A TWO-DIMENSIONAL TRANSFER MODEL

by

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CHAPTER I - THE TRANSFER EQUATION

1. INTRODUCTION

One of the fundamental problem areas of military operations is logistics. For many operations a vital factor in logistics is mobility. Mobility is dependent upon differing transportation systems. Some of these systems form well defined transportation networks such as a railroad-freight system, while others like maritime-freight form no less distinguished systems. However, the ability of an army to expand or contract its region of occupancy depends largely upon the dissemination of supplies and troops by motor carriers. These carriers, as they traverse open terrain as well as fixed networks, differ from the previous mentioned carriers in that they do not traverse a well defined network with definite terminals and in that the underlying phenomena is not distinguished by being relatively independent of position and velocity. To predict the distribution of motor carriers in space and in time operating in field conditions, it is necessary to construct models which implicitly account for velocity as well as position. The Transportation Research Command at Fort Eustis, Virginia has been interested in this problem, particularly when the magnitude of the operation is sufficiently large to justify a stochastic model. We shall present in what follows an introduction to transfer theory adapted for two dimensions, the construction of a model derived for the Army, and some consequences of interest.

In the present chapter we shall define the fundamental quantities with which the subject of radiative transfer deals and derive the basic equation of transfer which governs the radiative field in a medium which absorbs, emits and scatters radiation. However, we shall not attempt to attain maximum generality throughout our discussion, but rather, our formulations will be guided by the geometric considerations which the problems we are to consider actually require. In Chapter I our development follows Anselone (2) and Chandrasekhar (3).

2. FUNDAMENTAL DEFINITIONS

Consider entities which have four degrees of freedom, two space components, and their two corresponding velocity components. Assume that there are no arbitrary interactions between the entities and consequently define the μ phase space to be the four dimensional linear space with coordinates (x, y, v_x, v_y) where x and y are position and v_x and v_y the corresponding components of velocity. For convenience we will use a system where the independent variables are:

$\vec{R}$ = position vector,
$\vec{\Omega}$ = direction vector, $\|\vec{\Omega}\| = 1$, and
t = time.

Now suppose these entities, which we will call units, are radiating through a medium with which such a system has been associated. We shall consider the number of units, dE_v, with a fixed speed v which are transported across an element of length $d\sigma$ with normal $\vec{\Omega}$ and in the direction confined to an element of angle $d\omega$ about $\vec{\Omega}$, during the time dt. The number of units or radiation, dE_v, is expressed in terms of the principle dependent variable the specific intensity, I_v, which we define to be

(1.2.1.) $I_v(\vec{R}, \vec{\Omega})$ = number of units at $\vec{R}$ traveling in the direction $\vec{\Omega}$ per unit length perpendicular to $\vec{\Omega}$, per unit angle about $\vec{\Omega}$, per second.

Consequently,

(1.2.2.) $dE_v = I_v d\sigma d\omega\, dt$.

The construction which we have employed in the definition defines a pencil of radiation. For a given $\vec{R}$ and $\vec{\Omega}$ construct an element $d\sigma$ perpendicular to $\vec{\Omega}$. With each point of $d\sigma$ as vertex, construct an angle $d\omega$ about $\vec{\Omega}$. The resultant figure is called a pencil.

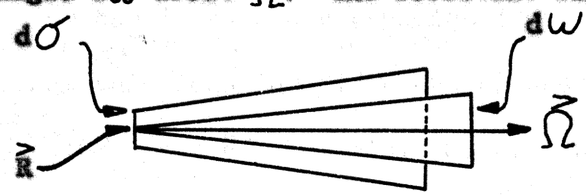
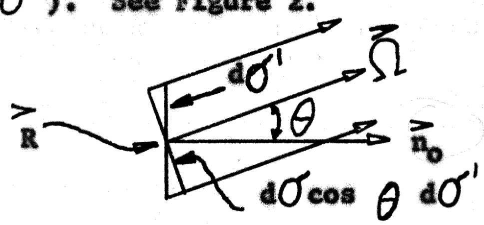


Figure 1 - Pencil

Let $d\sigma'$ be a element not necessarily perpendicular to $\vec{\Omega}$ (the normal to $d\sigma$). See Figure 2.



$\|\vec{n}_o\| = 1$

Figure 2 - Position of $d\sigma$ and $d\sigma'$

The radiation passing through $d\sigma$ which passed through $d\sigma'$ is

(1.2.3.) $I_v(\vec{R}, \vec{\Omega})\cos\theta\, d\sigma' d\omega\, dt$ = units at $\vec{R}$ traveling through points of $d\sigma'$ with direction in $d\omega$ about $\vec{\Omega}$ in time dt.

3. NET FLUX

If we define a vector intensity,

(1.3.1.) $\vec{I}_v(\vec{R}, \vec{\Omega}) = I_v(\vec{R}, \vec{\Omega})\vec{\Omega}$, which is a vector at $\vec{R}$ in the $\vec{\Omega}$ direction having a length equal to the intensity in that direction, the net vector flux, $\pi\vec{F}_v(\vec{R})$, becomes

(1.3.2.) $$\pi\vec{F}_v(\vec{R}) = \int_\omega \vec{I}_v \cdot \vec{n}_o\vec{n}_o d\omega = \int_\omega I_v\vec{\Omega} \cdot \vec{n}_o\vec{n}_o d\omega.$$

The net transfer of units across an element of length in any arbitrary direction $\vec{N}$ ($\|\vec{N}\|$ =1) at $\vec{R}$ is given by

(1.3.3.) $\pi\vec{F}_v \cdot \vec{N} = \int_\omega \vec{I}_v \cdot \vec{n}_o\vec{n}_o \cdot \vec{N}\ d\omega = \int_\omega I_v\vec{\Omega} \cdot \vec{n}_o\vec{n}_o \cdot \vec{N}\ d\omega$ and the net flux is given by

(1.3.4.) $$\|\pi\vec{F}_v\| = \int_\omega I_v \cos\theta\, d\omega$$

where θ is the angle between the chosen direction $\vec{N}$ and the variable direction of integration.

4. THE DENSITY OF UNITS

The density u_v of units of the radiation at the fixed speed v at any given point is the number of units per unit area that are in transit in the immediate neighborhood of the point considered.

Adapting Chandrasekhar's derivation ((3), Chapter 1, section 2.3) to a two dimensional geometry, to find the density at a point P, construct around P an infinitesimal area V which is a convex polyhedron with combinatorial boundary σ . With coincident centroids of their boundary vertices construct another such polyhedron with boundary Σ

such that the distances of the points of Σ from P are large compared with those for the points of σ and such that Σ lies in a spherical neighborhood, S, in which the difference in the intensity in a given direction for any two points in S is negligible.

Now the radiation traversing the area V must have crossed some element of the boundary Σ. Let $d\Sigma$ be such an element with length small compared to r, the distance between the centers of $d\Sigma$ and $d\sigma$, an element of σ. Further, let θ and ϑ denote the angles which the normals to $d\Sigma$ and to $d\sigma$ make with the line joining the two elements. From equation (1.2.3.), the units streaming across $d\Sigma$ that also flow across $d\sigma$ are

(1.4.1) $$I_v \cos\theta \; d\Sigma \, d\omega' = I_v \frac{\cos\vartheta \cos\theta \, d\sigma \, d\Sigma}{r}$$

since the angle $d\omega'$ subtended by $d\sigma$ at $d\Sigma$ is $\frac{d\sigma \cos\vartheta}{r}$ where r is the distance between $d\Sigma$ and $d\sigma$.

