SOME NONPARAMETRIC TESTS FOR CONSTANCY OF REGRESSION RELATIONSHIPS OVER TIME

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I. INTRODUCTION

This dissertation is a preliminary investigation in the development of non-parametric procedures for testing the constancy of regression relationships over time. Assume that Y_1, Y_2, ... is a sequence of independent random variables such that

$$Y_i = \underline{\beta}_i \underline{X}_i + \varepsilon_i, \quad i = 1, 2, \ldots \tag{1.1}$$

where $\underline{\beta}_i$ is a k-dimensional vector of unknown regression constants, $\underline{X}_i$ is a k-dimensional vector of known regressor variables, and ε_i is an error random variable having an absolutely continuous cumulative distribution function $F(z) = P[\varepsilon_i \leq z]$ with location parameter (mean or median) zero. Further assumptions about F(z) will be made depending upon the procedure under investigation. For instance, one procedure requires that F(z) be symmetric, another procedure requires that the first two moments exist.

When observations are made from a stochastic system over time where a regression model is proposed, the assumption may be made that the regression constants $\underline{\beta}_i$ do not change. That is, the assumption may be made that $\underline{\beta}_1 = \underline{\beta}_2 = \ldots = \underline{\beta}$, say. The development of suitable non-parametric procedures to test this assumptions is important in view of the many disciplines employing statistical techniques with data for which the assumption of even near-normality is not valid. This investigation develops essentially one procedure which is examined under three different sets of assumptions on the distribution function $F(z)$.

Although the assumption of constancy, $\underline{\beta}_1 = \beta_i = \ldots$ is being tested, the proposed procedure requires that the model is stable for some initial segment of time. That is, the assumption is made that $\underline{\beta}_1 = \ldots = \underline{\beta}_m = \underline{\beta}$, for some $m \geq k$, where $\underline{\beta}$ is unknown. Such an assumption may be quite reasonable in, say, a quality control situation where the first m observations are closely monitored for proper operation and then allowed to operate under automatic controls.

The first step of the procedure is to estimate $\underline{\beta}$ based on the first m observations $Y_1, \ldots, Y_m$. Call this estimate $\hat{\underline{\beta}}_0$. For the next N observations $Y_{m+1}, \ldots, Y_{m+N}$ the residuals $Y_{m+n} - \hat{\underline{\beta}}_0' \underline{x}_{m+i}$ are calculated. As the n^{th} residual becomes available a score, $\psi_n(Y_{m+i} - \hat{\underline{\beta}}_0 \underline{x}_{m+i})$, $1 \leq i \leq n \leq N$, is assigned to all residuals observed to that point. To detect a shift in the model we examine the statistic

$$R_1 = \max_{1 \leq n \leq N} \left| \sum_{i=1}^{n} \psi_n (Y_{m+i} - \hat{\underline{\beta}}_0' \underline{x}_{m+i}) - \mu_n^* \right| , \qquad (1.2)$$

where μ_n^* is a centering constant depending on n and ψ. A one-sided test could of course be done using

$$R_1 = \max_{1 \leq n \leq N} \left(\sum_{i=1}^{n} \psi(Y_{m+i} - \hat{\underline{\beta}}_0' \underline{x}_{m+i}) - \mu_n^* \right) \qquad (1.3)$$

or

$$R_1 = \min_{1 \leq n \leq N} \left(\sum_{i=1}^{n} \psi(Y_{m+i} - \hat{\underline{\beta}}_0' \underline{x}_{m+i}) - \mu_n^* \right) \qquad (1.4)$$

If R_1 indicates no shift in the model then a new pooled estimate of $\underline{\beta}$, $\hat{\underline{\beta}}_1$, is made based on all m + N observations, and for the next N observations the statistic

$$R_2 = \max_{1 \leq n \leq N} \left| \sum_{i=1}^{n} \psi_n(Y_{m+N+i} - \hat{\underline{\beta}}_1{}' \underline{x}_{m+N+i}) - \mu_n^* \right|$$

is examined for indications of a shift in the model. The procedure is continued as long as R_j, j = 1, 2, ..., indicates no shift in the model. Henceforth R_j will be referred to as the statistic of the jth stage of the test, and the values $Y_{m+(j-1)N+1}$, ..., Y_{m+jN} will be called the observations of the jth stage. Figure 1.1 illustrates this procedure through the first two stages.

The remainder of the introduction is a brief outline of the remaining chapters.

Chapter II contains a review of previous work in both parametric and non-parametric procedures for detecting a model shift. Parametric procedures for detecting a shift in regression coefficients have been developed, but the non-parametric literature concerns only detection of a shift in the median.

Chapter III is a review of the theoretical background used in the main body of the text.

Chapters IV-VII contain the main results of this work. In Chapter IV, Brownian motion results are derived in the general linear regression case for two scores functions:

i) $\psi_n(z) = \eta(z) = 1$ if $z \geq 0$, 0 otherwise, and

ii) $\psi_n(z) = z$.

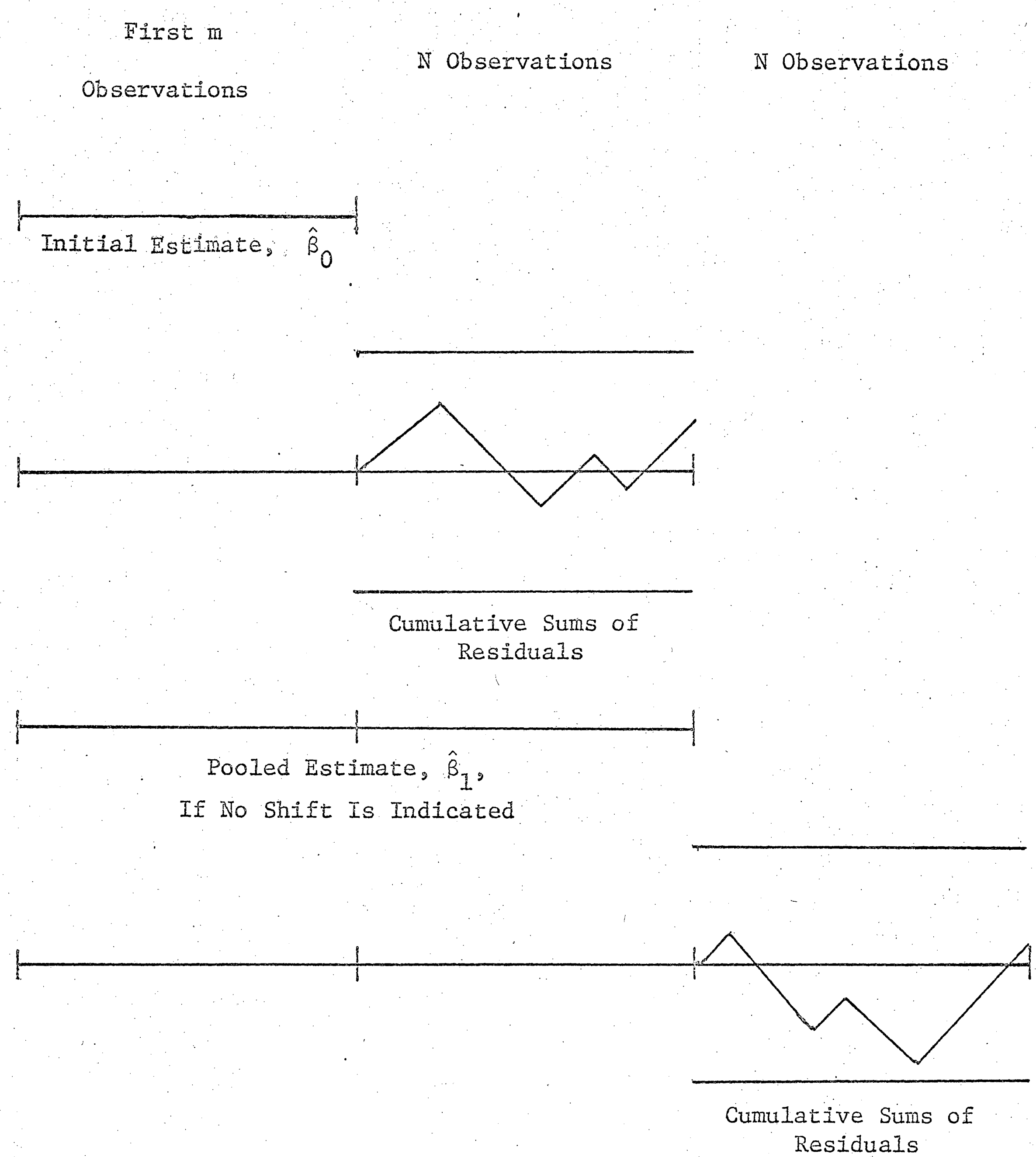


Figure 1.1. Illustration of First Two Stages of Proposed Procedure for Testing the Constancy of Regression Relationships Over Time.

The procedure described by i) is a cumulative sum sign test, and the procedure in ii) is a cumulative sum test. The Brownian motion results of theorems 4.1 and 4.2 enable the user to obtain approximate critical values for each stage of the test.

Chapter V contains Brownian motion results for a sequential Wilcoxon procedure for testing the constancy of an unknown median. That is, under the model $Y_i = \alpha_0 + \varepsilon_i$, $i = 1, \ldots, m$ and $Y_{m+i} = \alpha_i + \varepsilon_{m+i}$, $i = 1, \ldots, N$, Brownian motion results are obtained for the procedure with

$$\psi_n (Y_{m+i} - \hat{\alpha}_0) = \eta(Y_{m+i} - \hat{\alpha}_0)\, R_n(|Y_{m+i} - \hat{\alpha}_0|) ,$$

where $R_n(|Y_{m+i} - \hat{\alpha}_0|)$ is the rank of $|Y_{m+i} - \hat{\alpha}_0|$ among $|Y_{m+1} - \hat{\alpha}_0|$.

In Chapters IV and V the asymptotic theory shows that under the condition $N/m \to K^2$ as $N \to \infty$, approximate critical values can be obtained from Brownian motion critical values which are corrected for the factor K^2.

Chapter VI contains the derivation of some asymptotic results under alternatives; in particular a definition of asymptotic relative efficiency is given and some comparisons are made.

In Chapter VII exact results are obtained for the cumulative sum sign test of Chapter IV when testing for constant median.

Chapter VIII is concerned with the programming used to generate the tables of corrected Brownian motion critical values and the tables of comparisons of actual stagewise α-levels.

The final chapter is devoted to a discussion of the results and suggestions for further research in the area. As might be expected, this dissertation raises more questions than it answers.

II. LITERATURE REVIEW

Procedures for detecting a shift in model parameters fall into three classes

i) Tests for change at an unknown time in an infinite sequence of observations;

ii) Tests for change in a fixed length, finite sequence of observations;

iii) Standard hypothesis tests where the change, if any, occurs at the start.

The procedures outlined below fall in the second and third categories, whereas the proposed procedures fall in the first category.

In a recent survey article by Brown, Durbin, and Evans (1975), five techniques are outlined for testing the constancy of regression relationships over time. All of these techniques are presented under the assumption that the error terms are independent, identically distributed normal variates with zero mean. Two of the techniques proposed utilize recursive residuals - a transformation of the data to a sequence of independent normal variates with mean zero and common variance σ^2. This is done as follows:

As in the previous chapter the model is written $Y_i = \underline{\beta}_i' \underline{x}_i + \varepsilon_i$, $i = 1, \ldots, T$, where $\underline{\beta}_i$ is a k-dimensional vector of unknown regression constants. Note that the indexing set here is finite. Assuming $\underline{\beta}_1 = \ldots = \underline{\beta}_T = \underline{\beta}$, the least squares estimate of $\underline{\beta}$ based on the first r observations is $\underline{b}_r = (X_r' X_r)^{-1} X_r' \underline{Y}_r$, where $X_r = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_r)^1$, $\underline{Y}_r = (Y_1, \ldots, Y_r)'$, $k \leq r \leq T$. The recursive residuals are defined as

$$W_r = \frac{Y_r - \underline{b}_{r-1}' \underline{x}_r}{\sqrt{1 + \underline{x}_r'(X_{r-1}' X_{r-1})^{-1} \underline{x}_r}} , \quad k + 1 \leq r \leq T . \qquad (2.1)$$

Under the hypothesis $\underline{\beta}_1 = \ldots = \underline{\beta}_T = \underline{\beta}$ the W_r are independent normal variates with mean zero and variance σ^2.

The first of the procedures outlined by Brown, et al., is the plot of the CUSUM quantity

$$W_r = \frac{1}{\hat{\sigma}} \sum_{j=k+1}^{r} W_r \; , \; r = k+1, \ldots, T \; ,$$

where $\hat{\sigma}^2$ is the least squares estimate of σ^2 based on the residuals $Z_i = Y_i - \underline{b}_T' \underline{x}_i$, $1 \leq i \leq T$. Critical values for the process are obtained by Brownian motion approximation methods.

The other proposed procedure involving recursive residuals is the CUSUM of squares test:

$$S_r = \left(\sum_{j=k+1}^{r} W_j^2 \right) / \left(\sum_{j=k+1}^{T} W_j^2 \right) \; , \; r = k+1, \ldots, T$$

which the authors claim is "... a useful complement to the CUSUM test, particularly when the departure from constancy of the $\underline{\beta}$'s is haphazard rather than systematic". Under the hypothesis of constancy of the $\underline{\beta}$'s S_r follows a beta distribution with mean $(r-k)/(T-k)$, and using this, critical values can be obtained.

A third proposed procedure is the moving regression technique where the interval 1, ..., T is assumed to consist of p non-overlapping intervals of length n. The usual F-test for parallelism of the regression planes is suggested for detecting a shift in the model.

Another technique is called time-trending regressions in which a list of models having regression coefficients expressed as coefficients of polynomials of time is proposed. This list of models is fitted sequentially to determine which polynomial fit is best described by the data. That is, for $1 \leq t \leq T$ we have the following list of models:

$$Y_t = \underline{\beta}_0' \, \underline{x}_t + \varepsilon_t$$

$$Y_t = (\underline{\beta}_0 + \underline{\beta}_1 t)' \, \underline{x}_t + \varepsilon_t$$

$$\vdots$$

$$Y_t = (\underline{\beta}_0 + \underline{\beta}_1 t + \ldots + \underline{\beta}_e t^e)' \, x_t + \varepsilon_t \; .$$

Using all T observations the model list is fitted sequentially in order of increasing degree and each successive fit is compared to the previous one to determine whether the higher degree polynomial gives a better fit than the lesser degree polynomial.

The last procedure, due to Quandt (1958, 1960) and outlined by Brown, et al., is used to detect a single abrupt shift in the model at some unknown time point r, $1 \leq r \leq T$. For each r the likelihood is maximized for the null hypothesis (no change in regression constants at time r) and for the alternative hypothesis (a single shift in the model has occurred at time r). Then λ_r, the logarithm of the ratio of these two likelihoods is calculated, this logarithm being the statistic for detecting a shift. If λ_r is significant, the point r is taken as the point of shift in the mean.

Farley and Hinich (1970) have also proposed a likelihood ratio test to detect a shift in the model $Y_k = \alpha_k + \gamma_k X_k + \varepsilon_k$, $k = 1, \ldots, T$. This test is limited to the case of a single regressor variable. Farley, Hinich, and McGuire (1975) extend this test to the multivariate case. Unlike Quandt's likelihood ratio test in which a number of likelihood ratios must be calculated, the Farley-Hinich test assumes that the shift point is uniformly distributed over the time interval $1, \ldots, T$.

That is, the probability of shift at time r is assumed to be 1/T. Using this prior distribution on the shift point, a single likelihood ratio test statistic is calculated.

Nonparametric tests in the literature are sparse; apparently, the only nonparametric work in the literature concerns tests for shift of location.

The first such test appearing in the literature was proposed by Page (1955) who considered a test of the form

$$m = \max_{0 \le r \le n} (S_r - \min_{0 \le i < r} S_i)$$

where

$$S_r = \sum_{i=1}^{r} \eta(Y_i - \theta) ,$$

with $\eta(x) = 1$ if $x \geq 0$, 0 otherwise, and $Y_1, \ldots, Y_n$ are observed for a shift in known initial mean, θ.

A fixed sample test for a shift in unknown mean, similar to Page's test, was proposed by McGilchrist and Woodyer (1975). This test examines the statistic

$$\mu = \max_{1 \le n \le N} \sum_{i=1}^{n} \eta(Y_i - \hat{\theta})$$

where $\hat{\theta}$ is the median of the Y values.

Bhattacharyya and Johnson (1968) investigate tests for shift in median at an unknown time point from a locally best invariant test view point, both when the initial level is known and unknown. The approach is to assume a prior distribution on the location of the shift and then to develop tests which maximize local average power against specific translation alternatives. Assuming a uniform distribution on the point of shift, when the initial process level is known to be zero, the locally best invariant tests are

$$N^{-1} \sum_{i=1}^{N} i \operatorname{sgn}(Y_i)$$

for the double exponential distribution;

$$N^{-1} \sum_{i=1}^{N} i \operatorname{sgn}(Y_i) R_i$$

for the logistic distribution, when R_i is the rank of $|Y_i|$ among $|Y_1|, \ldots, |Y_N|$; and

$$N^{-1} \sum_{i=1}^{N} i \operatorname{sgn}(Y_i) E(W^{(R_i)})$$

for the normal distribution, where $W^{(1)} < \ldots < W^{(N)}$ is an ordered sample from a χ_1 distribution.

A sequential nonparametric test for shift of a known median has been introduced by Miller (1970) and further developed by Reynolds (1975). The procedure proposed by Reynolds is a truncated sequential test which examines the statistic

$$\max_{1 \leq n \leq N} \left| \sum_{i=1}^{n} \frac{1}{i+1} \eta(Y_i - \theta) R_{ii}^{+} \right|$$

where R_{ii}^{+} is the rank of $|Y_i - \theta|$ among $|Y_1 - \theta|, \ldots, |Y_i - \theta|$. The statistic proposed by Miller was

$$\max_{1 \leq n \leq N} \left| \sum_{i=1}^{n} \eta(Y_i - \theta) R_{ii}^{+} \right|$$

where R_{in}^{+} is the rank of $|Y_i - \theta|$ among $|Y_1 - \theta|, \ldots, |Y_n - \theta|$. The Reynolds procedure uses sequential signed ranks, whereas the Miller procedure uses the usual Wilcoxon statistic calculated sequentially. The sequential signed ranks have the property of being independent under the null hypothesis.

Nonparametric tests for a shift in regression coefficients have not yet appeared in the literature, although an ad hoc procedure for certain situations may employ such results as Adiche's (1975) which is a test for equality of c regression equations.

The procedures developed in this work are sequential in nature, yet there is a subtle difference between the testing situation here and the testing situation of classical sequential analysis as introduced by Wald (1945). The classical approach is to choose between two or more hypotheses. For instance, choosing between H_1: $\theta = \theta_1$ and H_2: $\theta = \theta_2$, or H_1: $\theta > \theta_1$ and H_2: $\theta < \theta_2$. The decision after stopping is to accept either H_1 or H_2. In the procedures proposed in this dissertation we are interested in testing H_0: $\theta = \theta_0$ against H_1: $\theta > \theta_0$, say, however, the decision after stopping is that we reject H_0. The procedure continues sampling as long as H_0 is not rejected. The classical sequential procedures are not applied to the classical hypothesis testing problem of H_0: $\theta = \theta_0$ vs. H_1: $\theta > \theta_0$, say, because, as stated by Rao (1950), "It is, clearly, not possible to accept the null hypothesis before an indefinitely large sample is observed." This was part of his justification for the study of truncated sequential tests in the hypothesis testing situation. However, the problem with truncating the sequential test is: What does one do if one wishes to determine whether or not observations taken beyond the truncation point will refute H_0? Clearly, the situation we have in mind is to continue sampling as long as H_0 is not refuted, and to stop sampling only when it is refuted.

A further point should be made that in the testing situation of this study the null hypothesis of no shift can never be accepted because the alternative hypothesis states that a shift can occur at any time.

Recall that the null hypothesis under consideration is that the model describing the underlying process remains constant, against the alternative hypothesis that the model shifts at some point in time. One of the difficulties encountered in simply adapting classical sequential tests to our hypothesis testing situation is that if the shift occurs late in a long sequence of observations, then the statistic may well be overburdened with past history: the preshift information outweighs the shift information, having a tendency to mask the shift.

The procedures proposed in this dissertation attempt to remedy the problem of preshift information masking the effect of shifting. The initial parameters are unknown and must be estimated. The estimates continue to utilize all past information so that a shift occurring late in the observations will not shift the estimates considerably--the estimates will still be essentially estimating preshift parameters. However, the testing is done with only the most recent observations, and so will not be burdened by the preshift observations.

III. MATHEMATICAL PRELIMINARIES

This Chapter is a summary of the theoretical material used in the following chapters on asymptotic results. Recall the definition

$$R_1 = \max_{1 \le n \le N} \left| \sum_{i=1}^{n} \psi_n(Y_{m+i} - \hat{\underline{\beta}}_0' \underline{x}_{m+i}) - \mu_n^* \right| .$$

The asymptotic distribution of R_1 can be obtained using the theory of weak convergence and the properties of Brownian motion. Let

$$S_{mn} = \sum_{i=1}^{n} \psi(Y_{m+i} - \hat{\underline{\beta}}_0' \underline{x}_{m+i}) \; , \; 1 \le n \le N \quad ,$$

and define $S_{mN}(t)$ as the random polygon on the interval [0, 1] with vertices at n/N defined by $S_{mn} - \mu_n^*$. The main results of Chapter IV are those proving that $S_{mN}(t)$, properly standardized, converges to the sum of two independent stochastic processes on [0, 1], one being the Brownian motion process and the other being a straight line through the origin with random slope.

3.1 The Hájék Projection Lemma

The first step in the proof involves obtaining an analytically tractable approximation, $\hat{S}_{mN}(t)$, to the random function $S_{mN}(t)$. To accomplish this a functional form of the Projection Lemma of Hájék (1968) is used. The Lemma as presented in Hájék is presented here:

Projection Lemma (Hájék, 1968). Let $Y_1, \ldots, Y_n$ be independent random variables and $S = S(Y_1, \ldots, Y_n)$ be a statistic such that $E[S^2] < \infty$. Let

$$\hat{S} = \sum_{i=1}^{n} E[S|Y_i] - (n-1)E[S] . \tag{3.1.1}$$

Then $E[\hat{S}] = E[S]$, and

$$E[(S-\hat{S})^2] = \mathrm{Var}[S] - \mathrm{Var}[\hat{S}] . \tag{3.1.2}$$

Moreover, if $L = \sum_{i=1}^{n} \ell_i(Y_i)$, where ℓ_i is chosen arbitrarily such that $E[\ell_i^2(Y_i)] < \infty$, $1 \leq i \leq n$, then

$$E[(S-L)^2] = E[(S-\hat{S})^2] + E[(\hat{S}-L)^2] .$$

A multivariate extension of this Lemma is given in Pirie (1974), where the $Y_1, \ldots, Y_n$ are independent random vectors. The multivariate form will be used in the proof of Theorem 4.1.

§3.2 Weak Convergence

The next step in the proof demonstrates that $\hat{S}_{mN}(t)$ converges weakly to the sum of Brownian motion and a random straight line. The weak convergence results used in the proofs are given. With the exception of a Lemma of Prohorov (1956), all the results of this Section can be found in Billingsley (1968).

Let $C[0, 1]$ be the space of all continuous functions on the interval $[0, 1]$ with the metric

$$\rho(\alpha(t),\ y(t)) = \sup_t\ |x(t) - y(t)|\ , \qquad (3.2.1)$$

$$x(t),\ y(t)\ \varepsilon\ C[0,\ 1]\ .$$

Let P_n and P be probability measures on C[0, 1], then P_n is said to converge weakly to P, written $P_n \Rightarrow P$, if $\int f dP_n \to \int f dP$ for every bounded, continuous function f on C[0, 1].

To show that $P_n \Rightarrow P$, Theorems 8.1 and 8.2 of Billingsley provide the following sufficient conditions:

A. The finite dimensional distributions of P_n converge to those of P;

B. i) For each $\eta > 0$, there exists an $a > 0$ such that $P_n\{x:\ |x(0)| > a\} \leq \eta,\ n \geq 1$;

ii) For each $\eta > 0$ and $\varepsilon > 0$, there exist a δ satisfying $0 < \delta < 1$, and an integer n_0 such that

$$P_n\{x:\ w_x(\delta) \geq \varepsilon\} \leq \eta\ ,\ n \geq n_0 \qquad (3.2.2)$$

where

$$w_x(\delta) = \sup_{|s-t|<\delta} |x(s) - x(t)|\ .$$

A sequence $\{P_n\}$ satisfying B is said to be tight. If X_n is a random element (random vector, function, etc.) with probability distribution P_n, then $\{X_n\}$ is also said to be tight. A complete discussion of tightness can be found in Billingsley.

Wiener measure is a probability on the space C[0, 1] such that

i) For each $t\ \varepsilon\ [0,\ 1]$, X(t) is normally distributed with mean μt and variance $\sigma^2 t$;

ii) If $0 \leq t_0 \leq t_1 \leq \ldots \leq t_k \leq 1$, then the random variables $X(t_1) - X(t_0), \ldots, X(t_k) - X(t_{k-1})$ are independent. To denote that some random function $X(\cdot)$ is a Wiener process as defined above, the notation

$$X \sim W(\mu, \sigma^2) \tag{3.2.3}$$

will be used.

The following theorem due to Donsker (1951) will be used to establish weak convergence of the projection statistic $\hat{S}_{mN}(t)$. The form of the theorem as stated is found in Billingsley (1968), Theorem 10.1.

Donsker's Theorem: Let $\xi_1, \xi_2, \ldots$ be a sequence of random variables which are independent and identically distributed with mean zero and variance σ^2. Let $S_n = \xi_1 + \ldots + \xi_n$ and define a random polygon in $C[0, 1]$ by

$$X_n(t) = \frac{1}{\sigma\sqrt{n}} S_{[nt]} + (nt - [nt]) \frac{1}{\sigma\sqrt{n}} \xi_{[nt+1]} \tag{3.2.4}$$

where $[x]$ denotes the largest integer less than or equal to x.

Then the function $X_n(t)$ converges weakly to $X(t)$ where $X(t) \sim W(0, 1)$.

The following theorem provides a sufficient condition for tightness which is used in the proof of Theorem 4.1.

Theorem 3.1 (Billingsley (1968), p. 59): Suppose $\{X_n(t)\}$ is defined as in (3.2.4). The sequence $\{X_n(t)\}$ is tight if for each $\varepsilon > 0$ there exists a $\lambda > 1$ and an integer n_0 such that if $n \geq n_0$ then

$$P\{\max_{i\leq n} |S_{k+i} - S_k| \geq \lambda\sigma\sqrt{n}\} \leq \varepsilon/\lambda^2 \tag{3.2.5}$$

holds for all k.

Another lemma which proves useful in the proof of Theorem 4.1, also found in Billingsley, p. 69, will be presented along with its proof. The technique used in the proof is the important point in the proof of Theorem 4.1.

<u>Lemma 3.2.1</u> Let $\xi_1, \ldots, \xi_m$ be independent random variables with mean 0 and finite variances σ_i^2; put $S_i = \xi_1 + \ldots + \xi_i$ and $s_i^2 = \sigma_1^2 + \ldots + \sigma_i^2$. Then

$$P\{\max_{1\leq m} |S_i| \geq \lambda s_m\} \leq 2P\{|S_m| \geq (\lambda - \sqrt{2})s_m\} . \tag{3.2.6}$$

<u>Proof</u>: Consider the sets

$$E_i = \{\max_{j<i} |S_j| < \lambda s_m \leq |S_i|\} .$$

Clearly,

$$P\{\max_{i\leq m} |S_i| \geq \lambda s_m\} \leq P\{|S_m| \geq (\lambda - \sqrt{2})s_m\}$$

$$+ \sum_{i=1}^{m-1} P[E_i \cap \{|S_m| < (\lambda - \sqrt{2})s_m\}] . \tag{3.2.7}$$

Since $|S_i| \geq \lambda s_m$ and $|S_m| < (\lambda - \sqrt{2})s_m$ together imply $|S_m - S_i| \geq \sqrt{2}\, s_m$, it follows by Chebyshev's inequality and the assumed independence of the ξ_i that the sum in (3.2.7) is at most

$$\sum_{i=1}^{m-1} P(E_i)P[|S_m - S_i| \geq \sqrt{2}\, s_m]$$

$$\leq \sum_{i=1}^{m-1} P(E_i) \frac{1}{2s_m^2} \sum_{k=i+1}^{m} \sigma_k^2$$

$$\leq \frac{1}{2} \sum_{i=1}^{m-1} P(E_i) \leq \frac{1}{2} P[\max_{i \leq m} |S_i| \geq \lambda\, s_m] \,. \qquad (3.2.8)$$

Now (3.2.7) and (3.2.8) combine to give (3.2.6).

Another theorem which will be useful in proving weak convergence is due to Prohorov (1956). Prohorov's Theorem establishes weak convergence of the random polygon $X_n(t)$ when instead of a single sequence $\xi_1, \xi_2, \ldots$ one must use a double sequence $\xi_{n1}, \ldots, \xi_{nk_n}$ of random variables which are independent for each n, and $\lim_{n\to\infty} k_n = \infty$. The following conditions are assumed:

i) The random variables ξ_{nk} are subject to the condition of asymptotic negligibility: for all $\varepsilon > 0$

$$\lim_{n\to\infty} \max_{1\leq k\leq k_n} P[|\xi_{nk}| > \varepsilon] = 0 \;; \qquad (3.2.9)$$

ii) $E[\xi_{nk}] = 0$, $Var[\xi_{nk}] = \sigma_{nk}^2 < \infty$,

$$\sum_{k=1}^{k_n} \sigma_{nk}^2 = 1 \,. \qquad (3.2.10)$$

<u>Prohorov's Theorem</u>: If the requirements (3.2.9) and (3.2.10) are met, then in order that

$$X_n(t) = X(t) \ , \ X(t) \sim W(0, 1)$$

the condition of Lindeberg is necessary and sufficient: For every $\lambda > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{[|x|>\lambda]} x^2 \, dF_{n,k}(x) = 0 \ ,$$

where

$$F_{n,k}(x) = P[\xi_{n,k} \leq x] \quad .$$

§ 3.3 Asymptotic Normality of Parameter Estimates

A fundamental assumption of the asymptotic results of Chapter IV is that the estimates $\hat{\underline{\beta}}_0$ are asymptotically normal. For the non-parametric procedures, estimates of $\underline{\beta}$ will be of the type proposed by Hodges and Lehmann (1963) for location parameters. This type of estimator was extended to the simple linear regression case, $y = \alpha + \beta x + \varepsilon$, by Adiche (1967), and to the general linear regression case by Jurecková (1971). Two theorems on asymptotic normality of parameter estimates are presented here, the first by Bickel (1964) gives a concise statement of asymptotic normality of the median of the observations and of the Hodges and Lehmann estimator of location, the second by Jurecková (1971) is presented to indicate conditions on the $\underline{x}_i$ values sufficient for asymptotic normality of $\hat{\underline{\beta}}_0$ in the general linear regression case.

The essence of the Hodges-Lehmann type estimator is that if one has a statistic $T(Y_1, \ldots, Y_n)$ for testing hypotheses about an unknown location parameter θ, and if $E[T(Y_1 - \theta, \ldots, Y_n - \theta)] = 0$, then the Hodges-Lehmann type estimator based on T for θ is a value $\hat{\theta}$ such that $|T(Y_1 - \hat{\theta}, \ldots, Y_n - \hat{\theta})|$ is minimum. For the general linear regression case this is generalized as follows by Jurecková:

The model (1.1) is assumed to hold with $\underline{\beta}_1 = \ldots = \underline{\beta}_m = \underline{\beta}$. Another way to write this model is

$$Y_i = \beta_1 x_{i1} + \ldots + \beta_k x_{ik} + \varepsilon_i \, . \tag{3.3.1}$$

A vector of statistics for testing H_0: $\underline{\beta} = \underline{\beta}^*$ is

$$\underline{S} = (S_{N1}(Y_1 - \underline{\beta}^{*\prime}\underline{x}_1, \ldots, Y_n - \underline{\beta}^{*\prime}\underline{x}_n), \ldots, S_{Nk}(Y_1 - \underline{\beta}^{*\prime}\underline{x}_1, \ldots, Y_n - \underline{\beta}^{*\prime}\underline{x}_n)) \tag{3.3.2}$$

where

$$S_{Nj}(Y_1 - \underline{\beta}^{*\prime}\underline{x}_1, \ldots, Y_n - \underline{\beta}^{*\prime}\underline{x}_n) = \sum_{i=1}^{n} (x_{ij} - \bar{x}_j) a_n(R_i^*) \ , \tag{3.3.3}$$

$a_n(\cdot)$ is a scores function and R_i^* is the rank of $Y_i - \underline{\beta}^{*\prime}\underline{x}_i$ among $Y_1 - \underline{\beta}^{*\prime}\underline{x}_1, \ldots, Y_n - \underline{\beta}^{*\prime}\underline{x}_n$. Under the hypothesis H_0: $\underline{\beta} = \underline{\beta}^*$, $E[a_n(R_1^*)] = \ldots = E[a_n(R_n^*)]$, so $E[S_{Nj}(Y_1 - \underline{\beta}^{*\prime}\underline{x}_1, \ldots, Y_n - \underline{\beta}^{*\prime}\underline{x}_n)] = 0$.

Jurecková proposes that any point in the set

$$D_n = \{\dot{\underline{\beta}} = (\dot{\beta}_1, \ldots, \dot{\beta}_k): \sum_{j=1}^{k} |S_{Nj}(Y_1 - \dot{\underline{\beta}}'\underline{x}_1, \ldots, Y_n - \dot{\underline{\beta}}'\underline{x}_n)| = \min\} \tag{3.3.4}$$

is a suitable estimate of $\underline{\beta}$.

Let $\hat{\underline{\beta}}$ be any point in D_n, then Jurecková establishes asymptotic normality of $\hat{\underline{\beta}}$ under the following three assumptions:

<u>Assumption 1</u>: $Y_1, \ldots, Y_n$ are independent random variables having absolutely continuous distribution function

$$P(Y_i \leq y) = F(y - \alpha - \underline{\beta}'\underline{x}_i) \; , \; i = 1, \ldots, n$$

where F has finite Fisher's information. That is, $\int_{-\infty}^{\infty} \left(\frac{f'(x)}{f(x)}\right)^2 f(x)dx < \infty$, with $f(x)$ being the density of F;

<u>Assumption 2</u>: The matrix of regressor variables $X_n = (x_{ij})$ $1 \leq i \leq n$, $i \leq j \leq k$ satisfies four conditions:

i) $x_{ij} = x^1_{ij} + \alpha^2_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq k$;

ii) The vectors $\underline{x}^1_{(j)} = (x^1_{1j}, \ldots, x^1_{nj})$ satisfy either

$$(\underline{x}^1_{(j)} - \bar{x}^1_j\underline{1})'(\underline{x}^1_{(j)} - \bar{x}^1_j\underline{1}) = 0 \qquad (3.3.5)$$

for all but a finite number of n, or

$$(\underline{x}^1_{(j)} - \bar{x}^1_j\underline{1})'(\underline{x}^1_{(j)} - \bar{x}^1_j\underline{1}) > 0 \qquad (3.3.6)$$

for all but a finite number of n; further

$$n^{-1}(x^1_{(j)} - \bar{x}^1_j\underline{1})'(\underline{x}^1_{(j)} - \bar{x}^1_j\underline{1}) \leq M_1, \; n = 1, 2, \ldots \qquad (3.3.7)$$

where $\underline{1}$ is an n-dimensional vector of 1's, and if (3.3.6) is satisfied, then

$$\lim_{n \to \infty} \max_{1 \leq i \leq n} (x^1_{ij} - \bar{x}^1_j)^2 / [\sum_{i=1}^{n} (x^1_{ij} - \bar{x}^1_j)^2] = 0 \; , \qquad (3.3.8)$$

where $\bar{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_{ij}$ and M_1 is a constant independent of n. Analogous assumptions are to be satisfied for the vectors $x^2_{(j)}$;

iii) For all pairs $j, \ell = 1, \ldots, k$,

$$(x_{rj}^1 - x_{sj}^1)(x_{r\ell}^1 - x_{s\ell}^1) \geq 0 \tag{3.3.9}$$

$$(x_{rj}^1 - x_{sj}^1)(x_{r\ell}^2 - x_{s\ell}^2) \leq 0 \tag{3.3.10}$$

$$(x_{rj}^2 - x_{sj}^2)(x_{r\ell}^2 - x_{s\ell}^2) \geq 0 \tag{3.3.11}$$

for all r, s = 1, ..., n, n = 2, 3, ...

iv) $\lim_{n\to\infty} n^{-1}(x_{(j)} - \bar{x}_j)'(x_{(\ell)} - \bar{x}_\ell) = \sigma_{j\ell}$ and $\Sigma = (\sigma_{j\ell})$ is a positive definite matrix.

Assumption 3: The scores $a_n(\cdot)$ in S_{nj} can be generated by either of the following:

i) $a_n(i) = E\ \psi(u_n^{(i)})$ (3.3.12)

ii) $a_n(i) = \psi(i/(n+1))$, (3.3.13)

where $u_n^{(i)}$ denotes the i^{th} order statistic in a sample of size n from a uniform distribution on (0, 1) and ψ is a scores-generating function which is square integrable on (0, 1).

Jurecková's Theorem: Under Assumptions 1-3,

$$N^{\frac{1}{2}}\ (\hat{\underline{\beta}} - \underline{\beta}) \sim AN(0,\ \gamma^{-2}A^2\Sigma^{-1})$$

where

$$\gamma^2 = (\textstyle\int_0^1 \psi(u)\psi(u,\ f)du)^2$$

$$A^2 = \textstyle\int_0^1 (\psi(u) - \bar{\psi})^2\ du$$

for

$$\psi(u,\ f) = -f'(F^{-1}(u))/f(F^{-1}(u))$$

$$\bar{\psi} = \textstyle\int_0^1 \psi(u)du\quad .$$

A brief statement about some of the conditions in Jureckovà's Theorem is in order. Note that (3.3.7) limits the rate of growth of the regressor variables in each of the k factors. What (3.3.9)-(3.3.11) state is that if the regressor variables $f_j(r)$, $r = 1, 2$ for the j^{th} factor are increasing with i then so must the $x_\ell(r)$, $r = 1, 2$ ℓ^{th} factor variables; further, if the x_j' variables are increasing, then the x_j^2 variables must be decreasing.

In the particular case of the constant model (1.9), when $T(Y_1 - \alpha, \ldots, Y_n - \alpha)$ is the ordinary sign test, the Hodges-Lehmann estimate is the median M_n of the observations $Y_1, \ldots, Y_n$. When $T(Y_1 - \alpha, \ldots, Y_n - \alpha)$ is the Wilcoxon signed rank test, the Hodges-Lehmann estimate is the median W_n of the $\frac{1}{2}n(n+1)$ averages $\frac{1}{2}(Y_i+Y_j)$, $i, j = 1, \ldots, N$. Then from Bickel (1964) we have:

Theorem: Given M_n and W_n as defined above,

i) if F(y) is absolutely continuous with density f(x), then

$$n^{\frac{1}{2}}(M_n - \alpha) \sim AN\left(0, \frac{1}{4f^2(0)}\right); \tag{3.3.14}$$

ii) $$n^{\frac{1}{2}}(W_n - \alpha) \sim AN\left(0, \frac{1}{12(\int_{-\infty}^{\infty} f^2(x)\,dx)^2}\right) \tag{3.3.15}$$

§3.4 A Lemma on U-Statistics

The material in this section is essential for the proof of the asymptotic results for the Wilcoxon procedure in the next Chapter.

Let $Y_1, \ldots, Y_n$ be independent, identically distributed random variables with cumulative distribution function $F(x)$. Let f be the class of all distribution functions and let $\theta(F)$ be a real valued functional defined on f. The functional $\theta(F)$ is called regular if there is an unbiased estimator, say $\phi(Y_1, \ldots, Y_n)$, for θ. That is,

$$E[\phi(Y_1, \ldots, Y_n)] = \theta(F) \text{ for each } F \in f. \qquad (3.4.1)$$

If (3.4.1) holds, $\theta(F)$ is said to be estimable. If $\theta(F)$ is estimable then there is a smallest sample size m for which (3.4.1) holds: m is called the degree of $\theta(F)$ and $\phi(Y_1, \ldots, Y_m)$ is called the kernel of $\theta(F)$. The kernel is regarded as being symmetric in its m arguments.

A symmetric, unbiased estimator, or U-statistic is defined by

$$U_n = \binom{n}{m}^{-1} \sum_S \phi(Y_{\alpha_1}, \ldots, Y_{\alpha_m}) \qquad (3.4.2)$$

where the region of summation, S, extends over all m-tuples $(\alpha_1, \ldots, \alpha_m)$ such that $1 \leq \alpha_1 < \ldots < \alpha_m \leq n$. Following the argument of Miller and Sen (1972), U_n can be decomposed as follows:

Write $\phi_h(Y_1, \ldots, Y_n) = E[\phi(y_1, \ldots, y_h, Y_{h+1}, \ldots, Y_m)]$, where the expectation is conditional on $y_1, \ldots, y_h$, then

$$U_n = \binom{n}{m}^{-1} \sum_S \phi(Y_{\alpha_1}, \ldots, Y_{\alpha_m})$$

$$= \binom{n}{m}^{-1} \frac{1}{m!} \sum_P \phi(Y_{\alpha_1}, \ldots, Y_{\alpha_m})$$

where $P = \{(\alpha_1, \ldots, \alpha_m): 1 \leq \alpha_1 \neq \ldots \neq \alpha_m \leq n\}$. Recall that

$\eta(x) = 1$ if $x \geq 0$, 0 otherwise, so U_n can be expressed in integral form as

$$U_n = \binom{n}{m}^{-1} \frac{1}{m!} \sum_P \int_m \dots \int \phi(y_1, \dots, y_m) d \prod_{j=1}^{m} \eta(y_j - Y_{\alpha_j})$$

$$= \frac{(n-m)!}{n!} \sum_P \int_m \dots \int \phi(y_1, \dots, y_m) d \prod_{j=1}^{m} [\eta\ (y_j - Y_{\alpha_j}) - F(y_j) + F(y_j)] \qquad (3.4.3)$$

Now the integrator in (3.4.3) is

$$\prod_{j=1}^{m} [\{\eta(y_j - Y_{\alpha_j}) - F(y_j)\} + F(y_j)]$$

$$= \prod_{j=1}^{m} [\eta(y_j - Y_{\alpha_j}) - F(y_j)] + \sum_{i=1}^{m} \{ \prod_{\substack{j=1 \\ j \neq i}}^{m} [\eta(y_j - Y_{\alpha_j}) - F(y_j)]\} F(y_i)$$

$$+ \sum_{i \neq i'} \sum \{ \prod_{\substack{j=1 \\ j \neq i, j \neq i'}}^{m} [\eta(y_j - Y_{\alpha_j}) - F(y_j)]\} F(y_i) F(y_{i'}) + \dots$$

$$+ \sum_{j=1}^{m} \{ \prod_{\substack{i=1 \\ i \neq j}}^{m} [\eta(y_j - Y_{\alpha_j}) - F(y_j)]\} F(y_i)$$

$$+ \prod_{i=1}^{m} F(y_i) .$$

When $\phi(y_1, \dots, y_m)$ is integrated with respect to each of these components the result is

$$\underset{m}{\int\cdots\int} \phi(y_1, \ldots, y_m) d \prod_{j=1}^{m} [\{\eta(y_j - Y_{\alpha_j}) - F(y_j)\} + F(y_j)]$$

$$= \underset{m}{\int\cdots\int} \phi(y_1, \ldots, y_m) d \prod_{j=1}^{m} [\eta(y_j - Y_{\alpha_j}) - F(y_j)]$$

$$+ \sum_{i=1}^{m} \underset{m}{\int\cdots\int} \phi(y_1, \ldots, y_m) d\{ \prod_{\substack{j=1 \\ j\neq i}}^{m} [\eta(y_j - Y_{\alpha_j}) - F(y_j)]\} F(y_i)$$

$$+ \sum_{i\neq i'}\sum \underset{m}{\int\cdots\int} \phi(y_1, \ldots, y_m) d\{ \prod_{\substack{j=1 \\ j\neq i, j\neq i'}}^{m} [\eta(y_j - Y_{\alpha_j}) - F(y_j)]\} F(y_i) F(y_i')$$

$$+ \ldots + \underset{m}{\int\cdots\int} \phi(y_1, \ldots, y_m)\ d \prod_{i=1}^{m} F(y_i)$$

$$= \underset{m}{\int\cdots\int} \phi(y_1, \ldots, y_m) d \prod_{j=1}^{m} [\eta(y_j - Y_{\alpha_j}) - F(y_j)]$$

$$+ m \underset{m-1}{\int\cdots\int} \phi_{m-1}(y_1, \ldots, y_{m-1}) d \prod_{j=1}^{m-1} [\eta(y_j - Y_{\alpha_j}) - F(y_j)]$$

$$+ \binom{m}{2} \underset{m-2}{\int\cdots\int} \phi_{m-2}(y_1, \ldots, y_{m-2}) d \prod_{j=1}^{m-2} [\eta(y_j - Y_{\alpha_j}) - F(y_j)]$$

$$+ \ldots + \theta(F)$$

$$= \theta(F) + \sum_{h=1}^{m} \binom{m}{h} \underset{h}{\int\cdots\int} \phi_h(y_1, \ldots, y_h) d \prod_{j=1}^{h} [\eta(y_j - Y_{\alpha_j}) - F(y_j)] \quad .$$

Now U_n can be written as

$$U_n = \frac{(n-m)!}{n!} \sum_{P} \int \cdots \int \phi(y_1, \ldots, y_m) d \prod_{j=1}^{m} [\eta(y_j - Y_{\alpha_j}) - F(y_j) + F(y_j)]$$

$$= \frac{(n-m)!}{n!} \sum_{P} \{\theta(F) + \sum_{h=1}^{m} \binom{m}{h} \underbrace{\int \cdots \int}_{h} \phi_h(y_1, \ldots, y_h) d \prod_{j=1}^{h} [\eta(y_j - Y_{\alpha_j}) - F(y_j)]\}$$

$$= \theta(F) + \sum_{h=1}^{m} \binom{m}{h} \left\{ \frac{(n-m)!}{n!} \sum_{P} \int \cdots \int \phi_h(y_1, \ldots, y_h) d \prod_{j=1}^{h} [\eta(y_j - Y_{\alpha_j}) - F(y_j)] \right\}$$

$$= \theta(F) + \sum_{h=1}^{m} \binom{m}{h} \left\{ \frac{(n-m)!}{n!} \frac{(n-h)!}{(n-m)!} \sum_{P_h} \underbrace{\int \cdots \int}_{h} \phi_h(y_1, \ldots, y_h) d \prod_{j=1}^{h} [\eta(y_j - Y_{\alpha_j}) - F(y_j)] \right\}$$

where $P_h = \{(\alpha_1, \ldots, \alpha_h): 1 \leq \alpha_1 \neq \cdots \neq \alpha_h \leq n\}$ and the appearance of the factor $\frac{(n-h)!}{(n-m)!}$ is due to the fact that a given index set $(\alpha_1, \ldots, \alpha_h)$ in P_h can be identified with $(n-h)(n-h-1) \ldots (n-h-(m-h)+1) = \frac{(n-h)!}{(n-m)!}$ distinct index sets $(\alpha_1, \ldots, \alpha_h, \alpha_{h+1}, \ldots, \alpha_m)$ in P.

