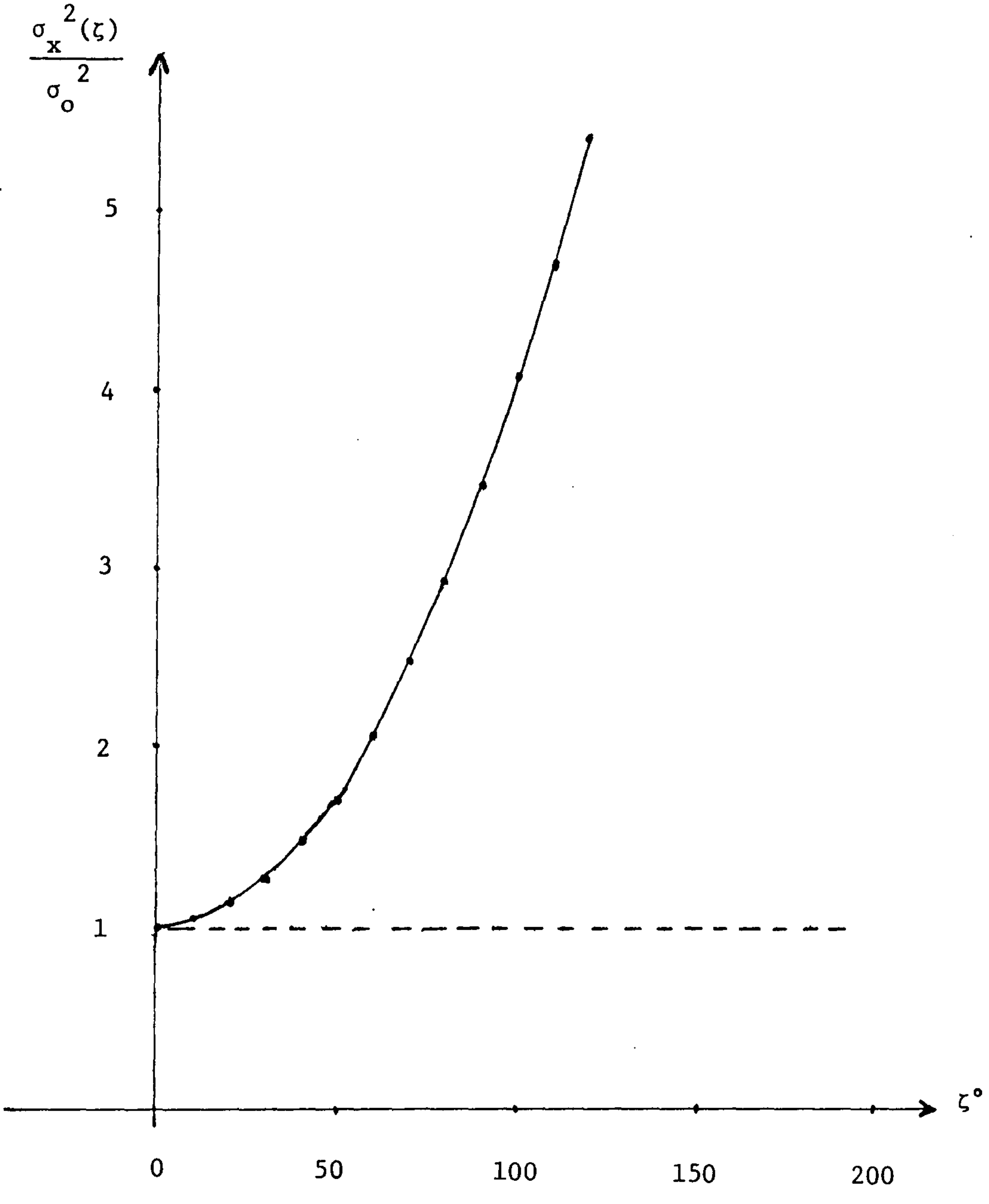

Figure 9-1: No profile or lens (ω=0, m=0).

