DETERMINANTS OF MATRICES OVER LATTICES

by

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I. INTRODUCTION

In recent years, there has been an ever increasing amount of work done in Lattice Theory. In the beginning of his investigation, the author found much work done in the field of Boolean algebras, taking notice of three different definitions for the determinant of a Boolean matrix. It became his task to examine these definitions given in Wedderburn [12], Rutherford [8] and [9], and Sokolov [10] in the light of arbitrary lattices and determine which properties and relations were reminiscent of the determinant or permanent of elementary algebra. Due to the recent developments in Lattice Theory and the desire to present a self-contained paper, the author has included a preliminary chapter on the background necessary for the main discussions. Each determinant has been given a separate chapter, but in many cases there are corresponding definitions and properties. In each determinant there are properties concerning: the number of elements of a matrix in the expansion of its determinant; the determinant of a matrix and its transpose; a principle of duality for rows and columns; the interchange of a row or column; the determinant of a matrix formed from another by a row or column meet of an element; and evaluations of certain special matrices.

The First Determinant as we defined it has added properties concerning the join of a row or column with certain elements, an expansion by row or column, and counterexamples to other related properties of determinants of elementary algebra. Our Second Determinant also has an interesting lemma on its relation to the First Determinant. In our discussion on the

Third Determinant, we have defined a new matrix and its determinant in terms of the First with an added property concerning rows and columns. We concluded this chapter with a lemma on the relation of the last two determinants and a sufficient condition for a matrix to have an inverse.

II. PRELIMINARIES

The first section of this chapter contains the basic definitions and remarks needed to give the reader an understanding of the theory behind the lattices that will be used in this paper, while the second section is devoted to pertinent matrix theory. Finally, the third section develops the subject of permutations to an extent that will be sufficient for our purposes.

2.1 Basic Lattice Theory

The elementary concepts employed in the first part of this section can be found in Szász [11], Rutherford [8], and Birkhoff [3], while the latter part was formulated from the notes of Bevis [2].

Definition 2.1.1. A partially ordered set is an algebraic system in which a binary relation $x \leq y$ is defined, which satisfies the following postulates.

P_1 For all x, $x \leq x$. (reflexive property)

P_2 If $x \leq y$ and $y \leq x$, then $x = y$. (antisymmetric property)

P_3 If $x \leq y$ and $y \leq z$, then $x \leq z$. (transitive property)

Definition 2.1.2. A lattice L is a partially ordered set in which every pair of elements $\{x,y\}$ of L have a least upper bound or join, denoted by $x \vee y$; and a greatest lower bound or meet, denoted by $x \wedge y$.

Remark 2.1.3. For x,y, and z in any lattice L, the following identities hold:

(1) $x \vee y = y \vee x$; $x \wedge y = y \wedge x$, (commutative laws)

(2) $(x \vee y) \vee z = x \vee (y \vee z)$; $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, (associative laws)

(3) $x \vee (x \wedge y) = x$; $x \wedge (x \vee y) = x$, (absorptive laws)

(4) $x \vee x = x$; $x \wedge x = x$. (idempotent laws)

Lemma 2.1.4. If $x, y \in L$, then

(1) $x \leq y$ if and only if $x \wedge y = x$;

(2) $x \leq y$ if and only if $x \vee y = y$.

Definition 2.1.5. If a lattice L has an element o such that any element x of L satisfies the inequality $o \leq x$, then o is called the least element of L. If a lattice L has an element 1 such that any element x of L satisfies the inequality $1 \geq x$, then 1 is called the greatest element of L. These elements will be called the bound elements of L and we will say that L is bounded provided it has a least and greatest element.

Remark 2.1.6. If a partially ordered set does not already have a least element and a greatest element, it may be equipped with them, therefore, we will assume throughout the rest of this paper that L is bounded.

Remark 2.1.7. By the definition of the ordering of lattices, the least element o and the greatest element 1 of the lattice L satisfy the identities:

(1) $o \wedge x = o$, $o \vee x = x$;

(2) $1 \wedge x = x$, $1 \vee x = 1$ for all $x \in L$.

Definition 2.1.8. An involution poset is a partially ordered set L, together with a mapping or unary operation $' : L \longrightarrow L$, called an involution, such that:

(1) $x \leq y$ implies $y' \leq x'$ for all $x,y \in L$.

(2) $x'' = (x')' = x$ for all $x \in L$.

We see by (2) of Definition 2.1.6 that the involution is an onto mapping and also by (1) that a generalized DeMorgan Law holds in the involution poset, that is,

$$(1) \quad \left(\bigvee_{i=1}^{n} x_i\right)' = \bigwedge_{i=1}^{n} x_i' \text{ and}$$

$$(2) \quad \left(\bigwedge_{i=1}^{n} x_i\right)' = \bigvee_{i=1}^{n} x_i'.$$

Definition 2.1.9. By a complement of an element x of a lattice L with o and 1 is meant an element $y \in L$ such that $x \wedge y = o$ and $x \vee y = 1$. A lattice in which every element has at least one complement, which may not be unique, is called a complemented lattice.

Definition 2.1.10. If $' : L \longrightarrow L$ is an involution and if x' is a complement of x for each x in L, then $'$ is called an orthocomplementation and L is an orthocomplemented lattice.

Definition 2.1.11. A lattice L will be called distributive if and only if it satisfies

(1) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and

(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for $x,y,z \in L$.

Definition 2.1.12. A lattice L is called modular if and only if its elements satisfy the <u>modular identity</u>:

$$\text{if } x \leq z, \text{ then } x \vee (y \wedge z) = (x \vee y) \wedge z \text{ for } x,y,z \in L.$$

<u>Definition 2.1.13</u>. A lattice L is <u>orthomodular</u> provided it is orthocomplemented and satisfies the <u>orthomodular identity</u>:

$$\text{if } x \leq y, \text{ then } x \vee (x' \wedge y) = y \text{ for } x,y \in L.$$

We now continue with some useful results that will be pertinent in the development of the determinant.

<u>Definition 2.1.14</u>. By the <u>direct product</u> of the lattices $L_1,\ldots,L_n$, denoted by $L_1 \otimes \ldots \otimes L_n$, we mean the algebra defined on the product set $L_1 \times \ldots \times L_n$, in which

(1) $(x_1,\ldots,x_n) = (y_1,\ldots,y_n)$ if and only if $x_i = y_i$ for all $i=1,\ldots,n$;

(2) $(x_1,\ldots,x_n) \wedge (y_1,\ldots,y_n) = (x_1 \wedge y_1,\ldots,x_n \wedge y_n)$, and

(3) $(x_1,\ldots,x_n) \vee (y_1,\ldots,y_n) = (x_1 \vee y_1,\ldots,x_n \vee y_n)$.

