SOLUTION OF THE LAMINAR BOUNDARY LAYER OF
A SEMI-INFINITE FLAT PLATE GIVEN AN
IMPULSIVE CHANGE IN VELOCITY AND TEMPERATURE

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NOMENCLATURE

Symbol	
a, b, c, d	Constants in equation (3-7)
a_1, b_1, c_1, d_1	Constants in equation (3-8)
B	Constant defined by equation (4-36a)
h	Convection heat-transfer conductance
K_1, K_2	Constants for equations (4-31), (4-32), (4-54), and (4-55)
K_3	Constant for equation (4-58)
K_4	Constant for equation (4-59)
k	Thermal conductivity
Nu_x	Local Nusselt number, hx/k
Pr	Prandtl number, ν/α
Re_x	Reynolds number, $u_\infty x/\nu$
r	Parameter used in method of characteristics
s	Parameter used in method of characteristics
t	Time
T	Nondimensionalized temperature, $(\theta-\theta_W)/(\theta_\infty-\theta_W)$
u	Velocity component in x direction
U	Nondimensionalized velocity component in x direction, u/u_∞
v	Velocity component in y direction
V	Nondimensionalized velocity component in y direction, v/u_∞

Symbol	
x	Coordinate parallel to plate
X	Nondimensionalized x coordinate, $u_\infty x/\nu$
y	Coordinate vertical to plate
Y	Nondimensionalized y coordinate, $u_\infty y/\nu$
α	Thermal diffusivity
δ	Velocity boundary-layer thickness
δ_t	Thermal boundary-layer thickness
Δ	Nondimensionalized velocity boundary-layer thickness
Δ_t	Nondimensionalized thermal boundary-layer thickness
ν	Kinematic viscosity
τ	Nondimensionalized time
θ	Temperature

Subscripts

i	Increment in x direction used in finite-difference solution
j	Increment in y direction used in finite-difference solution
w	Evaluated at the plate
o	Refers to value along a boundary curve in method of characteristics
∞	Evaluated in the free stream

Superscript

n	Time increment used in finite-difference solution

CHAPTER I

INTRODUCTION

There are many variations to the problem of determining the velocity and temperature profiles and the heat transfer in the boundary layer of an incompressible fluid moving over a semi-infinite flat plate. Solutions exist for the following cases:

a. The flow is steady and the plate temperature is constant.

b. The fluid free-stream velocity is changing continuously with time and the plate temperature is constant.

c. The fluid free-stream velocity is constant and the plate temperature is changing continuously with time.

d. Both the free-stream velocity and the plate temperature are changing continuously with time.

e. The fluid free-stream velocity is constant and there is a sudden step change in plate temperature.

There are no solutions which describe the velocity and temperature profiles at all times for the case where both the free-stream velocity and the plate temperature are given a sudden step change. Sarma (13) developed a partial solution to this problem. He developed a method which can be used to solve the thermal boundary layer for small times and/or large times for the problem of an impulsively changed free-stream velocity and a sudden change in plate temperature. However, Sarma's solution does not cover the transition period between the times when the velocity and temperature profiles are mainly time-dependent and mainly location dependent.

This thesis shows the derivation of an approximate solution for the temperature in the thermal boundary layer over a semi-infinite flat plate which is set impulsively in motion in an incompressible fluid and which has a simultaneous step change in temperature. The solution differs from previous solutions by covering the entire time period from when the plate is initially set in motion until both the velocity and temperature profiles are fully developed.

CHAPTER II

REVIEW OF LITERATURE

Steady-state solutions for the velocity, thermal and mass-concentration boundary layers for laminar incompressible flow over semi-infinite flat plates are available in numerous references. Kays (8) presented exact solutions using similarity techniques for all three boundary-layer problems. He also presented approximate solutions using a third-order polynomial for the thermal and velocity boundary-layer profiles. These solutions include provisions for fluid suction from the boundary layer or injection into the boundary layer. The Karman-Pohlhausen method of solving the velocity boundary layer using a fourth-order polynomial for the velocity profile is documented in Schlichting (14). This method allows one to determine boundary-layer properties when the free-stream velocity varies in the direction of flow. Sparrow (16) gave solutions for the local Nusselt number for cases where the velocity of fluid injection or suction is constant, as well as solutions for cases where this velocity is inversely proportional to the square root of the distance from the leading edge of the plate. Low (10) solved the problem for steady-state flow of a compressible fluid over a flat plate where the fluid injection velocity is proportional to the inverse square root of the distance from the leading edge.

Solutions of the velocity boundary layer for various nonsteady-state conditions have been presented by several authors. Cheng (4) developed a solution for the problem of a semi-infinite flat plate accelerating continuously from rest in an incompressible fluid. His solution was a series solution resulting from the perturbation of the quasi-steady-state solution. The same problem was solved for a compressible fluid by Moore (11). Stewartson (17) analyzed the problem of the impulsive motion of a semi-infinite flat plate in an incompressible fluid. First he presented Rayleigh's method which linearizes the boundary-layer equations and is only valid at the outer edge where the fluid velocity is approximately the same as the main stream velocity. This method gives a solution for the local velocity dependent only on time, t, and the distance normal to the plate for $u_\infty t < x$ where u_∞ is the free stream velocity and x is the distance from the leading edge. For $u_\infty t > x$ the local velocity is dependent only on x and the distance normal to the plate. Stewartson then used a momentum integral method, assumed the velocity profile to be a sine curve, and found a solution for the velocity dependent only on time and the distance normal to the plate for $u_\infty t \leq 2.65x$ and dependent only on the distance normal to the plate and the distance from the leading edge for $u_\infty t \geq 2.65x$. He concluded that at the outer edge of the boundary layer the changeover of the principal independent variable occurs when the distance from the leading edge equals the product of time and free-stream velocity and spreads down through the boundary layer, with the changeover being nearly complete at the plate when $u_\infty t = 2.65x$. Using similarity techniques Stewartson then analyzed the boundary-layer equations and found that the velocity in the boundary layer is independent of the distance from the leading edge if $u_\infty t < x$. At $u_\infty t = x$ the flow has an essential singularity, depending on the distance from the leading edge as well as time for $u_\infty t > x$. For very large times the influence of time dies out exponentially. Akamatsu (2) studied the same problem as Stewartson and developed an approximate solution which connects Rayleigh's unsteady-state solution to Blasius' steady-state solution. Using Meksyn's method for the steady-state boundary layer with pressure gradient Akamatsu reduced the third-order partial differential equation of Stewartson to a higher order ordinary differential equation.

There are also reports on investigations of the transient thermal boundary layer over a flat plate for various boundary conditions. Sarma (13) studied the unsteady two-dimensional thermal boundary-layer equation as linearized by Lighthill and developed series solutions for small times and for large times for the case where the main-stream temperature is constant and either the plate temperature or plate heat-transfer rate is unsteady. Ostrach (12) obtained series solutions for the laminar compressible boundary layer over a semi-infinite flat plate with a continuous but otherwise arbitrary time-dependent velocity. By neglecting all derivatives with respect to the x-coordinate (in other words assuming an infinite plate) Yang (18) also developed a method for solving the unsteady laminar compressible boundary layer. He used the integral method with either exponential or fourth-degree polynomial profiles. Cess (3) obtained a solution for the thermal boundary layer for steady, laminar, incompressible flow over a flat plate with a sudden change in surface temperature. He obtained series solutions for small times and for large times and used these to construct an approximate solution for all times. As in the case for the velocity boundary layer the solution for small times is a function only of time and the distance normal to the plate. Goodman (7) solved the same problem as Cess using the integral method with a linear velocity profile and a third-order polynomial for the temperature profile. Adams (1) developed a solution to the problem of fully-developed laminar flow over a flat plate with a sudden change in heat generation by using a third-order polynomial for the velocity profile and a second-order polynomial for the temperature profile. A finite-difference method for computing the velocity and temperature in the unsteady, incompressible, laminar boundary layer around a two-dimensional cylinder of arbitrary cross section was developed by Farn (6). His method of solution can include blowing or suction and is applicable to impulsive changes in velocity, surface temperature or surface heat generation. He presented solutions for the velocity boundary layer over a 45° wedge with an impulsive change in velocity and for the thermal boundary layer over a 45° wedge at a steady velocity with a sudden change in surface temperature.

CHAPTER III

DERIVATION OF THE APPROXIMATE DIFFERENTIAL EQUATIONS

Statement of the Problem

A flat plate is assumed to be initially at rest in an incompressible fluid which is also at rest with the temperature of the plate and fluid the same. At a certain instant of time, considered as zero time, the fluid is suddenly given a constant free-stream velocity, u_∞, relative to the plate and in a direction parallel to the plate, as shown in Figure 3-1. At the same time the temperature of the plate is abruptly changed to a value different from that of the fluid, and is held constant thereafter. The flow is assumed to be laminar and the fluid to have constant properties. It is desired to determine the velocity and temperature of the fluid at any position in the boundary layer at any time.

Equations Used

The governing continuity, momentum and energy equations for a boundary layer are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3\text{-}1)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (3\text{-}2)$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2}$$

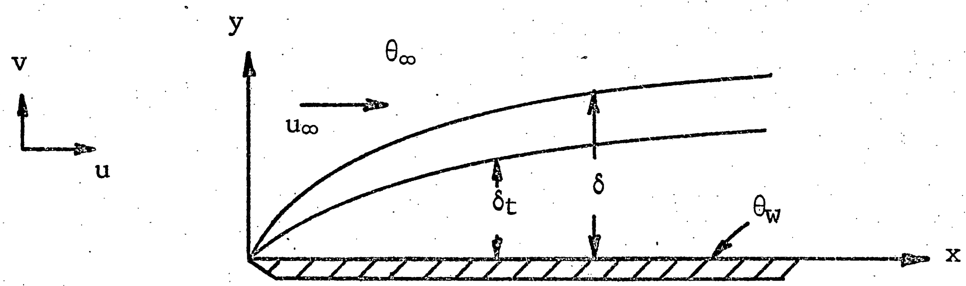

FIGURE 3-1. Sketch of the boundary layer.

By letting

$$T = \frac{\theta - \theta_w}{\theta_\infty - \theta_w}$$

where θ_∞ is the initial temperature of the plate and fluid and θ_w is the temperature to which the plate is raised, the energy equation can be written as:

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \qquad (3\text{-}3)$$

The boundary conditions for the above equations are:

$$\text{At } y = 0; \quad u = 0,\ \frac{\partial u}{\partial t} = 0,\ v = 0,\ \frac{\partial^2 u}{\partial y^2} = 0, \quad T = 0,\ \frac{\partial T}{\partial t} = 0,\ \frac{\partial^2 T}{\partial y^2} = 0\,. \qquad (3\text{-}4)$$

$$\text{As } y \to \infty; \quad u = u_\infty\,,\ \frac{\partial u}{\partial y} = 0\,,\ \frac{\partial^2 u}{\partial y^2} = 0\,, \quad T = 1,\ \frac{\partial T}{\partial y} = 0\,,\ \frac{\partial^2 T}{\partial y^2} = 0\,. \qquad (3\text{-}5)$$

At $x = 0$, $y > 0$; $u = u_\infty$, $T = 1$.

In other words, at the plate the fluid velocity components parallel and normal to the plate, the dimensionless fluid temperature excess, the first derivatives of velocity and temperature with respect to time and the second derivatives of velocity and temperature with respect to distance normal to the plate are all zero. In the free stream the fluid velocity component parallel to the plate is the free-stream velocity, the dimensionless fluid temperature excess is equal to unity, and all derivatives of velocity and temperature with respect to distance normal to the plate are zero. At the leading edge the component of fluid velocity parallel to the plate is the free stream velocity and the dimensionless temperature excess equals unity.

The initial conditions are given by:

At $t = 0$; $u = 0$, $T = 1$. (3-6)

In other words, before the fluid velocity and plate temperature are suddenly changed the fluid velocity is zero and the dimensionless fluid temperature excess equals unity.

Following the method of Schlichting (14) and many others, a fourth-order polynomial is assumed for both the velocity and temperature profiles in the boundary layer:

$$u = a\left(\frac{y}{\delta}\right) + b\left(\frac{y}{\delta}\right)^2 + c\left(\frac{y}{\delta}\right)^3 + d\left(\frac{y}{\delta}\right)^4 \qquad (3\text{-}7)$$

$$T = a_1\left(\frac{y}{\delta_t}\right) + b_1\left(\frac{y}{\delta_t}\right)^2 + c_1\left(\frac{y}{\delta_t}\right)^3 + d_1\left(\frac{y}{\delta_t}\right)^4 \qquad (3\text{-}8)$$

The terms δ and δ_t denote the velocity boundary-layer thickness and thermal boundary-layer thickness respectively.

When the constants in the above equations are evaluated using boundary conditions (3-4) and (3-5), equations (3-7) and (3-8) become:

$$\frac{u}{u_\infty} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4 \qquad (3\text{-}9)$$

$$T = 2\left(\frac{y}{\delta_t}\right) - 2\left(\frac{y}{\delta_t}\right)^3 + \left(\frac{y}{\delta_t}\right)^4 \qquad (3\text{-}10)$$

where δ and δ_t are functions of both the distance from the leading edge, x, and time, t.

Momentum Equation

In order to determine the expression for δ the momentum equation (3-2) is integrated over the velocity boundary-layer thickness using (3-9) as the expression for the fluid velocity parallel to the plate, u:

$$\int_o^\delta \left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}\right)\ dy = \int_o^\delta \nu\,\frac{\partial^2 u}{\partial y^2}\ dy \qquad (3\text{-}11)$$

The expression for the fluid velocity normal to the plate, v, is determined from the continuity equation (3-1):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$v = -\int_o^y \frac{\partial u}{\partial x}\ dy \qquad (3\text{-}12)$$

Inserting equation (3-12) into equation (3-11) gives:

$$\int_o^\delta \left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\int_o^y \frac{\partial u}{\partial x}\ dy\right)\ dy = \int_o^\delta \nu\,\frac{\partial^2 u}{\partial y^2}\ dy \qquad (3\text{-}13)$$

Integrating the third term in the left-hand side of the above equation by parts:

$$\int_o^\delta \left(\frac{\partial u}{\partial y}\int_o^y \frac{\partial u}{\partial x}\ dy\right)\ dy = u_\infty \int_o^\delta \frac{\partial u}{\partial x}\ dy - \int_o^\delta u\,\frac{\partial u}{\partial x}\ dy$$

Equation (3-13) can then be written:

$$\int_o^\delta \left(\frac{\partial u}{\partial t} + 2\,u\,\frac{\partial u}{\partial x} - u_\infty\,\frac{\partial u}{\partial x}\right)\ dy = \int_o^\delta \nu\,\frac{\partial^2 u}{\partial y^2}\ dy \qquad (3\text{-}14)$$

Integrating equation (3-14) term by term:

$$\int_o^\delta \frac{\partial u}{\partial t}\, dy = u_\infty \int_o^\delta \frac{\partial}{\partial t}\left[2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4\right] dy$$

$$= u_\infty \frac{\partial \delta}{\partial t} \int_o^\delta \left(-2\,\frac{y}{\delta^2} + 6\,\frac{y^3}{\delta^4} - 4\,\frac{y^4}{\delta^5}\right) dy$$

$$= -\frac{3}{10}\, u_\infty \frac{\partial \delta}{\partial t}$$

$$\int_o^\delta 2\,u \frac{\partial u}{\partial x}\, dy = \int_o^\delta \frac{\partial u^2}{\partial x} dy$$

$$= u_\infty^2 \int_o^\delta \frac{\partial}{\partial x}\left[2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4\right]^2 dy$$

$$= u_\infty^2 \frac{\partial \delta}{\partial x} \int_o^\delta \left(-8\,\frac{y^2}{\delta^3} + 32\,\frac{y^4}{\delta^5} - 20\,\frac{y^5}{\delta^6} - 24\,\frac{y^6}{\delta^7}\right.$$

$$\left. + 28\,\frac{y^7}{\delta^8} - 8\,\frac{y^8}{\delta^9}\right) dy$$

$$= -\frac{263}{630}\, u_\infty^2 \frac{\partial \delta}{\partial x}$$

$$\int_o^\delta u_\infty \frac{\partial u}{\partial x}\, dy = u_\infty^2 \int_o^\delta \frac{\partial}{\partial x}\left[2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4\right] dy$$

$$= u_\infty^2 \frac{\partial \delta}{\partial x} \int_o^\delta \left[-2\,\frac{y}{\delta^2} + 6\,\frac{y^3}{\delta^4} - 4\,\frac{y^4}{\delta^5}\right] dy$$

$$= -\frac{3}{10}\, u_\infty^2 \frac{\partial \delta}{\partial x}$$

$$\int_o^\delta \nu \frac{\partial^2 u}{\partial y^2}\, dy = \nu\left(\frac{\partial u}{\partial y}\right)_o^\delta = -\nu\left(\frac{\partial u}{\partial y}\right)_o$$

$$= -2\,\nu \frac{u_\infty}{\delta}$$

With these substitutions equation (3-14) becomes:

$$-\frac{3}{10}\, u_\infty \frac{\partial \delta}{\partial t} - \frac{263}{630}\, u_\infty^2 \frac{\partial \delta}{\partial x} + \frac{3}{10}\, u_\infty^2 \frac{\partial \delta}{\partial x} = -2\,\nu \frac{u_\infty}{\delta}$$

Simplifying this:

$$\frac{3}{20}\,\frac{\partial \delta^2}{\partial t} + \frac{37}{630}\, u_\infty \frac{\partial \delta^2}{\partial x} = 2\,\nu \qquad (3\text{-}15)$$

The boundary and initial conditions for equation (3-15) are: at $x = 0$; $\delta^2 = 0$ and at $t = 0$; $\delta^2 = 0$.

Energy Equation for Prandtl Number Greater Than Unity

The expression for the thermal boundary-layer thickness, δ_t, is obtained in a manner similar to that used for the velocity boundary-layer thickness. The energy equation (3-3) is integrated over the thermal boundary-layer thickness using equations (3-9) and (3-10) as expressions for the fluid velocity parallel to the plate, u, and the dimensionless fluid temperature excess, T:

$$\int_0^{\delta_t} \left(\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y}\right) dy = \int_0^{\delta_t} \alpha\,\frac{\partial^2 T}{\partial y^2}\, dy \qquad (3\text{-}16)$$

If the Prandtl number is greater than unity the thermal boundary-layer thickness is less than the velocity boundary thickness, and equation (3-16) can be integrated term by term as follows:

$$\int_0^{\delta_t} \frac{\partial T}{\partial t}\, dy = \int_0^{\delta_t} \frac{\partial}{\partial t}\left[\, 2\left(\frac{y}{\delta_t}\right) - 2\left(\frac{y}{\delta_t}\right)^3 + \left(\frac{y}{\delta_t}\right)^4 \right] dy$$

$$= -\,\frac{3}{10}\,\frac{\partial \delta_t}{\partial t}$$

$$\int_0^{\delta_t} u\,\frac{\partial T}{\partial x}\, dy = \int_0^{\delta_t} u\,\frac{\partial}{\partial x}\left[\, 2\left(\frac{y}{\delta_t}\right) - 2\left(\frac{y}{\delta_t}\right)^3 + \left(\frac{y}{\delta_t}\right)^4 \right] dy$$

$$= u_\infty\,\frac{\partial \delta_t}{\partial x}\int_0^{\delta_t}\left[\, 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4 \right] \times$$

$$\left[-2\,\frac{y}{\delta_t^2} + 6\,\frac{y^3}{\delta_t^4} - 4\,\frac{y^4}{\delta_t^5}\right] dy$$

$$= -u_\infty \left[\,\frac{4}{15}\left(\frac{\delta_t}{\delta}\right) - \frac{3}{35}\left(\frac{\delta_t}{\delta}\right)^3 + \frac{1}{36}\left(\frac{\delta_t}{\delta}\right)^4 \right]\frac{\partial \delta_t}{\partial x}$$

Substituting equation (3-12) for v in the third term of equation (3-16):

$$\int_0^{\delta_t} v \frac{\partial T}{\partial y} dy = \int_0^{\delta_t} \frac{\partial T}{\partial y} \left(-\int_0^y \frac{\partial u}{\partial x} dy\right) dy$$

$$v = -\int_0^y \frac{\partial u}{\partial x} dy = -u_\infty \int_0^y \frac{\partial}{\partial x} [2 (\tfrac{y}{\delta}) - 2 (\tfrac{y}{\delta})^3 + (\tfrac{y}{\delta})^4] dy$$

$$= u_\infty \frac{\partial \delta}{\partial x} [(\tfrac{y}{\delta})^2 - \tfrac{3}{2} (\tfrac{y}{\delta})^4 + \tfrac{4}{5} (\tfrac{y}{\delta})^5] \qquad (3\text{-}17)$$

$$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y} [2 (\tfrac{y}{\delta_t}) - 2 (\tfrac{y}{\delta_t})^3 + (\tfrac{y}{\delta_t})^4]$$

$$= \frac{2}{\delta_t} - 6 \frac{y^2}{\delta_t^3} + 4 \frac{y^3}{\delta_t^4} \qquad (3\text{-}17a)$$

$$\int_0^{\delta_t} v \frac{\partial T}{\partial y} dy = u_\infty \frac{\partial \delta}{\partial x} \int_0^{\delta_t} [(\tfrac{y}{\delta})^2 - \tfrac{3}{2} (\tfrac{y}{\delta})^4 + \tfrac{4}{5}(\tfrac{y}{\delta})^5] \times$$

$$[\frac{2}{\delta_t} - 6 \frac{y^2}{\delta_t^3} + 4 \frac{y^3}{\delta_t^4}] dy$$

$$= u_\infty \frac{\partial \delta}{\partial x} [\tfrac{2}{15}(\tfrac{\delta_t}{\delta})^2 - \tfrac{9}{140} (\tfrac{\delta_t}{\delta})^4 + \tfrac{1}{45}(\tfrac{\delta_t}{\delta})^5]$$

$$\int_0^{\delta_t} \alpha \frac{\partial^2 T}{\partial y^2} dy = \alpha \left(\frac{\partial T}{\partial y}\right)_0^{\delta_t} = -\alpha \left(\frac{\partial T}{\partial y}\right)_0$$

$$= -2 \frac{\alpha}{\delta_t}$$

Inserting the above terms into equation (3-16):

$$-\frac{3}{10} \frac{\partial \delta_t}{\partial t} - u_\infty [\tfrac{4}{15}(\tfrac{\delta_t}{\delta}) - \tfrac{3}{35}(\tfrac{\delta_t}{\delta})^3 + \tfrac{1}{36} (\tfrac{\delta_t}{\delta})^4] \frac{\partial \delta_t}{\partial x}$$

$$+ u_\infty \frac{\partial \delta}{\partial x}[\tfrac{2}{15} (\tfrac{\delta_t}{\delta})^2 - \tfrac{9}{140} (\tfrac{\delta_t}{\delta})^4 + \tfrac{1}{45} (\tfrac{\delta_t}{\delta})^5] = -2 \frac{\alpha}{\delta_t}$$

and simplifying:

$$\frac{3}{20}\frac{\partial \delta_t^2}{\partial t} + \frac{u_\infty}{2}\left[\frac{4}{15}\left(\frac{\delta_t}{\delta}\right) - \frac{3}{35}\left(\frac{\delta_t}{\delta}\right)^3 + \frac{1}{36}\left(\frac{\delta_t}{\delta}\right)^4\right]\frac{\partial \delta_t^2}{\partial x}$$

$$-\frac{u_\infty}{2}\frac{\partial \delta^2}{\partial x}\left[\frac{2}{15}\left(\frac{\delta_t}{\delta}\right)^3 - \frac{9}{140}\left(\frac{\delta_t}{\delta}\right)^5 + \frac{1}{45}\left(\frac{\delta_t}{\delta}\right)^6\right] = 2\alpha \qquad (3\text{-}18)$$

The boundary and initial conditions for equation (3-18) are: At $x = 0$; $\delta_t^2 = 0$ and at $t = 0$; $\delta_t^2 = 0$.

