THE FACTORADIC INTEGERS

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(ABSTRACT)

The arithmetic progressions, considered as functions $x \mapsto \lambda x + \phi$ equipped with addition and composition, are examined from an algebraic standpoint as "left outer-distributive rings", i.e. objects satisfying the ring axioms with a weakened left-distributive law: namely, $x(y+z) = xy + xz + \mathscr{D}(x)$, where $\mathscr{D}$ is a function dependent only on x. Under appropriate constraints on λ and ϕ, the images of these functions give bases for topologies on $\mathbb{Z}$ (or any unique factorization domain); the ring of factoradic integers $\bar{\mathbb{Z}}$ is defined as the completion of the topology generated on $\mathbb{Z}$ by the constraints $\lambda \in \mathbb{N}$, $\phi \in \mathbb{Z}$ (i.e. the evenly spaced integer topology) under the metric $d_!(n,m) = \frac{1}{N!}$, N maximal such that $N! \mid (n-m)$. It is shown that $\bar{\mathbb{Z}}$ is ring-isomorphic to the direct product of the p-adic integers $\mathbb{Z}_p$ over all primes p, i.e. to the profinite completion of the integers. Analogies between $\bar{\mathbb{Z}}$ and $\mathbb{Z}$ are exploited to allow for general factoradic integers to be used as exponents; this results in a unique factorization theorem which completely characterizes multiplication in $\bar{\mathbb{Z}}$, giving the multiplicative group of units as $U \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_2 \times \prod_p \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$, where the direct product is over all odd primes p. Similarly, bounds of summation are extended to take general values in $\bar{\mathbb{Z}}$, and continuous functions are examined in terms of these factoradic series and the finite difference operator. It is found that a continuous function is equal to its own Newton series if and only if it can be decomposed as a direct product of functions $\mathbb{Z}_p \to \mathbb{Z}_p$ over all primes p. The relationship between $\bar{\mathbb{Z}}$ and certain completions of polynomial rings is then examined, and it is shown that $\bar{\mathbb{Z}} \cong R/xR$, where $R = \varprojlim \mathbb{Z}[x]/x(x+1)...(x+n)\mathbb{Z}[x]$.

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(GENERAL ABSTRACT)

The arithmetic progressions under addition and composition satisfy the usual rules of arithmetic with a modified distributive law. The basic algebra of such mathematical structures is examined; this leads to the consideration of the integers as a metric space under the "factoradic metric", i.e., the integers equipped with a distance function defined by $d(n, m) = 1/N!$, where N is the largest positive integer such that $N!$ divides $n - m$. Via the process of metric completion, the integers are then extended to a larger set of numbers, the factoradic integers. The properties of the factoradic integers are developed in detail, with particular attention to prime factorization, exponentiation, infinite series, and continuous functions, as well as to polynomials and their extensions. The structure of the factoradic integers is highly dependent upon the distribution of the prime numbers and relates to various topics in algebra, number theory, and non-standard analysis.

Contents

Introduction

Motivation

The prime numbers are easily defined in terms of multiplication, but their additive distribution is extremely complex; a similar situation holds for many sets of integers of number-theoretic interest. Indeed, multiplying a number by a sum is trivial in light of the distributive law $a(b+c) = ab+ac$, but there is no such general formula for adding a number to a product; intuitively, addition is too fundamental to be so easily described. The structure of the integers, however, is entirely determined by the interplay between addition and multiplication. The simplest functions involving both addition and multiplication are the "waves", i.e. the functions $f(x) = \lambda x + \phi$ for some fixed $\lambda, \phi \in \mathbb{Z}$, whose images are fundamental periodic sets which correspond to the bidirectionally infinite arithmetic progressions.

It is the object of this paper to examine these waves from two perspectives. First, it is observed that as functions under addition and composition they satisfy a modification of the ring axioms with an altered distributive law, and the basic algebra of such structures is investigated. The case of waves over unique factorization domains is given special attention, resulting in a collection of topologies over any unique factorization domain R whose basis elements are wave images, i.e. sets of the form $\{\lambda x + \phi\}_{x \in R}$. Second, the topology generated by all non-singleton wave images over the integers is given a particular metric and completed, extending $\mathbb{Z}$ to the larger topological ring of factoradic integers $\bar{\mathbb{Z}}$, and the properties of $\bar{\mathbb{Z}}$ are investigated. In particular, a unique factorization theorem is proven for $\bar{\mathbb{Z}}$ which characterizes multiplication in $\bar{\mathbb{Z}}$ completely, including the structure of the multiplicative group of units, and infinite summations over $\bar{\mathbb{Z}}$ are developed in some detail. As a ring, $\bar{\mathbb{Z}}$ is found to be isomorphic to the profinite completion of the integers, i.e. to the direct product of the p-adic integers over all primes p; several alternative constructions are given, including the profinite construction via projective limits and a construction as a quotient of a certain completion of the polynomial ring $\mathbb{Z}[x]$.

Notation and Background

Notationally, the natural numbers $\mathbb{N}$ include all positive integers but not 0; $\mathbb{N}_0$ denotes $\mathbb{N} \cup \{0\}$. As usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Q}$ the set of rational numbers, and $\mathbb{R}$ the set of real numbers. When the meaning is clear, subscripts may be used to modify sets in obvious ways, e.g. $\mathbb{Z}_{\neq 0}$ denotes the nonzero integers, $\mathbb{R}_{<0}$ denotes the negative real numbers, etc. The symbol $\mathbb{P}$ always denotes the set of prime numbers, i.e. the positive integers each of which is divisible only by itself and by 1. The number 1 itself does not count as prime. Basic facts and definitions from algebra, modular arithmetic, and topology are assumed as common knowledge; this section elucidates some standard definitions and basic results from these fields.

Topology. Given a set X, a topology is a collection T of subsets of X such that $X, \emptyset \in T$, T is closed under finite intersections of elements, and T is closed under arbitrary unions of elements; the sets in T are called open sets, and their complements are called closed sets. If a set is both open and closed, it is called clopen. A basis for a topology T is a set $B \subset T$ such that every element of T is a union of elements of B; a set B is a basis for a topology on X whenever $\forall x \in X\ \exists b \in B$ such that $x \in b$, and $\forall a, b \in B$ if $x \in a \cap b$ then $\exists c \in B$ such that $x \in c \subset a \cap b$. A set X equipped with a topology T is called a topological space.

Given two topological spaces X and Y, a function $f \colon X \to Y$ is continuous if preimages of open sets are open; that is, if whenever $S \subset Y$ is an open set in Y, $f^{-1}(S) = \{x \in X : f(x) \in S\}$ is an open set in X. If there exists a continuous bijection $f \colon X \to Y$ which has a continuous inverse, we say f is a homeomorphism between X and Y, which are consequently homeomorphic topological spaces.

A metric space X is a set X together with a function $d \colon X \times X \to \mathbb{R}_{\geq 0}$, called a metric, which satisfies the following:

(1) (Identity) $\forall x, y \in X\ d(x, y) = 0$ if and only if $x = y$.
(2) (Symmetry) $\forall x, y \in X\ d(x, y) = d(y, x)$.
(3) (Triangle Inequality) $\forall x, y, z \in X\ d(x, z) \leq d(x, y) + d(y, z)$.

An ultrametric is a metric satisfying $d(x, z) \leq \max\{d(x, y),\, d(y, z)\}$, a stronger form of the triangle inequality. If X is a metric space, we can define the open metric balls as follows: if $r \in \mathbb{R}_{>0}$ and $x \in X$, then the open metric ball of radius r centered at x is $B_r(x) := \{y \in X : d(x, y) < r\}$. It is also convenient to consider the closed metric balls, which allow $d(x, y) = r$; the closed metric ball of radius r centered at x is $B_r[x] = \{y \in X : d(x, y) \leq r\}$. Now the set of all open metric balls $\{B_r(x) : r \in \mathbb{R}_{>0},\, x \in X\}$ is always a basis for a topology on X, and we say the metric d induces this topology; thus a metric space is always a topological space.

A Cauchy sequence is a sequence $\{x_n\}_{n \in \mathbb{N}}$ in a metric space X such that $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $n, m > N \longrightarrow d(n, m) < \epsilon$. That is, going far enough out in the sequence we find that all terms are arbitrarily close to one another. If there is an $x \in X$ such that $\forall \epsilon > 0$ $d(x, x_n) < \epsilon$ for sufficiently large n, we say x is the limit of $\{x_n\}$ and $\{x_n\}$ converges to x as $n \to \infty$, and we write $\lim\limits_{n \to \infty} x_n$.

A metric space in which every Cauchy sequence converges is called a complete metric space. If a metric space X is not complete, it can be extended to a complete metric space via metric completion: first, put the "metric" $d(\{x_n\}, \{y_n\}) = \lim\limits_{n \to \infty} d(x_n, y_n)$ on the set of Cauchy sequences in X. This is not a true metric, because two Cauchy sequences may be distinct and yet have distance 0; however, the Cauchy sequences at distance 0 from one another form an equivalence class, so letting $\{x_n\} \sim \{y_n\} \iff d(\{x_n\}, \{y_n\}) = 0$ we have that $\sim$ is an equivalence relation; if the set of Cauchy sequences is designated C, then $C/\sim$, i.e. the set of Cauchy sequences with $\{x_n\}$ considered equal to $\{y_n\}$ whenever $\{x_n\} \sim \{y_n\}$, gives the metric completion $\bar{X}$ of X. If X is a ring with addition and multiplication continuous, $\bar{X}$ will be as well.

There are several other useful definitions from topology which may be utilized in this paper; given a topological space X we have the following:

(1) Neighborhood: A neighborhood of a point $x \in X$ is any open set containing x.
(2) Limit point: The set of limit points of a subset $S \subset X$ is denoted by S' and defined as the set of all $x \in X$ such that every neighborhood of x contains an element of $S \backslash \{x\}$. A closed set contains all of its own limit points.
(3) Closure: The closure of a subset $S \subset X$ is defined by $\bar{S} = S \cup S'$.
(4) Continuous function: On a metric space, there is an alternative characterization of continuous functions. If X and Y are metric spaces and $f\colon X \to Y$, then f is continuous at $x_0 \in X$ if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in X$ with $0 < d(x, x_0) < \delta$ we have $d(f(x),\, f(x_0)) < \epsilon$. If for each ϵ there exists a valid choice of δ which holds for *all* x, then f is uniformly continuous.
(5) Hausdorff: X is Hausdorff if $\forall x, y \in X$ with $x \neq y$ there exist neighborhoods N, M of x and y, respectively, such that $N \cap M = \emptyset$.
(6) Open covering: An open covering of a subset $S \subset X$ is a collection C of open sets in X such that $S \subset \bigcup_{c \in C} c$. Given an open covering C of S, a subcovering is a subset of C such that we still have $S \subset \bigcup_{c \in C} c$.
(7) Compact: A subset $S \subset X$ is compact if every open covering of S has a finite subcovering. X itself is compact if it satisfies this criterion; every closed subset of a compact set is compact.

Algebra. A group is a set G equipped with a binary operation $*$ such that:

(1) $*$ is associative. That is, $\forall a, b, c \in G$ we have $a * (b * c) = (a * b) * c$.
(2) There is an identity on G. That is, $\exists i \in G$ such that $\forall g \in G\ g * i = i * g = g$.
(3) Every element has an inverse. That is, $\forall g \in G\ \exists h \in G$ such that $g * h = h * g = i$.

If G is commutative under $*$ (that is, $g * h = h * g\ \forall g, h \in G$), we say that G is abelian, and we will normally write $+$ for the binary operation and 0 for the identity, denoting the additive inverse of a group element g by $-g$.

Similarly, a ring is an abelian group G written additively and equipped with a second binary operation corresponding to multiplication, such that:

(1) Multiplication is associative. That is, $\forall a, b, c \in R$ we have $a(bc) = (ab)c$.
(2) There is a multiplicative identity. That is, $\exists 1 \in R$ such that $\forall r \in R\ r1 = 1r$.
(3) The right distributive law holds. That is, $\forall a, b, c \in R\ (a + b)c = ac + bc$.
(4) The left distributive law holds. That is, $\forall a, b, c \in R\ a(b + c) = ab + ac$.

A ring in which multiplication is commutative is called a commutative ring. An integral domain is a commutative ring R in which the only zero-divisor is 0, where $x \in R$ is called a zero divisor if there exists $y \neq 0$ in R such that $xy = 0$; a field is a commutative ring R in which every nonzero element is a unit, where $x \in R$ is a unit if it has a multiplicative inverse, i.e. if $\exists y \in R$ such that $xy = 1$. Any field is thus also an integral domain, because units and zero divisors are mutually exclusive.

An element x of an integral domain is called irreducible if $x = yz$ implies that either y or z must be a unit. An element x of an integral domain is called prime if $x = yz$ implies $x \mid y$ or $x \mid z$. An integral domain X is a unique factorization domain (UFD) if every element $x \in X$ can be written as a product of irreducible elements and a unit; in a UFD, all irreducibles are prime, and vice versa.

An ideal of a commutative ring R is a subset I of R such that I under addition is a subgroup of R, and such that for all $r \in R$ and $i \in I$ we have $ir \in I$. An ideal I is called principal if it is generated by a single element, i.e. if $\exists i \in I$ such that $I = iR$; if every ideal in an integer domain is principal, it is called a principal ideal domain (PID); every PID is a UFD. Given an ideal I of a ring R, the quotient ring R/I is composed of elements of the form $\{r + I : r \in R\}$, where $r + I = \{r + i : i \in I\}$, which form a ring.

Finally, given rings R and S, a ring homomorphism is any function $f\colon R \to S$ such that $f(r_1) + f(r_2) = f(r_1 + r_2)$ and $f(r_1)f(r_2) = f(r_1 r_2)$ $\forall r_1, r_2 \in R$. If a ring homomorphism is also a bijection, it is called an isomorphism, and R and S are said to be isomorphic, and we write $R \cong S$. The kernel of a homomorphism $R \to S$ is $ker(f) = \{x \in R : f(x) = 0 \,\text{in}\, S\}$. Note that there are also group homomorphisms, which are functions between groups $G \to H$ which have $f(x) * f(y) = f(x * y)$; in this case the kernel of f is the set of $x \in G$ such that $f(x)$ is the identity in H. The First Homomorphism Theorem asserts that for a ring homomorphism $f\colon R \to S$, the quotient ring $R/ker(f)$ is isomorphic to the image $f(R)$ of f, which is a subring of S.

Modular Arithmetic. $\mathbb{Z}$ is a principal ideal domain; the quotient rings $\mathbb{Z}/n\mathbb{Z}$ for n a fixed natural number are known as the integers modulo n. If two integers x, y map natually via $x \mapsto x + n\mathbb{Z}$, $y \mapsto y + n\mathbb{Z}$ to the same element of $\mathbb{Z}/n\mathbb{Z}$, we say that x and y are congruent modulo n, and write $x \equiv y\,(mod\,n)$. Two integers are congruent modulo n if and only if they give the same remainder when divided by n; every integer can thus be associated with its remainder modulo n, and these give equivalence classes corresponding to the elements of $\mathbb{Z}/n\mathbb{Z}$. Thus there are n elements of $\mathbb{Z}/n\mathbb{Z}$, asociated with the integers $\{0, 1, .., n-1\}$, and $x \equiv 0\,(mod\,n)$ if and only if $n \mid x$. x is congruent to a unit modulo $\mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(x, n) = 1$.

As an additive group, $\mathbb{Z}/n\mathbb{Z}$ is cyclic, i.e. it can be generated by repeated addition of a single element. The multiplicative structure of $\mathbb{Z}/n\mathbb{Z}$ is elucidated by the Chinese Remainder Theorem and the existence of primitive roots, as follows:

Theorem. *(Chinese Remainder Theorem) If $\{n_1, ...n_N\}$ are integers and $\{m_1, ..., m_N\}$ are integers which are pairwise coprime, then the system of congruences $\{x \equiv n_k\,(mod\,m_k)\}$ has an integer solution; in particular, it has a unique solution X modulo $M = \prod_{k=1}^{N} m_k$, and an integer is a solution if and only if it is $\equiv X\,(mod\,M)$.*

This can be recast in algebraic language: let $\prod_{n=1}^{N} p_n^{x_n}$ be the prime factorization of a natural number ν; then $\mathbb{Z}/\nu\mathbb{Z} \cong (\mathbb{Z}/p_1^{x_1}\mathbb{Z}) \times (\mathbb{Z}/p_2^{x_2}\mathbb{Z}) \times ... \times (\mathbb{Z}/p_3^{x_3}\mathbb{Z})$, with an explicit isomorphism given by $f(x+\nu\mathbb{Z}) = (x + p_1^{x_1}\mathbb{Z},\, x + p_2^{x_2}\mathbb{Z},\, ...,\, x + p_N^{x_N}\mathbb{Z})$. This form of the statement holds in any principal ideal domain P, if the p_n are replaced by prime elements of P.

From the Chinese Remainder Theorem we conclude that the residues of integers modulo powers of distinct primes are independent, i.e. an integer's residue modulo a power of 5 puts no constraints whatsoever on its residues modulo powers of 7. In considering the multiplicative structure of $\mathbb{Z}/n\mathbb{Z}$ it is thus sufficient to assume n is a prime power.

The existence of primitive roots modulo all powers of odd primes was proven by Gauss in 1801 (in Article 57 of his *Disquisitiones Arithmeticae*) [4]. A primitive root modulo p^n is a number γ that generates all the units modulo p^n by repeated multiplication. The multiplicative group of units modulo p^n is always cyclic, and a primitive root is the element such that every unit moduo p^n corresponds to some element of $\{\gamma, \gamma^2, ...\}$. The order of the multiplicative group of units modulo p^n is always given by Euler's totient function ϕ, defined so that $\phi(n)$ gives the number of naturals coprime to n and not greater than n . Thus a primitive root γ has $\gamma^x \equiv 1\,(mod\,p^n)$ if and only if $\phi(n) = (p-1)p^{n-1} \mid x$. Note that an integer which is a primitive root modulo p^2 is a primitive root modulo p^n $\forall n$. Note also that there is no primitive root modulo 2^n when $n > 2$; however, for each $n > 2$ a unit can be represented as the product of a unique power of -1 and a unique power of an element of order 2^{n-2}. We can fix an integer choice for this second element valid modulo 16 and it will be valid for all higher powers of 2.

P-Adic Numbers

The fields of p-adic numbers were introduced by Kurt Hensel in his 1897 paper *Über eine neue Begründung der Theorie der algebraischen Zahlen* [5]. They have become common tools in number theory, largely because they allow power series methods to be brought to bear on number-theoretical problems. The p in the term "p-adic" is a placeholder for a choice of prime; for any choice of $p \in \mathbb{P}$ there is a corresponding field $\mathbb{Q}_p$ of p-adic numbers. The simplest approach to their construction is to first consider the rings $\mathbb{Z}_p$ of p-adic integers, which are more immediately relevant and which relate to $\mathbb{Q}_p$ in the same way that $\mathbb{Z}$ relates to $\mathbb{Q}$ (specifically, $\mathbb{Q}_p$ is the field of fractions of $\mathbb{Z}_p$).

For a fixed prime p, $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ under the p-adic metric, defined by $d_p(x,y) = \begin{cases} 0 & : x = y \\ \frac{1}{p^n} & : x \neq y \end{cases}$, where n is the largest integer such that $p^n \mid (x-y)$. Thus two p-adic numbers are considered close to one another if their difference is divisible by a large power of p. We can define an absolute value on $\mathbb{Z}_p$ by $|x|_p := \frac{1}{p^n}$, where n is the maximal integer such that $p^n \mid x$ and where $|0|_p = 0$, and we will have $d(x,y) = |x-y|_p$ $\forall x, y \in \mathbb{Z}_p$; this is called the p-adic norm.

Now since $d_p(x,y) \leq \frac{1}{p^n}$ is equivalent for $x, y \in \mathbb{Z}$ to $x \equiv y \,(mod\, p^n)$, we see that if x and y are nonnegative integers then $d_p(x,y) = \frac{1}{p^N}$ where N is the largest number such that x and y share their first N digits when written in base-p notation. More formally, writing $x = \sum_{n=0}^{\infty} d_n p^n$ and $y = \sum_{n=0}^{\infty} \delta_n p^n$ for $\{d_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ sequences in $\{0, 1, ..., p-1\}$ which are both 0 for sufficiently large n, we have $x - y = \sum_{n=0}^{\infty} (d_n - \delta_n)p^n$; if $d_n - \delta_n = 0$ for all n in $\{0, ..., N-1\}$ but not for $n = N$, then $x - y = p^N(d_N - \delta_N) + p^{N+1}(d_{N+1} - \delta_{N+1}) + ...$, which is divisible by p^N but but not p^{N+1}, so $d_p(x,y) = \frac{1}{p^N}$.

This property holds true in general: the p-adic integers correspond bijectively with series of the form $\sum_{n=0}^{\infty} d_n p^n$ with each $d_n \in \{0, ..., p-1\}$, i.e. to base-p "integers" allowed to extend infinitely far to the left of the radix point. Addition and multiplication can be performed up to any desired precision according to the usual rules (addition can be performed by adding digits term-by-term and carrying ones, and multiplication can be performed by hand via base-p long multiplication or formally via the Cauchy product). The p-adic norm is simply $|x|_p = \frac{1}{p^n}$, where n is the number of leading 0s in the base-p representation of x.

It is worth noting that $\sum_{n=0}^{\infty}(p-1)p^n = -1$, as can easily be seen by adding 1 and observing that every digit carries. This is in fact the crucial difference between base-p notation extended to the left ($\mathbb{Z}_p$) and base-p notation extended to the right ($\mathbb{R}$): the more familiar real series $\sum_{n=1}^{\infty}(p-1)p^{-n}$ is the base-p analogue of 0.999... in base 10, and is a second representation of $1p^0 + 0p^{-1} + 0p^{-2} + ... = 1$; unlike in $\mathbb{R}$, where a number with a terminating base-p digit sequence has a second base-p representation, the digit sequence for an element of $\mathbb{Z}_p$ is truly unique. A similar phenomenon will occur in $\bar{\mathbb{Z}}$.

A p-adic integer x is a unit if and only if $|x|_p = 1$; thus every p-adic integer (and thus every integer) is invertible in $\mathbb{Z}_p$ unless it is divisible by p. We also have that for any $x \in \mathbb{Z}_p \backslash \{0\}$ there is some unique $n \in \mathbb{N}_0$ and some unique unit $u \in \mathbb{Z}_p$ such that $x = up^n$. Now $\mathbb{Q}_p$ is the field of fractions of $\mathbb{Z}_p$, so any nonzero p-adic number $q \in \mathbb{Q}_p$ can therefore be written in the form $q = up^n$, where u is a unit in $\mathbb{Z}_p$ and n is a (possibly negative) integer. In base-p representation, this means that $\mathbb{Q}_p$ consists of elements of the form $x = \sum_{n=N}^{\infty} d_n p^n$, where N can be any fixed integer; that is, in $\mathbb{Q}_p$ we are permitted a finite number of digits to the *right* of the radix as well as infinitely many to the left. One may also construct $\mathbb{Q}_p$ directly by completing $\mathbb{Q}$ under the p-adic metric.

Factoradic Integers

The construction of the factoradic integers proceeds similarly to that of the p-adic integers, but makes use of the properties of factorials rather than prime powers. The earliest directly related result is Hillel Furstenberg's topological proof of the infinitude of the primes, which was published during his undergraduate studies in 1955 [3]. This short and elegant proof of Euclid's classical result relies on the introduction of a topology on the integers with basis the bidirectionally infinite arithmetic progressions (i.e. wave images), sometimes called the evenly-spaced integer topology.

Theorem. *(Furstenberg, 1955) There are infinitely many primes.*

Proof. Endow $\mathbb{Z}$ with the topology whose basis is the set of bidirectionally infinite arithmetic progressions. Each basis element is clopen, since $\mathbb{Z}\backslash\{\lambda x+\phi : x\in\mathbb{Z}\}$ is equal to $\bigcup\limits_{\psi\not\equiv\phi\,(mod\,\lambda)}\{\lambda x+\psi : x\in\mathbb{Z}\}$; thus any finite union of basis elements is closed. Now consider $S:=\bigcup\limits_{p\in\mathbb{P}}\{px+0 : x\in\mathbb{Z}\}$. The complement $\mathbb{Z}\backslash S=\{-1,\,1\}$ is clearly not a union of arithmetic progressions and therefore is not open, so S is not closed. Therefore S is not the union of finitely many arithmetic progressions; so there are infinitely many primes. □

As Furstenberg mentions in passing, this topological space is metrizable; the most natural metric is in many ways the factoradic metric. We define this metric by $d_!(x,y)=\begin{cases}0 & : x=y\\ \frac{1}{n!} & : x\neq y\end{cases}$, where n is the largest integer such that $n!\mid(x-y)$, and we denote by $\bar{\mathbb{Z}}$ the corresponding metric completion of $\mathbb{Z}$. The metric balls are thus the bidirectionally infinite arithmetic progressions of factorial common difference, i.e. $\{n!x+\phi : x\in\mathbb{Z}\}$ for fixed ϕ and n (and every bidirectionally infinite arithmetic progression is clearly a union of these).

Now just as every element $x\in\mathbb{N}_0$ has a unique radix expansion in any base, so also is there a unique representation of the form $x=\sum\limits_{n=1}^{\infty}n!d_n$ with each $d_n\in\{0,1,...,n\}$. These are called factoradic representations, whence the term "factoradic integers" for $\bar{\mathbb{Z}}$, and the d_n are the called the factoradic digits of x. On $\mathbb{N}_0$ these representations terminate with $d_n=0$ for all sufficiently large n; similarly to the p-adic case, we have that the metric ball of radius $\frac{1}{N!}$ consists of all such series which share the same values for $\{d_1,...,d_{N-1}\}$, since this gives $\sum\limits_{n=1}^{\infty}n!d_n$-$\sum\limits_{n=1}^{\infty}n!\delta_n=N!(d_N-\delta_N)+(N+1)!(d_{N+1}-\delta_{N+1})+....$. Since $\mathbb{N}$ is clearly dense, in the metric completion $\bar{\mathbb{Z}}$ we thus obtain a canonical series $\sum\limits_{n=1}^{\infty}n!d_n$ for each element of $\bar{\mathbb{Z}}$, and the distance between two elements of $\bar{\mathbb{Z}}$ is given by $d_!(x,y)=\frac{1}{N!}$, where N is the least natural such that the Nth respective digits d_N and δ_N of x and y differ. (Note that we only have agreement of the first $N-1$ digits, not the first N, because unlike in the p-adic case there is no d_0.) Moreover, addition and multiplication are continuous functions, and so we can multiply and add such series in the usual way, as in the p-adic case performing carries whenever digits become too large.

In $\bar{\mathbb{Z}}$ we have the identity $-1=\sum\limits_{n=1}^{\infty}n!n$, which is readily verified since $1+(1!1+2!2+3!3...)$ $=1!2+2!2+3!3...=2!3+3!3+...=3!4+...$, a sequence which has limit $1!0+2!0+3!0+...=0$. Similarly to the p-adic case, we can represent real numbers in $[0,1]$ by series of the form $\sum_{n=1}^{\infty} \frac{d_n}{(n+1)!}$ with each $d_n \in \{0, ..., n\}$, and these representations are unique unless they terminate in a string of 0 digits, in which case if d_n is the last nonzero digit we have $\frac{d_1}{2!} + \frac{d_2}{3!} + ... + \frac{d_{n-1}}{n!} + \frac{d_n}{(n+1)!} = \frac{d_1}{2!} + \frac{d_2}{3!} + ... + \frac{d_{n-1}}{n!} + \frac{d_n - 1}{(n+1)!} + \frac{n+1}{(n+2)!} + \frac{n+2}{(n+3)!} + ...$, i.e. there are two representations. In this case, however, with the exceptions of 0 and 1 the numbers in $[0, 1]$ with terminating factoradic representations are exactly the rational numbers (just write $\frac{p}{q} = \frac{p(q-1)!}{q!}$ and then perform digit carries as many times as necessary). Some attention is given to this phenomenon in Section 5, where it is used to show that $\bar{\mathbb{Z}} \backslash \mathbb{Z}$ is homeomorphic to $[0, 1] \backslash \mathbb{Q}$.