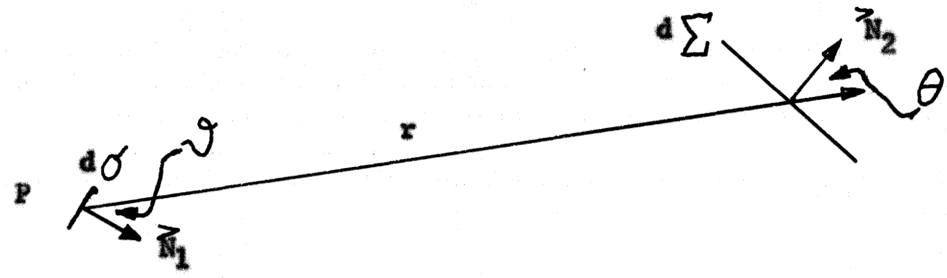


Figure 3 - Positions of $d\Sigma$ and $d\sigma$

Let ℓ be the length traversed by the pencil of radiation through the element of area V. The radiation (1.4.1.) incident on $d\sigma$ per second will traverse the element in a time $\frac{\ell}{v}$, where v is the speed of the units. The contribution to the total amount of radiating units in course of transit through V by the pencil is

$$(1.4.2.) \quad I_v \frac{\cos\vartheta \cos\theta \quad d\sigma \; d\Sigma}{r} \; \frac{\ell}{v} = \frac{1}{v} I_v dV d\omega$$

where $d\omega = d\Sigma \frac{\cos\theta}{r}$ is the angle subtended by $d\Sigma$ at P and $dV = \ell d\sigma \cos\vartheta$ is the area intercepted in V by the pencil of radiation. Thus, the total number of units in course of transit through V due to streaming in from all directions can be obtained by integrating (1.4.2.) over all V and ω.

Hence,

$$(1.4.3.) \quad \frac{1}{v} \int dV \int d\omega \, I_v = \frac{V}{v} \int I_v d\omega.$$

Thus,

$$(1.4.4.) \quad u_v = \frac{1}{v} \int I_v d\omega$$

, which is the number of units at $\vec{R}$ per unit area.

Let $J_v = \frac{1}{2\pi} \int I_v d\omega$ be the average intensity. Then

$$(1.4.5.) \quad u_v = \frac{2\pi}{v} J_v.$$

5. ABSORPTION AND SCATTERING

The intensity of a pencil of radiating units traversing a medium will be weakened by its interaction with "matter". Consider the units associated with $I_v(\vec{R}, \vec{\Omega})$ for a particular $\vec{R}$ and $\vec{\Omega}$. As the units proceed in the $\vec{\Omega}$ direction, they may be absorbed, or else scattered in a direction or velocity other than the direction and velocity of interest.

Thus, the intensity in the $\vec{\Omega}$ direction is diminished as it travels through the medium by absorption and scattering out of this direction.

Both of these processes will here be called absorption although later it will be necessary to distinguish between a scattering density (matter) and absorption density. In a later paragraph we will deal with the emission of units into a given direction $\vec{\Omega}$.

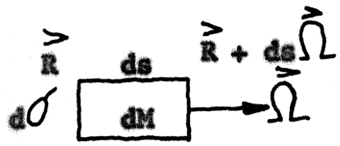


Figure 4 - Element of Mass

Let $I_v(\vec{R} + s\vec{\Omega}, \vec{\Omega})$ denote the number of units of $I_v(\vec{R}, \vec{\Omega})$ that remain in the ray after traveling a distance s in $\vec{\Omega}$ direction. The difference $dI_v = I_v(\vec{R} + ds\vec{\Omega}, \vec{\Omega}) - I_v(\vec{R}, \vec{\Omega}) \leqq 0$ is proportional to ds, to I_v, and to the density ρ of the medium. Thus, write

(1.5.1.) $dI_v = - K_v \rho I_v ds$

where K_v is the "<u>mass absorption coefficient</u>". Consider the loss from $I_v(\vec{R}, \vec{\Omega})$ due to absorption in an element of "mass",
$dM = \rho d\sigma ds$ (see Figure 4).
Recalling the definitions of $I_v(\vec{R}, \vec{\Omega})$ and equation (1.2.1.), the loss of units in dM is given by

(1.5.2.) $I_v d\sigma K_v \rho ds\, d\omega dt = K_v I_v dM\, d\omega dt$

the loss of units at $\vec{R}$ of velocity v, in direction $d\omega$ about $\vec{\Omega}$ in dM .

Strictly, the density is defined as

$$\rho \equiv \lim_{A \to 0} \frac{M}{A}$$

where A is an area and M is the mass associated with this area. As we are interested in a stochastic model, average values and probabilities apply, so define

$$\rho \equiv \frac{\bar{M}}{\bar{A}} \equiv \lim_{A \to 0} \frac{M}{A}$$

where $\bar{M}$ is the average total mass associated with a fixed area $\bar{A}$.
Recall that $I(\vec{R}, \vec{\Omega})d\sigma\ d\omega\ dt$ = number of units at $\vec{R}$ traveling through $d\sigma$ with direction $d\omega$ about $\vec{\Omega}$, and recall equation (1.5.2.).
Thus, take $K\rho ds$ to be dimensionless.
Now $\rho \equiv$ mass/unit area, where mass is defined in terms of intensity of enemy action, terrain properties, etc.
K is thus seen to be a probability that in a unit area an absorption will take place per unit mass, per unit length.

Suppose that a unit traveling in the $\vec{\Omega}'$ direction is absorbed at $\vec{R}$ and scattered i.e. emitted in another direction. We define the "phase function", $p(\vec{\Omega}, \vec{\Omega}')$ such that

(1.5.3.) $\frac{p(\vec{\Omega}, \vec{\Omega}')}{2\pi}$ = the probability per unit angle about $\vec{\Omega}'$ at $\vec{R}$ that the unit will be scattered into $d\omega$ about the $\vec{\Omega}$ direction.

Thus,

(1.5.4.) $\frac{p(\vec{\Omega}, \vec{\Omega}')}{2\pi} K_v I_v(\vec{R}, \vec{\Omega}')dt\ dM\ d\omega\ d\omega'$

= units scattered from directions $d\omega'$ about $\vec{\Omega}'$ into the directions $d\omega$ about $\vec{\Omega}$.

Now (1.5.4.) agrees with (1.5.2.) if $\int_{\omega'} p(\vec{\Omega},\vec{\Omega}') \frac{d\omega'}{2\pi} = 1.$

However, in general $\int p(\vec{\Omega},\vec{\Omega}') \frac{d\omega}{2\pi} = \overline{\omega}_0 \leqq 1.$ This is clear when one considers that absorption includes scattering as $\overline{\omega}_0$ is the probability that the unit which was absorbed will be scattered. Pi, $\overline{\omega}_0$, is called the albedo for single scattering and represents the fraction of the intensity of the incident pencil lost due to scattering.

Integrating over all directions $\vec{\Omega}'$ of the incident ray,

(1.5.5.) $$j_v^{(s)} = \frac{K_v}{2\pi} \int_{\omega'} p(\vec{\Omega},\vec{\Omega}') \, I_v(\vec{R},\vec{\Omega}') \, d\omega'$$

= units scattered from all directions $\vec{\Omega}'$ into the particular direction $\vec{\Omega}$ per unit angle about $\vec{\Omega}$, per second, per unit mass at $\vec{R}$.

$j_v^{(s)}$ is the "emission coefficient" due to scattering. The "general emission coefficient" is

(1.5.6.) $j_v = j_v(\vec{R},\vec{\Omega})$ = units emitted in the direction $\vec{\Omega}$ per unit angle about $\vec{\Omega}$, per second, per unit mass at $\vec{R}$.

The ratio

(1.5.7.) $j_v/K_v = \mathcal{J}_v = \mathcal{J}_v(\vec{R},\vec{\Omega})$ is the "source function".

$\mathcal{J}_v$ has the units of j_v/K_v.

6. THE EQUATION OF TRANSFER

The equation which governs the variation of intensity in a medium characterized by absorption and emission can be derived as follows:

Consider an element of mass dM emitting in the $\vec{\Omega}$ direction, as shown in Figure 5.

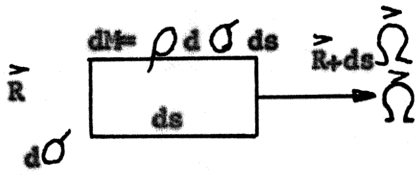


Figure 5 - Element of Area

By use of the emission coefficient, we have

(1.6.1.) $j_v dM = j_v \rho\, d\sigma\, ds$ = units emitted by dM in the $\vec{\Omega}$ direction per unit angle about $\vec{\Omega}$, per second.