Furthermore, the ordering of the lattice $L_1 \otimes \ldots \otimes L_n$ is described by the formula $(x_1,\ldots,x_n) \leq (y_1,\ldots,y_n)$ if and only if $x_i \leq y_i$ for all $i = 1,\ldots,n$. We shall denote the direct product of L with itself n times by $\underset{n}{\otimes} L$.

<u>Remark 2.1.15</u>. If L is an orthomodular lattice, then $\underset{n}{\otimes} L$ is also an orthomodular lattice.

<u>Definition 2.1.16</u>. In an orthocomplemented lattice L, we say that x <u>commutes with</u> y and write $x \, C \, y$ if $(x \vee y') \wedge y = x \wedge y$.

<u>Remark 2.1.17</u>. If L is an orthomodular lattice, the C is a symmetric relation, that is, $x \, C \, y$ if and only if $y \, C \, x$.

<u>Foulis-Holland Theorem 2.1.18</u>. In an orthomodular lattice, if two of the three relations $x \, C \, y$, $x \, C \, z$, $y \, C \, z$ hold, then

$$(1) \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) \text{ and}$$

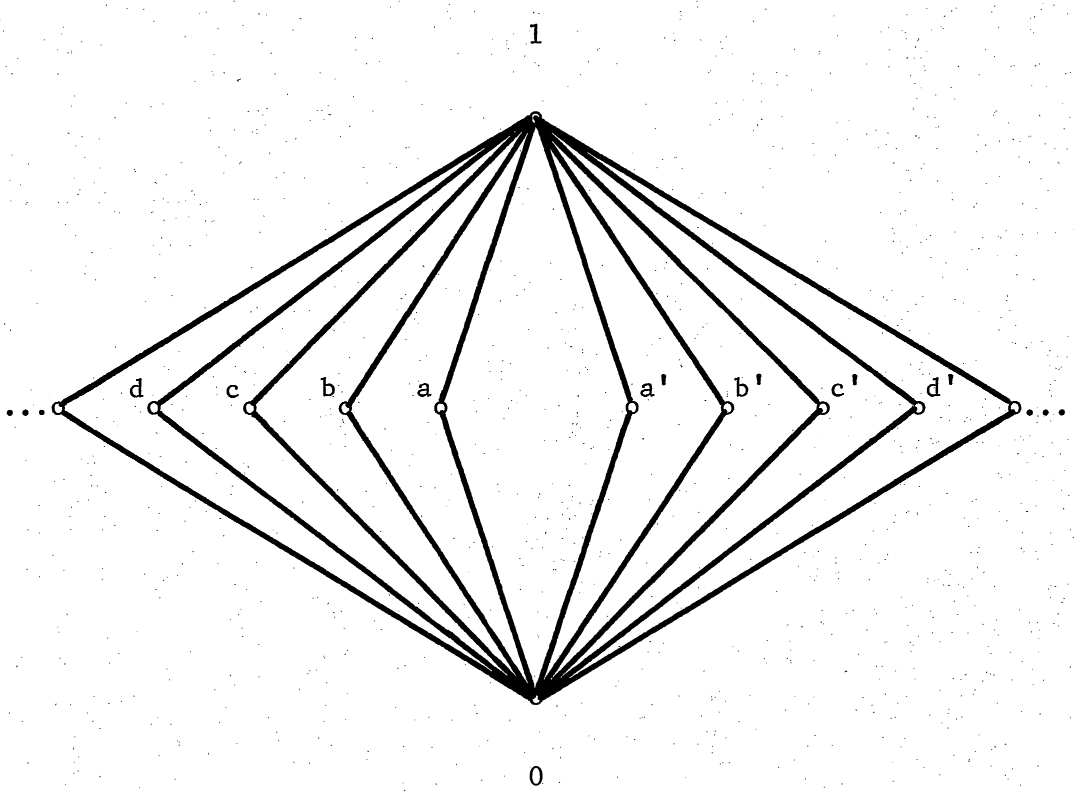
$$(2) \quad (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z).$$

<u>Definition 2.1.19</u>. Let L be an orthomodular lattice. If $N \subseteq L$, then $C(N) = \{x \in L : x \, C \, y \text{ for all } y \in N\}$. $C(N)$ is called the <u>centralizer</u> of N. If $N = L$, $C(L)$ is called the <u>center of L.</u>

<u>Remark 2.1.20</u>. Let L be an orthomodular lattice, then $C(L)$ is closed under the operations of meet, join, and orthocomplementation.

<u>Remark 2.1.21</u>. If $L = L_1 \times \ldots \times L_n = \{(x_1, \ldots, x_n) : x_i \in L_i \text{ for all } i\}$, then we can say $(x_1, \ldots, x_n) \in C(L)$ if and only if $x_i \in C(L_i)$ for all $i = 1, \ldots, n$.

<u>Example 2.1.22</u>. An interesting example that can be used to show many of the results obtained in the forthcoming chapters is the lattice of subspaces of Euclidean 2-space, whose Hasse diagram is the following.



Any finite portion of a lattice can be represented by a Hasse diagram in which distinct elements of L are represented by distinct circles and in which each relation $x \geq y$ is represented by a line or lines which descend steadily from x to y. The above example is an orthocomplemented modular lattice, and hence, it is an orthomodular lattice.

2.2 Matrices : Fundamental Concepts and Operations

The fundamental definitions and concepts concerning matrices can be found in Eves [4] and Marcus and Minc [6]. Our development of matrices over lattices is similar to the work done in Boolean matrices found in Luce [6], Rutherford [8], Birkhoff [3], and Flegg [5].

<u>Definition 2.2.1</u>. A <u>matrix A of order m × n</u> is a rectangular array of mn elements from a given set arranged in m rows and n columns:

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where $A_{ij} = a_{ij}$ denotes the element in the i^{th} row and j^{th} column and $i = 1,\ldots,m$; $j = 1,\ldots,n$. If $m = n$, A is called an <u>n-square matrix</u>. If matrix A is square of order n, the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are said to constitute the <u>principal diagonal</u> of A.

A matrix of order $n \times 1$ is called a <u>column matrix</u> and a matrix of order $1 \times n$ is called a <u>row matrix</u>.

