THE COMPARISON OF THE SENSITIVITIES OF EXPERIMENTS USING DIFFERENT SCALES OF MEASUREMENT

by

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I: INTRODUCTION

1.1 The problem.

The problem of obtaining the best method of measuring the effects of experimental treatments is a difficult one in certain fields of research. In subjective testing no natural unit of measurement is available and it is common practice to use ranking methods or to invent a scoring scale on which the flavor or quality of a food is 'measured', through the opinion of judges. In field experimentation decisions on the relative efficiencies of different plot sizes may cause considerable concern. Means are required for comparing the efficiencies of scoring systems or of plot sizes. It is with some aspects of this problem that this paper is concerned.

The consideration of statistical methods applicable in the sensory testing of foodstuffs and in particular the development of methods for comparing the sensitivities of taste panel experiments, led to the research discussed in the paper. The uses of both ranking and ~~of~~ scoring methods in taste testing have been considered by various persons. R. A. Bradley (1953) gives a fairly extensive bibliography of papers on methods used in sensory testing, listing these under headings which include Paired Comparisons, Rank Order Methods and Scoring Techniques. A recent list of references of scoring methods in sensory difference testing is given by Bradley and Somerville (1955). A comprehensive bibliography of ranking methods in general was compiled by Savage (1953), while references to more recent papers on ranking methods applied to taste testing problems are given in the Bi-Annual Reports on Statistical Methods for Sensory Difference Tests of Food Quality(1) (1953,1954). R. A. Bradley (1955) compared a method of paired comparisons quite widely used in sensory testing with the analysis of variance based on ideal scores. He found that the ideal scoring method was more efficient but remarked that it is not clear that ranking methods are inferior to the scoring methods commonly used.

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(1) Prepared by the Virginia Agricultural Experiment Station of the Virginia Polytechnic Institute under a Research and Marketing Act Contract.

The main advantages of ranking methods are their wide applicability and their computational simplicity. The usual criticism of ranking methods stems from a loss in efficiency to which reference was made in the last paragraph. If an ideal scoring scale is available it could result in more sensitive analyses. An ideal scoring scale in sensory testing would be one whereon observations were (i) on a continuous scale, (ii) independent in probability, (iii) homoscedastic, (iv) such that random fluctuations are normally distributed, and (v) capable of explanation as to location by an additive model. This ideal scale is necessary if methods of analysis of variance are to be applied with strict validity. In practice, scoring scales often depart considerably from the ideal. If several scoring methods are advanced having at least some of the desirable features listed above, it should be possible to choose between the scales on the basis of relative sensitivity or accuracy.

W. G. Cochran (1943) referred to this problem of comparing the sensitivities of scoring scales. He assumed that analysis of variance techniques were applicable and confined his discussion to the case in which all scales measure the same experiment. He indicated that a comparison of the relative sensitivity of two scales should depend both on the experimental errors associated with them and on the magnitudes of the treatment effects in the scales. In the concluding section of his paper he mentioned that a method was available to evaluate the relative sensitivity of two equivalent scales if there are only two treatments (see section 1.3 of this paper). In the case of more than two treatments no method of comparison was available and Cochran suggested that a test of significance of the hypothesis that the parameters of two non-central variance ratios were equal was needed. If one accepts this line of reasoning, the distribution function of two non-central F-variates or F'-variates (the notation F' will in future be used) must be sought and tests of significance formulated.

In most applications the comparison of scales of measurement will refer to variants of the same experiment. We will primarily be concerned with this situation and will generally assume that two or more independent experiments similar in size and design have been conducted. When we do not assume similar experiments, this will be explicitly stated.

It will throughout this paper be assumed that analysis of variance techniques are applicable. The observations scored on two continuous scales (which may be linearly related) will therefore be assumed to have the characteristics of independence, homoscedasticity, normality and additivity listed above.

Suppose for the moment that model I of the analysis of variance holds (though we need not restrict ourselves to this model only) and that randomized block experiments are run. (Other designs may be treated in a similar way.) Let the model be

$$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \quad (i = 1,2,\ldots, m; \; j = 1,2,\ldots, n)$$

where μ refers to the mean effect, τ_i to the effect of the i-th treatment, β_j to the j-th block effect, and ε_{ij} to the random effect in y_{ij} and where $\sum_i \tau_i = 0$ and $\sum_j \beta_j = 0$. Then experimental data scored on the p-th scale (p = 1, 2) will furnish the material for the following analysis of variance table:

Variation	D.F.	Mean Sum of Squares
Mean	1	A
Treatments	m-1	B
Blocks	n-1	C
Error	(m-1)(n-1)	D

It is also known that the expected mean sum of squares of treatments and error [written as E(M.S.T.) and E(M.S.E.)] are given by $[\sigma^2 + n \sum_i \tau_i^2/(m-1)]$ and σ^2, respectively.

It will be assumed that real treatment differences exist. It would be senseless to compare the sensitivities of experiments if no treatment differences are present to be detected.

Under the hypothesis of real treatment effects, $(m-1)B/\sigma^2$ is distributed as a non-central X^2-variate or X'^2-variate with (m-1) degrees of freedom independent of $(m-1)(n-1)D/\sigma^2$, which is distributed as X^2 with (m-1)(n-1) degrees of freedom. The variance ratio B/D is accordingly distributed as F' with (m-1) and (m-1)(n-1) degrees of freedom.

The density function of F' is known. If we write 2a for the degrees of freedom associated with the treatment mean square and 2b for the degrees of freedom associated with the error mean square, the density of F' is given by[2]

$$(1.1)\quad f(F') = e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!B(a+r,b)} \left(\frac{a}{b}\right)^{a+r} (F')^{a+r-1}$$

$$\cdot\ (1 + aF'/b)^{-a-b-r}, \qquad 0 \leq F' \leq \infty.$$

To simplify the writing of the function we set $aF'/b = u$ and obtain

$$(1.2)\quad f(u) = e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!B(a+r,b)}\, u^{a+r-1} (1+u)^{-a-b-r},$$

$$0 \leq u \leq \infty.$$

Here λ is a parameter of non-centrality, which equals $n_o \sum_i \tau_i^2/2\sigma^2$, where n_o is the number of observations in a treatment mean. In the example given above we had

$$E(M.S.T.) = \sigma^2 + n \sum_i \tau_i^2/(m-1),$$

and it easily follows that

$$\lambda = \frac{m-1}{2}\left[\frac{E(M.S.T.)}{E(M.S.E.)} - 1\right] \text{ in this case.}$$

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(2) f(x) is generally used to denote 'density function of x', whatever its form may be. Other symbols may be defined for special functions.

From the above it is obvious that the magnitudes of real treatment differences in relation to the true experimental error associated with a scale, depend on the parameter λ/n_o. If we now define as the more sensitive scale that one which accentuates treatment differences and regard λ/n_o (or λ when we consider similar experiments) as a measure of 'average' sensitivity, it becomes obvious that what is needed in a comparison of the sensitivities of two scoring scales is a test of the hypothesis that the parameters in the distributions of two independent F''s are equal. If, in the case of the similar experiments 1 and 2, the null hypothesis H_o: $\lambda_1 = \lambda_2$, represents the hypothesis of equal sensitivity, we need to know the density function of the statistic F_1'/F_2' (the ratio of two independent F'-variates with parameters λ_1 and λ_2) to be able to conduct tests of significance of the hypothesis. (In the case of experiments differing in size or design we will be interested in testing the hypothesis $\lambda_1/n_1 = \lambda_2/n_2$, where n_1 and n_2 refer to the numbers of observations in treatment means.)

It is proposed to derive the density function of the ratio of two independent F'-variates under the assumptions stated in this section, and to show how percentage points of the function can be calculated. Before doing so, however, it will be useful to consider some properties of the F'-distribution of which future use will be made and to review those results that are available from the literature on this subject.

1.2 The F'-distribution.

The distribution which is today generally associated with the non-central F-distribution was first derived by R. A. Fisher (1928). If we set $\beta^2/2 = \lambda$, $n_1 = 2a$, $n_2 = 2b$ and $R^2 = u/(1 + u)$ in Fisher's (C)-function, it can easily be reduced to the form of equation (1.2).

J. Wishart (1932) considered some properties of this function. Wishart assumed p arrays of the variate y, which is normally distributed with a common variance, and wrote $\overline{y}_i = \sum_i y_i/n_i$ for the mean of the observations in the i-th array and $\overline{y}$ for the general mean. Assuming the number n_i in each array to be the same for all samples, he defined $E^2 = \dfrac{\sum_{i=1}^{p}[n_i(\overline{y}_i-\overline{y})^2]}{\sum_{j=1}^{N}(y_i-\overline{y})^2}$, where $N = \sum_{i=1}^{p} n_i$. Wishart indicated that the distribution of E^2 was that of R^2 in Fisher's (C)-distribution. Writing $E^2 = x$, and observing that $2a = p - 1$ and $2b = N - p$, the density function of x takes the form,

$$f(x) = e^{-\lambda} [B(a,b)]^{-1} x^{a-1} (1 - x)^{b-1} [1 + \frac{(a+b)}{a}(\lambda x) + \frac{(a+b)(a+b+1)}{a(a+1)2!}(\lambda x)^2 + \quad \ldots \quad]$$

$$= e^{-\lambda} [B(a,b)]^{-1} x^{a-1} (1 - x)^{b-1} M(a + b, a; \lambda x) \quad ,$$

where $M(a, b; x) = {}_1F_1(a, b; x)$

$$= 1 + \frac{ax}{b} + \frac{a(a+1)x^2}{b(b+1)2!} + \quad \ldots$$

is the confluent hypergeometric series. This series converges for all finite values of the parameters a and b and the variable x. Applying Kummer's relation(3) $M(a, b; x) = e^x M(b - a, b; - x)$, we obtain for the distribution of x,

$$f(x) = e^{-\lambda} [B(a,b)]^{-1} x^{a-1} (1 - x)^{b-1} e^{\lambda x} M(- b, a; - \lambda x) \quad ,$$

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(3) See, for instance, Erdelyi a.o. (1953a) p. 253.

where $M(-b, a; -\lambda x)$ represents a finite series when b is an integer.

Wishart found methods of evaluating the probability integral $\int_0^{x_o} f(x)dx$. He also derived the first two moments of x using the function, $f(x)$.

Both Fisher and Wishart found, what is easily related to the F'-distribution, by considering distributions of the multiple correlation coefficient. P. C. Tang (1938) used a different approach. He found the distribution of the ratio of two independent X'^2-variates with parameters of non-centrality λ_1 and λ_2. By letting $\lambda_2 \longrightarrow 0$, the F'-distribution is obtainable from the form,

$$p(E^2|\lambda) = \sum_{r=o}^{\infty} \frac{\lambda^r e^{-\lambda}}{r!}[B(f_1/2 + r, f_2/2)]^{-1}(E^2)^{f_1/2+r-1} \cdot (1 - E^2)^{f_2/2-1} .$$

This is the same as the function $f(x)$ considered by Wishart as can be seen by writing x for E^2, $2a$ for f_1, and $2b$ for f_2. Tang showed that this is the power function of the usual analysis of variance test. He investigated methods of evaluating the probability integral and tabled values of E_α^2 with the corresponding values of $P_{II}(\alpha)$, where $P_{II}(\alpha) = \int_0^{E_\alpha^2} p(E^2|\lambda)dE^2$, and α, the significance level of the analysis of variance test, is .05 or .01.

Additional tables were subsequently published by Emma Lehmer (1944) and S. Ura (1954). Charts of the power function for the analysis of variance test, derived from the F'-distribution, were prepared and published by E. S. Pearson and H. O. Hartley (1951).

Both Wishart's and Tang's method of evaluating the probability integral involve a considerable amount of labor. P. B. Patnaik (1949) proposed a simple, though approximate, method of obtaining the integral $\int_0^{F'} f(F')dF'$. In Patnaik's approach it is assumed that F'/k with parameter of non-centrality λ and 2a and 2b degrees of freedom, approximately follows an F-distribution with $2a'$ and 2b degrees of freedom. Then k and a' are found by equating the first two moments of the F- and F'-distributions. Thus $k = (a + \lambda)/a$ and $a' = (a + \lambda)^2/(a + 2\lambda)$. Then

$$\int_0^{F'} f(F')dF' = \int_0^{F'/k} f(F'/k)d(F'/k) \text{ approximately}$$

equals $\int_0^{F'/k} f(F)dF$, where F has the ordinary F-distribution with 2a' and 2b degrees of freedom.

Patnaik also found the characteristic function of the F'-distribution which takes the form of an infinite sum of confluent hypergeometric functions.

Reference should be made to a further property of the F'-distribution. It is well known that an F-variate with 2a and 2b degrees of freedom [written here as F(2a,2b)] has the density function,

$$g[F(2a,2b)] = [B(a,b)]^{-1}(a/b)^a F^{a-1}(1 + aF/b)^{-a-b}$$

$$= [B(a,b)]^{-1} x^{a-1}(1 + x)^{-a-b} ,$$

where $x = aF/b$. It follows in the functional notation of (1.2), that

$$e^{\lambda} f(u) = [B(a,b)]^{-1} u^{a-1}(1 + u)^{-a-b}$$

$$+ \frac{\lambda}{1!}[B(a + 1, b)]^{-1} u^a (1 + u)^{-a-b-1} + \ldots$$

$$= g[F(2a,2b)] + \frac{\lambda}{1!} g[F(2a + 1, 2b)]$$

$$+ \frac{\lambda^2}{2!} g[F(2a + 2, 2b)] + \ldots ,$$

where $u = aF'/b$.

1.3 Results now available.

In two papers, published as far back as 1938 and 1939, results which are related to the problem under consideration, were derived. These were due to P. C. Tang (1938) and E. J. G. Pitman (1939).

It was indicated in the previous section that Tang obtained the density function of the ratio of two independent non-central X^2-variates. This can be regarded as a special case of the density function of the ratio of two independent F''s. If we consider two independent F'-variates both with 2a and 2b degrees of freedom and with parameters of non-centrality λ_1 and λ_2 and if we let $b \longrightarrow \infty$, we obtain the distribution of the ratio of two non-central X^2's with 2a and 2a degrees of freedom and the specified parameters of non-centrality.

Pitman (1939) showed that

$(F/\gamma - 1)/\sqrt{(F/\gamma + 1)^2 - 4r^2F/\gamma}$, follows the distribution of a sample correlation coefficient from (n+1) pairs of observations. Here $\gamma = \sigma_2^2/\sigma_1^2$ is the ratio of the true error variances associated with scales 2 and 1, $F = s_2^2/s_1^2$ is the ratio of the mean squares for error (both based on n degrees of freedom) of the two scales and r is the observed correlation coefficient of the two scales. This result can be used in tests of significance of the hypothesis that the scales 1 and 2, which are assumed equivalent except perhaps for a constant difference, have the same sensitivity, when we have only two treatments. Cochran (1943) indicated how this can be done. With only two treatments, t_1 and t_2, the true sensitivity of the i-th scale ($i = 1,2$) can appropriately be defined as $(\mu_{it_1} - \mu_{it_2})^2/2\sigma_i^2$, where μ_{it} is the true mean of treatment t as measured on the i-th scale. When the scales are equivalent, i.e., when $\mu_{1t} = \mu_{2t}$, the sensitivity of scale 1 relative to scale 2 is measured by $\gamma = \sigma_2^2/\sigma_1^2$ of which $F = s_2^2/s_1^2$, is a sample estimate. In comparing the sensitivities of the two equivalent scales in the case of only two treatments, Pitman's result is therefore applicable.

