

THE APPLICATION OF THE DUGDALE MODEL

TO AN ORTHOTROPIC PLATE

by

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NOTATION

E	Young's modulus for isotropic material
E_x, E_y	Young's modulus along x, y axis
T	applied tensile stress
Y	yield stress
a	$s + \ell$
ℓ	crack half length
s	plastic zone length
u, v	displacement in x, y direction
x, y	cartesian coordinates
z	complex variable, $x + iy$
z_1	complex variable, $x + i\beta_1 y$
z_2	complex variable, $x + i\beta_2 y$
β_1, β_2	orthotropic parameters (Equation 3.1.14)
ζ	transformed complex variable
ϵ, n	coordinates in transformed plane
θ	angular coordinate
θ_2	$\cos^{-1} \ell/a$
σ_2	$e^{i\theta_2}$
w, w_1, w_2	complex transformation functions for z, z_1, z_2 (sec. 3.3)
$\phi(z_1), \psi(z_2)$	stress functions in x, y plane
$\phi'(z_1), \psi'(z_2)$	derivative with respect to z_1, z_2
$\overline{\phi(z_1)}, \overline{\psi(z_2)}$	conjugate of stress functions
$\Phi(\zeta), \Psi(\zeta)$	stress functions in conformal plane

$\phi'(\zeta), \psi'(\zeta)$

derivative with respect to ζ

Re

real part

1. INTRODUCTION

1.1 General

Fracture may be defined as the phenomenon of structural failure by catastrophic crack propagation at average stresses well below yield strength. Why a crack propagates and under what conditions it is most likely to occur are the two most important questions engineers working in fracture mechanics have been trying to solve since Coulomb, in 1773, expressed the view that fracture of a solid would occur if the maximum shear strain at some point surpassed a critical value characterizing the mechanical strength of the material.

Most of the modern day crack propagation theories are based on the work of A. A. Griffith [7].* In his classic paper he concluded that an existing crack would propagate in a cataclysmic fashion, if the available elastic strain energy exceeded the increase in surface energy of the crack. There have been many different reactions to this concept since the time of its conception.

It wasn't until the late 1940's that Orowan and Irwin showed independently that the Griffith type energy balance must be between the strain energy stored in the specimen and the surface energy plus the work done in plastic deformation. For ductile materials the work done against surface tension is generally not significant in comparison with the work done against plastic deformation.

*Numbers in brackets refer to appended references.

Dugdale [3] proposed that the yielding associated with fracture in ductile materials must occur over some zone measured directly in front of the crack, and this zone would be a very narrow band lying along the line of the crack. Considering the sheet he was investigating to deform elastically under the action of the external stress together with a tensile stress Y distributed over part of the surface of a hypothetical cut, which would include the plastic region; he was able to obtain a relation between the extent of plastic yielding and external load applied.

Goodier and Field [6] were then able to calculate, using the results of Dugdale, the work done against plastic deformation.

Ang and Williams [1] have investigated combined axial and bending stresses in an orthotropic plate having a finite crack. Qualitatively no major difference in behavior due to orthotropy was found; although certain quantitative features were noted, mainly as a function of the characteristic rigidity ratio $(E_x/E_y)^{1/2}$.

In the following chapter these theories will be expounded.

1.2 Specific Objective

It is the purpose of this paper to find the stress distribution and displacements about a crack in an orthotropic infinite plate loaded in tension at infinity in a direction perpendicular to the line of the crack. A Dugdale model is assumed, and stress functions for the problem are obtained by use of Muskhelishvili complex variable methods.

2. LITERATURE REVIEW

2.1 Classic Theory

The stresses in a plate due to a crack were first calculated by C. E. Inglis [9]. His main work consisted of determining the stresses in a plate with an elliptical hole. The results he obtained were exact; and therefore, could be applied to the limiting case when one diameter of the ellipse approached zero. The case where one diameter is small can be interpreted as a crack, and he noted that fracture of a material with a crack occurs when a tensile load is applied which is small compared to the fracturing load of a material without a crack.

For the elliptical hole shown in Figure 2.1 when $a/b = 1000$, Inglis stated that the tension at A was 2001 times the mean tension. Thus; at a small tension applied to the plate across the crack, there would be set up at the end of the crack a sufficient tension to start a tear in the material. Inglis concluded as the length increased the situation became worse and the crack continued to propagate.

In 1920, A. A. Griffith [7] formulated his famous fracture theory, which is really the foundation of the structure of fracture as it is known today. Griffith was investigating the effect of surface scratches on the mechanical strength of solids. From his work he reached certain general conclusions on rupture and intermolecular cohesion.

The two basic hypothesis in use up to Griffith's time were: 1) the maximum tensile stress, or 2) the maximum extension exceeded a certain critical value at fracture. Using the work of Inglis he calculated the stresses and strains due to the scratches. The results he obtained

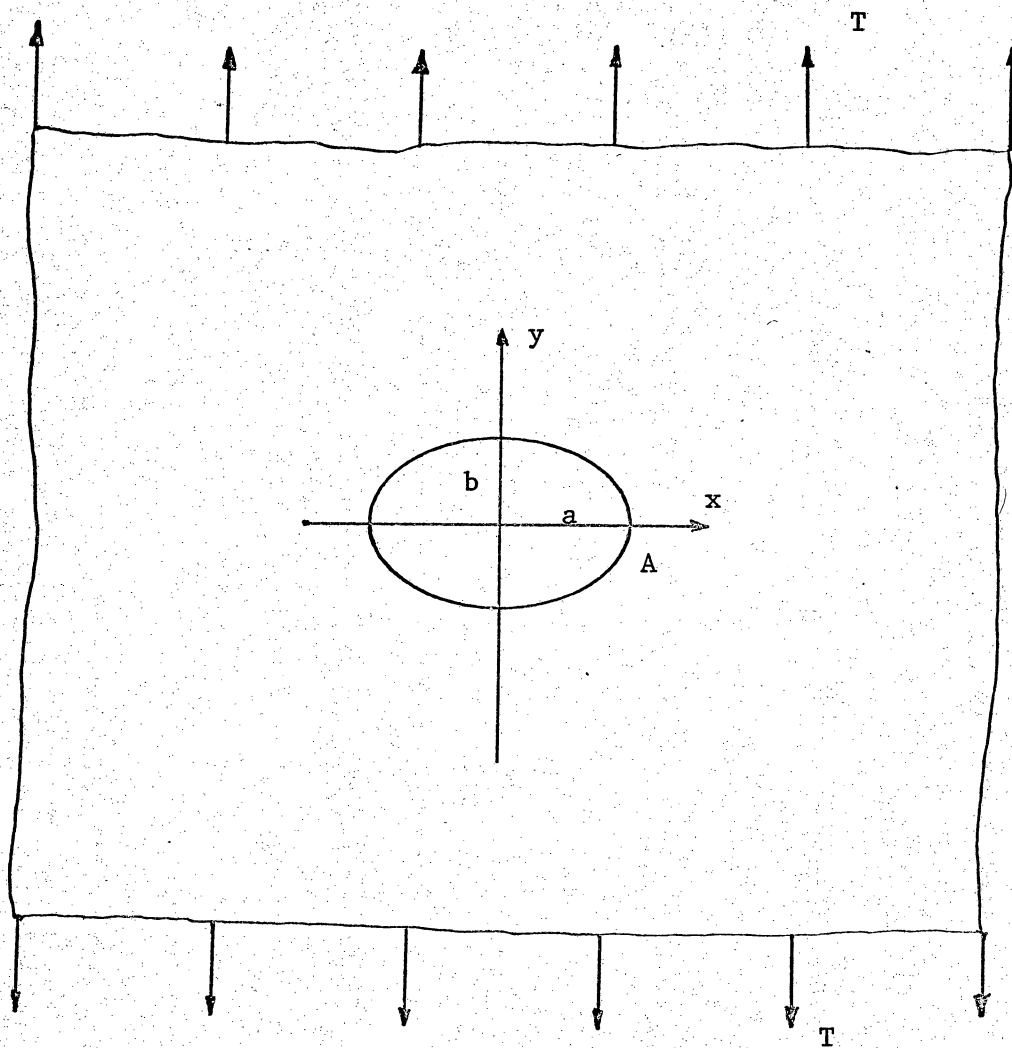


FIGURE 2.1
INGLIS PLATE

from fatigue test were in conflict with the basic hypothesis. Assuming that his data was not at fault he concluded that the basic theory was incorrect.

From this, Griffith formulated a new view on the theory of fracture. To formulate his theory he used the theorem of minimum potential energy which states that the equilibrium state of an elastic body defined by specific surface forces is such that the potential energy of the whole system is a minimum. Griffith then postulated that the equilibrium position, if equilibrium is possible, must be one in which fracture of the solid has occurred, if the system can pass from the unbroken to the broken condition by a process involving a continuous decrease in potential energy.

Account had to be taken of the increase in potential energy due to the creation of new surfaces along the crack in the interior of the body. Griffith's fundamental concept was that the bounding surfaces of a solid possess a surface tension, just as liquids do, and when a crack spreads the decrease in the strain energy is balanced by an increase in the potential energy due to surface tension (i.e., the surface tension of a substance is the work done in forming a unit area of new surface). If the width of the crack is greater than the radius of molecular action, then the energy per unit area is a constant, namely the surface tension. The total decrease in potential energy due to the formation of a crack is equal to the increase in strain energy less the increase in surface energy.

The calculation of the energy of the body due to the crack was based on Inglis' solution [9] of the two dimensional equations of

elastic equilibrium for an elliptical hole in a plate, the crack being taken to be an ellipse of zero eccentricity.

Griffith showed the crack spread when a stress R applied normal to the crack exceeded a certain critical stress. For plane stress the critical stress is,

$$R_c = \sqrt{\frac{2Ew}{\pi\ell}}, \quad (2.1.1)$$

and for plane strain the critical stress is,

$$R_c = 2 \sqrt{\frac{\mu w}{\pi\ell}}, \quad (2.1.2)$$

where 2ℓ is the length of the crack, w is the surface energy per unit surface area, E is Young's modulus, and μ is the modulus of rigidity.

Griffith drew the general conclusion that the weakness of isotropic solids, as ordinarily met with, is due to the presence of discontinuities or flaws, as they may be more correctly called, where ruling dimensions are large compared with molecular distances. The elimination of these flaws could increase the strength of materials ten or twenty times.

Griffith reasoned the largest crack like flaw would become self-propagating when the rate of release of strain energy became greater than the increase of surface energy of the extending crack.

Sack [19] extended Griffith's work to three dimensions. He noted that the length of internal cracks do not generally exceed their width, and he calculated the conditions of fracture for a solid containing a penny-shaped crack when one of the principal stresses is acting normally to the plane of the crack.

2.2 Irwin and Orowan Modification

In the late 1940's a new view of Griffith's energy balance was independently proposed by George Irwin [10] and Egon Orowan [17].

Irwin tried to add some clarity to the dynamics of rapid fracturing in ductile metals. He based his work on Griffith's comparison of the work required to extend a crack with the release of stored elastic energy which accompanies crack extension. He assumed a thin straight crack through a plate so loaded that it had a bi-axial tensile stress J . Stating the Griffith crack relations to be,

$$\frac{de}{dA} = \frac{d}{d\ell} \frac{\pi J^2 \ell^2}{E}, \quad (2.2.1)$$

$$\frac{dW}{dA} = \frac{d}{d\ell} (4S\ell). \quad (2.2.2)$$

where $\frac{de}{dA}$ is the release of strain energy and $\frac{dW}{dA}$ is the work done per unit increment of fracture area dA . Griffith said that the work dW was expended against surface tension (S), where the total work done in enlarging the crack to length 2ℓ , proportional to the surface area is $4S\ell$.

From the crack propagation criteria fracturing would occur when $de/dA > dW/dA$. Irwin noted that theoretically this instability to fracturing on strain energy would occur for a long enough crack since de/dA is proportional to ℓ . Practically he noted, for ductile materials, dW must include work done in plastic deformation. If this interpretation were to be used, the work done against surface tension would be insignificant. The term equivalent to $4S\ell$ might be approximately represented by two terms, one proportional to the area of fracture and one proportional to the volume of metal affected by plastic flow.

