Nonlinear Analysis and Control of Aeroelastic Systems

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ABSTRACT

Presence of nonlinearities may lead to limit cycle oscillations (LCOs) in aeroelastic systems. LCOs can result in fatigue in wings leading to catastrophic failures. Existence of LCOs for velocities less than the linear flutter velocity has been observed during flight and wind tunnel tests, making such subcritical behavior highly undesirable. The objective of this dissertation is to investigate the existence of subcritical LCOs in aeroelastic systems and develop state feedback controllers to suppress them. The research results are demonstrated on a two degree of freedom airfoil section model with stiffness nonlinearity.

Three different approaches are developed and discussed. The first approach uses a feedback linearization controller employing the aeroelastic modal coordinates. The use of modal coordinates results in a system which is linearly decoupled making it possible to avoid cancellation of any linear terms when compared to existing feedback linearization controllers which use the physical coordinates. The state and control costs of the developed controller are compared to the costs of the traditional feedback linearization controllers. Second approach involves the use of nonlinear normal modes (NNMs) as a tool to predict LCO amplitudes of the aeroelastic system. NNM dynamics along with harmonic balance method are used to generate analytical estimates of LCO amplitude and its sensitivities with respect to the introduced control parameters. A multiobjective optimization problem is solved to generate optimal control parameters which minimize the LCO amplitude and the control cost. The third approach uses a nonlinear state feedback control input obtained as the solution of a multiobjective optimization problem which minimizes the difference between the LCO commencement velocity and the linear flutter velocity. The estimates of LCO commencement velocity and its sensitivities are obtained using numerical continuation methods and harmonic balance methods. It is shown that the developed optimal controller eliminates any existing subcritical LCOs by converting them to supercritical LCOs.
Aeroelasticity deals with the problem of fluid-structure interaction in aerial systems. The desire for high speed travel and increased maneuverability has led to the development of complex designs, use of advanced materials and integration of control systems in aerial systems. All these factors cause the system to exhibit diverse response behavior which cannot be captured by linear models in general. The need to analyze and predict the behavior of such systems for design optimization and control design resulted in the use of nonlinear models to represent such systems mathematically. Flutter is a catastrophic phenomenon where the system absorbs energy from the surrounding air and leads to oscillations with increasing amplitudes. Presence of nonlinearities results in limited amplitude flutter or limit cycle oscillations (LCO). The objective of this dissertation is to analyze the effects of stiffness nonlinearities on the response of aeroelastic systems and development of nonlinear controllers to suppress LCOs. A two degree of freedom airfoil section model with quasi-steady aerodynamics is used for this purpose. Three different approaches are discussed and developed.

The first approach focuses on development of a feedback linearization controller highlighting the advantages of using modal coordinates in the control design process. The developed controller is optimal for small disturbances and guarantees stability of response for large disturbances. The second and third approaches develop analysis tools to capture response behavior of aeroelastic systems. Using the developed framework analytical estimates of LCO amplitude, LCO commencement velocity and their sensitivities to control parameters are derived. These are used to solve an optimization problem to generate strictly nonlinear feedback control laws to control the LCO amplitudes and eliminate the subcritical LCOs.
Attribution

Mayuresh J. Patil is co-author for the manuscripts of Chapters 3, 4 and 5. His contributions include strategic advice and review of the articles for technical accuracy, completeness and grammatical correctness.
Dedicated to

my parents, sister and brother
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Chapter 1

Introduction

The very first time Wright brothers flew their craft, they chose to twist/warp the tips of their aircraft in order to control its rolling motion [1]. Study of aeroelasticity and control of aeroelastic responses involves diverse disciplines dealing with problems such as wing flutter, divergence, control effectiveness and control reversal [2]. Flutter, which is considered the most catastrophic is a dynamic instability where the wing absorbs energy from air resulting in vibrations leading to sudden failure.

The demand for increasing flight speed and the increasing domain of applications of aerial systems over the period of aviation history has led to the development of complex structural designs, use of advanced materials and the enhanced use of avionics. All these factors result in increased nonlinear effects in the response characteristics of aeroelastic systems. The presence of nonlinearities in aeroelastic systems tends to cause a limited amplitude flutter or limit cycle oscillations (LCOs). These correspond to a case of neutral stability where the nonlinearities, such as geometric, aerodynamic, stiffness or structural damping tend to limit the amplitude of vibration. Essentially these are periodic oscillations with constant amplitude and frequency. Being subjected to LCOs over extended period of time can lead to structural fatigue which might lead to failure. Existence of LCOs for subcritical velocities, i.e., velocities less than the linear flutter velocity for nonlinear aeroelastic systems has been
verified during flight and wind tunnel tests. The destabilizing nonlinearities of subcritical LCOs can result in a dangerously large amplitudes for velocities above and below the flutter point. These are much more catastrophic compared to the amplitudes of supercritical LCOs which are relatively benign [3].

Over the years, aeroelasticity related problems have been attempted by designing airframes so as to keep the flight envelope free from any destructive phenomena. This generally requires addition/distribution of mass, or constraining the flight envelope. These methods are referred to as passive control methods and involve redesigning the structure which is followed by extensive wind tunnel and flight testing and are hence quite time consuming and costly. These undesirable phenomena can be eliminated through the use of active control methods leading to an increase in performance without the addition of any extra mass.

A lot of work has been done on active control methods for the nonlinear aeroelastic control problem over the decades using principles of classical control, modern control, robust control and nonlinear control design methodologies. A review of some of these methods is summarized in Chapter 2 on literature review.

1.1 Aims and Objectives

In the present research, a two degree of freedom airfoil section model is used to investigate the response characteristics of aeroelastic systems and develop nonlinear feedback control laws to suppress/eliminate the presence of limit cycle oscillations. Quasi steady aerodynamic model is used to represent the aerodynamic forces and moments. It is shown that transforming the system to modal coordinates before the control design phase results in lower state and control costs than the traditional nonlinear controller which utilize physical coordinates. Using modal coordinates also allows the ease of extension of the control design approach to large scale aeroelastic systems. A framework using nonlinear normal modes, harmonic
balance method and numerical continuation techniques is developed to predict aeroelastic system response and identify important response characteristics such as LCO commencement velocity. Analytical estimates of LCO amplitude, LCO commencement velocity and their sensitivities with respect to the introduced design parameters are generated. Knowledge of analytical sensitivities has vast applications ranging from use in control design methods to design optimization studies. Use of the estimated analytical sensitivities to design purely nonlinear state feedback controllers is demonstrated.

The objectives are summarized as below:

1. Develop a nonlinear feedback linearization based controller in modal coordinates which behaves like an optimal linear controller for small disturbances.

2. Generate analytical estimates of LCO amplitude and their sensitivities to control parameters using nonlinear normal modes and harmonic balance method.

3. Use analytical sensitivity estimates to design a nonlinear state feedback control law to control LCO amplitudes.

4. Investigate the nature of flutter boundary for the nonlinear aeroelastic systems identifying the subcritical and supercritical LCO regions.

5. Generate analytical estimates of LCO commencement velocity and its sensitivity to introduced control parameters using numerical continuation techniques and harmonic balance method.

6. Develop a nonlinear optimization framework to generate optimal control parameters which eliminates the presence of any subcritical LCOs by converting them to supercritical LCOs.
1.2 Outline

The dissertation is divided into 6 chapters. The current, Chapter 1 talks about the aims and objectives focused in the dissertation.

Chapter 2 presents a review of past work on analysis of aeroelastic system response and existing methodologies used for nonlinear aeroelastic systems. The literature review comprises of four sections which give an overview on the literature on aeroelasticity and aeroservoelasticity, effects of nonlinearities on aeroelastic systems, predicting response characteristics of nonlinear aeroelastic systems and the existing active control methods to avoid instability for aeroelastic systems.

Chapter 3 is based on the development of a feedback linearization controller using modal coordinates. The chapter starts with system definition, followed by linear analysis, linear modal transformation to obtain linearly decoupled modal dynamics and a nonlinear transformation to partially linearize the system. The results compare the state and control costs for a set of initial conditions with a linear controller and a feedback linearization controller which uses physical coordinates.

Chapter 4 is based on development of nonlinear control laws to control the LCO amplitudes of the aeroelastic system using nonlinear normal modes and harmonic balance method. The chapter starts with system definition, followed by an introduction of nonlinear normal modes with a short review of the asymptotic method for computing NNMs. This is followed by a section on harmonic balance method which is used to estimate LCO amplitude and their sensitivities to introduced control parameters. A multiobjective optimization framework is formulated to generate optimal control parameters. The results demonstrate that the generated optimal control parameters can be used to control the LCO amplitude of the aeroelastic system.

Chapter 5 is based on work towards development of a nonlinear control laws to eliminate
subcritical LCOs by converting them to supercritical LCOs. This includes sections on Moore Penrose inverse continuation technique, harmonic balance method and generating analytical estimates of LCO commencement velocity and its sensitivities to control parameter. The results show that no subcritical LCOs exist for the closed loop system dynamics.

Finally, Chapter 6 presents the conclusions and directions for future work.
Bibliography


Chapter 2

Literature Review

This chapter presents a review of past work on analysis of aeroelastic system response and control methodologies used for nonlinear aeroelastic systems. The literature review comprises of four sections. The first section talks about topics of aeroelasticity and aeroservoelasticity in general. The second section reviews the studies on effects of nonlinearities on the response of aeroelastic systems. The third section surveys the various methods used to characterize and predict the response of aeroelastic systems. The last section present a detailed overview of existing active control strategies used for avoiding instability in aeroelastic systems.

2.1 Aeroelasticity

A number of texts [1, 2, 3] give the conventional description of the field of aeroelasticity as the study of mutual interaction of three main forces: elastic, aerodynamic and inertial, each of which is a broad field of study in itself. Principles of solid mechanics are used to describe the elastic forces of the structure however complex the structural model might be. Fluid mechanics provides the description of the fluid surrounding the structure which can be
defined using a simple 2D potential flow model or a complex 3D Navier-Stokes model. Study of dynamics is then utilized to provide a framework for the fluid-structural interaction. Attempts to control aeroelastic systems for increasing flight envelope and performance have led to overlap of field of aeroelasticity with the domains of control theory and nonlinear dynamic theory, often called aeroservoelasticity as shown in Fig. 2.1. The term aeroservoelasticity was first coined by Collar [4] in 1946. Aeroservoelasticity deals with problems such as active control of aeroelastic systems [5, 6, 7], flutter suppression [8], gust load alleviation [8] and LCO suppression [9, 10].
2.2 Effects of nonlinearities on aeroelastic systems

Almost all physical systems contain some elements of nonlinearity. Linearization of the governing equations is often employed to generate linear approximations of these physical systems due to the existence of a number of tools to analyze and predict the behavior of linear systems. For small oscillations the response of a deformable body can be described adequately by linear equations and boundary conditions. The results generated using these linear approximations are valid and accurate while the assumptions are valid. However, with increasing amplitude of oscillations, nonlinear effects come into effect [11]. Introduction of nonlinearities not only changes the linearized solution but may also introduce certain complex response phenomena [12, 13] which cannot be explained using linearized approximations.

Fig 2.2 shows the response of a linearly stable nonlinear system when subjected to small and high disturbances. Depending on the disturbance level, the nonlinearities can either remain dormant or contribute significantly to the nonlinear system dynamics. For small disturbances, nonlinearities have no visible effect on the system dynamics making the system behave like a linearly stable system, hence the stable response. However for large disturbances nonlinearities contribute significantly to the system dynamics resulting in a range of response phenomena. Therefore, the notion of linear stability cannot be used for nonlinear systems without carrying out extensive analysis of the region of attraction.

Aeroelastic systems are inherently nonlinear with nonlinearities arising from both structural and aerodynamic terms [1, 2, 3]. Terms coming from unsteady aerodynamic models, large strain-displacement conditions or the partial loss of structural or control integrity account for sources of nonlinearities. Linearized models of such complex, safety critical systems cannot capture the complex nonlinear response phenomenon and hence are not reliable for accurate prediction of response characteristics [11, 14]. Failure of linear models to do so resulted in the necessity to develop more accurate prediction and analysis methods for aeroelastic systems.
Figure 2.2: Linearly stable nonlinear systems can exhibit stable, periodic, unstable and chaotic response depending on system parameters and disturbances.

Effects of nonlinearities on the dynamic stability of the aeroelastic system and its response has been a problem of primary interest among researchers over the past decades. An important design criterion for an aircraft and determination of its flight envelope is the prediction of its divergent solution or the flutter boundary. Nonlinearities arising in aeroelastic systems are often divided into distributed nonlinearities which are continuously activated throughout the whole structure by elasto-dynamic deformations or concentrated ones which act locally lumped in control mechanisms or in the connecting parts between wing and external stores [15]. Van Dyke [16] investigated the effects of aerodynamic nonlinearities introduced by airfoil thickness effects on a harmonically oscillating two dimensional airfoil using a sec-
ond order solution. He concluded that nonlinear thickness effects reduce torsional damping thereby enlarging the range of velocities in which instability occurs. The study of nonlinear effects in aeroelasticity remained almost unexplored until the 1950s when Woolston et al. [17, 18] presented results on the effects of nonlinear structural terms on the flutter of a two and three degree of freedom airfoil section model using Theodorsen’s unsteady aerodynamics [19]. He investigated three types of nonlinearities namely a flat spot, hysteresis and cubic spring. It was found that for small disturbance cases the linear flutter speed was not affected, however for large disturbance cases the instability speed usually decreased. He observed that for nonlinearities modeled as cubic hard spring, limited amplitude flutter or LCOs were observed for velocities well above the linear flutter velocity.

Inspired by Woolston’s work, Shen [20, 21] used the nonlinear mechanics theory of Kryloff and Bogoliuboff [22] to look into the flutter problem of airfoil section. They balanced the fundamental harmonic ignoring the effects of higher harmonics introduced by the nonlinearity to obtain a linear system with terms depending on the amplitude of the periodic oscillation. This is equivalent to generating an equivalent stiffness of the nonlinear stiffness functions and using it for the flutter problem.

Breitbach [15] conducted a detailed survey of various types of nonlinearities encountered in aeroelastic systems. An attempt to understand their physical origin and classify them based on their influence on flight and ground tests and analytical flutter predictions. Various types of concentrated and distributed nonlinearities are included in the study.

Price et al. [23] considered a bilinear and cubic structural nonlinearity in a two dimensional airfoil model in pitch degree of freedom. Using Theodorsen’s aerodynamic model, existence of LCOs for velocities beyond the linear flutter velocity was demonstrated. Existence of LCO was shown to depend strongly on the initial conditions. Comparisons are shown between the direct simulation and describing function approach showing that the describing function approach accurately predicts the magnitude of LCOs for different velocities, but cannot capture the dependence of LCOs on the initial conditions. It is noted by Price et al. [23]
that the describing function performs well when the number of harmonics in the describing function are approximately equal to those in the actual response. Existence of chaotic motions was shown for the cases with small structural preloads and was confirmed by power spectral densities, phase plane plots and a positive Lyapunov exponent.

A review paper by Lee et al. [24] discusses in detail the various structural and aerodynamic nonlinearities encountered in aeroelastic systems. They derived the equations of motion of a two dimensional airfoil oscillating in pitch and plunge under the influence of a structural nonlinearity modeling the lift and moments using subsonic aerodynamics. Studies focused on three different types of nonlinearities, cubic, freeplay and hysteresis are conducted to identify the onset of Hopf bifurcation along with the amplitude and frequencies of the exhibited LCOs. Application of Poincare maps and Lyapunov methods is demonstrated to determine stability of LCOs. Numerical simulations are conducted to identify the effects of aerodynamic nonlinearities. Bifurcation diagrams are used to determine the existence of chaos. The usefulness of describing function approach to predict the amplitudes of LCOs is demonstrated.

Figure 2.3: Classification of LCOs as good (supercritical) and bad (subcritical) LCOs depending on how their amplitudes vary with velocity [12].
Dowell [12] gives a broader treatment of structural nonlinearities beyond simple lumped parameter systems accounting for the state of the art developments in theoretical, computational and experimental studies for aeroelastic systems. LCOs are identified as one of the simplest response behavior exhibited by the aeroelastic system. Apart from LCO, theoretical studies have been able to identify the existence of complex behavior such as higher harmonic and subharmonic resonances, resonances, beating and period doubling oscillations. As shown in Fig. 2.3, LCOs are identified as ‘good’ and ‘bad’ based on how their amplitudes vary with the flight speed [12].

2.3 Predicting response characteristics of aeroelastic systems

A number of methods have been used to predict the behavior of aeroelastic systems. The most common and simplest method is via time marching. The system is simulated in time for a range of initial conditions. Phase plane plots can be employed to check if the response is diverging or converging to a LCO. The steady state response can also be used to estimate the LCO amplitudes and how it varies with changing velocities. Patil et al. [25] analyzed the LCO behavior in high aspect ratio wings using numerical time marching methods. Using numerical methods such as time marching are quite computationally expensive and time consuming and involve exploring a range of initial conditions. To avoid conducting numerous simulations, analytical methods such as method of multiple scales [26] and nonlinear normal modes [27, 28] have been developed based on normal form analysis and central manifold theory respectively. Harmonic balance methods [29, 30, 31] have been utilized to estimate approximate periodic solutions. Central manifold theory and normal form analysis are often used to study the behavior of the aeroelastic system in the vicinity of the flutter boundary. On the other hand harmonic balance method can be used globally over the entire range of
velocities.

2.3.1 Harmonic balance method

An airfoil placed in an incompressible flow may lead to self induced oscillations. Computational fluid dynamic simulations [32, 33] are often computationally expensive and time consuming limiting the analysis mainly towards generating system responses.

In contrast, investigations using simple airfoil section models have progressed towards identification of exhibited limit cycles, stability predictions and bifurcation analysis. Yang and Zhao [34] conducted theoretical and experimental investigations of self excited oscillations for a two dimensional wing model with nonlinear freeplay pitch stiffness. A case of double limit cycles was identified during wind tunnel testing which was also validated using harmonic balance method. The applicability of harmonic balance method towards predicting unstable LCOs was also established. Price et al. [23] analyzed a two dimensional airfoil section with bilinear or cubic nonlinearity in pitch using Wagner’s function to compute the aerodynamic forces. It is shown that describing function approach gives reasonable predictions of LCO commencement velocity and LCO amplitudes. Lee et al. [31] used higher order harmonic balance methods to derive analytical expression for LCO amplitude and frequency for the airfoil section model in the presence of cubic pitch nonlinearity. The generated results are validated by comparing them to numerical simulation results. It was concluded that high number of harmonics are required to capture some phenomenon which result as a consequence of secondary bifurcation.

Dimitriadis and Cooper [35] introduced a harmonic shifting technique while using higher order harmonic balance methods to predict both branches of limit cycles extending from the secondary bifurcation point. In a separate work Dimitriadis [30] used a coupled harmonic balance and continuation framework to follow the response of dynamic systems from bifurca-
tion point to any desired parameter value. The developed framework is applied to a generic transport aircraft model with polynomial or freeplay stiffness nonlinearity.

2.3.2 Nonlinear normal modes

Linear normal modes are an important tool to analyze vibrations in linear systems allowing one to obtain decoupled normal modes. This property forms the basis of theory of superposition of fundamental motions. The linear modal responses are invariant in the sense that a motion initiated in one mode always stays in the respective mode without exciting any other mode. These properties of linear normal modes do not hold in general for nonlinear systems.

Nonlinear normal modes (NNMs) are an extension of the linear modal theory for nonlinear systems. Introduced by Rosenberg in 1960 [36], NNMs have been identified as an efficient tool for analyzing the behavior of nonlinear dynamic systems [37, 11]. Shaw and Pierre [28] extended the concept of normal modes of vibration to general nonlinear systems, i.e., both conservative and non-conservative systems, defining NNMs as invariant subspaces for nonlinear equations of motion. The defined invariant subspaces are tangential to linear normal modes at the equilibrium point. The invariant manifold can be characterized by a set of differential equations in terms of the master coordinates. The displacements and velocities of all other degrees of freedom can be expressed as nonlinear functions of the displacements and velocity of the master coordinate.

Asymptotic method [28] and Galerkin formulation [38] are two most commonly used methods to compute NNMs for dynamic systems. Asymptotic method is local in nature and works well for weakly nonlinear systems while the galerkin formulation can be used to generate accurate NNMs for large amplitude, strongly nonlinear dynamic systems. Bellizi and Bouc [39] introduced a amplitude phase formulation of nonlinear normal modes to characterize periodic orbits and limit cycles in multi degree of freedom systems.
Emory and Patil [40] used the concept of NNMs to predict the growth of LCOs and its amplitude for nonlinear aeroelastic systems. The results demonstrate that NNMs can be used to model the aeroelastic system response using lesser number of equations.

2.4 Active control of aeroelastic systems

Aeroservoelasticity (ASE) is the interaction of unsteady aerodynamics, structural dynamics, and control systems and is an important interdisciplinary topic in aerospace industry [41]. Aeroelastic interactions have been widely studied over the past century; the impact of active control system on dynamic aeroelasticity has been studied over the past few decades. The formal beginning of aeroservoelasticity can be traced to the 1970s when measures to address the flutter instability were being investigated owing to the newer aircraft designs. Highly maneuverable aircrafts such as Lockheed F-16 and passenger aircrafts such as Boeing-767 and Airbus A-320 were being developed to achieve active flutter suppression and gust load alleviation using inbuilt flight control systems. Fig. 2.4 shows a block diagram of a typical aeroservoelastic system. Measurements from sensors capture the aeroelastic system response and are used by the flight control block to generate actuator commands to meet a specific goal.

Passive methods such as added structural stiffness, mass balancing, and speed restrictions have been used to address this problem [42]. However, the use of traditional passive methods to enlarge the operational range of flexible lifting structures and to enhance the aeroelastic response results in significant weight penalties or in unavoidable reduction of performances [43, 44]. These issues highlight the necessity of active control systems to fulfill the two basic objectives, i.e., 1) enhance subcritical aeroelastic response by suppression or alleviation of oscillations in shortest possible time, and 2) flight envelope expansion by suppressing flutter instability, resulting in significant increase in the allowable operational speed [45].
Significant amount of sustained research on aeroservoelasticity [46], active modal control, flutter suppression, and robust control design [47] were conducted by the Air Force Wright Aeronautical Laboratory. Kehoe [48] presents a detailed historical overview of flight flutter testing at NASA Dryden Flight Research Center. Mukhopadhyay [49] gives a detailed historical overview of aeroelasticity and its control. Benchmark Active Control Technologies (BACT) is a two degrees of freedom aeroelastic system experimental prototype developed at NASA Langley Research Centre [50]. It comprises of a NACA 0012 airfoil mounted on a platform allowing it to pitch and plunge. Equipped with a trailing edge control surface along with upper and lower surface spoilers, pressure transducers and accelerometers, BACT is an ideal setup to validate the performance of the developed flutter suppression and gust.
alleviation control systems for the two degree of freedom airfoil section model. Another similar experimental setup is the Nonlinear Aeroelastic Test Apparatus (NATA) [51] located at Texas A&M University. NATA has interchangeable airfoils mounted on a carriage placed underneath the wind tunnel allowing it to pitch and plunge. Pitch and plunge displacements are measured using encoders placed on the carriage and controlled using the leading and trailing edge control surfaces actuated with servo motors. NATA includes a nonlinear torsional spring for evaluating the effects of structural nonlinearity.

