Influence of cubic and dipolar anisotropies on the static and dynamic coexistence anomalies of the time-dependent Ginzburg-Landau models

U.C. TÄUBER and F. SCHWABL
Institut für Theoretische Physik
Physik-Department der Technischen Universität München
James-Franck-Str., D-W 8046 Garching, Germany

In isotropic systems below the transition temperature, the massless Goldstone modes imply critical infrared singularities in the statics and dynamics along the entire coexistence curve. We examine the important question whether these coexistence anomalies are of relevance also in more realistic systems displaying anisotropies. By applying a generalized renormalization scheme to the time-dependent Ginzburg-Landau models, we treat two quite different but characteristic cases, namely the influence of (i) weak cubic anisotropies, and (ii) long-range dipolar interactions. In the presence of cubic terms, the transverse excitations acquire a mass and thus one expects the theory to approach an uncritical "Gaussian" regime in the limit $\vec{q} \to 0$ and $\omega \to 0$. Therefore, we first consider the one-component case in order to show that our formalism also provides a consistent description of the crossover into an asymptotically Gaussian theory. In the case of (weak) cubic anisotropies, the fact that for $n < 4$ the fluctuations tend to restore the $O(n)$-symmetry at the critical point proves to be most important, since under these circumstances coexistence-type singularities may be found in an intermediate wavenumber and frequency range. The dipolar interaction induces an anisotropy in momentum space, which does not completely destroy the massless character of the transverse fluctuations, but only effectively reduces the number of Goldstone modes by one. Remarkably, similar to the isotropic case the asymptotic theory can be treated exactly. For $n \geq 3$ we find coexistence anomalies governed by the isotropic power laws. However, the amplitudes of the respective scaling functions depend on the angle between order parameter and external wavevector.

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I. INTRODUCTION

In this paper, we study the static and dynamic susceptibilities of the n-component time-dependent Ginzburg-Landau models (also called relaxational models A and B according to Hohenberg and Halperin's classification [1]) below the transition temperature, where in addition to an underlying $O(n)$-symmetry of the order parameter field two characteristic anisotropies are taken into account, namely (i) weak cubic anharmonic terms, and (ii) long-range dipolar interactions.

Our considerations have the following background. In the ordered phase of ideally isotropic systems, as a consequence of the spontaneously broken symmetry there appear $n - 1$ massless Goldstone modes which lead to infrared singularities in certain correlation functions for all temperatures below the critical temperature T_c. These so-called coexistence anomalies have been studied on the basis of several exponentiation and renormalization schemes for the static ϕ^4-model [2-9] as well as the dynamic relaxational models [5,10]. Lawrie's approach [8] to take into account both critical and Goldstone fluctuations within a suitable crossover theory, has recently been extended by the present authors to the $O(n)$-symmetric time-dependent Ginzburg-Landau models [11]. Generally, our calculations are based on the path-integral formulation of dynamical perturbation theory for Langevin-type equations of motion as developed by Bausch, Janssen, Wagner [12] and De Dominicis [13], supplemented by Amit and Goldschmidt's generalized renormalization procedure especially designed for crossover phenomena [14]. To our opinion, the formalism is rather transparent, the exact results in the coexistence limit can be easily implemented, and the connections to different approaches, e.g. the $1/n$-expansion [10,15,16], can be very clearly demonstrated. One of the major additional advantages is the possible applicability to even more complicated models. Some of the central results for the dynamic susceptibilities and correlation functions of the relaxational models A and B are listed in Table I. We also remark that lately a direct comparison with experimental findings of ultrasonic attenuation measurements within the incommensurate phase of the A_2BX_4-family crystals has been accomplished [17].

Having established a theory for ideally $O(n)$-symmetric systems, it is quite natural a question to ask whether the striking infrared anomalies found in the coexistence regime completely disappear when (weak) anisotropies are introduced, or certain remnants of the Goldstone excitations may still be experimentally relevant. This issue is even more important as for many cases, especially for structural and magnetic phase transformations with an underlying discrete lattice symmetry, the assumption of an isotropic order parameter field is certainly an idealization which is only approximately justified. On the other hand, one would expect a smooth transition of the physical properties from an $O(n)$-symmetric model system to a real, anisotropic crystal. Wallace [18] and Nelson [4] have noted that below but close to T_c even in weakly anisotropic sytems very strong fluctuations induced by quasi-Goldstone modes should be prominent. The dynamics in the high-frequency region of such cubic crystals in the ordered phase was studied by Meissner, Menyhárd, and Szépfalusy [19]. Our focus, however, will be the crossover from static and dynamic critical behavior into the hydrodynamic regime of vanishing external wavenumbers and frequencies.

The outline of our paper is as follows. In the following Section we briefly review the main results obtained for the $O(n)$-symmetric relaxational models below the transition temperature. The information presented here only serves for comparison with the chapters below. For a thorough explanation and discussion of the renormalization technique used, we refer the interested reader to Ref.[11]. By specializing to the case $n = 1$, we shall demonstrate in Section III that our procedure also provides a consistent description of a crossover to an asymptotically uncritical theory which can be described by a Gaussian model. This constitutes an inevitable preliminary, as the existence of cubic anisotropies in the free energy for the order parameter field will eventually (i.e.: for $\vec{q} \to 0$ and $\omega \to 0$) cause all the fluctuations to cease. Under certain circumstances, however, namely if in the vicinity of the critical point the $O(n)$-symmetry is dynamically restored, coexistence-type singularities may be found in an intermediate wavenumber and frequency region, as we are going to point out in Section IV. There, for $n < 4$ the crossover from the critical behavior to the asymptotically Gaussian model via a Goldstone-mode dominated regime will be discussed in detail, based on a one-loop approximation. In Section V we shall turn to our second generic example, namely long-range dipolar interactions introducing anisotropies in momentum space. Our investigation of the asymptotic theory, which remarkably can be treated exactly on very much the same basis as the isotropic model, shows that effectively the number of massless excitations is reduced by one. Hence for $n \geq 3$ coexistence anomalies persist, characterized by just the same power laws as summarized in Table I. However, the amplitudes of the associated scaling functions are generally lower, the reduction factors depending on the angle between the direction of the static order parameter and the external wavevector. At last we shall discuss and summarize our results. In the Appendix we list the diagrams and the corresponding analytical results for the two-point vertex functions up to one-loop order for the time-dependent Ginzburg-Landau models taking into account cubic terms.