The idea of locating the point of fracturing instability by equating de/dA to dW/dA remained basic. Working with an externally notched specimen Irwin formed the following picture of fast fracturing; as the specimen bends the notched region reaches its limit of localized plastic flow. The crack opens in the central portion of a relatively large region under the notch. As the crack deepens it acts as a sharper notch so the region subject to stress relief by plastic flow is steadily and considerably reduced. In many cases the reduction of dW/dA leads to fast fracturing. The fracture creeping effect, the sharpening of the crack head contour, and the release of stored elastic energy as the point of instability is reached appear to be the major features in the development of fast fracturing.

From Irwin's experimental work he was able to conclude that in the case of initially slow fractures on mild steel, dW/dA becomes sufficiently small for instability by gradual development of smaller volumes undergoing plastic flow at the head of the advancing crack. The transition to fast fracturing includes a creeping advance of the crack, as might be seen in exaggerated form in the fracturing of centrally notched foils.

Orowan [17] referred to X-ray work which showed extensive plastic deformation on the fracture surface of materials which had failed in a brittle fashion. Orowan stated essentially the same conclusions as Irwin.

Orowan was mainly concerned with fracture which occurs in the boundary case between ductility and brittleness (i.e., notch brittleness). He noted three types of fractures: 1) brittle fracture - which occurs at a critical magnitude of the highest tensile stress, 2) cleavage

fracture - which occurs in a crystal when the tensile stress in the cleavage plane reaches a critical value, and 3) ductile fracture - which cannot take place unless the condition of plastic yielding is satisfied. The complete fracture criterion of the ductile fracture was not known.

Referring to his earlier X-ray work, Orowan observed that the brittle fracture of ductile metals was always accompanied by a slight plastic deformation in a thin layer at the surface of the fracture. As the crack propagated the elastic energy of the specimen had to provide not only the surface energy but also the cold work required for the creation of a unit area of the surface of fracture. If the cold work were concentrated in a layer whose thickness was small compared with the length of the initial crack, the plastic surface work could be treated on the same basis as the surface energy, and its presence could be taken into account by the addition of the quantity p to w in the Griffith formula (2.1.1) to give,

$$R_c = \sqrt{\frac{2E(w + p)}{\pi l}} . \quad (2.2.3)$$

Orowan said that the presence of some plastic deformation even when the fracture appeared completely brittle could be expected. Orowan showed for a low carbon steel the plastic surface work per unit crack extension, p , was 1000 times greater than the surface energy. Therefore, the critical stress for fracture from the Griffith formula would be,

$$R_c = \sqrt{\frac{2Ep}{\pi l}} , \quad (2.2.4)$$

the plastic surface work is only this large in comparison with the surface energy for ductile materials. A critical value R_c of the applied

tensile stress at which a crack of depth l starts to propagate spontaneously is given by the Griffith formula modified (2.2.4).

From the work of Irwin and Orowan we conclude that in the fracture of ductile materials the Griffith energy balance is primarily between the elastic energy and the plastic work in crack propagation, which overshadows the energy requirements for the creation of new surfaces.

2.3 Dugdale's Model

While investigating yielding of steel sheets containing slits, D. S. Dugdale [3] observed the yielded region was shaped as a thin extension of the crack. The three hypothesis that he formulated were:

1. The material in the yielded zone is under a uniform tensile yield stress Y . (See Figure 2.2).
2. The thickness of the yielded zone is so small that the elastic region outside may be regarded as bounded internally by a flattened ellipse of length $2(l + s)$, where l is the half length of the crack and s the length of the plastic zone. (See Figure 2.3).
3. The length s is such that there is no stress singularity at the ends of the flattened ellipse.

With these hypotheses and noting the problem of a straight cut loaded over part of its edge had been examined by Muskhelishvili [11], Dugdale was able to get a relation between the extend of plastic yielding and the external load. Muskhelishvili's stress functions were found to assume a simple form when account was taken of the symmetry of the problem. Introducing variables α and θ_2 defined by,

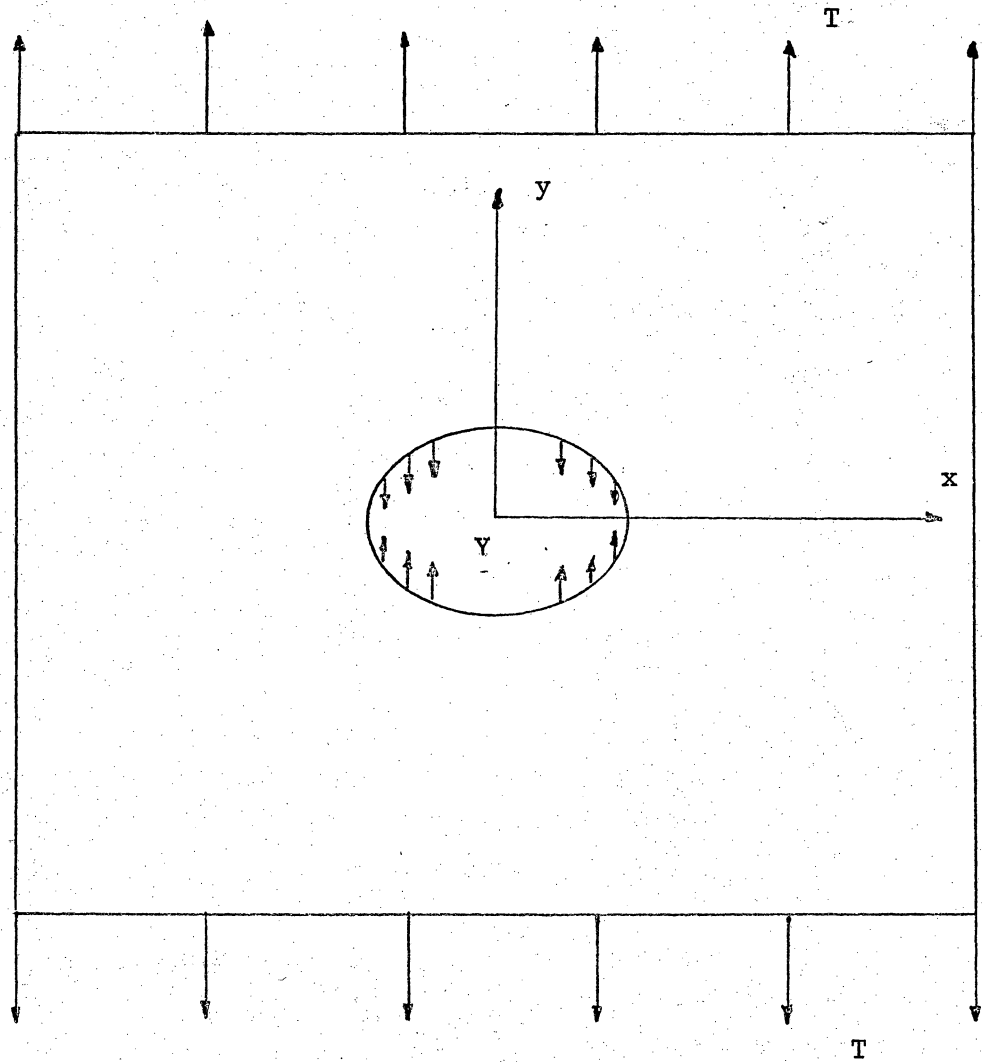


FIGURE 2.2

PLASTIC ZONE REPLACED BY YIELD STRESS

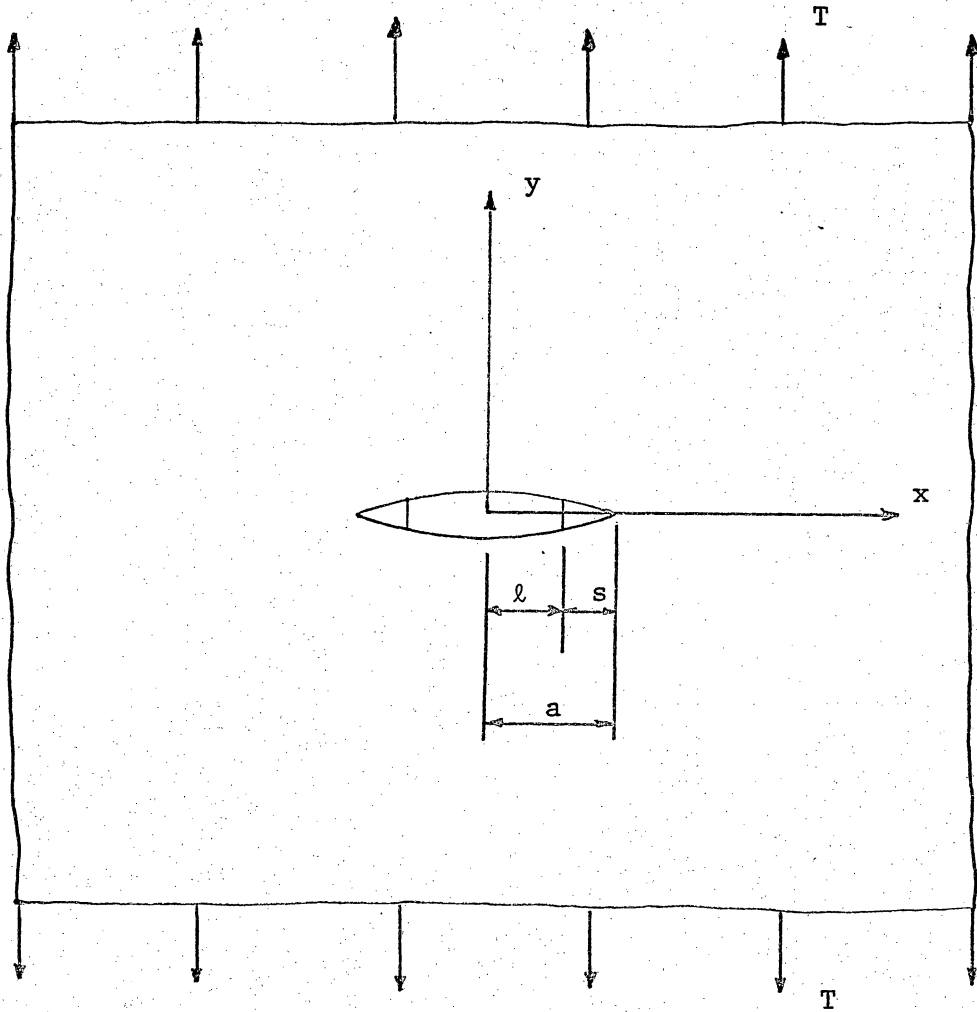


FIGURE 2.3

DUGDALE MODEL

$$x = a \cos h \alpha, \quad (2.3.1)$$

$$l = a \cos \theta_2, \quad (2.3.2)$$

the stress σ_y acting at points on $y = 0$ was determined in the form of a series in ascending powers of α having a leading term,

$$\sigma_y = - \frac{2Y\theta_2}{\pi\alpha}. \quad (2.3.3)$$

The analogous expression for stress due to the external loading was,

$$\sigma_y = \frac{T}{\alpha}. \quad (2.3.4)$$

When these stresses were superposed, use was made of the third hypothesis that the stress at the point $\alpha = 0$, (i.e., $x = a$) should not be infinite, so the coefficient of $1/\alpha$ must vanish. Therefore,

$$T - \frac{2Y\theta_2}{\pi} = 0, \quad (2.3.5)$$

or

$$\theta_2 = \frac{\pi}{2} \frac{T}{Y}. \quad (2.3.6)$$

This readily leads to the relation with T/Y very small,

$$s/l = 1.23 (T/Y)^2. \quad (2.3.7)$$

Using the results of Dugdale, Goodier and Field [5] were able to evaluate the plastic energy dissipation by the methods of elastic perfectly-plastic continuum mechanics. A slowly extending internal slit crack under tension and a rapidly extending semi-infinite crack under traveling wedge pressures were examined by them. Assuming that the quantity l/a was constant they obtained the following for the plastic work of the slowly extending crack.

$$dW_p = 4Yd\ell \int_{x=\ell}^{x=a} \frac{\partial v(x, \ell)}{\partial \ell} dx, \quad (2.3.8)$$

The quantity, v , was obtained by the method of Muskhelishvili.

2.4 Orthotropic Effects on the Classical Theory

Ang and Williams [1] using a formulation in integral equations, presented a solution for the combined extension-classical bending stress and displacement for the case of an infinite orthotropic flat plate containing a finite crack. While this solution could have been expressed in closed form for the entire field, primary emphasis was placed upon the stresses near the crack point. Qualitatively no major difference in behavior due to orthotropy was found, although certain quantitative features were noted, mainly as a function of the characteristic rigidity ratio $(E_x/E_y)^{1/2}$.

