ANALYSIS OF VARIANCE OF A GROUP DIVISIBLE SINGULAR DESIGN

WITH TWO ASSOCIATE CLASSES WITH MISSING OBSERVATIONS

by

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I. INTRODUCTION

1.1 Partially Balanced Incomplete Block Designs with Two Associate Classes

Partially balanced incomplete block designs have been used widely in many fields of research, since their introduction by Bose and Nair (1939). The distinguishing features of a PBIB design are (1) homogenenous plots are grouped into blocks which are too small to accommodate a complete replicate of the treatments, and (2) any two treatments do not occur together equally often in a block. PBIB designs may be classified according to the number of associate classes they possess.

Bose, Clatworthy, and Shrikhande (1954) published a monograph of PBIB designs with two associate classes and their notation will be adhered to in this thesis. Additional designs of this classification were published by Clatworthy (1955, 1956).

An incomplete block design is said to be partially balanced with two associate classes if it satisfies the following requirements:

(1) The experimental material is divided into b blocks of k units each, different treatments being applied to the units in the same block.

(2) There are v(7k) treatments each which occur in r blocks.

(3) There can be established a relation of association between any two treatments satisfying the following requirements.

(a) Two treatments are either first associates or second associates.

(b) Each treatment has exactly n_i i-th associates $(i = 1,2)$.

(c) Given any two treatments which are i-th associates, the number of treatments common to the j-th associate of the first and the k-th associate of the second is p^i_{jk} and is independent of the pair of treatments we start with. Also $p^i_{jk} = p^i_{kj}$ $(i,j,k = 1,2)$.

(4) Two treatments which are i-th associates occur together in exactly λ_i blocks $(i = 1,2)$.

The numbers v, r, k, b, n_1, n_2, λ_1, and λ_2 are called parameters of the first kind, whereas the numbers p^i_{jk} are called the parameters of the second kind.

The following relations between the parameters are known to hold:

$$vr = bk,$$

$$n_1 + n_2 = v-1,$$

$$n_1\lambda_1 + n_2\lambda_2 = r(k-1),$$

$$p^1_{11} + p^1_{12} = n_1-1,$$

$$p^1_{21} + p^1_{22} = n_2,$$

$$p^2_{11} + p^2_{12} = n_1,$$

$$p^2_{21} + p^2_{22} = n_2-1,$$

$$n_1p^1_{12} = n_2p^2_{11},$$

$$n_1p^1_{22} = n_2p^2_{12}.$$

The simplest type of partially balanced incomplete block design with two associate classes is denoted by GDS (Group Divisible Singular). This class of designs has the following properties:

(a) $v = mn$, and the treatments can be divided into m groups on n each, such that any two treatments of the same group are first associates while two treatments from different groups are second associates.

(b) The following relations amongst the parameters are:

$$r = \lambda_1,$$

$$rk - \lambda_2 v \geqslant 0,$$

$$n_1 = n-1,$$

$$n_2 = n(m-1),$$

$$(n-1)\lambda_1 + n(m-1)\lambda_2 = r(k-1),$$

$$P_1 = \begin{bmatrix} n-2 & 0 \\ 0 & n(m-1) \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{bmatrix}.$$

1.2 Missing Values in GDS Designs

In experimental work one or more observations may be missing. An agricultural plot may be trampled, an animal may die, a test tube may be dropped, or a machine may break down. When missing values occur, the usual method of computing the various sums of squares cannot be used unless the missing values are first estimated from the existing data.

In this thesis the analysis of GDS designs with missing observations will be discussed. The procedure that will be followed was first discussed by Yates (1933). It envolves estimating the missing values by minimizing the error sum of squares, subtracting one degree of freedom from the error degrees of freedom for each missing value, and correcting for the positive bias in the treatment sum of squares.

The above procedure has been used by Cornish (1943) for lattice designs, by Wilkinson (1958) for complete and incomplete block designs, by Glenn and Kramer (1958) for the randomized block design, by Glass and Kramer (1960) for the Latin square design, and by Bairdand Kramer (1961) for the balanced incomplete block design.

Specific formulae for one missing value and for two missing values occuring in various configurations will be developed. The case of n missing values will be dealt with for a few special cases. A method of analysis will be developed to obtain an unbiased treatment sum of squares. Finally, a numerical example illustrating the calculations envolved will be given.

II THEORETICAL BACKGROUND

2.1 Intra-block Estimation of Treatment Effects and Analysis of Variance

The mathematical model of the GDS design is

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}, \qquad (2.1)$$

$$(i = 1,\ldots,v;\ j = 1,\ldots,b)$$

where y_{ij} is the response of the i-th treatment in the j-th block (if treatment i occurs in block j), μ is the overall mean, τ_i is the effect of the i-th treatment, β_j is the effect of the j-th block, and ϵ_{ij} is a random effect assumed to be normally and independently distributed with zero mean and homogeneous variance, σ^2. The restrictions on the parameters in (2.1) are

$$\sum_i \tau_i = \sum_j \beta_j = 0. \qquad (2.2)$$

To test

$$H_0: \tau_1 = \tau_2 = \cdots = \tau_v = 0, \qquad (2.3)$$

by means of the F-test, first calculate

$$T_i = \sum_j y_{ij}, \qquad (2.4)$$

$$B_j = \sum_i y_{ij}, \qquad (2.5)$$

$$G = \sum_i T_i = \sum_j B_j, \qquad (2.6)$$

$$Q_i = T_i - 1/k\, B_{i.}, \quad (B_{i.} \text{ sum of blocks containing treatment } i) \tag{2.7}$$

$$S_1(Q_i) = \text{sum of Q-values of first associates of treatment } i, \tag{2.8}$$

$$G_i = Q_i + S_1(Q_i). \tag{2.9}$$

Then calculate the minimum-variance unbiased linear estimate t_i of τ_i by

$$t_i = \frac{(k-c_1)}{r(k-1)}\, Q_i + \frac{c_1-c_2}{r(k-1)}\, G_i. \tag{2.10}$$

The following sums of squares may then be obtained:

$$\text{Blocks (unadj)} = 1/k \sum_j B_j^2 - G^2/N = B, \tag{2.11}$$

$$\text{Treatments (adj)} = \sum_i t_i\, Q_i = A, \tag{2.12}$$

$$\text{Total} = \sum_i \sum_j y_{ij}^2 - G^2/N = T, \tag{2.13}$$

$$\text{Error} = (\text{by subbraction}) = E \tag{2.14}$$

The intra-block analysis of variance is given in Table 2.1, which is on page 8.

To test (2.3), the calculated F-ratio is compared with the tabular value of the F-distribution with (v-1), and (N-b-v-1) degrees of freedom of the pre-assigned level of significance.

Table 2.1

Intra-block Analysis of Variance

Source of Variation	d.f.	Sum of Squares	Mean Square	F
Blocks (unadj)	b-1	B		
Treatments (adj)	v-1	A	$s_A^2 = \frac{A}{v-1}$	s_A^2/s_E^2
Error	N-b-v-1	E	$s_E^2 = \frac{E}{N-b-v-1}$	
Total	N-1	T		

2.2 Missing Observations

In this section it will be shown that the error sum of squares obtained by the standard least squares procedure for the known data is equal to the error sum of squares obtained by first estimating the missing values and then using the formulae of Section 2.1.

Let m, t_i, and b_j be the least squares estimates of the parameters μ, τ_i, and β_j respectively taken over known data only, then

$$E' = \sum_{\text{known data}}\sum (y_{ij} - m - t_i - b_j)^2, \quad (2.15)$$

is a minimum. If we now denote missing data by x_{ij}'s, and use Yates' criterion to obtain estimates of the missing values we have to minimize E*, the error sum of squares taken over both known and estimated data. If m*, t_i^*, and

b_j^*, denote the least squares estimates of μ, τ_i, and β_j respectively ~~than~~ over both known and augmented data then

$$E^* = \sum_{\substack{\text{known}\\\text{data}}}\sum (y_{ij} - m^* - t_i^* - b_j^*)^2 + \sum_{\substack{\text{missing}\\\text{data}}}\sum (x_{ij} - m^* - t_i^* - b_j^*)^2, \tag{2.16}$$

is a minimum.

If we obtain the partial derivative of E^* with respect to x_{ij}, set it equal to zero, we obtain

$$x_{ij} = m^* + t_i^* + b_j^* . \tag{2.17}$$

Substituting (2.17) into (2.16) and comparing it with (2.15) it is seen that

$$E^* = E'. \tag{2.18}$$

However, to obtain the correct partitioning of the degrees of freedom, the error and total degrees of freedom must be reduced by one for each missing observation.

III ESTIMATION OF MISSING VALUES

3.1 Single Missing Value:

The procedure discussed in Section 2.2, will be used to estimate a single missing value, i.e., partially differentiating the error sum of squares with respect to the missing observation X_{cd} (say), and equating the resulting expression to zero.

Block and treatment totals which involve a missing value may be defined as,

$$B_d = B_d' + X_{cd}$$

$$T_c = T_c' + X_{cd} \; . \qquad (3.1)$$

X_{cd} will be involved in three types of Q - values and in three types of G - values, Q and G values for the treatment c, Q and G values for the first associates of treatment c, (denoted by 1c), Q and G values for the second associates of treatment c, (denoted by 2c).

Q and G values involving a missing observation may be written as

$$Q_c = Q_c' + \frac{k-1}{k} X_{cd} \; ,$$

$$Q_p^{1c} = Q'{}_p^{1c} - \frac{1}{k} X_{cd} \; , \qquad (3.2)$$

$$Q_q^{2c} = Q'{}_q^{2c} - \frac{1}{k} X_{cd} \; ,$$

and

$$G_c = G_c' + \frac{k-n}{k} X_{cd} ,$$

$$G_p^{1c} = G'{}_p^{1c} + \frac{k-n}{k} X_{cd} , \qquad (3.3)$$

$$G_q^{2c} = G'{}_q^{2c} - \frac{n}{k} X_{cd} .$$

The error sum of squares, involving one missing value, X_{cd}, may be written as, $E = \left[\sum\sum y_{ij}^2 - \frac{1}{k}\sum B_j^2 - \sum t_i Q_i\right]$

$$+\left[X_{cd}^2 - \frac{1}{k}(B_d' + X_{cd})^2 - t_c Q_c - \sum_{p \neq c}^{n-1} t_p^{1c} Q_p^{1c} - \sum^{k-n} t_q^{2c} Q_q^{2c}\right] ,$$

(3.4)

where the first quantity is taken from the existing data and the second quantity from the data with a missing observation, X_{cd}. We may write

$$t_c Q_c = \frac{X_{cd}}{rk(k-1)}\left\{\left[2(k-1)(k-c_1) + (k-n)(c_1-c_2)\right] Q_c' + (k-1)(c_1-c_2)G_c' \right.$$

$$\left. + \frac{1}{k}\left[(k-1)^2(k-c_1) + (k-1)(k-n)(c_1-c_2)\right] X_{cd}\right\} ,$$

$$t_p^{1c} Q_p^{1c} = \frac{X_{cd}}{r\,k(k-1)}\left\{\left[(k-n)(c_1-c_2) - 2(k-c_1)\right] Q'{}_p^{1c} - \right.$$

$$\left. (c_1 - c_2)\, G_p'{}^{1c} + \frac{1}{k}\left[(k-c_1) - (k-n)(c_1-c_2)\right] X_{cd}\right\} ,$$

(3.5)

$$t_q^{2c} Q_q^{2c} = \frac{X_{cd}}{rk(k-1)} \Big\{ \Big[2(k-c_1) + n(c_1-c_2) \Big] Q_q^{\prime 2c} + (c_1-c_2) G_q^{\prime 2c}$$

$$+ \frac{1}{k} \Big[(k-c_1) + n(c_1-c_2) \Big] X_{cd} \Big\} ,$$

where Q's, G's and t's are defined in (3.2), (3.3) and (2.10) respectively.

Now equation (3.4) may be rearranged as,

$$E^* = \Big[\sum\sum y^2_{ij} - \frac{1}{k} \sum B^2_j - \sum t_i Q_i \Big] + \Big[X^2_{cd} - \frac{1}{k} (B'_d + X_{cd})^2$$

$$- \frac{X_{cd}}{rk(k-1)} \Big\{ \Big[(k-n)(c_1-c_2) - 2(k-c_1) \Big] \Big[Q'_c + \sum_{p \neq c}^{n-1} Q'^{\,1c}_p \Big]$$

$$+ 2k(k-c_1) Q'_c + (c_1-c_2)(kG'_c - G'_c - \sum_{p \neq c}^{n-1} G'^{\,1c}_p)$$

$$- \Big[2(k-c_1) + n(c_1-c_2) \Big] \sum^{k-n} Q_q^{\prime 2c} - (c_1-c_2) \sum^{k-n} G'^{\,2c}_q$$

$$+ \Big[(k-1)(k-c_1) + (k-n)(c_1-c_2) \Big] X_{cd} \Big\} \Big] . \qquad (3.6)$$

Simplifying equation (3.6) with the help of the following definitions,

$$\begin{aligned} Q'_c + \sum_{p \neq c}^{n-1} Q_q^{\prime 1c} &= G'_c , \\ G'_c + \sum^{k-n} G_q^{\prime 1c} &= n\, G'_c , \end{aligned} \qquad (3.7)$$

and then partially differentiating the resulting expression with respect to X_{cd} , one obtains,

$$\frac{\partial E^*}{\partial X_{cd}} = 2X_{cd} - \frac{2}{k} B'_d - \frac{2}{k} X_{cd} - \frac{2}{rk(k-1)} \left[(k-n)(c_1-c_2)-(k-c_1)\right] G'_c$$

$$- \frac{2}{rk(k-1)} k(k-c_1) Q'_c + \frac{1}{rk(k-1)} \left[2(k-c_1)+n(c_1-c_2)\right] \sum_q^{k-n} Q'^{2c}_q$$

$$+ \frac{1}{rk(k-1)} (c_1-c_2) \sum_q^{k-n} G'^{2c}_q - \frac{2X_{cd}}{rk(k-1)} \left[(k-1)(k-c_1)+(k-n)(c_1-c_2)\right] . \quad (3.8)$$

Now, equating (3.8) to zero, one obtains the result as,

$$\left[r(k-1)^2 - (k-1)(k-c_1) - (k-n)(c_1-c_2)\right] X_{cd} = r(k-1)B'_d$$

$$+k(k-c_1) Q'_c + \left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_c$$

$$- \frac{1}{2} \left[2(k-c_1)+n(c_1-c_2)\right] \sum_q^{k-n} Q'^{2c}_q - \frac{1}{2} (c_1-c_2) \sum_q^{k-n} G'^{2c}_q . \quad (3.9)$$

Denoting the coefficient of the missing value under consideration, i.e., of X_{cd}, by m and the right hand side of the equation (3.9) by C_{cd}, (or the constant term involving B'_d, Q'_c, G'_c, Q'^{2c}_q, and G'^{2c}_q), we have,

$$m X_{cd} = C_{cd} , \quad (3.10)$$

where

$$m = r(k-1)^2 - (k-1)(k-c_1) - (k-n)(c_1-c_2)$$

$$C_{cd} = r(k-1)B'_d + k(k-c_1)Q'_c + \left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_c - \frac{1}{2}\left[2(k-c_1) + n(c_1-c_2)\right]\sum^{k-n} Q'^{2c}_q - \frac{1}{2}(c_1-c_2)\sum^{k-n} G'^{2c}_q. \qquad (3.11)$$

In (3.11) Q'^{2c}_q = Q's, which are second associates of treatments c, (3.12)

G'^{2c}_q = Sum of Q's, which are second associates of treatment c, in the association scheme, (3.13)

and B'_d, Q'_c, G'_c, Q'^{2c}_q, and G'^{2c}_q are the numerical values taken from the existing data.