Dividing by $d\sigma$, we obtain

(1.6.2.) $j_v \rho\, ds$ = units emitted along the line ($\vec{R}$, $\vec{R} + ds\vec{\Omega}$) in the $\vec{\Omega}$ direction per unit angle about $\vec{\Omega}$, per unit length perpendicular to $\vec{\Omega}$, per second.

By definition of j_v,

$j_v \rho\, ds$ = intensity in direction $\vec{\Omega}$ emitted along the segment ($\vec{R}$, $\vec{R} + ds\vec{\Omega}$).

Considering absorption equation (1.5.1.) and emission, the net change in the intensity is

(1.6.3.) $dI_v(\vec{R}, \vec{\Omega}) = - K_v \rho\, I_v ds + j_v \rho\, ds.$

Thus, the transfer equation is

(1.6.4.) $-\dfrac{1}{K_v\rho}\dfrac{dI_v(\vec{R}, \vec{\Omega})}{ds} = I_v(\vec{R}, \vec{\Omega}) - \mathcal{J}_v(\vec{R}, \vec{\Omega}).$

The derivation in (1.6.4.) is an operator for $\vec{R}$ for a fixed direction $\vec{\Omega}$ as (1.6.3.) accounts for the gains and losses in the pencil of radiation during its traversal of the element of area with which dM is associated. In a Cartesian coordinate system equation (1.6.4.) can be written

$$(1.6.5.)\quad -\frac{1}{K_v \rho}\,\vec{\Omega}\cdot\nabla I_v(\vec{R},\vec{\Omega}) = I_v(\vec{R},\vec{\Omega}) - \mathcal{J}_v(R,\vec{\Omega})$$

where ∇ is the del vector operator, $\underset{(s)}{\nabla} = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + K\frac{\partial}{\partial z}$.

When the emission term $\mathcal{J}_v = j_v / K_v$, i.e., when emission consists of scattering from all directions $\vec{\Omega}'$ into the fixed direction $\vec{\Omega}$, equation (1.6.4.) takes the form

$$(1.6.6.)\quad -\frac{1}{K_v \rho}\,\frac{dI_v(\vec{R},\vec{\Omega})}{ds} = I_v(\vec{R},\vec{\Omega}) - \frac{1}{2\pi}\int_{\omega'} p(\vec{\Omega},\vec{\Omega}')\,I_v(\vec{R},\vec{\Omega}')\,d\omega'.$$

Equation (1.6.4.) can be solved formally by writing it in the form

$$\left(\frac{d}{ds} + K_v\rho\right) I = K_v \rho\,\mathcal{J}_v .$$

If we use the integrating factor $\ell^{\int_0^s K_v \rho\,dt}$

we obtain

$$\frac{d}{ds}\left[I\,\ell^{\int_0^s \rho K_v\,dt}\right] = \ell^{\int_0^s \rho K_v\,dt}\,K_v\,\rho\,\mathcal{J}.$$

Integrating this equation from o to s we find that

$$(1.6.7.)\quad I(s) = I(o)\,\ell^{-\tau(s,o)} + \int_0^s \mathcal{J}(s')\,\ell^{-\tau(s,s')}\,K\rho\,ds$$

where $\tau(s,s') = \int_{s'}^{s} K_v \rho\,dt$ is the <u>optical thickness</u> of the material between points s and s'. Chandrasekhar uses this result to discuss the 3-dimensional plane-parallel case, (3) .

CHAPTER II - TWO-DIMENSIONAL TWO-INTENSITIES TRANSFER THEORY

1. INTRODUCTION

In this chapter we shall characterize the type of transfer problem of principal interest to us and shall derive the governing transfer equations. Consider two different intensities I_1 and I_2 traversing a medium where each is subjected to different mass effects and where transfer takes place between the two intensities. One of the simplest such models is characterized by the following equation:

$$(2.1.1.)\quad -\frac{d}{ds}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} K_1\rho_1 I_1 \\ K_2\rho_2 I_2 \end{pmatrix} - \begin{pmatrix} j_1\rho_1 \\ j_2\rho_2 \end{pmatrix} - \begin{pmatrix} K_{21}\,\rho_2 I_2 \\ K_{12}\,\rho_1 I_1 \end{pmatrix}$$

where the last term accounts for an isotropic cross scattering, i.e. a scattering between the two intensities where the emission is independent of $\vec{\Omega}$, and where

$$(2.1.2.)\quad \rho_i j_i = \frac{\rho_i K_i}{2\pi}\int_{\omega'} p_i(\vec{\Omega}, \vec{\Omega}')\, I_i(\vec{R}, \vec{\Omega}')\, d\omega',\ i=1,2.$$

Thus

$$(2.1.3.)\quad -\frac{d}{ds} I = KI - \rho j$$

where

$$I = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \qquad K = \begin{pmatrix} K_1\rho_1 & -K_{21}\rho_2 \\ K_2\rho_2 & -K_{12}\rho_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},\ \text{and } j = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix}.$$

From (2.1.2.) and (2.1.3.) we obtain

$$(2.1.4.)\quad -\frac{d}{ds} I(\vec{R},\vec{\Omega}) = KI(\vec{R},\vec{\Omega}) - \frac{1}{2\pi}\int_{\omega'} p(\vec{\Omega},\vec{\Omega}')I(\vec{R},\vec{\Omega}')\,d\omega'$$

where

$$p(\vec{\Omega},\vec{\Omega}') = \begin{pmatrix} \rho_1 K_1 p_1(\vec{\Omega},\vec{\Omega}') & 0 \\ 0 & \rho_2 K_2 p_2(\vec{\Omega},\vec{\Omega}') \end{pmatrix} .$$

2. THE GENERAL TWO-DIMENSIONAL TWO-INTENSITY TRANSFER MODEL

Consider two intensities I_i, $i=1$ or 2 and a medium characterized by different mass effects. Let ρ_{ij} be the different densities of the medium where $j=1, 2,--, n$, n is fixed. Similarly define K_{ij}, and also define $p_{ij}(\vec{\Omega},\vec{\Omega}')$ and $p_{ij}(v_i \rightarrow v_{i'},\vec{\Omega},\vec{\Omega}')$

where

$$\text{if } i = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{then } i' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

This is to be interpreted as a generalization of the previous section with the refinement that the cross scattering is no longer considered isotropic in $\vec{\Omega}$. The phase functions for the cross scattering are the $p_{ij}(v_i \rightarrow v_{i'},\vec{\Omega},\vec{\Omega}')$, $j=1,2,--,n$.

Geometrically we have the following situation. There are two intensities I_1 and I_2. These intensities account for radiation through a medium with independent mass effects arising from different densities ρ_{1j} and ρ_{2j} where $j=1,2,--$, n, n fixed and where the subscripts 1 and 2 refer to the respective intensities. Thus we allow for different independent mass effects for the different intensities which may annihilate, scatter or produce a unit. Scattering here must for a given intensity and for each independent mass effect include cross scattering from the given intensity to the other intensity as well as the usual scattering in direction for the given intensity. Further it is convenient to include the possibility of additional emission terms of the form of scattering and cross scattering which are not accompanied by an absorption term. Consequently our events can be grouped according to:

(1) Mass effects leading to an absorption which results in either (a) a loss of a unit, (b) a regular scattering, or (c) a cross scattering;

(2) Mass effects producing a unit appearing as a scattered unit or cross-scattered unit.

The transfer equations become

$$(2.2.1.)\quad -\frac{d}{ds} I_i = \left(\sum_{r=1}^{m} K_{ir}\,\rho_{ir}\right) I_i$$

$$-\frac{1}{2\pi}\int_{\omega'}\left(\sum_{j=1}^{n} K_{ij}\,\rho_{ij}\,P_{ij}(\vec{\Omega},\vec{\Omega}')\right) I_i(\vec{R},\vec{\Omega}')\,d\omega'$$

$$-\frac{1}{2\pi}\int_{\omega'}\left(\sum_{j=1}^{n} K_{i'j}\,\rho_{i'j}\,P_{i'j}(v_{i'}\rightarrow v_i,\vec{\Omega},\vec{\Omega}')\right) I_{i'}(\vec{R},\vec{\Omega}')\,d\omega'$$

where $0 < m \leqq j$.