II. COEXISTENCE ANOMALIES IN THE ISOTROPIC RELAXATIONAL MODELS

For convenience we start with a brief review of the static and dynamic properties of the $O(n)$-symmetric time-dependent Ginzburg-Landau models below the transition temperature. We shall merely repeat the basic definitions of the models under consideration and present the central results of the theory for later comparison with the anisotropic cases. Further details, especially concerning the dynamical perturbation expansion in the ordered phase and the generalized renormalization scheme used to describe the crossover behavior, may be found in our recent paper [11].

A. Model equations and perturbation theory

We are interested in the statics and dynamics near a second-order phase transition which phenomenologically can be described by an n-component order parameter field ϕ_0^α, $\alpha = 1, \ldots, n$, and the following $O(n)$-symmetric expansion of the Ginzburg-Landau free-energy functional ("Hamiltonian") with respect to ϕ_0^α,

$$H[\{\phi_0^\alpha\}] = \int d^d x \left[\frac{r_0}{2} \sum_{\alpha=1}^{n} \phi_0^\alpha(\vec{x})^2 + \frac{1}{2} \sum_{\alpha=1}^{n} \left[\vec{\nabla} \phi_0^\alpha(\vec{x}) \right]^2 + \frac{u_0}{4!} \left(\sum_{\alpha=1}^{n} \phi_0^\alpha(\vec{x})^2 \right)^2 \right] \quad . \tag{2.1a}$$

The parameter r_0 is proportional to the separation from the mean-field transition temperature T_c^0, and hence can be written as a sum of the fluctuation-induced T_c-shift r_{0c} and the reduced temperature variable with respect to the critical point T_c

$$r_0 = r_{0c} + \frac{T - T_c}{T} \quad . \tag{2.1b}$$

The positive coupling u_0 defines the strength of the isotropic anharmonic term. In thermal equilibrium, the probability density for a specific configuration $\phi_0^\alpha(\vec{x})$ is

$$P[\{\phi_0^\alpha\}] = \frac{e^{-H[\{\phi_0^\alpha\}]}}{\int \mathcal{D}[\{\phi_0^\alpha\}] \, e^{-H[\{\phi_0^\alpha\}]}} \quad . \tag{2.2}$$

To constitute the dynamics of our model we assume that (i) the only "slow" variable in the system is the order parameter itself, i.e.: there is no conserved quantity besides perhaps ϕ_0^α, and (ii) no reversible forces appear in the generalized Langevin equation, which is the simplest conceivable case. Therefore we have purely relaxational behavior where the damping $\propto \lambda_0$ may be considered as originating from the action of the "fast" degrees of freedom subsummed in the fluctuating forces ζ^α. Thus our basic equation of motion reads

$$\frac{\partial}{\partial t} \phi_0^\alpha(\vec{x},t) = -\lambda_0 \, (i \, \vec{\nabla})^a \, \frac{\delta H[\{\phi_0^\alpha\}]}{\delta \phi_0^\alpha(\vec{x},t)} + \zeta^\alpha(\vec{x},t) \quad . \tag{2.3}$$

Here two situations have to be distinguished, namely either there is no conservation law for the order parameter field, and the system simply relaxes into its equlibrium state after a distortion [$a = 0$ in Eq.(2.3)], or ϕ_0^α is a conserved quantity itself and hence follows a diffusion equation ($a = 2$). These two cases are referred to as model A or model B, respectively, according to the classification by Hohenberg and Halperin [1].

If one takes the probability distribution of the stochastic forces ζ^α to be of a Gaussian form, one finds for the first two moments

$$\langle \zeta^\alpha(\vec{x},t) \rangle = 0 \tag{2.4a}$$

$$\langle \zeta^\alpha(\vec{x},t) \, \zeta^\beta(\vec{x}',t') \rangle = 2 \, \lambda_0 \, (i \, \vec{\nabla})^a \delta^{\alpha\beta} \, \delta(\vec{x}-\vec{x}') \, \delta(t-t') \quad , \tag{2.4b}$$

which completes the definition of the time-dependent Ginzburg-Landau models. The Einstein relation (2.4b) ensures that the equilibrium probability density for a certain configuration $\phi_0^\alpha(\vec{x})$ is indeed given by Eq.(2.2).

Dynamical correlation functions are now calculated within the field-theoretical approach to generalized Langevin equations as developed by Bausch, Janssen, Wagner [12] and De Dominicis [13]. Starting from the assumed Gaussian distribution for the stochastic forces, one eliminates the ζ^α using the equation of motion (2.3). After a Gaussian transformation and the accompanying introduction of Martin-Siggia-Rose auxiliary fields $\tilde{\phi}_0^\alpha$, the probability density for the order parameter fluctuations ϕ_0^α finally reads

$$P[\{\phi_0^\alpha\}] = \int \mathcal{D}[\{i\tilde{\phi}_0^\alpha\}] \, P[\{\tilde{\phi}_0^\alpha\},\{\phi_0^\alpha\}] \propto \int \mathcal{D}[\{i\tilde{\phi}_0^\alpha\}] \, e^{J[\{\tilde{\phi}_0^\alpha\},\{\phi_0^\alpha\}]} \quad , \tag{2.5a}$$

where the statistical weight for a specific configuration is determined by the so-called Janssen-De Dominicis functional J. For the time-dependent Ginzburg-Landau models one finds

$$J[\{\tilde{\phi}_0^\alpha\},\{\phi_0^\alpha\}] = \int d^d x \int dt \sum_\alpha \left[\tilde{\phi}_0^\alpha \, \lambda_0 \, (i\,\vec{\nabla})^a \, \tilde{\phi}_0^\alpha - \tilde{\phi}_0^\alpha \left(\frac{\partial \phi_0^\alpha}{\partial t} + \lambda_0 \, (i\,\vec{\nabla})^a \, \frac{\delta H[\{\phi_0^\alpha\}]}{\delta \phi_0^\alpha} \right) \right] \; ; \tag{2.5b}$$

here we have omitted a term stemming from the functional derivative of the transformation from the ζ^α to the ϕ_0^α, because it exactly cancels those contributions to the perturbation series which violate causality [12].