3.2 Two Missing Values

By the application of the procedure discussed in Section 2.2, and setting up the error sum of squares in the same fashion as in the case of a single missing observation in Section 3.1, it is easy to obtain explicit formulae for every possible configuration, in the GDS designs with two missing values, referred to as X_{cd} and X_{ef}.

In a similar fashion, the first associates of treatments c and e will be denoted by 1c and 1e respectively.

The second associates of treatments c and e will be referred to as $\overline{2c}$ and $\overline{2e}$ respectively, and in the case where second associates of treatments c and e are common to blocks d and f, or other treatments are replicated in blocks d and f, will be referred to as $\underline{2\ell}$.

In this section, the configuration of X_{cd} and X_{ef} will be identified, (Table 4.1) and estimates of these missing values will be obtained.

For expediency of results, the coefficient of the missing value under consideration will be denoted by m, as before, and the constant term ~~or~~ the right hand side involving B'_d, Q'_c, G'_c, $Q'^{\overline{2c}}_q$, $Q'^{\underline{2\ell}}_q$, $G'^{\overline{2c}}_q$, $G'^{\underline{2\ell}}_q$, and B'_f, Q'_e, G'_e, $Q'^{\overline{2e}}_q$, $Q'^{\underline{2\ell}}_q$, $G'^{\overline{2e}}_q$, $G'^{\underline{2\ell}}_q$, will be referred to as C_{cd} and C_{ef} respectively.

E'^{*} will be referred to as the error sum of squares involving the missing values only. In this case, involving X_{cd} and X_{ef} only. Since the first quantity in the error sum of squares will always vanish under differentiation, i.e., the quantity taken from the existing data.

3.2.a <u>No two are missing in the same block or treatment.</u>

1 - Treatments c and e are first associate, but second associates of treatments c and e are not replicated in blocks

d and f.

In this case, Q and G values involving the missing observations, X_{cd} and X_{ef}, may be written as,

$$Q_c = Q'_c + \frac{k-1}{k} X_{cd} - \frac{1}{k} X_{ef} ,$$

$$Q_e = Q'_e - \frac{1}{k} X_{cd} + \frac{k-1}{k} X_{ef} ,$$

$$Q_p^{(1)} = Q'^{(1)}_p - \frac{1}{k} X_{cd} - \frac{1}{k} X_{ef} , \qquad (3.14)$$

$$Q_q^{2\bar{c}} = Q'^{2\bar{c}}_q - \frac{1}{k} X_{cd} ,$$

$$Q_q^{2\bar{e}} = Q'^{2\bar{e}}_q - \frac{1}{k} X_{ef},$$

and

$$G_c = G'_c + \frac{k-n}{k} X_{cd} - \frac{n}{k} X_{ef},$$

$$G_e = G'_e - \frac{n}{k} X_{cd} + \frac{k-n}{k} X_{ef} ,$$

$$G_p^{(1)} = G'^{(1)}_p + \frac{k-n}{k} X_{cd} + \frac{k-n}{k} X_{ef} , \qquad (3.15)$$

$$G_q^{2\bar{c}} = G'^{2\bar{c}}_q - \frac{n}{k} X_{cd} ,$$

$$G_q^{2\bar{e}} = G'^{2\bar{e}}_q - \frac{n}{k} X_{ef}.$$

The error sum of squares, involving two missing values, X_{cd} and X_{ef}, is,

$$E^* = \left[\sum\sum y_{ij}^2 - \frac{1}{k}\sum B_j^2 - \sum t_i Q_i\right] + \left[X_{cd}^2 + X_{ef}^2 - \frac{1}{k}(B'_d + X_{cd})^2 - \frac{1}{k}(B'_f + X_{ef})^2 - (t_c Q_c + t_e Q_e) - \sum_{p \neq 1}^{n-2} t_p^{(1)} Q_p^{(1)} - \sum^{k-n} (t_q^{\overline{2c}} Q_q^{\overline{2c}} + t_q^{\overline{2e}} Q_q^{\overline{2e}})\right]. \qquad (3.16)$$

The first quantity in E^* will vanish under differentiation. Now, according to the definitions made in (2.10), (3.14) and (3.15) and after some algebraic manipulation, the error sum of squares involving missing values only, i.e., E'^* involving X_{cd} and X_{ef}, may be written as,

$$E'^* = X_{cd}^2 + X_{ef}^2 - \frac{1}{k}(B'_d + X_{cd})^2 - \frac{1}{k}(B'_f + X_{ef})^2$$

$$- \frac{1}{rk(k-1)}\Big\{\Big[(k-n)(c_1-c_2) - 2(k-c_1)\Big]\Big[Q'_c + Q'_e + \sum^{n-2} Q_p^{(1)}\Big]$$

$$(X_{cd} + X_{ef}) + \Big[2k(k-c_1)Q'_c + (c_1-c_2)(kG'_c - G'_c - G'_e -$$

$$\sum^{n-2} G'^{(1)}_p) - \Big[2(k-c_1) + n(c_1-c_2)\Big]\sum^{k-n} Q'^{\overline{2c}}_q -$$

$$(c_1-c_2)\sum^{k-n} G'^{\overline{2c}}_q\Big] X_{cd}$$

$$+ \left[2k(k-c_1)Q'_e + (c_1-c_2)(kG'_e - G'_e - G'_c - \sum^{n-2} G'^{(1)}_p)\right.$$

$$- \left[2(k-c_1) + n(c_1-c_2)\right] \sum^{k-n} Q'^{\overline{2e}}_q - (c_1-c_2) \sum^{k-n} G'^{\overline{2e}}_q \Big] X_{ef}$$

$$+ \left[(k-1)(k-c_1) + (k-n)(c_1-c_2)\right]\left[X^2_{cd} + X^2_{ef}\right]$$

$$+2 \left[(k-n)^2(c_1-c_2) - (2k-n)(c_1-c_2)\right] X_{cd} X_{ef} \Big\} . \quad (3.17)$$

We now define

$$G'_c = G'_e = G'_\ell \; ,$$

$$Q'_c + Q'_e + \sum^{n-2} Q'^{(1)}_p = G'_\ell \; , \qquad (3.18)$$

$$2G'_\ell + \sum^{n-2} G'^{(1)}_p = nG'_\ell \; .$$

Simplifying (3.17) with the help of (3.18), partially differentiating the resulting expression, with respect to X_{cd} and X_{ef}, and equating the partial derivatives to zero, one obtains

$$m X_{cd} - \frac{1}{k} \left[(k-n)^2(c_1-c_2) - (2k-n)(k-c_1)\right] X_{ef} = C_{cd}, \qquad (3.19)$$

$$-\frac{1}{k} \left[(k-n)^2(c_1-c_2) - (2k-n)(k-c_1)\right] X_{cd} + mX_{ef} = C_{ef} \; . \quad (3.20)$$

m is as defined in (3.11) and

$$C_{cd} = \Big\{ r(k-1)B'_d + k(k-c_1)Q'_c + \big[(k-n)(c_1-c_2) - (k-c_1)\big] G'_\ell$$

$$-\frac{1}{2}\big[2(k-c_1) + n(c_1-c_2)\big]\sum^{k-n} Q'^{\,\overline{2c}}_q - \frac{1}{2}(c_1-c_2)\sum^{k-n} G'^{\,\overline{2c}}_q \Big\}, \tag{3.21}$$

$$C_{ef} = \Big\{ r(k-1)B'_f + k(k-c_1)Q'_e + \big[(k-n)(c_1-c_2) - (k-c_1)\big] G'_\ell$$

$$-\frac{1}{2}\big[2(k-c_1) + n(c_1-c_2)\big]\sum^{k-n} Q'^{\,\overline{2e}}_q - \frac{1}{2}(c_1-c_2)\sum^{k-n} G'^{\,\overline{2e}}_q \Big\}. \tag{3.22}$$

2. Treatments c and e are first associate and in addition, an second associates of treatments c and e are replicated in blocks d and f (or an second associates of treatment c occurring in block f or of treatment e occurring in block d.) It is not possible that a = 0. G and Q values in this case are,

$$Q_c = Q'_c + \frac{k-1}{k} X_{cd} - \frac{1}{k} X_{ef} ,$$

$$Q_e = Q'_e - \frac{1}{k} X_{cd} + \frac{k-1}{k} X_{ef} ,$$

$$Q^{(1)}_p = Q'^{(1)}_p - \frac{1}{k} X_{cd} - \frac{1}{k} X_{ef} ,$$

$$
\begin{aligned}
Q_q^{\underline{2\ell}} &= Q'^{\underline{2\ell}}_q - \frac{1}{k} X_{cd} - \frac{1}{k} X_{ef}\,, \\
Q_q^{\overline{2c}} &= Q'^{\overline{2c}}_q - \frac{1}{k} X_{cd}\,, \\
Q_q^{\overline{2e}} &= Q'^{\overline{2e}}_q - \frac{1}{k} X_{ef}\,,
\end{aligned}
\tag{3.23}
$$

and

$$
\begin{aligned}
G_c &= G'_c + \frac{k-n}{k} X_{cd} + \frac{k-n}{k} X_{ef}\,, \\
G_e &= G'_e + \frac{k-n}{k} X_{cd} + \frac{k-n}{k} X_{ef}\,, \\
G_p^{(1)} &= G'^{(1)}_p + \frac{k-n}{k} X_{cd} + \frac{k-n}{k} X_{ef}\,, \\
G_q^{\underline{2\ell}} &= G'^{\underline{2\ell}}_q - \frac{n}{k} X_{cd} - \frac{n}{k} X_{ef}\,, \\
G_q^{\overline{2c}} &= G'^{\overline{2c}}_q - \frac{n}{k} X_{cd}\,, \\
G_q^{\overline{2e}} &= G'^{\overline{2e}}_q - \frac{n}{k} X_{ef}\,.
\end{aligned}
\tag{3.24}
$$

The error sum of squares involving missing values only may be arranged in a similar fashion as in (3.17), using (2.10), (3.23), (3.24), and (3.18) and after some algebraic manipulation, one obtains,

$$
\begin{aligned}
E^{*\prime} = {} & X^2_{cd} + X^2_{ef} - \frac{1}{k}(B'_f + X_{cd})^2 - \frac{1}{k}(B'_f + X_{ef})^2 \\
& -2\,\frac{(X_{cd} + X_{ef})}{rk(k-1)}\left\{\left[(k-n)(c_1-c_2) - (k-c_1)\right]\right\} G'_{\ell} \\
& - \frac{X_{cd}}{rk(k-1)}\Big\{ 2k(k-c_1)Q'_c - \left[2(k-c_1) + n(c_1-c_2)\right] \\
& \left[\sum^{k-n-an} Q'^{\,\overline{2c}}_q + \sum^{an} Q'^{\,\underline{2\ell}}_q\right] - (c_1-c_2)\Big[\sum^{k-n-an} G'^{\,\overline{2c}}_q \\
& + \sum^{an} G'^{\,\underline{2\ell}}_q\Big] + \left[(k-1)(k-c_1) + (k-n)(c_1-c_2)\right] X_{cd}\Big\} \\
& - \frac{X_{ef}}{rk(k-1)}\Big\{ 2k(k-c_1)\,Q'_e - \left[2(k-c_1) + n(c_1-c_2)\right] \\
& \left[\sum^{k-n-an} Q'^{\,\overline{2e}}_q + \sum^{an} Q'^{\,\underline{2\ell}}_q\right] - (c_1-c_2)\Big[\sum^{k-n-an} G'^{\,\overline{2e}}_q \\
& + \sum^{an} G'^{\,\underline{2\ell}}_q\Big] + \left[(k-1)(k-c_1) + (k-n)(c_1-c_2)\right] X_{ef}\Big\} \\
& - \frac{2X_{cd}\,X_{ef}}{rk^2(k-1)}\Big[(k-n)^2(c_1-c_2) - (2k-n-an)(k-c_1) \\
& + an^2\,(c_1-c_2)\Big] \,. \qquad (3.25)
\end{aligned}
$$

Differentiating partially (3.25) with respect to X_{cd} and X_{ef}, and equating the partial derivatives to zero, one obtains the equations as,

$$mX_{cd} - \frac{1}{k}\left[(k-n)^2(c_1-c_2) - (2k-n-an)(k-c_1) + an^2(c_1-c_2)\right]X_{ef}$$

$$= C_{cd}, \qquad (3.26)$$

$$-\frac{1}{k}\left[(k-n)^2(c_1-c_2) - (2k-n-an)(k-c_1) + an^2(c_1-c_2)\right] X_{cd}$$

$$+ m\,X_{ef} = C_{ef} . \qquad (3.27)$$

m is defined in (3.11) and

$$C_{cd} = r(k-1)B'_d + k(k-c_1)Q'_c + \left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_{\ell}$$

$$-\frac{1}{2}\left[2(k-c_1) + n(c_1-c_2)\right]\left[\sum^{k-n-an} Q'^{\,\overline{2c}}_q + \sum^{an} Q'^{\,\underline{2\ell}}_q\right]$$

$$-\frac{1}{2}(c_1-c_2)\left[\sum^{k-n-an} G'^{\,\overline{2e}}_q + \sum^{an} G'^{\,\underline{2\ell}}_q\right], \qquad (3.28)$$

$$C_{ef} = r(k-1)B'_f + k(k-c_1)Q'_e + \left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_{\ell}$$

$$-\frac{1}{2}\left[2(k-c_1) + n(c_1-c_2)\right]\left[\sum^{k-n-an} Q'^{\,\overline{2e}}_q + \sum^{an} Q'^{\,\underline{2\ell}}_q\right]$$

$$-\frac{1}{2}\,(c_1-c_2)\Big[\sum^{k-n-an} G'^{\,2e}_{q} + \sum^{an} G'^{\,2\ell}_{q}\Big]. \qquad (3.29)$$