II: THE DENSITY FUNCTION OF THE RATIO OF TWO INDEPENDENT F''S

2.1 The density function of the ratio of two F''s with equal degrees of freedom.

Considering two independent F''s with the same degrees of freedom we seek the distribution of their ratio.

Let F_1' and F_2' follow the F'-distribution with parameters of non-centrality λ_1 and λ_2, respectively, and both with 2a and 2b degrees of freedom. Assume F_1' independent of F_2' and let $w = F_1'/F_2'$. Since it is easier to work with the variates u and v, where $u = 2aF_1'/2b$ and $v = 2aF_2'/2b$ we write $w = u/v$.

The density functions of u and v as given by (1.2) are

$$f(x) = e^{-\lambda_x} \sum_{r=o}^{\infty} \frac{\lambda_x^r}{r!}[B(a + r, b)]^{-1}x^{a+r-1}(1 + x)^{-a-b-r}$$

for $0 \leq x \leq \infty$, $x = u, v$ and $\lambda_x = \lambda_1, \lambda_2$. Since u and v are independent their joint density has element of probability

$$f(u,v)dudv = e^{-\lambda_1-\lambda_2} \sum_{r=o}^{\infty} \sum_{s=o}^{\infty} \frac{\lambda_1^{\,r}\lambda_2^{\,s}}{r!\ s!}[B(a+r,b)B(a+s,b)]^{-1}$$

$$\cdot\ u^{a+r-1}v^{a+s-1}(1+u)^{-a-b-r}(1+v)^{-a-b-s}dudv\ .$$

If we set $u/v = w$ and $du = vdw$, it follows on integrating termwise with respect to v (which is permissible since the series converges uniformly) that

$$(2.1)\qquad f(w) = e^{-\lambda_1-\lambda_2} \sum_{r} \sum_{s} \frac{\lambda_1^{\,r}\lambda_2^{\,s}}{r!\ s!}[B(a+r,b)B(a+s,b)]^{-1}$$

$$\cdot\ w^{a+r-1} \int_o^{\infty} v^{2a+r+s-1}(1+wv)^{-a-b-r}(1+v)^{-a-b-s}dv\ .$$

The summation over r and s will here and in future be taken to be from 0 to ∞.

Now, let $y = w + (1-w)/(1+v)$, for $w \neq 1$, and transform from v to the new variable y. This gives

$1 + v = (w-1)/(w-y)$, $v = (y-1)/(w-y)$,

$1 + wv = (yw-y)/(w-y)$ and $dv = (w-1)dy/(w-y)^2$.

Therefore

$$(2.2)\quad f(w) = e^{-\lambda_1-\lambda_2} \sum_{r} \sum_{s} \frac{\lambda_1^{\,r}\lambda_2^{\,s}}{r!\ s!}[B(a+r,b)B(a+s,b)]^{-1}$$

$$\cdot\ w^{a+r-1}(w-1)^{-2a-2b-r-s+1}H(w|a,b) \text{ for } w \neq 1,$$

where $H(w|a,b) = \int_1^w (y-1)^{2a+r+s-1}(w-y)^{2b-1}y^{-a-b-r}dy$.

A further transformation yields

$$(2.3) \qquad H(w|a,b) = \int_1^w (y-1)^{2a+r+s-1}(w-1)^{2b-1}$$

$$\cdot\ [1-(y-1)/(w-1)]^{2b-1}y^{-a-b-r}dy \ ,$$

upon writing $(w-1)[1-(y-1)/(w-1)]$ for $(w-y)$.

It can be shown that $f(w)$ is also defined at $w=1$. Consider

$$\lim_{w\to 1} [w^{a+r-1}(w-1)^{-2a-2b-r-s+1}H(w|a,b)] \ .$$

By repeated application of L'Hospital's rule, we obtain easily that this limit equals $B(2a + r + s, 2b)$. Hence

$$(2.4) \qquad \lim_{w\to 1} f(w) = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r\lambda_2^s}{r!\ s!}B(2a+r+s,2b)$$

$$\cdot\ [B(a+r,b)B(a+s,b)]^{-1} \ .$$

By setting $w=1$ in (2.1) we can show that $f(1)$ also equals the expression on the right hand side of (2.4). Hence, $\lim_{w\to 1} f(w) = f(1)$, and (2.2) defines $f(w)$ for all positive values of w. Expressions (2.1) and (2.2) are different forms of the density function of w, the ratio of two independent F's with the same degrees of freedom, defined over the range $0 \leq w \leq \infty$.

The integral appearing in (2.1) can be written in the form of a hypergeometric function, since

$$F(a',b';\ c';\ 1-z) = \Gamma(c')[\Gamma(b')\Gamma(c'-b')]^{-1}$$
$$\int_0^{\infty} t^{b'-1}(1+t)^{a'-c'}(1+zt)^{-a'}dt\ ,$$

for $|\arg z| < \pi$ and real $c' >$ real $b' > 0$.[1] Here $F(a',b';\ c';\ z)$ represents the hypergeometric function which is usually associated with the series

$$1 + \frac{a'b'}{c'}z + \frac{a'(a'+1)b'(b'+1)}{c'(c'+1)}\frac{z^2}{2!} + \ \cdots \quad \text{and}$$

which is often written as ${}_2F_1(a',b';\ c';\ z)$. Therefore,

$$\int_0^{\infty} v^{2a+r+s-1}(1+wv)^{-a-b-r}(1+v)^{-a-b-s}dv$$

$$= B(2a+r+s,2b)F(a+b+r,2a+r+s;2a+2b+r+s;1-w)$$

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(1) See Erdelyi a.o. (1953a) page 115, equation 2.12(5).

and (2.1) gives

$$f(w) = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!\, s!} B(2a+r+s, 2b) \tag{2.5}$$

$$\cdot\ [B(a+r,b)B(a+s,b)]^{-1} w^{a+r-1}$$

$$\cdot\ F(a+b+r, 2a+r+s; 2a+2b+r+s; 1-w)\ ,$$

$$0 \leq w \leq \infty\ .$$

Some further remarks should be made regarding equations (2.2), (2.3) and (2.5). In (2.2), we have 2a, 2b, r and s as integers. Hence, $H(w|a,b)$ can obviously be expressed as an elementary function of w by expanding $(y-1)^{2a+r+s-1}$ and $(w-y)^{2b-1}$ in finite (finite for given r and s) series and integrating termwise. As regards (2.3) we use the binomial expansion of $[1-(y-1)/(w-1)]^{2b-1}$ to write

$$H(w|a,b) = \sum_{j=o}^{2b-1} \frac{(-1)^j}{j!} \Gamma(2b)[\Gamma(2b-j)]^{-1}(w-1)^{2b-j-1}$$

$$\cdot \int_1^w (y-1)^{2a+r+s+j-1} y^{-a-b-r} dy\ .$$

Integrals of the type $\int_1^w (y-1)^n y^{-p} dy \equiv I'(n,p)$, where n is an integer and p is an integer or half-integer, can be reduced to elementary functions of w by using recursion formulae. By writing $I'(n,p) = I'(n-1,p-1) - I'(n-1,p)$ and integrating by parts the following formulae are easily obtained:

$$I'(n,p) = (p-1)^{-1}[- (w-1)^n w^{-p+1} + nI'(n-1,p-1)] , \quad p \neq 1,$$

$$I'(n,p) = (n+1-p)^{-1}[(w-1)^n w^{-p+1} - nI'(n-1,p)] , \quad p \neq n+1,$$

$$I'(n,p) = (p-1)^{-1}[(w-1)^{n+1} w^{-p+1} + (p-n-2)I'(n,p-1)], \quad p \neq 1.$$

Now, when p is a half-integer we can set $p = k + 1/2$ where k is an integer and obtain

$$I'(n,k+1/2) = - [\frac{(w-1)^n}{(k-1/2)w^{k-1/2}}$$

$$+ \frac{n(w-1)^{n-1}}{(k-1/2)(k-3/2)w^{k-3/2}} + \frac{n(n-1)(w-1)^{n-2}}{(k-1/2)(k-3/2)(k-5/2)w^{k-5/2}}$$

$$+ \cdots + \frac{n(n-1) \ldots 2.1}{(k-1/2)(k-3/2) \ldots (k-n+1/2)} I'(0,k-n+1/2)]$$

where $I'(0,k-n+1/2) = \int_1^w y^{-k+n-1/2} dy$

$$= (k-n-1/2)^{-1}(1-w^{-k+n+1/2}) .$$

A similar expression can be obtained for $I'(n,p)$ when p is an integer.

Since the parameters (2a+r+s) and 2a+2b+r+s) of the hypergeometric function appearing in equation (2.5) are integers, F(a+b+r, 2a+r+s; 2a+2b+r+s; 1-w) represents a finite series. The forms in which this series can be written will depend on whether the remaining parameter (a+b+r) is an integer or not. Since r is an integer the two cases (i) (a+b) ~~is~~ not an integer (which is equivalent to the sum of the degrees of freedom of a non-central F being odd), and (ii) (a+b) an integer (i.e., the sum of the degrees of freedom is even) will be considered.

(i) <u>(a+b) not an integer</u>: Since the hypergeometric function F(a,b; c; x) is symmetric in a and b, the degenerate case with a and c as integers and b not an integer, can be considered here. The hypergeometric function $F(2a+r+s,\ a+b+r;\ 2a+2b+r+s;\ 1-w)$ can then be written as[2]

$$A_1(w-1)^{-2a-r-s}F[2a+r+s,1-2b;a-b+s+1;(1-w)^{-1}]$$

$$+\ A_2(1-w)^{1-2a-2b-r-s}w^{b-a-r}$$

$$\cdot\ F(1-2a-r-s,1-a-b-r;b-a-r+1;w)$$

where

$$A_1 = \Gamma(2a+2b+r+s)\Gamma(b-a-r)[\Gamma(2b)\Gamma(a+b+s)]^{-1}\ ,$$

and

$$A_2 = \Gamma(2a+2b+r+s)\Gamma(a-b+r)[\Gamma(2a+r+s)\Gamma(a+b+r)]^{-1}\ .$$

Since the parameters $(1-2b)$, appearing in the first of these hypergeometric functions, and $(1-2a-r-s)$, appearing in the second, are integers, both functions represent finite series. Therefore, the distribution function of w can be written

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(2) See Erdelyi a.o. (1953a) page 72, case 17; also pages 105, 106 and 109.

$$f(w) = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!\, s!}\Big[\Gamma(a+b+r)\Gamma(b-a-r)$$

$$\cdot\ [B(a+r,a+s)]^{-1}[\Gamma(b)]^{-2} w^{a+r-1}(w-1)^{-2a-r-s}$$

$$\cdot\ F[2a+r+s,1-2b;a-b+s+1;(1-w)^{-1}]$$

$$+\ \Gamma(a+b+s)\Gamma(a-b+r)[B(b,b)\Gamma(a+r)\Gamma(a+s)]^{-1}$$

$$\cdot\ (1-w)^{1-2a-2b-r-s} w^{b-1} F(1-2a-r-s,1-a-b-r;b-a-r+1;w)\Big] .$$

(ii) <u>(a+b) an integer</u>: When $(a+b+r)$ is an integer, it can be shown that $F(2a+r+s,\ a+b+r;\ 2a+2b+r+s;\ 1-w)$ can be written in the form[3]

$$(2.6) \qquad (-1)^{b-a+r}[B(a+b+s,a+b+r)\Gamma(2a+r+s)\Gamma(2b)]^{-1}$$

$$\cdot\ D^{a+b+r-1}[w^{2b-1}D^{a+b+s-1}F(1,1;2;1-w)] ,$$

where $F(1,1;2;1-w) = \ln w/(w-1)$ and D is the operator $\frac{d}{dw}$. (Here $\ln w$ is written for $\log_e w$.) Now,

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(3) See Erdelyi a.o. (1953a) pages 102 and 105, especially equations 2.8(20), 2.8(25) and 2.9(2).

$$D^{a+b+s-1}[\ln w/(w-1)]$$

$$= (-1)^{a+b+s-1}[\Gamma(a+b+s)(w-1)^{-a-b-s} \ln w$$

$$- \sum_{t=o}^{a+b+s-1} \Gamma(a+b+s)t^{-1}w^{-t}(w-1)^{-a-b-s+t}] .$$

By applying Leibniz's theorem, viz.,

$$D^{a+b+r-1}(U.V) = \sum_{j=o}^{a+b+r-1} \Gamma(a+b+r)[\Gamma(j+1)\Gamma(a+b+r-j)]^{-1}$$

$$(D^jU)(D^{a+b+r-1-j}V)$$

and setting $U = w^{2b-1}$, $V = D^{a+b+s-1}[\ln w/(w-1)]$ it is possible to evaluate (2.6) which takes the form of a finite series.

2.2 Properties of the density function of the ratio of two independent F''s.

(i) Convergence of the series defined by f(w):

Consider equation (2.2) which gives the distribution function of the ratio of two F''s with parameters λ_1 and λ_2 and 2a, 2b degrees of freedom, and assume $b > a$. (This will correspond with situations encountered in practice.) Restrict w to be greater than one and to have a finite upper bound, and apply the mean value theorem to the integral $H(w|a,b)$. This integral reduces to

$$(w-1)(\zeta-1)^{2a+r+s-1}(w-\zeta)^{2b-1}\zeta^{-a-b-r} ,$$

where $1<\zeta \leq w$. Hence, $H(w|a,b)\leq (w-1)^{2a+2b+r+s-1}$.

Therefore,

$$(2.7) \qquad f(w)\leq e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!\, s!} \cdot [B(a+r,b)B(a+s,b)]^{-1} w^{a+r-1} ,$$

and it is easy to show, by applying a ratio test that this double infinite series converges. Similarly, it can be shown that the double infinite series defined by $f(w)$ converges for $0 \leq w < 1$, and it has already been established that $f(1)$ exists. Therefore, $f(w)$ converges uniformly in the interval $0 \leq w \leq w_o$, for any w_o.