The inverse square-root character of the isotropic bending and extension stress was not changed by the orthotropy, although amplitudes and distributions were affected.

3. ANALYSIS

3.1 Development of the General Orthotropic Problem [18]

Beginning with the usual plane stress assumptions, $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$, the stress-strain relations for an orthotropic medium are,

$$\begin{aligned}\epsilon_x &= \frac{\sigma_x}{E_x} - \frac{\nu_x}{E_x} \sigma_y, \\ \epsilon_y &= -\frac{\nu_y}{E_y} \sigma_x + \frac{\sigma_y}{E_y}, \\ \gamma_{xy} &= \frac{\tau_{xy}}{G_{xy}},\end{aligned}\tag{3.1.1}$$

where because of symmetry of the stress-strain matrix,

$$\frac{\nu_x}{E_x} = \frac{\nu_y}{E_y}.\tag{3.1.2}$$

The equilibrium and compatibility equations are independent of mechanical properties and are, assuming no body forces,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0,\tag{3.1.3}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0,$$

$$\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}.\tag{3.1.4}$$

A stress function can be chosen in the usual form,

$$\sigma_x = \frac{\partial^2 \chi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \chi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}.\tag{3.1.5}$$

Combining equations (3.1.1) through (3.1.5) leads to,

$$\frac{\partial^4 \chi}{\partial x^4} + 2A \frac{\partial^2 \chi}{\partial x^2 \partial y^2} + B \frac{\partial^4 \chi}{\partial y^4} = 0, \quad (3.1.6)$$

where,

$$A = \frac{Ey}{2Gxy} - \nu y \quad B = \frac{Ey}{Ex} = \frac{\nu y}{\nu x}. \quad (3.1.7)$$

The general solution of (3.1.6) depends on the roots of the characteristic equation,

$$Bm^4 + 2Am^2 + 1 = 0. \quad (3.1.8)$$

The roots are,

$$\begin{aligned} m_1 &= i\sqrt{\frac{A-C}{B}} \\ m_2 &= i\sqrt{\frac{A+C}{B}} \\ m_3 &= -i\sqrt{\frac{A-C}{B}} \\ m_4 &= -i\sqrt{\frac{A+C}{B}} \end{aligned} \quad (3.1.9)$$

where,

$$C = \sqrt{A^2 - B}, \quad (3.1.10)$$

and therefore,

$$A > C. \quad (3.1.11)$$

In the case of unequal roots the solution of (3.1.6) is,

$$\begin{aligned} \chi(x,y) = F_1(x + m_1 y) + F_2(x + m_2 y) + F_3(x + m_3 y) + \\ F_4(x + m_4 y). \end{aligned} \quad (3.1.12)$$

Lekhnitskii [13] has shown that the roots of (3.1.6) have to be complex. Let us assume therefore,

$$\begin{aligned} m_1 &= \alpha_1 + i\beta_1, & m_2 &= \alpha_2 + i\beta_2, \\ m_3 &= \alpha_1 - i\beta_1, & m_4 &= \alpha_2 - i\beta_2, \end{aligned} \quad (3.1.13)$$

where from (3.1.9) and taking $\beta_1 \neq \beta_2$; $\beta_1, \beta_2 > 0$, we have,

$$\begin{aligned} \alpha_1 &= \alpha_2 = 0, \\ \beta_1 &= \sqrt{\frac{A-C}{B}}, \\ \beta_2 &= \sqrt{\frac{A+C}{B}}. \end{aligned} \quad (3.1.14)$$

Denoting,

$$\begin{aligned} z_1 &= x + m_1 y = x + i\beta_1 y, \\ z_2 &= x + m_2 y = x + i\beta_2 y, \end{aligned} \quad (3.1.15)$$

Equation (3.1.12) can be written as,

$$\chi(x, y) = F_1(z_1) + F_2(z_2) + \overline{F_1(z_1)} + \overline{F_2(z_2)}, \quad (3.1.16)$$

where F_1 and F_2 are analytic functions and $\overline{F_1}$ and $\overline{F_2}$ are respectively their conjugates. Now let,

$$\frac{dF_1}{dz_1} = \phi(z_1), \quad \frac{dF_2}{dz_2} = \psi(z_2), \quad (3.1.17)$$

and from (3.1.17) we see that,

$$\frac{d\overline{F_1}}{d\overline{z_1}} = \overline{\phi(z_1)}, \quad \frac{d\overline{F_2}}{d\overline{z_2}} = \overline{\psi(z_2)} \quad (3.1.18)$$

Inserting (3.1.16) into (3.1.5) and using (3.1.17) we obtain,

$$\begin{aligned}\sigma_x &= 2\text{Re} [m_1^2 \phi'(z_1) + m_2^2 \psi'(z_2)], \\ \sigma_y &= 2\text{Re} [\phi'(z_1) + \psi'(z_2)], \\ \tau_{xy} &= -2\text{Re} [m_1 \phi'(z_1) + m_2 \psi'(z_2)].\end{aligned}\tag{3.1.19}$$

The strain displacement relations are independent of the mechanical properties and they are,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \epsilon_x = \frac{\sigma_x}{E_x} - \frac{\nu_x}{E_x} \sigma_y, \\ \frac{\partial v}{\partial y} &= \epsilon_y = -\frac{\nu_y}{E_y} \sigma_x + \frac{\sigma_y}{E_y}, \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= \gamma_{xy} = \frac{\tau_{xy}}{G_{xy}}.\end{aligned}\tag{3.1.20}$$

Substituting (3.1.19) into (3.1.20) and integrating yields,

$$\begin{aligned}u &= 2\text{Re} [p_1 \phi(z_1) + p_2 \psi(z_2)], \\ v &= 2\text{Re} [g_1 \phi(z_1) + g_2 \psi(z_2)],\end{aligned}\tag{3.1.21}$$

where,

$$\begin{aligned}p_1 &= \frac{m_1^2 - \nu_x}{E_x}, & p_2 &= \frac{m_2^2 - \nu_x}{E_x}, \\ g_1 &= \frac{-\nu_x m_1^2 + \frac{1}{E_y}}{m_1}, & g_2 &= \frac{-\nu_x m_2^2 + \frac{1}{E_y}}{m_2},\end{aligned}\tag{3.1.22}$$

and where we have neglected constant terms which account for rigid body displacements.

The stress boundary conditions are,

$$\begin{aligned} X_n &= \sigma x \cos(n, x) + \tau xy \cos(n, y), \\ Y_n &= \tau xy \cos(n, x) + \sigma y \cos(n, y), \end{aligned} \quad (3.1.23)$$

where,

$$\cos(n, x) = \frac{dy}{d\lambda} \quad \cos(n, y) = -\frac{dx}{d\lambda}, \quad (3.1.24)$$

therefore,

$$\begin{aligned} X_n &= \frac{d}{d\lambda} \left(\frac{\partial \chi}{\partial y} \right), \\ Y_n &= -\frac{d}{d\lambda} \left(\frac{\partial \chi}{\partial x} \right), \end{aligned} \quad (3.1.25)$$

and,

$$\begin{aligned} \frac{\partial \chi}{\partial x} &= -\int_0^\lambda Y_n d\lambda + C_1, \\ \frac{\partial \chi}{\partial y} &= \int_0^\lambda X_n d\lambda + C_2. \end{aligned} \quad (3.1.26)$$

Using (3.1.16) and (3.1.17) the boundary conditions for $\phi(z_1)$ and $\psi(z_2)$ become,

$$\begin{aligned} 2\operatorname{Re} [\phi(z_1) + \psi(z_2)] &= -\int_0^\lambda Y_n d\lambda + C_1 = f_1, \\ 2\operatorname{Re} [m_1 \phi(z_1) + m_2 \psi(z_2)] &= \int_0^\lambda X_n d\lambda + C_2 = f_2. \end{aligned} \quad (3.1.27)$$

Thus the solution of problems where external forces are given is reduced to the determination of the two functions $\phi(z_1)$ and $\psi(z_2)$, and the application of the boundary conditions given by (3.1.27).

3.2 Statement of Problem

Using Dugdale's hypothesis the stress functions for the Dugdale crack, Figure 3.1, may be determined by superposition of the three stress states shown in Figure 3.2. These stress functions may be found from the stress functions given by Muskhelishvili and adopted to the orthotropic case by Savin [18] for an infinite sheet containing an elliptical hole which is loaded by uniform normal forces acting over a portion of its surface.

Superposing the three stress states in Figure 3.3, and taking the limit as the ellipse flattens to a crack, yields the stress functions for the orthotropic Dugdale crack. The problem then reduces to the following: Given an infinitely large orthotropic elastic plate containing an elliptical hole with forces acting on the contour of the hole given, what are the stress functions for this particular geometry?

3.3 Elliptical Hole Part of Which is Subjected to a Normal Pressure

To get the stress function for states 2 and 3 we must therefore, first have the stress function for the case of an elliptical hole part of which is subjected to normal pressure, as in Figure 3.4. It is assumed that a uniform pressure p is applied only to the portion AB of the hole contour and that no external forces are applied to the remaining part BCA.

The coordinate axes Ox and Oy are chosen in the directions of the axes of the ellipse and the semi-axis are a and b , where for a crack along the x -axis of length $2a$, $b = 0$.