Thus the representation of u_n as given in Miller and Sen has been derived:

$$U_n = \theta(F) + \sum_{h=1}^{m} \binom{m}{h} U_{n,h} \tag{3.4.4}$$

where

$$U_{n,h} = \frac{(n-h)!}{n!} \sum_{P_h} \underbrace{\int \cdots \int}_{h} \phi_h(y_1, \ldots, y_h) d \prod_{j=1}^{h} [\eta(y_j - Y_{\alpha_j}) - F(y_j)] . \tag{3.4.5}$$

The following Lemma is due to Miller and Sen, and it plays an important role in the asymptotic results for the Wilcoxon procedure:

Lemma 3.4.1 Under the condition

$$\underset{m}{\int \ldots \int} \phi^2(y_1, \ldots, y_m) d \prod_{j=1}^{m} F(y_j) < \infty ,$$

for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P[\max_{m \le k \le n} k \left| \sum_{h=2}^{m} \binom{m}{h} U_{k,h} \right| > n^{\frac{1}{2}} \varepsilon] = 0 \qquad (3.4.6)$$

3.5 Miscellaneous Theorems

The central limit theorem is well known, however, the form given in Chung (1968) is not usually found in most references. This form is the Lindeberg-Feller theorem for double arrays. That is, for each $n \ge 1$ let there be k_n random variables $\{X_{nj}, 1 \le j \le k_n\}$, where $k_n \to \infty$ as $n \to \infty$. The random variables with n as first subscript will be referred to as being in the n^{th} row.

Let F_{nj} be the distribution function of X_{nj} and let $S_n = \sum_{j=1}^{k_n} X_{nj}$. Assume that the random variables in each row are independent. Furthermore, assume

$$E X_{nj} = 0 ; \text{Var } X_{nj} = \sigma_{nj}^2 \qquad (3.5.1)$$

and for each $\varepsilon > 0$

$$\lim_{n \to \infty} \max_{1 \le j \le k_n} P[|X_{nj}| > \varepsilon] = 0 . \qquad (3.5.2)$$

Theorem 3.5.1 Assume σ_{nj}^2 for each n and j, and

$$\sum_{j=1}^{k_n} \sigma_{nj}^2 = 1, n = 1, 2, \ldots . \qquad (3.5.3)$$

In order that as $n \to \infty$ the two conclusions below are both satisfied:

i) S_n converges in distribution to Φ,

ii) (3.5.2) holds;

it is necessary and sufficient that for each $\eta > 0$, we have

$$\sum_{j=1}^{k_n} \int_{|x|>\eta} x^2 \, dF_{nj}(x) \to 0 \ . \tag{3.5.4}$$

Condition (3.5.4) is known as Lindeberg's condition.

The next theorem provides a sufficient condition for convergence in the r^{th} mean:

<u>Theorem 3.5.2</u> (Chung, 1968, p.64) Let X_n, $n = 1, 2, \ldots, X$, and Y be random variables such that

$$P[|X_n| \leq |Y|] = 1, \ n = 1, 2, \ldots; \tag{3.5.5}$$

$$E|Y|^r < \infty; \tag{3.5.6}$$

$$X_n \xrightarrow{P} X \ ; \tag{3.5.7}$$

then

$$E[|X_n - X|^r] \to 0 \text{ as } n \to \infty \ . \tag{3.5.8}$$

IV. ASYMPTOTIC RESULTS FOR R_1: THE GENERAL REGRESSION CASE

§4.0 Statement of the Main Results

There are two theorems proven in this Chapter. Theorems 4.1 and 4.2 are weak convergence results for the cumulative sum sign procedure and the parametric theory cumulative sum procedure. In both of these tests asymptotic results are obtained for the general regression case.

For the sign test procedure, let Y_i, $i = 1, \ldots, m+N$ obey the following model:

$$Y_i = \underline{\beta}_0' \underline{x}_i + \varepsilon_i, \ i = 1, \ldots, m \tag{4.0.1}$$

$$Y_{m+i} = \underline{\beta}_1' \underline{x}_{m+i} + \varepsilon_{m+i}, \ i = 1, \ldots, N, \tag{4.0.2}$$

and let

$$S_{mn} = \sum_{i=1}^{n} \eta(Y_{m+i} - \hat{\underline{\beta}}_0' \underline{x}_{m+i}), \ 1 \le n \le N, \tag{4.0.3}$$

where $\hat{\underline{\beta}}_0$ is the Jurecková estimate for $\underline{\beta}_0$. Jurecková has shown that

$$m^{\frac{1}{2}}(\hat{\underline{\beta}}_0 - \underline{\beta}_0) \sim \text{MVN}\ (\underline{0}, \Sigma) \text{ as } m \to \infty \tag{4.0.4}$$

where Σ is some variance-covariance matrix. Assume that the following conditions hold:

i) $N/m \to \kappa^2$ as $N \to \infty$; (4.0.5)

ii) The set of regressor variables $\{\underline{x}_{m+1}, \ldots, \underline{x}_{m+N}\}$ is chosen from a finite set of distinct values as $N \to \infty$;

iii) The ε_i, $i = 1, \ldots, m+N$ are independent and identically distributed with absolutely continuous cumulative distribution function $F(x)$ possessing a continuous density $f(x)$;

iv) $\frac{1}{N} \sum_{i=1}^{N} \underline{x}_{m+i} \, f((\underline{\beta}_0 - \underline{\beta}_1)' \underline{x}_{m+i}) \to \underline{v}$ as $N \to \infty$, (4.0.6)

where $\underline{v}$ is some non-zero k-vector.

Define a random polygon for $t \in [0, 1]$ as

$$S_{mN}(t) = S_{m[Nt]} + (Nt-[Nt])\eta(Y_{m+[Nt+1]} - \hat{\underline{\beta}}_0' x_{m+[N+1]}); \quad (4.0.7)$$

also define

$$\mu_n^* = \sum_{i=1}^{n} \{1 - F((\underline{\beta}_0 - \underline{\beta}_1)' \underline{x}_{m+i})\}, \quad 1 \leq n \leq N \quad (4.0.8)$$

and

$$\mu_N^*(t) = \mu_{[Nt]}^* + (Nt-[Nt])\{1-F((\underline{\beta}_0-\underline{\beta}_1)' \underline{x}_{m+[Nt+1]})\}. \quad (4.0.9)$$

<u>Theorem 4.1</u>: Under the conditions i) - iv)

$$N^{-\frac{1}{2}}(S_{mN}(t) - \mu_N^*(t)) \Rightarrow W_1 + L_1 \quad (4.0.10)$$

where

$$W_1 \sim W(0, \gamma^2) \quad (4.0.11)$$

with

$$\gamma^2 = \lim_{N\to\infty} \bar{\gamma}_N^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \gamma_i^2 , \quad (4.0.12)$$

$$\gamma_i^2 = \{1 - F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})\} - \{1 - F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})\}^2 \; ; \quad (4.0.13)$$

and L_1 is a straight line through the origin with random slope

$$b_1 \sim N(0, \kappa^2 \, \underline{v}'\Sigma\underline{v}) \, . \qquad (4.0.14)$$

Furthermore, W_1 and L_1 are independent

For a comparable parametric procedure consider

$$T_{mn} = \sum_{i=1}^{n} (Y_{m+i} - \hat{\underline{\beta}}_0'\underline{x}_{m+i}), \; 1 \le n \le N, \qquad (4.0.15)$$

where $\hat{\underline{\beta}}_0$ is the usual least squares estimate of $\underline{\beta}_0$ based on $Y_1, \ldots,$ Y_m.

Define the following two polygons in $C[0, 1]$:

$$T_{mN}(t) = T_{m[Nt]} + (Nt-[Nt])(Y_{m+[Nt+1]} - \hat{\underline{\beta}}_0'\underline{x}_{m+[Nt+1]}); \qquad (4.0.16)$$

$$\mu_N(t) = \sum_{i=1}^{[Nt]} (\underline{\beta}_1-\underline{\beta}_0)'\underline{x}_{m+i} + (Nt-[Nt])(\underline{\beta}_1-\underline{\beta}_0)'\underline{x}_{m+[Nt+1]} \, . \qquad (4.0.17)$$

In addition to conditions i) - iv) the following addition is made to condition iii):

$$\text{iii)}' \; E[\varepsilon_i] = 0; \, \mathrm{Var}[\varepsilon_i] = \sigma^2 < \infty. \qquad (4.0.18)$$

Theorem 4.2: Under conditions i) - iv) and iii)'

$$N^{-\frac{1}{2}} (T_{mN}(t) - \mu_N(t)) \Rightarrow W_2 + L_2 \qquad (4.0.19)$$

where

$$W_2 \sim W(0,\ \sigma^2) \qquad (4.0.20)$$

and L_2 is a straight line through the origin with random slope

$$b_2 \sim N(0,\ \kappa^2 \underline{v}'(X'X)^{-1}\underline{v}\sigma^2)\ . \qquad (4.0.21)$$

Furthermore, W_2 and L_2 are independent

§4.1: Proof of Theorem 4.1

The proof of Theorem 4.1 will be accomplished in two parts. The first part proves that the finite dimensional distributions of $N^{-\frac{1}{2}}(S_{mN}(t) - \mu_N^*(t))$ converge to those of $W_1 + L_1$; the second part contains the proof that $\{N^{-\frac{1}{2}}(S_{mN}(t) - \mu_N^*(t))\}$ is tight.

§4.1.1: Convergence of the Finite Dimensional Distributions of $N^{-\frac{1}{2}}(S_{mN}(t) - {}^*_N(t))$ to Those of $W_1 + L_1$

The partial sums used for constructing the random process $N^{-\frac{1}{2}}(S_{mN}(t) - \mu_N^*(t))$ are, from (4.0.3), (4.0.7), (4.0.8), and (4.0.9)

$$S_{mn} - \mu_n^* = \sum_{i=1}^{n} \eta(Y_{m+i} - \underline{\hat{\beta}}_0' \underline{x}_{m+i}) - \sum_{i=1}^{n} \{1 - F((\beta_0 - \beta_1)' x_{m+i})\}\ , \qquad (4.1.1)$$

and the random polygon $S_{mN}(t)$ is defined as

$$S_{mN}(t) = S_{M[Nt]} + (Nt - [Nt])(S_{m[N\,t+1]} - S_{m[Nt]})\ , \qquad (4.1.12)$$

The following theorem states the main result of §4.1:

<u>Theorem 4.1.1</u>: Let $0 \leq t_1 < \ldots < t_k \leq 1$, then

$$\{ N^{-\frac{1}{2}}(S_{mN}(t_1) - \mu_N^*(t_1)), \ldots, N^{-\frac{1}{2}}(S_{mN}(t_k) - \mu_N^*(t_k))\}$$

converges in distribution as $N \to \infty$ to a multivariate normal random vector with mean $\underline{0}_k$ and variance-covariance structure (σ_{ij}), $1 \leq i, j \leq k$, where

$$\sigma_{ij} = \begin{cases} t_2\gamma + t_i^2\kappa^2\underline{v}'\Sigma\underline{v} & \text{if } i = j \\ t_i\gamma + t_it_j\kappa^2\underline{v}'\Sigma\underline{v} & \text{if } i < j \\ t_j\gamma + t_it_j\kappa^2\underline{v}'\Sigma\underline{v} & \text{if } i > j \end{cases} . \tag{4.1.3}$$

That is, the finite dimensional distributions of $N^{-\frac{1}{2}}(S_{mN}(t) - \mu_N^*(t))$ converge to those of $W_1 + L_1$.

Before proving Theorem 4.1.1 some preliminary calculations are presented and some further notation developed.

Define

$$\mu_{m+i} = E\,\eta(Y_{m+i} - \underline{\hat{\beta}}_0'\underline{x}_{m+i}) ,$$

then

$$\mu_{m+i} = P[Y_{m+i} - \underline{\hat{\beta}}_0'\underline{x}_{m+i} \geq 0] = P[Y_{m+i} \geq \underline{\hat{\beta}}_0'\underline{x}_{m+i}]$$

$$= 1 - P[Y_{m+i} < \underline{\hat{\beta}}_0'\underline{x}_{m+i}] = 1 - EP[Y_{m+i} < y|\underline{\hat{\beta}}_0'\underline{x}_{m+i} = y].$$

By independence of Y_{m+i} and $\underline{\hat{\beta}}_0$,

$$\mu_{m+i} = 1 - \int_{-\infty}^{\infty} F_{m+i}(y)dG_{m+i}(y) \qquad (4.1.4)$$

where

$$G_{m+i}(y) = P[\hat{\underline{\beta}}_0'\underline{x}_{m+i} \le y] \; ; \qquad (4.1.5)$$

$$F_{m+i}(y) = P[Y_{m+i} \le y] = F(y - \underline{\beta}_1'\underline{x}_{m+i}) \; . \qquad (4.1.6)$$

We state and prove the following lemma:

Lemma 4.1.1: Without loss of generality, let $\underline{x}_{m+i} = \underline{x}$ for each $m=k,k+1,\ldots,$ then

$$\mu_{m+i} \to 1 - F((\underline{\beta}_0 - \underline{\beta}_1)'\underline{x}) \quad \text{as } m \to \infty. \qquad (4.1.7)$$

Proof: Let $X_m = F_{m+i}(\hat{\underline{\beta}}_0'\underline{x}) = F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x})$; $X =$ $F((\underline{\beta}_0 - \underline{\beta}_1)'\underline{x})$ and $Z = 1$. Since

$$|X_m| = X_m \le |Z| \; ;$$

$$\int_{-\infty}^{\infty} |Z| dG_{m+i} = 1 \; ;$$

and since F is continuous and $\hat{\underline{\beta}}_0 \xrightarrow{P} \underline{\beta}_0$ as $m \to \infty$, $X_m \xrightarrow{P} X$, as $m \to \infty$, so by Theorem 3.5.2, $\int X_m dG_{m+i} - \int X dG_{m+i} \to 0$, or $\int F_{m+i} dG_{m+i} - F((\underline{\beta}_0 - \underline{\beta}_1)'\underline{x}) \to 0$ as $m \to \infty$. Thus $\mu_{m+i} \to$ $1 - F((\underline{\beta}_0 - \underline{\beta}_1)'\underline{x})$ as $m \to \infty$.

Now consider the variance of S_{mn}:

$$\mathrm{Var}\ S_{mn} = E\ S_{mn}^2 - [ES_{mn}]^2$$

$$= E[\sum_{i=1}^{n} \eta(Y_{m+i} - \hat{\underline{\beta}}_0'\underline{x}_{m+i})]^2 - [\sum_{i=1}^{n} \mu_{m+i}]^2$$

$$= \sum_{i=1}^{n} E\ \eta^2(Y_{m+i} - \hat{\underline{\beta}}_0'\underline{x}_{m+i})$$

$$+ 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[\eta(Y_{m+i} - \hat{\underline{\beta}}_0'\underline{x}_{m+i})\eta(Y_{m+j} - \hat{\underline{\beta}}_0'\underline{x}_{m+j})]$$

$$- [\sum_{i=1}^{n} \mu_{m+i}]^2 . \tag{4.1.8}$$

Now for $i \neq j$,

$$E[\eta(Y_{m+i} - \hat{\underline{\beta}}_0'\underline{x}_{m+i})\eta(Y_{m+j} - \hat{\underline{\beta}}_0'\underline{x}_{m+j})]$$

$$= P[Y_{m+i} - \hat{\underline{\beta}}_0'\underline{x}_{m+i} \geq 0 \ ; \ Y_{m+j} - \hat{\underline{\beta}}_0'\underline{x}_{m+j} \geq 0]$$

$$= 1 - P[Y_{m+i} - \hat{\underline{\beta}}_0\underline{x}_{m+i} < 0] - P[Y_{m+j} - \hat{\underline{\beta}}_0'\underline{x}_{m+j} < 0]$$

$$+ P[Y_{m+i} - \hat{\underline{\beta}}_0'\underline{x}_{m+i} < 0 \ ; \ Y_{m+j} - \hat{\underline{\beta}}_0'\underline{x}_{m+j} < 0]$$

$$= \mu_{m+i} + \mu_{m+j} - 1 + \underbrace{\int \cdots \int}_{k} F_{m+i}(\underline{b}'\underline{x}_{m+i})F_{m+j}(\underline{b}'\underline{x}_{m+j})dH_m(\underline{b}) ,$$

where $H_m(\underline{b}) = P[\hat{\underline{\beta}}_0 \leq \underline{b}]$. Since $\eta^2(x) = \eta(x)$,

$$\mathrm{Var}\ S_{mn} = \sum_{i=1}^{n} \mu_{m+i} + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \{\mu_{m+i} + \mu_{m+j}$$

$$+ \underbrace{\int \cdots \int}_{k} F_{m+i}(\underline{b}'\underline{x}_{m+i})F_{m+j}(\underline{b}'\underline{x}_{m+j})dH_m(\underline{b}) - 1\} - [\sum_{i=1}^{n} \mu_{m+i}]^2 . \tag{4.1.9}$$

Note that

$$2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\mu_{m+i} + \mu_{m+j}) = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} (\mu_{m+i} + \mu_{m+j})$$

$$= 2(n-1) \sum_{i=1}^{n} \mu_{m+i} \ .$$

Thus

$$\text{Var } S_{mn} = (2n-1) \sum_{i=1}^{n} \mu_{m+i} - (\sum_{i=1}^{n} \mu_{m+i})^2$$

$$+ 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \cdot\{\underbrace{\int\ldots\int}_{k} F_{m+i}(\underline{b}'\underline{x}_{m+i})F_{m+j}(\underline{b}'\underline{x}_{m+j})dH_m(b) - 1\} \ . \tag{4.1.10}$$

The projection statistic for S_{mn} is obtained by regarding S_{mn} as a function of the (n+1) independent random elements Y_{m+1}, ..., Y_{m+n}, $\hat{\underline{\beta}}_0$. Thus

$$\hat{S}_{mn} = \sum_{i=1}^{n} E[S_{mn}|Y_{m+1}] + E[S_{mn}|\hat{\underline{\beta}}_0] - n\, ES_{mn} \ . \tag{4.1.11}$$

Taking conditional expectations,

$$E[\eta(Y_{m+i} - \hat{\underline{\beta}}_0'\underline{x}_{m+i})|Y_{m+i} = y] = G_{m+i}(y) \ ; \tag{4.1.12}$$

$$E[\eta(Y_{m+i} - \hat{\underline{\beta}}_0'\underline{x}_{m+i})|\hat{\underline{\beta}}_0 = \underline{b}] = 1 - F_{m+i}(\underline{b}'\underline{x}_{m+i})$$

$$= 1 - F((\underline{b} - \underline{\beta}_1)'x_{m+i}) \ , \tag{4.1.13}$$

so

$$S_{mn} = \sum_{i=1}^{n} \{G_{m+i}(Y_{m+i}) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \mu_{m+j}\}$$

$$+ \sum_{i=1}^{n} \{1 - F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+i})\} - n \sum_{i=1}^{n} \mu_{m+i}$$

$$= \sum_{i=1}^{n} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\} + \sum_{i=1}^{n} \{1 - F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+i})\}. \quad (4.1.14)$$

The variance of $\hat{S}_{mn}$ is easily found from (4.1.14):

$$\text{Var } \hat{S}_{mn} = \sum_{i=1}^{n} \text{Var } G_{m+i}(Y_{m+i}) + \text{Var} \sum_{i=1}^{n} \{1 - F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+i})\}$$

$$= \sum_{i=1}^{n} \{\int G^2_{m+i}\, dF_{m+i} - \mu^2_{m+i}\}$$

$$+ \text{Var} \sum_{i=1}^{n} F((\hat{\beta}_0 - \underline{\beta}_1)'x_{m+i}) .$$

$$= \sum_{i=1}^{n} \{\int G^2_{m+i}\, dF_{m+i} - \mu^2_{m+i}\} + \sum_{i=1}^{n} \text{Var } F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'x_{m+i})$$

$$+ 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \text{Cov}(F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+i}), F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'x_{m+j}))$$

$$= \sum_{i=1}^{n} \{\int G^2_{m+i}\, dF_{m+i} - \mu^2_{m+i}\} + \sum_{i=1}^{n} \{\int F^2_{m+i}\, dG_{m+i} - (\int F_{m+i} dG_{m+i})^2\}$$

$$+ 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \underbrace{\int \ldots \int}_{k} F((\underline{b} - \underline{\beta}_1)'\underline{x}_{m+i}) F((\underline{b} - \underline{\beta}_1)'\underline{x}_{m+j}) dH_m(\underline{b})$$

$$- (1 - \mu_{m+i})(1 - \mu_{m+j})$$

$$= \sum_{i=1}^{n} \{\int G_{m+i}^2 \, dF_{m+i} - \mu_{m+i}^2\} + \sum_{i=1}^{n} \{\int F_{m+i}^2 \, dG_{m+i} - \{\int F_{m+i} \, dG_{m+i})^2\}$$

$$+ 2 \sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ j\neq i}}^{n} \{\int_k \dots \int F((\underline{b}-\underline{\beta}_1)'\underline{x}_{m+i})F((\underline{b}-\underline{\beta}_1)'x_{m+j}) \, dH_m(\underline{b}) - 1\} \quad (4.1.15)$$

$$+ (2n-1) \sum_{i=1}^{n} \mu_{m+i} - 2 \sum_{i=1}^{n-1} \sum_{\substack{j=i+1 \\ j\neq i}}^{n} \mu_{m+i} \mu_{m+j} \quad .$$

By combining the Projection Lemma, (4.1.10), and (4.1.15),

$$E(S_{mn} - \hat{S}_{mn})^2 = \text{Var } S_{mn} - \text{Var } \hat{S}_{mn}$$

$$= [(2n-1) \sum_{i=1}^{n} \mu_{m+i} - (\sum_{i=1}^{n} \mu_{m+i})^2$$

$$+ 2 \sum_{i=1}^{n-1} \sum_{\substack{j=i+1 \\ j\neq i}}^{n} \{\int_k \dots \int F((\underline{b}-\underline{\beta}_1)'x_{m+i})F((\underline{b}-\underline{\beta}_1)'\underline{x}_{m+j}) dH_m(\underline{b}) - 1\} \,]$$

$$- [\sum_{i=1}^{n} \{\int G_{m+i}^2 \, dF_{m+i} - \mu_{m+i}^2\} + \sum_{i=1}^{n} \{\int F_{m+i}^2 \, dG_{m+i} - (\int F_{m+i} \, dG_{m+i})^2\}$$

$$+ 2 \sum_{i=1}^{n-1} \sum_{\substack{j=i+1 \\ j\neq i}}^{n} \{\int_k \dots \int F((\underline{b}-\underline{\beta}_1)'\underline{x}_{m+i})F((\underline{b}-\underline{\beta}_1)'\underline{x}_{m+j}) \, dH_m(\underline{b}) - 1\}$$

$$+ 2(n-1) \sum_{i=1}^{n} \mu_{m+i} - 2 \sum_{i=1}^{n-1} \sum_{\substack{j=i+1 \\ j\neq i}}^{n} \mu_{m+i}\mu_{m+j}]$$

$$= (2n-1) \sum_{i=1}^{n} \mu_{m+i} - \sum_{i=1}^{n} \mu_{m+i}^2 - 2 \sum_{i=1}^{n-1} \sum_{\substack{j=1 \\ j\neq i}}^{n} \mu_{m+i}\mu_{m+j}$$

$$- \sum_{i=1}^{n} \{\int G_{m+i}^2\, dF_{m+i} - \mu_{m+i}^2\} - \sum_{i=1}^{n} \{\int F_{m+i}^2\, dG_{m+i} - (\int F_{m+i}\, dG_{m+i})^2\}$$

$$- (2n-1) \sum_{i=1}^{n} \mu_{m+i} + \sum_{i=1}^{n} \mu_{m+i} - 2 \sum_{i=1}^{n-1} \sum_{\substack{j=i+1 \\ j\neq i}}^{n} \mu_{m+i}\mu_{m+j}$$

$$= \sum_{i=1}^{n} \mu_{m+i} - \sum_{i=1}^{n} \mu_{m+i}^2 - \sum_{i=1}^{n} \{\int G_{m+i}^2\, dF_{m+i} - \mu_{m+i}^2\}$$

$$- \sum_{i=1}^{n} \{\int F_{m+i}^2\, dG_{m+i} - (\int F_{m+i}\, dG_{m+i})^2\}$$

$$= \sum_{i=1}^{n} \{\mu_{m+i} - \int G_{m+i}^2\, dF_{m+i}\}$$

$$- \sum_{i=1}^{n} \{\int F_{m+i}^2\, dG_{m+i} - (\int F_{m+i}\, dG_{m+i})^2\}$$

$$= \sum_{i=1}^{n} \{\int G_{m+i}\, dF_{m+i} - \int G_{m+i}^2\, dF_{m+i}\}$$

$$- \sum_{i=1}^{n} \{\int F_{m+i}^2\, dG_{m+i} - (\int F_{m+i}\, dG_{m+i})^2\} \qquad (4.1.16)$$

$$= \sum_{i=1}^{n} \psi_n(\underline{x}_{m+i}), \text{ where}$$

$$\psi_n(\underline{x}_{m+i}) = \int G_{m+i}\, dF_{m+i} - \int G_{m+i}^2\, dF_{m+i}$$

$$- \int F_{m+i}^2\, dG_{m+i} - (\int F_{m+i}\, dG_{m+i})^2 . \qquad (4.1.17)$$

For a fixed $\underline{x}$, let $F_{\underline{x}}(y) = F(y - \underline{\beta}_1'\underline{x})$, then as in Lemma 4.1.1 the following convergences hold:

$$\lim_{m\to\infty} \int G_{m+i}\, dF_{\underline{x}} = 1 - F_{\underline{x}}(\hat{\underline{\beta}}_0'\underline{x}) = 1 - F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}) \; ; \qquad (4.1.18)$$

$$\lim_{m\to\infty} \int G^2_{m+i}\, dF_{\underline{x}} = 1 - F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}) \; ; \qquad (4.1.19)$$

$$\lim_{m\to\infty} \int F^2_{\underline{x}}\, dG_{m+i} = [F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x})]^2 \; ; \qquad (4.1.20)$$

$$\lim_{m\to\infty} (\int F_{\underline{x}}\, dG_{m+i})^2 = [F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x})]^2 \; , \qquad (4.1.21)$$

hence $\lim_{m\to\infty} \psi_n(\underline{x}) = 0.$ (4.1.22)

Since the variables $\underline{x}_{m+i}$ take on only a finite number of values, the convergence in (4.1.22) is uniform, so

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{n} \sum_{i=1}^{n} \psi_n(\underline{x}_{m+i}) = 0 \qquad (4.1.23)$$

Combining (4.1.16), (4.1.17), and (4.1.23),

$$E[(S_{mn} - \hat{S}_{mn})^2] = 0(n) \; . \qquad (4.1.24)$$

Now for fixed $t \in [0, 1]$,

$$(S_{mN}(t) - \hat{S}_{mN}(t)) = (S_{m[Nt]} - \hat{S}_{m[Nt]})$$

$$+ (Nt - [Nt])((S_{m[Nt+1]} - S_{m[Nt]}) - (\hat{S}_{m[Nt+1]} - \hat{S}_{m[Nt]})) \qquad (4.1.25)$$

and $(Nt - [Nt]) \to 0$ as $N \to \infty$, so

$$E[(S_{mN}(t) - \hat{S}_{mN}(t))^2] = 0(n) \; . \qquad (4.1.26)$$

By fundamental probability theory, a sequence of k-dimensional random vectors converges in probability to zero if and only if each coordinate converges in probability to zero. Thus by (4.1.26) the following lemma is proven:

Lemma 4.1.2: Let $0 \leq t_1 < \ldots < t_k \leq 1$, then

$$\{N^{-\frac{1}{2}}(S_{mN}(t_1) - \hat{S}_{mN}(t_1)), \ldots, N^{-\frac{1}{2}}(S_{mN}(t_k) - \hat{S}_{mN}(t_k))\} \rightarrow \underline{0}_k \tag{4.1.27}$$

as $N \rightarrow \infty$. That is, the asymptotic finite dimensional distributions of $N^{-\frac{1}{2}}(S_{mN}(t)-\mu_N^*(t))$ are the same as those of $N^{-\frac{1}{2}}(\hat{S}_{mN}(t)-\mu_N^*(t))$.

Now it will be necessary to show that the finite dimensional distributions of $N^{-\frac{1}{2}}(\hat{S}_{mN}(t)-\mu_N^*(t))$ converge to those of $W_1 + L_1$. This will be established by proving weak convergence of $N^{-\frac{1}{2}}(\hat{S}_{mN}(t)-\mu_N^*(t))$ to $W_1 + L_1$, and convergence of the finite dimensional distributions follows automatically.

To conclude the proof of Theorem 4.1.1 the following lemma will be established:

Lemma 4.1.3: Under the conditions i)-iv)

$$N^{-\frac{1}{2}}(\hat{S}_{mN}(t) - \mu_N^*(t)) \Rightarrow W_1 + L_1 \tag{4.1.28}$$

where $W_1 + L_1$ are described in the statement of Theorem 4.1.

Proof: From (4.1.14) and (4.0.8), $\hat{S}_{mn} - \mu_n^*$ is the sum of two independent sums:

$$\sum_{i=1}^{n} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\} \; ; \tag{4.1.29}$$

and

$$\sum_{i=1}^{n} \{[1 - F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+i})] - [1 - F((\underline{\beta}_0 - \underline{\beta}_1)'\underline{x}_{m+i})]\}$$

$$= - \sum_{i=1}^{n} \{F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+i}) - F((\underline{\beta}_0 - \underline{\beta}_1)'\underline{x}_{m+i})\} \; , \tag{4.1.30}$$

and hence $N^{-\frac{1}{2}}(\hat{S}_{mN}(t) - \mu_N^*(t))$ can be written as the sum of two independent random polygons:

$$N^{-\frac{1}{2}} \left\{ \sum_{i=1}^{[Nt]} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\} - (Nt-[Nt])\{G_{m+[Nt+1]}(Y_{m+[Nt+1]}) - \mu_{m+[Nt+1]}\} \right\}, \tag{4.1.31}$$

and

$$-N^{-\frac{1}{2}} \left\{ \sum_{i=1}^{[Nt]} \{F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+i}) - F((\underline{\beta}_0 - \underline{\beta}_1)'\underline{x}_{m+i})\} \right.$$
$$\left. -(Nt-[Nt])\{F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+[Nt+1]}) - F((\underline{\beta}_0 - \underline{\beta}_1)'\underline{x}_{m+[Nt+1]})\} \right\}. \tag{4.1.32}$$

To show that (4.1.31) converges weakly to W_1, the condition of Lindeberg, as stated in Prohorov's Theorem will be established. To see that this condition is met, let

$$X_{Ni} = N^{-\frac{1}{2}} \gamma_i^{-1}\{G_{m+i}(Y_{m+i}) - \mu_{m+i}\} \; , \; 1 \leq i \leq N \; , \tag{4.1.33}$$

where γ_i^2 is defined by (4.0.13), and

$$F_{Ni}(x) = P[\xi_{ni} \leq x] \ . \qquad (4.1.34)$$

then $E\xi_{Ni} = 0$, and $\sigma^2_{Ni} = \mathrm{Var}\ \xi_{Ni} = \dfrac{\gamma_i^2}{n\bar{\gamma}_N^2}$, so $\sum_{i=1}^{N} \sigma^2_{Ni} = 1$ satisfying (3.2.10). Since $|G_{m+i}(Y_{m+i}) - \mu_{m+i}| \leq 1$ and $\bar{\gamma}_N^2 \to \gamma^2$ as $N \to \infty$, for every $\eta > 0$ there exists an n_0 such that

$$P[|\xi_{Ni}| > \eta] = 0, \ n \geq n_0 \ . \qquad (4.1.35)$$

from which (3.2.9) follows. Also from (4.1.35),

$$\int_{[|x|\leq\eta]} x^2 \, dF_{Ni}(x) = \int x^2 \, dF_{Ni}(x) = \frac{\gamma_i^2}{n\bar{\gamma}_N^2} \qquad (4.1.36)$$

from which it follows that

$$\lim_{N\to\infty} \sum_{i=1}^{N} \int_{[|x|\leq\eta]} x^2 \, dF_{Ni}(x) = 1 \ . \qquad (4.1.37)$$

That is, the ξ_{Ni} satisfy the Lindeberg condition, and so (4.1.31) converges weakly to W_1.

Now we need to show that (4.1.32) converges weakly to L_1.

The sum in (4.1.32) is a function of the asymptotically normal estimate $\hat{\beta}_0$. Recall that by (4.0.4), $m^{\frac{1}{2}}(\hat{\underline{\beta}}_0 - \underline{\beta}_0) \sim \mathrm{AMVN}(\underline{0}, \Sigma)$ as $m \to \infty$.

Proceeding as in Rao (1973, p. 386), the Taylor expansion of $F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'x_{m+i})$ can be written

$$F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+i}) - F((\underline{\beta}_0 - \underline{\beta}_1)'x_{m+i})$$

$$= [(\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+i} - (\underline{\beta}_0 - \underline{\beta}_1)'\underline{x}_{m+i}][f((\underline{\beta}_0 - \underline{\beta}_1)'x_{m+i} + \varepsilon_i]$$

where $[f((\underline{\beta}_0 - \underline{\beta}_1)'\underline{x}_{m+i}) + \varepsilon_i] = f(z)$ for some z between $(\hat{\underline{\beta}}_0 - \underline{\beta}_1)'x_{m+i}$ and $(\underline{\beta}_0 - \underline{\beta}_1)'x_{m+i}$. Since $f(\cdot)$ is continuous and $(\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+i} \overset{p}{\to} (\underline{\beta}_0 - \underline{\beta}_1)'\underline{x}_{m+i}$, $\varepsilon_i \overset{p}{\to} 0$. Hence

$$n^{\frac{1}{2}}\{[F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+i}) - F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})]$$

$$- [(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}\ f((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})]\}$$

$$= n^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}\ \varepsilon_i, \text{ and}$$

$$n^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}\ \varepsilon_i \xrightarrow{p} 0$$

since

$$n^{\frac{1}{2}}(\hat{\beta}_0-\beta_0)'\underline{x}_{m+i} = \frac{n}{m}^{\frac{1}{2}}\ m^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'x_{m+i} \sim AN(0,\ \kappa^2\underline{x}'_{m+i}\Sigma\ \underline{x}_{m+i})$$

and $\varepsilon_p \xrightarrow{p} 0$. Thus the asymptotic distribution of $n^{\frac{1}{2}}[F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+i}-$ $F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})]$ is the same as that of $n^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}\ f((\underline{\beta}_0-\underline{\beta}_1)'$ $\underline{x}_{m+i})$. The asymptotic normality of the terms of summation in (4.1.32) is now established. Now write

$$n^{-\frac{1}{2}}\sum_{i=1}^{n}\{F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+i}) - F((\underline{\beta}_0-\underline{\beta}_1)'x_{m+i})\} \tag{4.1.38}$$

$$= \frac{1}{n}\sum_{i=1}^{n} n^{\frac{1}{2}}[F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+i}) - F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})]$$

$$= \frac{1}{n}\sum_{i=1}^{n} n^{\frac{1}{2}}\ [(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}\ f((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})]$$

$$+ \frac{1}{n}\sum_{i=1}^{n} n^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}\ \varepsilon_i$$

$$= n^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'[\frac{1}{n}\sum_{i=1}^{n}\underline{x}_{m+i}\ f((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})] \tag{4.1.39a}$$

$$+ n^{\frac{1}{2}}(\hat{\beta}_0-\underline{\beta}_0)'[\frac{1}{n}\sum_{i=1}^{n}\underline{x}_{m+i}\ \varepsilon_i]\ . \tag{4.1.39b}$$

By condition ii) for Theorem 4.1 the ε_i converge to zero in probability uniformly in i, so

$$n^{\frac{1}{2}}(\hat{\beta}_0-\beta_0)'\left(\frac{1}{n}\sum_{i=1}^{n} x_{m+i}\ \varepsilon_i\right) \xrightarrow{p} 0 . \tag{4.1.40}$$

Then from (4.1.39a),

$$n^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'[\frac{1}{n}\sum_{i=1}^{n}\underline{x}_{m+i}\ f((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})]$$

$$= \left(\frac{n}{m}\right)^{\frac{1}{2}} m^{\frac{1}{2}}\ (\hat{\underline{\beta}}_0-\underline{\beta}_0)'[\frac{1}{n}\sum_{i=1}^{n}\underline{x}_{m+i}\ f((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})]$$

$$\sim AN(0,\ \kappa^2\ \underline{v}'\ \Sigma\ \underline{v}) \text{ as } n \to \infty. \tag{4.1.41}$$

Thus

$$N^{-\frac{1}{2}}\left(\sum_{i=1}^{[Nt]}\{F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+i}) - F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})\}\right.$$

$$\left. - (Nt-[Nt])\{F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+[Nt+1]}) - F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+[Nt+1]})\}\right)$$

$$= \left(\frac{[Nt]}{N}\right)^{\frac{1}{2}}[Nt]^{-\frac{1}{2}}\sum_{i=1}^{[Nt]}\{F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+i})-F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})\} \tag{4.1.43a}$$

$$-N^{-\frac{1}{2}}(Nt-[Nt])\{F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+[Nt+1]})-f((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+[Nt+1]})\}. \tag{4.1.43b}$$

By (4.1.41), for fixed t, (4.1.43a) is asymptotically normal with mean 0 and variance $t\ \kappa^2\ \underline{v}'\ \Sigma\ \underline{v}$; and (4.1.43b) converges in probability to zero. Since $\frac{1}{n}\sum_{i=1}^{n}\underline{x}_{m+i}\ f((\underline{\beta}_0-\beta_1)'\underline{x}_{m+i})$, i = 1, 2, ... are all in a closed and bounded neighborhood of $\underline{v}$, the convergences of (4.1.43a) and (4.1.43b) are uniform in t. Thus concludes the proof of Lemma 4.1.3.

§4.1.2 The Sequence $\{N^{-\frac{1}{2}}(S_{mN}(t) - \mu_N^*(t))\}$ is Tight

In this subsection, tightness of the sequence $\{N^{-\frac{1}{2}}(S_{mN}(t) - \mu_N^*(t))\}$ will be demonstrated using Billingsley's sufficiency criterion stated in Chapter III. That is, for each $\varepsilon > 0$ there exists a $\lambda > 1$ and an integer n_0 such that if $n \geq n_0$ then

$$P[\max_{n\leq N} |(S_{m,k+n} - \mu_{k+n}^*) - (S_{m,k} - \mu_k^*)| \geq \lambda N^{\frac{1}{2}}] \leq \varepsilon/\lambda^2 \qquad (4.1.44)$$

holds for all k.

Since the choice of the $\underline{x}$ values is arbitrary, the argument for (4.1.44) in this proof is equivalent to the argument for

$$P[\max_{n\leq N} |S_{m,n} - \mu_n^*| \geq \lambda N^{\frac{1}{2}}] \leq \varepsilon/\lambda^2 \ . \qquad (4.1.45)$$

That is, arguing

$$P[\max_{n\leq N} |\sum_{i=k+1}^{k+n} [\eta(Y_{m+i}-\hat{\underline{\beta}}_0'\underline{x}_{m+i})-(1-F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+i}))]| \geq \lambda N^{\frac{1}{2}}] \leq \varepsilon/\lambda^2 \qquad (4.1.46)$$

is no different than arguing

$$P[\max_{n\leq N} |\sum_{i=1}^{n} [\eta(Y_{m+i}-\hat{\underline{\beta}}_0'x_{m+i})-(1-F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+i}))]| \geq \lambda N^{\frac{1}{2}}] \leq \varepsilon/\lambda^2 \qquad (4.1.47)$$

except that (4.1.47) has one less letter in the index of summation to carry through the presentation. Also, whether λ or 2λ, say, is used in (4.1.45) does not matter. Define

$$U_{mn} = S_{mn} - \mu_n^* \; ; \; \hat{U}_{mn} = \hat{S}_{mn} - \mu_n^* \; . \tag{4.1.48}$$

Then

$$P[\max_{n \le N} |S_{mn} - \mu_n^*| \ge 4\lambda N^{\frac{1}{2}}] = P[\max_{n \le N} |U_{mn}| \ge 4\lambda N^{\frac{1}{2}}]$$

$$= P[\max_{n \le N} |\hat{U}_{mn} + U_{mn} - \hat{U}_{mn}| \ge 4\lambda N^{\frac{1}{2}}]$$

$$\le P[\max_{n \le N} |\hat{U}_{mn}| \ge 2\lambda N^{\frac{1}{2}}] \tag{4.1.49a}$$

$$+ P[\max_{n \le N} |U_{mn} - \hat{U}_{mn}| \ge 2\lambda N^{\frac{1}{2}}] \; . \tag{4.1.49b}$$

The tightness argument will proceed by first showing that (4.1.49a) is appropriately bounded, and then showing that (4.1.49b) is appropriately bounded.

From (4.1.48), (4.1.14), and (4.0.8),

$$\hat{U}_{mn} = \sum_{i=1}^{n} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\}$$

$$+ \sum_{i=1}^{n} \{(1 - F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+i})) - (1 - F((\underline{\beta}_0 - \underline{\beta}_1)'\underline{x}_{m+i}))\}$$

$$= \sum_{i=1}^{n} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\} \tag{4.1.50a}$$

$$- \sum_{i=1}^{n} \{F((\hat{\underline{\beta}}_0 - \underline{\beta}_1)'\underline{x}_{m+i}) - F((\underline{\beta}_0 - \underline{\beta}_1)'\underline{x}_{m+i})\} \; . \tag{4.1.50b}$$

Thus (4.1.49a) is bounded above by

$$P[\max_{n\leq N} |\sum_{i=1}^{n} \{G_{m+i}(Y_{m+i})-\mu_{m+i}\}| \geq \lambda N^{\frac{1}{2}}] \tag{4.1.51a}$$

$$+ P[\max_{n\leq N} |\sum_{i=1}^{n} \{F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+i})-F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})\}| \geq \lambda N^{\frac{1}{2}}] . \tag{4.1.51b}$$

Lemma 4.1.4: For every $\varepsilon > 0$ there is a $\lambda > 1$ and an integer N_0 such that if $N \geq N_0$,

$$P[\max_{n\leq N} |\sum_{i=1}^{n} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\}| \geq \lambda N^{\frac{1}{2}}] \leq \varepsilon/\lambda^2 . \tag{4.1.52}$$

Proof: The proof of Lemma 4.1.4 follows as in Billingsley's proof of tightness in Donsker's Theorem. The random variables $G_{m+i}(Y_{m+i})-\mu_{m+i}$, $i = 1, \ldots, N$ are independent random variables with mean zero and variance

$$\gamma_i^{*2} = \int G_{m+i}^2 \, dF_{m+i} - (\int G_{m+i} \, dF_{m+i})^2 \to \gamma_i^2 \text{ as } m \to \infty \tag{4.1.53}$$

where γ_i^2 is given by (4.0.13). Let

$$S_N^2 = \sum_{i=1}^{N} \gamma_i^* , \; N = 1, 2, \ldots, \tag{4.1.54}$$

then by Lemma 3.2.1, for $\lambda > 1$,

$$P[\max_{n\leq N} |\sum_{i=1}^{n} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\}| \geq \lambda S_N]$$

$$\leq 2P[|\sum_{i=1}^{N} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\}| \geq (\lambda - \sqrt{2})S_N] . \tag{4.1.55}$$

In the proof of Lemma 4.1.3 the Lindeberg condition was established for the sum in (4.1.55), and so

$$S_N^{-1} \sum_{i=1}^{N} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\} \sim AN(0, 1) \text{ as } N \to \infty. \qquad (4.1.56)$$

Therefore, for $\varepsilon > 0$ and λ sufficiently large,

$$\lim_{N\to\infty} P[S_N^{-1} | \sum_{i=1}^{N} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\}| \leq (\lambda - \sqrt{2})] \leq \varepsilon/2\lambda^2 . \qquad (4.1.57)$$

That is, from (4.1.55),

$$\lim_{N\to\infty} P[\max_{n\leq N} | \sum_{i=1}^{n} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\}| \geq \lambda S_N] \leq \varepsilon/\lambda^2 . \qquad (4.1.58)$$

Since γ_i^* is bounded above by 1, S_N is bounded above by $N^{\frac{1}{2}}$, so

$$P[\max_{n\leq N} | \sum_{i=1}^{n} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\}| \geq \lambda S_N]$$

$$\geq P[\max_{n\leq N} | \sum_{i=1}^{n} \{G_{m+i}(Y_{m+i}) - \mu_{m+i}\}| \geq \lambda N^{\frac{1}{2}}] . \qquad (4.1.59)$$

Combining (4.1.58) and (4.1.59), (4.1.52) follows.