The factoradic integers preserve all of modular arithmetic, whereas the p-adic integers preserve only arithmetic modulo powers of p; that is, no natural number except for 1 is a unit, and $\bar{\mathbb{Z}}/n\bar{\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}\ \forall n \in \mathbb{N}$. This is evident from the form of canonical series; $x = \sum_{n=1}^{\infty} n! d_n$ is always congruent to $\sum_{n=1}^{N-1} n! d_n \,(mod\, N!)$, and since elements of $\bar{\mathbb{Z}}$ are limits of sequences in $\mathbb{Z}$ which are eventually constant modulo any factorial (and hence modulo any natural number), properties of addition and multiplication modulo n which hold true in $\mathbb{Z}$ also hold true in $\bar{\mathbb{Z}}$. Indeed, if the sequence of partial sums of each canonical series is identified with a sequence of residues modulo increasing factorials, then the set of all such sequences is maximal such that the Chinese Remainder Theorem and the consistency law, $x \equiv c\,(mod\, p^n) \longrightarrow \exists d \,\text{such that}\, x \equiv c + p^n d\,(mod\, p^{n+1})$, hold. This is equivalent to the characterization of $\bar{\mathbb{Z}}$ algebraically as the profinite completion of the integers in Section 10. Consequently, because for a natural with prime factorization $\prod_n p_n^{x_n}$ we have $\mathbb{Z}/\prod_n p_n^{x_n}\mathbb{Z} \cong (\mathbb{Z}/p_1^{x_1}\mathbb{Z}) \times (\mathbb{Z}/p_2^{x_2}\mathbb{Z}) \times ....$ by the Chinese Remainder Theorem, we obtain $\bar{\mathbb{Z}} \cong \prod_{p \in \mathbb{P}} \mathbb{Z}_p$. This establishes the non-unit half of a unique factorization theorem characterizing multiplication in $\bar{\mathbb{Z}}$; the characterization of the multiplicative group of units is performed along similar lines by considering primitive roots modulo powers of primes, but requires first the establishment of a notion of exponentiation to factoradic powers.

Not only exponentiation but summation can also be extended; for a large class of functions it is possible to continuously extend $\sum_{n=a}^{b} f(n)$ from $a, b \in \mathbb{Z}$ with $a \leq b$ to a, b general factoradic integers. These factoradic series behave analogously to finite series in most ways, and in some ways behave like line integrals. For instance, if Δ is the forward difference operator, i.e. $(\Delta f)(x) = f(x+1) - f(x)$, then for any continuous function we have $f(x) = f(0) + \sum_{n=1}^{x} (\Delta f)(n)$; likewise, when f is summable we have "path-independence", i.e. $\sum_{n=a+1}^{b} f(n) + \sum_{n=b+1}^{c} f(n) = \sum_{n=a+1}^{c} f(n)\ \forall a, b, c \in \bar{\mathbb{Z}}$, and therefore closed paths $\sum_{n=a+1}^{a} f(n)$ sum to zero. The theory of finite differences thus has a deep relationship to $\bar{\mathbb{Z}}$; indeed, it is shown in Theorem 13 that a continuous function $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ decomposes into a direct product of functions

$\mathbb{Z}_p \to \mathbb{Z}_p$ over all $p \in \mathbb{P}$ if and only if it is equal to its own Newton series, which is true if and only if $(\Delta^n f)(0) \to 0$ as $n \to \infty$.

These considerations lead via the rising and falling factorials to the consideration of polynomial ring completions. A ring of polynomials $\mathbb{F}[x]$ over a field $\mathbb{F}$ is a principal ideal domain, and consequently its competion under the topology with basis all wave images decomposes into a direct product over the polynomial analogues of the rings of p-adic integers, just as $\bar{\mathbb{Z}} \cong \prod_{p\in\mathbb{P}} \mathbb{Z}_p$. It is also shown that $\bar{\mathbb{Z}}$ is ring-isomorphic to the quotient ring R/xR, where R is the projective limit $\varprojlim \mathbb{Z}/x(x+1)...(x+n)\mathbb{Z}$, $n \in \mathbb{N}$ ordered by magnitude. Elements of this ring are given by series of the form $a_0 + \sum_{n=0}^{\infty} a_{n+1}(x+0)...(x+n)$ with all coefficients in $\mathbb{Z}$.

$\bar{\mathbb{Z}}$ and Related Constructions

There are two notions of "infinite integers" to which $\bar{\mathbb{Z}}$ is deeply related and which provide excellent intuitions for working with $\bar{\mathbb{Z}}$. The first is the set of supernatural numbers, or Steinitz numbers. These were introduced to field theory by Ernst Steinitz in 1910 [11]; they generalize the prime factorizations of the natural numbers, allowing an infinite number of prime factors as well as factors of p^∞. Supernatural numbers cannot be added, but can be multiplied, and the gcd and lcm operations extend naturally to them; they are useful extensions of $\mathbb{N}$ for describing the orders and indices of elements of profinite groups [10]. These correspond exactly to the elements of $\bar{\mathbb{Z}}/U = \{zU : z \in \bar{\mathbb{Z}}\}$, where U is the multiplicative group of units in $\bar{\mathbb{Z}}$, which are used in Section 8.2 to characterize the multiplicative group of units in $\bar{\mathbb{Z}}$ as a product of topological $\bar{\mathbb{Z}}$-modules.

The second notion is that of the hyperintegers from non-standard analysis. Non-standard analysis was developed by Abraham Robinson in the 1960s; it is based on a foundation of mathematical logic which gives extensions to the standard model of arithmetic in order to obtain a rigorous theory of infinite and infinitesimal numbers. The most relevant result here is the extension of the integers $\mathbb{Z}$ to the uncountably infinite set $*\mathbb{Z}$ of hyperintegers, which is done in such a way that all first-order propositions that hold in $\mathbb{Z}$ continue to hold in $*\mathbb{Z}$, and vice versa. The hypernatural numbers $*\mathbb{N} = *\mathbb{Z}_{>0}$ contain infinitely large but well-defined numbers, and prime factorizations of hypernaturals correspondingly can involve infinitely many primes, possibly raised to general hypernatural powers; indeed, there are infinitely large *primes* in $*\mathbb{N}$ distinct from the primes in $\mathbb{N}$, and there are hypernaturals all prime factors of which are infinite. Sets in $*\mathbb{N}$ can be divided into two types: there are *internal* sets, and statements of the form "for all sets of numbers..." or "there exists a set of numbers such that..." true in $\mathbb{N}$ are also true in $*\mathbb{N}$ if "set" is replaced by "internal set"; and there are also *external sets*, to which such statements need not extend. $\mathbb{N}$ itself is an external set within $*\mathbb{N}$, as is its complement $*\mathbb{N}\backslash\mathbb{N}$. Note that the well-ordering property of $\mathbb{N}$ extends *a priori* only to internal sets: in particular, there is no least element of $*\mathbb{N}\backslash\mathbb{N}$. [9]

Carter Waid in 1974 published a short article entitled *On Dirichlet's Theorem and Infinite Primes* which is of great interest [12]. Therein he follows Robinson in constructing a ring isomorphic to $\bar{\mathbb{Z}}$ from $*\mathbb{N}$. Let $\mu = \bigcap_{n \in \mathbb{N}} n(*\mathbb{Z})$, so that μ is the set of all hypernatural integers divisible by every natural number; then μ is an ideal of $*\mathbb{Z}$ and is an external set. In fact, $*\mathbb{Z}/\mu$ is isomorphic to $\bar{\mathbb{Z}}$. In this sense the elements of $\bar{\mathbb{Z}}$ are equivalence classes of all hyperintegers with the same "finite part"; it is important to note, however, that μ is *not* a principal ideal–there is no least nonzero element m of μ, just as there is no least element of $*\mathbb{N}\backslash\mathbb{N}$.

The main results of the paper, however, deal with the relationship between $*\mathbb{N}$, $\bar{\mathbb{Z}}$, and Dirichlet's Theorem. Dirichlet's Theorem on Arithmetic Progressions is a celebrated result in number theory; it had long been conjectured, but was first proven by Johann Dirichlet in 1837 [6]. The statement of the theorem is this: every arithmetic progression $\{\lambda n + \phi : n \in \mathbb{N}_0\}$ with $\gcd(\lambda, \phi) = 1$ contains an infinite number of primes. In $\bar{\mathbb{Z}}$ this statement corresponds to the assertion $\bar{\mathbb{P}} = \mathbb{P} \cup U$, where $^-$ denotes topological closure and U denotes the multiplicative group of units in $\bar{\mathbb{Z}}$. Waid proves two theorems, giving three important results. Firstly, every infinite prime $p \in *\mathbb{N}\backslash\mathbb{N}$ maps via the natural homomorphism to a unit in $\bar{\mathbb{Z}}$. Secondly, given Dirichlet's Theorem, every unit in $\bar{\mathbb{Z}}$ is the image of an infinite prime. Lastly, any proof that every unit in $\bar{\mathbb{Z}}$ is the image of an infinite prime would also constitute a proof of Dirichlet's Theorem.

Part 1. **Left Outer-Distributive Rings**

The ring axioms are nearly symmetrical in the constraints they place upon addition and multiplication. There are only two asymmetries, namely: (1) the axioms require additive inverses to exist for all elements, but do not require multiplicative inverses; and (2) the distributive law fixes an asymmetrical relationship between addition and multiplication. The first asymmetry is merely a lack of constraint, and ring theory includes the study of rings with all possible multiplicative groups of units; but the second asymmetry is also mutable, i.e. there are consistent sets of axioms which are identical to the ring axioms except for the form of the distributive law. As shall be shown, the arithmetic progressions, considered as increasing functions $\mathbb{Z} \to \mathbb{Z}$ and equipped with addition via $(f+g)(x) = f(x) + g(x)$ and with "multiplication" defined as composition, i.e. $(fg)(x) = f(g(x))$, give a simple example of such an algebraic object. It will therefore be useful to examine the basic principles of algebra on "rings" with a modification of the distributive law akin to that found in the case of the arithmetic progressions; these are the *left outer-distributive rings.* Their most striking feature is the behavior of the additive identity: no longer does 0 absorb all elements under both left- and right-multiplication, but instead its action from the right encodes information about the distributive law, and it absorbs only when acting from the left.

1. Definition & General Properties

Definition 1. We say a set L equipped with binary operations $+, \cdot$ is a *left outer-distributive ring (LODR)* if:

1. L is an abelian group under $+$;
2. $\cdot$ is associative;
3. $\forall x, y, z \in L\ (x+y)z = xz + yz$;
4. $\forall x, y, z \in L\ x(y+z) = xy + xz + \mathscr{D}(x)$ for some function $\mathscr{D}$, called the *distributor,* dependent only on x.

We will write ϵ_R, or simply ϵ, for the additive identity in L.

Proposition 1. *Let L be a LODR; then $\mathscr{D}(x) = -x\epsilon$.*

Proof. $x\epsilon = x(\epsilon+\epsilon) = x\epsilon + x\epsilon + \mathscr{D}(x) \implies \mathscr{D}(x) = -x\epsilon$. ∎

Proposition 2. *Let L be a LODR; then $\epsilon x = \epsilon\ \forall x$.*

Proof. $\epsilon x = (\epsilon+\epsilon)x = \epsilon x + \epsilon x \implies \epsilon x = \epsilon$. ∎

Corollary. *Let L be a LODR; then $\epsilon^2 = \epsilon$.*

Proposition 3. *Let L be a LODR; then the following negativity relations hold:*

1. $(-x)y = -xy$
2. $x(-y) = -xy + x\epsilon + x\epsilon$

Proof. For each part:

(1) Suppose $(-x)y = -xy + f(x,y)$. Then $f(x,y) = (-x)y + xy = (-x+x)y = \epsilon y = \epsilon$.
(2) Suppose $x(-y) = -xy + f(x,y)$. Then $f(x,y) = x(-y) + xy = x(-y+y) + x\epsilon = x\epsilon + x\epsilon$.

□

Corollary. *Let L be a LODR and let $x \in L$; then $\mathscr{D}(\mathscr{D}(x)) = -\mathscr{D}(x)$.*

Proof. $\mathscr{D}(\mathscr{D}(x)) = -(-x\epsilon)\epsilon = -(-x\epsilon^2) = -\mathscr{D}(x)$. □

Proposition 4. *Let L be a LODR; then $X = \{x \in L : -x\epsilon = \epsilon\}$ is a ring under the inherited LODR operations. If there is a left-identity in L, it is in X.*

Proof. X consists precisely of those x whose distributor is ϵ. Therefore multiplication is properly distributive within X. X is additively closed since $\forall x, y \in X\ \forall a, b \in L\ (x+y)(a+b) = x(a+b) + y(a+b) = xa + ya + xb + yb = (x+y)a + (x+y)b$, hence $\mathscr{D}(x+y) = \epsilon$ so $(x+y) \in X$, and X is multiplicatively closed since $\forall x, y \in X\ \forall a, b \in L\ xy(a+b) = x(ya+yb) = xya + xyb$. Additive inverses of elements of X are in X since $\forall x \in X\ \forall a, b \in L$ $(-x)(a+b) = -(x(a+b)) = -xa - xb$ by Proposition 3, and $\epsilon \in X$ since $\mathscr{D}(\epsilon) = -\epsilon^2 = \epsilon$. If there is a left-identity e_- in L, then $e_-(a+b) = a + b = e_-a + e_-b\ \forall a, b \in L$, so $e_- \in X$. □

Definition 2. Suppose L is a LODR; then X as in Proposition 4 is called the *distributive subring of* L and is denoted $DSR(L)$. If $DSR(L)$ is commutative we say L is *DSR-commutative.*

Proposition 5. *Let L be a LODR; then $\forall x \in L$, $x - x\epsilon \in DSR(L)$.*

Proof. $\mathscr{D}(x - x\epsilon) = -(x - x\epsilon)\epsilon = -x\epsilon - (-x\epsilon)\epsilon = -x\epsilon + x\epsilon = \epsilon$. □

Definition 3. Suppose L is a LODR. Then a *left L-module* is an abelian group A and an operation $\cdot : L \times A \to A$ such that $\forall r, \rho \in L$ and $\forall a, b \in A$ the following hold:

(1) $r \cdot (a+b) = r \cdot a + r \cdot b - r \cdot 0_A$
(2) $(r + \rho) \cdot a = r \cdot a + \rho \cdot a$
(3) $(r\rho) \cdot a = r \cdot (\rho \cdot a)$
(4) $e_- \cdot a = a$, if e_- is a left-identity in L.

Proposition 6. *Let L be a LODR and let M be a left L-module. Then $\forall m \in M$, $\forall r \in L$ we have:*

(1) $\epsilon_R \cdot m = 0_M$
(2) $(-r) \cdot m = -(r \cdot m)$

Proof. Proceeding for each part:

(1) $\epsilon \cdot m = (\epsilon + \epsilon) \cdot m = \epsilon \cdot m + \epsilon \cdot m$;
(2) $0 = \epsilon \cdot m = (r - r) \cdot m = r \cdot m + (-r) \cdot m$

□

2. Wave Spaces

Definition 4. Suppose L is a LODR. Then we define the *LODR of waves over L* to be the set $\underset{L}{\widetilde{DSR(L)}} := DSR(L) \times L$ equipped with the operations $(\lambda,\phi)+(\rho,\psi) = (\lambda+\rho,\, \phi+\psi)$ and $(\lambda,\phi)(\rho,\psi) = (\lambda\rho,\, \lambda\psi+\phi)$; the elements of a LODR of waves are called waves. We will utilize the notation $\underset{\phi}{\widetilde{\lambda}} := (\lambda,\phi)$, and so we have $\underset{\phi}{\widetilde{\lambda}}+\underset{\psi}{\widetilde{\rho}} = \underset{\phi+\psi}{\widetilde{\lambda+\rho}}$ and $\underset{\phi}{\widetilde{\lambda}}\underset{\psi}{\widetilde{\rho}} = \underset{\lambda\psi+\phi}{\widetilde{\lambda\rho}}$ $\forall \underset{\phi}{\widetilde{\lambda}}, \underset{\psi}{\widetilde{\rho}} \in \underset{L}{\widetilde{DSR(L)}}$. In general, if Λ and Φ are subsets of a LODR L, then we write $\underset{\Phi}{\widetilde{\Lambda}} := \left\{ \underset{\phi}{\widetilde{\lambda}} \,:\, \lambda \in \Lambda,\, \phi \in \Phi \right\} \subset \underset{L}{\widetilde{DSR(L)}}$.

Proposition 7. *Let L be a LODR. Then $\underset{L}{\widetilde{DSR(L)}}$ is a LODR.*

Proof. We verify the axioms:

(1) Additivity: Under addition $\underset{L}{\widetilde{DSR(L)}} \cong DSR(L) \oplus L$, so it is clearly an abelian group.

(2) Associativity: $\underset{\phi}{\widetilde{\lambda}}\left(\underset{\psi}{\widetilde{\rho}}\underset{F}{\widetilde{\mu}}\right) = \underset{\phi}{\widetilde{\lambda}}\,\underset{\rho F+\psi}{\widetilde{\rho\mu}} = \underset{\lambda(\rho F+\psi)+\phi}{\widetilde{\lambda\rho\mu}}$ and $\left(\underset{\phi}{\widetilde{\lambda}}\underset{\psi}{\widetilde{\rho}}\right)\underset{F}{\widetilde{\mu}} = \underset{\lambda\psi+\phi F}{\widetilde{\lambda\rho}}\,\underset{F}{\widetilde{\mu}} = \underset{\lambda\rho F+\lambda\psi+\phi}{\widetilde{\lambda\rho\mu}}$, and these are equal precisely when $\lambda \in DSR(L)$.

(3) Right-Distributivity: $\left(\underset{\phi}{\widetilde{\lambda}}+\underset{\psi}{\widetilde{\rho}}\right)\underset{F}{\widetilde{\mu}} = \underset{\phi+\psi}{\widetilde{\lambda+\rho}}\,\underset{F}{\widetilde{\mu}} = \underset{(\lambda+\rho)F+\phi+\psi}{\widetilde{(\lambda+\rho)\mu}} = \underset{\lambda F+\rho F+\phi+\psi}{\widetilde{\lambda\mu+\rho\mu}}$ since $(\lambda+\rho) \in DSR(L)$, which is $= \underset{\lambda F+\phi}{\widetilde{\lambda\mu}} + \underset{\rho F+\psi}{\widetilde{\rho\mu}} = \underset{\phi}{\widetilde{\lambda}}\underset{F}{\widetilde{\mu}} + \underset{\psi}{\widetilde{\rho}}\underset{F}{\widetilde{\mu}}$.

(4) Left Outer-Distributivity: $\underset{\phi}{\widetilde{\lambda}}\left(\underset{\psi}{\widetilde{\rho}}+\underset{F}{\widetilde{\mu}}\right) = \underset{\phi}{\widetilde{\lambda}}\,\underset{\psi+F}{\widetilde{\rho+\mu}} = \underset{\lambda(\psi+F)+\phi}{\widetilde{\lambda(\rho+\mu)}} = \underset{\lambda\psi+\lambda F+\phi}{\widetilde{\lambda\rho+\lambda\mu}} = \underset{\lambda\psi+\phi}{\widetilde{\lambda\rho}} + \underset{\lambda F}{\widetilde{\lambda\mu}}$ $= \underset{\lambda\psi+\phi}{\widetilde{\lambda\rho}} + \underset{\lambda F+\phi}{\widetilde{\lambda\mu}} - \underset{\phi}{\widetilde{\epsilon}}$.

∎

Proposition 8. *Let L be a (properly distributive) ring with identity. Then:*

(1) $\underset{0_L}{\widetilde{1_L}}$ *is the identity in* $\underset{L}{\widetilde{DSR(L)}} = \underset{L}{\widetilde{L}}$.

(2) $\underset{\phi}{\widetilde{\lambda}}$ *is left-invertible in* $\underset{L}{\widetilde{L}}$ *if and only if λ is left-invertible in L; if it is, then* $\underset{-\lambda^{-1}\phi}{\widetilde{\lambda^{-1}}}\,\underset{\phi}{\widetilde{\lambda}} = \underset{0}{\widetilde{1}}$.

(3) $\underset{\phi}{\widetilde{\lambda}}$ *is right-invertible in* $\underset{L}{\widetilde{L}}$ *if and only if λ is right-invertible in L; if it is, then* $\underset{\phi}{\widetilde{\lambda}}\,\underset{-\lambda^{-1}\phi}{\widetilde{\lambda^{-1}}} = \underset{0}{\widetilde{1}}$.

Proof. $\underset{\phi}{\widetilde{\lambda}}\underset{0}{\widetilde{1}} = \underset{\lambda 0+\phi}{\widetilde{\lambda 1}} = \underset{\phi}{\widetilde{\lambda}} = \underset{1\phi+0}{\widetilde{1\lambda}} = \underset{0}{\widetilde{1}}\underset{\phi}{\widetilde{\lambda}}$. If λ is left-invertible, $\underset{-\lambda^{-1}\phi}{\widetilde{\lambda^{-1}}}\underset{\phi}{\widetilde{\lambda}} = \underset{\lambda^{-1}\phi-\lambda^{-1}\phi}{\widetilde{\lambda^{-1}\lambda}} = \underset{0}{\widetilde{1}}$, and if it is right-invertible, $\underset{0}{\widetilde{1}} = \underset{\lambda(-\lambda^{-1}\phi)+\phi}{\widetilde{\lambda\lambda^{-1}}} = \underset{\phi}{\widetilde{\lambda}}\underset{-\lambda^{-1}\phi}{\widetilde{\lambda^{-1}}}$. If λ is not right- invertible, $\underset{\phi}{\widetilde{\lambda}}\underset{\psi}{\widetilde{\rho}} = \underset{\lambda\psi+\phi}{\widetilde{\lambda\rho}}$ which is never $\underset{0}{\widetilde{1}}$ since $\lambda\rho \neq 1$, and similarly $\underset{\psi}{\widetilde{\rho}}\underset{\phi}{\widetilde{\lambda}}$ is never $\underset{0}{\widetilde{1}}$ if λ is not left-invertible. So $\underset{\phi}{\widetilde{\lambda}}$ is invertible from a given side if and only if λ is. □

Definition 5. Let L be a LODR. Then for each $n \in \mathbb{N}_0$ we define the *nth order wave space* $\mathcal{W}_L^n$ (or simply $\mathcal{W}^n$) over R inductively by setting $\mathcal{W}^0 = R$ and setting $\mathcal{W}^n = \underset{\mathcal{W}^{n-1}}{\widetilde{DSR(\mathcal{W}^{n-1})}}$ for each $n > 0$, and we consider each $\mathcal{W}^n$ a $\mathcal{W}^{n+1}$-module under the action $\underset{\phi}{\widetilde{\lambda}} \cdot x = \lambda x + \phi$. This does indeed make $\mathcal{W}^n$ into a $\mathcal{W}^{n+1}$-module, as the following proposition shows.

Proposition 9. *Let L be a LODR. Then $\forall n \in \mathbb{N}$, $\mathcal{W}_L^{n-1}$ is a left $\mathcal{W}_L^n$-module under the action $\underset{\phi}{\widetilde{\lambda}} \cdot x = \lambda x + \phi$.*

Proof. $\mathcal{W}_L^{n-1}$ is certainly an abelian group under addition, since it is a LODR; we verify the other axioms:

(1) Left Outer-Distributivity: $\underset{\phi}{\widetilde{\lambda}}(x+y) = \lambda x+\lambda y+\phi = \lambda x+\phi+\lambda y+\phi-\phi = \underset{\phi}{\widetilde{\lambda}}x+\underset{\phi}{\widetilde{\lambda}}y-\underset{\phi}{\widetilde{\lambda}}0$.

(2) Right Proper-Distributivity: $\left(\underset{\phi}{\widetilde{\lambda}} + \underset{\psi}{\widetilde{\rho}}\right) x = \underset{\phi+\psi}{\widetilde{\lambda+\rho}}x = (\lambda+\rho)x + \phi + \psi = \lambda x + \phi + \rho x + \psi = \underset{\phi}{\widetilde{\lambda}}x + \underset{\psi}{\widetilde{\rho}}x$.

(3) Associativity: $\left(\underset{\phi}{\widetilde{\lambda}}\underset{\psi}{\widetilde{\rho}}\right) x = \underset{\lambda\psi+\phi}{\widetilde{\lambda\rho}}\, x = \lambda\rho x + \lambda\psi + \phi = \lambda(\rho x + \psi) + \phi = \underset{\phi}{\widetilde{\lambda}}\left(\underset{\psi}{\widetilde{\rho}}x\right)$.

(4) Identity: If $\underset{b}{\widetilde{a}}$ is a left-identity in $\mathcal{W}_L^{n-1}$ then $\underset{b}{\widetilde{a}}\underset{x}{\widetilde{0}} = \underset{ax+b}{\widetilde{0}} = \underset{x}{\widetilde{0}}$, so $\underset{b}{\widetilde{a}}x = ax + b = x$.

□

Remark. For any LODR L, $\mathcal{W}_L^1$ is isomorphic to the set

$\{f \colon L \to L$ such that $\exists \lambda \in DSR(L)$ and $\phi \in L$ with $f(x) = \lambda x + \phi\}$

under the operations $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(g(x))$.

Definition 6. Let L be a LODR, and let $S \subset \underset{L}{\widetilde{DSR(L)}}$. Then we define the set of wave images over L by $S/L = \left\{\underset{\phi}{\widetilde{\lambda}}L \,:\, \underset{\phi}{\widetilde{\lambda}} \in S\right\}$.

Definition 7. Suppose L is a unique factorization domain (UFD). Then $\Omega_\Phi(L)$, or simply Ω_Φ, denotes the set of *coprime waves over* L, defined by $\Omega_\Phi = \left\{\underset{\phi}{\widetilde{\lambda}} \,:\, \gcd(\lambda, \phi) = 1\right\}$.

Remark. Note that if $\gamma \in L_{\neq 0}$, $\gamma\Omega_\Phi = \left\{\gamma \underset{\phi}{\widetilde{\lambda}} \,:\, \gcd(\lambda,\phi) = 1\right\} = \left\{\underset{\phi}{\widetilde{\lambda}} \,:\, \gcd(\lambda,\phi) = \gamma\right\}$, since $\gcd(\lambda,\phi) = 1 \implies \gcd(\gamma\lambda,\gamma\phi) = \gamma$ and if $\gcd(\lambda,\phi) = \gamma$ we can write $\underset{\phi}{\widetilde{\lambda}} = \gamma \underset{\phi/\gamma}{\widetilde{\frac{\lambda}{\gamma}}}$.