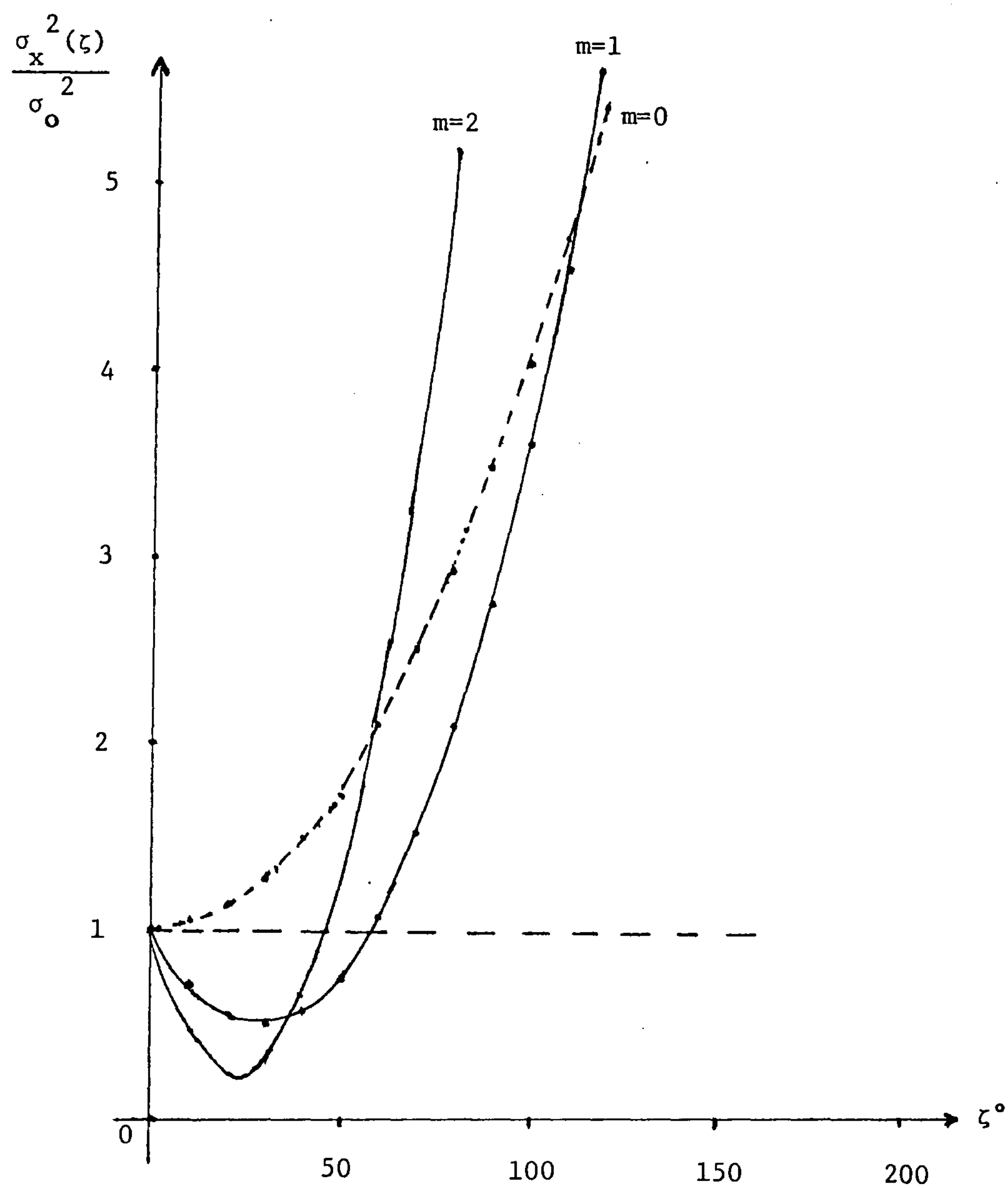

Figure 9-2: No profile; perfect lens (ω=0, D=1).

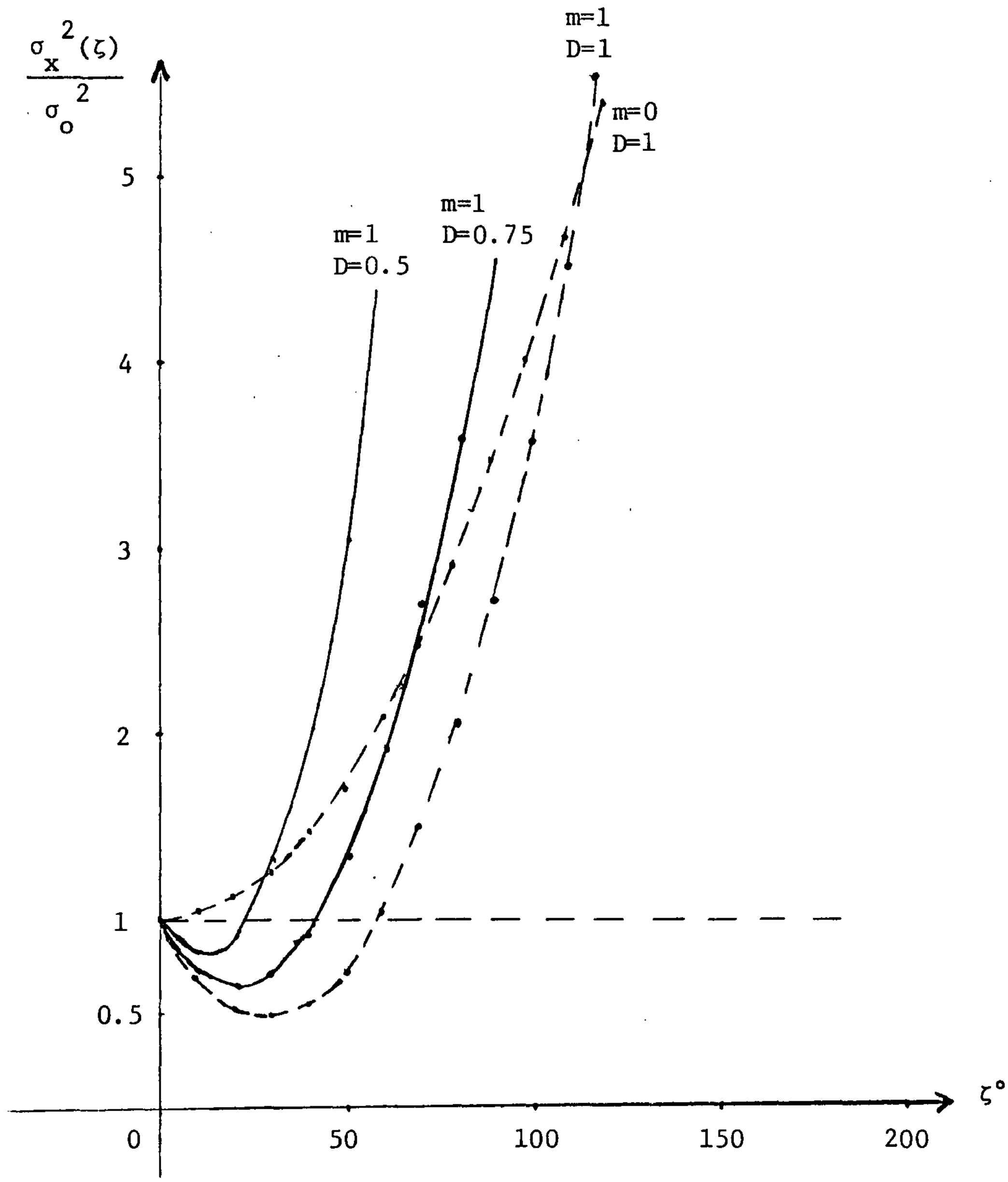

Figure 9-3: No profile; imperfect lens (ω=0).