<u>Lemma 4.1.5</u>: For every $\varepsilon > 0$ there is a $\lambda > 1$ and an integer N_0 such that if $N \geq N_0$,

$$P[\max_{n\leq N} | \sum_{i=1}^{n} \{F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+i}) - F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})\}| \geq N^{\frac{1}{2}}] \leq \varepsilon/\lambda^2 .$$
(4.1.60)

<u>Proof</u>: As in (4.1.39a) and (4.1.39b) the sum in (4.1.60) can be expressed as

$$\sum_{i=1}^{n} \{F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+i}) - F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})\}$$

$$= n(\hat{\underline{\beta}}_0-\underline{\beta}_0)'[\frac{1}{n} \sum_{i=1}^{n} \underline{x}_{m+i}\ f((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})] \qquad (4.1.61a)$$

$$+ n(\hat{\underline{\beta}}_0-\underline{\beta}_0)'[\frac{1}{n}\sum_{i=1}^{n} \underline{x}_{m+i}\ \varepsilon_i] \qquad (4.1.61b)$$

where $\varepsilon_i \xrightarrow{p} 0$ uniformly in i as $m \to \infty$. The probability in (4.1.60) is equivalent to

$$P[\max_{n\leq N} N^{-\frac{1}{2}}|n(\hat{\underline{\beta}}_0-\underline{\beta}_0)'[\frac{1}{n}\sum_{i=1}^{n} \underline{x}_{m+i}\ f((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})]$$

$$+ n(\hat{\underline{\beta}}_0-\underline{\beta}_0)'[\frac{1}{n}\sum_{i=1}^{n} \underline{x}_{m+i}\ \varepsilon_i]| \geq \lambda] \qquad (4.1.62)$$

$$\leq P[\max_{n\leq N} N^{-\frac{1}{2}}|n(\hat{\underline{\beta}}_0-\underline{\beta}_0)'[\frac{1}{n}\sum_{i=1}^{n} \underline{x}_{m+i}\ f((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})]| \geq \lambda/2] \qquad (4.1.63a)$$

$$+ P[\max_{n\leq N} N^{-\frac{1}{2}}|n(\hat{\underline{\beta}}_0-\underline{\beta}_0)'[\frac{1}{n}\sum_{i=1}^{h} \underline{x}_{m+i}\ \varepsilon_i]| \geq \lambda/2] . \qquad (4.1.63b)$$

Since the $\varepsilon_i \xrightarrow{p} 0$ uniformly and the $\underline{x}_{m+i}$'s are bounded, $\frac{1}{n}\sum_{i=1}^{n}\underline{x}_{m+i}\varepsilon_i \xrightarrow{p} \underline{0}$ uniformly in n, so for any $\lambda > 0$ the probability (4.1.63b) converges to zero. For any $\underline{x} \in R^k$

$$m^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x} \sim AN(0, \underline{x}'\ \Sigma\ \underline{x}) \text{ as } m \to \infty. \qquad (4.1.64)$$

Since

$$\underline{v}_n = \frac{1}{n}\sum_{i=1}^{n}\underline{x}_{m+i}\ f((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i}) \to \underline{v} \text{ as } n \to \infty \qquad (4.1.65)$$

there is a closed and bounded neighborhood of $\underline{v}$, $B_{\underline{v}}$, which contains all the $\underline{v}_n$, n = 1, 2, ... Let $\underline{v}_0 \in B_{\underline{v}}$ be chosen so that

$$\underline{v}_0'\ \Sigma\ \underline{v}_0 = \sup_{\underline{v}^*\in B_{\underline{v}}} \{\underline{v}^{*\prime}\ B_{\underline{v}}\ \underline{v}^*\} . \qquad (4.1.66)$$

Then for N sufficiently large,(4.1.63a) is bounded above by

$$P[N^{\frac{1}{2}}|(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\,\underline{v}_0| \geq \lambda/2] \;. \qquad (4.1.67)$$

Hence by (4.1.64), λ can be chosen large enough so that (4.1.67) is bounded above by ε/λ^2. Thus the sum of (4.1.63a) and (4.1.63b) is bounded above by ε/λ^2 for N sufficiently large, and Lemma 4.1.5 is proved.

Lemmas 4.1.4, 4.1.5 together establish the appropriate boundedness condition for (4.1.49a). That is, for each $\varepsilon > 0$ there exists a $\lambda > 1$ and an N_0 such that

$$P[\max_{n \leq N} |\hat{U}_{mn}| \geq 2\lambda N^{\frac{1}{2}}] \leq \varepsilon/4\lambda^2 \;. \qquad (4.1.68)$$

Note that (4.1.68) establishes tightness, and hence weak convergence to $W_1 + L_1$, for the process $N^{-\frac{1}{2}}(\hat{S}_{mN}(t) - \mu_N^*)$.

The remainder of this section will demonstrate that for any $\varepsilon > 0$ there is a $\lambda > 1$ and an N_0 such that if $N \geq N_0$,

$$P[\max_{n \leq N} |U_{mn} - \hat{U}_{mn}| \geq 2\lambda N^{\frac{1}{2}}] \leq \varepsilon/\lambda^2 \;, \qquad (4.1.69)$$

which is the appropriate boundedness condition for (4.1.49b). Before proving (4.1.69) two lemmas useful in the proof will be given.

<u>Lemma 4.1.6</u>: Let $Z_1, Z_2, \ldots$ be a sequence of random variables such that

$$N^{\frac{1}{2}} Z_N \sim AN(0, 1) \text{ as } N \to \infty \qquad (4.1.70)$$

and let $\varepsilon > 0$. Then for arbitrary positive constants a and b, constants $\kappa > 0$ and $\lambda > 0$ can be chosen so that the following two conditions are met for N sufficiently large:

$$P[|N^{\frac{1}{2}} Z_N| \geq \kappa] < \varepsilon/\lambda^2 \; ; \tag{4.1.71}$$

$$a\lambda - b\kappa > 0 \quad . \tag{4.1.72}$$

By (4.1.70) and (4.1.71) the following inequality must hold:

$$2 \cdot \frac{1}{(2\Pi)^{1/2}} \int_{-\infty}^{-\kappa} e^{-t^2/2} \, dt < \varepsilon/\lambda^2 \; . \tag{4.1.73}$$

The continued fraction of Laplace (1839) provides the following representation for the error integral in (4.1.73):

$$\frac{1}{(2\Pi)^{1/2}} \int_{-\infty}^{-K} e^{-t^2/2} \, dt = \frac{1}{(2\Pi)^{1/2}} e^{-K^2/2} [\frac{1}{K+} \frac{1}{K+} \frac{2}{K+} \frac{3}{K+} \frac{4}{K+} \cdots] \; . \tag{4.1.74}$$

Note that the sequence of convergents $\frac{1}{K}$, $\frac{1}{K+} \frac{1}{K}$, $\frac{1}{K+} \frac{1}{K+} \frac{2}{K}$, ... is such that the terms of the sequence alternate above and below the limit of the continued fraction in (4.1.74). Thus

$$\frac{1}{K+} \frac{1}{K} \leq [\frac{1}{K+} \frac{1}{K+} \frac{2}{K+} \cdots] \leq \frac{1}{K} \; . \tag{4.1.75}$$

Also note that as $K \to \infty$,

$$\frac{1}{K} \to [\frac{1}{K+} \frac{1}{K+} \frac{2}{K+} \cdots] \; : \tag{4.1.76}$$

This allows the choice of K so that, by (4.1.73)

$$\frac{2}{(2\Pi)^{1/2}} \int_{-\infty}^{-K} e^{-t^2/2} \, dt < \frac{2}{(2\Pi)^{1/2}} e^{-K^2/2} \cdot \varepsilon/\lambda^2 \; . \tag{4.1.77}$$

By inverting the terms of the right hand inequality and solving for λ, an upper bound can be placed on λ. That is:

$$\lambda < \left(\frac{\varepsilon(2\Pi)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}} e^{K^2/2} \cdot K^{\frac{1}{2}} = cK^{\frac{1}{2}} e^{K^2/2} \tag{4.1.78}$$

where $c = \left(\frac{\varepsilon(2\Pi)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}$.

Now substitute the supremum of the λ values in (4.1.72): That is

$$\sup\{a\lambda - bK\} = acK^{\frac{1}{2}} e^{K^2/2} - bK . \tag{4.1.79}$$

Thus the right hand side is positive when

$$K^{-\frac{1}{2}} e^{K^2/2} > \frac{b}{ac} \tag{4.1.80}$$

which is clearly true for K sufficiently large. Thus Lemma 4.1.6 is proven.

Next consider the following function of Y_{m+i}:

$$\zeta(Y_{m+i}) = \eta(Y_{m+i} - \underline{\beta}_0' \underline{x}_{m+i} + KN^{-\frac{1}{2}}) - G_{m+i}(Y_{m+i}) \tag{4.1.81}$$

$$+ \mu_{m+i} - [1 - F((\underline{\beta}_0 - \underline{\beta}_1)' \underline{x}_{m+i} - KN^{-\frac{1}{2}})] .$$

Lemma 4.1.7

$$E\,\zeta(Y_{m+i}) = 0 ; \tag{4.1.82}$$

$$\mathrm{Var}\,\zeta(Y_{m+i}) \to 0 \text{ as } N \to \infty . \tag{4.1.83}$$

Proof: Since

$$E\,\zeta(Y_{m+i}) = P[Y_{m+i} \geq \underline{\beta}_0'\underline{x}_{m+i} - KN^{-\frac{1}{2}}] - \mu_{m+i}$$

$$+ \mu_{m+i} - [1 - F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i}) - KN^{-\frac{1}{2}}]$$

$$= [1-F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i} - KN^{-\frac{1}{2}}] - [1-F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i}) - KN^{-\frac{1}{2}}]$$

$$- \mu_{m+i} + \mu_{m+i}$$

$$= 0\ ,$$

(4.1.82) follows immediately. Since

$$\eta(Y_{m+i} - \underline{\beta}_0'\underline{x}_{m+i} + KN^{-\frac{1}{2}}) \xrightarrow{P} \eta(Y_{m+i} - \underline{\beta}_0'\underline{x}_{m+i}) \qquad (4.1.84)$$

and

$$G_{m+i}(Y_{m+i}) \xrightarrow{P} \eta(Y_{m+i} - \underline{\beta}_0'\underline{x}_{m+i}) \text{ as } N \to \infty\ , \qquad (4.1.85)$$

and

$$|\zeta(Y_{m+i})| \leq 4\ , \qquad (4.1.86)$$

(4.1.83) follows as a result of Theorem 3.5.2.

With Lemmas 4.1.6 and 4.1.7 proved the main line of the tightness argument can now be resumed.

For any fixed value of $\lambda > 1$ and regressor $\underline{x}$, by (4.1.64) a constant K can be chosen so that

$$P[|N^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}| > K] < \varepsilon/8\lambda^2 \qquad (4.1.87)$$

for N sufficiently large. Since the number of values of $\underline{x}_{m+i}$ is finite, (4.1.87) can be changed to

$$P[\max_{i\leq N} |N^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}| > K] < \varepsilon/8\lambda^2\ . \qquad (4.1.88)$$

To establish (4.1.69) let

$$P[\max_{n\le N} |U_{mn}-\hat{U}_{mn}| \le 2\lambda N^{\frac{1}{2}}]$$

$$= P[\max_{n\le N} |U_{mn}-\hat{U}_{mn}| \ge 2\lambda N^{\frac{1}{2}} | \max_{i\le N} |N^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}| \le K] \qquad (4.1.89a)$$

$$\times P[\max_{i\le N} |N^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}| \le K] \qquad (4.1.89b)$$

$$+ P[\max_{n\le N} |U_{mn}-\hat{U}_{mn}| \ge 2\lambda N^{\frac{1}{2}} | \max_{i\le N} |N^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}| > K] \qquad (4.1.89c)$$

$$\times P[\max_{i\le n} |N^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}| > K] \qquad (4.1.89d)$$

$$\le P[\max_{n\le N} |U_{mn}-\hat{U}_{mn}| \ge 2\lambda N^{\frac{1}{2}} | \max_{i\le N} |N^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}| \le K]$$

$$+ \varepsilon/8\lambda^2$$

$$\le P[\max_{n\le N} (U_{mn}-\hat{U}_{mn}) \ge 2\lambda N^{\frac{1}{2}} | \max_{i\le N} |N^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}| \le K] \qquad (4.1.90a)$$

$$+ P[\min_{n\le N} (U_{mn}-\hat{U}_{mn}) \le -2\lambda N^{\frac{1}{2}} | \max_{i\le N} |N^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}| \le K] \qquad (4.1.90b)$$

$$+ \varepsilon/8\lambda^2 .$$

The lines of proof following (4.1.90a) and (4.1.90b) are similar, and so only the line following (4.1.90a) will be presented. Now from (4.1.48), (4.0.7), (4.0.8), (4.1.14),

$$P[\max_{n\le N}(U_{mn}-\hat{U}_{mn}) \ge 2\lambda N^{\frac{1}{2}} | \max_{i\le N} |N^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}| \le K]$$

$$= P[\max_{n\le N} \sum_{i=1}^{n} \{\eta(Y_{m+i}-\hat{\underline{\beta}}_0{}'x_{m+i}) - G_{m+i}(Y_{m+i}) + \mu_{m+i}$$

$$- [1-F((\hat{\underline{\beta}}_0-\underline{\beta}_1)'\underline{x}_{m+i})]\} \ge 2\lambda N^{\frac{1}{2}} | \max_{i\le N} |N^{\frac{1}{2}}(\hat{\underline{\beta}}_0-\underline{\beta}_0)'\underline{x}_{m+i}| \le K]$$

$$\le P[\max_{n\le N} \sum_{i=1}^{n} \{\eta(Y_{m+i}-\underline{\beta}_0{}'\underline{x}_{m+i}+KN^{-\frac{1}{2}}) - G_{m+i}(Y_{m+i}) + \mu_{m+i}$$

$$- [1-F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i}+KN^{-\frac{1}{2}}]\} \geq 2\lambda N^{\frac{1}{2}}]$$

$$= P[\max_{n\leq N} \sum_{i=1}^{n} \{\zeta(Y_{m+i}) + F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i} + KN^{-\frac{1}{2}})$$

$$- F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i} - KN^{-\frac{1}{2}})\} \geq 2\lambda N^{\frac{1}{2}}] \quad , \qquad (4.1.91)$$

where $\zeta(Y_{m+i})$ is defined in (4.1.81). Recall that by Lemma 4.1.7, $E\,\zeta(Y_{m+i}) = 0$ and $\mathrm{Var}\,\zeta(Y_{m+i}) \to 0$ as $N \to \infty$.

For sufficiently large N,

$$\left| \frac{F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i}+KN^{-\frac{1}{2}})-F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i}-KN^{-\frac{1}{2}})}{2KN^{-\frac{1}{2}}} - f((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i}) \right| < \varepsilon \ , \qquad (4.1.92)$$

or

$$F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i}+KN^{-\frac{1}{2}})-F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i}-KN^{-\frac{1}{2}}) \leq (f((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i})+\varepsilon)2KN^{-\frac{1}{2}} \ , \qquad (4.1.93)$$

so

$$P[\max_{n\leq N} \sum_{i=1}^{n} \{\zeta(Y_{m+i})+F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i}+KN^{-\frac{1}{2}}) - F((\underline{\beta}_0-\underline{\beta}_1)'\underline{x}_{m+i}-KN^{-\frac{1}{2}})\} \geq 2\lambda N^{\frac{1}{2}}]$$

$$\leq P[\max_{n\leq N} \sum_{i=1}^{n} \zeta(Y_{m+i}) \geq \lambda N^{\frac{1}{2}}-2N^{\frac{1}{2}}f^*K-2\varepsilon N^{\frac{1}{2}} K]$$

$$= P[\max_{n\leq N} \sum_{i=1}^{n} \zeta(Y_{m+i}) \geq (\lambda-2(f^*+\varepsilon)K)N^{\frac{1}{2}}] \ , \tag{4.1.94}$$

where $f^* = \sup_x\{f(x)\}$. By Lemma 4.1.6, K and λ can be chosen so that $(\lambda-2(f^*+\varepsilon)K) > 0$.

In fact, $(\lambda-2(f^*+\varepsilon)K)$ can be chosen arbitrarily large, as can be seen from (4.1.79) and (4.1.80). For the proof of the next lemma, let

$$\rho = (\lambda- 2(f^*+\varepsilon)K) > \sqrt{2} \ . \tag{4.1.95}$$

Lemma 4.1.8 For N sufficiently large,

$$P[\max_{n\leq N} \sum_{i=1}^{n} \zeta(Y_{m+i}) > \rho N^{\frac{1}{2}}] \leq \varepsilon/8\lambda^2 \ . \tag{4.1.96}$$

Proof: As in the proof of Lemma 3.1, let

$$E_n = [\max_{r<n} \sum_{i=1}^{r} \zeta(Y_{m+i}) \leq \rho N^{\frac{1}{2}} \leq \sum_{i=1}^{n} \zeta(Y_{m+i})] \ . \tag{4.1.97}$$

Then

$$\begin{aligned} P[\max_{n\leq N} \sum_{i=1}^{n} \zeta(Y_{m+i}) \geq \rho N^{\frac{1}{2}}] & \\ &\leq P[\sum_{i=1}^{N} \zeta(Y_{m+i}) \geq (\rho-\sqrt{2})N^{\frac{1}{2}}] \\ &+ \sum_{n=1}^{N-1} P[E_n \ [\sum_{i=1}^{N} \zeta(Y_{m+i}) < (\rho-\sqrt{2})N^{\frac{1}{2}}] \ . \end{aligned} \tag{4.1.98}$$

Since $\sum_{i=1}^{n} \zeta(Y_{m+i}) \geq \rho N^{\frac{1}{2}}$ and $\sum_{i=1}^{N} \zeta(Y_{m+i}) < (\rho-\sqrt{2})N^{\frac{1}{2}}$ imply

$$\left|\sum_{i=n+1}^{N} \zeta(Y_{m+i})\right| \leq (2N)^{\frac{1}{2}} \ ,$$

$$\sum_{n=1}^{N-1} P[E_n \ [\sum_{i=1}^{n} \zeta(Y_{m+i}) < (\rho-\sqrt{2})N^{\frac{1}{2}}]]$$

$$\leq \sum_{n=1}^{N-1} P[E_n \; [|\sum_{i=n+1}^{N} \zeta(Y_{m+i})| \leq (2N)^{\frac{1}{2}}]]$$

$$= \sum_{n=1}^{N-1} P(E_n)P[|\sum_{i=1}^{N} \zeta(Y_{m+i})| \leq (2N)^{\frac{1}{2}}] \qquad (4.1.99)$$

by the independence of $(\zeta(Y_{m+i}), \ldots, \zeta(Y_{m+n}))$ and $(\zeta(Y_{m+N+1}), \ldots, \zeta(Y_{m+N}))$. The sum (4.1.99) is bounded above by

$$\sum_{n=1}^{N-1} P(E_n) \frac{\sum_{i=r+1}^{N} \text{Var } \zeta(Y_{m+i})}{2N}$$

$$\leq \frac{1}{2} \sum_{n=1}^{N-1} P(E_n) ,$$

since Var $\zeta(Y_{m+i})$ converges to zero uniformly in i by the finiteness of the number of $\underline{x}_{m+i}$ values. But then

$$\frac{1}{2} \sum_{n=1}^{N-1} P(E_n) \leq \frac{1}{2} P[\max_{n\leq N}|\zeta(Y_{m+i})| \geq \rho N^{\frac{1}{2}}] . \qquad (4.1.100)$$

Combining (4.1.98) - (4.1.100),

$$P[\max_{n\leq N} \sum_{i=1}^{n} \zeta(Y_{m+i}) \geq \rho N^{\frac{1}{2}}]$$

$$\leq 2P[\sum_{i=1}^{N} \zeta(Y_{m\_i}) \geq (\rho-\sqrt{2})N^{\frac{1}{2}}] . \qquad (4.1.101)$$

Since

$$N^{-1} \sum_{i=1}^{N} \text{Var } \zeta(Y_{m+i}) \to 0 \text{ as } m \to \infty; \text{ then} \qquad (4.1.102)$$

$$N^{-\frac{1}{2}} \sum_{i=1}^{N} \zeta(Y_{m+i}) \overset{P}{\to} 0 \qquad (4.1.103)$$

and the following holds:

$$P[\sum_{i=1}^{N} \zeta(Y_{m+i}) \geq (\rho-\sqrt{2})N^{\frac{1}{2}} \to 0 \text{ as } N \to \infty . \quad (4.1.104)$$

Thus by (4.1.101),

$$P[\max_{n\leq N} \sum_{i=1}^{n} \zeta(Y_{m+i}) \geq \rho N^{\frac{1}{2}}] \leq \varepsilon/8\lambda^2$$

for N sufficiently large, which was to be shown.

Thus (4.1.69) holds, concluding the argument that $N^{-\frac{1}{2}}(S_{mN}(t)-\mu_N^*(t))$ is tight, and therefore concluding the proof of Theorem 4.1.

§4.2 Proof of Theorem 4.2

Recall from (4.0.15) that

$$\begin{aligned} T_{mn} &= \sum_{i=1}^{n} (Y_{m+i}-\hat{\underline{\beta}}_0'\underline{x}_{m+i}) \\ &= \sum_{i=1}^{n} (Y_{m+i}-\underline{\beta}_1'\underline{x}_{m+i}) + \sum_{i=1}^{n} (\underline{\beta}_1'\underline{x}_{m+i}-\hat{\underline{\beta}}_0'\underline{x}_{m+i}) \\ &= \sum_{i=1}^{n} \varepsilon_{m+i} + (\underline{\beta}_1'-\hat{\underline{\beta}}_0') \sum_{i=1}^{n} \underline{x}_{m+i} \\ &= E_{mn} + (\underline{\beta}_1'-\hat{\underline{\beta}}_0') \sum_{i=1}^{n} \underline{x}_{m+i} . \end{aligned} \quad (4.2.1)$$

Define the random polygon

$$E_{nN}(t) = E_{m[Nt]} + (Nt-[Nt])\ \varepsilon_{m+[Nt+1]},\ t \in [0, 1] . \quad (4.2.2)$$

By Donsker's Theorem,

$$N^{-\frac{1}{2}} E_{mN}(t) \Rightarrow W(0, \sigma^2) \text{ as } N \to \infty . \quad (4.2.3)$$

Now from (4.2.1) and (4.0.17)

$$T_{mN}(t) - \mu_N(t) = E_{mN}(t) + R_{mN}(t), \text{ where}$$

$$R_{mN}(t) = (\underline{\hat{\beta}}_0' - \underline{\beta}_0')\left[\sum_{i=1}^{[Nt]} \underline{x}_{m+i} + (Nt-[Nt])\underline{x}_{m+[Nt+1]}\right]. \tag{4.2.4}$$

Recall that

$$m^{\frac{1}{2}}(\underline{\hat{\beta}}_0 - \underline{\beta}_0)'\underline{r} \sim AN(0, \underline{r}'(X'X)^{-1}\underline{r}m\sigma^2) , \tag{4.2.5}$$

so
$$N^{-\frac{1}{2}}(\underline{\hat{\beta}}_0 - \underline{\beta}_1)' \sum_{i=1}^{[Nt]} \underline{x}_{m+i} = N^{\frac{1}{2}}(\underline{\hat{\beta}}_0 - \underline{\beta}_0)' \frac{1}{N} \sum_{i=1}^{[Nt]} \underline{x}_{m+i}$$

$$= \frac{N}{m}^{\frac{1}{2}} m^{\frac{1}{2}} (\underline{\hat{\beta}}_0 - \underline{\beta}_0)' \cdot \frac{[Nt]}{N} \cdot \frac{1}{[Nt]} \sum_{i=1}^{[Nt]} \underline{x}_{m+i}$$

$$\sim AN(0, K^2\sigma^2\underline{r}'D\underline{r}) , \tag{4.2.6}$$

where $D = \lim_{m\to\infty} m(X'X)^{-1}$.

Thus, the finite dimensional distributions of $R_{mN}(t)$ converge to those of L_3. The proof that $R_{mN}(t)$ is tight follows the same argument for (4.1.63a) in Lemma (4.1.5) and is therefore omitted. Thus Theorem 4.2 is proven.

V. ASYMPTOTIC RESULTS FOR R_1: A WILCOXON PROCEDURE

§5.0 Statement of the Main Result

There is one theorem proven in this chapter. Theorem 5.1 is the weak convergence theorem for a cumulative Wilcoxon procedure to test for the constancy of an unknown median.

Let Y_i, $i = 1, \ldots, m + N$ obey the following model:

$$Y_i = \alpha_0 + \varepsilon_i, \; i = 1, \ldots, m \tag{5.0.1}$$

$$Y_{m+i} = \alpha_1 + \varepsilon_{m+i}, \; i = 1, \ldots, N \tag{5.0.2}$$

and let

$$W^*_{mn} = \sum_{i=1}^{n} \eta(Y_{m+i} - \hat{\alpha}_0) \, R_n \, (|Y_{m+i} - \hat{\alpha}_0|), \; 1 \leq n < N, \tag{5.0.3}$$

where R_n $(|Y_{m+i} - \hat{\alpha}_0|)$ is the rank of $|Y_{m+i} - \hat{\alpha}_0|$ among $|Y_{m+1} - \hat{\alpha}_0|$, $\ldots$, $|Y_{m+n} - \hat{\alpha}_0|$, and $\hat{\alpha}_0$ is the Hodges-Lehmann estimate of α_0 based on the first m observations. By (3.3.14)

$$m^{1/2} \, (\hat{\alpha}_0 - \alpha_0) \sim AN \left(0, \frac{1}{12 \, (\int_{-\infty}^{\infty} f^2 \, (x) dx)^2}\right) . \tag{5.0.4}$$

Define a random polygon for $t \varepsilon [0, 1]$ as

$$W^*_{mN}(t) = W^*_{m[Nt]} + (Nt - [Nt]) \, (W^*_{m[Nt+1]} - W^*_{m[Nt]}). \tag{5.0.5}$$

Also let

$$\theta(\alpha_0) = 1 - \int_{-\infty}^{\infty} F(2(\alpha_0 - \alpha_1) - x)\, dF(x). \qquad (5.0.6)$$

The following assumptions are made:

i) $N/m \to \kappa^2$ as $N \to \infty$, $0 \leq \kappa \leq \infty$; (5.0.7)

ii) The ε_i, $i = 1, \ldots, m + N$, are independent and identically distributed with absolutely continuous cumulative distribution function $F(x)$ possessing a continuous density $f(x)$ satisfying $f(x) = f(-x)$.

Theorem 5.1: Under the conditions i) and ii),

$$t^{-1} N^{-3/2} (W^*_{mN}(t) - \frac{Nt(Nt-1)}{2} \theta(\alpha_0) \Rightarrow W_3 + L_3 \qquad (5.0.8)$$

where

$$W_3 \sim W(0, \delta^2), \qquad (5.0.9)$$

$$\delta^2 = \int_{-\infty}^{\infty} [F(2(\alpha_0 - \alpha_1) - x)]^2\, dF(x) - (\int_{-\infty}^{\infty} F(2(\alpha_0 - \alpha_1)$$

$$- x)\, dF(x))^2, \qquad (5.0.10)$$

L_3 is a straight line through the origin with random slope,

$$b_3 \sim N(0, \tau^2), \qquad (5.0.11)$$

and

$$\tau^2 = \frac{\kappa^2}{12} \left(\frac{\int_{-\infty}^{\infty} f(2(\alpha_0 - \alpha_1) - x)\, f(x)\, dx}{\int_{-\infty}^{\infty} f^2(x)\, dx}\right) \qquad (5.0.12)$$

Furthermore, W_3 and L_3 are independent.

§5.1 Proof of Theorem 5.1

The proof of Theorem 5.1 will follow the same format as the proof of Theorem 4.1. First, the finite-dimensional distributions of $t^{-1} N^{-3/2} (W^*_{mN}(t) - \frac{Nt(Nt-1)}{2} \theta(\alpha_0))$ converge to those of $W_3 + L_3$; and second, the process is tight.

§5.1.1 Convergence of the Finite-Dimensional Distributions of $t^{-1} N^{-3/2} (W^*_{mN}(t) - \frac{Nt(Nt+1)}{2} \theta(\alpha_0))$ to those of $W_3 + L_3$

Recall that the partial sums used for constructing the random polygon $W^*_{mN}(t) - \frac{Nt(Nt+1)}{2} \theta(\alpha_0)$ are from (5.0.3), (5.0.5), and (5.0.6),

$$W^*_{mn} - \frac{n(n-1)}{2} \theta(\alpha) = \sum_{i=1}^{n} \eta(Y_{m+i} - \hat{\alpha}_0) R_n (|Y_{m+i} - \hat{\alpha}_0|) \qquad (5.1.1)$$

$$- \frac{n(n-1)}{2} [1 - \int_{-\infty}^{\infty} F(2(\alpha_0 - \alpha_1) - x) \, dF(x)], \qquad (5.1.2)$$

where $R_n (|Y_{m+i} - \hat{\alpha}_0|)$ is the rank of $|Y_{m+i} - \hat{\alpha}_0|$ among $|Y_{m+1} - \hat{\alpha}_0|$, ..., $|Y_{m+n} - \hat{\alpha}_0|$. The following lemma provides an alternate form for (5.1.1) which is easier to work with.

Lemma 5.1.1

$$\sum_{i=1}^{n} \eta(Y_{m+i} - \hat{\alpha}_0) R_n (|Y_{m+i} - \hat{\alpha}_0|) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \eta(Y_{m+i} + Y_{m+j}$$

$$- 2\hat{\alpha}_0) + \sum_{i=1}^{n} \eta(Y_{m+i} - \hat{\alpha}_0). \qquad (5.1.3)$$

Furthermore, let $R_{mn} = \sum_{i=1}^{n} \eta(Y_{m+i} - \hat{\alpha}_0)$, and let $R_{mN}(t) = R_{m[Nt]} + (Nt - [Nt]) \eta(Y_{m+[Nt+1]}) - \hat{\alpha}_0)$, $t \in [0, 1]$, then

(5.1.4)

$$N^{-3/2} R_{mN}(t) \to 0(t) \text{ as } N \Rightarrow \infty, \tag{5.1.5}$$

where $0(t)$ is the zero function in $C[0, 1]$.

Proof: To see that (5.1.3) holds, write

$$R_n(|Y_{m+i} - \hat{\alpha}_0|) = \sum_{j=1}^{n} \eta(|Y_{m+i} - \hat{\alpha}_0| - |Y_{m+j} - \hat{\alpha}_0|),$$

so

$$\sum_{i=1}^{n} \eta(Y_{m+i} - \hat{\alpha}_0) R_n(|Y_{m+i} - \hat{\alpha}_0|)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \eta(Y_{m+i} - \hat{\alpha}_0) \eta(|Y_{m+i} - \hat{\alpha}_0| - |Y_{m+j} - \hat{\alpha}_0|)$$

$$= \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq 1}}^{n} \eta(Y_{m+i} - \hat{\alpha}_0) \eta(|Y_{m+i} - \hat{\alpha}_0| - |Y_{m+j} - \hat{\alpha}_0|)$$

$$+ \sum_{i=1}^{n} \eta(Y_{m+i} - \hat{\alpha}_0) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} [\eta(Y_{m+i} - \hat{\alpha}_0) \eta(|Y_{m+i} - \hat{\alpha}_0|$$

$$- |Y_{m+j} - \hat{\alpha}_0|) + \eta(Y_{m+j} - \hat{\alpha}_0) \eta(|Y_{m+j} - \hat{\alpha}_0| - |Y_{m+i} - \hat{\alpha}_0|)]$$

$$+ \sum_{i=1}^{n} \eta(Y_{m+i} - \hat{\alpha}_0) \tag{5.1.6}$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \eta(Y_{m+i} + Y_{m+j} - 2\hat{\alpha}_0) + R_{mn}. \tag{5.1.7}$$

Since $N^{-3/2} R_{mN}(t_1) \to 0$ for any fixed $t_1 \varepsilon [0, 1]$, the finite-dimensional distributions of $N^{-3/2} R_{mN}(t)$ converge to those of $0(t)$. For any $\varepsilon > 0$ and $\eta > 0$

$$P[\sup_{|s-t|<\delta} |N^{-3/2}\,(R_{mN}(s) - R_{mN}(t))| \geq \varepsilon]$$

$$\leq P[\sup_{|s-t|<\delta} |N^{-3/2}\,(N)| \geq \varepsilon] = 0 \text{ for } N^{-1/2} < \varepsilon,$$

hence by (3.6) $N^{-3/2}\, R_{mN}(t)$ is tight and (5.1.5) is shown.

Now let $W_{mn} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \eta(Y_{m+i} + Y_{m+j} - 2\alpha_0)$,

$$W_{mN}(t) = W_{m[Nt]} + (Nt - [Nt])\,(W_{m[Nt+1]} - W_{m[Nt]})\ , \tag{5.1.8}$$

then

$$W^*_{mN}(t) = W_{mN}(t) + R_{mN}(t) \tag{5.1.9}$$

and by (5.1.5) $N^{-3/2}\,(W^*_{mN}(t) - W_{mN}(t)) \Rightarrow 0(t)$ as $N \to \infty$. Thus weak convergence of $t^{-1}\, N^{-3/2}\,(W^*_{mN}(t) - \frac{Nt(Nt-1)}{2}\,\theta(\alpha_0)$ to $W_3 + L_3$ can be obtained by showing weak convergence of $t^{-1}\, N^{-3/2}\,(W_{mN}(t) - \frac{Nt(Nt-1)}{2}\,\theta(\alpha_0))$ to $W_3 + L_3$.

The following theorem states the main result of §5.1:

<u>Theorem 5.1.1</u>: Let $0 \leq t_2 < \ldots < t_k \leq 1$, then

$$\{N^{-3/2}\, t^{-1}\,(W_{mN}(t_1) - \frac{Nt_1(Nt_1-1)}{2}\,\theta(\alpha_0)),\ \ldots,\ N^{-3/2}\, t_k^{-1}\,(W_{mN}(t_k) - \frac{Nt_k(Nt_k-1)}{2}\,\theta(\alpha_0))\}$$

converges in distribution as $N \to \infty$ to a multivariate normal random vector with mean $\underline{0}_k$ and variance-covariance structure (σ_{ij}), $1 \leq i, j \leq k$, where

$$\sigma_{ij} = \begin{cases} t_i\ \delta^2 + t_i^{\ 2}\ \tau^2 \text{ if } i = j \\ t_i\ \delta^2 + t_i t_j \tau^2 \text{ if } i < j \\ t_j\ \delta^2 + t_i t_j \tau^2 \text{ if } i > j. \end{cases} \tag{5.1.11}$$

That is, the finite-dimensional distributions of $N^{-3/2}\ t^{-1}\ (W_{mN}(t) - \frac{Nt(Nt-1)}{2}\ \theta(\alpha_0))$ converge to those of $W_3 + L_3$.

Before proving Theorem 5.1.1 some preliminary calculations are in order. First define

$$H(t) = \int_{-\infty}^{\infty} F(t - 2\alpha_1 - x)\ dF(x) \tag{5.1.12}$$

$$G_{2\alpha_0}(a) = P[2\hat{\alpha}_0 \leq a] \tag{5.1.13}$$

$$\omega_m = 1 - \int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t). \tag{5.1.14}$$

Then for $i \neq j$ $E\ \eta(Y_{m+i} + Y_{m+j} - 2\hat{\alpha}_0)$

$$= 1 - P[Y_{m+i} + Y_{m+j} \leq 2\hat{\alpha}_0]$$

$$= 1 - E_{\hat{\alpha}_0}\ P[Y_{m+i} + Y_{m+j} \leq t \,|\, 2\hat{\alpha}_0 = t]$$

$$= 1 - \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t - y - \alpha_1)\ dF(y - \alpha_1))\ dG_{2\hat{\alpha}_0}(t)$$

$$= 1 - \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t - y - 2\alpha_1)\ dF(y))\ dG_{2\hat{\alpha}_0}(t)\ .$$

$$= 1 - \int_{-\infty}^{\infty} H(t)\quad dG_{2\hat{\alpha}_0}(t) = \omega_m \tag{5.1.15}$$

and as in Lemma 4.1.1,

$$\omega_m \to 1 - \int_{-\infty}^{\infty} F(2(\alpha_0 - \alpha_1) - y)\, dF(y) = \theta(\alpha_0) \qquad (5.1.16)$$

as $m \to \infty$. Hence

$$EW_{mn} = \binom{n}{2} \omega_m \qquad (5.1.17)$$

Now consider the variance of W_{mn}. First,

$$EW_{mn}^2 = E[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \eta(Y_{m+i} + Y_{m+j} - 2\hat{\alpha}_0)]^2$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[\eta(Y_{m+i} + Y_{m+j} - 2\hat{\alpha}_0)]^2 \qquad (5.1.18a)$$

$$+ 2 \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{r=j+1}^{n} E[\eta(Y_{m+i} + Y_{m+j} - 2\hat{\alpha}_0)\, \eta(Y_{m+j} + Y_{m+r} - 2\hat{\alpha}_0)]$$

(5.1.18b)

$$+ 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{\substack{r=i+1 \\ r\neq j}}^{n} E[\eta(Y_{m+i} + Y_{m+j} - 2\hat{\alpha}_0)\, \eta(Y_{m+i}$$

$$+ Y_{m+r} - 2\hat{\alpha}_0)] \qquad (5.1.18c)$$

$$+ 2 \sum_{i=2}^{n-1} \sum_{j=i+1}^{n} \sum_{r=1}^{i-1} E[\eta(Y_{m+i} + Y_{m+j} - 2\hat{\alpha}_0)\, \eta(Y_{m+r} + Y_{m+j} - 2\hat{\alpha}_0)]$$

(5.1.18d)

$$+ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{\substack{r=1 \\ r\neq i \\ r\neq j}}^{n-1} \sum_{\substack{s=r+1 \\ s\neq i \\ s\neq j}}^{n} E[\eta(Y_{m+i} + Y_{m+j} - 2\hat{\alpha}_0)$$

$$\eta(Y_{m+r} + Y_{m+s} - 2\hat{\alpha}_0)] \qquad (5.1.18e)$$

Taking expected values in (5.1.18b),

$$E[\eta(Y_{m+i} + Y_{m+j} - 2\hat{\alpha}_0)\ \eta(Y_{m+j} + Y_{m+r} - 2\hat{\alpha}_0)]$$

$$= P[Y_{m+i} + Y_{m+j} \geq 2\hat{\alpha}_0;\ Y_{m+j} + Y_{m+r} \geq 2\hat{\alpha}_0]$$

$$= 1 - P[Y_{m+i} + Y_{m+j} < 2\hat{\alpha}_0] - P[Y_{m+j} + Y_{m+r} < 2\hat{\alpha}_0]$$

$$+ P[Y_{m+i} + Y_{m+j} < 2\hat{\alpha}_0;\ Y_{m+j} + Y_{m+r} < 2\hat{\alpha}_0]$$

$$= 1 - 2\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t)$$

$$+ E_{2\hat{\alpha}_0}\ E_{Y_{m+j}}\ P[Y_{m+i} < t-y;\ Y_{m+r} < t-y \mid 2\hat{\alpha}_0 = t;\ Y_{m+j} - y]$$

$$= 1 - 2\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t) + \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} F^2(t-y-2\hat{\alpha}_1)\ dF(y)\right) dG_{2\hat{\alpha}_0}(t) \tag{5.1.19}$$

Similarly, in (5.1.18e),

$$E[\eta(Y_{m+i} + Y_{m+j} - 2\hat{\alpha}_0)\eta(Y_{m+r} + Y_{m+s} - 2\hat{\alpha}_0)]$$

$$= 1 - 2\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t) + \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} F(t-y-2\hat{\alpha}_1)\ dF(y)\right)^2 dG_{2\hat{\alpha}_0}(t)$$

$$= 1 - 2\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t) + \int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t) \tag{5.1.20}$$

Thus

$$\mathrm{Var}\ W_{mn} = \binom{n}{2}\left[1 - \int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t)\right]$$

$$+ 2\binom{n}{2}(n-2)\left[1 - 2\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t) + \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} F^2(t-y-2\hat{\alpha}_1)\right.\right.$$

$$\left.\left.\cdot\ dF(y)\right) dG_{2\hat{\alpha}_0}(t)\right]$$

$$+ \binom{n}{2}\binom{n-2}{2}\left[1 - 2\int_{-\infty}^{\infty} H(t)\, dG_{2\hat{\alpha}_0}(t) + \int_{-\infty}^{\infty} H^2(t)\, dG_{2\hat{\alpha}_0}(t)\right]$$

$$- \binom{n}{2}^2 \left[1 - \int_{-\infty}^{\infty} H(t)\, dG_{2\hat{\alpha}_0}(t)\right]^2$$

$$= \binom{n}{2} \omega_m + 2\binom{n}{2}(n-2)\left[2\omega_m - 1 + \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F^2(t-y-2\alpha_1)\, dF(y)\right)\right.$$

$$\left. \cdot\, dG_{2\hat{\alpha}_0}(t)\right]$$

$$+ \binom{n}{2}\binom{n-2}{2}\left[2\omega_m - 1 + \int H^2(t)\, dG_{2\hat{\alpha}_0}(t)\right] - \binom{n}{2}^2 \omega_m^2. \qquad (5.1.21)$$

From (3.3) the Hájek projection statistic is

$$\hat{W}_{mn} = \sum_{i=1}^{n} E[W_{mn}|Y_{m+i}] + E[W_{mn}|\hat{\alpha}_0] - nE\, W_{mn}. \qquad (5.1.22)$$

Further, since

$$E[\eta(Y_{m+i} + Y_{m+j} - 2\hat{\alpha}_0)|Y_{m+i} = y] = 1 - \int_{-\infty}^{\infty} F(t-y-\hat{\alpha}_1)\, dG_{2\hat{\alpha}_0}(t), \qquad (5.1.23)$$

and

$$E[\eta(Y_{m+i} + Y_{m+j} - 2\hat{\alpha}_0)|2\alpha_0 = t] = 1 - \int_{-\infty}^{\infty} F(t-y-2\hat{\alpha}_1)\, dF(y), \qquad (5.1.24)$$

the projection statistic may be written

$$\hat{W}_{mn} = \sum_{i=1}^{n} \{(n-1)\left(1 - \int_{-\infty}^{\infty} F(t-Y_{m+i}-\alpha_1)\, dG_{2\hat{\alpha}_0}(t)\right)$$

$$+ \left[\binom{n}{2} - (n-1)\right]\omega_m\}$$

$$+ \binom{n}{2} \{1 - \int_{-\infty}^{\infty} F(2\ (\alpha_0 - \alpha_1) - y)\ dF(y)\} - n\binom{n}{2}\ \omega_m$$

$$= \sum_{i=1}^{n} (n-1)\ [1 - \int_{-\infty}^{\infty} F(t - Y_{m+i} - \alpha_1)\ dG_{2\hat{\alpha}_0}(t) - \omega_m]$$

$$+ \binom{n}{2}\ [1 - \int_{-\infty}^{\infty} F(2(\alpha_0 - \alpha_1) - y)\ dF(y)]$$

$$= \sum_{i=1}^{n} (n-1)\ \psi(Y_{m+i}) + \binom{n}{2}\ \theta(\alpha_0) \qquad (5.1.25)$$

where

$$\psi(Y_{m+i}) = 1 - \int_{-\infty}^{\infty} F(t - Y_{m+i} - \alpha_1)\ dG_{2\hat{\alpha}_0}(t) - \omega_m \qquad (5.1.26)$$

$$\theta(\alpha_0) = 1 - \int_{-\infty}^{\infty} F(2(\alpha_0 - \alpha_1) - y)\ dF(y). \qquad (5.1.27)$$

Now the following lemma is shown:

Lemma 5.1.2: Let $0 \leq t, < \ldots < t_k \leq 1$, then

$$\{N^{-3/2}\ t_1^{-1}\ (W_{mN}(t_1) - W_{mN}(t_1)),\ \ldots,\ N^{-3/2}\ t_k^{-1}\ (W_{mN}(t_k)$$

$$- W_{mN}(t_k))\} \xrightarrow{p} \underline{0}_k \qquad (5.1.28)$$

Proof: From (3.4), $E(W_{mn} - W_{mn})^2 = \mathrm{Var}\ W_{mn} - \mathrm{Var}\ W_{mn}$, and

$$\mathrm{Var}\ W_{mn} = n(n-1)^2\ \mathrm{Var}\ \psi(Y_{m+1}) + \binom{n}{2}^2\ \mathrm{Var}\ \theta\ (\alpha_0)$$

$$= n(n-1)^2 \{\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t-y-\alpha_1)\ dG_{2\hat{\alpha}_0}(t))^2\ dF(y-\alpha_1)$$

$$- (\int_{-\infty}^{\infty} H\ dG_{2\alpha_0})^2\} + \binom{n}{2}^2 \{\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t-y-2\hat{\alpha}_1)\ dF(y))^2\ dG_{2\hat{\alpha}_0}(t)$$

$$- (\int_{-\infty}^{\infty} H \, dG_{2\hat{\alpha}_0})^2\} = n(n-1)^2 \{\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t-y-\alpha_1) \, dG_{2\hat{\alpha}_0}(t))^2 \, dF(y-\alpha_1)$$

$$- (\int_{-\infty}^{\infty} H(t) \, dG_{2\hat{\alpha}_0}(t)^2 + \binom{n}{2}^2 \{\int_{-\infty}^{\infty} H^2(t) \, dG_{2\hat{\alpha}_0}(t)$$

$$- (\int_{-\infty}^{\infty} H(t) \, dG_{2\hat{\alpha}_0}(t))^2\} . \qquad (5.1.29)$$

Thus, combining (5.1.21) and (5.1.29),

$$\text{Var } W_{mn} - \text{Var } W_{mn} = \binom{n}{2} \omega_m + 2\binom{n}{2} (n-2) \{ 2\omega_m - 1$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^2(t-y-2\hat{\alpha}_1) \, dF(y) \, dG_{2\hat{\alpha}_0}(t)\} + \binom{n}{2} \binom{n-2}{2} \{ 2\omega_m - 1$$

$$+ \int_{-\infty}^{\infty} H^2(t) \, dG_{2\ 0}(t)\} - \binom{n}{2}^2 \omega_m{}^2$$

$$- n(n-1)^2 \{\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t-y-\alpha_1) \, dG_{2\hat{\alpha}_0}(t))^2 \, dF(y - \alpha_1)$$

$$- (\int_{-\infty}^{\infty} H(t) \, dG_{2\hat{\alpha}_0}(t))^2\}$$

$$- \binom{n}{2}^2 \{\int_{-\infty}^{\infty} H^2(t) \, dG_{2\hat{\alpha}_0} - (\int_{-\infty}^{\infty} H(t) \, dG_{2\hat{\alpha}_0}(t))^2\}$$

$$= [\binom{n}{2} + 4\binom{n}{2} (n-2) + 2\binom{n}{2} \binom{n-2}{2}] \omega_m - \binom{n}{2} \omega_m{}^2$$

$$+ 2\binom{n}{2} (n-2) \{\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F^2(t-y-2\hat{\alpha}_1) \, dF(y)) \, dG_{2\hat{\alpha}_0}(t) - 1$$

$$+ \binom{n}{2} \binom{n-2}{2} \{\int_{-\infty}^{\infty} H^2(t) \, dG_{2\hat{\alpha}_0}(t) - 1\}$$

$$- n(n-1)^2 \{\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t-y-\alpha_1) \, dG_{2\hat{\alpha}_0}(t))^2 \, dF(y -\alpha_1)$$

$$- (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2\} - \binom{n}{2}^2 \{\int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t)$$

$$- (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2\}. \tag{5.1.30}$$

Now $\binom{n}{2} + 2\binom{n}{2}(n-2) + \binom{n}{2}\binom{n-2}{2} = \binom{n}{2}^2$, so the coefficient of ω_m in (5.1.30) is

$$\binom{n}{2} + 4\binom{n}{2}(n-2) + 2\binom{n}{2}\binom{n-2}{2}$$

$$= 2[\binom{n}{2} + 2\binom{n}{2}(n-2) + \binom{n}{2}\binom{n-2}{2}] - \binom{n}{2} = 2\binom{n}{2}^2 - \binom{n}{2}.$$

Expanding the factorials in (5.1.30) gives

$$\text{Var } W_{mn} - \text{Var } W_{mn}$$

$$= 1/2\ (n^4 - 2n^3 + n)\ \omega_m - 1/2\ (n^4 - 2n^3 + n^2)\ \omega_m^{\ 2}$$

$$+ (n^3 - 3n^2 + 2n) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^2(t-y-2\alpha_1)\ dF(y)\ dG_{2\hat{\alpha}_0}(t)$$

$$+ 1/4\ (n^4 - 6n^3 + 11n^2 - 6n) \int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t)$$

$$- 1/4\ (n^4 - 2n^3 - n^2 + 2n) \tag{5.1.31}$$

$$- (n^3 - 2n^2 + n) \{\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t-y-\alpha_1)\ dG_{2\hat{\alpha}_0}(t))^2\ dF(y - \alpha_1)$$

$$- (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2\}$$

$$- 1/4\ (n^4 - 2n^3 + n^2) \{\int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t) - (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2\}$$

Regarding (5.1.31) as a polynomial in n of order 4, the quartic term is:

$$n^4 \{1/2\ \omega_m - 1/4\ \omega_m^{\ 2} + 1/4 \int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t) - 1/4$$

$$- 1/4 \int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t) + 1/4 (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2\}$$

$$= n^4 \{1/2\ (1 - \int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t)) - 1/4\ (1 - 2 \int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t)$$

$$+ (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2) - 1/4 + 1/4 (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2\}$$

$$= n^4 \{1/2 - 1/2 \int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t) - 1/4 + 1/2 \int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t)$$

$$- 1/4 (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2 - 1/4 + 1/4 (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2\} = 0,$$

so the quartic term vanishes. The n^3 term is

$$n^3 \{-\omega_m + 1/2\ \omega_m^{\ 2} + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^2(t-y-2\alpha_1)\ dF(y)\ dG_{2\hat{\alpha}_0}(t)$$

$$- 3/2 \int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t) + 1/2 - \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t-y-\alpha_1)\ dG_{2\hat{\alpha}_0}(t))^2$$

$$\cdot\ dF(y-\alpha_1) + (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2 + 1/2 \int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t)$$

$$- 1/2 (\int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t))^2\} = n^3 \{-1 + \int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t)$$

$$+ 1/2\ (1 - 2 \int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t) + (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2)$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^2(t-y-2\alpha_1)\ dF(y)\ dG_{2\hat{\alpha}_0}(t) - 3/2 \int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t) + 1/2$$

$$- \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t-y-\alpha_1)\ dG_{2\hat{\alpha}_0}(t))^2\ dF(y-\alpha_1) + 1/2 (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2$$

$$+ 1/2 \int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t)\} = n^3 \{ (\int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t))^2$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^2(t-y-2\hat{\alpha}_1)\ dF(y)\ dG_{2\hat{\alpha}_0}(t) - \int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t)$$

$$- \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t-y-\alpha_1)\ dG_{2\hat{\alpha}_0}(t))^2\ dF(y-\alpha_1)\}. \qquad (5.1.32)$$

Since $\hat{\alpha}_0 \to \alpha_0$ and F is absolutely continuous, as in the proof of Lemma 4.1.1 the following are true:

i) $$\lim_{m\to\infty} \int_{-\infty}^{\infty} H(t)\ dG_{2\hat{\alpha}_0}(t) = \int_{-\infty}^{\infty} F(2(\alpha_0-\alpha_1)-y)\ dF(y); \qquad (5.1.33)$$

ii) $$\lim_{m\to\infty} \int_{-\infty}^{\infty} H^2(t)\ dG_{2\hat{\alpha}_0}(t) = (\int_{-\infty}^{\infty} F(2(\alpha_0-\alpha_1)-y)\ dF(y))^2; \qquad (5.1.34)$$

iii) $$\lim_{m\to\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^2(t-y-2\hat{\alpha}_1)\ dF(y)\ dG_{2\hat{\alpha}_0}(t)$$

$$= \int_{-\infty}^{\infty} F^2(2(\alpha_0-\alpha_1)-y)\ dF(y); \qquad (5.1.35)$$

iv) $$\lim_{m\to\infty} \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t-y-\alpha_1)\ dG_{2\hat{\alpha}_0}(t))^2\ dF(y-\alpha_1)$$

$$= \int_{-\infty}^{\infty} F^2(2(\alpha_0-\alpha_1)-y)\ dF(y). \qquad (5.1.36)$$

Hence, the coefficient of n^3 in (5.1.32) converges to zero as $m\to\infty$, and so $E(W_{mn} - W_{mn})^2 = 0(n^3)$.

The remainder of the proof follows as in Lemma 4.1.2, and so will be omitted.

To complete the proof of Theorem 5.1.1, one need only state the following lemma:

Lemma 5.1.3: Under the conditions for Theorem 5.1,

$$N^{-3/2}\ t^{-1}\ (W_{mN}(t) - \frac{Nt(Nt-1)}{2}\ \beta(\alpha_0)) \Rightarrow W_3 + L_3 \qquad (5.1.38)$$

as $N\to\infty$.

Proof: The proof is similar to the proof of Lemma 4.1.3 and will be omitted.

§5.1.2 Closeness of $W_{mN}(t)$ and $\hat{W}_{mN}(t)$:

Rather than prove tightness of $\hat{W}_{mN}(t)$ directly, as was done for $S_{mN}(t)$ in Theorem 4.1, the approach here will be to show first that $\hat{W}_{mN}(t)$ is tight, and then that the distance between $W_{mN}(t)$ and $\hat{W}_{mN}(t)$ converges to zero.

Lemma 5.1.4 The sequence

$$\{t^{-1}\, N^{-3/2}\, (\hat{W}_{mN}(t) - \frac{Nt(Nt-1)}{2}\,\theta(\alpha_0))\}$$

is tight.

Proof: From (5.1.25), (5.1.26), (5.1.27) and (5.0.5)

$$t^{-1}\, N^{-3/2}\, (\hat{W}_{mN}(t) - \frac{Nt(Nt-1)}{2}\,\theta(\alpha_0))$$

$$= t^{-1}\, N^{-3/2}\{\sum_{i=1}^{[Nt]} [Nt-1]\, \psi(Y_{m+i}) \tag{5.1.41a}$$

$$+ (Nt - [Nt])\, [[Nt]\, \psi(Y_{m+[Nt+1]}) + \sum_{i=1}^{[Nt]} \psi(Y_{m+i})]\} \tag{5.1.41b}$$

$$+ t^{-1}\, N^{-3/2}\, \{ \binom{[Nt]}{2} (\beta(\hat{\alpha}_0) - \beta(\alpha_0))$$

$$+ (Nt-[Nt])\, [\binom{[Nt+1]}{2} - \binom{[Nt]}{2}]\, (\beta(\hat{\alpha}_0) - \beta(\alpha_0))\}. \tag{5.1.41c}$$

Establishing tightness for (5.1.41b) and (5.1.41c) separately is sufficient for establishing tightness for (5.1.41a), since for $x(t)$, $y(t) \in C[0, 1]$, using the tightness criterion (3.6),

$$P[\sup_{|s-t|<\delta} |(x(s) + y(s)) - (x(t) + Y(t))| \geq 2\varepsilon]$$

$$\leq P[\sup_{|s-t|<\delta} |x(s) - x(t)| + \sup_{|s-t|<\delta} y(s) - y(t)| \geq 2\varepsilon]$$

$$\leq P[\sup_{|s-t|<\delta} |x(s) - x(t)| \geq \varepsilon]$$

$$+ P[\sup_{|s-t|<\delta} |y(s) - y(t)| \geq \varepsilon].$$

Rewriting (5.1.41b) as

$$\frac{[Nt-1]}{Nt} \cdot N^{-1/2} \{ \sum_{i=1}^{[Nt]} \psi(Y_{m+i})$$

$$+ (Nt - [Nt]) [\frac{[Nt]}{[Nt-1]} \psi(Y_{m+[Nt+1]} + \frac{1}{[Nt-1]} \sum_{i=1}^{[Nt]} \psi(Y_{m+i})]\}$$

$$= \frac{[Nt-1]}{Nt} N^{-1/2} \{ \sum_{i=1}^{[Nt]} \psi(Y_{m+i}) + (Nt - [Nt]) \Psi(Y_{m+[Nt+1]})\} \tag{5.1.42a}$$

$$+ \frac{[Nt-1]}{Nt} N^{-1/2} (Nt - [Nt]) \frac{1}{[Nt-1]} \sum_{i=1}^{[Nt+1]} \psi(Y_{m+i}). \tag{5.1.42b}$$

Arguing as in the proof of (5.1.5), the remainder process (5.1.42b) converges weakly to the zero function. Tightness of (5.1.42a) is established by applying Theorem 3.2.1 to the partial sums $\sum_{i=1}^{n} \psi(Y_{m+i})$, $1 \leq n \leq N$. That is,

$$P[\max_{n\leq N} | \sum_{i=1}^{n} \psi(Y_{m+i})| \geq \lambda N^{1/2}]$$

$$\leq 2P[| \sum_{i=1}^{N} \psi(Y_{m+i})| \geq (\lambda - \sqrt{2}) N^{1/2}]. \tag{5.1.43}$$

By the central limit theorem,

$$N^{-1/2} \sum_{i=1}^{N} \psi(Y_{m+i}) \sim AN(0, \delta^2), \tag{5.1.44}$$

where δ^2 is defined by (5.0.10). Thus λ can be chosen sufficiently large so that

$$P[\max_{n \le N} | \sum_{i=1}^{n} \psi(Y_{m+i}) | \ge \lambda N^{1/2}] < \varepsilon/\lambda^2, \tag{5.1.45}$$

and tightness of (5.1.41b) is established.

Now note that (5.1.41c) can be written

$$t^{-1} N^{-3/2} [(\binom{[Nt]}{2}) + (Nt - [Nt]) ((\binom{[Nt+1]}{2})$$

$$- (\binom{[Nt]}{2}))] (\theta(\hat{\alpha}_0) - (\theta (\alpha_0)) = t^{-1} N^{-3/2} \binom{[Nt]}{2} (\theta(\hat{\alpha}_0)$$

$$- \theta(\alpha_0)) + X(t), \tag{5.1.46}$$

where $X(t)$ is a random function converging weakly to the zero function in $C[0, 1]$. Now

$$t^{-1} N^{-3/2} \binom{[Nt]}{2} (\theta(\hat{\alpha}_0) - \theta(\alpha_0) = \frac{[Nt-1]}{Nt} \frac{[Nt]}{N^{1/2}} (\theta(\hat{\alpha}_0) - \theta(\alpha_0)).$$

As in Rao (1973, p. 386), the Taylor expansion of $\theta(\hat{\alpha}_0) = 1 - \int_{-\infty}^{\infty} F(2(\hat{\alpha}_0 - \alpha_1)-y)\, dF(y)$ can be written

$$[1 - \int_{-\infty}^{\infty} F(2(\hat{\alpha}_0 - \alpha_1)-y)\, dF(y)] - [1 - \int_{-\infty}^{\infty} F(2(\alpha_0 - \alpha_1)-y)\, dF(Y)]$$

$$= 2(\hat{\alpha}_0 - \alpha_0) [\int_{-\infty}^{\infty} f(2(\alpha_0 - \alpha_1)-y)\, f(y)\, dy + \varepsilon] \tag{5.1.47}$$

where $\varepsilon \xrightarrow{p} 0$ as $N \to \infty$. Also the asymptotic distribution of $N^{1/2} (\theta(\hat{\alpha}_0) - \theta(\alpha_0))$ is the same as $N^{1/2}\, 2(\hat{\alpha}_0 - \alpha_0) [\int_{\infty}^{\infty} f(2(\alpha_0 - \alpha_1)-y)\, f(y)\, dy]$, which is asymptotically normal with mean zero and variance $\tau^2 < \infty$.

Thus, (5.1.41c) describes a straight line through the origin with an asymptotically normal slope, and so (5.1.41c) is tight. Thus Lemma 5.1.4 is proven.

By Lemma 5.1.4 and the results of the last section on convergence of the finite-dimensional distributions, the following is proven:

Lemma 5.1.5 Under the conditions of Theorem 5.1,

$$t^{-1}\ N^{-3/2}\ (\hat{W}_{mN}(t)\ -\ \frac{Nt(Nt-1)}{2}\ \theta(\alpha_0) \Rightarrow W_3 + L_3 \qquad (5.1.48)$$

To conclude the proof of Theorem 5.1, the closeness of $W_{mN}(t)$ and $W_{mN}(t)$, as defined below needs to be established.

For any two random functions $X(t) \in C[0, 1]$, $Y(t) \in C[0, 1]$, define the distance between $X(t)$ and $Y(t)$ as

$$\rho(X(t), Y(t)) = \sup_{t\in[0,1]} |X(t) - Y(t)|. \qquad (5.1.49)$$

Then the following lemma will conclude the proof of Theorem 5.1:

Lemma 5.1.6 Under the conditions of Theorem 5.1

$$\rho(t^{-1}\ N^{-3/2}\ \hat{W}_{mN}(t),\ t^{-1}\ N^{-3/2}\ W_{mN}(t)) \xrightarrow{p} 0 \qquad (5.1.50)$$

Proof: By definition,

$$\rho(t^{-1}\ N^{-3/2}\ \hat{W}_{mN}(t),\ t^{-1}\ N^{-3/2}\ W_{mN}(t))$$

$$= \sup_{t\in[0,1]}\ t^{-1}\ N^{-3/2} |\hat{W}_{mN}(t) - W_{mN}(t)|$$

$$- \sup_{t\varepsilon[0,1]} t^{-1} N^{-3/2} |\hat{W}_{m[Nt]} + (Nt-[Nt]) (\hat{W}_{m[Nt+1]} - \hat{W}_{m[Nt]})$$

$$- W_{m[Nt]} - (Nt-[Nt]) (W_{m[Nt+1]} - W_{m[Nt]})|$$

$$\leq \sup_{t\varepsilon[1/N,1]} t^{-1} N^{-3/2} |\hat{W}_{m[Nt]} - W_{m[Nt]}|$$

$$+ \sup_{t\varepsilon[0,1]} t^{-1} N^{-3/2} (Nt-[Nt]) | (\hat{W}_{m[Nt+1]} - \hat{W}_{m[Nt]}) -$$

$$- (W_{m[Nt+1]} - W_{m[Nt]})| = \max_{1\leq n\leq N} N^{-1/2} n^{-1} |\hat{W}_{mn} - W_{mn}| \qquad (5.1.51a)$$

$$+ \sup_{t\varepsilon[0,1]} t^{-1} N^{-3/2} (Nt-[Nt]) | (\hat{W}_{m[Nt+1]} - \hat{W}_{m[Nt]}) -$$

$$- (W_{m[Nt+1]} - W_{m[Nt]})| \qquad (5.1.51b)$$

Consider first (5.1.51b): On the interval [0, 1/N), (5.1.51b) is

$$\sup_{t\varepsilon[0,1/N]} t^{-1} N^{-3/2} (Nt) |\hat{W}_{m1} - W_{m1}|$$

$$= \sup_{t\varepsilon[0,1/N]} N^{-1/2} |\hat{W}_{m1} - W_{mi}| = 0_p(1); \qquad (5.1.52)$$

For t ε [1/N, 1], (5.1.51b) is bounded above by

$$\sup_{t\varepsilon[i/N,1]} N^{-1/2} (Nt)^{-1} | (\hat{W}_{m[Nt+1]} - \hat{W}_{m[Nt]}) - (W_{m[Nt+1]}$$

$$- W_{m[Nt]})| \qquad (5.1.53)$$

Now recall from (5.1.8),

$$W_{m[Nt+1]} - W_{m[Nt]} = \sum_{i=1}^{[Nt]} \sum_{j=i+1}^{[Nt+1]} \eta(Y_{m+i} + Y_{m+j} - \alpha_0)$$

$$- \sum_{i=1}^{[Nt-1]} \sum_{j=1}^{[Nt]} \eta(Y_{m+i} + Y_{m+j} - \alpha_0)$$

$$= \sum_{i=1}^{[Nt]} \eta(Y_{m+i} + Y_{m+[Nt+1]} - 2\alpha_0); \qquad (5.1.54)$$

and from (5.1.25) - (5.1.27),

$$\hat{W}_{m[Nt+1]} - \hat{W}_{m[Nt]} = \sum_{i=1}^{[Nt+1]} [Nt]\ \psi(Y_{m+i}) + \binom{[Nt+1]}{2}\theta(\alpha_0)$$

$$- \sum_{i=1}^{[Nt]} [Nt-1]\ \psi(Y_{m+i}) - \binom{[Nt]}{2}\ \theta(\alpha_0)$$

$$= [Nt]\ \psi(Y_{m+[Nt+1]}) + \sum_{i=1}^{[Nt]} \psi(Y_{m+i}) + [Nt]\ \theta(\alpha_0)$$

$$= [Nt]\ (1 - \int_{-\infty}^{\infty} F(t - Y_{m+[Nt+1]} - \alpha_1)\ dG_{2\alpha_0}(t))$$

$$+ \sum_{i=1}^{[Nt]} (1 - \int_{-\infty}^{\infty} F(t - Y_{m+i} - \alpha_1)\ dG_{2\alpha_0}(t))$$

$$+ [Nt]\ (\ 1 - \int_{-\infty}^{\infty} F(2(\alpha_0 - \alpha_1)-y)\ dF(y)). \qquad (5.1.55)$$

From (5.1.54) and (5.1.55), an upper bound for (5.1.53) is immediately seen to be

$$N^{-1}\ (Nt)^{-1}\ 4\ [Nt] = o(1) \qquad (5.1.56)$$

The proof of Lemma 5.1.6 now rests in showing, from (5.1.51a), that

$$\max_{1\leq n\leq N} N^{-1/2}\ n^{-1}\ |\hat{W}_{mn} - W_{mn}| = o_p(1), \qquad (5.1.57)$$

or equivalently, that for any $\varepsilon > 0$,

$$\lim_{N\to\infty} P[\max_{1\leq n\leq N} N^{-1/2}\ n^{-1}\ |\hat{W}_{mn} - W_{mn}| > \varepsilon] = 0 \qquad (5.1.58)$$

In order to utilize the lemma for U-statistics given in Chapter III, (5.1.58) will be altered slightly to conform to U-statistic notation. Let

$$U_{mn} = \binom{n}{2}^{-1} W_{mn}; \; \hat{U}_{mn} = \binom{n}{2}^{-1} \hat{W}_{mn}, \tag{5.1.59}$$

then

$$\max_{1 \le n \le N} N^{-1/2} n^{-1} |W_{mn} - \hat{W}_{mn}|$$

$$= \max_{1 \le n \le N} N^{-1/2} n^{-1} \binom{n}{2} |U_{mn} - \hat{U}_{mn}|$$

$$= \max_{1 \le n \le N} N^{-1/2} \frac{n-1}{2} |U_{mn} - \hat{U}_{mn}|$$

$$= \max_{1 \le n \le N} N^{-1/2} n \cdot 1/2 \, |U_{mn} - \hat{U}_{mn}| + o_p(1). \tag{5.1.60}$$

Ignoring the factor 1/2 in (5.1.60), showing that (5.1.58) holds is equivalent to showing that

$$\lim_{N \to \infty} P[\max_{1 \le n \le N} n \, |U_{mn} - \hat{U}_{mn}| > \varepsilon N^{1/2}] = 0 \tag{5.1.61}$$

Now from (5.1.25) - (5.1.27),

$$\hat{U}_{mn} = \frac{2}{n} \sum_{i=1}^{n} \psi(Y_{m+i}) + \theta(\alpha_0), \tag{5.1.62}$$

and let $U_{mn}(a)$ be the U-statistic with symmetric kernel.

$$\phi(Y_{m+i}, Y_{m+j}) = \eta(Y_{m+i} + Y_{m+j} - 2a). \tag{5.1.63}$$

From (5.0.6), the functional for $U_{mn}(a)$ is

$$E\phi\ (Y_{m+i}, Y_{m+j}) = \theta(a), \tag{5.1.64}$$

so by (3.4.4),

$$U_{mn}(a) = \theta(a) + 2U_{mn,1}(a) + U_{mn,2}(a) \tag{5.1.65}$$

Before proceeding with the proof of Lemma 5.1.6, the following lemma is proven.

Lemma 5.1.7

$$U_{mn,1}(a) = \frac{1}{n} \sum_{i=1}^{n} [1-F(2a - \alpha_1 - Y_{m+i})] - \theta(a); \tag{5.1.66}$$

$$U_{mn,2}(a) = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \eta(Y_{m+i} + Y_{m+j} - 2a)$$

$$- \frac{2}{n} \sum_{i=1}^{n} [1-F(2a - \alpha_1 - Y_{m+i})] + \theta(a) \tag{5.1.67}$$

Proof: By (3.4.5)

$$U_{mn,1}(a) = \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{\infty} \phi_1(y_1)\ d[\eta(y_1 - Y_{m+i}) - F(y_1 - \alpha_1)] \tag{5.1.68}$$

$$\phi_1(y_1) = E\ \phi(y_1, Y_2) = E\ \eta(_y1 + Y_2 - 2a)$$

$$= 1 - F(2a - \alpha_1 - y_1), \text{ so}$$

$$U_{mn,1}(a) = \frac{1}{n} \sum_{i=1}^{n} \{\int_{-\infty}^{\infty} [1 - F(2a - \alpha_1 = y_1)]\ d\ \eta(y_1 - Y_{m+i})$$

$$- \int_{-\infty}^{\infty} [1 - F(2a - \alpha_1 - y_1)\ dF(y_1 - \alpha_1)\}$$

$$= \frac{1}{n} \sum_{i=1}^{n} [1 - F(2a - \alpha_1 - Y_{m+i})] - \theta(a), \tag{5.1.69}$$

which establishes (5.1.66). Again by (3.4.5),

$$U_{mn,2}(a) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_2 (y_1, y_2) \tag{5.1.70}$$

$$\cdot\ d[\eta(y_1 - Y_{m+i}) - F(y_1 - \alpha_1)]\ [\eta(y_2 - Y_{m+j}) - F(y_2 - \alpha_1)],$$

$$\phi_2(y_1, y_2) = E\ \phi(y_1, y_2) = \eta(y_1 + y_2 - 2a), \tag{5.1.71}$$

so

$$U_{mn,2}(a) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta(y_1 + y_2 - 2a)\ d[\eta(y_1 - Y_{m+i})$$

$$\cdot\ \eta(y_2 - Y_{m+j})] - \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta(y_1 + y_2 - 2a)$$

$$\cdot\ d[\eta(y_1 - Y_{M+i})\ F(y_2 - \alpha_1)] + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta(y_1 + y_2 - 2a)$$

$$\cdot\ d[\eta(y_2 - Y_{m+j})\ F(y_1 - \alpha_1)]\}$$

$$+ \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta(y_1 + y_2 - 2a)\ d[F(y_1 - \alpha_1)\ F(y_2 - \alpha_1)]$$

$$= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \eta(Y_{m+i} + Y_{m+j} - 2a)$$

$$- \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \{[1 - F(2a - \alpha_1 - Y_{m+i})] + [1 - F(2a - \alpha_1$$

$$- Y_{m+j})]\} + \theta(a)$$

$$= U_{mn}(a) - \frac{2}{n} \sum_{i=1}^{n} [1 - F(2a - \alpha_1 - Y_{m+i})] + \theta(a), \qquad (5.1.72)$$

which could also have been established by combining (5.1.65) and (5.1.66). This concludes the proof of Lemma 5.1.7.

Now putting α_0 in place of a in (5.1.65), and using (5.1.62), (5.1.61) can be written

$$\lim_{N\to\infty} P[\max_{1\leq n\leq N} n|\theta(\alpha_0) + 2U_{mn,1}(\hat{\alpha}_0) + U_{mn,2}(\hat{\alpha}_0)$$

$$- \frac{2}{n} \sum_{i=1}^{n} \psi(Y_{m+i}) - \theta(\hat{\alpha}_0)| \geq \varepsilon N^{1/2}]$$

$$= \lim_{N\to\infty} P[\max_{1\leq n\leq N} n|2U_{mn,1}(\hat{\alpha}_0) - \frac{2}{n} \sum_{i=1}^{n} \psi(Y_{m+i}) + U_{mn,2}(\hat{\alpha}_0)$$

$$\geq \varepsilon N^{1/2}] \qquad (5.1.73a)$$

$$\leq \lim_{N\to\infty} P[\max_{1\leq n\leq N} n|2U_{mn,1}(\hat{\alpha}_0) - \frac{2}{n} \sum_{i=1}^{n} \psi(Y_{m+i})| \geq 1/2\ \varepsilon N^{1/2}] \qquad (5.1.73b)$$

$$+ \lim_{N\to\infty} P[\max_{1\leq n\leq N} n|U_{mn,2}(\hat{\alpha}_0)| \geq 1/2\ \varepsilon N^{1/2}]. \qquad (5.1.73c)$$

By the lemma on U-statistics in Chapter III, the term (5.1.73c) is zero.

Lemma 5.1.8

$$\lim_{N\to\infty} P[\max_{1\leq n\leq N} n|U_{mn,1}(\alpha_0) - \frac{1}{n}\sum_{i=1}^{n} \psi(Y_{m+i})| \geq \varepsilon N^{1/2}] = 0 \qquad (5.1.74)$$

Proof: Let K be chosen such that if

$$A_N = P[N^{1/2}|\hat{\alpha}_0 - \alpha_0| \leq K], \text{ then } P[\tilde{A}_N] < \varepsilon$$

then by (5.1.14), (5.1.26) and (5.1.66), the probability in (5.1.74) equals

$$P[\max_{1\leq n\leq N} n|U_{mn,1}(\hat{\alpha}_0) - \frac{1}{n}\sum_{i=1}^{n} \psi(Y_{m+i})| \geq \varepsilon N^{1/2}|A_n]\ P[A_N]$$

$$+ P[\max_{1\leq n\leq N} n|U_{mn,1}(\hat{\alpha}_0) - \frac{1}{n}\sum_{i=1}^{n} \psi(Y_{m+i})| \geq \varepsilon N^{1/2}|\tilde{A}_N]\ P[\tilde{A}_N]$$

$$\leq P[\max_{1\leq n\leq N} n|U_{mn,1}(\hat{\alpha}_0) - \frac{1}{n}\sum_{i=1}^{n} \psi(Y_{m+i})| \geq \varepsilon N^{1/2}\ A_N] + \varepsilon$$

$$= E_{\hat{\alpha}_0|A}\ P[\max_{1\leq n\leq N} n|U_{mn,1}(a) - \frac{1}{n}\sum_{i=1}^{n} \psi(Y_{m+i})| \geq \varepsilon N^{1/2}|\hat{\alpha}_0 = a\varepsilon A_N]$$

$$+ \varepsilon = E_{\hat{\alpha}_0|A}\ P[\max_{1\leq n\leq N} n|\frac{1}{n}\sum_{i=1}^{n} \{[1 - F(2a - \alpha_1 - Y_{m+i})]$$

$$- [1 - \int_{-\infty}^{\infty} F(2a - \alpha_1)-y)\ dF(y)]\} - \frac{1}{n}\sum_{i=1}^{n} \{[1 - \int_{-\infty}^{\infty} F(t - \alpha_1$$

$$- Y_{m+i})\ dG_{2\alpha_0}(t)] + [1 - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(t-2\alpha_1-y)\ dF(y)\ dG_{2\alpha_0}(t)]\}| \qquad (5.1.75)$$

$$\geq \varepsilon N^{1/2} | \hat{\alpha}_0 = a \varepsilon A_N]$$

$$= E_{\hat{\alpha}_0 | A} \; P[\max_{1 \leq n \leq N} n \left| \frac{1}{n} \sum_{i=1}^{n} \gamma_a \, (Y_{m+i}) \right| \geq \varepsilon N^{1/2} | \hat{\alpha}_0 = a \varepsilon A_N], \tag{5.1.76}$$

where

$$\gamma_a \, (Y_{m+i}) = [1 - F(2a - \alpha_1 - Y_{m+i})] - [1 - \int_{-\infty}^{\infty} F(2(a-\alpha_1)-y) \, dF(y)]$$

$$- [1 - \int_{-\infty}^{\infty} F(t - \alpha_1 - Y_{m+i}) \, dG_{2\ 0}(t)] + [1 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(t - 2\alpha_1 - y)$$

$$\cdot \, dG_{2\alpha_0}(t)]. \tag{5.1.77}$$

Taking expectations,

$$E\gamma_a \, (Y_{m+i}) = 0, \text{ and} \tag{5.1.78}$$

$$\text{Var } \gamma_a \, (Y_{m+i}) = \text{Var } F(2a - \alpha_1 - Y_{m+i}) + \text{Var} \int_{-\infty}^{\infty} F(t - \alpha_1$$

$$- Y_{m+i}) \, dG_{2\alpha_0}(t)$$

$$-2 \text{ Cov } (F(2a - \alpha_1 - Y_{m+i}), \int_{-\infty}^{\infty} F(t - \alpha_1 - Y_{m+i}) \, dG_{2\alpha_0}(t))$$

$$= \int_{-\infty}^{\infty} F^2(2(a-\alpha_1)-y) \, dF(y) - (\int_{-\infty}^{\infty} F(2(a-\alpha_1)-y) \, dF(y))^2$$

$$+ \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} F(t - \alpha_1 - y) \, dG_{2\alpha_0}(t))^2 \, dF(y - \alpha_1)$$

$$- (\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(t - \alpha_1 - y) \, dG_{2\alpha_0}(t) \, dF(y - \alpha_1))^2 \tag{5.1.79}$$

$$- 2 \int_{-\infty}^{\infty} F(2a - \alpha_1 - y) \int_{-\infty}^{\infty} F(t - \alpha_1 - y)\, dG_{2\alpha_0}(t)\, dF(y - \alpha_1)$$

$$+ 2 \int_{-\infty}^{\infty} F(2(a - \alpha_1)-y)\, dF(y) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(t - \alpha_1 - y)\, dG_{2\alpha_0}(t)$$

$$\cdot\ dF(y - \alpha_1).$$

As $n \to \infty$, (5.1.79) converges to

$$\int_{-\infty}^{\infty} F^2(2(\alpha_0 - \alpha_1)-y)\, dF(y) - (\int_{-\infty}^{\infty} F(2(\alpha_0 - \alpha_1)-y)\, dF(y))^2$$

$$+ \int_{-\infty}^{\infty} F^2(2(\alpha_0 - \alpha_1)-y)\, dF(y) - (\int_{-\infty}^{\infty} F(2(\alpha_0 - \alpha_1)-y)\, dF(y))^2$$

$$- 2 \int_{-\infty}^{\infty} F^2(2(\alpha_0 - \alpha_1)-y)\, dF(y) - 2(\int_{-\infty}^{\infty} F(2(\alpha_0 - \alpha_1)-y)\, dF(y))^2$$

$$= 0$$

Now following the notation of Lemma 3.2.1, from (5.1.76),

$$P[\max_{1 \le n \le N} | \sum_{n=1}^{n} \gamma_a(Y_{m+i}) | \ge N^{1/2} (\text{Var } \gamma_a)^{1/2}]$$

$$\le 2\, P[| \sum_{i=1}^{N} \gamma_a (Y_{m+i}) | \ge (\gamma - \sqrt{2})\, N^{1/2} (\text{Var } \gamma_a)^{1/2}], \qquad (5.1.80)$$

and since

$$N^{-1/2} (\text{Var } \gamma_a)^{-1/2} \sum_{i=1}^{N} \gamma_a (Y_{m+i}) \sim AN(0, 1) \text{ as } N \to \infty, \qquad (5.1.81)$$

λ can be chosen large enough to make (5.1.80) less than ε. Since $\text{Var } \gamma_a \to 0$ as $N \to \infty$, N_o can be chosen large enough so that $\lambda(\text{Var } \gamma_a)^{1/2} < \varepsilon$, $N \ge N_o$. Thus for $N \ge N_o$

$$P[\max_{1\le n\le N} \sum_{i=1}^{n} |\gamma_a (Y_{m+i})| \ge \varepsilon N^{1/2}]$$

$$\le P[\max_{1\le n\le N} \sum_{i=1}^{n} |\gamma_a (Y_{m+i})| \ge \lambda N^{1/2}(\text{Var } \gamma_a)^{1/2}] < \varepsilon. \qquad (5.1.82)$$

Since F is uniformly continuous, Var γ_a is uniformly convergent in the compact set A_N, so from (5.1.76),

$$\lim_{N\to\infty} E_{\alpha_0|A} P[\max_{1\le n\le N} n|\frac{1}{n} \sum_{i=1}^{n} \gamma_a (Y_{m+i})| \ge \varepsilon N^{1/2}|\alpha_0 = a\varepsilon A_N] = 0$$

from which (5.1.74) follows, and so lemma (5.1.8) is proved.

VI. ASYMPTOTIC PROPERTIES UNDER ALTERNATIVES

§6.0: Statement of the Main Results

One way to compare sequential truncated sequential procedures is to find the asymptotic relative efficiency (A.R.E.) of the truncated sequential portion; that is, the A.R.E. of a single stage. This method of comparison has the drawback that the sequential character of the test is ignored.

A standard method for comparing sequential procedures is to adjust the operating parameters so that the average run length (A.R.L.) under H_0 is the same for each procedure and then compare A.R.L.'s under alternatives. For the procedures proposed in this dissertation the A.R.L. can not be calculated in general and hence must be left for simulation. One should expect, however, that the better of two procedures compared by A.R.E. will also be the better of the two compared by A.R.L.

The A.R.E. results apply to an important problem in its own right. A single stage in the current procedures is equivalent to a truncated sequential test for a shift in model parameters when those parameters must be estimated from an independent sample. The A.R.E. results provide a measure of the relative performance of the competing procedures under alternatives.

For the procedures of the previous two chapters the asymptotic efficiency is derived under alternatives such as the Pitman alternatives where, for example, in (5.0.2) $\alpha_1 = \alpha_0 + k/N^{1/2}$.

The statistic under consideration is of the form

$$C_0(S_N(t) - K_0),\ n = 1, 2, \ldots,\ t \in [0, 1]\ , \tag{6.0.1}$$

where C_0 and K_0 are chosen so that under the null hypothesis

$$C_0(S_N(t) - K_0) \Rightarrow W + L \tag{6.0.2}$$

where $W \sim W(0, 1)$, L is a straight line through the origin with random slope $b \sim N(0, \kappa^2)$, independent of W, and C_0 and K_0 may depend on t. Under the appropriate alternatives there exist two sequences of constants $\{C_N\}$, $\{K_N\}$ such that

a) $C_N(S_N(t) - K_N) \Rightarrow W + L;$ (6.0.3)

b) $\dfrac{C_0}{C_N} \to 1;\ C_0(K_N - K_0) \to \mu t,\ 0 \le t \le 1,$ (6.0.4)

where μ is some constant.

Examples of $S_N(t)$ are $S_{mN}(t)$, $T_{mN}(t)$, and $W_{mN}(t)$ of Chapters IV and V.

The following theorem states the asymptotic power of $C_0(S_N(t)-K_0)$ under the above conditions:

Theorem 6.1 Under alternatives such that (6.0.2), (6.0.3), and (6.0.4) hold, and given that

$$\lim_{N\to\infty} P[C_0(S_N(t) - K_0) \geq C^*_\alpha] = \alpha \tag{6.0.5}$$

or

$$\lim_{N\to\infty} P[|C_0(S_N(t) - K_0)| \geq C_\alpha] = \alpha \tag{6.0.6}$$

then the asymptotic power for the one-sided test (6.0.5) is

$$\Phi\left(\frac{-C^*_\alpha+\mu}{\sqrt{1+\kappa^2}}\right) + e^{2C^*_\alpha(\mu+C^*_\alpha\kappa^2)}\,\Phi\left(\frac{-C^*_\alpha-\mu-2C^*_\alpha\kappa^2}{\sqrt{1+\kappa^2}}\right). \tag{6.0.7}$$

For the two-sided test, (6.0.6), the asymptotic power is

$$\begin{aligned}
&\sum_{S=0}^{\infty} (-1)^S\{e^{-2SC_\alpha(\mu-SC_\alpha\kappa^2)}\,\Phi\left(\frac{\mu-2SC_\alpha\kappa^2-(2S+1)C_\alpha}{\sqrt{1+\kappa^2}}\right)\\
&+ e^{-2(S+1)C_\alpha(\mu-(S+1)C_\alpha\kappa^2)}\,\Phi\left(\frac{\mu-2(S+1)C\kappa^2-(2S+1)C_\alpha}{\sqrt{1+\kappa^2}}\right)\\
&+ e^{2SC_\alpha(\mu+SC_\alpha\kappa^2)}\,\Phi\left(\frac{-\mu-2SC_\alpha\kappa^2-(2S+1)C_\alpha}{\sqrt{1+\kappa^2}}\right)\\
&+ e^{2(S+1)C_\alpha(\mu+(S+1)C_\alpha\kappa^2)}\,\Phi\left(\frac{-\mu-2(S+1)C_\alpha\kappa^2-(2S+1)C_\alpha}{\sqrt{1+\kappa^2}}\right)\}.
\end{aligned} \tag{6.0.8}$$

One method by which the proposed procedures can be compared analytically is to obtain asymptotic relative efficiencies of the Pitman type for a single stage.

Toward this end, define from (4.0.10), (5.0.8), and (4.0.19),

$$T_N^{(1)} = 2N^{-\frac{1}{2}}(S_{mN}(t) - \frac{Nt}{2}) \tag{6.0.9}$$

$$T_N^{(2)} = \sqrt{12}\, N^{-\frac{3}{2}} t^{-1}\left(W_{mN}(t) - \frac{Nt(Nt-1)}{4}\right) \tag{6.0.10}$$

$$T_N^{(3)} = \sigma^{-1} N^{-\frac{1}{2}} T_{mN}(t) \; . \tag{6.0.11}$$

In the model (5.0.1), (5.0.2) the Pitman alternatives for $T_N^{(i)}$, $i = 1, 2, 3$, are $\alpha_1 = \alpha_0 + k_i N_i^{-\frac{1}{2}}$, where k_i and N_i are chosen so that

$$k_i N_i^{-\frac{1}{2}} = k_{i'} N_{i'}^{-\frac{1}{2}} \; ; \tag{6.0.12}$$

and

$$\lim_{N_i \to \infty} P[|T_N^{(i)}| \geq C_\alpha] = \lim_{N_{i'} \to \infty} P[|T_N^{(i')}| \geq C_\alpha] \; , \quad i, i' = 1, 2, 3, \tag{6.0.13}$$

where the probabilities in (6.0.13) are under alternatives.
The Pitman asymptotic relative efficiency of $T_N^{(i)}$ with respect to $T_N^{(i')}$ is defined as

$$e_{ii'} = \lim_{N_i \to \infty} N_{i'}/N_i \; , \; i, i' = 1, 2, 3 \tag{6.0.14}$$

Theorem 6.2

i) $$e_{12} = \frac{1}{3}\left(\frac{f(0)}{\int_{-\infty}^{\infty} f^2(x)\,dx}\right)^2 \; ; \tag{6.0.15}$$

ii) $$e_{13} = 4\sigma^2 f^2(0) \; . \tag{6.0.16}$$

iii) $$e_{23} = 12\sigma^2\left(\int_{-\infty}^{\infty} f^2(x)\,dx\right)^2 \; . \tag{6.0.17}$$

Note that the A.R.E. expressions of Theorem 6.2 are identical to those for the one and two sample problems. This is due to the fact that by comparing single stages we are essentially comparing fixed sample procedures.

The following corollary to Theorem 6.2 gives e_{13} under the model (4.0.1) and (4.0.2). Suppose that the set$\{ \underline{x}_{m+i}, \underline{x}_{m+2}, \ldots\}$ takes on the finite list of values $\underline{z}_1, \ldots, \underline{z}_r$ with asymptotic relative frequencies $\lambda_1, \ldots, \lambda_r$, respectively.

Recall that the model (4.0.1) and (4.0.2) is

$$Y_i = \underline{\beta}_0'\underline{x}_i + \varepsilon_i = \sum_{j=1}^{k} \beta_{0j}\underline{x}_{ij} + \varepsilon_i , \; i = 1, \ldots, m$$

$$Y_{m+i} = \underline{\beta}_1'\underline{x}_{m+i} + \varepsilon_{m+i} = \sum_{j=1}^{k} \underline{\beta}_{1j}\underline{x}_{m+ij} + \varepsilon_{m+ij} , \; i = 1, \ldots, N .$$

Corollary 6.2 For some j, $1 \leq j \leq k$, let

$$\beta_{1j} = \beta_{0j} + k_i N_i^{-1/2} , \; i = 1, 3$$

$$\beta_{1j'} = \underline{\beta}_{0j'} , \; j' \neq j ,$$

then again

$$e_{13} = 4\sigma^2 f^2(0) . \tag{6.0.20}$$

That is, the asymptotic relative efficiency e_{13} is the same under either the constant or the regression model.

Following the proofs of Theorem 6.2 and Corollary 6.2, selected asymptotic relative efficiencies are presented in Table 6.2.1.