Based on the path-integral formulation (2.5) for non-linear Langevin dynamics, a perturbation expansion with respect to the anharmonicity $\propto u_0$ and its diagrammatic representation by propagators and vertices can be constructed in complete analogy to the static case (see e.g. Ref.[20]). As usual, one defines a generating functional for N-point Green functions and cumulants by inserting source terms into (2.5b); furthermore, after a Legendre transform one may also obtain the vertex functions by appropriate functional derivatives. Details may be found in Refs.[20,12,11].

In order to evaluate the dynamic (linear) response functions within our formalism, we have to take into account the action of an external field $\tilde{h}^\alpha$. This leads to an additional term $-\sum_\alpha \tilde{h}^\alpha \, \phi_0^\alpha$ in the Hamiltonian (2.1a), and thus to a modified Janssen-De Dominicis functional [12]

$$J^{\tilde{h}}[\{\tilde{\phi}_0^\alpha\},\{\phi_0^\alpha\}] = J[\{\tilde{\phi}_0^\alpha\},\{\phi_0^\alpha\}] + \int d^d x \int dt \sum_\alpha \tilde{h}^\alpha \, \lambda_0 \, (i\,\vec{\nabla})^a \, \tilde{\phi}_0^\alpha \quad . \tag{2.6}$$

Hence the dynamic susceptibility is given by the expression

$$\chi_0^{\alpha\beta}(\vec{x},t;\vec{x}',t') = \frac{\delta \langle \phi_0^\alpha(\vec{x},t) \rangle}{\delta \tilde{h}^\beta(\vec{x}',t')} \bigg|_{\tilde{h}^\beta = 0} = \lambda_0 \, \langle \phi_0^\alpha(\vec{x},t) \, (i\,\vec{\nabla})^a \, \tilde{\phi}_0^\beta(\vec{x}',t') \rangle \quad , \tag{2.7a}$$

and its Fourier transform is intimately connected with the so-called response propagator

$$\chi_0^{\alpha\beta}(\vec{q},\omega) = \lambda_0 \, q^a \, G_{0\,\tilde{\phi}^\alpha \phi^\beta}(\vec{q},\omega) \quad . \tag{2.7b}$$

The dynamical correlation function can either be calculated directly or derived from the following (classical) fluctuation-dissipation theorem [12]

$$G_{0\,\phi^\alpha \phi^\beta}(\vec{q},\omega) = \frac{2\,\lambda_0 \, q^a}{\omega} \, \mathrm{Im}\, G_{0\,\tilde{\phi}^\alpha \phi^\beta}(\vec{q},\omega) \quad . \tag{2.8}$$

B. Renormalization in the ordered phase and coexistence anomalies

Below the critical temperature T_c a spontaneous order parameter $\bar{\phi}_0$ appears, which we assume to point in the n-th direction of the order parameter space. For convenience we define new transverse ($\alpha = 1, \ldots, n-1$) and longitudinal fields

$$\begin{pmatrix} \tilde{\phi}_0^\alpha \\ \tilde{\phi}_0^n \end{pmatrix} = \begin{pmatrix} \tilde{\pi}_0^\alpha \\ \tilde{\sigma}_0 \end{pmatrix} \quad , \qquad \begin{pmatrix} \phi_0^\alpha \\ \phi_0^n \end{pmatrix} = \begin{pmatrix} \pi_0^\alpha \\ \sigma_0 + \bar{\phi}_0 \end{pmatrix} \tag{2.9a}$$

with vanishing thermal average

$$\langle \pi_0^\alpha \rangle = \langle \sigma_0 \rangle = 0 \quad . \tag{2.9b}$$

In addition we shall use the parametrization

$$\bar{\phi}_0 = \sqrt{\frac{3}{u_0}}\, m_0 \quad , \tag{2.10}$$

where the quantity m_0 has the dimension of a mass or inverse length. By evaluating the condition (2.9b) for the longitudinal field one arrives at the equation of state

$$r_0 + \frac{m_0^2}{2} = A + \mathcal{O}(u_0^2) \quad , \tag{2.11}$$

which allows the elimination of the temperature variable r_0 in favor of the parameter m_0 [11,8]; to express the perturbation expansion in terms of m_0 has the considerable advantage that the T_c-shift is already properly included (compare the discussion in Ref.[21]).

From a general dynamic Ward-Takahashi identity [11] the following exact relation for the transverse two-point vertex function can be derived

$$\bar{\phi}_0 \, \Gamma_{0\,\tilde{\pi}\pi}(\vec{q} = 0, \omega = 0) = \tilde{h}^n \quad , \tag{2.12}$$

implying the massless character of the $n-1$ transverse excitations for vanishing external field $\tilde{h}^n \to 0$. These Goldstone modes induce infrared singularities that are not restricted to the vicinity of the critical point, but persist in the entire ordered phase.

Very remarkably, however, in contrast to the critical region ($m_0 \approx 0$) the coexistence regime $T < T_c$, $\vec{q} \to 0$, $\omega \to 0$, $\tilde{h}^n \to 0$ allows for an exact treatment [8,15,11]. Heuristically, the above limits may be replaced by fixed wavenumbers and frequencies and $m_0 \to \infty$ instead. The renormalization group analysis a posteriori confirms the assertion that the infrared behavior is indeed characterized by a diverging longitudinal mass parameter m. By introducing new longitudinal fields

$$\varphi_0(\vec{q},\omega) = m_0\,\sigma_0(\vec{q},\omega) + \frac{\sqrt{3\,u_0}}{6}\int_{q'}\int_{\omega'}\sum_{\alpha=1}^{n-1}\pi_0^\alpha(\vec{q}',\omega')\,\pi_0^\alpha(\vec{q}-\vec{q}',\omega-\omega') + \sqrt{\frac{3}{u_0}}\,A\,\delta(\vec{q})\,\delta(\omega) \tag{2.13a}$$

$$\tilde{\varphi}_0(\vec{q},\omega) = m_0\tilde{\sigma}_0(\vec{q},\omega) + \frac{\sqrt{3\,u_0}}{3}\int_{q'}\int_{\omega'}\frac{q'^a}{q^a}\sum_{\alpha=1}^{n-1}\tilde{\pi}_0^\alpha(\vec{q}',\omega')\,\pi_0^\alpha(\vec{q}-\vec{q}',\omega-\omega') \tag{2.13b}$$