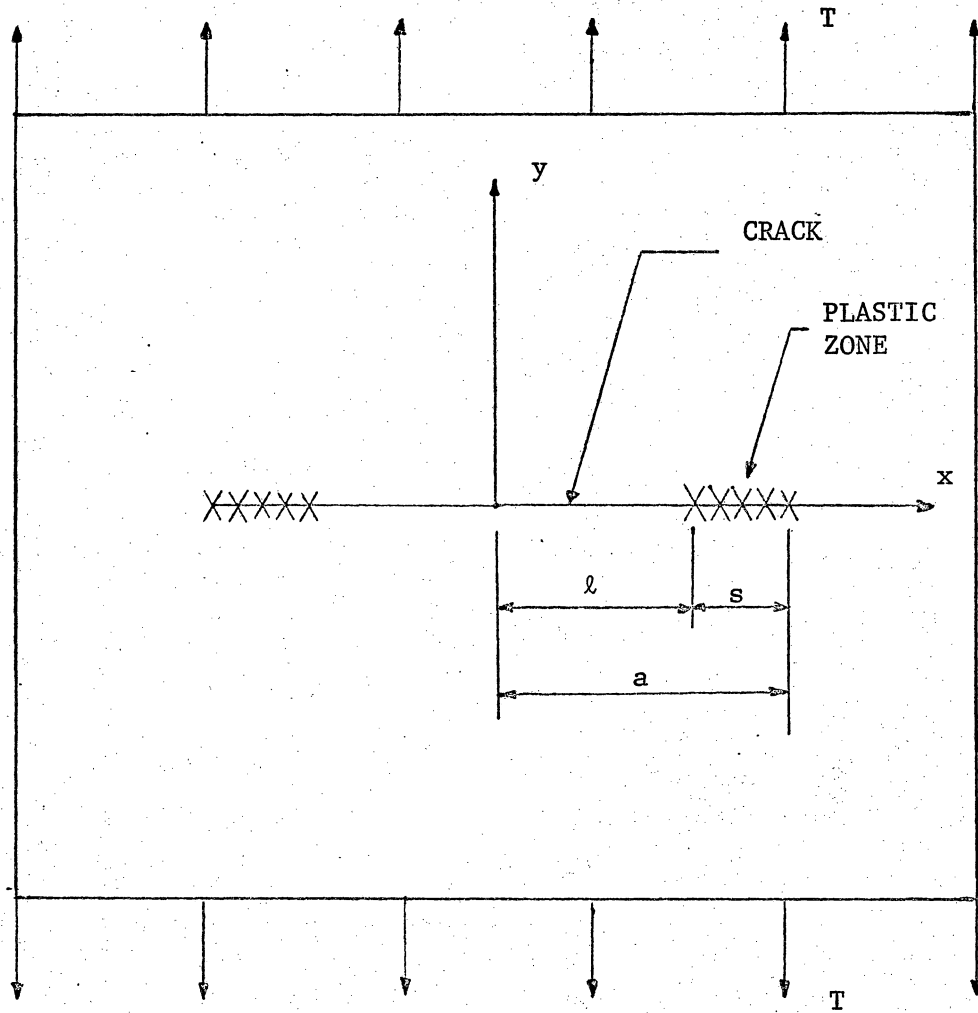


FIGURE 3.1

DUGDALE CRACK

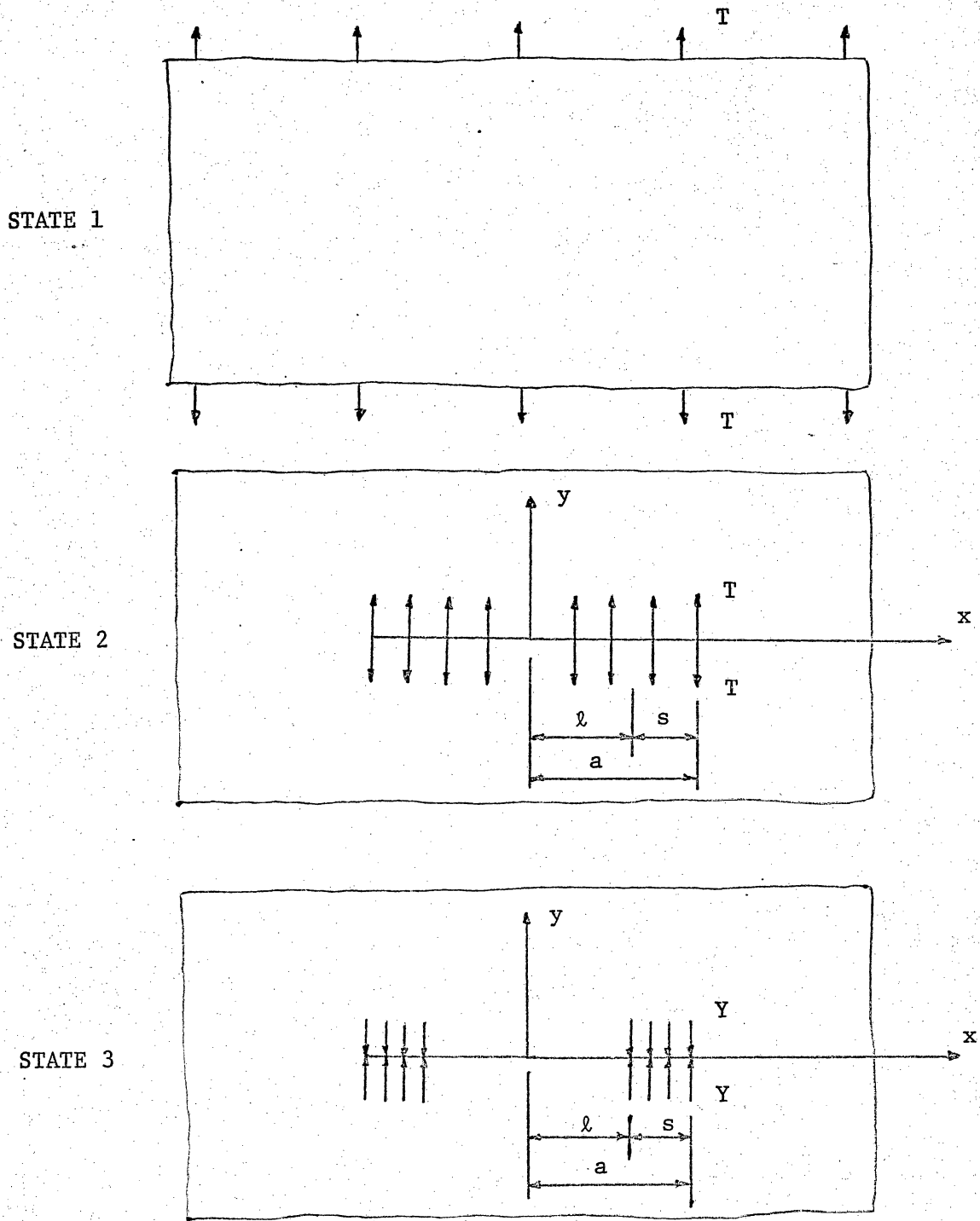


FIGURE 3.2

COMPONENT STRESS STATES FOR DUGDALE CRACK

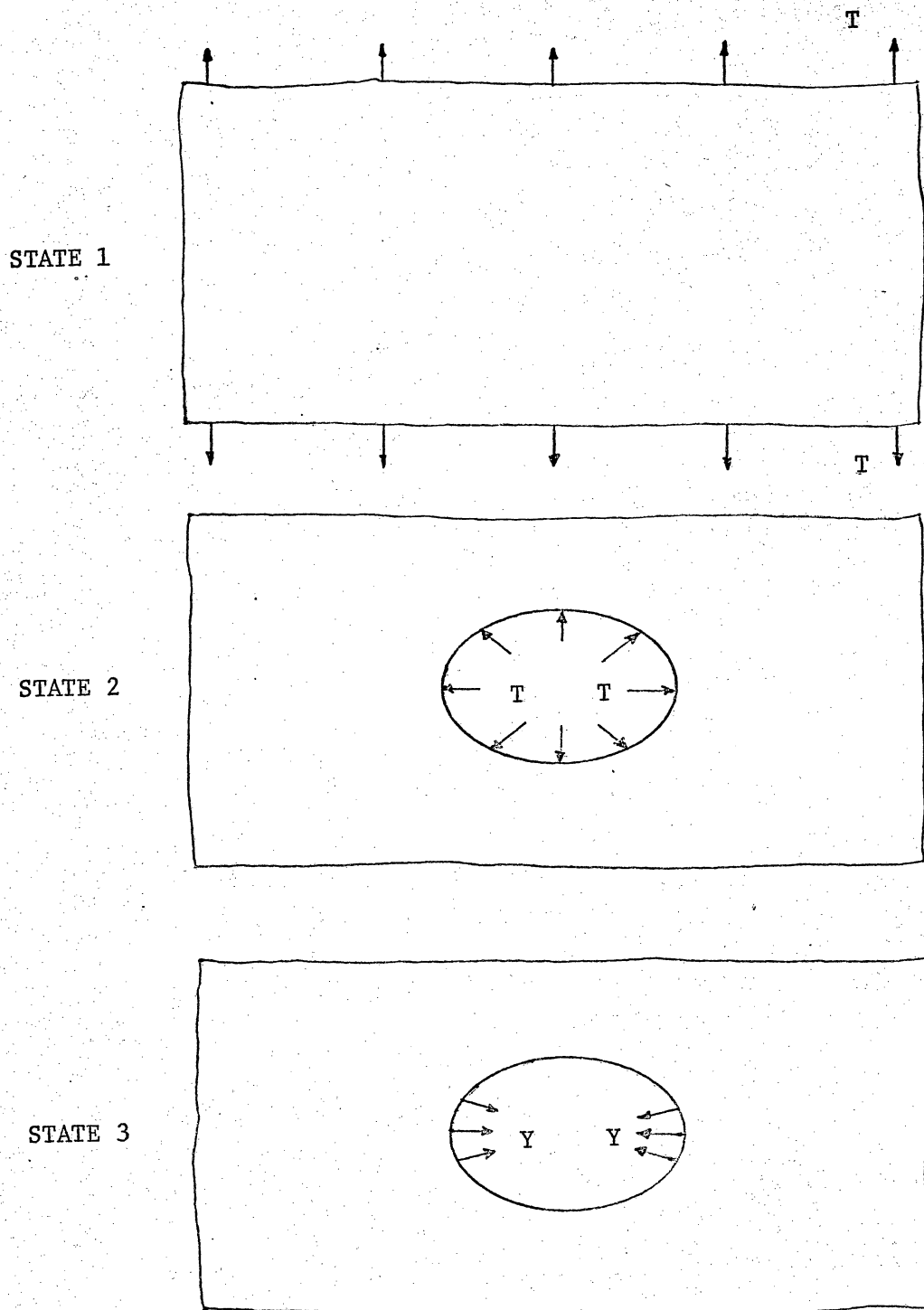


FIGURE 3.3

COMPONENT STRESS STATES FOR DUGDALE CRACK

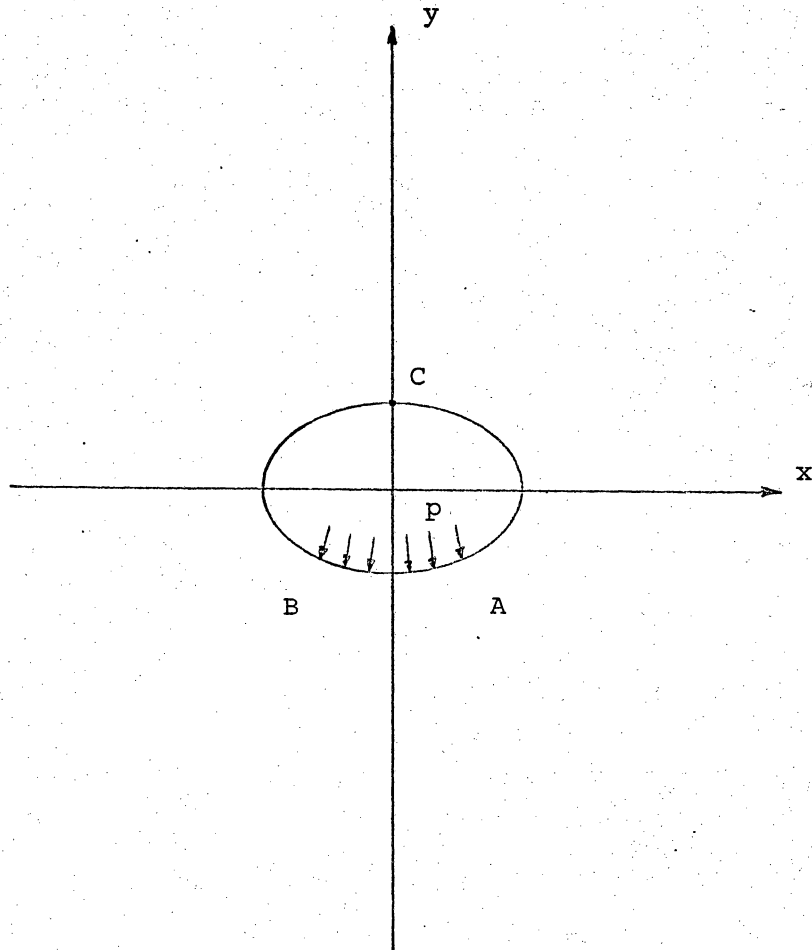


FIGURE 3.4

LOADING ALONG ELLIPTICAL
HOLE

In addition to the given plane $z = x + iy$, the planes z_1 and z_2 obtained from this plane by an affine transformation (3.1.15) will also be investigated.

The ellipse in the z plane is transformed into ellipses in the z_1 and z_2 planes. The areas outside these ellipses will be denoted by S , S^1 , S^2 respectively and the functions which give a conformal representation of the areas on the inside of the unit circle γ are to be determined.

Now,

$$z = w(\zeta) = \frac{a-b}{2} \zeta + \frac{a+b}{2} \frac{1}{\zeta}, \quad (3.3.1)$$

gives the conformal representation of the area S on the inside of the unit circle, where the coordinates of the contour points of the ellipse of area S are,

$$x = a \cos \theta, \quad y = -b \sin \theta, \quad (3.3.2)$$

but,

$$z_1 = x + m_1 y = a \cos \theta - m_1 b \sin \theta, \quad (3.3.3)$$

therefore,

$$z_1 = w_1(\zeta) = \frac{a + im_1 b}{2} \zeta + \frac{a - im_1 b}{2} \frac{1}{\zeta}, \quad (3.3.4)$$

and similarly,

$$z_2 = w_2(\zeta) = \frac{a + im_2 b}{2} \zeta + \frac{a - im_2 b}{2} \frac{1}{\zeta}. \quad (3.3.5)$$

Now our two functions $\phi(z_1)$ and $\psi(z_2)$ can be written in the conformal plane as,

$$\phi(z_1) = \phi(w_1(\zeta)) = \Phi(\zeta), \quad (3.3.6)$$

$$\psi(z_2) = \psi(w_2(\zeta)) = \Psi(\zeta).$$

The derivatives of these functions are,

$$\phi'(z_1) = \frac{\Phi'(\zeta)}{w_1'(\zeta)}, \quad (3.3.7)$$

$$\psi'(z_2) = \frac{\Psi'(\zeta)}{w_2'(\zeta)},$$

Equation (3.1.19) becomes,

$$\begin{aligned} \sigma_x &= 2\text{Re} \left[m_1^2 \frac{\Phi'(\zeta)}{w_1'(\zeta)} + m_2^2 \frac{\Psi'(\zeta)}{w_2'(\zeta)} \right], \\ \sigma_y &= 2\text{Re} \left[\frac{\Phi'(\zeta)}{w_1'(\zeta)} + \frac{\Psi'(\zeta)}{w_2'(\zeta)} \right], \end{aligned} \quad (3.3.8)$$

$$\tau_{xy} = -2\text{Re} \left[m_1 \frac{\Phi'(\zeta)}{w_1'(\zeta)} + m_2 \frac{\Psi'(\zeta)}{w_2'(\zeta)} \right],$$

and (3.1.21) becomes,

$$\begin{aligned} u &= 2\text{Re} [p_1 \Phi(\zeta) + p_2 \Psi(\zeta)], \\ v &= 2\text{Re} [g_1 \Phi(\zeta) + g_2 \Psi(\zeta)]. \end{aligned} \quad (3.3.9)$$

The points A and B in the conformal plane are σ_1 and σ_2 where $\sigma_1 = e^{i\theta_1}$ and $\sigma_2 = e^{i\theta_2}$, Figure 3.5.