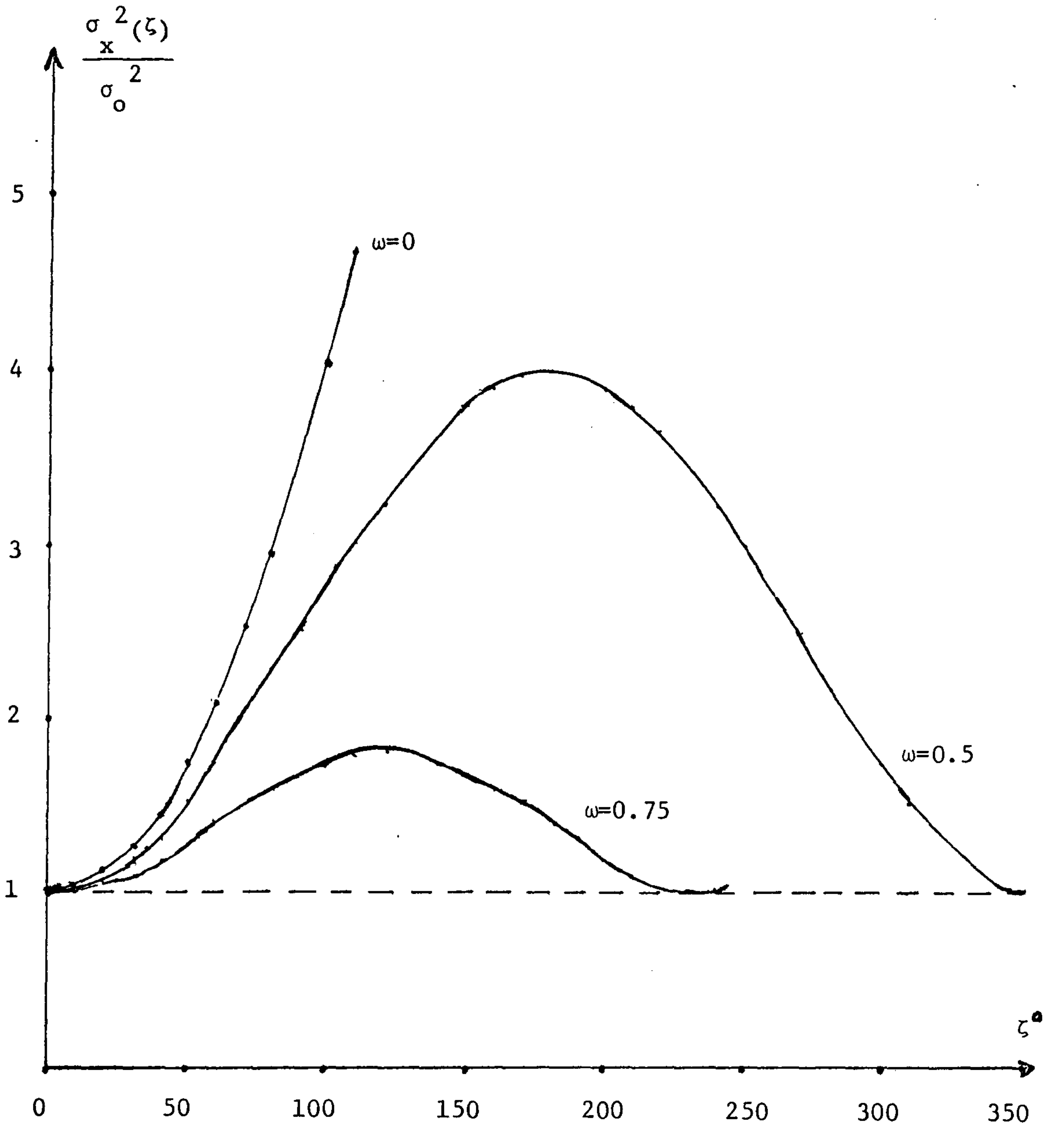

Figure 9-4a: Profile; no lens (m=0).

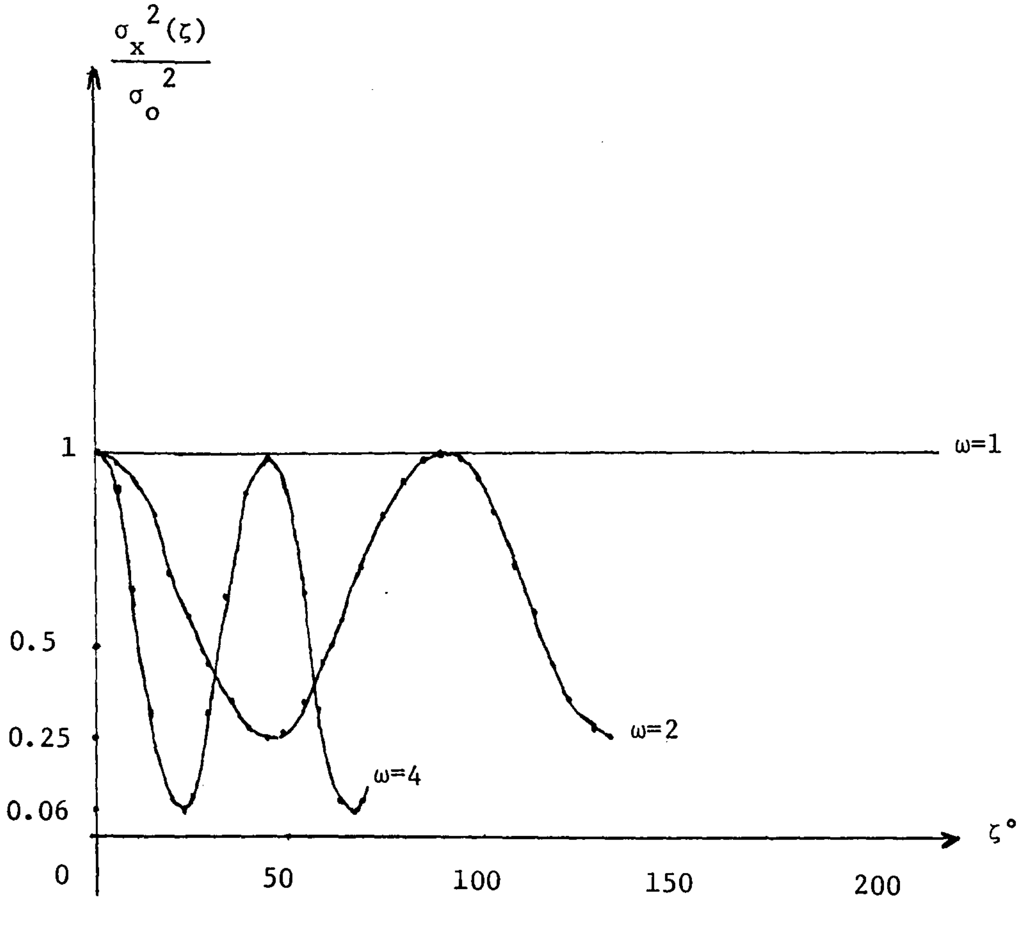

Figure 9-4b: Profile; no lens (m=0)