Solving the equations (3.28) and (3.29), one obtains the estimates of X_{cd} and X_{ef} as,

$$X_{cd} = \frac{mC_{cd} + \frac{1}{k}\Big[(k-n)^2(c_1-c_2) - (2k-n-an)(k-c_1) + an^2(c_1-c_2)\Big]C_{ef}}{m^2 - \frac{1}{k^2}\Big[(k-n)^2(c_1-c_2) - (2k-n-an)(k-c_1) + an^2(c_1-c_2)\Big]^2}, \qquad (3.30)$$

$$X_{ef} = \frac{mC_{ef} + \frac{1}{k}\Big[(k-n)^2(c_1-c_2) - (2k-n-an)(k-c_1) + an^2(c_1-c_2)\Big]C_{cd}}{m^2 - \frac{1}{k^2}\Big[(k-n)^2(c_1-c_2) - (2k-n-an)(k-c_1) + an^2(c_1-c_2)\Big]^2}. \qquad (3.31)$$

3 - Treatments c and e are second associate, but no group of treatments are common to blocks d and f. Q and G values, in this case are

$$Q_c = Q'_c + \frac{k-1}{k}\,X_{cd}\;,$$

$$Q_e = Q'_e + \frac{k-1}{k}\,X_{ef}\;,$$

$$Q^{1c}_{p} = Q'^{\,1c}_{p} - \frac{1}{k} X_{cd} ,$$

$$Q^{1e}_{p} = Q'^{\,1e}_{p} - \frac{1}{k} X_{ef} , \qquad (3.32)$$

$$Q^{\overline{2c}}_{q} = Q'^{\,\overline{2c}}_{q} - \frac{1}{k} X_{cd} ,$$

$$Q^{\overline{2e}}_{q} = Q'^{\,\overline{2e}}_{q} - \frac{1}{k} X_{ef} ,$$

and

$$G_{c} = G'_{c} + \frac{k-n}{k} X_{cd} ,$$

$$G_{e} = G'_{e} + \frac{k-n}{k} X_{ef} ,$$

$$G^{1c}_{p} = G'^{\,1c}_{p} + \frac{k-n}{k} X_{cd} , \qquad (3.33)$$

$$G^{1e}_{p} = G'^{\,1e}_{p} + \frac{k-n}{k} X_{ef} ,$$

$$G^{\overline{2c}}_{q} = G'^{\,\overline{2c}}_{q} - \frac{n}{k} X_{cd} ,$$

$$G^{\overline{2e}}_{q} = G'^{\,\overline{2e}}_{q} - \frac{n}{k} X_{ef} .$$

The error sum of squares involving missing values, i.e., X_{cd} and X_{ef}, only, is

$$
\begin{aligned}
E'^{*} = {} & X^2_{cd} + X^2_{ef} - \frac{1}{k}(B'_d + X_{cd})^2 - \frac{1}{k}(B'_f + X_{ef})^2 \\
& - \frac{1}{rk(k-1)} \Bigg\{ \Bigg[\big[(k-n)(c_1-c_2) - 2(k-c_1)\big]\Big[Q'_c + \sum^{n-2} Q'^{\,1c}_{p}\Big] \\
& + 2k(k-c_1)\, Q'_c + (c_1-c_2)\Big(kG'_c - G'_c - \sum^{n-2} G'^{\,1c}_{p}\Big) \\
& - \big[2(k-c_1) + n(c_1-c_2)\big] \sum^{k-n} Q'^{\,\overline{2c}}_{q} - (c_1-c_2) \sum^{k-n} G'^{\,\overline{2c}}_{q} \Bigg] X_{cd} \\
& + \Bigg[\big[(k-n)(c_1-c_2) - 2(k-c_1)\big]\Big[Q'_e + \sum^{n-2} Q'^{\,1e}_{p} \Big] \\
& + 2k(k-c_1)Q'_e + (c_1-c_2)\Big(kG'_e - G'_e - \sum^{n-2} G'^{\,1e}_{p}\Big) \\
& - \big[2(k-c_1) + n(c_1-c_2)\big] \sum^{k-n} Q'^{\,2e}_{q} - (c_1-c_2) \sum^{k-n} G'^{\,2e}_{q} \Bigg] X_{ef} \\
& + \frac{1}{k}\big[k(k-1)(k-c_1) - (k-c_1) + (k+1)(k-n)(c_1-c_2)\big]\big[X^2_{cd} + X^2_{ef}\big] \Bigg\}.
\end{aligned}
\tag{3.34}
$$

Now defining,

$$Q'_c + \sum^{n-2} Q'^{\,1c}_{p} = G'_c \,,$$

$$G'_c + \sum^{n-2} G'^{\,1c}_{p} = nG'_c \,,$$

$$Q'_e + \sum^{n-2} Q'^{\,1e}_p = G'_e , \qquad (3.35)$$

$$G'_e + \sum^{n-2} G'^{\,1e}_p = nG'_e ,$$

equation (3.34) reduces to,

$$E'^* = X^2_{cd} + X^2_{ef} - \frac{1}{k}(B'_d + X_{cd})^2 - \frac{1}{k}(B'_f + X_{ef})^2$$

$$- \frac{2}{rk(k-1)} \Bigg\{ \Bigg[\Big[\big[(k-n)(c_1-c_2) - (k-c_1)\big]\big[G'_c\big] + k(k-c_1)\, Q'_e$$

$$- \frac{1}{2} \big[2(k-c_1) + n(c_1-c_2)\big] \sum^{k-n} Q'^{\,\overline{2e}}_q - \frac{1}{2}(c_1-c_2)$$

$$\sum^{k-n} G'^{\,\overline{2e}}_q \Big] X_{cd} + \Big[\big[(k-n)(c_1-c_2) - (k-c_1)\big] \; G'_e$$

$$+ k(k-c_1)\, Q'_e - \frac{1}{2}\big[2(k-c_1) + n(c_1-c_2)\big] \sum^{k-n} Q'^{\,\overline{2e}}_q$$

$$- \frac{1}{2}(c_1-c_2) \sum^{k-n} G'^{\,\overline{2e}}_q \Big] X_{ef} + \frac{1}{k}\Big[k(k-1)(k-c_1)$$

$$-(k-c_1) + (k+1)(k-n)(c_1-c_2)\Big]\Big[X^2_{cd} + X^2_{ef}\Big] \Bigg\} . \qquad (3.36)$$

Differentiating partially equation (3.36), with respect to X_{cd} and X_{ef}, and simplifying the resulting expressions, one obtains

$$m\, X_{cd} = C_{cd} , \qquad (3.37)$$

$$m X_{ef} = C_{ef} , \qquad (3.38)$$

where

$$m = rk(k-1)^2 - k(k-1)(k-c_1) + (k-c_1) - (k+1)(k-n)(c_1-c_2) , \qquad (3.39)$$

$$C_{cd} = rk(k-1)B'_d + k^2(k-c_1)Q'_c + k\left[(k-n)(c_1-c_2) - (k-c_1)\right]G'_c$$
$$-\frac{k}{2}\left[2(k-c_1) + n(c_1-c_2)\right]\sum^{k-n} Q'_q{}^{\overline{2c}} - \frac{k}{2}(c_1-c_2)\sum^{k-n} G'_q{}^{\overline{2c}}, \qquad (3.40)$$

$$C_{ef} = rk(k-1)B'_f + k^2(k-c_1)Q'_e + k\left[(k-n)(c_1-c_2) - (k-c_1)\right]G'_e$$
$$-\frac{k}{2}\left[2(k-c_1) + n(c_1-c_2)\right]\sum^{k-n} Q'_q{}^{\overline{2e}} - \frac{k}{2}(c_1-c_2)\sum^{k-n} G'_q{}^{\overline{2e}}. \qquad (3.41)$$

Solving (3.37) and (3.38), the estimates of X_{cd} and X_{ef} are,

$$X_{cd} = \frac{1}{m} C_{cd} , \qquad (3.42)$$

$$X_{ef} = \frac{1}{m} C_{ef} , \qquad (3.43)$$

which are identical to the estimate of a single missing observation given in Section 3.1.

4 - Treatments c and e are second associates, and in addition, an group of treatments, (other than the groups containing the missing values) are common to blocks d and f.

Setting up the error sum of squares containing missing values in a similar fashion as in equation (3.36), one obtains,

$$E'^{*} = X^2_{cd} + X^2_{ef} - \frac{1}{k}(B'_d + X_{cd})^2 - \frac{1}{k}(B'_f + X_{ef})^2$$

$$- \frac{2}{rk(k-1)}\Bigg\{\Bigg[\Big[(k-n)(c_1-c_2) - (k-c_1)\Big] G'_c + k(k-c_1)Q'_c$$

$$- \frac{1}{2}\Big[2(k-c_1)+n(c_1-c_2)\Big]\Big[\sum^{k-n-an} Q'_q{}^{\bar{2}c} + \sum^{an} Q'_q{}^{\underline{2l}}\Big]$$

$$- \frac{1}{2}(c_1-c_2)\Big[\sum^{k-n-an} G'_q{}^{\bar{2}c} + \sum^{an} G'_q{}^{\underline{2l}}\Big]\Bigg] X_{cd}$$

$$+ \Bigg[\Big[(k-n)(c_1-c_2) - (k-c_1)\Big] G'_e + k(k-c_1) Q'_e$$

$$- \frac{1}{2}\Big[2(k-c_1) + n(c_1-c_2)\Big]\Big[\sum^{k-n-an} Q'_q{}^{\bar{2}c} + \sum^{an} Q'_q{}^{\underline{2l}}\Big]$$

$$- \frac{1}{2}(c_1-c_2)\Big[\sum^{k-n-an} G'_q{}^{\bar{2}c} + \sum^{an} Q'_q{}^{\underline{2l}}\Big]\Bigg] X_{ef}$$

$$+ \frac{1}{k}\Big[k(k-1)(k-c_1)+(k-c_1) + (k+1)(k-n)(c_1-c_2)\Big]\Big[X^2_{cd} + X^2_{ef}\Big]$$

$$+ \frac{2}{k}\Big[an(k-c_1) + an^2(c_1-c_2)\Big] X_{cd} X_{ef}\Bigg\}. \qquad (3.44)$$

Differentiating partially (3.44) with respect to X_{cd} and X_{ef}, one obtains the equations as,

$$m X_{cd} - an \left[(k-c_1) + n(c_1-c_2)\right] X_{ef} = C_{cd} , \quad (3.45)$$

$$-an \left[(k-c_1) + n(c_1-c_2)\right] X_{cd} + m X_{ef} = C_{ef} . \quad (3.46)$$

Solving (3.45) and (3.46), the estimates of X_{cd} and X_{ef} are,

$$X_{cd} = \frac{m C_{cd} + an \left[(k-c_1) + n(c_1-c_2)\right] C_{ef}}{m^2 - a^2n^2 \left[(k-c_1) + n(c_1-c_2)\right]^2} , \quad (3.47)$$

$$X_{ef} = \frac{m C_{ef} + an \left[(k-c_1) + n(c_1-c_2)\right] C_{cd}}{m^2 - a^2n^2 \left[(k-c_1) + n(c_1-c_2)\right]^2} , \quad (3.48)$$

where m, C_{cd} , and C_{ef} are defined in (3.39), (3.40), and (3.41).

5 - Treatments c and e are second associate.

Treatments c and e are replicated in blocks d and f, but no groups of treatments (other than the first associates of treatments c and e) are common to blocks d and f.

Q and G values, in this case, are

$$Q_c = Q'_c + \frac{k-1}{k} X_{cd} - \frac{1}{k} X_{ef} ,$$

$$
\begin{aligned}
Q_e &= Q'_e - \frac{1}{k} X_{cd} + \frac{k-1}{k} X_{ef}\,, \\
Q^{1c}_p &= Q'^{\,1c}_p - \frac{1}{k} X_{cd} - \frac{1}{k} X_{ef}\,, \\
Q^{1e}_p &= Q'^{\,1e}_p - \frac{1}{k} X_{cd} - \frac{1}{k} X_{ef}\,, \\
Q^{\overline{2c}}_q &= Q'^{\,\overline{2c}}_q - \frac{1}{k} X_{cd}\,, \\
Q^{\overline{2e}}_q &= Q'^{\,\overline{2e}}_q - \frac{1}{k} X_{ef}\,,
\end{aligned}
\tag{3.49}
$$

and

$$
\begin{aligned}
G_c &= G'_c + \frac{k-n}{k} X_{cd} - \frac{1}{k} X_{ef}\,, \\
G_e &= G'_e - \frac{n}{k} X_{cd} + \frac{k-n}{k} X_{ef}\,, \\
G^{1c}_p &= G'^{\,1c}_p + \frac{k-n}{k} X_{cd} - \frac{n}{k} X_{ef}\,, \\
G^{1e}_p &= G'^{\,1e}_p - \frac{n}{k} X_{cd} + \frac{k-n}{k} X_{ef}\,, \\
G^{\overline{2c}}_q &= G'^{\,\overline{2c}}_q - \frac{n}{k} X_{cd}\,, \\
G^{\overline{2e}}_q &= G'^{\,\overline{2e}}_q - \frac{n}{k} X_{ef}\,.
\end{aligned}
\tag{3.50}
$$

The error sum of squares, involving missing values, X_{cd} and

X_{ef}, only, may be written as,

$$E^{*'} = X^2_{cd} + X^2_{ef} - \frac{1}{k}(B'_d + X_{cd})^2 - \frac{1}{k}(B'_f + X_{ef})^2$$

$$- \frac{1}{rk(k-1)} \Bigg\{ \Bigg[\Big[(k-n)(c_1-c_2) - 2(k-c_1)\Big]\Big[Q'_c + \sum^{n-1} Q'_p{}^{1c}\Big]$$

$$+ 2k(k-c_1)Q'_c + (c_1-c_2)(kG'_c - G'_c - \sum^{n-1} G'_p{}^{1c})$$

$$- \Big[2(k-c_1) + n(c_1-c_2)\Big]\Big[Q'_c + \sum^{n-1} Q'_p{}^{1c}\Big] - (c_1-c_2)$$

$$(G'_c + \sum^{n-1} G'_p{}^{1c}) - \Big[2(k-c_1) + n(c_1-c_2)\Big] \sum^{k-2n} Q'_q{}^{\overline{2c}}$$

$$- (c_1-c_2) \sum^{k-2n} G'_q{}^{\overline{2c}} \Bigg] X_{cd} + \Bigg[\Big[(k-n)(c_1-c_2) - 2(k-c_1)\Big]$$

$$\Big[Q'_e + \sum^{n-1} Q'_p{}^{1e}\Big] + 2k(k-c_1)Q'_e + (c_1-c_2)(kG'_e - G'_e - \sum G'_p{}^{1e})$$

$$- \Big[2(k-c_1) + n(c_1-c_2)\Big]\Big[Q'_e + \sum^{n-1} Q'_p{}^{1e}\Big] - (c_1-c_2)$$

$$(G'_e + \sum^{n-1} G'_p{}^{1e}) - \Big[2(k-c_1) + n(c_1-c_2)\Big] \sum^{k-2n} Q'_q{}^{\overline{2e}}$$

$$- (c_1-c_2) \sum^{k-2n} G'_q{}^{\overline{2e}} \Bigg] X_{ef} + \Big[(k-1)(k-c_1) + (k-n)(c_1-c_2)\Big]$$

$$\Big[X^2_{cd} + X^2_{ef}\Big] - \frac{2}{k}\Big[n(k-c_1)(k-n) + 2n(k-n)(c_1-c_2)\Big] X_{cd}X_{ef} \Bigg\}. \tag{3.51}$$