Assuming $b = 0$, the stress functions are,

$$\Phi(\zeta) = \frac{pa\beta_2}{4\pi i(\beta_1 - \beta_2)} \left[-\frac{1}{\zeta} \ln \frac{\sigma_2}{\sigma_1} + \left\{ \zeta + \frac{1}{\zeta} \right\} \ln \frac{\sigma_2 - \zeta}{\sigma_1 - \zeta} + \right.$$

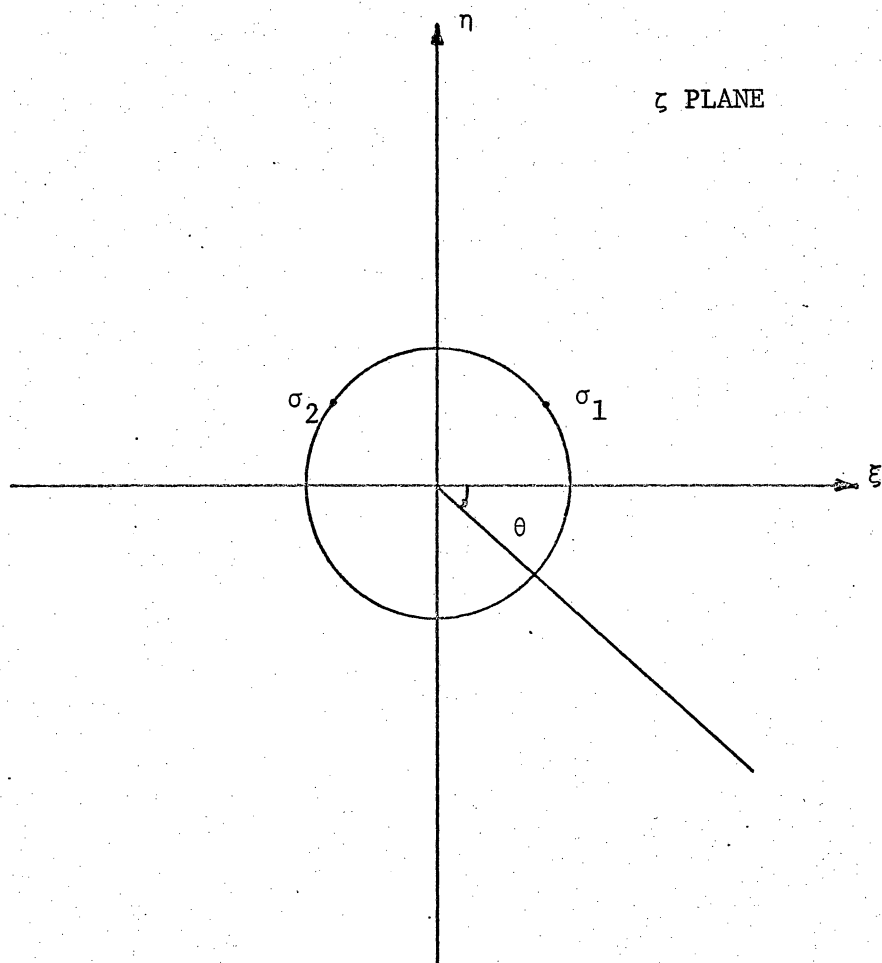


FIGURE 3.5

CONFORMAL PLANE

$$r \ln (\sigma_1 - \zeta) - t \ln (\sigma_2 - \zeta) - \left\{ \frac{(\nu x + \beta_2^2)}{2\beta_2(\beta_1 + \beta_2)} + 1 \right\} (r-t) \ln \zeta \Big], \quad (3.3.10)$$

$$\Psi(\zeta) = \frac{pa\beta_1}{4\pi i(\beta_1 - \beta_2)} \left[\frac{1}{\zeta} \ln \frac{\sigma_2}{\sigma_1} - \left\{ \zeta + \frac{1}{\zeta} \right\} \ln \frac{\sigma_2 - \zeta}{\sigma_1 - \zeta} - r \ln (\sigma_1 - \zeta) + \right. \\ \left. t \ln (\sigma_2 - \zeta) + \left\{ \frac{(\nu x + \beta_1^2)}{2\beta_1(\beta_1 + \beta_2)} + 1 \right\} (r-t) \ln \zeta \right],$$

where,

$$r = \sigma_1 + \frac{1}{\sigma_1}, \quad (3.3.11)$$

$$t = \sigma_2 + \frac{1}{\sigma_1}.$$

3.4 Superposition of State 2 and State 3

For state 2 the stress functions (3.3.10) reduce with the entire contour loaded and therefore with $\sigma_1 = \sigma_2$, $r = t$, $\ln \frac{\sigma_2}{\sigma_1} = 2\pi i$, $p = T$ to,

$$\phi_2(\zeta) = \frac{+Ta\beta_2}{4\pi i(\beta_1 - \beta_2)} \left[\frac{2\pi i}{\zeta} \right] = - \frac{Ta\beta_2}{2(\beta_1 - \beta_2)\zeta}, \quad (3.4.1)$$

$$\Psi_2(\zeta) = \frac{Ta\beta_1}{4\pi i(\beta_1 - \beta_2)} \left[\frac{2\pi i}{\zeta} \right] = \frac{Ta\beta_1}{2(\beta_1 - \beta_2)\zeta}.$$

For state 3 we have Figure 3.6,

$$p = -Y, \quad (3.4.2)$$

$$\theta_2 = \cos^{-1} (\lambda/a), \quad (3.4.3)$$

$$\sigma_2^R = e^{-i\theta_2} = \sigma_2,$$

$$\sigma_1^R = e^{i\theta_2} = \frac{1}{\sigma_2},$$

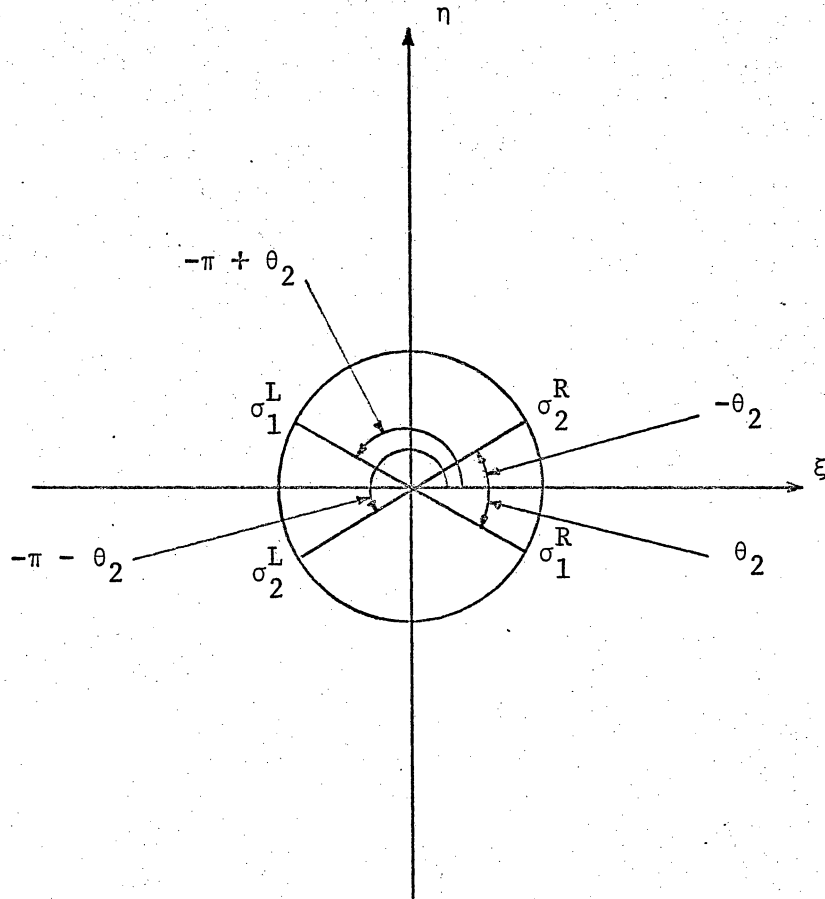


FIGURE 3.6

LOADING IN CONFORMAL PLANE

$$\begin{aligned}\sigma_1^L &= e^{-i(\pi-\theta_2)} = -\bar{\sigma}_2, \\ \sigma_2^L &= e^{-i(\pi+\theta_2)} = -\sigma_2,\end{aligned}\quad (3.4.4)$$

and the stress functions (3.3.10) reduce to,

$$\begin{aligned}\phi_3(\zeta) &= -\frac{Y\alpha\beta_2}{4\pi i(\beta_1-\beta_2)} \left[-\frac{4i\theta_2}{\zeta} + \left(\zeta + \frac{1}{\zeta}\right) \ln \frac{(\sigma_2-\zeta)(\sigma_2+\zeta)}{(\bar{\sigma}_2-\zeta)(\bar{\sigma}_2+\zeta)} + \right. \\ &\quad \left. \frac{2\ell}{a} \ln \frac{(\sigma_2+\zeta)(\bar{\sigma}_2-\zeta)}{(\sigma_2-\zeta)(\bar{\sigma}_2+\zeta)} \right], \quad (3.4.5) \\ \psi_3(\zeta) &= \frac{Y\alpha\beta_1}{4\pi i(\beta_1-\beta_2)} \left[-\frac{4i\theta_2}{\zeta} + \left(\zeta + \frac{1}{\zeta}\right) \ln \frac{(\sigma_2-\zeta)(\sigma_2+\zeta)}{(\bar{\sigma}_2-\zeta)(\bar{\sigma}_2+\zeta)} + \right. \\ &\quad \left. \frac{2\ell}{a} \ln \frac{(\sigma_2+\zeta)(\bar{\sigma}_2-\zeta)}{(\sigma_2-\zeta)(\bar{\sigma}_2+\zeta)} \right].\end{aligned}$$

Superposing the stress functions for states 2 and 3 and changing the positive direction of θ from clockwise to counter-clockwise* we get,

$$\begin{aligned}\phi(\zeta) &= \left[\frac{Y\alpha\beta_2\theta_2}{\pi(\beta_1-\beta_2)} - \frac{Ta\beta_2}{2(\beta_1-\beta_2)} \right] \frac{1}{\zeta} + \frac{Y\alpha\beta_2}{4\pi i(\beta_1-\beta_2)} \left[\left(\zeta + \frac{1}{\zeta}\right) \ln \frac{(\sigma_2-\zeta)(\sigma_2+\zeta)}{(\bar{\sigma}_2-\zeta)(\bar{\sigma}_2+\zeta)} + \right. \\ &\quad \left. \frac{2\ell}{a} \ln \frac{(\sigma_2+\zeta)(\bar{\sigma}_2-\zeta)}{(\sigma_2-\zeta)(\bar{\sigma}_2+\zeta)} \right], \quad (3.4.6) \\ \psi(\zeta) &= \left[-\frac{Y\alpha\beta_1\theta_2}{\pi(\beta_1-\beta_2)} + \frac{Ta\beta_1}{2(\beta_1-\beta_2)} \right] \frac{1}{\zeta} \\ &\quad - \frac{Y\alpha\beta_1}{4\pi i(\beta_1-\beta_2)} \left[\left(\zeta + \frac{1}{\zeta}\right) \ln \frac{(\sigma_2-\zeta)(\sigma_2+\zeta)}{(\bar{\sigma}_2-\zeta)(\bar{\sigma}_2+\zeta)} + \frac{2\ell}{a} \ln \frac{(\sigma_2+\zeta)(\bar{\sigma}_2-\zeta)}{(\sigma_2-\zeta)(\bar{\sigma}_2+\zeta)} \right].\end{aligned}$$