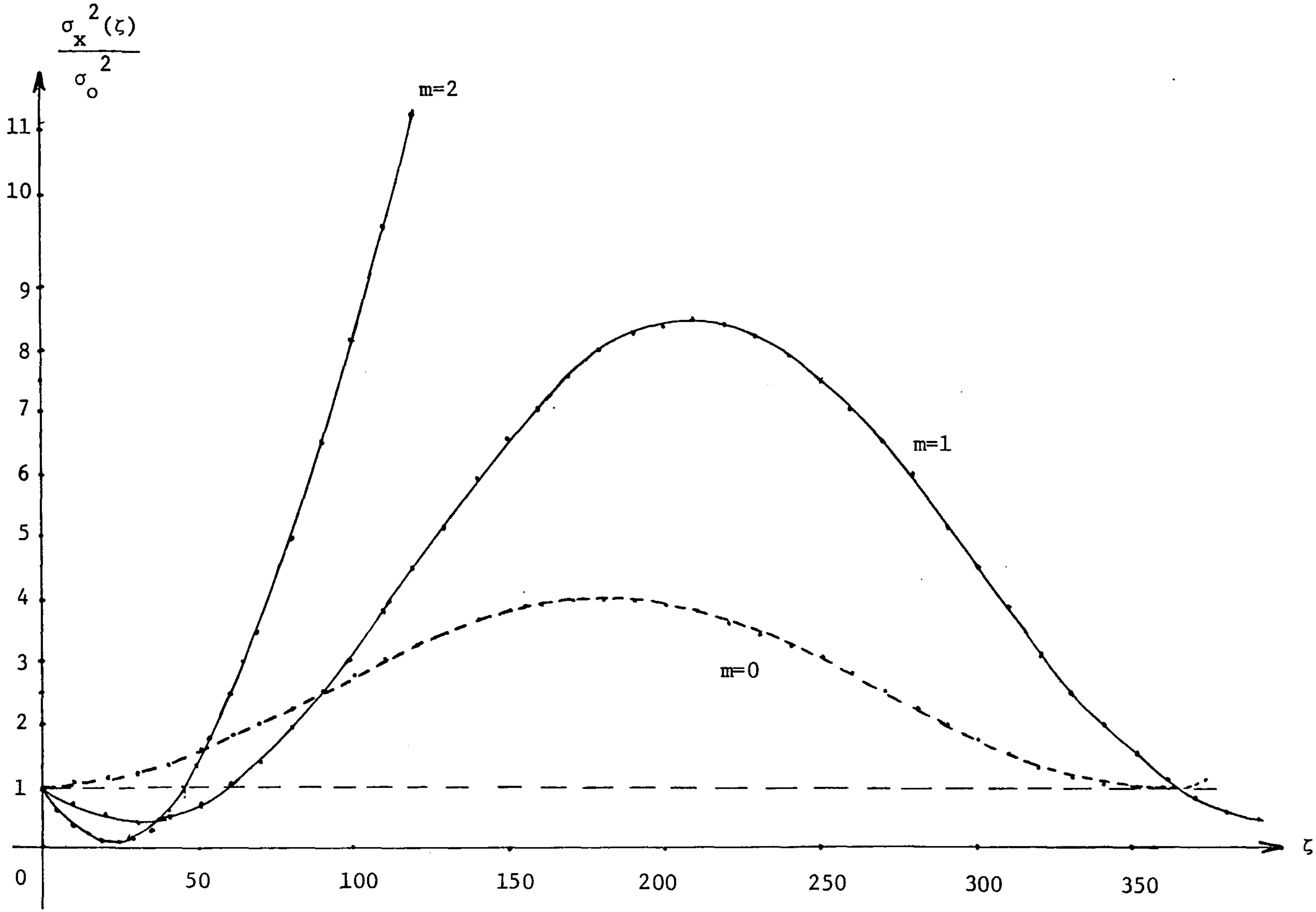

Figure 9-5a: Profile; perfect lens (ω=0.5; D=1)