Energy Equation for Prandtl Number Less Than Unity

If the Prandtl number is less than unity the thermal boundary-layer thickness is greater than the velocity boundary-layer thickness. In integrating equation (3-3), expression (3-9) is used for u inside the velocity boundary layer and u_∞ is used outside the velocity boundary layer. Integrating equation (3-16) term by term as before:

$$\int_0^{\delta_t} \frac{\partial T}{\partial t}\, dy = -\frac{3}{10}\frac{\partial \delta_t}{\partial t}$$

$$\int_0^{\delta_t} u\,\frac{\partial T}{\partial x}\, dy = u_\infty \int_0^{\delta}\left[2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4\right] \times$$

$$\frac{\partial}{\partial x}\left[2\left(\frac{y}{\delta_t}\right) - 2\left(\frac{y}{\delta_t}\right)^3 + \left(\frac{y}{\delta_t}\right)^4\right] dy$$

$$+ u_\infty \int_{\delta}^{\delta_t} \frac{\partial}{\partial x}\left[2\left(\frac{y}{\delta_t}\right) - 2\left(\frac{y}{\delta_t}\right)^3 + \left(\frac{y}{\delta_t}\right)^4\right] dy$$

$$= u_\infty \frac{\partial \delta_t}{\partial x}\int_0^{\delta}\left[2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4\right] \times$$

$$\left[-2\frac{y}{\delta_t^2} + 6\frac{y^3}{\delta_t^4} - 4\frac{y^4}{\delta_t^5}\right] dy$$

$$+ u_\infty \frac{\partial \delta_t}{\partial x} \int_\delta^{\delta_t} [-2 \frac{y}{\delta_t^2} + 6 \frac{y^3}{\delta_t^4} - 4 \frac{y^4}{\delta_t^5}] dy$$

$$= u_\infty \frac{\partial \delta_t}{\partial x} [-\frac{3}{10} + \frac{2}{15} (\frac{\delta}{\delta_t})^2 - \frac{9}{140} (\frac{\delta}{\delta_t})^4 + \frac{1}{45} (\frac{\delta}{\delta_t})^5]$$

In $\int_0^{\delta_t} v \frac{\partial T}{\partial y} dy$ the expression for v for $y > \delta$ is the same as that for $y = \delta$ since the velocity normal to the plate is constant in the free stream. From equation (3-17) for $y \geq \delta$, $v = \frac{3}{10} u_\infty \frac{\partial \delta}{\partial x}$. Using equations (3-17) and (3-17a):

$$\int_0^{\delta_t} v \frac{\partial T}{\partial y} dy = u_\infty \frac{\partial \delta}{\partial x} \int_0^\delta [(\frac{y}{\delta})^2 - \frac{3}{2} (\frac{y}{\delta})^4 + \frac{4}{5} (\frac{y}{\delta})^5] \times$$

$$[\frac{2}{\delta_t} - 6 \frac{y^2}{\delta_t^3} + 4 \frac{y^3}{\delta_t^4}] dy$$

$$+ \frac{3}{10} u_\infty \frac{\partial \delta}{\partial x} \int_\delta^{\delta_t} [\frac{2}{\delta_t} - 6 \frac{y^2}{\delta_t^3} + 4 \frac{y^3}{\delta_t^4}] dy$$

$$= u_\infty \frac{\partial \delta}{\partial x} [\frac{3}{10} - \frac{4}{15} (\frac{\delta}{\delta_t}) + \frac{3}{35} (\frac{\delta}{\delta_t})^3 - \frac{1}{36} (\frac{\delta}{\delta_t})^4]$$

$$\int_0^{\delta_t} \alpha \frac{\partial^2 T}{\partial y^2} dy = -2 \frac{\alpha}{\delta_t}$$

Inserting the above terms in equation (3-16):

$$-\frac{3}{10} \frac{\partial \delta_t}{\partial t} + u_\infty [-\frac{3}{10} + \frac{2}{15} (\frac{\delta}{\delta_t})^2 - \frac{9}{140} (\frac{\delta}{\delta_t})^4 + \frac{1}{45} (\frac{\delta}{\delta_t})^5] \frac{\partial \delta_t}{\partial x}$$

$$+ u_\infty \frac{\partial \delta}{\partial x} [\frac{3}{10} - \frac{4}{15} (\frac{\delta}{\delta_t}) + \frac{3}{35} (\frac{\delta}{\delta_t})^3 - \frac{1}{36} (\frac{\delta}{\delta_t})^4] = -2 \frac{\alpha}{\delta_t}$$

This can be simplified to:

$$\frac{3}{20}\frac{\partial \delta_t^2}{\partial t} + \frac{u_\infty}{2}\left[\frac{3}{10} - \frac{2}{15}\left(\frac{\delta}{\delta_t}\right)^2 + \frac{9}{140}\left(\frac{\delta}{\delta_t}\right)^4 - \frac{1}{45}\left(\frac{\delta}{\delta_t}\right)^5\right]\frac{\partial \delta_t^2}{\partial x}$$

$$- \frac{u_\infty}{2}\frac{\partial \delta^2}{\partial x}\left[\frac{3}{10}\left(\frac{\delta_t}{\delta}\right) - \frac{4}{15} + \frac{3}{35}\left(\frac{\delta}{\delta_t}\right)^2 - \frac{1}{36}\left(\frac{\delta}{\delta_t}\right)^3\right] = 2\,\alpha \qquad (3\text{-}19)$$

The boundary and initial conditions for equation (3-19) are that the thermal boundary-layer thickness equals zero at the leading edge and at time zero, respectively.

Nondimensionalized Equations

Equations (3-10), (3-15), (3-18), and (3-19) can be nondimensionalized using the following quantities.

$$\Delta^2 = \delta^2/(\nu/u_\infty)^2,\quad \Delta_t^2 = \delta_t^2/(\nu/u_\infty)^2,\quad \tau = t/(\nu/u_\infty^2),$$

$$X = x/(\nu/u_\infty),\quad Y = y/(\nu/u_\infty),\quad Pr = \nu/\alpha \qquad (3\text{-}19a)$$

Equation (3-10) becomes:

$$T = 2\left(\frac{Y}{\Delta_t}\right) - 2\left(\frac{Y}{\Delta_t}\right)^3 + \left(\frac{Y}{\Delta_t}\right)^4 \qquad (3\text{-}20)$$

Equation (3-15) becomes:

$$\frac{3}{20}\frac{\partial \Delta^2}{\partial \tau} + \frac{37}{630}\frac{\partial \Delta^2}{\partial X} = 2 \qquad (3\text{-}21)$$

Equation (3-18) becomes:

$$\frac{3}{20}\frac{\partial \Delta_t^2}{\partial \tau} + \frac{1}{2}\left[\frac{4}{15}\left(\frac{\Delta_t}{\Delta}\right) - \frac{3}{35}\left(\frac{\Delta_t}{\Delta}\right)^3 + \frac{1}{36}\left(\frac{\Delta_t}{\Delta}\right)^4\right]\frac{\partial \Delta_t^2}{\partial X}$$

$$- \frac{1}{2} \frac{\partial \Delta^2}{\partial X} \left[\frac{2}{15} \left(\frac{\Delta_t}{\Delta}\right)^3 - \frac{9}{140} \left(\frac{\Delta_t}{\Delta}\right)^5 + \frac{1}{45} \left(\frac{\Delta_t}{\Delta}\right)^6 \right] = \frac{2}{Pr} \qquad (3\text{-}22)$$

Equation (3-19) becomes:

$$\frac{3}{20} \frac{\partial \Delta_t^2}{\partial \tau} + \frac{1}{2} \left[\frac{3}{10} - \frac{2}{15} \left(\frac{\Delta}{\Delta_t}\right)^2 + \frac{9}{140} \left(\frac{\Delta}{\Delta_t}\right)^4 - \frac{1}{45} \left(\frac{\Delta}{\Delta_t}\right)^5 \right] \frac{\partial \Delta_t^2}{\partial X}$$

$$- \frac{1}{2} \frac{\partial \Delta^2}{\partial X} \left[\frac{3}{10} \left(\frac{\Delta_t}{\Delta}\right) - \frac{4}{15} + \frac{3}{35} \left(\frac{\Delta}{\Delta_t}\right)^2 - \frac{1}{36} \left(\frac{\Delta}{\Delta_t}\right)^3 \right] = \frac{2}{Pr} \qquad (3\text{-}23)$$

CHAPTER IV

SOLUTION OF THE APPROXIMATE DIFFERENTIAL EQUATIONS

The differential equations (3-21), (3-22), and (3-23) were solved by the method of characteristics as described in Courant (5).

Velocity Boundary Layer

The solution starts with equation (3-21):

$$\frac{3}{20}\frac{\partial \Delta^2}{\partial \tau} + \frac{37}{630}\frac{\partial \Delta^2}{\partial X} = 2 \tag{3-21}$$

This equation can be solved in terms of an arbitrary parameter s. A one-parameter family of curves

$$\tau = \tau(s), \; X = X(s) \text{ and } \Delta^2 = \Delta^2(\tau(s), X(s))$$

is defined by equation (3-21) in the following manner:

$$\frac{d\Delta^2}{ds} = \frac{\partial \Delta^2}{\partial X}\frac{dX}{ds} + \frac{\partial \Delta^2}{\partial \tau}\frac{d\tau}{ds}$$

By comparing with equation (3-21):

$$\frac{dX}{ds} = \frac{37}{630}$$

$$\frac{d\tau}{ds} = \frac{3}{20}$$

$$\frac{d\Delta^2}{ds} = 2$$

These can be integrated to obtain:

$$X = \frac{37}{630}s + X_o \tag{4-1}$$

$$\tau = \frac{3}{20}s + \tau_o \tag{4-2}$$

$$\Delta^2 = 2s + \Delta_o^2 \tag{4-3}$$

where X_o, τ_o and Δ_o^2 are constants of integration.

Referring to Figure 4-1, let $\tau = 0$ be the boundary curve C_1. On C_1 $X_\circ = r$, $\tau_\circ = 0$ and from the initial conditions for Δ^2, $\Delta_\circ^2 = 0$. Therefore equations (4-1) through (4-3) become:

$$X = \frac{37}{630} s + r \qquad (4\text{-}4)$$

$$\tau = \frac{3}{20} s \qquad (4\text{-}5)$$

$$\Delta^2 = 2s \qquad (4\text{-}6)$$

Substituting s from equation (4-5) into equation (4-6):

$$\Delta^2 = \frac{40}{3} \tau \qquad (4\text{-}7)$$

Combining equations (4-4) and (4-5), the characteristic curves are given by:

$$\tau = \frac{189}{74} (X-r) \approx 2.554(X-X_\circ) \qquad (4\text{-}8)$$

Since $X_\circ \geq 0$ it is seen from equation (4-8) that equation (4-7) is the solution for $\tau \leq 2.554X$. This is shown in Figure 4-1.

The solution for $\tau \geq 2.554X$ is obtained by using $X = 0$ as the boundary curve C_2. On C_2 $X_\circ = 0$, $\tau_\circ = r$, and from the boundary condition on Δ^2 at $X = 0$ $\Delta_\circ^2 = 0$. Therefore, equations (4-1) through (4-3) become:

$$X = \frac{37}{630} s \qquad (4\text{-}9)$$

$$\tau = \frac{3}{20} s + r \qquad (4\text{-}10)$$

$$\Delta^2 = 2 s \qquad (4\text{-}11)$$

Combining equations (4-9) and (4-11):

$$\Delta^2 = \frac{1260}{37} X \approx 34.0 X \qquad (4\text{-}12)$$

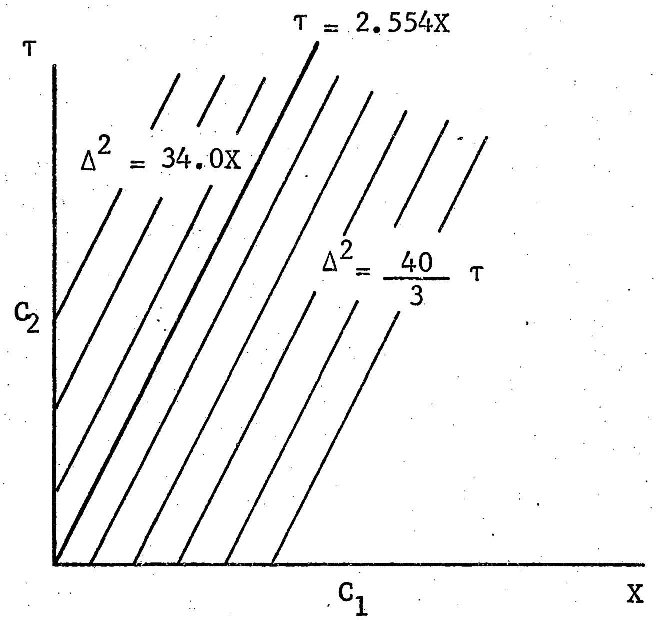

FIGURE 4-1. Solution for the velocity boundary-layer thickness

Combining equations (4-9) and (4-10) the characteristic curves are given by:

$$\tau - \tau_o = \frac{189}{74} X \approx 2.554 X \tag{4-13}$$

Since $\tau_o \geq 0$ it is seen from equation (4-13) that equation (4-12) is the solution for $\tau \geq 2.554$ X. In summary:

$$\Delta^2 = \frac{40}{3} \tau \text{ for } \tau \leq 2.554 X \tag{4-7}$$

$$\Delta^2 = 34.0 X \text{ for } \tau \geq 2.554 X \tag{4-12}$$

Thermal Boundary Layer for Prandtl Number Less Than Unity

The solution starts with equation (3-23):

$$\frac{3}{20} \frac{\partial \Delta_t^2}{\partial \tau} + \frac{1}{2} [\frac{3}{10} - \frac{2}{15} (\frac{\Delta}{\Delta_t})^2 + \frac{9}{140} (\frac{\Delta}{\Delta_t})^4 - \frac{1}{45} (\frac{\Delta}{\Delta_t})^5] \frac{\partial \Delta_t^2}{\partial X}$$

$$- \frac{1}{2} \frac{\partial \Delta^2}{\partial X} [\frac{3}{10}(\frac{\Delta_t}{\Delta}) - \frac{4}{15} + \frac{3}{35}(\frac{\Delta}{\Delta_t})^2 - \frac{1}{36} (\frac{\Delta}{\Delta_t})^3] = \frac{2}{Pr} \tag{3-23}$$

Solving in the same manner as for the velocity boundary-layer thickness:

$$\frac{d\Delta_t^2}{ds} = \frac{\partial \Delta_t^2}{\partial X} \frac{dX}{ds} + \frac{\partial \Delta_t^2}{\partial \tau} \frac{d\tau}{ds}$$

$$\frac{dX}{ds} = \frac{3}{20} - \frac{1}{15}(\frac{\Delta}{\Delta_t})^2 + \frac{9}{280} (\frac{\Delta}{\Delta_t})^4 - \frac{1}{90} (\frac{\Delta}{\Delta_t})^5 \tag{4-14}$$

$$\frac{d\tau}{ds} = \frac{3}{20} \tag{4-15}$$

$$\frac{d\Delta_t^2}{ds} = \frac{2}{Pr} + \frac{1}{2} \frac{\partial \Delta^2}{\partial X} [\frac{3}{10}(\frac{\Delta_t}{\Delta}) - \frac{4}{15} + \frac{3}{35} (\frac{\Delta}{\Delta_t})^2 - \frac{1}{36}(\frac{\Delta}{\Delta_t})^3] \tag{4-16}$$

Letting $\tau = 0$ be the boundary curve C_1, and using equation (4-7) for the region $\tau \leq 2.554$ X shown in Figure 4-1 equation (4-16) becomes:

$$\frac{d\Delta_t^2}{ds} = \frac{2}{Pr} \tag{4-17}$$

Integrating equations (4-15) and (4-17):

$$\tau = \frac{3}{20} s + \tau_o$$

$$\Delta_t^2 = \frac{2}{Pr} s + \Delta_{t_o}^2$$

Along C_1, $\tau_o = 0$ and from the initial conditions for Δ_t^2, $\Delta_{t_o}^2 = 0$. Therefore:

$$\tau = \frac{3}{20} s \tag{4-18}$$

$$\Delta_t^2 = \frac{2}{Pr} s \tag{4-19}$$

Combining equations (4-18) and (4-19):

$$\Delta_t^2 = \frac{40}{3} \frac{\tau}{Pr} \tag{4-20}$$

Inserting equations (4-7) and (4-20) into equation (4-14):

$$\frac{dX}{ds} = \frac{3}{20} - \frac{1}{15} \frac{(\frac{40}{3} \tau)}{(\frac{40}{3} \frac{\tau}{Pr})} + \frac{9}{280} \frac{(\frac{40}{3} \tau)^2}{(\frac{40}{3} \frac{\tau}{Pr})^2} - \frac{1}{90} \frac{(\frac{40}{3} \tau)^{5/2}}{(\frac{40}{3} \frac{\tau}{Pr})^{5/2}}$$

Cancelling terms and integrating:

$$X = (\frac{3}{20} - \frac{1}{15} Pr + \frac{9}{280} Pr^2 - \frac{1}{90} Pr^{5/2}) s + X_o \tag{4-21}$$

Combining equations (4-18) and (4-21), the expression for the characteristic curves becomes:

$$\tau = \frac{3}{20} \frac{X - X_o}{\frac{3}{20} - \frac{1}{15} Pr + \frac{9}{280} Pr^2 - \frac{1}{90} Pr^{5/2}} \tag{4-22}$$

From the above equation for $Pr \leq 1$, τ is always less than 2.554 X. Therefore:

$$\Delta_t^2 = \frac{40}{3}\,\frac{\tau}{Pr} \quad \text{if} \quad \tau \leq \frac{3}{20}\,\frac{X}{\frac{3}{20} - \frac{1}{15}Pr + \frac{9}{280}Pr^2 - \frac{1}{90}Pr^{5/2}}$$

Figure 4-2 shows this region of solution.