At the same time (2.7) enables us to find an upper bound for $f(w)$ in the following form:

$$f(w) \le e^{-\lambda_1-\lambda_2} w^{a-1} [\Gamma(b)]^{-2} \sum_r \Gamma(a+b+r) [\Gamma(r+1)\,\Gamma(a+r)]^{-1}$$

$$\cdot\ (\lambda_1 w)^r \sum_s \Gamma(a+b+s) [\Gamma(s+1)\,\Gamma(a+b)]^{-1} \lambda_2^{\,s}$$

$$\le e^{-\lambda_1-\lambda_2} w^{a-1} [B(a,b)]^{-2} M(a+b,a;\lambda_1 w) M(a+b,a;\lambda_2)$$

$$\le e^{-\lambda_1(1-w)} w^{a-1} [B(a,b)]^{-2} M(-b,a;-\lambda_1 w) M(-b,a;-\lambda_2),$$

where both confluent hypergeometric functions are finite when b is an integer. For example, when b = 1,

$$f(w) \le e^{-\lambda_1(1-w)} w^{a-1} (a+\lambda_1 w)(a+\lambda_2)\ .$$

(ii) General properties of f(w):

It follows from (2.1) that for any given r and s

$$f(w)=\text{const.}\ w \int_0^\infty (wv)^{a+r-2}(1+wv)^{-a-b-r} v^{a+s+1}(1+v)^{-a-b-s}\,dv$$

$$\le \text{const.}\ w \int_0^\infty v^{a+s+1}(1+v)^{-a-b-s}\,dv, \text{ since } \frac{(wv)^{a+r-2}}{(1+wv)^{a+b+r}} \le 1,$$

$$\le \text{const.}\ w\, B(a+s+2,b-2), \qquad b > 2.$$

$\therefore$ f(0) = 0 for b > 2.

Again, if we consider (2.5), which gives f(w) in the form of hypergeometric functions, it can be shown that $f(\infty) = 0$. When $|z| \to \infty$ and $(a'-b')$ is not an integer the hypergeometric function $F(a',b';c';z)$ can be put in the form

$$k_1 z^{-a'} + k_2 z^{-b'} + O(z^{-a'-1}) + O(z^{-b'-1})$$

where k_1 and k_2 are constants. When $(a'-b')$ is an integer $z^{-a'}$ or $z^{-b'}$ has to be multiplied by ln z.[4] Therefore, when $(a+b)$ is not an integer, we have in (2.5),

$$\lim_{w\to\infty} w^{a+r-1} F(2a+r+s, a+b+r; 2a+2b+r+s; 1-w)$$

$$\leq k_1 \lim_{w\to\infty} w^{a+r-1}(1-w)^{-2a-r-s}$$

$$+ k_2 \lim_{w\to\infty} w^{a+r-1}(1-w)^{-a-b-r}$$

$= 0$, for $(a+s+1) > 0$ and $(b+1) > 0$. A similar result holds when $(a+b)$ is an integer.

In summary:

$$f(0) = 0$$

$$f(1) = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!\, s!} B(2a+r+s, 2b)$$

$$\cdot\, [B(a+r,b)B(a+s,b)]^{-1}$$

$f(\infty) = 0$, for $b > a > 1$.

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(4) See Erdelyi a.o. (1953a) p. 76, equation (9).

From equation (2.1), it follows that

$$\int_0^\infty f(w)dw = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!\, s!} [B(a+r,b)B(a+s,b)]^{-1}$$

$$\cdot \int_0^\infty w^{a+r-1} K(w|a,b)dw$$

where

$$K(w|a,b) = \int_0^\infty v^{2a+r+s-1}(1+wv)^{-a-b-r}(1+v)^{-a-b-s}dv \ .$$

Since $K(w|a,b)$ can be transformed to $(w-1)^{-2a-2b-r-s+1}$ $H(w|a,b)$, it follows from the work done in part (i) of this section that $K(w|a,b)$ converges for $0 \leq w \leq w_0$. Again,

$$w^{a+r-1}K(w|a,b) = \text{constant } w^{a+r-1}$$

$$\cdot F(2a+r+s, a+b+r; 2a+2b+r+s; 1-w)$$

$$\leq k_1 O(w^{-a-s-1}) + k_2 O(w^{-b-1}) \ ,$$

when w is large. Hence, $\int_0^\infty w^{a+r-1}K(w|a,b)dw$ converges for $(a+s) > 0$ and $b > 0$. Since both a and b are positive it follows that we can interchange the order of integration, above.

$$(2.8)\quad \int_0^{\infty} w^{a+r-1} K(w|a,b)\,dw$$

$$= \int_0^{\infty} v^{2a+r+s-1}(1+v)^{-a-b-s} \int_0^{\infty} w^{a+r-1}(1+wv)^{-a-b-r}\,dw\,dv$$

$$= \int_0^{\infty} v^{2a+r+s-1}(1+v)^{-a-b-s} B(a+r,b)\, v^{-a-r}\,dv$$

$$= B(a+r,b)B(a+s,b) \ .$$

Hence

$$\int_0^{\infty} f(w)\,dw = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!\ s!} = 1 \ .$$

Since w represents the ratio of two non-central F's with the same degrees of freedom, $Pr[\ F_1' \leq F_2'\ ]$ $= Pr\ [\ F_2' \leq F_1'\ ] = 1/2$ when $\lambda_1 = \lambda_2$. Assuming $\lambda_1 = \lambda_2$ we, therefore, have $Pr[\ w \leq 1\ ] = 1/2$, i.e., $\int_0^1 f(w)\,dw = 1/2$.

(iii) Moments.

By using the same procedure as in the derivation of result (2.8) it can be shown that

$$\int_0^\infty w^{a+k+r-1}\int_0^\infty v^{2a+r+s-1}(1+wv)^{-a-b-r}(1+v)^{-a-b-s}dvdw$$

$$= B(a-k+s,b+k)B(a+k+r,b-k) \quad ; \; b > k, a+s > k \; .$$

Therefore,

$$\mu_k' = \int_0^\infty f(w)w^k dw$$

$$= e^{-\lambda_1-\lambda_2}\sum_r\sum_s \frac{\lambda_1^r\lambda_2^s}{r!\;s!}[B(a+r,b)B(a+s,b)]^{-1}\int_0^\infty w^{a+k+r-1}$$

$$\cdot\int_0^\infty v^{2a+r+s-1}(1+wv)^{-a-b-r}(1+v)^{-a-b-s}dvdw$$

$$= \Gamma(b-k)\Gamma(b+k)[\Gamma(b)]^{-2}e^{-\lambda_1-\lambda_2}\sum_r\sum_s\frac{\lambda_1^r\lambda_2^s}{r!\;s!}$$

$$\cdot\,\Gamma(a+k+r)\Gamma(a-k+s)[\Gamma(a+r)\Gamma(a+s)]^{-1}$$

$$= \Gamma(b-k)\Gamma(b+k)[\Gamma(b)]^{-2}\Gamma(a+k)\Gamma(a-k)[\Gamma(a)]^{-2}$$

$$\cdot\, e^{-\lambda_1-\lambda_2}M(a+k,a;\lambda_1)M(a-k,a;\lambda_2) \quad ; \; a > k, \; b > k \; .$$

By using Kummer's transformation the confluent hypergeometric functions $M(a+k,a;\lambda_1)$ and $M(a-k,a;\lambda_2)$ can be written as $e^{\lambda_1}M(-k,a;-\lambda_1)$ and $e^{\lambda_2}M(k,a;-\lambda_2)$, respectively, resulting in some further simplification of the expression for μ_k'. These confluent hypergeometric functions can be expressed as finite combinations of elementary functions of λ_1 and λ_2, respectively.[5]

In particular

$$\mu_1' = ab(a-1)^{-1}(b-1)^{-1}(1+\lambda_1/a)$$

$$\cdot\ (1-\lambda_2/a+\lambda_2^{\,2}/a(a+1) - \dots\ ),$$

$$a > 1, \quad b > 1.$$

For $a = 2$, this gives after some simplification

$$\mu_1' = 2b(b-1)^{-1}(1+\lambda_1/2)\lambda_2^{-1}(1-e^{-\lambda_2}).$$

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(5) Refer Erdelyi a.o. (1953a) p. 268.

$$\mu_2' = a(a+1)b(b+1)[(a-1)(a-2)(b-1)(b-2)]^{-1}$$

$$\cdot [1+2\lambda_1/a+\lambda_1^2/a(a+1)]$$

$$\cdot [1-2\lambda_2/a+3\lambda_2^2/a(a+1) - \dots] , a > 2, b > 2 .$$

When a and/or b is ≤ 1 no moments exist. When a and/or b is ≤ 2 only the first moment is defined.

2.4 The density function of the ratio of two F''s with different degrees of freedom.

When the restriction of similar experiments is lifted, that is, when it is no longer assumed that the two independent variates, F_1' and F_2', have the same degrees of freedom, the density function of $z = F_1'/F_2'$ can be found in a manner similar to that used above.

Let F_1' and F_2' be independent, non-central F-variates with degrees of freedom 2a, 2b and 2α, 2β and parameters of non-centrality λ_1 and λ_2, respectively. The distributions of $u = aF_1'/b$ and $v = \alpha F_2'/\beta$ can be obtained from (1.2) and their joint element of probability, will be

$$f(u,v)dudv = e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1{}^r \lambda_2{}^s}{r!\ s!}$$

$$\cdot\ [B(a+r,b)B(\alpha+s,\beta)]^{-1} u^{a+r-1} v^{\alpha+s-1}$$

$$\cdot\ (1+u)^{-a-b-r}(1+v)^{-\alpha-\beta-s} dudv\ .$$

Now, write $z = F_1'/F_2' = cu/v$, where $c = b\alpha/a\beta$. After substituting in the above expression and integrating out the remaining variable, we obtain

$$(2.9)\quad f_1(z) = c^{-a}\ e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1{}^r \lambda_2{}^s}{r!\ s!}[B(a+r,b)B(\alpha+s,\beta)]^{-1} c^{-r}$$

$$\cdot\ z^{a+r-1} \int_0^{\infty} v^{a+\alpha+r+s-1}(1+zv/c)^{-a-b-r}(1+v)^{-\alpha-\beta-s} dv\ ,$$

$$0 \leq z \leq \infty\ .$$

This can again be written in the form of hypergeometric functions.

Alternately, upon substituting $y-z/c = (1-z/c)(1+v)^{-1}$, the distribution function is obtained in the form

$$f_1(z) = c^{\alpha} \quad e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1{}^r \lambda_2{}^s}{r!\ s!} [B(a+r,b)B(\alpha+s,\beta)]^{-1} c^s$$

$$\cdot\ z^{a+r-1}(z-c)^{-\alpha-a-\beta-b-r-s+1} \int_1^{z/c} (y-1)^{a+\alpha+r+s-1}$$

$$\cdot\ (z-cy)^{b+\beta-1} y^{-a-b-r} dy \ .$$

In this form we are again concerned with the evaluation of integrals, similar to $H(w|a,\alpha,b,\beta)$ defined in (2.2). The problem of obtaining $f_1(z)$ in terms of elementary functions of z is therefore quite similar to the problem we had in the case of the function, f(w).

III: THE DENSITY FUNCTION OF THE RATIO OF TWO INDEPENDENT F'S

3.1 The density function, g(w), of the ratio of two F's with equal degrees of freedom.

Following Patnaik (1949) it will be assumed that the F'-distribution can effectively be approximated by an ordinary F-distribution. The density function, g(w), of the ratio of two independent F's with equal degrees of freedom will be derived and regarded as an approximation to the distribution of the ratio of two independent F''s with equal degrees of freedom and the same parameters of non-centrality. The extent to which $\int_0^{w_o} g(w)dw$ can be used to obtain approximations to comparable probability integrals of the distribution of the ratio of two independent F''s will subsequently be investigated.

Consider equation (2.1) and assume $\lambda_1 = \lambda_2 = \lambda$. When $\lambda \longrightarrow 0$ the density function, f(w), becomes

$$(3.1) \qquad g(w) = w^{a-1}[B(a,b)]^{-2} \int_0^{\infty} v^{2a-1}(1+wv)^{-a-b}(1+v)^{-a-b}dv \,,$$

for $0 \leq w \leq \infty$, where $w = F_1/F_2$ is the ratio of two independent F's both with 2a and 2b degrees of freedom.

This density function can again be written as a hypergeometric function similar to (2.5), viz.,

$$(3.2) \qquad g(w) = B(2a,2b)[B(a,b)]^{-2}w^{a-1}$$

$$\cdot F(a+b,2a;2a+2b;1-w) .$$

When (a+b) is not an integer [2a and (2a+2b) are integers], the properties of hypergeometric functions given in Section 2.1 can again be applied to write (3.2) in the form,

$$(3.3) \quad g(w) = \Gamma(2a)\Gamma(a+b)[\Gamma(a)\Gamma(b)]^{-2}$$

$$\cdot [\Gamma(b-a)w^{a-1}(w-1)^{-2a}F(2a,1-2b;a-b+1;(1-w)^{-1})$$

$$+ \Gamma(a-b)w^{b-1}(1-w)^{-2a-2b+1}$$

$$\cdot F(1-2a,1-a-b;b-a+1;w)] .$$

Both of the hypergeometric series appearing in this expression are finite since (1-2b) and (1-2a) are negative integers or zero. When (a+b) is an integer, (3.2) takes the form,

$$(3.4) \qquad g(w) = (-1)^{b-a}[\Gamma(a)\Gamma(b)]^{-2}w^{a-1}$$

$$\cdot\; D^{a+b-1}[w^{2b-1}D^{a+b-1}(\ln w/(w-1)\,)\,] ,$$

as in Section 2.1. Here D is again the operator d/dw.

The density function, $g(w)$, can also be obtained in a form similar to (2.2) by setting $y-w = (1-w)/(1+v)$ in (3.1). This gives

$$(3.5) \qquad g(w) = [B(a,b)]^{-2}w^{a-1}(w-1)^{1-2a-2b}H'(w) ,$$

where $H'(w) = \int_1^w (y-1)^{2a-1}(w-y)^{2b-1}y^{-a-b}dy$ and $w \neq 1$. It can again be shown that $g(w)$ is defined also for $w = 1$, and that (3.5) is the density function of w for $0 \leq w \leq \infty$.

3.2 Special cases of the function, g(w).

By applying (3.3), (3.4) or (3.5), $g(w)$ can always be obtained as a finite sum of elementary functions of w. For example, when $a = 1/2$ and $b = 1$, either (3.3) or (3.5) will give $g(w) = [2\sqrt{w}\,]^{-1}[\sqrt{w}+1]^{-2}$. When $a = 1/2$ and $b = 3/2$, either (3.4) or (3.5) will give after some simplification

$$g(w) = 4\pi^{-2}w^{-1/2}(w-1)^{-3}(w^2-1-2w\ln w) .$$

In a few special cases the hypergeometric function $F(a+b,2a;2a+2b;1-w)$ can be equated to other well known functions. Thus, the relationship $\ln w = (w-1)F(1,1;2;1-w)$ immediately gives us the result $g(w) = \pi^{-2}\ln w/(w-1)\sqrt{w}$ when $a = b = 1/2$.