Theorem 1. (Wave Topology Theorem) *Let L be a LODR such that $DSR(L)$ is commutative, and let $\Lambda = \{\lambda \in DSR(L) \,:\, \lambda$ is not a zero divisor$\}$. Then $\underset{L}{\widetilde{\Lambda}}/L$ is a basis for a topology on L. Moreover, if L is a UFD, then $\Lambda = L_{\neq 0}$, and:*

(1) *If $\gamma,\kappa \in L_{\neq 0}$, $\underset{\phi}{\widetilde{\lambda}} \in \gamma\Omega_\Phi$, $\underset{\psi}{\widetilde{\rho}} \in \kappa\Omega_\Phi$, and $\underset{F}{\widetilde{\mu}}L = \underset{\phi}{\widetilde{\lambda}}L \cap \underset{\psi}{\widetilde{\rho}}L$, then $\underset{F}{\widetilde{\mu}} \in lcm(\gamma,\kappa)\Omega_\Phi$.*

(2) *If Γ is a subset of $L_{\neq 0}$ such that $\gamma_1,\gamma_2 \in \Gamma \longrightarrow lcm(\gamma_1,\gamma_2) \in \Gamma$, then:*
   (a) *$\bigcup_{\gamma\in\Gamma} \gamma\Omega_\Phi/L$ is a basis for a topology on $\bigcup_{\gamma\in\Gamma} \underset{0}{\widetilde{\gamma}}L$.*
   (b) *$\bigcup_{\gamma\in\Gamma} \underset{R}{\widetilde{\gamma}}/L$ is a basis for a topology on L.*

Proof. First, we shall show that $\underset{L}{\widetilde{\Lambda}}/L$ is a basis for a topology on L. $\Lambda \subset DSR(L)$ so for all $\lambda \in \Lambda$ and $\phi \in \Phi$ we have $\phi = \underset{\phi}{\widetilde{\lambda}}0$, so every $r \in L$ is in some element of $\underset{L}{\widetilde{\Lambda}}/L$. Moreover, suppose $x \in \underset{\phi}{\widetilde{\lambda}}L \cap \underset{\psi}{\widetilde{\rho}}L$. Then $\exists n$ such that $\lambda n + \phi = x$, so $\lambda m + x = \lambda n + \lambda m + \phi$, and since $\lambda \in DSR(L)$ this means $\lambda m + x = \lambda(n+m) + \phi$, so $\underset{\phi}{\widetilde{\lambda}}L = \underset{x}{\widetilde{\lambda}}L$; similarly, $\underset{\psi}{\widetilde{\rho}}L = \underset{x}{\widetilde{\rho}}L$. Therefore $x \in \underset{x}{\widetilde{\lambda}}L \cap \underset{x}{\widetilde{\rho}}L = \underset{\phi}{\widetilde{\lambda}}L \cap \underset{\psi}{\widetilde{\rho}}L$; but subtracting x, this is equivalent to $0 \in \underset{0}{\widetilde{\lambda}}L \cap \underset{0}{\widetilde{\rho}}L$. If L is a UFD this intersection of principal ideals is itself a principal ideal $\underset{0}{\widetilde{\mu}}L$, so $x \in \underset{x}{\widetilde{\mu}}L = \underset{\phi}{\widetilde{\lambda}}L \cap \underset{\psi}{\widetilde{\rho}}L$; if not, we can take $\underset{0}{\widetilde{\mu}} = \underset{0}{\widetilde{\lambda\rho}}$; then since $\underset{0}{\widetilde{\mu}}L \subset \underset{0}{\widetilde{\lambda}}L \cap \underset{0}{\widetilde{\rho}}L$, taking $F = x$ we have $\underset{F}{\widetilde{\mu}}L \subset \underset{x}{\widetilde{\lambda}}L \cap \underset{x}{\widetilde{\rho}}L = \underset{\phi}{\widetilde{\lambda}}L \cap \underset{\psi}{\widetilde{\rho}}L$. Now λ,ρ are not zero-divisors so $\mu \neq 0$; therefore $\forall \underset{\phi}{\widetilde{\lambda}}L, \underset{\psi}{\widetilde{\rho}}L \in \underset{L}{\widetilde{\Lambda}}/L$ $\forall x \in \underset{\phi}{\widetilde{\lambda}}L \cap \underset{\psi}{\widetilde{\rho}}L$ $\exists \underset{F}{\widetilde{\mu}}L \in \underset{L}{\widetilde{\Lambda}}$ such that $x \in \underset{F}{\widetilde{\mu}}L \subset \underset{\phi}{\widetilde{\lambda}}L \cap \underset{\psi}{\widetilde{\rho}}L$, and so $\underset{L}{\widetilde{\Lambda}}/L$ is a basis for a topology on L. Now, suppose L is a UFD. Then we will clearly have $\mu = \mathrm{lcm}(\lambda,\rho)$. If, moreover, $\gamma = \gcd(x,\lambda) = \gcd(x,\rho)$, then $\gamma = \gcd(x,\mathrm{lcm}(\lambda,\rho)) = \gcd(x,\mu)$; so for each $\gamma \in L_{\neq 0}$, $\gamma\Omega_\Phi/L$ is closed under nonempty intersection, and therefore gives a basis for a topology on $\underset{0}{\widetilde{\gamma}}L$ by the argument above. Indeed, $\gcd(x,\mathrm{lcm}(\lambda,\rho)) = \mathrm{lcm}(\gcd(x,\lambda),\gcd(x,\rho))$, so if $\underset{\phi}{\widetilde{\lambda}}L \in \gamma_\lambda\Omega_\Phi/L$ and $\underset{\psi}{\widetilde{\rho}}R = L \in \gamma_\rho\Omega_\Phi/L$, we have $\underset{\phi}{\widetilde{\lambda}}L \cap \underset{\psi}{\widetilde{\rho}}L = \underset{F}{\widetilde{\mu}}L \in \mathrm{lcm}(\gamma_\lambda,\gamma_\rho)\Omega_\phi/L$ as claimed. So if Γ is a subset of $L_{\neq 0}$ closed under taking least common multiples, then $\bigcup_{\gamma\in\Gamma} \gamma\Omega_\Phi/L$ is closed under nonempty intersection, and therefore the above argument applies, and $\bigcup_{\gamma\in\Gamma} \gamma\Omega_\Phi/L$ is a basis for a topology on the subset of L included in the basis elements, i.e. in the union $\bigcup_{\gamma\in\Gamma}\bigcup_{\underset{\phi}{\widetilde{\lambda}}\in\Omega_\Phi}\gamma\underset{\phi}{\widetilde{\lambda}}L$. Since $\bigcup_{\underset{\phi}{\widetilde{\lambda}}\in\Omega_\Phi}\underset{\phi}{\widetilde{\lambda}}L = L$ (it would be $L_{\neq 0}$, but $\underset{0}{\widetilde{1}}\in\Omega_\Phi$), we have $\bigcup_{\gamma\in\Gamma}\bigcup_{\underset{\phi}{\widetilde{\lambda}}\in\Omega_\Phi}\gamma\underset{\phi}{\widetilde{\lambda}}L = \bigcup_{\gamma\in\Gamma}\gamma L = \bigcup_{\gamma\in\Gamma}\underset{0}{\widetilde{\gamma}}L$. As for $\bigcup_{\gamma\in\Gamma}\underset{R}{\widetilde{\gamma}}/L$, we see that it is closed under nonempty intersections as well, but that the union of its elements contains all of L. This establishes claims (2a) and (2b). $\square$

Corollary. *If L is a DSR-commutative LODR, then $\underset{\mathcal{W}_L^n}{\widetilde{DSR(L)}}/\mathcal{W}_L^n$ is a basis for a topology on $\mathcal{W}_L^n$. In particular, if R is a commutative (properly distributive) ring which is an integral domain, then by definition $\mathcal{W} = \underset{R}{\widetilde{R}}$ and $DSR(R)\backslash\{\text{zero divisors}\} = R_{\nmid 0}$, so*

$$\underset{\mathcal{W}}{\widetilde{R_{\nmid 0}}}/\mathcal{W} = \underset{\underset{R}{\widetilde{R}}}{\widetilde{R_{\nmid 0}}}/\underset{R}{\widetilde{R}} = \left\{ \underset{\underset{\psi}{\widetilde{\rho}}}{\widetilde{\left(\underset{0}{\widetilde{\lambda}}\right)}}\underset{R}{\widetilde{R}} \,:\, \lambda\in R\backslash\{0\},\ \underset{\psi}{\widetilde{\rho}}\in\underset{R}{\widetilde{R}} \right\}$$

is a basis for a topology on $\mathcal{W}$. We have $\underset{\underset{\psi}{\widetilde{\rho}}}{\widetilde{\left(\underset{0}{\widetilde{\lambda}}\right)}}\underset{F}{\widetilde{\mu}} = \underset{\lambda F+\psi}{\widetilde{\lambda\mu+\rho}} = \underset{\underset{\psi}{\widetilde{\lambda F}}}{\widetilde{\underset{\rho}{\widetilde{\lambda}}}}\mu$, so each basis element is of the form $\underset{\underset{\beta}{\widetilde{\lambda R}}}{\widetilde{\underset{\alpha}{\widetilde{\lambda}}}}R$ for some $\lambda\in R_{\nmid 0}$ and $\alpha,\beta\in R$, i.e., it is of the form $\left\{\underset{\psi}{\widetilde{\rho}} \,:\, \rho\equiv\alpha,\ \psi\equiv\beta\,(mod\,\lambda)\right\}$. Thus the basis elements are precisely the sets of waves with a common image in $\underset{R/\lambda R}{\widetilde{R/\lambda R}}$ for some $\lambda\in R_{\nmid 0}$ under the natural homomorphism $\underset{\psi}{\widetilde{\rho}}\mapsto\underset{\psi+\lambda R}{\widetilde{\rho+\lambda R}}$.

Definition 8. Suppose L is a DSR-commutative LODR and $\mathscr{T}$ is a topology on $S\subset W_n$. We say $\mathscr{T}$ is *a wave topology if $\exists W\subset\underset{L}{\widetilde{DSR(L)}}$ such that W/L is a basis for $\mathscr{T}$.* We say $\mathscr{T}$ is the *standard wave topology* (or simply *the* wave topology where unambiguous) on L if we may take $W = \underset{L}{\widetilde{DSR(L)\backslash\{\text{zero divisors}\}}}$.

Theorem 2. *Let $\mathscr{T}$ be a wave topology on a subset S of a UFD R, and let W be a subset of $\underset{R}{\widetilde{R}}$ such that W/R is a basis for $\mathscr{T}$. Then:*

(1) *$\mathscr{T}$ is the discrete topology if and only if for each $s\in S$ there exists a finite set $D\subset R$ such that $\prod D = 0$ and $\underset{s}{\widetilde{d}}R\in W/R$ $\forall d\in D$.*
(2) *$\mathscr{T}$ is the trivial topology if and only if, given $\Delta S = \{s_2 - s_1 \,:\, s_1, s_2\in S\}$ and $\Lambda = \left\{\lambda \,:\, \exists s\in S \text{ with } \underset{s}{\widetilde{\lambda}}R\in W/R\right\}$, we have $lcm(\hat{\Lambda})\mid\gcd(\hat{\Delta S})$ for all finite subsets $\hat{\Lambda}$, $\hat{\Delta S}$ of Λ and ΔS, respectively.*

Proof. Since R is a UFD, nonempty finite intersections of wave images are wave images; so $\mathscr{T}$ is the discrete topology if and only if for each $s\in S$ there exists a finite set of wave images

$\left\{\underset{s}{\widetilde{\lambda_n R}}\right\}_{n\in\{1,..,N\}}$ with intersection $\{s\} = \underset{s}{\widetilde{0}}R$. Since $\bigcap\left\{\underset{s}{\widetilde{\lambda_n R}}\right\}_{n\in\{1,..,N\}} = \mathrm{lcm}(\underset{s}{\widetilde{\lambda_1,...,\lambda_N}})R$, and since $\mathrm{lcm}(\lambda_1,...,\lambda_N) = 0 \iff \lambda_1\cdots\lambda_N = 0$, this occurs if and only if for each $s \in S$ there exists a finite set $D \subset R$ such that $\prod D = 0$ and $\underset{s}{\widetilde{d}}R \in W/R\ \forall d \in D$. Regarding the trivial topology, and using the definitions of Λ and ΔS above, we have that $\mathscr{T}$ is the trivial topology$\iff \forall\sigma \in S\ \forall\underset{\phi}{\widetilde{\lambda}}R \in \Lambda\ \sigma \in \underset{\phi}{\widetilde{\lambda}}R \iff \forall\sigma \in S\ S \subset \bigcap\left(\underset{\sigma}{\widetilde{\Lambda}}/R\right) \iff \lambda_n \mid (s_2 - s_1)$ for each $s_1, s_2 \in S$ and each $\lambda_n \in \Lambda \iff \gcd(\hat{\Delta S})$ is divisible by $\mathrm{lcm}(\hat{\Lambda})$ for all finite subsets $\hat{\Lambda} \subset \Lambda$ and $\hat{\Delta S} \subset \Delta S$. □

Corollary. *Take $S = R$ in the preceding theorem, and suppose W contains no wave of zero-divisor wavelength. Then:*

1. *$\mathscr{T}$ is the discrete topology if and only if R is the zero ring.*
2. *$\mathscr{T}$ is the trivial topology if and only if every element of R is a zero-divisor or a unit.*

Definition 9. Given a UFD R, let $\mathbb{P}_R$ denote the set of primes in R and let $\mathbb{P}_R^{\geq n}$ denote all elements of R which are the product of at least n primes, counted with multiplicity. If $W \subset \underset{R}{\widetilde{R}}$ generates a topology on $\bigcup W/R$, we write that topology $\mathscr{T}(W)$. We name the topologies generated by the following bases accordingly (these are all bases for topologies on the union of the basis elements by the Wave Topology Theorem (Theorem 1); the even-numbered topologies below are all derived from part (2a), the odd-numbered from part (2b) of the theorem):

1. (Standard Wave Topology) $\underset{R}{\widetilde{R_{\neq 0}}}/R = R_{\neq 0}\Omega_\Phi/R = \mathbb{P}_R^{\geq 0}\Omega_\Phi/R = \underset{R}{\widetilde{\mathbb{P}_R^{\geq 0}}}/R$
2. (GCD$=\gamma$ Wave Topology, Coprime Wave Topology) $\gamma\Omega_\Phi/R$ for $\gamma \in R_{\neq 0}$
3. (Composite Multiple Wave Topology of Order n) $\underset{R}{\widetilde{\mathbb{P}_R^{\geq n}}}/R$ for $n \in \mathbb{N}_0$
4. (Coprimal Composite Multiple Wave Topology of Order n) $\mathbb{P}_R^{\geq n}\Omega_\Phi/R$ for $n \in \mathbb{N}_0$
5. (r-adic Topology) $\underset{R}{\widetilde{r^{\mathbb{N}_0}}}/R$ for $r \in R_{\neq 0}$
6. (Coprimal r-adic Topology) $r^{\mathbb{N}_0}\Omega_\Phi/R$ for $r \in R_{\neq 0}$

Part 2. The Factoradic Integers

3. Construction

In light of the vocabulary and notation introduced in the preceding section, it is here worthwhile to repeat Furstenberg's 1955 proof of the infinitude of the primes, which was presented in the introduction.

Theorem 3. *(Furstenberg's Proof) There are infinitely many primes.* [3]

Proof. Consider $\mathbb{Z}$ with the wave topology. Each wave image $\underset{\phi}{\widetilde{\lambda}}\mathbb{Z}$ is clopen since $\mathbb{Z}\backslash\underset{\phi}{\widetilde{\lambda}}\mathbb{Z} = \bigcup\limits_{\psi\not\equiv\phi\,(mod\,\lambda)} \underset{\psi}{\widetilde{\lambda}}\mathbb{Z}$, so any finite union of wave images is closed. Let $C = \bigcup\limits_{p\in\mathbb{P}} \underset{0}{\widetilde{p}}\mathbb{Z}$; then $\mathbb{Z}\backslash C = \{-1, 1\}$ is finite hence clearly not open, so C cannot be closed; therefore $\mathbb{P}$ is infinite. $\square$

Definition 10. We define the *factoradic metric* $d_! : \mathbb{Z} \to \mathbb{R}_{\geq 0}$ by setting $d_!(n,m) = \begin{cases} \frac{1}{N!} & : n \neq m \\ 0 & : n = m \end{cases}$, where $N = \max\{N \in \mathbb{N} : N! \mid (n-m)\}$; to emphasize the metric, we may write $\mathbb{Z}_!$ for the integers considered as a metric space under $d_!$. Additionally, let $B_r(x)$ denote the open ball $\{y \in X : d(x,y) < r\}$ of radius r centered at x in a metric space X, and let $B_r[x]$ denote the closed ball $\{y \in X : d(x,y) \leq r\}$.

Proposition 10. *The factoradic metric is an ultrametric on $\mathbb{Z}$; the induced topology is the wave topology (generated by basis $\underset{\mathbb{Z}}{\widetilde{\mathbb{N}}}/\mathbb{Z}$), and the closed & open metric balls are given by $B_{\frac{1}{N!}}[x] = B_{\frac{1}{(N-1)!}}(x) = \underset{x}{\widetilde{N!}}\mathbb{Z}$.*

Proof. That $d_!(n,m) = d_!(m,n)$ is obvious and that $d_!(n,n) = 0$ follows by definition. Now for integers a, b, c, if $N_{ab}!$ is the greatest factorial dividing $b-a$ and $N_{bc}!$ is the greatest factorial dividing $c-b$, then write $m = \min\{N_{ab}, N_{bc}\}$, $M = \max\{N_{ab}, N_{bc}\}$ and we have for some $k, j \in \mathbb{Z}$ as follows: $c-a = c-b+b-a = m!k+M!j = m!(k+(m+1)(N_{ab}+2)\cdots(M)j)$. Therefore $m! \mid (c-a)$, so the greatest factorial $N_{ac}!$ dividing $c-a$ has $N_{ac} \geq m = \min\{N_{ab}, N_{bc}\}$. Consequently $d_!(a,c) = \frac{1}{N_{ac}!} \leq \frac{1}{\min\{N_{ab},N_{bc}\}!} = \max\left\{\frac{1}{N_{ab}!}, \frac{1}{N_{bc}!}\right\} = \max\{d_!(a,b),\, d_!(b,c)\}$, and so $d_!$ is an ultrametric. The open ball $B_{\frac{1}{N!}}(n)$ of radius $\frac{1}{N!}$ centered at $n \in \mathbb{Z}$ is therefore $\left\{m \in \mathbb{Z} : d_!(n,m) < \frac{1}{N!}\right\} = \{m \in \mathbb{Z} : \exists k \in \mathbb{N} \text{ such that } (N+k)! \mid (n-m)\}$ $= \{m \in \mathbb{Z} : (N+1)! \mid (n-m)\} = \underset{n}{\widetilde{(N+1)!}}\mathbb{Z}$. Had we used a closed ball we would have had $\{m \in \mathbb{Z} : d_!(n,m) \leq \frac{1}{N!}\} = \underset{n}{\widetilde{N!}}\mathbb{Z}$. Since the open balls are a basis, and each open ball is in $\underset{\mathbb{Z}}{\widetilde{\mathbb{N}}}/\mathbb{Z}$, it remains to show only that the remaining elements of $\underset{\mathbb{Z}}{\widetilde{\mathbb{N}}}/\mathbb{Z}$ are unions of open balls; but this is immediately clear, for $\underset{\phi}{\widetilde{\lambda}}\mathbb{Z} = \bigcup\limits_{k=0}^{(\lambda-1)!-1} \underset{\phi+k\lambda}{\widetilde{\lambda!}}\mathbb{Z}$ for all waves $\underset{\phi}{\widetilde{\lambda}} \in \underset{\mathbb{Z}}{\widetilde{\mathbb{N}}}$. $\square$

Lemma 1. *$\forall m \in \mathbb{N}\ \forall \phi \in \mathbb{N}_0\ \underset{\phi}{\widetilde{m!}}\mathbb{Z} = \underset{d_1}{\widetilde{2}}\underset{d_2}{\widetilde{3}}\underset{d_3}{\widetilde{4}}\underset{d_4}{\widetilde{5}}\cdots\underset{d_m}{\widetilde{m}}\mathbb{Z}$ for some set of $d_k \in \mathbb{Z}$, which is unique under the constraint $d_k \in \{0, 1, ..., k\}$.*

Proof. $\underset{d_1}{\widetilde{2}}\underset{d_2}{\widetilde{3}}\underset{d_3}{\widetilde{4}}\underset{d_4}{\widetilde{5}}\cdots\underset{d_{m-1}}{\widetilde{m}} \equiv \underset{2!d_2+d_1}{\widetilde{3!}}\underset{d_3}{\widetilde{4}}\underset{d_4}{\widetilde{5}}\cdots\underset{d_m}{\widetilde{m}} \equiv \underset{3!d_3+2!d_2+1!d_1}{\widetilde{4!}}\underset{d_4}{\widetilde{5}}\cdots\underset{d_{m-1}}{\widetilde{m}} \equiv \cdots \equiv \underset{\sum_{k=1}^{m-1} k!d_k}{\widetilde{m!}}$. The d_k exist and are unique because ϕ has a unique residue modulo 2! (hence a unique $d_1 \in \{0, 1\}$), a unique residue modulo 3! (hence a unique $d_2 \in \{0, 1, 2\}$ since d_1 is already fixed), etc. $\square$

Theorem 4. *Put the metric* $d(\{x_n\},\{y_n\}) = \lim_{n\to\infty} d_!(x_n, y_n)$ *on the set* $\mathcal{C}$ *of Cauchy sequences in* $\mathbb{Z}$*, and define an equivalence relation* $\{x_n\} \sim \{y_n\} \iff d(\{x_n\},\{y_n\}) = 0$*. Then* $\mathcal{C}/\sim$ *defines the metric completion* $\bar{\mathbb{Z}}$ *of* $\mathbb{Z}$ *under the factoradic metric, and the* $z \in \bar{\mathbb{Z}}$ *correspond bijectively to infinite series of the form* $\left\{\sum_{n=1}^{\infty} n!d_n \;:\; \text{each } d_n \in \{0,...,n\}\right\}$ *via the correspondence* $z = \sum_{n=1}^{\infty} n!d_n := \lim_{N\to\infty}\sum_{n=1}^{N} n!d_n$*. Two such series can be added and multiplied up to the* N*th term and then truncated to yield the first* N *terms of the series corresponding to their sum or product, respectively.*

Proof. A sequence is Cauchy precisely if by omitting a finite number of terms from the beginning we can obtain a sequence contained in metric balls of arbitrarily small size; i.e. a sequence $\{x_n\}$ in $\mathbb{Z}_!$ is Cauchy if and only if $\forall N \in \mathbb{N}\ \exists \nu \in \mathbb{N}$ such that $\forall n > \nu$ we have $x_n \in \underset{x_\nu}{\widetilde{N}!\mathbb{Z}}$. For a particular N, let ϕ_N denote the corresponding x_ν. Then we see from Lemma 1 that for each $N > 1$ there is a unique sequence $\{d_n\}_{n\in\mathbb{N}}$ with each $d_n \in \{0,...,n\}$ such that $\phi_N \equiv \sum_{n=1}^{N-1} n!d_n \,(mod\, N!)$. Furthermore, we must have $\phi_{N+1} \equiv \phi_N \,(mod\, N!)\ \forall N$ since $\underset{\phi_N}{\widetilde{N}!\mathbb{Z}} \cap \underset{\phi_{N+1}}{\widetilde{(N+1)!}} \neq \emptyset$, and so if $N, M \in \mathbb{N}\backslash\{1\}$ and $\{d_n\}_{n\in\{1,..,N-1\}}$ and $\{\delta_n\}_{n\in\{1,...,M-1\}}$ are the unique sequences corresponding to N and M, respectively, we must have $d_n = \delta_n$ $\forall n < \min\{N, M\}$. So fix the sequence $\{d_n\}_{n\in\mathbb{N}}$ such that $\forall N\ \phi_N \equiv \sum_{n=1}^{N-1} n!d_n \,(mod\, N!)$. Since the $\underset{\phi_N}{\widetilde{N}!\mathbb{Z}}$ are metric balls with diameters approaching 0 as $N \to \infty$, $d(\{x_n\},\{y_n\}) \neq 0$ if the same procedure applied to $\{y_n\}$ does not give the same $\phi_N\ \forall N$, and $d(\{x_n\},\{y_n\}) = 0$ if it does give the same $\phi_N\ \forall N$. Consequently the sequences $\{d_n\}_{n\in\mathbb{N}}$ correspond bijectively to the equivalence classes of Cauchy series induced by $\sim$, and we can take $\{\phi_N\}_{N\in\mathbb{N}}$ as a canonical representative of the equivalence class containing $\{x_n\}$. So considering the equivalence classes as elements of the metric completion $\bar{\mathbb{Z}}$, we have $\lim_{n\to\infty} x_n = \lim_{N\to\infty}\phi_N = \lim_{N\to\infty}\sum_{n=1}^{N} n!d_n$, which yields a canonical representative each $z \in \bar{\mathbb{Z}}$. That these series can be added and multiplied via truncation as claimed follows since we can add or multiply their respective ϕ_N (say ϕ_N and $\hat{\phi}_N$) and extract the first $N-1$ terms; since for $K \geq N$ the first $N-1$ terms of the series for $\phi_K + \hat{\phi}_K$ and $\phi_K\hat{\phi}_K$ are completely determined by arithmetic modulo $N!$, and since we must have $\phi_K \equiv \phi_N$ and $\hat{\phi}_K \equiv \hat{\phi}_N$ modulo $N!$, this operation is well-defined. □

Definition 11. We define the *ring of factoradic integers* to be the metric completion of $\mathbb{Z}$ under the factoradic metric and denote it by $\bar{\mathbb{Z}}$, as in Theorem 4. For a given $z \in \bar{\mathbb{Z}}$, *the canonical series for* z is the unique series of the form $\sum_{n=1}^{\infty} n!d_n$ with each $d_n \in \{0,...,n\}$ which sums to z, and for each $n \in \mathbb{N}$ we say that d_n is the *nth factoradic digit* (or simply *nth digit* when the meaning is clear) of $\bar{\mathbb{Z}}$.

Proposition 11. $\bar{\mathbb{Z}}$ *has the following properties:*

(1) *$\bar{\mathbb{Z}}$ is a complete, commutative topological ring;*
(2) *$\mathbb{N}$ is dense in $\bar{\mathbb{Z}}$;*
(3) *$\bar{\mathbb{Z}}$ has the cardinality of the continuum;*
(4) *$\bar{\mathbb{Z}}$ is totally bounded, totally disconnected, and sequentially compact;*
(5) *The maximum distance between any two points in $\bar{\mathbb{Z}}$ is 1;*
(6) *The inherited topology on $\bar{\mathbb{Z}}$ is $\mathscr{T}\left(\underset{\mathbb{Z}}{\widetilde{\mathbb{N}}}/\bar{\mathbb{Z}}\right) = \mathscr{T}\left(\underset{\bar{\mathbb{Z}}}{\widetilde{\mathbb{N}}}/\bar{\mathbb{Z}}\right)$, and the open metric balls are $B_{\frac{1}{N!}}(x) = \underset{x}{\widetilde{(N+1)!}}\bar{\mathbb{Z}}$, i.e. the set of $z \in \bar{\mathbb{Z}}$ such that the first N factoradic digits of z are the same as those of x.*

Proof. Proceeding for each part:

(1) $\bar{\mathbb{Z}}$ is the metric completion of a commutative topological ring.
(2) Every wave image contains positive integers.
(3) As a set, $\bar{\mathbb{Z}} = \{0, 1\} \times \{0, 1, 2\} \times \{0, 1, 2, 3\} \times ....$
(4) $\bar{\mathbb{Z}}$ is totally bounded since the open metric balls of radius $\frac{1}{(N-1)!}$ are given by $\underset{x}{\widetilde{N!}}\bar{\mathbb{Z}}$, $x \in \{0, ..., N! - 1\}$, and cover $\bar{\mathbb{Z}}$. $\bar{\mathbb{Z}}$ is totally disconnected for the same reason: given any subset $S \subset \bar{\mathbb{Z}}$, choose $s \in S$ and $N \in \mathbb{N}$ large enough that the metric ball $\underset{s}{\widetilde{N!}}\bar{\mathbb{Z}}$ does not contain all of S. Then $\underset{s}{\widetilde{N!}}\bar{\mathbb{Z}}$ is closed as well as open since $\bar{\mathbb{Z}} \backslash \underset{s}{\widetilde{N!}}\bar{\mathbb{Z}} = \bigcup_{\phi \not\equiv s \, (mod \, N!)} \underset{\phi}{\widetilde{N!}}\bar{\mathbb{Z}}$ is an open set, so $\underset{s}{\widetilde{N!}}\bar{\mathbb{Z}}$ and $\bar{\mathbb{Z}} \backslash \underset{s}{\widetilde{N!}}\bar{\mathbb{Z}}$ separate S. $\bar{\mathbb{Z}}$ is closed and totally bounded, so it is compact; it is a compact metric space, so it is sequentially compact.
(5) By the definition of the factoradic metric, the distance is always 0 or $\frac{1}{n!}$ for some $n \in \mathbb{N}$.
(6) These claims follow directly from the construction of $\bar{\mathbb{Z}}$ as the metric completion of $\mathbb{Z}$ and the continuity of addition and multiplication by constants.

□

Proposition 12. *$S \subset \bar{\mathbb{Z}}$ is clopen if and only if S is a finite union of wave images.*

Proposition 13. $-1 = \sum_{n=1}^{\infty} n!n.$

Proof. Since $n!(n+1) = (n+1)!1$, adding 1 to the proposed canonical series causes every digit to carry, giving $1+1!1+2!2+3!3+... = 1!2+2!2+3!3+... = 2!3+3!3+... = 3!4+... \to 0$ in the limit, so subtract 1 from both sides of $1 + \sum_{n=1}^{\infty} n!n = 0$. □

Definition 12. Suppose $p \in \mathbb{P}$. Then we define $p^{\overset{0}{\infty}}$ to be the unique element of $\bar{\mathbb{Z}}$ such that $p^{\overset{0}{\infty}} \equiv \begin{cases} 0 \, (mod \, q^n) & : q = p \\ 1 \, (mod \, q^n) & : q \neq p \end{cases}$. If $z \in \bar{\mathbb{Z}}$ and P is the set of primes dividing $\bar{\mathbb{Z}}$ (i.e., the set of p such that $z \in \underset{0}{\widetilde{p}}\bar{\mathbb{Z}}$) then we define $z^{\overset{0}{\infty}} = \prod_{p \in P} p^{\overset{0}{\infty}}$ (where we take a limit if the product is infinite). This is a sound definition, as the following proposition shows.