§6.1: Proof of Theorem 6.1

The proof of Theorem 6.1 uses a theorem of Dinges (1962) which states that if $W \sim W(\mu, \sigma^2)$, then for any constant $C > 0$,

$$P[\sup W \geq C] = \exp\left(\frac{2\mu C}{\sigma^2}\right)\Phi\left(\frac{-\mu-C}{\sigma}\right) + \Phi\left(\frac{\mu-C}{\sigma}\right) \tag{6.1.1}$$

and Corollary 4.2 of Anderson (1960)

$$P[\sup|W| \geq C] = \sum_{S=0}^{\infty} (-1)^S\{\Phi\left(\frac{\mu-(2S+1)C}{\sigma}\right) \exp\left(\frac{-2SC\mu}{\sigma^2}\right)$$

$$\times \left\{1 + \exp\left(\frac{-2C\mu}{\sigma^2}\right)\right\} \tag{6.1.2}$$

$$+ \Phi\left(\frac{-\mu-(2S+1)C}{\sigma}\right) \exp\left(\frac{2SC\mu}{\sigma^2}\right)$$

$$\times \left\{1 + \exp\left(\frac{2C\mu}{\sigma^2}\right)\right\} \}.$$

Repeated use will also be made of a fact from the theory of convolutions of distribution functions: If X and Y are independent random variables with distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, respectively, then

$$P[X+Y \le C] = \int_{-\infty}^{\infty} F_X(C-y)dF_Y(y) \ . \tag{6.1.3}$$

As stated in the introductory section of this chapter, the statistics under consideration are of the form $C_0(S_N(t)-K_0)$. Under the null hypothesis, this statistic converges weakly to W + L, where $W \sim W(0, 1)$ and L is a straight line through the origin with random slope $B \sim N(0, \kappa^2)$, independent of W. Combining (6.0.2), (6.0.3), and (6.0.4), under alternatives

$$C_0(S_N(t) - K_0)$$

$$= \frac{C_0}{C_N} \cdot C_N(S_N(t) - K_N) + C_0(K_N - K_0)$$

$$\Rightarrow W + L + \mu t \text{ as } N \to \infty \tag{6.1.4}$$

or

$$C_0(S_N(t) - K_0) \Rightarrow W^* + L \text{ as } N \to \infty \ , \tag{6.1.5}$$

where $W^* \sim W(\mu, 1)$.

Thus the purpose of Theorem 6.1 is to show that (6.0.7) is $P[|W^* + L| \geq C_\alpha]$.

The following lemma will be useful in the proof of Theorem 6.1:

Lemma 6.1.1

$$\int_{-\infty}^{\infty} \Phi((\mu+b)-(2S+1)C)\exp(2(\mu+b)C^*)d\Phi(b/\kappa)$$
$$= \exp[2C^*(\mu+C^*\kappa^2)]\Phi\left(\frac{\mu+2C^*\kappa^2-(2S+1)C}{(1+\kappa^2)^{\frac{1}{2}}}\right) ; \qquad (6.1.6)$$

$$\int_{-\infty}^{\infty} \Phi(-(\mu+b)-(2S+1)C)\exp(2(\mu+b)C^*)d\Phi(b/\kappa)$$
$$= \exp[2C^*(\mu+C^*\kappa^2)]\Phi\left(\frac{-\mu-(2S+1)C-2C^*\kappa^2}{(1+\kappa^2)^{\frac{1}{2}}}\right); \qquad (6.1.7)$$

Proof of (6.1.6):

$$\int_{-\infty}^{\infty} \Phi((\mu+b)-(2S+1)C)\exp(2(\mu+b)C^*)d\Phi(b/\kappa)$$
$$= \int_{-\infty}^{\infty} (2\Pi\kappa^2)^{-\frac{1}{2}}\exp[-\frac{1}{2}\frac{b^2}{\kappa 2} + 2(\mu+b)C^*]\Phi((\mu+b)-(2S+1)C)db$$

$$= \exp[2C^*(\mu+C^*\kappa^2)]\int_{-\infty}^{\infty}(2\Pi\kappa^2)^{-\frac{1}{2}}\exp[\frac{-1}{2\kappa^2}(b-2C^*\kappa^2)^2]$$
$$\times \Phi((\mu+b)-(2S+1)C)db$$

$$= \exp[2C^*(\mu+C^*\kappa^2)] \ \{1-\int_{-\infty}^{\infty}\Phi((2S+1)C-\mu-b)d\Phi\left(\frac{b-2C^*\kappa^2}{\kappa}\right)\} \qquad (6.1.8)$$

$$= \exp[2C^*(\mu+C^*\kappa^2)]\{1-P[X+Y \le (2S+1)C-\mu]\} \tag{6.1.9}$$

where $X \sim N(0, 1)$, $Y \sim N(2C^*\kappa^2, \kappa^2)$, or

$X+Y \sim N(2C^*\kappa^2, 1+\kappa^2)$, so (4.4.9) equals

$$\exp[2C^*(\mu+C^*\kappa^2)]\{1-\Phi\left(\frac{(2S+1)C-\mu-2C^*\kappa^2}{(1+\kappa^2)^{\frac{1}{2}}}\right)$$

$$= \exp[2C^*(\mu+C^*\kappa^2)]\Phi\left(\frac{\mu+2C^*\kappa^2-(2S+1)C}{(1+\kappa^2)^{\frac{1}{2}}}\right) . \tag{6.1.10}$$

Proof of (6.1.7): As in obtaining (6.1.8),

$$\int_{-\infty}^{\infty}\Phi(-(\mu+b)-(2S+1)C)\exp(2(\mu+b)C^*)d\Phi(b/\kappa)$$

$$= \exp[2C^*(\mu+C^*\kappa^2)]\int_{-\infty}^{\infty}\Phi(-(\mu+b)-(2S+1)C)d\Phi\left(\frac{b-2C^*\kappa^2}{\kappa}\right)$$

$$= \exp[2C^*(\mu+C^*\kappa^2)]P[X+Y \le -\mu-(2S+1)C] \tag{6.1.11}$$

where $X \sim N(0, 1)$, $Y \sim N(2C^*\kappa^2, \kappa^2)$, or $X+Y \sim N(2C^*\kappa^2, 1+\kappa^2)$, so (6.1.11) equals

$$\exp[2C^*(\mu+C^*\kappa^2)]\Phi\left(\frac{-\mu-(2S+1)C-2C^*\kappa^2}{(1+\kappa^2)^{\frac{1}{2}}}\right) , \tag{6.1.12}$$

and Lemma 6.1.1 is proved.

To prove the theorem, consider first the one-sided case:

$$P[W^* + L \ge C] = P[W^* + Bt \ge C]$$

$$= E_B P[W + (\mu+b)t \ge C \mid B = b]$$

$$= \int_{-\infty}^{\infty}\{\exp(2(\mu+b)C)\Phi(-(\mu+b)-C) + \Phi((\mu+b)-C)\}d\Phi(b/\kappa)$$

$$= \int_{-\infty}^{\infty} \exp(2(\mu+b)C)\Phi(-(\mu+b)-C)d\Phi(b/\kappa) \tag{6.1.13a}$$

$$+ \int_{-\infty}^{\infty}\Phi((\mu+b)-C)d\Phi(b/\kappa)\ . \tag{6.1.13b}$$

Now (6.1.13a) is of the form (6.1.7) with $C^* = C$ and $s = 0$, so by (6.1.12), (6.1.13) equals

$$\exp[2C(\mu+c\kappa^2)]\Phi\left(\frac{-\mu-C-2C\kappa^2}{(1+\kappa^2)^{\frac{1}{2}}}\right). \tag{6.1.14}$$

Similarly (6.1.13b) is of the form (6.1.6) with $S = 0$ and $C^* = 0$, so by (6.1.10), (6.1.13b) equals

$$\Phi\left(\frac{\mu-C}{(1+\kappa^2)^{\frac{1}{2}}}\right)\ . \tag{6.1.15}$$

Combining (6.1.14) and (6.1.15),

$$P[W^*+L \geq C] = \exp[2C(\mu+C\ ^2)]\Phi\left(-\ \frac{\mu+C+2C\kappa^2}{(1+\kappa^2)^{\frac{1}{2}}}\right)$$

$$+\Phi\left(\frac{\mu-C}{(1+\kappa^2)^{\frac{1}{2}}}\right)$$

which establishes (6.0.7).

Now consider the two-sided case:

$$P[|W^* + L| \geq C] = P[|W^* + Bt| \geq C]$$

$$= E_B\ P[|W + (\mu+b)t| \geq C]$$

$$= \int_{-\infty}^{\infty} \sum_{S=0}^{\infty} (-1)^S \{\Phi(\mu+b-(2S+1)C)\exp(-2SC(\mu+b))$$

$$\times (1+\exp(-2C(\mu+b)))$$

$$+ \Phi(-(\mu+b)-(2S+1)C)\exp(2SC(\mu+b)) \tag{6.1.17}$$

$$\times (1+\exp(2C(\mu+b)))\}d\Phi(b/\kappa) \ .$$

Formally interchanging the order of integration and summation, and collecting the exponents in (6.1.17) yields

$$\sum_{S=0}^{\infty} (-1)^S \{\int_{-\infty}^{\infty} \Phi((\mu+b)-(2S+1)C)\exp(-2SC(\mu+b))d\Phi(b/\kappa) \tag{6.1.18a}$$

$$+ \int_{-\infty}^{\infty} \Phi((\mu+b)-(2S+1)C)\exp(-2(S+1)C(\mu+b))d\Phi(b/\kappa) \tag{6.1.18b}$$

$$+ \int_{-\infty}^{\infty} \Phi(-(\mu+b)-(2S+1)C)\exp(2SC(\mu+b))d\Phi(b/\kappa) \tag{6.1.18c}$$

$$+ \int_{-\infty}^{\infty} \Phi(-(\mu+b)-(2S+1)C)\exp(2(S+1)C(\mu+b))d\Phi(b/\kappa)\} \ . \tag{6.1.18d}$$

Showing that the interchanging of integration and summation is justified will be left for Lemma 6.1.2, which will be presented after showing that (6.0.8) holds.

The integral in (6.1.18a) is of the form (6.1.6), where $C^* = -SC$, so (6.1.18a) equals

$$\exp[-2SC(\mu-SC\kappa^2)\Phi\left(\frac{\mu-2SC\kappa^2-(2S+1)C}{(1+\kappa^2)^{\frac{1}{2}}}\right) . \tag{6.1.19}$$

The integral in (6.1.18b) is of the form (6.1.6), where $C^* = -(S+1)C$, so (6.1.18b) equals

$$\exp[-2(S+1)C(\mu-(S+1)C\kappa^2)]\Phi\left(\frac{\mu-2(S+1)C\kappa^2-(2S+1)C}{(1+\kappa^2)^{\frac{1}{2}}}\right) . \tag{6.1.20}$$

The integral in (6.1.18c) is of the form (6.1.7), where C* = SC, so (6.18c) equals

$$\exp[2SC(\mu+SC\kappa^2)]\Phi\left[\frac{-\mu-2SC\kappa^2-(2S+1)C}{(1+\kappa^2)^{\frac{1}{2}}}\right]. \qquad (6.1.21)$$

Similarly the integral in (6.1.18d) is of the form (6.1.7), where C* = (S+1)C, so (6.1.18d) equals

$$\exp[2(S+1)C(\mu+(S+1)C\kappa^2)]\Phi\left[\frac{-\mu-2(S+1)C\kappa^2-(2S+1)C}{(1+\kappa^2)^{\frac{1}{2}}}\right]. \qquad (6.1.22)$$

Combining (6.1.19) - (6.1.22) and (6.1.18) equals

$$\begin{aligned}\sum_{S=0}^{\infty} (-1)^S\{&\exp[-2SC(\mu-SC\kappa^2)]\Phi\left[\frac{\mu-2SC\kappa^2-(2S+1)C}{(1+\kappa^2)^{\frac{1}{2}}}\right]\\ &+\exp[-2(S+1)C(\mu-(S+1)C\kappa^2)]\left[\frac{\mu-2(S+1)C\kappa^2-(2S+1)C}{(1+\kappa^2)^{\frac{1}{2}}}\right]\\ &+\exp[2SC(\mu+SC\kappa^2)]\Phi\left[\frac{-\mu-2SC\kappa^2-(2S+1)C}{(1+\kappa^2)^{\frac{1}{2}}}\right]\\ &+\exp[2(S+1)C(\mu+(S+1)C\kappa^2)]\Phi\left[\frac{-\mu-2(S+1)C\kappa^2-(2S+1)C}{(1+\kappa^2)^{\frac{1}{2}}}\right].\end{aligned} \qquad (6.1.23)$$

Before proceeding to Lemma 6.1.2, note that under the null hypothesis, $\mu = 0$, and (6.1.23) can be written in the following form:

$$\begin{aligned}&P[|W + L| \geq C]\\ &= 2\sum_{S=0}^{\infty} (-1)^S\{\exp[2(SC\kappa)^2]\Phi\left[\frac{-2SC\kappa^2-(2S+1)C}{(1+\kappa^2)^{\frac{1}{2}}}\right]\end{aligned} \qquad (6.1.24)$$

$$+ \exp[2[(S+1)C\kappa]^2\Phi\left(\frac{-2(S+1)C\kappa^2-(2S+1)C}{(1+\kappa^2)^{1/2}}\right)].$$

Using (6.1.24) and (6.0.7) with $\mu = 0$, the tables of Brownian motion corrected critical values in the appendix, Tables A.1 and A.2 have been generated. Figures A.1 and A.2 in the appendix are graphs of Brownian motion critical values as a function of κ^2. These graphs provide a rough idea of how the critical values increase as κ^2 increases.

For notational simplicity in the following lemma, define $\theta_S(b) = \theta_S^{(1)}(b)+\theta_S^{(2)}(b)+\theta_S^{(3)}(b)+\theta_S^{(4)}(b)$, where

$$\theta_S^{(1)}(b) = \Phi((\mu+b)-(2S+1)C)\exp[-2SC(\mu+b)]\phi(b/\kappa) \qquad (6.1.25a)$$

$$\theta_S^{(2)}(b) = \Phi((\mu+b)-(2S+1)C)\exp[-2(S+1)C(\mu+b)]\phi(b/\kappa) \qquad (6.1.25b)$$

$$\theta_S^{(3)}(b) = \Phi(-(\mu+b)-(2S+1)C)\exp[2SC(\mu+b)]\phi(b/\kappa) \qquad (6.1.25c)$$

$$\theta_S^{(4)}(b) = \Phi(-(\mu+b)-(2S+1)C)\exp[2(S+1)C(\mu+b)]\phi(b/\kappa)\ . \qquad (6.1.25d)$$

Then (6.1.18a)-(6.1.18d) are $\int_{-\infty}^{\infty}\theta_S^{(i)}(b)db$, $i = 1, 2, 3, 4$, respectively, and the following lemma will establish equality between (6.1.17) and (6.1.23):

<u>Lemma 6.1.2</u> Given $\theta_S(b)$ as defined above,

$$\int_{-\infty}^{\infty} \sum_{S=0}^{\infty} (-1)^S\ \theta_S(b)db = \sum_{S=0}^{\infty}\ (-1)^S \int_{-\infty}^{\infty} \theta_S(b)db\ . \qquad (6.1.26)$$

Proof: The proof will establish three conditions, given in Apostol (1964, p. 451), sufficient for (6.1.26) to hold:

i) The series $\sum_{S=0}^{\infty} \theta_S(b)$ converges uniformly for b in any closed, bounded interval $[a,d]$, $-\infty < a < b < \infty$. (6.1.27)

ii) Each $\theta_S(b)$ is integrable on $[a, d]$; (6.1.28)

iii) $\sum_{S=0}^{\infty} \int_{-\infty}^{\infty} \theta_S(b)db$ is convergent. (6.1.29)

Condition ii) is obvious.

Proof of condition i): Proving that $\sum_{S=0}^{\infty} \theta_S(b)$ converges uniformly in any interval [a,d] is equivalent to proving that $\sum_{i=0}^{\infty} \theta_S^{(i)}(b)$ converges uniformly in [a,d] for each i = 1, ..., 4. The argument for showing that $\sum_{i=0}^{\infty} \theta_S^{(i)}(b)$ converges uniformly is essentially the same for each i, so the proof of i) will involve only showing uniform convergence $\sum_{i=0}^{\infty} \theta_S^{(1)}(b)$. From (6.1.25a) this involves showing

$$\sum_{S=0}^{\infty} \Phi((\mu+b)-(2S+1)C)\exp[-2SC(\mu+b)] \qquad (6.1.30)$$

converges uniformly in [a,d].

An upper bound on the terms of (6.1.30) can be obtained by using the continued fraction of Laplace, as in (4.1.84). That is, $\Phi(-x) \leq \phi(x) \cdot \frac{1}{x}$, x > 0, so

$$\begin{aligned} &\sum_{S=0}^{\infty} \Phi((\mu+b)-(2S+1)C)\exp[-2SC(\mu+b)] \\ &\leq \sum_{S=0}^{\infty} \phi((\mu+b)-(2S+1)C)((\mu+b)-(2S+1)C)^{-1}\exp(-2SC(\mu+b)) \\ &= \sum_{S=0}^{\infty} (2\Pi)^{-\frac{1}{2}}((2S+1)C-(\mu+b))^{-1} \qquad (6.1.31) \\ &\times \exp(-\tfrac{1}{2}((\mu+b)-(2S+1)C)^2-2SC(\mu+b)) \; . \end{aligned}$$

Assuming C > 0, for S sufficiently large (2S+1)C-(μ+b) > 1, then (6.1.31) is bounded above by

$$\sum_{S=0}^{\infty} \exp(-\tfrac{1}{2}((\mu+b)-(2S+1)C)^2)$$

$$\leq \sum_{S=0}^{\infty} \exp[-(\mu+b)^2]\exp[-((2S+1)C)^2] \;, \qquad (6.1.32)$$

and this series is obviously convergent. Since b is arbitrary the series is also uniformly convergent in [a,d]. Thus condition i) is established.

The proof of condition iii) is similar to the proof of condition i), since by (6.1.23) $\sum_{S=0}^{\infty} \theta_S(b)$ and $\sum_{S=0}^{\infty} \int_{-\infty}^{\infty} \theta_S(b)db$ have the same form.

Thus Lemma 6.1.2 is established, and with it the conclusion of the proof of Theorem 6.1.

§6.2 Proof of Theorem 6.2

The Brownian motion theorems in Chapters IV and V establish as an immediate consequence that under the constant model, (5.0.1) and (5.0.2),

$$T_{N_1}^{(1)} = 2N_1^{-\frac{1}{2}}(S_{mN_1}(t) - \frac{N_1 t}{2}); \qquad (6.2.1)$$

$$T_{N_2}^{(2)} = \sqrt{12}\, N_2^{-\frac{3}{2}} t^{-1}(W_{mN_2}(t) - \frac{N_2 t(N_2 t-1)}{2}) \;; \text{ and} \qquad (6.2.2)$$

$$T_{N_3}^{(3)} = \sigma^{-1} N_3^{-\frac{1}{2}} T_{mN_3}(t) \qquad (6.2.3)$$

all converge weakly to W+L under H_0, where W ~ W(0, 1) and L is a straight line through the origin with random slope b ~ N(0, κ^2). Furthermore, W and L are independent, so that (6.2.1)-(6.2.3) all satisfy condition (6.0.2). That the conditions (6.0.3) and (6.0.4) are met under the Pitman alternatives is established in the next three lemmas.

<u>Lemma 6.2.1</u> Under the model (5.0.1), (5.0.2) with alternatives $\alpha_1 = \alpha_0 + k_1/N_1^{\frac{1}{2}}$,

$$2N_1^{-\frac{1}{2}}(S_{mN_1}(t) - \frac{N_1 t}{2}) \Rightarrow W + L \qquad (6.2.4)$$

under H_0, and under H_1,

$$(\gamma^2 N_1)^{-\frac{1}{2}}(S_{mN_1}(t) - Nt\mu^*_{N_1}) \Rightarrow W + L , \qquad (6.2.5)$$

where

$$\gamma^2 = F(k_1 N_1^{-\frac{1}{2}})[1-F(k_1 N_1^{-\frac{1}{2}})] \qquad (6.2.6)$$

$$\mu^*_{N_1} = 1-F(-k_1 N_1^{-\frac{1}{2}}) = F(k_1 N_1^{-\frac{1}{2}}) \quad . \qquad (6.2.7)$$

Furthermore,

$$\lim_{N_1 \to \infty} 2N_1^{-\frac{1}{2}}(NtF(k_1 N_1^{-\frac{1}{2}}) - Nt \cdot \tfrac{1}{2}) = 2k_1 f(0)t \; . \qquad (6.2.8)$$

Proof: The proof of (6.2.4) and (6.2.5) is an immediate consequence of Theorem 4.1 when the appropriate models and hypotheses are used. Since

$$\lim_{N_1 \to \infty} \frac{F(k_1 N_1^{-\frac{1}{2}}) - F(0)}{k_1 \; N_1^{-\frac{1}{2}}} = f(0) , \qquad (6.2.9)$$

(6.2.8) follows immediately.

<u>Lemma 6.2.2</u> Under the model (5.0.1), (5.0.2) with alternatives $\alpha_1 = \alpha_0 + k_2 N_2^{-\frac{1}{2}}$

$$\sqrt{12}\, N^{-\frac{3}{2}} t^{-1}(W_{mN}(t) - \frac{Nt(Nt-1)}{2}) \Rightarrow W + L \qquad (6.2.10)$$

under H_0, and under H_1,

$$\zeta^{-1} N^{-\frac{3}{2}} t^{-1}(W_{mN}(t) - \frac{Nt(Nt-1)}{2}\beta(\alpha_0)) \Rightarrow W + L \quad (6.2.11)$$

where

$$\zeta^2 = \int_{-\infty}^{\infty}[F(-2k_2N_2^{-\frac{1}{2}}-x)]^2 dF(x)$$
$$- [\int_{-\infty}^{\infty}F(-2k_2N_2^{-\frac{1}{2}}-x)dF(x)]^2 \quad (6.2.12)$$

$$\beta(\alpha_0) = 1-\int_{-\infty}^{\infty}F(-2k_2N_2^{-\frac{1}{2}}-x)dF(x)$$
$$= \int_{-\infty}^{\infty}F(2k_2N_2^{-\frac{1}{2}}+x)dF(x) \quad . \quad (6.2.13)$$

Furthermore,

$$\lim_{N_2\to\infty} \sqrt{12}\, N^{-\frac{3}{2}}t^{-1} \left(\frac{Nt(Nt-1)}{2}\right)\int_{-\infty}^{\infty}F(2k_2N_2^{-\frac{1}{2}}+x)dF(x)-\left(\frac{Nt(Nt-1)}{4}\right) \quad (6.4.14)$$

$$= \sqrt{12}\, k_2\int_{-\infty}^{\infty}f^2(x)dx\, t \quad .$$

<u>Proof</u>: Similar to the proof of Lemma 6.2.1.

<u>Lemma 6.2.3</u> Under the model (5.0.1), (5.0.2) with alternatives $\alpha_1 = \alpha_0 + k_3N_3^{-\frac{1}{2}}$

$$(\sigma^2N_3)^{-\frac{1}{2}} T_{mN}(t) \Rightarrow W + L \quad (6.2.15)$$

under H_0, and under H_1,

$$(\sigma^2N_3)^{-\frac{1}{2}}(T_{mN}(t) - N_3tk_3N_3^{-\frac{1}{2}}) \Rightarrow W + L \quad . \quad (6.2.16)$$

Furthermore,

$$\lim_{N_2\to\infty} (\sigma^2N_3)^{-\frac{1}{2}} N_3tk_3N_3^{-\frac{1}{2}} = \sigma^{-1}k_3t \quad . \quad (6.2.17)$$

Conditions (4.31) can be written

$$\frac{C_0^{(i)}}{C_N^{(i)}} \to 1;\; C_0^{(i)}(K_N^{(i)} - K_N^{(i)}) \to \mu^{(i)}\, t \tag{6.2.18}$$

as $N_i \to \infty$, $0 \leq t \leq 1$, $i = 1, 2, 3$, in which case, from Lemmas 6.2.1 - 6.2.3,

$$\mu^{(1)} = 2k_1 f(0) \tag{6.2.19}$$

$$\mu^{(2)} = \sqrt{12}\, k_2 \int_{-\infty}^{\infty} f^2(x)dx \tag{6.2.20}$$

$$\mu^{(3)} = \sigma^{-1} k_3 . \tag{6.2.21}$$

In comparing tests $T_{N_i}^{(i)}(t)$ and $T_{N_{i'}}^{(i')}(t)$ using asymptotic relative efficiency, the alternatives must be chosen so that

$$k_i N_i^{-\frac{1}{2}} = k_{i'} N_{i'}^{-\frac{1}{2}} \text{ ; and} \tag{6.2.22}$$

$$\lim_{N_i \to \infty} P[|T_N^{(i)}(t)| \geq C\,] = \lim_{N_{i'} \to \infty} P[|T_N^{(i')}(t)| \geq C_\alpha] . \tag{6.2.23}$$

Given that κ^2 is the same for all $i = 1, 2, 3$, by Theorem 6.1, condition (6.2.23) is satisfied if

$$\mu^{(i)} = \mu^{(i')} . \tag{6.2.24}$$

That is, e_{12} is obtained by the conditions

$$k_1 N_1^{-\frac{1}{2}} = k_2 N_2^{-\frac{1}{2}};\; 2k_1 f(0) = \sqrt{12}\, k_2 \int_{-\infty}^{\infty} f^2(x)dx , \tag{6.2.25}$$

or

$$\lim_{n_1 \to \infty} \frac{N_2}{N_1} = \left(\frac{k_2}{k_1}\right)^2 ;\; \frac{k_2}{k_1} = \frac{f(0)}{\sqrt{3}\int_{-\infty}^{\infty} f^2(x)dx} \tag{6.2.26}$$

from which

$$e_{12} = \lim_{N_1 \to \infty} \frac{N_2}{N_1} = \frac{1}{3} \left(\frac{f(0)}{\int_{-\infty}^{\infty} f^2(x)\,dx} \right)^2 \tag{6.2.27}$$

which establishes (6.0.15). Equation (6.0.16) is similarly established. From (6.0.14), the asymptotic relative efficiency of $T_{N_2}^{(2)}$ with respect to $T_{N_3}^{(3)}$ is

$$e_{23} = \lim_{N_2 \to \infty} \frac{N_3}{N_2} = \lim_{N_2 \to \infty} \frac{N_3}{N_1} \cdot \frac{N_1}{N_2} = \frac{e_{13}}{e_{12}}$$

thus establishing (6.0.17), which concludes the proof of Theorem 6.2. The corollary to Theorem 6.2 is established by stating the following two lemmas first:

<u>Lemma 6.2.4</u> Under the model (4.0.1), (4.0.2) with alternatives $\beta_{1,j} = \beta_{0,j} + k_1 N_1^{-\frac{1}{2}}$ for some $j = 1, \ldots, k$, and $\beta_{1j'} = \beta_{0j'}$ for all $j' \neq j$,

$$2N_1^{-\frac{1}{2}}(S_{mN_1}(t) - \frac{N_1 t}{2}) \Rightarrow W + L \tag{6.2.28}$$

under H_0, and under H_1,

$$(\gamma_{N_1}^2 N_1)^{-\frac{1}{2}}(S_{mN_1}(t) - \mu_{N_1}^*(t)) \Rightarrow W + L\ , \tag{6.2.29}$$

where

$$\gamma_{N_1}^2 = \frac{1}{N_1} \sum_{i=1}^{N_1} F(k_1 N_1^{-\frac{1}{2}} x_{m+i,j})[1 - F(k_1 N_1^{-\frac{1}{2}} x_{m+i,j})]\ , \tag{6.2.30}$$

and

$$\mu_{N_1}^*(t) = \sum_{i=1}^{[Nt]} F(k_1 N_1^{-\frac{1}{2}} x_{m+i,j})$$

$$+ (N_1t-[N_1t])F(k_1N_1^{-\frac{1}{2}}x_{m+[N_1t+1],j}) \ . \tag{6.2.31}$$

Furthermore,

$$\lim_{N_1\to\infty} 2N_1^{-\frac{1}{2}}(\mu^*_{N_1}(t) - \frac{N_1t}{2}) = 2k_1f(0) \sum_{p=1}^{r} \lambda_p z_{p,j} \ t \ . \tag{6.2.32}$$

Proof: Similar to the proof of Lemma 6.2.1.

Lemma 6.2.5 Under the model (4.0.1), (4.0.2) with alternatives $\beta_{1,j} = \beta_{0,j} + k_3N_3^{-\frac{1}{2}}$ for some $j = 1, \ldots, k$ and $\beta_{1j'} = \beta_{0j'}$ for all $j \neq j'$,

$$(\sigma^2N_3)^{-\frac{1}{2}} T_{mN_3}(t) \Rightarrow W + L \tag{6.2.33}$$

under H_0, and under H_1,

$$(\sigma^2N_3)^{-\frac{1}{2}} T_{mN_3}(t) - \mu_{N_3}(t)) \Rightarrow W + L \tag{6.2.34}$$

where

$$\mu_{N_3}(t) = \sum_{i=1}^{[Nt]} k_3N_3^{-\frac{1}{2}}x_{m+i,j} + (N_3t-[N_3t])k_3N_3^{-\frac{1}{2}}x_{m+[N_3t+1],j} \ . \tag{6.2.35}$$

Furthermore,

$$\lim_{N_3\to\infty} \sigma^2N_3^{-\frac{1}{2}}\mu_N(t) = \sigma^2k_3 \sum_{p=1}^{r} \lambda_p z_{p,j} \ t \quad . \tag{6.2.36}$$

Proof: Similar to the proof of Lemma 6.2.3.

Since in the general regression case

$$\mu^{(1)} = 2k_1f(0) \sum_{p=1}^{r} \lambda_p z_{p,j} \tag{6.2.37}$$

and

$$\mu^{(3)} = k_3 \sigma^2 \sum_{p=1}^{r} \lambda_p z_{p,j} \tag{6.2.38}$$

e_{13} is the same as with the model (5.0.1), (5.0.2). Thus the corollary is established.

Table 6.2.1 contains specific ARE values for some commonly discussed distributions. The values are well known since they are duplicate ARE values for the one and two sample location shift problems.

Table 6.2.1 Pitman Asymptotic Relative Efficiencies Comparing

$$T_N^{(1)} = 2N^{-\frac{1}{2}}(S_{mN}(t) - \frac{Nt}{2})$$

$$T_N^{(2)} = \sqrt{12}\, N^{-\frac{3}{2}} t^{-1}(W_{mN}(t) - \frac{Nt(Nt-1)}{4})$$

$$T_N^{(3)} = \sigma^{-1} N^{-\frac{1}{2}} T_{mN}(t) .$$

DENSITY OF ε_i	e_{12}	e_{13}	e_{23}
Normal	$\frac{2}{3}$	$\frac{2}{\Pi}$	$\frac{3}{\Pi}$
Logistic	$\frac{3}{4}$	$\frac{\Pi^2}{12}$	$\frac{\Pi^2}{9}$
Double Exponential	$\frac{4}{3}$	2	$\frac{3}{2}$
Uniform	$\frac{1}{3}$	$\frac{1}{3}$	1
Cauchy	$\frac{4}{3}$	∞	∞

VII. THE CUSUM SIGN TEST FOR SHIFT IN MEDIAN

The procedure of Theorem 4.1 when applied to the problem of testing for a constant median, and with $m = 2k+1$, $N = 2q$, has the property of being distribution-free and analytically tractable. That is, exact probabilities under H_0 for paths and exact expected sample sizes can be calculated. This provides a way of measuring how well the corrected Brownian motion critical values perform when obtaining approximate boundaries for a size α test. The purpose of the present chapter is to explore this facet of the cumulative sum sign test for constancy of an unknown median.

§7.1 Exact Sample Path Probabilities

Definition: A sample path for the first stage is the sequence of partial sums,

$$2 \sum_{i=1}^{n} [\eta(Y_{m+i} - \hat{\alpha}_0) - 1/2], \ 1 \leq n \leq N \ . \qquad (7.1.1)$$

The above definition of a sample path is equivalent to defining a sample path as the vector

$$(2[\eta(Y_{m+1} - \hat{\alpha}_0) - 1/2], \ \ldots, \ 2[\eta(Y_{m+N} - \hat{\alpha}_0) - 1/2]) \qquad (7.1.2)$$

which is a random vector with entries ± 1. A similar definition, of course, applies to a sample path in any stage, but for present purposes only the first stage is considered.

Since under $H_0:\alpha_1 = \alpha_0$, for any vector $(\nu_1,\ldots,\nu_N)$ of ± 1 values,

$$P[2\,[\eta(Y_{m+1} - \hat{\alpha}_0) - 1/2] = \nu_1,\ \ldots,\ 2\,[\eta(Y_{m+N} - \hat{\alpha}_0) - 1/2] = \nu_N]$$

$$= \int_{-\infty}^{\infty} P[2\,[\eta(Y_{m+1} - a) - 1/2] = \nu_1,\ \ldots,\ 2\,[\eta(Y_{m+N} - a) - 1/2]$$

$$= \nu N]\,dG_{\hat{\alpha}_0}(a) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} P[2\,[\eta(Y_{m+i} - a) - 1/2] = \nu_i]\; d\,G_{\hat{\alpha}_0}(a), \tag{7.1.3}$$

where $G_{\hat{\alpha}_0}(a) = P[\hat{\alpha}_0 \le a]$, the probability of any particular path is the same as the probability of any other path with the same number of +1 values.

Let $S^{(j)} = 2 \sum_{i=1}^{N} [\eta(Y_{m+(j-1)N+i} - \hat{\alpha}_{j-1}) - 1/2]$. That is, $S^{(j)}$ is the final sum of the sample path in stage j. Suppose that $S^{(1)}=s$, then the sample path consists of $1/2(N+s)$ plus 1's and $1/2(N-s)$ minus 1's. There are $\binom{N}{1/2(N+s)}$ distinct paths with sum s; and each one, under H_0, has the same probability.

Since $m = 2k+1$, $\hat{\alpha}_0 = Y_m^{(k+1)}$, where $Y_m^{(k+1)}$ is the $(k+1)^{\underline{st}}$ order statistic among $Y_1, \ldots, Y_m$. Thus the density of $\hat{\alpha}_0$ is (Gibbons, 1971, p. 27)

$$g_{\hat{\alpha}_0}(a) = \frac{m!}{k!k!}\,[F_Y(a)]^k\,[1 - F_Y(a)]^k\,f_Y(a), \tag{7.1.4}$$

where $F_Y(a) = P[Y_i \le a]$, $i = 1, \ldots, m$. Thus

$$P[S^{(1)} = s] = \binom{N}{1/2(N+s)} \int_{-\infty}^{\infty} \prod_{i=1}^{1/2(N+s)} P[\eta(Y_{m+i} - a) = 1] \prod_{i=1/2(N+s)+1}^{N}$$

$$\cdot\, P[\eta(Y_{m+i} - a) = 0]\; g_{\hat{\alpha}_0}(a)\,da = \binom{N}{1/2(N+s)} \int_{-\infty}^{\infty} [1 - F_Y(a)]^{1/2(N+s)}$$

$$\cdot\ [F_Y(a)]^{1/2(N-s)} \cdot \frac{m!}{k!k!}[F_Y(a)]^k\,[1 - F_Y(a)]^k\, f_Y(a)\, da$$

$$= \binom{N}{1/2(N+s)} \frac{m!}{k!k!} \int_{-\infty}^{\infty} [F_Y(a)]^{1/2(N+s)+k}\,[1-F_Y(a)]^{1/2(N-s)+k} f_Y(a)\, da$$

$$= \binom{N}{1/2(N+s)} \frac{m!}{k!k!} \frac{(1/2(N+s)+k)!(1/2(N-s)+k)!}{(N+m)!} \qquad (7.1.5)$$

which shows that $P[S^{(1)} = s]$ is distribution free.

Since $\hat{\alpha}_0 = Y_m^{(k+1)}$, all the information for calculating the probability of a given sample path sum for stage 1, $P[S^{(1)} = s]$, is contained in the two sets of order statistics $Y_m^{(1)}, \ldots, Y_m^{(m)}$ and $Y_{m+N}^{(1)}, \ldots, Y_{m+N}^{(m+N)}$, where $Y_{m+N}^{(i)}$ is the i^{th} smallest among $Y_1, \ldots, Y_{m+N}$. If each $Y_{m+N}^{(i)}$ is classified as a 0 or a 1 depending on whether $Y_{m+N}^{(i)} \in \{Y_1, \ldots, Y_m\}$ or $Y_{m+N}^{(i)} \in \{Y_{m+1}, \ldots, Y_{m+N}\}$, respectively, and if the order statistics are mapped to the vector of these 0 and 1 values, the classification vector, then the path sum $S^{(1)}$ can be obtained by counting the number of 1's to the right (or left) of the $(k+1)\underline{^{st}}$ zero.

For example, suppose m = 5, N = 4 and the following numbers are observed:

$$(y_1, \ldots, y_9) = (4, 3, 5, 7, 9, 8, 1, 2, 6) \qquad (7.1.6)$$

The order statistic is

$$(y^{(1)}, \ldots, y^{(9)}) = (1, 2, 3, 4, 5, 6, 7, 8, 9)$$

Since $1 \in \{y_6, y_7, y_8, y_9\}$, $2 \in \{y_6, y_7, y_8, y_9\}$, etc., the classification vector of the order statistic is

$$(C^{(1)}, \ldots, C^{(9)}) = (1, 1, 0, 0, 0, 1, 0, 1, 0)$$

Since there are two 1's to the right of the middle zero, the middle zero being the classification of $\hat{\alpha}_0$, and two 1's to the left of the middle zero, the path sum is +1+1-1-1=0. This can be ascertained also from the particular realization (7.1.6). Since $\hat{\alpha}_0=5$, the vector of ± 1's generated by (y_6, y_7, y_8, y_9) is $(1,-1,-1,1)$, which sums to zero.

In the general case, to count the number of paths summing to a specified s, note that the location of the $(k+1)^{\underline{st}}$ zero in the classification vector of the order statistic is fixed. To the left of $(k+1)^{\underline{st}}$ zero there are k zeros and 1/2(N-s) 1's, and to the right there are k zeros and 1/2(N+s) 1's, and so the number of distinct classification vectors such that $S^{(1)} = s$ is

$$\binom{1/2(N+s)+k}{k}\binom{1/2(N-s)+k}{k},$$

and under the null hypothesis, each classification vector is equally likely. Thus

$$P[S^{(1)} = s] = \binom{N+m}{m}^{-1}\binom{1/2(N+s)+k}{k}\binom{1/2(N-s)+k}{k} \qquad (7.1.7)$$

which is the same as (7.1.5).

The following lemma concludes §7.1:

<u>Lemma 7.1.1</u>: As $m \to \infty$, N constant,

$$P[S^{(1)} = s] = \binom{N}{1/2(N+s)}\left[2^{-N} + 0\,(m^{-1})\right] \qquad (7.1.8)$$

<u>Proof</u>: From (7.1.5), $\binom{N}{1/2(N+s)}^{-1} P[S^{(1)} = s]$

$$= \frac{m!}{(N+m)!}\,\frac{(1/2N+s)+k)!}{k!}\,\frac{(1/2(N-s)+k)!}{k!}$$

$$= \frac{[(1/2(N+s)+k)\ (1/2(N+s)+k-1)\ \ldots\ (k+1)]\ [(1/2(N-s)+k)\ \ldots\ (k+1)]}{(N+m)\ (N+m-1)\ \ldots\ (m+1)}$$

$$= \frac{1/2(N+s)+k}{N+m}\ \frac{1/2(N+s)+k-1}{N+m-1} \cdots \frac{k+1}{1/2(N-s)+m+1} \times \frac{1/2(N-s)+k}{1/2(N-s)+m} \cdots \frac{k+1}{m+1}$$

$$= \prod_{i=1}^{1/2(N+s)} \left(\frac{1/2(N+s)+k-i+1}{N+m-i+1}\right) \prod_{i=1}^{1/2(N-s)} \frac{1/2(N-s)+k-i+1}{1/2(N-s)+m-i+1} \qquad (7.1.9)$$

Since m = 2k+1, the factors in 7.1.9 are

$$\frac{1/2(N+s)+k-i+1}{N+m-i+1} = 2 - \left(2 - \frac{1/2(N+s)+k-i+1}{N+m-i+1}\right)$$

$$= 1/2 - \left(\frac{(N+m-i+1) - (N+s+2k+1-2i+1)}{2(N+m-i+1)}\right)$$

$$= 1/2 - \frac{i-s}{2(N+m-i+1)} = 1/2 + 0(m^{-1}), \qquad (7.1.10)$$

and

$$\frac{1/2(N-s)+k-i+1}{1/2(N-s)+m-i+1} = 2 - \left(2 - \frac{1/2(N-s)+k-i+1}{1/2(N-s)+m-i+1}\right)$$

$$= 1/2 - \left(\frac{(1/2(N-s)+m-i+1) - (N-s+2k+1-2i+1)}{N-s+2m-2i+2}\right)$$

$$= 1/2 - \left(\frac{i-1/2(N-s)}{N-s+2m-2i+2}\right) = 1/2 + 0(m^{-1}). \qquad (7.1.11)$$

Thus (7.1.9) equals

$$\prod_{i=1}^{1/2(N+s)} \left(1/2 + \frac{s-i}{2(N+m-i+1)}\right) \prod_{i=1}^{1/2(N-s)} \left(1/2 + \frac{1/2(N-s)-i}{N-s+2(m-i+1)}\right)$$

$$= 2^{-N} + 0(m^{-1}),$$

which proves (7.1.8). Note that if the cumulative sign procedure were to be performed to test for a shift in a known null median, then under the null hypothesis the probability of any path of length N is 2^{-N}, whereas by Lemma 7.1.1, the probability of any path in the procedure where α_0 must be estimated is $2^{-N} + O(m^{-1})$, hence as $m\to\infty$, N fixed, the path probabilities converge to those of the ordinary cumulative sum sign test.

§7.2 Exact Boundary Crossing Probabilities

The topic of this section is the derivation of exact values for

$$P[\max_{1\le n\le N} 2\sum_{i=1}^{N} [\eta(Y_{m+i} - \hat{\alpha}_0) - 1/2] \ge K]; \tag{7.2.1}$$

$$P[\max_{1\le n\le N} \left|2\sum_{i=1}^{n} [\eta(Y_{m+i} - \hat{\alpha}_0) - 1/2]\right| \ge K] \tag{7.2.2}$$

where K is an integer between 1 and N. The approach taken is to evaluate

$$1 - \sum_{s=-N}^{N} P[\max_{1\le n\le N} 2\sum_{i=1}^{n} [\eta(Y_{m+i} - \hat{\alpha}_0) - 1/2] < K;\ S^{(1)} = s], \tag{7.2.3}$$

and

$$1 - \sum_{s=-N}^{N} P[\max_{1\le n\le N} \left| 2\sum_{i=1}^{n} [\eta(Y_{m+i} - \hat{\alpha}_0) - 1/2]\right| < K;\ S^{(1)} = s] \tag{7.2.4}$$

The terms of summation in (7.2.3) and (7.2.4) are obtained by multiplying the number of paths with $S^{(1)} = s$ that cross the boundary by the probability of a single path such that $S^{(1)} = s$.

To count the number of paths with $S^{(1)} = s$ that cross K, the results of Fray and Roselle (1971) are used. These results concern the counting of lattice paths bounded above and/or below by lines of the form $y = x \pm K$.

Definition: A lattice path is a path from (0, 0) to (n_1, n_2) where a lattice point (i, j) may be approached from any of the lattice points (i-1, j), (i, j-1).

Equivalently, a lattice path may be thought of as an $(n_1 + n_2)$-vector of ordered pairs:

$$((0, 0), (i_1, j_1), \ldots, (n_1, n_2))$$

where $(i_1, j_1) \in \{(0, 1), (1, 0)\}$, $(i_{r+1}, j_{r+1}) \in \{(i_r+1, j_r), (i_r, j_r+1)\}$, $r = 1, \ldots, n_1 + n_2$.

In order to use the Fray-Roselle results to count sample paths, a simple transformation to lattice paths will be required. The sample path starts at (0, 0) and the first step is to either (1, 1) or (1, -1); the lattice path starts at (0, 0) also, but the first step is to either (0, 1) or (1, 0), respectively. The transformation is completely determined by the mapping of ((1, 1), (1, -1)) to ((0, 1), (1, 0)). The matrix representation of this transformation is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{7.2.5}$$

where x_1, x_2 represents the time and partial sum coordinates, respectively, of the sample path, and y_1, y_2 represents the transformed values in lattice path coordinates. Figure 7.2.1 illustrates this transformation of sample path coordinates to lattice path coordinates.

The original problem is to determine the number of sample paths from (0, 0) to (N, s) which lie strictly within the boundaries $\pm K$(for the two sided test). The equivalent lattice path problem is to count the number of lattice paths from (0, 0) to (1/2(N-s), 1/2(N+s)) which stay between the lines y = x + K and y = x - K; and this count is

$$D\,(1/2(N-s),\ 1/2(N+s);\ K) = \binom{N}{1/2(N+s)}$$
$$- \sum_{h=0}^{a_1} \binom{N}{1/2(N+s)-K-2Kh} - \sum_{h=0}^{a_2} \binom{N}{1/2(N+s)+K+2Kh}$$
$$+ \sum_{h=1}^{a_3} \binom{N}{1/2(N+s)+2Kh} + \sum_{h=1}^{a_4} \binom{N}{1/2(N+s)-2Kh}, \qquad (7.2.6)$$

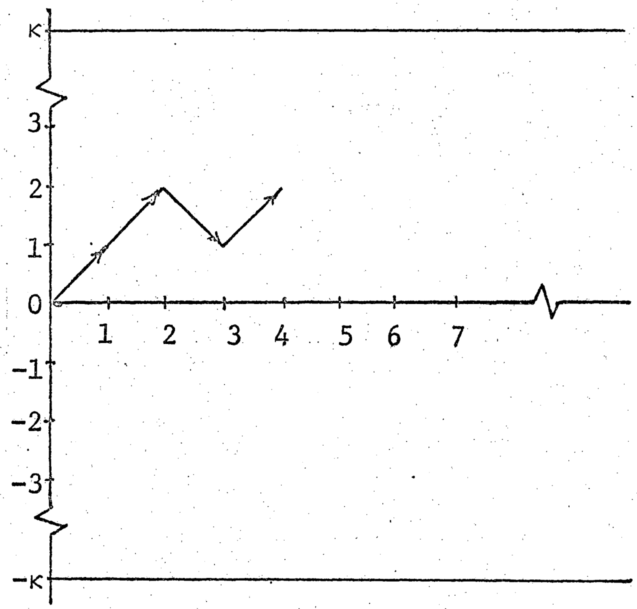
where a_1, a_2, a_3, a_4 are non-negative integers such that the combinatoric denominators remain between zero and N. If, for instance, all values of a_1 yield denominators which are negative or greater than N, the sum associated with that index is zero.