Using X = 0 as the boundary curve C_2 and solving for the region $\tau \geq 2.554$ X, where Δ^2 is taken from equation (4-12), equations (4-14) through (4-16) become:

$$\frac{dX}{ds} = \frac{3}{20} - \frac{1}{15}\,\frac{(34.0X)}{\Delta_t^2} + \frac{9}{280}\,\frac{(34.0X)^2}{\Delta_t^4} - \frac{1}{90}\,\frac{(34.0X)^{5/2}}{\Delta_t^5} \qquad (4\text{-}23)$$

$$\frac{d\tau}{ds} = \frac{3}{20} \qquad (4\text{-}24)$$

$$\frac{d\Delta_t^2}{ds} = \frac{2}{Pr} + 17.0\left[\frac{3}{10}\,\frac{\Delta_t}{(34.0X)^{1/2}} - \frac{4}{15} + \frac{3}{35}\,\frac{(34.0X)}{\Delta_t^2} - \frac{1}{36}\,\frac{(34.0X)^{3/2}}{\Delta_t^3}\right] \qquad (4\text{-}25)$$

Integrating equation (4-24):

$$\tau = \frac{3}{20}s + \tau_o \qquad (4\text{-}26)$$

Assume that $\Delta_t^2 = K_1 s + \Delta_{t_o}^2$ and $X = K_2 s + X_o$. On C_2, $X_o = 0$ and from the boundary conditions at X = 0, $\Delta_{t_o}^2 = 0$. Therefore:

$$\Delta_t^2 = K_1 s \qquad (4\text{-}27)$$

$$X = K_2 s \qquad (4\text{-}28)$$

Inserting these into equations (4-23) and (4-25):

$$K_2 = \frac{3}{20} - \frac{34.0}{15}\left(\frac{K_2}{K_1}\right) + \frac{9(34.0)^2}{280}\left(\frac{K_2}{K_1}\right)^2 - \frac{(34.0)^{5/2}}{90}\left(\frac{K_2}{K_1}\right)^{5/2} \qquad (4\text{-}29)$$

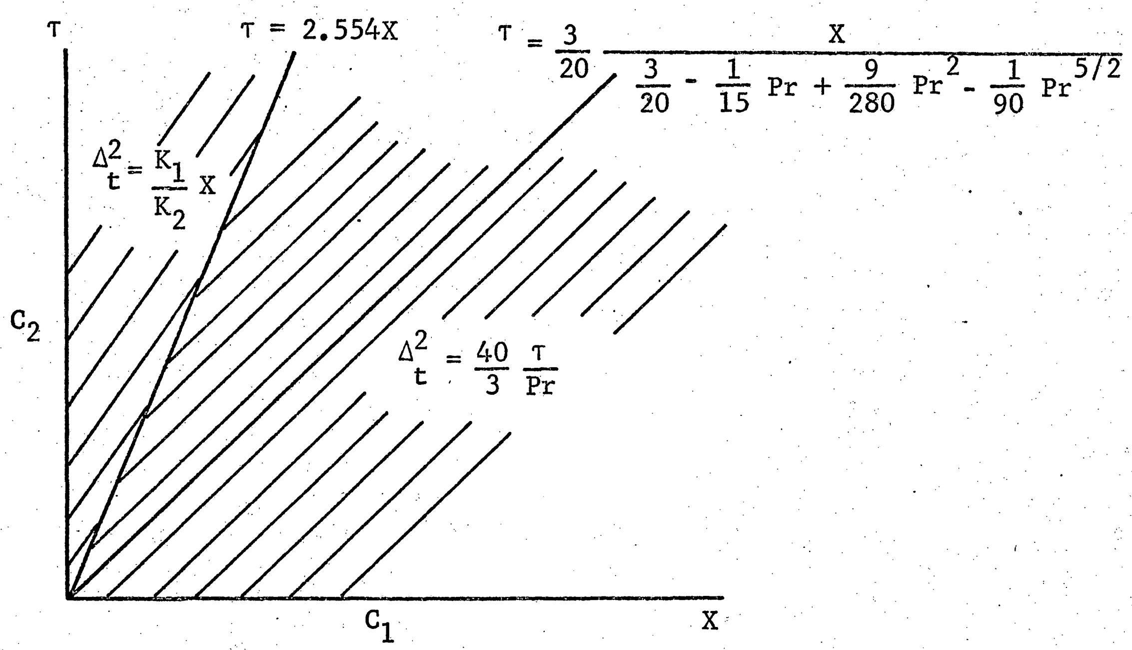

FIGURE 4-2. Typical solution for $Pr \leq 1$

$$K_1 = \frac{2}{Pr} + 17.0\left[\frac{3}{10(34.0)^{1/2}}\left(\frac{K_1}{K_2}\right)^{1/2} - \frac{4}{15} + \frac{3(34.0)}{35}\left(\frac{K_2}{K_1}\right) - \frac{(34.0)^{3/2}}{36}\left(\frac{K_2}{K_1}\right)^{3/2}\right] \tag{4-30}$$

Combining equations (4-27) and (4-28):

$$\Delta_t^2 = \frac{K_1}{K_2} X \tag{4-31}$$

From equations (4-26) and (4-28) the expression for the characteristic curves is

$$\tau - \tau_0 = \frac{3}{20} \frac{X}{K_2} \tag{4-32}$$

Therefore $\Delta_t^2 = \frac{K_1}{K_2} X$ if $2.554X \leq \tau$ and $\frac{3}{20} \frac{X}{K_2} \leq \tau$.

Equations (4-29) and (4-30) were solved numerically and curves of K_1/K_2 as a function of Prandtl number and $3/20K_2$ as a function of Prandtl number are shown in Appendix A as Figure A-1 and A-2 respectively. The expression $3/20K_2$ is always less than 2.554 so that $\Delta_t^2 = \frac{K_1}{K_2} X$ for $\tau \geq 2.554X$.

The expression for Δ_t^2 remains to be solved in the region:

$$\frac{3}{20} \frac{X}{\frac{3}{20} - \frac{1}{15} Pr + \frac{9}{280} Pr^2 - \frac{1}{90} Pr^{5/2}} \leq \tau \leq 2.554X$$

The applicable characteristic equations are:

$$\frac{dX}{ds} = \frac{3}{20} - \frac{1}{15}\left(\frac{\Delta}{\Delta_t}\right)^2 + \frac{9}{280}\left(\frac{\Delta}{\Delta_t}\right)^4 - \frac{1}{90}\left(\frac{\Delta}{\Delta_t}\right)^5 \tag{4-14}$$

$$\frac{d\tau}{ds} = \frac{3}{20} \tag{4-15}$$

$$\frac{d\Delta_t^2}{ds} = \frac{2}{Pr} \tag{4-17}$$

with $\Delta^2 = \frac{40}{3}\,\tau$

Integrating equations (4-15) and (4-17):

$$\tau = \frac{3}{20}\,s + \tau_o \tag{4-33}$$

$$\Delta_t^2 = \frac{2}{Pr}\,s + \Delta_{t_o}^2 \tag{4-34}$$

The boundary curve for this region will be the line $\tau = 2.554X$. On this curve $\tau_o = 2.554X_o$ and from the solution for $\tau \geq 2.554X$, $\Delta_{t_o}^2 = \frac{K_1}{K_2}\,X_o$.

Inserting equations (4-7), (4-33) and (4-34) into equation (4-14):

$$\frac{dX}{ds} = \frac{3}{20} - \frac{(40/3)}{15}\frac{(\frac{3}{20}\,s + \tau_o)}{(\frac{2}{Pr}\,s + \Delta_{t_o}^2)} + \frac{9}{280}\left(\frac{40}{3}\right)^2\frac{(\frac{3}{20}\,s + \tau_o)^2}{(\frac{2}{Pr}\,s + \Delta_{t_o}^2)^2}$$

$$- \frac{(40/3)^{5/2}}{90}\,\frac{(\frac{3}{20}\,s + \tau_o)^{5/2}}{(\frac{2}{Pr}\,s + \Delta_{t_o}^2)^{5/2}}$$

Integrating:

$$X = \frac{3}{20}\,s + \frac{(40/3)}{15}\left[\frac{\frac{3}{20}\,s}{2/Pr} + \frac{2\,\frac{\tau_o}{Pr} - \frac{3}{20}\,\Delta_{t_o}^2}{4/Pr^2}\,\ln\left(\Delta_{t_o}^2 + \frac{2}{Pr}\,s\right)\right]$$

$$- \frac{9}{280}\left(\frac{40}{3}\right)^2\left(\frac{Pr}{2}\right)\left\{\frac{(\frac{3}{20}\,s + \tau_o)^2}{\frac{2}{Pr}\,s + \Delta_{t_o}^2} - \frac{3}{10}\left[\frac{\frac{3}{20}\,s}{2/Pr} + \right.\right.$$

$$\left.\left.\frac{2\,\frac{\tau_o}{Pr} - \frac{3}{20}\,\Delta_{t_o}^2}{4/Pr^2}\,\ln\left(\Delta_{t_o}^2 + \frac{2}{Pr}\,s\right)\right]\right\} + \frac{(40/3)^{5/2}Pr}{270}\left\{\frac{(\frac{3}{20}\,s + \tau_o)^{5/2}}{(\frac{2}{Pr}\,s + \Delta_{t_o}^2)^{3/2}}\right.$$

$$+ \frac{3Pr}{8}\left[\frac{(\frac{3}{20} s + \tau_o)^{3/2}}{(\frac{2}{Pr} s + \Delta_{t_o}^2)^{1/2}} - \frac{9}{40} \{\frac{Pr}{2}(\frac{3}{20} s + \tau_o)^{1/2} (\frac{2}{Pr} s + \Delta_{t_o}^2)^{1/2}\right.$$

$$- \frac{(\frac{3}{20} \Delta_{t_o}^2 - 2 \tau_o/Pr)}{2} (\frac{10Pr}{3})^{1/2} Pr \ln [(\frac{3}{10Pr})^{1/2} (\frac{2}{Pr} s + \Delta_{t_o}^2)^{1/2}$$

$$+ \frac{2}{Pr} (\frac{3}{20} s + \tau_o)^{1/2}]\}]\} - \frac{(40/3)}{15} [\frac{2\frac{\tau_o}{Pr} - \frac{3}{20} \Delta_{t_o}^2}{4/Pr^2} \ln \Delta_{t_o}^2]$$

$$+ \frac{9}{280} (\frac{40}{3})^2 (\frac{Pr}{2}) \{\frac{\tau_o^2}{\Delta_{t_o}^2} - \frac{3}{10} [\frac{2\frac{\tau_o}{Pr} - \frac{3}{20} \Delta_{t_o}^2}{4/Pr^2} \ln \Delta_{t_o}^2]\}$$

$$- \frac{(40/3)^{5/2} Pr}{270} \{\frac{\tau_o^{5/2}}{\Delta_{t_o}^3} + \frac{3Pr}{8} [\frac{\tau_o^{3/2}}{\Delta_{t_o}} - \frac{9}{40}\{(\frac{Pr}{2}) (\tau_o^{1/2}) (\Delta_{t_o})$$

$$- \frac{(\frac{3}{20} \Delta_{t_o}^2 - 2 \frac{\tau_o}{Pr})}{2} (\frac{10\ Pr}{3})^{1/2} Pr \ln [(\frac{3}{10\ Pr})^{1/2} (\Delta_{t_o}) + \frac{2}{Pr}\tau_o^{1/2}]\}]\}$$

$$+ X_o \qquad (4\text{-}35)$$

Using $\tau_o = 2.554X_o$, $s = \frac{20}{3} (\tau - \tau_o)$ and $\Delta_{t_o}^2 = \frac{K_1}{K_2} X_o$

equations (4-34) and (4-35) were solved numerically for X and Δ_t^2 in terms of X_o and τ. It was found that the characteristic curves are nearly parallel to the line

$$\tau = \frac{\frac{3}{20} X}{\frac{3}{20} - \frac{1}{15} Pr + \frac{9}{280} Pr^2 - \frac{1}{90} Pr^{5/2}}$$

although they vary slightly from a straight line, and that Δ_t^2 can be predicted with less than one percent error by:

$$\Delta_t^2 = \frac{40}{3Pr} (\frac{\tau - 2.554X}{1-17B}) + \frac{K_1}{K_2} (\frac{X - \frac{20}{3} B\tau}{1-17B}) \qquad (4\text{-}36)$$

where $B = \frac{3}{20} - \frac{1}{15} Pr + \frac{9}{280} Pr^2 - \frac{1}{90} Pr^{5/2}$ (4-36a)

In summary, for a Prandtl number less than unity:

$$\Delta_t^2 = \frac{40}{3} \frac{\tau}{Pr} \quad \text{for } \tau \leq \frac{3}{20} \frac{X}{B}$$

$$\Delta_t^2 = \frac{40}{3Pr} \left(\frac{\tau - 2.554X}{1-17B} \right) + \frac{K_1}{K_2} \left(\frac{X - \frac{20}{3} B\tau}{1-17B} \right)$$

$$\text{for } \frac{3}{20} \frac{X}{B} \leq \tau \leq 2.554X$$

$$\Delta_t^2 = \frac{K_1}{K_2} X \quad \text{for } \tau \geq 2.554X$$

Thermal Boundary Layer for Prandtl Number Greater Than Unity

The solution starts with equation (3-22):

$$\frac{3}{20} \frac{\partial \Delta_t^2}{\partial \tau} + \frac{1}{2} \left[\frac{4}{15} \left(\frac{\Delta_t}{\Delta}\right) - \frac{3}{35} \left(\frac{\Delta_t}{\Delta}\right)^3 + \frac{1}{36} \left(\frac{\Delta_t}{\Delta}\right)^4 \right] \frac{\partial \Delta_t^2}{\partial X}$$

$$- \frac{1}{2} \frac{\partial \Delta^2}{\partial X} \left[\frac{2}{15} \left(\frac{\Delta_t}{\Delta}\right)^3 - \frac{9}{140} \left(\frac{\Delta_t}{\Delta}\right)^5 + \frac{1}{45} \left(\frac{\Delta_t}{\Delta}\right)^6 \right] = \frac{2}{Pr} \tag{3-22}$$

Solving in the same manner as before:

$$\frac{d\Delta_t^2}{ds} = \frac{\partial \Delta_t^2}{\partial X} \frac{dX}{ds} + \frac{\partial \Delta_t^2}{\partial \tau} \frac{d\tau}{ds}$$

$$\frac{dX}{ds} = \frac{2}{15} \left(\frac{\Delta_t}{\Delta}\right) - \frac{3}{70} \left(\frac{\Delta_t}{\Delta}\right)^3 + \frac{1}{72} \left(\frac{\Delta_t}{\Delta}\right)^4 \tag{4-37}$$

$$\frac{d\tau}{ds} = \frac{3}{20} \tag{4-38}$$

$$\frac{d\Delta_t^2}{ds} = \frac{2}{Pr} + \frac{1}{2} \frac{\partial \Delta^2}{\partial X} \left[\frac{2}{15} \left(\frac{\Delta_t}{\Delta}\right)^3 - \frac{9}{140} \left(\frac{\Delta_t}{\Delta}\right)^5 + \frac{1}{45} \left(\frac{\Delta_t}{\Delta}\right)^6 \right] \tag{4-39}$$

Using $\tau = 0$ as the boundary curve C_1, and solving for the region $\tau \leq 2.554X$ shown in Figure 4-1, where Δ^2 is given by:

$$\Delta^2 = \frac{40}{3} \tau \qquad (4\text{-}7)$$

then equation (4-39) becomes:

$$\frac{d\Delta_t^2}{ds} = \frac{2}{Pr} \qquad (4\text{-}40)$$

Integrating equations (4-38) and (4-40):

$$\tau = \frac{3}{20} s + \tau_o$$

$$\Delta_t^2 = \frac{2}{Pr} s + \Delta_{t_o}^2$$

Along C_1, $\tau_o = 0$ and $\Delta_{t_o}^2 = 0$.

Therefore $\tau = \frac{3}{20} s$ (4-41)

and $\Delta_t^2 = \frac{2}{Pr} s$ (4-42)

Combining equations (4-41) and (4-42):

$$\Delta_t^2 = \frac{40}{3} \frac{\tau}{Pr} \qquad (4\text{-}43)$$

Inserting equations (4-7) and (4-43) into equation (4-37):

$$\frac{dX}{ds} = \frac{2}{15} \frac{(\frac{40}{3} \frac{\tau}{Pr})^{1/2}}{(\frac{40}{3} \tau)^{1/2}} - \frac{3}{70} \frac{(\frac{40}{3} \frac{\tau}{Pr})^{3/2}}{(\frac{40}{3} \tau)^{3/2}} + \frac{1}{72} \frac{(\frac{40}{3} \frac{\tau}{Pr})^{2}}{(\frac{40}{3} \tau)^{2}}$$

Integrating:

$$X = (\frac{2}{15} Pr^{-1/2} - \frac{3}{70} Pr^{-3/2} + \frac{1}{72} Pr^{-2}) s + X_o \qquad (4\text{-}44)$$

Combining equations (4-41) and (4-44) the expression for the characteristic curves becomes:

$$\tau = \frac{3}{20} \frac{X - X_o}{\frac{2}{15} Pr^{-1/2} - \frac{3}{70} Pr^{-3/2} + \frac{1}{72} Pr^{-2}} \tag{4-45}$$

Therefore, $\Delta_t^2 = \frac{40}{3} \frac{\tau}{Pr}$ if:

$$\tau \leq \frac{3}{20} \frac{X}{\frac{2}{15} Pr^{-1/2} - \frac{3}{70} Pr^{-3/2} + \frac{1}{72} Pr^{-2}} \quad \text{and} \quad \tau \leq 2.554X \tag{4-45a}$$

Figure 4-3 shows this region of solution.

Using X = 0 as the boundary curve C_2 and solving for the region $\tau \geq 2.554X$, where Δ^2 is given by:

$$\Delta^2 = 34.0X \tag{4-12}$$

equations (4-37) through (4-39) become:

$$\frac{dX}{ds} = \frac{2}{15} \frac{\Delta_t}{(34.0X)^{1/2}} - \frac{3}{70} \frac{\Delta_t^3}{(34.0X)^{3/2}} + \frac{1}{72} \frac{\Delta_t^4}{(34.0X)^2} \tag{4-46}$$

$$\frac{d\tau}{ds} = \frac{3}{20} \tag{4-47}$$

$$\frac{d\Delta_t^2}{ds} = \frac{2}{Pr} + 17.0 \left[\frac{2}{15} \frac{\Delta_t^3}{(34.0X)^{3/2}} - \frac{9}{140} \frac{\Delta_t^5}{(34.0X)^{5/2}} + \frac{1}{45} \frac{\Delta_t^6}{(34.0X)^3} \right] \tag{4-48}$$

Integrating equation (4-47):

$$\tau = \frac{3}{20} s + \tau_o \tag{4-49}$$

Assume that $\Delta_t^2 = K_1 s + \Delta_{t_o}^2$ and $X = K_2 s + X_o$. On the boundary curve C_2, $X_o = 0$ and from the boundary conditions at $X = 0$, $\Delta_{t_o}^2 = 0$.