Lyons et al. [52] presented methods to represent unsteady aerodynamic loads for the two and three dimensional airfoil section models with leading or trailing edge flaps. Analytic continuation of Theodorsen function into the complex plane guided by the time domain approximation of the Wagner indicial function was used to model the loads in two dimensional case. Pade approximations were used to approximate oscillatory generalized aerodynamic forces for the three dimensional case. Feedback control laws are selected based on quadratic error criteria and implemented on the typical section model to illustrate the application of modern control theory and to develop design techniques applicable to three-dimensional cases. Full state feedback controller using a Kalman filter to estimate unknown states is developed based on a quadratic error criterion while evaluating the response of system states and control effort to gust disturbances.

Mukhopadhyay et al. [53] laid out a method to synthesize a low order optimal feedback control law for a high order system. A nonlinear programming algorithm is used to identify control law design variables that minimize a performance index defined as a weighted sum of mean square steady state responses and control inputs. The controller is shown to be equivalent to a partial state estimator. Lower order controllers are compared with the full order optimal control law. It is shown that with a proper selection of states to be estimated, a lower order control law with good stability margins can be obtained.

Comparing the unsteady aerodynamic approximations by Lyons et al. [52], Vepa [54] and Edwards [55], Karpel [8] developed a more accurate reduced order model enabling him to
generate simpler partial state feedback controllers for the purpose of flutter suppression and gust load alleviation. Horikawa and Dowell [56] did flutter analysis along with a proportional gain feedback control obtained using root locus plots. The control surface deflection involved feedback of four different terms, the bending displacement, bending acceleration, torsion displacement and torsion acceleration. Another example of typical section aeroelasticity control is the controller synthesized by Ozbay and Bachmann [57]. They developed a finite dimensional controller stabilizing the infinite dimensional aeroelastic model using $H_{\infty}$ control technique. The gust alleviation problem was modeled as a disturbance rejection problem using a mixed $H_2/H_{\infty}$ framework.

Mukhopadhyay [58] presented a flutter suppression control law along with wind tunnel results in transonic flow for a NACA 0012 airfoil using the BACT. A control law based on classical control theory and a robust control law using minimax techniques were designed and tested on the BACT. An increase of 50% in the flutter dynamic pressure was demonstrated for the closed loop system.

Waszak [59] implemented and compared several $H_{\infty}$ and $\mu$ based controllers, including single input single output (SISO) and multiple input and multiple output (MIMO) variations and validated the performance on BACT. A $P - \Delta$ framework was used, where $P$ represents the nominal plant and $\Delta$ represents the uncertainty block acting as an upper LFR (linear fractional representation) perturbation. Resulting control laws are shown to be able to suppress flutter across the usable wind tunnel dynamic pressure and Mach number.

Blue and Balas [60] transformed the BACT dynamics using a LFT (linear fractional transformation), formulating the feedback term $\Delta$ as a function of dynamic pressure and Mach number. The developed model was used by Barker et al. [61] to develop a gain scheduled controller depending in a linear fractional manner on the dynamic pressure and Mach number. Closed loop stability was demonstrated using simulations in the presence of disturbances for varying dynamic pressures and Mach numbers. The developed gain scheduled controller was compared with an optimal linear controller and was shown to outperform the
linear controller throughout the tunnel operating conditions. Barker and Balas [62] extended their earlier work further by using a direct LPV (linear parameter varying) framework and compared it to the previously developed LFR scheduled framework [61]. The control laws generated using the direct LPV framework were shown to perform better compared to the gain scheduled LFR approach.

NATA was used for the first time to experimentally validate flutter suppression control laws by Block et al. [6]. A few results from this work were published later by Block and Strganac [63]. The work uses a two dimensional airfoil section model. The unsteady aerodynamics is modeled using the Jones approximation [64] to Theodorsens model. Using a full state feedback controller in conjunction with an optimal observer, the closed loop system was shown to be stabilized above the open loop flutter velocity. Simulation results in MATLAB are compared to the experimental results to show a correlation between theory and experiments. A feedback linearization controller was developed by Ko et al. [65, 66] using a two degree of freedom airfoil model and quasi steady aerodynamics to represent the aerodynamic lifts and moments. Ko et al. [67] further applied the feedback linearization controller to a theoretical model equipped with two trailing edge control surfaces at opposite ends of airfoil span. An adaptive feedback linearization controller was developed by Ko et al. [68] using a two degree of freedom model based on NATA and which formed the basis of work by Strganac et al. [69]. A two degree of freedom airfoil model with quasi steady aerodynamics to represent the lift and moments was used to develop an adaptive feedback linearization based controller aimed to suppress limit cycle oscillations. The experimental results demonstrate effective suppression of LCOs in both pitch and plunge degrees of freedom. These experimental results on NATA used an airfoil with a single trailing edge control surface with polynomial type nonlinearity in the pitch degree of freedom.

A different airfoil model equipped with both a leading and trailing edge control surface was used by Platanitis and Strganac [70] to conduct theoretical and experimental studies. This was based on work by Xing and Singh [71] who used a two degree of freedom model based on NATA with quasi steady aerodynamics with a single trailing edge control surface.
They used a backstepping technique to develop a new adaptive controller using output feedback assuming unknown system parameters with only plunge displacement and pitch angle measurements. The simulation results demonstrate the state vector asymptotically converging to the equilibrium state for the closed loop system inspite of the uncertainty in the aerodynamic and structural parameters. Platanitis and Strganac [72] extended their earlier work [70] to include an additional leading edge control surface. Experimental results demonstrated saturation of control surfaces before suppressing the limit cycle oscillations.

Controllers based on adaptive backstepping technique for the two dimensional airfoil section model were developed by Singh and Wang [73], Reddy et al. [74] and Lee and Singh [75]. Quasi steady aerodynamics was used to model the aerodynamic forces and moments. Singh and Wang [73] showed that an adaptive backstepping output feedback controller can be developed using pitch angle as output. The results demonstrate control of pitch angle trajectory along with asymptotic convergence of the state vector in the presence of uncertainties and unknown model parameters. Reddy et al. [74] developed an adaptive output feedback controller using the pitch and plunge displacement measurements for an airfoil section with both leading and trailing edge control surfaces. The proposed controller is shown to achieve global asymptotic stability in the presence of uncertainties. Simulation results demonstrate suppression of flutter and LCOs and reduction in vibration levels for subcritical flight speed.

Singh and Brenner [76] proposed a modular adaptive control scheme for the two degree of freedom airfoil section model with a trailing edge control surface. Unsteady aerodynamics was used to model the aerodynamic forces and moments. Nonlinearities in both pitch and plunge degrees of freedom are considered. All the parameters of the model are assumed to be unknown except the sign of control input coefficient and its lower bound. A input-to-state stabilizing control law is developed demonstrating input-to-state stability with respect to parameter estimation error treated as a disturbance input. Simulation results demonstrate flutter suppression when parameter adaptation module is included. In the absence of parameters adaptation it is shown that even though flutter suppression is not achieved, the plunge and pitch motions remain bounded.
Baranyi [77, 78] used the tensor product model defined in [79, 80] to transform the airfoil section state space model into a tensor product model. Parallel distributed compensation [81] framework is then used to check the feasibility of obtained linear matrix inequalities. The method guarantees asymptotic stability and is capable of easy modification based on design specifications. Takarics and Baranyi [82] further demonstrate that the designed controller is resilient to a variety of perturbations.

Prime et al. [83, 84] develop a linear parameter varying (LPV) controller for a three dimensional airfoil section model which includes the servo dynamics. Using an $H_2$ representation of the standard linear quadratic regulator (LQR) control problem, a state feedback controller is synthesized using linear matrix inequalities. The developed controller is scheduled with airspeed and effectively suppresses limit cycle oscillations over a range of speeds.
Bibliography


Chapter 3

Feedback Linearization based Control of Aeroelastic System represented in Modal Coordinates

3.1 Abstract

The paper presents a nonlinear control law based on partial feedback linearization for an aeroelastic system equipped with a trailing edge control flap in the presence of polynomial type of stiffness nonlinearity. A modal transformation is utilized to linearly decouple the system, followed by a nonlinear transformation to make the transformed system partially feedback linearizable. A nonlinear control law is then derived to suppress limit cycle oscillations. Emphasis is laid on use of modal coordinates which enable us to avoid cancellation of any linear terms when using feedback linearization. It is demonstrated that the derived nonlinear control law behaves exactly like a linear optimal controller for small disturbances and can guarantee stability for large disturbances. Simulations results comparing the state and control costs for a set of randomly generated initial conditions using the proposed controller
and a traditional feedback linearization controller in physical coordinates are presented highlighting the advantages of using modal coordinates in the control design phase.

### 3.2 Introduction

Aeroelasticity is the interaction of structural, inertial and aerodynamic forces [1]. Combined effects of these loads may often cause the aeroelastic system to go unstable and exhibit a range of phenomena such as flutter, divergence, control reversal, limit cycle oscillations (LCOs) and even chaotic vibrations. Active suppression of such undesired phenomena, often called aeroservoelasticity has been a topic of interest among researchers over the past few decades. Many control methodologies have been applied to delay the onset of flutter, suppress limit cycle oscillations, mitigate the effects of gust and extend the flight envelope of an aircraft. Mukhopadhyay [2] presents a historical review of the various aspects of the nonlinear aeroelastic control problem and its development over time.

There has been considerable work done on the aeroservoelastic problem, given below are few efforts relevant to the present work. Lyons et al. [3] developed a full state feedback controller using a Kalman filter to get an estimate of the unknown states for the purpose of flutter suppression. The paper uses approximate models to represent unsteady aerodynamics in a two dimensional airfoil section model with trailing edge flap and a three dimensional airfoil section with both trailing and leading edge flaps. Feedback controllers are selected based on quadratic error minimization. Mukhopadhyay et al. [4] laid out a method to synthesize low order optimal feedback control for a higher order system. A nonlinear programming algorithm is used to search for optimal control design variables to minimize a performance index defined by weighted sum of squared steady state response and control inputs. Comparing the unsteady aerodynamic approximations by Lyons et al. [3], Vepa [5] and Edwards [6], Karpel [7] developed a more accurate reduced order model resulting in simpler partial
state feedback controllers for the purpose of flutter suppression and gust load alleviation. Horikawa & Dowell [8] did flutter analysis along with a proportional gain feedback control obtained using root locus plots.

The above works highlight applications of linear control theory to the aeroelastic control problem. However, most aeroelastic systems inherently include a number of nonlinearities which are often discarded while developing linear feedback controllers. Common sources of nonlinearities occurring in aeroelastic systems are actuator saturation which occurs when the actuator output is limited, freeplay nonlinearities which are associated with control surface linkages and hysteresis which is encountered when friction affect linkage dynamics. Apart from these control surface nonlinearities, structural and aerodynamic nonlinearities are common. Dowell [9] sheds light on the importance of considering nonlinearities in aeroelastic analysis. Woolston et al. [10] looks into the effects of nonlinear structural terms on flutter of a wing showing that these may play a dominant role in causing the onset of flutter. Freeplay, hysteresis and a cubic hardening and softening nonlinearity are considered to demonstrate that the stability of an aeroelastic system is highly dependent on the magnitude of initial displacements of the system from equilibrium. Tang and Dowell [11] considered a freeplay nonlinearity and concluded that limit cycle motion is dependent of free stream velocity, initial conditions and magnitude of the nonlinearity. O'Neil et al. [12] investigated the characteristics of aeroelastic response of a wing section with structural nonlinearity. The paper accurately predicts nonlinear response using an analytical model.

The first work focused on experimental active control of nonlinear aeroelastic system was presented by Block and Gilliatt [13] using a three degree of freedom model with unsteady aerodynamics approximated using Jones approximation of the Theodorsen's function. The work was later extended by Block and Strganac [14] to derive an LQG controller which was shown to suppress LCOs if initiated before the onset of LCOs. It was observed that the controller fails if initiated after the system has settled into LCOs. Ko et al. used a theoretical aeroelastic section model with single and double trailing edge control surfaces and applied feedback linearization [15] and adaptive feedback linearization [16, 17] to the
aeroelastic system. Adaptive control methods seem useful because of the uncertainties in modeling the coefficients of the structural stiffness function and the system parameters. Strganac et al. [18] verified the realizability of the controller developed by Ko et al. [16] by conducting experiments demonstrating effective suppression of limit cycle oscillations. Xing and Singh [19] developed a nonlinear adaptive backstepping controller assuming unknown system parameters for a two degree of freedom model with quasisteady aerodynamics. Their work was extended by Platanitis and Strganac [20] by incorporating a leading edge flap. An output feedback controller has been developed for suppressing flutter and reducing vibrations in subcritical flight range by Behal et al. [21]. Lee and Singh [22] developed a robust control law for the regulation of two degree of freedom aeroelastic system considering polynomial type nonlinearity and assuming only pitch feedback and unknown parameters. Wang et al. [23] proposed a modular output feedback controller to suppress aeroelastic vibrations subject to external disturbances. Wang et al. [24] gives a good review of the various approaches used for the aeroelastic control problem with structural nonlinearities.

In the present work a linear modal transformation matrix is used to linearly decouple the aeroelastic system in modal coordinates, followed by a nonlinear transformation to render the modal system partially feedback linearizable. The resulting dynamics allow derivation of a nonlinear control law to cancel out only the strictly nonlinear terms. Applying feedback linearization in modal coordinates guarantees that our derived feedback linearization control law is completely nonlinear and we do not end up canceling any linear component while attempting to control the aeroelastic system. The resultant nonlinear internal dynamics are shown to be asymptotically stable. Simulations are used to generate weighted control and state costs for a set of randomly generated initial conditions to compare the performance of the developed controller to a feedback linearization controllers in physical coordinates.
3.3 Equations of motion

A typical section airfoil is the simplest and most commonly used aeroelastic model. It consists of a two dimensional rigid plate connected via a torsional spring and a translational spring mounted parallel to a uniform freestream flow. We consider a variant of the typical section airfoil which includes a trailing edge flap with deflection, $\beta$ as shown in Fig. 3.1.

![Figure 3.1: Representation of two dimensional airfoil section model equipped with a trailing edge flap.](image)

The governing equations of motion can be derived using Lagrange's equations and are given below

\[
\begin{bmatrix}
 m & mx_a b \\
 mx_a b & I_a
\end{bmatrix}
\begin{bmatrix}
 \ddot{h} \\
 \dot{\alpha}
\end{bmatrix}
+ \begin{bmatrix}
 c_h & 0 \\
 0 & c_\alpha
\end{bmatrix}
\begin{bmatrix}
 \dot{h} \\
 \dot{\alpha}
\end{bmatrix}
+ \begin{bmatrix}
 K_h & 0 \\
 0 & K_\alpha(\alpha)
\end{bmatrix}
\begin{bmatrix}
 h \\
 \alpha
\end{bmatrix}
= \begin{bmatrix}
 -L \\
 M
\end{bmatrix}
\]  

(3.1)

where $h$ and $\alpha$ represent the plunge displacement and the pitch angle respectively, $m$ represents the total mass, $b$ is the semi chord, $I_a$ is the moment of inertia, $x_\alpha$ is the non-dimensional distance of the center of mass from the elastic axis, $c_h$ and $c_\alpha$ are the plunge
and pitch damping coefficients. $K_h$ is the constant translational stiffness in plunge while $K(\alpha)$ is the polynomial type stiffness nonlinearity in pitch and is stated below

$$K_\alpha(\alpha) = K_{\alpha 0} + K_{\alpha 1}\alpha + K_{\alpha 2}\alpha^2 + K_{\alpha 3}\alpha^3 + K_{\alpha 4}\alpha^4$$ (3.2)

$L$ and $M$ represent the aerodynamic lift force and moments acting on the elastic axis respectively. A quasi-steady aerodynamic model stated in Eq. 3.3 is used to represent the aerodynamic forces and moment.

$$L = \rho U^2 bs C_{L\alpha} \left[ \frac{h}{U} + \left( \frac{1}{2} - a \right) b \frac{\dot{\alpha}}{U} \right] + \rho U^2 bs C_{L\beta}$$

$$M = \rho U^2 bs C_{M\alpha} \left[ \alpha + \frac{\dot{h}}{U} + \left( \frac{1}{2} - a \right) b \frac{\dot{\alpha}}{U} \right] + \rho U^2 bs C_{M\beta}$$ (3.3)

where $a$ is the non-dimensional distance of the elastic axis from the mid-chord, $s$ is the span of the airfoil and $U$ is the freestream velocity. $C_{L\alpha}$ and $C_{L\beta}$ are the lift coefficient per unit angle of attack and flap deflection. Similarly, $C_{M\alpha}$ and $C_{M\beta}$ are the moment coefficients per unit angle of attack and flap deflection.

Choosing $q = \begin{bmatrix} h & \alpha \end{bmatrix}^T$ the aeroelastic system using the quasi steady aerodynamics can be represented as a system of two second order differential equations

$$M_0\ddot{q} + C\dot{q} + K(q)q = F_q(u, \dot{u})$$

$$F_q(u, \dot{u}) = F_q + F_{\dot{q}}q + F_c\beta$$ (3.4)

\begin{align*}
M_0 &= \begin{bmatrix} m & mx_ab \\ mx_ab & I_a \end{bmatrix}, C = \begin{bmatrix} c_h & 0 \\ 0 & c_\alpha \end{bmatrix}, K(q) = \begin{bmatrix} k_h & 0 \\ 0 & K_\alpha(q_2) \end{bmatrix}, F_c = \begin{bmatrix} -\rho U^2 bs C_{L\beta} \\ \rho U^2 b^2 s C_{M\beta} \end{bmatrix} \\
F_q &= \begin{bmatrix} 0 & -\rho U^2 bs C_{L\alpha} \\ -\rho U^2 b^2 s C_{L\beta} \end{bmatrix}, F_{\dot{q}} = \begin{bmatrix} -\rho Ubs C_{L\alpha} & -\rho U b^2 s C_{L\alpha} (0.5 - a) \\ -\rho U b^2 s C_{M\alpha} & -\rho U b^3 s C_{M\alpha} (0.5 - a) \end{bmatrix}
\end{align*}
where \( q^2 = \alpha \). The terms \( M_0, C, K(q) \) represent mass, damping and stiffness matrices respectively. The forcing term, \( F_{qs}(q, \dot{q}) \) comes from the quasi-steady aerodynamic and the control terms. Collecting the displacement and velocity terms from both sides, the aeroelastic system can be represented as

\[
M_0 \ddot{q} + (C - F_q) \dot{q} + (K(q) - F_q) q = F_c \beta
\]  

A state vector composed of displacements and velocities of \( h \) and \( \alpha \) can be defined as

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} h \\ \alpha \\ \dot{h} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}
\]  

Using the state vector defined in Eq. 3.6 the set of 2\(^{nd}\) order differential equations given in Eq. 3.5 can be transformed into a set of four first order differential equations given below

\[
\begin{bmatrix} I_n & 0_n \\ 0_n & M_0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0_n & I_n \\ - (K(q) - F_q) - (C - F_q) \end{bmatrix} x + \begin{bmatrix} 0_{n \times 1} \\ F_c \end{bmatrix} \beta
\]  

where \( n = 2 \). \( I_n \) and \( 0_n \) represent identity and zero matrices of order \( n \times n \). Separating out the linear and nonlinear terms, the above equation is represented in the following form

\[
B \dot{x} = A_0 x + A_{nl}(x) + F \beta
\]  

where \( A_0 \in R^{n \times n} \) represents the linear state matrix, \( A_{nl}(x) \) is a \( 4 \times 1 \) state dependent vector containing strictly nonlinear terms, \( F \) represents the control matrix for the aeroelastic system represented as a first order system of equations

\[
B = \begin{bmatrix} I_n & 0_n \\ 0_n & M_0 \end{bmatrix}, A_0 = \begin{bmatrix} 0_n & I_n \\ - (K(0) - F_q) - (C - F_q) \end{bmatrix},
\]
Considering the eigenvalue problem based on the homogeneous linear part of Eq. 3.8, the linear flutter velocity can be determined for the aeroelastic system. It is seen that matrix $A_0$ depends on parameters, $a$ and $U$. For a constant value of $a$, varying $U$, root locus plots are constructed to identify the linear flutter velocity, $U_F$.

Fig. 3.2 shows the root locus plot for the linearized aeroelastic system as velocity, $U$ is varied for a fixed value of the elastic axis location parameter, $a = -0.5$ and the system parameters given in Table 3.1. It is seen that at low values of $U$ all the eigenvalues lie on the left half plane (stable system), however as the velocity is increased a pair of complex conjugate eigenvalues cross over to the right half plane making the system exhibit an unstable behavior. The velocity at which a pair of complex conjugate eigenvalues have their real part equal to zero corresponds to the linear flutter velocity, $U_F$. For the system parameters given in Table 3.1, $U_F$ comes out to be 15.85 m/s. This can be seen clearly in Fig. 3.3 (a). Fig. 3.3(b) shows the locus of the imaginary part of the eigenvalues as $U$ is varied.