Since there are $\binom{N}{1/2(N+s)}$ lattice paths between (0, 0) and (1/2(N-s), 1/2(N+s)), from (7.2.4) and (7.1.5)

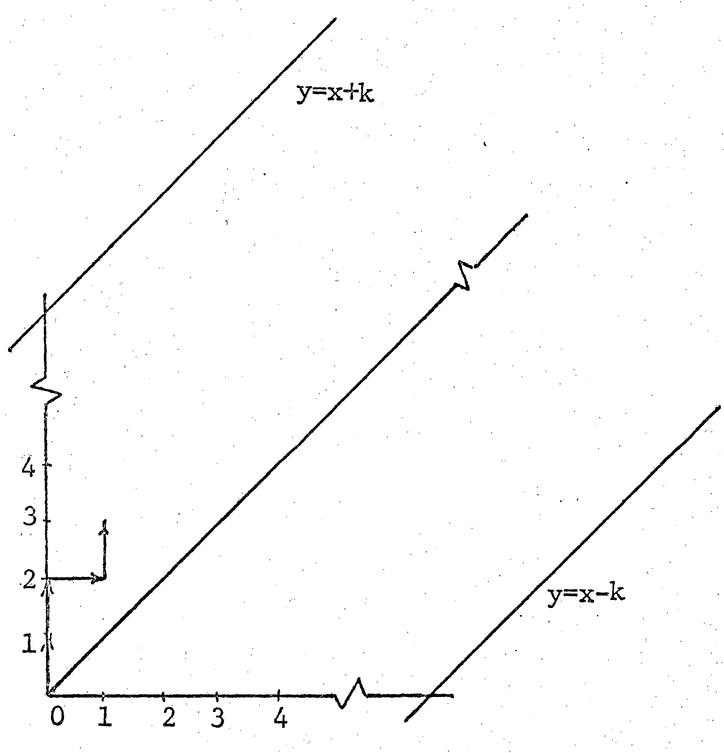
$$P[\max_{1\leq n\leq N} \left| 2 \sum_{i=1}^{n} [\eta(Y_{m+i} - \hat{\alpha}_0) - 1/2] \right| \geq K]$$

$$= 1 - \sum_{s=-K}^{K} D(1/2(N-s),\ 1/2(N+s);\ K)\ \frac{m!}{k!k!}$$

$$\cdot\ \frac{(1/2(N+s)+k)!\,(1/2(N-s)+k)!}{(N+m)!}\ . \qquad (7.2.7)$$



First Four Steps of Sample Path



First Four Steps Of Lattice Path

Figure 7.2.1 Transformation of Sample Paths To Lattice Paths.

For the one-sided test, the number of lattice paths between (0, 0) and (1/2(N-s), 1/2(N+s)) which remain below the line y = x + K is

$$\binom{N}{1/2(N+s)} - \binom{N}{1/2(N-s)+K} . \qquad (7.2.8)$$

By (7.2.3) and (7.1.5),

$$P[\max_{1\leq n\leq N} 2 \sum_{i=1}^{n} [\eta(Y_{m+i} - \hat{\alpha}_0) - 1/2] \geq K] = 1 - \sum_{s=-N}^{K} \{\binom{N}{1/2(N+s)}$$

$$- \binom{N}{1/2(N-s)+K}\} \frac{m!}{k!k!} \frac{(1/2(N+s)+k)!(1/2(N-s)+k)!}{(N+m)!} \qquad (7.2.9)$$

As an example, consider the case in which m = 17, N = 16, and K = 9. The exact probabilities will be calculated for a two-sided test, where

$$P[\max_{1\leq n\leq N} \left| 2 \sum_{i=1}^{n} [\eta(Y_{m+i} - \hat{\alpha}_0) - 1/2]\right| < K;\ S^{(1)} = s]$$

$$= P[\max_{1\leq n\leq N} \left| 2 \sum_{i=1}^{n} [\eta(Y_{m+i} - \hat{\alpha}_0) - 1/2]\right| < K;\ S^{(1)} = -s]. \qquad (7.2.10)$$

Since N is even, s must also be even, so the only values of s for which D (1/2(N-s), 1/2(N+s); K) needs to be calculated are s = 0, 2, 4, 6, 8.

<u>For s = 0:</u> D (1/2(16-0), 1/2(16+0); 9) = D (8, 8; 9)

$$= \binom{16}{8} - \sum_{h=0}^{a1} \binom{16}{8-9-18h} - \sum_{h=0}^{a2} \binom{16}{8+9+18h}$$

$$+ \sum_{h=1}^{a_3} \binom{16}{8+18h} + \sum_{h=1}^{a_4} \binom{16}{8-18h} = \binom{16}{8} - 0 - 0 + 0 + 0 = \binom{16}{8}. \tag{7.2.11}$$

That is, all sample paths summing to zero remain between ± 9 since a minimum of 9 1's or 9 -1's is required to reach the boundary.

<u>For s = 2:</u> $D\ (7,\ 9;\ 9) = \binom{16}{9} - \sum_{h=0}^{a_1} \binom{16}{9-9-18h}$

$$- \sum_{h=0}^{a_1} \binom{16}{9+9+18h} + \sum_{h=1}^{a_3} \binom{16}{9+18h} + \sum_{h=1}^{a_4} \binom{16}{9-18h} = \binom{16}{9} - \binom{16}{0} \tag{7.2.12}$$

Similarly, for

<u>r = 4:</u> $D\ (6,\ 10;\ 9) = \binom{16}{10} - \binom{16}{1};$ (7.2.13)

<u>r = 6:</u> $D\ (5,\ 11;\ 9) = \binom{16}{11} - \binom{16}{2};$ (7.2.14)

<u>r = 8:</u> $D\ (4,\ 12;\ 9) = \binom{16}{12} - \binom{16}{3}.$ (7.2.15)

Thus under $H_0 : \alpha_1 = \alpha_0$, by (7.2.7)

$$P\left[\max_{1 \le n \le 16} \left| 2 \sum_{i=1}^{n} [\ (Y_{17+i} - \hat{\alpha}_0) - 1/2] \right| \ge 9 \right]$$

$$= 1 - \binom{16}{8} \frac{17!}{8!8!} \frac{16!16!}{33!} - 2 \left\{ \binom{16}{9} - \binom{16}{0} \right\} \frac{17!}{8!8!} \frac{17!15!}{33!}$$

$$- 2 \left\{ \binom{16}{10} - \binom{16}{1} \right\} \frac{17!}{8!8!} \frac{18!14!}{33!} - 2 \left\{ \binom{16}{11} - \binom{16}{2} \right\} \frac{17!}{8!8!} \frac{19!13!}{33!}$$

$$- 2 \left\{ \binom{16}{12} - \binom{16}{3} \right\} \frac{17!}{8!8!} \frac{20!12!}{33!}$$

$$= 1.0 - 0.14196 - 0.26812 - 0.22479 - 0.16522 - 0.11099$$

$$= 0.08992. \tag{7.2.16}$$

A critical value of $K=9$ when $m=17$, $N = 16$ has a first-stage actual α-level of $0.08992 \simeq 0.09$. If calculations are done as above for $K = 12$, the actual α-level is $0.03365 \simeq 0.034$.

The critical values 9 and 12 were not chosen at random. The value 9 arises as the integer critical value for a 5% test when uncorrected Brownian motion approximate critical values are used for the process

$$\max_{1 \leq n \leq N} N^{-1/2}\, 2 \sum_{i=1}^{n} [\eta(Y_{m+i} - \hat{\alpha}_0) - 1/2]. \tag{7.2.17}$$

That is, if the truncated sequential sign test is performed to test for a change of a prior known median α_0, using the statistic

$$\max_{1 \leq n \leq N} N^{-1/2}\, 2 \sum_{i=1}^{n} [\eta(Y_{m+i} - \alpha_0) - 1/2], \tag{7.2.18}$$

then the appropriate asymptotic critical values are uncorrected Brownian motion critical values. However, if the true median were assumed known but had in reality been estimated by 17 observations, then the true α-level is almost double the putative α-level of 0.05.

The value $K = 12$ is the integer critical value when using Brownian motion critical values corrected for $\kappa^2 = 16/17$. From Table A.2 the Brownian motion critical value is approximately 2.924 for $\kappa^2 \simeq .94$, and $\sqrt{16} \cdot 2.924=11.696$, and the integer value closest to this is 12.

Table 7.2.1 is a table of comparisons of exact α-levels when corrected and uncorrected Brownian motion approximate critical values are used. In this table the integer critical value selection process which chooses the integer closest to $N^{1/2}\ C_\alpha$ is used, where C_α is the critical value, corrected or uncorrected, from Table A.2. Table 7.2.2 is also a table of comparisons of α-levels, but the integer critical value selection process in the program chooses the smallest integer greater than or equal to $N^{1/2}\ C_\alpha$.

From the tables it is obvious that neither the corrected nor the uncorrected values perform well for small values of m. In such a case one would need to calculate exact crossing probabilities for various critical values and choose the most suitable critical value accordingly. As m increases the approximate critical values become very good, even for small N. The lower left corner of the table indicates that for κ^2 very small there is no significant difference between corrected and uncorrected critical values.

A comment is in order concerning the lack of monotonicity in the tables of α-level comparisons. The lack of monoticity is due to the fact that there can be relatively large differences in α-levels between two adjacent integer critical values, even for large values of N. Consider, for example, N = 100, m = 199. Between the two tables there is a difference of 0.005 for the corrected critical values, and a difference of 0.022 for uncorrected critical values. This can effect the monotonicity in the table as follows:

For two adjacent N values, the lower value $N^{1/2}\ C_{\alpha}$ might be below the mid-point between two integer critical values, and the "value-closest" procedure will select the integer just below the value $N^{1/2}\ C_{\alpha}$, causing an inflation of the α-level. The upper N value could then have $N^{1/2}\ C_{\alpha}$ above the mid-point between two integer critical values, and the "value-closest" procedure will select the integer value just above $N^{1/2}\ C_{\alpha}$, which will cause the integer critical value to be conservative (notice that all the corrected entries in Table 7.2.2 are conservative). A similar argument can be stated using the "value above" selection procedure.

Table 7.2.1. Comparisons of Exact Alpha Levels for a Putative 0.05 Level Two-Sided Test When Corrected and Uncorrected Approximate Brownian Motion Critical Values are Used. For Each Pair (m,N) the Top Entry is the Corrected Value. m is the Number of Observations for Estimating the Median; N is the Length of the Test. The Integer Chosen as Critical Value is the Integer Closest to the Approximate Value.

N =	10	20	30	40	70	100
m = 5	0.00000	0.00870	0.00334	0.00545	0.00557	0.00555
	0.19580	0.35175	0.44088	0.48953	0.57304	0.65192
15	0.01750	0.04208	0.03610	0.02815	0.03434	0.03278
	0.10149	0.19111	0.25794	0.29757	0.37703	0.47119
25	0.05145	0.03028	0.03642	0.03855	0.03942	0.03890
	0.07851	0.14090	0.19119	0.22081	0.28548	0.37642
35	0.04312	0.04684	0.04179	0.04511	0.04194	0.04293
	0.06838	0.11697	0.15712	0.17981	0.23188	0.31613
55	0.03571	0.04852	0.03551	0.04222	0.04921	0.05026
	0.05910	0.09415	0.12311	0.13756	0.17238	0.24344
75	0.05477	0.04152	0.04917	0.04096	0.05307	0.04540
	0.05477	0.08325	0.10636	0.11626	0.14061	0.20142
99	0.05189	0.03697	0.04193	0.05254	0.04834	0.04595
	0.05189	0.07594	0.09495	0.10161	0.11808	0.16993
199	0.04739	0.06446	0.04447	0.05011	0.04826	0.05249
	0.04739	0.06446	0.07681	0.07816	0.08114	0.11476
∞	0.04297	0.05321	0.05890	0.05507	0.04493	0.05618

Table 7.2.2. Comparisons of Exact Alpha Levels for a Putative 0.05 Level Two-Sided Test When Corrected and Uncorrected Approximate Brownian Motion Critical Values are Used. For Each Pair (m,N) the Top Entry is the Corrected Value. m is the Number of Observations for Estimating the Median; N is the length of the Test. The Integer Chosen as Critical Value is the Smallest Integer Greater Than or Equal to the Approximate Value.

N =	10	20	30	40	70	100
m = 5	0.00000	0.00000	0.00334	0.00545	0.00296	0.00362
	0.15784	0.26378	0.37494	0.43784	0.57304	0.62973
15	0.01750	0.01892	0.03052	0.02814	0.03434	0.03095
	0.07105	0.11988	0.19468	0.24339	0.37703	0.44143
25	0.01121	0.03028	0.03642	0.03250	0.02544	0.03890
	0.05145	0.08108	0.13496	0.17100	0.28548	0.34482
35	0.04312	0.04684	0.04179	0.03707	0.04194	0.04293
	0.04312	0.06393	0.19620	0.13409	0.23189	0.28445
55	0.03571	0.03392	0.03551	0.04222	0.03744	0.04117
	0.03571	0.04852	0.07888	0.09767	0.17238	0.21313
75	0.03234	0.04152	0.04917	0.04096	0.03998	0.04540
	0.03234	0.04152	0.06602	0.08007	0.14061	0.17282
99	0.03012	0.03697	0.04193	0.03327	0.04007	0.04595
	0.03012	0.03697	0.05751	0.06829	0.11808	0.14317
199	0.02671	0.03007	0.04447	0.03695	0.03863	0.04095
	0.02671	0.03007	0.04447	0.05011	0.08114	0.09266
∞	0.02344	0.02364	0.033225	0.03318	0.04493	0.04196

For example, in Table 7.2.1 in the row for $m = \infty$, the α-levels are monotone increasing until $N = 70$, at which point there is a drop to 0.04493 from 0.05507 at $N = 40$. The uncorrected critical value for a 5% level test is $C_\alpha = 2.241$, and $\sqrt{40}\,(2.241) = 14.17$; $\sqrt{70}\,(2.241) = 18.75$. For $N = 40$ the "value-closest" procedure selected 14; for $N = 70$ it selected 19. For $N = 40$ the α-level is inflated, for $N = 70$ the α-level is conservative.

§7.3 Expected Sample Size Calculations

The aim of this section is to derive an expression for calculating the expected sample size, when it exists, under the null hypothesis of a constant median. That the exact expected sample size can be calculated for the sequential sign test is due to the following lemma.

Lemma 7.3.1 Let $[S^{(j)} = s_j]$ be the event that the path sum at stage j is s_j and let $N_j = 2q_j$ for all $j = 1, 2, \ldots$; $m = 2k+1$, then for any $r = 1, 2, \ldots$, when H_0 is true,

$$P[S^{(1)} = s_1, \ldots, S^{(r)} = s_r] = \prod_{i=1}^{r} P[S^{(i)} = s_i] \;. \qquad (7.3.1)$$

Proof: By the multiplication rule for probabilities (see, for example, Tucker, p. 7),

$$\begin{aligned} &P[S^{(1)} = s_1, \ldots, S^{(r)} = s_r] \\ &\quad = P[S^{(1)} = s_1 | S^{(2)} = s_2, \ldots, S^{(r)} = s_r] \\ &\quad \times P[S^{(2)} = s_2 | S^{(3)} = s_3, \ldots, S^{(r)} = s_r] \qquad (7.3.2) \\ &\quad \times \ldots \times P[S^{(r)} = s_r] \;. \end{aligned}$$

Given the order statistic $(y^{(1)}, \ldots, y^{(m+N.)})$ for the observation vector

$$(y_1, \ldots, y_m; y_{m+1}, \ldots, y_{m+N_1}, \ldots, y_{m+N.}),\ N. = \sum_{j=1}^{r} N_j ,$$

define the classification vector $(C^{(1)}, \ldots, C^{(m+N.)}$ by

$$C^{(i)} = \begin{cases} r \text{ if } y^{(i)} \varepsilon \{y_{m+N_1+\ldots+N_{r-1}+j},\ j = 1, \ldots, N_r\} \\ 0 \text{ if } y^{(i)} \varepsilon \{y_1, \ldots, y_m\} \ . \end{cases} \tag{7.3.3}$$

Under the hypothesis of constant median, all distinct permutations of the classification vector are equally likely, and under the conditions $N_j = 2q_j$, $j = 1, 2, \ldots,$ and $m = 2k+1$, the classification vector contains all the information about the path sums for the first r stages. That is, $(s_1, \ldots, s_r)$ is completely determined by $(C^{(1)}, \ldots, C^{(m+N.)})$. Let $\zeta = \{(C^{(1)}, \ldots, C^{(m+N.)}):\ S^{(2)} = s_2, \ldots, S^{(r)} = s_r\}$. If $\underline{C} \,\varepsilon\, \zeta$, and $\underline{C^*}$ is a classification vector differing from $\underline{C}$ only in having 0's and 1's interchanged, then $\underline{C^*} \,\varepsilon\, \zeta$. That is, the fact that $S^{(2)} = s_2, \ldots, S^{(r)} = s_r$ is independent of the path sum $S^{(1)}$. Thus

$$P[S^{(1)} = s_1 | S^{(s)} = s_2, \ldots, S^{(r)} = s_r] = P[S^{(1)} = s_1] \ . \tag{7.3.4}$$

A similar result also holds for the remaining probabilities in (7.3.2), and so (7.3.1) holds.

Define the random variable H as the time at which the procedure crosses the critical value for the first time. Then, H is the number of observations taken before the null hypothesis is rejected. Also define the random variable S as the number of stages until the null hypothesis is rejected. Then

$$E[H] = \sum_{r=1}^{\infty} E[H|S = r]P[S = r] \ , \tag{7.3.5}$$

Let $A^{(j)}(s_j)$, $B^{(j)}(s_j)$ be the events, that the null hypothesis is, respectively, accepted, rejected in stage j and that the path sum $S^{(j)} = s_j$. Then

$$P[S = r] = \sum_{s_1,\ldots,s_r} \cdots \sum P[A^{(1)}(s_1)\ldots,A^{(r-1)}(s_{r-1})B^{(r)}(s_r)]$$

$$= \sum_{s_1,\ldots,s_r} \cdots \sum P[S^{(r)} = s_1,\ldots, S^{(r)} = s_r]p[A^{(1)}(s_1)\ldots B^{(r)}(s_r)], \tag{7.3.6}$$

where $p[A^{(1)}(s_1)\ldots A^{(r-1)}(s_{r-1})B^{r}(s_r)]$ is the proportion of paths for the first r stages satisfying $A^{(1)}(s_1)\ldots A^{(r-1)}(s_{r-1})B^{(r)}(s_r)$.

By Lemma 7.3.1,

$$P[S^{(1)} = s_1,\ldots,S^{(r)} = s_r] = \prod_{i=1}^{r} P[S^{(i)} = s_i] \ . \tag{7.3.7}$$

Also, arguing as in Lemma 7.3.1,

$$p[A^{(1)}(s_1)\ldots A^{(r-1)}(s_{r-1})B^{(r)}(s_r)] = [\prod_{i=1}^{r-1} p(A^{(i)}(s_i))]p(B^{(r)}(s_r)), \tag{7.3.8}$$

so (7.3.6) becomes

$$P[S = r] = \sum_{s_1,\ldots,s_r} \cdots \sum [\prod_{i=1}^{r-1} P[S^{(i)} = s_i]p(A^{(i)}(s_i))]P[S^{(r)}=s_r]p(B^{(r)}(s_r))$$

$$= \sum_{s_1,\ldots,s_r} \cdots \sum [\prod_{i=1}^{r-1} P(A^{(i)}(s_i))]P(B^{(r)}(s_r)) \ . \tag{7.3.9}$$

The probabilities $P(A^{(i)}(s_i))$, $P(B^{(r)}(s_r))$ are those derived in (7.2.7) and (7.2.9): $P(A^{(i)}(s_i))$ is the probability of remaining between the critical values and having path sum s_j in stage j. Thus the form (7.3.9) is amenable to programming expected sample size using the program to calculate exact α-levels.

Let

$$\Pi_r = \sum_{s_1,\ldots,s_r} \cdots \sum \prod_{i=1}^{r} P(A^{(i)}(s_i)) \ . \tag{7.3.10}$$

Then

$$\Pi_r = \Pi_{r-1}\left(\sum_{s_r} P(A^{(r)}(s_r))\right) , \tag{7.3.11}$$

and by (7.3.9)

$$P[S = r] = [\sum_{s_1,\ldots,s_{r-1}} \cdots \sum \prod_{i=1}^{r-1} P(A^{(i)}(s_i))][\sum_{s_r} P(B^{(r)}(s_r))]$$

$$= \Pi_{r-1}[\sum_{s_r} P(B^{(r)}(s_r))] \ . \tag{7.3.12}$$

Define $B^{(r)}(\cdot) = \sum_{s_r} B^{(r)}(s_r)$; $A^{(r)}(\cdot) = \sum_{s_r} A^{(r)}(s_r)$, then since $B^{(r)}(s_r)$ and $B^{(r)}(s_r')$ are disjoint, and $A^{(r)}(s_r)=A^{(r)}(s_r')$ for any $s_r \neq s_r'$,

$$P[B^{(r)}(\cdot)] = \sum_{s_r} P[B^{(r)}(s_r)] \ ; \tag{7.3.13}$$

$$P[A^{(r)}(\cdot)] = \sum_{s_r} P[A^{(r)}(s_r)] \ ; \tag{7.3.14}$$

$$P[A^{(r)}(\cdot)] + P[B^{(r)}(\cdot)] = 1 \ . \tag{7.3.15}$$

Thus

$$P[S = r] = \Pi_{r-1}(1-P(A^{(r)}(\cdot))) \tag{7.3.16}$$

$$= \Pi_{r-1} - \Pi_r \ . \tag{7.3.17}$$

The forms (7.3.16) and (7.3.17) indicate the recursive relation that can be used for calculating $P[S = r]$ in the successive terms of (7.3.5).

To complete this section, a computational form for $E[H|S = r]$ of (7.3.5) will be presented.

First, let $\kappa^{(r)}(n, s_r)$ be the event that a path in stage r summing to s_r crosses the r^{th} stage critical value for the first time at the n^{th} observation of that stage. Then

$$E[H|S = r] = \sum_{i=1}^{r-1} N_i + \sum_{n=1}^{N_r} n\left[\sum_{s_r} P(\kappa^{(r)}(n, s_r))\right]. \tag{7.3.18}$$

A program for calculating $P(\kappa^{(r)}(n, s_r))$ is a slight modification of the program for calculating $P(A^{(r)}(s_r))$ and will be discussed further in Chapter VIII.

The results of this section can be summarized in the following theorem:

<u>Theorem 7.3.1</u> From (7.3.5), (7.3.17), and (7.3.18), under the hypothesis of no shift in median, the sequential cumulative sign procedure with $m = 2k+1$, $N_i = 2q_i$ has exact expected sample size

$$\sum_{r=1}^{\infty} \{ \sum_{i=1}^{r-1} N_i + \sum_{n=1}^{N_r} n(\sum_{s_r} P(\kappa^{(r)}(n, s_r))\} (\Pi_{r-1} - \Pi_r) \tag{7.3.19}$$

when the expected sample size exists.

A discussion of the existence of the expected sample size will be presented in Chapter IX.

Although exact expected sample sizes can be calculated the cost of computation is high. Actual computations have not been carried out in this study for that reason. An extension of this study is planned which would include computations of these exact results in addition to some simulation.

VIII. PROGRAMMING CONCEPTS

There were two main objectives of the computer programming for this work. The first objective is general--that of generating corrected Brownian motion critical values for use as approximate critical values in any of the procedures discussed in Chapters IV and V. The second objective is specific to the exact results of the cumulative sum sign test as presented in Chapter VII: To compare exact stagewise α-levels when corrected and uncorrected Brownian motion approximate critical values are used. Toward these ends, five primary subroutines were written. The purpose of this chapter is to discuss briefly the concepts of these subroutines and to present the listings for these and for a main program which can be used to calculate exact expected sample sizes.

The five subroutines that will be discussed are:

i) QUOTNT: Calculates the quotient of two products of powers and factorials;

ii) PATH: Obtains the probability that a path summing to a particular value crosses an integer boundary value;

iii) CROSS: A subroutine for calculating first crossing probabilities for sample paths;

iv) CRITL: Finds the corrected Brownian motion critical values;

v) GAUSS7: Calculates the normal cumulative distribution function to a log odds error of 10^{-7}.

All the programming was done for an IBM 370 system in double precision FORTRAN IV G language.

Subroutine QUOTNT: Subroutine QUOTNT was written specifically for calculating expressions such as those found in (7.2.9), with the additional feature that real numbers raised to an integer power may also be included in the numerator and denominator expressions. This added feature was used for calculating probabilities such as (7.1.8).

More specifically, suppose the quotient

$$\left(\prod_{i=1}^{n_1} A_i!\right)\left(\prod_{i=1}^{n_2} B_i^{b_i}\right)\Big/\left(\prod_{i=1}^{n_3} C_i!\right)\left(\prod_{i=1}^{n_i} D_i^{d_i}\right) \tag{8.1}$$

is to be calculated. Subroutine QUOTNT does the calculation by switching between two loops--the multiplication loop and the division loop. Calculation starts in the multiplication loop with, say, $A_1!$, where successive multiplications $A_1(A_1-1)(A_1-2)$ etc. are performed until the product reaches a preprogrammed overflow point. The program then switches to the division loop with, say, $C_1!$, where successive divisions $\frac{1}{C_1(C_1-1)\ldots}$ are performed on the overflow product until an underflow limit is reached. Control is then returned to the multiplication loop where successive multiplication begins with the factor following the one giving an overflow. In the calculation of $B_1^{b_1}$ and $\frac{1}{D_1^{d_1}}$ the program does successive multiplications $B_1B_1B_1 \ldots$ and divisions $\frac{1}{D_1D_1\ldots}$ switching between loops as overflows and underflows are encountered.

Th input/output list for QUOTNT consists of four variable dimensioned arrays DVDND, DVSOR, DVDNDP, and DVSORP, the output variable ANSWR, and the dimensioning variable for the four input vectors, $|D|M$.

The input arrays DVDND and DVSOR are defined in the calling program and contain the information, $A_1,\ldots,A_{n_1}$; $B_1,b_1,\ldots,B_{n_2},b_{n_2}$, etc. The array DVDND, for instance, is an $|D|M\times2$ array, which can be viewed as $|D|M$ pairs of numbers. Similarly for the array DVSOR. The arrays DVDNDP and DVSORP are program arrays that are not defined in the calling program but are used by the subroutine to handle 0!.

Suppose that in (8.1) $n_1 = n_2 = n_3 = n_4 = 1$. That is, the quotient $A!B^b/C!D^d$ is desired, then $|D|M$ is 2, and the pairs of numbers entered for DVDND are

$$(\text{DVDND}(1,1),\ \text{DVDND}(1,2)) = (A,0.000) \tag{8.2}$$

$$(\text{DVDND}(2,1),\ \text{DVDND}(2,2)) = (B,b)\ . \tag{8.3}$$

The pairs of numbers entered for DVSOR are

$$(\text{DVSOR}(1,1),\ \text{DVSOR}(1,2)) = (C,0.000) \tag{8.4}$$

$$(\text{DVSOR}(2,1),\ \text{DVSOR}(2,2)) = (D,d)\ . \tag{8.5}$$

The zeros in (8.2) and (8.4) indicate to QUOTNT that A! and 1/C!, respectively, are to be calculated. The variables b and d in (8.3) and (8.5) are integer valued REAL*8 variables greater than zero. If (0.000, 0.000) is defined by the calling program in either DVDND or DVSOR, QUOTNT converts it to (1.000, 0.000), since 0! = 1 = 1!.

After programming QUOTNT to obtain exact boundary crossing probabilities, the decision was made to try approximately ANSWR by using DLGAMA in a dummy QUOTNT subroutine. That is, instead of calculating A!/C! exactly as done above, the Fortran IV routines DEXP and DLGAMA were used as follows

$$\text{ANSWR} = \text{DEXP (DLGAMA(A+1.000) - DLGAMA(C+1.000))} \;. \tag{8.6}$$

making use of the fact that $\Gamma(n+1) = n!$ Tables 7.2.1 and 7.2.2 were generated using this approximation, and the output was identical to the tables generated by the exact method.

<u>Subroutine PATH</u>: From (7.2.7) and (7.2.9), this subroutine calculates, for the two-sided test,

$$D(\tfrac{1}{2}(N-s),\ \tfrac{1}{2}(N+s);\ \kappa)\ \frac{m!}{k!k!}\ \frac{(\frac{1}{2}(N+s)+k)!\,(\frac{1}{2}(N-s)+k)!}{(N+m)!}\ ; \tag{8.7}$$

and for the one-sided test,

$$\left\{\binom{N}{\frac{1}{2}(N+s)} - \binom{N}{\frac{1}{2}(N-s)+\kappa}\right\}\ \frac{m!}{k!k!}\ \frac{(\frac{1}{2}(N+s)+k)!\,(\frac{1}{2}(N-s)+k)!}{(N+m)!}\ . \tag{8.8}$$

These two values are the probabilities that a path of length N, with m initial observations for estimating the median, does not cross the boundary κ and sums to s. By the proper indexing of ISTOP, PATH will also calculate

$$\binom{N}{\frac{1}{2}(N+s)}\frac{m!}{k!k!}\ \frac{(\frac{1}{2}(N+s)+k)!\,(\frac{1}{2}(N-s)+k)!}{(N+m)!} \tag{8.9}$$

which is the unconditional probability, (5.1.5), that the path will sum to s. In any case, PATH will calculate (8.7) by calling QUOTNT to calculate (8.9) first, and then calling QUOTNT to calculate such factors, from (7.2.6), as

$$\binom{N}{\frac{1}{2}(N+s)-\kappa-2\kappa h} \frac{m!}{k!k!} \frac{(\frac{1}{2}(N+s)+k)!(\frac{1}{2}(N-s)+k)!}{(N+m)!} . \tag{8.10}$$

The input/output list consists of

i) PROB: The output variable;

ii) NS: The path sum, s;

iii) MEST: The number of initial observations for estimating the median, m = 2k+1;

iv) NTEST: The length of the path, N;

v) KRV: The boundary value, κ;

vi) KSIDE: A variable controlling whether a one-sided or two-sided test is being calculated:

KSIDE = 1 - one-sided test;

KSIDE = 2 - two-sided test;

vii) ISTOP: A program control variable primarily intended to stop the main program in case of an error in the subroutine. If an error occurs, the subroutine will define ISTOP = 1 and return control to the main program. If the calling program defines ISTOP = 2, PATH calculates (8.9).

<u>Subroutine CROSS</u>: As defined for (7.3.18), $\kappa^{(r)}(n, s_r)$ needs to be calculated to obtain the exact expected sample size. That is, for a path of length N, with m initial observations for estimating the median, CROSS calculates the probability of remaining within

the critical value(s) up to time (n-1), $1 \leq n \leq N$, touching the boundary at time n, and then summing to s. With critical value K, for a two-sided test, the calculation is done as

$$D(\tfrac{1}{2}((n-1)-(K-1)),\ \tfrac{1}{2}((n-1)+(K-1);\ K)\times\binom{N-n}{\tfrac{1}{2}(N-n-s+K)}$$
$$\times\ \frac{m!}{k!k!}\ \frac{(\tfrac{1}{2}(N-s)+k)!\,(\tfrac{1}{2}(N+s)+k)!}{(N+m)!}\ . \qquad (8.11)$$

The factor $D(\frac{1}{2}((n-1)-(\ -1)),\ \frac{1}{2}((n-1)+(K-1));\ K)$ counts the number of paths from (0, 0) to (n-1, K-1) which do not cross or touch the boundary values $\pm$ K. The factor $\begin{pmatrix} N-n \\ \frac{1}{2}(N-n-s+K) \end{pmatrix}$ counts the number of paths from (n, K) to (N, s). The final factor, of course, is the probability of a single path summing to s.

The input/output list for CROSS is the same as for PATH except for the variable NT, the crossing time, n, of the path.

Subroutine CRITL: From (6.1.16) and (6.1.24) the Brownian motion approximate critical value for a one-sided size α test is the value C_α such that

$$\alpha = \exp[2(C_\alpha\ )^2]\Phi\left(\frac{-C_\alpha(1+2\kappa^2)}{(1+\kappa^2)^{\frac{1}{2}}}\right) + \left(\frac{-C_\alpha}{(1+\kappa^2)^{\frac{1}{2}}}\right)\ ; \qquad (8.12)$$

and for a two-sided test the critical value is found by solving

$$\alpha = 2\sum_{S=0}^{\infty}(-1)^S\{\exp[2(SC_\alpha\kappa)^2]\Phi\left(\frac{-2SC_\alpha\kappa^2-(2S+1)C_\alpha}{(1+\kappa^2)^{\frac{1}{2}}}\right)$$
$$\exp[2[(S=1)C_\alpha\kappa]^2\Phi\left(\frac{-2(S+1)C_\alpha\kappa^2-(2S+1)C_\alpha}{(1+\kappa^2)^{\frac{1}{2}}}\right)\} \qquad (8.13)$$

The procedure begins with $C_\alpha = 1$ and increments by +1 until (8.12) (or (8.13)) is less than α, then the increment is $-\frac{1}{2}$ until α is exceeded, in which case the increment becomes $+\frac{1}{4}$. That is, when the direction of incrementing changes the absolute value of the increments is halved. The procedure stops when (8.12) (or (8.13)) is within 10^{-5} of α.

The input/output list consists of CRIT, the output variable, or C_α; ALPHA, the input α-level; RKSQ, the correction parameter κ^2; and KSIDE, the variable indicating whether the critical region is one or two sided.

Function GAUSS7: This subroutine calculates the normal cumulative distribution function values for CRITL to a log-odds accuracy of 10^{-7}. Log-odds error is defined as

$$\left| \ln \left(\frac{\Phi(x)}{1-\Phi(x)} \cdot \frac{1-\hat{\Phi}(x)}{\hat{\Phi}(x)} \right) \right|$$

where $\hat{\Phi}(x)$ is a numerical estimate of $\Phi(\alpha)$. For arguments between zero and 2.5, $\hat{\Phi}(x)$ is obtained via the continued fraction of Shenton (1954) which was developed from Gauss (1866). That is,

$$\Phi(x) = \frac{1}{2} + \phi(x)\left[\frac{x}{1-}\,\frac{x^2}{3+}\,\frac{2x^2}{5-}\,\frac{3x^2}{7+}\,\cdots\right] , \qquad (8.14)$$

and the number of partial quotients to attain a log-odds error of 10^{-7} is determined by taking the smallest integer greater than or equal to

$$y = 2.90 + 5.32\,x . \qquad (8.15)$$

For arguments between 2.5 and 8.0 the continued fraction of Laplace (1839) is used:

$$\Phi(x) = 1 - \phi(x)\left[\frac{1}{x+}\,\frac{1}{x+}\,\frac{2}{x+}\,\frac{3}{x+}\,\frac{4}{x+}\cdots\right] , \tag{8.16}$$

and the appropriate number of partial quotients to be computed is the smallest integer greater than or equal to

$$y = 2.0 + 60x^{-1.45} . \tag{8.17}$$

For arguments greater than 8.0 the Laplace representation is used computing 5 partial quotients.

A table of coefficients for (8.15) and (8.17) can be found in Good and Roller (1977) which will provide sufficient lengths of convergents for accuracies from 10^{-4} to 10^{-9}.

The listings of the above subroutines, along with a listing of a program for calculating the exact expected sample size is found in Appendix 2.

IX. DISCUSSION AND FURTHER RESEARCH

As mentioned in the introduction, this dissertation is a preliminary investigation in the development of non-parametric procedures for testing the constancy of regression relationships over time. The procedures discussed in the text are designed for independent, identically distributed errors, and intuitively would be suitable to situations where one might expect a sudden single shift in the model rather than, say, a slow drift. In the case of a slow drift the estimates of the parameters would tend to follow the drift, and the test would not be likely to detect such a change.

Primarily the procedures are for use in sampling which is done over time rather than testing for shift in a single record. For example, one might be sampling for pollution levels in a river at one sampling station once per week for the summer months. Assuming that during the first year of sampling the model is stable, the interest might be to detect model changes in the following years. Estimates of the model parameters are made after the first summer, and in the following summer, the procedures for detecting a shift can be employed as the weekly observations become available. In such a situation the detection of change may be a primary objective of the sampling.

There are, of course, situations in which detection of a shift is only secondary. For example one might have an economic model describing some commodity price in terms of some other market factors and be using the model for forecasting. The forecaster would be interested in knowing whether the original model still remains valid as the model becomes older. The price of beef, for instance, may well have the price of corn written in the model initially, and the regression on this factor may be very significant. But over time, for some reason the price of corn may no longer affect beef prices, in which case the price of corn might be eliminated from the model.

As to the procedures themselves, there is the question of choosing the critical values at each stage, and also the question of choosing the sample size at each stage.

Obtaining the stage-wise critical values for the procedure is a primary objective of the asymptotic results of this thesis. Assuming that the number of observations remains constant from stage to stage, two modes of choosing a stagewise critical value are considered. One mode is that the critical value be fixed from stage to stage, using uncorrected Brownian motion approximate values. The second mode is that the critical value at the j^{th} stage is adjusted for the ratio $\kappa_j^2 = \frac{N}{m+(j-1)N}$ using the results of Chapters IV-VI. In either of these modes, as the number of stages increases the sequence of stagewise alpha levels will converge to some constant α.

The first mode is the simplest mode. The test lengths and stagewise critical values are held constant. If the initial number of observations for estimation is large with respect to the number of observations for testing in each stage, then the first mode is appropriate because the corrected critical value will not differ significantly from the uncorrected value, even in the early stages.

If the ratio $\frac{N}{m+(j-1)N} = \kappa_j^2$ is large, then the actual α-level of that stage may be quite inflated when employing the first mode. This is a problem mainly in the first stage, since $\kappa_j^2 < 1$, $j = 2, 3, \ldots$. This inflation of first stage α-levels can be seen in Tables 7.2.1 and 7.2.2 where uncorrected Brownian motion approximate critical values are used.

For example, in the first stage, if N/m is about 2.0 then the true α-level for that stage can be in excess of 0.20 when the putative α level is 0.05.

The second mode is an attempt to maintain a reasonably stable stagewise α-level. Since the number of observations for estimating the parameters is changing from stage to stage it is not possible to keep the stagewise α-levels equal, but by maintaining some degree of control one can prevent severely inflated α-levels in the early stages, and this is an important factor when there is a small number of initial observations for estimating the parameters.

A severely inflated first stage α-level can have an effect on the expected number of stages that the procedure will go through before indicating a shift in the model. A rapid calculation will suffice to demonstrate why this is so.

By Lemma 7.3.1, under the hypothesis of no shift in median the stages of the CUSUM sign test for shift of median are pairwise independent, thus if all the stages were exactly level α, then the expected number of stages would be

$$\sum_{r=1}^{\infty} r(1-\alpha)^{r-1}\alpha = 1 + \sum_{r=1}^{\infty} (1 - \alpha)^r = \frac{1}{\alpha} \quad . \tag{9.1}$$

If the first stage level were α_1 and the remaining stages were all level α then the expected number of stages would be

$$\alpha_1 + \sum_{r=2}^{\infty} r(1-\alpha_1)(1-\alpha)^{r-2}\alpha = \alpha_1 + (1-\alpha_1)(2+\sum_{r=1}^{\infty}(1-\alpha)^r)$$

$$= 1 + \frac{(1-\alpha_1)}{\alpha} \simeq (1-\alpha_1) \cdot \frac{1}{\alpha} \, . \tag{9.2}$$

Thus for α = .05, if all stages have roughly the same level the expected number of stages is approximately 20. If, however, the first stage level is, say, 0.35, then the expected number of stages is roughly (0.65)(20) = 13, a considerable reduction.

Note should be made of the fact that the above calculations depend upon the independence of the stages, a condition which is not true in general. The calculations are intended as a yardstick which can provide some insight to the behaviour of more general procedures. How well the yardstick performs is left as part of a planned simulation study.

Another mode of sampling is possible wherein the number of observations per stage changes and κ_j^2 remains constant. A property of this mode is that the stage lengths grow so rapidly that the expected sample size under H_0 may not be finite. To see this, the stage lengths for fixed κ^2 are approximately:

1. $\kappa^2 m$ for the first stage;

2. $\kappa^2(m+\kappa^2 m)$ for the second stage;

3. $\kappa^2 m+2(\kappa^2)^2 m+(\kappa^2)^3$ for the third stage;

$\vdots$

r. $\kappa^2 m \sum_{k=1}^{r-1} \binom{r}{k}(\kappa^2)^k = \kappa^2 m(1+\kappa^2)^{r-1}$ for the r^{th} stage. (9.3)

Assuming that all stages have approximately the same level α, the expected sample size of the CUSUM sign test in Chapter VII is bounded below by

$$\sum_{r=1}^{\infty} \left(\sum_{j=1}^{r-1} N_j \right) (1-\alpha)^{r-1} \alpha$$

$$= \sum_{r=1}^{\infty} \left(\sum_{j=1}^{r-1} \kappa^2 m(1+\kappa^2)^{j-1} \right) (1-\alpha)^{r-1} \alpha$$

$$= \kappa^2 m\alpha \sum_{r=1}^{\infty} \sum_{s=r}^{\infty} (1-\alpha)^s (1+\kappa^2)^r$$

$$= \kappa^2 m\alpha \sum_{r=1}^{\infty} (1+\kappa^2)^r (1-\alpha)^r \cdot \frac{1}{\alpha}$$

$$= \kappa^2 m \sum_{r=1}^{\infty} [(1+\kappa^2)(1-\alpha)]^r \qquad (9.4)$$

The sum in (9.4) converges only if

$$\kappa^2 < \frac{1}{1-\alpha} - 1 = \frac{\alpha}{1-\alpha} \qquad (9.5)$$

So if $\alpha = 0.05$, κ^2 must be less than .0526 to guarantee a finite expected sample size under H_0. For each observation used in testing there must be twenty observations for estimation. When the value of κ^2 is as small as 0.05 there is little to choose between corrected and uncorrected critical values.

To clarify this point, recall that the first stage statistic for the two-sided CUSUM sign test of Chapter VII is

$$R_1 = \max_{1 \le n \le N} 2N^{-\frac{1}{2}} \left| \sum_{i=1}^{n} \eta(Y_{m+i} - \hat{\theta}_0) - n/2 \right| . \qquad (9.6)$$

The corrected critical value for R_1 is approximated by c_α, the Brownian motion critical value corrected for the factor $N/m = \kappa^2$.

By (7.1.1), the sequence of partial sums,

$$2\left|\sum_{i=1}^{n} \eta(Y_{m+i} - \hat{\theta}_0) - \frac{n}{2}\right| , \quad 1 \leq n \leq N ,$$

is the sample path for this procedure. The integer critical value is defined as either the integer closest to or the greatest integer less than or equal to $N^{\frac{1}{2}} c_\alpha$, depending upon the selection procedure used. If N = 100 and $\kappa^2 = 0.05$, then from Table A.2, the corrected integer critical value of the sample path is 23 (using the integer closest to $\sqrt{100}\, c_\alpha$) for a 5% level test. The uncorrected value is 22. With so many observations required for the initial estimate of the parameters one could, without significant loss, employ the uncorrected critical values.

There are two points to be made about the third mode when the expected sample size may not be finite, for example when κ^2 is fixed at 1. Note that under the first two modes the expected sample size is always finite.

The first point is that the expected sample size under alternatives in the third mode might tend to be much larger than under the first two modes, making the procedure under the third mode less sensitive than under the other modes.

The second point, which lends credence to the first point, is that as κ^2 increases there can be a significant increase in the number of initial observations required in a stage before a shift can be detected. Again, let N = 100, $\alpha = 0.05$. If $\kappa^2 = 1.0$ then a two-sided test requires a minimum of 30 observations, whereas if $\kappa^2 = 4.0$ the same test requires a minimum of 45 observations.

Note that in the above example for $\kappa^2 = 4.0$ nearly half the observations in a stage of length 100 are required before a shift can be detected. In the second stage of this example, under the third mode, 500 observations would be taken and a minimum of 100 observations would be required to detect a shift. In the second mode, however, 100 observations would be taken in the second stage, and corrected for $\kappa^2 = 100/125 = 0.8$, the integer critical value would be 28. That is, although the first stage required a minimum of 45 observations to detect a shift, the second stage is much more responsive, requiring a minimum of 28 observations.

A mixture of sampling modes may be appropriate in some situations. Suppose, for example, an experimenter would like to run stages of length 50 once the first 25 or more observations have been taken. However, the observations may come in slowly, and the experimenter may not want to wait until all twenty-five observations are in before checking for a shift in the model. Then using a small number of observations for a fixed estimate of the parameters, and fixing κ^2, the third sampling mode might be utilized until 25 or more observations have been taken, after which time a shift to the second sampling mode is effected.

For example, it may be reasonable to wait for 5 observations, then with $\kappa^2 = 1$, test with the next 5; then test with the next 10, then with the next 20. From then on use the mode two sampling scheme with $m = 40$ and $N_1 = 50$.

The choice of m and N, of course, depends upon the situation in which the procedure will be used. The user may have no choice in the matter. If possible, though, the largest number of observations that can be obtained for the first estimate should be used so that the parameter estimates are as accurate as possible. The selection of N depends upon the magnitude of shift one wishes to detect. Just as in the fixed sample case, the larger the sample size the more sensitive the test is to small changes.