Using the relationships[(1)]

$$F(a',a'-1/2;2a';1-w)$$

$$= \sqrt{w}\,F(a',a'+1/2;2a';1-w)$$

$$= 2^{2a'-1}(1+\sqrt{w})^{1-2a'}$$

it is possible to write down $g(w)$ at once when $b = a \pm 1/2$. For example, when $b = a + 1/2$, we have from (3.2) that

$$g(w) = B(2a,2a+1)[B(a,a+1/2)]^{-2}w^{a-1}$$

$$\cdot\ F(2a+1/2,2a;4a+1;1-w)\ ,$$

and upon applying the appropriate relationship given above, it follows easily that

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(1) See Erdelyi a.o. (1953a) p. 101.

$$g(w) = B(2a,2a+1)[B(a,a+1/2)]^{-2}$$

$$\cdot\ w^{a-1}2^{4a}(1+\sqrt{w})^{-4a}\ .$$

By using the relations between contiguous hypergeometric series in the case where two parameters are constant, viz.,

$$(a-b\ +\ aw-bw)F(a+b,2a;2a+2b;1-w)$$

$$=\ 2awF(a+b,2a+1;2a+2b;1-w)$$

$$-2bF(a+b,2a-1;2a+2b;1-w)$$

together with the foregoing relationships, $g(w)$ can also be obtained when $b = a \pm 3/2$, $b = a \pm 5/2$ and $b = a \pm 7/2$ without much trouble. For example, after obtaining

$$F(2a+1/2,2a;4a+1;1-w)\ =\ 2^{4a}(1+\sqrt{w})^{-4a} \quad \text{and}$$

$$F(2a+1/2,2a+1;4a+1;1-w)\ =\ 2^{4a}w^{-1/2}(1+\sqrt{w})^{-4a}\ ,$$

we find, after some simplification,

$$F(2a+1/2,2a-1;4a+1;1-w)$$

$$= (2a+1)^{-1}2^{4a-1}(1+\sqrt{w})^{-4a}(w-1-4a\sqrt{w}) ,$$

by using the relations between contiguous hypergeometric functions. Now,

$$g(w) = B(2c-1,2c+1)[B(c-1/2,c+1)]^{-2}w^{c-3/2}$$

$$\cdot F(2c+1/2,2c-1;4c+1;1-w)$$

$$= B(2c-1,2c+1)[B(c-1/2,c+1)]^{-2}w^{c-3/2}$$

$$\cdot [(2c+1)^{-1}2^{4c-1}(1+\sqrt{w})^{-4c}(w-1-4c\sqrt{w})] ,$$

when we take $a = c-1/2$ and $b = c+1$, i.e., $b = a+3/2$.

By setting $x = 1/2\cdot\ln w$ we found it possible to derive the following general expression for $p(x)dx$ in the case where $a = 1/2$ and $(a+b)$ is an integer:

$$p(x)dx = [\frac{\cosh x}{\pi}\sum_{i=1}^{b-1/2}(-1)^{i+1}\frac{B(b+1/2,b+1/2-i)}{B(b,b+1-i)}\sinh^{-2i}x$$

$$+\frac{(-1)^{b-1/2}2x}{\pi B(1/2,b)\sinh^{2b}x}]dx .$$

When a = 1/2 and b = 1/2 this readily gives

$$p(x)dx = 2x\pi^{-2}\sinh^{-1}x dx\ , \quad \text{or}$$

$$g(w)dw = \pi^{-2}\ln w/\sqrt{w}(w-1)dw\ , \quad \text{as before.}$$

3.3 Properties of the density function, g(w).

(i) Symmetry:

A useful property of g(w) is its symmetry with respect to a and b. Write $g(w|a,b)$ for the distribution of the ratio of two independent F's with 2a and 2b degrees of freedom in numerator and denominator, respectively. Then, by (3.2)

$$(3.6)\quad g(w|\ b,a) = B(2b,2a)[B(b,a)]^{-2}w^{b-1}$$

$$\cdot\ F(a+b,2b;2a+2b;1-w)$$

$$= B(2a,2b)[B(a,b)]^{-2}w^{a-1}$$

$$\cdot\ F(a+b,2a;2a+2b;1-w)$$

$$= g(w|a,b)\ ,$$

upon application of a well-known property of hypergeometric functions given, for example, in Erdelyi a.o. (1953a) p. 105.

(ii) General properties of $g(w)$.

Since $F(a+b,2a;2a+2b;1)$
$= \Gamma(2a+2b)\Gamma(b-a)[\Gamma(a+b)\Gamma(2b)]^{-1}$, it follows immediately from (3.2) that $g(0) = 0$ for $b > a > 1$. When $a = 1$ and $b > 1$, $g(0) = b/(b-1)$.

It is also obvious from equation (3.1) that $g(1) = B(2a,2b)[B(a,b)]^{-2}$. Again, by following a line of reasoning similar to that given in the case of the distribution of the ratio of two F''s, we can show that $g(\infty) = 0$ for positive a and b.

From (3.1) it follows that

$$\int_0^\infty g(w)dw$$

$$= [B(a,b)]^{-2}\int_0^\infty w^{a-1}\int_0^\infty v^{2a-1}(1+wv)^{-a-b}(1+v)^{-a-b}dvdw$$

$$= [B(a,b)]^{-2}\int_0^\infty v^{2a-1}(1+v)^{-a-b}\int_0^\infty w^{a-1}(1+wv)^{-a-b}dwdv$$

$$= [B(a,b)]^{-2}\int_0^\infty v^{2a-1}(1+v)^{-a-b}B(a,b)v^{-a}dv$$

$$= [B(a,b)]^{-1}\int_0^\infty v^{a-1}(1+v)^{-a-b}dv$$

$$= 1 .$$

Since $Pr[F_1 \leq F_2] = Pr[F_2 \leq F_1] = 1/2$, it follows that $\int_0^1 g(w)dw = Pr[w \leq 1] = 1/2$. This result can be proved mathematically in the following way:

$$[B(a,b)]^2\int_0^1 g(w)dw$$

$$= \int_0^\infty v^{a-1}(1+v)^{-a-b}\int_0^1 (wv)^{a-1}(1+wv)^{-a-b}vdwdv$$

$$= \int_0^\infty v^{a-1}(1+v)^{-a-b}\int_0^{v/(1+v)} y^{a-1}(1-y)^{b-1}dydv ,$$

where $y = wv/(1+wv)$. [The change in the order of integration is permissible here and in the previous paragraph in the same way as it was permissible under similar circumstances in Section 2.2, part (ii).] Upon expanding $(1-y)^{b-1}$ in a binomial and integrating out the y's we obtain,

$$\int_0^\infty v^{a-1}(1+v)^{-a-b}\left[\frac{y^a}{a} - \frac{(b-1)y^{a+1}}{1!(a+1)} + \dots\right]_0^{v/(1+v)} dv$$

$$= \frac{1}{a}B(2a,b) - \frac{(b-1)}{(a+1)}B(2a+1,b) + \dots$$

$$= a^{-1}B(2a,b)\,{}_3F_2\left[{2a,a,-b+1 \atop a+1,2a+b};1\right] ,$$

where ${}_3F_2(a_1,a_2,a_3;b_1,b_2;x)$ represents the generalized hypergeometric function with five parameters. In our case the parameters are such that $b_1 = 1+a_1-a_2 = 1+a$, and $b_2 = 1+a_1-a_3 = 2a+b$, and Dixon's theorem [see Erdelyi a.o. (1953a) p. 189] can be applied. The above generalized hypergeometric function is equal to $[\Gamma(a+1)]^2\Gamma(2a+b)\Gamma(b)[\Gamma(2a+1)]^{-1}[\Gamma(a+b)]^{-2}$ and we obtain,

$$\int_0^1 g(w)dw = a^{-1}\Gamma(2a)[\Gamma(a+1)]^2[\Gamma(a)]^{-2}[\Gamma(2a+1)]^{-1}$$

$$= 1/2 \; .$$

It is obvious that $g(w)$ is continuous for $0 \leq w \leq \infty$, and it can easily be shown, for any specific values of a and b, that the function has a single maximum. The maximum value lies at some value w' where $0 \leq w' < 1$ when $a \geq 1$.

(iii) Moments.

The moments about zero can easily be obtained. If we write μ_k' for the k-th moment about zero, it follows from (3.1) that

$$[B(a,b)]^2 \mu_k'$$

$$= \int_0^{\infty} w^{a+k-1} \int_0^{\infty} v^{2a-1}(1+wv)^{-a-b}(1+v)^{-a-b} dv dw$$

$$= \int_0^{\infty} v^{2a-1}(1+v)^{-a-b} \int_0^{\infty} w^{a+k-1}(1+wv)^{-a-b} dw dv$$

$$= \int_0^{\infty} v^{a-k-1}(1+v)^{-a-b} B(a+k,b-k) dv$$

$$= B(a+k,b-k)B(a-k,b+k) \ , \ a > k \ , \ b > k \ .$$

$$\mu_k' = \Gamma(a+k)\Gamma(b-k)\Gamma(a-k)\Gamma(b+k)[\Gamma(a)\Gamma(b)]^{-2} \ ,$$

where $a > k$, $b > k$. In particular,

$$\mu_1' = ab/(a-1)(b-1) \ , \ a > 1 \ , \ b > 1 \ , \text{ and}$$

$$\mu_2' = ab(a+1)(b+1)[(a-1)(b-1)(a-2)(b-2)]^{-1} \ , \ a > 2 \ , \ b > 2 .$$

Hence,

$$\mathrm{Var}(w) = ab(2a^2b+2ab^2-a^2-b^2-4ab+1)$$
$$\cdot (a-1)^{-2}(b-1)^{-2}(a-2)^{-1}(b-2)^{-1} \ .$$

It is of interest to compare the first few moments of the functions g(w) and f(w) since it was suggested in Section 3.1 that f(w) can be approximated by g(w).

Here $f(w)$ is the density function of the ratio of two F''s both with $2a$ and $2b$ degrees of freedom and with parameter of non-centrality λ, and $g(w)$ is the density function of the ratio of two ordinary F's both with $2a' = 2(a+\lambda)^2/(a+2\lambda)$ and $2b$ degrees of freedom. We will write $\mu_k'(f)$ and $\mu_k'(g)$, respectively, for the k-th moment about zero of the functions $f(w)$ and $g(w)$ defined above.

It follows from the preceding work and Section 2.2 part (iii) that,

$$\mu_1'(g) = b(b-1)^{-1}(a-1)^{-1}(a+\lambda)[1 + \frac{\lambda(a+\lambda-1)}{(a-1)(a+\lambda)}]^{-1}$$

$$\doteq b(b-1)^{-1}(a-1)^{-1}(a+\lambda)(1 + \lambda/a)^{-1} ,$$

for large a and λ considerably smaller than a.

$$\mu_1'(f) = b(b-1)^{-1}(a-1)^{-1}(a+\lambda)$$

$$\cdot (1 - \lambda/a + \lambda^2/a(a+1) \ldots)$$

$$\doteq b(b-1)^{-1}(a-1)^{-1}(a+\lambda)(1 + \lambda/a)^{-1}$$

for large a and $\lambda < a$. Hence $\mu_1'(g)$ and $\mu_1'(f)$ will be approximately equal, at least for large degrees of freedom.

Again,

$$\mu_2'(g) = b(b+1)(b-1)^{-1}(b-2)^{-1}(a-1)^{-1}(a-2)^{-1}$$

$$\cdot\ (a^2+2a\lambda+\lambda^2+a+2\lambda)$$

$$\cdot\ [1+2\lambda/a+\lambda^2/a^2+(3a\lambda-4\lambda^2)/a^2(a-1)(a-2)]^{-1}$$

$$\doteq b(b+1)(b-1)^{-1}(b-2)^{-1}(a-1)^{-1}(a-2)^{-1}$$

$$\cdot\ (a^2+2a\lambda+\lambda^2+a+2\lambda)(1+\lambda/a)^{-2}\ ,$$

for large a and $a > \lambda$.

$$\mu_2'(f) = b(b+1)(b-1)^{-1}(b-2)^{-1}(a-1)^{-1}(a-2)^{-1}$$

$$\cdot\ (a^2+2a\lambda+\lambda^2+a+2\lambda)$$

$$\cdot\ [1-2\lambda/a+3\lambda^2/a(a+1) - \dots]\ ,$$

which approximates $\mu_2'(g)$ for large a.

3.4 The density function of the ratio of two independent F's with different degrees of freedom.

When it is no longer assumed that the two independent F-variates have the same degrees of freedom, the density function of their ratio can be obtained from (2.9) by letting λ_1 and $\lambda_2 \to 0$. If the two F's are F_1 and F_2 with 2a, 2b and 2α, 2β degrees of freedom, respectively, and we write $z = F_1/F_2$, we obtain

$$(3.7) \qquad g_1(z) = [B(a,b)B(\alpha,\beta)]^{-1} z^{a-1} c^{-a} \cdot \int_0^{\infty} v^{a+\alpha-1}(1+v)^{-\alpha-\beta}(1+zv/c)^{-a-b}dv,$$

where $c = b\alpha/a\beta$ and $0 \leq z \leq \infty$. The integral appearing here is of the same form as the one appearing in (3.1).

Following Patnaik (1949) it will be assumed that the variate F_1' with parameter λ_1 and 2a, 2b degrees of freedom can be approximated by $(a+\lambda_1)F_1/a$ with $2a' = 2(a+\lambda_1)^2/(a+2\lambda_1)$ and 2b degrees of freedom. Similarly F_2' with parameter λ_2 and 2α, 2β degrees of freedom can be approximated by $(\alpha+\lambda_2)F_2/\alpha$ with $2\alpha' = 2(\alpha+\lambda_2)^2/(\alpha+2\lambda_2)$ and 2β degrees of freedom. We will, therefore, assume that the distribution of F_1'/F_2' can be approximated by the distribution of $(a+\lambda_1)\alpha F_1/(\alpha+\lambda_2)aF_2 = (a+\lambda_1)\alpha z/(\alpha+\lambda_2)a$, with which we will now associate the degrees of freedom $2a'$, 2b and $2\alpha'$, 2β.

IV: THE EVALUATION OF PROBABILITY INTEGRALS OF THE DISTRIBUTIONS OF THE RATIOS OF TWO F′'S AND TWO F'S

4.1 The probability integral, $\int_0^{w_o} f(w)dw$.

Consider equation (2.1), the distribution of the ratio of two independent F′'s both with 2a and 2b degrees of freedom and with parameters of non-centrality λ_1 and λ_2. From integration between the limits 0 and w_o and an interchange of the order of integration it follows that the probability integral,

$$(4.1) \int_0^{w_o} f(w)dw$$

$$= e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!\, s!}[B(a+r,b)B(a+s,b)]^{-1}$$

$$\cdot \int_0^{\infty} v^{a+s-1}(1+v)^{-a-b-s} \int_0^{w_o} (wv)^{a+r-1}(1+wv)^{-a-b-r} v\,dw\,dv .$$

We now set $x = wv/(1+wv)$, $dx = vdw/(1+wv)^2$ and obtain

$$(wv)^{a+r-1}(1+wv)^{-a-b-r}d(wv) = x^{a+r-1}(1-x)^{b-1}dx .$$

Upon expanding $(1-x)^{b-1}$ in a binomial series and integrating out x between the limits 0 and $w_o v/(1+w_o v)$, we obtain

$$(4.2) \quad \int_0^{w_o} f(w)dw$$

$$= e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1{}^r \lambda_2{}^s}{r!\, s!} [B(a+r,b)B(a+s,b)]^{-1}$$

$$\cdot \int_0^{\infty} v^{a+s-1}(1+v)^{-a-b-s}(w_o v)^{a+r}(1+w_o v)^{-a-r}$$

$$\cdot \left[\frac{1}{a+r} - \frac{(b-1)\, w_o v}{(a+r+1)(1+w_o v)} + \frac{(b-1)(b-2)(w_o v)^2}{2!(a+r+2)(1+w_o v)^2} \cdots \right]dv \; .$$

When b is an integer the series in square brackets will be finite. It will obviously be easier to evaluate the probability integral for values of b which are integers. In practice it is also simpler to evaluate this integral for two adjacent values of b which are integers and interpolate, whenever b is a half-integer. It can be shown that the value of the probability integral for $b = n+1/2$, where n is an integer, is bounded by the values for $b = n$ and $b = n+1$,[1] and interpolation should, therefore, always be possible.