(provided $\vec{q} \neq 0$ in the case of model B) and performing the limit $m_0 \to \infty$ — under the assumption $u_0/m_0^2 \to 0$ — one arrives at a purely harmonic asymptotic functional. (Here we have introduced the short-hand notation $\int_q\int_\omega \ldots = \frac{1}{(2\,\pi)^{d+1}}\int d^dq\int d\omega\ldots$.) In the coexistence regime merely transverse fluctuations exist, and the corresponding Goldstone propagators are identical to those from mean-field theory, as has been anticipated in earlier work [2,3,5]. On the other hand, by using (2.13) one finds for the (original) longitudinal correlation functions in the asymptotic limit [11] exact results (see Eq.(3.18) of Ref.[11]) that correspond to (i) the one-loop expressions for the two-point cumulants, and (ii) the leading order of the $1/n$-expansion for the two-point vertex functions [15,16], which can be summed up via a geometric series (see Fig.2 of Ref.[11]). In the asymptotic model the ultraviolet divergences can be removed by introducing renormalized counterparts m and u for the order parameter and coupling constant, respectively, where $m \propto u$ with a finite proportionality factor. Within the renormalized theory, the remaining infrared divergences are transcribed to an anomalous dimension of the mass m which eventually yields the correct behavior for low wavenumbers and frequencies.

Following Lawrie's work [8], we apply Amit and Goldschmidt's generalized minimal subtraction procedure [14] in order to describe the entire crossover from the critical behavior to the Goldstone regime. However, as we do not want to weaken the exact statements valid in the coexistence limit by introducing any further approximations, we refrain from ϵ-expansion [11], where

$$0 \leq \epsilon = 4 - d < 2 \tag{2.14}$$

is the difference to the upper critical dimension of the ϕ^4-model. Following the arguments of Schloms and Dohm [21], quite generally the extraction of the correct infrared behavior via the analysis of the ultraviolet divergences appearing as ϵ-poles within the dimensional regularization scheme does not require an expansion with respect to the distance from the upper critical dimension. The latter is needed if one wants to define a "small" expansion parameter in the perturbation series which can be avoided by a suitable resummation procedure. Our case is even simpler, though, for the theory asymptotically reduces to the one-loop expressions.

Neglecting the Fisher corrections connected with the static exponent η we thus use a one-loop approximation for the entire wavenumber and frequency region. On this level we can drop any field renormalizations; furthermore, because the Z-factor for the Onsager coefficient as a consequence of the fluctuation-dissipation theorem (2.8) can be expressed in terms of Z_ϕ and $Z_{\tilde{\phi}}$ [11,12] (see also Eq.(5.18) below), one finds $Z_\lambda = 1 + \mathcal{O}(u_0^2)$ for the relaxational models. Hence the only renormalized quantities we have to introduce are

$$m^2 = Z_m^{-1}\, m_0^2\, \mu^{-2} \tag{2.15a}$$

$$u = Z_u^{-1}\, u_0\, A_d\, \mu^{-\epsilon} \quad , \tag{2.15b}$$

where for convenience we have explicitly separated the naive dimensions and the geometric factor [21]

$$A_d = S_d\, \Gamma\left(3 - \frac{d}{2}\right)\, \Gamma\left(\frac{d}{2} - 1\right) = \frac{\Gamma(3 - \frac{d}{2})}{2^{d-2}\, \pi^{\frac{d}{2}}\, (d-2)} \quad . \tag{2.16}$$

From the explicit one-loop results for the Z-factors defined in (2.15) we proceed to calculate Wilson's flow functions

$$\zeta_m(u,m) = \mu \frac{\partial}{\partial \mu}\bigg|_0 \ln \frac{m^2}{m_0^2} = -2 + \frac{n-1}{6}\, u + \frac{3}{2}\, \frac{u}{(1+m^2)^{1+\epsilon/2}} \tag{2.17a}$$

$$\beta_u(u,m) = \mu \frac{\partial}{\partial \mu}\bigg|_0 u = u \left[-\epsilon + \frac{n-1}{6}\, u + \frac{3}{2}\, \frac{u}{(1+m^2)^{1+\epsilon/2}} \right] \quad . \tag{2.17b}$$

They enter the renormalization group equations for the cumulants, which are solved via the introduction of the characteristics $\mu(\ell) = \mu\, \ell$ with the result

$$G^c_{\tilde{N}N}(\mu, \lambda, m, u, \{\vec{q}\}, \{\omega\}) = G^c_{\tilde{N}N}(\mu\ell, \lambda, m(\ell), u(\ell), \{\vec{q}/\mu\ell\}, \{\omega/(\mu\ell)^2\}) \quad , \tag{2.18}$$

where the flow-dependent parameters are to be determined as solutions of the coupled first-order differential equations

$$\ell\, \frac{dm(\ell)}{d\ell} = \frac{1}{2}\, m(\ell)\, \zeta_m(\ell) \tag{2.19a}$$

$$\ell\, \frac{du(\ell)}{d\ell} = \beta_u(\ell) \tag{2.19b}$$

with the initial conditions $m(1) = m$, $u(1) = u$.