There are many directions of future research for this dissertation offering a variety of work from highly theoretical to purely simulation. Six extensions are outlined below:

Extension of the Wilcoxon Procedure

The procedure presented in Chapter V, the Wilcoxon procedure for constant median, should be extended to the general regression case. A possible approach to this extension is to assume a finite number of regressor variables, as in the sign test procedure of Chapter IV, and to modify the weak convergence techniques of Miller and Sen (1972) to obtain a proof similar to the proof of Theorem 5.1. This extension will be investigated by the author in the near future.

Estimating the Size and Location of a Shift

Should any of the procedures outlined in this dissertation indicate a shift in the model, one needs a procedure for identifying the size and location of the shift; the tacit assumption being that the shift is a single jump rather than, say, a slow drift.

As a first approach to this problem, consider the case of rejecting H_0 in the first stage only, and assume that the shift has occurred in that first age. For simplicity of notation, the two-sided test for a shift in median only will be discussed.

Suppose that the first stage of the procedure rejects H_0 at time $N^* \leq N_1$, and a shift in median occurred at time r in the first stage, $1 \leq r \leq N^*$. Recall from 1.2 that the statistic

$$R_1 = \max_{1 \leq n \leq N} \left| \sum_{i=1}^{n} \psi_n (Y_{m+i} - \hat{\alpha}_0) - \mu_n^* \right| \tag{9.7}$$

is being employed, and that

$$\left| \sum_{i=1}^{N^*} \psi_{N^*} (Y_{m+i} - \hat{\alpha}_0) - \mu_{N^*}^* \right| \geq K_\alpha , \tag{9.8}$$

where K_α is the critical value chosen for a size α test. Let Δ be the amount of shift in the model at time r, then the value of $\overline{\Delta}$ which minimizes

$$\left| \sum_{i=1}^{r-1} \psi_{N^*} (Y_{m+i} - \hat{\alpha}_0) + \sum_{i=r}^{N^*} \psi_{N^*} (Y_{m+i} - \hat{\alpha}_0 - \overline{\Delta}) - \mu_{N^*}^* \right| \tag{9.9}$$

is an estimate of Δ of the Hodges-Lehmann type. Let $\hat{\Delta}_r$ be the value of $\overline{\Delta}$ which minimizes (9.9) - the estimate of Δ assuming the shift occurred at time r. Now let $(r^*, \hat{\Delta}r^*)$ be the estimates of location and size of shift, respectively, where r* is the value of r for which (9.9) is minimized. That is,

$$\left| \sum_{i=1}^{r^*-1} \psi_N^* \, (Y_{m+i} - \hat{\alpha}_0) + \sum_{r^*}^{N^*} \psi_N^* \, (Y_{m+i} - \hat{\alpha}_0 - \hat{\Delta}_r^*) - \mu_{N^*}^* \right|$$

$$= \min_{1 \le r \le N^*} \left\{ \left| \sum_{i=1}^{r-1} \psi_N^* \, (Y_{m+i} - \hat{\alpha}_0) + \sum_{i=r}^{N^*} \psi_{N^*} \, (Y_{m+i} - \hat{\alpha}_0 - \hat{\Delta}_r) - \mu_N^* \right| \right\} \quad (9.9)$$

In studying this estimator there would be an extensive amount of mathematics involved showing, perhaps, that $(\hat{\Delta}_{r^*} - \Delta)$ is asymptotically normal, or in showing that $\hat{\Delta}_r^*$ is unbiased or asymptotically unbiased. Some useful references for this problem might be Hodges and Lehmann (1963); Adichie (1967), who studied rank estimates in the simple linear regression case; Jureckova (1971), who generalized Adichie's work.

Extension to Generalized Scores Functions

A problem that would be highly mathematical in nature would be to extend the procedures presented in this work to more generalized scores $\psi_n(\cdot)$, which would include as a special case the sign, rank, and identity scores dealt with in Chapters IV and V. An approach to this problem would involve a proof of the Chernoff-Savage (1958) type.

Modified Assumptions about the Error Terms

One drawback to the procedures proposed in this work is that the errors are assumed to be independent, making the procedures unsuitable when applied to more general time-series. An approach to rectifying this drawback is suggested in Vinod (1976), who worked with this problem in the normal theory setting. By modeling the type of departure from independent errors, bounds are established outside of which the decisions to reject or accept a null hypothesis are not effected by having non-independent errors.

Generalization of McGilchrist and Woodyer Procedure

The cumulative sum sign procedure of McGilchrist and Woodyer (1975) can be generalized in a number of directions. One possibility is that the procedure they propose be considered under a regression model. Following that, the procedure could be generalized to a Wilcoxon procedure, and then perhaps to a generalized scores procedure. As in the procedures of this dissertation, the McGilchrist and Woodyer procedure is proposed under the assumption of independent and identically distributed random errors, a restrictive condition if one wishes to use such tests of constancy in a more general time-series setting.

Development of Other Procedures

During a recent consulting session the author was confronted with the following problem:

Samples of ocean bottom are to be taken at a particular station each year to monitor for changes in levels of particular compounds. In the first year six observations per station were taken. The analysis of samples is very costly, and a reduction of number of samples per station in future years to 3 or 4 would provide considerable savings in time and money.

Now for the sake of simplicity, assume only one station is being sampled and that only one compound is being monitored for a shift in concentration. For this simplified situation the following possible procedures have been proposed by the author for investigation:

Suppose that in the second year only three observations are to be taken. To test for a shift in concentration, perform a two sample Wilcoxon test comparing the first and second sample. If no shift is indicated, in the third year take another sample of size three and perform a two sample test comparing the most recent three to the previous nine observations. The procedure is continued until a shift is indicated; testing the current three observations against all prior observations.

Another possible approach, which might be more sensitive to a gradual drift in concentration is that when the kth year's observations become available, perform a k-sample test for ordered alternatives, say, as in Pirie (1974). This test, of course, assumes that a particular direction of shift is of primary interest. For example one might be interested in monitoring for a rise in some compound in an area where the compound will become a factor due to recent manufacturing installations.

Now as a generalization, suppose that blocks of N observations are to be made in each successive period of time, and that the observations are taken serially rather than simultaneously. Then a functional form of the above procedures might be appropriate. An investigation of these procedures and perhaps extensions to more general regression models would provide competitors for the procedures of this dissertation.

A modification of the procedures proposed in this work might also provide increased sensitivity to a slow drift. Along with the sample paths generated as described in the Introduction, calculate the sample path statistics in the jth stage using residual estimates based on $Y_{m+(j-1)N+n} - \hat{\theta}_j^*$, $1 \leq n \leq N$, where $\hat{\theta}_j^*$ is an estimate of the median based on an initial segment of the data which is increasing more slowly than the entire data set.

For example, suppose that stages of size 10 are to be used, and that 10 observations are available for an initial estimate of θ. The suggested procedure would be to look at not only

$$R_j = \max_{1 \leq n \leq 10} \left| \sum_{i=1}^{n} \psi_n \left(Y_{n+10(j-1)+n} - \hat{\theta}_j\right) - \mu_n^* \right|, \tag{9.10}$$

as in (1.2), but to calculate also

$$R_j^* = \max_{1 \leq n \leq 10} \left| \sum_{i=1}^{n} \psi_n \left(Y_{m+10(j-1)+n} - \hat{\theta}_j^*\right) - \mu_n^* \right| \tag{9.11}$$

where $\hat{\theta}_j^*$ is an estimate of θ based on the first, say, $10 + j$ observations, or on the first $10 + \lambda 10j$ observations, where $0 < \lambda \leq 1$. The estimate of θ in (9.11) would lag increasingly further behind the time of the test observations, and convergence to θ would be slower than for $\hat{\theta}_j$, but the test should be more sensitive to a slow drift of the median.

An argument against the use of (9.11) is that some of the information in the sample is not being utilized; no use is made of the observations between the estimating segment and the testing segment. A counter to this argument is that there may be little more information about a slow drift in those eliminated observations. If a single sample of 100 observations is taken serially there may be more information about a slow drift in the median comparing the first 10 observations to the last 10 than would be contained in a test comparing the first 50 observations to the last 50, the assumption tacitly made that the drift is monotone rather than, say, periodic.

Simulation

There is a need for an extensive simulation to study the effectiveness of the procedures for testing model constancy. A number of factors would need to be considered:

i) Type of procedure: For example, the sign, Wilcoxon, or parametric procedures of Chapters IV and V;

ii) Mode of sampling: There is a need to compare the fixed stage length modes to the fixed κ^2 mode;

iii) Types of alternatives: Three types of alternatives possible are: The single jump of various sizes and times; The slow drift under a variety of drift functions; Random jumps of random size.

iv) Models: For the sake of simplicity one might wish to limit the study to the constant model and the simple linear regression model;

v) The distribution of the error term.

The list of factors above is quite extensive and an effective simulation is going to require careful thought concerning just what questions are to be answered.

One final point should be made regarding truncated non-parametric sequential CUSUM procedures. The procedure may reach a point prior to truncation where the critical value cannot be reached. In this case it makes no sense to continue sampling until the truncation point is reached. For example, if the cumulative sum of signs is zero with a critical value of 10 and only 9 observations left until the end of the stage, then it would be a waste of observations to continue in that stage: better to cease testing, re-estimate the parameters, and start testing with a new stage. The effect of this on average run length is to be studied in future work through simulation.

APPENDIX I

Tables and Graphs of Corrected Brownian Motion Critical Values

Table A.1. Brief Table of Brownian Motion Critical Values

Corrected for the Factor κ^2

	One Sided Critical Region			Two Sided Critical Region		
α-Level / κ^2	0.01	0.05	0.10	0.01	0.05	0.10
0.0*	2.57617	1.95996	1.64490	2.80713	2.24146	1.95996
0.1	2.67480	2.02539	1.69312	2.91895	2.32227	2.02539
0.2	2.77148	2.08911	1.74011	3.02686	2.40088	2.08899
0.3	2.86426	2.15112	1.78589	3.13184	2.47729	2.15100
0.4	2.95508	2.21143	1.83057	3.23389	2.55176	2.21155
0.5	3.04395	2.27051	1.87427	3.33301	2.62427	2.27051
0.6	3.12988	2.32813	1.91711	3.42969	2.69507	2.32825
0.7	3.21387	2.38452	1.95898	3.52344	2.76416	2.38464
0.8	3.29590	2.43970	2.00000	3.61523	2.83179	2.43982
0.9	3.37598	2.49390	2.04016	3.70508	2.89795	2.49377
1.0	3.45410	2.54688	2.07959	3.79297	2.96265	2.54675
1.1	3.53125	2.59863	2.11841	3.87891	3.02588	2.59875
1.2	3.60645	2.64966	2.15649	3.96289	3.08813	2.64966
1.3	3.67969	2.69971	2.19397	4.04492	3.14917	2.69971
1.4	3.75195	2.74902	2.23083	4.12500	3.20801	2.74890
1.5	3.82324	2.79736	2.26709	4.20410	3.26782	2.79724
1.6	3.89258	2.84473	2.30273	4.28223	3.32568	2.84485
1.7	3.96094	2.89136	2.33789	4.35840	3.38257	2.89136
1.8	4.02832	2.93774	2.37256	4.43359	3.43848	2.93774
1.9	4.09473	2.98315	2.40674	4.50781	3.49365	2.98315
2.0	4.16016	3.02783	2.44043	4.58008	3.54785	3.02783
2.1	4.22461	3.07178	2.47375	4.65234	3.60132	3.07190
2.2	4.28711	3.11548	2.50659	4.72266	3.65381	3.11548
2.3	4.34861	3.15820	2.53845	4.79199	3.70581	3.15833
2.4	4.41211	3.20068	2.57104	4.86035	3.75708	3.20068
2.5	4.47266	3.24243	2.60254	4.92773	3.80762	3.24243
2.6	4.53223	3.28369	2.63379	4.99414	3.85767	3.28369
2.7	4.59180	3.32446	2.66479	5.06055	3.90698	3.32446
2.8	4.65039	3.36475	2.69531	5.12500	3.95557	3.36475
2.9	4.70703	3.40454	2.72546	5.18945	4.00366	3.40454
3.0	4.76465	3.44385	2.75537	5.25293	4.05127	3.44397

*Entries for κ^2 = 0.0 are the usual Brownian motion critical values.

Table A.1 (Contd.). Brief Table of Brownian Motion Critical Values Corrected for the Factor κ^2

	One Sided Critical Region			Two Sided Critical Region		
α-Level / κ^2	0.01	0.05	0.10	0.01	0.05	0.10
3.1	4.82031	3.48291	2.78491	5.31641	4.09814	3.48291
3.2	4.87695	3.52148	2.81421	5.37793	4.14478	3.52148
3.3	4.93164	3.55957	2.84302	5.43945	4.19067	3.55957
3.4	4.98633	3.59717	2.87183	5.50000	4.23608	3.59717
3.5	5.04004	3.63452	2.90015	5.55957	4.28125	3.63452
3.6	5.09375	3.67139	2.92822	5.61914	4.32568	3.67139
3.7	5.14648	3.70801	2.95605	5.67773	4.36963	3.70801
3.8	5.19824	3.74414	2.98364	5.73535	4.41333	3.74414
3.9	5.25000	3.78027	3.01099	5.79297	4.45654	3.78003
4.0	5.30078	3.81543	3.03809	5.84961	4.49951	3.81567

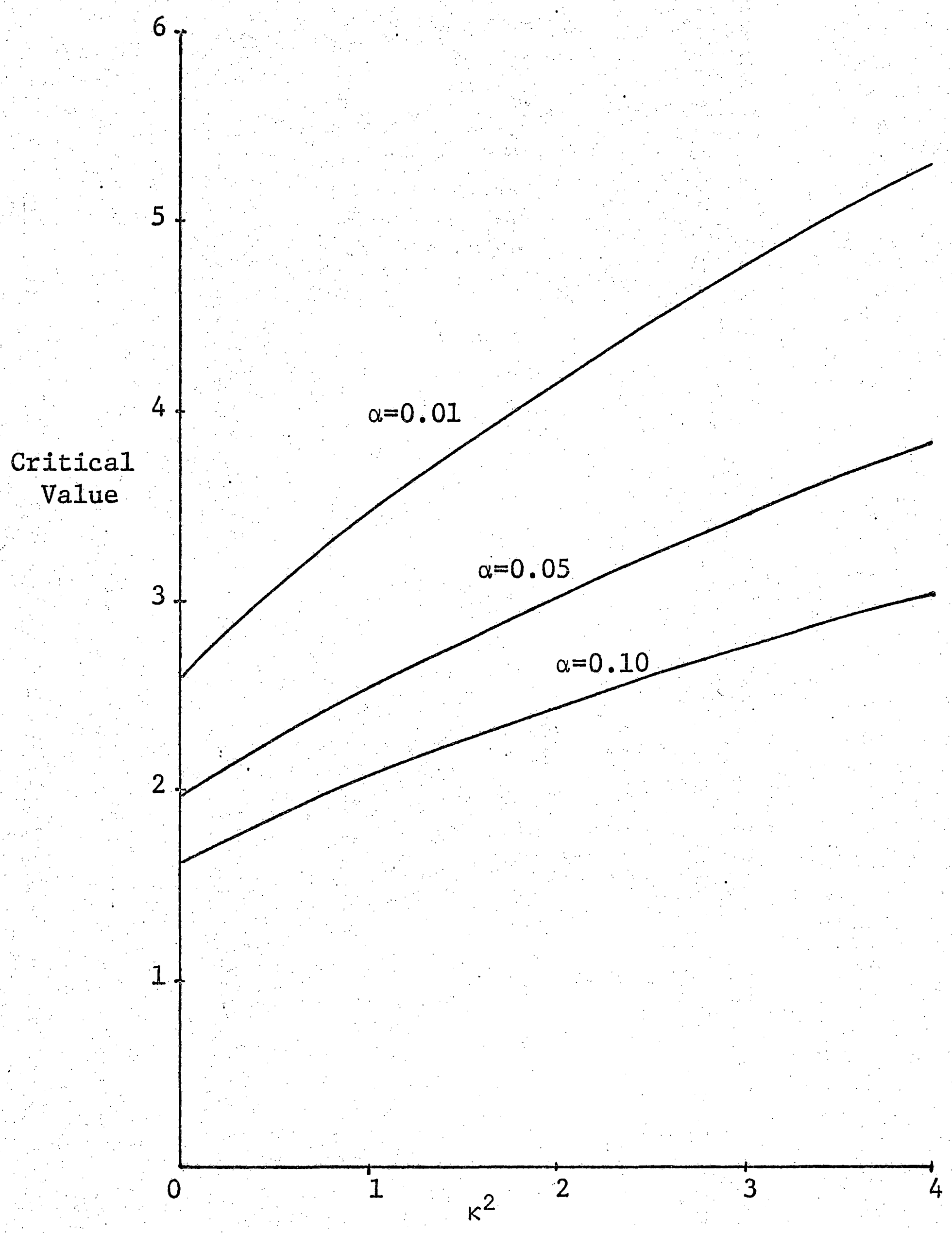


Figure A.1. Graph of Brownian Motion Critical Values Corrected for κ^2 One Sided Case.

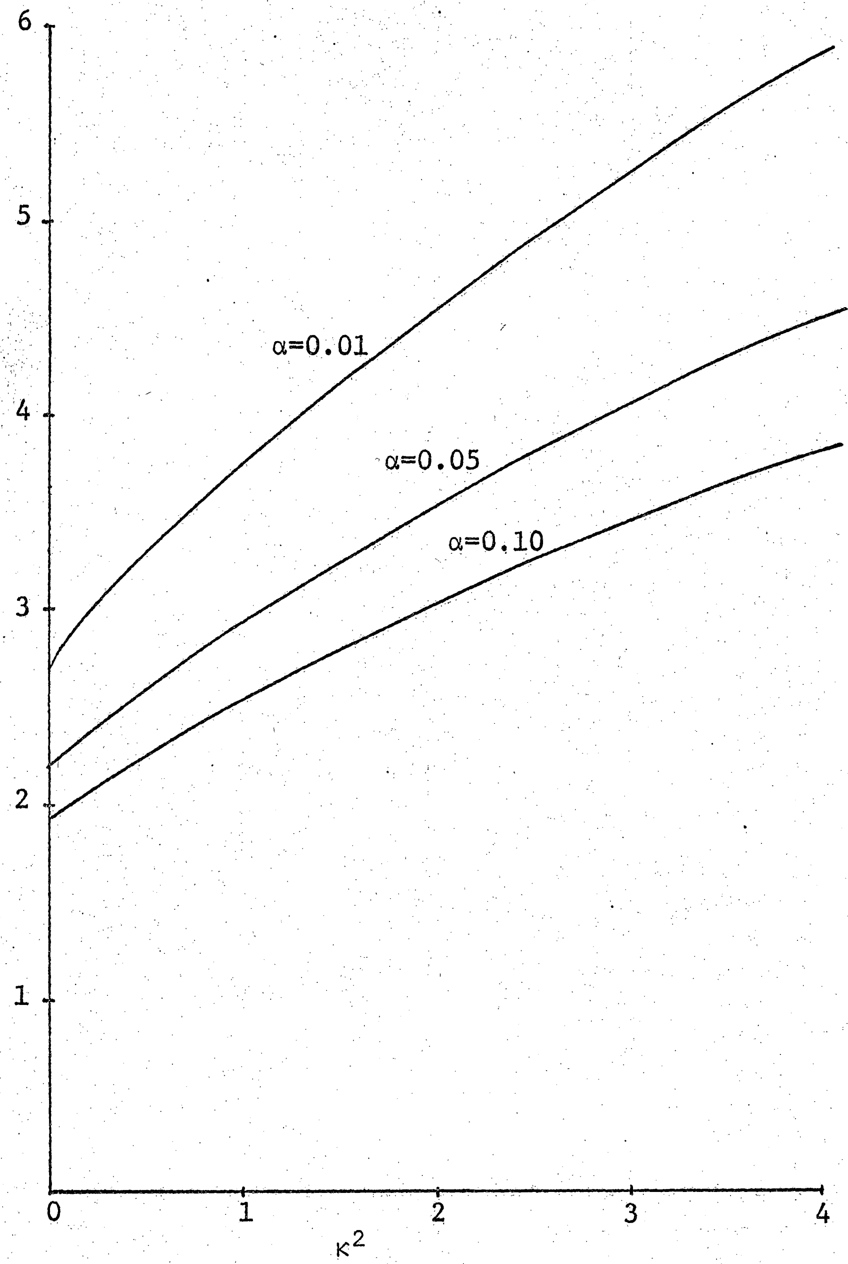


Figure A.2. Graph of Brownian Motion Critical Values Corrected for κ^2 Two Sided Case.

Table A.2. Brownian Motion Critical Values Corrected For The Factor κ^2

α-Level / κ^2	One Sided Critical Region 0.01	0.05	0.10	Two Sided Critical Region 0.01	0.05	0.10
0.0*	2.57617	1.95996	1.64490	2.80713	2.24146	1.95996
0.01	2.58594	1.96655	1.64978	2.81836	2.24963	1.96655
0.02	2.59570	1.97314	1.65460	2.82959	2.25781	1.97321
0.03	2.60596	1.97974	1.65942	2.84082	2.26587	1.97974
0.04	2.61572	1.98633	1.66431	2.85205	2.27405	1.98633
0.05	2.62598	1.99292	1.66919	2.86328	2.28210	1.99292
0.06	2.63574	1.99951	1.67395	2.87451	2.29016	1.99939
0.07	2.64551	2.00586	1.67877	2.88574	2.29822	2.00598
0.08	2.65527	2.01245	1.68359	2.89648	2.30627	2.01245
0.09	2.66504	2.01892	1.68835	2.90771	2.31421	2.01892
0.10	2.67480	2.02539	1.69312	2.91895	2.32227	2.02539
0.11	2.68457	2.03174	1.69788	2.92969	2.33020	2.03186
0.12	2.69434	2.03833	1.70264	2.94043	2.33813	2.03821
0.13	2.70410	2.04468	1.70734	2.95166	2.34607	2.04468
0.14	2.71387	2.05103	1.71204	2.96240	2.35400	2.05103
0.15	2.72363	2.05737	1.71680	2.97314	2.36182	2.05737
0.16	2.73291	2.06372	1.72144	2.98438	2.36963	2.06378
0.17	2.74268	2.07007	1.72607	2.99512	2.37744	2.07007
0.18	2.75195	2.07642	1.73083	3.00586	2.38525	2.07642
0.19	2.76172	2.08276	1.73547	3.01660	2.39307	2.08276
0.20	2.77148	2.08911	1.74011	3.02686	2.40088	2.08899
0.21	2.78027	2.09521	1.74475	3.03760	2.40857	2.09534
0.22	2.79004	2.10156	1.74939	3.04834	2.41626	2.10156
0.23	2.79980	2.10779	1.75391	3.05859	2.42395	2.10779
0.24	2.80859	2.11401	1.75854	3.06934	2.43164	2.11401
0.25	2.81836	2.12024	1.76318	3.08008	2.43921	2.12024
0.26	2.82715	2.12646	1.78770	3.09033	2.44690	2.12646
0.27	2.83691	2.13257	1.77222	3.10059	2.45459	2.13257
0.28	2.84570	2.13879	1.77686	3.11133	2.46216	2.13879
0.29	2.85547	2.14490	1.78137	3.12158	2.46973	2.14490
0.30	2.86426	2.15112	1.78589	3.13184	2.47729	2.15100

*Entries for $\kappa^2 = 0.0$ are the usual uncorrected Brownian motion critical values.

Table A.2 (Contd.). Brownian Motion Critical Values Corrected For the Factor κ^2

α-Level / κ^2	One Sided Critical Region 0.01	0.05	0.10	Two Sided Critical Region 0.01	0.05	0.10
0.31	2.87402	2.15723	1.79041	3.14258	2.48486	2.15717
0.32	2.88281	2.16333	1.79492	3.15234	2.49231	2.16327
0.33	2.89209	2.16943	1.79944	3.16309	2.49976	2.16931
0.34	2.90137	2.17542	1.80396	3.17285	2.50732	2.17542
0.35	2.91016	2.18140	1.80835	3.18359	2.51465	2.18152
0.36	2.91895	2.18750	1.81287	3.19336	2.52222	2.18750
0.37	2.92871	2.19360	1.81726	3.20361	2.52954	2.19354
0.38	2.93750	2.19946	1.82178	3.21387	2.53699	2.19958
0.39	2.94629	2.20557	1.82617	3.22363	2.54431	2.20557
0.40	2.95508	2.21143	1.83057	3.23389	2.55176	2.21155
0.41	2.96436	2.21753	1.83508	3.24414	2.55908	2.21753
0.42	2.97363	2.22339	1.83948	3.25391	2.56641	2.22345
0.43	2.98242	2.22937	1.84387	3.26367	2.57373	2.22937
0.44	2.99121	2.23535	1.84827	3.27393	2.58093	2.23535
0.45	3.00000	2.24121	1.85254	3.28418	2.58813	2.24121
0.46	3.00879	2.24707	1.85693	3.29395	2.59546	2.24707
0.47	3.01758	2.25293	1.86133	3.30371	2.60266	2.25299
0.48	3.02637	2.25879	1.86560	3.31348	2.60986	2.25885
0.49	3.03516	2.26465	1.87000	3.32324	2.61707	2.26471
0.50	3.04395	2.27051	1.87427	3.33301	2.62427	2.27051
0.51	3.05273	2.27637	1.87866	3.34277	2.63135	2.27637
0.52	3.06152	2.28223	1.88293	3.35254	2.63855	2.28223
0.53	3.06982	2.28809	1.88721	3.36230	2.64575	2.28796
0.54	3.07813	2.29370	1.89148	3.37207	2.65283	2.29382
0.55	3.08691	2.29956	1.89575	3.38184	2.65991	2.29956
0.56	3.09570	2.30530	1.90002	3.39160	2.66699	2.30530
0.57	3.10449	2.31104	1.90430	3.40137	2.67407	2.31104
0.58	3.11279	2.31689	1.90857	3.41016	2.68103	2.31677
0.59	3.12109	2.32251	1.91284	3.41992	2.68799	2.32251
0.60	3.12988	2.32813	1.91711	3.42969	2.69507	2.32825
0.61	3.13867	2.33398	1.92126	3.43945	2.70203	2.33386
0.62	3.14648	2.33960	1.92554	3.44873	2.70898	2.33960
0.63	3.15527	2.34521	1.92969	3.45801	2.71594	2.34521
0.64	3.16406	2.35083	1.93396	3.46777	2.72290	2.35095
0.65	3.17188	2.35657	1.93811	3.47705	2.72974	2.35657

Table A.2 (Contd.). Brownian Motion Critical Values Corrected For the Factor κ^2

α-Level / κ^2	One Sided Critical Region 0.01	0.05	0.10	Two Sided Critical Region 0.01	0.05	0.10
0.66	3.18066	2.36230	1.94226	3.48633	2.73682	2.36218
0.67	3.18848	2.36792	1.94641	3.49609	2.74365	2.36780
0.68	3.19727	2.37354	1.95068	3.50488	2.75049	2.37341
0.69	3.20508	2.37891	1.95483	3.51465	2.75732	2.37903
0.70	3.21387	2.38452	1.95898	3.52344	2.76416	2.38464
0.71	3.22168	2.39014	1.96301	3.53320	2.77100	2.39014
0.72	3.23047	2.39575	1.96716	3.54199	2.77783	2.39575
0.73	3.23828	2.40137	1.97131	3.55176	2.78467	2.40125
0.74	3.24707	2.40674	1.97546	3.56055	2.79138	2.40686
0.75	3.25488	2.41235	1.97949	3.57031	2.79810	2.41235
0.76	3.26367	2.41785	1.98364	3.57910	2.80493	2.41785
0.77	3.27148	2.42334	1.98779	3.58789	2.81165	2.42334
0.78	3.27930	2.42883	1.99182	3.59766	2.81836	2.42883
0.79	3.28809	2.43433	1.99585	3.60645	2.82495	2.43433
0.80	3.29590	2.43970	2.00000	3.61523	2.83179	2.43982
0.81	3.30371	2.44531	2.00403	3.62500	2.83838	2.44519
0.82	3.31152	2.45068	2.00806	3.63379	2.84509	2.45068
0.83	3.32031	2.45605	2.01208	3.64258	2.85181	2.45605
0.84	3.32813	2.46143	2.01611	3.65137	2.85840	2.46155
0.85	3.33594	2.46704	2.02014	3.66016	2.86499	2.46692
0.86	3.34375	2.47241	2.02417	3.66992	2.87158	2.47229
0.87	3.35254	2.47778	2.02820	3.67871	2.87817	2.47766
0.88	3.36035	2.48315	2.03223	3.68750	2.88477	2.48303
0.89	3.36816	2.48853	2.03613	3.69629	2.89136	2.48840
0.90	3.37598	2.49390	2.04016	3.70508	2.89795	2.49377
0.91	3.38379	2.49927	2.04419	3.71387	2.90430	2.49915
0.92	3.39160	2.50439	2.04810	3.72266	2.91089	2.50452
0.93	3.39941	2.50977	2.05212	3.73145	2.91748	2.50977
0.94	3.40723	2.51514	2.05603	3.74023	2.92383	2.51514
0.95	3.41504	2.52051	2.06006	3.74902	2.93042	2.52039
0.96	3.42285	2.52563	2.06396	3.75781	2.93677	2.52576
0.97	3.43066	2.53101	2.06787	3.76563	2.94336	2.53101
0.98	3.43848	2.53638	2.07178	3.77439	2.94971	2.53625
0.99	3.44629	2.54150	2.07568	3.78223	2.95605	2.54150
1.00	3.45410	2.54688	2.07959	3.79297	2.96265	2.54675

Table A.2 (Contd.). Brownian Motion Critical Values Corrected For the Factor κ^2

κ^2 \ α-Level	One Sided Critical Region 0.01	0.05	0.10	Two Sided Critical Region 0.01	0.05	0.10
1.01	3.46191	2.55200	2.08350	3.80176	2.96899	2.55200
1.02	3.46973	2.55713	2.08740	3.81055	2.97534	2.55725
1.03	3.47754	2.56250	2.09131	3.81836	2.98169	2.56250
1.04	3.48535	2.56763	2.09521	3.82715	2.98804	2.56763
1.05	3.49316	2.57275	2.09912	3.83594	2.99438	2.57288
1.06	3.50000	2.57813	2.10303	3.84473	3.00073	2.57800
1.07	3.50781	2.58325	2.10693	3.85254	3.00708	2.58325
1.08	3.51563	2.58838	2.11072	3.86133	3.01343	2.58838
1.09	3.52344	2.59351	2.11450	3.87012	3.01978	2.59351
1.10	3.53125	2.59863	2.11841	3.87891	3.02588	2.59875
1.11	3.53809	2.60376	2.12231	3.88672	3.03223	2.60388
1.12	3.54590	2.60889	2.12610	3.89551	3.03857	2.60901
1.13	3.55273	2.61401	2.12988	3.90430	3.04468	2.61414
1.14	3.56055	2.61914	2.13379	3.91211	3.05103	2.61926
1.15	3.56836	2.62427	2.13757	3.92090	3.05713	2.62427
1.16	3.57617	2.62939	2.14136	3.92871	3.06348	2.62939
1.17	3.58398	2.63452	2.14514	3.93750	3.06958	2.63452
1.18	3.59082	2.63965	2.14893	3.94531	3.07568	2.63953
1.19	3.59863	2.64453	2.15271	3.95410	3.08203	2.64465
1.20	3.60645	2.64966	2.15649	3.96289	3.08813	2.64966
1.21	3.61328	2.65479	2.16028	3.97070	3.09424	2.65479
1.22	3.62109	2.65967	2.16406	3.97949	3.10034	2.65979
1.23	3.62891	2.66479	2.16785	3.98730	3.10645	2.66479
1.24	3.63574	2.66992	2.17163	3.99512	3.11267	2.66980
1.25	3.64355	2.67480	2.17529	4.00391	3.11865	2.67480
1.26	3.65039	2.67969	2.17908	4.01172	3.12476	2.67981
1.27	3.65820	2.68481	2.18286	4.02051	3.13086	2.68481
1.28	3.66504	2.68970	2.18652	4.02832	3.13696	2.68982
1.29	3.78285	2.69482	2.19019	4.03613	3.14307	2.69482
1.30	3.67969	2.69971	2.19397	4.04492	3.14917	2.69971
1.31	3.68750	2.70459	2.19775	4.05273	3.15503	2.70471
1.32	3.69434	2.70972	2.20142	4.06055	3.16113	2.70959
1.33	3.70117	2.71460	2.20508	4.06934	3.17724	2.71460
1.34	3.70898	2.71948	2.20874	4.07715	3.17310	2.71948
1.35	3.71582	2.72437	2.21240	4.08496	3.17908	2.72437

Table A.2 (Contd.). Brownian Motion Critical Values Corrected For the Factor κ^2

α-Level / κ^2	One Sided Critical Region 0.01	0.05	0.10	Two Sided Critical Region 0.01	0.05	0.10
1.36	3.72363	2.72925	2.21619	4.09375	3.18494	2.72937
1.37	3.73047	2.73438	2.21985	4.10156	3.19080	2.73425
1.38	3.73828	2.73926	2.22351	4.10938	3.19675	2.73914
1.39	3.74512	2.74414	2.22717	4.11719	3.20258	2.74402
1.40	3.75195	2.74902	2.23083	4.12500	3.20801	2.74890
1.41	3.75977	2.75391	2.23450	4.13379	3.21442	2.75378
1.42	3.76660	2.75879	2.23804	4.14160	3.22070	2.75867
1.43	3.77344	2.76343	2.24170	4.14941	3.22461	2.76355
1.44	3.78027	2.76831	2.24536	4.15723	3.23267	2.76831
1.45	3.78809	2.77319	2.24902	4.16504	3.23853	2.77319
1.46	3.79492	2.77808	2.25269	4.17285	3.24438	2.77808
1.47	3.80176	2.78271	2.25623	4.18066	3.25024	2.78284
1.48	3.80859	2.78760	2.25989	4.18848	3.25610	2.78760
1.49	3.81641	2.79248	2.26343	4.19629	3.26196	2.79248
1.50	3.82324	2.79736	2.26709	4.20410	3.26782	2.79724
1.51	3.83008	2.80200	2.27063	4.21289	3.27368	2.80200
1.52	3.83691	2.80688	2.27429	4.22070	3.27954	2.80688
1.53	3.84375	2.81152	2.27783	4.22852	3.28540	2.81165
1.54	3.85156	2.81641	2.28149	4.23633	3.29102	2.81641
1.55	3.85840	2.82104	2.28491	4.24316	3.29688	2.82117
1.56	3.86523	2.82593	2.28857	4.25098	3.30273	2.82593
1.57	3.87207	2.83057	2.29211	4.25879	3.30835	2.83057
1.58	3.87891	2.83545	2.29565	4.26660	3.31421	2.83533
1.59	3.88574	2.84009	2.29919	4.27441	3.31982	2.84009
1.60	3.89258	2.84473	2.30273	4.28223	3.32568	2.84485
1.61	3.89941	2.84961	2.30640	4.29004	3.33130	2.84949
1.62	3.90625	2.85425	2.30981	4.29785	3.33716	2.85425
1.63	3.91309	2.85889	2.31335	4.30566	3.34277	2.85889
1.64	3.91992	2.86353	2.31689	4.31250	3.34863	2.86359
1.65	3.92676	2.86816	2.32043	4.32031	3.35425	2.86829
1.66	3.93359	2.87305	2.32397	4.32813	3.35986	2.87292
1.67	3.94043	2.87769	2.32739	4.33594	3.36548	2.87769
1.68	3.94727	2.88232	2.33093	4.34375	3.37134	2.88232
1.69	3.95410	2.88672	2.33447	4.35156	3.37695	2.88672
1.70	3.96094	2.89136	2.33789	4.35840	3.38257	2.89136

Table A.2 (Contd.). Brownian Motion Critical Values Corrected For the Factor κ^2

α-Level	One Sided Critical Region			Two Sided Critical Region		
κ^2	0.01	0.05	0.10	0.01	0.05	0.10
1.71	3.96777	2.89600	2.34143	4.36621	3.38818	2.89600
1.72	3.97461	2.90015	2.34497	4.37402	3.39380	2.90015
1.73	3.98145	2.90511	2.34839	4.38086	3.39941	2.90511
1.74	3.98828	2.90942	2.35181	4.38867	3.40503	2.90961
1.75	3.99512	2.91384	2.35535	4.39648	3.41064	2.91384
1.76	4.00195	2.91825	2.35889	4.40430	3.41626	2.91825
1.77	4.00879	2.92268	2.36230	4.41113	3.42188	2.92268
1.78	4.01563	2.92847	2.36572	4.41895	3.42725	2.92859
1.79	4.02148	2.93311	2.36914	4.42578	3.43286	2.93311
1.80	4.02832	2.93774	2.37256	4.43359	3.43848	2.93774
1.81	4.03516	2.94238	2.37598	4.44141	3.44409	2.94226
1.82	4.04199	2.94678	2.37952	4.44824	3.44946	2.94678
1.83	4.04883	2.95142	2.38293	4.45605	3.45508	2.95142
1.84	4.05469	2.95605	2.38635	4.46387	3.46069	2.95593
1.85	4.06152	2.96045	2.38977	4.47070	3.46606	2.96045
1.86	4.06836	2.96509	2.39319	4.47852	3.47168	2.96509
1.87	4.07520	2.96948	2.39661	4.48535	3.47705	2.96948
1.88	4.08203	2.97412	2.40002	4.49316	3.48267	2.97412
1.89	4.08789	2.97852	2.40332	4.50000	3.48804	2.97864
1.90	4.09473	2.98315	2.40674	4.50781	3.49365	2.98315
1.91	4.10156	2.98755	2.41016	4.51465	3.49902	2.98755
1.92	4.10742	2.99219	2.41357	4.52246	3.50439	2.99207
1.93	4.11426	2.99658	2.41699	4.52930	3.51001	2.99658
1.94	4.12109	3.00098	2.42041	4.53711	3.51538	3.00110
1.95	4.12793	3.00562	2.42371	4.54395	3.52075	3.00562
1.96	4.13477	3.01001	2.42700	4.55176	3.52612	3.01001
1.97	4.14063	3.01440	2.43042	4.55859	3.53174	3.01453
1.98	4.14746	3.01904	2.43384	4.56641	3.53711	3.01892
1.99	4.15332	3.02344	2.43713	4.57324	3.54248	3.02344
2.00	4.16016	3.02783	2.44043	4.58008	3.54785	3.02783
2.01	4.16602	3.03223	2.44385	4.58789	3.55322	3.03223
2.02	4.17285	3.03662	2.44727	4.59473	3.55859	3.03674
2.03	4.17969	3.04102	2.45044	4.60156	3.56396	3.04114
2.04	4.18555	3.04541	2.45386	4.60938	3.56934	3.04553
2.05	4.19238	3.04980	2.45715	4.61621	3.57471	3.04993

Table A.2 (Contd.). Brownian Motion Critical Values Corrected For the Factor κ^2

α-Level / κ^2	One Sided Critical Region 0.01	0.05	0.10	Two Sided Critical Region 0.01	0.05	0.10
2.06	4.19922	3.05444	2.46045	4.62305	3.58008	3.05444
2.07	4.20508	3.05884	2.46387	4.63086	3.58545	3.05884
2.08	4.21191	3.06323	2.46716	4.63770	3.59058	3.06323
2.09	4.21777	3.06763	2.47046	4.64453	3.59595	3.06763
2.10	4.22461	3.07178	2.47375	4.65234	3.60132	3.07190
2.11	4.23047	3.07617	2.47705	4.65918	3.60645	3.07629
2.12	4.23730	3.08057	2.48035	4.66602	3.61182	3.08069
2.13	4.24316	3.08496	2.48364	4.67285	3.61719	3.08496
2.14	4.25000	3.08936	2.48694	4.67969	3.62256	3.08936
2.15	4.25586	3.09375	2.49023	4.68750	3.62769	3.09375
2.16	4.26270	3.09814	2.49341	4.69434	3.63306	3.09814
2.17	4.26855	3.10254	2.49683	4.70117	3.63818	3.10242
2.18	4.27539	3.10669	2.50000	4.70898	3.64355	3.10669
2.19	4.28125	3.11108	2.50317	4.71582	3.64868	3.11108
2.20	4.28711	3.11548	2.50659	4.72266	3.65381	3.11548
2.21	4.29395	3.11963	2.50977	4.72949	3.65918	3.11975
2.22	4.30078	3.12402	2.51294	4.73633	3.66431	3.12402
2.23	4.30664	3.12842	2.51611	4.74316	3.66968	3.12842
2.24	4.31250	3.13257	2.51941	4.75000	3.67480	3.13257
2.25	4.31934	3.13696	2.52271	4.75781	3.67993	3.13696
2.26	4.32520	3.14111	2.52582	4.76465	3.68506	3.14124
2.27	4.33203	3.14551	2.52930	4.77148	3.69043	3.14551
2.28	4.33789	3.14990	2.53235	4.77832	3.69556	3.14978
2.29	4.34375	3.15405	2.53516	4.78516	3.70068	3.15405
2.30	4.34961	3.15820	2.53845	4.79199	3.70581	3.15833
2.31	4.35645	3.16260	2.54199	4.79883	3.71094	3.16260
2.32	4.36230	3.16699	2.54439	4.80566	3.71631	3.16675
2.33	4.36914	3.17114	2.54785	4.81250	3.72144	3.17114
2.34	4.37500	3.17529	2.55099	4.81934	3.72656	3.17529
2.35	4.38086	3.17969	2.55473	4.82617	3.73169	3.16957
2.36	4.38672	3.18384	2.55560	4.83301	3.73682	3.18384
2.37	4.39355	3.18799	2.56152	4.83984	3.74170	3.18799
2.38	4.39941	3.19238	2.56470	4.84668	3.74707	3.19226
2.39	4.40527	3.19629	2.56787	4.85352	3.75195	3.19641
2.40	4.41211	3.20068	2.57104	4.86035	3.75708	3.20068

Table A.2 (Contd.). Brownian Motion Critical Values Corrected For the Factor κ^2

	One Sided Critical Region			Two Sided Critical Region		
α-Level / κ^2	0.01	0.05	0.10	0.01	0.05	0.10
2.41	4.41797	3.20483	2.57422	4.86719	3.76221	3.20483
2.42	4.42383	3.20898	2.57739	4.87402	3.76733	3.20898
2.43	4.42969	3.21338	2.58057	4.88086	3.77246	3.21326
2.44	4.43555	3.21729	2.58374	4.88770	3.77734	3.21741
2.45	4.44238	3.22168	2.58691	4.89453	3.78247	3.22168
2.46	4.44824	3.22583	2.59009	4.90137	3.78760	3.22583
2.47	4.45410	3.22998	2.59326	4.90820	3.79248	3.22998
2.48	4.46094	3.23413	2.59631	4.91406	3.79761	3.23413
2.49	4.46680	3.23828	2.59949	4.92188	3.80273	3.23828
2.50	4.47266	3.24243	2.60254	4.92773	3.80762	3.24243
2.51	4.47852	3.24658	2.60571	4.93457	3.81274	3.24658
2.52	4.48438	3.25073	2.60889	4.94141	3.81787	3.25073
2.53	4.49023	3.25488	2.61206	4.94824	3.82275	3.25488
2.54	4.49609	3.25903	2.61523	4.95508	3.82764	3.25903
2.55	4.50195	3.26318	2.61829	4.96094	3.83276	3.26318
2.56	4.50781	3.26733	2.62134	4.96875	3.83789	3.26733
2.57	4.51465	3.27148	2.62451	4.97461	3.84277	3.27148
2.58	4.52051	3.27539	2.62769	4.98145	3.84766	3.27551
2.59	4.52637	3.27979	2.63074	4.98828	3.85254	3.27954
2.60	4.53223	3.28369	2.63379	4.99414	3.85767	3.28369
2.61	4.53809	3.28784	2.63696	5.00195	3.86255	3.28784
2.62	4.54395	3.29199	2.64014	5.00781	3.86768	3.29199
2.63	4.55078	3.29590	2.64307	5.01465	3.87256	3.29602
2.64	4.55664	3.30005	2.64624	5.02148	3.87744	3.30005
2.65	4.56250	3.30420	2.64941	5.02734	3.88232	3.30420
2.66	4.56836	3.30811	2.65234	5.03418	3.88721	3.30823
2.67	4.57422	3.31226	2.65552	5.04102	3.89209	3.31226
2.68	4.58008	3.31641	2.65869	5.04688	3.89697	3.31641
2.69	4.58594	3.32031	2.66162	5.05371	3.90210	3.32043
2.70	4.59180	3.32446	2.66479	5.06055	3.90698	3.32446
2.71	4.59766	3.32861	2.66772	5.06738	3.91187	3.32861
2.72	4.60352	3.33252	2.67090	5.07422	3.91675	3.33252
2.73	4.60938	3.33667	2.67395	5.08008	3.92163	3.33667
2.74	4.61523	3.34082	2.67700	5.08691	3.92651	3.34070
2.75	4.62109	3.34473	2.68005	5.09375	3.93140	3.34473