The evaluation of $\int_0^{w_o} f(w)dw$ will be considered for values of b which are integers, only. The

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(1) By differentiating $\int_0^{w_o} f(w)\, dw$ or $1 - \int_{w_o}^{\infty} f(w)dw$ with respect to b it can be shown that the probability integral is a monotone function of b.

probability integral can then be found by (a) evaluating integrals of the type,

$$K_1(w_o|a,b) = \int_o^{\infty} v^{2a+s+r+j-1}(1+v)^{-a-b-s}(1+w_o v)^{-a-r-j}dv \ ,$$

$(j = 1,2,3 \ldots)$, as indicated by (4.2), and then by summing linear combinations of these over all values of r and s, (b) first eliminating the summation over r and s, (i) through an approximation method or (ii) by applying properties of confluent hypergeometric and exponential functions, and then evaluating integrals of a different type. These methods will be considered in more detail.

<u>(a)</u>. $K_1(w_o|a,b)$ can be written in the form of a hypergeometric function,[2] viz.,

$$\Gamma(2a+r+s+j)\Gamma(b)[\Gamma(2a+b+r+s+j)]^{-1}$$

$$F(a+r+j,2a+r+s+j;2a+b+r+s+j;1-w_o) \ .$$

Tables of the hypergeometric function--if these were available--could then be used to evaluate the

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(2) Again see Erdelyi a.o. (1953a) p. 115.

probability integral. Alternatively, by setting $y-w_o = (1-w_o)/(1+v)$, $1+v = (w_o-1)/(w_o-y)$, $dv = (w_o-1)dy/(w_o-y)^2$ we see that

$$K_1(w_o|a,b) = (w_o-1)^{-2a-b-r-s-j+1} \int_1^w (y-1)^{2a+r+s+j-1}$$

$$\cdot \ (w_o-y)^{b-1} y^{-a-r-j} dy$$

$$= H_1(w_o|a,b) \ ,$$

and that the recursion formulae given in Section 2.1 can be used to evaluate these integrals. Since the intergrand in $K_1(w_o|a,b)$ decreases as r and/or s increase(s), for all values of v, it is obvious that the integrals will become smaller with increasing r and s. By repeating the arguments used in Section 2.2, part (i), we can also show that the series defined by (4.2) converges. Therefore, the summation of a finite number of terms, where these terms consist of linear combinations of the integrals $K_1(w_o|a,b)$ will give the value of the probability integral to a given degree of accuracy. A large number of terms will, in most instances, have to be taken to attain 2 or 3 decimal accuracy.

<u>(b) (i)</u>. The summation over r and s in equations (4.1) and (4.2) can be eliminated and the probability integral simplified by approximating it by $\int_0^{w_0} g(w)dw$, where g(w) is the distribution of the ratio of two independent F's both with $2(a+\lambda)^2/(a+2\lambda)$ and 2b degrees of freedom. Methods of evaluating $\int_0^{w_0} g(w)dw$ will be considered in the next section. It should be remarked here that this approximation will improve with increasing degrees of freedom. In a few cases which have been investigated, the approximation was found to be quite good even for small a and b. The results obtained are given in Table I, which gives the values of $\int_0^{w_0} g(w)dw$ and $\int_0^{w_0} f(w)dw$ in columns (5) and (6), respectively. Unfortunately, limitations in time and labor made it impossible to evaluate the latter probability integral with great accuracy and the values given in the last column are no more than bounds to the true value of the integral.

TABLE I

Values of the Probability Integrals
$\int_0^{w_o} g(w)dw$ and $\int_0^{w_o} f(w)dw$ for Certain
Values of a, b, λ and w_o

(1) a	(2) b	(3) λ	(4) w_o	(5) $\int_0^{w_o} g(w)dw$	(6) $\int_0^{w_o} f(w)dw$
1	1	8	23.4	.950	.945-.950
1	2	4	11.0	.950	.935-.940
1	2	6	9.6	.950	.940-.945
1	2	8	8.8	.950	.940-.945
2	2	4	9.8	.950	.945-.950
2	2	6	8.9	.950	.945-.950
2	2	8	8.4	.950	.945-.950
1	3	4	8.3	.950	.940-.945
1	3	6	7.0	.950	.940-.945
1	3	8	6.3	.950	.940-.945

(b) (ii). When b is small (say $b \leq 6$) the following method which does away with the summation over r and s can be used to obtain values of $\int_0^{w_o} f(w)dw$ to any given degree of accuracy:

Write equation (4.2) in the form

$$\int_0^{w_o} f(w)dw$$

$$= e^{-\lambda_1-\lambda_2}[\Gamma(b)]^{-2}\int_0^{\infty} v^{a-1}(1+v)^{-a-b}(w_o v)^a(1+w_o v)^{-a}$$

$$\cdot \sum_s \left(\frac{\lambda_2 v}{1+v}\right)^s \frac{\Gamma(a+s+b)}{s!\ (a+s)} \sum_r \left(\frac{\lambda_1 w_o v}{1+w_o v}\right)^r \frac{\Gamma(a+r+b)}{r!\ (a+r)}$$

$$\cdot \left[\frac{1}{a+r} - \frac{(b-1)(w_o v)}{(a+r+1)(1+w_o v)} + \ldots\right]dv \ .$$

Observe that

$$\sum_{s=o}^{\infty} \left(\frac{\lambda_2 v}{1+v}\right)^s \frac{\Gamma(a+s+b)}{s!\ (a+s)}$$

$$= \Gamma(a+b)[\Gamma(a)]^{-1}M(a+b,a;\frac{\lambda_2 v}{1+v})$$

$$= \Gamma(a+b)[\Gamma(a)]^{-1}\exp\left(\frac{\lambda_2 v}{1+v}\right)M(-b,a;-\frac{\lambda_2 v}{1+v}) \ ,$$

with the application of Kummer's relation. This confluent hypergeometric series is finite.

Again,

$$\sum_{r=o}^{\infty}\left(\frac{\lambda_1 w_o v}{1+w_o v}\right)^r \frac{\Gamma(a+r+b)}{r!\Gamma(a+r)}\left[\frac{1}{(a+r)} - \frac{(b-1)w_o v}{(a+r+1)(1+w_o v)} + \cdots\right.$$

$$\left.\cdots \pm \frac{1}{(a+r+b-1)}\left(\frac{w_o v}{1+w_o v}\right)^{b-1}\right]$$

$$= \sum_r \left(\frac{\lambda_1 w_o v}{1+w_o v}\right)^r \frac{1}{r!}[(a+r+1) \cdots (a+r+b-1)$$

$$- (b-1)(a+r)(a+r+2) \cdots (a+r+b-1)\left(\frac{w_o v}{1+w_o v}\right) + \cdots]$$

$$= \sum_r \left(\frac{\lambda_1 w_o v}{1+w_o v}\right)^r \frac{1}{r!}[P_o + rP_1 + r(r-1)P_2 + \cdots]$$

$$= \exp\left(\frac{\lambda_1 w_o v}{1+w_o v}\right)\left[P_o + \left(\frac{\lambda_1 w_o v}{1+w_o v}\right)P_1 + \cdots \right],$$

where $P_i = P_i[a,b,w_o v/(1+w_o v)]$, $i = 0,1,\ldots,(b-1)$, is some polynomial in $w_o v/(1+w_o v)$ independent of r. Hence

$$(4.3) \int_o^{w_o} f(w)dw = \Gamma(a+b)[\Gamma(a)]^{-1}[\Gamma(b)]^{-2} w_o^{\,a}$$

$$\cdot\int_o^{\infty} v^{2a-1}(1+v)^{-a-b}(1+w_o v)^{-a}\exp\left(-\frac{\lambda_1}{1+w_o v} - \frac{\lambda_2}{1+v}\right)$$

$$\cdot\, M\left(-b,a; -\frac{\lambda_2 v}{1+v}\right)\left[P_o + \left(\frac{\lambda_1 w_o v}{1+w_o v}\right)P_1 + \cdots\right.$$

$$\left.\cdots + \left(\frac{\lambda_1 w_o v}{1+w_o v}\right)^{b-1} P_{b-1}\right]dv \,.$$

For example, when b = 2,

$$M[-\,2,a;\,-\,\lambda_2 v/(1+v)]$$

$$= 1 + 2\lambda_2 v/a(1+v) + \lambda_2^{\,2} v^2/a(a+1)(1+v)^2$$

and $$\sum_r \left(\frac{\lambda_1 w_o v}{1+w_o v}\right)^r \frac{\Gamma(a+r+2)}{r!\,\Gamma(a+r)}\left[\frac{1}{a+r} - \frac{w_o v}{(a+r+1)(1+w_o v)}\right]$$

$$= \sum_r \left(\frac{\lambda_1 w_o v}{1+w_o v}\right)^r \frac{1}{r!}\left[a+r+1 - \frac{(a+r) w_o v}{(1+w_o v)}\right]$$

$$= \exp\left(\frac{\lambda_1 w_o v}{1+w_o v}\right)\left[\left(a+1 - \frac{a w_o v}{1+w_o v}\right)\right.$$

$$\left. + \left(1 - \frac{w_o v}{1+w_o v}\right)\left(\frac{\lambda_1 w_o v}{1+w_o v}\right)\right] .$$

After some simplification we obtain (when b = 2)

$$\int_o^{w_o} f(w)\,dw = w_o^{\,a} \int_o^{\infty} \exp[-\,\lambda_1/(1+w_o v) - \lambda_2/(1+v)] v^{2a-1}$$

$$\cdot\ (1+v)^{-a-2}(1+w_o v)^{-a}[1 + (a+\lambda_1)/(1+w_o v) - \lambda_1/(1+w_o v)^2]$$

$$.\ [a^2 + a + 2(a+1)\lambda_2 v/(1+v) + \lambda_2^{\,2} v^2/(1+v)^2]\,dv\ .$$

Upon writing $v^r = (1+v-1)^r$ and $(w_o v)^r = (1+w_o v-1)^r$, for $r = 1,2,3,\ \ldots$ and expanding in powers of $(1+v)$

and $(1+w_o v)$, respectively, the probability integral can easily be written in terms of a relatively small number of integrals of the type,

$$I(w_o|\lambda,p,q) = \int_o^\infty \exp[-\lambda\ /(1+w_o v) - \lambda\ /(1+v)] \cdot (1+v)^{-p}(1+w_o v)^{-q} dv\ ,$$

when $\lambda_1 = \lambda_2 = \lambda$.
Here p and q are integers or half-integers. Upon first evaluating five basic integrals, viz., $I(w_o|\lambda,0,2)$, $I(w_o|\lambda,1,1)$, $I(w_o|\lambda,3/2,1/2)$, $I(w_o|\lambda,1/2,3/2)$ and $I(w_o|\lambda,5/2,1/2)$, it is possible for us to find all other integrals of this type by using recursion formulae. For example, for integral p and q, the following recursion formulae hold:

$$I(w_o|\lambda,p,q) = (w_o-1)^{-1}[w_o I(w_o|\lambda,p-1,q) - I(w_o|\lambda,p,q-1)]$$

when both p and q are greater than or equal to 1,

$$I(w_o|\lambda,0,q)$$

$$= \lambda^{-1}[(q-2)I(w_o|\lambda,0,q-1) - w_o^{-1}e^{-2\lambda}]$$

$$- w_o^{-1}I(w_o|\lambda,2,q-2) \ , \quad q > 2 \ ,$$

$$I(w_o|\lambda,p,0)$$

$$= \lambda^{-1}[(p-2)I(w_o|\lambda,p-1,0) - e^{-2\lambda}]$$

$$- w_oI(w_o|\lambda,p-2,2) \ , \quad p > 2 \ .$$

When $p = 2$ and $q = 0$, we have

$$I(w_o|\lambda,2,0) = \lambda^{-1}[1 - e^{-2\lambda}] - w_oI(w_o|\lambda,0,2) \ .$$

The basic integrals can be found by expanding $I(w_o|\lambda,p,q) = h(w_o)$ in Taylor series about 1 since $h(1)$ and the derivatives of $h(w_o)$ at $w_o = 1$ are known for all values of λ, p and q. Thus, $h(w_o) = h(1+w_o-1)$

$$= h(1) + (w_o-1)h'(1) + \frac{(w_o-1)^2}{2!}h''(1) + \dots \ .$$

When $w_o \geq 2$ and $\lambda_1 = \lambda_2$ it may prove advantageous to expand $h(1/w_o)$ in a Taylor series instead of $h(w_o)$. Then

$$h(1/w_o) = h[1 - (w_o-1)/w_o]$$

$$= h(1) - (w_o-1)h'(1)/w_o + \dots .$$

When $h(1/w_o)$ is known, $h(w_o) = I(w_o|\lambda,q,p)$ can easily be obtained, since

$$h(1/w_o) = I(1/w_o|\lambda,p,q)$$

$$= \int_o^\infty (1+v)^{-p}(1+v/w_o)^{-q}\exp[-\lambda/(1+w_o/v) - \lambda/(1+v)]dv$$

$$= w_o\int_o^\infty (1+w_ov)^{-p}(1+v)^{-q}\exp[-\lambda/(1+w_ov) - \lambda/(1+v)]dv$$

$$= w_oI(w_o|\lambda,q,p) = w_oh(w_o) .$$

Observe that when $\lambda_1 = \lambda_2$,

$$h(1) = \int_o^\infty (1+v)^{-p-q}\exp[-2\lambda/(1+v)]dv , \quad \text{and}$$

$$h^{(r)}(1) = D^r[h(w_o)]_{w_o=1}$$

$$= (-1)^r \int_0^{\infty} v^r(1+v)^{-p-q-r}$$

$$\exp[-\, 2\lambda/(1+v)]\Gamma(q+r)[\Gamma(q)]^{-1}$$

$$\cdot\, [1 - \binom{r}{1}\lambda q^{-1}(1+v)^{-1} + \ldots$$

$$\ldots - r\,\lambda^{r-1}[q(q+1)\,\ldots\,(q+r-2)]^{-1}(1+v)^{-r+1}$$

$$+\,\lambda^r[q(q+1)\,\ldots\,(q+r-1)]^{-1}(1+v)^{-r}]dv\ ,$$

where the latter result is obtained by differentiating under the integral sign with respect to w_o and setting $w_o = 1$.

It has been found that the Taylor series converge slowly, especially for small values of λ. As many as 18 terms of the series expansions were required to obtain 6 decimal accuracy in some of the basic integrals. Since the five basic integrals are used in recursion formulae to evaluate others of the same type, computational errors are magnified many times. When combining sets of these integrals in the evaluation of $\int_0^{w_o} f(w)dw$ positive and negative errors compensate for each other to a very large extent, and reasonable accuracy could be obtained. The basic integrals should, however, for most purposes, be evaluated to eight or ten decimal accuracy which would require Taylor series expansions of more than 20 terms.