Now by defining

$$Q'_c + \sum^{n-1} Q'_p{}^{1c} = G'_c \,,$$

$$G'_c + \sum^{n-1} G'_p{}^{1c} = nG'_c \,,$$

$$Q'_e + \sum^{n-1} Q'_p{}^{1e} = G'_e \,, \qquad (3.51a)$$

$$G'_e + \sum^{n-1} G'_p{}^{1e} = nG'_e \,,$$

equation (3.51) reduces to

$$E'^* = X^2_{cd} + X^2_{ef} - \frac{1}{k}(B'_d + X_{cd})^2 - \frac{1}{k}(B'_f + X_{ef})^2$$

$$- \frac{2}{rk(k-1)} \Bigg\{ \Bigg[\Big[(k-n)(c_1-c_2) - (k-c_1)\Big] G'_c + k(k-c_1)\, Q'_c$$

$$- \Big[(k-c_1) + n(c_1-c_2)\Big] G'_e - \frac{1}{2} \Big[2(k-c_1) + n(c_1-c_2)\Big]$$

$$\sum^{k-2n} Q'_q{}^{\bar{2c}} - \frac{1}{2}(c_1-c_2) \sum^{k-2n} G'_q{}^{\bar{2c}} \Bigg] X_{cd}$$

$$+ \Bigg[\Big[(k-n)(c_1-c_2) - (k-c_1)\Big] G'_e + k(k-c_1)\, Q'_e$$

$$- \Big[(k-c_1) + n(c_1-c_2)\Big] G'_c - \frac{1}{2} \Big[2(k-c_1) + n(c_1-c_2)\Big]$$

$$\sum^{k-2n} Q'_q{}^{\bar{2e}} - \frac{1}{2}(c_1-c_2) \sum^{k-2n} G'_q{}^{\bar{2e}} \Bigg] X_{ef} +$$

This is a numbering mistake.

$$\left[(k-1)(k-c_1) + (k-n)(c_1-c_2)\right]\left[X^2_{cd} + X^2_{ef}\right]$$

$$- \frac{2}{k}\left[n(k-n)(k+c_1-2c_2)\right] X_{cd}\, X_{ef} \Big\} . \qquad (3.52)$$

Differentiating partially (3.52) with respect to X_{cd} and X_{ef}, one obtains the equations as,

$$m\, X_{cd} + \frac{1}{k}\left[n(k-n)(k+c_1-2c_2)\right] X_{ef} = C_{cd} , \qquad (3.53)$$

$$\frac{1}{k}\left[n(k-n)(k+c_1-2c_2)\right] X_{cd} + m\, X_{ef} = C_{ef}, \qquad (3.54)$$

where

$$m = r(k-1)^2 - (k-1)(k-c_1) - (k-n)(c_1-c_2) ,$$

$$C_{cd} = r(k-1)B'_d + k(k-c_1)\, Q'_c + \left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_c$$

$$- \left[(k-c_1) + n(c_1-c_2)\right] G'_e - \frac{1}{2}\left[2(k-c_1) + n(c_1-c_2)\right]$$

$$\sum^{k-2n} Q'_q{}^{\overline{2e}} - \frac{1}{2}(c_1-c_2) \sum^{k-2n} G'_q{}^{\overline{2e}} , \qquad (3.55)$$

$$C_{ef} = r(k-1)B'_f + k(k-c_1)\, Q'_e + \left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_e$$

$$- \left[(k-c_1) + n(c_1-c_2)\right] G'_c - \frac{1}{2}\left[2(k-c_1) + n(c_1-c_2)\right]$$

$$\sum^{k-2n} Q'_q{}^{\overline{2e}} - \frac{1}{2}(c_1-c_2) \sum^{k-2n} G'_q{}^{\overline{2e}} . \qquad (3.56)$$

In (k-2n), 2 stands for the number of missing values and in the case of $z = n$ missing values, it will be $(k - zn)$, where $z = 1, 2, 3, 4, \ldots, n$.

Solving equations (3.53) and (3.54), the estimates of X_{cd} and X_{ef} are,

$$X_{cd} = \frac{m\, C_{cd} - \frac{1}{k}\left[n(k-n)(k+c_1-2c_2)\right] C_{ef}}{m^2 - \frac{1}{k^2}\left[n(k-n)(k+c_1-2c_2)\right]^2}, \tag{3.57}$$

$$X_{ef} = \frac{m\, C_{ef} - \frac{1}{k}\left[n(k-n)(k+c_1-2c_2)\right] C_{cd}}{m^2 - \frac{1}{k^2}\left[n(k-n)(k+c_1-2c_2)\right]^2}. \tag{3.58}$$

6 - Treatments c and e are second associate.

Treatments c and e are replicated in blocks d and f and in addition, an group of treatments are common to blocks d and f. It is not possible that $a = 0$. The error of sum of squares involving missing values may be written in a similar fashion as in equation (3.52), following the definitions given in (3.51a). After algebraic manipulation, one obtains,

$$E'^{*} = X^2_{cd} + X^2_{ef} - \frac{1}{k}(B'_d + X_{cd})^2 - \frac{1}{k}(B'_f + X_{ef})^2$$

$$- \frac{2}{rk(k-1)}\Bigg\{\Bigg[\Big[(k-n)(c_1 - c_2) - (k-c_1)\Big] G'_c + k(k-c_1)Q'_c$$

$$- \left[k(k-c_1) + n(c_1 - c_2)\right] G'_e - \frac{1}{2}\left[2(k-c_1) + n(c_1-c_2)\right]$$

$$\left[\sum^{k-2n-an} Q'_q{}^{\overline{2c}} + \sum^{an} Q'_q{}^{\underline{2\ell}}\right] - \frac{1}{2}(c_1-c_2)\left[\sum^{k-2n-an} G'_q{}^{\overline{2c}}\right.$$

$$\left.+ \sum^{an} G'_q{}^{\underline{2\ell}}\right]\Bigg] X_{cd} + \Bigg[\left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_e + K(k-c_1)Q'_e$$

$$- \left[(k-c_1) + n(c_1-c_2)\right] G'_e - \frac{1}{2}\left[2(k-c_1) + n(c_1-c_2)\right]$$

$$\left[\sum^{k-2n-an} Q'_q{}^{\overline{2c}} + \sum^{an} Q'_q{}^{\underline{2\ell}}\right] - \frac{1}{2}(c_1-c_2)\left[\sum^{k-2n-an} G'_q{}^{\overline{2c}}\right.$$

$$\left.+ \sum^{an} G'_q{}^{\underline{2\ell}}\right]\Bigg] X_{ef} + \left[(k-1)(k-c_1) + (k-n)(c_1-c_2)\right]\left[X^2_{cd} + X^2_{ef}\right]$$

$$- \frac{2}{k}\left[n(k-c_1)(k-n+a) + n(c_1-c_2)(2k-2n+an)\right] X_{cd} X_{ef}\Bigg\} . \tag{3.59}$$

Differentiating partially (3.59) with respect to X_{cd} and X_{ef}, one obtains the equations,

$$m X_{cd} + \frac{1}{k}\left[n(k-c_1)(k-n+a) + n(c_1-c_2)(2k+an-2n)\right] X_{ef} = C_{cd} , \tag{3.60}$$

$$\frac{1}{k}\left[n(k-c_1)(k-n+a)\ +n(c_1-c_2)(2k-2n+an)\right]X_{cd}+m\,X_{ef}=C_{ef}\ ,$$

(3.61)

where m, C_{cd} and C_{ef} are defined in (3.55) and (3.56).

Solving (3.60) and (3.61), the estimates of X_{cd} and X_{ef} are,

$$X_{cd}=\frac{m\,C_{cd}-\frac{n}{k}\left[(k-c_1)(k-n+a)+(c_1-c_2)(2k-2n+an)\right]C_{ef}}{m^2-\frac{n^2}{k^2}\left[(k-c_1)(k-n+a)+(c_1-c_2)(2k-2n+an)\right]^2}\ ,$$

(3.62)

$$X_{ef}=\frac{m\,C_{ef}-\frac{n}{k}\left[(k-c_1)(k-n+a)+(c_1-c_2)(2k-2n+an)\right]C_{cd}}{m^2-\frac{n^2}{k^2}\left[(k-c_1)(k-n+a)+(c_1-c_2)(2k-2n+an)\right]^2}\ .$$

(3.63)

3.2.b. Two are missing in the same block, i.e., d = f.

7 - Treatments c and e are just first associates.

The Q and C values involving X_{cd} and X_{ed} , may be written as,

$$Q_c=Q'_c+\frac{k-1}{k}X_{cd}-\frac{1}{k}X_{ed}\ ,$$

$$Q_e=Q'_e-\frac{1}{k}X_{cd}+\frac{k-1}{k}X_{ed},$$

$$Q^{(1)}_p = Q'^{(1)}_p - \frac{1}{k} X_{cd} - \frac{1}{k} X_{ed},$$

$$Q^{(2)}_q = Q'^{(2)}_q - \frac{1}{k} X_{cd} - \frac{1}{k} X_{ed}, \qquad (3.64)$$

and

$$G_c = G'_c + \frac{k-n}{k} X_{cd} + \frac{k-n}{k} X_{ed},$$

$$G_e = G'_e + \frac{k-n}{k} X_{cd} + \frac{k-n}{k} X_{ed}, \qquad (3.65)$$

$$G^{(1)}_p = G'^{(1)}_p + \frac{k-n}{k} X_{cd} + \frac{k-n}{k} X_{ed},$$

$$G^{(2)}_q = G'^{(2)}_q - \frac{n}{k} X_{cd} - \frac{n}{k} X_{ed}.$$

The error sum of squares involving two missing values, X_{cd} and X_{ed}, may be written as,

$$E'^{*} = \Big[X^2_{cd} + X^2_{ed} - \frac{1}{k}(B'_d + X_{cd} + X_{ed})^2 - (t_c Q_c + t_e Q_e) - \sum^{n-2} t^{(1)}_p Q^{(1)}_p - \sum^{k-n} t^{(2)}_q Q^{(2)}_q \Big], \qquad (3.66)$$

where (1) and (2) are referred to as first and second associates of treatments c and e.

Using (2.10), (3.64) and (3.65) and after some algebraic manipulation, (3.66) may be written as,

$$E'^* = X^2_{cd} + X^2_{ed} - \frac{1}{k}(B'_d + X_{cd} + X_{ed})^2 - \frac{1}{rk(k-1)}\Bigg\{\Bigg[\Big[(k-n)(c_1-c_2)$$

$$- 2(k-c_1)\Big]\Big[Q'_c + Q'_e + \sum^{n-2} Q'_p{}^{(1)}\Big] + 2k(k-c_1)\, Q'_c$$

$$+ (c_1-c_2)(kG'_c - G'_c - G'_e - \sum^{n-2} G'_p{}^{(1)})$$

$$- \Big[2(k-c_1) + n(c_1-c_2)\Big] \sum^{k-n} Q'_q{}^{(2)} - (c_1-c_2) \sum^{k-n} G'_q{}^{(2)}\Bigg] X_{cd}$$

$$+ \Bigg[\Big[(k-n)(c_1-c_2) - 2(k-c_1)\Big]\Big[Q'_c + Q'_e + \sum^{n-2} Q'_p{}^{(1)}\Big]$$

$$+ 2k(k-c_1)\, Q'_e + (c_1-c_2)(kG'_e - G'_e - G'_c - \sum^{n-2} G'_p{}^{(1)})$$

$$- \Big[2(k-c_1) + n(c_1-c_2)\Big] \sum^{k-n} Q'_q{}^{(2)} - (c_1-c_2) \sum^{k-n} G'_q{}^{(2)}\Bigg] X_{ed}$$

$$+ \Big[(k-1)(k-c_1) + (k-n)(c_1-c_2)\Big]\Big[X^2_{cd} + X^2_{ed}\Big]$$

$$+ 2\Big[(k-c_1) + (k-n)(c_1-c_2) + r(k-1)\Big] X_{cd}\, X_{ed}\Bigg\}. \quad (3.67)$$

Using definitions in (3.18), equation (3.67) reduces to,

$$E'^* = X^2_{cd} + X^2_{ed} - \frac{1}{k}(B'_d + X_{ed} + X_{cd})^2$$

$$- \frac{2}{rk(k-1)}\Bigg\{\Bigg[\Big[(k-n)(c_1-c_2) - (k-c_1)\Big] G'\ell - \frac{1}{2}\Big[2(k-c_1)$$

$$+\, n(c_1-c_2)\Big] \sum^{k-n} Q'^{(2)}_q \; - \frac{1}{2}(c_1-c_2) \sum^{k-n} Q'^{(2)}_q \Big] (X_{cd} + X_{ed})$$

$$+\, k(k-c_1)Q'_c\, X_{cd} + k(k-c_1)Q'_e\, X_{ed} + \big[(k-1)(k-c_1) + (k-n)(c_1-c_2)\big]$$

$$\big[X^2_{cd} + X^2_{ed}\big] + \big[(k-c_1) + (k-n)(c_1-c_2) + r(k-1)\big]\, X_{cd}\, X_{ed} \Big\}. \tag{3.68}$$

Differentiating partially (3.68) with respect to X_{cd} and X_{ed}, equating the resulting expressions to zero, one obtains the equations as,

$$m\, X_{cd} - \big[(k-c_1) + (k-n)(c_1-c_2) + r(k-1)\big]\, X_{ed} = C_{cd}\,, \tag{3.69}$$

$$-\big[(k-c_1) + (k-n)(c_1-c_2) + r(k-1)\big]\, X_{cd} + m\, X_{ed} = C_{ed}\,, \tag{3.70}$$

where m, C_{cd} and C_{ed} are as defined in section 3.1.