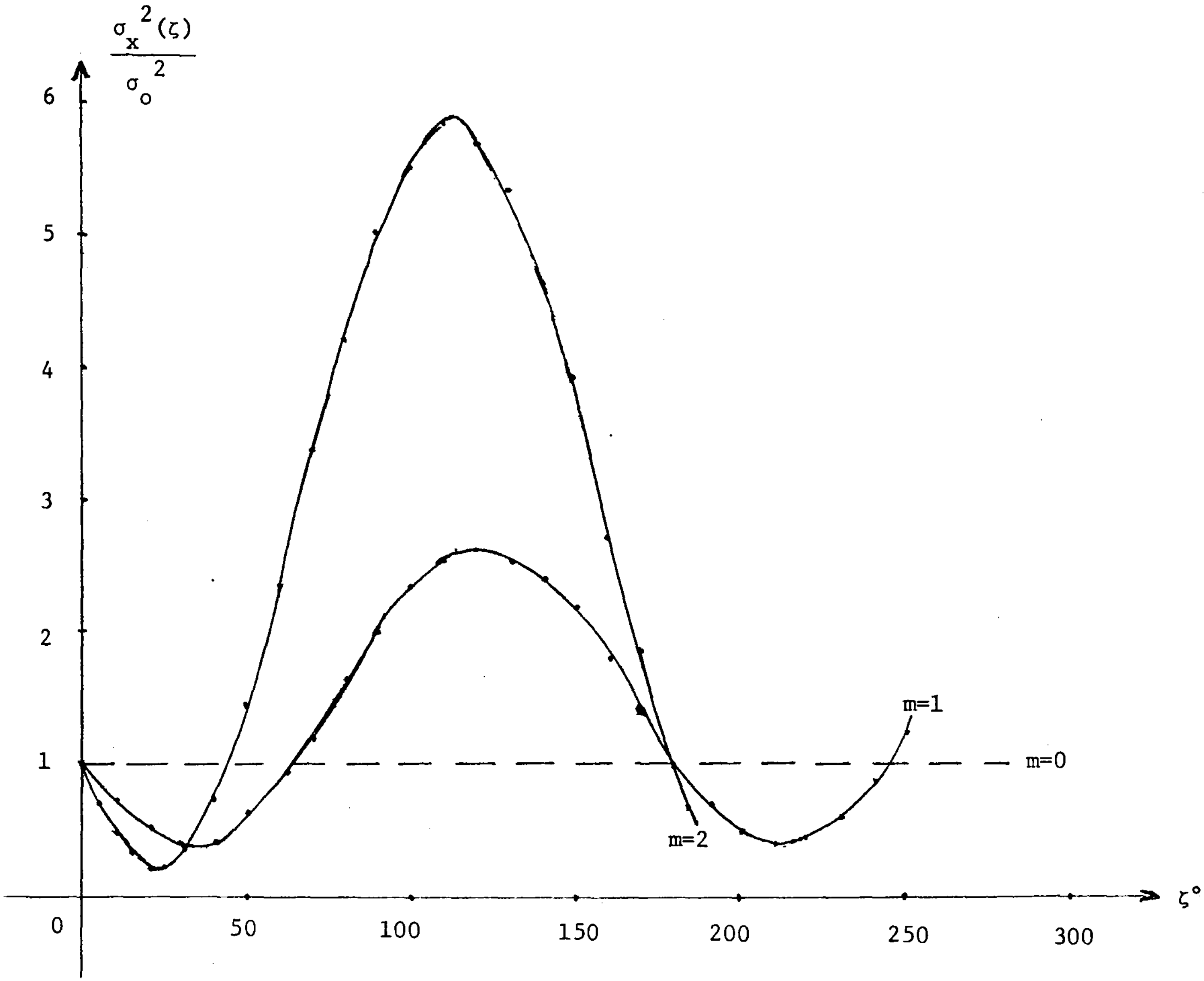

Figure 9-5b: Profile; perfect lens (ω=1; D=1)

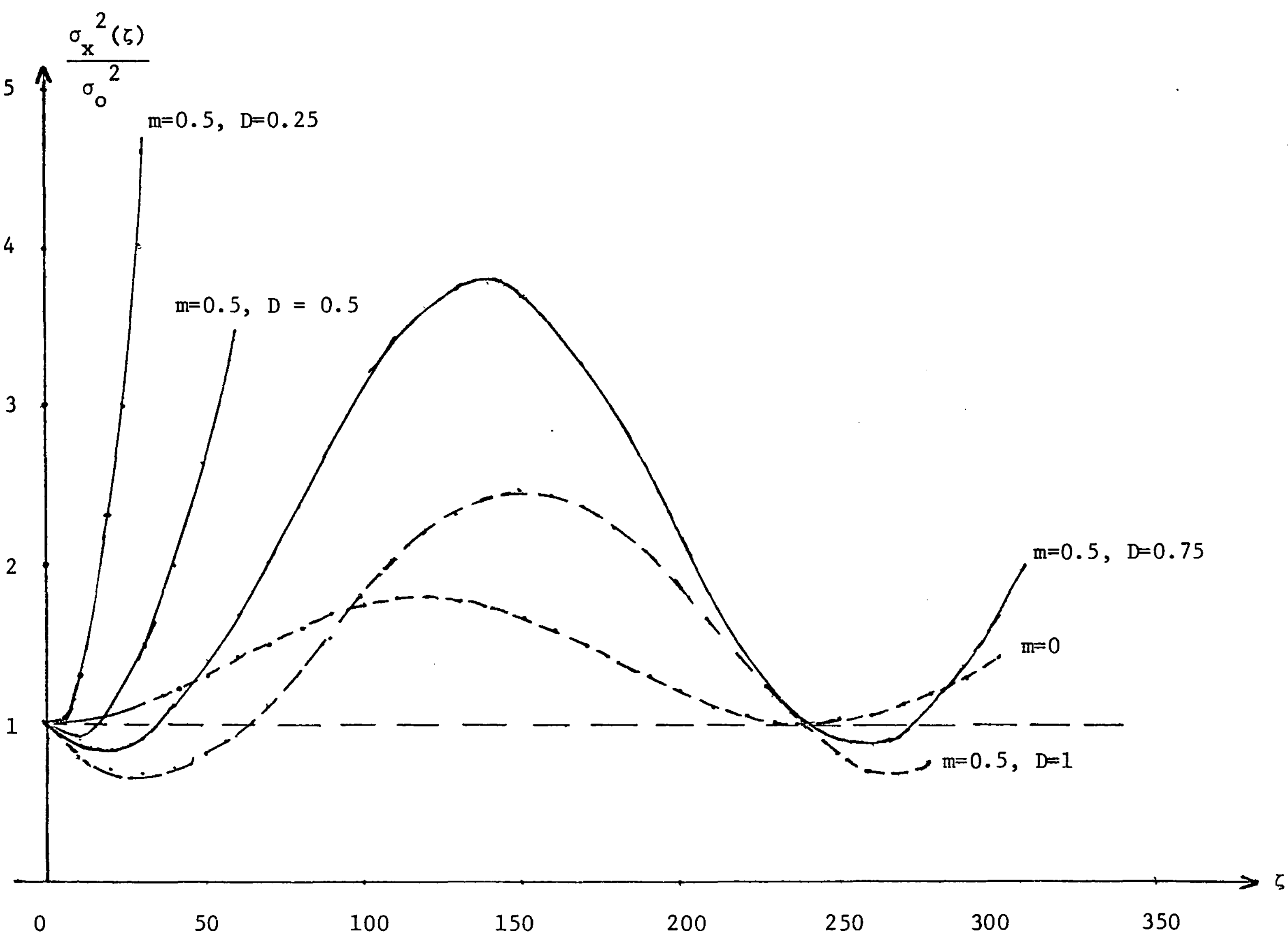

Figure 9-6a: Profile; imperfect lens (ω=0.75)