Figure 3.2: Root locus plot for the linearized aeroelastic system as $U$ is varied for $a = -0.5$; the eigenvalues corresponding to $U = U_F$ are marked by $\times$. 

\[
A_{nl}(x) = \begin{bmatrix}
0_{n \times 1} \\
-(K(\alpha) - K(0)) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{bmatrix}, 
F = \begin{bmatrix} 0_{n \times 1} \\ F_c \end{bmatrix}
\]
Table 3.1: Parameters considered for the nonlinear aeroelastic system representing the design parameters of Nonlinear Aeroelastic Testing Apparatus (NATA) at Texas A & M University.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$-0.5$</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>$0.135$</td>
<td>m</td>
</tr>
<tr>
<td>$x_{\alpha}$</td>
<td>$(0.0873 - (b + ab))/b$</td>
<td></td>
</tr>
<tr>
<td>$s$</td>
<td>$0.6$</td>
<td>m</td>
</tr>
<tr>
<td>$m$</td>
<td>$12.287$</td>
<td>kg</td>
</tr>
<tr>
<td>$I_{\alpha}$</td>
<td>$0.065$</td>
<td>kgm$^2$</td>
</tr>
<tr>
<td>$k_h$</td>
<td>$2844$</td>
<td>Nm</td>
</tr>
<tr>
<td>$c_h$</td>
<td>$27.43$</td>
<td>Ns/m</td>
</tr>
<tr>
<td>$c_{\alpha}$</td>
<td>$0.036$</td>
<td>Ns</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$1.225$</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>$C_{L\alpha}$</td>
<td>$6.28$</td>
<td></td>
</tr>
<tr>
<td>$C_{L\beta}$</td>
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<td></td>
</tr>
<tr>
<td>$C_{M\alpha}$</td>
<td>$(0.5 + a)C_{L\alpha}$</td>
<td></td>
</tr>
<tr>
<td>$C_{M\beta}$</td>
<td>$-0.635$</td>
<td></td>
</tr>
<tr>
<td>$K_{\alpha 0}$</td>
<td>$2.82$</td>
<td>Nm/rad</td>
</tr>
<tr>
<td>$K_{\alpha 1}$</td>
<td>$-62.232$</td>
<td>Nm/rad$^2$</td>
</tr>
<tr>
<td>$K_{\alpha 2}$</td>
<td>$3709.71$</td>
<td>Nm/rad$^3$</td>
</tr>
<tr>
<td>$K_{\alpha 3}$</td>
<td>$23970$</td>
<td>Nm/rad$^4$</td>
</tr>
<tr>
<td>$K_{\alpha 4}$</td>
<td>$48756.954$</td>
<td>Nm/rad$^5$</td>
</tr>
</tbody>
</table>
3.4 Control synthesis formulation

We use the method of feedback linearization to derive nonlinear control laws to make the aeroelastic system asymptotically stable. Ko et al. [15, 25, 26] used feedback linearization for control of the pitch angle and plunge displacement. Ko et al. [27] extend their work to make it adaptive and robust to errors arising due to poor approximation of the nonlinear stiffness functions. In the current work, a feedback linearization based approach in modal coordinates for the aeroelastic control problem is introduced. A modal transformation, similar to the one described by Meirovitch [28] and Patil [29], is employed to obtain a system of equations which give a relative degree of two with the flutter mode as the chosen output function. Feedback linearization is then applied on this modal system to derive nonlinear control laws.
3.4.1 Nonlinear aeroelastic system

A nonlinear control input resulting from the summation of both linear and nonlinear state dependent terms is assumed as below

\[ \beta = \beta_l + \beta_{nl} \]  \hspace{1cm} (3.9)

The linear part of the control input, \( \beta_l \) is realized using linear control theory. In the current work, linear quadratic regulator (LQR) control strategy is utilized for the linear controller to obtain, \( \beta_l = -K_{LQR}x \). The linear gain \( K_{LQR} \) can be easily obtained by solving the algebraic Riccati equation. The nonlinear aeroelastic system given in Eq. 3.8 using the control input defined in Eq. 3.9 can be represented in a form as shown

\[ B\dot{x} = Ax + A_{nl}(x) + F\beta_{nl} \]  \hspace{1cm} (3.10)

where \( A = A_0 - FK_{LQR} \).

3.4.2 Modal transformation to obtain linearly decoupled second order system

To identify a transformation to generate a linearly decoupled second order system, the eigenvalue problem and its adjoint problem are considered based on the linear terms of Eq. 3.10.

\[ \lambda_j B u_j = A v_j \]
\[ \lambda_j B^T v_j = A^T v_j \hspace{1cm} \text{for} \hspace{0.5cm} j = 1, 2, ..., 2n \]  \hspace{1cm} (3.11)

where \( \lambda_j, u_j \) and \( v_j \) represent the eigenvalues, the right eigenvector and the left eigenvector of the \((A, B)\) system respectively. The left and right eigenvectors being biorthonormal can
be normalized to satisfy the following conditions.

\[ v_j^T B u_k = 2\delta_{jk} \]
\[ v_j^T A u_k = 2\lambda_j \delta_{jk} \]  

(3.12)

For real systems, \( \lambda_j \) can be real or exist as pair of complex conjugate numbers. Considering the eigenvalues to be two complex conjugate pairs, the eigenvalues and eigenvectors are written in terms of their real and imaginary parts

\[ \lambda_j = \alpha_j + i\beta_j \quad \lambda_{j+2} = \bar{\lambda}_j = \alpha_j - i\beta_j \]
\[ u_j = y_j + i\bar{z}_j \quad u_{j+2} = \bar{u}_j = y_j - i\bar{z}_j \quad \text{for } j = 1, 2 \]
\[ v_j = r_j + is_j \quad v_{j+2} = \bar{v}_j = r_j - is_j \]

(3.13)

Patil [29] and Meirovitch [28] talk about complex eigenvalue problems and using real basis of eigenvectors to obtain linearly decoupled second order differential equations. To decouple the modes in the absence of any forcing terms, Patil [29] suggests using a basis of vectors comprising of real quantities as below

\[
[U] = \begin{bmatrix}
y_1 - (\alpha_1/\beta_1)z_1 & (1/\beta_1)\bar{z}_1 \\
r_1 & \alpha_1 r_1 - \beta_1 s_1 \end{bmatrix}
\]
\[
[V] = \begin{bmatrix}
y_2 - (\alpha_2/\beta_2)z_2 & (1/\beta_2)\bar{z}_2 \\
r_2 & \alpha_2 r_2 - \beta_2 s_2 \end{bmatrix}
\]

(3.14)

The biorthogonality conditions for this new basis become

\[ [V]^T [B][U] = I_4 \]
\[ [V]^T [A][U] = A^* \]

(3.15)

where \( A^* \) is a block diagonal matrix with two blocks of the following form

\[
[A_j^*] = \begin{bmatrix}
0 & 1 \\
-(\alpha_j^2 + \beta_j^2) & 2\alpha_j
\end{bmatrix}
\]

(3.16)

A modal expression in terms of \([U]\) is defined as below

\[ x = [U]\{w\} = [U]\left\{\begin{array}{c}
\xi_1 \\
\eta_1 \\
\xi_2 \\
\eta_2
\end{array}\right\}^T \]

(3.17)
The defined modal expression is substituted in Eq. 3.10 to obtain

\[ B[U] \dot{w} = A[U] w + A_{nl} ([U] w) + F \beta_{nl} \]  \hspace{1cm} (3.18)

Premultiplying the above relation by \([V]^T\) and using the biorthogonality conditions given in Eq. 3.12, the following modal dynamics are obtained

\[ \dot{w} = A^* w + [V]^T A_{nl} ([U] w) + [V]^T F \beta_{nl} \]  \hspace{1cm} (3.19)

The obtained modal dynamics are linearly decoupled. However, nonlinear coupling terms are still present due to the presence of nonlinear terms in the original system. The modal dynamics given in Eq. 3.19 can be represented as two sets of two linearly decoupled first order equations

\[
\begin{bmatrix}
\dot{\xi}_j \\
\dot{\eta}_j
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-(\alpha_j^2 + \beta_j^2) & 2\alpha_j
\end{bmatrix}
\begin{bmatrix}
\xi_j \\
\eta_j
\end{bmatrix} +
\begin{bmatrix}
F^\xi_j \\
F^\eta_j
\end{bmatrix} \beta_{nl} +
\begin{bmatrix}
X^\xi_j \\
X^\eta_j
\end{bmatrix}
\]  \hspace{1cm} (3.20)

where

\[
F^\xi_j = r_j^T F \\
F^\eta_j = (\alpha_j r_j - \beta_j s_j)^T F \\
X^\xi_j (w) = r_j^T A_{nl} ([U] w) \\
X^\eta_j (w) = (\alpha_j r_j - \beta_j s_j)^T A_{nl} ([U] w)
\]  \hspace{1cm} (3.21)

The terms represented by \(F_j^{(\xi)}\) and \(X_j^{(\xi)}\) come from the control and nonlinear part of Eq. 3.10.

The above system has a relative degree of one. However, we want to modify the modal transformation matrix so as to obtain a system with relative degree of two. Patil [29] shows how we can transform the system to obtain uncoupled second order equations in the presence of a single forcing vector. This is done by rotating the eigenvectors by a certain complex number to obtain a resulting system such that \(F_j^{\xi} = 0\) for \(j = 1, 2\). Let the original and
The modified left eigenvector matrix be as follows

\[
[V] = \begin{bmatrix} v_1 & \bar{v}_1 & v_2 & \bar{v}_2 \end{bmatrix} \\
[\hat{V}] = \begin{bmatrix} c_1 v_1 & \bar{c}_1 \bar{v}_1 & c_2 v_2 & \bar{c}_2 \bar{v}_2 \end{bmatrix}
\]

(3.22)

In above, \([V]\) and \([\hat{V}]\) represent the original and the modified left eigenvector matrices respectively. \(v_1\) and \(v_2\) represent the 1\(^{st}\) and 3\(^{rd}\) columns of left eigenvector matrix, \(\bar{v}_1\) and \(\bar{v}_2\) represent their conjugates. \(c_1 = c_1^R + i c_1^I, c_2 = c_2^R + i c_2^I \in \mathbb{C}\) are the complex numbers used to rotate the eigenvector matrix. Using the modified left eigenvector matrix, \([\hat{V}]\) the modified right eigenvector matrix can be accordingly computed such that the biorthogonality conditions are satisfied. The new basis of vectors for the modal transformation is now given as

\[
[\hat{U}] = \begin{bmatrix} \hat{y}_1 - (\alpha_1/\beta_1) \hat{z}_1 & (1/\beta_1) \hat{z}_1 & \hat{y}_2 - (\alpha_2/\beta_2) \hat{z}_2 & (1/\beta_2) \hat{z}_2 \end{bmatrix} \\
[\hat{V}] = \begin{bmatrix} \hat{r}_1 & \alpha_1 \hat{r}_1 - \beta_1 \hat{s}_1 & \hat{r}_2 & \alpha_2 \hat{r}_2 - \beta_2 \hat{s}_2 \end{bmatrix}
\]

(3.23)

For \(i = 1, 2\), the vectors \(\hat{y}_i, \hat{z}_i, \hat{r}_i\) and \(\hat{s}_i\) correspond to \(y_i, z_i, r_i\) and \(s_i\) for the modified eigenvector matrices given in Eq. 3.13. The new forcing terms can be expressed as

\[
\hat{F}_j^\xi = \hat{r}_j^T F \\
\hat{F}_j^\eta = (\alpha_j \hat{r}_j - \beta_j \hat{s}_j)^T F \\
\hat{X}_j^\xi (w) = \hat{r}_j^T A_{nl} \left( [\hat{U}] w \right) \\
\hat{X}_j^\eta (w) = (\alpha_j \hat{r}_j - \beta_j \hat{s}_j)^T A_{nl} \left( [\hat{U}] w \right)
\]

(3.24)

The value of \(c_1\) and \(c_2\) can be calculated by setting \(\hat{F}_j^\xi\) to zero. \(\hat{F}_j^\xi\) can be expanded as

\[
\hat{F}_j^\xi = \hat{r}_j^T F = \sum_{k=1}^{4} \left( c_j^R \hat{r}_{jk} - c_j^I \hat{s}_{jk} \right) F_k \\
= c_j^R \sum_{k=1}^{4} \hat{r}_{jk} F_k - c_j^I \sum_{k=1}^{4} \hat{s}_{jk} F_k \quad \text{for} \quad j = 1, 2
\]

(3.25)
Defining $c_1$ and $c_2$ as below

$$c_j = \frac{(b_j + ia_j)}{\sqrt{a_j^2 + b_j^2}} \quad \text{for} \quad j = 1, 2$$  \hspace{1cm} (3.26)

results in $\hat{F}_1^\xi = 0$ and $\hat{F}_2^\xi = 0$. The resulting system in modal coordinates is

$$\dot{w} = A^* w + \begin{bmatrix} X_1^\xi (w) \\ X_1^\eta (w) \\ X_2^\xi (w) \\ X_2^\eta (w) \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{F}_1^\eta \\ 0 \\ \hat{F}_2^\eta \end{bmatrix} \beta_{nl}$$  \hspace{1cm} (3.27)

The above system has a relative degree of two with respect to $w_1$ or $w_3$.

### 3.4.3 Feedback Linearization

Having transformed the nonlinear aeroelastic system to modal coordinates given in Eq. 3.27, the method of partial feedback linearization is utilized to generate the nonlinear control input. Feedback linearization has been used in previous works by Ko et al. [15, 25] in physical coordinates to control the nonlinear aeroelastic system, but the approach involves cancellation of some linear terms as well. The motive to apply feedback linearization in modal coordinates is to prevent cancellation of any linear terms, thereby preventing the disruption of any linear dynamics in the resulting closed loop system. Equation 3.27 can be represented as below in its expanded form

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \\ \dot{w}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(\alpha_1^2 + \beta_1^2) & 2\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(\alpha_2^2 + \beta_2^2) & 2\alpha_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} + \begin{bmatrix} 0 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} \beta_{nl} - \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} Q[f(w)]$$  \hspace{1cm} (3.28)
where

\[
Q(x) = (K_{1a}x + K_{2a}x^2 + K_{3a}x^3 + K_{4a}x^4)x
\]

\[
f(w) = \sum_{i=1}^{4} \hat{U}_{2i}w_i
\]

\\[p_i = \dot{V}_{4i} \text{ for } i = 1, ..., 4\]

(3.29)

It can be shown with ease that relative degree of above system in modal coordinates is two with respect to \(w_1\) and \(w_3\), implying that two degrees of freedom of the system can be linearized. Let \(w_1\) and \(w_2\) represent the two states of the linearly stabilized flutter mode which is likely to go unstable due to nonlinear coupling. Introducing the transformation, \(\phi = T(w)\), and choosing \(\phi_1\) and \(\phi_2\) as below

\[
\phi_1 = h(w) = w_1
\]

\[
\phi_2 = L_f [h(w)] = w_2 - p_1Q [f(w)]
\]

(3.30)

\(\phi_3\) and \(\phi_4\) are chosen such that \([\partial \phi_i / \partial w] g = 0\) for \(i = 1, 3\). One such choice for \(\phi_3\) and \(\phi_4\) is stated below

\[
\phi_3 = w_3
\]

\[
\phi_4 = g_2w_4 - g_4w_2
\]

(3.31)

Representing \(f(w)\) defined earlier in Eq. 3.29 in terms of \(\phi\) by applying the transformation, we have

\[
f(w) = \hat{U}_{21}\phi_1 + \hat{U}_{22}\{\phi_2 + p_1Q[f(w)]\} + \hat{U}_{22}\phi_3 + \frac{\hat{U}_{24}}{g_2}\{g_4[\phi_2 + p_1Q[f(w)]] + \phi_4\}
\]

\[
= \hat{U}_{21}\phi_1 + \left(\hat{U}_{22} + \hat{U}_{24}g_4/g_2\right)\{\phi_2 + p_1Q[f(w)]\} + \hat{U}_{23}\phi_3 + \frac{\hat{U}_{24}}{g_2}\phi_4
\]

\[
= \hat{U}_{21}\phi_1 + \hat{U}_{23}\phi_3 + \frac{\hat{U}_{24}}{g_2}\phi_4
\]

\[
= d(\phi)
\]

(3.32)
It is to be noted that, \( \left( \hat{U}_{22} + \hat{U}_{24}g_4/g_2 \right) = 0 \), which enables us to cancel out the \( Q[f(w)] \) term in the right hand side of Eq. 3.32. The inverse transformation, \( w = T^{-1}(\phi) \), is defined as below

\[
\begin{align*}
    w_1 &= \phi_1 \\
    w_2 &= \phi_2 + p_1 Q[d(\phi)] \\
    w_3 &= \phi_3 \\
    w_4 &= \frac{1}{g_2} \{ \phi_4 + g_4\phi_2 + g_4p_1 Q[f(w)] \}
\end{align*}
\] (3.33)

Admissibility of the defined transformation and its inverse can be checked by verifying the non singularity of the Jacobian matrix for the transformation and the inverse transformation. Both the transformation and the inverse transformation have non-zero determinants, proving them to be admissible. The modal equations given in Eq. 3.28 are transformed using the forward transformation resulting in the following \( \phi \) dynamics

\[
\begin{bmatrix}
    \dot{\phi}_1 \\
    \dot{\phi}_2 \\
    \dot{\phi}_3 \\
    \dot{\phi}_4
\end{bmatrix} = \begin{bmatrix}
    \phi_2 \\
    k_1\phi_1 + k_2\phi_2 + q_1(\phi) \beta_{nl} + q_2(\phi) \\
    (g_4/g_2) \phi_2 + (1/g_2) \phi_4 + (g_4p_1/g_2 - p_3) Q[d(\phi)] \\
    -(g_4k_1) \phi_1 - (g_4k_2 - g_4k_4) \phi_2 + (g_2k_3) \phi_3 + k_4\phi_4 + k_5Q[d(\phi)]
\end{bmatrix}
\] (3.34)

where \( q_1(\phi) \) and \( q_2(\phi) \) contain purely nonlinear terms and are expressed as below

\[
\begin{align*}
    q_1(\phi) &= g_2 - p_1a_6Q'[d(\phi)] \\
    q_2(\phi) &= (k_2p_1 - p_2) Q[d(\phi)] - p_1Q'[d(\phi)] \left\{ \sum_{j=1}^{4} a_j\phi_j - a_5Q[d(\phi)] \right\}
\end{align*}
\] (3.35)
The introduced variables $k_j$, \((j = 1, \ldots, 4)\) are constants defined as

\[
\begin{align*}
k_1 &= -(\alpha_1^2 + \beta_1^2) \\
k_2 &= 2\alpha_1 \\
k_3 &= -(\alpha_2^2 + \beta_2^2) \\
k_4 &= 2\alpha_2 \\
k_5 &= (g_4p_2 - g_2p_4 + k_4p_1 - g_4k_2p_1)
\end{align*}
\]

and the $a_j$, \((j = 1, \ldots, 6)\) are defined as

\[
\begin{align*}
a_1 &= \hat{U}_{22}k_1 \\
a_2 &= \hat{U}_{22}k_2 + \hat{U}_{21} + \left(\hat{U}_{24}k_4 + \hat{U}_{23}\right)g_4/g_2 \\
a_3 &= \hat{U}_{24}k_3 \\
a_4 &= \left(\hat{U}_{24}k_4 + \hat{U}_{23}\right)/g_2 \\
a_5 &= \sum_{j=1}^4 \hat{U}_{2j}p_j - \left(\hat{U}_{22}k_2 + \hat{U}_{21}\right)p_1 - \left(\hat{U}_{24}k_4 + \hat{U}_{23}\right)p_1g_4/g_2 \\
a_6 &= \hat{U}_{22}g_2 + \hat{U}_{24}g_4
\end{align*}
\]

Looking at the $\phi$ dynamics, it is seen that the \((\phi_1, \phi_2)\) subsystem dynamics can be linearized by choosing $\beta_{nl}$ as

\[
\beta_{nl} = \frac{\nu - g_2(\phi)}{q_1(\phi)}
\]  
(3.36)

where $\nu$ is an auxiliary control input which can be defined to change the linearized \((\phi_1, \phi_2)\) dynamics. Substituting the above defined $\beta_{nl}$ in Eq. 3.34 we obtain,

\[
\begin{bmatrix}
\dot{\phi}_1 \\
\dot{\phi}_2 \\
\dot{\phi}_3 \\
\dot{\phi}_4
\end{bmatrix} =
\begin{bmatrix}
\phi_2 \\
k_1\phi_1 + k_2\phi_2 + \nu \\
(g_4/g_2)\phi_2 + (1/g_2)\phi_4 + (g_4p_1/g_2 - p_3)Q[d(\phi)] \\
-(g_4k_1)\phi_1 - (g_4k_2 - g_4k_4)\phi_2 + (g_2k_3)\phi_3 + k_4\phi_4 + k_5Q[d(\phi)]
\end{bmatrix}
\]  
(3.37)
It is seen that the \((\phi_1, \phi_2)\) dynamics are linear and acceptable even if \(\nu = 0\). This is because a linear controller was used for the aeroelastic system before introducing the modal transformation. In addition, the \((\phi_3, \phi_4)\) subsystem dynamics are not affected by \(\beta_{nl}\) and remain nonlinear. To guarantee stability of the overall \(\phi\) system, it is mandatory to show that the zero dynamics are stable. The zero dynamics are obtained by substituting \(\phi_1, \phi_2 \to 0\) in the \((\phi_3, \phi_4)\) dynamics and are stated below

\[
\begin{bmatrix}
\dot{\phi}_3 \\
\dot{\phi}_4 
\end{bmatrix} = \begin{bmatrix}
\frac{1}{g_2} \phi_4 + \left(g_4p_1/g_2 - p_3\right) Q [d_0 (\phi_3, \phi_4)] \\
\left(g_2k_3\right) \phi_3 + k_4\phi_4 + k_5Q [d_0 (\phi_3, \phi_4)]
\end{bmatrix}
\tag{3.38}
\]

where \(d_0 (\phi_3, \phi_4) = d (0, 0, \phi_3, \phi_4)\). It can be easily shown that the zero dynamics are linearly stable. However, to prove their stability in general, Lyapunov function is employed as shown subsequently.