The Dugdale's finiteness condition requires,

*This is done in order to check the results with Goodier and Field.

$$\frac{\beta_2}{(\beta_1 - \beta_2)} \left(\frac{Y\theta_2}{\pi} - \frac{T}{2} \right) = - \frac{\beta_1}{(\beta_1 - \beta_2)} \left(\frac{Y\theta_2}{\pi} - \frac{T}{2} \right) = 0, \quad (3.4.7)$$

or

$$\theta_2 = \frac{\pi}{2} \frac{T}{Y}. \quad (3.4.8)$$

The stress functions corresponding to the Dugdale crack, (exclusive of the uniform stress field $\sigma_y = T$) are

$$\begin{aligned} \Phi(\zeta) &= \frac{Ya\beta_2}{4\pi i(\beta_1 - \beta_2)} \left[\left(\zeta + \frac{1}{\zeta} \right) \ln \frac{(\sigma_2 - \zeta)(\sigma_2 + \zeta)}{(\bar{\sigma}_2 - \zeta)(\bar{\sigma}_2 + \zeta)} + \frac{2l}{a} \ln \frac{(\sigma_2 + \zeta)(\bar{\sigma}_2 - \zeta)}{(\sigma_2 - \zeta)(\bar{\sigma}_2 + \zeta)} \right] \\ \Psi(\zeta) &= - \frac{Ya\beta_1}{4\pi i(\beta_1 - \beta_2)} \left[\left(\zeta + \frac{1}{\zeta} \right) \ln \frac{(\sigma_2 - \zeta)(\sigma_2 + \zeta)}{(\bar{\sigma}_2 - \zeta)(\bar{\sigma}_2 + \zeta)} + \frac{2l}{a} \ln \frac{(\sigma_2 + \zeta)(\bar{\sigma}_2 - \zeta)}{(\sigma_2 - \zeta)(\bar{\sigma}_2 + \zeta)} \right]. \end{aligned} \quad (3.4.9)$$

3.5 Stresses and Displacements

For the stresses along the line of the crack, letting $\zeta = \epsilon$, (i.e., $y = 0$), we substitute (3.4.9) into (3.3.8) to get,

$$\begin{aligned} \sigma_x &= \frac{2Y\beta_1\beta_2}{\pi} \tan^{-1} \left(\frac{\sin 2\theta_2}{\epsilon^2 - \cos 2\theta_2} \right), \\ \sigma_y &= \frac{2Y}{\pi} \tan^{-1} \left(\frac{\sin 2\theta_2}{\epsilon^2 - \cos 2\theta_2} \right) \leftarrow T \end{aligned} \quad (3.5.1)$$

Taking $\epsilon = 1$, (the tip of the plastic zone) and recalling the uniform stress $\sigma_y = T$, we get,

$$\sigma_y = Y, \quad (3.5.2)$$

$$\sigma_x = \beta_1\beta_2 (Y - T) = \frac{E_x}{E_y} (Y - T).$$

At this point as well as the rest of the x-axis, τ_{xy} vanishes by symmetry.

Displacements at the surface of the crack and the elastic plastic interface on which $\zeta = e^{i\theta}$ and $\sigma_2 = e^{i\theta_2}$ are derived by substituting (3.4.9) into (3.3.9).

$$v = \frac{\gamma a (\beta_1 + \beta_2)}{2 \epsilon_y \beta_1 \beta_2} \left[\cos \theta \ln \left[\frac{\sin (\theta - \theta_2)}{\sin (\theta + \theta_2)} \right]^2 + \cos \theta_2 \ln \left[\frac{\sin \theta + \sin \theta_2}{\sin \theta - \sin \theta_2} \right]^2 \right] \quad (3.5.3)$$

$$u = \frac{2\gamma a (\nu x - \beta_1 \beta_2)}{\pi \epsilon_x} \theta_2 \cos \theta.$$

4. RESULTS

4.1 Yield Criteria

Up to now the material in the plastic zone has been assumed to be perfectly plastic and we have replaced it by its equivalent yield stress Y . We have made no mention of what this yield stress is other than it is constant. The relation between the yield stress and the material properties of the plate will now be investigated.

The criteria for deciding which combination of multi-axial stresses will cause yielding are called yield criteria. Numerous criteria have been proposed for the yielding of solids going as far back as Coulomb in 1773. Many of these were originally suggested as criteria for failure of brittle materials and were later adopted as yield criteria for ductile materials. One of the inadequacies of these theories is that they are not applicable for materials with different tensile strengths in various directions.

The yield criteria of von Mises has been shown to be in excellent agreement with experiment for many isotropic ductile materials, for example copper, nickel, aluminum, iron, cold-worked mild steel, medium carbon and alloy steels. For an isotropic material the von Mises' criterion is,

$$\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_x\sigma_z = \sigma_{yp}^2, \quad (4.1.1)$$

where σ_{yp} is the yield stress in simple tension.

The simplest yield criterion for orthotropic ductile materials is, therefore, one which reduces to von Mises' law when the orthotropy is

small. If, then, the yield criterion is assumed to be a quadratic in the stress components it must be of the form,

$$F(\sigma_y - \sigma_z)^2 + G(\sigma_z - \sigma_x)^2 + H(\sigma_x - \sigma_y)^2 + 2L\tau_{yz}^2 + 2M\tau_{zx}^2 + 2N\tau_{xy}^2 = 1, \quad (4.1.2)$$

where F, G, H, L, M, N, are parameters characteristic of the current state of orthotropy.

If T_x , T_y , T_z , are the tensile yield stresses in the orthotropic directions we have,

$$\frac{1}{T_x^2} = G + H$$

$$\frac{1}{T_y^2} = H + F \quad (4.1.3)$$

$$\frac{1}{T_z^2} = F + G,$$

$$2F = \frac{1}{T_y^2} + \frac{1}{T_z^2} - \frac{1}{T_x^2}$$

$$2G = \frac{1}{T_z^2} + \frac{1}{T_x^2} - \frac{1}{T_y^2} \quad (4.1.4)$$

$$2H = \frac{1}{T_x^2} + \frac{1}{T_y^2} - \frac{1}{T_z^2}.$$

If τ_{yz} , τ_{zx} , τ_{xy} , are the yield stresses in shear with respect to the orthotropic axis then,

$$2L = \frac{1}{\tau_{yz}^2}$$

$$2M = \frac{1}{\tau_{zx}^2}$$

$$2N = \frac{1}{T_{xy}^2} \quad (4.1.5)$$

We consider now the case of a two dimensional plate where the natural axis of the material are represented by x and y. The yield criterion for this case is therefore,

$$\frac{\sigma_x^2}{T_x^2} - \left(\frac{1}{T_x^2} + \frac{1}{T_y^2} - \frac{1}{T_z^2} \right) \sigma_x \sigma_y + \frac{\sigma_y^2}{T_y^2} + \frac{\tau_{xy}^2}{T_{xy}^2} = 1. \quad (4.1.6)$$

The yield stresses T_x , T_y , T_z , T_{xy} , are considered to be the same in tension and compression. If σ_x is positive, for example, then the tensile value of T_x is used; if it is negative the compressive value of T_x is used.

As a load is applied to a structure, the stresses at any point in the material increase numerically and remain proportional to one another so that they all can be expressed in terms of one of them.

Considering the yield criterion along the line of the crack we have,

$$\tau_{xy} = 0, \quad (4.1.7)$$

due to symmetry and also,

$$\sigma_x = \frac{2Y\beta_1\beta_2}{\pi} \tan^{-1} \left(\frac{\sin 2\theta_2}{\epsilon^2 - \cos 2\theta_2} \right), \quad (4.1.8)$$

and

$$\sigma_y = \frac{2Y}{\pi} \tan^{-1} \left(\frac{\sin 2\theta_2}{\epsilon^2 - \cos 2\theta_2} \right) + T. \quad (4.1.9)$$

The relation between the σ_x and σ_y is therefore,

$$\sigma_x = \beta_1\beta_2 (\sigma_y - T). \quad (4.1.10)$$

Substituting this into (4.1.6) and solving for σ_y we find,

$$\sigma_y = \frac{-d + \sqrt{d^2 - 4e}}{2}, \quad (4.1.11)$$

where,

$$d = \frac{-T\beta_1\beta_2 (2\beta_1\beta_2 - 1) T_y^2 T_z^2 + T_x^2 (T_y^2 - T_z^2)}{(\beta_1^2 \beta_2^2 - \beta_1 \beta_2) T_y^2 T_z^2 + T_x^2 T_z^2 (1 - \beta_1 \beta_2) + \beta_1 \beta_2 T_x^2 T_y^2} \quad (4.1.12)$$

and

$$e = \frac{[\beta_1^2 \beta_2^2 T^2 - T_x^2] T_y^2 T_z^2}{[(\beta_1^2 \beta_2^2 - \beta_1 \beta_2) T_y^2 T_z^2 + T_x^2 T_z^2 (1 - \beta_1 \beta_2) + \beta_1 \beta_2 T_x^2 T_y^2]} \quad (4.1.13)$$

If we therefore set,

$$Y = \frac{-d + \sqrt{d^2 - 4e}}{2} \quad (4.1.14)$$

the stress distribution will follow a von Mises' yield criterion for plane stress conditions.

Because $\beta_1\beta_2$ can vary depending on the state of orthotropy the von Mises' conditions continue to be satisfied as long as $\sigma_x < \sigma_y$. If σ_x becomes greater than σ_y , σ_x becomes the maximum principal stress. The material then tends to deform in a direction normal to the crack line [22]. Hence the value of $\beta_1\beta_2$ at which $\sigma_x = \sigma_y$ must be the greatest value for which the present analysis is valid. Then from (3.5.2) this greatest value is determined by,

$$\beta_1\beta_2(Y-T) = Y \quad (4.1.15)$$

or

$$\beta_1\beta_2 = \frac{Y}{Y-T} \quad (4.1.16)$$

Therefore our analysis is valid only if,

$$\beta_1 \beta_2 \leq \frac{Y}{Y-T} \quad (4.1.17)$$

or

$$\frac{T}{Y} > \frac{(\beta_1 \beta_2 - 1)}{\beta_1 \beta_2} \quad (4.1.18)$$

4.2 Orthotropic Effects

Having found the stress functions, stresses along the line of the crack, and the displacements at the surface of the crack, we are now in a position to see what effects the orthotropy of the problem produces.

All orthotropic properties will be denoted by superscripts or subscripts o and all isotropic properties will be denoted by superscripts or subscripts i.

a. Stresses Along the Crack Line (See Figure 4.1)

$$\begin{aligned} \sigma_x^i &= \frac{2Y}{\pi} \tan^{-1} \left(\frac{\sin 2\theta_2}{\epsilon^2 - \cos 2\theta_2} \right) \\ \sigma_y^i &= \frac{2Y}{\pi} \tan^{-1} \left(\frac{\sin 2\theta_2}{\epsilon^2 - \cos 2\theta_2} \right) \end{aligned} \quad [4] \quad (4.2.1)$$

$$\sigma_x^o = \beta_1 \beta_2 \sigma_x^i \quad (4.2.2)$$

$$\sigma_y^o = \sigma_y^i$$

b. Displacements Along Crack and Elastic-Plastic Interface (See Figures 4.2, 4.3)

$$v^i = \frac{Ya}{\pi E} \left[\cos \theta \ln \left[\frac{\sin (\theta - \theta_2)}{\sin (\theta + \theta_2)} \right]^2 + \cos \theta_2 \ln \left[\frac{\sin \theta + \sin \theta_2}{\sin \theta - \sin \theta_2} \right]^2 \right] \quad [6] \quad (4.2.3)$$