We will interchange the use of A_{ij} and a_{ij} to denote the ij^{th} element of the matrix A whichever is most convenient in the context.

<u>Definition 2.2.2</u>. Let $M_n(L)$ be the <u>set of n-square matrices</u> A,B,... whose elements $a_{ij}, b_{ij}, \ldots,$ $i,j = 1,\ldots,n$ belong to the lattice L.

We now consider some basic operations and relations of members of $M_n(L)$.

<u>Definition 2.2.3</u>. The two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be <u>equal</u> if and only if $a_{ij} = b_{ij}$ for all $i,j = 1,\ldots,n$.

<u>Definition 2.2.4</u>. The <u>join of two matrices</u> $A = (a_{ij})$ and $B = (b_{ij})$ is

defined to be the matrix $C = (c_{ij})$, where $c_{ij} = a_{ij} \vee b_{ij}$.

Definition 2.2.5. The meet of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined to be the matrix $C = (c_{ij})$, where $c_{ij} = a_{ij} \wedge b_{ij}$.

Definition 2.2.6. Let $A = (a_{ij})$, $c \in L$, then we define a scalar meet and join as follows:

(1) $c \wedge A = C = (c_{ij})$, where $c_{ij} = c \wedge a_{ij}$ for all i,j.

(2) $c \vee A = C = (c_{ij})$, where $c_{ij} = c \vee a_{ij}$ for all i,j.

Definition 2.2.7. By matrix inclusion, we mean $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i,j = 1,\ldots,n$.

An immediate consequence of the above is the following lemma.

Lemma 2.2.8. $M_n(L)$ is lattice isomorphic to $\otimes_{n^2} L$.

We now define and exhibit certain matrices that will be of importance in our later developments of the determinant.

Definition 2.2.9. A matrix each of whose elements is one is called the universal matrix, and will be denoted by

$$I = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \cdot & \cdot & \cdots & \cdot \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Definition 2.2.10. A matrix each of whose elements is zero is called the zero matrix, and will be denoted by

$$0 = \begin{bmatrix} o & o & \cdots & o \\ o & o & \cdots & o \\ \cdot & \cdot & \cdots & o \\ o & o & \cdots & o \end{bmatrix}.$$

Definition 2.2.11. A matrix whose elements along the principal diagonal are ones with zeroes elsewhere is called the unit matrix, and will be denoted by

$$E = \begin{bmatrix} 1 & o & \cdots & o \\ o & 1 & \cdots & o \\ \cdot & \cdot & \cdots & \cdot \\ o & o & \cdots & 1 \end{bmatrix}.$$

Definition 2.2.12. A square matrix $A = (a_{ij})$ such that $a_{ij} = o$ if $i \neq j$ and $a_{ij} = c \in L$ if $i = j$ is called a scalar matrix, and $A = c \wedge E =$

$$\begin{bmatrix} c & o & \cdots & o \\ o & c & \cdots & o \\ \cdot & \cdot & \cdots & \cdot \\ o & o & \cdots & c \end{bmatrix}.$$

Definition 2.2.13. A square matrix $A = (a_{ij})$ such that $a_{ij} = o$ if $i \neq j$ is called a diagonal matrix, and the principal diagonal elements are $a_{11}, a_{22}, \ldots a_{nn}$.

Definition 2.2.14. In a square matrix $U^t = (a_{ij})$, if $a_{ij} = o$, $i > j$, so that

$$U^t = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ o & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ o & o & \cdots & a_{nn} \end{bmatrix},$$

then U^t is called an upper triangular matrix. A square matrix $U = (a_{ij})$ such that, $a_{ij} = o$ if $i \geq j$, is called an upper matrix, and

$$U = \begin{bmatrix} o & a_{12} & \cdots & a_{1n} \\ o & o & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ o & o & \cdots & o \end{bmatrix}.$$

Similarly, if $a_{ij} = o$, $i < j$, then U_t is called a lower triangular matrix and if $a_{ij} = o$, $i \leq j$, then U is called a lower matrix.

Definition 2.2.15. Associated with the matrix $A = (a_{ij})$ is the complement of A, defined by $A' = (a'_{ij})$ where a'_{ij} is the complement of a_{ij} for all i,j.

Definition 2.2.16. Associated with the matrix $A = (a_{ij})$ is the matrix A^T, the transpose of A, whose ij^{th} entry is the ji^{th} entry of A. Thus $(A^T)_{ij} = a_{ji}$.

Now we proceed to define matrix multiplication in order to be able to define the inverse of a matrix. Then we close this section with the statement of a lemma concerning necessary and sufficient conditions for matrices over a lattice to have an inverse. This lemma and its proof was originally done for Boolean matrices by Luce [6] and has recently been shown to be true for matrices over arbitrary lattices with 0 and 1 by Bevis [1].

Definition 2.2.17. If A and B are members of $M_n(L)$, then the matrix product of A and B is the matrix C, where $c_{ij} = \bigvee_{k=1}^{n} (a_{ik} \wedge b_{kj})$.

Remark 2.2.18. In general, matrix multiplication is neither associative nor commutative.

Definition 2.2.19. Let $A \in M_n(L)$, an inverse of A, if it exists, denoted by A^{-1}, is a square matrix such that $AA^{-1} = A^{-1}A = E$.

Definition 2.2.20. The matrix A is said to be orthogonal if $A^{-1} = A^T$.

Lemma 2.2.21. For a matrix $A \in M_n(L)$,

(1) $AA^T = E$ if and only if $\bigvee_{k=1}^{n} a_{ik} = 1$ for all i and $a_{ik} \wedge a_{jk} = o$ for all i,j,k with $i \neq j$, and

(2) $A^TA = E$ if and only if $\bigvee_{k=1}^{n} a_{kj} = 1$ for all j and $a_{kj} \wedge a_{ki} = o$ for all i,j,k with $i \neq j$.

2.3 Permutations

The following discussion is similar to those found in Eves [4] and Marcus and Minc [7]. We need to develope the idea of permutations in order to have a better understanding of the definitions of our determinants.

<u>Definition 2.3.1</u>. A <u>permutation</u> on n objects, labeled $1,\ldots,n$, is a one-one mapping of the set $\{1,\ldots,n\}$ onto itself.

We shall denote the image of i under a permutation σ by $\sigma(i)$.

<u>Remark 2.3.2</u>. We shall denote the set of all permutations of $1,\ldots,n$ by P_n.

<u>Lemma 2.3.3</u>. There are n! distinct permutations in P_n.