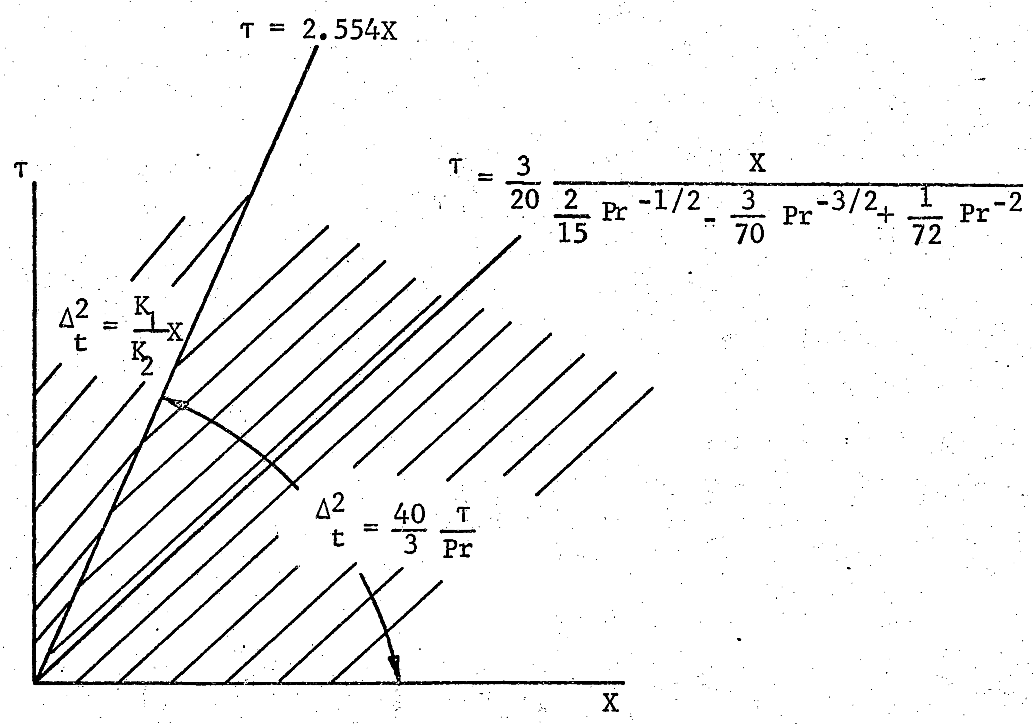

FIGURE 4-3. Typical solution for $1 \leq Pr \leq 4.55$

Therefore:

$$\Delta_t^2 = K_1 s \tag{4-50}$$

$$X = K_2 s \tag{4-51}$$

Inserting these into equations (4-46) and (4-48):

$$K_2 = \frac{2}{15(34.0)^{1/2}}\left(\frac{K_1}{K_2}\right)^{1/2} - \frac{3}{70(34.0)^{3/2}}\left(\frac{K_1}{K_2}\right)^{3/2} + \frac{1}{72(34.0)^2}\left(\frac{K_1}{K_2}\right)^2 \tag{4-52}$$

$$K_1 = \frac{2}{Pr} + 17.0\left[\frac{2}{15(34.0)^{3/2}}\left(\frac{K_1}{K_2}\right)^{3/2} - \frac{9}{140(34.0)^{5/2}}\left(\frac{K_1}{K_2}\right)^{5/2} + \frac{1}{45(34.0)^3}\left(\frac{K_1}{K_2}\right)^3\right] \tag{4-53}$$

Combining equations (4-50) and (4-51):

$$\Delta_t^2 = \frac{K_1}{K_2} X \tag{4-54}$$

From equations (4-49) and (4-51) the expression for the characteristic curves is:

$$\tau - \tau_o = \frac{3}{20} \frac{X}{K_2} \tag{4-55}$$

Therefore, $\Delta_t^2 = \frac{K_1}{K_2} X$ if:

$$2.554X \leq \tau \text{ and } \frac{3}{20} \frac{X}{K_2} \leq \tau$$

Equations (4-52) and (4-53) were solved numerically and curves of K_1/K_2 as a function of Prandtl number and $3/20K_2$ as a function of Prandtl number are shown in Appendix A as Figures A-3 and A-4 respectively.

Two regions remain to be solved for Prandtl numbers greater than unity. One of these is:

$$\frac{3}{20}\ \frac{X}{\frac{2}{15}\,Pr^{-1/2} - \frac{3}{70}\,Pr^{-3/2} + \frac{1}{72}\,Pr^{-2}} \le \tau \le 2.554X \tag{4-56}$$

This region occurs for Prandtl numbers between 1.0 and 4.55 and is shown in Figure 4-3.

Initially it was attempted to solve this problem by making the solution continuous across $\tau = 2.554X$. However, this procedure does not produce a valid solution in this region because the characteristic curves cross each other very close to the boundary curve, whereas no solution is valid beyond the point where the characteristic curves cross. When the line $\tau = 2.554X$ is used as a boundary curve, any value of $\Delta_{t_o}^2$ greater than the transient value, $\frac{40}{3}\frac{\tau}{Pr}$, when chosen as the boundary value for Δ_t^2 along the line $\tau = 2.554X$ fails to produce a solution for the same reason. This is shown in the following manner.

The applicable equations are:

$$\frac{dX}{ds} = \frac{2}{15}\left(\frac{\Delta_t}{\Delta}\right) - \frac{3}{70}\left(\frac{\Delta_t}{\Delta}\right)^3 + \frac{1}{72}\left(\frac{\Delta_t}{\Delta}\right)^4 \tag{4-37}$$

$$\frac{d\tau}{ds} = \frac{3}{20} \tag{4-38}$$

$$\frac{d\Delta_t^2}{ds} = \frac{2}{Pr} \tag{4-40}$$

with $\Delta^2 = \frac{40}{3} \tau$

Integrating equations (4-38) and (4-40):

$$\tau = \frac{3}{20} s + \tau_o$$

$$\Delta_t^2 = \frac{2}{Pr} s + \Delta_{t_o}^2$$

Use $\tau = 2.554X$ as the boundary curve with $X_o = r$, $\tau_o = 2.554r$.

Let $\Delta_{t_o}^2 = Nr$ where N is an unknown constant. Then:

$$\tau = \frac{3}{20} s + 2.554r$$

$$\Delta_t^2 = \frac{2}{Pr} s + Nr$$

Substituting these into equation (4-37):

$$\frac{dX}{ds} = \frac{2}{15(\frac{40}{3})^{1/2}} \frac{(\frac{2}{Pr} s + Nr)^{1/2}}{(\frac{3}{20} s + 2.554r)^{1/2}} - \frac{3}{70(\frac{40}{3})^{3/2}} \frac{(\frac{2}{Pr} s + Nr)^{3/2}}{(\frac{3}{20}s + 2.554r)^{3/2}}$$

$$+ \frac{1}{72(\frac{40}{3})^2} \frac{(\frac{2}{Pr} s + Nr)^2}{(\frac{3}{20} s + 2.554r)^2}$$

It must now be determined how the term

$$\frac{\frac{2}{Pr} s + Nr}{\frac{3}{20} s + 2.554r} \qquad (4\text{-}56a)$$

varies with r. To find this the derivative with respect to r is taken:

$$\frac{d}{dr}\left(\frac{\frac{2}{Pr}s + Nr}{\frac{3}{20}s + 2.554r}\right) = \frac{N}{\frac{3}{20}s + 2.554r} - \frac{2.554(\frac{2}{Pr}s + Nr)}{(\frac{3}{20}s + 2.554r)^2}$$

$$= \frac{[\frac{3}{20}N - \frac{2(2.554)}{Pr}]s}{(\frac{3}{20}s + 2.554r)^2}$$

It is seen that if

$$N > \frac{40}{3}\ \frac{(2.554)}{Pr}$$

then the derivative of the term (4-56a) with respect to r is positive and the term (4-56a) will increase if r increases. It can be shown numerically that as the term (4-56a) increases, dX/ds increases. Since $d\tau/ds$ is constant, and

$$d\tau/dX = (d\tau/ds)/(dX/ds),$$

$d\tau/dX$ will decrease as dX/ds increases. Along the line $\tau = 2.554X$ the transient solution of

$$\Delta_t^2 = \frac{40}{3}\ \frac{\tau}{Pr}$$

becomes

$$\Delta_t^2 = \frac{40}{3}\ \frac{(2.554)}{Pr}X$$

Therefore, if

$$N > \frac{40}{3}\ \frac{(2.554)}{Pr}$$

the solution along $\tau = 2.554X$ is greater than the transient solution. It follows that if the solution of Δ_t^2 along $\tau = 2.554X$ is greater than the transient solution, dX/ds increases and $d\tau/dX$ decreases as r increases. Therefore, the characteristic curves will cross infinitesimally close to the boundary curve and the solution obtained will not be valid. This proves that the value of Δ_t^2 along $\tau = 2.554X$ must be equal to or less than the value of Δ_t^2 obtained from the transient solution.

In the Prandtl number range considered, $1.0 \leq Pr \leq 4.55$ the solution for $\tau = 2.554X$ is given by equation (4-54):

$$\Delta_t^2 = \frac{K_1}{K_2} X$$

and the transient solution is given by equation (4-43):

$$\Delta_t^2 = \frac{40}{3} \frac{\tau}{Pr}$$

Since the value of $(K_1/K_2)X$ is greater than $(40/3)(\tau/Pr)$ for $\tau = 2.554X$, there is a discontinuity in Δ_t^2 along the line $\tau = 2.554X$ if Δ_t^2 is equal to or less than the transient solution along this line. This discontinuity will become greater as the value of Δ_t^2 along this line becomes smaller. Therefore, the discontinuity in Δ_t^2 along the line $\tau = 2.554X$ will be a minimum if the solution for Δ_t^2 along this line is the transient solution. Figure 4-4 may clarify this.

It is possible to find an infinite number of solutions to the differential equation (3-22) in the region defined by the inequalities (4-56) which are continuous across the curve:

$$\tau = \frac{3}{20} \frac{X}{\frac{2}{15} Pr^{-1/2} - \frac{3}{70} Pr^{-3/2} + \frac{1}{72} Pr^{-2}}$$

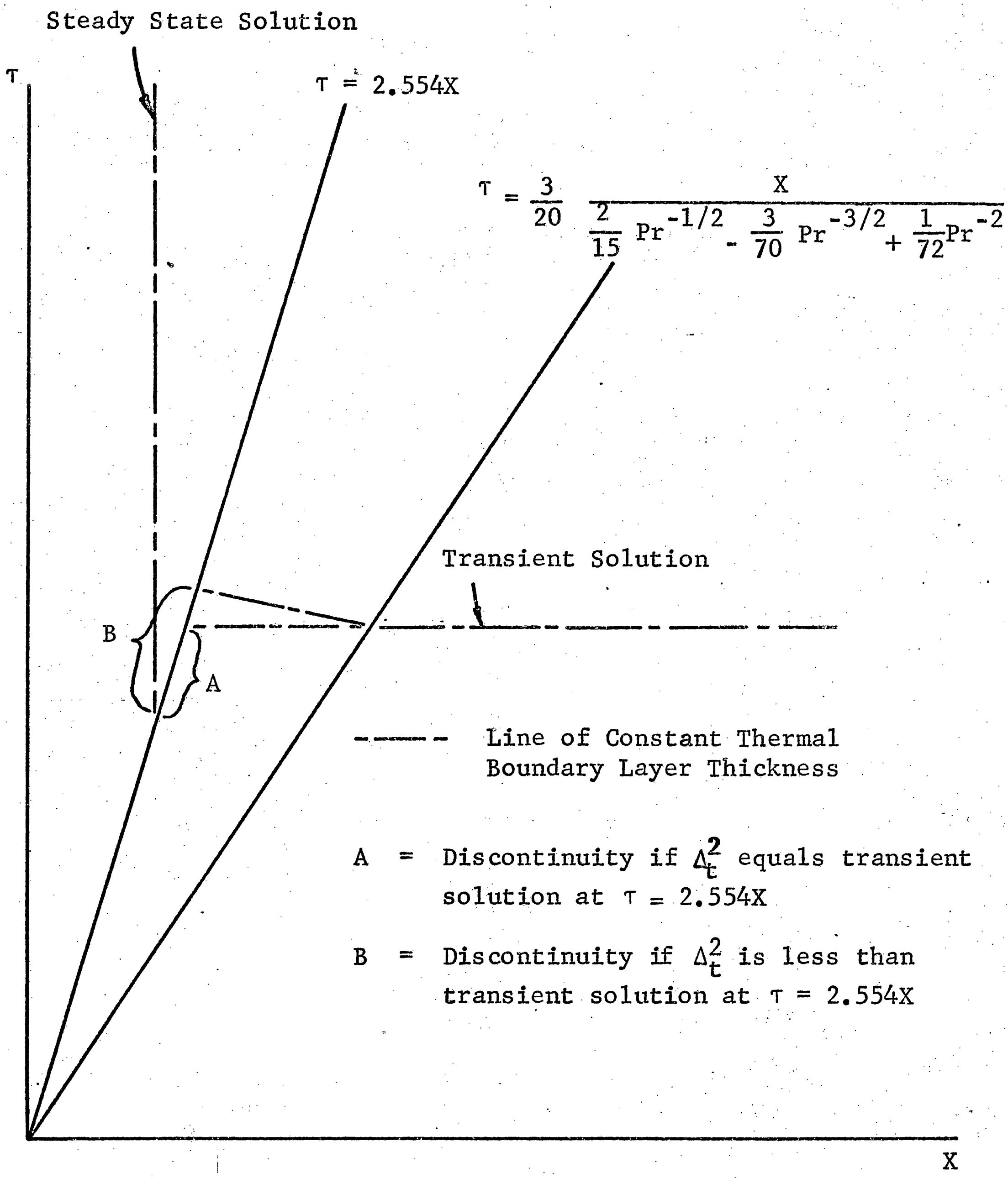

FIGURE 4-4. Line of constant thermal boundary layer thickness for $1.0 \leq Pr \leq 4.55$

and for which Δ_t^2 is less than or equal to the transient solution along the line $\tau = 2.554X$. For reasons discussed in the following paragraphs, it appears reasonable to take the transient solution as the correct solution in this region. However, no mathematical proof of the uniqueness of the transient solution in this region has been discovered.

It is known from equation (4-45a) that for Prandtl numbers greater than 4.55 the transient solution is valid for all $\tau \leq 2.554X$. Figure 4-5 shows the characteristics for Prandtl numbers between 4.45 and 8.86 and Figure 4-6 shows them for Prandtl numbers greater than 8.86. It is known from the solution for Prandtl numbers less than unity that at Prandtl number unity the transient solution is valid for $\tau \leq 2.554X$. The discontinuity for Δ_t^2 along $\tau = 2.554X$ is zero for Prandtl number unity and increases with Prandtl number. If the boundary value along this line is chosen to be equal to the transient value, the percent discontinuity described by

$$\frac{\frac{K_1}{K_2} X - \frac{40}{3} \frac{\tau}{Pr}}{\frac{K_1}{K_2} X} \times 100$$

varies smoothly with Prandtl number as shown in Figure 4-7.

If it is assumed that the value of Δ_t^2 along $\tau = 2.554X$ is given by the transient solution, the solution obtained for the region indicated by inequality (4-56) is the transient solution.

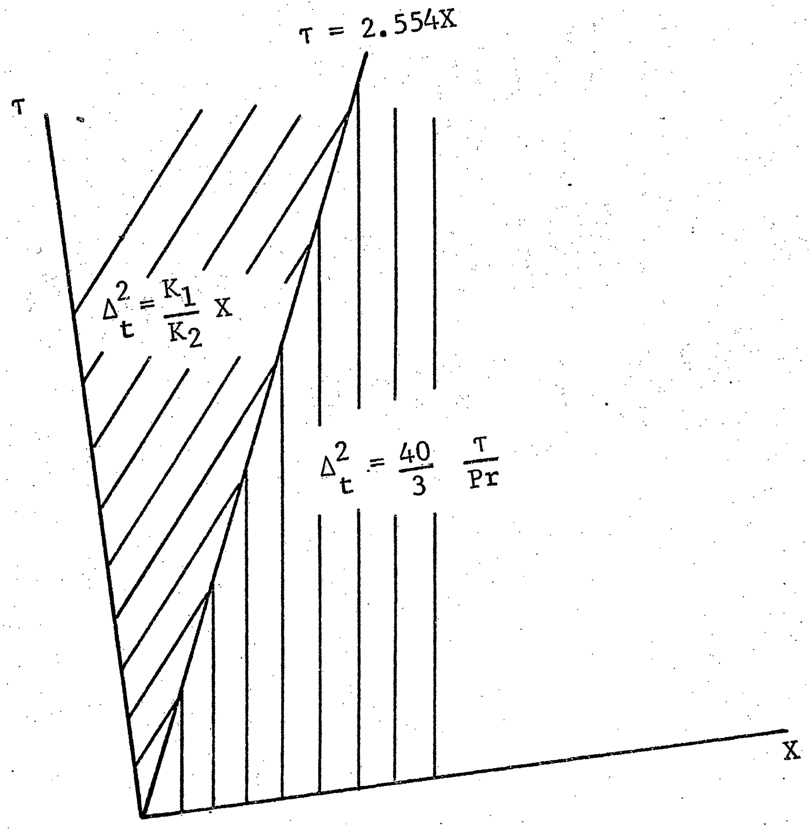

FIGURE 4-5. Typical Solution for $4.55 \leq Pr \leq 8.86$

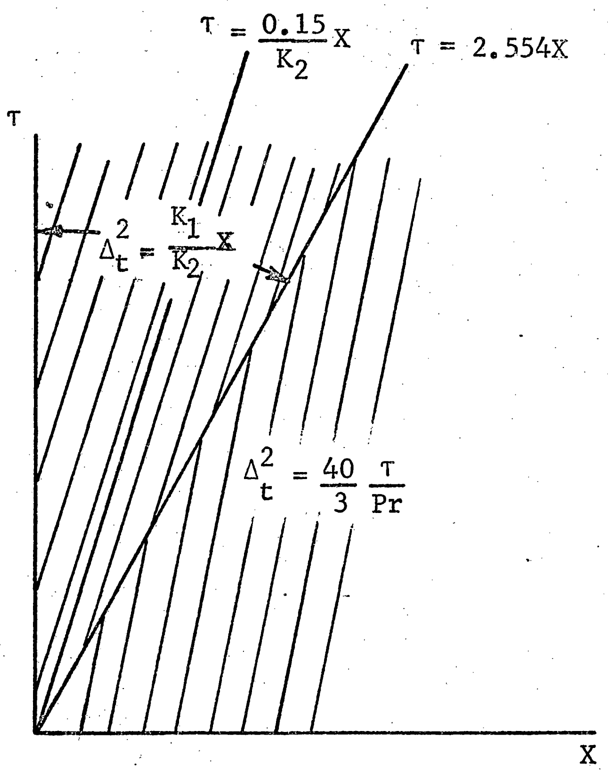

FIGURE 4-6. Typical Solution for Pr $\gtrsim$ 8.86

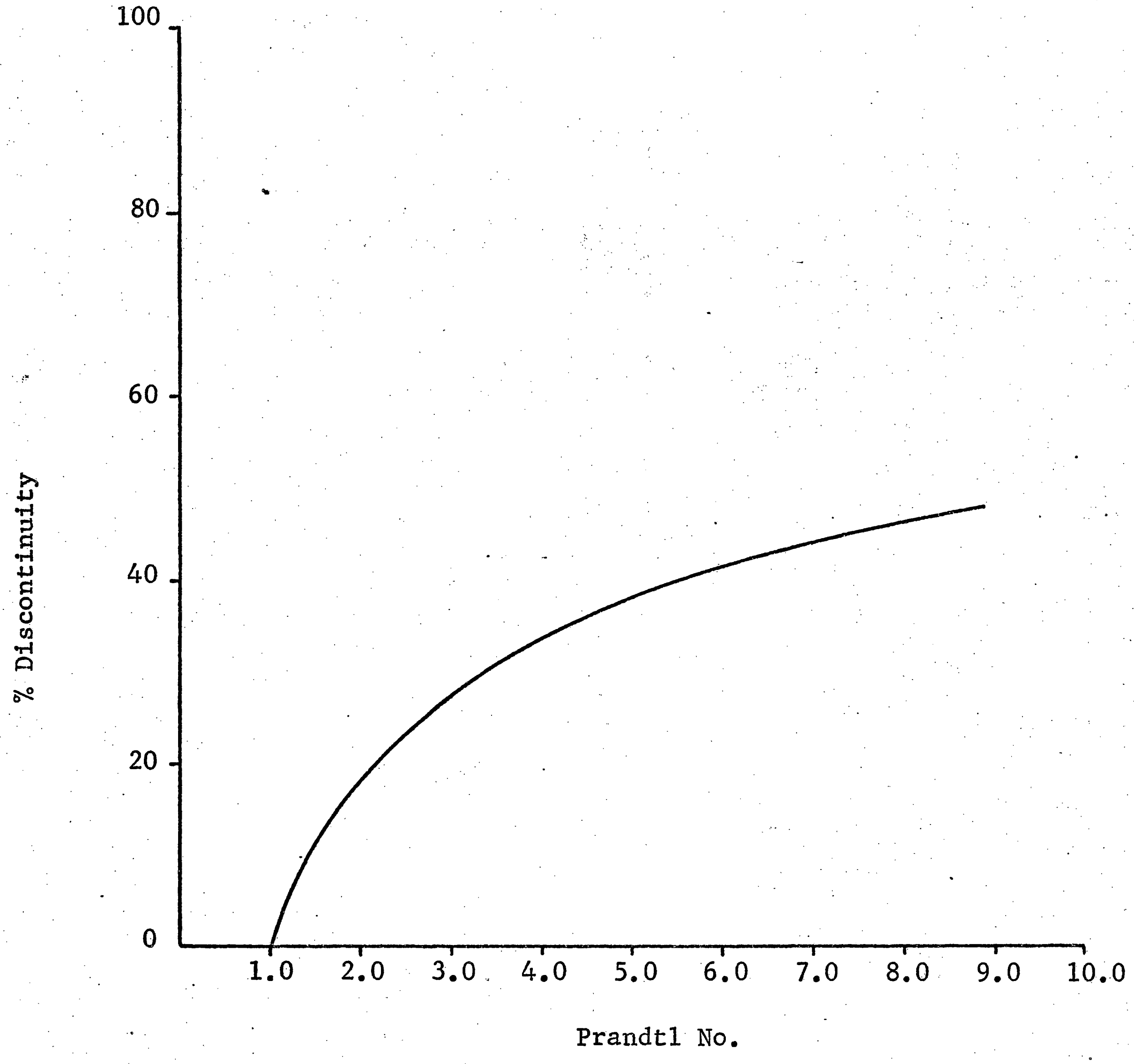

FIGURE 4-7. Discontinuity assuming transient solution for $\tau \leq 2.554X$

This solution is obtained in the same manner as before using equations (4-37) through (4-39). Since the differential equation (3-22) is the same for all $\tau \leq 2.554X$, it is reasonable to expect that the solution for Δ_t^2 is the same for all $\tau \leq 2.544X$.