### 3.4.4 Stability of the zero dynamics

Representing the zero dynamics stated in Eq. 3.38 as below

\[
\begin{align*}
\dot{\phi}_3 &= m_1\phi_4 + m_2Q [d_0 (\phi_3, \phi_4)] \\
\dot{\phi}_4 &= m_3\phi_3 + m_4\phi_4 + m_5Q [d_0 (\phi_3, \phi_4)]
\end{align*}
\tag{3.39}
\]

where \(m_1 = 1/g_2, m_2 = g_4p_1/g_2 - p_3, m_3 = g_2k_3, m_4 = k_4\) and \(m_5 = k_5\). For a range of LQR controllers for the given aeroelastic system the coefficients of the obtained zero dynamics have the following signs

\[
m_1 > 0, \quad m_2 > 0, \quad m_3 < 0, \quad m_4 < 0, \quad m_5 > 0
\tag{3.40}
\]

To prove the stability of zero dynamics, it is sufficient to show that a Lyapunov function \((V)\) exists, such that
\( V(\phi_3, \phi_4) \geq 0 \quad \forall (\phi_3, \phi_4) \in R^2 \)

\( V(0, 0) = 0 \)

\( \dot{V} < 0 \quad \forall (\phi_3, \phi_4) \in R^2 - (0, 0) \)

Assuming a Lyapunov function of the form below

\[
V(\phi_3, \phi_4) = \frac{1}{2} \left[ s_1 \phi_3^2 + s_2 \phi_4^2 \right] \quad (3.41)
\]

where \( s_1, s_2 \in R^+ \) and will be chosen subsequently. For the assumed \( V \), \( \dot{V} \) can be represented in the following form

\[
\dot{V} = s_1 \phi_3 \dot{\phi}_3 + s_2 \phi_4 \dot{\phi}_4 \\
= s_1 \phi_3 (m_1 \phi_4 + m_2 Q[d_0(\phi_3, \phi_4)]) + s_2 \phi_4 (m_3 \phi_3 + m_4 \phi_4 + m_5 Q[d_0(\phi_3, \phi_4)]) \\
= (s_1 m_1 + s_2 m_3) \phi_3 \phi_4 + (s_1 m_2 \phi_3 + s_2 m_5 \phi_4) Q[d_0(\phi_3, \phi_4)] + (s_2 m_4) \phi_4^2 
\]

(3.42)

Since, \( m_1 > 0 \) and \( m_3 < 0 \), to make the coefficient of \( \phi_3 \phi_4 \) term in \( \dot{V} \) vanish, we choose \( s_1 = -m_3 > 0 \) and \( s_2 = m_1 > 0 \) resulting in

\[
\dot{V} = (-m_2 m_3 \phi_3 + m_1 m_5 \phi_4) Q[d_0(\phi_3, \phi_4)] + (m_1 m_4) \phi_4^2 
\]

(3.43)

The last term in above expression of \( \dot{V} \) is always negative since \( m_1 > 0 \) and \( m_4 < 0 \). To determine the sign of the first term in above expression some analysis is needed. Figure 3.4 shows the variation of \( Q[d_0(\phi_3, \phi_4)] \) against \( d_0(\phi_3, \phi_4) \). It is seen from the figure that \( d_0(\phi_3, \phi_4) Q[d_0(\phi_3, \phi_4)] > 0 \quad \forall \phi_3, \phi_4 \in R \). The function \( d_0(\phi_3, \phi_4) \) can be defined using \([\hat{U}]\) as below

\[
d_0(\phi_3, \phi_4) = \hat{U}_{23} \phi_3 + \frac{\hat{U}_{24}}{g_2} \phi_4 
\]

(3.44)
Considering the first term in the expression of $\dot{V}$ given in Eq. 3.43 and writing it in terms of $d_0(\phi_3, \phi_4)$ and noting that $|\hat{U}_{23}\phi_3| >> |\hat{U}_{24}\phi_4/g_2|

\dot{V}_1 = (-m_2m_3\phi_3 + m_1m_5\phi_4) Q[d_0(\phi_3, \phi_4)] \\
\approx \left(-m_2m_3/\hat{U}_{23}\right) d_0(\phi_3, \phi_4) Q[d_0(\phi_3, \phi_4)]

Using the knowledge of signs given in Eq. 3.40 and knowing $\hat{U}_{23} < 0$, it can be shown that $\dot{V}_1 < 0$. This guarantees that the derivative of the Lyapunov function, $\dot{V}$ is always negative semi definite, i.e, $\dot{V} \leq 0 \forall (\phi_3, \phi_4) \in \mathbb{R}^2$. This guarantees the stability of the zero dynamics, thereby establishing asymptotic stability of the $\phi$ dynamics.

3.5 Results

Limit cycle oscillations are periodic oscillations with constant amplitude and frequency and are typically a characteristic of nonlinear systems. The existence of LCOs has been verified during flight tests of aircrafts [30] as well as in experimental prototypes during wind tunnel
testing. It is shown in previous papers by Ko [15], Block & Strganac [14], that the considered two degrees of freedom aeroelastic model exhibits limit cycle oscillations (LCOs) even at velocities below the linear flutter velocity.

3.5.1 Open loop system behavior

To verify that the considered system exhibits LCOs for velocities less than the linear flutter velocity, numerical integration was performed using Runge Kutta’s method at a flow velocity of $U = 0.9 U_F$ subject to the initial conditions, $x_0 = \begin{bmatrix} 0.01 & 0.1 & 0 & 0 \end{bmatrix}^T$. From the open loop response of the state variables and corresponding phase plane plots shown in Fig. 3.5, the nonlinear aeroelastic system clearly exhibits LCOs for $U < U_F$.

3.5.2 Closed loop response - Linear control

Linear control methods have been used in studies to control the aeroelastic system, however it only works well close to the operating point and it is unreliable when the nonlinear terms contribute significantly to the system dynamics at higher disturbance levels. Based on whether the initial conditions lie inside or outside the region of attraction of the linear controller, the linear controller can or cannot guarantee the stability of the aeroelastic system.

The chosen state and control penalty matrices $Q$ and $R$ used in the algebraic Riccati equation
Figure 3.5: State response and phase plane plots of the open loop nonlinear aeroelastic system exhibiting LCOs at a velocity of $U = 0.9 \ U_F$ when subjected to initial conditions, $x_0 = [0.01 \ 0.1 \ 0 \ 0]^T$ marked by $\times$.

are stated below

$$Q = \begin{bmatrix} k_h & 0 & 0 & 0 \\ 0 & K_{a0} & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & I_{\alpha} \end{bmatrix}, \quad R = 10\gamma \quad (3.46)$$

where $\gamma$ is a control parameter. Varying $\gamma$, low and high authority controllers can be developed. Fig. 3.6 compares the open and closed loop response of the aeroelastic system using a linear quadratic controller ($\gamma = 1.8$) subject to the initial conditions, $x_{\text{mid}} = 0.1 \ x_0$ where $x_0 = [0.01 \ 0.1 \ 0 \ 0]^T$. It is seen that the linear controller seems to make the nonlinear aeroelastic system stable while the open loop system attains a limit cycle oscillation. From
the stable closed loop response, it can be concluded that the initial condition, \( x_{\text{mid}} \), lies in the region of attraction of the considered linear controller.

![Figure 3.6: Comparison of open loop (---) and closed loop (- - -) state response and phase plane plots of the nonlinear aeroelastic system at \( U = 0.9 \) \( U_F \) subjected to initial conditions, \( x_{\text{mid}} = 0.1 \) \( x_0 \) using a linear controller (\( \gamma = 1.8 \)).](image)

Keeping the parameters of the linear controller fixed, Fig. 3.7 compares the open and closed loop response of the aeroelastic system at higher initial conditions, i.e., \( x_{\text{high}} = x_0 \). It is seen that the linear controller fails to stabilize the system when subjected to higher initial conditions. In this case because of higher initial conditions, i.e., initial conditions far from the operating point, the nonlinearities actively participate in system dynamics which makes the closed loop system using a linear controller attain a different limit cycle oscillation instead of the equilibrium state. It can be concluded that the initial condition, \( x_{\text{high}} \), lies outside the region of attraction of the considered linear controller. These two cases perfectly demonstrate the disadvantages of using a linear controller to stabilize a nonlinear system.
Figure 3.7: Comparison of open loop (—) and closed loop (-----) state response and phase plane plots of the aeroelastic system at $U = 0.9 \: U_F$ subjected to initial conditions, $x_0 = [\begin{bmatrix} 0.01 & 0.1 & 0 & 0 \end{bmatrix}]^T$ using a linear controller ($\gamma = 1.8$).

Computing the region of attraction analytically is a cumbersome task and is therefore carried out numerically. For the linear controller with state and control penalty matrices given in Eq. 3.46, simulations are conducted for a set of initial conditions to identify the region of attraction for varying $\gamma$. To generate the initial conditions, $h$ and $\alpha$ are varied while $\dot{h}$ and $\dot{\alpha}$ are kept zero. The initial conditions at which the closed loop response exhibits stable behavior and LCOs can be identified as shown in Fig. 3.8. It is seen that an increase in $\gamma$ results in a decrease in the region of attraction. To guarantee the stability of a nonlinear system using a linear controller involves extensive analysis to identify the region of attraction.
3.5.3 Comparison of controller performance

To compare the performance of the three controllers, i) feedback linearization in modal coordinates (FLMC), ii) linear quadratic regulator (LQR) and, iii) feedback linearization in...
physical coordinates (FLPC), measures of control cost and state cost are defined as below

\[
cc = \sum_{i=1}^{t_k} u_i^T [R] u_i (\Delta t_i)
\]

\[
sc = \sum_{i=1}^{t_k} x_i^T [Q] x_i (\Delta t_i)
\]

(3.47)

cc and sc represent the control cost and state cost respectively. In the above formulation the control cost is a measure of the control effort used while the state cost is essentially an approximate of time integral of the total energy of the system. \(Q\) and \(R\) are state and control penalization matrices for the LQR controller, which is also used as the first step for the FLMC controllers and are defined in Eq. 3.46. Variants of LQR and FLMC controllers can be generated by varying the parameter, \(\gamma\). Variants of the FLPC controller are generated by varying the pole locations of the linearized subsystem from the values stated in [15]. Latin hypercube sampling is used to generate two different sets of initial conditions with 100 samples each, labeled \(x_0^{low}\) and \(x_0^{high}\). The maximum absolute values of the pitch and plunge displacements and velocities in the considered sets of initial conditions are defined in Table 3.2.

Table 3.2: Maximum values of pitch and plunge displacements and velocities considered in the initial conditions

<table>
<thead>
<tr>
<th>State</th>
<th>(x_0^{low})</th>
<th>(x_0^{high})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(h)</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>(\dot{h})</td>
<td>0.001</td>
<td>0.01</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>0.01</td>
<td>0.1</td>
</tr>
<tr>
<td>(\dot{\alpha})</td>
<td>0.01</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Mean control cost (\(\bar{cc}\)) and mean state cost (\(\bar{sc}\)) can be computed over the set of initial
Figure 3.9: Comparison of mean state cost against the mean control cost for FLPC, LQR and FLMC controllers for the aeroelastic system subject to set of initial conditions $x_0^{low}$.

conditions with $N$ samples as shown below

\[
\begin{align*}
\bar{cc} & = \frac{1}{N} \sum_{i=1}^{N} cc \\
\bar{sc} & = \frac{1}{N} \sum_{i=1}^{N} sc
\end{align*}
\]  

(3.48)

Fig. 3.9 shows the variation of mean state cost against the mean control cost for $x_0^{low}$ case. This is essentially a Pareto front for variants of FLPC, LQR and FLMC controllers. It is clearly seen that the LQR controller and FLMC controller have almost the same mean state and control costs. This is expected for the $x_0^{low}$ initial conditions. For this set of initial conditions, the nonlinearities remain dormant and do not contribute significantly to the system dynamics. This ensures that the nonlinear term of the control input of FLMC controller is left unused leaving only the linear control terms to ensure convergence. It is therefore concluded that the proposed FLMC controller behaves exactly like a linear controller for small initial condition cases and can be considered optimal. The FLPC controller can be seen to
have much higher mean control costs than both the LQR and FLMC controllers. This is because it involves cancellation of some linear terms along with the nonlinear terms while choosing the nonlinear control law to make the system partially linearizable. Choosing an appropriate value of $\gamma$, lower mean state and control costs for FLMC and LQR controllers can be guaranteed when compared to the investigated cases of FLPC controller.

Similarly, Fig. 3.10 shows a Pareto front with the mean state cost plotted against the mean control cost for the $x_0^{high}$ case for variants of FLPC, LQR and FLMC controllers. For the LQR and FLMC controllers, the parameter $\gamma$ is varied between $-0.3$ and 2.5. It is seen that

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure3.10.png}
\caption{Comparison of mean state cost against the mean control cost for FLPC, LQR and FLMC controllers for the aeroelastic system subject to set of initial conditions $x_0^{high}$.}
\end{figure}

the FLMC and LQR controllers behave differently for these set of larger initial conditions, $x_0^{high}$. For higher values of the parameter, $\gamma$, i.e., $\gamma \geq 1.5$ it can be seen that the mean state and control costs of the LQR controller are much higher than the costs for the cases for which $\gamma < 1.5$. With increasing $\gamma$ the region of attraction of the LQR controller decreases resulting in the closed loop response exhibiting LCOs for initial conditions lying outside the region of attraction. An average of the costs over the set of 100 samples results in much
higher mean state and control costs.

Fig. 3.10 shows that LQR controller gives the minimum mean state and control costs for the initial conditions, $x_0^{high}$ for $\gamma < 1.5$. For these values of $\gamma$, the mean state and control costs of the LQR controller are comparable to the mean costs of the FLMC controller. For the considered set of initial conditions, $x_0^{high}$, the nonlinearities participate actively in the system dynamics making the contribution of the nonlinear control term significant for the FLMC controller. This results in slightly higher values of mean control cost for the FLMC controller when compared to those of the LQR controller. Inspite of the lower costs of the LQR controller, it cannot guarantee the stability of the closed loop system for all the initial conditions, until a detailed analysis of the region of attraction is carried out to guarantee that all the samples in $x_0^{high}$ lie inside the region of attraction. In contrast the FLMC controller guarantees the stability of the closed loop system response for all the considered initial conditions for all values of $\gamma$ with a small added control effort. The mean state and control costs of the FLPC controller are much higher than the costs of LQR ($\gamma < 1.5$) and FLMC ($\gamma \in R$) controllers.

![Graph](image)

Figure 3.11: Comparison of the closed loop state response of the aeroelastic system subject to the initial conditions, $x_0 = [0.01 \ 0.1 \ 0 \ 0]^T$ using the FLMC controller (---) at $\gamma = 1.0$ and the FLPC controller (-----).
Figure 3.11 compares the closed loop response of the nonlinear aeroelastic system using FLMC controller, designed using $\gamma = 1.0$ with the FLPC controller used in Ko et al. [15] for the initial condition, $[0.01 \ 0.1 \ 0 \ 0]^T$. Comparing the absolute area under the response curves for both the controllers, it can be seen that the closed loop response has lower state cost for the FLMC controller than the FLPC controller. Figure 3.12 shows the comparison between the generated control inputs (in radians) using the FLMC ($\gamma = 1$) and FLPC controllers when the nonlinear aeroelastic system is subjected to the initial condition, $[0.01 \ 0.1 \ 0 \ 0]^T$. It can clearly be seen from Fig. 3.12 that the control cost (proportional to the area under the $\beta$ response curve), using the FLMC controller is less than that of FLPC controller. The performance of the described FLMC controller can be improved by considering the actuator dynamics along with the nonlinear aeroelastic system.
3.6 Conclusion

A feedback linearization based controller is developed in the modal coordinates. Modal transformation is done to linearly decouple the system and feedback linearization is used to cancel the nonlinearities in the flutter mode. Feedback linearization in modal coordinates does not disrupt the linear dynamic terms. The controller developed is compared with a standard linear quadratic controller and a feedback linearization controller which uses the physical coordinates. The proposed controller is shown to behave optimally similar to a linear controller for small disturbances. For large disturbances, it is shown that the costs of the proposed controller are comparable with that of linear controller with the added advantage of guaranteed stability.
Bibliography


Chapter 4

Controlling Limit Cycle Oscillations Amplitudes in Nonlinear Aeroelastic Systems

4.1 Abstract

The paper focuses on the design of state feedback controllers to control the amplitude of limit cycle oscillations exhibited by nonlinear aeroelastic systems. A purely nonlinear state feedback control law is designed to improve the closed loop system dynamics. Nonlinear normal modes are computed for the closed loop system to identify the flutter mode dynamics. The effectiveness of nonlinear normal modes as a tool to capture the limit cycle oscillation growth is demonstrated. Harmonic balance method is utilized to estimate the amplitude and frequency of the limit cycle oscillations exhibited by the flutter mode. Analytical estimates of sensitivities of limit cycle amplitude with respect to the introduced control parameters are generated and shown to match closely to sensitivities computed numerically using finite difference method on time marching simulation. A multiobjective optimization problem
with the costs chosen to minimize the estimate of limit cycle oscillation amplitude and an approximate measure of control cost is solved using the estimated sensitivities. Numerical simulation results are used to verify that minimizing the limit cycle amplitude of flutter nonlinear normal mode corresponds closely to minimizing the limit cycle amplitude of the aeroelastic system.

4.2 Introduction

Aeroelastic systems are inherently nonlinear thereby exhibiting quite diverse response phenomena. Combination of geometric, freeplay, structural and aerodynamic nonlinearities lead to complex behavior [1, 2, 3] including existence of multiple equilibria, bifurcations, chaos [4], limit cycle oscillations (LCOs) and various types of resonances [5]. To enhance the flight envelope and increase the maneuverability of aerial vehicles, suppression of such undesired phenomena is quite critical and has been a topic of active interest among researchers for the past few decades. Woolston et al. [6, 7] and Shen [8] looked into the effects of structural nonlinearities on the flutter of a wing. An excellent review of the types of nonlinearities encountered in aeroelastic systems and their effects is given by Breitbach [9, 10]. LCOs are constant amplitude and frequency periodic oscillations whose existence has verified in experimental studies involving airfoil section models [11, 12] as well as in modern aircrafts such as F-16 and F-18 [13]. Being subjected to LCOs over extended periods of time can cause structural fatigue which might lead to system failure and catastrophic damage. This makes LCOs undesirable and their occurrence must be suppressed within the flight envelope.

Several works in literature concentrate on controlling the amplitude of LCOs for various nonlinear systems, for example, power systems [14], van der pol oscillator [15, 16, 17], and aeroelastic systems [18, 19] . The process involves predicting the LCO amplitude and using nonlinear state feedback control laws to modify the amplitude as desired. Method of normal
forms [20, 2, 19], harmonic balance methods [21, 22] and nonlinear normal modes [23, 24, 25, 26] have been employed to predict the LCO amplitudes of nonlinear dynamic systems. Harmonic balance method derived by Krylov and Bogoliubov [27] can be used to estimate LCO amplitudes and bifurcation behavior of nonlinear systems. Lee et al [28] used harmonic balance to investigate the dynamic response of a two degree of freedom airfoil section model coupled with a cubic nonlinearity. They obtained amplitude frequency relations and analyzed the stability of equilibrium points. A good correlation between the results from harmonic balance and numerical simulations was demonstrated. Yang and Zhao [29] analyzed the self excited oscillations for a two dimensional wing model with nonlinear pitch stiffness using experimental data and harmonic balance method. It is shown that the harmonic balance method is able to predict the unstable LCOs as well as the stable LCOs observed in the experimental data. Dimitriadis [21] used a combination of harmonic balance method and numerical continuation techniques to study the bifurcation behavior of two dimensional airfoil section model in the presence of freeplay nonlinearity. Emory and Patil [25] demonstrated that nonlinear normal modes (NNMs) [30, 31] can be used as an effective tool to predict the LCO amplitudes exhibited by an aeroelastic system using lesser number of equations. It is shown that the predictions of LCO amplitude obtained using NNMs are close to those obtained from time integration using Runge Kutta methods.

In the present work, the flutter mode for the aeroelastic system is identified using asymptotic method for computing NNMs. Instead of applying harmonic balance method directly to the aeroelastic system as done in a number of works in literature, harmonic balance method is used to estimate the LCO amplitude of the flutter NNM along with its sensitivities to the introduced control parameters. Generating accurate analytical sensitivity equations can result in reduction of computational effort in comparison to numerically estimating sensitivities from finite difference method for large scale systems. Furthermore, finite difference sensitivities have significant errors due to finite precision in a computational analysis framework. A multiobjective optimization problem is solved to minimize the estimated LCO amplitude of the flutter NNM and an approximate measure of control cost. It is shown that minimizing
the LCO amplitude estimate of the flutter NNM results in the same percentage reduction in
the LCO amplitude of the aeroelastic system.

4.3 Nonlinear Aeroelastic System

A variant of the typical airfoil section equipped with a trailing edge flap is considered in the
present work. The airfoil section is free to move up and down along the plunge degree of
freedom, and rotate in the pitch degree of freedom. The governing equations of motion can
be obtained using Lagrange’s equations. The non-dimensional form of governing equations
of motion [25] are expressed as below

\[ \ddot{h} + x_\alpha \dot{\alpha} + \ddot{\omega} \bar{h} = -\bar{L} \]
\[ x_\alpha \ddot{h} + r_\alpha^2 \dot{\alpha} + r_\alpha^2 (1 + G_\alpha \alpha^2) \alpha = \bar{M} \]  

These differ from the classic airfoil section equations given by Bisplinghoff et al. [32] in
terms of the nonlinear stiffness term, \(G_\alpha\). \(\bar{h}\) is the non-dimensional plunge which is plunge
displacement \(h\) divided by the half chord \(b\), \(\alpha\) is the pitch angle, \(I_\alpha\) is the pitch inertia, \(m\) is
the mass, \(x_\alpha\) is the dimensionless static imbalance, \(\dot{\omega} = (\omega_h/\omega_\alpha)\) is the pitch plunge natural
frequency ratio where \(\omega_h = \sqrt{K_h/m}\) and \(\omega_\alpha = \sqrt{K_\alpha/I_\alpha}\), \(r_\alpha\) is the dimensionless radius of
gyration \(I_\alpha/mb^2\), the overdot represents a derivative with respect to non dimensional time
\((\tau = t\omega_\alpha)\). \(\bar{L}\) and \(\bar{M}\) represent non dimensional aerodynamic force and moment. Nonlinear
pitch stiffness function with a cubic nonlinearity is considered and is defined as below

\[ K_\alpha [\alpha] = r_\alpha^2 (1 + G_\alpha \alpha^2) \]

Theodorsen [33] derived the expressions for lift and moment assuming harmonic motion of
the airfoil. In the current work, a quasi steady model obtained by setting the Theodorsen's
Table 4.1: Parameters used for the airfoil section model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>11</td>
</tr>
<tr>
<td>$a$</td>
<td>-0.35</td>
</tr>
<tr>
<td>$x_\alpha$</td>
<td>0.2</td>
</tr>
<tr>
<td>$r_\alpha$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\bar{\omega}$</td>
<td>0.5</td>
</tr>
<tr>
<td>$G_\alpha$</td>
<td>0.5</td>
</tr>
<tr>
<td>$T_4$</td>
<td>-0.4104</td>
</tr>
<tr>
<td>$T_{10}$</td>
<td>1.6798</td>
</tr>
</tbody>
</table>

function $C[k] = 1$ is used to model the lift forces and moments acting on the airfoil section. The lift and moment acting per unit span are

$$\bar{L} = \frac{1}{\mu} \left[ \dot{h} + \bar{u} \dot{\alpha} - a \ddot{\alpha} \right] + \frac{2 \bar{u}}{\mu} \left[ \left( \dot{h} + \bar{u} \alpha + \left( \frac{1}{2} - a \right) \dot{\alpha} + \bar{u} \frac{T_{10}}{\pi \beta} \right) \right]$$

(4.3)

$$\bar{M} = \frac{1}{\mu} \left[ a \ddot{h} - \left( \frac{1}{2} - a \right) \bar{u} \ddot{\alpha} - \left( \frac{1}{8} + a^2 \right) \ddot{\alpha} - \frac{\left( T_4 + T_{10} \right)}{\pi} \bar{u}^2 \beta \right] + \frac{2 \bar{u}}{\mu} \left[ \left( \frac{1}{2} + a \right) \left( \dot{h} + \bar{u} \alpha + \left( \frac{1}{2} - a \right) \dot{\alpha} + \bar{u} \frac{T_{10}}{\pi \beta} \right) \right]$$

(4.4)

Here, $\bar{u} = U_\infty/b\omega_\alpha$ is the dimensionless freestream velocity, $a$ is the elastic axis location and $\mu$ is the density ratio. $T_4$ and $T_{10}$ are two of the Theodorsen’s coefficients and $\beta$ represents the flap deflection. Choosing a state vector composed of displacements and velocities of $h$ and $\alpha$, the governing equations can be transformed to a set of four first order differential equations and represented in a state space form. Varying $\bar{u}$, a root locus plot can be used to compute the linear flutter velocity and frequency for the aeroelastic system. For the considered aeroelastic system with parameters shown in Table 4.1, the dimensionless linear flutter velocity is 0.807 and dimensionless linear flutter frequency is 0.1598. Fig. 4.1 shows
the locus of the real part of the eigenvalue of the linearized aeroelastic system with variations in $\bar{u}$.