<u>Definition 2.3.4</u>. A situation in a permutation $(\sigma(1), \sigma(2),\ldots,\sigma(n))$ of $1,2,\ldots,n$ in which $\sigma(r)$ preceeds $\sigma(s)$ and $\sigma(r) > \sigma(s)$, is called an <u>inversion</u>. The permutation is said to be <u>even</u> or <u>odd</u>, according as it possesses an even or odd number of inversions.

<u>Remark 2.3.5</u>. We shall denote the set of all even permutations of $1,\ldots,n$ by P_n^+ and the set of all odd permutations of $1,\ldots,n$ by P_n^-.

<u>Definition 2.3.6</u>. The operation of interchanging any two distinct elements of a permutation $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ of $1,\ldots,n$ is called a <u>transposition</u>.

<u>Lemma 2.3.7</u>. A transposition converts an even (odd) permutation of $1,\ldots,n$ into and odd (even) permutation of $1,\ldots,n$.

<u>Lemma 2.3.8</u>. Of the n! permutations of $1,\ldots,n$ where $n > 1$, exactly half are odd and half are even.

III. THE FIRST DETERMINANT

We now proceed to develope the definitions and properties of a certain scalar-valued function of the matrices in $M_n(L)$, which we will call the First Determinant. A similar definition and brief discussion done for the determinant of a Boolean matrix can be found in Rutherford [8] and [9], while Flegg [5] mentions a definition and some properties of a determinant of a switching matrix done in Boolean algebra. The statements of some definitions and theorems are similar to those in Marcus and Minc [7] and Eves [4].

<u>Definition 3.1</u>. Let $A \in M_n(L)$. If $\sigma \in P_n$ then the <u>diagonal</u> corresponding to σ is the n-tuple of elements from L, $(a_{1\sigma(1)}, a_{2\sigma(2)}, \ldots, a_{n\sigma(n)})$, and for any given permutation σ, $\bigwedge_{i=1}^{n} a_{i\sigma(i)} = a_{1\sigma(1)} \wedge a_{2\sigma(2)} \wedge \ldots \wedge a_{n\sigma(n)}$ is called a <u>joinand</u> of A.

<u>Definition 3.2</u>. Let $A \in M_n(L)$. We define the <u>first determinant of A</u> as the join over all $\sigma \in P_n$ of the joinands of A and we denote it by $|A|_1$. Thus

$$|A|_1 = \bigvee_{\sigma \in P_n} (a_{1\sigma(1)} \wedge a_{2\sigma(2)} \wedge \ldots \wedge a_{n\sigma(n)}) = \bigvee_{\sigma \in P_n} \left(\bigwedge_{i=1}^{n} a_{i\sigma(i)}\right).$$

<u>Remark 3.3</u>. There are n! joinands in the expansion of a determinant of $A \in M_n(L)$.

<u>Lemma 3.4</u>. Each joinand of $|A|_1$ contains one and only one element from each row and column of A.

Proof: Immediate consequence of the definition of the first determinant.

Theorem 3.5. If $A \in M_n(L)$, then $|A^T|_1 = |A|_1$.

Proof: Let $A = (a_{ij})$, then $(A^T)_{ij} = a_{ji}$ and $|A^T|_1 = \bigvee_{\sigma \in P_n} (\bigwedge_{i=1}^{n} (A^T)_{i\sigma(i)}) = \bigvee_{\sigma \in P_n} (\bigwedge_{i=1}^{n} a_{\sigma(i)i})$, but observe that by the previous lemma, a joinand of $|A|_1$ is $\bigwedge_{i=1}^{n} a_{\sigma(i)i}$. So, if we write this joinand in the form $\bigwedge_{i=1}^{n} a_{i\varphi(i)}$, by rearranging the elements so that the second suffixes come into natural order and denote by φ the permutation $(\varphi(1), \varphi(2), \ldots, \varphi(n))$ where both σ and φ range over P_n, then we have $|A^T|_1 = \bigvee_{\varphi \in P_n} (\bigwedge_{i=1}^{n} a_{i\varphi(i)}) = |A|_1$.

The next corollary follows immediately from the above theorem and gives us a principle of duality.

Corollary 3.6. Any theorem concerning the rows, columns, and value of a first determinant remains valid if the words "row" and "column" are everywhere interchanged in the statement of the theorem.

Remark 3.7. Due to Corollary 3.6, the proofs concerning rows or columns of the first determinant will be done either for the rows or the columns but not both.

Notation 3.7. Let $A \in M_n(L)$, we shall denote the matrix A whose i^{th} and j^{th} rows have been interchanged by $A[i:j]$.

<u>Theorem 3.8</u>. Let $A \in M_n(L)$; if matrix B is obtained from A by the interchange of two rows (columns), then $|B|_1 = |A|_1$.

<u>Proof</u>: Let $B = A[r:s]$. Then, in passing from the joinands of $|A|_1$ to the corresponding joinands of $|B|_1$, the natural order of the column suffixes is not altered, but the row suffixes receive one transposition, that is, the joinands are only rearranged, hence $|A|_1 = |A[r:s]|_1 = |B|_1$.

<u>Theorem 3.9</u>. Let $A \in M_n(L)$, where L is an orthomodular lattice; if the matrix B is formed from A by the meet of each element of a row (column) of A with an element $c \in C(L)$, then $|B|_1 = c \wedge |A|_1$.

<u>Proof</u>: By Lemma 3.4, each joinand of $|B|_1$ contains one and only one element from each row and column of B. Suppose B is obtained from A by the scalar meet of the j^{th} row of A by c. Then

$$\begin{aligned}|B|_1 &= \bigvee_{\sigma \in P_n} (a_{1\sigma(1)} \wedge \ldots \wedge c \wedge a_{j\sigma(j)} \wedge \ldots \wedge a_{n\sigma(n)}) \\ &= c \wedge \bigvee_{\sigma \in P_n} (a_{1\sigma(1)} \wedge \ldots \wedge a_{n\sigma(n)}) \quad (\text{since } c \in C(L)) \\ &= c \wedge |A|_1.\end{aligned}$$

<u>Remark 3.10</u>. We now give statements and examples of a few properties which do not hold, in general, for the first determinant of matrices over arbitrary lattices.

(1) Two rows (columns) of a matrix A may be identical, but this does not imply that $|A|_1 = \hat{0}$.