For the reasons described in the previous two paragraphs it will be assumed that in the region described by inequality (4-56) the solution is:

$$\Delta_t^2 = \frac{40}{3}\ \frac{\tau}{Pr}$$

The final region to be solved is defined by:

$$2.554X \leq \tau \leq \frac{3}{20}\ \frac{X}{K_2} \tag{4-57}$$

This region occurs for Prandtl numbers greater than 8.86 and is shown in Figure 4-6.

The applicable equations are:

$$\frac{dX}{ds} = \frac{2}{15}\ \frac{\Delta_t}{(34.0X)^{1/2}} - \frac{3}{70}\ \frac{\Delta_t^3}{(34.0X)^{3/2}} + \frac{1}{72}\ \frac{\Delta_t^4}{(34.0X)^2} \tag{4-46}$$

$$\frac{d\tau}{ds} = \frac{3}{20} \tag{4-47}$$

$$\frac{d\Delta_t^2}{ds} = \frac{2}{Pr} + 17.0\left[\frac{2}{15}\ \frac{\Delta_t^3}{(34.0X)^{3/2}} - \frac{9}{140}\ \frac{\Delta_t^5}{(34.0X)^{5/2}} + \frac{1}{45}\ \frac{\Delta_t^6}{(34.0X)^3}\right] \tag{4-48}$$

Integrating equation (4-47):

$$\tau = \frac{3}{20}\, s + \tau_o$$

Assume $\Delta_t^2 = K_3 s + \Delta_{t_o}^2$ and $X = K_4 s + X_o$.

Along the boundary curve $\tau = 2.554X$, $X_o = r$,

$\tau_o = 2.554r$ and $\Delta^2_{t_o} = Nr$ where N is an unknown constant.

Therefore:

$$\Delta^2_t = K_3\, s + Nr \tag{4-58}$$

$$X = K_4\, s + r \tag{4-59}$$

$$\tau = \frac{3}{20}\, s + 2.554r \tag{4-60}$$

At $r = 0$ the characteristic curve defined by equations (4-59) and (4-60) is $\tau = \frac{3}{20}\frac{X}{K_4}$. In order for the solution to be continuous across $\tau = \frac{3}{20}\frac{X}{K_2}$ the characteristic curve for $r = 0$ must fall along $\tau = \frac{3}{20}\frac{X}{K_2}$. Therefore, $K_4 = K_2$. From equations (4-58) and (4-59) at $r = 0$, $\Delta^2_t = \frac{K_3}{K_4} X = \frac{K_3}{K_2} X$. If the solution is to be continuous across $\tau = \frac{3}{20}\frac{X}{K_2}$, from equation (4-54) $\Delta^2_t = \frac{K_1}{K_2} X$. Therefore, $K_3 = K_1$. Equations (4-58) and (4-59) become:

$$\Delta^2_t = K_1 s + Nr \tag{4-61}$$

$$X = K_2 s + r \tag{4-62}$$

As in the preceding case it was initially attempted to make the solution continuous across $\tau = 2.554X$. However, once again the characteristic curves cross very close to the boundary curve and a valid solution is not obtained. It can be shown that when the line $\tau = 2.554X$ is used as a boundary curve, any value below the steady state value, $\Delta^2_t = (K_1/K_2)X$, chosen as the boundary value for Δ_t^2 along $\tau = 2.554X$, fails to produce a solution for the same reason. This can be shown mathematically using the same procedure as in the previous section. Therefore, in equation (4-61):

$$N \geq \frac{K_1}{K_2}$$

The argument for making $N = K_1/K_2$ and thereby creating a solution which is the same as the steady-state solution is almost identical to that used for the preceding case to show that the transient solution exists for $\tau < 2.554X$. If N is chosen to be greater than K_1/K_2 the amount of discontinuity is increased. If the boundary value along this line is chosen to be equal to the steady-state value, the percent discontinuity for Prandtl numbers greater than 8.86 varies smoothly from the known values between Prandtl numbers of 4.55 and 8.86. The solution obtained for the region indicated by the inequality (4-57) is the steady-state solution if it is assumed that the value of Δ_t^2 along $\tau = 2.554X$ is given by the steady-state solution. All available information indicates that in the area defined by (4-57) the solution is the steady-state solution, although efforts to prove that this is the unique solution have been unsuccessful. Therefore, it will be assumed that in this area the solution is:

$$\Delta_t^2 = \frac{K_1}{K_2} X$$

where K_1/K_2 is given in Figure A-3 of Appendix A.

In summary for a Prandtl number greater than unity:

$$\Delta_t^2 = \frac{40}{3}\frac{\tau}{Pr} \quad \text{for } \tau \leq 2.554X$$

$$\Delta_t^2 = \frac{K_1}{K_2} X \quad \text{for } \tau \geq 2.554X$$

Determination of the Nusselt Number

The heat transfer from the plate to the fluid can be determined either from Fourier's heat conduction equation:

$$q = -kA\left(\frac{\partial\theta}{\partial y}\right)_{y=0}$$

or from the convection equation:

$$q = hA(\theta_w - \theta_\infty)$$

where A is the surface area of the plate. Equating the above two expressions:

$$-kA\left(\frac{\partial\theta}{\partial y}\right)_{y=0} = hA(\theta_w - \theta_\infty)$$

$$\frac{h}{k} = \frac{\left(\frac{\partial\theta}{\partial y}\right)_{y=0}}{\theta_\infty - \theta_w}$$

$$\frac{hx}{k} = x\,\frac{\left(\frac{\partial\theta}{\partial y}\right)_{y=0}}{\theta_\infty - \theta_w}$$

Nondimensionalizing the above equation using the substitutions defined previously by equation (3-19a):

$$Nu_x = \frac{hx}{k} = X\left(\frac{\partial T}{\partial Y}\right)_{Y=0} \tag{4-63}$$

From equation (3-20):

$$T = 2\left(\frac{Y}{\Delta_t}\right) - 2\left(\frac{Y}{\Delta_t}\right)^3 + \left(\frac{Y}{\Delta_t}\right)^4 \tag{3-20}$$

$$\left(\frac{\partial T}{\partial Y}\right)_{Y=0} = \frac{2}{\Delta_t}$$

Substituting into equation (4-63):

$$Nu_x = \frac{2X}{\Delta_t} \tag{4-64}$$

CHAPTER V

FINITE-DIFFERENCE SOLUTION OF THE BOUNDARY-LAYER EQUATIONS

To obtain results which can be compared with the approximate solutions of Chapter IV the continuity, momentum and energy equations (3-1) through (3-3) were solved numerically on a digital computer using the finite-difference equations of Farn (6).

When equations (3-1) through (3-3) are non-dimensionalized using the substitutions (3-19a), they become:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{5-1}$$

$$\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2} \tag{5-2}$$

$$\frac{\partial T}{\partial \tau} + U\frac{\partial T}{\partial X} + V\frac{\partial T}{\partial Y} = \frac{1}{Pr}\frac{\partial^2 T}{\partial Y^2} \tag{5-3}$$

where $U = u/u_\infty$ and $V = v/u_\infty$.

The boundary conditions (3-4) and (3-5) become:

for $X = 0,\ Y > 0,\ \tau > 0;\ U = 1,\ T = 1$

for $X \geq 0,\ Y = 0,\ \tau > 0;\ U = 0,\ V = 0,\ T = 0$

for $X \geq 0,\ Y \to \infty,\ \tau > 0;\ U = 1,\ T = 1$ (5-4)

The initial conditions (3-6) become:

at $\tau = 0;\ U = 0,\ T = 1$ (5-5)

Using the notation that any physical property φ at the coordinate (X_i, Y_j, τ_n) is uniquely determined by $\varphi^n_{i,j}$, Farn (6) wrote equations (5-1) through (5-3) in the following finite-difference form:

$$\frac{U^{n+1}_{i,j-1} - U^{n+1}_{i-1,j-1} + U^{n+1}_{i,j} - U^{n+1}_{i-1,j}}{2(\Delta X)} + \frac{V^{n+1}_{i,j} - V^{n+1}_{i,j-1}}{\Delta Y} = 0 \qquad (5\text{-}6)$$

$$\frac{U^{n+1}_{i,j} - U^{n}_{i,j}}{\Delta\tau} + U^{n}_{i,j}\left(\frac{U^{n}_{i,j} - U^{n}_{i-1,j}}{\Delta X}\right) + V^{n}_{i,j}\left(\frac{U^{n}_{i,j+1} - U^{n}_{i,j-1}}{2 \cdot \Delta Y}\right)$$

$$-\left(\frac{U^{n}_{i,j+1} - 2\,U^{n}_{i,j} + U^{n}_{i,j-1}}{(\Delta Y)^2}\right) = 0 \qquad (5\text{-}7)$$

$$\frac{T^{n+1}_{i,j} - T^{n}_{i,j}}{\Delta\tau} + U^{n}_{i,j}\left(\frac{T^{n}_{i,j} - T^{n}_{i-1,j}}{\Delta X}\right) + V^{n}_{i,j}\left(\frac{T^{n}_{i,j+1} - T^{n}_{i,j-1}}{2 \cdot \Delta Y}\right)$$

$$-\frac{T^{n}_{i,j+1} - 2\,T^{n}_{i,j} + T^{n}_{i,j-1}}{Pr \cdot (\Delta Y)^2} = 0 \qquad (5\text{-}8)$$

The above three equations when rewritten and presented in the following order are easily solvable by computer:

$$U^{n+1}_{i,j} = U^{n}_{i,j} + \Delta\tau\left[-U^{n}_{i,j}\left(\frac{U^{n}_{i,j} - U^{n}_{i-1,j}}{\Delta X}\right) - V^{n}_{i,j}\left(\frac{U^{n}_{i,j+1} - U^{n}_{i,j-1}}{2 \cdot \Delta Y}\right)\right.$$

$$\left. + \left(\frac{U^{n}_{i,j+1} - 2\,U^{n}_{i,j} + U^{n}_{i,j-1}}{(\Delta Y)^2}\right)\right] \qquad (5\text{-}9)$$

$$T^{n+1}_{i,j} = T^{n}_{i,j} + \Delta\tau\left[-U^{n}_{i,j}\left(\frac{T^{n}_{i,j} - T^{n}_{i-1,j}}{\Delta X}\right) - V^{n}_{i,j}\left(\frac{T^{n}_{i,j+1} - T^{n}_{i,j-1}}{2 \cdot \Delta Y}\right)\right.$$

$$\left. + \left(\frac{T^{n}_{i,j+1} - 2\,T^{n}_{i,j} + T^{n}_{i,j-1}}{Pr \cdot (\Delta Y)^2}\right)\right] \qquad (5\text{-}10)$$

$$V^{n+1}_{i,j} = V^{n+1}_{i,j-1} - \frac{\Delta Y}{2(\Delta X)}\left[U^{n+1}_{i,j-1} - U^{n+1}_{i-1,j-1} + U^{n+1}_{i,j} - U^{n+1}_{i-1,j}\right] \qquad (5\text{-}11)$$

Equations (5-9) through (5-11) were solved by computer for Prandtl numbers of 0.7 and 3.0 using the following increments:

$\Delta X = 750$

$\Delta Y = 30$

$\Delta \tau = 200$

The nodal network of the X and Y coordinates is shown in Figure 5-1.

The local Nusselt number in nondimensionalized form is given in equation (4-63):

$$Nu_x = X \left(\frac{\partial T}{\partial Y}\right)_{Y=0} \tag{4-63}$$

Using the network of Figure 5-1, the finite-difference form of this equation is:

$$Nu_{x_i} = X_i \left(\frac{T_{i,2} - T_{i,1}}{\Delta Y}\right) \tag{5-12}$$

Solutions to equations (5-9) through (5-11) are presented in Figures 6-3 through 6-6. The temperature of the fluid is plotted against the nondimensionalized distance above the plate, for nondimensionalized distances from the leading edge of 15,000 and 22,500 and for various values of time. The solution to equation (5-12) is presented in Figures 6-8 and 6-9 for Prandtl numbers of 0.7 and 3.0 respectively.

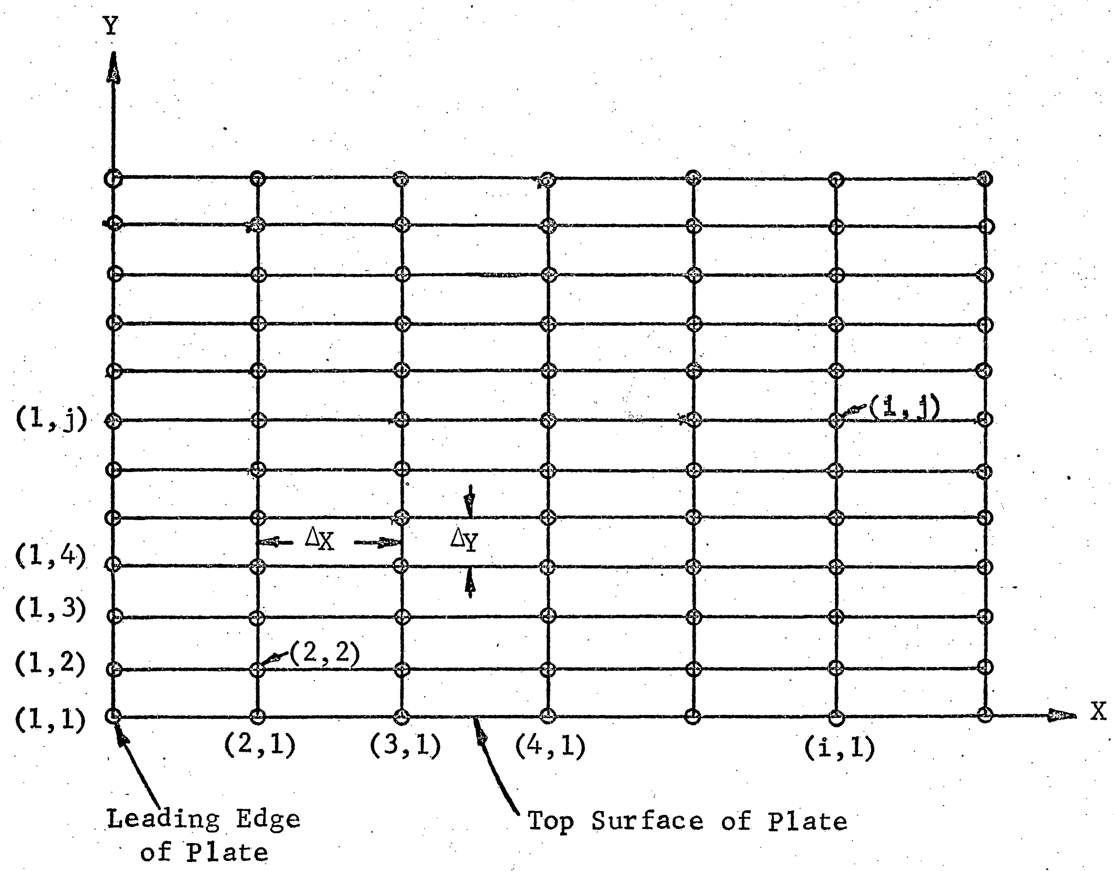

FIGURE 5-1. Nodal network for finite-difference solution.

CHAPTER VI

COMPARISON OF APPROXIMATE

SOLUTION WITH INDEPENDENTLY DERIVED RESULTS

Steady-State Temperature Profiles

As seen from equations (4-31) and (4-54) the approximate solution of the steady-state thermal boundary-layer thickness is given by:

$$\Delta_t^2 = \frac{K_1}{K_2} X$$

or

$$\Delta_t = \left(\frac{K_1}{K_2}\right)^{1/2} X^{1/2} \tag{6-1}$$

where K_1/K_2 is given in Figures A-1 and A-3 of Appendix A.

From equation (3-20):

$$T = 2\left(\frac{Y}{\Delta_t}\right) - 2\left(\frac{Y}{\Delta_t}\right)^3 + \left(\frac{Y}{\Delta_t}\right)^4 \tag{3-20}$$

Substituting equation (6-1) into equation (3-20):

$$T = \frac{2}{(K_1/K_2)^{1/2}} \frac{Y}{X^{1/2}} - \frac{2}{(K_1/K_2)^{3/2}} \left(\frac{Y}{X^{1/2}}\right)^3 + \frac{1}{(K_1/K_2)^2} \left(\frac{Y}{X^{1/2}}\right)^4 \tag{6-2}$$

The solution to the above equation was compared to the solution of Schlichting (14) of the steady-state energy equation for fully developed flow over a flat plate. In this solution the continuity, momentum and energy equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6-3}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \tag{6-4}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{6-5}$$

are rewritten by defining the new set of independent variables:

$$\eta = y \left(\frac{u_\infty}{\nu x}\right)^{1/2}, \quad \psi = (\nu x u_\infty)^{1/2} f(\eta) \tag{6-6}$$

These are the well-known substitutions used by Blasius in solving the boundary-layer equation for flow over a flat plate. Using the variables (6-6) the partial differential equations (6-3) through (6-5) can be reduced to two ordinary differential equations. These equations can then be solved by numerical integration. The solution was first given by E. Pohlhausen and solutions for particular Prandtl numbers are given in Schlichting (14).

The approximate solution given by equation (6-2) is compared in Figure 6-1 to the steady-state solution in Schlichting (14) for several Prandtl numbers. It is seen from this figure that the steady-state temperature profiles obtained by the approximate solution derived herein compare closely to the temperature profiles obtained from the "exact" solution presented in Schlichting (14). The maximum difference in the two solutions presented in Figure 6-1 occurs for a Prandtl number of unity and is about six per cent.

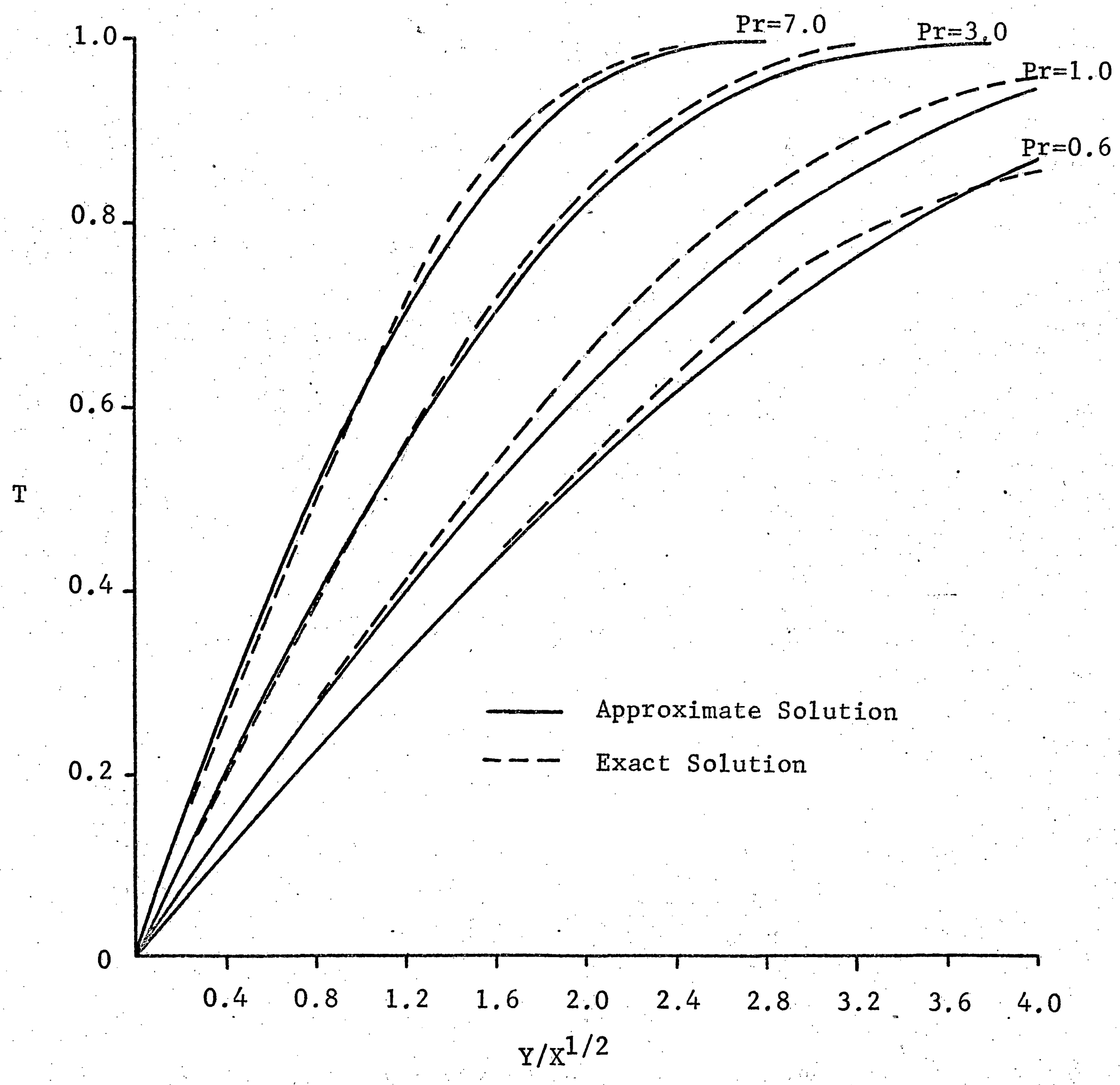

FIGURE 6-1. Comparison of steady-state temperature profiles.