Now solving equations (3.69) and (3.70), one obtains the estimates of X_{cd} and X_{ed} as

$$X_{cd} = \frac{m\, C_{cd} + \big[(k-c_1) + (k-n)(c_1-c_2) + r(k-1)\big]\, C_{ed}}{m^2 - \big[(k-c_1) + (k-n)(c_1-c_2) + r(k-1)\big]^2}\,, \tag{3.71}$$

$$X_{ed} = \frac{m\, C_{ed} + \big[(k-c_1) + (k-n)(c_1-c_2) + r(k-1)\big]\, C_{cd}}{m^2 - \big[(k-c_1) + (k-n)(c_1-c_2) + r(k-1)\big]^2}\,. \tag{3.72}$$

The estimates are identical to the estimate of a single missing value discussed in Section 3.1.

8 - Treatments c and e are second associates.

Q and G values involving X_{cd} and X_{ed} may be written as,

$$\begin{aligned}
Q_c &= Q'_c + \frac{k-1}{k} X_{cd} - \frac{1}{k} X_{ed} , \\
Q_e &= Q'_e - \frac{1}{k} X_{cd} + \frac{k-1}{k} X_{ed} , \\
Q^{1c}_p &= Q'^{1c}_p - \frac{1}{k} X_{cd} - \frac{1}{k} X_{ed} , \\
Q^{1e}_p &= Q'^{1e}_p - \frac{1}{k} X_{cd} - \frac{1}{k} X_{ed} ,
\end{aligned} \tag{3.73}$$

and

$$\begin{aligned}
G_e &= G'_e - \frac{n}{k} X_{cd} + \frac{k-n}{k} X_{ed} , \\
G_c &= G'_c + \frac{k-n}{k} X_{cd} - \frac{n}{k} X_{ed} , \\
G_p^{1c} &= G'^{1c}_p + \frac{k-n}{k} X_{cd} - \frac{n}{k} X_{ed} , \\
G_p^{1e} &= G'^{1e}_p - \frac{n}{k} X_{cd} + \frac{k-n}{k} X_{ed} .
\end{aligned} \tag{3.74}$$

In this situation, there are no second associates of treatments c and e, which involves the missing values X_{cd} and X_{ed}.

The error sum of squares, involving X_{cd} and X_{ed} may be written as,

$$E'^{*} = \Big[X^2_{cd} + X^2_{ed} - \frac{1}{k}(B'_d + X_{cd} + X_{ed})^2 - (t_c Q_c + t_e Q_e) - \sum^{n-1}(t_p^{\,lc}\, Q_p^{\,lc} + t_p^{\,le}\, Q_p^{\,le})\Big]. \qquad (3.75)$$

Using (2.10), (3.73) and (3.74) and doing some algebraic manipulation, (3.75) may be written as,

$$E'^{*} = X^2_{cd} + X^2_{ed} - \frac{1}{k}(B'_d + X_{cd} + X_{ed})^2$$

$$- \frac{1}{rk(k-1)}\Bigg\{\Bigg[\Big[(k-n)(c_1-c_2) - 2(k-c_1)\Big]\Big[Q'_c + \sum^{n-1} Q'^{\,lc}_p\Big]$$

$$+ 2k(k-c_1)\, Q'_c + (c_1-c_2)(kG'_c - G'_c - G'_e - \sum^{n-1} G'^{\,lc}_p$$

$$- \sum^{n-1} G'^{\,le}_p) - \Big[2(k-c_1) + n(c_1-c_2)\Big]\Big[Q'_e + \sum^{n-1} Q'^{\,le}_p\Big]\Bigg] X_{cd}$$

$$+ \Big[\Big[(k-n)(c_1-c_2) - 2(k-c_1)\Big]\Big[Q'_e + \sum^{n-1} Q'^{\,le}_p\Big] + 2k(k-c_1)Q'_e$$

$$+ (c_1-c_2)(kG'_e - G'_e - G'_c - \sum^{n-1} G'^{\,le}_p - \sum^{n-1} G'^{\,lc}_p)$$

$$- \left[2(k-c_1) + n(c_1-c_2)\right]\left[Q'_c + \sum^{n-1} Q'_p{}^{lc}\right]\Bigg] X_{ed}$$

$$+ \frac{1}{k} \Big[(k-1)^2(k-c_1) + (2n-1)(k-c_1) + (k-n)^2(c_1-c_2)$$

$$+ n^2(c_1-c_2)\Big]\left[X^2_{cd} + X^2_{ed}\right] - \frac{2}{k} \Big[(2k-n-1)(k-c_1)$$

$$+ 2n(k-n)(c_1-c_2)\Big] \; X_{cd}\, X_{ed} \Bigg\} . \qquad (3.76)$$

Using definitions in (3.51a), simplifying and then partially differentiating the resulting expression with respect to X_{cd} and X_{ed}, one obtains the equations,

$$m\, X_{cd} - \frac{1}{k} \left[rk(k-1) - (2k-n-1)(k-c_1) - 2n(k-n)(c_1-c_2)\right] X_{ed} = C_{cd} ,$$

$$(3.77)$$

$$- \frac{1}{k} \left[rk(k-1) - (2k-n-1)(k-c_1) - 2n(k-n)(c_1-c_2)\right] X_{cd}$$

$$+ m\, X_{ed} = C_{ed} , \qquad (3.78)$$

where

$$m = rk(k-1)^2 - (k-c_1) \left[(k-1)^2 + 2n - 1\right] - (c_1-c_2) \left[(k-n)^2 + n^2\right],$$

$$(3.79)$$

$$C_{cd} = rk(k-1)B'_d + k\left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_c$$

$$+ k^2(k-c_1)\, Q'_c - k\left[(k-c_1) + n(c_1-c_2)\right] G'_e, \qquad (3.80)$$

$$C_{ed} = rk(k-1)B'_d + k\left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_e$$

$$+ k^2(k-c_1)Q'_e - k\left[(k-c_1) + n(c_1-c_2)\right] G'_c\,. \qquad (3.81)$$

Solving (3.77) and (3.78), one obtains the estimate of X_{cd} and X_{ed} as,

$$X_{cd} = \frac{m\,C_{cd} + \frac{1}{k}\left[rk(k-1) - (2k-n-1)(k-c_1) - 2n(k-n)(c_1-c_2)\right]C_{ed}}{m^2 - \frac{1}{k^2}\left[rk(k-1) - (2k-n-1)(k-c_1) - 2n(k-n)(c_1-c_2)\right]^2}, \qquad (3.82)$$

$$X_{ed} = \frac{m\,C_{ed} + \frac{1}{k}\left[rk(k-1) - (2k-n-1)(k-c_1) - 2n(k-n)(c_1-c_2)\right]C_{cd}}{m^2 - \frac{1}{k^2}\left[rk(k-1) - (2k-n-1)(k-c_1) - 2n(k-n)(c_1-c_2)\right]^2}\,. \qquad (3.83)$$

3.2.c <u>Two are missing in the same treatment, i.e., $c = f$.</u>

9 - Second associates of treatment c are not replicated in blocks d and f.

Q and G values in this case are,

$$Q_c = Q'_c + \frac{k-1}{k} X_{cd} + \frac{k-1}{k} X_{ef} ,$$

$$Q_p^{(1)} = Q'_p{}^{(1)} - \frac{1}{k} X_{cd} - \frac{1}{k} X_{cf} , \tag{3.84}$$

$$Q_q^{\overline{2c}} = Q'_q{}^{\overline{2c}} - \frac{1}{k} X_{cd} ,$$

$$Q_q^{\overline{2c}} = Q'_q{}^{\overline{2c}} - \frac{1}{k} X_{cf} ,$$

and

$$G_c = G'_c + \frac{k-n}{k} X_{cd} + \frac{k-n}{k} X_{ef} ,$$

$$G_p^{(1)} = G'_p{}^{(1)} + \frac{k-n}{k} X_{cd} + \frac{k-n}{k} X_{cf} ,$$

$$G_q^{\overline{2c}} = G'_q{}^{\overline{2c}} - \frac{n}{k} X_{cd} , \tag{3.85}$$

$$G_q^{\overline{2c}} = G'_q{}^{\overline{2c}} - \frac{n}{k} X_{cf} .$$

The error sum of squares involving the missing values X_{cd} and X_{cf}, may be written as,

$$E'^{*} = X^2_{cd} + X^2_{cf} - \frac{1}{k} (B'_d + X_{cd})^2 - \frac{1}{k} (B'_f + X_{cf})^2 - t_c Q_c$$

$$- \sum_{p \neq c}^{n-1} t_p^{(1)} Q_p^{(1)} - \sum^{k-n} t_q^{\bar{2}c} Q_q^{\bar{2}c} - \sum^{k-n} t_q^{\bar{2}c} Q_q^{\bar{2}c} . \tag{3.86}$$

Using (2.10), (3.84) and (3.85), (3.86) may be rearranged as,

$$E'^{*} = X^2_{cd} + X^2_{cf} - \frac{1}{k}(B'_d + X_{cd})^2 - \frac{1}{k}(B'_f + X_{cf})^2$$

$$- \frac{1}{rk(k-1)} \Bigg[\Big[(k-n)(c_1-c_2) - 2(k-c_1)\Big]\Big[Q'_c +$$

$$\sum^{n-1} Q'^{(1)}_p\Big](X_{cd} + X_{cf}) + 2k(k-c_1)Q'_c\,(X_{cd} + X_{cf})$$

$$+ (c_1-c_2)(kG'_c - G'_c - \sum^{n-1} G'^{(1)}_p)(X_{cd} + X_{cf})$$

$$- \Big[2(k-c_1) + n(c_1-c_2)\Big] \sum^{k-n} Q'^{\bar{2}c}_q\,(X_{cd} + X_{cf})$$

$$- (c_1-c_2) \sum^{k-n} G'^{\bar{2}c}_q\,(X_{cd} + X_{cf})$$

$$+ \Big[(k-1)(k-c_1) + (k-n)(c_1-c_2)\Big]\Big[X^2_{cd} + X^2_{cf}\Big]$$

$$+ \frac{2}{k}\Big[(k-1)^2(k-c_1) + (k-c_1)(n-1) + (k-n)^2(c_1-c_2)\Big] X_{cd}X_{cf} \Bigg] . \tag{3.87}$$

As before, making use of definitions in (3.51a) for further simplification, and then partially differentiating (3.87) with respect to X_{cd} and X_{cf}, one obtains the equations as,

$$m\,X_{cd} - \frac{1}{k}\left[(k-1)^2(k-c_1) + (k-c_1)(n-1) + (k-n)^2(c_1-c_2)\right] X_{cf} = C_{cd}, \tag{3.88}$$

$$-\frac{1}{k}\left[(k-1)^2(k-c_1) + (k-c_1)(n-1) + (k-n)^2(c_1-c_2)\right] X_{cd} + m\,X_{cf} = C_{cf}, \tag{3.89}$$

where m, C_{cd} and C_{cf} are as defined in Section 3.1.

Now solving the equations (3.88) and (3.89), one obtains the estimates of X_{cd} and X_{cf} as,

$$X_{cd} = \frac{m\,C_{cd} + \frac{1}{k}\left[(k-1)^2(k-c_1) + (k-c_1)(n-1) + (k-n)^2(c_1-c_2)\right] C_{cf}}{m^2 - \frac{1}{k^2}\left[(k-1)^2(k-c_1) + (k-c_1)(n-1) + (k-n)^2(c_1-c_2)\right]^2}, \tag{3.90}$$

$$X_{cf} = \frac{m\,C_{cf} + \frac{1}{k}\left[(k-1)^2(k-c_1) + (k-c_1)(n-1) + (k-n)^2(c_1-c_2)\right] C_{cd}}{m^2 - \frac{1}{k^2}\left[(k-1)^2(k-c_1) + (k-c_1)(n-1) + (k-n)^2(c_1-c_2)\right]^2}. \tag{3.91}$$

10 - There are αn second associates of treatments c replicated

in blocks d and f. It is not possible that $a = 0$. In this situation Q and G values are

$$Q_c = Q'_c + \frac{k-1}{k} X_{cd} \quad \frac{k-1}{k} X_{cf} ,$$

$$Q_p^{(1)} = Q_p'^{(1)} - \frac{1}{k} X_{cd} - \frac{1}{k} X_{cf} ,$$

$$Q_q^{2\ell} = Q'^{2\ell}_q - \frac{1}{k} X_{cd} - \frac{1}{k} X_{cf} , \qquad (3.92)$$

$$Q_q^{\overline{2c}} = Q'^{\overline{2c}}_q - \frac{1}{k} X_{cd} ,$$

$$Q_q^{\overline{2c}} = Q'^{\overline{2c}}_q - \frac{1}{k} X_{cf} ,$$

and

$$G_c = G'_c + \frac{k-n}{k} X_{cd} + \frac{k-n}{k} X_{cf} ,$$

$$G^{(1)}_p = G'^{(1)}_p + \frac{k-n}{k} X_{cd} + \frac{k-n}{k} X_{cf} ,$$

$$G^{2\ell}_q = G'^{2\ell}_q - \frac{n}{k} X_{cd} - \frac{n}{k} X_{cf} , \qquad (3.93)$$

$$G_q^{\overline{2c}} = G'^{\overline{2c}}_q - \frac{n}{k} X_{cd} ,$$

$$G_q^{2c} = G'^{\overline{2c}}_q - \frac{n}{k} X_{cf} .$$

In a similar fashion as in equation (3.87), setting up the error sum of squares, and for algebraic manipulation, using (3.51a), one obtains E'^{*} as,

$$E'^{*} = X^2_{cd} + X^2_{cf} - \frac{1}{k}(B'_d + X_{cd})^2 - \frac{1}{k}(B'_f + X_{cf})^2$$

$$- t_c Q_c - \sum^{n-1} t_p^{(1)} Q_p^{(1)} - \sum^{an} t_q^{2\ell} Q_q^{2\ell} - \sum^{k-n-an} t_q^{\overline{2c}}\, Q_q^{\overline{2c}}$$

$$- \sum^{k-n-an} t_q^{\overline{2c}}\, Q_q^{\overline{2c}} \;. \tag{3.94}$$

Partially differentiating (3.94) with respect to X_{cd} and X_{cf}, one obtains the equations as,

$$m\,X_{cd} - \frac{1}{k}\left[\left\{(k-1)^2 + n(a+1)\right\}(k-c_1) + \left\{(k-n)^2 + an^2\right\}(c_1-c_2)\right]X_{cf}$$

$$= C_{cd}\,, \tag{3.95}$$

$$-\frac{1}{k}\left[\left\{(k-1)^2 + n(a+1)\right\}(k-c_1) + \left\{(k-n)^2 + an^2\right\}(c_1-c_2)\right]X_{cd}$$

$$+ \; m\,X_{cf} = C_{cf}\;. \tag{3.96}$$

Solving equations (3.95) and (3.96), one obtains the estimates of X_{cd} and X_{cf} as,

$$X_{cd} = \frac{m\, C_{cd} + \frac{1}{k}\left[\left\{(k-1)^2 + n(a+1)\right\}(k-c_1) + \left\{(k-n)^2 + an^2\right\}(c_1-c_2)\right] C_{cf}}{m^2 - \frac{1}{k^2}\left[\left\{(k-1)^2 + n(a+1)\right\}(k-c_1) + \left\{(k-n)^2 + an^2\right\}(c_1-c_2)\right]^2}, \tag{3.97}$$

$$X_{cf} \quad \frac{m\, C_{cf} + \frac{1}{k}\left[\left\{(k-1)^2 + n(a+1)\right\}(k-c_1) + \left\{(k-n)^2 + an^2\right\}(c_1-c_2)\right] C_{cd}}{m^2 - \frac{1}{k^2}\left[\left\{(k-1)^2 + n(a+1)\right\}(k-c_1) + \left\{(k-n)^2 + an^2\right\}(c_1-c_2)\right]^2}, \tag{3.98}$$

where m, C_{cd} and C_{cf} are as defined in (3.55) and (3.56).