Now these equations are seen to take the form

$$(2.2.2.)\quad -\frac{d}{ds} I(\vec{T},\vec{\Omega}) = AI(\vec{R},\vec{\Omega}) - \frac{1}{2\pi}\int_{\omega'} pI(\vec{R},\vec{\Omega}')\,d\omega'$$

where $I(\vec{R},\vec{\Omega}) = \begin{pmatrix} I_1(\vec{R},\vec{\Omega}) \\ I_2(\vec{R},\vec{\Omega}) \end{pmatrix},$

$$A = \begin{pmatrix} \sum_{r=1}^{m} K_{1r}\,\rho_{1r} & 0 \\ 0 & \sum_{r=1}^{m} K_{2r}\,\rho_{2r} \end{pmatrix},$$ and

$$p = \begin{pmatrix} \sum_{r=1}^{m} K_{1j}\rho_{1j}\, p_{1j}(\vec{\Omega}\cdot\vec{\Omega}') & \sum_{r=1}^{n} K_{2j}\rho_{2j}\, p_{2j}(v_2 \to v_1, \vec{\Omega}\cdot\vec{\Omega}') \\ \sum_{j=1}^{n} K_{1j}\rho_{1j}\, p_{1j}(v_1 \to v_2, \vec{\Omega}\cdot\vec{\Omega}') & \sum_{j=1}^{n} K_{2j}\rho_{2j}\, p_{2j}(\vec{\Omega}\cdot\vec{\Omega}') \end{pmatrix}.$$

We shall be interested in a time dependent intensity which consequently is governed by

$$(2.2.3.)\quad \frac{1}{v}\frac{\partial}{\partial t}I = -\frac{\partial}{\partial s}I - AI + \frac{1}{2\pi}\int_{\omega'} pI(\vec{R}, \vec{\Omega}')\, d\omega'$$

where $\frac{1}{v} = \begin{pmatrix} \frac{1}{v_1} & 0 \\ 0 & \frac{1}{v_2} \end{pmatrix}$ and where we have evaluated the total derivative.

Equation (2.2.3.) characterizes the type of transfer model in which we shall be interested. There are two approaches open to investigation. Either we can investigate equation (2.2.3.) directly obtaining general and formal solutions for different classes of problems corresponding to different types of kernels. Or we can carry through the solution of this equation corresponding to a given kernel. We shall elect to do the latter for in obtaining the kernel one indicates how diverse transport phenomena can be formulated in terms of a transfer model, and because a formal solution does not always conclude an investigation.

CHAPTER III - ANALYSIS OF A SPECIFIC MODEL

1. INTRODUCTION

Consider the following problem: A commander is conducting a battle in a theater of operation and his efforts are supplied by trucks which must traverse a medium in which they are subjected to the effects of enemy action and terrain. The commander desires to know a reasonable estimate of supply in terms of loaded trucks which he can expect under various conditions. In a real sense he desires to know the number of loaded trucks in transit per unit area at a given place at a given time which, in the course of carrying out a mission, have been subjected to being put out of operation, deflected, or forced to change their speed by terrain, enemy action or decision. We shall assume a distinction between traffic on a road (in a road net) and off road. Further unpublished performance data of the Transportation Research Command permits the assumption that these two different traffic flows can be represented by two different constant speeds v_1 and v_2, respectively.

When this problem is formulated in terms of transfer theory we construct models one of which is characterized by the following table. We have to relate the factors occurring in Table 1 to the problem at hand.

Table 1

Events and Interactions Cross Sections

	I_1	I_2
Absorptions		
Annihilation by enemy action	$K_{11}\rho_{11}$	$K_{21}\rho_{21}$
Scattering (enemy action)	$K_{12}\rho_{12}$	$K_{22}\rho_{22}$
Scattering (terrain)	$K_{13}\rho_{13}$	$K_{23}\rho_{23}$
Speed transfer (enemy action)	$K_{14}\rho_{14}$	$K_{24}\rho_{24}$
Speed transfer (terrain)	$K_{15}\rho_{15}$	$K_{25}\rho_{25}$
Gains		
Scattering (enemy action)	$K_{12}\rho_{12}P_{12}(\vec{\Omega},\vec{\Omega}')$	$K_{22}\rho_{22}P_{22}(\vec{\Omega},\vec{\Omega}')$
Scattering (terrain)	$K_{13}\rho_{13}P_{13}(\vec{\Omega},\vec{\Omega}')$	$K_{23}\rho_{23}P_{23}(\vec{\Omega},\vec{\Omega}')$
Speed transfer (enemy action)	$K_{24}\rho_{24}P_{24}(v_2 \to v_1,\vec{\Omega},\vec{\Omega}')$	$K_{14}\rho_{14}P_{14}(v_1 \to v_2,\vec{\Omega},\vec{\Omega}')$
Speed transfer (terrain)	$K_{25}\rho_{25}P_{25}(v_2 \to v_1,\vec{\Omega},\vec{\Omega}')$	$K_{15}\rho_{15}P_{15}(v_1 \to v_2,\vec{\Omega},\vec{\Omega}')$

where I_1 is the intensity of on-road traffic and I_2 the intensity of off-road traffic.

1.1 OPERATIONAL DEFINITION OF THE DENSITIES AND MASS ABSORPTION COEFFICIENT

We now consider separately the different absorption terms given in Table 1.

(a) Annihilation by Enemy Action

Define

$\rho_{11} \equiv$ occurrences per unit area, where an occurrence could be the detonation of a bomb. As discussed,

$$(3.1.1.)\ \rho_{11} = \frac{\text{total number of expected events (occurrences) in a region.}}{\text{area of the region}}$$

Then K_{11} is the probability per unit occurrence that a unit will be annihilated in terms of length perpendicular to $\vec{\Omega}$.

ρ_{21} and K_{21} are similarly defined.

(b) Speed Transfer by Enemy Action

Let $\rho_{14} = \rho_{11}$ and $\rho_{24} = \rho_{21}$. Then K_{14} and K_{24} are just the probabilities per unit occurrence that a speed transfer will take place.

(c) Speed Transfer by Terrain

The nature of terrain is fundamentally different in the cases of on-road and off-road.

Off-Road: From terrain features such as slope and soil conditions, a go/no-go function can be evaluated for an area which will determine whether a specific vehicle type will or will not stall in this area. If we consider a region R (see Figure 6) and impose a grid on it giving go/no-go values for different cells,

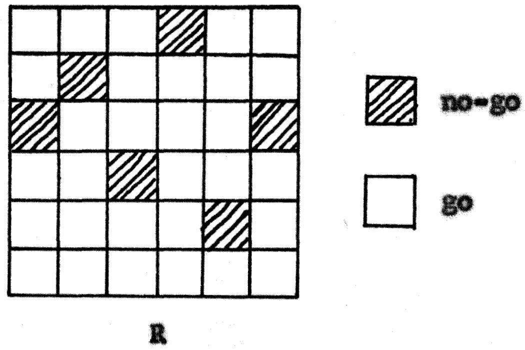


Figure 6 - Region of Terrain

a density ρ_{25} can be defined as

(3.1.2.) $\rho_{25} = \dfrac{\text{number of no-go cells in the region.}}{\text{area of the region}}$

Then K_{25} becomes the probability per unit no-go that a unit will be transferred given in terms of length perpendicular to $\vec{\Omega}$.