Proof: Consider $A \in M_2(L)$ for L of Example 2.1.22, then for

$$A = \begin{bmatrix} a & 1 \\ a & 1 \end{bmatrix}, \ |A|_1 = (a \wedge 1) \vee (1 \wedge a) = a \vee a = a.$$

(2) If one row (column) of a matrix A is joined to another row (column) of A to form B, then $|B|_1$ does not necessarily equal $|A|_1$.

Proof: Consider $A \in M_2(L)$ for L of Example 2.1.22, then if

$$A = \begin{bmatrix} 1 & a \\ o & o \end{bmatrix} \text{ and B is such that } B = \begin{bmatrix} 1 & a \\ 1 \vee o & a \vee o \end{bmatrix}, \text{ then } |A|_1 = o \text{ and}$$

$|B|_1 = a$.

(3) If an element of L meet one row of a matrix A is joined to another row of A resulting in the matrix B, the $|B|_1$ does not necessarily equal $|A|_1$.

Proof: Follows from (2).

(4) We do not have the property that the first determinant of the product of two matrices is equal to the meet of their first determinants.

Proof: Consider $A \in M_2(L)$ for L of Example 2.1.22, then if

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \text{ and } B = \begin{bmatrix} c & d \\ d & c \end{bmatrix} \text{ then } AB = \begin{bmatrix} o & o \\ o & o \end{bmatrix} \text{ and } |AB|_1 = o \text{ but}$$

$|A|_1 = 1$ and $|B|_1 = 1$, hence $|A|_1 \wedge |B|_1 = 1$.

Definition 3.11. Given $A \in M_n(L)$ and $C = (c_1,\ldots,c_n)$ with $c_i \in L$, then we define A(k:C) to be the matrix A with the k^{th} row replaced by C and A(C:j) to be the matrix A with the j^{th} column replaced by C.

Theorem 3.12. Let $A \in M_n(L)$, where L is an orthomodular lattice; if the matrix B is formed from A by the join of any row k (column j) of A with

$C = (c_1, \ldots, c_n)$, where $c_i \in C(L)$, then $|B|_1 = |A(k:C)|_1 \vee |A|_1$ $(|B|_1 = |A(C:j)|_1 \vee |A|_1)$.

Proof: By Lemma 3.4, each joinand of $|B|_1$ contains one and only one element from each row and column of B. Suppose B is obtained from A by the join of the k^{th} row of A by C. Then $|B|_1 = \bigvee_{\sigma \in P_n} (a_{1\sigma(1)} \wedge \ldots \wedge (c_{\sigma(k)} \vee a_{k\sigma(k)}) \wedge \ldots \wedge a_{n\sigma(n)}) = \bigvee_{\sigma \in P_n} (a_{1\sigma(1)} \wedge \ldots \wedge a_{k-1\sigma(k-1)} \wedge c_{\sigma(k)} \wedge a_{k+1\sigma(k+1)} \wedge \ldots \wedge a_{n\sigma(n)}) \vee \bigvee_{\sigma \in P_n} (a_{1\sigma(1)} \wedge \ldots \wedge a_{k\sigma(k)} \wedge \ldots \wedge a_{n\sigma(n)})$

(since $c_i \in C(L)$ and by use of the associative laws)

$= |A(k:C)|_1 \vee |A|_1$.

Definition 3.13. If we delete the i^{th} row and j^{th} column from A, the first determinant of the resulting (n-1)-square matrix is called the minor of the element a_{ij} and is denoted by $|A_{(i)(j)}|_1$.

Theorem 3.14. Let $A \in M_n(L)$, where L is an orthomodular lattice. If for any row i (column j), a_{ij} commutes with the joinands of $|A_{(i)(j)}|_1$ for all j (for all i), then the first determinant $|A|_1$ can be expanded by row i(column j), where the expansion by the i^{th} row yields

$$|A|_1 = \bigvee_{j=1}^{n} (a_{ij} \wedge |A_{(i)(j)}|_1) \text{ (for all i)}$$

(expansion by j^{th} column yields $|A|_1 = \bigvee_{i=1}^{n} (a_{ij} \wedge |A_{(i)(j)}|_1)$ for all j).

Proof: Choose an arbitrary row k, then by Lemma 3.4, an element $a_{kj} \in A$ can occur at most one time in any joinand of $|A|_1$ for each j.

Collect all the joinands of $|A|_1$ that contain the element a_{kj} for each j. Since a_{kj} commutes with the joinands of $|A_{(k)(j)}|_1$ for all j, the join of the collected terms may be written, using Theorem 2.1.18, as $(a_{k1} \wedge d_{k1}) \vee (a_{k2} \wedge d_{k2}) \vee \dots \vee (a_{kn} \wedge d_{kn})$, where d_{kj} $(j = 1,\dots,n)$ represents the join of all the joinands of $|A|_1$ except those of row k and column j for each a_{kj} but these are precisely the $|A_{(k)(j)}|_1$, therefore $d_{kj} = |A_{(k)(j)}|_1$ and hence $|A|_1 = (a_{k1} \wedge d_{k1}) \vee \dots \vee (a_{kn} \wedge d_{kn})$ $= \bigvee_{j=1}^{n} (a_{kj} \wedge d_{kj}) = \bigvee_{j=1}^{n} (a_{kj} \wedge |A_{(k)(j)}|_1)$, where k is arbitrary, so the theorem follows for all rows.

<u>Corollary 3.15</u>. Let $A \in M_n(L)$, where L is an orthomodular lattice. If for any row i (column j), $a_{ij} \in C(L)$ for all j, (for all i) then $|A|_1$ has an expansion by minors of the i^{th} row (j^{th} column).

<u>Proof</u>: Follows from above theorem and definition of $C(L)$.

<u>Corollary 3.16</u>. If all the elements of a row (column) of the matrix A are zero, or if all the minors in $|A|_1$ of the elements of any row (column) of A are zero, then $|A|_1 = o$.

<u>Proof</u>: The joinands of a minor that is zero, are zero, and thus, are in $C(L)$. Hence, the result follows from the expansion formula.

<u>Lemma 3.17</u>. The following are evaluations of the first determinant of certain special matrices:

(1) $|I|_1 = 1$.

(2) $|0|_1 = o$.

(3) $|E|_1 = 1$.