IV TERMS UNAFFECTED AND AFFECTED BY CONFIGURATION

The results of the estimates of two missing observations, discussed in Section 3.2, will obviously yield the configuration of the missing values in the design. The equations obtained after partially differentiating the error sum of squares with respect to a particular missing value, X_{cd}, show that the configuration is neither affecting the coefficient of X_{cd}, nor the constant term involving B'_d, Q'_c, G'_c, Q'^{2c}_q, and G'^{2c}_q. The only term affected by the configuration is the coefficient of the missing value not under consideration, i.e., coefficient of X_{ef}. In general, the sum from the missing values not under consideration, will be denoted by N.

The $z = n$ missing values will be referred to as X_{cd}, X_{ef}, ..., X_{uv}, and the particular missing value as X_{cd}. Two particular missing values will be denoted by X_{cd} and X_{ef}.

The error sum of squares involving missing values only can be written as,

$$E'^{*} = \sum\sum X^2_{ij} - \frac{1}{k}\sum B^2_j - \sum t_i Q_i - \sum_{p \neq i}^{n-1} t^{1i}_{p} Q^{1i}_{p} - \sum^{k-n} t^{2i}_{q} Q^{2i}_{q} \, . \tag{4.1}$$

The result obtained after partially differentiating $\sum\sum X^2_{ij}$, with respect to a particular missing value X_{cd}, is,

$$\frac{\partial}{\partial X_{cd}}\left[X^2_{cd} + X^2_{ef} + \cdots + X^2_{uv}\right] = 2X_{cd} \cdot \qquad (4.2)$$

Obviously, this result will be the same under any configuration.

The block total involving the missing value, X_{cd}, is of the form,

$$B_d = (B'_d + X_{cd}) , \qquad (4.3)$$

and, in case, two or more values are missing in the same block, then,

$$B_d = B'_d + X_{cd} + N. \qquad (4.4)$$

Partially differentiating the block sum of squares, involving missing values only, one obtains the result,

$$\frac{\partial}{\partial X_{cd}}\left[-\frac{1}{k}\left\{(B'_d + X_{cd})^2 + \sum_{j \neq d}(B_j + N)^2\right\}\right] = -\frac{2}{k}B'_d - \frac{2}{k}X_{cd}, \qquad (4.5)$$

if $j = d$

$$\frac{\partial}{\partial X_{cd}}\left[-\frac{1}{k}\sum(B'_j + X_{cj} + N)^2\right] = -\frac{2}{k}B'_d - \frac{2}{k}X_{cd} - \frac{2}{k}N \cdot \qquad (4.6)$$

The only term affected by the configuration is N.

By definition,

$$Q_c = T_c - \frac{1}{k}B_c. , \qquad (4.7)$$

and $$t_c = \frac{k-c_1}{r(k-1)} Q_c + \frac{c_1-c_2}{r(k-1)} G_c \, . \tag{4.8}$$

Since treatment c involves X_{cd} , (4.7) may be written as,

$$Q_c = Q'_c + \frac{k-1}{k} X_{cd} + N \, , \tag{4.9}$$

and

$$G_c = G'_c + \frac{k-n}{k} X_{cd} + N \, . \tag{4.10}$$

The Q and G values containing first and second associates of treatment c, will be of the form,

$$\begin{aligned} Q^{1c}{}_p &= Q'{}_p{}^{1c} - \frac{1}{k} X_{cd} + N, \\ Q^{2c}{}_q &= Q'{}_q{}^{2c} - \frac{1}{k} X_{cd} + N, \end{aligned} \tag{4.11}$$

$$\begin{aligned} G^{1c}{}_p &= G'{}_p{}^{1c} \quad \frac{k-n}{k} X_{cd} + N \, , \\ G^{2c}{}_q &= G'{}_q{}^{2c} - \frac{n}{k} X_{cd} + N. \end{aligned} \tag{4.12}$$

The treatment sum of squares involving X_{cd} and other missing values may be written as,

$$\sum t_i Q_i + \sum_{p \neq i}^{n-1} t_p{}^{1i} \, Q_p{}^{1i} + \sum^{k-n} t_q{}^{2i} \, Q_q{}^{2i} = E'^{*}{}_1$$

$$= \frac{1}{rk(k-1)} \Bigg\{ 2k(k-c_1)\, Q'_c + 2\left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_c$$

$$- \left[2(k-c_1) + n(c_1-c_2)\right] \sum^{k-n} Q'^{\,2c}_q$$

$$-(c_1-c_2) \sum^{k-n} G'^{\,2c}_q \Bigg\} (X_{cd} + X_{ef} + \cdots + X_{uv})$$

$$+ \frac{1}{rk^2(k-1)} \Bigg\{ \left[k(k-1)(k-c_1) - k(k-n)(c_1-c_2)\right] (X^2_{cd} + X^2_{ef} + \cdots + X^2_{uv})$$

$$+ 2X_{cd}\, N \Bigg\}. \qquad (4.13)$$

Partially differentiating (4.13), with respect to X_{cd}, one obtains,

$$\frac{\partial E'^*_1}{\partial X_{cd}} = \frac{2}{rk(k-1)} \Bigg\{ k(k-c_1)\, Q'_c + \left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_c$$

$$- \frac{1}{2}\left[2(k-c_1) + n(c_1-c_2)\right] \sum^{k-n} Q'^{\,2c}_q - \frac{1}{2}(c_1-c_2) \sum^{k-n} G'^{\,2c}_q$$

$$+ \left[(k-1)(k-c_1) - (k-n)(c_1-c_2)\right] X_{cd} + \frac{1}{k} N \Bigg\}. \qquad (4.14)$$

Again, the only expression whose form is affected by configuration is N. Consequently, combining results (4.2), (4.5) and (4.14), one gets

$$\frac{\partial E^*}{\partial X_{cd}} = 2X_{cd} - \frac{2}{k} B'_d - \frac{2}{k} X_{cd} - \frac{2}{rk(k-1)} \Big\{ k(k-c_1)Q'_c$$

$$+ \left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_c - \frac{1}{2} \Big[2(k-c_1)$$

$$+ n(c_1-c_2)\Big] \sum^{k-n} Q'^{\,2c}_{q} - \frac{1}{2}(c_1-c_2) \sum^{k-n} G'^{\,2c}_{q}$$

$$+ \left[(k-1)(k-c_1) - (k-n)(c_1-c_2)\right] X_{cd} \Big\} + N \, . \qquad (4.15)$$

Equating equation (4.15) to zero, and simplifying, one obtains the general result,

$$m\, X_{cd} + N = C_{cd} \, . \qquad (4.16)$$

The equation system involving z missing values may be written as,

$$\begin{aligned} m\, X_{cd} + N &= C_{cd}, \\ m\, X_{ef} + N &= C_{ef}\,, \\ - - - - - - & \quad - - \\ m\, X_{uv} + N &= C_{uv}\,, \end{aligned} \qquad (4.17)$$

i.e., the coefficient of the missing value under differentiation is the same, i.e., m, in every case and the right hand side or the constant term is almost the same, with the exception of those cases, where second associates of treatment c are replicated in any two of the involved blocks. Explicit formulae have been discussed in Section 3.2, explaining the constant terms, where necessary. Possible configuration of X_{cd} and X_{ef} are given in Table 4.1 along with the off diagonal coefficients.

TABLE 4.1

OFF-DIAGONAL COEFFICIENT

Configuration of X_{cd} and X_{ef}	Coefficient of X_{ef}
No two are missing in the same block or treatment	
1 - Treatment c and e are first associates but second associates of c and e are not replicated in blocks d and f.	$-\frac{1}{k}\left[(k-n)^2(c_1-c_2) - (2k-n)(k-c_1)\right]$
2 - Treatments c and e are first associates, an second associates of treatment c occurring in block f or of treatment e occurring in block d.	$-\frac{1}{k}\left[(k-n)^2(c_1-c_2) - (2k-n-an)(k-c_1) + an^2(c_1-c_2)\right]$
3 - Treatments c and e are second associates but no groups of treatments are common to blocks d and f.	0
4 - Treatments c and e are second associates but an groups of treatments are common to blocks d and f.	$-an\left[(k-c_1) + n(c_1-c_2)\right]$

TABLE 4.1

OFF-DIAGONAL COEFFICIENT

Configuration of X_{cd} and X_{ef}	Coefficient of X_{ef}
5 - Treatments c and e are second associates, treatments cand e are replicated in blocks d and f, but no groups of treatments (other than first associates of c and e) are replicated in blocks d and f.	$\frac{1}{k}\left[n(k-n)(k+c_1-2c_2)\right]$
6 - Treatments c and e are second associates, treatments c and e are common to blocks d and f, in addition an group of treatments are common to blocks d and f.	$\frac{1}{k}\left[n(k-c_1)(k-n+a) + n(c_1-c_2)(2k-2n+an)\right]$
<u>X_{cd} and X_{ef} are missing in the same block i.e., d = f</u>	
7 - Treatments c and e are first associates.	$-\left[r(k-1)+(k-c_1) + (k-n)(c_1-c_2)\right]$
8 - Treatments c and e are second associates.	$-\frac{1}{k}\left[r(k-1) - (k-c_1)(2k-n-1) - 2n(k-n)(c_1-c_2)\right]$
<u>X_{cd} and X_{ef} are missing in the same treatment i.e., c = f</u>	

TABLE 4.1

OFF-DIAGONAL COEFFICIENTS

Configuration of X_{cd} and X_{ef}	Coefficient of X_{ef}
9 - Second associates of treatment c are not replicated in blocks d and f.	$-\frac{1}{k}\left[(k-1)^2(k-c_1) + (n-1)(k-c_1) + (k-n)^2(c_1-c_2)\right]$
10 - There are an second associates of treatment c, replicated in blocks d and f.	$-\frac{1}{k}\left[(k-c_1)\left\{(k-1)^2 + a(n+1) - 1\right\} + (c_1-c_2)\left\{(k-n)^2 + an^2\right\}\right]$

V ESTIMATION OF SEVERAL MISSING VALUES

5.1 General Case

In this section, the procedure discussed in Section 2.2 will be used for the estimation of several missing values, i.e., partially differentiating the error sum of squares with respect to each missing value and equating the resulting expression with zero, which will yield a set of equations in unknowns.

The error sum of squares may be written as,

$$E^* = \left[\sum\sum y^2{}_{ij} - \frac{1}{k}\sum B^2{}_j - \sum t_i Q_i - \sum_{p\neq i}^{n-1} t^{1i}{}_p\, Q^{1i}{}_p - \sum^{k-n} t^{2i}{}_q\, Q^{2i}{}_q\right]$$

$$+\left[\sum\sum x_{ij}{}^2 - \frac{1}{k}\sum B_j{}^2 - \sum t_i Q_i - \sum_{p\ i}^{n-1} t_p{}^{1i} Q_p{}^{1i} - \sum^{k-n} t_q{}^{2i}\, Q_q{}^{2i}\right].$$

(5.1)

The first term in (5.1) is taken from the existing data, while second term is obtained from the data involving missing values.

For expediency, the following definitions are made for the second quantity in (5.1), since the first quantity in (5.1) will vanish under differentiation.

$$\sum\sum x_{ij}{}^2 = R_1 ,$$

$$-\frac{1}{k}\sum B_j{}^2 = R_2 , \qquad (5.2)$$

$$-\left[\sum t_i Q_i + \sum_{p \neq i}^{n-1} t_p^{1i} \, Q_p^{1i} + \sum^{k-n} t_q^{2i} \, Q_q^{2i}\right] = R_3 .$$

Now, the error sum of squares, involving missing values, only, reduces to

$$E^{*\prime} = R_1 + R_2 + R_3 . \tag{5.3}$$

In R_3, the product terms may be defined as,

$$t_i Q_i = \left\{\frac{k-c_1}{r(k-1)} \left[Q'_i + \frac{k-1}{k} X_{ij} + N\right] + \frac{c_1-c_2}{r(k-1)} \left[G'_i + \frac{k-n}{k} X_{ij} + N\right]\right\} \left\{Q'_i + \frac{k-1}{k} X_{ij} + N\right\}, \tag{5.4}$$

$$t^{1i}_p \, Q^{1i}_p = \left\{\frac{k-c_1}{r(k-1)} \left[Q'^{\,1i}_p - \frac{1}{k} X_{ij} + N\right] + \frac{c_1-c_2}{r(k-1)} \left[G'^{\,1i}_p + \frac{k-n}{k} X_{ij} + N\right]\right\} \left\{Q'^{\,1i}_p - \frac{1}{k} X_{ij} + N\right\}, \tag{5.5}$$

$$t^{\overline{2i}}_q Q^{\overline{2i}}_q = \left\{\frac{k-c_1}{r(k-1)} \left[Q'^{\,\overline{2i}}_q - \frac{1}{k} X_{ij} + N\right] + \frac{c_1-c_2}{r(k-1)} \left[G'^{\,\overline{2i}}_q - \frac{n}{k} X_{ij} + N\right]\right\} \left\{Q'^{\,\overline{2i}}_q - \frac{1}{k} X_{ij} + N\right\}. \tag{5.6}$$

In case the second associates are replicated in any two of the involved blocks, we will have,

$$t_q^{\underline{2\ell}} Q_q^{\underline{2\ell}} = \left\{ \frac{k-c_1}{r(k-1)} \left[Q'_q{}^{\underline{2\ell}} - \frac{1}{k} X_{ij} + N \right] + \frac{c_1-c_2}{r(k-1)} \left[Q'_q{}^{\underline{2\ell}} - \frac{n}{k} X_{ij} + N \right] \right\} \left\{ Q'_q{}^{\underline{2\ell}} - \frac{1}{k} X_{ij} + N \right\} . \tag{5.7}$$

For the estimation purposes, the quantity to be considered is given in (5.3). The block and treatment totals involving missing values, may be defined as,

$$\begin{aligned} B_j &= B'_j + \text{(missing values in } B_j\text{)}, \\ T_i &= T'_i + \text{(missing values in treatment } i\text{)}. \end{aligned} \tag{5.8}$$

5.1.a <u>Particular case</u>:

In the case of a single missing value, X_{cd}, (say), X_{cd} will be involved in three types of Q values and three types of G values, as defined in Section 3.1. Two particular missing values will be referred to as X_{cd} and X_{ef} and z (= n) missing values will be referred to as X_{cd}, X_{ef}, ..., X_{uv} .