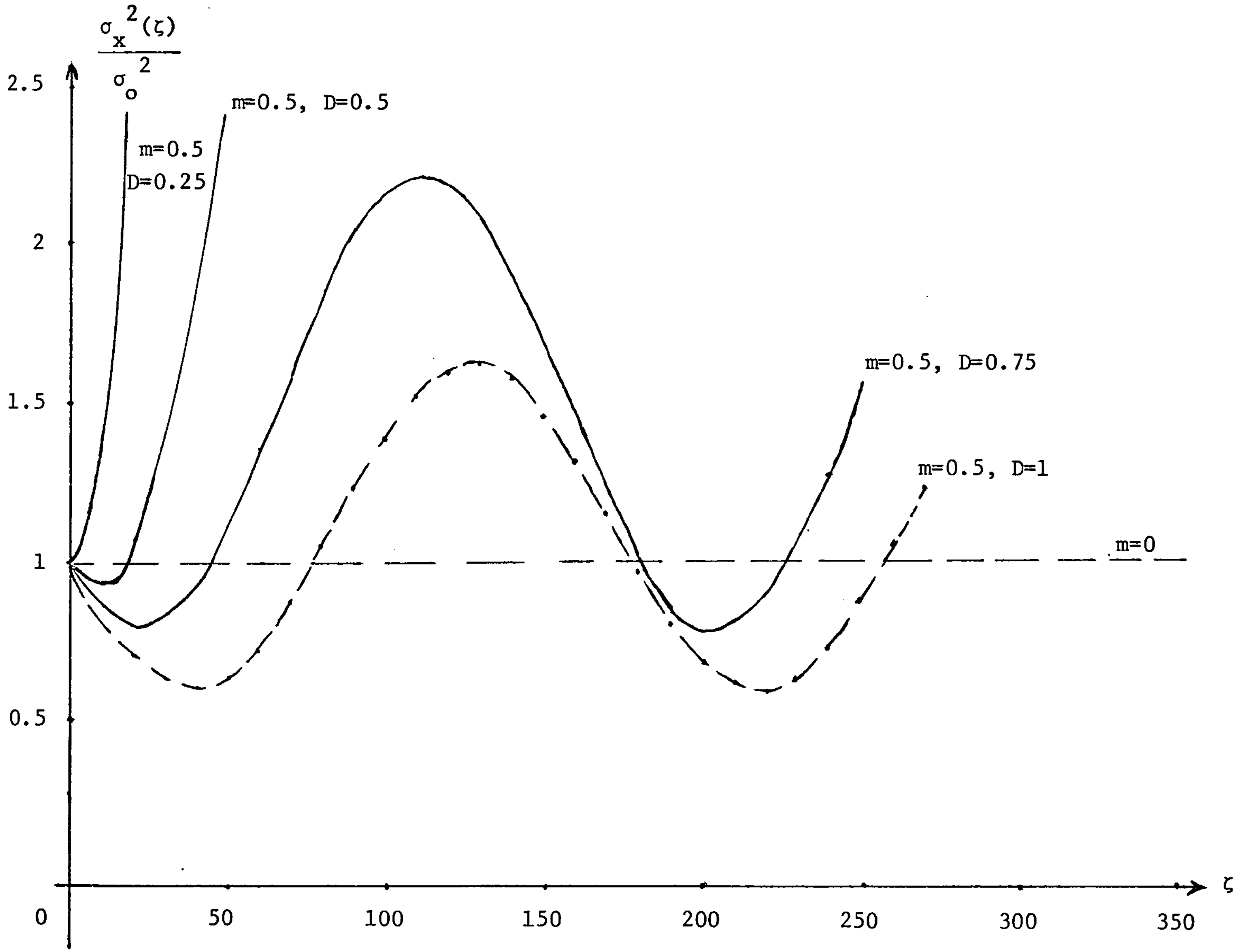

Figure 9-6b: Profile; imperfect lens (ω=1)

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APPENDIX A

On the basis of the Schwartz inequality,

$$|f(\underline{x},\underline{\theta},z)|^2 \leq (\frac{k}{2\pi})^4 [\int_{R^2} d\underline{y} |\psi^*(\underline{x} + \frac{1}{2}\underline{y},z)|^2]$$
$$\cdot [\int_{R^2} d\underline{y} |\psi(\underline{x} - \frac{1}{2}\underline{y},z)|^2] \ . \qquad \text{(A-1)}$$

Consider the integral

$$I_+ \equiv \int_{R^2} d\underline{y} |\psi^*(\underline{x} + \frac{1}{2}\underline{y},z)|^2 = 4 \int_{R^2} d\underline{x} |\psi(\underline{x},z)|^2. \qquad \text{(A-2)}$$

The total intensity, however, is conserved and normalized to unity (cf. Sec. III). Therefore,

$$I_+ = 4 \ . \qquad \text{(A-3)}$$

Similarly,

$$I_- \equiv \int_{R^2} d\underline{y} |\psi(\underline{x} - \frac{1}{2}\underline{y},z)|^2 = 4 \ . \qquad \text{(A-4)}$$

Using (A-3) and (A-4) in conjunction with (A-1) we obtain, finally,

$$|f(\underline{x},\underline{\theta},z)|^2 \leq (\frac{k}{\pi})^4 \qquad \text{(A-5)}$$

or

$$|f(\underline{x},\underline{\theta},z)| \leq (\frac{k}{\pi})^2 \ . \qquad \text{(A-6)}$$