<u>On-Road</u>: Consider a road net in a given region. Impose upon it a rectangular grid (see Figure 7) such that the map in each cell is connected and such that each crossroad falls within a cell. Now, in each cell of this grid, construct a linear approximation to the road map such that the maximum distance from the road to the linear segment is a preassigned value, say ϵ (see Figure 8), and such that the segment is tangent to the road at its contact points. Now consider the following

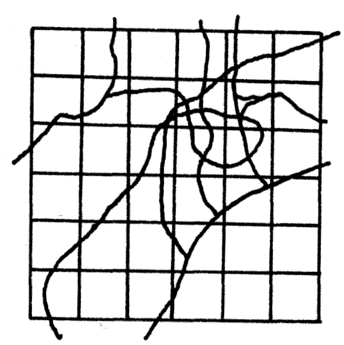


Figure 7 - A Road Net with Grid

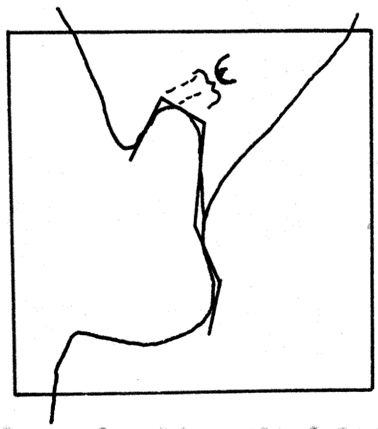


Figure 8 - Linearized Road

two quantities:

ϕ = sum of the angles that adjacent segments made with each other when considered as vectors located in first and second quadrant with respect to a preferred direction .

Ψ = the absolute value of the signed sum of the angles that the segments make with a preferred direction when located as above.

Further consider the conditions of the road net in a cell. Because of such factors as a bridge out, road demolished, or mud, traffic may be stalled. From these three quantities can be determined estimates of a transfer/no-transfer function for each cell. Thus, as in the case of off-road traffic, a density ρ_{15} and a mass absorption coefficient can be evaluated.

(d) Scattering by Enemy Action

Let $\rho_{12} = \rho_{11}$ and $\rho_{22} = \rho_{21}$. Then K_{12} and K_{22} are just the probabilities per unit occurrence that a scattering event will take place.

(e) Scattering by Terrain

Let $\rho_{23} = \rho_{25}$. Then K_{23} is the probability per unit no-go that a scattering event will take place. The on-road density ρ_{13} is, however, in general different from ρ_{15}. Consider Figure 9; a scattering event occurs if there are two or more branches issuing from a cell with different tangent vectors. We will assume that the case in which two road branches issue from a cell is fundamentally the same as that in which multiple branches issue from a cell. A further refinement would be to make this distinction. Thus we could, in principle, determine a scatter/no-scatter function for each cell and then derive a density ρ_{13} and corresponding coefficient K_{13}.

2. OPERATIONAL DEFINITIONS OF THE PHASE FUNCTIONS

We now consider separately the different phase functions given in the table.

(a) Speed Transfer by Enemy Action

Here the on-road transfer phase function differs from the off-road transfer phase function. Realistic phase functions can be constructed in the following manner:

First recall that

$\frac{p(\vec{\Omega}, \vec{\Omega}')}{2\pi} d\omega$ = the probability per unit solid angle about $\vec{\Omega}'$ that the unit which was absorbed at $\vec{R}$ will be emitted into $d\omega$ about $\vec{\Omega}$, given that an absorption-scattering (transfer) event has occurred. Now, consider $p_{14}(v_1 \rightarrow v_2, \vec{\Omega}, \vec{\Omega}')$ and consider the situation illustrated in Figure 9.

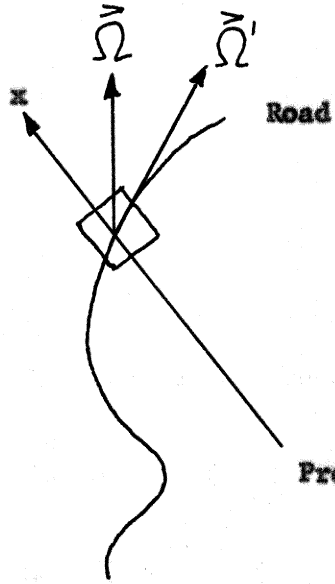


Figure 9 - Cross-Scattering on Road

Once a transferring event has occurred on the road, a unit traveling in the $\vec{\Omega}'$ direction becomes reoriented in the $\vec{\Omega}$ direction, with an assumed symmetric distribution about the preferred direction of travel, x . As with the phase function for pure scattering to be considered later, we assume that $p_{14}(v_1 \rightarrow v_2, \vec{\Omega}, \vec{\Omega}') = p_{14}(v_1 \rightarrow v_2, \vec{\Omega})$, or the phase function is independent of $\vec{\Omega}'$. Thus,

$$(3.2.1.)\quad \frac{1}{2\pi}\int_{\omega'} p_{14}(v_1 \rightarrow v_2, \vec{\Omega})\, I_1(\vec{R}, \vec{\Omega}')\, d\omega' = \frac{p_{14}(v_1 \rightarrow v_2, \vec{\Omega})}{2} \int_{\omega'} I_1(\vec{R}, \vec{\Omega}')\, d\omega'.$$

Now consider $p_{24}(v_2 \rightarrow v_1, \vec{\Omega}, \vec{\Omega}')$ and consider the situation illustrated in Figure 10.

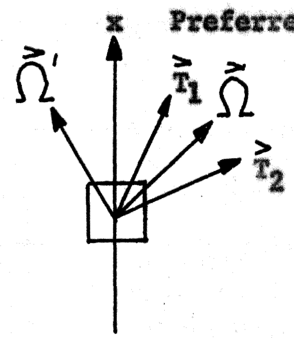


Figure 10 - Cross-Scattering Off-Road

Once a transferring event has occurred, assume that a unit traveling in the $\vec{\Omega}'$ direction becomes reoriented in the $\vec{\Omega}$ direction independent of $\vec{\Omega}'$. Let the $\vec{T}_i$'s be the tangent vectors to the roads as explained in 1 (c) under "Speed Transfer by Terrain", where the road has been "linearized", and let $\vec{\Omega}$ be the resultant on-going direction of the units. We assume that the phase function for each cell is of the following form:

$$P^j_{24}\,(v_2 \rightarrow v_1, \vec{\Omega}, \vec{\Omega}') = \sum_{i=1}^{\mu(j)} \bar{\delta}^{\,j}_{a}\,(\vec{T}_{ij}, \vec{\Omega})$$

where $$\bar{\delta}^{\,j}_{a}\,(\vec{T}_{ij}, \vec{\Omega}) = \begin{cases} 0 \text{ if } \dfrac{\vec{T}_{ij}}{\|\vec{T}_{ij}\|} \neq \vec{\Omega} \\ a_{ij} \text{ if } \dfrac{\vec{T}_{ij}}{\|\vec{T}_{ij}\|} = \vec{\Omega} \end{cases}$$
, where j refers to the jth cell in some ordering of the cells.

where $\mu(j)$ is the number of road branches, and

$$\int_{\omega} \sum_{i=1}^{\mu(j)} \bar{\delta}^{\,j}_{a}\,(\vec{T}_{ij}, \vec{\Omega})\, d\omega = 1.$$

A realistic determination of the $\bar{\delta}^{\,j}_{a}\,(\vec{T}_{ij}, \vec{\Omega})$ is indicated in Figure 11,

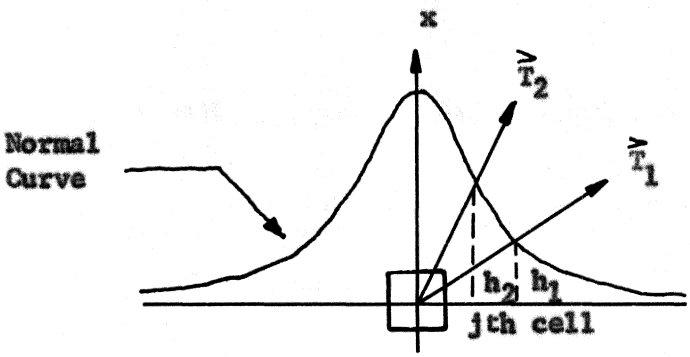


Figure 11 - Scattering Probabilities $\bar{\delta}_a^{\,j}$

Where $\bar{\delta}_a^{\,j}(\vec{T}_{1j}, \vec{\Omega})$ is to $\bar{\delta}_a^{\,j}(\vec{T}_{2j}, \vec{\Omega})$ as h_1 is to h_2

and where $\int_\omega \left\{ \bar{\delta}_a^{\,j}(\vec{T}_{1j}, \vec{\Omega}) + \bar{\delta}_a^{\,j}(\vec{T}_{2j}, \vec{\Omega}) \right\} d\omega = 1.$ Thus,

$$\frac{1}{2\pi} \int_{\omega'} P_{24}(v_2 \rightarrow v_1, \vec{\Omega}, \vec{\Omega}') \, I_2(\vec{R}, \vec{\Omega}') \, d\omega' =$$

(3.2.2.)
$$= \sum_{j=1}^{n} \frac{\sum_{i=1}^{\mu(j)} \bar{\delta}_a^{\,j}(\vec{T}_{ij}, \vec{\Omega})}{2\pi n} \int_{\omega'} I_2(\vec{R}, \vec{\Omega}') \, d\omega'.$$

Note $\dfrac{\sum_{j=1}^{n} \sum_{i=1}^{\mu(j)} \bar{\delta}_a^{\,j}(\vec{T}_{ij}, \vec{\Omega}')}{2\pi n}$ is a regional function in agreement with K_{24} and ρ_{24}.