Infinite Plate Temperature Profile

For a very short time after motion begins or for a very large distance from the leading edge the boundary layer acts as if there were no leading edge. The problem becomes the same as that of an infinite plate moving impulsively in an incompressible fluid with a sudden change in plate temperature. All derivatives with respect to the x coordinate are negligible as is the velocity in the y direction. The energy equation (3-3) then becomes:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{6-7}$$

with the boundary conditions of T equal to zero at the plate and T equal to unity in the free stream and initial condition of T equal to unity. Equation (6-7) and the above-stated boundary and initial conditions are the same as that for heat conduction in a semi-infinite solid with a suddenly changed surface temperature. The exact solution is derived in Schneider (15) using Fourier's integral and is:

$$T = \text{erf} \left(\frac{y}{2\sqrt{\alpha\, t}}\right)$$

When the above equation is nondimensionalized by the substitutions of equation (3-19a) it becomes:

$$T = \text{erf}\left(\frac{1}{2}\,\text{Pr}^{1/2}\,\frac{Y}{\text{T}^{1/2}}\right) \tag{6-8}$$

The approximate solution for the thermal boundary-layer thickness for short times is derived in Chapter IV and is given by equations (4-20) and (4-43). The solution is:

$$\Delta_t^2 = \frac{40}{3}\frac{\tau}{Pr}$$

or

$$\Delta_t = (\frac{40}{3})^{1/2}\frac{\tau^{1/2}}{Pr^{1/2}}$$

Inserting the above expression for Δ_t into equation (3-20), the temperature in the boundary layer is given by:

$$T = \frac{2}{(\frac{40}{3})^{1/2}}(Pr^{1/2}\frac{Y}{\tau^{1/2}}) - \frac{2}{(\frac{40}{3})^{3/2}}(Pr^{1/2}\frac{Y}{\tau^{1/2}})^3 + \frac{1}{(\frac{40}{3})^2}(Pr^{1/2}\frac{Y}{\tau^{1/2}})^4 \qquad (6\text{-}9)$$

Solutions to equations (6-8) and (6-9) are compared in Figure 6-2. The approximate solution is almost identical to the exact solution except near the edge of the boundary layer where the maximum error is about 2.5 per cent.

Approximate Versus Finite-Different Temperature Profile

In order to get an idea of the accuracy of the approximate solution at other than strictly transient or steady-state conditions, the temperature profile obtained from the approximate solution was compared to that obtained from a finite-difference solution of the continuity, momentum and energy equations. The approximate temperature profile was obtained by substituting the solutions for the thermal boundary layer thickness, equations (4-20), (4-31), and (4-36) for a Prandtl number less than unity and equations (4-43) and (4-54) for a Prandtl number greater than

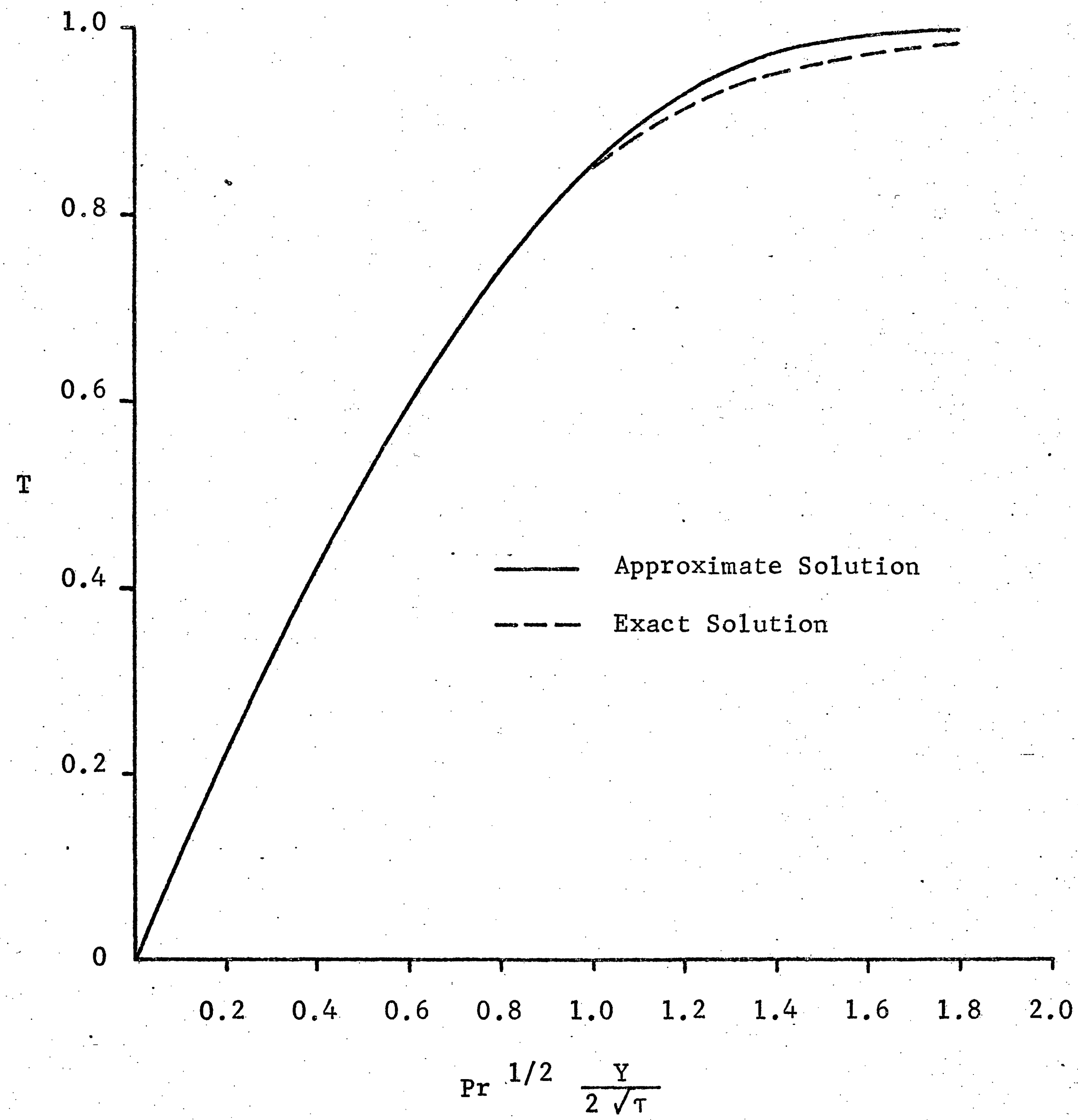

FIGURE 6-2. Comparison of purely transient (infinite plate solution) temperature profiles.

unity, into equation (3-20) which is the expression for the temperature in the boundary layer. The finite-difference solution was described in Chapter V. Comparisons of the two solutions for a Prandtl number of 0.7 are given in Figures 6-3 and 6-4 and for a Prandtl number of 3.0 in Figures 6-5 and 6-6. Comparisons are made at nondimensionalized distances from the leading edge of 15,000 and 22,500.

From Figures 6-3 and 6-4 it is seen that for a Prandtl number of 0.7 the approximate and finite-difference solutions agree closely with a maximum difference of about three per cent. Figures 6-5 and 6-6 show that for a Prandtl number of 3.0 the difference in the two solutions is greater. A maximum difference of about eight per cent occurs near the edge of the boundary layer for $X = 15,000$ and $\tau = 32,000$. It should be mentioned that part of this difference is due to errors in the finite-difference solution caused by the size of the distance and time increments used in solving the finite-difference equations (5-9) through (5-11). If very small increments had been used the solution would have approached the exact solution of equations (5-1) through (5-3) and the temperatures obtained from the finite-difference solution would have been greater than those shown in Figures 6-3 through 6-6. Smaller increments were not used because excessive computer time would have been required.

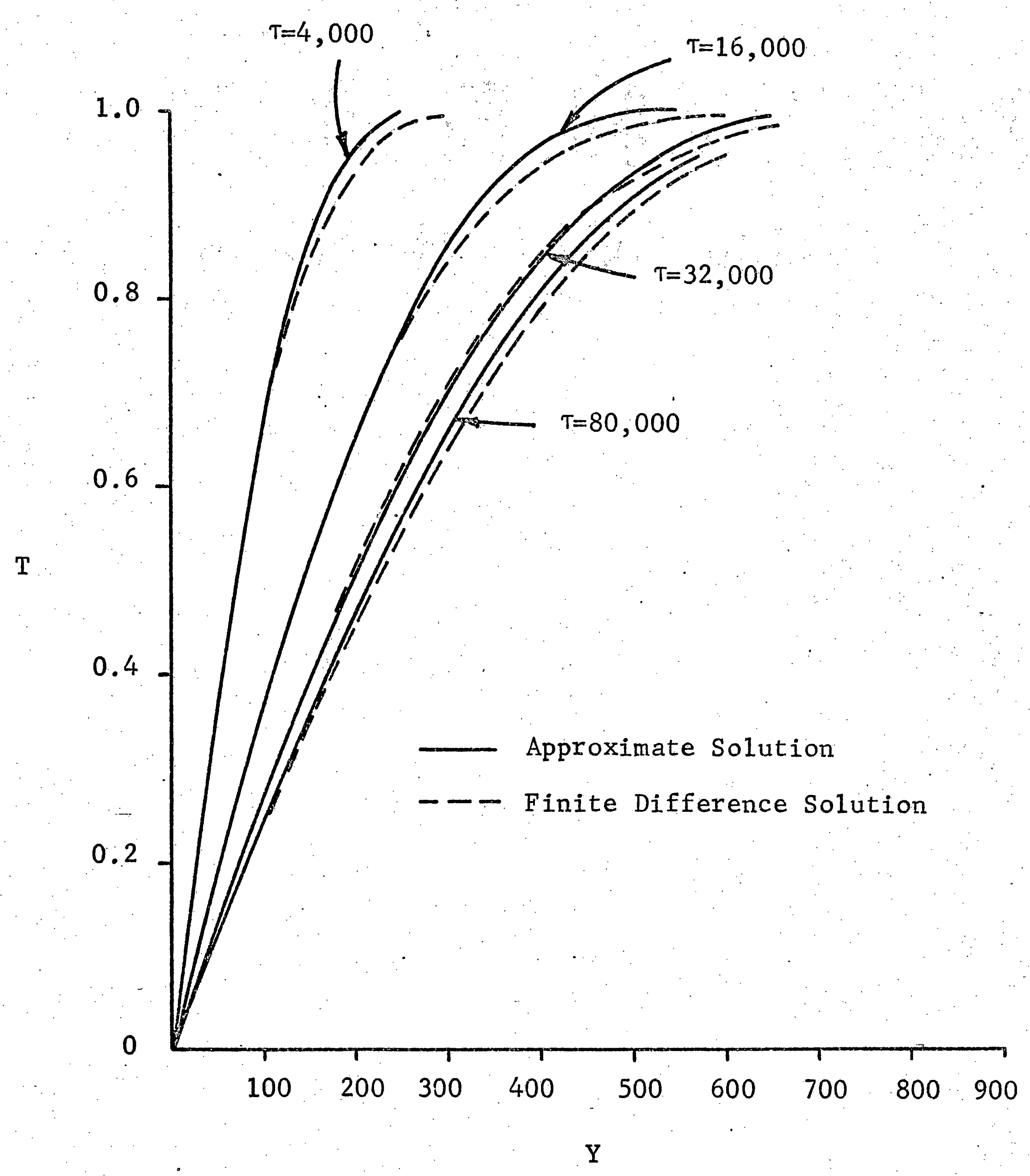

FIGURE 6-3. Comparison of temperature profiles for Pr=0.7 and X=15,000.

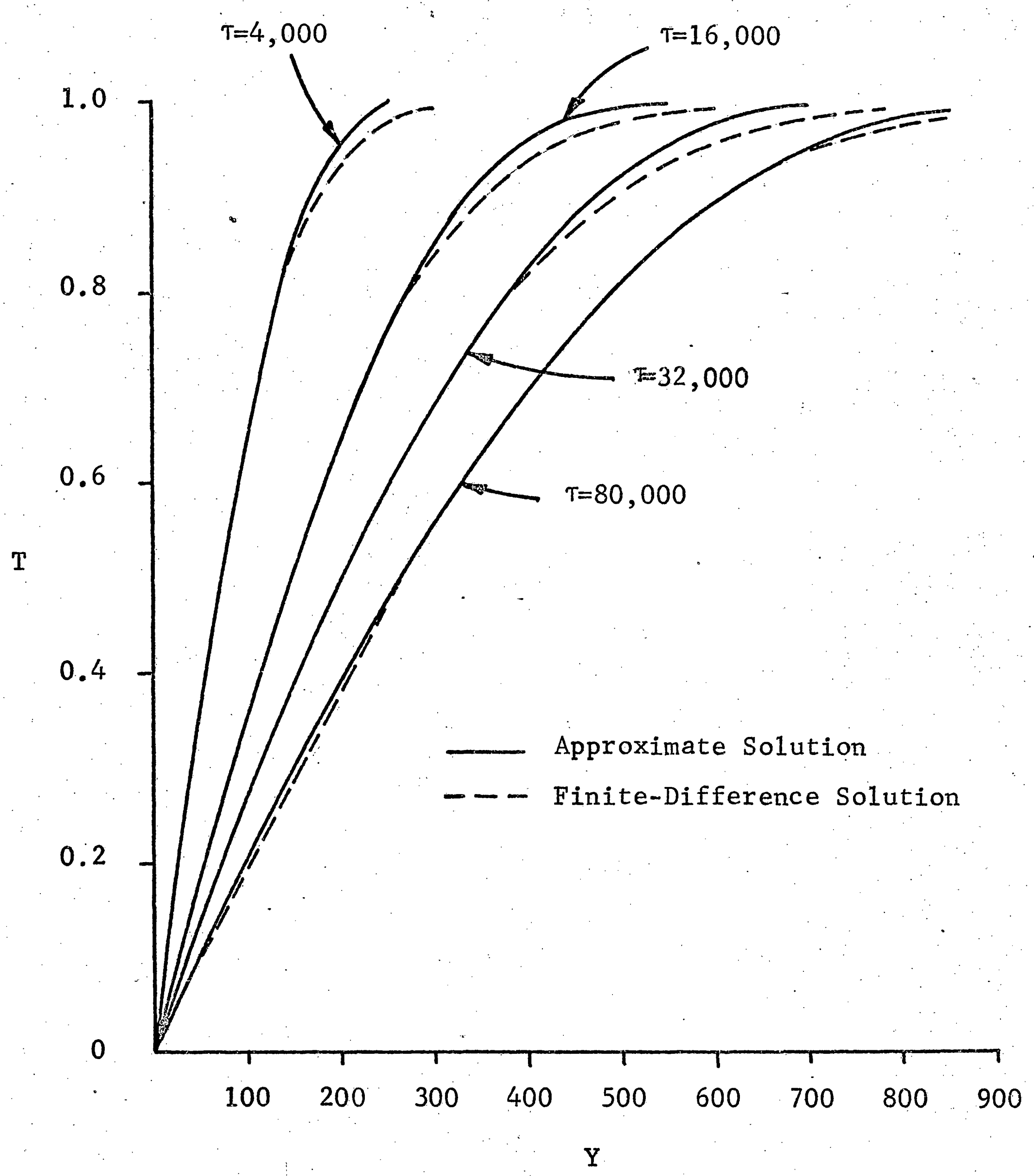

FIGURE 6-4. Comparison of temperature profiles for Pr=0.7 and X=22,500.

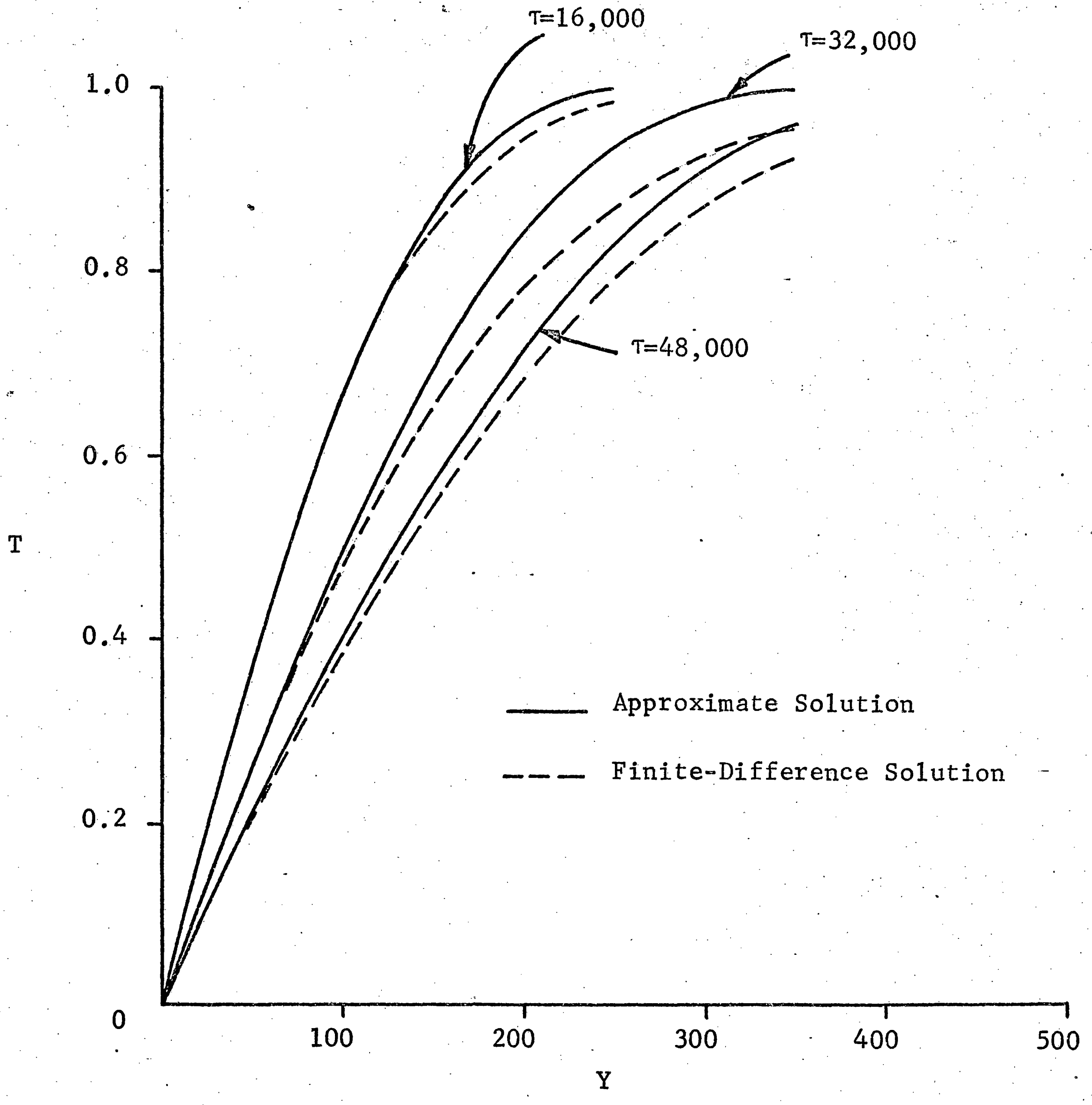

FIGURE 6-5. Comparison of temperature profiles for Pr=3.0 and X=15,000.