Table A.2 (Contd.). Brownian Motion Critical Values Corrected For the Factor κ^2

α-Level / κ^2	One Sided Critical Region 0.01	0.05	0.10	Two Sided Critical Region 0.01	0.05	0.10
2.76	4.62695	3.34863	2.68311	5.09961	3.93628	3.34875
2.77	4.63281	3.35278	2.68616	5.10645	3.94092	3.35278
2.78	4.63867	3.35693	2.68921	5.11230	3.94580	3.35669
2.79	4.64453	3.36084	2.69214	5.11914	3.95068	3.36084
2.80	4.65039	3.36475	2.69531	5.12500	3.95557	3.36475
2.81	4.65625	3.36865	2.69824	5.13184	3.96045	3.36877
2.82	4.66211	3.37280	2.70142	5.13867	3.96533	3.37280
2.83	4.66699	3.37695	2.70435	5.14453	3.97021	3.37671
2.84	4.67285	3.38086	2.70740	5.15137	3.97485	3.38074
2.85	4.67871	3.38477	2.71045	5.15820	3.97974	3.38477
2.86	4.68457	3.38867	2.71338	5.16406	3.98462	3.38867
2.87	4.69043	3.39258	2.71655	5.16992	3.98926	3.39270
2.88	4.69531	3.39673	2.71948	5.17676	3.99414	3.39673
2.89	4.70117	3.40063	2.72241	5.18359	3.99902	3.40063
2.90	4.70703	3.40454	2.72546	5.18945	4.00366	3.40454
2.91	4.71289	3.40869	2.72852	5.19629	4.00854	3.40857
2.92	4.71875	3.41260	2.73145	5.20215	4.01318	3.41260
2.93	4.72461	3.41650	2.73438	5.20898	4.01807	3.41650
2.94	4.73047	3.42041	2.73755	5.21484	4.02295	3.42041
2.95	4.73633	3.42432	2.74048	5.22168	4.02759	3.42432
2.96	4.74219	3.42822	2.74341	5.22754	4.03223	3.42822
2.97	4.74805	3.43213	2.74634	5.23438	4.03711	3.43213
2.98	4.75293	3.43604	2.74939	5.24023	4.04175	3.43616
2.99	4.75879	3.43994	2.75244	5.24707	4.04639	3.44006
3.00	4.76465	3.44385	2.75537	5.25293	4.05127	3.44397
3.01	4.76953	3.44775	2.75830	5.25977	4.05591	3.44788
3.02	4.77539	3.45166	2.76123	5.26563	4.06055	3.45178
3.03	4.78125	3.45557	2.76416	5.27148	4.06543	3.45569
3.04	4.78711	3.45947	2.76721	5.27832	4.07007	3.45959
3.05	4.79297	3.46338	2.77026	5.28516	4.07471	3.46350
3.06	4.79883	3.46729	2.77319	5.29102	4.07959	3.46729
3.07	4.80469	3.47119	2.77612	5.29688	4.08423	3.47119
3.08	4.80957	3.47510	2.77905	5.30273	4.08887	3.47510
3.09	4.81543	3.47900	2.78198	5.30957	4.09351	3.47900
3.10	4.82031	3.48291	2.78491	5.31641	4.09814	3.48291

Table A.2 (Contd.). Brownian Motion Critical Values Corrected For the Factor κ^2

α-Level / κ^2	One Sided Critical Region 0.01	One Sided Critical Region 0.05	One Sided Critical Region 0.10	Two Sided Critical Region 0.01	Two Sided Critical Region 0.05	Two Sided Critical Region 0.10
3.11	4.82617	3.48682	2.78784	5.32227	4.10303	3.48682
3.12	4.83203	3.49072	2.79077	5.32813	4.10742	3.49060
3.13	4.83789	3.49463	2.79370	5.33398	4.11230	3.49451
3.14	4.84375	3.49829	2.79663	5.34082	4.11694	3.49829
3.15	4.84863	3.50220	2.79956	5.34668	4.12158	3.50220
3.16	4.85449	3.50586	2.80249	5.35352	4.12622	3.50610
3.17	4.85938	3.50977	2.80542	5.35938	4.13086	3.50989
3.18	4.86523	3.51367	2.80835	5.36523	4.13550	3.51367
3.19	4.87109	3.51758	2.81128	5.37109	4.14014	3.51758
3.20	4.87695	3.52148	2.81421	5.37793	4.14478	3.52148
3.21	4.88184	3.52539	2.81714	5.38379	4.14941	3.52515
3.22	4.88770	3.52905	2.82007	5.39063	4.15381	3.52905
3.23	4.89258	3.53271	2.82288	5.39648	4.15869	3.53284
3.24	4.89844	3.53662	2.82568	5.40234	4.16309	3.53662
3.25	4.90430	3.54053	2.82861	5.40820	4.16772	3.54053
3.26	4.91016	3.54443	2.83154	5.41406	4.17236	3.54431
3.27	4.91504	3.54810	2.83447	5.41992	4.17700	3.54810
3.28	4.91992	3.55176	2.83740	5.42676	4.18164	3.55188
3.29	4.92578	3.55566	2.84021	5.43262	4.18604	3.55566
3.30	4.93164	3.55957	2.84302	5.43945	4.19067	3.55957
3.31	4.93750	3.56323	2.84595	5.44531	4.19531	3.56323
3.32	4.94238	3.56689	2.84888	5.45117	4.19971	3.56714
3.33	4.94727	3.57080	2.85181	5.45703	4.20435	3.57080
3.34	4.95313	3.57471	2.85449	5.46289	4.20898	3.57458
3.35	4.95898	3.57837	2.85742	5.46875	4.21338	3.57837
3.36	4.96484	3.58203	2.86035	5.47461	4.21802	3.58215
3.37	4.96973	3.58594	2.86328	5.48145	4.22266	3.58594
3.38	4.97461	3.58984	2.86597	5.48730	4.22705	3.58960
3.39	4.98047	3.59326	2.86890	5.49316	4.23169	3.59351
3.40	4.98633	3.59717	2.87183	5.50000	4.23608	3.59717
3.41	4.99121	3.60107	2.87451	5.50586	4.24072	3.60083
3.42	4.99609	3.60449	2.87744	5.51172	4.24512	3.60474
3.43	5.00195	3.60840	2.88037	5.51758	4.24976	3.60840
3.44	5.00781	3.61230	2.88306	5.52344	4.25415	3.61206
3.45	5.01367	3.61572	2.88599	5.52930	4.25879	3.61597

Table A.2 (Contd.). Brownian Motion Critical Values Corrected For the Factor κ^2

α-Level / κ^2	One Sided Critical Region 0.01	0.05	0.10	Two Sided Critical Region 0.01	0.05	0.10
3.46	5.01855	3.61963	2.88892	5.53516	4.26318	3.61963
3.47	5.02344	3.62329	2.89160	5.54102	4.26758	3.62329
3.48	5.02930	3.62695	2.89453	5.54688	4.27222	3.62695
3.49	5.03516	3.63086	2.89722	5.55273	4.27661	3.63086
3.50	5.04004	3.63452	2.90015	5.55957	4.28125	3.63452
3.51	5.04492	3.63818	2.90308	5.56543	4.28564	3.63818
3.52	5.05078	3.64209	2.90576	5.57129	4.29004	3.64185
3.53	5.05664	3.64551	2.90869	5.57715	4.29443	3.64551
3.54	5.06055	3.64941	2.91138	5.58301	4.29907	3.64929
3.55	5.06641	3.65283	2.91431	5.58887	4.30347	3.65308
3.56	5.07227	3.65674	2.91699	5.59473	4.30786	3.65674
3.57	5.07715	3.66040	2.91992	5.60156	4.31250	3.66040
3.58	5.08203	3.66406	2.92261	5.60742	4.31689	3.66406
3.59	5.08789	3.66772	2.92554	5.61328	4.32129	3.66772
3.60	5.09375	3.67139	2.92822	5.61914	4.32568	3.67139
3.61	5.09863	3.67505	2.93115	5.62500	4.33008	3.67505
3.62	5.10352	3.67871	2.93384	5.63086	4.33447	3.67871
3.63	5.10938	3.68262	2.93652	5.63672	4.33887	3.68237
3.64	5.11426	3.68604	2.93945	5.64258	4.34326	3.68604
3.65	5.11914	3.68994	2.94214	5.64844	4.34766	3.68970
3.66	5.12500	3.69336	2.94507	5.65430	4.35205	3.69336
3.67	5.13086	3.69702	2.94775	5.66016	4.35645	3.69702
3.68	5.13477	3.70068	2.95044	5.66602	4.36084	3.70068
3.69	5.14063	3.70435	2.95337	5.67188	4.36523	3.70435
3.70	5.14648	3.70801	2.95605	5.67773	4.36963	3.70801
3.71	5.15039	3.71143	2.95898	5.68359	4.37402	3.71167
3.72	5.15625	3.71533	2.96167	5.68945	4.37842	3.71533
3.73	5.16211	3.71875	2.96436	5.69531	4.38281	3.71887
3.74	5.16699	3.72266	2.96716	5.70117	4.38721	3.72241
3.75	5.17188	3.72607	2.96997	5.70703	4.39160	3.72607
3.76	5.17773	3.72974	2.97266	5.71191	4.39600	3.72974
3.77	5.18262	3.73340	2.97534	5.71777	4.40039	3.73340
3.78	5.18750	3.73682	2.97827	5.72363	4.40479	3.73706
3.79	5.19336	3.74072	2.98096	5.72949	4.40918	3.74060
3.80	5.19824	3.74414	2.98364	5.73535	4.41333	3.74414

Table A.2 (Contd.). Brownian Motion Critical Values Corrected For the Factor κ^2

α-Level / κ^2	One Sided Critical Region 0.01	0.05	0.10	Two Sided Critical Region 0.01	0.05	0.10
3.81	5.20313	3.74780	2.98633	5.74121	4.41772	3.74780
3.82	5.20898	3.75146	2.98926	5.74609	4.42212	3.75146
3.83	5.21387	3.75488	2.99194	5.75195	4.42627	3.75500
3.84	5.21875	3.75879	2.99463	5.75781	4.43066	3.75854
3.85	5.22461	3.76221	2.99731	5.76367	4.43506	3.76221
3.86	5.22852	3.76563	3.00012	5.76953	4.43945	3.76575
3.87	5.23438	3.76953	3.00293	5.77539	4.44385	3.76929
3.88	5.24023	3.77295	3.00562	5.78125	4.44800	3.77295
3.89	5.24414	3.77637	3.00830	5.78711	4.45215	3.77649
3.90	5.25000	3.78027	3.01099	5.79297	4.45654	3.78003
3.91	5.25488	3.78369	3.01367	5.79883	4.46094	3.78369
3.92	5.25977	3.78711	3.01636	5.80469	4.46533	3.78711
3.93	5.26563	3.79077	3.01929	5.81055	4.46948	3.79077
3.94	5.27051	3.79443	3.02197	5.81543	4.47363	3.79431
3.95	5.27539	3.79785	3.02466	5.82129	4.47803	3.79785
3.96	5.28125	3.80127	3.02734	5.82715	4.48242	3.80151
3.97	5.28516	3.80518	3.03003	5.83203	4.48657	3.80493
3.98	5.29102	3.80859	3.03271	5.83789	4.49072	3.80859
3.99	5.29590	3.81201	3.03540	5.84375	4.49512	3.81201
4.00	5.30078	3.81543	3.03809	5.84961	4.49951	3.81567

APPENDIX II

Listings of Subroutines Discussed in Chapter VIII

```
      IMPLICIT REAL*8 (A-H,O-Z)
C
C
C     THE EXACT EXPECTED SAMPLE SIZE IS CALCULATED FOR THE
C     CUSUM SIGN TEST FOR A SHIFT IN MEDIAN. DATA CARDS ARE
C     READ IN THE FOLLOWING ORDER AND FORMAT:
C       #1 ISTOP  1I5  PROGRAM CONTROL:
C          ISTOP=1 STOP THE PROGRAM;
C          ISTOP=2 SR PATH AND SR PATHM CALCULATE
C                  UNCONDITIONAL PROBABILITIES;
C          ISTOP=3 INDICATES THE ADJUSTED CRITICAL
C                  VALUE MODE.
C       #2 ALPHA  1F10.7  PUTATIVE STAGEWISE SIGNIFICANCE
C          LEVEL.
C       #3 MEST  1I5  INITIAL NUMBER OF OBSERVATIONS TO
C          ESTIMATE THE MEDIAN (MUST BE ODD).
C       #4 NTEST  1I5  NUMBER OF OBSERVATIONS FOR FIRST
C          STAGE TEST (MUST BE EVEN).
C       #5 KSIDE  1I5  SIDEDNESS OF TEST (1 OR 2).
C
C     SUBROUTINES AND FUNCTIONS REQUIRED:
C       PATH, CROSS, QUOTNT, CRITL, GAUSS7, CFL, CFGS.
C
C     PROGRAM WILL READ ANY NUMBER OF DATA SETS. LAST
C     DATA SET MUST BE A SINGLE CARD: ISTOP=1.
C
C     PRINTOUT IS STAGE BY STAGE LIST WITH:
C       STAGE NUMBER
C       CUMULATIVE EXPECTED SAMPLE SIZE (ESS) TO THAT STAGE
C       CONTRIBUTION OF STAGE TO ESS
C       ACTUAL ALPHA LEVEL OF STAGE
C       INTEGER CRITICAL VALUE KRV
```

```
C         CORRECTION FACTOR RKSQ
C         CORRECTED BROWNIAN MOTION CRITICAL VALUE WCRIT.
C
C
C       INDEX PROGRAM AND READ INPUT
C
C
C       PASTAC IS THE PROBABILITY OF ACCEPTING THE NULL
C       HYPOTHESIS THROUGH STAGE (NSTAGE-1).
   10 PASTAC=1.0D0
C
      READ(5,100) ISTOP
  100 FORMAT(1I5)
      IF(ISTOP.EQ.1) GO TO 1000
C
      READ(5,101) ALPHA
  101 FORMAT(1F10.7)
C
      READ(5,100) MEST
      RMEST=DFLOAT(MEST)
C
C       NSUM IS THE NUMBER OF OBSERVATIONS TAKEN THROUGH
C       STAGE (NSTAGE-1).
      NSUM=MEST
      RNSUM=DFLOAT(NSUM)
C
      READ(5,100) NTEST
      RNTEST=DFLOAT(NTEST)
C
      READ(5,100) KSIDE
C
C       EXPN IS THE OUTPUT VARIABLE.
```

```
      EXPN=0.0D0
C
      NSTAGE=1
C
      WRITE(6,110) MEST
  110 FORMAT('1       MEST=',1I5)
      WRITE(6,111) NTEST
  111 FORMAT('        NTEST=',1I5)
      WRITE(6,112) ALPHA
  112 FORMAT('        ALPHA=',1F5.2)
      IF(ISTOP.EQ.3) WRITE(6,113)
  113 FORMAT('       ADJUSTED CRITICAL VALUE MODE')
      IF(ISTOP.NE.3) WRITE(6,114)
  114 FORMAT('        CONSTANT CRITICAL VALUE MODE')
      IF(KSIDE.EQ.1) WRITE(6,115)
  115 FORMAT('        ONE SIDED TEST')
      IF(KSIDE.EQ.2) WRITE(6,116)
  116 FORMAT('        TWO SIDED TEST')
      WRITE(6,102)
  102 FORMAT('0STAGE   ESS TO STAGE   ESS OF STAGE',
     C'   ALPHA OF STAGE   KRV    RKSQ     WCRIT')
C
      RKSQ=0.0D0
      IF(ISTOP.NE.3) CALL CRITL(WCRIT,ALPHA,RKSQ,KSIDE)
C
C     START OF THE LOOP
C
C     DETERMINE NNOW, THE NUMBER OF OBSERVATIONS TO BE
C     TAKEN IN STAGE NSTAGE AND DETERMINE THE INTEGER
C     CRITICAL VALUE KRV FOR NSTAGE.
C
    1 NNOW=NTEST
```

```
      RNNOW=DFLOAT(NNOW)
      RKSQ=RNNOW/RNSUM
      IF(ISTOP.EQ.3) CALL CRITL(WCRIT,ALPHA,RKSQ,KSIDE)
      CRIT=WCRIT*DSQRT(RNNOW)
      KRV=CRIT
      RKRV=DFLOAT(KRV)
      IF(CRIT.GT.RKRV+0.50D0) KRV=KRV+1
C
C     CALCULATE THE MARGINAL PROBABILITY OF ACCEPTING THE
C     NULL HYPOTHESIS AT STAGE NSTAGE.
C
      RMARG=0.0D0
      IF(KSIDE.EQ.1) GO TO 2
      LIM=KRV+1
      DO 3 I=1,LIM,2
      NPATH=I-1
      CALL PATH(PROB,NPATH,NSUM,NNOW,KRV,KSIDE,ISTOP)
      IF(ISTOP.EQ.1) GO TO 1000
      IF(I.GT.1) PROB=2.0D0*PROB
      RMARG=RMARG+PROB
    3 CONTINUE
      GO TO 5
C
    2 KRV=IABS(KRV)
      LIM=KRV+1+NNOW
      DO 4 I=1,LIM,2
      NPATH=I-1-NNOW
      CALL PATH(PROB,NPATH,NSUM,NNOW,KRV,KSIDE,ISTOP)
      IF(ISTOP.EQ.1) GO TO 1000
      RMARG=RMARG+PROB
    4 CONTINUE
C
```

```
C     CALCULATE THE PROBABILITY THISAC OF ACCEPTING THE
C     NULL HYPOTHESIS THROUGH STAGE NSTAGE.
    5 THISAC=RMARG*PASTAC
C
C     CALCULATE THE PROBABILITY REJECT OF REJECTING THE
C     NULL HYPOTHESIS AT STAGE NSTAGE AFTER HAVING
C     ACCEPTED THROUGH (NSTAGE-1).
      REJECT=PASTAC-THISAC
C
C
C     CALCULATE THE STAGEWISE EXPECTED SAMPLE SIZE AT
C     NSTAGE GIVEN THAT THE NULL HYPOTHESIS IS REJECTED
C     AT NSTAGE.
C
      EXPSTG=0.0D0
      DO 6 NT=KRV,NNOW
      RNT=DFLOAT(NT)
      LIM=2*NNOW+1
      DO 7 I=1,LIM,2
      NPATH=I-NNOW-1
      CALL CROSS(PROB,NT,NPATH,NSUM,NNOW,KRV,KSIDE,ISTOP)
      IF(ISTOP.EQ.1) GO TO 1000
      IF(KSIDE.EQ.2) PROB=2.0D0*PROB
      EXPSTG=EXPSTG+RNT*PROB
    7 CONTINUE
    6 CONTINUE
      EXPSTG=EXPSTG/(1.0D0-RMARG)
C
C     CALCULATE THE OUTPUT VARIABLE.
C
      SUM=REJECT*(RNSUM+EXPSTG)
      EXPN=EXPN+SUM
```

```
      ALPH =1.0D0-RMARG
      WRITE(6,103) NSTAGE,EXPN,SUM,ALPH ,KRV,RKSQ,WCRIT
  103 FORMAT(' ',1I4,4X,1F10.3,5X,1F10.3,5X,1F12.10,5X,1I4,
     C1F9.4,1F9.5)
C
C     CHECK SUM FOR STOPPING CRITERION
C
      IF(SUM.LT.0.01D0) GO TO 10
C
C     INDEX FOR RETURN TO BEGINNING OF LOOP.
C
      NSTAGE=NSTAGE+1
      IF(NSTAGE.EQ.501) GO TO 1000
      NSUM=NSUM+NNOW
      RNSUM=DFLOAT(NSUM)
      PASTAC=THISAC
      GO TO 1
 1000 STOP
      END
```

```
      SUBROUTINE CROSS(PROB,NT,NS,MEST,NTEST,KRV,KSIDE,
     CISTOP)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION DVDND(8,2),DVSOR(8,2),DVDNDP(8,2),
     CDVSORP(8,2)
      IDIM=8
      PROB=0.0D0
C
C     SUBROUTINE CROSS CALCULATES THE MARGINAL PROBABILITY,
C     PROB, THAT A CUSUM SIGN TEST PATH OF LENGTH NTEST
C     WITH MEST OBSERVATIONS FOR ESTIMATING THE MEDIAN DOES
C     NOT CROSS THE CRITICAL VALUE KRV UNTIL TIME NT AND
C     THEN SUMS TO NS. THE PATH IS REGARDED AS BEING
C     GENERATED BY PARTIAL SUMS OF +1 AND -1 VALUES.
C     KSIDE INDICATES THE SIDEDNESS OF THE TEST (1 OR 2).
C     ISTOP IS A PROGRAM CONTROL VARIABLE. ISTOP=1 RETURNS
C     CONTROL TO MAIN PROGRAM FOR ERROR IN SR CROSS.
C     THIS SR REQUIRES SR QUOTNT.
C     ONE SIDED TESTS REQUIRE POSITIVE CRITICAL VALUE.
C
C
C     INITIAL CHECKS
C
      KRV=IABS(KRV)
C
C     RANGE CHECK FOR (NS-KRV)
      IF(IABS(NS-KRV).GT.(NTEST-NT)) GO TO 34
C
C     RANGE CHECK FOR NT
      IF((NT.LT.KRV).OR.(NT.GT.NTEST)) GO TO 34
C
C     PARITY CHECK FOR (NT-KRV)
```

```
      IF(2*(IABS(NT-KRV)/2).LT.IABS(NT-KRV)) GO TO 34
C
C     PARITY CHECK FOR MEST
      IF((2*((MEST-1)/2)+1).EQ.MEST) GO TO 30
C
      WRITE(6,31)
   31 FORMAT(' ILLEGAL CONSTANT IN SR CROSS. MEST MUST BE',
     C' ODD')
      ISTOP=1
      GO TO 34
   30 IF(KSIDE.EQ.1) GO TO 35
      NS=IABS(NS)
C
C     PARITY CHECK FOR (NTEST-NS)
   35 IF(2*((NTEST-NS)/2).EQ.(NTEST-NS)) GO TO 32
C
      WRITE(6,33)
   33 FORMAT(' NS HAS ILLEGAL PARITY. PARITY OF NTEST AND',
     C' NS MUST BE THE SAME.')
      ISTOP=1
      GO TO 34
C
C     INITIALIZATION OF DIMENSIONED VECTORS
C
   32 DO 2 I=1,IDIM
      DVDND(I,1)=1.0D0
      DVDND(I,2)=0.0D0
      DVDNDP(I,1)=1.0D0
      DVDNDP(I,2)=0.0D0
      DVSOR(I,1)=1.0D0
      DVSOR(I,2)=0.0D0
      DVSORP(I,1)=1.0D0
```

```
      DVSORP(I,2)=0.0D0
    2 CONTINUE
C
      RNTEST=DFLOAT(NTEST)
      RKRV=DFLOAT(KRV)
      RMEST=DFLOAT(MEST)
      S=DFLOAT(NS)
      T=DFLOAT(NT)
C
C     INDEX FIRST FOUR PLACES OF DVDND AND DVSOR FOR
C     CALCULATING THE PROBABILITY OF A PATH OF SUM NS.
C
      DVDND(1,1)=RMEST
      DVSOR(1,1)=(RMEST-1.0D0)/2.0D0
      DVSOR(2,1)=DVSOR(1,1)
C
      DVSOR(3,1)=RNTEST+RMEST
      DVDND(3,1)=(RNTEST+RMEST-1.0D0+S)/2.0D0
      DVDND(4,1)=(RNTEST+RMEST-1.0D0-S)/2.0D0
C
C     INDEX TWO POSITIONS OF DVDND AND DVSOR FOR NUMBER OF
C     PATHS FROMH RKRV AT TIME T TO NS AT TIME RNTEST.
C
      DVDND(7,1)=RNTEST-T
      DVSOR(7,1)=(DVDND(7,1)-S+RKRV)/2.0D0
      DVSOR(8,1)=DVDND(7,1)-DVSOR(7,1)
C     INDEX NEXT TWO POSITIONS OF DVDND AND DVSOR FOR
C     FRAY-ROSELLE COUNT AND CALL SR QUOTNT.
C
      DVDND(5,1)=T-1.0D0
      NDEX=1
    3 GO TO (40,41,42,43,44), NDEX
```

```
C
   40 DVSOR(5,1)=(T+RKRV-2.0D0)/2.0D0
      IF(DVSOR(5,1).LT.0.0D0) GO TO 34
      DVSOR(6,1)=DVDND(5,1)-DVSOR(5,1)
      IF(DVSOR(6,1).LT.0.0D0) GO TO  34
      CALL QUOTNT(DVDND,DVSOR,DVDNDP,DVSORP,Z,IDIM)
      PROB=PROB+Z
      NDEX=2
      COUNT=0.0D0
      IF(KSIDE.EQ.2) GO TO 3
      DVSOR(5,1)=(T-RKRV)/2.0D0+RKRV
      IF(DVSOR(5,1).LT.0.0D0) GO TO 34
      DVSOR(6,1)=DVDND(5,1)-DVSOR(5,1)
      IF(DVSOR(6,1).LT.0.0D0) GO TO 34
      CALL QUOTNT(DVDND,DVSOR,DVDNDP,DVSORP,Z,IDIM)
      PROB=PROB-Z
      GO TO 34
C
   41 DVSOR(5,1)=(T+RKRV-2.0D0)/2.0D0-RKRV-2.0D0*RKRV*COUNT
      GO TO 45
   42 DVSOR(5,1)=(T+RKRV-2.0D0)/2.0D0+RKRV+2.0D0*RKRV*COUNT
      GO TO 45
   43 DVSOR(5,1)=(T+RKRV-2.0D0)/2.0D0+2.0D0*RKRV*COUNT
      GO TO 45
   44 DVSOR(5,1)=(T+RKRV-2.0D0)/2.0D0-2.0D0*RKRV*COUNT
   45 IF(DVSOR(5,1).LT.0.0D0) GO TO 51
      DVSOR(6,1)=DVDND(5,1)-DVSOR(5,1)
      IF (DVSOR(6,1).LT.0.0D0) GO TO 51
C
      CALL QUOTNT(DVDND,DVSOR,DVDNDP,DVSORP,Z,IDIM)
      GO TO (34,46,46,47,47),NDEX
   46 PROB=PROB-Z
```

```
      GO TO 48
   47 PROB=PROB+Z
C
   48 COUNT=COUNT+1.0D0
      GO TO 3
   51 NDEX=NDEX+1
      GO TO (34,34,52,53,53,34), NDEX
   52 COUNT=0.0D0
      GO TO 3
   53 COUNT=1.0D0
      GO TO 3
   34 RETURN
      END
```

```
      SUBROUTINE PATH(PROB,NS,MEST,NTEST,KRV,KSIDE,ISTOP)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION DVDND(6,2),DVSOR(6,2),DVDNDP(6,2),
     CDVSORP(6,2)
      IDIM=6
      PROB=0.0D0
C
C     SUBROUTINE PATH CALCULATES THE MARGINAL PROBABILITY,
C     PROB, THAT A CUSUM SIGN TEST PATH OF LENGTH NTEST
C     WITH MEST OBSERVATIONS FOR ESTIMATING THE MEDIAN DOES
C     NOT CROSS THE CRITICAL VALUE KRV AND SUMS TO NS.
C     THE PATH IS REGARDED AS GENERATED BY PARTIAL SUMS OF
C     +1 AND -1 VALUES.
C     KSIDE IS THE SIDEDNESS OF THE TEST (1 OR 2).
C     ISTOP IS A PROGRAM CONTROL VARIABLE:
C       ISTOP=1 RETURN CONTROL TO MAIN PROGRAM FOR ERROR IN
C               SR PATH;
C       ISTOP=2 CALCULATE THE UNCONDITIONAL PROBABILITY OF
C               A PATH SUMMING TO NS.
C     NOTE: DO NOT INPUT AN NS VALUE IN THE CRITICAL REGION
C     UNLESS ISTOP IS 2.
C     THIS SUBROUTINE REQUIRES SR QUOTIENT.
C
C
C     INITIAL CHECKS
C
      K=(MEST-1)/2
      K=2*K+1
      IF(K.EQ.MEST) GO TO 30
      WRITE(6,31)
   31 FORMAT(' ILLEGAL CONSTANT MEST IN SR PATH. MEST',
     C' MUST BE ODD.')
```

```
      ISTOP=1
      GO TO 34
   30 IF(KSIDE.EQ.1) GO TO 35
      KRV=IABS(KRV)
      NS=IABS(NS)
   35 K=(NTEST-NS)/2
      K=2*K
      IF(K.EQ.NTEST-NS) GO TO 32
      WRITE(6,33)
   33 FORMAT(' NS HAS ILLEGAL PARITY. PARITY OF NTEST AND',
     C' NS MUST BE THE SAME.')
      ISTOP=1
      GO TO 34
C
C     INITIALIZATION OF DIMENSIONED VECTORS
C
   32 DO 2 I=1,IDIM
      DVDND(I,1)=1.0D0
      DVDND(I,2)=0.0D0
      DVDNDP(I,1)=1.0D0
      DVDNDP(I,2)=0.0D0
      DVSOR(I,1)=1.0D0
      DVSOR(I,2)=0.0D0
      DVSORP(I,1)=1.0D0
      DVSORP(I,2)=0.0D0
    2 CONTINUE
C
      RNTEST=DFLOAT(NTEST)
      RKRV=DFLOAT(KRV)
      RMEST=DFLOAT(MEST)
      S=DFLOAT(NS)
C
```

```
C     INDEX FIRST FOUR PLACES OF DVDND AND DVSOR FOR
C     CALCULATING THE PROBABILITY OF A PATH WITH SUM NS.
C
      DVDND(1,1)=RMEST
      DVSOR(1,1)=(RMEST-1.0D0)/2.0D0
      DVSOR(2,1)=DVSOR(1,1)
C
      DVSOR(3,1)=RNTEST+RMEST
      DVDND(3,1)=(RNTEST+RMEST-1.0D0+S)/2.0D0
      DVDND(4,1)=(RNTEST+RMEST-1.0D0-S)/2.0D0
C
C     INDEX NEXT TWO POSITIONS OF DVDND AND DVSOR FOR
C     FRAY-ROSELLE COUNT AND CALL SR QUOTNT.
C
      DVDND(5,1)=RNTEST
      NDEX=1
    3 GO TO (40,41,42,43,44), NDEX
C
   40 DVSOR(5,1)=(RNTEST+S)/2.0D0
      IF(DVSOR(5,1).LT.0.0D0) GO TO 34
      DVSOR(6,1)=DVDND(5,1)-DVSOR(5,1)
      IF(DVSOR(6,1).LT.0.0D0) GO TO  34
      CALL QUOTNT(DVDND,DVSOR,DVDNDP,DVSORP,Z,IDIM)
      PROB=PROB+Z
      IF(ISTOP.EQ.2) GO TO 34
      NDEX=2
      COUNT=0.0D0
      IF(KSIDE.EQ.2) GO TO 3
      DVSOR(5,1)=(RNTEST-S)/2.0D0+RKRV
      IF(DVSOR(5,1).LT.0.0D0) GO TO 34
      DVSOR(6,1)=DVDND(5,1)-DVSOR(5,1)
      IF(DVSOR(6,1).LT.0.0D0) GO TO 34
```

```
      CALL QUOTNT(DVDND,DVSOR,DVDNDP,DVSORP,Z,IDIM)
      PROB=PROB-Z
      GO TO 34
C
   41 DVSOR(5,1)=(RNTEST+S)/2.0D0-RKRV-2.0D0*RKRV*COUNT
      GO TO 45
   42 DVSOR(5,1)=(RNTEST+S)/2.0D0+RKRV+2.0D0*RKRV*COUNT
      GO TO 45
   43 DVSOR(5,1)=(RNTEST+S)/2.0D0+2.0D0*RKRV*COUNT
      GO TO 45
   44 DVSOR(5,1)=(RNTEST+S)/2.0D0-2.0D0*RKRV*COUNT
   45 IF(DVSOR(5,1).LT.0.0D0) GO TO 51
      DVSOR(6,1)=DVDND(5,1)-DVSOR(5,1)
      IF (DVSOR(6,1).LT.0.0D0) GO TO 51
C
      CALL QUOTNT(DVDND,DVSOR,DVDNDP,DVSORP,Z,IDIM)
      GO TO (34,46,46,47,47),NDEX
   46 PROB=PROB-Z
      GO TO 48
   47 PROB=PROB+Z
C
   48 COUNT=COUNT+1.0D0
      GO TO 3
   51 NDEX=NDEX+1
      GO TO (34,34,52,53,53,34), NDEX
   52 COUNT=0.0D0
      GO TO 3
   53 COUNT=1.0D0
      GO TO 3
   34 RETURN
      END
```

```
      SUBROUTINE QUOTNT(DVDND,DVSOR,DVDNDP,DVSORP,ANSWR,
     CIDIM)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION DVSOR(IDIM,2),DVDND(IDIM,2),DVSORP(IDIM,2),
     CDVDNDP(IDIM,2)
C
C     SUBROUTINE QUOTNT IS DESIGNED TO TAKE THE QUOTIENT OF
C     TWO PRODUCTS, THE DIVIDEND AND THE DIVISOR, WHERE
C     THESE PRODUCTS ARE STRINGS OF FACTORIALS AND POWERS.
C
C     THE INPUT IS TWO TWO-DIMENSIONAL ARRAYS (DVSOR AND
C     DVDND) AND A DIMENSION VARIABLE (IDIM).
C     DVDND(I,1)=X IS A POSITIVE REAL VARIABLE;
C     DVDND(I,2)=N IS AN INTEGER VALUED REAL VARIABLE.
C     IF N=0.0D0 THEN QUOTNT WILL COMPUTE X-FACTORIAL;
C     IF N IS POSITIVE QUOTNT WILL COMPUTE THE N-TH POWER
C     OF X.
C     A SIMILAR RULE FOLLOWS FOR DVSOR.
C     THE INPUT DVDNDP AND DVSORP ARE ARRAYS OF THE SAME
C     DIMENSION AS DVDND AND DVSOR. THEY ARE FOR
C     SUBROUTINE USE ONLY,BUT NEED TO BE DIMENSIONED IN
C     THE MAIN PROGRAM.
C
C
C
C     PRESERVE THE INPUT ARAYS
C
      DO 42 I=1,IDIM
      DVSORP(I,1)=DVSOR(I,1)
      DVSORP(I,2)=DVSOR(I,2)
      DVDNDP(I,1)=DVDND(I,1)
      DVDNDP(I,2)=DVDND(I,2)
```

```
   42 CONTINUE
C
C     INDEX AND INPUT CONVERSION
C
      DO 40 I=1,IDIM
      IF(DVDNDP(I,1).GT.0.0) GO TO 41
      DVDNDP(I,1)=1.0
   41 IF(DVSORP(I,1).GT.0.0) GO TO 40
      DVSORP(I,1)=1.0
   40 CONTINUE
      I=0
      J=1
      INDXB=1
      ANSWR=1.0
C
C     MULTIPLICATION LOOP
C
    4 I=I+1
      INDXA=1
      IF(I.LE.IDIM) GO TO 2
      IF(J.LE.IDIM) GO TO 6
      IF(J.GT.IDIM) GO TO 30
    2 NDXA=INDXA
      IF(DVDNDP(I,2).EQ.0.0) GO TO 10
C     COMPUTE DVDND(I,1)**DVDND(I,2)
      KV=DVDND(I,2)
      IF(INDXA.GT.KV) GO TO 4
      DO 12 M=NDXA,KV
      INDXA=INDXA+1
      ANSWR=ANSWR*DVDNDP(I,1)
      IF(J.GT.IDIM) GO TO 12
      IF(ANSWR.GT.1000000.0) GO TO 6
```

```
   12 CONTINUE
      GO TO 4
C
C     COMPUTE DVOND(I,1)
   10 KV=DVONDP(I,1)
      IF(INDXA.GT.KV) GO TO 4
      DO 11 M=NDXA,KV
      RM=DFLOAT(M)
      INDXA=INDXA+1
      ANSWR=ANSWR*RM
      IF(J.GT.IDIM) GO TO 11
      IF(ANSWR.GT.1000000.0) GO TO 6
   11 CONTINUE
      GO TO 4
C
C     DIVISION LOOP
C
    5 J=J+1
      INDXB=1
      IF(J.LE.IDIM) GO TO 6
      IF(I.LE.IDIM) GO TO 2
      IF(I.GT.IDIM) GO TO 30
    6 NDXB=INDXB
      IF(DVSORP(J,2).EQ.0.0) GO TO 20
C     COMPUTE DVSOR(J,1)**(-DVSOR(J,2))
      KS=DVSORP(J,2)
      IF(INDXB.GT.KS) GO TO 5
      DO 22 L=NDXB,KS
      INDXB=INDXB+1
      ANSWR=ANSWR/DVSORP(J,1)
      IF(I.GT.IDIM) GO TO 22
      IF(ANSWR.LT.0.0000001) GO TO 2
```

```
   22 CONTINUE
      GO TO 5
C
C     COMPUTE (DVSOR(J,1) )**(-1)
   20 KS=DVSORP(J,1)
      IF(INDXB.GT.KS) GO TO 5
      DO 21 L=NDXB,KS
      RL=DFLOAT(L)
      INDXB=INDXB+1
      ANSWR=ANSWR/RL
      IF(I.GT.IDIM) GO TO 21
      IF(ANSWR.LT.0.0000001) GO TO 2
   21 CONTINUE
      GO TO 5
   30 RETURN
      END
```

```
      SUBROUTINE CRITL(CRIT,ALPHA,RKSQ,KSIDE)
      IMPLICIT REAL*8 (A-H,O-Z)
C
C
C     SUBROUTINE CRIT FINDS THE BROWNIAN MOTION CORRECTED
C     CRITICAL VALUE CRIT WHEN THE CRITICAL REGION IS SIZE
C     ALPHA AND THE CORRECTION FACTOR IS RKSQ.
C     KSIDE IS THE SIDEDNESS OF THE CRITICAL REGION (1 OR 2)
C     THIS SUBROUTINE REQUIRES FUNCTION GAUSS7
C
C
      PI=3.14159265358973D0
      F=1.0D0/DSQRT(2.0D0*PI)
      LAST=1
      COMPAR=0.0D0
      DELTA=1.0D0
      CRIT=1.0D0
      D=DSQRT(1.0D0+RKSQ)
C
C     CALCULATE THE PROBABILITY AT CRIT.
C
    1 S=1.0D0
      IF(CRIT.EQ.COMPAR) RETURN
      COMPAR=CRIT
      ARG=CRIT*(1.0D0+2.0D0*RKSQ)/D
      IF(ARG.GT.8.0D0) GO TO 4
      Z=GAUSS7(-CRIT/D)+DEXP(2.0D0*(CRIT**2)*RKSQ)*
     CGAUSS7(-(CRIT*(1.0D0+2.0D0*RKSQ))/D)
    5 IF(KSIDE.EQ.1) GO TO 2
      Z=2.0D0*Z
C     USE FIRST TERM OF SUMMATION ONLY IF RKSQ IS GREATER
C     THAN 10.
```

```
      IF(RKSQ.GT.10.0D0) GO TO 2
C
      SIGN=-1.0D0
    3 ARG1=CRIT*(1.0D0+2.0D0*S*(1.0D0+RKSQ))/D
      IF(ARG1.GT.8.0D0) GO TO 6
      SUM=2.0D0*SIGN*(DEXP(2.0D0*((S*CRIT)**2)*RKSQ)*
     CGAUSS7(-CRIT*(1.0D0+2.0D0*S*(1.0D0+RKSQ))/D)+
     CDEXP(2.0D0*(((S+1.0D0)*CRIT)**2)*RKSQ)*
     CGAUSS7(-CRIT*(2.0D0*(S+1.0D0)*(1.0D0+RKSQ)-1.0D0)/D))
    7 Z=Z+SUM
      IF(DABS(SUM).LE.0.000001D0) GO TO 2
      SIGN=-SIGN
      S=S+1.0D0
      GO TO 3
C
C     ----------------------------------------------------------------
C     THIS BLOCK IS A MODIFIED PROCEDURE FOR CALCULATING
C     THE TERMS OF SUMMATION WHEN THE NORMAL CDF ARGUMENT
C     IS EXCESSIVE.
C
    4 XNT=DEXP(2.0D0*(CRIT**2)*RKSQ-(ARG**2)/2.0D0)
      N=5
       CALL CFL(N,ARG,Y)
      Z=XNT*Y*F+GAUSS7(-CRIT/D)
      GO TO 5
C
    6 ARG2=CRIT*(2.0D0*(S+1.0D0)*(1.0D0+RKSQ)-1.0D0)/D
      XNT1=DEXP(2.0D0*((S*CRIT)**2)*RKSQ-(ARG1**2)/2.0D0)
      N=5
      CALL  CFL(N,ARG1,Y1)
      XNT2=DEXP(2.0D0*(((S+1.0D0)*CRIT)**2)*RKSQ-(ARG2**2)
     C/2.0D0)
```

```
      CALL CFL(N,ARG2,Y2)
      SUM=2.0D0*SIGN*F*(XNT1*Y1+XNT2*Y2)
      GO TO 7
C     ---------------------------------------------------------
C
C     COMPARE COMPUTED PROBABILITY TO ALPHA.
C
    2 IF(DABS(Z-ALPHA).LT.0.00001D0) RETURN
      NOW=1
      IF(Z.LT.ALPHA) NOW=-1
      IF(NOW*LAST.EQ.-1) DELTA=-0.5D0*DELTA
      LAST=NOW
      CRIT=CRIT+DELTA
      IF(S.GE.0.0D0) GO TO 1
      RETURN
      END
```

```
      FUNCTION GAUSS7(X)
      IMPLICIT REAL*8 (A-H,O-Z)
C
C     FUNCTION GAUSS7 CALCULATES THE NORMAL CDF AT ARGUMENT
C     X TO A LOG-ODDS ACCURACY OF 10**(-7).
C
      Z=DABS(X)
      PI=3.14159265358979300
      R=DEXP(-(Z**2)/2.0D0)/DSQRT(2.0D0*PI)
      IF(Z.GT.2.5D0) GO TO 1
      U=3.9D0+5.32D0*Z
      N=U
      CALL CFGS(N,Z,Y)
      Y=0.50D0+R*Y
      IF(X.LE.0.0D0) Y=1.0D0-Y
      GAUSS7=Y
      RETURN
C
    1 U=5.0D0
      IF(Z.LE.8.0D0) U=3.0D0+60.0D0*DEXP(-1.45D0*DLOG(Z))
      N=U
      CALL CFL(N,Z,Y)
      Y=1.0D0-R*Y
      IF(X.LE.0.0D0) Y=1.0D0-Y
      GAUSS7=Y
      RETURN
      END
C
C
      SUBROUTINE CFL(N,X,Y)
      IMPLICIT REAL*8 (A-H,O-Z)
C     SUBROUTINE CFL COMPUTES THE N-TH CONVERGENT IN THE
```

```
C        LAPLACE CONTINUED FRACTION APPROXIMATING THE NORMAL
C        CDF.
C        INPUT: N IS THE LENGTH OF THE PARTIAL CONVERGENT;
C               X IS THE ARGUMENT;
C               Y IS THE PARTIAL CONVERGENT.
         Y=X
         DO 1 K=1,N
         M=N-K+1
         RM=DFLOAT(M)
         Y=X+RM/Y
       1 CONTINUE
         Y=1.0D0/Y
         RETURN
         END
C
C
         SUBROUTINE CFGS(N,X,Y)
         IMPLICIT REAL*8 (A-H,O-Z)
C        SUBROUTINE CFGS COMPUTES THE N-TH CONVERGENT IN THE
C        GAUSS-SHENTON CONTINUED FRACTION APPROXIMATING THE
C        NORMAL CDF.
C        INPUT: N IS THE LENGTH OF THE PARTIAL CONVERGENT;
C               X IS THE ARGUMENT;
C               Y IS THE PARTIAL CONVERGENT.
         RN=DFLOAT(N)
         Y=2.0D0*RN+1.0D0
         I=N/2
         I=2*I
         W=X**2
         IF (N-I) 1,2,1
       1 SGN=-1.0D0
         GO TO 3
```

```
2 SGN=1.0D0
3 DO 4 K=1,N
  M=N-K+1
  RM=DFLOAT(M)
  Y=2.0D0*RM-1.0D0+(SGN*RM*W)/Y
  SGN=-SGN
4 CONTINUE
  Y=X/Y
  RETURN
  END
```

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SOME NONPARAMETRIC TESTS FOR CONSTANCY OF REGRESSION RELATIONSHIPS OVER TIME

by

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(ABSTRACT)

Let Y_1, Y_2, ... be a sequence of random variables obeying the law $Y_i = \underline{\beta}_i' \underline{x}_i + \varepsilon_i$, where $\underline{\beta}_1$, $\underline{\beta}_2$, ... is a sequence of unknown k-dimensional regression vectors; $\underline{x}_1$, $\underline{x}_2$, ... is a sequence of known k-dimensional regressor vectors; and ε_1, ε_2, ... is a sequence of independent and identically distributed random variables. Assume that $\underline{\beta}_1 = \ldots = \underline{\beta}_m = \underline{\beta}$, $m \geq k$, and that $\hat{\underline{\beta}}_0$ is an asymptotically normal estimate of $\underline{\beta}$ based on $Y_1, \ldots, Y_m$. This study develops nonparametric procedures for testing H_0: $\underline{\beta} = \underline{\beta}_{m+1} = \underline{\beta}_{m+2} = \ldots$.

The proposed tests involve sequences of truncated sequential tests. That is, a function of the residuals $Y_{m+1} - \hat{\underline{\beta}}_0' \underline{x}_{m+1}, \ldots,$ $Y_{m+N} - \hat{\underline{\beta}}_0' x_{m+N}$ is examined for a shift in the model. If no shift is indicated all m+N observations are pooled and a new estimate of $\underline{\beta}$, $\hat{\underline{\beta}}_1$, is formed. The next N residuals are then examined for a shift. The procedure continues until a shift is indicated.

Brownian motion results are used to obtain approximate critical values when the function of the residuals is: the cumulative sum of the signs of the residuals; the sequential Wilcoxon scores; the ordinary cumulative sums of residuals.

Exact results are obtained for the cumulative sum of signs procedure when testing for a shift in median.

Asymptotic relative efficiency results are also obtained.