APPENDIX B

Differentiating both sides of (4.1) with respect to z and using the parabolic equation (3.3a), it follows that the two-(transverse) point field density function obeys the equation

$$\frac{i}{k}\frac{\partial}{\partial z}\rho(\underline{x}_2,\underline{x}_1,z) = [\frac{1}{2k^2}(\frac{\partial^2}{\partial \underline{x}_2{}^2} - \frac{\partial^2}{\partial \underline{x}_1{}^2}) + \frac{1}{2}g^2(\underline{x}_1{}^2 - \underline{x}_2{}^2)]$$

$$\cdot\ \rho(\underline{x}_2,\underline{x}_1,z). \tag{B-1}$$

We resort next to the "center-of-mass" and "difference" coordinates

$$\underline{x} = \frac{1}{2}(\underline{x}_2 + \underline{x}_1),\ \underline{y} = \underline{x}_2 - \underline{x}_1\ . \tag{B-2}$$

In terms of these new variables, (B-1) becomes

$$\frac{\partial}{\partial z}\rho(\underline{x} + \frac{1}{2}\underline{y},\ \underline{x} - \frac{1}{2}\underline{y},z) \equiv -\ ik[\frac{1}{k^2}(\frac{\partial}{\partial \underline{x}}\cdot\frac{\partial}{\partial \underline{y}}) - g^2(\underline{x}\cdot\underline{y})]$$

$$\cdot\ \rho(\underline{x} + \frac{1}{2}\underline{y},\ \underline{x} - \frac{1}{2}\underline{y},z)\ . \tag{B-3}$$

We differentiate next both sides of (4.2) with respect to z and use (B-3) to obtain

$$\frac{\partial}{\partial z}f(\underline{x},\underline{\theta},z) = -\ ik(\frac{k}{2\pi})^2\int_{R^2} d\underline{y}\ e^{ik\underline{\theta}\cdot\underline{y}}\ [\frac{1}{k^2}(\frac{\partial}{\partial \underline{x}}\cdot\frac{\partial}{\partial \underline{y}}) - g^2(\underline{x}\cdot\underline{y})]$$

$$\cdot\ \rho(\underline{x} + \frac{1}{2}\underline{y},\ \underline{x} - \frac{1}{2}\underline{y},z)\ . \tag{B-4}$$

The field density function ρ in the integrand can be expressed in terms of the Wigner distribution function f by means of the relationship

$$\rho(\underline{x} + \frac{1}{2}\,\underline{y},\ \underline{x} - \frac{1}{2}\,\underline{y}, z) = \int_{R^2} d\underline{\theta}'\, e^{-ik\underline{\theta}'\cdot\underline{y}}\, f(\underline{x},\underline{\theta}',z)\ . \qquad \text{(B-5)}$$

Introducing (B-5) into (B-4), we have

$$\frac{\partial}{\partial z} f(\underline{x},\underline{\theta},z) = -\, ik(\frac{k}{2\pi})^2 \int_{R^2} d\underline{y}\ e^{ik\underline{\theta}\cdot\underline{y}}[\frac{1}{k^2}(\frac{\partial}{\partial\underline{x}} \cdot \frac{\partial}{\partial\underline{y}}) - g^2(\underline{x}\cdot\underline{y})]$$

$$\cdot\, [\int_{R^2} d\theta'\ e^{-k\underline{\theta}'\cdot\underline{y}}\, f(\underline{x},\underline{\theta}',z)]$$

$$\equiv I_1 + I_2\ . \qquad \text{(B-6)}$$

Here,

$$I_1 = -\, ik(\frac{k}{2\pi})^2 \frac{1}{k^2}\int_{R^2} d\underline{y}\ e^{ik\underline{\theta}\cdot\underline{y}}\, (\frac{\partial}{\partial\underline{x}} \cdot \frac{\partial}{\partial\underline{y}})[\int_{R^2} d\underline{\theta}'\ e^{-ik\underline{\theta}'\cdot\underline{y}} f(\underline{x},\underline{\theta}',z)]$$

$$= (-ik)^2(\frac{k}{2\pi})^2\, \frac{1}{k^2}\frac{\partial}{\partial\underline{x}} \cdot [\int_{R^2} d\underline{y} \int_{R^2} d\underline{\theta}'\ e^{ik(\underline{\theta}-\underline{\theta}')\cdot\underline{y}}\underline{\theta}' f(\underline{x},\underline{\theta}',z)]$$

$$= (-ik)^2\, \frac{1}{k^2}\frac{\partial}{\partial\underline{x}} \cdot [\int_{R^2} d\underline{\theta}'\ \delta(\underline{\theta} - \underline{\theta}')\underline{\theta}' f(\underline{x},\underline{\theta}',z)]$$

$$= -\,\underline{\theta} \cdot \frac{\partial}{\partial\underline{x}} f(\underline{x},\underline{\theta},z) \qquad \text{(B-7)}$$

and

$$I_2 = ik(\frac{k}{2\pi})^2\, g^2\underline{x} \cdot [\int_{R^2} d\underline{y} \int_{R^2} d\underline{\theta}'\ \underline{y}\ e^{ik(\underline{\theta} - \underline{\theta}')\cdot\underline{y}}\, f(\underline{x},\underline{\theta}',z)]$$

$$= (\frac{k}{2\pi})^2\, g^2\underline{x} \cdot \frac{\partial}{\partial\underline{\theta}}[\int_{R^2} d\underline{y} \int_{R^2} d\underline{\theta}'\ e^{ik(\underline{\theta} - \underline{\theta}')\cdot\underline{y}}\, f(\underline{x},\underline{\theta}',z)]$$

$$= g^2\underline{x} \cdot \frac{\partial}{\partial\underline{\theta}}[\int_{R^2} d\underline{\theta}'\ \delta(\underline{\theta} - \underline{\theta}')f(\underline{x},\underline{\theta}'z)]$$

$$= g^2\underline{x} \cdot \frac{\partial}{\partial\underline{\theta}}\, f(\underline{x},\underline{\theta},z)\ . \qquad \text{(B-8)}$$