(b) <u>Speed Transfer by Terrain</u>

As with speed transfer in part (a) above, we obtain:

(3.2.3.)
$$\frac{1}{2\pi} \int_{\omega'} P_{15}(v_1 \rightarrow v_2, \vec{\Omega}, \vec{\Omega}') \, I_2(\vec{R}, \vec{\Omega}') \, d\omega' = \frac{P_{15}(v_1 \rightarrow v_2, \vec{\Omega})}{2} \int_{\omega'} I_1(\vec{R}, \vec{\Omega}') \, d\omega'$$

and

(3.2.4.)
$$\frac{1}{2\pi} \int_{\omega'} P_{25}(v_2 \rightarrow v_1, \vec{\Omega}, \vec{\Omega}') \, I_2(\vec{R}, \vec{\Omega}') \, d\omega'$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{\mu(j)} \frac{\overline{\delta_b}^{\,j}\,(\vec{T}_{ij}\,,\,\vec{\Omega})}{2\pi n} \int_{\omega'} I_2\,(\vec{R}, \vec{\Omega}')\, d\omega'$$

where the $\overline{\delta_b}$'s, refer to p_{25} ($v_2 \rightarrow v_1$) and p_{25} is also an unspecified function of $\vec{R}$. Let p_{25} ($\vec{\Omega}$) be the appropriate regional phase function.

(c) <u>Scattering by Enemy Action</u>

Here the on-road case differs from the off-road case.

<u>Off-Road</u>: We assume, considering that a unit is goal-oriented, that a realistic phase function can be determined in the following manner: To fix discussion, let the positive x direction in a rectangular system be the "ideal" direction of traffic (see Figure 12).

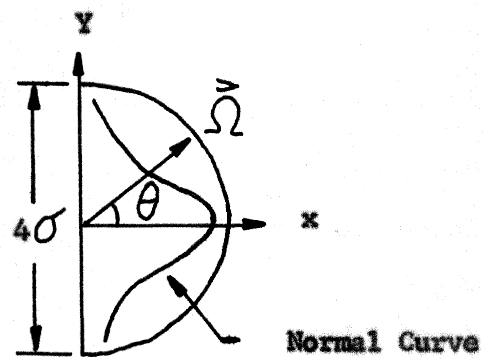


Figure 12 - Off-Road Scattering Probability

Assume that once a scattering "event" has taken place, the unit independent of $\vec{\Omega}'$ seeks to orient itself in the positive x direction resulting in an $\vec{\Omega}$ orientation.

Thus,

(3.2.5.) $\dfrac{p(\vec{\Omega}, \vec{\Omega}')}{2\pi} = \dfrac{p(\vec{\Omega})}{2\pi}$ and therefore we obtain

$$(3.2.6.)\quad \frac{1}{2\pi}\int_{\omega'} P_{22}(\vec{\Omega})\, I_2(\vec{R},\vec{\Omega}')\, d\omega' = \frac{P_{22}(\vec{\Omega})}{2\pi}\int_{\omega'} I_2(\vec{R},\vec{\Omega}')\, d\omega'.$$

<u>On-Road</u>: To a first approximation, on-road scattering by enemy action does not take place.

Therefore,

$$(3.2.7.)\quad \frac{1}{2\pi}\int_{\omega'} P_{12}(\vec{\Omega},\vec{\Omega}')\, I_1(\vec{R},\vec{\Omega}')\, d\omega' \equiv 0.$$

(d) <u>Scattering by Terrain</u>

Here the on-road case differs from the off-road case.

<u>Off-Road</u>: We assume that, fundamentally, the phase function for terrain scattering is of the same form as that for enemy action.

Thus,

$$(3.2.8.)\quad \frac{1}{2\pi}\int_{\omega'} P_{23}(\vec{\Omega})\, I_2(\vec{R},\vec{\Omega}')\, d\omega' = \frac{P_{23}(\vec{\Omega})}{2\pi}\int_{\omega'} I_2(\vec{R},\vec{\Omega}')\, d\omega'.$$

<u>On-Road</u>: The on-road scattering phase function might be viewed as in the cases of speed transfer due to enemy action and due to terrain. Consequently, we obtain

$$(3.2.9.)\quad \frac{1}{2\pi}\int_{\omega'} P_{13}(\vec{\Omega},\vec{\Omega}')\, I_1(\vec{R},\vec{\Omega}')\, d\omega' =$$

$$\frac{\sum_{j=1}^{n}\sum_{i=1}^{\mu(j)} \overline{\sigma}_d^{\,j}(\vec{T}_{ij},\vec{\Omega})}{2\;n}\int_{\omega'} I_1(\vec{R},\vec{\Omega}')\, d\omega'.$$

3. <u>THE MODIFIED GENERAL EQUATIONS</u>

Clearly, the definitions and assumptions introduced in this chapter mathematically simplify the general transfer equations. The appropriate simplified equations will be given below.

(a) Time-Independent Equations

If we take into account the functional dependency of our emission terms derived in this chapter and group together the various emission and absorption terms, after considerable manipulation, we obtain equations of the form:

$$\frac{dI_1(\vec{R},\vec{\Omega})}{ds} = -K_1 I_1(\vec{R},\vec{\Omega}) + \frac{\sum_2^T}{2\pi}\int_{\omega'} I_2(\vec{R},\vec{\Omega}')\,d\omega' + \frac{\sum_1}{2\pi}\int_{\omega'} I_1(\vec{R},\vec{\Omega}')\,d\omega' \tag{3.3.1.}$$

$$\frac{dI_2(\vec{R},\vec{\Omega})}{ds} = -K_2 I_2(\vec{R},\vec{\Omega}) + \frac{\sum_1^T}{2\pi}\int_{\omega'} I_1(\vec{R},\vec{\Omega}')\,d\omega' + \frac{\sum_2}{2\pi}\int_{\omega'} I_2(\vec{R},\vec{\Omega}')\,d\omega'$$

where

$$\textstyle\sum_2^T = \sum_2^T (v_2 \rightarrow v_1\,,\vec{\Omega})$$

$$\textstyle\sum_1 = \sum_1 (v_1\,,\vec{\Omega})$$

$$\textstyle\sum_1^T = \sum_1^T (v_1 \rightarrow v_2\,,\vec{\Omega})$$

$$\textstyle\sum_2 = \sum_2 (v_2\,,\vec{\Omega}).$$

As in Chapter II, these equations can be written:

$$\frac{d}{ds} I(\vec{R},\vec{\Omega}) = -AI(\vec{R},\vec{\Omega}) + \frac{1}{2\pi}\int_{\omega'} p(\vec{\Omega})\, I(\vec{R},\vec{\Omega}')\,d\omega', \tag{3.3.2.}$$

where

$$I = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}.$$

$$A = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}, \quad \text{and } p(\vec{\Omega}) = \begin{pmatrix} \Sigma_1 & \Sigma_2^T \\ \Sigma_1^T & \Sigma_2 \end{pmatrix}.$$