(4) The first determinant of a scalar matrix for $c \in L$ is:

$|c \wedge E|_1 = c.$

(5) The first determinant of a diagonal matrix $A = (a_{ij})$ is:

$|A|_1 = a_{11} \wedge a_{22} \wedge \ldots \wedge a_{nn}.$

(6) Let U^t and U_t be defined as in Definition 2.2.14, then

$|U^t|_1 = |U_t|_1 = a_{11} \wedge a_{22} \wedge \ldots \wedge a_{nn}.$

(7) Let U be an upper matrix, then $|U|_1 = o.$

(8) Let U be a lower matrix, then $|U|_1 = o.$

<u>Remark 3.18</u>. Since $|A|_1$ is a monotone function of its elements, then $A \leq B$ implies $|A|_1 \leq |B|_1$.

IV. THE SECOND DETERMINANT

During the author's review of the literature, he found a second definition for the determinant of a square matrix whose elements are in a Boolean algebra in Sokolov [10]. The paper gives the definition and states some properties of the determinant. It is the purpose of this fourth chapter to give a similar definition for a determinant of matrices over orthocomplemented lattices and to consider briefly a few properties and relations of this determinant, which we shall call the Second Determinant.

Definition 4.1. For any given even permutation P_n^+, $\bigwedge_{k=1}^{n} a_{k\sigma(k)}$ is called an even joinand, and for any given odd permutation P_n^-, $\bigwedge_{k=1}^{n} a_{k\sigma(k)}$ is called and odd joinand.

Definition 4.2. Let $A \in M_n(L)$, where L is an orthocomplemented lattice. We define the second determinant of A to be the symmetric difference of a_A and b_A, where $a_A = \bigvee_{\sigma \in P_n^+} (a_{1\sigma(1)} \wedge \ldots \wedge a_{n\sigma(n)})$, and $b_A = \bigvee_{\sigma \in P_n^-} (a_{1\sigma(1)} \wedge \ldots \wedge a_{n\sigma(n)})$. Thus

$$|A|_2 = a_A \Delta b_A = (a_A \wedge b_A') \vee (b_A \wedge a_A')$$

$$= [(\bigvee_{\sigma \in P_n^+} (a_{1\sigma(1)} \wedge \ldots \wedge a_{n\sigma(n)})) \wedge (\bigwedge_{\sigma \in P_n^-} (a_{1\sigma(1)}' \vee \ldots \vee a_{n\sigma(n)}'))] \vee$$

$$[(\bigvee_{\sigma \in P_n^-} (a_{1\sigma(1)} \wedge \ldots \wedge a_{n\sigma(n)})) \wedge (\bigwedge_{\sigma \in P_n^+} (a_{1\sigma(1)}' \vee \ldots \vee a_{n\sigma(n)}'))].$$

<u>Lemma 4.3</u>. Each even and odd joinand and their orthocomplements in the expansion of $|A|_2$ contains one and only one element from each row and column of A.

<u>Proof</u>: Follows immediately from the definition and the symmetric difference of a_A and b_A.

Theorem 4.4. If $A \in M_n(L)$, where L is an orthocomplemented lattice, then $|A^T|_2 = |A|_2$.

<u>Proof</u>: By the previous lemma, each even (odd) joinand of $|A|_2$ contains one and only one element from each row and column of A, likewise for their orthocomplements, and so each even (odd) joinand and their orthocomplements in the expansion of $|A^T|_2$ contains one and only one element from each column and row of A. Hence, each even (odd) joinand and its orthocomplement in the expansion of $|A|_2$ is an even (odd) joinand and orthocomplement in the expansion of $|A^T|_2$, and vice versa, therefore, all the joinands are identical and $|A^T|_2 = |A|_2$.

<u>Corollary 4.5</u>. For every theorem concerning the rows of the second determinant there is a corresponding theorem concerning the columns and vice versa.

<u>Theorem 4.6</u>. Let $A \in M_n(L)$, where L is an orthocomplemented lattice; if matrix B is obtained from A by the interchange of two rows (columns) then $|B|_2 = |A|_2$.

<u>Proof</u>: If $B = A[i:j]$, the natural order of the column suffixes is

not altered and the row suffixes receive one transposition. By Lemma 2.37, the number of even and odd permutations remain the same, hence, if $|A|_2 = a_A \Delta b_A$, where a_A and b_A are as in Definition 4.2, then $b_A(a_A)$ is the join of the even (odd) joinands for B, therefore, $|B|_2 = b_A \Delta a_A = a_A \Delta b_A = |A|_2$.

Theorem 4.7. Let $A \in M_n(L)$, where L is an orthomodular lattice; if the matrix B is formed from A by the meet of each element of a row (column) of A with an element $c \in C(L)$, then $|B|_2 = c \wedge |A|_2$.

Proof: Let a_A and b_A be as is defined in Definition 4.2, then if $c \in C(L)$ meets with each element of a row of A to form B, we have $a_B = c \wedge a_A$ and $b_B = c \wedge b_A$ by properties of $C(L)$. Hence, $|B|_2 = a_B \Delta b_B =$ $(c \wedge a_A) \Delta (c \wedge b_A) = [(c \wedge a_A) \wedge (c' \vee b_A')] \vee [(c \wedge b_A) \wedge (c' \vee a_A')] =$ $[((c \wedge a_A) \wedge c') \vee ((c \wedge a_A) \wedge b_A')] \vee [((c \wedge b_A) \wedge c') \vee ((c \wedge b_A) \wedge a_A')] =$ $c \wedge [(a_A \wedge b_A') \vee (b_A \wedge a_A')] = c \wedge |A|_2$.

Corollary 4.8. If all the elements of a row (column) of the matrix $A \in M_n(L)$ are zero, $|A|_2 = o$.

Proof: $o \in C(L)$, therefore, in previous theorem we would have $o \wedge |A|_2 = o$.

Lemma 4.9. The following are evaluations of the second determinant of certain special matrices:

(1) $|I|_2 = o$.

(2) $|0|_2 = o$.

(3) $|E|_2 = 1$.

(4) The second determinant of a scalar matrix for $c \in L$ is:

$|c \wedge E|_2 = c$.

(5) The second determinant of a diagonal matrix $A = (a_{ij})$ is:

$|A|_2 = a_{11} \wedge a_{22} \wedge \ldots \wedge a_{nn}$.

(6) Let U^t and U_t be defined as in Definition 2.2.14, then $|U^t|_2 =$

$|U_t|_2 = a_{11} \wedge a_{22} \wedge \ldots \wedge a_{nn}$.

(7) Let U be an upper (lower) matrix, then $|U|_2 = o$.

Lemma 4.10. Let $A \in M_n(L)$, then $|A|_2 \leq |A|_1$.