The fixed points of the renormalization group characterizing the asymptotic behavior are given by the zeros of the β-function (2.19b). At the critical point ($m = 0$) the Heisenberg fixed point

$$u_H^* = \frac{6\,\epsilon}{n+8} \tag{2.20a}$$

becomes stable, and the associated anomalous dimension for the mass parameter is

$$\zeta_{m\,H}^* = -2 + \epsilon \quad . \tag{2.20b}$$

On the contrary, in the limit $m \to \infty$ Lawrie's coexistence fixed point [8] is approached

$$u_C^* = \frac{6\,\epsilon}{n-1} \tag{2.21a}$$

$$\zeta_{m\,C}^* = -2 + \epsilon \quad ; \tag{2.21b}$$

hence indeed $m(\ell)^2 \propto \ell^{-2+\epsilon} \to \infty$ for $\ell \to 0$ in the range $0 \le \epsilon < 2$, and the above treatment of the asymptotic model becomes meaningful. For $d > 4$, the Gaussian fixed point is stable and the coexistence anomalies (as well as any non-trivial critical behavior) disappear. At the critical dimension $d_c = 4$ there will be logarithmic corrections. On approaching $d \to 2$, $m(\ell)$ no longer diverges and the assumption of a homogeneous long-range order parameter turns out to be inconsistent, which is to be expected according to the theorem by Mermin, Wagner [22], and Hohenberg [23].

In Ref.[11] approximate analytical as well as numerical solutions of the flow equations (2.19) are discussed in detail (see Fig.5 for $y = 0$). Here we just quote that because of $Z_m = Z_u$ the ratio

$$\frac{m(\ell)^2}{u(\ell)}\,\ell^{2-\epsilon} = \frac{m(1)^2}{u(1)} \tag{2.22}$$

is invariant under the renormalization group, and the important property that for small initial values $m(1) < 0.1$, the flows coincide when expressed in terms of the scaling variable

$$x = \frac{\ell}{m(1)^{2/(2-\epsilon)}} \quad , \tag{2.23}$$

which turns out to be the origin for the general scaling laws obeyed by the static and dynamic susceptibilities. In order to determine the scaling functions we finally have to use a matching condition fixing at least one of the arguments on the right-hand side of

(2.18b) at a sufficiently large value such that a conventional perturbation approach is applicable. In this context a very suitable choice will be

$$\ell^2 = \left| \frac{q^2}{\mu^2} - \frac{i\,\omega}{\lambda\,\mu^2\,(q/\mu)^a} \right| \quad , \tag{2.24}$$

i.e.: $\ell^2 \approx \mid \chi_T^{-1}(\vec{q},\omega) \mid$ is precisely identified with the distance to the critical surface of the Goldstone singularities.

Taking advantage of the scaling property (2.23) and inserting (2.24) into the perturbational expressions for the dynamic susceptibilities, one finds that the response functions are subject to the general scaling law

$$\chi_{T/L}^{-1}(\tau,\vec{q},\omega) = q^{2-\eta}\,\hat{\chi}_{T/L}^{-1}(\vec{q}\xi\,,\,\omega/\omega_c) \quad , \tag{2.25}$$

where

$$\xi^{-1} = \mu\, m(1)^{2/(2-\epsilon+\eta)} \propto \mid \tau \mid^{\nu} \tag{2.26a}$$

is the inverse static correlation length [remember that $m(1) \propto \bar{\phi} \propto \mid \tau \mid^{\beta}$ and the scaling relation $\nu = 2\,\beta/(2-\epsilon+\eta)$] with $\eta = 0$ to one-loop order, and

$$\omega_c = \lambda \mu^{2-z}\, \xi^z \propto \mid \tau \mid^{z\,\nu} \tag{2.26b}$$

denotes the characteristic frequency scale. The asymptotic power laws for the coexistence singularities are fixed by the anomalous dimension (2.21b); remarkably one finds to leading order for $\vec{q} \to 0$ and $\omega \to 0$

$$\chi_L(\vec{q},\omega) \propto \bar{\phi}^2 \left|\chi_T(\vec{q},\omega)\right|^{\epsilon/2} \quad . \tag{2.27}$$

This means that the longitudinal correlation function displays an anomalous line-shape, considerably stretched on the frequency scale in comparison with a Lorentzian, and corresponding to an algebraic decay of $G_L(\vec{x},t)$ for long times. Of course, Eq.(2.24) can be specialized to the static limit ($\omega = 0$) or the case $\vec{q} \to 0$, $\omega/q^a =$ const, for which the resulting coexistence anomalies are summarized in Table I. For more information about the crossover features compare Figs.7-9 and the discussion in Ref.[11].

III. CROSSOVER TO AN ASYMPTOTICALLY GAUSSIAN THEORY: THE CASE $n = 1$

In the case of a single-component order parameter there will be no massless modes in the low-temperature phase, of course, and hence infrared singularities are confined to the vicinity of T_c. However, one expects not only the critical region itself, but also the crossover into the asymptotic "Gaussian" regime to display a universal character. As an example, we therefore extrapolate our generalized renormalization scheme to the situation $T < T_c$ for $n = 1$, because the formalism allows the separation of the universal behavior from any features depending on the specific initial values of the flow parameters. This approach to the crossover towards a Gaussian model will turn out to be a necessary prerequisite for an appropriate discussion of the anisotropic n-component relaxational models.

Hence we put $n = 1$ in the flow equations (2.19) using the explicit one-loop results (2.17). Again, the m-dependent term describes the freezing-out of the fluctuations on leaving the critical region. For $m \to \infty$ only the mean-field contributions to Wilson's ζ- and β-functions remain, leading to a power-law behavior of the flow parameters according to their canonical dimensions

$$m(\ell) \propto \ell^{-1} \tag{3.1a}$$

$$u(\ell) \propto \ell^{-\epsilon} \quad . \tag{3.1b}$$

Thus the anharmonicity $u(\ell)$ diverges (see also Eq.(2.21a) for $n \to 1$); however, for large $m(\ell)$ the relevant effective coupling entering the flow functions vanishes according to

$$u_{eff}(\ell) = \frac{u(\ell)}{[1 + m(\ell)^2]^{1+\epsilon/2}} \propto \ell^2 \quad . \tag{3.2}$$

This eventually leads to mean-field exponents, which is precisely what should be expected for a Gaussian model. On the other hand, the loop-corrections to the scaling functions are of the order $u\, m^{-\epsilon} \propto \ell^0$, i.e.: a non-universal constant. Hence the asymptotic perturbation theory, although non-trivial with respect to the anharmonicity, does not contain any infrared singularities. Altogether, this seems to provide a consistent description for the crossover to an asymptotically uncritical regime.