4.4 Solution Approach

A strictly nonlinear state feedback control law is chosen with the coefficients of nonlinear terms considered as the control parameters. The control parameters are represented as a vector, $k$. Asymptotic method [30] is used to compute the nonlinear normal modes for the closed loop system dynamics. From the computed NNMs, the flutter mode is identified and fundamental harmonic balance method is used to obtain a set of algebraic nonlinear equations. For fixed value of control parameters these equations can be solved using Newton’s method to compute the LCO amplitude ($c_1$) and frequency ($\omega$). The set of equations is further differentiated with respect to each control parameter, to obtain a linear equation for the sensitivities. The estimate of LCO amplitude and the corresponding analytical estimates
of sensitivities are used to solve a multiobjective optimization problem to generate optimal control parameters which minimize the LCO amplitude of flutter NNM and an approximate measure of the control cost.

Fig. 4.2 shows an outline of the solution approach. The accompanying sections describe the approach in detail.

Figure 4.2: Flowchart outlining the solution approach.
4.5 Nonlinear Normal Modes

Nonlinear normal modes extend the concept of linear normal modes to nonlinear systems. Introduced in 1960s by Rosenberg [34, 35, 36] as the synchronous vibration of the system, i.e., all material points of the system reach their extreme values and pass through zero simultaneously allowing all displacements to be expressed in terms of a single reference displacement. Shaw and Pierre [37, 30, 31] generalized Rosenberg’s definition of NNM as a two dimensional invariant manifold in phase space. An invariant set for a dynamical system is defined as a subset $S$ of the phase space, such that starting from an initial condition in $S$ the solution of governing equations of motions remains in $S$ for all times. Moreover, this manifold can be parameterized in terms of a single pair of state variables, i.e., displacements and velocities of the chosen variable.

Shaw and Pierre’s approach has been applied for both continuous and discrete mechanical systems. A short overview of computation of NNMs using the approach of Shaw and Pierre [30] as applied to the nonlinear aeroelastic system is shown in this section. The nonlinear aeroelastic system’s governing equations of motion given in Eq. 4.1 can be represented in the form below

$$
\begin{bmatrix}
\dot{\bar{h}} \\
\dot{\bar{\alpha}} \\
\dot{\bar{y}} \\
\dot{\alpha}
\end{bmatrix} = 
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} =
\begin{bmatrix}
y_1 \\
y_2 \\
f_1(x_1, x_2, y_1, y_2) \\
f_2(x_1, x_2, y_1, y_2)
\end{bmatrix}
$$

(4.5)

where $x_i$ and $y_i$ represent the displacements and velocities in the nondimensional plunge and pitch degrees of freedom respectively. The displacements and velocities of all degrees of freedom are represented as nonlinear function of displacements and velocities in the master coordinate. Pitch, $\alpha$ is chosen as the master coordinate and the displacements and velocities
in plunge are represented as nonlinear functions of the master coordinate.

\[
\begin{align*}
x_1 &= X_1[u, v] \\
x_2 &= u \\
y_1 &= Y_1[u, v] \\
y_2 &= v
\end{align*}
\]

(4.6)

\(X_1[u, v]\) and \(Y_1[u, v]\) represent nonlinear functions of the master coordinates, \(u\) and \(v\). From Eq. 4.6 a normal mode can be defined as a motion taking place on a two dimensional invariant manifold in the system’s phase space. This manifold passes through the equilibrium point of the system and is tangential to the eigen space of the linear system obtained by linearizing the nonlinear system about the equilibrium point. Differentiating Eq. 4.6 using chain rule, we obtain

\[
\begin{align*}
\dot{x}_1 &= \frac{\partial X_1}{\partial u} \dot{u} + \frac{\partial X_1}{\partial v} \dot{v} \\
\dot{y}_1 &= \frac{\partial Y_1}{\partial u} \dot{u} + \frac{\partial Y_1}{\partial v} \dot{v}
\end{align*}
\]

(4.7)

From Eq. 4.5 and Eq. 4.6 we obtain the following relations

\[
\begin{align*}
\dot{u} &= v \\
\dot{v} &= f_2 (X_1[u, v], u, Y_1[u, v], v)
\end{align*}
\]

(4.8)

Substituting Eq. 4.8 in Eq. 4.7 the following relations are obtained

\[
\begin{align*}
Y_1[u, v] &= \frac{\partial X_1[u, v]}{\partial u} v + \frac{\partial X_1[u, v]}{\partial v} f_2(X_1[u, v], u, Y_1[u, v], v) \\
f_1(X_1[u, v], u, Y_1[u, v], v) &= \frac{\partial Y_1[u, v]}{\partial u} v + \frac{\partial Y_1[u, v]}{\partial v} f_2(X_1[u, v], u, Y_1[u, v], v)
\end{align*}
\]

(4.9)

The modal laws for the nonlinear system are expanded in the form of a Taylor series in the master coordinates upto the order equal to the degree of nonlinearity of the original
equations of motion as below

\[
X_1[u, v] = a_1u + a_2v + a_3u^2 + a_4uv + a_5v^2 + a_6u^3 + a_7u^2v + a_8uv^2 + a_9v^3 \\
Y_1[u, v] = b_1u + b_2v + b_3u^2 + b_4uv + b_5v^2 + b_6u^3 + b_7u^2v + b_8uv^2 + b_9v^3
\] (4.10)

Substituting Eq. 4.10 in Eq. 4.9, the coefficients of similar terms are collected and set to zero to obtain a set of nonlinear algebraic equations. Solving these equations the coefficients \( (a_i, b_i) \) in the defined modal laws become known and can be substituted into the equations corresponding to the master coordinate to obtain NNMs. The equations obtained by setting the coefficients to zero generally have multiple solutions with each solution set corresponding to a different NNM. For the aeroelastic system two sets of solutions are obtained. Defining a modal vector, \( w = [u_1 \quad v_1 \quad u_2 \quad v_2]^T \), a transformation \( M_{M2P}(w) \) can be defined from the modal coordinates to the physical coordinates, \( z = [x_1 \quad x_2 \quad y_1 \quad y_2]^T \) as below

\[
z = M_{M2P}(w) \\
= [M_0 + M_1(w) + M_2(w)]w
\] (4.11)

where \( M_0, M_1 \) and \( M_2 \) are defined as below

\[
M_0 = \begin{bmatrix}
a_{11} & a_{21} & a_{12} & a_{22} \\
1 & 0 & 1 & 0 \\
b_{11} & b_{21} & b_{12} & b_{22} \\
0 & 1 & 0 & 1
\end{bmatrix}, \\
M_1(w) = \begin{bmatrix}
a_{31}u_1 + a_{41}v_1 & a_{51}v_1 & a_{32}u_2 + a_{42}v_2 & a_{52}v_2 \\
0 & 0 & 0 & 0 \\
b_{31}u_1 + b_{41}v_1 & b_{51}v_1 & b_{32}u_2 + b_{42}v_2 & b_{52}v_2 \\
0 & 0 & 0 & 0
\end{bmatrix}, \text{ and}
\]

\[
M_2(w) = \begin{bmatrix}
a_{61}u_1^2 + a_{81}v_1^2 & a_{71}u_1^2 + a_{91}v_1^2 & a_{62}u_2^2 + a_{82}v_2^2 & a_{72}u_2^2 + a_{92}v_2^2 \\
0 & 0 & 0 & 0 \\
b_{61}u_1^2 + b_{81}v_1^2 & b_{71}u_1^2 + b_{91}v_1^2 & b_{62}u_2^2 + b_{82}v_2^2 & b_{72}u_2^2 + b_{92}v_2^2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
where $a_{ij}$ and $b_{ij}$ represent the coefficients $a_i$ and $b_i$ in Eq. 4.10 for the $j^{th}$ NNM. Since the considered system has only cubic nonlinearity terms, $M_1(w) = 0$. Assuming $M_0^{-1}M_2 << I$, an inverse transformation from physical to modal coordinates, $M_{P2M}(z)$ is defined as below

$$ w = [M(w)]^{-1} z $$

$$ = [M_0 + M_2(w)]^{-1} z $$

$$ = [I + M_0^{-1}M_2(w)]^{-1} M_0^{-1} z $$

$$ = [I - M_0^{-1}M_2(M_0^{-1}z)] M_0^{-1} z $$

$$ = M_{P2M}(z) $$

(4.12)

The computed NNMs for the aeroelastic system with parameter values as given in Table 4.1 and at a velocity 1.05 times the non dimensional linear flutter velocity are stated below. The flutter NNM dynamics are given by

$$ \ddot{u}_1 = u_1 (-1.00094 - 0.5541u_1^2 + 0.005475\dot{u}_1^2) + \dot{u}_1 (0.004313 - 0.043356u_1^2 - 0.03092\dot{u}_1^2) $$

(4.13)

A destabilizing linear damping term ($0.004313\dot{u}_1$) can be clearly seen in Eq. 4.13 along with two stabilizing nonlinear damping terms ($-0.043356u_1^2\dot{u}_1 - 0.03092\dot{u}_1^2$). The second NNM represents a damped oscillator whose dynamics are given as

$$ \ddot{u}_2 = u_2 (-0.23632 - 0.15194u_2^2 + 2.2921\dot{u}_2^2) + \dot{u}_2 (-0.20603 + 1.1469u_2^2 + 0.1142\dot{u}_2^2) $$

(4.14)

The stabilizing linear term is clearly seen in Eq. 4.14. The process enables us to obtain completely decoupled NNMs represented in Eq. 4.13 and Eq. 4.14 for the nonlinear aeroelastic system. Fig. 4.3 compares the state response obtained by simulating the nonlinear aeroelastic system dynamics given in Eq. 4.5 and those obtained by transforming the modal response generated from the computed NNMs using the transformation defined in Eq. 4.11. Fig 4.3 highlights the effectiveness of using NNM as a tool to capture the growth and amplitude of LCOs in aeroelastic systems.
Figure 4.3: Comparison of state response obtained by simulating the complete aeroelastic system (—) and the state response obtained by transforming the modal response generated by simulation of NNM dynamics (—).

4.6 Estimation of LCO amplitudes

Harmonic balance method (HBM) is used to estimate the amplitude of the LCOs exhibited by the flutter NNM given in Eq. 4.13. The main idea behind HBM is to balance the Fourier components of dominant harmonics assuming the existence of a periodic limit cycle. HBM has been directly applied to aeroelastic systems to estimate LCO amplitudes in a number of works, such as Yang and Zhao [29], Liu and Dowell [38] and Dmitriadis [21]. In the current work, HBM is used to estimate the LCO amplitude of the flutter NNM only.
4.6.1 Harmonic Balance method

This section reviews the application of HBM for a second order oscillator represented as

\[ \ddot{u} = f (u, \dot{u}) = f_l (u, \dot{u}) + f_{nl} (u, \dot{u}) \]  

(4.15)

where \( f_l (u, \dot{u}) \) and \( f_{nl} (u, \dot{u}) \) are strictly linear and nonlinear functions of \( u, \dot{u} \) respectively. It is assumed that the system exhibits a periodic response in steady state which is approximated by a series of harmonic functions as below

\[ u_p = c_0 + \sum_{i=1}^{n} (c_i \cos (\omega t) + s_i \sin (\omega t)) \]  

(4.16)

where \( c_0 \) is a constant term, \( c_i \) and \( s_i \) are coefficients of the cosine and sine terms of the \( i^{th} \) harmonic respectively. For \( n = 1 \), the method is called fundamental harmonic balance. \( n \) can be increased to generate higher order HB methods to get more accurate periodic solution approximations. The assumed solution is substituted back into the original system equations to obtain the following relation

\[ -\omega^2 \sum_{i=1}^{n} (c_i \cos (\omega t) + s_i \sin (\omega t)) = f_l (u_p, \dot{u}_p) + f_{nl} (u_p, \dot{u}_p) \]  

(4.17)

The nonlinear terms are approximated using Fourier laws to obtain the required harmonic coefficient. The nonlinear function in Eq. 4.17, \( f_{nl} (u_p, \dot{u}_p) \) is approximated as a summation of harmonics as shown below

\[ f_{nl} (u_p, \dot{u}_p) = c_{nl,0} + \sum_{k=1}^{n} [c_{nl,k} \cos (k \omega t) + s_{nl,k} \sin (k \omega t)] \]  

(4.18)
where $c_{nl,0}$, $c_{nl,k}$ and $s_{nl,k}$ can be computed in terms of previously defined variables, $\omega$, $c_0$, $c_1$, ..., $c_n$, and $s_1$, ..., $s_n$ using the following integrals [21]

\[
\begin{align*}
    c_{nl,0} &= \frac{\omega}{2\pi} \int_0^{2\pi} f_{nl}(u_p, \dot{u}_p) \, dt \\
    c_{nl,k} &= \frac{\omega}{\pi} \int_0^{2\pi} f_{nl}(u_p, \dot{u}_p) \cos(k\omega t) \, dt \\
    s_{nl,k} &= \frac{\omega}{\pi} \int_0^{2\pi} f_{nl}(u_p, \dot{u}_p) \sin(k\omega t) \, dt
\end{align*}
\] (4.19)

Coefficients of constant term, sine and cosine terms of the same harmonics are collected and set to zero to obtain a set of nonlinear equations in the variables, $\omega$, $c_0$, $c_1$, ..., $c_n$, and $s_2$, ..., $s_n$. For an autonomous system, $s_1 = 0$ to avoid multiple solutions for the same LCO with different phase values. The obtained set of nonlinear equations can be solved using Newton’s method to compute the approximate periodic solution of the second order oscillator. The number of harmonics considered determines the accuracy of the solution. In the current work, fundamental harmonic balance method ($n = 1$) is employed to compute the periodic solution of the flutter NNM.

### 4.6.2 Fundamental Harmonic balance for LCO Nonlinear normal mode

The NNM exhibiting LCO at a nondimensional velocity of 1.05 times the linear flutter velocity is given in Eq. 4.13. The modal dynamics can be represented in a shorthand form as below by separating the linear terms in $u$ and $\dot{u}$

\[
\ddot{u} = p_1 u + p_2 \dot{u} + f_{nl}(u, \dot{u})
\] (4.20)

where $p_1$, $p_2$ represent the coefficients of terms linear in $u$ and $\dot{u}$ respectively while $f_{nl}(u, \dot{u})$ contains all the nonlinear terms of the LCO nonlinear normal mode. A periodic solution of
the form given in Eq. 4.16 with \( n = 1 \) is assumed

\[
\begin{align*}
\dot{u} &= c_0 + c_1 \cos(\omega t) \\
\ddot{u} &= c_{0l} + c_{1l} \cos(\omega t) + s_{1l} \sin(\omega t) \\
\end{align*}
\]  \hspace{1cm} (4.21)

Using the defined periodic solution, first harmonic representation of the nonlinear terms in modal dynamics, \( f_{nl}(u, \dot{u}) \) is obtained as shown in the previous section

\[
\begin{align*}
f_{nl}(u, \dot{u}) &= c_{nl,0} + c_{nl,1} \cos(\omega t) + s_{nl,1} \sin(\omega t) \\
\end{align*}
\]  \hspace{1cm} (4.22)

where \( c_{nl,0}, c_{nl,1} \) and \( s_{nl,1} \) can be computed in terms of \( c_0, c_1 \) and \( \omega \) using Eq. 4.16. The periodic solution is substituted in the modal dynamics, Eq. 4.20 to obtain

\[
\begin{align*}
-\omega^2 c_1 \cos(\omega t) &= p_1 (c_0 + c_1 \cos(\omega t)) + p_2 \omega (-c_1 \sin(\omega t)) + (c_{nl,0} + c_{nl,1} \cos(\omega t) + s_{nl,1} \sin(\omega t)) \\
\end{align*}
\]  \hspace{1cm} (4.23)

Collecting the constant, first harmonic sine and cosine terms in Eq. 4.23, we obtain a set of nonlinear algebraic equations as below

\[
\begin{align*}
p_1 c_0 + c_{nl,0} &= 0 \\
\omega^2 c_1 + p_1 c_1 + c_{nl,1} &= 0 \\
-p_2 \omega c_1 + s_{nl,1} &= 0 \\
\end{align*}
\]  \hspace{1cm} (4.24)

Since \( c_{nl,0}, c_{nl,1} \) and \( s_{nl,1} \) are nonlinear functions of \( c_0, c_1 \), and \( \omega \), Eq. 4.24 represents a set of three nonlinear algebraic equations which can be solved for the parameters \( c_0, c_1 \) and \( \omega \). The parameters \( c_1 \) and \( \omega \) represent the amplitude and frequency of the LCO exhibited by the flutter NNM.

### 4.7 Closed loop system

To analyze the closed loop system, a nonlinear state feedback function is defined as below

\[
\beta = k_1 \dot{h}^3 + k_2 \dot{\alpha}^3 + k_3 h^3 + k_4 \dot{\alpha}^3
\]  \hspace{1cm} (4.25)
where, \( k_i \forall i = 1, ..., 4 \) represents the control parameters. It is to be noted that linear control terms are not included to prevent the linear flutter velocity from changing. Furthermore, a linear control is active over the entire range of operating conditions while the nonlinear controller does not lead to significant actuation for low state response. Finally, a purely nonlinear controller also enables us to use the same linear modal solution while computing the nonlinear normal modes. The procedure of computation of NNMs shown in the previous section is carried out for the closed loop system to obtain control parameter dependent nonlinear normal modal dynamics. The linear part of the computed NNMs remains unaltered, however, the nonlinear terms are now a function of the introduced control parameters, \( k_i (i = 1, 2, 3, 4) \). The NNM corresponding to flutter mode can be represented in a general form as below

\[
\ddot{u} = p_1 u + p_2 \dot{u} + F_{nl}(u, \dot{u}, k) \tag{4.26}
\]

Having obtained the NNMs dependent on the control parameters, fundamental harmonic balance method is used to obtain a set of three nonlinear equations represented as below

\[
\begin{align*}
    p_1 c_0 + c_{nl,0} (c_0, c_1, \omega, k_1, k_2, k_3, k_4) & = 0 \\
    \omega^2 c_1 + p_1 c_1 + c_{nl,1} (c_0, c_1, \omega, k_1, k_2, k_3, k_4) & = 0 \\
    -p_2 \omega c_1 + s_{nl,1} (c_0, c_1, \omega, k_1, k_2, k_3, k_4) & = 0
\end{align*}
\tag{4.27}
\]

It was earlier stated that the terms \( c_{nl,0}, c_{nl,1} \) and \( s_{nl,1} \) are nonlinear functions of \( c_0, c_1 \) and \( \omega \). For the closed loop system, these terms will also be functions of the control parameters, \( k_1, k_2, k_3 \) and \( k_4 \). For fixed values of control parameters these equations can be solved using Newton’s method to compute \( c_0, c_1 \) and \( \omega \) which define the fundamental component of the periodic solution for the flutter mode NNM with amplitude, \( c_1 \) and frequency, \( \omega \).
4.7.1 Sensitivity of LCO amplitude to control parameters

Differentiating Eq. 4.27 with respect to the control parameters, we obtain three sensitivity equations for each control parameter, \( k_i \). These can be represented in the form of a linear matrix equation as below

\[
\begin{bmatrix}
    p_1 + \frac{\partial c_{nl,0}}{\partial c_0} & \frac{\partial c_{nl,0}}{\partial c_1} & \frac{\partial c_{nl,0}}{\partial \omega} \\
    \frac{\partial c_{nl,1}}{\partial c_0} & \omega^2 + p_1 + \frac{\partial c_{nl,1}}{\partial c_1} & 2\omega c_1 + \frac{\partial c_{nl,1}}{\partial \omega} \\
    \frac{\partial c_{nl,1}}{\partial s_{nl,1}} & -p_2\omega + \frac{\partial s_{nl,1}}{\partial c_1} & -p_2c_1 + \frac{\partial s_{nl,1}}{\partial \omega}
\end{bmatrix}
\begin{bmatrix}
    \frac{\partial c_0}{\partial k_i} \\
    \frac{\partial c_1}{\partial k_i} \\
    \frac{\partial \omega}{\partial k_i}
\end{bmatrix}
= - \begin{bmatrix}
    \frac{\partial c_{nl,0}}{\partial k_i} \\
    \frac{\partial c_{nl,1}}{\partial k_i} \\
    \frac{\partial s_{nl,1}}{\partial k_i}
\end{bmatrix}
\]

(4.28)

The above equations can be easily solved using linear solvers to generate analytical estimates of sensitivities. \([\partial(\cdot)/\partial k_i]\) represents the sensitivity of the response variables with respect to the control parameter, \( k_i \). The computed analytical sensitivities of \( c_1 \) with respect to control parameters, i.e., \( \partial c_1/\partial k_i \) for \( i = 1, ..., 4 \) are used for control design.

The estimate of LCO amplitude, \( c_1 \) uses NNM and HBM to approximate the amplitude of the flutter NNM. The analytical sensitivities of the approximate amplitude, \( \partial c_1/\partial k_i \) are compared with the numerically computed sensitivities of the LCO amplitude calculated from simulation of the original system and the flutter NNM. The numerical sensitivities are computed by employing finite difference on the LCO amplitudes estimated from the steady state time response of the flutter mode NNM dynamics and the aeroelastic system dynamics.

\[
\frac{\Delta A}{\Delta k_i} = \frac{A(k_i + \Delta k_i) - A(k_i)}{\Delta k_i}
\]

(4.29)

where \( A \) represents the LCO amplitude exhibited by the flutter NNM. \( A \) can be estimated numerically by simulating the flutter mode NNM dynamics using Runge Kutta method. Also, the state response of the aeroelastic system can be transformed to modal coordinates to obtain \( A \) in a different way.
Figure 4.4: Comparison of numerical sensitivities $\Delta A/\Delta k_i$ obtained by applying finite difference on amplitudes estimated from time simulation of NNM dynamics and aeroelastic system dynamics with analytical sensitivities obtained using HBM, $\partial c_1/\partial k_i$ while varying (a) $k_1$, (b) $k_2$, (c) $k_3$ and (d) $k_4$. 
Fig. 4.4 compares the sensitivities of LCO exhibited by flutter NNM with respect to the control parameters, $k_i$ obtained using finite difference method, $\Delta A/\Delta k_i$ and the analytical estimates, $\partial c_1/\partial k_i$ obtained using Eq. 4.28. The finite difference method to compute sensitivities involves numerically estimating the amplitude of flutter NNM followed by application of finite difference on the estimated amplitude ($A_{u1}$) making it prone to machine precision errors. Moreover, it becomes computationally expensive if used in an optimization problem which requires the sensitivities at each step.

From Fig. 4.4 it is clear that analytical estimates of sensitivities obtained on the amplitude estimated by harmonic balance method match closely with the finite difference sensitivities obtained on numerically estimated amplitudes. The next section describes a multiobjective optimization framework which uses the estimated analytical sensitivities at each step.