Proof: By definition, $|A|_2 = a_A \Delta b_A = (a_A \wedge b_A') \vee (b_A \wedge a_A') \leq a_A \vee b_A = |A|_1$, therefore, $|A|_2 \leq |A|_1$.

V. THE THIRD DETERMINANT

In 1934, J. H. M. Wedderburn [12] introduced a quite different definition for the determinant of a Boolean matrix and stated a necessary and sufficient condition for one such matrix to possess an inverse. In 1963, D. E. Rutherford [9] recalled Wedderburn's definition and noted a few properties that were desirable, while stating that this definition did not permit row or column expansion of the matrix according to the familiar formula. Rutherford then showed a relationship between his remarks on inverses and that of Wedderburn's with regard to the determinant.

We now consider Wedderburn's definition in the light of arbitrary lattices and give a short development of the properties and relations of this determinant, which we shall call the Third Determinant. Then we will prove a lemma concerning inverses and give a counterexample to its converse.

Definition 5.1. Let $A \in M_n(L)$, where L is an orthocomplemented lattice. We define $\overline{A} = (\overline{a}_{ij})$, where $\overline{a}_{ij} = a_{ij} \wedge (\bigwedge_{k \neq j} a'_{ik})$ and $\underline{A} = (\underline{a}_{ij})$, where $\underline{a}_{ij} = a_{ij} \wedge (\bigwedge_{k \neq i} a'_{kj})$.

Lemma 5.2. Let $A \in M_n(L)$, where L is an orthocomplemented lattice, then $|\overline{A}|_1 = |\underline{A}|_1$.

Proof: Immediate from Definition 3.2 and 5.1, since $|\overline{A}|_1 = \bigvee_{\sigma \in P_n} (a_{1\sigma(1)} \wedge \ldots \wedge a_{n\sigma(n)} \wedge q(\sigma)) = |\underline{A}|_1$, where $q(\sigma)$ is the meet of the complements of the elements which are not of the form $a_{i\sigma(i)}$, i=1,...,n.

Definition 5.3. Let $A \in M_n(L)$, where L is an orthocomplemented lattice. We define the third determinant of A as $|A|_3 = |\overline{A}|_1 = |\underline{A}|_1$.

Definition 5.4. For any given permutation σ, $\bigwedge_{i=1}^{n} \overline{a}_{i\sigma(i)} = (a_{1\sigma(1)} \wedge \ldots \wedge a_{n\sigma(n)} \wedge q(\sigma)) = \bigwedge_{i=1}^{n} \underline{a}_{i\sigma(i)}$ are called the joinands of $\overline{A}$ and $\underline{A}$.

Lemma 5.5. Each joinand of $|A|_3$ is the meet of one element from each row and column of A and the complements of all the other elements in A.

Proof: Follows immediately from Definition 5.3.

Lemma 5.6. Let $A \in M_n(L)$, where L is orthocomplemented lattice, then $(\overline{A^T}) = (\underline{A})^T$.

Proof: $(\overline{A^T})_{ij} = A^T_{ij} \wedge (\bigwedge_{k \neq j} A^{T'}_{ik}) = A_{ji} \wedge (\bigwedge_{k \neq j} A'_{ki}) = \underline{A}_{ji} = (\underline{A})^T_{ij}$.

Theorem 5.7. Let $A \in M_n(L)$, where L is an orthocomplemented lattice, then $|A|_3 = |A^T|_3$.

Proof: $|A|_3 = |\underline{A}|_1 = |(\underline{A})^T|_1 = |(\overline{A^T})|_1 = |A^T|_3$.

Remark 5.8. Due to Theorem 5.7, we have a corresponding principle of duality for the third determinant of A as in Corollary 3.6.

Theorem 5.9. Let $A \in M_n(L)$, where L is an orthocomplemented lattice. If matrix B is obtained from A by the interchange of two rows (columns), then $|B|_3 = |A|_3$.

Proof: Let $B = A[r:s]$, then $|B|_3 = |A[r:s]|_3 = |\overline{A[r:s]}|_1 = |\overline{A}\,[r:s]|_1 = |\overline{A}|_1 = |A|_3$.

<u>Theorem 5.10</u>. Let $A \in M_n(L)$ for $n > 1$ and L an orthocomplemented lattice. If one column (row) is less than or equal to another column (row), then $\bar{A}$ ($\underline{A}$) has a column (row) of zeroes, and $|A|_3 = o$.

<u>Proof</u>: Without loss of generality, suppose column one is less than or equal to column two, that is $a_{i1} \le a_{i2}$ for $i = 1,\ldots,n$. By Definition 5.1, the elements in the first column of $\bar{A}$ are $a_{i1} \wedge (a'_{i2} \bigwedge_{k\neq 1,2} a'_{ik}) = (a_{i1} \wedge a'_{i2}) \wedge (\bigwedge_{k\neq 1,2} a'_{ik}) = o$ for $i = 1,\ldots,n$.

The following two corollaries for $A \in M_n(L)$, where L is an orthocomplemented lattice, are immediate consequences of the previous theorem.

<u>Corollary 5.11</u>. If two rows (columns) of A are identical then $|A|_3 = o$.

<u>Corollary 5.12</u>. If all the elements of a row (column) of A are zero, then $|A|_3 = o$.

<u>Theorem 5.13</u>. Let $A \in M_n(L)$, where L is an orthomodular lattice; if the matrix B is formed from A by the meet of each element of a row (column) of A with an element $c \in C(L)$, then $|B|_3 = c \wedge |A|_3$.

<u>Proof</u>: Suppose B is obtained from A by c meet the k^{th} row of A, then we first need to show that $\bar{b}_{kj} = c \wedge \bar{a}_{kj}$. Now $\bar{b}_{kj} = (a_{kj} \wedge c) \wedge \bigwedge_{i\neq j} (c \wedge a_{ki})' = (a_{kj} \wedge c) \wedge \bigwedge_{i\neq j} (c' \vee a'_{ki}) = [(c' \wedge (a_{kj} \wedge c)) \vee ((c \wedge a_{kj}) \wedge \bigwedge_{i\neq j} a'_{ki})] = c \wedge a_{kj} \wedge (\bigwedge_{i\neq j} a'_{ki}) = c \wedge \bar{a}_{kj}$. Since $\bar{b}_{ij} = \bar{a}_{ij}$ for $i\neq k$, the proof follows from Theorem 3.9.