These considerations can be confirmed by the investigation of an approximate solution of Eq.(2.19b) simply using (3.1a) for the complete crossover region. Instead of

Eqs.(4.26a),(4.27a) of Ref.[11] one finds for $\epsilon > 1$

$$u(\ell) = \frac{u\,\ell^{-\epsilon}}{1 + \frac{3\,u}{2\,\epsilon}\left[(m^2+\ell^2)^{-\epsilon/2} - (m^2+1)^{-\epsilon/2}\right]} \quad , \tag{3.3a}$$

while in the case $\epsilon = 0$ the result is identical to Eq.(4.27b) of Ref.[11] for $n = 1$

$$u(\ell) = \frac{u}{1 - \frac{3}{4}\,u\,\ln\frac{m^2+\ell^2}{m^2+1}} \quad . \tag{3.3b}$$

Hence at $d_c = 4$ even the logarithmic corrections in the ordered phase disappear, and anomalous exponents are confined to the immediate vicinity of the critical point. Note that according to (3.1a) $m(\ell)$ diverges for $\epsilon = 2$, too, which implies that the above mentioned inconsistency when assuming a phase with spontaneously broken continuous symmetry for $n \geq 2$ at two dimensions, does not appear in the Ising case. Of course, contrary to the systems where the Mermin-Wagner-Hohenberg theorem [22,23] applies, the two-dimensional Ising model displays a second-order phase transition at a finite temperature.

Fig.1a shows the numerical flow diagram for the effective coupling $u_{eff}(\ell)$ vs. the quantity $m(\ell)^2/[1+m(\ell)^2]$; compare with Fig.4 of Ref.[11]. Again, for $m(1) < 0.1$ one finds scaling behavior with respect to the variable x of Eq.(2.23), which is not surprising as this is essentially a property of the critical point. Hence for temperatures not too far away from T_c universal crossover features can be studied, see Fig.1b (and compare Fig.5 of Ref.[11]), starting from the Ising fixed point

$$u_I^* = \frac{2\,\epsilon}{3} \tag{3.4a}$$

$$\zeta_{m\,I}^* = -2 + \epsilon \tag{3.4b}$$

and terminating at the asymptotically stable Gaussian fixed point with

$$u_G^* = \infty \quad , \qquad u_{eff}^* = 0 \tag{3.5a}$$

$$\zeta_{m\,G}^* = -2 \quad . \tag{3.5b}$$

We illustrate the ensuing crossover within the $\vec{q}$- and ω-dependence of the correlation functions for the case of the static susceptibility. Using the numerical solutions for $u(\ell)$ and $m(\ell)$, and inserting $n = 1$ into Eq.(5.11b) of Ref.[11], one arrives at the one-loop cumulant (C). The corresponding vertex function (V) and the zero-loop expression follow by expanding with respect to u or putting $u = 0$, respectively. For comparison of the different results the effective exponent

$$2 - \eta_{eff} = \frac{\partial \ln \chi^{-1}(\vec{q}, 0)}{\partial \ln q} \tag{3.6}$$

is plotted in Fig.2. In all three cases shown, one finds similar features because grossly the flow of the parameter $m(\ell)$ describes most of the crossover behavior. For large values of q one has $\eta_{eff} = 0$ (to one-loop order), before eventually the static susceptibility becomes finite for $\vec{q} \to 0$. This is to be contrasted with the divergences induced by the transverse massless modes for the models with $n \geq 2$ (see Fig.7 with $y = 0$). We finally remark that our interpretation for the minimum of the effective exponent $2 - \eta^L_{eff}$ apparent for the time-dependent Heisenberg models at intermediate wavenumbers [11], namely that it is due to the freezing-out of the longitudinal fluctuations before the Goldstone modes become the predominant excitations, is quantitatively supported by Fig.2.

IV. THE INFLUENCE OF WEAK CUBIC ANISOTROPIES ON THE CRITICAL DYNAMICS OF MODELS A AND B BELOW T_c

A. Model and renormalization

We now start our investigations of the influence of characteristic anisotropies with the discussion of the crossover properties when cubic anharmonicities have to be taken into account. Thus we add the term

$$\Delta H_{cub}[\{\phi_0^\alpha\}] = \int d^d x \, \frac{v_0}{4!} \sum_{\alpha=1}^{n} \phi_0^\alpha(\vec{x})^4 \tag{4.1}$$

to the Hamiltonian (2.1a). Its static properties above and below T_c were examined by Wallace [18] and Aharony [24] on the basis of a $1/n$-expansion, and by Ketley and Wallace [25] using the ϵ-expansion. The phase diagram of the cubic model was described in more detail by Brézin, Le Guillou and Zinn-Justin [26], Nattermann and Trimper [27], Lyuksyutov and Pokrovsky [28], Rudnick [29], Iacobson and Amit [30]. For a summary of their main results we refer to Chap.4 of Ref.[20]. We shall restrict ourselves to the case $v_0 < 0$, for then the spontaneous order parameter stays along one of the principal axes of the n-dimensional hypercube. For stability reasons we furthermore have to require

$$b_0 = u_0 + v_0 > 0 \quad ; \tag{4.2a}$$

for $b_0 = 0$ the transition becomes first-order [27-30]. Finally we are only interested in a small deviation from the $O(n)$-symmetric case and hence tiny anisotropy parameters

$$0 \leq y_0 = \frac{-v_0}{b_0} \ll 1 \quad , \tag{4.2b}$$

such that the continuous character of the transition is guaranteed [28].

Eq.(2.3) again constitutes the Langevin dynamics of the system, supplemented by the moments (2.4) for the stochastic forces. For the spontaneous order parameter of Eq.(2.9a), however, we write

$$\bar{\phi}_0 = \sqrt{\frac{3}{b_0}}\, m_0 \tag{4.3}$$

instead of (2.10), and the fluctuation-induced T_c-shift as determined from Eq.(2.11) for $m_0 = 0$ acquires an additional contribution $\propto v_0$

$$r_{0c} = \left(\frac{n+2}{6\,\epsilon}\, u_0\, A_d + \frac{1}{2\,\epsilon}\, v_0\, A_d \right)^{2/\epsilon} \tag{4.4}$$

with respect to the isotropic case.