4.8 Optimization Framework

The objective of optimization problem is to find optimal control parameters which minimize the LCO amplitude of the nonlinear aeroelastic system. The problem is formulated as a weighted sum multiobjective optimization with the objective to minimize the estimate of LCO amplitude of the flutter NNM, i.e., $c_1$ and an approximate measure of the control cost ($cc_{approx}$). It is subsequently shown that minimizing $c_1$ ends up in minimization of the LCO amplitude of the aeroelastic system by approximately the same percentage.

4.8.1 Approximate control cost

The measure of approximate control cost used involves considering the linear flutter mode at the linear flutter velocity. The eigen vector corresponding to the linear flutter mode at
$U_F$ normalized with respect to the $\tilde{h}$ coordinate is stated below

$$
\begin{bmatrix}
\tilde{h} \\
\alpha \\
\dot{\tilde{h}} \\
\dot{\alpha}
\end{bmatrix} = (\gamma_r \pm i\gamma_i)
\begin{bmatrix}
1 \\
\lambda_r \pm i\lambda_i \\
\pm i\omega_f \\
\pm (\lambda_r \pm \lambda_i) i\omega_f
\end{bmatrix}
$$

(4.30)

Based on the normalized linear flutter modes, the amplitudes of the state variables can be approximated as below

$$
\begin{bmatrix}
A_h \\
A_\alpha \\
A_{\tilde{h}} \\
A_{\dot{\alpha}}
\end{bmatrix} = h_o
\begin{bmatrix}
1 \\
\bar{\alpha} \\
\omega_f \\
\bar{\alpha}\omega_f
\end{bmatrix} = \sqrt{\gamma_r^2 + \gamma_i^2}
\begin{bmatrix}
1 \\
\sqrt{\lambda_r^2 + \lambda_i^2} \\
\omega_f \\
\omega_f\sqrt{\lambda_r^2 + \lambda_i^2}
\end{bmatrix}
$$

(4.31)

For the considered system, the parameters, $h_0$, $\bar{\alpha}$ and $\omega_f$ are 0.1808, 3.7645 and 1.0085 respectively. The approximate control cost is defined as

$$
c_{cc_{approx}} = \{k\}^T [Q] \{k\}
$$

(4.32)

where $Q = diag\ [d_1, d_2, d_3, d_4] \in R^{4 \times 4}$ is a diagonal matrix with the diagonal terms ($d_i$) equal to the corresponding amplitude ratios as shown below

$$
\begin{bmatrix}
d_1 \\
d_2 \\
d_3 \\
d_4
\end{bmatrix} =
\begin{bmatrix}
1 \\
\bar{\alpha}^6 \\
\omega_f^6 \\
(\bar{\alpha}\omega_f)^6
\end{bmatrix}
$$

(4.33)
4.8.2 Optimization Problem

The optimization problem is represented as below

\[
\min_k \quad \lambda c_1(k) + (1 - \lambda) cc_{\text{approx}}(k)
\]
\[
\text{s.t.} \quad lb \leq k \leq ub
\]

(4.34)

where, \(k\) represents the vector of control parameters, \(k = [k_1 \ k_2 \ k_3 \ k_4]^T\), \(lb\) and \(ub\) represent the lower and upper bounds on the control parameter vector respectively and \(\lambda\) is a weight parameter to choose the amount of control used from low authority \((\lambda = 0)\) to high authority \((\lambda = 1)\). \(k\) needs to be bounded to guarantee the stability of the response of both the aeroelastic system and the NNMs. These bounds can be computed by guaranteeing that the overall stiffness of the closed loop system remains positive subject to variations in the state variables. In the current work, numerical simulations are conducted to identify these bounds based on the response of the aeroelastic system and the NNM dynamics. These bounds are not related to any physical actuator constraints.

For a fixed value of \(k\), both \(c_1\) and \(cc_{\text{approx}}\) can be computed by solving the HBM equations given in Eq. 4.27 using Newton’s method and Eq. 4.28 using a linear solver respectively. \(k_{\text{opt}}(\lambda)\) represents the solution of the optimization problem for a given value of the weighing parameter, \(\lambda \in [0, 1]\). Choosing the lower and upper bounds as below

\[
lb = \begin{bmatrix} -230 \\ -230 \\ -60.0 \\ -230 \end{bmatrix}^T, \quad ub = \begin{bmatrix} 230 \\ 1 \\ 230 \\ 5 \end{bmatrix}^T
\]

(4.35)

the sequential quadratic programming (SQP) algorithm in Matlab is utilized to solve the optimization problem using the analytically generated sensitivities. The cost functions in
the multiobjective optimization problem are scaled to obtain the following problem

\[
\begin{align*}
\min_k & \quad \lambda \left[ \frac{c_1 (k)}{c_1 (k_{opt} (\lambda = 0))} \right] + (1 - \lambda) \left[ \frac{cc_{approx} (k)}{cc_{approx} (k_{opt} (\lambda = 1))} \right] \\
\text{s.t.} & \quad lb \leq k \leq ub
\end{align*}
\]

(4.36)

In the current work, \( c_1 (k_{opt} (\lambda = 0)) = 0.3498 \) and \( cc_{approx} (k_{opt} (\lambda = 1)) = 5.8966 \times 10^3 \). Solving the optimization problem with the scaled cost functions for varying values of \( \lambda \), a set of optimal controllers can be generated.

### 4.9 Results

Depending on the requirements on the amplitude of LCO, \( \lambda \) can be chosen between 0 and 1 to obtain the corresponding optimal control parameters. \( \lambda = 1 \) will result in parameters which minimize the amplitude of LCO NNM while \( \lambda = 0 \) minimizes only the approximate control cost and will lead to no control. Fig. 4.5 shows the variations of the normalized cost functions for varying values of \( \lambda \in [0, 1] \). It is seen that increasing \( \lambda \) results in decrease of \( c_1 \) and an increase of \( cc_{approx} \). Both the normalized costs have a maximum value of one because of the scaling factors used.

To minimize the LCO amplitude, \( \lambda \) close to one is chosen. For \( \lambda = 0.97 \), the optimal control parameters obtained are

\[
k_{opt} (\lambda = 0.97) = \begin{bmatrix}
2.86 \\
-201.42 \\
9.13 \\
-63.60
\end{bmatrix}^T
\]

(4.37)

The periodic solution obtained from the harmonic balance equations for the open loop, \( k = 0 \)
Figure 4.5: Pareto front obtained using weighted sum multi-objective optimization approach.

and closed loop, \( k = k_{opt} \) cases are stated below

\[
\begin{bmatrix}
  c_1 \\
  \omega
\end{bmatrix}
\bigg|_{k=0} = \begin{bmatrix}
  0.3499 \\
  1.0254
\end{bmatrix}, \quad
\begin{bmatrix}
  c_1 \\
  \omega
\end{bmatrix}
\bigg|_{k=k_{opt}} = \begin{bmatrix}
  0.0337 \\
  1.0007
\end{bmatrix}
\]

The percentage decrease in \( c_1 \) from the open loop case to the closed loop case comes out to be 90.369 %.

4.9.1 NNM response

The open loop \((k = 0)\) and closed loop \((k = k_{opt})\) NNM dynamics are simulated subject to initial conditions, \( w_0 = [-0.084 \ -0.0017 \ 0.0157 \ 0.0017]^T \). Table 4.2 compares the numerically estimated amplitudes for the LCO exhibited by flutter NNM. It can be seen that the closed loop LCO amplitude is around 90 % less than that of the open loop LCO.
amplitude.

Table 4.2: Comparison of LCO amplitudes for open and closed loop response of the flutter NNM dynamics

<table>
<thead>
<tr>
<th>State</th>
<th>Open Loop</th>
<th>Closed Loop</th>
<th>% Reduction in amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Amplitude</td>
<td>Frequency</td>
<td>Amplitude</td>
</tr>
<tr>
<td>$u_1$</td>
<td>0.3042</td>
<td>1.0195</td>
<td>0.02822</td>
</tr>
<tr>
<td>$v_1$</td>
<td>0.3082</td>
<td>1.0195</td>
<td>0.02824</td>
</tr>
</tbody>
</table>

Figure 4.6 compares the open loop and closed loop time response and phase plane plots of the NNM dynamics. It is seen that the flutter NNM exhibits LCOs while the damped mode attains the equilibrium state. The amplitude of LCO in the closed loop case is clearly much smaller than that of the open loop case as seen in the phase plane plots.

4.9.2 Nonlinear Aeroelastic system response

The open loop ($k = 0$) and closed loop ($k = k_{opt}$) nonlinear aeroelastic system dynamics are simulated subject to initial conditions, $x_0 = [0.01, 0.1, 0, 0]^T$ as shown in Figure 4.7 which compares the time response as well as the phase plane plots. It can be seen that the system states are executing LCOs for both the open loop and closed loop cases, however, the amplitude of LCO is much less for the closed loop response. This can be seen more clearly in the phase plane plots for pitch and non dimensional plunge degree of freedom shown in Fig. 4.7. Table 4.3 compares the numerically estimated LCO amplitudes for pitch and non dimensional plunge degree of freedom for open loop and closed loop case. The LCO amplitudes for the closed loop case are around 91% less than the LCO amplitudes in the open loop case. This is quite close to the percentage decrease in the numerically estimated
Figure 4.6: Comparison of open loop, $k = 0$ (---) and closed loop, $k = k_{opt}$ (—) response of the NNM dynamics subject to the initial conditions, $w_0 = M_{P2M}(z_0)$ where $z_0 = [0.01 \ 0.1 \ 0 \ 0]^T$. $(u_1, v_1)$ and $(u_2, v_2)$ are the flutter mode and damped mode states respectively.
Figure 4.7: Comparison of open loop, $k = 0$ (dashed) and closed loop, $k = k_{opt}$ (solid) state response and phase plane plot for the aeroelastic system subject to initial conditions, $z_0 = [0.01 \ 0.1 \ 0 \ 0]^T$. 
Table 4.3: Comparison of LCO amplitudes in pitch and non dimensional plunge degrees of freedom of the nonlinear aeroelastic system for open and closed loop cases.

<table>
<thead>
<tr>
<th>State</th>
<th>Open Loop</th>
<th>Closed Loop</th>
<th>% Reduction in amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Amplitude</td>
<td>Frequency</td>
<td>Amplitude</td>
</tr>
<tr>
<td>$\bar{h}$</td>
<td>0.0903</td>
<td>1.0245</td>
<td>0.00829</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.3414</td>
<td>1.0243</td>
<td>0.0304</td>
</tr>
<tr>
<td>$\dot{\bar{h}}$</td>
<td>0.0930</td>
<td>1.0245</td>
<td>0.00832</td>
</tr>
<tr>
<td>$\dot{\alpha}$</td>
<td>0.3470</td>
<td>1.0245</td>
<td>0.0305</td>
</tr>
</tbody>
</table>

LCO amplitude of flutter NNM and LCO amplitude estimate of flutter NNM obtained from HBM, $c_1$.

Using the time response of the displacements and velocities in pitch and non dimensional plunge degree of freedom, the nonlinear feedback control input signal can be easily computed. The control input for a section of simulation time when the aeroelastic system states are executing LCOs is shown in Fig. 4.8. It can be seen that after the system states start exhibiting LCOs, the nonlinear control input is essentially a periodic signal with its maximum absolute value being $5.7 \times 10^{-3}$ radians for the control parameter vector, $k = k_{opt}$.

4.10 Conclusions

A methodology to compute analytical estimates of LCO amplitudes and its sensitivities to the introduced control parameters is developed for a nonlinear aeroelastic system. The process involves the computation of nonlinear normal modes and use of harmonic balance method. The analytical estimates of sensitivity are shown to be similar/consistent with the numerically generated sensitivities using finite difference method. This is quite useful as the
Figure 4.8: The generated periodic control input using the nonlinear feedback control function with the control parameter vector, $k = k_{opt}$.

Sensitivities obtained using finite difference are prone to machine precision errors and requires a lot of computational effort for large scale systems. The analytical estimates of sensitivities are utilized to solve a multiobjective optimization problem which generates optimal control parameters to minimize LCO amplitudes of the flutter NNM. Numerical simulations are used to verify that minimizing the LCO amplitude of flutter NNM corresponds directly to minimizing the LCO amplitude of the aeroelastic system.

Higher order harmonic balance methods can be used to come up with more accurate prediction of the periodic solution, thereby giving more accurate estimates of LCO amplitude and their corresponding sensitivities.
Bibliography


Chapter 5

Nonlinear Control Design to Eliminate Subcritical LCOs in Aeroelastic Systems

5.1 Abstract

A strictly nonlinear state feedback control law is designed for an aeroelastic system to eliminate the presence of subcritical limit cycle oscillations. Numerical continuation techniques and harmonic balance methods are employed to generate analytical estimates of limit cycle oscillation commencement velocity and its sensitivity with respect to the introduced control parameters. The obtained estimates are used in a multiobjective optimization framework to generate optimal control parameters which minimize the difference between limit cycle oscillation commencement velocity and the linear flutter velocity along with an approximate measure of the control cost. Numerical simulations are used to show that the assumed nonlinear state feedback law with the optimal control parameters successfully eliminates any existing subcritical limit cycle oscillations by converting them to supercritical limit cycle os-
cillations thereby guaranteeing safe operation of the system in its operational flight envelope.

5.2 Introduction

Presence of nonlinearities have been deemed responsible for the existence of limit cycle oscillations (LCOs) in aeroelastic systems. Nonlinearities are an inherent part of an aeroelastic system typically arising from structural stiffness terms, unsteady aerodynamics and control surface effects such as actuator freeplay and deadzone. Lee et al. [1] and Dowell et al. [2] give a detailed review of structural and aerodynamic nonlinearities encountered in aeroelastic systems. Woolston et al. [3, 4] first investigated the effects of structural stiffness terms on the flutter of a wing in 1955. Their results indicated that when a nonlinear system becomes unstable, its flutter may become self-limited resulting in LCOs. Existence of limit cycle oscillations for velocities less than the linear flutter velocity, i.e., subcritical LCO has been known and verified theoretically as well as in experiments. Such LCOs are highly undesirable as they can cause ride discomfort and might lead to catastrophic failures even for flight conditions well within the flight envelope and hence need to suppressed.

A number of works in literature focus on controlling the amplitude of LCOs exhibited by nonlinear systems. This in general involves two major aspects i.) predicting the LCO amplitude ii.) using nonlinear control laws to change the nature of flutter boundary and thereby the amplitude of LCOs. Linear controllers have been employed to suppress limit cycle oscillations in aeroelastic systems but they can not guarantee stability in presence of large disturbances/ initial conditions. Moreover, they can lead to very high control inputs in regions where nonlinearities contribute dominantly to the system dynamics. A review of the advances in linear/nonlinear control for aeroelastic systems is given by Librescu and Marcozza [5]. Method of normal forms [6, 7, 8], harmonic balance methods [9, 10] and nonlinear normal modes [11, 12, 13, 14] have been employed in the past to predict the LCO ampli-
Shahrzad and Mahzoon [15] used method of normal forms, central manifold theory and harmonic balance method to forecast the flutter speed of a two dimensional wing and compared the result with those obtained using numerical integration. The comparison shows that results obtained using the method of normal forms and central manifold theory are satisfactory only in the proximity of the critical points while those obtained from harmonic balance can forecast flutter speed with a good accuracy.

Liu and Zhao [16] studied the bifurcations of airfoils in incompressible flow with nonlinear pitching stiffness by a harmonic balance method, an asymptotic expansion method, an averaging method and a numerical integration method. Lee et al. [17] used harmonic balance to come up with the conclusion that the soft spring property destabilizes the subcritical Hopf bifurcation in a 2-DOF airfoil system. Lee et al. [18] investigated the LCOs of a 2 degree of freedom airfoil with cubic nonlinearity in the restoring forces. Analytical expressions were derived for the amplitudes of the pitch and plunge motions.

Dimitriadis [10] presents a detailed review of various harmonic balance methods and a continuation framework to follow the response of nonlinear dynamic systems from the bifurcation point to any desired parameter value. The described methods were applied to a nonlinear aeroelastic system with polynomial or freeplay stiffness nonlinearity in the presence of a control surface. It is shown that high-order harmonic balance solutions accurately capture the complete bifurcation behavior of the system for both types of nonlinearity. Low-order solutions become inaccurate in the presence of multiple folds in the limit cycle oscillation branch but can still prove useful to obtain useful information at a much lesser computational cost than the cost of higher-order solutions. Guo et al. [19] used harmonic balance method and multivariable Floquet theory to analyze the LCOs of the airfoil.

The current work utilizes the fundamental harmonic balance and numerical continuation methods to estimate LCO amplitudes and LCO commencement velocity as a function of free stream velocity. These estimates are further used to compute analytical sensitivities of LCO amplitude and LCO commencement velocity to the introduced control parameters. A
nonlinear optimization framework with the objective to minimize the difference between the function chosen to make the LCO commencement velocity at least equal to the linear flutter velocity is solved using the analytical sensitivity estimates. To generate optimal control parameters an approximate measure of control cost is used as the second objective in a multiobjective optimization problem. Numerical simulations results are presented to verify that the generated optimal control parameters successfully eliminate any existing subcritical behavior in the open loop system resulting in a closed loop system with behavior analogous to supercritical behavior.

5.3 System Model

A variant of the two degree of freedom airfoil section equipped with a trailing edge flap is considered as the physical model as shown in Fig. 5.1. This is basically a flat plate supported by a spring with linear stiffness in plunge degree of freedom and nonlinear stiffness in pitch degree of freedom. The flat plate is free to move up and down along the plunge degree of freedom and rotate about the pitch degree of freedom.

The governing equations of motion can be obtained using Lagrange's equations and are stated as below

\begin{align}
    m_T \ddot{h} + m_W x_\alpha b \dot{\alpha} + c_h \dot{h} + k_h (h) h &= -L \\
    m_W x_\alpha b \dot{h} + I_\alpha \dot{\alpha} + c_\alpha \dot{\alpha} + k_\alpha (\alpha) \alpha &= M
\end{align}

(5.1)

where \( h \) and \( \alpha \) represent the displacements along plunge and pitch degree of freedom respectively. An overdot represents a derivative with respect to time. \( c_h \) and \( c_\alpha \) are damping coefficients while \( k_h \) and \( k_\alpha \) represents stiffness functions along plunge and pitch degrees of freedom respectively. \( m_W \) represents the mass of the wing while \( m_T \) represents the total mass of the system, i.e., wing and its support structure. \( b \) is the half chord length and \( x_\alpha \)
is the static imbalance, i.e., non dimensional distance between the center of mass and the elastic axis. $L$ and $M$ are the aerodynamic lift forces and moments per unit span respectively acting at the elastic axis. Quasi steady aerodynamics obtained by setting $C[k] = 1$ and neglecting apparent mass effects in the Theordorsen’s aerodynamics are used to model the aerodynamic lift forces and moments. The lift forces and moments per unit span are as stated below

\[ L = \rho U^2 b C_{L\alpha} \alpha_{\text{eff}} + \rho U^2 b C_{L\beta} \beta \]
\[ M = \rho U^2 b^2 C_{M\alpha} \alpha_{\text{eff}} + \rho U^2 b^2 C_{M\beta} \beta \] (5.2)

where

\[ \alpha_{\text{eff}} = \left\{ \alpha + \frac{\dot{h}}{U} + \left( \frac{1}{2} - a \right) b \frac{\dot{\alpha}}{U} \right\} \] (5.3)

$U$ represents the free stream velocity, $a$ represents the non dimensional distance from the mid chord to the elastic axis. $C_{L\alpha}$ and $C_{M\alpha}$ are derivatives of lift and moment coefficients.
with respect to angle of attack. Similarly, $C_{L\beta}$ and $C_{M\beta}$ are derivatives of lift and moment coefficients with respect to flap deflection, $\beta$. We choose linear stiffness in plunge and a polynomial type stiffness in pitch degree of freedom in the current work as given below

$$k_\alpha (\alpha) = k_{\alpha 0} + k_{\alpha 1} \alpha + k_{\alpha 2} \alpha^2$$  \hspace{1cm} (5.4)

The system parameters for the aeroelastic system used in the current work are stated in Table 5.1

### 5.3.1 Linear Analysis

Combining Eq. 5.1 and Eq. 5.2 the aeroelastic system can be represented in terms of mass, stiffness and damping matrices as below

$$M_0 \ddot{q} + C\dot{q} + K(q) \dot{q} = F_{qs}(q, \dot{q})$$

$$F_{qs}(q, \dot{q}) = F_q q + F_\dot{q} \dot{q} + F_c \beta$$  \hspace{1cm} (5.5)

where $q = [ h \quad \alpha ]^T$ and the other terms defined accordingly. A state vector, $z$ composed of displacements and velocities of plunge and pitch degrees of freedom is defined,

$$z = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h \\ \alpha \\ \dot{h} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$  \hspace{1cm} (5.6)

Choosing $z$ as the state vector, Eq. 5.5 which represents a set of two second order differential equations can be transformed into a system of four first order differential equations system represented as below

$$\begin{bmatrix} I_n & 0_{n \times n} \\ 0_{n \times n} & M_0 \end{bmatrix} \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0_{n \times n} & I_n \\ F_q - K(q) & F_\dot{q} - C \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0_{n \times 1} \\ F_c \end{bmatrix} \beta$$  \hspace{1cm} (5.7)
Table 5.1: System Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$-0.6847$</td>
<td></td>
</tr>
<tr>
<td>$m_T$</td>
<td>$12.387$</td>
<td>$kg$</td>
</tr>
<tr>
<td>$m_W$</td>
<td>$2.0490$</td>
<td>$kg$</td>
</tr>
<tr>
<td>$b$</td>
<td>$0.135$</td>
<td>$m$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$1.225$</td>
<td>$kg/m^3$</td>
</tr>
<tr>
<td>$x_\alpha$</td>
<td>$[0.0873 - (b + ab)]/b$</td>
<td></td>
</tr>
<tr>
<td>$I_\alpha$</td>
<td>$m_W x_\alpha^2 b^2 + 0.0517$</td>
<td>$kgm^2$</td>
</tr>
<tr>
<td>span</td>
<td>$0.6$</td>
<td>$m$</td>
</tr>
<tr>
<td>$C_{L\alpha}$</td>
<td>$6.28$</td>
<td></td>
</tr>
<tr>
<td>$C_{M\alpha}$</td>
<td>$(0.5 + a) C_{L\alpha}$</td>
<td></td>
</tr>
<tr>
<td>$C_{L\beta}$</td>
<td>$3.358$</td>
<td></td>
</tr>
<tr>
<td>$C_{M\beta}$</td>
<td>$-1.94$</td>
<td></td>
</tr>
<tr>
<td>$c_h$</td>
<td>$27.43$</td>
<td>$Ns/m$</td>
</tr>
<tr>
<td>$c_\alpha$</td>
<td>$0.036$</td>
<td>$Ns$</td>
</tr>
<tr>
<td>$k_h$</td>
<td>$2844.4$</td>
<td>$N/m$</td>
</tr>
<tr>
<td>$k_{\alpha 0}$</td>
<td>$6.833$</td>
<td>$Nm/rad$</td>
</tr>
<tr>
<td>$k_{\alpha 1}$</td>
<td>$9.967$</td>
<td>$Nm/rad^2$</td>
</tr>
<tr>
<td>$k_{\alpha 2}$</td>
<td>$667.685$</td>
<td>$Nm/rad^3$</td>
</tr>
</tbody>
</table>

where $n = 2$. The generalized eigenvalue problem based on the homogeneous linear part of the above equations can be used to construct root locus plots by varying the freestream velocity, $U$. From the root locus plot, the linear flutter velocity can be identified as the velocity at which at least a pair of complex conjugate eigenvalues become purely imaginary.