<u>Theorem 5.14</u>. Let $A \in M_n(L)$, where L is an orthomodular lattice; if the matrix B is formed from A by the join of each element of a row (column) of A with an element $c \in C(L)$, then $|B|_3 = c' \wedge |A|_3$.

<u>Proof</u>: Suppose B is obtained from A by the join of the k^{th} row of A by c. We will show that $\bar{b}_{kj} = c' \wedge \bar{a}_{kj}$. Now $\bar{b}_{kj} = (a_{kj} \vee c) \wedge \bigwedge_{i \neq j} (c \vee a_{ki})' = (a_{kj} \vee c) \wedge \bigwedge_{i \neq j} (c' \wedge a'_{ki}) = (a_{kj} \vee c) \wedge c' \wedge (\bigwedge_{i \neq j} a'_{ki}) = [(a_{kj} \wedge c') \vee (c \wedge c')] \wedge (\bigwedge_{i \neq j} a'_{ki}) = c' \wedge a_{kj} \wedge (\bigwedge_{i \neq j} a'_{ki}) = c' \wedge \bar{a}_{kj}$. Since $\bar{b}_{ij} = \bar{a}_{ij}$ for $i \neq k$, the result follows from Theorem 3.9.

<u>Lemma 5.15</u>. The following are evaluations of the third determinant of certain special matrices:

(1) $|I|_3 = o$.

(2) $|0|_3 = o$.

(3) $|E|_3 = 1$.

(4) The third determinant of a scalar matrix for $c \in L$ is : $|c \wedge E|_3 = c$.

(5) The third determinant of a diagonal matrix $A = (a_{ij})$ is: $|A|_3 = a_{11} \wedge a_{22} \wedge \ldots \wedge a_{nn}$.

(6) Let U^t and U_t be defined as in Definition 2.2.14, then $|U^t|_3 = (a_{11} \wedge \ldots \wedge a_{nn}) \wedge (\bigwedge_j \bigwedge_{i<j} a'_{ij})$ and $|U_t|_3 = (a_{11} \wedge \ldots \wedge a_{nn}) \wedge (\bigwedge_j \bigwedge_{i>j} a'_{ij})$.

(7) Let U be an upper (lower) matrix, then $|U|_3 = o$.

Remark 5.16. According to the definitions, $\bar{a}_{ij} \leq a_{ij}$, so from the monotone property, it is evident that $|A|_3 \leq |A|_1$.

Lemma 5.17. Let $A \in M_n(L)$, then $|A|_3 \leq |A|_2$.

Proof: By definition $|A|_3 = \bigvee_{\sigma \in P_n^+} (a_{1\sigma(1)} \wedge \ldots \wedge a_{n\sigma(n)} \wedge q(\sigma)) \vee \bigvee_{\sigma \in P_n^-} (a_{1\sigma(1)} \wedge \ldots \wedge a_{n\sigma(n)} \wedge q(\sigma))$, also, $b_A' = \bigwedge_{\sigma \in P_n^-} (a'_{1\sigma(1)} \vee \ldots \vee a'_{n\sigma(n)})$ and $a_A' = \bigwedge_{\sigma \in P_n^+} (a'_{1\sigma(1)} \vee \ldots \vee a'_{n\sigma(n)})$, where $q(\sigma) = \bigwedge_{i,j \nmid \sigma} a'_{ij}$. If σ is even, $q(\sigma) \leq b_A'$, and if σ is odd, $q(\sigma) \leq a_A'$. Hence, $|A|_3 \leq \bigvee_{\sigma \in P_n^+} (a_{1\sigma(1)} \wedge \ldots \wedge a_{n\sigma(n)} \wedge b_A') \vee \bigvee_{\sigma \in P_n^-} (a_{1\sigma(1)} \wedge \ldots \wedge a_{n\sigma(n)} \wedge a_A') \leq (a_A \wedge b_A') \vee (b_A \wedge a_A') = |A|_2$.

Lemma 5.18. Let $A \in M_n(L)$. If $|A|_3 = 1$, then $AA^T = A^TA = E$.

Proof: Since $|A|_3 \leq |A|_1$, we have $1 \leq |A|_1 \leq \bigvee_{k=1}^{n} a_{ik}$ for all i and likewise $|A|_1 \leq \bigvee_{k=1}^{n} a_{kj}$ for all j and hence we have for all i,j $\bigvee_{k=1}^{n} a_{ik} = 1 = \bigvee_{k=1}^{n} a_{kj}$. Now consider $\underline{a}_{mk} = a_{mk} \wedge (\bigwedge_{i \neq m} a'_{ik}) \leq a'_{ik}$ for $i \neq m$, whereas $\underline{a}_{ik} \leq a'_{jk}$ for $i \neq j$, thus $1 = \bigvee_{i=1}^{n} \underline{a}_{ik} \leq a'_{ik} \vee a'_{jk} = (a_{ik} \wedge a_{jk})'$ for $j \neq i$ which implies that $o = a_{ik} \wedge a_{jk}$ for all k and $j \neq i$, therefore, by Lemma 2.2.21 $AA^T = E$.

Now consider $\bar{a}_{km} = a_{km} \wedge (\bigwedge_{i \neq m} a'_{ki}) \leq a'_{ki}$ for all $i \neq m$, but $\bar{a}_{ki} \leq a'_{kj}$ for any $i \neq j$, thus $1 = \bigvee_{k=1}^{n} \bar{a}_{ik} \leq a'_{kj} \vee a'_{ki} = (a_{kj} \wedge a_{ki})'$ for $i \neq j$ implies $o = a_{kj} \wedge a_{ki}$ for all k and $j \neq i$, therefore by same lemma $A^TA = E$.

<u>Remark 5.19</u>. We now give a counterexample to the converse of Lemma 5.18. Consider $A \in M_2(L)$ for L of Example 2.1.22, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Now $AA^T = E$, but $|A|_3 = |A|_1 = o$.

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DETERMINANTS OF MATRICES OVER LATTICES

by

Daniel Sprigg Chesley, IV

ABSTRACT

Three different definitions for the determinant of a matrix over arbitrary lattices have been developed to determine which properties and relations were reminiscent of the determinant or permanent of elementary algebra. In each determinant there are properties concerning: the elements of the matrix in the expansion of its determinant; the determinant of a matrix and its transpose; a principle of duality for rows and columns; the interchange of rows and columns; the determinant of a matrix formed from another by a row or column meet of certain elements; and evaluations of certain special matrices. An expansion by row or column is given for one determinant and a lemma on inverses is proven in light of another. A preliminary section on Lattice Theory is also included.