Proposition 14. *Let $p \in \mathbb{P}$. Then the given definition for $p^{\otimes}$ defines a unique element of $\bar{\mathbb{Z}}$, and products of infinitely many such elements are always well-defined factoradic integers invariant under rearrangement of factors.*

Proof. $p^{\otimes} := \bigcap_{n\in\mathbb{N}} \left(\underset{0}{\widetilde{p_0^n}}\bar{\mathbb{Z}} \cap \bigcap_{k=1}^{n-1} \underset{1}{\widetilde{p_k^n}}\bar{\mathbb{Z}} \right)$, where $\{p_k\}_{k\in\mathbb{N}_0}$ is an enumeration of $\mathbb{P}$ with $p_0 = p$. Since we always have $\underset{\phi}{\widetilde{p_i^{n+1}}}\bar{\mathbb{Z}} \subset \underset{\phi}{\widetilde{p_i^n}}\bar{\mathbb{Z}}$, this is an intersection of nested closed sets and is hence nonempty; moreover, we have for each finite intersection $\bigcap_{n=1}^{N} \left(\underset{0}{\widetilde{p_0^n}}\bar{\mathbb{Z}} \cap \bigcap_{k=1}^{n-1} \underset{1}{\widetilde{p_k^n}}\bar{\mathbb{Z}} \right) = \underset{\phi_N}{\widetilde{p_0^N \prod_{k=1}^{N-1} p_k^N}}\bar{\mathbb{Z}}$, where ϕ_N satisfies $\phi_N \equiv \begin{cases} 0 & (mod\, p^N) \\ 1 & (mod\, p_k^N),\ k \in \{1, ..., N\} \end{cases}$ (such ϕ_N exist since by the Chinese Remainder Theorem we can take each $\phi_N \in \mathbb{Z}$). For each N, define $\lambda_N = p_0^N \prod_{k=1}^{N-1} p_k^N$ and we have for the Nth partial intersection $I_N := \underset{\phi_N}{\widetilde{\lambda_N}}\bar{\mathbb{Z}}$. Now we see that $\bigcap_{n\in\mathbb{N}} \left(\underset{0}{\widetilde{p_0^n}}\bar{\mathbb{Z}} \cap \bigcap_{k=1}^{n-1} \underset{1}{\widetilde{p_k^n}}\bar{\mathbb{Z}} \right)$ $= \bigcap_{n=1}^{\infty} I_n$ uniquely defines a factoradic integer as follows: $\forall \kappa \in \mathbb{N}$ there exists an $N \in \mathbb{N}$ such that $\forall n > N\ \kappa! \mid \lambda_n$, since letting $\pi(x) =$ (the number of primes $\le x$) and $p_k =$ (the kth prime) we can write the prime factorization $\kappa! = \prod_{k=1}^{\pi(\kappa)} p_k^{m_k}$, and then choose $N > \max\{\pi(\kappa), m_1, m_2, ..., m_{\pi(\kappa)}\}$. Thus we need only the first N intersections to fully determine a single value of $x\,(mod\,\kappa!)$ which must hold for all $x \in \bigcap_{n=1}^{\infty} I_n$, i.e. the value of the first $\kappa - 1$ terms $\sum_{i=1}^{\kappa-1} i!d_i$ of the canonical series for x; since there exists an N for every κ, every digit of the canonical series is determined by the intersection. So the intersection is not only nonempty, but uniquely determines a single point in $\bar{\mathbb{Z}}$. Though this shows that a unique canonical series will correspond to the intersection, it does not give the canonical series directly. However, we do know which waves of prime-power order do and do not contain $p^{\otimes}$; in particular, for any $q \in \mathbb{P}$ and $n \in \mathbb{N}$ we have $p^{\otimes} \equiv \begin{cases} 0\,(mod\, q^n) & : q = p \\ 1\,(mod\, q^n) & : q \ne p \end{cases}$. Now for any set $P \subset \mathbb{P}$ enumerated by $\{p_n\}$ we have $\prod_{p\in P} p^{\otimes} = \lim_{N\to\infty} \prod_{n=1}^{N} p_n^{\otimes} \equiv \begin{cases} 0\,(mod\, q^n) & : q \in P \\ 1\,(mod\, q^n) & : q \notin P \end{cases}$ $\forall n \in \mathbb{N}$, which clearly does not depend on the enumeration but rather is entirely determined modulo q^n by whether $P \cap \{q\} = \emptyset$; so the product over infinite sets of this form always exists and is invariant under rearrangement of factors. $\square$

Theorem 5. *$\bar{\mathbb{Z}} \cong \prod_{p \in \mathbb{P}} \bar{\mathbb{Z}}/p^{\circledast}\bar{\mathbb{Z}}$, where the isomorphism is of rings and the product is a direct product. Moreover, for each $p \in \mathbb{P}$, $\bar{\mathbb{Z}}/p^{\circledast}\bar{\mathbb{Z}}$ is isomorphic to the ring of p-adic integers, that is, $\bar{\mathbb{Z}}/p^{\circledast}\bar{\mathbb{Z}} \cong \mathbb{Z}_p$.*

Definition 13. For each $p \in \mathbb{P}$, let $g_p \in \bar{\mathbb{Z}}$ be defined by $g_p \equiv \begin{cases} p\,(mod\, q^{\circledast}) & : q = p \\ 1\,(mod\, q^{\circledast}) & : q \neq p \end{cases}$, where q ranges over $\mathbb{P}$. We clearly have $\lim\limits_{n \to \infty} g_p^n = p^{\circledast}$ since $g_p^n \equiv \begin{cases} p^n \equiv 0\,(mod\, q^N) & : q = p,\ N \leq n \\ p^n\,(mod\, q^N) & : q = p,\ N > n \\ 1\,(mod\, q^N) & : q \neq p \end{cases}$ gives us $g_p^n \equiv p^{\circledast}\,(mod\, q^n)\ \forall q \in \mathbb{P}$; therefore we define $g_p^{\infty} = g_p^{\circledast} = p^{\circledast}$.

Proposition 15. *Let $z \in \bar{\mathbb{Z}}$. Then there exists a unit ω and a unique prime-indexed sequence $\{x_p\}_{p \in \mathbb{P}}$ of elements of $\mathbb{N}_0 \cup \{\infty\}$ such that $z = \omega \prod_{p \in \mathbb{P}} g_p^{x_p}$.*

Remark. Proposition 15 is a unique factorization theorem, albeit in terms of an infinite product. However, the familiar primes $\{p\}$ from $\mathbb{N}$ have been replaced by $\{g_p\}$; for each p, g_p is an associate of p that is not in $\mathbb{Z}$. Somewhat surprisingly, given that $\bar{\mathbb{Z}}$ arises very naturally from $\mathbb{N}$, we *cannot* use $\mathbb{P}$ in place of $\{g_p\}$, because modulo $q^{\circledast}$ for any $q \in \mathbb{P}\backslash\{p\}$ we will not in general have $p \equiv 1$. Indeed, the values of the primes modulo the powers of other primes vary widely, and consequently the infinite product $\prod_{p \in \mathbb{P}} p^{x_p}$ will not converge for all sequences of natural exponents; and unlike in the case of g_p, we cannot rely on $\lim\limits_{n \to \infty} p^n$ to converge either (though we do have convergence of $\lim\limits_{n \to 0} p^n = p^{\circledast} = g_p^{\infty}$). The g_p are in this sense a naturally privileged choice of prime associates, and the choice in our definition that they be $\equiv 1\,(mod\, q^{\circledast})$ for $q \neq p$ is *not* merely a notational convenience.

Proposition 16. *The* gcd *and lcm extend naturally to $\bar{\mathbb{Z}}$ and are well-defined up to multiplication by a unit. If $Z = \left\{ \omega_n \prod_{p \in \mathbb{P}} g_p^{x_{n,p}} \right\}_{n \in \mathbb{N}}$ is a set of factoradic integers expressed as in Proposition 15, then:*

(1) $\gcd(Z) = U \prod_{p \in \mathbb{P}} g_p^{x_p}$ *where for each $p \in \mathbb{P}$ we have $x_p = \min\{x_{n,p} : n \in \mathbb{N}\}$ (and ∞ is the largest possible value of $x_{n,p}$).*
(2) *$lcm(Z) = U \prod_{p \in \mathbb{P}} g_p^{x_p}$ where for each $p \in \mathbb{P}$ we have $x_p = \max\{x_{n,p} : n \in \mathbb{N}\}$ (and ∞ is the largest possible value of $x_{n,p}$).*

4. The Factoradic Rationals

Definition 14. We shall denote by $\hat{\mathbb{Q}}$ the *ring of factoradic rational numbers,* i.e. the ring of fractions $\bar{\mathbb{Z}}\left(\bar{\mathbb{Z}}\backslash\{\text{zero divisors}\}\right)^{-1}$. We denote by $\bar{\mathbb{Q}}$ the ring of fractions $\bar{\mathbb{Z}}\mathbb{N}^{-1}$.

Proposition 17. *$\bar{\mathbb{Q}} \cong \bar{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\otimes$ denotes the tensor product.*

Proposition 18. *Each element of $\hat{\mathbb{Q}}$ can be written $u\prod_{p\in\mathbb{P}} p^{n_p}$, where each $n_p \in \mathbb{Z}\cup\{\infty\}$ and u is a unit in $\bar{\mathbb{Z}}$.*

5. The Digit-Flip Function

Proposition 19. *Consider $[0,1]\subset\mathbb{R}$. For each $r\in[0,1]$ there is a sequence $\{d_n\}$ with each $d_n\in\{0,...,n\}$ such that $r=\sum_{n=1}^{\infty}\frac{d_n}{(n+1)!}$. Moreover, the sequence $\{d_n\}$ corresponding to r is unique unless $r\in\mathbb{Q}\backslash\{0,1\}$, in which case thare are exactly two corresponding sequences $\{d_n\}$ and $\{\delta_n\}$, which are such that the following hold:*

(1) *d_n is zero except at a finite set of values of n;*

(2) *Let $N=\max\{N\in\mathbb{N} : d_N\neq 0\}$. Then $\delta_n=\begin{cases} d_n & : n<N \\ d_n-1 & : n=N \\ n & : n>N \end{cases}$.*

Definition 15. The *digit-flip function* is the function $DF\colon\bar{\mathbb{Z}}\to[0,1]$ defined on canonical series by $DF\left(\sum_{n=1}^{\infty} n!d_n\right)=\sum_{n=1}^{\infty}\frac{d_n}{(n+1)!}$.

Proposition 20. *The digit-flip function is bijective restricted to $\bar{\mathbb{Z}}\backslash(\mathbb{Z}\backslash\{-1,0\})\to[0,1]\backslash(\mathbb{Q}\backslash\{0,1\})$, and is 2-to-1 restricted to $\mathbb{Z}\backslash\{-1,0\}\to\mathbb{Q}\cap(0,1)$. We have $DF(-1)=1$ and $DF(0)=0$.*

Proof. This follows since the only elements of $[0,1]$ with two representations are the elements of $\mathbb{Q}\backslash\{0,1\}$, whose digit sequences are given by Proposition 19 as $\delta_n=\begin{cases} d_n & : n<N \\ d_n-1 & : n=N \\ n & : n>N \end{cases}$, where d_n is the nth digit of the terminating digit sequence and δ_n is the nth digit of the non-terminating equivalent, and $N=\max\{N\in\mathbb{N} : d_N\neq 0\}$. The two preimages of a rational number in $(0,1)$ under the digit-flip function are thus a factoradic integer with terminating digit sequence (i.e. an element of $\mathbb{N}_0$) and a factoradic integer with digit sequence terminating in maximal digits $d_n=n$ (i.e. an element of $\mathbb{Z}\backslash\mathbb{N}_0$). Clearly we have $DF(\mathbb{Z})\subset(\mathbb{Q}\cap[0,1])$ as well as $(\mathbb{Q}\cap[0,1])\subset DF(\mathbb{Z})$, so $DF(\mathbb{Z})=(\mathbb{Q}\cap[0,1])$. $0\in(\mathbb{Q}\cap[0,1])$ has the single representation $\sum_{n=1}^{\infty}\frac{0}{(n+1)!}$ and hence the single preimage $\sum_{n=1}^{\infty} n!0=0\in\bar{\mathbb{Z}}$, and 1 has the single representation $\sum_{n=1}^{\infty}\frac{n}{(n+1)!}$ and hence the single preimage $\sum_{n=1}^{\infty} n!n=-1$, but every other rational in $[0,1]$ has two representations and hence two preimages. □

Lemma 2. $-(n!)=\sum_{k=n}^{\infty} k!k$.

Proof. $n! + \sum_{k=n}^{\infty} k!k = n!(1+n) + \sum_{k=n+1}^{\infty} k!k = (n+1)! + \sum_{k=n+1}^{\infty} k!k = ... = (n+m)! + \sum_{k=n+m}^{\infty} k!k$ for any $m \in \mathbb{N}$. This goes to 0 as $m \to \infty$ and so must be equal to 0; therefore $-(n!) = \sum_{k=n}^{\infty} k!k$. ☐

Proposition 21. *If $\sum_{n=1}^{\infty} \frac{d_n}{(n+1)!}$, each $d_n \in \{0, ..., n\}$, is the terminating representation of an element x of $\mathbb{Q} \cap (0,1)$, let $N := \max\{N \in \mathbb{N} : d_N \neq 0\}$. Then $DF^{-1}(x) = \{\nu, \tau\}$ where $\tau = \sum_{n=1}^{N} n!d_n$ and $\nu = \tau - N!(N+2)$.*

Proof. The two representations of x are $\sum_{n=1}^{N} \frac{d_n}{(n+1)!}$ and $\sum_{n=1}^{N-1} \frac{d_n}{(n+1)!} + \frac{d_N - 1}{(N+1)!} + \sum_{n=N+1}^{\infty} \frac{n}{(n+1)!}$, and consequently the preimages of x under the digit-flip function are $\tau = \sum_{n=1}^{N} n!d_n$ and $\nu = \sum_{n=1}^{N-1} n!d_n + N!(d_N - 1) + \sum_{n=N+1}^{\infty} n!n$. This gives $\nu - \tau = \left(\sum_{n=N+1}^{\infty} n!n\right) - N!$, which by Lemma 2 is equal to $-(N+1)! - N! = -N!(N+1+1) = -N!(N+2)$. ☐

Proposition 22. *The digit-flip function is continuous.*

Proof. Metric balls of factoradic integers correspond to sets of factoradic integers sharing the same first n digits, where $n \to \infty$ as the radius of the metric ball goes to 0. For any $z, \zeta \in \bar{\mathbb{Z}}$ sharing their first n digits we have that $DF(z)$ and $DF(\zeta)$ also share their first n digits, and therefore $|DF(z) - DF(\zeta)| \leq \frac{n+1}{(n+2)!} + \frac{n+2}{(n+3)!} + ... = \frac{1}{(n+1)!}$, so DF is continuous. ☐

Proposition 23. *Let $\sim$ be the equivalence relation which differs from identity only to equate the two digit-flip preimages of each rational in $(0,1)$. The inverse of the digit-flip function is continuous considered as a function $[0,1] \to (\bar{\mathbb{Z}}/\sim)$ with the quotient topology inherited from $\bar{\mathbb{Z}}$.*

Proof. Like the previous proposition, this follows from the fact that in both $[0,1] \backslash (\mathbb{Q} \backslash \{0,1\})$ and $\bar{\mathbb{Z}} \backslash (\mathbb{Z} \backslash \{-1,0\})$ a sequence $\{x_n\}$ approaches a point x if and only if $\forall N\ \exists K$ such that if $n > K$ then the first N digits of x_n agree with the first N digits of x. For points in $\mathbb{Q} \cap (0,1)$ the two digit sequences representing a point correspond to the two points making up a given equivalence class in $\bar{\mathbb{Z}}/\sim$, so any sequence of elements of $[0,1]$ approaching a chosen rational x will give a sequence via digit-flip approaching an element of $\bar{\mathbb{Z}}/\sim$. ☐

Theorem 6. *Under their respective subspace topologies, $\bar{\mathbb{Z}} \backslash \mathbb{Z}$ is homeomorphic to $[0,1] \backslash \mathbb{Q}$. More strongly, $\bar{\mathbb{Z}} \backslash (\mathbb{Z} \backslash \{-1,0\})$ is homeomorphic to $[0,1] \backslash (\mathbb{Q} \backslash \{0,1\})$.*

Proof. DF is a homeomorphism by the preceding propositions. ☐

6.1. **Infinite Series.**

Proposition 24. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\bar{\mathbb{Z}}$. *Then* $\sum_{n=1}^{\infty} x_n$ *converges if and only if* $\lim_{n\to\infty} x_n = 0$.

Proof. For each n write the canonical series $x_n = \sum_{k=1}^{\infty} k! d_{n,k}$, and suppose $\lim_{n\to\infty} x_n = 0$. Then we must have for each k only finitely many n with $d_{n,k} \neq 0$; if we define for each k the value $N_k = \max\{N \in \mathbb{N} : d_{N,k} \neq 0\}$ and write the partial sums $S_K = \sum_{k=1}^{K} k! d_{n,k}$, we therefore have $S_K \equiv S_{\max\{N_k : k\leq K\}} \,(mod\,(K+1)!)$ for all $K \geq \max\{N_k : k \leq K\}$. Therefore $\forall n, m \geq \max\{N_k : k \leq K\}$ we have $S_n - S_m \equiv 0 \,(mod\,(K+1)!)$, i.e. $d(S_n, S_m) \leq \frac{1}{(K+1)!}$. So $\{S_n\}_{n\in\mathbb{N}}$ is Cauchy, and the series converges. For the other half of the claim, suppose $\lim_{n\to\infty} x_n \neq 0$. Then since $x_n = S_n - S_{n-1}$ (setting $S_0 = 0$), $\lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \neq 0$; but if the S_n converged to L both of these limits would be L, and we have a contradiction. $\square$

Proposition 25. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\bar{\mathbb{Z}}$ *with* $\lim_{n\to\infty} x_n = 0$, *and let* σ *be any permutation of* $\mathbb{N}$. *Then* $\sum_{n=1}^{\infty} x_{\sigma n} = \sum_{n=1}^{\infty} x_n$. *Furthermore, let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\bar{\mathbb{Z}}$ *with* $\lim_{n\to\infty} x_n = 0$, *and let* $\{K_n\}_{n\in\mathbb{N}}$ *be a sequence of sets that partition* $\mathbb{N}$, *and let* $\sum_{k\in K_n} x_k$ *denote* $\lim_{M\to\infty} \sum_{i=1}^{M} y_{n,i}$ *for* $\{y_{n,i}\}_{i\in\mathbb{N}}$ *some fixed enumeration of* K_n. *Then* $\sum_{n=1}^{\infty} \sum_{k\in K_n} x_k = \sum_{n=1}^{\infty} x_n$.

Proof. Since $x_n \to 0$, the Nth factoradic digit of $\sum_{n=1}^{\infty} x_n$ depends on only the finitely many terms of $\{x_n\}$ that have $x_n \not\equiv 0 \,(mod\,(N+1)!)$. Under any permutation of the sequence the Nth digit of the sum will still depend only on this same finite set of numbers, and therefore $\sum_{n=1}^{\infty} x_{\sigma n}$ converges to a sum with the same canonical series as $\sum_{n=1}^{\infty} x_n$, as claimed. Similarly, if $\{K_n\}_{n\in\mathbb{N}}$ is any sequence of sets that partion $\mathbb{N}$, then the elements of $\{K_n\}$ give a disjoint cover of the finite set $S = \{n \in \mathbb{N} : x_n \not\equiv 0 \,(mod\,(N+1)!)\}$, and since this applies $\forall N \in \mathbb{N}$ and therefore we obtain the same factoradic digit sequence for both sums, $\sum_{n=1}^{\infty} \sum_{k\in K_n} x_k$ converges to the same value as $\sum_{n=1}^{\infty} x_n$ regardless of the order of summation over the K_n. $\square$

Corollary. *If* S *is a countable subset of* $\bar{\mathbb{Z}}$ *and the only limit point of* S *is* 0, *then* $\sum_{s\in S} s$ *converges to a well-defined factoradic integer* z *invariant under all rearrangements and groupings of terms.*

Corollary. *In particular, if* $\lim_{n\to\infty} x_n = 0$ *and* $[a,b]$ *is a bijection* $\{(a,b) \in \mathbb{N}_0^2 : a \geq b\} \to \mathbb{N}$ *, then* $\sum_{a=0}^{\infty}\sum_{b=0}^{a} x_{[a,b]} = \sum_{b=0}^{\infty}\sum_{a=b}^{\infty} x_{[a,b]}$.

Proposition 26. *Let* $\{f_n\}_{n\in\mathbb{N}_0}$ *be a sequence of continuous functions* $\bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ *such that* f_n *converges uniformly to the zero-function as* $n \to \infty$, *and let* $z \in \bar{\mathbb{Z}}$. *Then* $\sum_{n=0}^{\infty} f_n(x)$ *is a continuous function* $\bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$.

Proof. The series converges for any x since the f_n converge to 0, so the claim is equivalent to the assertion that $\forall z \in \bar{\mathbb{Z}}$ $\lim_{x\to z}\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} f_n(z)$. Now $\lim_{x\to z}\sum_{n=0}^{\infty} f_n(x) - \sum_{n=0}^{\infty} f_n(z)$ $= \lim_{x\to z}\sum_{n=0}^{\infty}(f_n(x) - f_n(z))$. Note that $d(x,y) \leq \frac{1}{m!} \iff m! \mid (y-x)$. Given $m!$, let $N \in \mathbb{N}$ such that $\forall n > N$ $m! \mid f_n(x)$ $\forall x$, which is possible since $f_n \to 0$ uniformly. Then modulo $m!$, $\sum_{n=0}^{\infty}(f_n(x) - f_n(z)) = \sum_{n=0}^{N}(f_n(x) - f_n(z))$. So choose

$\delta = \max_{n\in\mathbb{N}}\{\delta \in \mathbb{N} : \delta! \mid (z-x) \longrightarrow m! \mid (f_n(z) - f_n(x))\}$, and $\forall x \in \underset{z}{\widetilde{\delta!}}\bar{\mathbb{Z}}$ we have $f_n(x) - f_n(z) \equiv 0 \,(mod\, m)$ $\forall n \in \{0,...,N\}$ and hence $0 \equiv \sum_{n=0}^{N}(f_n(x) - f_n(z))$

$\equiv \sum_{n=0}^{\infty}(f_n(x) - f_n(z))$ $(mod\, m)$. In other words, $d(x,z) \leq \frac{1}{\delta!}$ implies $d(0, \sum_{n=0}^{\infty}(f_n(x) - f_n(z))) \leq \epsilon$. Therefore we have $\lim_{x\to z}\sum_{n=0}^{\infty}(f_n(x) - f_n(z)) = 0$, so $\lim_{x\to z}\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} f_n(z)$. ☐

6.2. **Factoradic Limits.**

Definition 16. Let X be a metric space and let$\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. We say $\{x_n\}$ *converges factoradically to* $L \in X$ *as* n *approaches* $z \in \bar{\mathbb{Z}}$, and write $\underset{n\to z}{\mathrm{lim}} x_n = L$, if $\lim_{n\to z} x_n = L$ when x_n is considered a function $x_n : (\mathbb{N}_! \subset \bar{\mathbb{Z}}) \to X$. That is, if $\forall\epsilon > 0$ $\exists\delta > 0$ such that $\forall n \in \mathbb{N}$ $0 < d_{\bar{\mathbb{Z}}}(n,z) < \delta \longrightarrow d_X(x_n, L) < \epsilon$, or equivalently, $\forall\epsilon > 0$ $\exists N \in \mathbb{N}$ such that $\forall n \in \underset{z}{\widetilde{N!}}\bar{\mathbb{Z}}$ $d_X(x_n, L) < \epsilon$.

Proposition 27. *Let* X *be a metric space and let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in* X, *let* $L \in X$, *and let* $z \in \bar{\mathbb{Z}}$. *Then* $\underset{n\to z}{\mathrm{lim}} x_n = L$ *if and only if* $\lim_{k\to\infty} x_{n_k} = L$ *for every subsequence* $\{x_{n_k}\}_{k\in\mathbb{N}}$ *with* $\lim_{k\to\infty} n_k = z$.

Proof. This follows immediately from the definition. ☐

Theorem 7. *Suppose* X *is a metric space,* $L \in X$, *and* $\{x_n\}_{n\in\mathbb{N}}$ *is a sequence in* X. *Then* $\lim_{n\to\infty} x_n = L$ *if and only if* $\underset{n\to z}{\mathrm{lim}} x_n = L$ $\forall z \in \bar{\mathbb{Z}}$.

Proof. $\lim_{n\to\infty} x_n = L$ implies that every subsequence of $\{x_n\}$ converges to L, and so in particular any subsequence $\{x_{n_k}\}$ where $n_k \to z \in \bar{\mathbb{Z}}$ converges to L; thus $\lim_{n\to z} x_n = L\ \forall z \in \bar{\mathbb{Z}}$. In the other direction, suppose $\lim_{n\to z} x_n = L\ \forall z \in \bar{\mathbb{Z}}$. Since $\bar{\mathbb{Z}}$ is sequentially compact, every sequence $\{n_k\}_{k\in\mathbb{N}}$ in $\bar{\mathbb{Z}}$ has a convergent subsequence; thus every subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}$ has a subsequence converging to L. Now suppose $\lim_{n\to\infty} x_n \neq L$; then there exists some subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}$ and some $\epsilon > 0$ such that $d_X(x_{n_k}, L) > \epsilon\ \forall k$. But then some subsequence $\left\{n_{k_j}\right\}_{j\in\mathbb{N}}$ of $\{n_k\}$ converges factoradically, and therefore the corresponding subsequence $\left\{x_{n_{k_j}}\right\}_{j\in\mathbb{N}}$ of $\{x_{n_k}\}$ converges to L. But this is a contradiction, for $d_X(x_{n_k}, L) > \epsilon\ \forall k$; therefore $\lim_{n\to\infty} x_n = L$. □

Definition 17. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in a metric space X. If $Z \subset \bar{\mathbb{Z}}$ and $\forall z \in Z$ $\lim_{n\to z} x_n$ exists, we say $\{x_n\}$ is *convergent on* Z. Moreover:

- If $\{x_n\}$ converges on $\mathbb{N}$ with $\lim_{n\to N} x_n = x_N$, we say $\{x_n\}$ is *consistent*. Equivalently, $\{x_n\}$ is consistent if the function $f\colon \mathbb{N}_! \to \bar{\mathbb{Z}}$ defined by $f(n) = x_n$ is continuous.
- If $\{x_n\}$ converges on $\bar{\mathbb{Z}}$, we say $\{x_n\}$ is *universally convergent.*
- Let $Z \subset \bar{\mathbb{Z}}$ be the maximal set whereupon $\{x_n\}$ is convergent; then *the function* $f : Z \to X$ *induced by* $\{x_n\}$ is defined by $f(z) = \lim_{n\to z} x_n$.