Fig. 5.2 shows the locus of eigenvalues as the freestream velocity is varied. It is clearly seen that at low values of $U$ all eigenvalues lie on the left side on the imaginary axis. However, as
the velocity is increased a pair of complex conjugate eigenvalues cross over to the right half plane. From Fig. 5.3 (a) the velocity at which the real part becomes zero is the linear flutter velocity and is equal to 9.124 m/s for the linearized aeroelastic system with the system parameters given in Table 5.1.

### 5.3.2 Open Loop behavior

To demonstrate the existence of subcritical LCOs, i.e., LCOs for velocities less than the linear flutter velocity, \( U < U_F \), the state response of the nonlinear aeroelastic system given in Eq. 5.7 is shown in Fig. 5.4 with the setting \( \beta = 0 \) and \( U = 0.9 U_F \) using Runge Kutta method for numerical integration. The initial condition used is \( z_0 = [ 0.01 \ 0.1 \ 0 \ 0 ]^T \). The state and phase plane response in Fig. 5.4 clearly shows that the system exhibits a periodic motion with constant amplitude and frequency.
5.4 Problem Statement

The linearized system of equations allows us to determine the flutter boundary [20] at which the aeroelastic instability occurs. However, linearized analysis is limited to cases when the perturbations are small depending on flight conditions. In order to determine the character of flutter boundary, either catastrophic or benign [21, 1], nonlinear equations of motion need to be employed owing to the considerable effects of nonlinearities in the vicinity of point of instability. In the case of benign flutter boundary the vehicle can be operated over the flutter velocity upto a certain limit without any catastrophic consequences. However in the case of catastrophic flutter boundary, the safety of vehicle cannot be guaranteed even for velocities which are less than the linear flutter velocity. Subcritical and supercritical LCOs are generally associated with catastrophic and benign flutter boundaries respectively [22, 23].

Fig. 5.5 shows the behavior of a general nonlinear aeroelastic system in the vicinity of the flutter boundary. Upto a certain velocity, $U_{LCO}$ referred to as LCO commencement velocity,
Figure 5.4: Open loop response and the phase plane plots of the aeroelastic system subject to initial conditions, $z_0 = [0.01 \ 0.1 \ 0 \ 0]^T$, denoted by $\times$ at a velocity of $U = 0.9U_F$. 
Figure 5.5: Behaviour of a general aeroelastic system in the vicinity of flutter boundary (adapted from Librescu and Marcozza [5]).

the system always exhibits a stable behavior marked as 1. This means irrespective of the initial conditions, the system always attains the equilibrium state in steady state motion. For velocities between $U_{LCO}$ and $U_F$, depending on the initial conditions the system can either exhibit stable behavior or start exhibiting subcritical LCOs as marked by 2. The dotted line in 2 represents unstable LCOs which can be attained only for short periods of time. The system jumps over to the top stable LCO branch or to the stable equilibrium state from the unstable LCO branch depending on the excitation. For velocities larger than $U_F$, if the system is exhibiting subcritical behavior it takes the branch 2 which has much higher amplitudes compared to branch 3 which is followed in the case of supercritical behavior.

Owing to the much higher amplitudes of stable LCOs in subcritical branch it corresponds to the case of catastrophic flutter while the lower amplitudes of supercritical branch ensures a benign flutter. Due to highly destructive effects of the flutter instability, LCO must not be allowed to occur within or on the boundary of operational envelope of a vehicle. In the current work, the aim is to eliminate any subcritical behavior present in the nonlinear
aeroelastic system. This is accomplished by using nonlinear state feedback controllers to alter the subcritical LCO path without altering the linear flutter velocity. A linear controller can always be used to change the linear flutter velocity, if required.

5.4.1 Solution Approach

Fig. 5.6 shows an outline of the solution approach used in the current work. A nonlinear state feedback function is used for controlling the trailing edge flap with the coefficients of nonlinear terms as the control parameters, \( k \). The harmonic balance (HB) method is employed to obtain a set of nonlinear algebraic equations \( (E) \) in terms of harmonic coefficients of the periodic solution along with the freestream velocity. Since a solution of the HB equations corresponding to the linear flutter velocity and with zero LCO amplitude is known, numerical continuation techniques are used to trace out the LCO boundary, i.e., LCO amplitude variation with the free stream velocity. Free stream velocity is expressed as a perturbation from the linear flutter velocity using a non dimensional parameter, \( \delta \), such that \( \delta > 0 \) implies velocities higher than the linear flutter velocity and vice versa. These HB equations are further employed to compute the LCO commencement velocity, \( U_{LCO} \) by differentiating the HB equations with respect to the harmonic coefficients corresponding to the LCO amplitude \( (amp) \) to obtain another set of nonlinear algebraic equations \( (S) \). These equations can be solved by using Newton’s method under the condition, \( \partial\delta/\partial(amp) \) to obtain an estimate of the LCO commencement velocity. Finally, HB equations are differentiated with respect to control parameters to obtain analytical sensitivities of \( U_{LCO} \) and the harmonic coefficients with respect to the introduced control parameters. The computed analytical sensitivities are used in a nonlinear optimization framework with the objective of minimizing \( (U_{LCO} - U_F)^2 \) and an approximate measure of control cost to generate optimal control parameters. Validation is done using numerical time integration to verify that no subcritical LCOs exist using the generated optimal control parameters for a set of randomly generated initial conditions.
Figure 5.6: Flowchart showing the solution approach involving the use of harmonic balance and numerical continuation methods.
The complete process is explained step by step in subsequent sections.

5.5 Estimation of LCO commencement velocity and amplitudes

In the present work, harmonic balance (HB) method in conjunction with numerical continuation is used to estimate LCO amplitudes as a function of freestream velocity. The main idea behind HBM is to balance the Fourier components of dominant harmonics assuming the existence of a periodic limit cycle. It is shown by Liu and Dowell [24] that a large number of harmonics are needed to accurately capture LCO amplitude and shape. The focus of the current work is on estimating LCOs amplitudes at velocities less than the linear flutter velocity, i.e., $U < U_F$. We use fundamental harmonic balance method (FHM) to generate an estimate of the frequency and amplitude of the LCOs. Liu and Dowell [9] give results comparing the amplitudes of LCO from time marching methods and FHM method for velocities less than the linear flutter velocity. FHM results are seen to closely match those generated by time marching methods. Harmonic balance methods highlighted in Lee et al. [18] are used to obtain pitch and plunge LCO amplitudes for the nonlinear aeroelastic system.

5.5.1 Harmonic balance method: Nonlinear Aeroelastic System

The second order differential equations for the nonlinear aeroelastic system given in Eq. 5.5 are represented as below

$$M_0\ddot{q} + C_0\dot{q} + K(q)q = F_0\beta$$

(5.8)
Substituting the control input, $\beta$ as a function of displacements and velocities of state variables, $\beta = \beta(q, \dot{q})$, the above equations can be represented as below

$$M_0\ddot{q} + C_0\dot{q} + K_0q + K_{nl}(q, \dot{q}) = 0 \quad (5.9)$$

where $K_{nl}$ is a $2 \times 1$ vector representing all the nonlinear terms coming from stiffness, aerodynamics or state feedback from the controller. To begin, the free stream velocity, $U$, is substituted as a perturbation from linear flutter velocity, $U_F$ using a non dimensional parameter, $\delta$ as below

$$U = (1 + \delta) U_F \quad (5.10)$$

Assuming a periodic solution exists for Eq. 5.9, $q = [h \quad \alpha]^{T}$ is approximated as a summation of harmonic functions.

$$q_1 = q_0 + \sum_{n=1}^{N} [q_{cn}\cos(n\omega t) + q_{sn}\sin(n\omega t)] \quad (5.11)$$

where $\omega$ is the unknown fundamental frequency, $q_0$, $q_{cn}$ and $q_{sn}$ are $2 \times 1$ vectors of unknowns representing the constant term, the coefficient of $n^{th}$ harmonic cosine and sine terms respectively in the assumed periodic solution. $N$ represents the order of approximation and is equal to the number of harmonic retained for approximating the periodic solution. Choosing $N = 1$ results in the first order harmonic balance scheme whereas $N > 1$ refers to higher order harmonic balance schemes [10]. The nonlinear term, $K_{nl}(q, \dot{q})$ are approximated using the well known relations of Fourier coefficients as below

$$K_{nl}(q_1, \dot{q}_1) = q_{0,nl} + \sum_{n=1}^{N} [q_{cn,nl}\cos(n\omega t) + q_{sn,nl}\sin(n\omega t)] \quad (5.12)$$
where

\[
q_{0, nl} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} K_{nl} (q_1, \dot{q}_1) \, dt \\
q_{cn, nl} = \frac{\omega}{\pi} \int_0^{2\pi/\omega} K_{nl} (q_1, \dot{q}_1) \cos (n\omega t) \, dt \\
q_{sn, nl} = \frac{\omega}{\pi} \int_0^{2\pi/\omega} K_{nl} (q_1, \dot{q}_1) \sin (n\omega t) \, dt
\]

(5.13)

The above relations are used to compute the harmonic approximations of the nonlinearities in terms of the harmonic balance variables introduced in Eq. 5.11. The assumed solution, along with the Fourier approximation of nonlinear terms are substituted in the system dynamics given in Eq. 5.9. For \( n = 1 \), the obtained relation is as below

\[
-M_0\omega^2 [q_{c1} \cos (\omega t) + q_{s1} \sin (\omega t)] + C_0\omega [-q_{c1} \sin (\omega t) + q_{s1} \cos (\omega t)] \\
+ K_0 [q_0 + q_{c1} \cos (\omega t) + q_{s1} \sin (\omega t)] + [q_{0, nl} + q_{c1, nl} \cos (\omega t) + q_{s1, nl} \sin (\omega t)] = 0
\]

(5.14)

From the above relation the constant term and the coefficients of sine and cosine terms are set to zero to obtain the set of equations stated below

\[
K_0 q_0 + q_{0, nl} = 0 \\
(K_0 - M_0\omega^2) q_{s1} - C_0\omega q_{c1} + q_{s1, nl} = 0 \\
(K_0 - M_0\omega^2) q_{c1} + C_0\omega q_{s1} + q_{c1, nl} = 0
\]

(5.15)

Since all harmonic balance variables, i.e., \( q_0, q_{c1} \) and \( q_{s1} \) are \( 2 \times 1 \) vectors, the above represents six equations in seven harmonic balance variables including the natural frequency, \( \omega \) and the free stream velocity parameter, \( \delta \). For a fixed value of \( \delta \), the problem of underdetermined system can be overcome without loss of generality by setting one of the harmonic coefficients, say, the second element of \( q_{s1} \) to zero. This process is called phase fixing. This reduces the system to six equations and six unknowns. Let the \( 2 \times 1 \) variables \( q_0, q_{c1} \) and \( q_{s1} \) be defined
as below

\[ q_0 = \begin{bmatrix} h_0 \\ \alpha_0 \end{bmatrix}, \quad q_{c1} = \begin{bmatrix} h_c \\ \alpha_c \end{bmatrix}, \quad q_{s1} = \begin{bmatrix} h_s \\ 0 \end{bmatrix} \]

In terms of the above defined variables, amplitude of LCO in \( \alpha \) and \( h \) is given by \( \alpha_c \) and \( \sqrt{h_c^2 + h_s^2} \). Representing the set of equations given by Eq. 5.15 in a shorthand form as below

\[ E_i [h_0, \alpha_0, h_c, \alpha_c, h_s, \omega, \delta] = 0 \quad \text{for} \quad i = 1, 2, ..., 6 \quad (5.16) \]

The above set of equations represent a set of parameterized algebraic nonlinear equations. Fixing one of the variables, Newton’s method can be used to solve for all the other variables.

### 5.5.2 Numerical Continuation

Having obtained a set of parameterized algebraic nonlinear equations as given in Eq. 5.16, numerical continuation is used to compute the harmonic balance coefficients for varying values of \( \delta \). Numerical continuation is a method of computing solutions of parameterized nonlinear equations [25]. These are solutions of the form \([u(s), \lambda(s)]\) for the parameterized nonlinear set of equations represented below

\[ G(u, \lambda) = 0, \quad u, G \in R^n, \quad \lambda \in R \quad (5.17) \]

where \( s \) denotes some parametrization. It is assumed that \( G \) is sufficiently smooth. Depending on the value of the parameter, \( \lambda \), the solution \( u \) varies as dictated by Eq. 5.17.

It is assumed that the solution \( u_0 \) of the system of equations is known for a given value of parameter, \( \lambda_0 \). Numerical continuation deals with how the solution varies under small changes in the parameter, \( \lambda \) from \( \lambda_0 \) to \( \lambda_0 + \Delta \lambda \). The basis of continuation methods is laid
by the Implicit Function Theorem (IFT) which states that for nearby solutions of the form, 

\[ u_0 + \Delta u \]  

to exist for parameter value \( \lambda_0 + \Delta \lambda \), non singularity of the Jacobian of \( G \) with respect to \( u \) at \( (u_0, \lambda_0) \) is a necessary condition [26].

\[
\det \left[ \frac{\partial G}{\partial u} \right]_{(u_0, \lambda_0)} \neq 0 \tag{5.18}
\]

If the above condition is satisfied, IFT guarantees the existence of a unique local map, 

\[ u = u(\lambda) \]

in a neighborhood of \( (u_0, \lambda_0) \) satisfying \( G(u(\lambda), \lambda) = 0 \).

Natural parameter continuation and pseudo arc length continuation methods are the two basic methods used for continuation problems introduced by Keller [27, 28]. Both the methods assume knowledge of a known solution and the tangent vector at the known solution. Parameter continuation uses the natural parameter, \( \lambda \) as the continuation parameter and requires the Jacobian, \( \partial G/\partial u \) to be non singular at each point on the solution path. This condition gets violated at a simple quadratic fold. This is a major disadvantage of using parameter continuation method which lead to the development of arc length continuation methods. Arc length method overcomes this issue by choosing arc length as the continuation parameter instead of the natural parameter. For a detailed review on theory and methods of numerical continuation user is directed to works of Keller [27, 28, 29].

In the current work, we use Moore Penrose numerical continuation to generate the solution path of parameterized nonlinear algebraic equations.

**Moore-Penrose numerical continuation**

Moore-Penrose continuation method [25] is slightly different from the pseudo arc length continuation method. This section reviews the Moore Penrose continuation methods as described in the MATCONT continuation package manual [30]. The method involves use of the Moore Penrose inverse. For a matrix \( A \in N \times (N + 1) \) with maximal rank, the Moore Penrose inverse is given by

\[
A^{\dagger} = \left( A^{\text{T}} A \right)^{-1} A^{\text{T}}
\]
Penrose inverse is defined as $A^+ = A^T (AA^T)^{-1}$. Consider the linear system

$$\begin{align*}
Ax &= b \\
v^T x &= 0
\end{align*}$$

(5.19)

where $x$ is a point on curve and $v$ is its tangent vector with respect to $A$, i.e., $Av = 0$. A solution of the above system can be easily identified in terms of Moore Penrose inverse as $x = A^+ b$.

Defining $x = [u \quad \lambda]^T$, we can write Eq. 5.17 in the form

$$G(x) = 0, \quad G : \mathbb{R}^{n+1} \to \mathbb{R}^n$$

(5.20)

which does not differentiate between state and parameter. A solution path to the Eq. 5.20 is denoted as $x(s)$. It is assumed we know a solution, $x_i$ and the tangent vector, $v_i$ to the solution path on the known point. A prediction of the next point is given as below

$$X^0 = x_i + (\Delta s) v_i$$

(5.21)

where $\Delta s$ represents the step size. The solution, $x$ is the point on the curve which is closest to $X^0$. This is equivalent to solving the system

$$\begin{align*}
G(x) &= 0 \\
w^T (x - X^0) &= 0
\end{align*}$$

(5.22)

where $w$ is the tangent vector at point $x$. Expanding the above equations about $X^0$ using Taylor series and neglecting the higher order terms, we obtain

$$\begin{align*}
G(x) &= G(X^0) + G_x (X^0) (x - X^0) \\
w^T (x - X^0) &= v^T (x - X^0)
\end{align*}$$

(5.23)
Comparing the above system of equations to that in Eq. 5.19, a solution of Eq. 5.23 in terms of Moore Penrose inverse can be given as below

$$x = X^0 - G_x^+ (X^0) G (X^0)$$  \hspace{1cm} (5.24)

The null vector of $G_x(X^0)$ is approximated by $V^0 = v_i$, the tangent vector at $x_i$. This is equivalent to solving $F(x) = 0$ in a hyperplane perpendicular to the previous tangent vector. Fig. 5.7 shows the geometrical representation of the Moore Penrose continuation method.

Figure 5.7: Graphical interpretation of Moore Penrose inverse continuation technique.

### 5.5.3 Application to Aeroelastic System

Moore Penrose inverse algorithm is used to generate the solution path corresponding to the equations given in Eq. 5.16. The solution of above set of equations corresponding to $\delta = 0$
for any fixed set of control parameters is known and is stated below

\[ x_0 = \begin{bmatrix} h_0 & \alpha_0 & h_c & \alpha_c & h_s & \omega & \delta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & \omega_F & 0 \end{bmatrix} \] (5.25)

where \( \omega_F \) represents the LCO frequency at the linear flutter velocity (\( \delta = 0 \)) which can be computed easily from the root locus plots. The tangent vector, \( v_0 \) can be easily solved from the following equations.

\[ \begin{bmatrix} \frac{\partial E}{\partial x} \end{bmatrix} x_0 = 0 \quad \text{and}, \quad ||v_0|| = 1 \] (5.26)

where \( x \) is a vector of unknowns defined as \( x = [ h_0 \ \alpha_0 \ h_c \ \alpha_c \ h_s \ \omega \ \delta ]^T \).

Figure 5.8: Comparison between the LCO boundary generated by Moore Penrose inverse continuation (\( \Delta s = 0.01 \)) and the LCO amplitudes obtained numerically using time integration for the open loop aeroelastic system.

Fig. 5.8 shows the variation of \( \alpha_c \) against \( \delta \) for the open loop system with the continuation process initiated with known solution \( x_0 \) and the corresponding tangent vector \( v_0 \). Negative
and positive values of $\delta$ correspond to velocities lower and higher than the linear flutter velocity respectively. The generated LCO boundary is compared with the LCO amplitudes obtained numerically using simulation results. It is clear that numerical results cannot capture the unstable LCOs while these can be easily predicted using harmonic balance method. The amplitudes for the stable LCOs are seen to match closely.

### 5.5.4 Determination of LCO commencement velocity

Having generated the LCO boundary using Moore Penrose inverse continuation, the next step is to compute the LCO commencement velocity, $U_{LCO}$ for the nonlinear aeroelastic system. It can be clearly seen from Fig. 5.8 that $U_{LCO}$ corresponds to a point of inflection on the curve where

$$\frac{d\delta}{d\alpha_c} = 0 \quad (5.27)$$

It is to be noted that another point on the curve satisfies the above condition. This second point corresponds to the known solution, $x_0$ stated in Eq. 5.25.

Differentiating each equation given in Eq. 5.16 with respect to $\alpha_c$ using chain rule we obtain the following equations

$$S_i \left[ h_0, \alpha_0, h_c, \alpha_c, h_s, \delta, \omega, h_0', \alpha_0', h_c', h_s', \delta', \omega' \right] = 0 \quad (5.28)$$

where $(\cdot)'$ represents $d(\cdot)/d\alpha_c$ and $i = 1, 2, \ldots, 6$. To compute $U_{LCO}$, we substitute $\delta' = 0$ in Eq. 5.28 to obtain the following relation

$$H_i \left[ h_0, \alpha_0, h_c, \alpha_c, h_s, \delta, \omega, h_0', \alpha_0', h_c', h_s', \omega' \right] = 0 \quad (5.29)$$
Figure 5.9: Determination of LCO commencement velocity \( (U_{LCO}) \) using the sensitivity equations and the solution from Moore Penrose inverse continuation method for the open loop aeroelastic system.

Eq. 5.16 and Eq. 5.29 together represent 12 equations in 12 variables which can be solved using Newton’s method. The \( \delta \) value in the obtained solution corresponds to \( U_{LCO} \).

Figure 5.9 shows a plot of \( \partial \delta / \partial \alpha_c \) against \( \delta \). It is clear that there are two points, \( P_1 \) and \( P_2 \) which correspond to the condition given in Eq. 5.27. \( P_1 \) and \( P_2 \) correspond to \( \delta = 0 \) and \( \delta = -0.3107 \) respectively. The inset of Fig. 5.9 shows the position of points \( P_1 \) and \( P_2 \) on the LCO boundary. \( P_1 \) corresponds to the known solution, \( x_0 \) whereas \( P_2 \) corresponds to the LCO commencement velocity, \( U_{LCO} \).

To verify the obtained estimate of \( U_{LCO} \), numerical integration was done employing Runge Kutta method to find the \( \delta \) values corresponding to the \( U_{LCO} \). Figure 5.10 shows the LCO amplitudes of the open loop aeroelastic system for varying values of \( \delta \) subject to four different initial conditions which are multiples of \( z_0 = [ 0 \ 0.01 \ 0.1 \ 0 \ 0 ]^T \). It can be seen that at initial condition, \( z_0 \), the onset of LCO starts at \( \delta = -0.31 \). However if we decrease the initial condition by a factor of 2 and 100 the onset of LCOs is delayed to \( \delta = -0.18 \) and
Figure 5.10: LCO amplitude in $\alpha$ obtained by numerical integration using Runge Kutta method for initial conditions which are multiples of $z_0 = [0.01, 0.1, 0, 0]^T$ for the open loop aeroelastic system.
\[ \delta = 0 \] respectively. This is expected as by decreasing the initial conditions, the contribution of nonlinearities to the system dynamics is decreased. When subjected to initial condition \( z_0/100 \) the system behaves almost like a linear system attaining stable behavior up to the linear flutter velocity and exhibiting LCO for velocities greater than the linear flutter velocity. If the initial conditions are increased by a factor of 10, it is seen that the system starts exhibiting LCO at \( \delta = -0.31 \). On further increasing the initial conditions, it is seen that the value of \( \delta \) at which system starts exhibiting LCOs remains equal to \(-0.31\). Hence, it can be concluded that \( \delta = -0.31 \) corresponds to \( U_{LCO} \).