Hence in the ordered phase one finds for the cubic Janssen-De Dominicis functional $J = J_0 + J_{int} + J_{CT}$, where the harmonic part explicitly reads

$$\begin{aligned} J_0[\{\tilde{\pi}_0^\alpha\}, \tilde{\sigma}_0, \{\pi_0^\alpha\}, \sigma_0] = \int_q \int_\omega \Bigg[& \sum_\alpha \lambda_0\, q^a\, \tilde{\pi}_0^\alpha(\vec{q},\omega)\, \tilde{\pi}_0^\alpha(-\vec{q},-\omega) + \lambda_0\, q^a\, \tilde{\sigma}_0(\vec{q},\omega)\, \tilde{\sigma}_0(-\vec{q},-\omega) \\ & - \sum_\alpha \tilde{\pi}_0^\alpha(\vec{q},\omega) \left[i\omega + \lambda_0\, q^a\, (\overline{m}_0^2 + q^2) \right] \pi_0^\alpha(-\vec{q},-\omega) \\ & - \tilde{\sigma}_0(\vec{q},\omega) \left[i\omega + \lambda_0\, q^a\, (m_0^2 + q^2) \right] \sigma_0(-\vec{q},-\omega) \Bigg] \ , \end{aligned} \tag{4.5a}$$

while the interactions are given by

$$\begin{aligned} J_{int}[\{\tilde{\pi}_0^\alpha\}, \tilde{\sigma}_0, \{\pi_0^\alpha\}, \sigma_0] = -\frac{1}{6}\, \lambda_0 \int_{q_1 q_2 q_3 q_4} \int_{\omega_1 \omega_2 \omega_3 \omega_4} q_1^a\, \delta\Big(\sum_i \vec{q}_i \Big)\, \delta\Big(\sum_i \omega_i \Big) \times \\ \times \Bigg[\sum_{\alpha\beta} u_0\, \tilde{\pi}_0^\alpha(\vec{q}_1,\omega_1)\, \pi_0^\alpha(\vec{q}_2,\omega_2)\, \pi_0^\beta(\vec{q}_3,\omega_3)\, \pi_0^\beta(\vec{q}_4,\omega_4) \\ + \sum_\alpha v_0\, \tilde{\pi}_0^\alpha(\vec{q}_1,\omega_1)\, \pi_0^\alpha(\vec{q}_2,\omega_2)\, \pi_0^\alpha(\vec{q}_3,\omega_3)\, \pi_0^\alpha(\vec{q}_4,\omega_4) \\ + \sum_\alpha u_0\, \tilde{\pi}_0^\alpha(\vec{q}_1,\omega_1)\, \pi_0^\alpha(\vec{q}_2,\omega_2)\, \sigma_0(\vec{q}_3,\omega_3)\, \sigma_0(\vec{q}_4,\omega_4) \\ + \sum_\alpha u_0\, \tilde{\sigma}_0(\vec{q}_1,\omega_1)\, \pi_0^\alpha(\vec{q}_2,\omega_2)\, \pi_0^\alpha(\vec{q}_3,\omega_3)\, \sigma_0(\vec{q}_4,\omega_4) \end{aligned}$$

$$+ b_0\, \tilde{\sigma}_0(\vec{q}_1,\omega_1)\, \sigma_0(\vec{q}_2,\omega_2)\, \sigma_0(\vec{q}_3,\omega_3)\, \sigma_0(\vec{q}_4,\omega_4) \Big] \tag{4.5b}$$

$$- \frac{1}{6}\, \lambda_0 \sqrt{\frac{3}{b_0}}\, m_0 \int_{q_1 q_2 q_3} \int_{\omega_1 \omega_2 \omega_3} q_1^a\, \delta\left(\sum_i \vec{q}_i\right)\, \delta\left(\sum_i \omega_i\right) \times$$

$$\times \Big[\sum_\alpha 2\, u_0\, \tilde{\pi}_0^\alpha(\vec{q}_1,\omega_1)\, \pi_0^\alpha(\vec{q}_2,\omega_2)\, \sigma_0(\vec{q}_3,\omega_3)$$

$$+ \sum_\alpha u_0\, \tilde{\sigma}_0(\vec{q}_1,\omega_1)\, \pi_0^\alpha(\vec{q}_2,\omega_2)\, \pi_0^\alpha(\vec{q}_3,\omega_3) + 3\, b_0\, \tilde{\sigma}_0(\vec{q}_1,\omega_1)\, \sigma_0(\vec{q}_2,\omega_2)\, \sigma_0(\vec{q}_3,\omega_3) \Big] ,$$

and the counter-term stemming from the equation of state (2.11) finally is

$$J_{CT}[\{\tilde{\pi}_0^\alpha\}, \tilde{\sigma}_0, \{\pi_0^\alpha\}, \sigma_0] = -\lambda_0\, A \int_q \int_\omega q^a \Big[\sum_\alpha \tilde{\pi}_0^\alpha(\vec{q},\omega) \pi_0^\alpha(-\vec{q},-\omega) + \tilde{\sigma}_0(\vec{q},\omega) \sigma_0(-\vec{q},-\omega) \Big]$$

$$- \lambda_0 \sqrt{\frac{3}{b_0}}\, m_0\, A\, \tilde{\sigma}_0(0,0)\, \delta_{a,0} \quad . \tag{4.5c}$$

The most important difference to the isotropic model, which of course emerges as the special case with $v_0 = 0$ and $b_0 = u_0$, is the appearance of the transverse mass (remember $v_0 < 0$)

$$\overline{m}_0^2 = \frac{-v_0}{2 b_0}\, m_0^2 \tag{4.6}$$

in the harmonic functional (4.5a). The basic elements of the perturbation theory to be constructed from Eq.(4.5) and their graphic representations are depicted in Fig.3.