Let N be defined as the sum of those terms, which involve the missing values not under consideration.

The error sum of squares, involving missing values, only, is,

$$E^{*'} = R_1 + R_2 + R_3 \,. \tag{5.9}$$

Differentiating partially (5.9) with respect to each missing value, in particular, say, X_{cd}, one gets the resulting equation of the form,

$$\frac{\partial R_1}{\partial X_{cd}} + \frac{\partial R_2}{\partial X_{cd}} + \frac{\partial R_3}{\partial X_{cd}} = \Big[(\text{terms involving } X_{cd}) + N$$

$$+ \text{ constant terms involving } Q'_c \,,\; G'_c \,,\; Q'^{\,2c}_q ,\; G'^{\,2c}_q ,$$

$$\text{and } (Q'^{\,\underline{2\ell}}_q \,,\; G'^{\,\underline{2\ell}}_q)\Big] \,. \tag{5.10}$$

Equating (5.10) to zero, one obtains,

$$\begin{aligned} m\, X_{cd} + N = {} & rk(k-1)B'_d + k^2(k-c_1)Q'_c \\ & + k\left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_c \\ & - \frac{k}{2}\left[2(k-c_1) + n(c_1-c_2)\right] \sum^{k-n} Q'^{\,\overline{2c}}_q \\ & - \frac{k}{2}\,(c_1-c_2) \sum^{k-n} G'^{\,\overline{2c}}_q \,, \end{aligned} \tag{5.11}$$

where

$$m = rk(k-1)^2 - (k-c_1)\left\{k(k-1) - (z-1)\right] - (k+z-1)(k-n)(c_1-c_2) \,.$$

$$z = 1,\, 2,\, 3,\, 4,\, \ldots,\, n. \tag{5.12}$$

In the case when second associates of the missing value under consideration or group of treatments, are replicated in any two of the involved blocks, the equation, (after partially differentiating (5.9) with respect to X_{cd} and equating the partial derivative to zero) will be of the form,

$$m\,X_{cd} + N = rk(k-1)B'_d + k^2(k-c_1)Q'_c$$
$$+ k\left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_c$$
$$- \frac{k}{2}\left[2(k-c_1) + n(c_1-c_2)\right]\left[\sum^{k-n-an} Q'^{\,\overline{2c}}_q + \sum^{an} Q'^{\,\underline{2\ell}}_q\right]$$
$$- \frac{k}{2}(c_1-c_2)\left[\sum^{k-n-an} G'^{\,\overline{2c}}_q + \sum^{an} G'^{\,\underline{2\ell}}_q\right], \qquad (5.13)$$

where (k-n-an) is the number of second associates or the treatments, not replicated in the involved blocks and an is the number of second associates or the treatments being replicated in the involved blocks. It is not possible that $a = 0$.

B'_d, Q'_c, G'_c, $Q'^{\,\overline{2c}}_q$, $Q'^{\,\underline{2\ell}}_q$, $G'^{\,\overline{2c}}_q$, and $G'^{\,\underline{2\ell}}_q$

are the numerical values taken from the existing data.

5.1.b Matrix Representation

If there are z missing values, X_{cd}, X_{ef}, ..., X_{uv}, and one partially differentiates the error sum of squares with respect to each missing value, equates the partial derivatives with zero, one obtains the following system of equations.

$$
\begin{aligned}
m X_{cd} + \ell_{12} X_{ef} + \cdots + \ell_{1z} X_{uv} &= C_{cd} , \\
\ell_{21} X_{cd} + m X_{ef} + \cdots + \ell_{2z} X_{uv} &= C_{ef} , \\
- - - - - - - - - - - - - - - - & - - \qquad (5.14) \\
- - - - - - - - - - - - - - - - & - - \\
\ell_{z1} X_{cd} + \ell_{z2} X_{ef} + \cdots + m X_{uv} &= C_{uv} ,
\end{aligned}
$$

where $\ell_{ij} \equiv \ell_{ji}$.

The ℓ_{ij} 's are obtained by considering the possible configurations of two missing values, X_{cd} and X_{ef} from Table 4.1. These configurations are not all mutually exclusive.

The above system of equations may be represented in matrix form

$$A \underline{X} = \underline{C} , \qquad (5.15)$$

where

$$A = \begin{bmatrix} m & \ell_{12} & ---- & \ell_{1z} \\ \ell_{21} & m & ---- & \ell_{2z} \\ - & - & - - - - - & - \\ - & - & - - - - - & - \\ \ell_{z1} & \ell_{z2} & ---- & m \end{bmatrix}, \tag{5.16}$$

$$\underline{X} = \begin{bmatrix} X_{cd} \\ X_{ef} \\ - \\ - \\ X_{uv} \end{bmatrix}, \tag{5.17}$$

and

$$\underline{C} = \begin{bmatrix} C_{cd} \\ C_{ef} \\ - \\ - \\ C_{uv} \end{bmatrix}. \tag{5.18}$$

The estimates of the missing values are then

$$\underline{X} = A^{-1}\,\underline{C}\ . \tag{5.19}$$

5.2. Special Cases

5.2.a Case of z missing values in z distinct blocks and z distinct treatments. Missing values are second assoc-

<u>iates and no groups of treatments are replicated in any two of the involved blocks.</u>

Obviously, each pair of missing values satisfies configuration 3, and the system of equations reduces to

$$
\begin{array}{l}
m X_{cd} + 0 + \cdot \cdot \cdot + 0 = C_{cd} , \\
0 + m X_{ef} + \cdot\cdot \cdot \cdot + 0 = C_{ef} , \\
- - - - - - - - - - \quad - - \\
0 + - + \cdot \cdot \cdot + m X_{uv} = C_{uv} .
\end{array}
\qquad (5.20)
$$

The estimate of missing values in treatment i, and block j, in this case, is

$$X_{ij} = \frac{1}{m} C_{ij} \qquad (5.21)$$

where

$$m = rk(k-1)^2 - (k-c_1)\left[k(k-1) - (z-1)\right] - (k+z-1)(k-n)(c_1-c_2),$$

$$z = 1, 2, 3, 4, \ldots$$

and

$$C_{ij} = rk(k-1)B'_j + k^2(k-c_1)Q'_i + k\left[(k-n)(c_1-c_2) - (k-c_1)\right]G'_i$$

$$- \frac{k}{2}\left[2(k-c_1) + n(c_1-c_2)\right]\sum^{k-n} Q'_q{}^{\overline{2i}} - \frac{k}{2}(c_1-c_2)\sum^{k-n} G'_q{}^{\overline{2i}} .$$

This is identical to the estimate of single missing observation, as shown by Baird and Kramer (1961) for the balanced incomplete block design.

5.2.b <u>Case of z missing values in z distinct treatments and z distinct blocks. Missing values are first associate and no second associates of the missing treatments are replicated in any two of the involved blocks.</u>

Here each possible pair of missing values satisfies configuration 1, so that

$$\ell_{ij} = -\frac{1}{k}\left[(k-n)^2(c_1-c_2) - (2k-n)(k-c_1)\right] = -\ell \text{ (say)}$$

In this case, the equation system is,

$$
\begin{aligned}
m X_{cd} - \ell X_{ef} - \cdots\cdots - \ell X_{uv} &= C_{cd}, \\
-\ell X_{cd} + m X_{ef} - \cdots\cdots - \ell X_{uv} &= C_{ef}, \\
\text{-------------------------------} & \\
-\ell X_{cd} - \ell X_{ef} - \cdots\cdots + m X_{uv} &= C_{uv}.
\end{aligned}
\tag{5.22}
$$

Performing the operation $A^{-1}\underline{C}$, one obtains

$$
A^{-1} = \frac{1}{L}\begin{bmatrix} m-(z-2)\ell & \ell & \cdots & \ell \\ \ell & m-(z-2)\ell & \cdots & \ell \\ \cdots & \cdots & \cdots & \cdots \\ \ell & \cdots & \cdots & m-(z-2)\ell \end{bmatrix},
\tag{5.23}
$$

$$\underline{C} \quad \begin{bmatrix} C_{cd} \\ C_{ef} \\ - \\ - \\ C_{uv} \end{bmatrix}, \tag{5.24}$$

the solution $\underline{X} = A^{-1}\,\underline{C}$.

Estimates of the missing values are

$$X_{cd} = \frac{1}{L}\left\{\left[m - (z-2)\ell\right]\, C_{cd} + \ell \sum_{i \neq d} C_{id}\right\},$$

$$X_{ef} = \frac{1}{L}\left\{\left[m - (z-2)\ell\right]\, C_{ef} + \ell \sum_{i \neq e} C_{if}\right\}, \tag{5.25}$$

- - - - - - - - - - - - - - - - - - - -

$$X_{uv} = \frac{1}{L}\left\{\left[m - (z-2)\ell\right]\, C_{uv} + \ell \sum_{i \neq u} C_{iv}\right\},$$

where

$$L = m^2 - m(z-2)\ell - (z-1)\ell^2 \;,$$

$$\ell = \frac{1}{k}\left[(k-n)^2(c_1-c_2) - (2k-n)(k-c_1)\right] . \tag{5.26}$$

5.2.c. <u>Case of z missing values in the same block. Missing values are first associates.</u>

In this case, each pair of missing values satisfies configuration 7, yielding the equation system

$$
\begin{aligned}
m\, X_{cd} - \ell_1 X_{ed} - \cdots - \ell_1 X_{ud} &= C_{cd}\,, \\
-\ell_1 X_{ed} + m\, X_{ed} - \cdots - \ell_1 X_{ud} &= C_{ed}\,, \\
\text{-----------------------} & \\
-\ell_1 X_{cd} - \ell_1 X_{ed} - \cdots + m\, X_{ud} &= C_{ud}\,,
\end{aligned}
\qquad (5.27)
$$

where

$$\ell_1 = r(k-1) + (k-c_1) + (k-n)(c_1-c_2)\;.$$

Again, using matrix notation, the equation system may be written as

$$A\,\underline{X} = \underline{C}\,, \qquad (5.28)$$

where

$$
A = \begin{bmatrix} m & -\ell_1 & --- & -\ell_1 \\ -\ell_1 & m & --- & -\ell_1 \\ - & - & - & - \\ -\ell_1 & -\ell_1 & --- & m \end{bmatrix}, \qquad (5.29)
$$

and $\underline{X}$ and $\underline{C}$ are as defined above.

Now

$$A^{-1} = \frac{1}{L_1} \begin{bmatrix} m-(z-2)\,\ell_1 & \ell_1 & \cdots & \ell_1 \\ \ell_1 & m-(z-2)\,\ell_1 & \cdots & \ell_1 \\ \cdots & \cdots & \cdots & \cdots \\ \ell_1 & \ell_1 & \cdots & m-(z-2)\,\ell_1 \end{bmatrix}, \tag{5.30}$$

where

$$L_1 = m^2 - m(z-2)\,\ell_1 - (z-1)\,\ell_1^2 \;, \tag{5.31}$$

and the solution is

$$\underline{X} = A^{-1}\,\underline{C} \;,$$

giving the missing value estimates,

$$X_{cd} = \frac{1}{L_1}\left\{\left[m-(z-2)\,\ell_1\right] C_{cd} + \ell_1 \sum_{i \neq d} C_{id}\right\},$$

$$X_{ed} = \frac{1}{L_1}\left\{\left[m-(z-2)\,\ell_1\right] C_{ed} + \ell_1 \sum_{i \neq e} C_{id}\right\}, \tag{5.31a}$$

$$- \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; -$$

$$X_{ud} = \frac{1}{L_1}\left\{\left[m-(z-2)\,\ell_1\right] C_{ud} + \ell_1 \sum_{i \neq u} C_{id}\right\}.$$

5.2.d <u>z missing values in the same treatment. No second associates of the missing treatment are replicated in any two of the involved blocks.</u>

In this case, every possible pair of the missing values satisfies configuration 9, yielding the equation system as

$$
\begin{aligned}
m X_{cd} - \ell_2 X_{cf} - - - - \ell_2 X_{cv} &= C_{cd} , \\
- \ell_2 X_{cd} + m X_{cf} - - - - \ell_2 X_{cv} &= C_{cf} , \\
- & \\
- \ell_2 X_{cd} - \ell_2 X_{cf} - - - + m X_{cv} &= C_{cv} ,
\end{aligned}
\qquad (5.32)
$$

where

$$\ell_2 = \frac{1}{k} \left[(k-1)^2(k-c_1) + (n-1)(k-c_1) + (k-n)^2(c_1-c_2)\right] . \qquad (5.33)$$

Performing the operation $A^{-1} \underline{C}$, where

$$A^{-1} = \frac{1}{L_2} \begin{bmatrix} m-(z-2)\,\ell_2 & \ell_2 & --- & \ell_2 \\ \ell_2 & m-(z-2)\,\ell_2 & - & \ell_2 \\ - - - & - - - & - - - & - - - \\ \ell_2 & \ell_2 & -- & m-(z-2)\,\ell_2 \end{bmatrix} , \qquad (5.34)$$

where

$$L_2 = m^2 - m(z-2)\,\ell_2 - (z-1)\,\ell_2^2 \quad , \qquad (5.35)$$

the solution $\underline{X} = A^{-1} \underline{C}$, gives the estimates of missing values.

$$
\left.
\begin{aligned}
X_{cd} &= \frac{1}{L_2}\left\{\left[m-(z-2)\,\ell_2\right] c_{cd}+\ell_2\sum_{i\neq c} c_{id}\right\},\\
X_{cf} &= \frac{1}{L_2}\left\{\left[m-(z-2)\,\ell_2\right] c_{cf}+\ell_2\sum_{i\neq c} c_{if}\right\},\\
&-\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\\
X_{cv} &= \frac{1}{L_2}\left\{\left[m-(z-2)\,\ell_2\right] c_{cv}+\ell_2\sum_{i\neq c} c_{iv}\right\}.
\end{aligned}
\right\}\quad (5.36)
$$

VI ANALYSIS OF GDS DESIGN WITH MISSING OBSERVATIONS

To analyse the GDS design with missing observations first find the blocks totals B_j 's and grand total G and insert them in the plan of table 6.1. For each treatment, obtain the treatment total T_i , and $B_{i.}$ the total of all blocks in which treatment i occurs. These quantities can be obtained simultaneously on a desk calculator by placing the individual observations on the left of the keyboard and the corresponding block totals on the right. The T_i's and $B_{i.}$'s are placed in the first two columns of Table 6.2, then the original Q_i-values are obtained by

$$Q_i = T_i - \frac{1}{k} B_{i.} \qquad (6.1)$$

The G_i-values are obtained by

G_i = sum of Q_i over all treatments in the same row in the association scheme.