Using (B-7) and (B-8) in conjunction with (B-6), we obtain, finally, the kinetic equation

$$\frac{\partial}{\partial z} f(\underline{x},\underline{\theta},z) + \underline{\theta} \cdot \frac{\partial}{\partial \underline{x}} f(\underline{x},\underline{\theta},z) - g^2 \underline{x} \cdot \frac{\partial}{\partial \underline{\theta}} f(\underline{x},\underline{\theta},z) = 0. \tag{B-9}$$

APPENDIX C

We shall integrate here the kinetic equation (4.3a), viz.,

$$\frac{\partial}{\partial z} f(\underline{x},\underline{\theta},z) + \underline{\theta} \cdot \frac{\partial}{\partial \underline{x}} f(\underline{x},\underline{\theta},z) - g^2 \underline{x} \cdot \frac{\partial}{\partial \underline{\theta}} f(\underline{x},\underline{\theta},z) = 0, \qquad z > 0, \qquad \text{(C-1a)}$$

with the boundary condition

$$f(\underline{x},\underline{\theta},0) = f_o(\underline{x},\underline{\theta}) \ , \qquad \text{(C-1b)}$$

using the method of characteristics. The initial data are specified on the "initial surface" z=0, and the characteristics "flow" from this surface and "fill up" the three-dimensional domain.

Introducing the parameters $\underline{\xi}$, $\underline{\eta}$, and λ, the characteristic equations for the problem under consideration here are given as follows:

$$\frac{d}{d\lambda} z(\underline{\xi},\underline{\eta},\lambda) = 1, \qquad \lambda > 0; \qquad z(\underline{\xi},\underline{\eta},0) \equiv z_o(\underline{\xi},\underline{\eta}) = 0, \qquad \text{(C-2a)}$$

$$\frac{d}{d\lambda} \underline{x}(\underline{\xi},\underline{\eta},\lambda) = \underline{\theta}(\underline{\xi},\underline{\eta},\lambda), \qquad \lambda > 0; \qquad \underline{x}(\underline{\xi},\underline{\eta},0) \equiv \underline{x}_o(\underline{\xi},\underline{\eta}) = \underline{\xi} \ , \qquad \text{(C-2b)}$$

$$\frac{d}{d\lambda} \underline{\theta}(\underline{\xi},\underline{\eta},\lambda) = -g^2 \underline{x}(\underline{\xi},\underline{\eta},\lambda), \qquad \lambda > 0; \qquad \underline{\theta}(\underline{\xi},\underline{\eta},0) \equiv \underline{\theta}_o(\underline{\xi},\underline{\eta}) = \underline{\eta}. \qquad \text{(C-2c)}$$

Viewing $\underline{\xi}$ and $\underline{\eta}$ as parameters, we note that

$$\frac{\partial}{\partial \lambda} f[\underline{x}(\underline{\xi},\underline{\eta},\lambda), \ \underline{\theta}(\underline{\xi},\underline{\eta},\lambda), \ z(\underline{\xi},\underline{\eta},\lambda)]$$

$$= \frac{\partial f}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial \lambda} + \frac{\partial f}{\partial \underline{\theta}} \cdot \frac{\partial \underline{\theta}}{\partial \lambda} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \lambda}$$

$$= \underline{\theta} \cdot \frac{\partial}{\partial \underline{x}} f - g^2 \underline{x} \cdot \frac{\partial}{\partial \underline{\theta}} f + \frac{\partial}{\partial z} f = 0 \; . \qquad \text{(C-3)}$$

Therefore, $f[\underline{x}(\underline{\xi},\underline{\eta},\lambda),\ \underline{\theta}(\underline{\xi},\underline{\eta},\lambda),\ z(\underline{\xi},\underline{\eta},\lambda)]$ is a constant along a characteristic. As a consequence,

$$f[\underline{x}(\underline{\xi},\underline{\eta},\lambda),\ \underline{\theta}(\underline{\xi},\underline{\eta},\lambda),\ z(\underline{\xi},\underline{\eta},\lambda)]$$
$$= f[\underline{x}_o(\underline{\xi},\underline{\eta}),\ \underline{\theta}_o(\underline{\xi},\underline{\eta}),\ z_o(\underline{\xi},\underline{\eta})] = f_o(\underline{\xi},\underline{\eta}) \; . \qquad \text{(C-4)}$$

The solution of the characteristic equations (C-2) can be obtained easily:

$$\underline{x} = \underline{\xi} \cos g\lambda + \frac{1}{g} \underline{\eta} \sin g\lambda, \qquad \text{(C-5a)}$$

$$\underline{\theta} = -g\underline{\xi} \sin g\lambda + \underline{\eta} \cos g\lambda, \qquad \text{(C-5b)}$$

$$z = \lambda. \qquad \text{(C-5c)}$$

Solving for $\underline{\xi}$ and $\underline{\eta}$ results in the expressions

$$\underline{\xi} = \underline{x} \cos gz - \frac{1}{g} \underline{\theta} \sin gz, \qquad \text{(C-6a)}$$

$$\underline{\eta} = \underline{x}g \sin gz + \underline{\theta} \cos gz \; . \qquad \text{(C-6b)}$$

Finally, substituting these solutions into (C-4), we obtain

$$f(\underline{x},\underline{\theta},z) = f_o(\underline{x} \cos gz - \frac{1}{g} \underline{\theta} \sin gz,\ \underline{x}g \sin gz + \underline{\theta} \cos gz) \; . \qquad \text{(C-7)}$$

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ABSTRACT

The wave-kinetic method, a technique developed recently by Tappert and Besieris[1-3] for extending conventional ray-tracing methods to incorporate diffractive, refractive, and stochastic effects associated with the propagation of beamed signals, is illustrated in this thesis by considering the propagation of coherent and partially coherent cw Gaussian laser beams in lens-like media (e.g., selfoc fibers, laser-induced plasmas, etc.). The main results presented here extend those reported recently in connection with vacuum propagation and deterministic propagation through media characterized by a parabolic profile. Special attention is paid on the irreversible degradation of the focusing ability of a beam due to random imperfections in a lens, with allowance for the presence of a medium having a parabolic profile.