(b) **The Time-Dependent Equations:**

The time dependent equations are:

$$\frac{1}{v_1}\frac{\partial I_1(\vec{R},\vec{\Omega})}{\partial t} = -\frac{\partial I_1(\vec{R},\vec{\Omega})}{\partial s} - K_1 I_1(\vec{R},\vec{\Omega}) + \frac{\Sigma_2^T}{2\pi}\int_{\omega'} I_2(\vec{R},\vec{\Omega}')\,d\omega'$$

$$(3.3.3.) \qquad + \frac{\Sigma_1}{2\pi}\int_{\omega'} I_1(\vec{R},\vec{\Omega}')\,d\omega'$$

$$\frac{1}{v_2}\frac{\partial I_2(\vec{R},\vec{\Omega})}{\partial t} = -\frac{\partial I_2(\vec{R},\vec{\Omega})}{\partial s} - K_2 I_2(\vec{R},\vec{\Omega}) + \frac{\Sigma_1^T}{2\pi}\int_{\omega'} I_1(\vec{R},\vec{\Omega}')\,d\omega'$$

$$+ \frac{\Sigma_2}{2\pi}\int_{\omega'} I_2(\vec{R},\vec{\Omega}')\,d\omega'.$$

If we set $\frac{1}{v} = \begin{pmatrix} \frac{1}{v_1} & 0 \\ 0 & \frac{1}{v_2} \end{pmatrix}$ these equations can be written:

$$(3.3.4.) \quad \frac{1}{v}\frac{\partial}{\partial t} I(\vec{R},\vec{\Omega}) = -\frac{\partial}{\partial s} I(\vec{R},\vec{\Omega}) - AI(\vec{R},\vec{\Omega}) + \frac{p(\vec{\Omega})}{2\pi}\int_{\omega'} I(\vec{R},\vec{\Omega}')\,d\omega'.$$

(c) **The Density-Equations:**

Following Section 4 of Chapter I, if we integrate with respect to ω we obtain:

$$\frac{1}{v_1}\frac{\partial u_1}{\partial t} = -\frac{\partial u_1}{\partial s} - K_1 u_1 + \bar{\Sigma}_2^T u_2 + \bar{\Sigma}_1 u_1$$

$$(3.3.5.) \quad \frac{1}{v_2}\frac{\partial u_2}{\partial t} = -\frac{\partial u_2}{\partial s} - K_2 u_2 + \bar{\Sigma}_1^T u_1 + \bar{\Sigma}_2 u_2,$$

where K's, $\bar{\Sigma}^T$'s and $\bar{\Sigma}$'s are now regional constants.

These equations can be written:

$$\frac{1}{v}\frac{\partial}{\partial t}u = -\frac{\partial}{\partial s}u + Bu \tag{3.3.6.}$$

where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$B = \begin{pmatrix} \overline{\overline{\Sigma}}_1 - K_1 & \overline{\overline{\Sigma}}_2^T \\ \overline{\overline{\Sigma}}_1^T & \overline{\overline{\Sigma}}_2 - K_2 \end{pmatrix}.$$

CHAPTER IV - THE CONCENTRIC ANNULI PROBLEM

In this chapter we will consider the system of equations (3.3.5.) and (3.3.6.) in a disk-geometry and derive the density function for the concentric annuli geometry.

1. THE GENERAL POLAR DENSITY EQUATION

In polar coordinates equation (3.3.5.) becomes

$$\frac{1}{v_1}\frac{\partial u_1}{\partial t} = -\left(\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta}\right) u_1 - K_1 u_1 + \overline{\sum_2}^{T} u_2 + \overline{\sum_1} u_1$$

(4.1.1.)

$$\frac{1}{v_2}\frac{\partial u_2}{\partial t} = -\left(\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta}\right) u_2 - K_2 u_2 + \overline{\sum_1}^{T} u_1 + \overline{\sum_2} u_2$$

(see Figure 13) where because of symmetry $\frac{1}{r}\frac{\partial}{\partial \theta}$ u_1 or $u_2 = 0$.

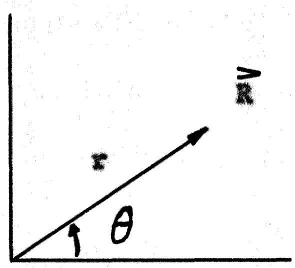


Figure 13 - Polar Coordinates

2. THE STEADY STATE DENSITY PROBLEM

The obvious significance of the steady state density is: given appropriate boundary conditions at a boundary $r = a$ and at another boundary $r = b$, we can determine the point function expressing the number of units per unit area which are in transit in the enclosed region or the number of units in transit in the region.

The steady state density equations with constant sources or loss terms are

$$\frac{\partial u_1}{\partial r} - \left(\overline{\Sigma}_1 - K_1 \right) u_1 - \overline{\Sigma}_2^{T} u_2 = S_1$$

(4.2.1.)

$$\frac{\partial u_2}{\partial r} - \left(\overline{\Sigma}_2 - K_2 \right) u_2 - \overline{\Sigma}_1^{T} u_1 = S_2.$$

In matrix form

$$\begin{pmatrix} \left(\frac{\partial}{\partial r} + K_1 - \overline{\Sigma}_1 \right) & -\overline{\Sigma}_2^{T} \\ -\overline{\Sigma}_1^{T} & \left(\frac{\partial}{\partial r} + K_2 - \overline{\Sigma}_2 \right) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}.$$

Thus by Cramer's rule

(4.2.2.)
$$\begin{vmatrix} \left(\frac{\partial}{\partial r} + K_1 - \overline{\Sigma}_1 \right) & -\overline{\Sigma}_2^{T}) \\ -\overline{\Sigma}_1^{T} & \left(\frac{\partial}{\partial r} + K_2 - \overline{\Sigma}_2 \right) \end{vmatrix} u_1 = \begin{vmatrix} S_1 & -\overline{\Sigma}_2^{T} \\ S_2 & \left(\frac{\partial}{\partial r} + K_2 - \overline{\Sigma}_2 \right) \end{vmatrix}.$$

Expanding (4.2.2.) we obtain

(4.2.3.)
$$\frac{\partial^2 u_1}{\partial r^2} + A \frac{\partial u_1}{\partial r} + B u_1 = C_1$$

where

$$A = K_1 + K_2 - \overline{\Sigma}_1 - \overline{\Sigma}_2$$

$$B = K_1 K_2 - K_1 \overline{\Sigma}_2 - K_2 \overline{\Sigma}_1 + \overline{\Sigma}_1 \overline{\Sigma}_2 - \overline{\Sigma}_2^{T} \overline{\Sigma}_1^{T}$$

and

$$C_1 = S_1 (K_2 - \overline{\overline{\sum_2}}) + S_2 \overline{\sum_2}^T \quad .$$

Similarly for u_2 we obtain

(4.2.4.)

$$\frac{\partial^2 u_2}{\partial r^2} + A \frac{\partial u_2}{\partial r} + Bu_2 = C_2 \quad \text{where}$$

$$C_2 = S_2 (K_1 - \overline{\overline{\sum_1}}) + S_1 \overline{\sum_1}^T \quad .$$

These equations are of the same form. When $A^2-4B \neq 0$ their formal solutions are:

(4.2.4.)

$$u_1 = C_{11} \exp\left[\frac{-A + \sqrt{A^2-4B}}{2} r \right] + C_{12} \exp\left[\frac{-A-\sqrt{A^2-4B}}{2} r \right] + \frac{C_1}{B}$$

$$u_2 = C_{21} \exp \left[\frac{-A + \sqrt{A^2-4B}}{2} r \right] + C_{22} \exp \left[\frac{-A- \sqrt{A^2-4B}}{2} r \right] + \frac{C_2}{B}$$

where C_{ij}'s are constants of integration to be determined from specific boundary conditions and the following constraints:

(4.2.5.)

$$C_{11} \left[\frac{-A+\sqrt{A^2-4B}}{2} + K_1 - \overline{\overline{\sum_1}}\right] - \overline{\sum_2}^T C_{21} = 0$$

$$C_{12} \left[\frac{-A-\sqrt{A^2-4B}}{2} + K_1 - \overline{\overline{\sum_1}}\right] - \overline{\sum_2}^T C_{22} = 0 \; .$$

If $A^2 - 4B = o$ then

(4.2.6.) $$u_1 = (C_{11} + C_{12} r) \exp \left[- \frac{A}{2} r\right] + \frac{C_1}{B}$$

$$u_2 = (C_{21} + C_{22} r) \exp \left[- \frac{A}{2} r\right] + \frac{C_2}{B}$$

where constraints like (4.2.5.) apply.