Bounds for the magnitude of the error resulting from taking a finite number of terms of the Taylor expansion could be found in the following way: It was observed in computational work that

$$t_r > t_{r+1} > t_{r+2} > \quad \ldots$$

where $t_r = (-1)^r h^{(r)}(1)/r!$, and that

$$1/2 < \ \ldots \ < t_{r+1}/t_r < t_{r+2}/t_{r+1} < \ \ldots \ < 1 \, .$$

Hence, if

$$h(1/w_o) = t_o + (w_o-1)t_1/w_o + (w_o-1)^2 t_2/w_o^{\,2} + \ldots + (w_o-1)^r t_r/w_o^{\,r} + R(r+1) \, ,$$

it should follow that upper and lower bounds for the remainder, $R(r+1)$, could be obtained by finding the sums of two infinite geometric series, one with common ratio $(w_o-1)/w_o$ and the other with common ratio $(w_o-1)/2w_o$. Since the ratio

$$(w_o-1)t_{n+1}/w_ot_n \ , \ n > r \ ,$$

of successive terms in R(r+1) is greater than $(w_o-1)t_r/w_ot_{r-1}$ but less than $(w_o-1)/w_o$, it should also follow that narrower bounds for the remainder will be given by the sums of two infinite geometric series with common ratios $(w_o-1)t_r/w_ot_{r-1}$ and $(w_o-1)/w_o$ and both with first terms equal to $(w_o-1)^{r+1}t_{r+1}/w_o{}^{r+1}$. Hence

$$(w_o-1)^{r+1}t_{r+1}w_o{}^{-r} > R(r+1) >$$
$$(w_o-1)^{r+1}t_{r+1}w_o{}^{-r}t_{r-1}[w_ot_{r-1} - (w_o-1)t_r]^{-1} \ .$$

It has now been shown that for integral and small values of b the probability integral can be obtained as a linear combination of a finite (and relatively small) number of integrals of the type $I(w_o|\lambda,p,q)$. The basic integrals of this type are evaluated by using Taylor series expansions and all others are obtained from these by using recursion formulae. Since the series expansions have to be taken to a large number of terms and these evaluations have to be made for $1 \leq a \leq b$ (for both integer and half-integer values of a), for $1 \leq b \leq 6$ (say) and for a large range of values of λ, the computations required to construct even a limited table of values of the probability integral are quite extensive.

4.2 The probability integral, $\int_0^{w_0} g(w)dw$.

Equation (3.1) gave the density function of the ratio of two independent F's, both with 2a and 2b degrees of freedom in the form

$$g(w) = [B(a,b)]^{-2} w^{a-1} \int_0^{\infty} v^{2a-1}(1+wv)^{-a-b}(1+v)^{-a-b} dv \ .$$

By integrating between the limits 0 and w_0 and changing the order of integration on the right hand side, we obtain

$$\int_0^{w_0} g(w)dw = [B(a,b)]^{-2} \int_0^{\infty} v^{a-1}(1+v)^{-a-b}$$

$$\cdot \int_0^{w_0} (wv)^{a-1}(1+wv)^{-a-b} vdwdv \ .$$

When we write $x = wv/(1+wv)$, it follows that $(wv)^{a-1}(1+wv)^{-a-b}vdw = x^{a-1}(1-x)^{b-1}dx$. By expanding $(1-x)^{b-1}$ in a binomial series and integrating termwise with respect to x between the limits 0 and $w_o v/(1+w_o v)$, we obtain

$$(4.4)\quad \int_0^{w_o} g(w)dw$$

$$= [B(a,b)]^{-2} \int_0^{\infty} v^{a-1}(1+v)^{-a-b}(w_o v)^a (1+w_o v)^{-a}$$

$$\cdot \left[\frac{1}{a} - \frac{(b-1)w_o v}{(a+1)(1+w_o v)} + \frac{(b-1)(b-2)(w_o v)^2}{2!(a+2)(1+w_o v)^2} \cdots \right]dv .$$

Again it must be pointed out that the series in square brackets is finite when b is an integer.

When a and b are both integers, the probability integral can be written very neatly in terms of the integrals

$$I(w_o | 0,m,n) = \int_0^{\infty} (1+v)^{-m}(1+w_o v)^{-n} dv ,$$

where m and n are both integers. [To facilitate the writing of this symbol we shall in future write $I(w_o|a,b)$ for $I(w_o|0,a,b)$.] Thus, for a = 1

$$\int_0^{w_o} g(w)dw = 1 - bI(w_o| b+1,b) .$$

When $a = 2$, the probability integral (4.4) is given by

$$1 - b(b+1)[(b+1)I(w_o|b+1,b) - (b+1)I(w_o|b+2,b)$$

$$- bI(w_o|b+1,b+1) + bI(w_o|b+2,b+1)] .$$

When $a = 3$, (4.4) takes the form,

$$1 - \frac{1}{2!}b(b+1)(b+2)\{(b+1)(b+2)[I(w_o|b+1,b)$$

$$- 2I(w_o|b+2,b) + I(w_o|b+3,b)]/2!$$

$$- b(b+2)[I(w_o|b+1,b+1) - 2I(w_o|b+2,b+1)$$

$$+ I(w_o|b+3,b+1)]/1!1! + b(b+1)[I(w_o|b+1,b+2)$$

$$- 2I(w_o|b+2,b+2) + I(w_o|b+3,b+2)]/2!\} .$$

Similar expressions can be obtained when $a = 4,5, \ldots$. Since $I(w_o|1,1) = \ln w_o/(w_o-1)$ and $I(w_o|m,n) = (w_o-1)^{-1}[w_oI(w_o|m-1,n) - I(w_o|m,n-1)]$, when m and n are unequal to one,

$$I(w_o|1,n) = [(n-1)^{-1} - I(w_o|1,n-1)] , \quad n > 1 , \text{ and}$$

$$I(w_o|m,1) = [w_oI(w_o|m-1,1) - (m-1)^{-1}] , \quad m > 1 ,$$

the evaluation of the probability integral presents little difficulty--although a considerable amount of work.

When a is not an integer and b is an integer, the probability integral can again be written as a linear combination of integrals of the type $I(w_o|p,q)$ where p and q are both half-integers. Thus, for a = 3/2,

$$\int_o^{w_o} g(w)dw = (3/2)w_o^{3/2}[I(w_o|1/2,3/2) - 2I(w_o|3/2,3/2) + I(w_o|5/2,3/2)]$$

when b = 1, and

$$\int_o^{w_o} g(w)dw = (3/2)(5/2)w_o^{3/2}[I(w_o|3/2,3/2) - 2I(w_o|5/2,3/2) + I(w_o|7/2,3/2) + 1.5I(w_o|3/2,5/2) - 3I(w_o|5/2,5/2) + 1.5I(w_o|7/2,5/2)] \quad \text{when } b = 2.$$

Here $\quad I(w_o|1/2,3/2) = 2w_o^{-1/2}(1+\sqrt{w_o})^{-1}$,

$I(w_o|3/2,1/2) = 2(1+\sqrt{w_o})^{-1}$ and

$$I(w_o|p,q) = (w_o-1)^{-1}[w_oI(w_o|p-1,q) - I(w_o|p,q-1)]$$

when p and q are both greater than 1/2,

$$I(w_o|1/2,q)$$
$$= w_o^{-1}(q-1)^{-1}[1 - 2^{-1}I(w_o|3/2,q-1)] , \quad q > 3/2 ,$$

and

$$I(w_o|p,1/2)$$
$$= (p-1)^{-1}[1 - 2^{-1}w_oI(w_o|p-1,3/2)] , \quad p > 3/2 .$$

Again, the probability integral can be found with relative ease especially when a and b are not large.

In certain special cases alternative methods of evaluating the probability integral are available. As was indicated in Section 3.2,

$$g(w) = B(2a,2a+1)[B(a,a+\tfrac{1}{2})]^{-2}2^{4a}w^{a-1}(1+\sqrt{w})^{-4a}$$

when $b = a+\frac{1}{2}$, and

$$g(w) = B(2a,2a-1)[B(a,a-\tfrac{1}{2})]^{-2}$$
$$\cdot\ 2^{4a-2}w^{a-3/2}(1+\sqrt{w})^{2-4a}$$

when $b = a-\frac{1}{2}$. It follows very easily that

$$\int_0^{w_o} g(w)dw = I_x(2a,2a) \quad \text{when} \quad b = a+\tfrac{1}{2} \quad \text{and}$$

$$\int_0^{w_o} g(w)dw = I_x(2a-1,2a-1) \quad \text{when} \quad b = a-\tfrac{1}{2},$$

where $I_x(p,q)$ is the incomplete Beta-function with parameters p and q and $x = \sqrt{w_o}/(1+\sqrt{w_o})$.

By obtaining expressions for $g(w)$, when $b = a \pm 3/2$, $b = a \pm 5/2$ and $b = a \pm 7/2$ in the manner indicated in Section 3.2, we were able to find similar expressions for the probability integral in terms of incomplete Beta-functions in these cases. For example,

$$\int_0^{w_o} g(w)dw = (4a+2)^{-1}[4aI_x(2a+1,2a+1)$$

$$+ I_x(2a+2,2a) - I_{1-x}(2a+2,2a) + 1]$$

when $b = a+3/2$.

4.3 The probability integrals $\int_0^{z_o} f_1(z)dz$ and $\int_0^{z_o} g_1(z)dz$.

The methods described in the previous sections can also be used in the evaluation of the probability integrals $\int_0^{z_0} f_1(z)dz$ and $\int_0^{z_0} g_1(z)dz$. Here $f_1(z)$ is the density function of the ratio of two independent F's with 2a,2b and $2\alpha,2\beta$ degrees of freedom and parameters of non-centrality, λ_1 and λ_2, and $g_1(z)$ is the density function of the ratio of two independent F's with 2a,2b and $2\alpha,2\beta$ degrees of freedom.

It follows from (2.9) and Section 4.1 that,

$$\int_0^{z_0} f_1(z)dz$$

$$= \int_0^{z_0/c} f_1(z/c)d(z/c)$$

$$= \int_0^{x_0} e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!s!} \left[B(a+r,b)B(\alpha+s,\beta)\right]^{-1}$$

$$\cdot x^{a+r-1} \int_0^{\infty} v^{a+\alpha+r+s-1} (1+xv)^{-a-b-r}$$

$$\cdot (1+v)^{-\alpha-\beta-s} dvdx$$

$$= e^{-\lambda_1-\lambda_2} \sum_r \sum_s \frac{\lambda_1^r \lambda_2^s}{r!s!} \left[B(a+r,b)B(\alpha+s,\beta)\right]^{-1}$$

$$\cdot \int_0^{\infty} v^{\alpha+s-1} (1+v)^{-\alpha-\beta-s} (vx_0)^{a+r} (1+vx_0)^{-a-r}$$

$$\cdot \left[\frac{1}{a+r} - \frac{(b-1)vx_0}{(a+r+1)(1+vx_0)} + \cdots\right] dv ,$$

where $x = z/c$, $x_o = z_o/c$ and $c = b\alpha/a\beta$. This probability integral takes the same form as $\int_o^{w_o} f(w)dw$ and the methods of evaluating the latter integral, considered in Section 4.1, can therefore be applied.

Similar remarks can be made regarding $\int_o^{z_o} g_1(z)dz$. It follows from (3.7) and Section 4.2 that,

$$\int_o^{z_o} g_1(z)dz$$

$$= \int_o^{z_o/c} g_1(z/c)d(z/c)$$

$$= \int_o^{x_o} [B(a,b)B(\alpha,\beta)]^{-1} x^{a\cancel{+\alpha}-1} \int_o^{\infty} v^{a+\alpha-1}$$

$$\cdot\ (1+v)^{-\alpha-\beta}(1+xv)^{-a-b} dvdx$$

$$= [B(a,b)B(\alpha,\beta)]^{-1} \int_o^{\infty} v^{\alpha-1}(1+v)^{-\alpha-\beta}$$

$$\cdot\ (vx_o)^a(1+vx_o)^{-a}[\frac{1}{a} - \frac{(b-1)vx_o}{(a+1)(1+vx_o)} + \ldots]dv$$

where $x = z/c$, $x_o = z_o/c$ and $c = b\alpha/a\beta$. This probability integral is of the same form as $\int_o^{w_o} g(w)dw$ considered in the previous section, and the methods of evaluating that integral can again be applied.

V: THE CONSTRUCTION AND USE OF TABLES OF PERCENTAGE POINTS OF THE DISTRIBUTIONS OF THE RATIOS OF TWO F''S AND TWO F'S

5.1 The construction of tables of percentage points for f(w).

To conduct tests of significance of the hypothesis that the parameters in f(w), the distribution of the ratio of two F''s with the same degrees of freedom, are equal, tables of percentage points for f(w) are required. When tests are conducted at the α-level of significance and the null hypothesis $H_o:\lambda_1 = \lambda_2$ is tested against the alternative, $H_a:\lambda_1 \neq \lambda_2$, it is of practical importance to know those values of w_o and w_1 such that $Pr[w_o \leq w \leq w_1] = 1-\alpha$. When the null hypothesis is tested against the alternative $H_a:\lambda_1 > \lambda_2$, those values of w_o such that $Pr[w \leq w_o] = 1-\alpha$, should be known.

The construction of accurate tables of 5%, 2.5%, 1% and 0.5% points for the function f(w) would require a vast amount of labor, since different values of λ as well as different degrees of freedom have to be considered and the computational methods are not simple. Some of the upper 5% points of the distribution function were found and are given in Table II. In calculating these percentage points it was assumed that $\lambda_1 = \lambda_2 = \lambda$ and that the degrees of freedom associated with the two F''s were 2a and 2b. Formula (4.3) was used after the basic integrals $I(w_o|\lambda,0,2)$ and $I(w_o|\lambda,1,1)$ were evaluated by using Taylor series expansions in the manner described in Section 4.1. The values tabulated are those values of w_o which made $\int_0^{w_o} f(w|\lambda,a,b)dw = .95$.

Some corresponding 5% points obtained by equating $f(w|\lambda,a,b)$ to $g(w|a',b)$, the density function of the ratio of two independent F's both with $2a' = 2(a+\lambda)^2/(a+2\lambda)$ and 2b degrees of freedom, are given in Table III for purposes of comparison. To construct Table III the degrees of freedom $2a'$ and 2b associated with $g(w|a',b)$, for the different values of a,b and λ shown in Table II were obtained. The 5% points were subsequently found from Table IV (see Section 5.2) by entering this latter table at $2a'$ and 2b degrees of freedom, interpolating between tabulated values where necessary.