Proposition 28. *If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in a metric space X convergent on $Z \subset \bar{\mathbb{Z}}$, then the induced function $f(z) = \lim_{n\to z} x_n$ is continuous $Z \to X$.*

Proof. Let $z \in Z$ and let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in Z converging strictly monotonically to z as $n \to \infty$, i.e. $m > n$ implies $d_!(z_n, z) > d_!(z_m, z)$. (If this is not possible, z is an isolated point so $f(z)$ satisfies the definition of continuity vacuously.) For each n let $\{m_{n,k}\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ converging to n as $k \to \infty$ and let $N_n \in \mathbb{N}$ such that $\forall k \geq N$ $d(f(z_n), x_{m_{n,k}}) < \frac{1}{2^n}$. Then $\lim_{n\to\infty} m_{n,N_n} = z \in Z$ and $\{x_n\}$ is convergent on Z so by definition $\lim_{n\to\infty} x_{m_{n,N_n}}$ converges to $f(z)$. Therefore $\lim_{n\to\infty} f(z_n) = f(z)$, since $d(x_{m,N_n}, f(z_n)) < \frac{1}{2^n} \to 0$ as $n \to \infty$. □

6.3. **Factoradic Sequences in $\bar{\mathbb{Z}}$.**

Remark. For convenience, let us rephrase the convergence condition in terms of $\epsilon, \delta \in \mathbb{N}$ rather than $\epsilon, \delta \in \mathbb{R}_{>0}$: given a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\bar{\mathbb{Z}}$ and $L \in \bar{\mathbb{Z}}$, $\lim_{n\to z} x_n = L$ if and only if $\forall \epsilon \in \mathbb{N}\ \exists \delta \in \mathbb{N}$ such that $n \in \underset{z}{\widetilde{\delta!}}\bar{\mathbb{Z}} \longrightarrow x_n \in \underset{L}{\widetilde{\epsilon!}}\bar{\mathbb{Z}}$.

Definition 18. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $\bar{\mathbb{Z}}$. We say that $\{x_n\}$ is *modularly eventually periodic (MEP)* if $\forall \kappa \in \mathbb{N}\ \exists \lambda, N \in \mathbb{N}$ such that $\forall n \geq N$ $x_n \equiv x_{n+\lambda} \,(mod\, \kappa)$. If $\{x_n\}$ satisfies the stronger condition that $\forall \kappa \in \mathbb{N}\ \exists \lambda \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ $x_n \equiv x_{n+\lambda} \,(mod\, \kappa)$, we say that $\{x_n\}$ is *modularly periodic (MP)*.

Proposition 29. *A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* $\bar{\mathbb{Z}}$ *is modularly periodic if and only if it is periodic modulo* p^m $\forall p \in \mathbb{P}$, $m \in \mathbb{N}$.

Proof. The Chinese Remainder Theorem says that for any $\kappa \in \mathbb{N}$, $\mathbb{Z}/\kappa\mathbb{Z}$ is a finite direct product of rings $\mathbb{Z}/p^m\mathbb{Z}$ over some set of prime powers p^m, so the least common multiple of the periods modulo p^m suffices as a period modulo κ. Thus periodicity modulo each p^m implies periodicity modulo every natural number; likewise, periodicity modulo every natural number implies periodicity modulo each p^m, so the two conditions are equivalent. $\square$

Proposition 30. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\bar{\mathbb{Z}}$ *and let* $f\colon \mathbb{N}_! \to \bar{\mathbb{Z}}$ *be defined by* $f(n) = x_n$. *Then* f *is uniformly continuous if and only if* $\{x_n\}$ *is modularly periodic.*

Proof. If f is uniformly continuous then $\forall \epsilon \in \mathbb{N}\ \exists \delta \in \mathbb{N}$ such that $n \equiv m \,(mod\, \delta!) \implies x_n \equiv x_m \,(mod\, \epsilon!)$; thus $\forall \kappa \in \mathbb{N}\ \forall n \in \mathbb{N}\ x_n \equiv x_{n+\delta!} \,(mod\, \kappa)$, so $\{x_n\}$ is modularly periodic. If, on the other hand, $\{x_n\}$ is modularly periodic, then $\forall \kappa \in \mathbb{N}\ \exists \lambda \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ $x_n \equiv x_{n+\lambda} \,(mod\, \kappa)$, so $\forall \epsilon \in \mathbb{N}\ \exists \lambda \in \mathbb{N}$ such that $n \equiv m \,(mod\, \lambda) \implies x_n \equiv x_m \,(mod\, \epsilon!) \iff f(n) \equiv f(m) \,(mod\, \epsilon!)$; take $\delta! = \lambda!$, and f is uniformly continuous. $\square$

Proposition 31. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a universally convergent sequence in* $\bar{\mathbb{Z}}$. *Then* $\{x_n\}$ *induces a uniformly continuous function if and only if it is MEP.*

Proof. Let $\{y_n\}_{n\in\mathbb{N}}$ be defined by $y_n = f(n)$, where $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ is the function induced by $\{x_n\}$.

If f is uniformly continuous then $\{y_n\}$ is uniformly continuous as a function $\mathbb{N}_! \to \bar{\mathbb{Z}}$, and therefore is a modularly periodic sequence inducing f by Proposition 30. Thus the sequence $\{z_n\}_{n\in\mathbb{N}}$, where $z_n = x_n - y_n\ \forall n$, induces the zero function; so modulo $\kappa \in \mathbb{N}$ the sequence $\{x_n\}$ is identical to the sequence $\{y_n\}$ except at a finite number of indices (otherwise there would be an infinite set of naturals n with $z_n = x_n - y_n$ taking on a particular value modulo κ, and this set of naturals would have a limit point in $\bar{\mathbb{Z}}$, so the function induced by $\{z_n\}$ could not be universally 0). Since $\{y_n\}$ is MP, it follows that $\{x_n\}$ is MEP.

In the other direction, suppose $\{x_n\}$ is MEP. Then given a prime p and a natural N, modulo p^N the sequence $\{x_n\}$ is MP except at a finite number of points; so let $\{y_{N,n}\}_{n\in\mathbb{N}}$ be a sequence which is MP and has for each $p \in \mathbb{P}$ that $y_{N,n} \equiv x_n \,(mod\, p^N)$ for all but a finite number of n. Clearly $y_n \coloneqq \lim\limits_{N\to\infty} y_{N,n}$ exists for each n since $y_{N+1,n} \equiv y_{N,n} \,(mod\, p^N)\ \forall p \in \mathbb{P}$, $N \in \mathbb{N}$; thus we have a sequence $\{y_n\}_{n\in\mathbb{N}}$ which is MP, and since $\{z_n \coloneqq x_n - y_n\}_{n\in\mathbb{N}}$ induces the zero function (because $\forall p \in \mathbb{P}$, $N \in \mathbb{N}$ we have $x_n \equiv y_n \,(mod\, p^N)$ at all but finitely many n) it follows that $\{x_n\}$ induces the same function f as does the MP sequence $\{y_n\}$, and so f must be uniformly continuous. $\square$

Considering $-1=\sum\limits_{n=1}^{\infty} n!n$ in light of exponentiation, the immediate question is whether for a unit u we have the multiplicative inverse u^{-1} of u equal to $\lim\limits_{N\to\infty} u^{\sum\limits_{n=1}^{\infty} n!n}$. Happily, the answer is a resounding *yes*: fixing a modulus $\kappa\in\mathbb{N}$, and writing the prime factorization $\kappa=\prod\limits_{k=1}^{K} p_{n_k}^{x_k}$ (where $\{p_{n_k}\}$ is a finite subsequence of $\{p_n\}=\{2,3,5,7,11,13,...\}$), we have $\bar{\mathbb{Z}}/N\bar{\mathbb{Z}}=\mathbb{Z}/N\mathbb{Z}$ $\forall N\in\mathbb{N}$ and we have by the Chinese Remainder Theorem $\bar{\mathbb{Z}}/\kappa\bar{\mathbb{Z}}\cong\prod\limits_{k=1}^{K}\left(\bar{\mathbb{Z}}/p_{n_k}^{x_k}\bar{\mathbb{Z}}\right)$. Therefore $\left(\bar{\mathbb{Z}}/\kappa\bar{\mathbb{Z}}\right)^{\times}\cong\prod\limits_{k=1}^{K}\left(\bar{\mathbb{Z}}/p_{n_k}^{x_k}\bar{\mathbb{Z}}\right)^{\times}$, and a multiplicative group of units $\left(\bar{\mathbb{Z}}/p_{n_k}^{x_k}\bar{\mathbb{Z}}\right)^{\times}$with p_{n_k} prime is cyclic of order $\phi(p_{n_k}^{x_k})=(p_{n_k}-1)p_{n_k}^{x_k-1}$, where ϕ is Euler's totient function. Since $\sum\limits_{n=1}^{\infty} n!n\equiv\sum\limits_{n=1}^{m-1} n!n\equiv-1\,(mod\, m)$ for every $m\in\mathbb{N}$ (because adding 1 will give 0 through digit carry) this holds in particular for $(p_{n_k}-1)p_{n_k}^{x_k-1}$ for each k. Thus $\lim\limits_{N\to\infty} u^{\sum\limits_{n=1}^{\infty} n!n}\equiv u^{-1}\,(mod\, p_{n_k}^{x_k})$ for each k and therefore $\lim\limits_{N\to\infty} u^{\sum\limits_{n=1}^{\infty} n!n}\equiv u^{-1}\,(mod\,\kappa)$. Since this holds $\forall\kappa$ it holds in particular for κ any factorial, defining a unique canonical series. We expand this argument to define general exponentition with factoradic exponents. To preserve the identity $z^0=1\ \forall z\in\bar{\mathbb{Z}}$ while also easily representing the quantity $\lim\limits_{n\to 0} z^n$, it will be necessary to make use of the symbol ⧝.

Definition 19. Suppose $b\in\bar{\mathbb{Z}}$, $x\in\bar{\mathbb{Z}}$. Then we define $b^{\text{⧝}+x}=\lim\limits_{n\to x} b^n$. We also define b^x to be the product of b with itself x times if $x\in\mathbb{N}$, or 1 if $x=0$. When b is 0 or a unit modulo $p^{\text{⧝}}$ for each prime p, we may simply write b^x rather than $b^{\text{⧝}+x}$ since, as shall be shown, the definitions coincide. We define $\text{⧝}\text{⧝}=\text{⧝}$, $\text{⧝}+\text{⧝}=\text{⧝}$, and $\forall z\in\bar{\mathbb{Z}}$ $\text{⧝}z=\begin{cases}0 & : z=0\\ \text{⧝} & : z\neq 0\end{cases}$.

Proposition 32. *$\forall b,x\in\bar{\mathbb{Z}}$ the limit $b^{\text{⧝}+x}$ is well-defined. Moreover, we have $\forall b\in\bar{\mathbb{Z}}$ that if $x,y\in\mathbb{N}_0\cup\left\{z+\text{⧝}\,:\,z\in\bar{\mathbb{Z}}\right\}$ then:*

(1) $1^x=1$
(2) $b^xb^y=b^{x+y}$
(3) $(b^x)^y=b^{xy}$

Proof. Modulo p^N where p is prime, either b is a unit hence b^n is modularly periodic hence uniformly continuous, or b is p^ku for some unit u, in which case $\lim\limits_{n\to\infty} b^n\equiv 0\,(mod\, p^{\text{⧝}})$, so $b^{\text{⧝}+x}$ is well-defined $\forall x$. Letting $x,y\in\mathbb{N}_0\cup\left\{z+\text{⧝}\,:\,z\in\bar{\mathbb{Z}}\right\}$, we have:

(1) $1^n = 1 \; \forall n \in \mathbb{N}$

(2) If $x, y \in \mathbb{N}_0$ we have the result. Without loss of generality, if $x \in \mathbb{N}_0$, $y = \circledast + \upsilon$ then $b^x b^y = b^x \lim\limits_{n \to \upsilon} b^n = \lim\limits_{n \to \upsilon} b^{x+n} = b^{\circledast + x + \upsilon} = b^{x+y}$. If $x = \xi + \circledast$ and $y = \upsilon + \circledast$, then $b^x b^y = \lim\limits_{n \to \xi} b^n \lim\limits_{m \to \upsilon} b^m = \lim\limits_{n \to \xi} \lim\limits_{m \to \upsilon} b^{n+m} = \lim\limits_{k \to \xi + \upsilon} b^{n+m} = b^{\circledast + \xi + \upsilon} = b^{x+y}$.

(3) If either x or y is 0 then from (1) we get $(b^x)^y = b^{xy} = 1$. If $x, y \in \mathbb{N}_0$ we have the result. If $x \in \mathbb{N}$, $y = \circledast + \upsilon$ then $(b^x)^y = \lim\limits_{n \to \upsilon} b^{nx} = b^{x\upsilon + \circledast}$, and $xy = x(\upsilon + \circledast) = x\upsilon + \circledast$. If $x = \circledast + \xi$, $y \in \mathbb{N}$ then $(b^x)^y = \lim\limits_{n \to \xi} b^{ny} = b^{\xi y + \circledast} = b^{xy}$. If $x = \circledast + \xi$ and $y = \circledast + \upsilon$ then $(b^x)^y = \lim\limits_{n \to \xi} \lim\limits_{m \to \upsilon} b^{nm} = b^{xy}$.

□

Proposition 33. *$\forall x \in \bar{\mathbb{Z}}$ the function $f_x \colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ defined by $f_x(z) = z^{\circledast + x}$ is uniformly continuous. $\forall b \in \bar{\mathbb{Z}}$ the function $g_b \colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ defined by $g_b(z) = b^{\circledast + z}$ is uniformly continuous.*

Proof. $f_x(z) = \lim\limits_{n \to x} f_n(z)$ along $n \in \mathbb{N}$, and $\forall n \in \mathbb{N}$ $f_n(z + N!k) - f_n(z) = (z + N!k)^n - z^n$ is in $\underset{0}{\widetilde{N}}!\bar{\mathbb{Z}}$ since every term will be divisible by $N!k$. $\underset{0}{\widetilde{N}}!\bar{\mathbb{Z}}$ is closed, so $f_x(z) \in \underset{0}{\widetilde{N}}!\bar{\mathbb{Z}}$, i.e. we can take $\delta = \epsilon$ and we have uniform continuity. As for $g_b(z)$, it is uniformly continuous because $\{g_b(n)\}_{n \in \mathbb{N}}$ is modularly periodic; to see this, observe that for any chosen $p \in \mathbb{P}$ and $m \in \mathbb{N}$ we will have $\forall n \in \mathbb{N}$ $b^{\circledast + n} \equiv \begin{cases} 0 \,(mod\, p^m) & : p \mid b \\ b^n \,(mod\, p^m) & : p \nmid b \end{cases}$, and both possibilities are periodic. □

8. The Units

8.1. Units and Zero-Divisors.

Theorem 8. *Let $z \in \bar{\mathbb{Z}}$. Then:*

(1) *z is a zero-divisor if and only if $p^{\circledast} \mid z$ for some $p \in \mathbb{P}$.*
(2) *z is a unit if and only if $p \nmid z$ $\forall p \in \mathbb{P}$.*

Proof. If no $p^{\circledast}$ divides z, then $xz = 0$ implies x is divisible by $p^{\circledast}$ for every $p \in \mathbb{P}$, i.e. $x = 0$; if $p^{\circledast}$divides z, then $(1 - p^{\circledast})z = 0$. If $p \mid z$ then z is not a unit since $z \equiv 0 \,(mod\, p)$ and $0x$ is never 1 modulo p; if no prime divides z, then z is unit modulo every $p^{\circledast}$ and is therefore a unit. □

Definition 20. Let U denote the set of units in $\bar{\mathbb{Z}}$, and let D denote the set of zero-divisors. For any $z \in \bar{\mathbb{Z}}$ let υ_z be the unique unit and $\sigma_z = \prod\limits_{p \in \mathbb{P}} g_p^{x_p}$ the unique product of g_p values such that $z = \upsilon_z \sigma_z$ (taking $\upsilon_z \equiv 1 \,(mod\, p^{\circledast})$ where $x_p = \infty$ to preserve uniqueness, since υ_z is actually unconstrained when $x_p = \infty$). Then υ_z is called the *unitary part of z* and is written *$upart(z)$*, and σ_z is called the *zero signature of z* and is written *$zsig(z)$*. Observing that

$z^{\circledast} \equiv \begin{cases} 0\,(mod\,p^{\circledast}) & : p \mid z \\ 1\,(mod\,p^{\circledast}) & : p \nmid z \end{cases}$ and $1 - z^{\circledast} \equiv \begin{cases} 1\,(mod\,p^{\circledast}) & : p \mid z \\ 0\,(mod\,p^{\circledast}) & : p \nmid z \end{cases}$, we call $z^{\circledast}z$ the *coprime part* and $(1 - z^{\circledast})z$ the *noncoprime part* of z. Note that $z^{\circledast}z + (1 - z^{\circledast})z = z$.

Proposition 34. *$z \in \bar{\mathbb{Z}}$ is idempotent (i.e. z satisfies $z^2 = z$ and hence $z^x = z^{\circledast + x} = z$ $\forall x \in \bar{\mathbb{Z}}\backslash\{0\}$) if and only if $z = \zeta^{\circledast}$ for some $\zeta \in \bar{\mathbb{Z}}$.*

Proof. $z^2 - z = z(z-1) \equiv 0\,(mod\,p^{\circledast})$. So either $z \equiv 0$ or $z \equiv 1$, since if neither z nor $z - 1$ is 0, neither is divisible by $p^{\circledast}$, so their product is divisible by at most a finite power of p, and therefore nonzero. Letting $P = \{p \in \mathbb{P} : z \equiv 0\,(mod\,p^{\circledast})\}$, we thus have $z = \prod_{p \in P} p^{\circledast} = \zeta^{\circledast}$ for any ζ which is divisible by the primes in P and by no other primes. □

Proposition 35. *$z \in \bar{\mathbb{Z}}$ is a zero-divisor if and only if $\exists p \in \mathbb{P}$ such that $p^{\circledast} \mid z$. and $yz = 0$ if and only if $\forall p \in \mathbb{P}$ $p^{\circledast} \mid x$ or $p^{\circledast} \mid y$.*

Proof. If z is not divisible by any $p^{\circledast}$, and $y \neq 0$, then y is divisible by at most a finite power of p for some $p \in \mathbb{P}$, so $zy \not\equiv 0\,(mod\,p^{\circledast})$ hence $zy \neq 0$, so z is not a zero divisor. On the other hand, since $\prod_{p \in \mathbb{P}} p^{\circledast} = 0$, if P is the set of primes p with $p^{\circledast}$ dividing x, and $Q = \mathbb{P}\backslash P$, then for y divisible by $q^{\circledast}$ for every $q \in Q$ we have $xy = 0$, and for y not divisible by some $q^{\circledast}$ with $q \in Q$ we have $xy \not\equiv 0\,(mod\,q^{\circledast})$ hence $xy \neq 0$, which establishes the claim. □

Proposition 36. *$\underset{\phi}{\widetilde{\lambda}}\bar{\mathbb{Z}} \in \underset{\bar{\mathbb{Z}}}{\widetilde{\mathbb{N}}}/\bar{\mathbb{Z}}$ contains a unit if and only if $\underset{\phi}{\widetilde{\lambda}}$ is a coprime wave, i.e.* $\gcd(\lambda, \phi) = 1$. *A coprime wave image contains infinitely many units.*

Proof. If $\underset{\phi}{\widetilde{\lambda}}$ is not coprime then any prime dividing $\gcd(\lambda, \phi)$ divides each element of $\underset{\phi}{\widetilde{\lambda}}\bar{\mathbb{Z}}$, so in particular $\underset{\phi}{\widetilde{\lambda}}\bar{\mathbb{Z}}$ must be unit-free. If $\underset{\phi}{\widetilde{\lambda}}$ is coprime, $x \equiv \begin{cases} \phi + \lambda z & (mod\,\lambda^{\circledast}) \\ \psi & (mod\,(1 - \lambda^{\circledast})) \end{cases}$ is a unit in $\underset{\phi}{\widetilde{\lambda}}\bar{\mathbb{Z}}$ for any $z \in \bar{\mathbb{Z}}$ and $\psi \in U$. □

Proposition 37. *U is perfect, closed, and compact.*

Proof. $U = \bar{\mathbb{Z}}\backslash \bigcup_{p \in \mathbb{P}0} \widetilde{p}\bar{\mathbb{Z}}$, so U is closed. $\bar{\mathbb{Z}}$ is compact, so closed sets are compact. Moreover, every neighborhood of a unit contains a coprime wave, and every coprime wave contains infinitely many units, so every point of U is a limit point of U. Therefore U is perfect. □

Theorem. *(Dirichlet's Theorem) $\bar{\mathbb{P}} = \mathbb{P} \cup U$.* [6]

Proof. The statement is equivalent to Dirichlet's Theorem, since each coprime wave contains infinitely many primes if and only if each coprime wave contains a prime if and only if $U \subset \bar{\mathbb{P}}$, and $\mathbb{P}$ has no limit points outside of U since any sequence of primes $\{p_n\}$ has $p_N \equiv 0\,(mod\,q)$ $\Longleftrightarrow q = p_N \implies p_n \not\equiv 0\,(mod\,q)$ $\forall n > N$, and therefore no sequence of primes converges to a limit divisible by any prime. □

8.2. **Topological $\bar{\mathbb{Z}}$-Modules and the Multiplicative Group of Units.**

Definition 21. Let$R$ be a topological ring, and let M be an R-module. We say M is a *topological R-module* if M is endowed with a topology under which addition $M \times M \to M$ and scalar multiplication $R \times M \to M$ are continuous (where the domains are given their respective product topologies). M is a *torsion R-module* (or simply torsion) is $\forall m \in M$ $\exists r \in R \backslash \{0\}$ such that $r \cdot m = 0$.

Lemma 3. *Let G be a torsion topological $\mathbb{Z}$-module, i.e. $\forall g \in G$ $\exists n \in \mathbb{Z}\backslash\{0\}$ such that $|g| = n$, i.e. n is minimal such that $n \cdot g = 0$. Then G is a $\bar{\mathbb{Z}}$-module under the action $z \cdot g = (z \, mod \, |g|) \cdot g$.*

Proof. We verify the axioms: $\square$

(1) $z \cdot (a+b) = (z \, mod \, |a+b|) \cdot (a+b) = (z \, mod \, |a+b|) \cdot a + (z \bmod |a+b|) \cdot b$. The order of $a+b$ must be divisible by the orders of a and b so this is $= (z \, mod \, |x|) \cdot x + (z \, mod \, |y|) \cdot y = z \cdot x + z \cdot y$.
(2) $(z + \zeta) \cdot a = r \cdot a + \rho \cdot a$ since $z + \zeta + |a|\bar{\mathbb{Z}} = z + |a|\bar{\mathbb{Z}} + \zeta + |a|\bar{\mathbb{Z}}$, i.e. since the natural map $\bar{\mathbb{Z}} \to \bar{\mathbb{Z}}/|a|\bar{\mathbb{Z}}$ is an additive homomorphism.
(3) $(z\zeta) \cdot a = z \cdot (\zeta \cdot a)$ since the natural map $\bar{\mathbb{Z}} \to \bar{\mathbb{Z}}/|a|\bar{\mathbb{Z}}$ is a multiplicative homomorphism.
(4) $1 \cdot a = a$ holds since $(1 \, mod \, |a|) \cdot a = a$.

Definition 22. Let M be a topological $\bar{\mathbb{Z}}$-module. Then we define $|M| = \operatorname{lcm}\{|m| \, : \, m \in M\}$, where $|m| = \begin{cases} \gcd\{z \in \bar{\mathbb{Z}} \, : \, z \cdot m = 0\} & : \exists z \in \bar{\mathbb{Z}}\backslash\{0\} \text{ with } z \cdot m = 0 \\ \infty & : \text{otherwise} \end{cases}$ and the lcm of any set including ∞ is ∞. Both $|M|$ and $|m|$ are defined only up to their zero-signatures, so $|M|$ and $|m|$ are not elements of $\{\infty\} \cup \bar{\mathbb{Z}}$ but of $\{\infty\} \cup \bar{\mathbb{Z}}/U = \{\infty\} \cup \{zU \, : \, z \in \bar{\mathbb{Z}}\}$. We say $|m|$ is the *order of* m and $|M|$ is the *elemental order* of M. Note that we have $|M| = \infty$ if M contains any element which is not $\bar{\mathbb{Z}}$-torsion, and $|M| = 0$ if every element of M is $\bar{\mathbb{Z}}$-torsion but $\forall z \in \bar{\mathbb{Z}}\backslash\{0\}$ $\exists m \in M$ such that $m^z \neq 1$. If $|M| = \infty$ we say M is of *non-factoradic* elemental order, and otherwise $|M| \in \bar{\mathbb{Z}}$ and we say M is of *factoradic* elemental order; so a $\bar{\mathbb{Z}}$-module of factoradic elemental order is precisely a torsion $\bar{\mathbb{Z}}$-module. If M is a topological $\bar{\mathbb{Z}}$-module of factoradic elemental order we say M is a *factoradically finite group.* If there is some $g \in M$ such that $\forall m \in M$ $\exists z \in \bar{\mathbb{Z}}$ such that $m = z \cdot g$, then we say M is *cyclic* and write $M =< g >= \left\{z \cdot g \, : \, z \in \bar{\mathbb{Z}}\right\}$.

Proposition 38. *Let M be a factoradically finite group. Then $\forall m \in M$, $|m| \cdot m = 0$.*

Proof. Write $|m| = \mu U$ for μ a zero signature, and let $X = \left\{x \in \bar{\mathbb{Z}} \, : \, x \cdot m = 0\right\}$. Then by definition $\mu U = \gcd(X)$. For each $p \in \mathbb{P}$ let x_p denote the element in X divisible by the same maximal power of p as is μ (where $\otimes\!\!\!\!\!\infty$ is considered the largest; such an element must exist since $\mu U = \gcd(X)$). Then $\sum_{p \in \mathbb{P}} (1 - p^{\otimes\!\!\!\!\!\infty}) x_p$ is $\equiv x_p \pmod{p^{\otimes\!\!\!\!\!\infty}}$ for each p and so is divisible by the same maximal power of each prime as is μ, i.e. $\sum_{p \in \mathbb{P}} (1 - p^{\otimes\!\!\!\!\!\infty}) x_p = \mu \omega$ for some unit ω.