The treatment sum of squares is then obtained by

$\sum \hat{t}_i Q_i$, where

$$\hat{t}_i = \frac{k-c_1}{r(k-1)} Q_i + \frac{c_1-c_2}{r(k-1)} G_i \qquad (6.2)$$

The treatment sum of squares will be biased if obtained from $\sum \hat{t}_i Q_i$ using the augmented data. To remove the bias and to obtain an exact F - test, the method given by Glenn and Kramer, will be followed.

To modify the analysis of variance technique for exact F - test, the following steps may be taken to calculate the sums of squares. After obtaining the estimates of the missing values and inserting them in the proper cells, calculate the sums of squares in the following fashion,

$$\text{Block S.S. (unadjusted)} = \frac{1}{k} \sum_{\substack{\text{known} \\ \text{data}}} B_j^2 + \frac{1}{k} \sum_{\substack{\text{missing} \\ \text{data}}} (B'_j + \sum_j X_{ij})^2 - \frac{(G' + \sum_i \sum_j X_{ij})^2}{N}$$

$$= B^*, \qquad (6.3)$$

$$\text{Total S.S.} = \sum_{\substack{\text{known} \\ \text{data}}}\sum y_{ij}^2 + \sum_{\substack{\text{missing} \\ \text{data}}}\sum X_{ij}^2 - \frac{(G' + \sum\sum X_{ij})^2}{N} \qquad (6.4)$$

B'_j is the block total for the original observations;(blocks containing the missing values) , G' is the grand total for the original values. The treatment sum of squares, A^* and the error sum of squares, E^*, are calculated from the augmented data. Now, in case the F - test is significant, this may be due to the bias in treatment sum of squares. To eliminate the bias, the following sum of squares are calculated.

$$\text{Total S.S.} = T' = \sum_{\substack{\text{known}\\ \text{data}}}\sum y_{ij}^2 - \frac{(G')^2}{N-z} , \tag{6.5}$$

$$\text{Block(unadjusted)} = B' = \frac{1}{k} \sum_{\substack{\text{known}\\ \text{data}}} B_j^2 + \sum_{\substack{\text{known}\\ \text{data}}} \frac{B'_j{}^2}{k_i} - \frac{G'^2}{N-z} \tag{6.6}$$

where z = number of missing values.

The unbiased treatment sum of squares is then found by

$$T' - B' - E^* . \tag{6.7}$$

Numerical calculations are shown by taking an example in Section 6.1. Exact and approximate analysis of variance is obtained for a GDS design with two missing observations in the same treatment, while no second associates are replicated in any of the involved blocks, by applying the formulae (3.90) and (3.91).

6.1 Example

The numerical example discussed in this section utilizes the data from an experiment in which a standard variety of corn was subjected to 10 types of fertilizers. The observations represent the yield of corn in pounds per plot. The design was a Group Divisible Singular, $r = \lambda_1$, i.e., GDS Design with $v = 10$, $r = 4$, $k = 4$, $b = 10$, $m = 5$, $n = 2$, $\lambda_1 = 4$, $\lambda_2 = 1$, $c_1 = 1$, $c_2 = 1/10$.

The Design is taken from "Table of Partially Balanced Designs with two Associate Classes" by Clatworthy and Shrikhade Design §17.

To discuss the method of Analysis presented in this paper, two observations are taken as missing in treatment 2, that is, X_{21} and X_{22}, treatment 2, block 1, and treatment 2, block 2, respectively. The two cells are circled in Table 6.1.

Since both values are missing in the same treatment, i.e., 2, and second associates of treatment 2, are not replicated in blocks 1 and 2 (the involved blocks), one would apply the formulae (3.90) and (3.91).

The formulae are:

$$X_{21} = \frac{m\,C_{21} + \frac{1}{k}\left[(k-1)^2(k-c_1) + (n-1)(k-c_1) + (k-n)^2(c_1-c_2)\right]C_{22}}{m^2 - \frac{1}{k^2}\left[(k-1)^2(k-c_1) + (n-1)(k-c_1) + (k-n)^2(c_1-c_2)\right]^2},$$

$$X_{22} = \frac{m\,C_{22} + \frac{1}{k}\left[(k-1)^2(k-c_1) + (n-1)(k-c_1) + (k-n)^2(c_1-c_2)\right]C_{21}}{m^2 - \frac{1}{k^2}\left[(k-1)^2(k-c_1) + (n-1)(k-c_1) + (k-n)^2(c_1-c_2)\right]^2}.$$

TABLE 6.1

YIELD OF CORN IN POUND PER PLOT

Block No					Block Totals Original	Augmented
1	1: 28	6: 42	2: 22	7: 29	99 B'_1	121
2	2: 13.44	7: 26	3: 23	8: 26	75 B'_2	84.44
3	3: 28	8: 24	4: 25	9: 32	117	117
4	4: 25	9: 26	5: 27	10: 24	102	102
5	5: 27	10: 25	1: 27	6: 26	105	105
6	6: 41	1: 26	8: 24	3: 25	116	116
7	7: 30	2: 25	9: 26	4: 32	113	113
8	8: 27	3: 23	10: 23	5: 26	99	99
9	9: 36	4: 32	6: 41	1: 29	138	138
10	10: 26	5: 28	7: 28	2: 24	106	106
					1070	1105.44 Total

Groups

1	6
2	7
3	8
4	9
5	10

where

$$m = r(k-1)^2 - (k-1)(k-c_1) - (k-n)(c_1-c_2)$$

$$= 25.2$$

and according to (3.11), C_{21} and C_{22} are defined as

$$C_{21} = r(k-1)B'_1 + k(k-c_1)Q'_2 + \left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_2$$

$$- \frac{1}{2}\left[2(k-c_1) + n(c_1-c_2)\right] \sum^{k-n} Q'_q{}^{\overline{22}}$$

$$- \frac{1}{2}(c_1-c_2) \sum^{k-n} G'_q{}^{\overline{22}},$$

$$C_{22} = r(k-1)B'_2 + k(k-c_1)Q'_2 + \left[(k-n)(c_1-c_2) - (k-c_1)\right] G'_2$$

$$-\frac{1}{2}\left[2(k-c_1) + n(c_1-c_2)\right] \sum^{k-n} Q'_q{}^{\overline{22}}$$

$$- \frac{1}{2}(c_1-c_2) \sum^{k-n} G'_q{}^{\overline{22}},$$

the (B'_d, B'_f), Q'_c, G'_c, $Q'_q{}^{\overline{2c}}$, $G'_q{}^{\overline{2c}}$ are defined in equations (3.1), (3.2), (3.7), (3.12) and (3.13) respectively.

Therefore,

$$C_{21} = 12(99) + 12(-49.25) + (\tfrac{9}{5} - 3)(34.5) - \tfrac{1}{2}(6 + \tfrac{9}{5})$$

$$\cdot(31 - 3.5 + 7 + 0) - \tfrac{9}{5}(31 - 3.5 + 7 + 0)$$

$$= 441.75,$$

$$G_{22} = 12(75) + 12(-49.25) + (\frac{9}{5} - 3)(-34.5) - \frac{1}{2}(6+\frac{9}{5})$$

$$(31 - 3.5 + 7 + 0) - \frac{9}{5}(31 - 3.5 + 7 + 0)$$

$$= 153.75 ,$$

The estimates of missing values are

$$X_{21} = \frac{(25.2)(441.75) + (8.4)(153.75)}{564.48} = \frac{12423.60}{564.48} = 22.00 ,$$

$$X_{22} = \frac{(25.2)(153.75) + (8.4)(441.75)}{564.48} = \frac{7585.20}{564.48} = 13.44 ,$$

where

$$\frac{1}{k}\left[(k-1)^2(k-c_1) + (n-1)(k-c_1) + (k-n)^2(c_1-c_2)\right]$$

$$= \frac{1}{4}\left[27 + 3 + \frac{18}{5}\right] = 8.4 ,$$

and

$$m^2 - \frac{1}{k^2}\left[(k-1)^2(k-c_1) + (n-1)(k-c_1) + (k-n)^2(c_1-c_2)\right]$$

$$= (25.2)^2 - (8.4)^2 = 564.48 .$$

These estimates are then entered in Table 6.1 and circled. The augmented quantities B_j, T_i, Q_i, and G_i are calculated and presented in Table 6.2 .

The approximate and exact analyses of variance are obtained as discussed in Section 5.

The approximate analysis is obtained in the following way. First find the correction factor

$$= \frac{(G' + \sum_i \sum_j x_{ij})^2}{N} = \frac{(116544)^2}{40} = \frac{1221997.5936}{40}$$

$$= 30549.9398 \text{ .}$$

Now partitioning the block sum of squares unadjusted into two parts, one part in which block totals does not involve missing values and the other part in which block totals involve missing values we obtain

$$B^* = \text{Block SS (unadjusted)} = \frac{1}{k} \sum B_j^2 + \frac{1}{k} (B_j + \sum_j x_{ij})^2 - \frac{(G' + \sum_i \sum_j x_{ij})^2}{N}$$

$$= \frac{101424}{4} + \frac{21771.1136}{4} - 30549.9398$$

$$= 25356 + 5442.7784 - 30549.9398$$

$$= 248.8386 \text{ .}$$

In the similar fashion the total sum of squares is partitioned, one part corresponding to existing observa-

TABLE 6.2

Table of T_i, $B_{i.}$, Q_i, and G_i

	T_i		$B_{i.}$		$\frac{1}{k} B_{i.}$		Q_i		G_i	
Treat.	Orig.	Aug.	Orig.	Aug.	Orig.	Aug.	Orig.	Aug.	Orig.	Aug.
1	110	110	458	480	114.5	120.0	-4.5	-10.0	31.0	20.0
2	49	84.44	393	428.44	98.25	107.11	-49.25	-22.67	-34.5	-16.78
3	99	99	407	420.44	101.75	105.11	-2.75	-6.11	-3.5	-10.22
4	122	122	470	470	117.5	117.5	4.5	4.5	7.0	7.0
5	108	108	412	412	103.0	103.0	5.0	5.0	0.0	0.0
6	150	150	458	480	114.5	120.0	35.5	30.0	31.0	20.0
7	113	113	393	428.44	98.25	107.11	14.75	5.89	-34.5	-16.78
8	101	101	407	420.44	101.75	105.11	-0.75	-4.11	-3.5	-10.22
9	120	120	470	470	117.5	117.5	2.5	2.5	7.0	7.0
10	98	98	412	412	103.0	103.0	-5.0	-5.0	0.0	0.0
Total	1070	1105.44	4280	4421.76	1070	1105.44	0.0	0.0	0.0	0.0

TABLE 6.3

Estimates of Treatment Effects

and Treatment Sum of Squares

	Augmented $\frac{1}{4} Q_i$	$\frac{3}{40} G_i$	$\hat{t}_i$	$\hat{t}_i Q_i$
1	-2.5000	1.5000	-1.0	10.0000
2	-5.6675	-1.2585	-6.926	157.0124
3	-1.5275	-0.7665	-2.294	14.0163
4	1.1250	0.5250	1.65	7.4250
5	1.2500	0.0000	1.25	6.2500
6	7.5000	1.5000	9.0	270.0000
7	1.4725	-1.2585	0.214	1.2604
8	-1.0275	-0.7665	-1.794	7.3733
9	0.6250	0.5250	1.15	2.8750
10	-1.2500	0.0000	1.25	6.2500
	0.0	0.0		482.4624

tions and the other part corresponding to estimates of missing observations.

$$T^* = \text{Total SS} = \sum_{\substack{\text{existing}\\ \text{data}}}\sum y_{ij}^2 + \sum_{\substack{\text{missing}\\ \text{data}}}\sum x_{ij}^2 - \frac{G^2}{N}$$

$$= 31016 + 664.6336 - 30549.9398$$

$$= 1130.6938 .$$

The adjusted treatments sum of squares is

$$A^* = \sum \hat{t}_i Q_i = 482.4624 ,$$

where

$$\hat{t}_i = \frac{k-c_1}{r(k-1)} Q_i + \frac{c_1-c_2}{r(k-1)} G_i = \frac{1}{4} Q_i + \frac{3}{40} G_i$$

and the error sum of squares (by subtraction)

$$= E^* = 399.3928 .$$

The Analysis of Variance (Approximate) is arranged in Table 6.4.

$$F_{.05}(9,19) = 2.43$$

Since the F-test is significant at 5 per cent level, this significance may be due to bias, so the exact analysis of variance is as follows:

$$\text{Correction Factor} = \frac{(G')^2}{N-z} = \frac{(1070)^2}{38} = 30128.948 ,$$

Table 6.4

Approximate Analysis of Variance

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Blocks (unadjusted	9	248.8386	27.6487	
Treatments (adjusted)	9	482.4624	53.6069	2.5502*
Error	19	399.3928	21.0206	
Total	37	1130.6938		

Error sum of squares will be the same, that is,

$$E^* = 399.3928 .$$

The total sum of squares, T' , as indicated in equation (6.5) is,

$$T' = 31016 - 30128.948 = 887.052 .$$

And, according to equation (6.6), the Block sum of squares unadjusted is

$$B' = \frac{101424}{4} + \frac{15426}{3} - 30128.948$$

$$= 30494 - 30128.948 = 369.052 .$$

The unbiased treatment sum of squares adjusted as given in equation (6.7), is obtained then by subtraction:

$$A' = 887.052 - 369.052 - 399.3928$$
$$= 118.6072 .$$

The exact analysis of variance is given in Table 6.5.

Table 6.5

Exact Analysis of Variance

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Blocks (unadjusted)	9	369.0520	41.0058	
Treatment (adjusted)	9	118.6072	13.1786	n.s.
Error	19	887.0520	21.0206	
Total	37			

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ABSTRACT

The problem discussed in this paper is the estimation of a single missing observation, two missing observations and several missing observations in a Group Divisible (Singular) partially balanced incomplete blocks design with two associate classes. Subsequently the analysis of variance, of the data augmented by the estimates of the missing observations, is derived.

The method, first employed by Yates (1933), was followed to minimize the error sum of squares. Explicit formulae were developed, for the estimates of one missing observation, two missing observations occurring in various configurations and general formulae for $z(= n)$ missing observations for certain particular configurations.

Analysis of the data augmented by the estimates of the missing observations leads to positive bias in the case of treatment sum of squares, a method of analysis was discussed to eliminate this bias.

A numerical example illustrating the technique of estimating missing observations in a GDS P.B.I.B. design was given. The approximate and exact tests were performed, for the null hypothesis of no treatment differences, using the intra-block error mean square.