TABLE II

Values of w_o Such That $\int_{w_o}^{\infty} f(w|\lambda,a,b)dw = .05$

D.F.		λ								
2a	2b	4	6	8	12	16	24	40	60	100
2	2	-	-	24.1	22.3	21.4	20.6	20.0	19.7	19.4
2	4	11.6	10.1	9.3	8.5	8.1	7.8	7.6	7.2	7.2
4	4	10.4	9.4	8.9	-	-	-	-	-	-
2	6	8.7	7.4	6.6	-	-	-	-	-	-

TABLE III

Values of w_o Such That $\int_{w_o}^{\infty} g(w|a',b)dw = .05$

As an Approximation to $\int_{w_o}^{\infty} f(w|\lambda,a,b)dw = .05$

D.F.		λ								
2a	2b	4	6	8	12	16	24	40	60	100
2	2	-	-	23.4	21.9	21.1	20.4	-	-	19.2
2	4	11.0	9.6	8.8	8.0	7.6	7.2	-	-	6.6
4	4	9.8	8.9	8.4	-	-	-	-	-	-
2	6	8.3	7.0	6.3	-	-	-	-	-	-

It should be pointed out that bounds for the exact probabilities, $\int_{o}^{w_o} f(w|\lambda,a,b)dw$, where w_o

takes the values 23.4, 11.0, 9.6, 8.8, 9.8, 8.9, 8.4, 8.3, 7.0 and 6.3 listed in Table III, are given in Table I (see Section 4.1).

Two things are clearly discernable from the above tables:

(i) The percentage points remain fairly stable even for considerable variations in the values of λ. This implies that it will have little effect on the test conducted if we enter the table at a value of λ somewhat different from its true (unknown) value.

(ii) The values given in Table III are close enough to the corresponding values in Table II (even for small degrees of freedom) to make their use meaningful. Since the construction of tables of percentage points for g(w) is much less arduous than the construction of tables for f(w), it is suggested that the former be used to conduct, at least approximate tests of significance of the hypotheses formulated above.

5.2 The construction of tables of percentage points for g(w).

A table of upper 5% points for $g(w|a,b)$, the distribution of two independent F's with 2a and 2b degrees of freedom, was constructed and is reproduced as Table IV. The formulae given in Section 4.2 were employed and 'trial-and-error' methods were used to arrive at the appropriate values of w_o such that $\int_o^{w_o} g(w|a,b)dw = .95$. Pearson's Tables of the Incomplete Beta Function were used to obtain the values of w_o for $b = a \pm 1/2$, $a + 3/2$, $a + 5/2$ and $a + 7/2$, in view of the results given towards the end of Section 4.2. These values were found to three decimal places and afterwards rounded off to two places. The intermediate values, i.e., for $b = a$, $a + 1$, $a + 2$ and $a + 3$ were obtained by interpolation except for small b where interpolation did not lead to accurate results.

Since $g(w|a,b)$ is symmetric in a and b (refer Section 2.3) the percentage points were essentially tabulated for a = b. The table gives the 5% points for

$a = 1, 3/2, 2, 5/2, \ldots (1/2) \ldots , b + 1/2$, when

$b = 1, 2, 3, \ldots\ldots\ldots\ldots (1) \ldots , 10$,

as well as for

$a = 1, 3/2, 2, \ldots\ldots\ldots (1/2) \ldots , 10, 15$, when

$b = 15$ and ∞ .

TABLE IV

Values of w_o Such That $\int_{w_o}^{\infty} g(w\ a,b)dw = .05$

	b	1	2	3	4	5	6	7	8	9	10	15	∞
	D.F.	2	4	6	8	10	12	14	16	18	20	30	∞
a	D.F.												
1	2	66.12	32.76	26.76	24.37	23.10	22.31	21.77	21.39	21.10	20.87	20.21	19.00
3/2	3	40.81	23.15	14.40	12.81	11.97	11.45	11.09	10.85	10.65	10.50	10.07	9.28
2	4		13.91	10.62	9.32	8.62	8.19	7.90	7.69	7.54	7.41	7.06	6.39
5/2	5		11.82	8.87	7.70	7.11	6.68	6.42	6.23	6.09	5.96	5.64	5.05
3	6			7.86	6.77	6.18	5.78	5.54	5.37	5.25	5.13	4.82	4.28
7/2	7			7.22	6.17	5.61	5.26	5.03	4.85	4.72	4.61	4.30	3.79
4	8				5.75	5.21	4.88	4.65	4.48	4.35	4.24	3.94	3.44
9/2	9				5.45	4.92	4.59	4.36	4.19	4.07	3.97	3.68	3.18
5	10					4.70	4.37	4.14	3.98	3.86	3.76	3.48	2.98
11/2	11					4.52	4.19	3.98	3.82	3.70	3.60	3.32	2.82
6	12						4.05	3.84	3.68	3.56	3.46	3.18	2.69
13/2	13						3.94	3.72	3.57	3.45	3.35	3.07	2.58
7	14							3.63	3.47	3.36	3.26	2.98	2.49
15/2	15							3.54	3.39	3.27	3.18	2.90	2.40
8	16								3.32	3.20	3.11	2.83	2.34
17/2	17								3.26	3.14	3.05	2.77	2.28
9	18									3.09	3.00	2.72	2.22
19/2	19									3.04	2.95	2.67	2.17
10	20										2.91	2.63	2.12
21/2	21										2.87	-	-
15	30											2.36	1.85

The percentage points for $b = \infty$ were obtained from an F-table, since $g(w|a,\infty)$ has an F-distribution with 2a and 2a degrees of freedom.

Tables of 2.5%, 1% and 0.5% points for $g(w|a,b)$ can be constructed in the same way.

5.3 The use of the tables of percentage points.

It was indicated at the beginning of this chapter that tables of the α% points for $f(w|\lambda,a,b)$ can be used in testing (at the α-level of significance) the hypothesis that the parameters, λ_1 and λ_2, in the distribution of the ratio of two F''s, are equal. When tables for $f(w|\lambda,a,b)$ are not available, the use of tables of the percentage points for $g(w|a',b)$ was suggested, at least for an approximate test of significance.

To illustrate the procedure let us consider the use of Table IV in a one-sided test at the 5% level of significance. For a given λ the null hypothesis that $\lambda_1 = \lambda_2 = \lambda$ is tested by comparing the observed value of $w = F_1'/F_2'$, where both F''s have degrees of freedom 2a and 2b, with the critical value of w obtained from a table of 5% points. An approximation to the critical value of w is obtained from

Table IV by entering this table at $2a'$ and $2b$ degrees of freedom, where $2a' = 2(a+\lambda)^2/(a+2\lambda)$. The use of linear interpolation in Table IV should suffice and the critical value obtained in this way is used for comparison with the observed value of w. The null hypothesis is rejected in favor of an alternative when the observed value of w exceeds the critical value.

Some remarks regarding the use of Table IV, in certain cases where we are concerned with two independent F''s with different degrees of freedom could be made here. We will want to test the null hypothesis $\lambda_1/n_1 = \lambda_2/n_2 = \lambda'$ (say), where n_1 and n_2 refer to the numbers of observations in treatment means. Now, assume as in Section 3.4, that the density function of the ratio of two F'-variates with parameters λ_1 and λ_2 and $2a$, $2b$ and 2α, 2β degrees of freedom can be approximated by $(a+\lambda_1)\alpha z/(\alpha+\lambda_2)a = (a+n_1\lambda')\alpha z/(\alpha+n_2\lambda')a = \gamma z$ (say). In all cases where γ is equal to or approximately equal to one the density given by equation (3.7) with which we will associate the degrees of freedom $2a' = 2(a+n_1\lambda')^2/(a+2n_1\lambda')$ and $2b$ and $2\alpha' = 2(\alpha+n_2\lambda')^2/(\alpha+2n_2\lambda')$ and 2β, and which we will write as $g_1(z|a',b,\alpha',\beta)$, will approximate the density function of the ratio of the two F'-variates.

A scrutiny of Table IV now reveals that the percentage points of $g(w|a,b)$ vary relatively little with small variations in a and b especially for large degrees of freedom. It is, therefore, reasonable to assume that we can approximate $\int_0^{z_o} g_1(z|a',b,\alpha',\beta)dz$ $= \int_0^{z_o/c} g_1(z/c|a',b,\alpha',\beta)d(z/c)$ by $\int_0^{w_o} g(w|a^*,b^*)dw$, which is of the same form and where $a^* = (a'+\alpha')/2$ and $b^* = (b+\beta)/2$. When $2a'$ and $2\alpha'$ and $2b$ and 2β differ only slightly and these degrees of freedom are not small, Table IV can be used to find an approximation to the upper 5% points for $g_1(z|a',b,\alpha',\beta)$. If $\gamma = 1$ (approximately) these will approximate the upper 5% points for the distribution of the ratio of the two F''s under the null hypothesis.

A further use of tables of percentage points for $f(w|\lambda,a,b)$ and $g(w|a,b)$ lies in the comparison of population multiple correlation coefficients based on values of R^2 observed in two independent samples under the usual assumptions of linear regression, viz., that the independent 'variables' are fixed.

Consider the observations

$$\begin{array}{lllll}
y_1, & x_{11}, & x_{21}, & \cdots\cdots\cdots & x_{p1} \\
y_2, & x_{12}, & x_{22}, & \cdots\cdots\cdots & x_{p2} \\
\cdots & \cdots & \cdots & \cdots\cdots\cdots & \cdots \\
y_N, & x_{1N}, & x_{2N}, & \cdots\cdots\cdots & x_{pN},
\end{array}$$

and let $E(y_i) = \beta_o + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_p x_{pi}$ for $i = 1,2,3, \dots , N$. Assume y to be normally distributed and $x_1, x_2, \dots , x_p$ to be fixed. Let the squares of the observed and population multiple correlation coefficients be $R^2 = R^2_{y \cdot x_1 x_2 \dots x_p}$ and $\rho^2 = \rho^2_{y \cdot x_1 x_2 \dots x_p}$, respectively, and assume $\rho^2 \neq 0$. The power function of R^2 can then be written in one of the forms referred to in Section 1.2, which are due to Fisher and Wishart. Using Wishart's form of the density function of R^2 we have

$$(5.1) \quad f(R^2)d(R^2) = e^{-\lambda}[B(a,b)]^{-1}(R^2)^{a-1}(1-R^2)^{b-1}$$

$$\cdot\ M(a+b,a;\lambda R^2)d(R^2), \quad 0 \leq R^2 \leq 1,$$

where $a = p/2$, $b = (N-p-1)/2$ and $\lambda = (a+b)\rho^2/(1-\rho^2)$. Upon setting $R^2 = u/(1+u)$, i.e., $u = R^2/(1-R^2)$ we easily obtain equation (1.2), i.e.,

$$f(u)du = e^{-\lambda} \sum_{r=o}^{\infty} \frac{\lambda^r}{r!B(a+r,b)} \cdot \frac{u^{a+r-1}}{(1+u)^{a+b+r}} du ,$$

$$0 \leq u \leq \infty ,$$

which was the form in which we wrote the F'-distribution. It follows that in the function, f(w), considered in this paper, w can be taken to be $\frac{R_1^2(1-R_2^2)}{R_2^2(1-R_1^2)}$ where R_1 and R_2 are sample multiple correlation coefficients from the independent experiments 1 and 2, which are equal in size and design. A test of significance of the hypothesis that two population multiple correlation coefficients are equal can, therefore, be based on the statistic w. In such a test the observed value of $w = \frac{R_1^2(1-R_2^2)}{R_2^2(1-R_1^2)}$ is compared with the critical value obtained from Table IV which is entered at

$$2a' = \left[p + (N-1)\rho^2/(1-\rho^2)\right]^2\left[p + 2(N-1)\rho^2/(1-\rho^2)\right]^{-1}$$

and 2b = N-p-1 degrees of freedom.

VI: APPLICATIONS TO SENSORY TESTING, FIELD EXPERIMENTATION AND THE COMPARISON OF CORRELATION COEFFICIENTS

6.1 Sensory testing.

Methods of comparing the sensitivities of any two independent experiments of the same size and with identical experimental treatments have now been developed so long as the experimental designs yield tests of significance dependent on the F-statistic with the usual null and non-null distributions. We now turn to applications of the theory developed. While the examples given are randomized block designs, valid applications of the new technique is by no means limited to these simple designs.

It was indicated in Section 1.1 that the development of methods for comparing the sensitivities of taste panel experiments based on different scoring techniques led to a consideration of the distribution function of the ratio of two independent F'-variates. The emphasis was placed on the comparison of scoring scales used in taste testing, the more sensitive scale being defined as the one which accentuates real treatment differences. The methods developed are, however, applicable over a wider range of subjective testing experiments and in various ways. Thus, two judges or panels of judges scoring differences in flavor or color may be compared as to discriminatory ability, the judge or the panel accentuating real treatment differences being regarded as the one with greater discriminatory power. A test of significance of the hypothesis that the judges or panels have equal discriminatory ability will again be given by a test for equality of the parameters in the distribution of the ratio of two F''s. Hence, the functions discussed in this paper and the tests of hypotheses formulated will enable research workers in the fields of the application of sensory testing to compare methods based on different scoring scales, different panels of judges, or different experimental techniques, thus adding to their knowledge and assisting in the search for the 'best' methods in sensory test design.

Let us consider an example in which the discriminatory power of two judges A and B are compared.

The two judges were asked to score the color of canned tomatoes in two independent experiments which may be regarded as randomized block experiments. The scale used ranged from 0 to 100, the scores being indicated to the nearest 5 units. Although the scale was discrete, it will be assumed that analysis of variance techniques apply and the observed scores will be used in the analysis. The ordered average scores accorded each of nine color treatments over four replications (blocks) were as follows:

Treatment:	2	5	9	3	1
Judge A :	75.00	68.75	65.00	60.00	52.50
Judge B :	87.50	85.00	78.75	75.00	61.25

Treatment:	7	4	8	6
Judge A :	45.00	40.00	40.00	38.75
Judge B :	50.00	48.75	48.75	47.50 .

A test of the hypothesis of equal discriminatory power of judges A and B now requires:

(i) analyses of variance for both experiments which are equal in size and design;

(ii) the estimation of the parameters λ_A and λ_B pertaining to the two experiments. Since Model I of the analysis of variance applies, an estimate of λ_i, $(i = A,B)$, is given by

$$a\left[\frac{M.S.T.}{M.S.E.} - 1\right],$$ where M.S.T. is the mean sum of squares of treatments based on 2a degrees of freedom and M.S.E. is the mean sum of squares of error based on 2b degrees of freedom. The mean of λ_A and λ_B is heuristically taken to be the 'best available' single estimate of the population parameter λ;

(iii) the formulation of the null hypothesis $H_o: \lambda_A = \lambda_B = \lambda$ and an alternative hypothesis;

(iv) the decision to test at a given level of significance, say α;

(v) the calculation of the observed value of w, the ratio of the two F''s;

(vi) the use of a table of percentage points (such as Table IV) to obtain the critical value of w;

(vii) a decision to accept or reject the null hypothesis.

For the example considered above, this procedure gave the following results:

(i) Analyses of variance resulted in the following tables:

Judge A

Source	D.F.	Mean Sum of Squares	F
Blocks	3	346.297	2.46
Treatments	8	761.633	5.42**
Error	24	140.567	

Judge B

Source	D.F.	Mean Sum of Squares	F
Blocks	3	167.593	2.83
Treatments	8	1134.078	19.14**
Error	24	59.259	

The F-values indicate that real treatment differences exist.