Then for any unit u, $\mu u \cdot m = u\omega^{-1} \cdot (\mu\omega \cdot m) = u\omega^{-1} \cdot \left(\sum_{p \in \mathbb{P}} (1 - p^{\infty}) x_p \cdot m \right) = u\omega^{-1} \cdot \sum_{p \in \mathbb{P}} 0$ $= 0$. Thus $|m| \cdot m = 0$. $\square$

Lemma 4. *Let $M = < g >$ be a cyclic factoradically finite group, and let $d \in \bar{\mathbb{Z}}/U$. If $d \nmid |M|$ then $\forall m \in M$ $|m| \neq d$, and if $d \mid |M|$ then:*

- *If for some $p \in \mathbb{P}$ $p^{\infty} \mid |M|$ and $p^{\infty} \nmid d$, then no $m \in M$ has $|m| = d$;*
- *Otherwise,$\exists m \in M$ such that $|m| = d$.*

Proof. If $d \nmid |M|$ then no element has order dU since $|M|$ is the least common multiple of the orders. On the other hand, if $d \mid |M| = |g|$, choose $c \in \bar{\mathbb{Z}}/U$ such that $cd = |g|$. Take $m = c \cdot g$; then for $n \in \bar{\mathbb{Z}}/U$, $n \cdot m = 0 \iff nc \cdot g = 0 \iff |g| \mid nc$. Now choosing $nU = dU$ we have $|g| \mid nd$. For the first case, suppose there is some prime p such that $p^{\infty} \mid |M|$ but $p^{\infty} \nmid d$; then since $cd = |M|$ and d is divisible by a finite maximal power of p, we have $p^{\infty} \mid c$.. Thus if δ is divisible by the same maximal powers of the same primes as d except that $p \nmid \delta$, $\delta c = dc$ so $\delta \cdot m = \delta c \cdot g = 0$, and d is not the order of m hence not the order of any element of M. For the second case, suppose there is no such prime, i.e. $\forall p \in \mathbb{P}$ either $p^{\infty} \nmid |M|$ or $p^{\infty} \mid d$. Then we are free to choose c such that c is divisible by only a finite maximal power of each prime, since if $|M| = cd$ is divisible by p^{∞} then so is d. Then $d \cdot m = 0$; moreover, for any δ dividing d with $\delta \neq d$, either for some prime p we have $p^{\infty} \mid d$ but $p^{\infty} \nmid \delta$, in which case $p^{\infty} \nmid c$ so $p^{\infty} \mid |M|$ but $p^{\infty} \nmid \delta c$ and hence $\delta \cdot m = \delta c \cdot g \neq 0$, or some prime p divides d with a greater finite maximal power than δ, and divides c only to a finite power, so δc is a proper divisor of $|M|$ and $\delta \cdot m \neq 0$. Therefore $d = \gcd(\delta : \delta \cdot m = 0) = |m|$, and both claims are established. $\square$

Proposition 39. *$\forall \kappa \in \bar{\mathbb{Z}}$ the additive group $\bar{\mathbb{Z}}/\kappa\bar{\mathbb{Z}}$ is a cyclic factoradically finite group of order κ.*

Proof. $\bar{\mathbb{Z}}$ acts naturally on $\bar{\mathbb{Z}}/\kappa\bar{\mathbb{Z}}$ via $z \cdot (\zeta + \kappa\bar{\mathbb{Z}}) = z\zeta + \kappa\bar{\mathbb{Z}}$. Thus $\bar{\mathbb{Z}} \cdot (1 + \kappa\bar{\mathbb{Z}}) = \bar{\mathbb{Z}}/\kappa\bar{\mathbb{Z}}$ and $z \cdot (1 + \kappa\bar{\mathbb{Z}}) = 0 + \kappa\bar{\mathbb{Z}}$ if and only if $\kappa \mid z$, as claimed. $\square$

Theorem 9. *(Multiplicative Modular Groups) For all odd primes p and $n \in \mathbb{N} \cup \{\infty\}$, $\left(\bar{\mathbb{Z}}/p^n\bar{\mathbb{Z}}\right)^{\times}$ is a cyclic factoradically finite group of order $(p-1)p^{n-1}$ under the action $z \cdot u = u^z$. (In the case of $n = \infty$, we have equivalently a group of order $(p-1)p^{\infty-1}$ or $(p-1)p^{\infty}$, since these are associates). For all $n \in \mathbb{N} \cup \{\infty\}$, $\left(\bar{\mathbb{Z}}/2^n\bar{\mathbb{Z}}\right)^{\times} \cong \left(\bar{\mathbb{Z}}/2\bar{\mathbb{Z}}\right)^{+} \times \left(\bar{\mathbb{Z}}/2^{n-2}\bar{\mathbb{Z}}\right)^{+}$ unless $n = 1$ in which case $\left(\bar{\mathbb{Z}}/2\bar{\mathbb{Z}}\right)^{\times}$ is the trivial group $\{1\}$.*

Proof. Let p be an odd prime; then for all $n \in \mathbb{N}$, $\left(\bar{\mathbb{Z}}/p^n\bar{\mathbb{Z}}\right)^{\times}$ is cyclic of order $\phi(p^n) = (p-1)p^{n-1}$ (so the result holds for $n \in \mathbb{N}$), and if γ_p is a primitive root modulo p^2, it is also a primitive root modulo p^n $\forall n$. So fix $\gamma_p \in \bar{\mathbb{Z}}/p^{\infty}\bar{\mathbb{Z}}$ such that γ_p is a primitive root modulo p^2. Now $\gamma_p^{x_n} \equiv \gamma_p^{x_m} \pmod{p^n}$ if and only if $x_n \equiv x_m \pmod{(p-1)p^{n-1}}$, so $\gamma_p^x = \gamma_p^y$ if and only if $x \equiv y \pmod{(p-1)p^n}$ $\forall n \in \mathbb{N}$, i.e. if and only if $x \equiv y \pmod{(p-1)p^{\infty}}$. Furthermore, $\gamma_p^{\bar{\mathbb{Z}}}$ contains all units: if $u \in \bar{\mathbb{Z}}/p^{\infty}\bar{\mathbb{Z}}$ write $u \equiv \gamma_p^{x_n} \pmod{p^n}$. We must have

$x_{n+1} \equiv x_n \pmod{(p-1)p^{n-1}}\ \forall n$, so in particular if $\xi_p \equiv \begin{cases} x_1 & : \pmod{p-1} \\ 1 & : \pmod{p^{\infty}} \end{cases}$ and we write $x_n = \xi_p \Xi_{p,n}$ for some $\Xi_{p,n} \equiv \begin{cases} 1 & : \pmod{p-1} \\ x_n & : \pmod{p^{n-1}} \end{cases}$, then $\Xi_{p,n+1} \equiv \Xi_n \pmod{p^{n-1}}\ \forall n$ so there is a p-adic Ξ_p with $\Xi_p \equiv \Xi_{p,n} \pmod{p^n}\ \forall n$, and we have $u = \gamma_p^{\xi_p \Xi_p}$. This establishes the claim for odd primes. For $p = 2$, choose a "primitive root" γ_2 such that γ_2 and -1 generate $\left(\mathbb{Z}/2^n\bar{\mathbb{Z}}\right)^\times$ for all n. Then by the same argument as for odd primes, every element of $\left(\mathbb{Z}/2^n\bar{\mathbb{Z}}\right)^\times$ is of the form $(-1)^0\gamma_2^x$ or $(-1)^1\gamma_2^x$ for some $x \in \bar{\mathbb{Z}}/2^{\infty}\bar{\mathbb{Z}}$, and no two such expressions give the same element since γ_2 must have order 2^{∞}. □

Definition 23. For each odd prime p, let $\gamma_p \equiv \begin{cases} \text{some primitive root} \pmod{q^{\infty}} & : q = p \\ 1 \pmod{q^{\infty}} & : q \neq p \end{cases}$, where q ranges over the primes, let $\tau \equiv \begin{cases} -1 \pmod{q^{\infty}} & : q = 2 \\ 1 \pmod{q^{\infty}} & : q \neq 2 \end{cases}$, and let γ_2 be $\equiv 1 \pmod{q^{\infty}}$ for $q \neq 2$ and have multiplicative order $2^{\infty}U$ modulo 2^{∞}, as in Theorem 9. Then we say $\{\gamma_p\}_{p\in\mathbb{P}}$ is a choice of primitive roots, and we will freely use the τ symbol where its meaning is clear.

Lemma 5. *(Unit Factorization Lemma) Given a choice of primitive roots $\{\gamma_p\}_{p\in\mathbb{P}}$, for each $\omega \in U$ there is a unique choice of $(\xi_2, \Xi_2, \xi_3, \Xi_3, \xi_5, \Xi_5, ...)$ in $\left(\bar{\mathbb{Z}}/2\bar{\mathbb{Z}}\right) \times \left(\bar{\mathbb{Z}}/2^{\infty}\bar{\mathbb{Z}}\right) \times \prod_{p\in\mathbb{P}\setminus\{2\}} \left(\bar{\mathbb{Z}}/(p-1)\bar{\mathbb{Z}}\right) \times \left(\bar{\mathbb{Z}}/p^{\infty}\bar{\mathbb{Z}}\right)$ such that $\omega = \tau^{\xi_2}\gamma_2^{\Xi_2} \prod_{p\in\mathbb{P}\setminus\{2\}} \gamma_p^{\xi_p\Xi_p}$, where the ξ_p and Ξ_p are interpreted as factoradic integers by taking any representatives such that $\xi_p \equiv \begin{cases} \xi_p & \pmod{p-1} \\ 1 & \pmod{p^{\infty}} \end{cases}$ and $\Xi_p \equiv \begin{cases} 1 & \pmod{p-1} \\ \Xi_p & \pmod{p^{\infty}} \end{cases}$ for odd p, and taking any representatives at all for ξ_2, Ξ_2. In particular, U under multiplication is therefore isomorphic to $\bar{\mathbb{Z}} \times \left(\bar{\mathbb{Z}}/2\bar{\mathbb{Z}}\right) \times \prod_{p\in\mathbb{P}} \left(\bar{\mathbb{Z}}/(p-1)\bar{\mathbb{Z}}\right)$ under addition.*

Proof. This follows directly from Theorem 9. □

Theorem 10. *(Unique Factorization Theorem) For each $p \in \mathbb{P}$ let $X_p = \{\infty\} \cup \left(\mathbb{N}_0 \times \left(\bar{\mathbb{Z}}/(p-1)\bar{\mathbb{Z}}\right) \times \left(\bar{\mathbb{Z}}/p^{\infty}\bar{\mathbb{Z}}\right)\right)$ and define addition on X_p componentwise, with $\infty + x = \infty\ \forall x$. Then $\bar{\mathbb{Z}}$ under multiplication is isomorphic to the cartesian product $\prod_{p\in\mathbb{P}} X_p$ under componentwise addition. That is, for each $z \in \bar{\mathbb{Z}}$, given a choice of primitive roots $(\gamma_2, \gamma_3, \gamma_5, ...)$ there is a unique factorization $z = g_2^{x_2}\tau^{\xi_2}\gamma_2^{\Xi_2}\prod_{p\in\mathbb{P}} g_p^{x_p}\gamma_p^{\xi_p\Xi_p}$ with the pth component of the corresponding element of $\prod X_p$ given by $(x_p, \xi_p, \Xi_p) \in X_p$, with the caveat* *that if* $(x_p, \xi_p, \Xi_p) = \infty$ *we must interpret* $g_p^{x_p} \gamma_p^{\xi_p \Xi_p} := p^{\infty\!\!\!\times}$ *for odd primes or* $g_2^{x_2} \tau^{\xi_2} \gamma_2^{\Xi_2} := 2^{\infty\!\!\!\times}$. *The product* $\xi_p \Xi_p$ *is to be interpreted as* $\equiv \begin{cases} \xi_p & (mod\, p-1) \\ \Xi_p & (mod\, p^{\infty\!\!\!\times}) \end{cases}$.

Proof. This follows from Theorem 9 and Proposition 15. $\square$

Theorem 11. *(p − adic Roots of Unity) If* p *is an odd prime and* $n \in \bar{\mathbb{Z}}$*, then* $x^n - 1 \equiv 0\,(mod\, p^{\infty\!\!\!\times})$ *has a nontrivial solution* $x = \nu$ *in* $\bar{\mathbb{Z}}/(p-1)p^{\infty\!\!\!\times}\bar{\mathbb{Z}}$ *if and only if* $n \mid (p-1)$ *or* $n = mp^{\infty\!\!\!\times}$ *for some* $m \mid (p-1)$*. Moreover:*

(1) *For* $n = p - 1$*, the roots of unity of order* $p - 1$ *are generated by* $\gamma_p^{p^{\infty\!\!\!\times}}$.
(2) *For* $n = p^{\infty\!\!\!\times}$*, the roots of unity of order* $p^{\infty\!\!\!\times}$ *are generated by* γ_p^{p-1}.

8.3. **The Total Logarithm.**

Definition 24. We define $\mathbb{L}_2 := \big(\mathbb{Z} \times (\bar{\mathbb{Z}}/2\bar{\mathbb{Z}}) \times (\bar{\mathbb{Z}}/2^{\infty\!\!\!\times}\bar{\mathbb{Z}})\big)$ and for each odd prime p we define

$\mathbb{L}_p := \big(\mathbb{Z} \times (\bar{\mathbb{Z}}/(p-1)\bar{\mathbb{Z}}) \times (\bar{\mathbb{Z}}/p^{\infty\!\!\!\times}\bar{\mathbb{Z}})\big)$. We then define $\mathbb{L} := \prod_{p \in \mathbb{P}} \mathbb{L}_p$, and all of these direct products we consider rings under componentwise addition and multiplication. For each prime p, $\hat{\mathbb{L}_p}$ denotes $\mathbb{L}_p$ extended by an element ∞ such that $\forall x \in \hat{\mathbb{L}_p}$ we have $x + \infty = \infty$ and $x\infty = \begin{cases} 0 & : x = 0 \\ \infty & : x \neq 0 \end{cases}$, and we denote $\hat{\mathbb{L}} := \prod_{p \in \mathbb{P}} \hat{\mathbb{L}_p}$. Given a fixed choice of primitive roots $\{\gamma_p\}_{p \in \mathbb{P}}$, the *total logarithm of base* $\prod_{p \in \mathbb{P}} \gamma_p$ is the unique map $\log\colon \bar{\mathbb{Z}} \to \hat{\mathbb{L}}$ defined by the Unique Factorization Theorem.

Proposition 40. $\hat{\mathbb{Q}}$ *under multiplication is isomorphic to* $\hat{\mathbb{L}}$ *under addition.*

Proof. This is an immediate consequence of the Unique Factorization Theorem and Proposition 18. $\square$

9. Continuous Functions

9.1. **Some Useful Functions.**

Definition 25. For $n \in \mathbb{Z}$, $d \in \mathbb{Z}\backslash\{0\}$, we define $n\%d$ to be the least representative of n modulo d in $\mathbb{N}_0$, and we define $\lfloor \frac{n}{d} \rfloor$ to be the unique integer such that $\lfloor \frac{n}{d} \rfloor d + n\%d = n$. We extend this definition to $n \in \bar{\mathbb{Z}}$ according to the following proposition:

Proposition 41. *For fixed* $d \in \mathbb{Z}\backslash\{0\}$, $\lfloor \frac{n}{d} \rfloor$ *is a uniformly continuous function* $\mathbb{Z} \to \mathbb{Z}$ *of* n*, and therefore uniquely extends to a uniformly continuous function* $\lfloor \frac{z}{d} \rfloor$ *taking* $z \in \bar{\mathbb{Z}}$ *to* $\bar{\mathbb{Z}}$.

Proof. Given $\epsilon!$, take $\delta = \epsilon + d$ and $\lfloor \frac{n+\delta!k}{d} \rfloor - \lfloor \frac{n}{d} \rfloor = \frac{\delta!}{d}k = \frac{(\epsilon+d)!k}{d}$ and since some number in $\{\epsilon+1, ..., \epsilon+d\}$ is divisible by d, we certainly have $\epsilon! \mid \frac{(\epsilon+d)!k}{d}$. Therefore the function is uniformly continuous on $\mathbb{Z}$, and thus extends uniquely to a uniformly continuous function on the metric completion $\bar{\mathbb{Z}}$ of $\mathbb{Z}$. $\square$

Definition 26. Define the *Pochhammer symbol* $(x)_n := \prod_{k=0}^{n-1}(x-k)\ \forall n \in \mathbb{N}_0$, with $(x)_0 = 1$ the empty product. Note that $(x)_n$ is always a polynomial of degree n, hence uniformly continuous as a function of x.

Proposition 42. *$\{(x)_n\}_{n\in\mathbb{N}_0}$ is a basis over $\mathbb{Z}$ for $\mathbb{Z}[x]$.*

Proof. A polynomial of zero degree $f(x) = c$ can be written in exactly one way, namely $c(x)_0$. Suppose all polynomials of degree less than n can be written in exactly one way, and let $f(x) = a_n x^n + ... + a_0 x^0$ be a polynomial of degree n. Then if $f(x) = \sum_{k=0}^{N} b_k(x)_k$ with $b_N \neq 0$, we must have $N = n$ since $(x)_k$ is a polynomial of degree k $\forall k$ and f is of degree n. Moreover, $(x)_k$ is always monic, so we must have $b_n = a_n$. So $f(x) - a_n(x)_n$ is a polynomial of degree $n-1$, which by induction can be represented in exactly one way, so there is exactly one way to write f as a linear combination of Pochhammer symbols over $\mathbb{Z}$. $\square$

Lemma 6. $\forall z \in \bar{\mathbb{Z}}\ n! \mid (z)_n$.

Proof. It suffices to consider $z \in \mathbb{N}$ since $\mathbb{N}$ is dense, $(z)_n$ is continuous, and $\underset{0}{\widetilde{n}}!\bar{\mathbb{Z}}$ is closed. We proceed by induction to show that for $z \in \mathbb{N}$, $\forall n \in \mathbb{N}$ $(z)_n \equiv 0 \pmod{n!}$; equivalently, the product of any n consecutive naturals is divisible by $n!$. Every natural is divisible by $1!$, so the proposition holds in the case $n = 1$. Now suppose the proposition holds for all naturals less than n, and let $I_\phi = \{\phi, \phi+1, ..., \phi+(n-1)\}$ be a set of n consecutive integers. We will proceed here as well by induction, this time on ϕ, in order to show that $\forall \phi \in \mathbb{N}$ $n! \mid \prod I_\phi$. For $\phi = 1$ the claim holds since $\prod I_\phi = n!$. Suppose $\phi > 1$ and the result holds for all naturals less than ϕ. Then distributing the last term, $\prod I_\phi = \phi(\phi+1)...(\phi-1+n)$ $= (\phi-1)\phi(\phi+1)...(\phi+n-2) + n\phi(\phi+1)...(\phi+n-2)$. The first expression is a product of n consecutive numbers starting from $\phi-1$, so by induction on ϕ it is $\equiv 0 \pmod{n!}$; the second expression is n times a product of $n-1$ consecutive numbers starting from ϕ, so by induction on n it is $\equiv 0 \pmod{n!}$. Therefore $\prod I_\phi \equiv 0 \pmod{n!}$. Thus $n! \mid \prod I_\phi\ \forall \phi \in \mathbb{N}$, as claimed, and thus the product of any n consecutive naturals is divisible by $n!$, as claimed. Thus by the continuity of $(z)_n$ with respect to z, $\forall z \in \bar{\mathbb{Z}}\ n! \mid (z)_n$. $\square$

Proposition 43. *$\forall z \in \bar{\mathbb{Z}}$, $\lim_{n\to\infty}(z)_n = 0$.*

Proof. $(z)_n \equiv 0 \pmod{n!}$ by Lemma 6. $\square$

Definition 27. For $k \in \mathbb{N}_0$, $z \in \bar{\mathbb{Z}}$, we define the *binomial coefficient* $\begin{pmatrix} z \\ k \end{pmatrix} := \frac{(z)_k}{k!}$. Considered for fixed k as a function of x, $\begin{pmatrix} x \\ k \end{pmatrix}$ is called the *binomial polynomial of order k*. Furthermore, let $\mathbb{Q}[x]_{\mathbb{Z}}$ denote the ring of integer-valued polynomials, i.e. $\{f \in \mathbb{Q}[x] : f(\mathbb{Z}) \subseteq \mathbb{Z}\}$. Note that $\mathbb{Q}[x]_{\mathbb{Z}}$ is an abelian group and so may be considered a $\mathbb{Z}$-module.

Proposition 44. *For fixed $k \in \mathbb{N}_0$, $f(z) := \left(\begin{array}{c} z \\ k \end{array} \right)$ is a continuous function $\bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$.*

Proof. $\forall z \in \bar{\mathbb{Z}}$ $(z)_k \equiv 0 \pmod{k!}$ so $\left(\begin{array}{c} z \\ k \end{array} \right) = \frac{(z)_k}{k!}$ is a factoradic integer $\forall z$. Furthermore, $\left(\begin{array}{c} z + N!m \\ k \end{array} \right) - \left(\begin{array}{c} z \\ k \end{array} \right) = \frac{1}{k!}\left((z+N!m)_k - (z)_k\right)$ and $(z)_k$ is continuous so this $\to 0$ as $N \to \infty$. $\square$

Proposition 45. *For $n \in \mathbb{N}$, $\left(\begin{array}{c} n \\ k \end{array} \right) = 0$ if $k > n$.*

Proof. $\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{(n)_k}{k!}$ and $(n)_k = \prod_{k=0}^{n-1}(n-k)$ which includes a factor of $(n-n) = 0$ if $k > n$. $\square$

9.2. **Finite Differences.**

Definition 28. Writing F for the set of functions $\bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$, for $z \in \bar{\mathbb{Z}}$ we define $\Delta_z : F \to F$ by $(\Delta_z f)(x) = f(x+z) - f(x)$. If the subscript is omitted, we define $\Delta = \Delta_1$. Similarly, $(\nabla_z f)(x) := f(x) - f(x-z)$ and $\nabla = \nabla_1$.

Proposition 46. *Fixing $k \in \mathbb{N}$, we have the following identities:*

(1) $\Delta\lfloor \frac{x}{k} \rfloor = \begin{cases} 1 & : k \mid (x+1) \\ 0 & : k \nmid (x+1) \end{cases}$

(2) $\Delta(x)_n = n(x)_{n-1}$

(3) $\Delta \left(\begin{array}{c} x \\ k \end{array} \right) = \left(\begin{array}{c} x \\ k-1 \end{array} \right)$

Proof. Proceding for each part:

(1) $\lfloor \frac{x+1}{k} \rfloor = \begin{cases} \lfloor \frac{x}{k} \rfloor + 1 & : k \mid (x+1) \\ \lfloor \frac{x}{k} \rfloor & : k \nmid (x+1) \end{cases}$.

(2) $(x+1)_n - (x)_n = (x+1)(x)(x-1)...(x+2-n) - (x)(x-1)...(x+1-n) = (x)(x-1)...(x+2-n)(x+1-(x+1-n)) = n(x)_{n-1}$. In the case $n = 0$ note that we still have $\Delta(x)_0 = 1 - 1 = 0 = 0(x)_{-1}$ regardless of the definition of $(x)_{-1}$.

(3) $\left(\begin{array}{c} x+1 \\ k \end{array} \right) - \left(\begin{array}{c} x \\ k \end{array} \right) = \frac{1}{k!}\left((x+1)_k - (x)_k\right) = \frac{1}{k!}\left((x)_{k-1}(x+1) - (x)_{k-1}(x+1-k)\right)$
$= \frac{k}{k!}(x)_{k-1} = \left(\begin{array}{c} x \\ k-1 \end{array} \right)$

$\square$

Proposition 47. *Given any function $f : \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ and a natural number n, we have*

$(\Delta^n f)(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k).$

Proof. For $n = 1$ the result holds since $(\Delta f)(x) = f(x+1) - f(x) = \binom{1}{0}(-1)^0 f(x+1) + \binom{1}{1}(-1)^1 f(x)$. Suppose the result holds for all naturals less than n; then $(\Delta^n f)(x) = (\Delta^{n-1} f)(x+1) - (\Delta^{n-1} f)(x)$

$= \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} f(x+1+k) - \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-1-k} f(x+k)$

$= \sum_{k=1}^{n} \binom{n-1}{k-1} (-1)^{n-k} f(x+k) + \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-k} f(x+k)$

$= \binom{n-1}{0} (-1)^n f(x) + \sum_{k=1}^{n-1} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) (-1)^{n-k} f(x+k)$

$+ \binom{n-1}{n-1} (-1)^0 f(x+n)$, and since $\binom{n-1}{0} = 1 = \binom{n}{0}$, $\binom{n-1}{n-1} = 1 = \binom{n}{n}$, and $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$, this is equal to $\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k)$. So the result holds for n, and therefore $\forall n \in \mathbb{N}$ by induction. ☐

Proposition 48. *(Newton Series on $\mathbb{N}$) For any function $f \colon \mathbb{N}_0 \to \bar{\mathbb{Z}}$ there is a unique sequence of coefficients $\{a_n\}_{n \in \mathbb{N}}$ such that $f(x) = \sum_{n=0}^{\infty} \binom{x}{n} a_n$, given by $a_n = (\Delta^n f)(0)$ $\forall n$.*

Proof. We adapt a proof given by Alain M. Robert [8]. The a_n are unique as follows: suppose $f(x) = \sum_{n=0}^{\infty} \binom{x}{n} a_n$. We have $\binom{x}{n} = 0$ when $n > x$ (Proposition 45), so $\sum_{n=0}^{\infty} \binom{x}{n} a_n = \sum_{n=0}^{x} \binom{x}{n} a_n$. Thus $f(0) = \sum_{n=0}^{0} \binom{0}{n} a_n = a_0$ uniquely determines a_0. Now suppose that $f(0)$, ..., $f(N-1)$ uniquely determine $a_0, ..., a_{N-1}$. Then $f(N) = \sum_{n=0}^{N} \binom{N}{n} a_n \Longleftrightarrow a_N = f(N) - \sum_{n=0}^{N-1} \binom{N}{n} a_n$, since $\binom{N}{N} = 1$, so $f(0)$, ..., $f(N)$ uniquely determine $a_0, ..., a_N$. Thus by induction the $\{a_N\}$ are uniquely determined by the values of f. Now we shall show existence: set $a_n = (\Delta^n f)(0)$ $\forall n \in \mathbb{N}_0$. We have $\sum_{n=0}^{\infty} \binom{x}{n} a_n = \sum_{n=0}^{x} \binom{x}{n} (\Delta^n f)(0)$. At $x = 0$ this is $(\Delta^0 f)(0) = f(0)$. Writing $g(x) = f(x) - \sum_{n=0}^{x} \binom{x}{n} (\Delta^n f)(0)$, we thus have $g(0) = 0$. Applying Proposition 47, $\Delta^N g(0) = \sum_{k=0}^{N} \binom{N}{k} (-1)^{N-k} g(k)$, so we clearly have the implication $(\forall k \in \{0, ..., N\}\ g(k) = 0) \longrightarrow \big(\forall k \in \{0, ..., N\}\, (\Delta^k g)(0) = 0\big)$; moreover, $(\forall k \in \{0, ..., N\}\ g(k) = 0) \wedge \big(\forall k \in \{0, ..., N\}\, (\Delta^k g)(0) = 0\big) \longrightarrow (g(N+1) = 0)$ since if we assume the antecedent then $g(N+1) = f(N+1) - \sum\limits_{n=0}^{N+1} \begin{pmatrix} N+1 \\ n \end{pmatrix} (\Delta^n f)(0) = f(N+1) - (\Delta^{N+1} f)(0) = f(N+1) - \sum\limits_{k=0}^{N+1} \begin{pmatrix} N+1 \\ k \end{pmatrix} (-1)^{N+1-k} f(k) = f(N+1) - \begin{pmatrix} N+1 \\ N+1 \end{pmatrix} (-1)^0 f(N+1) = 0$. Therefore since $g(0) = 0$, by induction $g(n) = 0\ \forall n \in \mathbb{N}_0$, and so $f(x) = \sum\limits_{n=0}^{x} \begin{pmatrix} x \\ n \end{pmatrix} (\Delta^n f)(0)\ \forall x \in \mathbb{N}_0$, as claimed. □

Proposition 49. $\mathbb{Q}[x]_{\mathbb{Z}}$ *is a free* $\mathbb{Z}$*-module with basis the binomial polynomials.*

9.3. **Direct & Indirect Continuity.**

Definition 29. Suppose $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ is continuous. Then we say f is *directly continuous* if $\forall p \in \mathbb{P}$ there exists a function $f_p\colon \bar{\mathbb{Z}}/p^{\infty}\bar{\mathbb{Z}} \to \bar{\mathbb{Z}}/p^{\infty}\bar{\mathbb{Z}}$ such that $f(x) \equiv f(x + p^{\infty}\bar{\mathbb{Z}}) \pmod{p^{\infty}}$ $\forall x \in \bar{\mathbb{Z}}$, i.e. if f is a direct product over $p \in \mathbb{P}$ of continuous p-adic functions. Otherwise, we say f is *indirectly continuous.*

Example.

(1) Every polynomial is directly continuous, since addition and multiplication can be performed independently modulo $\bar{\mathbb{Z}}/p^{\infty}\bar{\mathbb{Z}}\ \forall p$.
(2) Let $u \in U$. If there are exponents such that for some choice of primitive roots we have $u = \tau^{\xi_2}\gamma_2^{\Xi_2} \prod\limits_{p \in \mathbb{P}\setminus\{2\}} \gamma_p^{(p-1)\Xi_2}$, then $f(x) = u^x$ is directly continuous, since modulo 2^{∞} we have $u^x \equiv \tau^{\xi_2 x}\gamma_2^{\Xi_2 x}$ which depends only on $x \bmod 2^{\infty}$, and modulo any p^{∞} we have $\underline{u^x \equiv \gamma_p^{(p-1)\Xi_2 x}}$ which depends only on $x \bmod p^{\infty}$. If there are no such exponents, then $f(x) = u^x$ is indirectly continuous, since if $u \equiv \gamma_p^{\xi} \pmod{p^{\infty}}$ with $(p-1) \nmid \xi$, the value of $f(x) \bmod p^{\infty}$ depends on $x \bmod (p-1)$ as well as on $x \bmod p^{\infty}$. Note that in both cases, $f(x)$ is uniformly continuous by modular periodicity, so it is entirely possible for indirectly continuous functions to be uniformly continuous.
(3) For $x \in \bar{\mathbb{Z}}$ and p_n the nth prime write the p_n-adic expansion of $x \bmod p_n^{\infty}$ as $x \equiv \sum\limits_{k=0}^{\infty} d_{p_n,k} p_n^k$. Considering the $d_{p_n,k}$ to be elements of $\mathbb{N}$, define the function $f(x)$ according to $f(x) \equiv \begin{cases} 0 \pmod{p^{\infty}} & : p = 2 \\ \sum\limits_{k=0}^{\infty} d_{p_{n-1},k} p^k \pmod{p^{\infty}} & : p \in \mathbb{P}\setminus\{2\} \end{cases}$. Then f is continuous (its value modulo $\prod p_n^{x_n}$ is determined by the value of x modulo $\prod p_{n-1}^{x_n}$ as n ranges over any set of naturals), but it clearly does not reduce to a direct product of p-adic functions; thus f is indirectly continuous.