When $A^2-4B>0$ it is seen from (4.2.4.) that the densities are the result of a composition of exponential effects. With appropriate constants A and B this corresponds to an exponential depletion in the radial direction of the density of the units. If $A^2-4B>0$ we also have the possibility of trignometric terms with exponentially depleted amplitudes. This corresponds to exponentially depleted standing waves. When $A^2-4B<0$ in general the densities are complex.

3. THE TIME DEPENDENT DENSITY PROBLEM WITH NO SOURCE TERMS

The time dependent density equations with no arbitrary regional source terms are given in the system (4.1.1.). They are of the following form:

$$\frac{1}{\overline{\sum}_2^T} \left(\frac{1}{v_1} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + K_1 - \overline{\sum}_1 \right) u_1 = u_2$$

(4.3.1.)

$$\frac{1}{\overline{\sum}_1^T} \left(\frac{1}{v_1} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + K_2 - \overline{\sum}_2 \right) u_2 = u_1 \ .$$

Thus if we write $Au_1 = u_2$ and $Bu_2 = u_1$, we see that $A^{-1}=B$ and $B^{-1}=A$ or

$$BAu_1 = u_1$$

(4.3.2.)

$$ABu_2 = u_2 \ .$$

If the indicated operations were carried out we would obtain a general 2nd order partial differential equation. However, each of the equations (4.3.1.) are quasi-linear of 1st order i.e. of the form

(4.3.3.) $P_p + Q_q = R$.

The system (4.3.1.) can be solved by simultaneous application of the well-known method for the solution of (4.3.3.) (see a text on partial differential equations such as Sneddon (7)).

Consider the 1st equation of system (4.1.1.) in the form:

$$a_1 \frac{\partial u_1}{\partial t} + a_2 \frac{\partial u_1}{\partial r} = a_3 u_1 + a_4 u_2 \text{ where } a_4 = \overline{\sum_2^T} .$$

Thus $R = a_3 u_1 + a_4 u_2$ is a linear function of u_1.

Similarly let

$$b_1 \frac{\partial u_2}{\partial t} + b_2 \frac{\partial u_2}{\partial r} = b_3 u_2 + b_4 u_1 .$$

Now we obtain:

$$\frac{dt}{a_1} = \frac{dr}{a_2} = \frac{du_1}{a_3 u_1 + a_4 u_2} \quad \text{and}$$

$$\frac{dt}{b_1} = \frac{dr}{b_2} = \frac{du_2}{b_3 u_2 + b_4 u_1}.$$

Equating the first two terms in both equations we obtain

$$C_{11} = a_2 t - a_1 r$$

(4.3.4.) where C_{11} and C_{21} are constants of integration.

$$C_{21} = b_2 t - b_1 r$$

Similarly equating the first and last terms in the preceeding equations we obtain

$$\frac{du_1}{dt} = \frac{a_3}{a_1} u_1 + \frac{a_4}{a_1} u_2 \text{ and } \frac{du_2}{dt} = \frac{b_3}{b_1} u_2 + \frac{b_4}{b_1} u_1 .$$

These equations can be integrated simultaneously by determinants to give

$$u_1 = C_{12}\,\ell^{At} + C_{13}\,\ell^{Bt}$$

(4.3.5.)

$$u_2 = C_{22}\,\ell^{At} + C_{23}\,\ell^{Bt}$$

where the C_{ij}'s are constants of integration subject to the following constraints

$$C_{12} = \frac{b_1}{b_4}\left(A - \frac{b_3}{b_1}\right) C_{22}$$

(4.3.6.)

$$C_{13} = \frac{b_1}{b_4}\left(B - \frac{b_3}{b_1}\right) C_{23} \quad \text{and}$$

where

$$A \text{ or } B = \frac{\frac{b_3}{b_1} + \frac{a_3}{a_1} \pm \sqrt{\left(\frac{b_3}{b_1} + \frac{a_3}{a_1}\right)^2 - 4\left(\frac{a_3 b_3}{a_1 b_1} \quad \frac{a_4 b_4}{a_1 b_1}\right)}}{2} .$$

Now solve equations of (4.3.5.) for

$$C_{12} = \frac{u_1 - C_{13}\exp Bt}{\exp At} \quad \text{and} \quad C_{22} = \frac{u_2 - C_{23}\exp Bt}{\exp At} .$$

Thus applying the theorem applicable to equations of the type (4.3.3.) we obtain

$u_1 = (\exp At)\, f\,(t - a_1 r) + (\exp Bt)\, g\,(t - a_1 r)$

and

$u_2 = (\exp At)\, f\,(t - b_1 r) + (\exp Bt)\, g\,(t - b_1 r)$

where $b_2 = a_2 = 1$ for the general form of the solution.

Now recalling the constraints (4.3.6.), let

$u_2 = (\exp At)\ f\ (t - b_1 r) + (\exp Bt)\ g\ (t - b_1 r).$

Then we can determine u_1 as a function of u_2 i.e. we have

$F\left[C_{11}, C_{12}\right] = 0$ and $F\left[C_{21}, C_{22}\right] = 0$

where $C_{12} = f\ (C_{11})$ and $C_{22} = f\ (C_{21})$.

Thus let

$$F\left[C_{21}, C_{22}\right] = F\left[C_{21}\ \frac{b_4}{b_1}\left(A - \frac{b_3}{b_1}\right)^{-1} C_{12}\right] = 0$$

so that

$\frac{b_4}{b_1}\left(A - \frac{b_3}{b_1}\right)^{-1} C_{12} = f\ (C_{21})$, which also follows from (4.3.6.).

Hence

$$(4.3.7.)\quad u_1 = \frac{b_1}{b_4}\left(A - \frac{b_3}{b_1}\right)(\exp At)\ f\ (t - b_1 r) + \frac{b_1}{b_4}\left(B - \frac{b_3}{b_1}\right)(\exp Bt)\ g(t - b_1 r).$$

We have thus determined the general form of the solution for a prime case of interest.

The equations

$$a_1 \frac{\partial u_1}{\partial t} + a_2 \frac{\partial u_1}{\partial r} = a_3 u_1 + a_4 u_2$$

$$b_1 \frac{\partial u_2}{\partial t} + b_2 \frac{\partial u_2}{\partial r} = b_3 u_2 + b_4 u_1$$

can be transformed into the system

$$(4.3.8.)\quad \begin{aligned} \frac{\partial u_1}{\partial \xi} &= a_3 u_1 + a_4 u_2 \\ \frac{\partial u_2}{\partial \eta} &= b_3 u_2 + b_4 u_1 \end{aligned}$$

by the transformations

$$t = a_1 \xi + b_1 \eta$$

(4.3.9.)

$$r = a_2 \xi + b_2 \eta \, .$$

If we solve for one of the densities say u_2 we obtain an equation of the form

(4.3.10.) $$a_3 \frac{\partial u_2}{\partial \eta} - \frac{\partial^2 u_2}{\partial \xi \partial \eta} + b_3 \frac{\partial u_2}{\partial \xi} + (1 - a_3 b_3) u_2 = 0,$$

where $a_4 = b_4 = 1$.

System (4.3.8.) would appear to offer the greatest opportunity for digital programming where, as (4.3.10) is in the normal hyperbolic form, offers some analytical advantages as it has been extensively discussed in the literature. A solution of (4.3.10.) can be readily found by separation of variables. A thorough discussion of finite difference methods for equations of the form (4.3.8.) is to be found in reference (6).

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TABLE OF SYMBOLS

ABSTRACT

The fundamental definitions of radiative transfer theory are given and the two-dimensional equation of transfer is derived. A density of radiation is defined and a two-dimensional two-intensity transfer model is presented. An operational interpretation of the latter model is given in terms of military truck transport supply and the functional dependencies of the terms in the transfer equations are evaluated. For this interpretation the density equations are given and the steady state and time dependent solutions of the density equations are discussed in polar coordinates. This work was conducted for the U. S. Army Transportation Research Command, Fort Eustis, Virginia, 1961, Task 9R38-11-009-02.