(ii) The two independent estimates of the parameters λ_A and λ_B are $\lambda_A = 17.68$ and $\lambda_B = 72.56$. The average of λ_A and λ_B is, therefore, 45.12.

(iii) We test the null-hypothesis $H_o: \lambda_A = \lambda_B = 45.12$ against the alternative $\lambda_B > \lambda_A$, since the ability of judge A to discriminate effectively between colors was in some doubt.

(iv) The test is conducted at the 5% level of significance.

(v) The observed value of $w = F_B'/F_A'$ (both F''s having 8 and 24 degrees of freedom) is 19.14/5.42 = 3.53.

(vi) Assuming that the distribution of the ratio of two F''s with 8 and 24 degrees of freedom can effectively be approximated by the distribution of the ratio of two F's with $2(a+\lambda)^2/(a+2\lambda) = 51.2$ and 24 degrees of freedom, we use Table IV to find the critical value of w. Entering the table at 51 and 24 degrees of freedom, an approximation to the critical value is obtained, as 2.3.

(vii) The observed value of w is significant and the hypothesis of equal discriminatory powers for the judges is rejected in favor of greater discriminatory ability for judge B.

It should be observed that we would have arrived at the same decision if we had tested the null hypothesis $\lambda_A = \lambda_B = \lambda$, where λ is any value greater than or equal to 3.2, against the same alternative. This shows the stability of the test, as our estimates of λ would indicate a value in this range.

6.2 Field experimentation.

At various times research workers have considered the relative efficiency of different plot sizes in field experimentation, e.g., crop-raising experiments. These experiments are usually conducted under conditions which make the application of analysis of variance techniques valid.

With the theory developed in this paper, tests of significance can now be conducted to compare the efficiencies or sensitivities of different plot sizes. When two independent experiments are run on different sizes of plots and yields of corn, say, are used to obtain 'scores' (assuming real differences in yields), the sensitivities of the plots can be compared in the same way as discriminatory ability was compared in the previous section. The plot size which accentuates yield differences will be considered to be the more efficient or sensitive. It is conceivable that an 'optimum' plot size, that is, one which results in greatest efficiency, may be found in this way.

The theory developed here can be used to conduct tests of significance in many other instances relating to field experimentation, thus increasing our knowledge of the nature of such experiments. Several examples could be quoted. A comparison of the yields of different varieties of corn under irrigation and with no irrigation, can show whether irrigation accentuates variety differences. A comparison of grazing and clipping methods on the yields of different varieties of grasses in pasture experiments could again show whether one method accentuates variety differences. As a further example, data obtained in the following experiment[1] will be considered and a test of significance will actually be conducted: The yields of a mixture of Orchard Grass and Ladino Clover, subjected to eight different treatments (eight fertilizers) were obtained on plots which were either irrigated or not irrigated. The experiments relating to 'irrigation' and 'no-irrigation' were independent and four replications were run. The average yields in lbs/acre for treatments A to H over the four replications were:

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(1) This constitutes part of the data collected in a larger experiment run at the Virginia Agricultural Experiment Station, of the Virginia Polytechnic Institute, Blacksburg, Virginia. The research was conducted by the Department of Agronomy in 1954.

Treatment :	A	B	C	D	E	F	G	H
Irrigation :	499	627	642	756	884	494	597	377
No-irrigation:	831	836	931	1137	1346	730	903	583

Analyses of variance gave the following tables:

Irrigation (Experiment 1)

Source	D.F.	Mean Sum of Squares	F
Replications	3	17,899	1.55
Treatments	7	102,104	8.86**
Error	21	11,517	

No-irrigation (Experiment 2)

Source	D.F.	Mean Sum of Squares	F
Replications	3	93,155	2.94
Treatments	7	224,462	7.10**
Error	21	31,599	

The F-values obtained indicate the existence of real treatment differences.

Estimates of λ relating to experiments 1 and 2 are given by $\lambda_1 = 27.52$ and $\lambda_2 = 21.36$, respectively. Their mean value is 24.44. We test the hypothesis $H_o: \lambda_1 = \lambda_2 = 24.44$ against the alternative $H_a: \lambda_1 \neq \lambda_2$ at the 10%-level of significance. The observed value of $w = F_1'/F_2' = 8.86/7.10 = 1.25$ and the critical value obtained from Table IV (which can be used in a two-sided test at the 10%-level of significance) is approximately 2.6. Subject to the assumption that we can effectively use the approximation to the distribution of the ratio of two independent F''s we, therefore, have no reason to reject the null hypothesis, i.e., the hypothesis of equal sensitivity of the two methods, 'irrigation' and 'no-irrigation'.

<u>6.3 Comparison of multiple correlation coefficients</u>.

It was indicated in Section 5.3 that a direct application of the theory and tests developed in this paper lies in the comparison of multiple correlation coefficients. As an example we will consider data obtained from a Southern Regional Livestock Auction Market survey conducted in the fall and spring of 1953 and 1954.(2) Reports on cattle sold, obtained from auction markets, were used to obtain regression equations showing the dependence of y, the price per 100 pounds, on $2a = p = 5$ other factors, for four

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(2) Southern Regional Livestock Marketing Research Project SM-7, subproject II - grade-price differentials of slaughter cattle and calves.

different strata which were different sections of the southeastern part of the United States. The independent variables in the regression equations were:

x_1 = grade,

x_2 = dressing percentage,

x_3 = weight,

x_4 = size of market, and

x_5 = visit.

The correlation coefficients, R^2, were calculated for various classes of beef cattle. The results for steers for each of the four strata are shown here, along with the relevant degrees of freedom, i.e., values of 2b = N-p-1:

Stratum:	I	II	III	IV
R^2 :	.912	.813	.659	.923
D.F. :	54	50	54	50 .

It is possible to compare the population correlation coefficients for strata I and III and again for strata II and IV since the values of R^2 are based on equal degrees of freedom in these cases.

Consider strata I and III and use the adjusted value of R^2,[3] i.e.,

$$R^2 \text{ (adjusted)} = 1 - (1-R^2)(a+b)/b ,$$

as an estimate of ρ^2. Thus, two estimates of ρ^2 and two estimates of $\lambda = (a+b)\rho^2/(1-\rho^2)$ are obtained, viz.,

$$\rho_1^2 = .904 \quad , \quad \rho_3^2 = .627 \quad \text{and}$$

$$\lambda_1 = 277.8 \quad , \quad \lambda_3 = 49.6 \quad , \text{ where}$$

the subscripts refer to strata. We test the null hypothesis H_o: $\rho_1^2 = \rho_3^2$ or $\lambda_1 = \lambda_3 = \lambda$, where λ is the mean of λ_1 and λ_3, i.e., 163.7, against the alternative H_a: $\lambda_1 \neq \lambda_3$. The test is conducted at the 10%-level of significance. The observed value of $w = R_1^2(1-R_3^2)/R_3^2(1-R_1^2) = 5.36$. Assuming that the approximation to the distribution of the ratio of two F''s will give satisfactory results, the critical value of w is obtained from Table IV, by entering it at $2a' = 2(a+\lambda)^2/(a+2\lambda) = 167.4$ and $2b = 54$ degrees of freedom. The critical value of w is found to be less than 2.0. The observed value is

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(3) See Snedecor (1950) Section 13.4, p. 348.

therefore significant and the null hypothesis is rejected in favor of the alternative.

Comparing the population correlation coefficients for strata II and IV, we obtain the following estimates of ρ^2 and λ:

$$\rho_2{}^2 = .794 \quad , \quad \rho_4{}^2 = .915 \quad \text{and}$$

$$\lambda_2 = 106 \quad , \quad \lambda_4 = 296 \quad , \text{ where}$$

the subscripts again refer to strata. We test the null hypothesis of equal correlation, or $\lambda_2 = \lambda_4 = \lambda$ where $\lambda = (\lambda_2+\lambda_4)/2 = 201.0$ against the alternative $\lambda_2 \neq \lambda_4$, at the 10%-level of significance. The observed value of $w = R_4{}^2(1-R_2{}^2)/(1-R_4{}^2)R_2{}^2 = 2.76$. The critical value of w obtained from Table IV by entering it at $2a' = 2(a+\lambda)^2/(a+2\lambda) = 204$ and $2b = 50$ degrees of freedom, is less than 1.9. Hence, on the basis of a test of significance at the 10%-level we reject the null hypothesis of equal correlation coefficients.

Two things should be observed regarding these tests:

(i) We would have been led to the same decisions if quite different values of λ (or ρ^2) had been employed. In the first test any value of $\lambda > 0$ (or $\rho^2 > 0$) would have led us to reject the null hypothesis of equality of the parameters. In the second test values of $\lambda \geq 8$ (or $\rho^2 \geq .225$ approximately) would have led us to arrive at the same result as we did, and $\lambda = 8$ ($\rho^2 = .225$) is considerably smaller than either of the estimates λ_2 or λ_4 ($\rho_2{}^2$ or $\rho_4{}^2$).

(ii) Since the degrees of freedom, 2b, were as large as 50 and 54 the approximation to the distribution of the ratio of two F'’s should have been quite good and the tests can be regarded as quite accurate.

VII: SUMMARY

Assuming that analysis of variance techniques apply and that real treatment differences exist, we derived the distribution of w, the ratio of two independent F''s with equal degrees of freedom, and with parameters of non-centrality λ_1 and λ_2. Various forms in which the density function, f(w), can be written were considered and the properties of the function were investigated. Under the null hypothesis an approximation to f(w) was found by approximating the F'-distribution to an ordinary F-distribution and finding the density function, g(w), of the ratio of two independent F's with the same degrees of freedom. The moments of these functions were derived and it was shown that the lower moments of the density functions, f(w) and g(w), were approximately equal for large degrees of freedom, provided λ was not large.

Methods of evaluating the probability integral, $\int_0^{w_0} f(w)dw$, were considered. It was indicated that a practical and accurate method of evaluating the integral for small degrees of freedom was available, but this required a considerable amount of computational labor. For larger degrees of freedom the approximation of $\int_0^{w_o} f(w)dw$ by the integral $\int_0^{w_o} g(w)dw$, where g(w) is the density function of the ratio of two independent F's, was suggested. It was illustrated numerically that this approximation consistently led to values of $\int_0^{w_o} f(w)dw$ that were too high, for given w_o. This suggests that some correction factor should be sought, which when subtracted from the approximate values of this probability integral will give values equal to or very near the true values. No such correction factor can be suggested at this stage. Although it will require quite comprehensive numerical work to check the efficiency of such a factor which may be suggested, the resulting gain in accuracy together with the relative simplicity of the methods available to evaluate the approximation to the probability integral of two F''s, should make it worth the while.

A table of upper 5% points of the function, g(w), was constructed. The use of this table was considered and illustrated in examples relating to the sensory testing of foods, field experimentation and the comparison of multiple correlation coefficients. Thus, some indication was given of the ways in which the theory developed in this paper could be used to increase our knowledge of experiments which may be conducted in these fields.

It should be remarked that further tables of percentage points are required. Tables of both upper and lower 2.5%, 1% and 0.5% points would be the most important. Tables of exact percentage points of the distribution function of the ratio of two independent F''s would be preferable, but in their absence tables of percentage points of the approximation function could be used.

The theory was extended to cover the more complicated case where the degrees of freedom of the two F''s or two ordinary F's are no longer the same. It was shown that the evaluation of the probability integrals and the calculation of percentage points followed essentially the same lines as in the simpler case of equal degrees of freedom.

The tests of significance formulated in this paper can be conducted whenever two independent F-variates, in the null or non-null case, are available. The theory was based on certain assumptions, and investigations to ascertain whether such tests resulting in correct deductions, could still be conducted when all the assumptions are not met, should be undertaken. An answer should for instance be sought to the question whether the use of discrete scoring scales--which are commonly used in subjective testing--will lead to vastly different deductions and invalidate the procedures based on the assumption of continuous scales.

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The Comparison of the Sensitivities
of Experiments Using Different
Scales of Measurement

by

Daniel Eberhardt Waldemar Schumann

ABSTRACT

THE COMPARISON OF THE SENSITIVITIES OF
EXPERIMENTS USING DIFFERENT SCALES
OF MEASUREMENT

by

Daniel Eberhardt Waldemar Schumann, B. Com., M. Com.

Thesis submitted to the Graduate Faculty of the
Virginia Polytechnic Institute
in candidacy for the degree of

DOCTOR OF PHILOSOPHY

in

STATISTICS

In subjective testing where scoring scales are used to measure the effects of experimental treatments, methods are required for comparing the efficiencies or sensitivities of different scales. A comparison of the sensitivites of two scales should depend both on the experimental errors associated with the scales and on the magnitudes of the treatment effects scored on them.

We define as the more sensitive scale that one which accentuates treatment differences. If we assume that real treatment differences exist and that we may use analysis of variance to detect and measure them, the extent of the differences will be measured by the parameter of non-centrality, λ, of a variance ratio F'. If λ_1 and λ_2 are the parameters of non-centrality associated with two independent F'-variates, we need a test of significance of the null-hypothesis that $\lambda_1 = \lambda_2 = \lambda$. Such a test could be based on the statistic w, the ratio of two F'-variates.

Two independent F''s, both with 2a and 2b degrees of freedom are considered and $f(w)$, the density function of their ratio, is derived. This is written in various forms including that of hypergeometric functions. The properties of $f(w)$ and in particular the moments of w

are investigated. The results are subsequently generalized to the case where the two F''s have different degrees of freedom.

Under the null-hypothesis an approximation to $f(w)$ is suggested in the form of $g(w)$, the density function of the ratio of two independent central F's both with $2a' = 2(a+\lambda)^2/(a+2\lambda)$ and $2b$ degrees of freedom. The density function $g(w)$ is derived in various forms and its properties are investigated. The lower moments relating to the two density functions tend to equality for large degrees of freedom, provided λ is not large.

Methods of evaluating the probability integrals, $\int_0^{w_0} f(w)dw$ and $\int_0^{w_0} g(w)dw$, are considered. For small degrees of freedom a practical method of evaluating $\int_0^{w_0} f(w)dw$ is available. For larger degrees of freedom the approximation of this probability integral by $\int_0^{w_0} g(w)dw$ is suggested, since the evaluation of the latter integral is much easier. It is demonstrated numerically that the approximation gives good results even for small degrees of freedom. The evaluation of the probability integrals for the distributions of the ratios of two independent F''s or F's with different degrees of freedom follows essentially the same lines as the case of variates with equal degrees of freedom.

Some of the upper 5% points for f(w) are found and a comprehensive table of upper 5% points for g(w) is given. It is indicated that this table can be used to conduct, at least approximately, tests of significance of the hypotheses that two scoring scales are equally sensitive, or that two population correlation coefficients are equal under the usual assumptions of multiple regression, and similar hypotheses. Applications of the theory developed in the paper to the sensory testing of foods, in field experimentation and in the comparison of correlation coefficients are illustrated and some tests of significance are conducted. It is shown how the theory will increase our knowledge of experiments conducted under conditions where analysis of variance techniques apply.