Proposition 50. *An indirectly continuous function is not the pointwise limit of any sequence of directly continuous functions.*

Proof. Let $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$, for each $p \in \mathbb{P}$ define $f_p\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}/p^{\circledast}\bar{\mathbb{Z}}$ by $f_p(x) = f(x) + p^{\circledast}\bar{\mathbb{Z}}$, and let $X_p = \{q \in \mathbb{P} : \exists x, y \in \bar{\mathbb{Z}} \text{ such that } x \equiv y \,(\mathrm{mod}\,(1 - q^{\circledast}) \text{ and } f_p(x) \neq f_p(y)\}$. Now suppose f is indirectly continuous, and fix $p \in \mathbb{P}$ such that $X_p \backslash \{p\} \neq \emptyset$. Choose $q \in X_p \backslash \{p\}$ and choose $x, y \in \bar{\mathbb{Z}}$ such that $x \equiv y \,(\mathrm{mod}\,(1 - q^{\circledast}))$ but $f_p(x) \neq f_p(y)$. Then for all directly continuous $g\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ we have $g(x)+p^{\circledast}\bar{\mathbb{Z}} = g(y)+p^{\circledast}\bar{\mathbb{Z}}$. So if $\{F_n\}_{n\in\mathbb{N}}$ is a sequence of directly continuous functions and $F(x) = \lim\limits_{n\to\infty} F_n(x)\ \forall x$, we have $F(x) = F(y)$, but $f(x) \neq f(y)$. Therefore an indirectly continuous function is not the pointwise limit of any sequence of directly continuous functions. $\square$

There are two results particularly relevant to direct continuity in $\bar{\mathbb{Z}}$. The first is known as Mahler's Theorem, and was proven by Kurt Mahler in 1958 [7]; Mahler used properties of quadratic extension of $\mathbb{Z}_p$ in his original proof, but there is a very nice proof by R. Bojanic in 1974 based on the properties of factorials [2]. The second result is a refinement of Mahler's theorem in the case $f\colon \mathbb{N}_0 \to \mathbb{Z}_p$, and was proven by R. Ahlswede and R. Bojanic in 1975 [1]. These results follow:

Theorem 12. *(Mahler's Theorem, 1958) If $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ is continuous, then $\forall x \in \mathbb{Z}_p$ we have $f(x) = \sum\limits_{k=0}^{\infty} \begin{pmatrix} x \\ k \end{pmatrix} (\Delta^k f)(0)$.* [7][2]

Theorem. *(Ahlswede & Bojanic, 1975) Let $f\colon \mathbb{N}_0 \to \mathbb{Z}_p$ be any function and let $||_p$ denote the p-adic norm. Then $\lim\limits_{n\to\infty} |(\Delta^n f)(0)|_p = 0$ if and only if*

$\max\{|f(n + p^t) - f(n)|_p : n \in \mathbb{N}_0\} \to 0$ *as* $t \to \infty$. [1]

So a continuous function $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ must have $f(x + p^{\circledast+z}) \equiv f(x) \,(\mathrm{mod}\, p^{\circledast})\ \forall z \in \bar{\mathbb{Z}}$ and $\forall p \in \mathbb{P}$ in order to have $(\Delta^n f)(0) = 0$ (since a sequence converges as $n \to \infty$ if and only if it converges as $n \to z\ \forall z \in \bar{\mathbb{Z}}$). Then writing the induced function $\hat{f}\colon \bar{\mathbb{Z}} \to \mathbb{Z}_p$ as $\hat{f}(x) = f(x) + p^{\circledast}\bar{\mathbb{Z}}$, we see that $\hat{f}(x + np^{\circledast}) = \hat{f}(x)$ whenever $n \in \mathbb{N}$, since $\hat{f}(x + p^{\circledast}) = \hat{f}(x + p^{\circledast} + p^{\circledast}) = ...$, and therefore by the continuity of f we will have $\hat{f}(x + zp^{\circledast}) = f(x + zp^{\circledast}) + p^{\circledast}\bar{\mathbb{Z}} = f(x) + p^{\circledast}\bar{\mathbb{Z}} = \hat{f}(x)\ \forall z \in \bar{\mathbb{Z}}$, and f is directly continuous. Mahler's Theorem provides the converse, resulting in the following theorem:

Theorem 13. *Let $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ be continuous. Then $f(z) = \sum\limits_{k=0}^{\infty} \begin{pmatrix} z \\ k \end{pmatrix} (\Delta^k f)(0)\ \forall z \in \bar{\mathbb{Z}}$ if and only if f is directly continuous.*

Example. The most familiar and well-behaved indirectly continuous functions are the functions $f(x) = u^x$ for u some unit which fails to be a power of $\gamma_p^{(p-1)}$ modulo $p^{\circledast}$ for at least one $p \in \mathbb{P}$. For a unit u we have $(\Delta f)(x) = u^{x+1} - u^x = u^x(u - 1)$, and by induction it is easily shown that $(\Delta^n f)(x) = u^x(u - 1)^n$. So $(\Delta^n f)(0) = (u - 1)^n$ goes to 0 as $n \to \infty$ if and only if $(u - 1)^{\circledast} = 0$, i.e. if and only if $u - 1$ is divisible by every prime, which holds if and only if $u \equiv 1 \,(\mathrm{mod}\, p)\ \forall p \in \mathbb{P}$, which holds if and only if for some set of Ξ_p we have $u \equiv \gamma_p^{(p-1)\Xi_p} \,(\mathrm{mod}\, p^{\circledast})\ \forall p \in \mathbb{P}$, which is exactly the condition required in order for f to be directly continuous.

9.4. **Factoradic Series.**

Definition 30. Let $z \in \bar{\mathbb{Z}}$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\bar{\mathbb{Z}}$. Then we say the *factoradic series* $\sum\limits_{n=1}^{\text{ꝏ}+z} x_n$ *converges to* (or *sums to*) $\lim\limits_{N\to z}\sum\limits_{n=1}^{N} x_n$; if the limit does not exist we say the series diverges at $\text{ꝏ}+z$. Note that this is a factoradic limit over $\mathbb{N}$, i.e. N ranges over $\mathbb{N}\cap\left(\bar{\mathbb{Z}}\backslash\{z\}\right)$. Omitting the ꝏ, we have the slightly different definition $\sum\limits_{n=1}^{z} x_n = \begin{cases} \sum\limits_{n=1}^{\text{ꝏ}+z} x_n & : z \notin \mathbb{N}_0 \\ x_1 + x_2 + \ldots + x_z & : z \in \mathbb{N} \\ 0 & : z = 0 \end{cases}$. This distinction is unfortunately necessary if we are to make any sense of sequences of partial sums $\{S_n\}$ that are not consistent, i.e. for which we do not have $\lim\limits_{n\to N} S_n = S_N$ for $N \in \mathbb{N}_0$. In general, if $Z \subset \bar{\mathbb{Z}}$, $z \in \bar{\mathbb{Z}}$, $f\colon Z \to \bar{\mathbb{Z}}$ is continuous, and $a + \mathbb{N}_0 \subset Z$, then $\sum\limits_{n=a}^{z} f(n) := \lim\limits_{N\to(z-a)}\sum\limits_{n=a}^{a+N} f(n)$. A continuous function $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ is called *summable* if $\sum\limits_{n=1}^{\text{ꝏ}+z} f(n)$ converges $\forall z \in \bar{\mathbb{Z}}$. As shall be shown, we may omit the ꝏ when f is known to be a continuous function of n.

Remark. Observe that a factoradic series sums over exactly the same set of elements as the more familiar form of infinite series; the difference is in how the terms are grouped. For example, let $\{x_n\}$ be a sequence in $\bar{\mathbb{Z}}$, let $\sum\limits_{n=1}^{\infty} n!d_n$ be the canonical series for a factoradic integer z, and let $S_0 = 0$, $S_{N\in\mathbb{N}} = \sum\limits_{n=1}^{N} n!d_n$ be its partial sums; then either $\sum\limits_{n=1}^{\text{ꝏ}+z} x_n = \sum\limits_{k=1}^{\infty}\sum\limits_{n=S_{k-1}+1}^{S_k} x_n$ or $\sum\limits_{n=1}^{\text{ꝏ}+z} x_n$ diverges. The sequence of partial sums of the canonical series in this example can be replaced with *any* sequence converging to z, because either we have convergence for every such choice, and they all have the same value, or $\lim\limits_{N\to z}\sum\limits_{n=1}^{N} x_n$ fails to exist, and the series diverges at z. Thus we can define an equivalence relation $\sim$ on the partitions of $\mathbb{N}$ into contiguous regions by $P = \{\{1,\ldots,M_1\},\{M_1+1,\ldots,M_2\},\ldots\} \sim Q = \{\{1,\ldots,T_1\},\{T_1+1,\ldots,T_2\},\ldots\} \iff \lim\limits_{k\to\infty}(M_k - T_k) = 0$; now if a contiguous partition P has its maximal elements defined by $M_N = \sum\limits_{n=1}^{N} n!d_n$, then any other partition with maximal element sequence converging to the same factoradic will be equivalent to P, and the sums of a summable sequence will agree over any two equivalent partitions. On the other hand, for $\sum\limits_{n=1}^{\infty} x_n$ we have $P = \{\{1\},\{2\},\{3\},\ldots\}$; when this converges the sum is invariant under reorderings and regroupings, so convergent infinite series have sums invariant under changes to the underlying partition of $\mathbb{N}$; i.e., a series in $\bar{\mathbb{Z}}$ converges over $\{\{1\},\{2\},\{3\},\ldots\}$ if and only if it converges to the same value over all partitions.

Proposition 51. *$\forall z \in \bar{\mathbb{Z}}$ we have* $\sum_{n=1}^{z} 1 = z$.

Proof. $f(z) = z - \sum_{n=1}^{z} 1$ is identically 0 on $\mathbb{N}$ and therefore extends by continuity to $f(z) = 0$ everywhere. $\square$

Proposition 52. *Let $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ be summable, and let $a, b \in \bar{\mathbb{Z}}$. Then* $\sum_{n=a}^{b-1} f(n) = \sum_{n=0}^{b-1} f(n) - \sum_{n=0}^{a-1} f(n) = \sum_{n=a}^{b-1+\rlap{0}\infty} f(n)$.

Proof. Define the partial sum sequence $\left\{ S_N = \sum_{n=1}^{N} f(n) \right\}_{N \in \mathbb{N}}$. Since $\{S_N\}$ is universally convergent the induced function $F(x) = \lim_{N \to x} S_N$ is continuous $\bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$. Since for any fixed $N \in \mathbb{N}$ $(F(z+N) - F(z)) - \sum_{n=z+1}^{z+N} f(n)$ is continuous and equal to 0 on $\mathbb{N}$, it is equal to 0 on $\bar{\mathbb{Z}}$. Thus $\sum_{n=a}^{a+N-1} f(n) = F(a+N-1) - F(a-1)$. Since F is continuous, taking the limit as $N \to b - a$ we obtain $\sum_{n=a}^{b-1} f(n) = F(b-1) - F(a-1) = \sum_{n=1}^{b-1} f(n) - \sum_{n=1}^{a-1} f(n)$ $= \sum_{n=0}^{b-1} f(n) - \sum_{n=0}^{a-1} f(n)$, as claimed in the first equality. For the second equality, note that $F(z) - F(z-1) = \lim_{N \to z} (S_N - S_{N-1}) = \lim_{N \to z} f(N) = f(z)$. Consequently $F(b) - F(a-1) \to f(a) + f(z+1) + \ldots + f(b)$ when $b - a \in \mathbb{N}_0$, so we always have $\sum_{n=a}^{b-1} f(n) = \sum_{n=a}^{b-1+\rlap{0}\infty} f(n)$. $\square$

Theorem 14. *Let $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ be summable. Then $\sum_{n=1}^{z} f(n)$ is continuous, and $\forall a, b, c \in \bar{\mathbb{Z}}$ the following hold:*

(1) *Reindexing Invariance: $\forall a, b \in \bar{\mathbb{Z}}$ $\sum_{n=a}^{b} f(n) = \sum_{n=a-c}^{b-c} f(n+c)$*
(2) *"Path"-Invariance: $\forall a, b, c \in \bar{\mathbb{Z}}$ $\sum_{n=a+1}^{c} f(n) = \sum_{n=a+1}^{b} f(n) + \sum_{n=b+1}^{c} f(n)$*
(3) *Orientation Invariance: $\forall a, b \in \bar{\mathbb{Z}}$ $\sum_{n=0}^{b-a} f(a+n) = \sum_{n=0}^{b-a} f(b-n)$*

Proof. $\sum_{n=1}^{z} f(n)$ is the function induced by $\left\{ \sum_{n=1}^{N} f(n) \right\}_{N \in \mathbb{N}}$ which is universally convergent by assumption, so $\sum_{n=1}^{z} f(n)$ is continuous. Reindexing invariance follows since the partial sums of $\sum_{n=a}^{b} f(n)$ and $\sum_{n=a-c}^{b-c} f(n+c)$ give exactly the same sequence, and therefore have the same limits. Now $\sum_{n=a+1}^{c} f(n) = \sum_{n=a+1}^{b} f(n) + \sum_{n=b+1}^{c} f(n) = \sum_{n=a+1}^{b} f(n) + \sum_{n=1}^{c-b} f(n+b)$ for all $b \in \mathbb{N}$ and so taking the limit as b approaches a factoradic it still holds, giving path-invariance. For orientation-invariance, if $a, b \in \mathbb{N}$ with $a < b$ the left-hand sum $\sum_{n=0}^{b-a} f(a+n)$ evaluates f at $\{a, ..., b\}$ and the right-hand sum $\sum_{n=0}^{b-a} f(b-n)$ evaluates f at $\{a, ..., b\}$, so taking b to a factoradic they remain equal by continuity; then we have for any summable g that $\sum_{n=a}^{b} g(n) = \sum_{n=1}^{b} g(n) - \sum_{n=1}^{a-1} g(n)$, so we can also take a to a factoradic as well, and the two will remain equal by continuity. □

Corollary. *Let $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ be summable. Then f sums to 0 along "closed paths", i.e. $\sum_{n=a+1}^{a} f(n) = 0 \; \forall a \in \bar{\mathbb{Z}}$. We therefore have the useful (and equivalent) identities:*

(1) $f(b) - f(a) = \sum_{n=a}^{b} f(n) + \sum_{n=b}^{a} f(n) \; \forall a, b \in \bar{\mathbb{Z}}$.

(2) $\sum_{n=a+1}^{b} f(n) = -\sum_{n=b+1}^{a} f(n) \; \forall a, b \in \bar{\mathbb{Z}}$.

Proposition 53. *Suppose $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ is continuous. Then $f(z) = f(0) + \sum_{n=0}^{z-1} (\Delta f)(n)$.*

Proof. $\forall N \in \mathbb{N}$ $\sum_{n=0}^{N-1} (\Delta f)(n) = (f(1) - f(0)) + (f(2) - f(1)) + ... + (f(N) - f(N-1))$ $= f(N) - f(0)$, so $\lim_{N \to z} \sum_{n=0}^{N-1} (\Delta f)(n) = f(z) - f(0)$. □

Corollary. *If $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ satisfies $f(z) = (\Delta g)(z)$ for some continuous function g, then f is summable.*

Lemma 7. $\forall z \in \bar{\mathbb{Z}} \; \forall n \in \mathbb{N}$ $\sum_{k_1=0}^{z-1} ... \sum_{k_{n-1}=0}^{k_{n-2}-1} \sum_{k_n=0}^{k_{n-1}-1} 1 = \binom{z}{n}$.

Proof. We proceed by induction. We have $1 = \binom{z}{0}$. Now suppose the proposition holds for all naturals less than n; then $\sum_{k_1=0}^{x-1} ... \sum_{k_{n-1}=0}^{k_{n-2}-1} 1 = \binom{x}{n-1} \; \forall x$, so $\sum_{k_1=0}^{z-1} ... \sum_{k_n=0}^{k_{n-1}-1} 1$ $= \sum_{k_1=0}^{z-1} \binom{k_1}{n-1}$. Thus the claim follows if $\sum_{k=0}^{z-1} \binom{k}{n-1} = \binom{z}{n}$. We have $\Delta \binom{z}{n} =$

$\begin{pmatrix} z \\ n-1 \end{pmatrix}$ by Proposition 46, and $\begin{pmatrix} z \\ n \end{pmatrix}$ is a continuous function of z, so $\sum_{k=0}^{z-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} = \begin{pmatrix} 0 \\ n \end{pmatrix} + \sum_{k=0}^{z-1} \left(\Delta \begin{pmatrix} z \\ n \end{pmatrix}\right)(k) = 0 + \begin{pmatrix} z \\ n \end{pmatrix}$. □

Proposition 54. *Suppose $f\colon \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ is continuous. Then $\forall n \in \mathbb{N}$ we have $\forall z \in \bar{\mathbb{Z}}$ $f(z) = \sum_{k=0}^{n-1} \begin{pmatrix} z \\ k \end{pmatrix} (\Delta^k f)(0) + \sum_{k_1=0}^{z-1} \dots \sum_{k_n=0}^{k_{n-1}-1} (\Delta^n f)(k_n)$.*

Proof. We proceed by induction, invoking Proposition 53 repeatedly to prove the claim: $\forall n \in \mathbb{N}$ $f(z) = \sum_{k=0}^{n-1} \begin{pmatrix} z \\ k \end{pmatrix} (\Delta^k f)(0) + \sum_{k_1=0}^{z-1} \dots \sum_{k_n=0}^{k_{n-1}-1} (\Delta^n f)(k_n)$. This holds for $n = 1$ since $f(z) = f(0) + \sum_{k_1=0}^{z-1} (\Delta f)(k_1)$ by Proposition 53. Suppose the claim holds for $n-1$; then $f(z) = \sum_{k=0}^{n-2} \begin{pmatrix} z \\ k \end{pmatrix} (\Delta^k f)(0) + \sum_{k_1=0}^{z-1} \dots \sum_{k_{n-1}=0}^{k_{n-2}-1} (\Delta^{n-1} f)(k_{n-1})$, so expanding $(\Delta^{n-1} f)(k_{n-1})$ according to proposition Proposition 53 we have:

$$f(z) = \sum_{k=0}^{n-2} \begin{pmatrix} z \\ k \end{pmatrix} (\Delta^k f)(0) + \sum_{k_1=0}^{z-1} \dots \sum_{k_{n-1}=0}^{k_{n-2}-1} \left((\Delta^{n-1} f)(0) + \sum_{k_n=0}^{k_{n-1}-1} (\Delta^n f)(k_n) \right)$$

$= \sum_{k=0}^{n-2} \begin{pmatrix} z \\ k \end{pmatrix} (\Delta^k f)(0) + \sum_{k_1=0}^{z-1} \dots \sum_{k_{n-1}=0}^{k_{n-2}-1} (\Delta^{n-1} f)(0) + \sum_{k_1=0}^{z-1} \dots \sum_{k_n=0}^{k_{n-1}-1} (\Delta^n f)(k_n)$. Now $(\Delta^{n-1} f)(0)$ does not depend on any of the k_i, so we have:

$$\sum_{k_1=0}^{z-1} \dots \sum_{k_{n-1}=0}^{k_{n-2}-1} (\Delta^{n-1} f)(0) = (\Delta^{n-1} f)(0) \sum_{k_1=0}^{z-1} \dots \sum_{k_{n-1}=0}^{k_{n-2}-1} (1)$$

$= (\Delta^{n-1} f)(0) \begin{pmatrix} z \\ n-1 \end{pmatrix}$ by Lemma 7, so $f(z) = \sum_{k=0}^{n-1} \begin{pmatrix} z \\ k \end{pmatrix} (\Delta^k f)(0) + \sum_{k_1=0}^{z-1} \dots \sum_{k_n=0}^{k_{n-2}-1} (\Delta^n f)(k_n)$ as claimed, completing the induction. □

Theorem 15. *Suppose $f : \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ is uniformly continuous. Then f is summable and $s(z) \coloneqq \sum_{n=0}^{z-1} f(n)$ is uniformly continuous.*

Proof. Since f is uniformly continuous, it is modularly periodic. Fix $M \in \mathbb{N}$ and let $P \in \mathbb{N}$ denote the period of $f(x)$ modulo M. Then for any $N \in \mathbb{N}$, write $N = \lfloor \frac{N}{P} \rfloor P + N\%P$ and we have $\sum_{n=1}^{N} f(n) \equiv \lfloor \frac{N}{P} \rfloor \sum_{n=1}^{P} f(n) + \sum_{n=1}^{N\%P} f(n) \pmod{M}$. As N approaches some $z \in \bar{\mathbb{Z}}$, where $z = \lfloor \frac{z}{P} \rfloor P + z\%P$, N sufficiently large N will always have $\equiv z\%P \pmod{P}$, giving $\lfloor \frac{N}{P} \rfloor \sum_{n=1}^{P} f(n) - \sum_{n=1}^{z\%P} f(n)$ for the Nth partial sum, which clearly converges as $N \to z$ since the quotient is uniformly continuous; so $s(z)$ is summable. Given $\epsilon!$, take $\delta = \epsilon + P$ and we have $s(z+\delta!k)-s(z)=\left(\lfloor\frac{z+(\epsilon+P)!k}{P}\rfloor-\lfloor\frac{z}{P}\rfloor\right)\sum_{n=1}^{P}f(n)=\frac{(\epsilon+P)!}{P}k$, which is clearly an integer divisible by $\epsilon!$ since one of $(\epsilon+1),...,(\epsilon+P)$ must be divisible by P. Thus $s(z)$ is modularly periodic hence uniformly continuous. $\square$

9.5. **Geometric Series of Units.**

Definition 31. For $\omega\in U$, let Σ_ω denote in this section the function $\Sigma_\omega(x):=\sum_{n=1}^{x}\omega^n$.

Proposition 55. *$\forall\omega\in U$, Σ_ω is uniformly continuous.*

Proof. $f(x)=\Sigma_\omega(x)$ is a factoradic sum over $g(x)=\omega^x$, a uniformly continuous function. $\square$

Proposition 56. *$\forall\omega\in U$, $x\in\bar{\mathbb{Z}}$ $\Sigma_\omega(x)+\omega^x\Sigma_\omega(y)=\Sigma_\omega(x+y)$.*

Proof. $\sum_{n=1}^{x}\omega^n+\omega^x\sum_{n=1}^{y}\omega^n=\sum_{n=1}^{x}\omega^n+\sum_{n=1}^{y}\omega^{x+n}=\sum_{n=1}^{x}\omega^n+\sum_{n=x+1}^{x+y}\omega^n=\sum_{n=1}^{x+y}\omega^n$. $\square$

Proposition 57. *For $\omega=1$, $\Sigma_\omega(x)=x$. For $\omega=-1$, $\Sigma_\omega(x)=\begin{cases}-1 & : x\,odd\\ 0 & : x\,even\end{cases}$.*

Proposition 58. *If $\omega-1$ is not a zero divisor, the roots of $\Sigma_\omega(x)$ are precisely $x\in|\omega|\bar{\mathbb{Z}}$. If $\omega-1$ is a zero divisor and P is the set of primes with $p^{\rlap{\times}\infty}\mid(\omega-1)$, then for $x\in|\omega|\bar{\mathbb{Z}}$ we have $\Sigma_\omega(x)\equiv\begin{cases}0\,(mod\,p^{\rlap{\times}\infty}) & : p\notin P\\ x\,(mod\,p^{\rlap{\times}\infty}) & : p\in P\end{cases}$, and the roots are $|\omega|(\omega-1)^{\rlap{\times}\infty}\bar{\mathbb{Z}}$.*

Proof. If $z\in\bar{\mathbb{Z}}$ such that $\sum_{n=1}^{z}\omega^n=0$ then $0=(1-\omega)\sum_{n=1}^{z}\omega^n=\sum_{n=1}^{z}\omega^n-\sum_{n=2}^{z+1}\omega^n=\omega^1-\omega^{z+1}=\omega(1-\omega^z)$, so $\omega^z=1$, hence $z\in|\omega|\bar{\mathbb{Z}}$. In the other direction, if $\omega^z=1$ then working backwards we have $(1-\omega)\sum_{n=1}^{z}\omega^n=0$. If $\sum_{n=1}^{z}\omega^n\not\equiv 0\,(\text{mod}\,p^{\rlap{\times}\infty})$ for some $p\in\mathbb{P}$ then $\omega\equiv 1$ so $\omega-1$ is a zero-divisor and $\sum_{n=1}^{z}\omega^n\equiv z\,(\text{mod}\,p^{\rlap{\times}\infty})$, so a root must be $\equiv 0\,(\text{mod}\,p^{\rlap{\times}\infty})$. $\square$

10. Projective Limits

10.1. **Projective Limits.**

Definition. A poset is a set S together with a partial order $\leq$, i.e. $\leq: S\times S\to\{T,F\}$ which satisfies $\forall a,b,c\in S$:

1. $a\leq a$
2. $a\leq b\wedge b\leq a\to a=b$
3. $a\leq b\wedge b\leq c\to a\leq c$

A poset S is *directed* if $\forall a, b \in S\ \exists c \in S$ such that $a \leq c$ and $b \leq c$.

Definition 32. Let I be a directed poset, and let $\{S_i\}_{i\in I}$ be a collection of sets indexed by I. A *projective system* is an ordered pair $(\{S_i\}, \{f_{ij}\})$, where $\{f_{ij}\}$ is an indexed family of functions $\{f_{ij}\colon S_j \to S_i\}_{\substack{i,j \in I \\ i \leq j}}$ such that $\forall i \in I$ f_{ii} is the identity on S_i and $\forall i,j,k \in I$ with $i \leq j \leq k$ we have $f_{ik} = f_{ij} \circ f_{jk}$. The functions f_{ij} are called *bonding maps.* A *projective system of groups* (or rings, or modules) is a projective system $(\{S_i\}, \{f_{ij}\})$ such that the bonding maps are group (or ring, or module) homomorphisms; likewise for a *projective system of topological* groups (or rings, or modules) the bonding maps are continuous homomorphisms. If $i \leq j$ implies S_i is a quotient ring (or quotient group) of S_j and we do not specify the bonding maps, they are defined to be the natural homomorphisms, i.e. if $S_i = S_j/Q$ for Q an ideal (or a normal subgroup) in S_j, then $f_{ij}(s_j \in S_j) = s_j + Q$. The *projective limit* of the projective system $\left(\{S_i\}_{i\in I}, \{f_{ij}\}_{i\leq j\in I}\right)$ is defined by $\varprojlim_{i\in I} S_i := \left\{(s_i)_{i\in I} \in \prod_{i\in I} S_i \ :\ \forall i,j \in I \,\text{with}\, i \leq j,\ s_i = f_{ij}(s_j)\right\}$. If G is a group, the *profinite completion* of G is $\varprojlim_{N} G/N$, where N runs over all normal subgroups N of G ordered by inclusion, and the bonding maps are the natural homomorphisms.