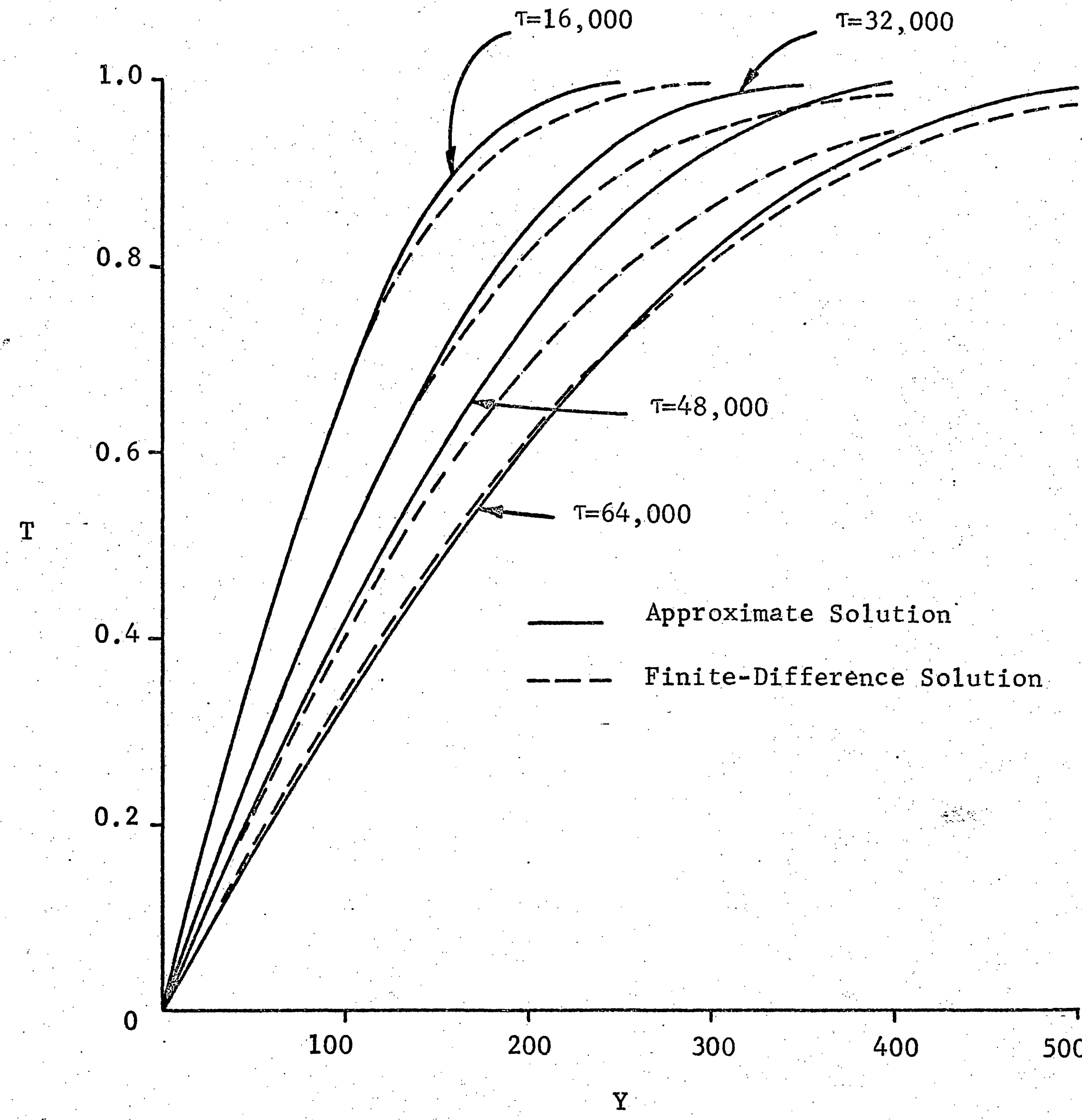

FIGURE 6-6. Comparison of temperature profiles for Pr=3.0 and X=22,500.

It can be concluded that the approximate solution for the temperature profile in the transition period between the transient condition and steady-state condition is a close approximation to the exact solution in this region.

Steady-State Nusselt Number

When the steady-state problem is solved by the method of changing the variables described briefly in the first section of this chapter the solution for the heat transfer can be obtained in the form of the Nusselt number. It is shown in Kays (8), Schlichting (14), and other publications that under steady-state conditions for a moderate Prandtl-number range the local Nusselt number can be approximated with good accuracy by:

$$Nu_x = 0.332\ Pr^{1/3}\ Re_x^{\ 1/2} \tag{6-10}$$

where:

$$Re_x = \frac{u_\infty x}{\nu} = X$$

Therefore:

$$\frac{Nu_x}{X^{1/2}} = 0.332\ Pr^{1/3} \tag{6-11}$$

Equation (4-64) gives the approximate solution for the Nusselt number:

$$Nu_x = \frac{2X}{\Delta_t} \tag{4-64}$$

For the steady-state condition:

$$\Delta_t = \left(\frac{K_1}{K_2}\right)^{1/2} X^{1/2}$$

Therefore:

$$\frac{Nu_x}{X^{1/2}} = \frac{2}{(\frac{K_1}{K_2})^{1/2}} \qquad (6\text{-}12)$$

Equations (6-11) and (6-12) are compared in Figure 6-7. From the figure it is seen that the difference in the two solutions for the Nusselt number increases steadily from near zero for a Prandtl number of 0.5 to about seven per cent for a Prandtl number of 10. In other words, for moderate Prandtl numbers the approximate solution gives a fairly accurate value of the steady-state local Nusselt number while for high Prandtl numbers the error may be large.

Approximate Versus Finite-Difference Solution for Nusselt Number

The accuracy of the approximate solution of the local Nusselt number at conditions other than steady state was determined by comparing it with the finite-difference solution described in Chapter V. The approximate solution for the Nusselt number is given by equation (4-64):

$$Nu_x = \frac{2X}{\Delta_t} , \qquad (4\text{-}64)$$

while the finite-difference expression for the Nusselt number is given by equation (5-12). Comparisons were made for Prandtl numbers of 0.7 and 3.0 at nondimensionalized distances from the leading edge of 15,000 and 22,500. These results are shown in Figures 6-8 and 6-9.

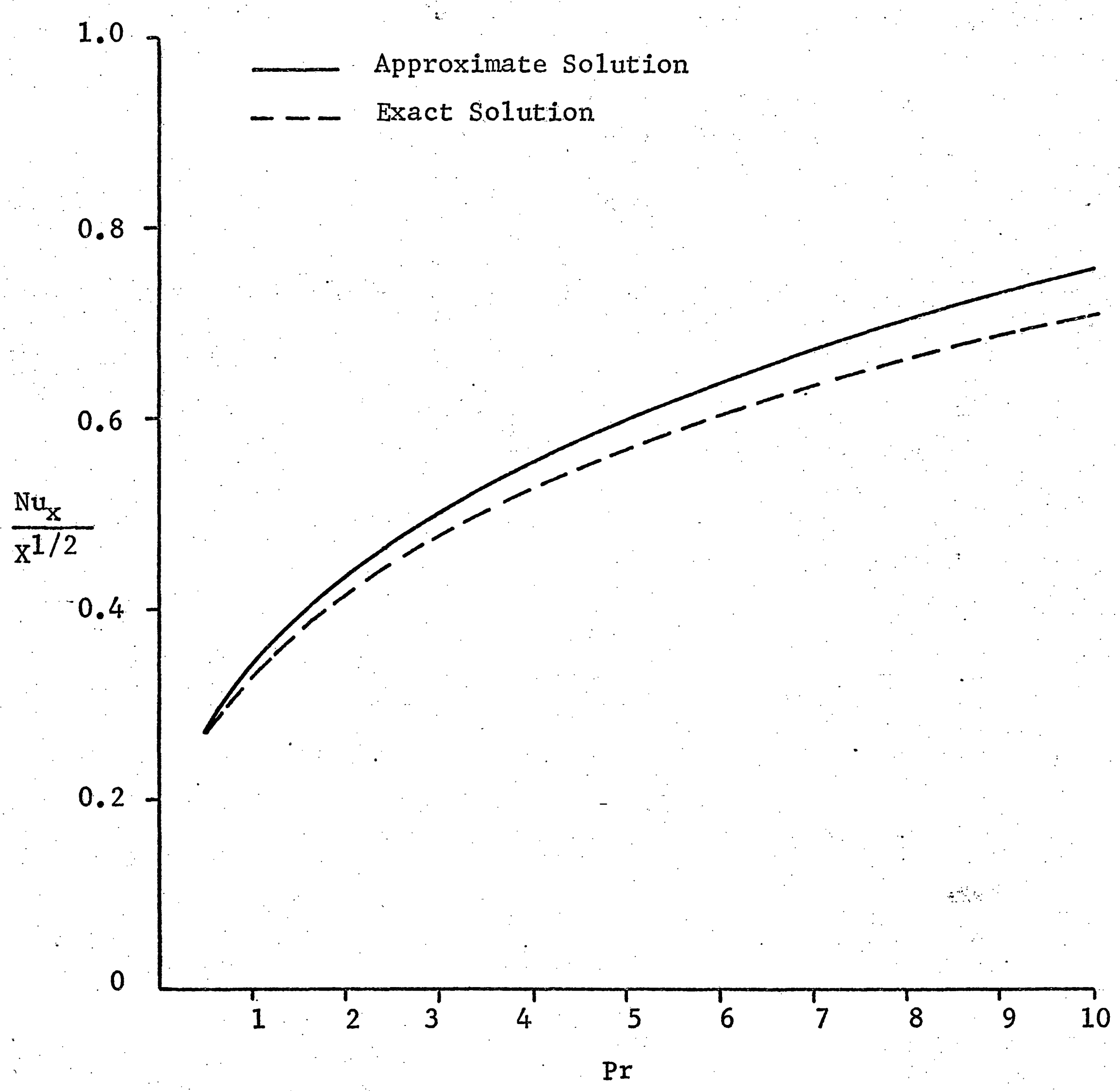

FIGURE 6-7. Comparison of steady-state Nusselt numbers.

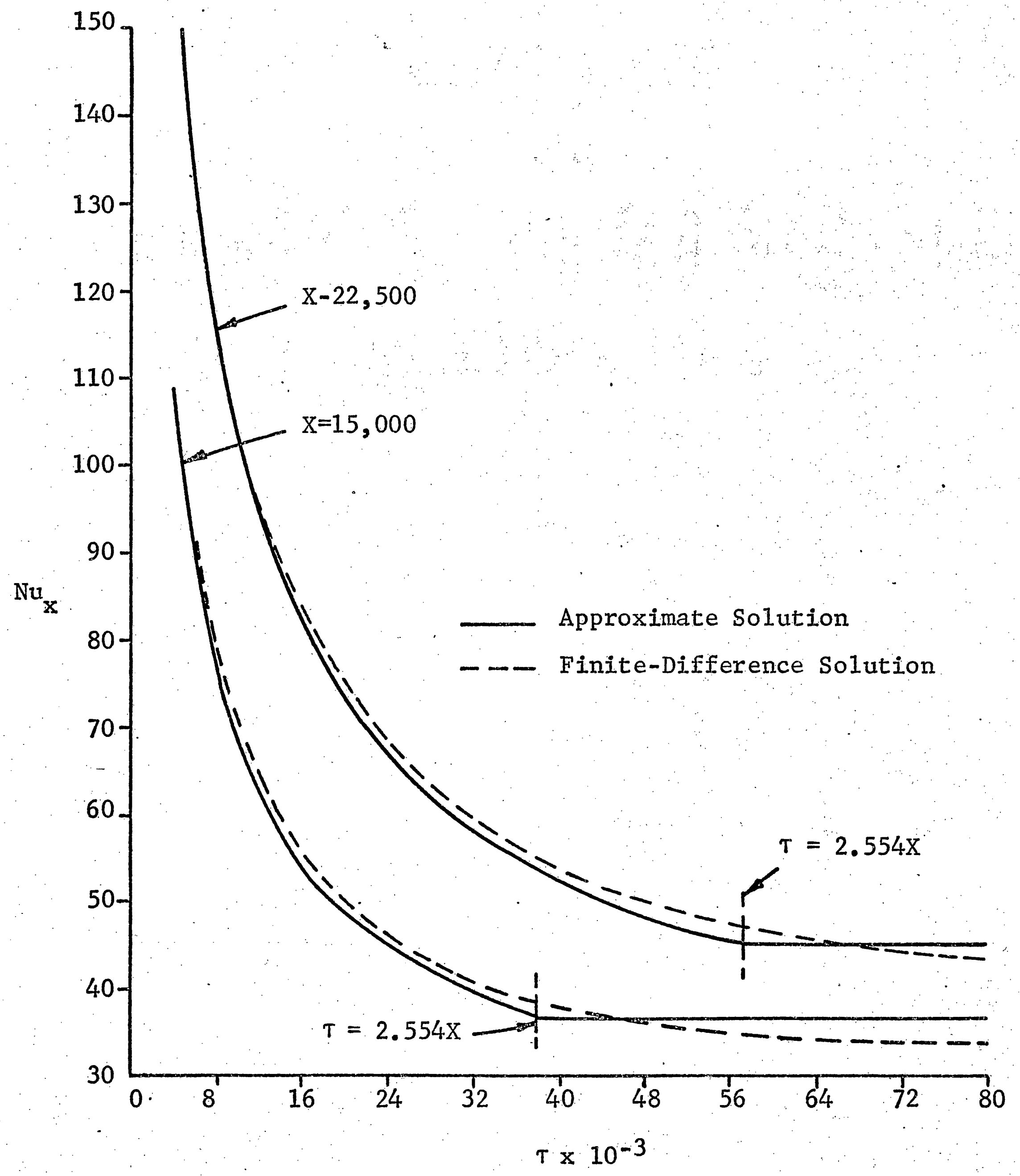

FIGURE 6-8. Comparison of Nusselt numbers for Pr=0.7.

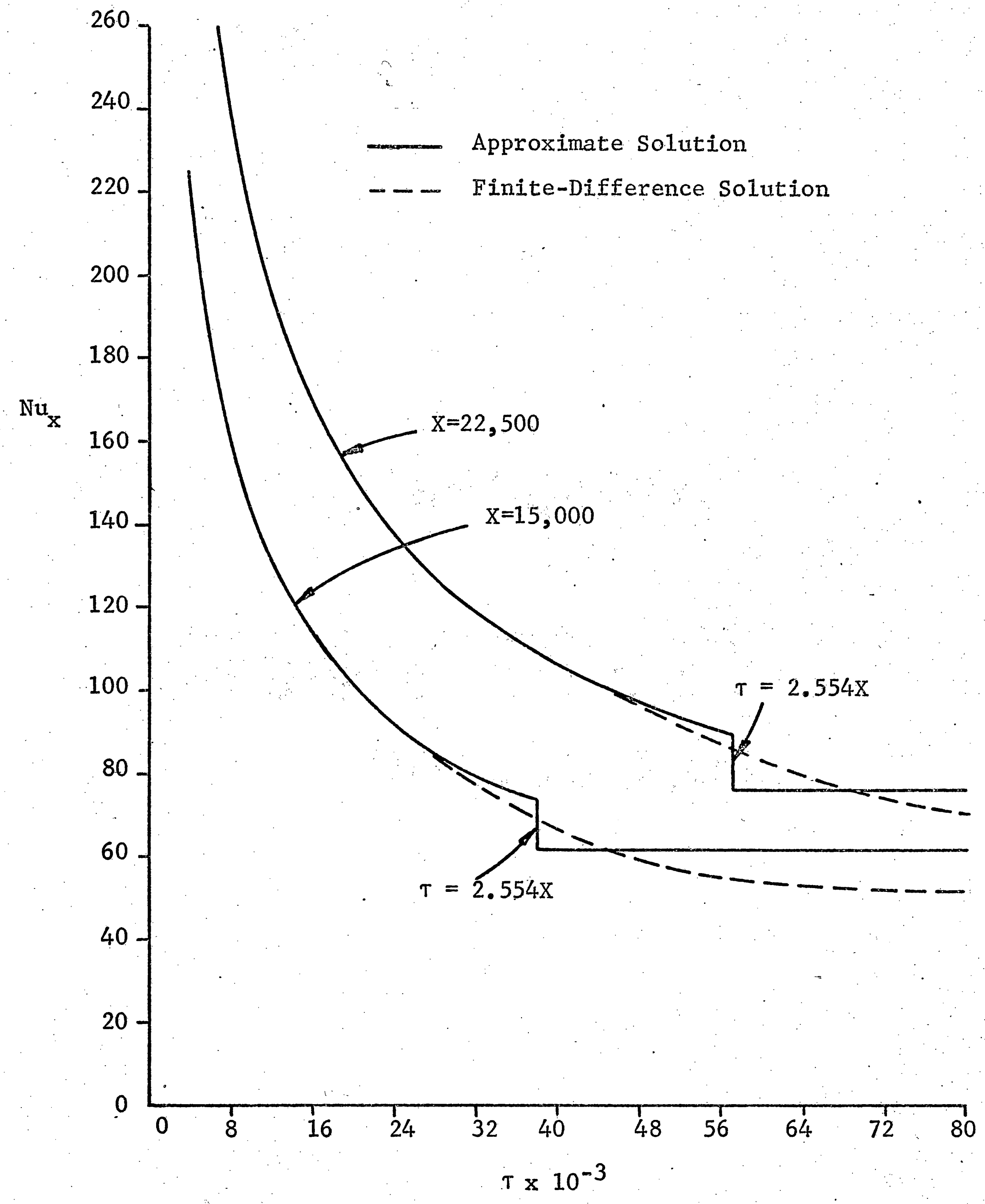

FIGURE 6-9. Comparison of Nusselt numbers for Pr=3.0.

The two solutions agree very closely for $\bar{\tau} \leq 2.554X$ but the difference increases after the approximate solution reaches the steady-state condition. The greatest difference in the solutions for the cases compared occurs for a Prandtl number of 3.0 at a nondimensionalized distance of 15,000 from the leading edge and is about ten per cent. For a Prandtl number of 0.7 the differences are not as large. As was the case for the temperature profiles, part of the difference can be attributed to errors in the finite-difference solution caused by the size of the increments used.

In Figure 6-9 the discontinuity in the Nusselt number in the approximate solution is caused by the discontinuity in the thermal boundary-layer thickness at $\tau = 2.554X$.

CHAPTER VII

DISCUSSION OF RESULTS

For a Prandtl number less than unity it was determined that the thermal boundary-layer thickness can be expressed by:

$$\Delta_t^2 = \frac{40}{3}\frac{\tau}{Pr} \text{ for } \tau \le \frac{3}{20}\frac{X}{B} \tag{4-20}$$

$$\Delta_t^2 = \frac{40}{3\,Pr}\left(\frac{\tau - 2.554X}{1 - 17B}\right) + \frac{K_1}{K_2}\left(\frac{X - \frac{20}{3}B\tau}{1 - 17B}\right) \tag{4-36}$$

$$\text{for } \frac{3}{20}\frac{X}{B} \le \tau \le 2.554X$$

$$\Delta_t^2 = \frac{K_1}{K_2} X \text{ for } \tau \ge 2.554X \tag{4-31}$$

where:

$$B = \frac{3}{20} - \frac{1}{15} Pr + \frac{9}{280} Pr^2 - \frac{1}{90} Pr^{5/2}$$

There are three regions of solution for this range of Prandtl numbers - one where the thermal boundary-layer thickness is a function of time only, one where it is a function only of the distance from the leading edge, and a transition region between these two where the boundary-layer thickness is a function of both time and distance from the leading edge.

The first region of solution occurs for a short time period after the plate has changed its velocity and temperature or at a large distance from the leading edge. The boundary layer begins building up at the leading edge and moves along the plate. At large distances from the leading edge this effect is not felt and the boundary layer builds up as if the plate were an infinite plate with no leading edge. It is well known that the exact solution of the infinite-plate problem gives a boundary-layer thickness which is a function of time only.

The region of solution expressed by equation (4-31) occurs at a short distance from the leading edge or after a large time has elapsed. The complete effect of the leading edge build-up has been felt and the thickness of the boundary layer is constant with time. In other words, this is the steady-state solution where the velocity and thermal boundary-layer thicknesses are functions only of the distance from the leading edge. The velocity and temperature profiles of the boundary layer are fully developed.

It is reasonable to expect that between these two regions of solution there exists a transition region wherein the thermal boundary-layer thickness is a function of both time and distance from the leading edge. The method of solution used in this thesis gives such a region for a Prandtl number less than unity and the solution in this region is given by equation (4-36). This solution mates smoothly with the solutions for the other two regions. There is no discontinuity in the thermal boundary layer thickness in passing from one region to another.

For a Prandtl number greater than unity it was determined that the thermal boundary-layer thickness can be expressed by:

$$\Delta_t^2 = \frac{40}{3}\frac{\tau}{Pr} \quad \text{for } \tau \leq 2.554X \tag{4-43}$$

$$\Delta_t^2 = \frac{K_1}{K_2} X \quad \text{for } \tau \geq 2.554X \tag{4-54}$$

There are only two regions of solution for this range of Prandtl numbers - one where the thermal boundary-layer thickness is a function of time only and one where it is a function only of the distance from the leading edge. It was attempted to obtain a solution in the transition region which would smoothly connect the solutions given by equations (4-43) and (4-54) as was done for a Prandtl number less than unity. However, these attempts were unsuccessful. There is a discontinuity in the thermal boundary-layer thickness across $\tau = 2.554X$, the point where the approximate solution used for the velocity boundary layer passes from the transient to the steady-state solution.

The solution for a Prandtl number less than unity is compatible with that for a Prandtl number greater than unity in that the two solutions are identical for a Prandtl number of unity.