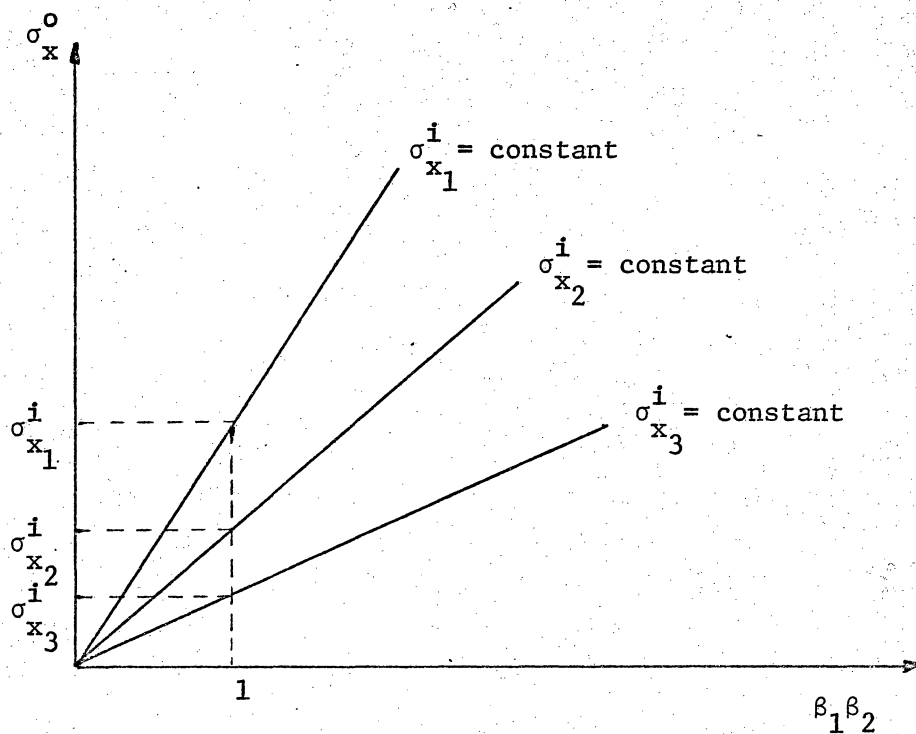


FIGURE 4.1
 ORTHOTROPIC STRESS IN
 x DIRECTION

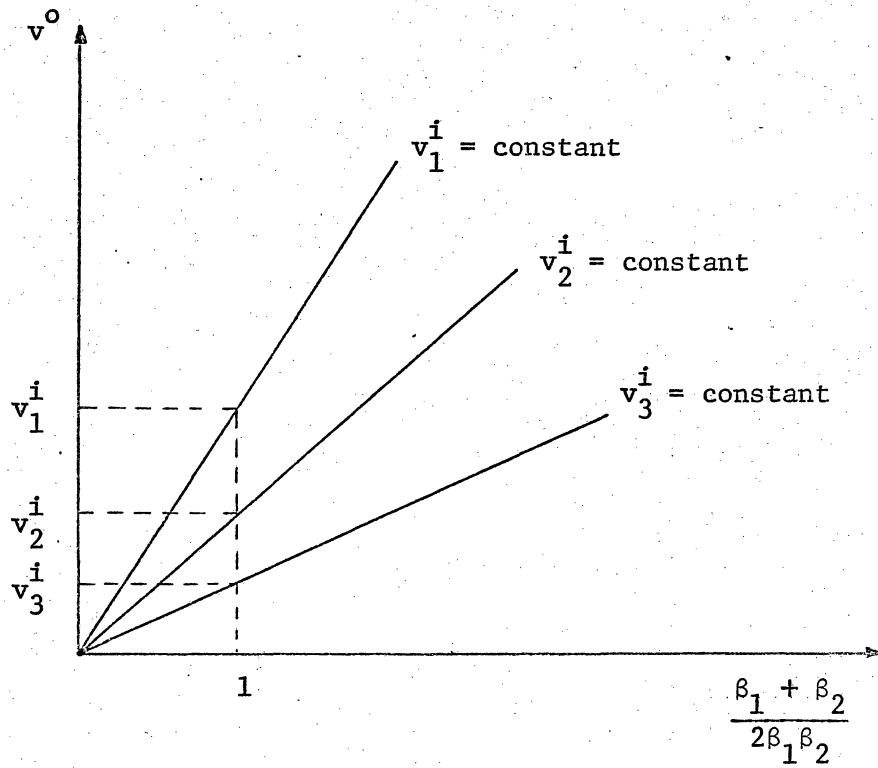


FIGURE 4.2

ORTHOTROPIC DISPLACEMENT

IN y DIRECTION

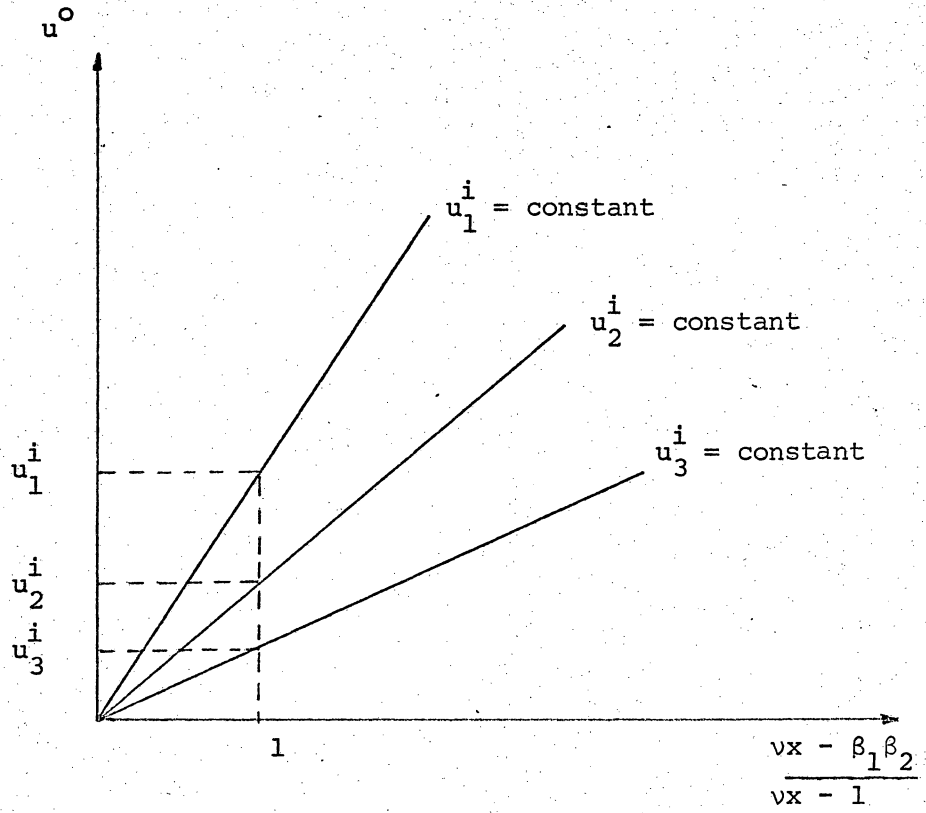


FIGURE 4.3

ORTHOTROPIC DISPLACEMENT

IN x DIRECTION

$$v^o = \frac{\beta_1 + \beta_2}{2\beta_1\beta_2} v^i \quad (4.2.4)$$

$$u^i = \frac{2Ya(v-1)}{E\pi} \theta_2 \cos \theta \quad (4.2.5)$$

$$u^o = \frac{vx - \beta_1\beta_2}{vx-1} u^i \quad (4.2.6)$$

where the E in (4.2.3) is E_y , and the E and v in (4.2.5) are E_x and v_x .

c. The Plastic Work Rate (See Figure 4.4)

$$\frac{dW_p^i}{d\ell} = \frac{16}{\pi} E \left[\frac{Y}{E} \right]^2 \ell f(\theta_2) \quad [6] \quad (4.2.7)$$

where

$$f(\theta_2) = \theta_2 \tan \theta_2 - \ln \sec \theta_2 \quad (4.2.8)$$

$$\frac{dW_p^o}{d\ell} = \frac{\beta_1 + \beta_2}{2\beta_1\beta_2} \frac{dW_p^i}{d\ell} \quad (4.2.9)$$

where the E in (4.2.5) is E_y .

d. Forman Correction Factor (See Figure 4.5)

Forman [5] has applied the Dugdale model to finding the strain energy release rate for a finite plate with a crack. He has found a correction factor which takes into account not only the finiteness of the plate but also the plastic deformation in front of the crack. This correction factor is,

$$\gamma_i^2 = \frac{C(v^i)}{2C(n)^2} \left[1 + C(n) + \frac{\tan \theta_2}{\theta_2(n\bar{N}-1)} \right], \quad (4.2.10)$$

where,

$$C(v^i) = \frac{Y}{T} \left[\ln \sec \theta_2 + \frac{\sin^{-1} p}{4p} \ln \frac{\sin \theta_2 + 1}{\sin \theta_2 - 1} \right], \quad (4.2.11)$$

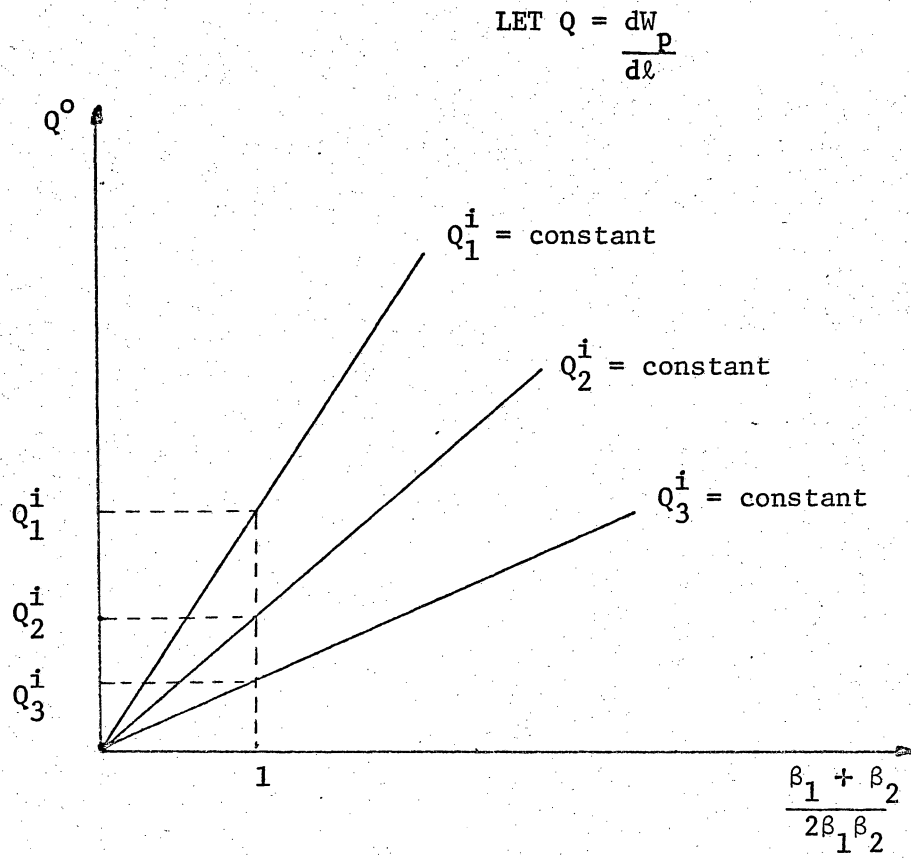


FIGURE 4.4

ORTHOTROPIC PLASTIC WORK RATE

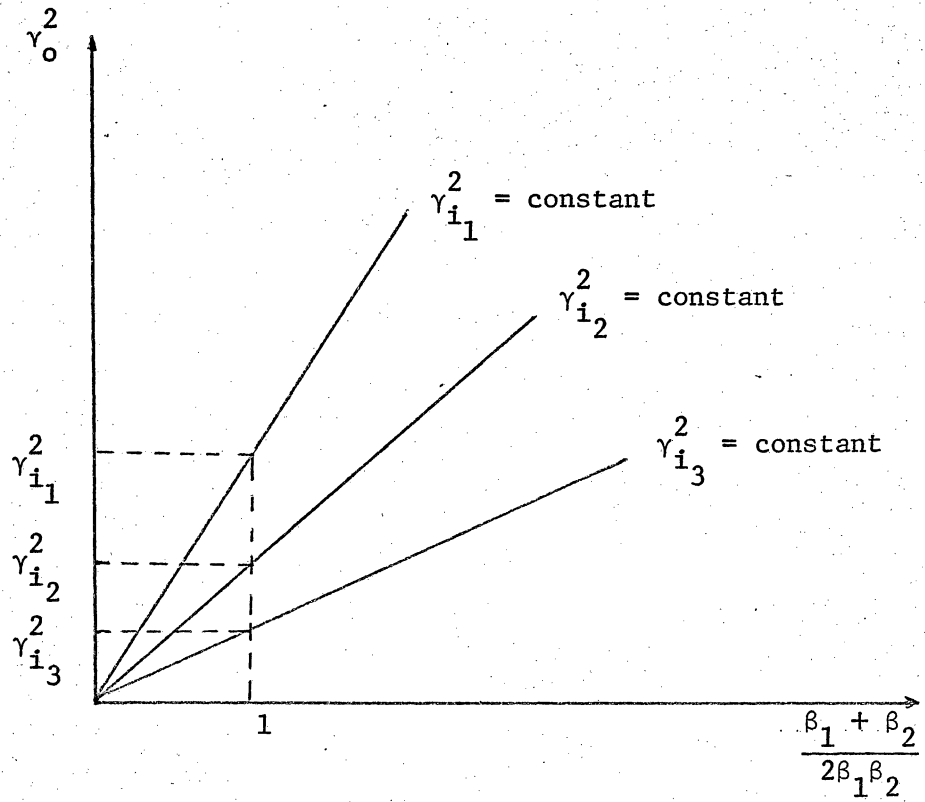


FIGURE 4.5

ORTHOTROPIC FORMAN CORRECTION FACTOR

$$p = \left[1 - \frac{4 \ln \bar{m}}{\ln \left(\frac{\sin \theta_2 + 1}{\sin \theta_2 - 1} \right)^2} \right]^{1/2}, \quad (4.2.12)$$