Comparing the \( \delta \) values corresponding to \( U_{LCO} \) obtained using the sensitivity equations, \( \delta = -0.3107 \) and the value obtained numerically, \( \delta = -0.31 \), it is seen that the approach described gives an accurate analytical estimate of \( U_{LCO} \).

### 5.6 Closed loop system analysis

A methodology to generate the flutter boundary and compute \( U_{LCO} \) for the open loop system is presented in the above section. To analyze the closed loop system, a nonlinear state feedback control law is assumed. A feedback law composed of cubic terms with all possible combinations of the state variables is considered below

\[
\beta (x) = k_1 h^3 + k_2 \alpha^3 + k_3 \dot{h}^3 + k_4 \alpha \dot{h} + k_5 h^2 \alpha + k_6 h^2 \dot{h} + k_7 \alpha \dot{h} + k_8 \alpha^2 h + k_9 \alpha^2 \dot{h} + k_{10} \alpha^2 \dot{\alpha} + k_{11} \dot{h}^2 h + k_{12} \dot{h}^2 \alpha + k_{13} h^2 \alpha + k_{14} \dot{\alpha}^2 h + k_{15} \dot{\alpha}^2 \alpha + k_{16} \dot{\alpha}^2 \dot{h} \quad (5.30)
\]

where \( k_i \) are the coefficients for the nonlinear state feedback terms, i.e., the control parameters. It is to be noted that the objective is to convert subcritical LCOs to supercritical LCOs which involves altering the nature of LCO boundary without changing the linear flutter velocity. This is the reason why no linear terms are considered in the state feedback control law. For fixed values of \( k_i \) the LCO boundary can be easily computed as shown in
the previous section. The harmonic balance equations and the results obtained from it get modified due to the control parameters as shown below

\[
E_i [h_0, \alpha_0, h_c, \alpha_c, h_s, \delta, \omega, k] = 0 \\
H_i [h_0, \alpha_0, h_c, \alpha_c, h_s, \delta, \omega, h'_0, \alpha'_0, h'_c, \alpha'_c, h'_s, \omega', k] = 0
\]

(5.31)

where \( i = 1, 2, \ldots, 6 \) and \( k \) represents the set of control parameters. For fixed value \( k \) the above equations can still be solved to trace the LCO boundary as well as to compute the \( U_{LCO} \). Fig. 5.11 shows the affect of varying \( k_2 \) while keeping all other control parameters zero on the LCO boundary. It is clear that on decreasing \( k_2 \) the \( \delta \) value corresponding to \( U_{LCO} \) increases (magnitude decreases) shifting \( U_{LCO} \) closer to the linear flutter velocity, thereby increasing the operational flight envelope. Using numerical simulations it can be shown that irrespective of the initial conditions, the closed loop system with \( k_2 = -10 \) will not exhibit any LCOs for all values of \( \delta < -0.0947 \).

![Figure 5.11](image.png)

Figure 5.11: Plot showing that the control parameters (\( k \)) can be used to bring \( U_{LCO} \) close to the linear flutter velocity.

The goal of the current work is to come up with optimal control parameters which guarantee that there are no subcritical LCOs, i.e., \( U_{LCO} = U_F \) or make the \( \delta \) value corresponding
to $U_{LCO}$ zero. In the following sections, a nonlinear optimization framework is developed to formalize this process and generate optimal control parameters. Such an optimization framework will require the knowledge of sensitivities with respect to the control parameters. One approach is to use finite difference measures to compute these sensitivities numerically. However, computing numerical sensitivities is quite time consuming and prone to machine precision errors. A methodology to compute analytical estimates of sensitivities to the control parameters is developed in the subsequent section. These analytical sensitivities are then utilized in the nonlinear optimization framework to generate optimal control parameters.

### 5.6.1 Sensitivity to control parameters

A vector composed of all the variables in Eq. 5.31 is defined below

$$y = \begin{bmatrix} h_0 & \alpha_0 & h_c & \alpha_c & h_s & \delta & \omega & h'_0 & \alpha'_0 & h'_c & h'_s & \omega' \end{bmatrix}^T \in \mathbb{R}^{12}$$  \hspace{1cm} (5.32)

Eq. 5.31 can now be represented in the following form

$$F(y, k) = 0$$  \hspace{1cm} (5.33)

where $F \in \mathbb{R}^{12}$ represents a set of 12 equations as a function of $y$ and the control parameters, $k \in \mathbb{R}^{16}$. Differentiating these equations with respect to $k_j$ ($j = 1, 2, \ldots, 16$), we obtain the following relation

$$\left[ \frac{\partial F(y, k)}{\partial y} \right] \begin{bmatrix} \frac{\partial y}{\partial k_j} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F(y, k)}{\partial k_j} \end{bmatrix}$$  \hspace{1cm} (5.34)

The above relation is a linear equation in terms of the sensitivity vector, $\partial y/\partial k_j$. Eq. 5.34 can be solved for each $k_j$ to compute sensitivities of all the variables in $y$. At the point on LCO boundary corresponding to LCO commencement velocity, $U_{LCO}$, the sensitivity $\partial \delta/\partial k_j$
tells us how $U_{LCO}$ is affected by the control parameter, $k_j$. It is this sensitivity value which is further utilized in the optimization framework.

### 5.6.2 Nonlinear optimization framework

The objective of the optimization framework is to utilize the estimate of analytical sensitivities to generate optimal control parameters which convert any existing subcritical LCO behavior to supercritical LCO behavior while minimizing a measure of the approximate control cost. This is equivalent to eliminating the presence of LCOs for velocities less than the linear flutter velocity.

For subcritical LCOs, $\delta_{LCO}$ is always negative while for supercritical LCOs, $\delta_{LCO}$ is always zero. The multi-objective nonlinear optimization problem is formulated using the $\epsilon$-constraint method [31, 32] with the approximate control cost ($cc_{approx}$) as the primary objective and $[\delta_{LCO}]^2$ as the secondary objective represented in the form of an inequality constraint subject to bounds on the control parameter as represented below

$$k_{opt} = \min_k c_{cc_{approx}}(k)$$

s.t.  
$$[\delta_{LCO}(k)]^2 < \epsilon$$

$$lb \leq k \leq ub$$  \hspace{1cm} (5.35)

where $\epsilon$ is an optimization parameter, $k$ represents the vector of control parameters, $lb$ and $ub$ represent upper and lower bounds on the control parameter vector. Bounding the control parameters is necessary to prevent the aeroelastic system from becoming unstable. Computing these bounds is a lengthy and tedious process which involves ensuring that the overall stiffness of the closed loop system remains positive subject to variations in the state feedback variables. In the current work, we choose the following bounds for the control
The above bounds constrain the control parameters, \( k_4, k_{10} \) and \( k_{16} \) to zero. This is done because even a small independent change in these parameters introduces undesired characteristics in the LCO boundary. By undesired characteristics it is meant that the LCO boundary moves from the \( \delta > 0 \) region to \( \delta < 0 \) region. It can be seen in Fig. 5.12 that introduction of \( k_4, k_{10} \) and \( k_{16} \) causes the LCO boundary to turn back towards the region with \( \delta < 0 \) which is the region LCOs need to be eliminated from. On being coupled with other control parameters it is seen that the LCO boundary starts exhibiting multiple turns in the presence of \( k_4, k_{10} \) and \( k_{16} \). A general way around this problem will be to employ second derivatives and ensure they remain positive at all points on the LCO boundary for the optimal solution which is a direction for future work.

![Figure 5.12: LCO boundary generated by variation in the control parameter \( k_4, k_{10} \) and \( k_{16} \) for the nonlinear aeroelastic system.](image)

**Approximate control cost**

Choosing an approximate measure of the control cost involves considering the linear flutter mode at the linear flutter velocity. The eigenvector corresponding to the linear flutter mode
at $U_F$ normalized with respect to the $h$ coordinate is stated below

$$
\begin{bmatrix}
h \\
\alpha \\
h \dot{h} \\
\dot{\alpha}
\end{bmatrix} = (\gamma_r \pm i \gamma_i)
\begin{bmatrix}
1 \\
\lambda_r \pm i \lambda_i \\
\pm i \omega_f \\
\pm (\lambda_r \pm \lambda_i) i \omega_f
\end{bmatrix}
$$

(5.37)

Based on the normalized linear flutter modes, the amplitudes of the state variables can be approximated as below

$$
\begin{bmatrix}
A_h \\
A_\alpha \\
A_{\dot{h}} \\
A_{\dot{\alpha}}
\end{bmatrix} = h_0
\begin{bmatrix}
1 \\
\bar{\alpha} \\
\omega_f \\
\bar{\alpha} \omega_f
\end{bmatrix} = \sqrt{\gamma_r^2 + \gamma_i^2}
\begin{bmatrix}
1 \\
\sqrt{\lambda_r^2 + \lambda_i^2} \\
\omega_f \\
\omega_f \sqrt{\lambda_r^2 + \lambda_i^2}
\end{bmatrix}
$$

(5.38)

where the parameters $h_0$, $\bar{\alpha}$ and $\omega_f$ are defined in Table 5.2.

Table 5.2: Parameters used to determine the approximate control costs

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_0$</td>
<td>0.0071</td>
</tr>
<tr>
<td>$\bar{\alpha}$</td>
<td>10.2778</td>
</tr>
<tr>
<td>$\omega_f$</td>
<td>13.61</td>
</tr>
</tbody>
</table>

The approximate control cost is defined as

$$
cc_{approx} = \{k\}^T [Q] \{k\}
$$

(5.39)

where $Q = \text{diag} [d_1, d_2, ..., d_{16}] \in \mathbb{R}^{16 \times 16}$ is a diagonal matrix with the diagonal terms $(d_i)$
equal to the corresponding amplitude ratios as shown below

\[
\begin{bmatrix}
  d_1 \\
  d_2 \\
  d_3 \\
  d_4 \\
  d_5 \\
  d_6 \\
  d_7 \\
  d_8 \\
  d_9 \\
  d_{10} \\
  d_{11} \\
  d_{12} \\
  d_{13} \\
  d_{14} \\
  d_{15} \\
  d_{16}
\end{bmatrix}
= h_0^6
\begin{bmatrix}
  1 \\
  \bar{\alpha}^6 \\
  \omega_f^6 \\
  (\bar{\alpha}\omega_f)^6 \\
  (1^2\bar{\alpha})^2 \\
  (1^2\omega_f)^2 \\
  (1^2\bar{\alpha}\omega_f)^2 \\
  (\bar{\alpha}^21)^2 \\
  (\bar{\alpha}^2\omega_f)^2 \\
  (\bar{\alpha}^2\bar{\alpha}\omega_f)^2 \\
  (\omega_f^21)^2 \\
  (\omega_f^2\bar{\alpha})^2 \\
  (\omega_f^2\bar{\alpha}\omega_f)^2 \\
  (\bar{\alpha}^2\omega_f^21)^2 \\
  (\bar{\alpha}^2\omega_f^2\bar{\alpha})^2 \\
  (\bar{\alpha}^2\omega_f^2\omega_f)^2
\end{bmatrix}
\] (5.40)

5.7 Results

The sequential quadratic programming (SQP) algorithm in MATLAB is used to solve the optimization problem stated in Eq. 5.35. The estimates of analytical sensitivities obtained using Eq. 5.34 are utilized to solve the above problem while varying the parameter, \( \epsilon \).

Fig. 5.13 plots the variation of the costs used in the multiobjective optimization problem as \( \epsilon \) is varied. It is seen from the generated pareto front that a decrease in \( cc_{approx} \) is accompanied by an increase in \( \delta_{LCO}^2 \) which is expected in a multiobjective optimization problem. Fig. 5.14
Figure 5.13: Pareto front generated by varying $\epsilon$ between $3.162 \times 10^{-7}$ and 0.01.

Figure 5.14: Variation of the approximate control cost against $\delta_{LCO}$. 

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shows the variation of $cc_{\text{approx}}$ against the corresponding $\delta_{LCO}$ values. The $\delta_{LCO}$ value for the open loop system was computed earlier as $-0.3107$. It can be seen that for all the cases in Fig. 5.14, $|\delta_{LCO}| < 0.3107$ implying that the generated control parameters shift the $U_{LCO}$ closer to the linear flutter velocity. The controllers which correspond to $\delta_{LCO}$ values very close to zero can be effectively used to eliminate subcritical LCOs.

To check if the measure of control cost used to approximate the actual control cost is a good estimate, time simulations of the aeroelastic system using the generated optimal controllers at $\delta = -0.2$ subject to initial conditions, $z_0 = \begin{bmatrix} 0.01 & 0.1 & 0 & 0 \end{bmatrix}^T$ were utilized to generate the control input response. The control input response was used to compute the actual control cost as shown below

$$cc_{\text{actual}} = \int_0^T \beta(t)^T \beta(t) \, dt$$

(5.41)

Figure 5.15: Plot of normalized actual control cost computed at $\delta = -0.2$ versus the approximate control cost used in the optimization process.
Fig. 5.15 compares the normalized actual control cost ($\delta = -0.2$) with the approximate control cost obtained for the optimal controllers generated by varying $\epsilon$ between $3.162 \times 10^{-7}$ and 0.01. It is concluded that the approximate control cost used is a good measure based on the almost linear dependence seen in the plot. Fig. 5.16 plots the variation of the actual control cost, $cc_{actual}$ computed at $\delta = -0.2$ versus $\delta_{LCO}^2$.

![Figure 5.16: Variation of the actual control cost, $cc_{actual}$ calculated using numerical simulation at $\delta = -0.2$ against $\delta_{LCO}^2$.](image)

For $\epsilon = 3.162 \times 10^{-7}$ the optimization results in $\delta_{LCO} = -5.6234 \times 10^{-4}$ for the optimal
control parameters given below

\[
k_{opt} = \begin{bmatrix}
0.01726 \\
-12.6713 \\
0.8592 \\
0 \\
-0.1657 \\
0.0128 \\
0.9922 \\
1.7124 \\
4.0210 \\
0 \\
1.0762 \\
-2.3883 \\
0.0058 \\
0.3064 \\
-0.0231 \\
0
\end{bmatrix}
\] (5.42)

The very small negative value of \( \delta_{LCO} \) corresponding to \( k_{opt} \) tells us that the \( U_{LCO} \) for the closed loop system is almost equal to the linear flutter velocity. This can be verified from the Fig. 5.17 which shows the LCO boundary for the closed loop system using \( k = k_{opt} \). The inset in Fig. 5.17 shows that there are still two points \( P_1 \) and \( P_2 \) which satisfy \( \partial \delta / \partial c_\alpha = 0 \). \( P_1 \) is the known solution used to initiate the continuation algorithm while \( P_2 \) is the point corresponding to the closed loop \( U_{LCO} = -5.6234 \times 10^{-4} \). This behavior of the closed loop system is almost analogous to supercritical behavior.

To verify that subcritical behavior of the aeroelastic system is successfully eliminated, time marching using Runge Kutta method is done for varying values of \( \delta \) subject to different initial conditions. Fig. 5.18 shows the numerically estimated LCO amplitude in \( \alpha \) for the closed
Figure 5.17: LCO boundary for the closed loop aeroelastic system using the optimal control parameter vector, \((k = k_{opt})\)

loop aeroelastic system for varying \(\delta\) values subject to four different initial conditions which are multiples of \(z_0 = \begin{bmatrix} 0.01 & 0.1 & 0 & 0 \end{bmatrix}\). It is seen that the system attains the equilibrium state for all \(\delta < 0\) in all the cases. For \(\delta > 0\) the system starts exhibiting LCOs which highlights the supercritical behavior of the closed loop aeroelastic system. From the Fig. 5.18 it can be concluded that the closed loop aeroelastic system does not exhibit subcritical behavior, i.e., no LCOs exist for velocities less than the linear flutter velocity.

Figure 5.19 compares the state response and phase plane plots of the open loop and closed loop aeroelastic system subject to initial condition, \(x_0 = \begin{bmatrix} 0.01 & 0.1 & 0 & 0 \end{bmatrix}\) for \(\delta = -0.05\). It is seen that the open loop response exhibits LCO which is expected as the open loop system subcritical behavior with \(\delta_{LCO} = -0.3107\). The closed loop response shows that the system attains the equilibrium state and no LCOs are observed. It can be concluded that the closed loop system will not exhibit LCOs for any value of \(\delta < -5.6234 \times 10^{-4}\).

Figure 5.20 shows the control input for the closed loop aeroelastic system subjected to initial
Figure 5.18: LCO amplitude in $\alpha$ for the closed loop aeroelastic system ($k = k_{opt}$) obtained by numerical integration using Runge Kutta method subject to initial conditions which are multiples of $z_0 = [0.01 \ 0.1 \ 0 \ 0]^T$. 
Figure 5.19: Comparison of the state response and phase plane plots of the open loop, $k = 0$ (---) and closed loop, $k = k_{opt}$ (---) aeroelastic system subject to the initial conditions, $z_0 = [0.01 \ 0.1 \ 0 \ 0]^T$ at $\delta = -0.05$. 
Figure 5.20: The required control input response for the closed loop system using the optimal control parameter vector, $k_{opt}$ when subjected to the initial conditions, $z_0 = [ 0.01 \ 0.1 \ 0 \ 0 ]^T$ at $\delta = -0.05$.

condition, $z_0 = [ 0.01 \ 0.1 \ 0 \ 0 ]^T$. The maximum absolute value of the control input is close to 0.013 radians.

5.8 Conclusion

A methodology employing fundamental harmonic balance method coupled with Moore Penrose inverse continuation is developed to obtain estimates of LCO amplitudes and LCO commencement velocity as a function of free stream velocity and the introduced control parameters. It is shown that even though the fundamental harmonic balance is not able to estimate the LCO amplitudes accurately it gives a good estimate of the LCO commencement velocity. Analytical estimates of sensitivities of LCO commencement velocity with respect to the control parameters are obtained to be used in a nonlinear optimization framework. The optimization framework generates the optimal control parameters for a nonlinear state feed-
back controller using a constrained multiobjective optimization problem which minimizes the magnitude of the difference between the LCO commencement velocity and the linear flutter velocity and an approximate measure of the control cost. This is equivalent to enhancing the flight envelope by converting any subcritical LCOs to supercritical LCOs. State response generated from time integration using Runge Kutta method is shown to verify the absence of LCO for velocities less than the linear flutter velocity for an initial conditions for which the open loop system exhibits LCOs.

The estimate of LCO amplitude and the sensitivities can be further improved with the use of higher order harmonic balance methods, the extension to which is straightforward. Moreover curvature constraints on the LCO boundary can be employed during the optimization process to eliminate the presence of multiple turns or points of inflection in the closed loop LCO boundary.
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Chapter 6

Conclusions and Future Work

This chapter summarizes the main contributions of this work along with a discussion on the future research directions.

6.1 Conclusions

Aeroelastic systems being inherently nonlinear exhibit limited amplitude flutter or limit cycle oscillations. LCOs are known to exist for velocities both lower and higher than the linear flutter velocity making them highly undesirable. Use of nonlinear controllers for aeroelastic systems to suppress instability is driven by the failure of linear controllers to guarantee the stability of closed loop system dynamics for initial conditions far from the operating point. The current research addresses the importance of involving nonlinear analysis methods in designing state feedback controllers to avoid instabilities in the operational flight envelope. The analysis methods used in this research along with the developed control law methodologies are summarized below.

1. A feedback linearization controller is developed using the modal coordinates instead
of the physical coordinates. The stability of the developed controller is established by guaranteeing the stability of the zero dynamics using an appropriate Lyapunov function. The use of modal coordinates enables us to avoid cancellation of any linear terms while choosing the feedback law to cancel the nonlinear terms. The state and control costs of the developed controller are much less when compared to the state and control costs of feedback linearization controller in physical coordinates. This is expected since the feedback linearization controller in physical coordinates involves cancellation of some linear terms as well and is not optimal in the linear limits. The use of modal coordinates makes the method easier to extend to higher order aeroelastic systems which are generally represented in modal coordinates.

2. The work highlights the effectiveness of using NNM as a tool to model the response of aeroelastic system with lesser number of equations. It is shown that fundamental harmonic balance method can be used to generate accurate estimates of LCO amplitudes along with corresponding sensitivities to the introduced control parameters. Optimal controllers are generated using a multi-objective optimization framework as a function of the weighing parameter. The generated pareto front can be utilized to choose the weighing parameter to obtain a desired LCO amplitude of the closed loop aeroelastic system. The work establishes that the LCO amplitude of the flutter NNM directly corresponds to the LCO exhibited by the aeroelastic system.

The methodology to obtain analytical estimates of sensitivities can be used as a substitute of sensitivities obtained from finite difference approach to save computational effort and time when using optimization for large scale aeroelastic systems.

3. Limit cycle oscillations in aeroelastic systems commence at velocities higher than the limit cycle oscillation commencement velocity, $U_{LCO}$. A methodology to estimate $U_{LCO}$ and its sensitivity to control parameters for nonlinear aeroelastic systems is developed. The approach involves the use of numerical continuation techniques and fundamental harmonic balance method. Generating numerical estimates of $U_{LCO}$ involves conduct-
ing a large number of simulations with varying velocities subject to a number of initial conditions and hence is computationally extensive and time consuming.

The optimal controller obtained as a solution of a multi-objective optimization problem is shown to eliminate subcritical LCOs by converting them to supercritical LCOs, thereby resulting the extension of flight envelope. The highlighted approach can be used in design optimization to generate designs which don’t exhibit subcritical LCOs.

6.2 Future Work

The nonlinearities used in the current research are modeled as a polynomial which are typically associated with nonlinear stiffness terms. A direction of future work is to analyze the effect of other types of nonlinearities associated with control surfaces such as freeplay or backlash using the developed approaches.

The optimal controller developed to convert subcritical LCOs to supercritical LCOs can be improved and the analysis made more general by incorporating curvature constraints so as to avoid the existence of multiple points of inflections in the LCO boundary of the closed loop system dynamics.

The current research uses fundamental harmonic balance method to estimate the LCO amplitudes and sensitivities of the aeroelastic system. Use of higher order harmonic balance method can result in more accurate estimates of the LCO amplitude and its corresponding sensitivities to the introduced control parameters. Moreover, higher order harmonic balance methods can accurately capture multiple existing LCOs and secondary bifurcations in aeroelastic systems as discussed by Dimitriadis [1]. The extension of developed methods to higher order harmonic balance methods is straightforward and leads to new research areas such as bifurcation analysis and control.
In the current work a simple two degree of freedom airfoil section model was used to analyze the response characteristic of aeroelastic systems. The model used is a good start to identify and explore new analysis methods for aeroelastic systems. A direction of future work lies in extension of the developed framework towards more realistic aeroelastic systems such as a wing modeled as a beam. A highly flexible wing model as given by Tang and Dowell [2] and used by Stanford and Beran [3] for flutter and limit cycle computations along with structural optimization can be considered as the aeroelastic system.
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