Proposition 59. *A projective limit of (topological) groups/rings/modules is a (topological) group/ring/module.*

Proposition 60. *$\bar{\mathbb{Z}}$ is ring-isomorphic to $\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n!\mathbb{Z}$, where $\mathbb{N}$ is ordered by magnitude.*

Proof. The set of canonical series $\sum_{n=1}^{\infty} n! d_n$ truncated to $n \in \{1, ..., N-1\}$ correspond bijectively via their sums in $\mathbb{Z}/N!\mathbb{Z}$ to the elements of $\mathbb{Z}/N!\mathbb{Z}$. Each bonding map $f_{ij}\colon \mathbb{Z}/j!\mathbb{Z} \to \mathbb{Z}/i!\mathbb{Z}$ is defined by $f_{ij}(x) = x + i!\mathbb{Z}$, so in terms of truncated canonical series this is simply $f_{ij}\left(\sum_{n=1}^{j-1} n! d_n\right) = \sum_{n=1}^{i-1} n! d_n$. Thus $\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n!\mathbb{Z}$ is the subset of the direct product $\prod_{n\in\mathbb{N}} \mathbb{Z}/n!\mathbb{Z}$ such that for every element there is a canonical series (i.e. a factoradic integer) whose residue modulo $n!$ is the *nth* entry of the element $\forall n$. Since multiplication and addition modulo the factorials completely determines multiplication and addition in $\bar{\mathbb{Z}}$, this correspondence is an isomorphism. $\square$

Proposition 61. *For a fixed prime p, the p-adic integers $\bar{\mathbb{Z}}/p^{\infty}\bar{\mathbb{Z}}$ are ring-isomorphic to $\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$, where $\mathbb{N}$ is ordered by magnitude.*

Proof. This follows by the same argument as Proposition 60; simply replace $\mathbb{Z}/n!\mathbb{Z}$ with $\mathbb{Z}/p^n\mathbb{Z}$ and the set of canonical series $\sum_{n=1}^{\infty} n! d_n$ with the set of p-adic series $\sum_{n=1}^{\infty} d_n p^n$. $\square$

Proposition 62. *$\bar{\mathbb{Z}}$ is ring-isomorphic to $\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$, where $\mathbb{N}$ is ordered by divisibility, i.e. $n \leq m \iff n \mid m$. This is the profinite completion of the integers.*

Proof. $\bar{\mathbb{Z}}$ is isomorphic to the subset of the direct product $\prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ obeying $(x + nm\mathbb{Z}) \in (x+n\mathbb{Z})$, since an element of $\bar{\mathbb{Z}}$ is uniquely determined by its residues modulo natural numbers and, $\mathbb{N}$ being dense, any finite set of residues must also correspond to some element of $\mathbb{N}$. This is exactly $\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$, since the natural bonding maps are exactly $(x+nm\mathbb{Z}) \mapsto (x+n\mathbb{Z})$. $\square$

10.2. **Direct Limits & Prüfer p-Groups.**

Definition 33. Let I be a directed poset, and let $\{S_i\}_{i\in I}$ be a collection of sets indexed by I. A *direct system* is an ordered pair $(\{S_i\}, \{f_{ij}\})$, where $\{f_{ij}\}$ is an indexed family of functions $\{f_{ij}\colon S_i \to S_j\}_{\substack{i,j\in I \\ i\leq j}}$ such that $\forall i \in I$ f_{ii} is the identity on S_i and $\forall i,j,k \in I$ with $i \leq j \leq k$ we have $f_{ik} = f_{jk} \circ f_{ij}$. The functions f_{ij} are called *structure maps*; note that the difference between projective and direct systems is the direction of the maps, i.e. the structure maps are $f_{ij} : S_i \to S_j$ for direct systems whereas the bonding maps are $f_{ij} : S_j \to S_i$ for projective systems. As in the case of projective systems, if the S_i are (topological) groups/rings/modules and the f_{ij} are (continuous) homomorphisms, then we say $(\{S_i\}, \{f_{ij}\})$ is a *direct system of (topological) groups/rings/modules.* The *direct limit* of the direct system $(\{S_i\}, \{f_{ij}\})$ is defined by $\varinjlim_{i\in I} S_i = \bigsqcup_{i\in I} S_i/\sim$, where $\bigsqcup$ denotes disjoint union and $\sim$ is the equivalence relation defined by $a \in S_i \sim b \in S_j \iff \exists k \geq i,j$ such that $f_{ik}(a) = f_{jk}(b)$. Informally, $\varinjlim_{i\in I} S_i$ is thus the set of all elements of all of the S_i, except that two elements are considered the same whenever the structure maps "eventually" take them to the same element.

Proposition 63. *Let $(\{S_i\}, \{f_{ij}\})$ be the direct system of groups indexed by $\mathbb{N}$ ordered by magnitude such that each $S_n = (\mathbb{Z}/p^n\mathbb{Z})^+$ and $f_{nm}(x) = p^{m-n}x$. Then $\varinjlim_{n\in\mathbb{N}} S_n$ is isomorphic to the Prüfer p-group, i.e. the subgroup of the circle group composed of all p-power roots of unity.*

Proposition 64. *For each prime p, $\mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic to the Prüfer p-group.*

Proposition 65. *Let $(\{S_i\}, \{f_{ij}\})$ be the direct system of groups indexed by $\mathbb{N}$ ordered by magnitude such that each $S_n = (\mathbb{Z}/n!\mathbb{Z})^+$ and $f_{nm}(x) = x(n+1)(n+2)...(m)$. Then $\varinjlim_{n\in\mathbb{N}} S_n$ is isomorphic to the torsion subgroup of the circle group, i.e. to the additive group $\mathbb{Q}/\mathbb{Z}$.*

Part 3. Projective Limits and Polynomial Completions

The fact that power series $\sum_{n=0}^{\infty} a_n x^n$ often fail to converge in $\bar{\mathbb{Z}}$ while factorial series $\sum_{n=0}^{\infty} a_n (x)_n$ always converge is no accident, but an artifact of the relationship between $\bar{\mathbb{Z}}$ and $\mathbb{Z}[x]$. The Wave Topology Theorem at the end of Part I is phrased in terms of unique factorization domains, and generalizations of the method of construction of $\bar{\mathbb{Z}}$ (or any of the $\mathbb{Z}_p$) are applicable to polynomial rings over UFDs; of particular interest are the various completions of $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$.

11. Power Series in $\bar{\mathbb{Z}}$

$\bar{\mathbb{Z}}$ is in many ways poorly suited for analytic methods based on power series. Intuitively, this is because $\bar{\mathbb{Z}}$ shares the characteristics of the factorials from which it was constructed: meaningful infinite iterations in $\bar{\mathbb{Z}}$ must generally involve addition in an intimate way, since they must encode meaningful patterns in each $\mathbb{Z}_p$, and these spaces are very different. The need for the symbol $\circledast$ was the first hint of this exponential-phobic tendency: except for units, which are the paragon of periodicity (and thus regularity) under repeated multiplication, and for units which have had some information destroyed already (that is, units multiplied by factoradics of the form $z^{\circledast}$), factoradic integers lose information when raised to unnatural powers, i.e. $f(x) = b^x$ cannot be extended continuously from $\mathbb{N}$ to $\bar{\mathbb{Z}}$. Hence $\lim_{n\to z} b^n$ cannot be written simply b^z, but must involve $\circledast$; thus the terms of power series $\sum_{n=0}^{\infty} a_n x^n$ are not in general continuous functions of n, even when each a_n is 1, and power series inherit the irregularity one might expect as a result. The hyperintegers $*\mathbb{Z}$ from non-standard analysis provide an interesting perspective on this phenomenon. Since $\bar{\mathbb{Z}}$ is the quotient ring of $*\mathbb{Z}$ which preserves only the residues of hyperintegers modulo naturals, and exponentiation "works" in $*\mathbb{Z}$ by analogy with exponentiation in $\mathbb{Z}$, we see that the need for $\circledast$ is a result of loss of information in traversing the natural map $*\mathbb{Z} \to \bar{\mathbb{Z}}$. More specifically, the hyperintegers allow natural primes to be raised to hypernatural powers, and the hypernaturals are totally ordered and contain infinite elements, whereas in $\bar{\mathbb{Z}}$ the zero-signature g_p of a prime p can only distinctly be raised to $\mathbb{N}_0 \cup \{\infty\}$ because $\bar{\mathbb{Z}} \cong \varprojlim \mathbb{Z}/n\mathbb{Z}$; the destruction of information by the $\circledast$ symbol thus corresponds to the map taking hypernatural to factoradic exponents, which is the identity on $\mathbb{N}_0$ but takes all elements of $*\mathbb{N}\backslash\mathbb{N}$ to ∞ when the base is factoradically a zero signature.

Definition 34. We designate by the symbol $G_{\mathbb{P}}$ the quantity $\equiv p \pmod{p^{\circledast}}$ $\forall p \in \mathbb{P}$, i.e. $G_{\mathbb{P}} = \prod_{p\in\mathbb{P}} g_p$.

Proposition 66. *Power series $\sum_{n=0}^{\infty} a_n x^n$ with coefficients in $\bar{\mathbb{Z}}$ satisfy the following:*

(1) $\sum_{n=0}^{\infty} a_n x^n$ *converges $\forall x \in \bar{\mathbb{Z}}$ if and only if* $\lim_{n\to\infty} a_n = 0$.

(2) *Every power series converges on $G_p\bar{\mathbb{Z}}$.*

Proof. The first claim follows since an infinite series converges if and only if its terms go to 0, and at $x \in U$ this occurs only if $a_n \to 0$. The second claim follows by the same reasoning, since $z^n \to 0$ as $z \to \infty$ only when $G_p \mid z$. $\square$

Proposition 67. $\forall p \in \mathbb{P}\ \bar{\mathbb{Z}}/p^{\infty}\bar{\mathbb{Z}} \cong \mathbb{Z}[[x]]/(p-x)\mathbb{Z}[[x]]$.

Proposition 68. *(Base $G_{\mathbb{P}}$ Representations of Factoradic Integers) Let D be the set of sequences $\{d_n\}_{n\in\mathbb{N}_0}$ in $\bar{\mathbb{Z}}$ such that for each $n \in \mathbb{N}_0$ and $p \in \mathbb{P}$ we have $d_n \equiv 0, 1, ..., p-2$, or $p-1 \pmod{p^{\infty}}$. Then $\sum\limits_{n=0}^{\infty} d_n G_{\mathbb{P}}^n$ takes D to $\bar{\mathbb{Z}}$ bijectively.*

Proof. Modulo each p^{∞}, the constraints on the d_n are precisely those that apply to p-adic digits, and $G_{\mathbb{P}} \equiv p$; moreover, the values of d_n modulo each p^{∞} can be chosen independently. So every choice of p-adic integers over all p corresponds to a unique element of D, and vice versa. $\square$

12. $\overline{\mathbb{F}[x]}$

Wave actions on a principal ideal domain behave much like those on $\mathbb{Z}$, i.e. if R is a principal ideal domain then $\underset{R}{\widetilde{R}}/R = \{\{r \in R : (r-\phi) \in I\} : \phi \in R \text{ and } I \text{ an ideal in } R\}$. As a consequence, completions of metrizations of the standard wave topology $\mathscr{T}\left(\underset{\mathbb{F}[x]}{\widetilde{\mathbb{F}[x]\backslash\{0\}}}/\mathbb{F}[x]\right)$ on a polynomial ring $\mathbb{F}[x]$ over a field $\mathbb{F}$ bear a striking algebraic resemblance to $\bar{\mathbb{Z}}$.

Definition 35. Let $\mathbb{F}$ be a field, let U denote the set of units in $\mathbb{F}[x]$, and let $\mathbb{M}$ be the set of monic irreducible polynomials in $\mathbb{F}[x]$. Then:

(1) We define the topological ring $\overline{\mathbb{F}[x]} = \varprojlim_{f\in F} \mathbb{F}[x]/f(x)\mathbb{F}[x]$, where F is the set of all products of elements of $\mathbb{M}$ with multiplicity (i.e. all monic polynomials), ordered by divisibility, and the topology is $\mathscr{T}\left(\underset{\overline{\mathbb{F}[x]}}{\widetilde{\mathbb{F}[x]\backslash\{0\}}}/\overline{\mathbb{F}[x]}\right)$.

(2) We define for each $m \in \mathbb{M}$ the topological ring $\mathbb{F}[[m]] = \varprojlim_{n\in\mathbb{N}} \mathbb{F}[x]/m^n\mathbb{F}[x]$, where $\mathbb{N}$ is ordered by magnitude and the topology is $\mathscr{T}\left(\underset{\mathbb{F}[[m]]}{\widetilde{m^{\mathbb{N}_0}}}/\mathbb{F}[[m]]\right)$. This is the m-adic completion of $\mathbb{F}[x]$. Note that for $u \in \mathbb{F}$, the definition readily extends to give $\mathbb{F}[[um]] \cong \mathbb{F}[[m]]$.

Proposition 69. *Let $\mathbb{F}$ be a field, and let $\mathbb{M}$ be the set of monic, irreducible polynomials in $\mathbb{F}[x]$. Then $\overline{\mathbb{F}[x]} \cong \prod\limits_{m\in\mathbb{M}} \mathbb{F}[[m]]$ as rings.*

Proof. $\prod_{m\in\mathbb{M}}\mathbb{F}[[m]]$ is by definition the subset of $D=\prod_{m\in\mathbb{M}}\prod_{n\in\mathbb{N}}\mathbb{F}[x]/m^n\mathbb{F}[x]$ such that if $x\in D$ with the component of x in $\mathbb{F}[x]/m^n\mathbb{F}[x]$ denoted x_{m^n} we have $x_{m^{n+1}}+m^n\left(\mathbb{F}[x]/m^{n+1}\mathbb{F}[x]\right)=x_{m^n}$ for each choice of m,n. $\mathbb{F}[x]$ is a principal ideal domain, so by the Chinese Remainder Theorem if $f,g\in\mathbb{F}[x]$ have no common factor in $\mathbb{M}$ then $\mathbb{F}[x]/fg\mathbb{F}[x]\cong\mathbb{F}[x]/f\mathbb{F}[x]\oplus\mathbb{F}[x]/g\mathbb{F}[x]$ via the isomorphism $h+fg\mathbb{F}[x]\mapsto$

$(h+f(x)\left(\mathbb{F}[x]/fg\mathbb{F}[x]\right))\oplus(h+g(x)\left(\mathbb{F}[x]/fg\mathbb{F}[x]\right))$. Therefore, given $x\in D$ define x_{m^n} for powers of monic irreducibles m^n as before, and for each monic $f\in\mathbb{F}[x]$ write the unique factorization $f=\prod_{m\in\mathbb{M}}m^{n_m}$(with all but finitely many n_m equal to 0) and define x_f to be the unique element of $\mathbb{F}[x]/f\mathbb{F}[x]$ such that $x_f+m^{n_m}\left(\mathbb{F}[x]/f\mathbb{F}[x]\right)=x_{m^{n_m}}\ \forall m\in\mathbb{M}$. Then for all monic $f,g\in\mathbb{F}[x]$ the mapping $x_{fg}\mapsto x_f$ is given by the natural homomorphism, i.e. the bonding map $\mathbb{F}[x]/fg\mathbb{F}[x]\to\mathbb{F}[x]/f\mathbb{F}[x]$ used in the definition of $\overline{\mathbb{F}[x]}$ as a projective limit; therefore the map $\prod_{m\in\mathbb{M}}\mathbb{F}[[m]]\to\prod_{f\,monic}\mathbb{F}[x]/f\mathbb{F}[x]$ defined by $x\mapsto(x_f)_{f\,monic}$ is a homomorphism $\prod_{m\in\mathbb{M}}\mathbb{F}[[m]]\to\overline{\mathbb{F}[x]}$. It is injective since $x\neq 0\to x_{f^n}\neq 0$ for some $f\in\mathbb{M}$, $n\in\mathbb{N}$, and it is surjective since for any $\xi\in\overline{\mathbb{F}[x]}$, writing ξ_f for the component of ξ in $\mathbb{F}[x]/f\mathbb{F}[x]$, the indexed set $(\xi_{m^n})_{m\in\mathbb{M},n\in\mathbb{N}}$ will have to satisfy $\xi_{m^{n+1}}+m^n\left(\mathbb{F}[x]/m^{n+1}\mathbb{F}[x]\right)=\xi_{m^n}$ and will consequently be a preimage of ξ. Therefore it is an isomorphism, and $\overline{\mathbb{F}[x]}\cong\prod_{m\in\mathbb{M}}\mathbb{F}[[m]]$. ☐

Corollary. *In particular,* $\overline{\mathbb{F}[x]}\cong\prod_{m\in\mathbb{M}}\mathbb{F}[[m]]$ *for* $\mathbb{F}=\mathbb{Q}$, $\mathbb{Q}_p$, *and* $\mathbb{F}_p$.

13. $\varprojlim\mathbb{Z}[x]/x^{!n}\mathbb{Z}[x]$

Definition 36. Define the rising factorial $x^{!n}:=\prod_{k=1}^{n}(x+k)$, and let $\mathbb{Z}[x]_{x^{!n}}$ denote the projective limit $\varprojlim_{n\in\mathbb{N}_0}\mathbb{Z}[x]/x^{!n}\mathbb{Z}[x]$, where $\mathbb{N}_0$ is ordered by magnitude.

Proposition 70. *To each element f of $\mathbb{Z}[x]_{x^{!n}}$ there corresponds a unique sequence of integers $\{a_n\}_{n\in\mathbb{N}}$ such that $f(x)=\sum_{n=0}^{\infty}a_nx^{!n}$, and vice versa.*

Proof. Since the bonding maps $\mathbb{Z}[x]/x^{!N}\mathbb{Z}[x]\to\mathbb{Z}[x]/x^{!(N-1)}\mathbb{Z}[x]$ are the natural homomorphisms and the $x^{!n}$ are a basis over $\mathbb{Z}$ for $\mathbb{Z}[x]$, the bonding maps simply take polynomials in $\left\{x^{!0},...,x^{!N}\right\}$ to those in $\left\{x^{!0},...,x^{!(N-1)}\right\}$ by omitting the term corresponding to $x^{!N}$, from which the claim follows immediately. ☐

Proposition 71. *Let $f=\sum_{n=0}^{\infty}a_nx^{!n}\in\mathbb{Z}[x]_{x^{!n}}$ and let $\phi\colon\mathbb{Z}[x]_{x^{!n}}\to Map(\bar{\mathbb{Z}},\bar{\mathbb{Z}})$ be defined by $\hat{f}:=(\phi(f))(x):=\sum_{n=0}^{\infty}a_nx^{!n}$ as a function $\bar{\mathbb{Z}}\to\bar{\mathbb{Z}}$. Then ϕ is an injective homomorphism of rings.*

Proof. Let $\hat{f} = \phi(f)$. Suppose $\hat{f}(x) = 0\ \forall x \in \bar{\mathbb{Z}}$ and the a_n are not all 0. Then for any $N \geq 0$ we have $(-N)^{!(N+1)} = 0$, so $\sum\limits_{n=0}^{\infty} a_n(-N)^{!n} = \sum\limits_{n=0}^{N} a_n(-N)^{!n}$, so in particular if N is the least natural such that $a_n \neq 0$ we have $\hat{f}(N) = a_N(-N)^{!N} \neq 0$, a contradiction. So the kernel of ϕ is trivial; ϕ is clearly an additive homomorphism since we may add coefficients termwise, and it is also a multiplicative homomorphism, because it clearly preserves the multiplicative identity $1x^{!0}$, and if $\sum\limits_{n=0}^{\infty} a_n x^{!n} \sum\limits_{n=0}^{\infty} b_n x^{!n} = \sum\limits_{n=0}^{\infty} c_n x^{!n}$ then since for $n \geq N$ we have $z^{!n} \equiv 0 \pmod{N!}\ \forall z \in \bar{\mathbb{Z}}$ and $x^{!n} \equiv 0 \pmod{(x)_N}$, we must have $\forall z \in \bar{\mathbb{Z}}$ that $\sum\limits_{n=0}^{\infty} a_n x^{!n} \sum\limits_{n=0}^{\infty} b_n x^{!n} \equiv \sum\limits_{n=0}^{\infty} c_n x^{!n}$ modulo every factorial, and so the two are equal. □

Lemma 8. *We have the following identities:*

(1) $\Delta x^{!n} = n(x+1)^{!(n-1)}$
(2) $\nabla x^{!n} = n x^{!(n-1)}$
(3) $(\nabla - 1)\sum\limits_{n=0}^{\infty} b_n x^{!n} = \sum\limits_{n=0}^{\infty} \left((n+1)b_{n+1} - b_n\right) x^{!n}$

Proof. Proceeding for each part: □

(1) $\Delta x^{!n} = (x+1)^{!n} - x^{!n} = n(x+1)^{!(n-1)}$
(2) $\nabla x^{!n} = x^{!n} - (x-1)^{!n} = x^{!(n-1)}(x+n-1-(x-1)) = n x^{!(n-1)}$
(3) $(\nabla - 1)\sum\limits_{n=0}^{\infty} b_n x^{!n} = \sum\limits_{n=0}^{\infty} b_n n x^{!n-1} - \sum\limits_{n=0}^{\infty} b_n x^{!n} = \sum\limits_{n=0}^{\infty} \left((n+1)b_{n+1} - b_n\right) x^{!n}$

Lemma 9. $f \mapsto (1-\nabla)f = f(x-1)$ *is a ring automorphism of* $\mathbb{Z}[x]_{x^{!n}}$.

Proof. From the preceding lemma we see that left-multiplication by $-(\nabla - 1) = 1 - \nabla$ is a function $\mathbb{Z}[x]_{x^{!n}} \to \mathbb{Z}[x]_{x^{!n}}$. It is a homomorphism since $(1-\nabla)(f)(x) + (1-\nabla)(g)(x) = f(x-1) + g(x-1) = (f+g)(x-1) = (1-\nabla)(f+g)(x)$, and $((1-\nabla)(f)(x))\,((1-\nabla)(g)(x)) = f(x-1)g(x-1) = (fg)(x-1) = (1-\nabla)(fg)(x)$. It is surjective since any $f(x)$ is equal to $(1-\nabla)(f(x+1))$, and it is injective since $f(x-1) = 0\ \forall x$ implies $f(x) = 0\ \forall x$. □

Theorem 16. *For each* $z \in \mathbb{Z}$ *let* $\phi_z \colon \mathbb{Z}[x]_{x^{!n}} \to \bar{\mathbb{Z}}$ *be defined by* $\phi_z(f) = (\phi f)(z)$, *where* $\phi \colon \mathbb{Z}[x]_{x^{!n}} \to Map(\bar{\mathbb{Z}}, \bar{\mathbb{Z}})$ *is the natural map* $\sum\limits_{n=0}^{\infty} a_n x^{!n} \mapsto \sum\limits_{n=0}^{\infty} a_n x^{!n}$. *Then:*

(1) $ker(\phi_0) = x\mathbb{Z}[x]_{x^{!n}}$ *and* ϕ_0 *is surjective; thus* $\bar{\mathbb{Z}} \cong \mathbb{Z}[x]_{x^{!n}}/x\mathbb{Z}[x]_{x^{!n}}$.
(2) $ker(\phi_1) = (x-1)\mathbb{Z}[x]_{x^{!n}}$ *and* ϕ_1 *is surjective; thus* $\bar{\mathbb{Z}} \cong \mathbb{Z}[x]_{x^{!n}}/(x-1)\mathbb{Z}[x]_{x^{!n}}$.
(3) *For general* $z \in \bar{\mathbb{Z}}$, $ker(\phi_z)$ *is the set of all elements* $\sum\limits_{n=0}^{\infty} a_n x^{!n}$ *for which there exists a sequence* $\{\hat{a}_n\}_{n\in\mathbb{N}_0}$ *such that* $\hat{a}_0 = 0$, $\hat{a}_1 = a_0$, *and* $\forall n \in \mathbb{N}_0$ *we have* $a_n \frac{z^{!n}}{n!} = (n+1)\hat{a}_{n+1} - \hat{a}_n$.

Proof. Fix z; ϕ_z is the composition of ϕ and evaluation at z, which are both ring homomorphisms; therefore ϕ_z is a ring homomorphism. Now suppose $f = \sum_{n=0}^{\infty} a_n x^{!n}$ is in the kernel of ϕ_z, i.e. $\sum_{n=0}^{\infty} a_n z^{!n} = 0$. It is immediately clear from $0^{!n} = \begin{cases} 1 & : n = 0 \\ 0 & : n > 0 \end{cases}$ that f is in $ker(\phi_0)$ exactly when $a_0 = 0$, i.e. $ker(\phi_0) = x\mathbb{Z}[x]_{x^{!n}}$, and since every canonical series is in the image ϕ_0 is surjective, establishing $\mathbb{Z}[x]_{x^{!n}}/x\mathbb{Z}[x]_{x^{!n}} \cong \bar{\mathbb{Z}}$. For general z, z is a root if and only if $\sum_{n=0}^{N-1} a_n z^{!n} \equiv 0 \pmod{N!}$ $\forall N \in \mathbb{N}$, so let us give an equivalent characterization of this condition by examing the case corresponding to each N. Recall that $n! \mid (z+n-1)_n = z^{!n}$ for all choices of n and z. The choices of constants $\hat{a}_n$ that follow can all be made in $\mathbb{Z}$:

(1) Set $\hat{a}_0 = 0$.

(2) At $N = 1$ we have $a_0 z^{!0} = a_0 \equiv 0 \pmod 1$, i.e. $\sum_{n=0}^{0} a_n z^{!n} = a_0 z^{!0} = \hat{a}_1$ for $\hat{a}_1 = \hat{a}_0 + a_0$.

(3) At $N = 2$ we have $a_0 z^{!0} + a_1 z^{!2} = \hat{a}_1 + a_1 z^{!1} \equiv 0 \pmod 2$, i.e. $\sum_{n=0}^{1} a_n z^{!n} = \hat{a}_1 + a_1 z^{!1} = 2\hat{a}_2$ for some $\hat{a}_2$.

(4) At $N = 3$ we have $a_0 z^{!0} + a_1 z^{!2} + a_2 z^{!2} = 2\hat{a}_1 + a_2 z^{!2} = 2\left(\hat{a}_2 + a_2 \frac{z^{!2}}{2}\right) \equiv 0 \pmod{3!}$, i.e. $\exists \hat{a}_3$ such that $3\hat{a}_3 = \hat{a}_2 + a_2 \frac{z^{!2}}{2}$, and we have $\sum_{n=0}^{2} a_n z^{!n} = 3!\hat{a}_3$.

Proceeding by induction, suppose $\forall K < N$ $\exists \hat{a}_{K+1}$ such that $(K+1)\hat{a}_{K+1} = \hat{a}_K + a_K \frac{z^{!K}}{K!}$, and we have $\sum_{n=1}^{K} a_n x^{!n} = (K+1)!\hat{a}_{K+1}$. Then $\sum_{n=0}^{N} a_n z^{!n} = N!\hat{a}_N + a_N z^{!n} = N!\left(\hat{a}_N + a_N \frac{z^{!N}}{N!}\right)$, so $\hat{a}_N + a_N \frac{z^{!N}}{N!} \equiv 0 \pmod{(N+1)!}$ so $\exists \hat{a}_{N+1}$ such that $(N+1)\hat{a}_{N+1} = \hat{a}_N + a_N \frac{z^{!N}}{N!}$, and we have $\sum_{n=0}^{N} a_n z^{!n} = (N+1)!\hat{a}_{N+1}$, as claimed; so the proposition holds for all naturals. Therefore f is in the kernel if and only if there exists a sequence $\{\hat{a}_n\}_{n\in\mathbb{N}}$ such that $\hat{a}_1 = a_0$ and $\forall n \in \mathbb{N}_0$ $(n+1)\hat{a}_{n+1} = \hat{a}_n + a_n \frac{z^{!n}}{n!}$. Now $(n+1)\hat{a}_{n+1} = \hat{a}_n + a_n \frac{z^{!n}}{n!}$ is equivalently $a_n z^{!n} = (n+1)!\hat{a}_{n+1} - n!\hat{a}_n$. Suppose $z = 1$; then $z^{!n} = n!$, so this is $n!a_n = (n+1)!\hat{a}_{n+1} - n!\hat{a}_n$, or equivalently $a_n = (n+1)\hat{a}_{n+1} - \hat{a}_n$, so $a_n x^{!n} = (n+1)\hat{a}_{n+1} x^{!n} - \hat{a}_n x^{!n}$. This is exactly the condition giving $(\nabla - 1)\sum_{n=0}^{\infty} \hat{a}_n x^{!n} = \sum_{n=0}^{\infty} \left((n+1)\hat{a}_{n+1} - \hat{a}_n\right) x^{!n}$, by Lemma 8; since $\hat{a}_0 = 0 \iff x \mid \sum_{n=0}^{\infty} \hat{a}_n x^{!n}$, we thus have $ker(\phi_1) = \{(\nabla - 1)(xf) : f \in \mathbb{Z}[x]_{x^{!n}}\} = \{(x-1)((\nabla - 1)f)(x) : f \in \mathbb{Z}[x]_{x^{!n}}\}$ $= \{(x-1)f : f \in \mathbb{Z}[x]_{x^{!n}}\}$, so $ker(\phi_1) = (x-1)\mathbb{Z}[x]_{x^{!n}}$. Moreover, ϕ_1 is clearly surjective since $1^{!n} = (n-1)!$ except at $n = 0$ where $1^{!0} = 1$, and hence series of the form $\sum_{n=0}^{\infty} a_n 1^{!n}$ include all canonical series. $\square$

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