$$\bar{N} = n + \sqrt{n^2 - m^2}, \quad (4.2.13)$$

$$\bar{m} = \sec \theta_2, \quad (4.2.14)$$

$$n = \frac{\bar{B}}{2l}, \quad (4.2.15)$$

where \bar{B} is the plate width,

for $0 < T/Y < 0.5$

$$C(n) = \frac{(\bar{m}-1)}{n} \frac{Y}{T} + \frac{(\bar{n}-m)}{n} + \frac{\bar{m} \tan 2\theta_2}{2n \theta_2} \left[1 - \frac{\bar{m}}{\bar{N}} - \frac{\sin^2 \theta_2}{\sqrt{\cos 2\theta_2}} \ln \frac{\left(\frac{\bar{N}-\bar{m}}{\bar{N}+\bar{m}} \sqrt{\cos 2\theta_2} \right) \left(1 + \sqrt{\cos 2\theta_2} \right)}{\left[1 - \sqrt{\cos 2\theta_2} \right]} \right] \quad (4.2.16)$$

for $0.5 < T/Y < 1$,

$$C(n) = \frac{(\bar{m}-1)}{n} \frac{Y}{T} + \frac{(\bar{n}-m)}{n} + \frac{\bar{m} \tan 2\theta_2}{2n \theta_2} \left[1 - \frac{\bar{m}}{\bar{N}} - \frac{2 \sin^2 \theta_2}{\sqrt{-\cos 2\theta_2}} \tan^{-1} \left(\frac{(\bar{N}-\bar{m}) \sqrt{-\cos 2\theta_2}}{\bar{N}-\bar{m} \cos 2\theta_2} \right) \right]. \quad (4.2.17)$$

Now the Forman orthotropic correction factor would be,

$$\gamma_o^2 = \frac{\beta_1 + \beta_2}{2\beta_1\beta_2} \gamma_i^2 \quad (4.2.18)$$

Figures 4.1 and 4.5 show plots of the orthotropic properties versus some orthotropic parameter for constant values of the isotropic properties. These figures may be generalized as in figure 4.6 where an orthotropic

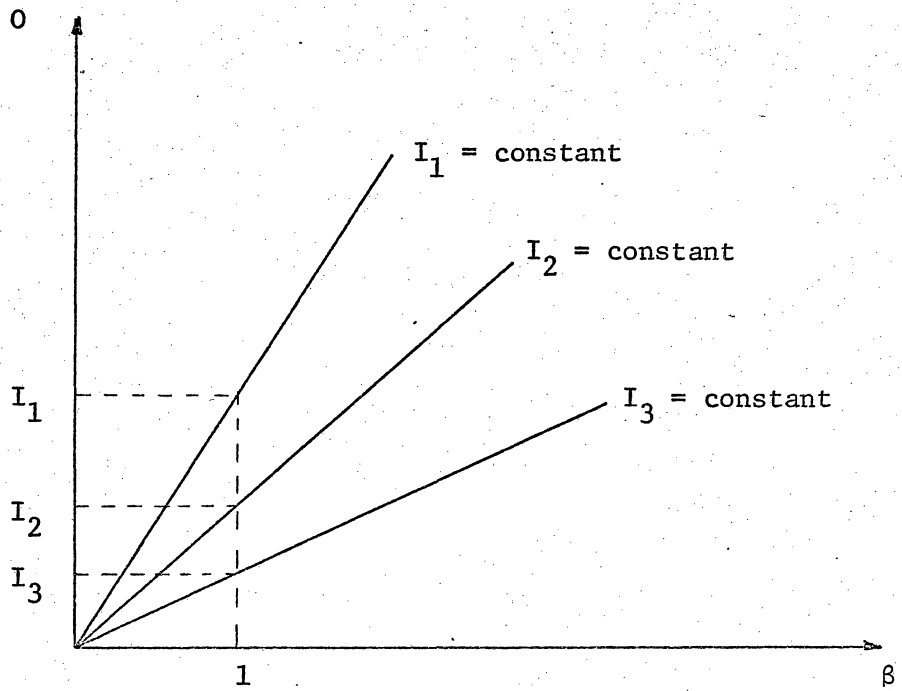


FIGURE 4.6

GENERAL ORTHOTROPIC - ISOTROPIC RELATIONSHIP

property 0, is plotted versus an orthotropic parameter, β , for constant values of the isotropic property, I.

Thus we see that any orthotropic property mentioned may be found if the isotropic property is known and the proper orthotropic parameter is known.

e. Plastic Zone Size

We have shown in equation (3.4.8) that the results from the Dugdale finiteness condition are independent of whether or not the material is orthotropic. This means that for an orthotropic material the plastic zone size is given by the same relation as for the isotropic case, providing that the inequality (4.1.17) is satisfied, that relation being,

$$\frac{s}{a} = 2 \sin^2 \left(\frac{\pi}{4} \frac{T}{Y} \right) \quad (4.2.17)$$

Brinson (2) has found the plastic zone size for polycarbonate, which is an orthotropic material. The crack was oriented along the strong direction and the strong direction's modulus was approximately 1.2 times the weak direction's modulus. His results compared with Dugdale's equation (4.2.17) are shown in Figure 4.7.

We can see that the experimental results are in reasonable agreement with the theory up to $s/a = 1/2$. Above this value the deviation can be attributed to the fact that the plate is finite and the finiteness affects the plastic zone size.

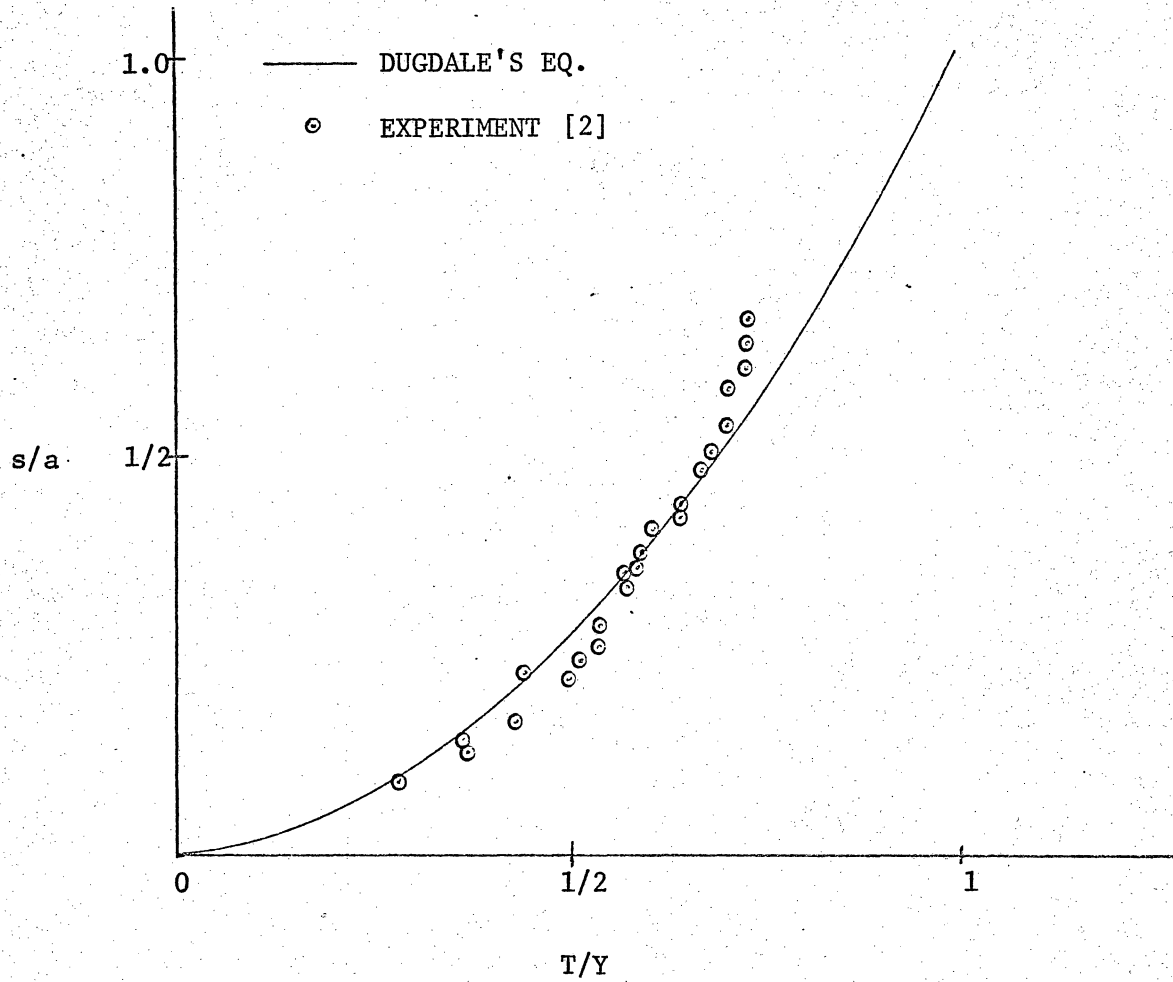


FIGURE 4.7

COMPARISON OF PLASTIC ZONE SIZE IN POLYCARBONATE TO DUGDALE'S EQ.

5. CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

We are now in a position to analyze the effect of orthotropy.

As far as the stresses along the crack line are concerned, we see that the stresses in the direction of the crack are the only ones affected. This effect is, as Ang and Williams pointed out, confined to the characteristic rigidity ratio $\sqrt{\frac{E_x}{E_y}}$. The stresses normal to the line of the crack are unaffected.

The other properties investigated were seen to be affected by only a multiplicative factor which will depend on the state of orthotropy. The plastic zone size was shown to be, both mathematically and experimentally, unaffected by orthotropy.

From Equations (4.1.17) and (4.1.18) we see that if these inequalities are not satisfied the material will tend to deform in a direction normal to the line of the crack.

For the Dugdale hypothesis to be satisfied the following must be true; for a given external tensile stress a material must satisfy (4.1.17). A material which is thus sufficiently orthotropic will tend to deform normal to the line of the crack. Similarly, for a given material the external stress must be of such a magnitude such that (4.1.18) is satisfied. For small enough external tensile stresses the material will tend to deform normal to the line of the crack.

5.2 Recommendations

The generalization of the problem to the general anisotropic case could be followed in an analogous manner to the present problem.

For the orthotropic case we saw that deformation did not necessarily proceed along the line of the crack. Deformation was expected either along the crack or normal to it depending on the state of orthotropy and external stress. In the anisotropic case deformation may be expected in any direction depending on the anisotropy and external stress.

An analysis, therefore, of the general anisotropic case might lead to a clearer picture of the plastic zone.

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THE APPLICATION OF THE DUGDALE
MODEL TO AN ORTHOTROPIC PLATE

Henry Gonzalez Jr.

ABSTRACT

The Dugdale model is applied to an orthotropic plate. Stresses along the crack line and displacements along the crack and elastic-plastic interface were found. The effect of orthotropy on several isotropic properties was found to be a multiplicative factor which is a function of the state of orthotropy.

The yield stress is assumed to follow a von Mises' yield criterion which was adopted to the orthotropic case. A limit on the severity of orthotropy for a given external load was found as well as a limit on the external load for a given state of orthotropy in order that the material would still follow the Dugdale hypothesis.

Finally, as long as the material satisfies the above mentioned limits, the plastic zone size was shown to be unaffected by orthotropy.