The temperature at any point and time in the thermal boundary layer can be determined by substituting the expression for the thermal boundary thickness into the expression:

$$T = 2 \left(\frac{Y}{\Delta_t}\right) - 2 \left(\frac{Y}{\Delta_t}\right)^3 + \left(\frac{Y}{\Delta_t}\right)^4 \tag{3-20}$$

This temperature profile was compared to known temperature profiles for the infinite-plate problem, the steady-state problem, and to a finite-difference solution of the temperature profile in the transition region. There was good agreement between this approximate solution and the other solutions.

CHAPTER VIII

SUMMARY

An approximate technique has been developed which can be used to determine the thermal boundary-layer thickness at any time and any distance from the leading edge of a semi-infinite flat plate which has been set impulsively in motion in an incompressible fluid and has a simultaneous step change in temperature. From the thermal boundary-layer thickness the temperature at any time and position in the boundary layer can be determined. The heat transfer rate through the boundary layer can also be determined by using the local Nusselt number which can be determined for any time.

Solutions obtained by the approximate technique have been compared with solutions for special situations which are documented elsewhere and with a solution obtained by the method of finite differences; the approximate solutions compare favorably with the other solutions.

CHAPTER IX

RECOMMENDATIONS

It is recommended that this problem be investigated using the same approach in arriving at a solution but using different expressions for the velocity and temperature profiles. By using different profiles a continuous solution might be obtained for the thermal boundary layer thickness for Prandtl numbers greater than unity as well as for Prandtl numbers less than unity.

This same approach could be used to solve the more general problem of flow over a suddenly accelerated wedge given a sudden change in temperature, rather than for flow over a flat plate.

A somewhat similar but more difficult problem for which a solution could probably be obtained using this technique is that of determining the concentration of a fluid in the boundary layer for impulsive flow of the fluid over a flat plate at which the concentration of another fluid evaporating or being injected is suddenly changed.

BIBLIOGRAPHY

1. Adams, D. E. and Gebhart, B., "Transient Forced Convection from a Flat Plate Subjected to a Step Energy Input", Transactions ASME, Journal of Heat Transfer, Vol. 86C, p. 253, May 1964.

2. Akamatsu, T. and Kamimoto, G., "On the Incompressible Laminar Boundary Layer for Impulsive Motion of a Semi-Infinite Flat Plate", Department of Aeronautical Engineering, Kyoto University, Current Papers CP 10, June 1966.

3. Cess, R. D., "Heat Transfer to Laminar Flow across a Flat Plate with a Nonsteady Surface Temperature", Transactions ASME, Journal of Heat Transfer, Vol. 83C, p. 274, August 1961.

4. Cheng, Sin I, "Some Aspects of Unsteady Laminar Boundary Layer Flows", Quarterly of Applied Mathematics, Vol. 14, pp. 337-352, 1957.

5. Courant, R. and Hilbert, D., Methods of Mathematical Physics - Volume II - Partial Differential Equations, Interscience Publishers, 1966.

6. Farn, C. L., Arpaci, V. S. and Clark, J. A., "A Finite Difference Method for Computing Unsteady, Incompressible, Laminar Boundary-Layer Flows", Aerospace Research Laboratories, Office of Aerospace Research Report ARL 66-0010, (Contract No. AF 33(657)-8368), January 1966.

7. Goodman, T. R., "Effect of Arbitrary Nonsteady Wall Temperature on Incompressible Heat Transfer", Transactions ASME, Journal of Heat Transfer, Vol. 84C, p. 347, November 1962.

8. Kays, W. M., Convective Heat and Mass Transfer, McGraw-Hill, Inc., 1966.

9. Lighthill, M. J., "The Response of Laminar Skin Friction and Heat Transfer to Fluctuations in the Stream Velocity", Proceedings of the Royal Society, Series A, Vol. 224, pp. 1-23, 9 June 1954.

10. Low, G., "The Compressible Laminar Boundary Layer with Fluid Injection", NACA TN 3404, March 1955.

11. Moore, F. K., "Unsteady Laminar Boundary-Layer Flow", NACA TN 2471, September 1951.

12. Ostrach, S., "Compressible Laminar Boundary Layer and Heat Transfer for Unsteady Motions of a Flat Plate", NACA TN 3569, November 1955.

13. Sarma, G. N., "Unified Theory of the Solutions of the Unsteady Thermal Boundary-Layer Equations", Proceedings of the Cambridge Philosophical Society, Vol. 61, Part 3, p. 809, 1965.

14. Schlichting, H., Boundary Layer Theory, McGraw-Hill, Inc., 1960.

15. Schneider, P. J., Conduction Heat Transfer, Addison-Wesley Publishing Company, Inc., 1957.

16. Sparrow, E. M. and Starr, J. B., "The Transpiration-Cooled Flat Plate with Various Thermal and Velocity Boundary Conditions", International Journal of Heat and Mass Transfer, p. 508, May 1966.

17. Stewartson, K., "On the Impulsive Motion of a Flat Plate in a Viscous Fluid", Quarterly Journal of Mechanics and Applied Mathematics, Vol. 4, pp. 182-198, 1951.

18. Yang, K. T., "Unsteady Laminar Compressible Boundary Layers on Infinite Plate with Suction or Injection", Journal of the Aero Space Sciences, Vol. 26, No. 10, p. 653, 1959.

APPENDIX

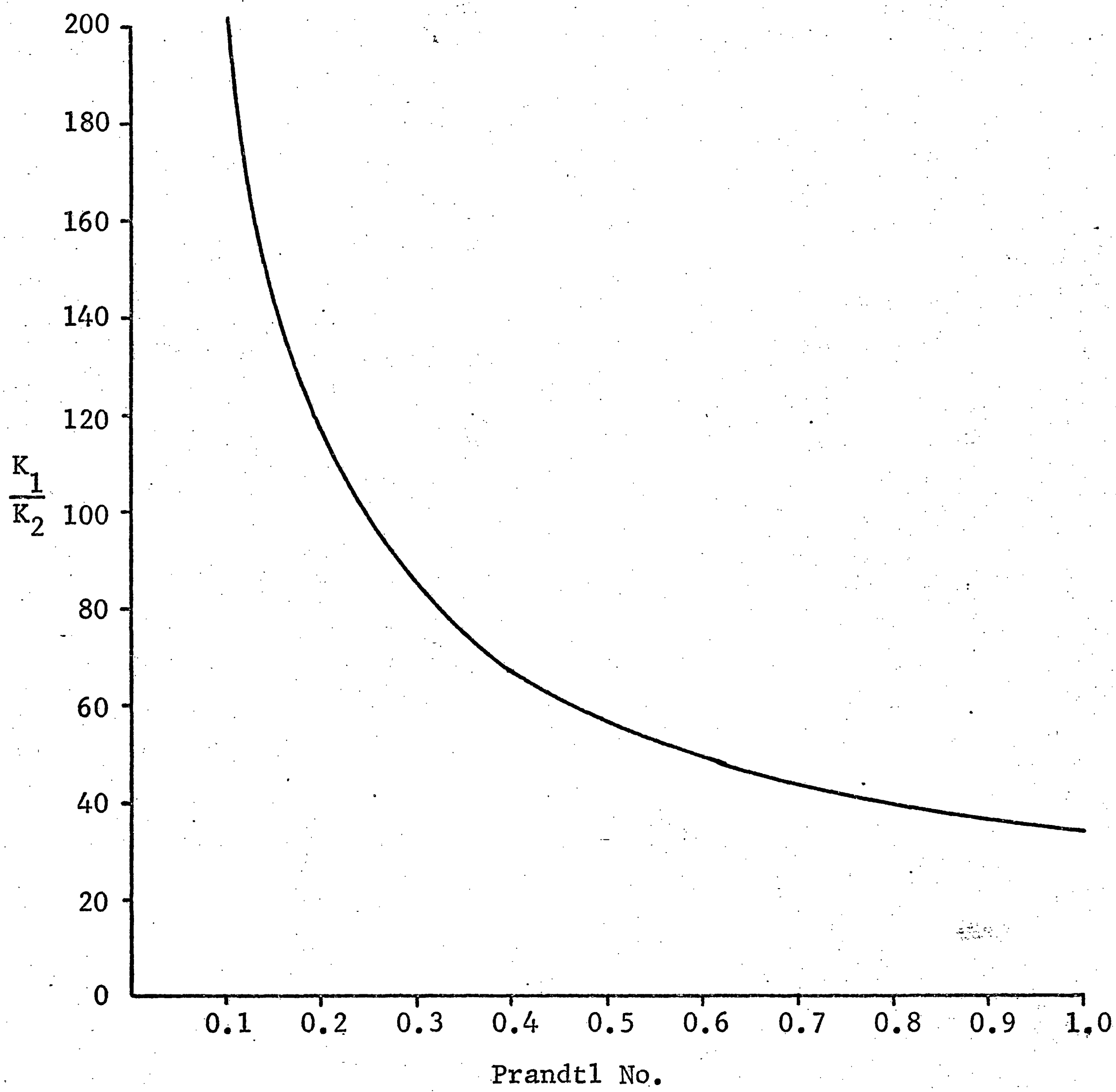

FIGURE A-1. Constant required for equation (4-31)

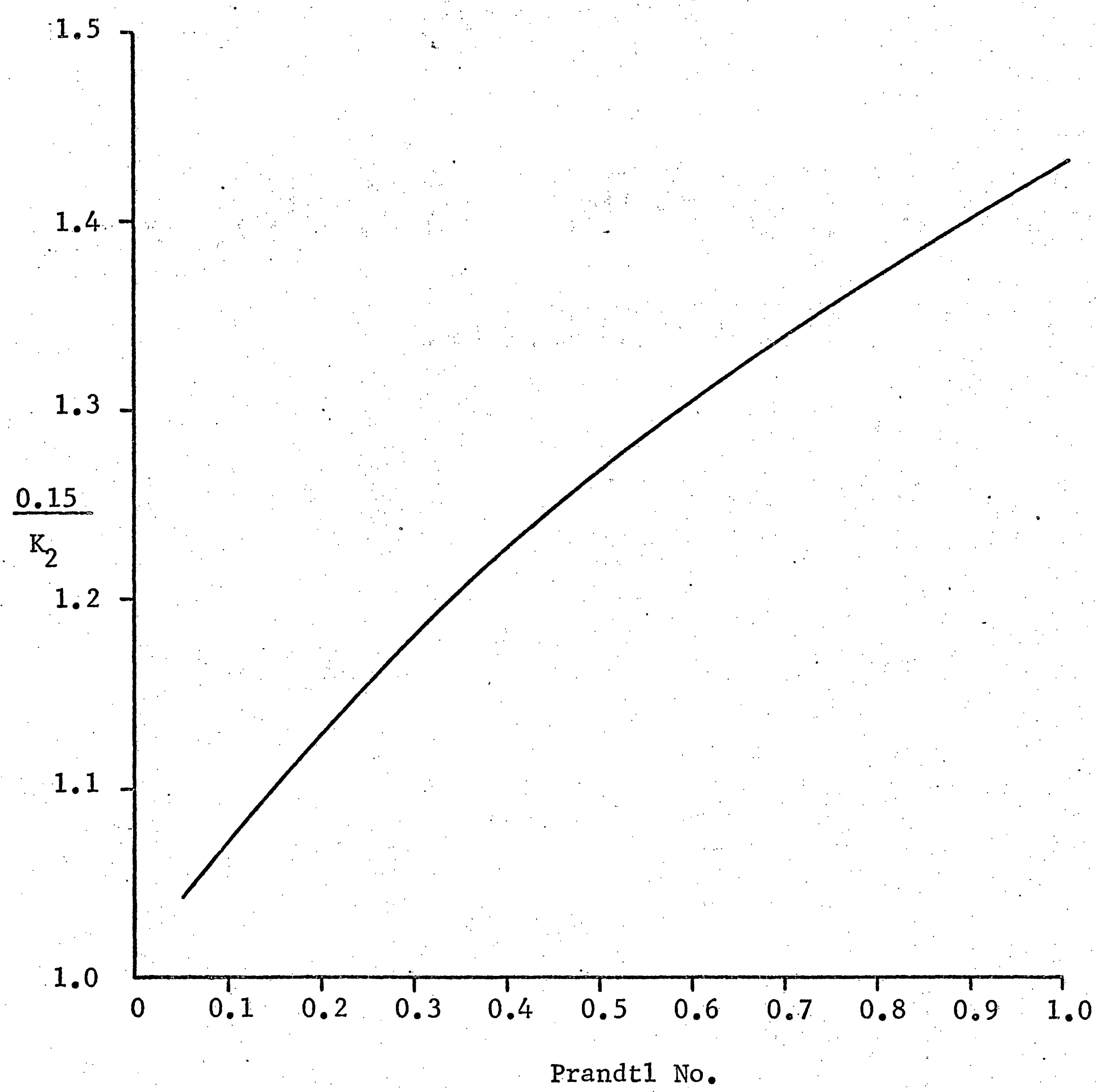

FIGURE A-2. Constant required for equation (4-32)

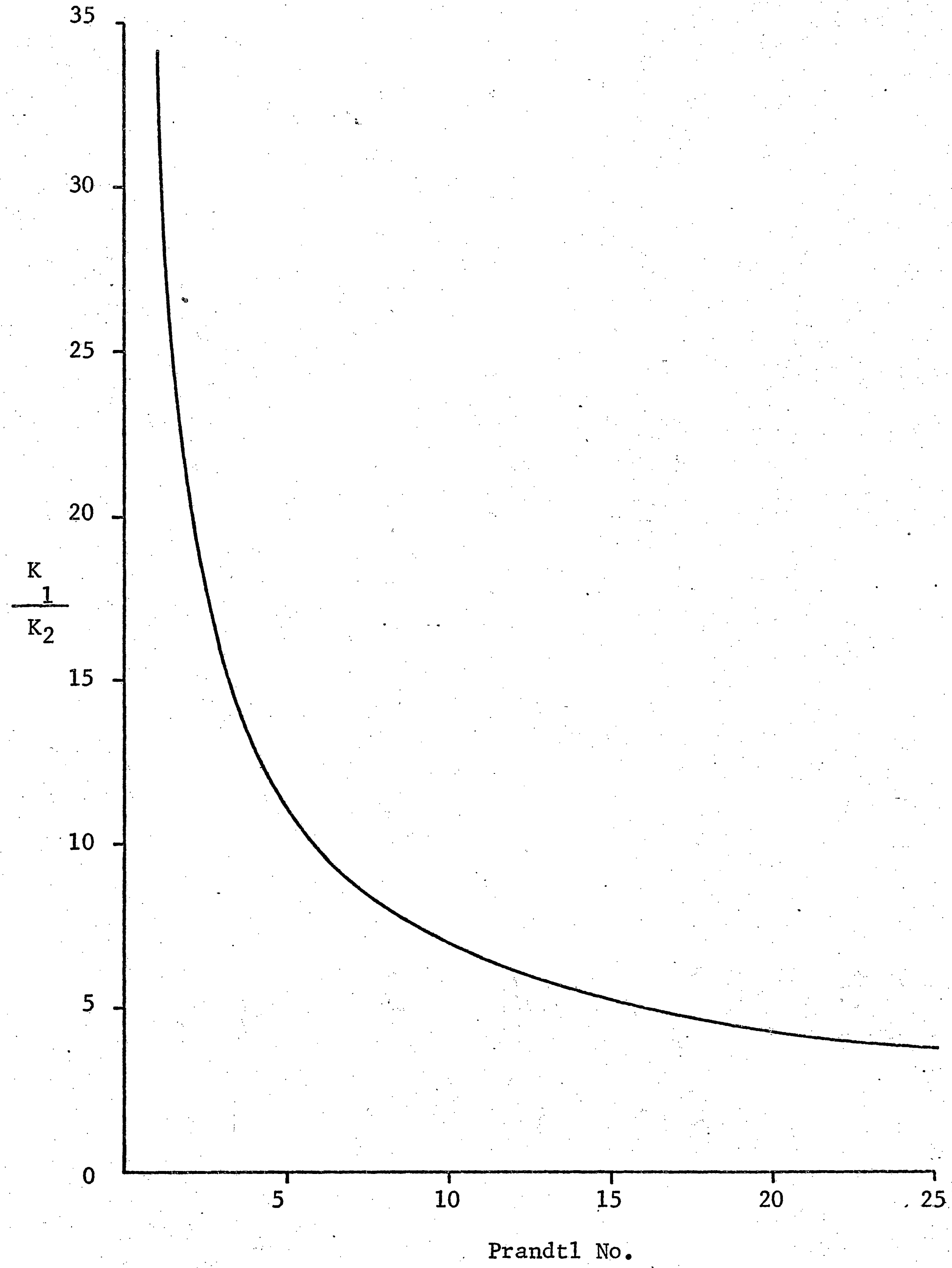

FIGURE A-3. Constant required for equation (4-54)

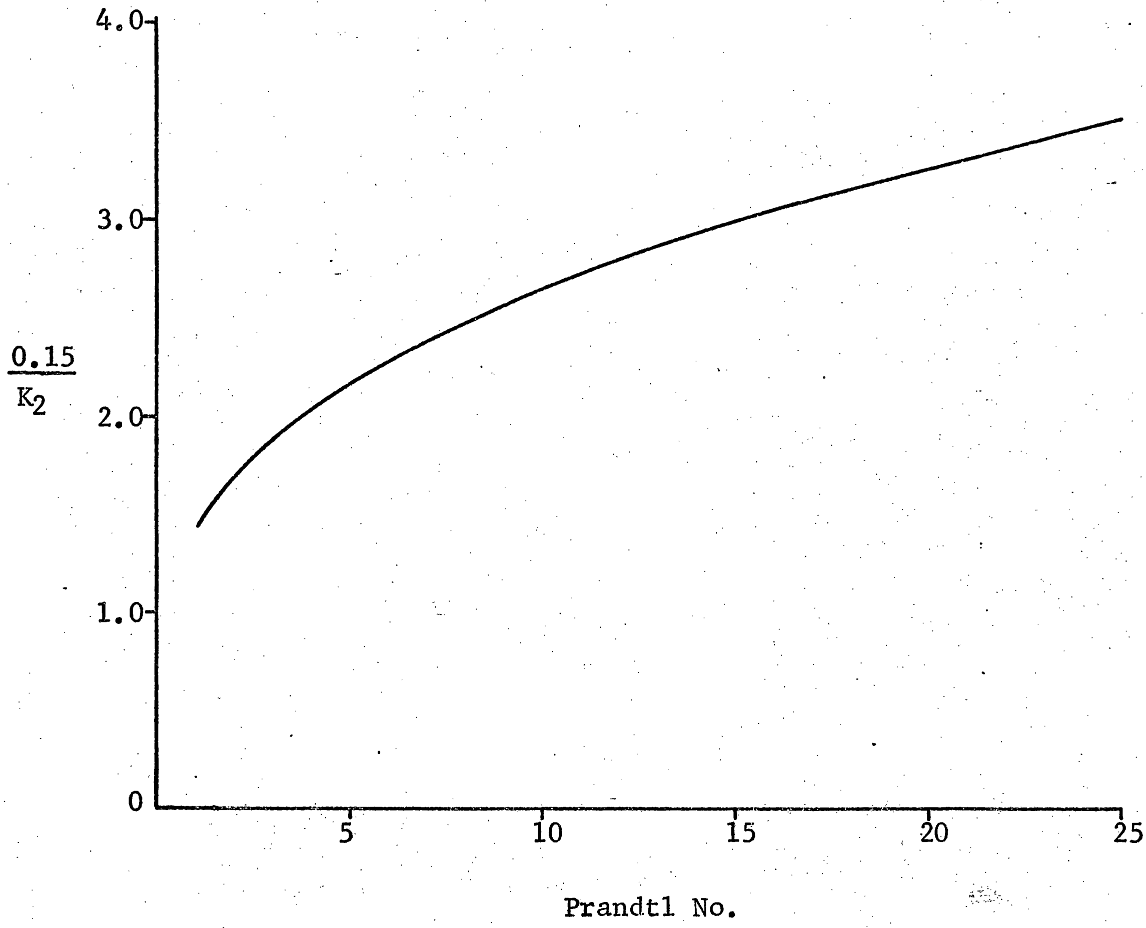

FIGURE A-4. Constant required for equation (4-55)

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SOLUTION OF THE LAMINAR BOUNDARY LAYER OF A SEMI-INFINITE FLAT PLATE GIVEN AN IMPULSIVE CHANGE IN VELOCITY AND TEMPERATURE

By

Michael D. Bare

The laminar boundary layer over a semi-infinite flat plate which is impulsively set in motion in an incompressible fluid and which has a simultaneous step change in surface temperature was studied. An approximate method was derived which can be used to determine the thermal boundary layer thickness as a function of the distance from the leading edge and of time. From the thermal boundary layer thickness the temperature of the fluid can be determined at any position in the boundary layer and at any time. The local Nusselt number can also be determined from the thermal boundary layer thickness.

The approximate solution was compared with exact steady-state and infinite-plate solutions of the energy equation and with a finite-difference solution of the unsteady continuity, momentum and energy equations. Agreement between the solutions was close enough to indicate that the approximate solutions for the temperature in the boundary layer and for the Nusselt number approximate the actual situation with reasonable accuracy.