The ultraviolet divergences of the dynamic correlation functions are multiplicatively renormalized by introducing the Z-factors

$$m^2 = Z_m^{-1}\, m_0^2\, \mu^{-2} \tag{4.7a}$$

$$\overline{m}^2 = Z_{\overline{m}}^{-1}\, \overline{m}_0^2\, \mu^{-2} \tag{4.7b}$$

$$b = Z_b^{-1}\, b_0\, A_d\, \mu^{-\epsilon} \tag{4.7c}$$

$$v = Z_v^{-1}\, v_0\, A_d\, \mu^{-\epsilon} \tag{4.7d}$$

in analogy to (2.15). In the framework of a one-loop theory we may omit the field renormalizations and hence the Z-factor for the Onsager coefficient λ. The renormalization constants introduced in (4.7) are not independent; e.g. from the definition (4.3) one easily derives

$$Z_b = Z_m\, Z_\sigma \quad , \tag{4.8a}$$

with $Z_\sigma = 1$ to one-loop order. Similarly, (4.6) leads to

$$Z_b\, Z_{\overline{m}} = Z_v\, Z_m \quad . \tag{4.8b}$$

The renormalization constants (4.7a,b) are conveniently determined by rendering the quantities $\left(\partial_{q^2}\right)^{a/2}\, \Gamma_{\tilde\sigma\sigma}(\vec q,\omega)$ and $\left(\partial_{q^2}\right)^{a/2}\, \Gamma_{\tilde\pi\pi}(\vec q,\omega)$, respectively, finite at the normalization point $q=\mu$, $\omega=0$. Using the explicit expressions for the two-point vertex functions listed in the Appendix, the one-loop results are

$$Z_m = Z_b = 1 + \frac{n-1}{6\,\epsilon}\, \frac{(b_0-v_0)^2}{b_0}\, \frac{A_d\,\mu^{-\epsilon}}{\left(1+\overline{m}_0^2/\mu^2\right)^{\epsilon/2}} + \frac{3}{2\,\epsilon}\, \frac{b_0\,A_d\,\mu^{-\epsilon}}{\left(1+m_0^2/\mu^2\right)^{\epsilon/2}} \qquad (4.9a)$$

$$Z_{\overline{m}} = Z_v = 1 + \frac{1}{2\,\epsilon}\, \frac{4\,b_0-v_0}{2\,b_0+v_0}\, \frac{v_0\,A_d\,\mu^{-\epsilon}}{\left(1+\overline{m}_0^2/\mu^2\right)^{\epsilon/2}} + \frac{1}{\epsilon}\, \frac{4\,b_0-v_0}{2\,b_0+v_0}\, \frac{b_0\,A_d\,\mu^{-\epsilon}}{\left(1+m_0^2/\mu^2\right)^{\epsilon/2}} \quad . \qquad (4.9b)$$

Obviously, for $v_0=0$ (4.9a) reduces to Eq.(4.11c) of Ref.[11] for the isotropic model.

The ζ- and β-functions following from (4.9) read

$$\zeta_m = \mu\,\frac{\partial}{\partial\mu}\Big|_0\, \ln\frac{m^2}{m_0^2} = -2 + \frac{n-1}{6}\, \frac{(b-v)^2/b}{(1-\frac{v}{2\,b}\,m^2)^{1+\epsilon/2}} + \frac{3}{2}\, \frac{b}{(1+m^2)^{1+\epsilon/2}} \qquad (4.10a)$$

$$\beta_b = \mu\,\frac{\partial}{\partial\mu}\Big|_0\, b = b\left[-\epsilon + \frac{n-1}{6}\, \frac{(b-v)^2/b}{(1-\frac{v}{2\,b}\,m^2)^{1+\epsilon/2}} + \frac{3}{2}\, \frac{b}{(1+m^2)^{1+\epsilon/2}}\right] \qquad (4.10b)$$

$$\beta_v = \mu\,\frac{\partial}{\partial\mu}\Big|_0\, v = v\left[-\epsilon + \frac{4\,b-v}{2\,b+v}\left(\frac{v/2}{(1-\frac{v}{2\,b}\,m^2)^{1+\epsilon/2}} + \frac{b}{(1+m^2)^{1+\epsilon/2}}\right)\right] \quad , \qquad (4.10c)$$

and the flow-dependent couplings are given by the following set of ordinary diffential equations

$$\ell\,\frac{dm(\ell)}{d\ell} = \frac{1}{2}\, m(\ell)\,\zeta_m(\ell) \qquad (4.11a)$$

$$\ell\,\frac{db(\ell)}{d\ell} = \beta_b(\ell) \qquad (4.11b)$$

$$\ell\,\frac{dv(\ell)}{d\ell} = \beta_v(\ell) \quad , \qquad (4.11c)$$

with the usual initial conditions $m(1)=m$, $b(1)=b$ and $v(1)=v$. The renormalization group equations for the two-point cumulants are finally solved by [compare Eq.(2.18b)]

$$G^c_{\tilde N N}(\mu,\lambda,m,b,v,\{\vec q\},\{\omega\}) = G^c_{\tilde N N}(\mu\ell,\lambda,m(\ell),b(\ell),v(\ell),\{\vec q/\mu\ell\},\{\omega/(\mu\ell)^2\}) \ . \qquad (4.12)$$

B. Discussion of the flow equations and fixed points

The common zeros of the β-functions (4.10b,c) yield the fixed points governing the universal critical behavior

$$\beta_b(b^*,v^*,m) = 0 = \beta_v(b^*,v^*,m) \quad . \qquad (4.13)$$

Their regions of stability can be inferred by investigation of the matrix

$$\Omega = \begin{pmatrix} \partial\beta_b/\partial b & \partial\beta_b/\partial v \\ \partial\beta_v/\partial b & \partial\beta_v/\partial v \end{pmatrix}\Bigg|_{b=b^*,\, v=v^*} \tag{4.14}$$

(with fixed m): if both eigenvalues of Ω are positive, the fixed point is (infrared-)stable, one positive and one negative eigenvalue correspond to a saddlepoint, and two negative eigenvalues lead to instability.

We begin with the case $m = 0$, i.e.: the properties of the critical point $T = T_c$. Besides the Gaussian fixed point $b_G^* = v_G^* = 0$, which is unstable below $d_c = 4$ because both eigenvalues of Ω_G are equal to $-\epsilon$, one finds three non-trivial solutions of Eq.(4.13), namely (i) the isotropic Heisenberg fixed point (see Eq.(2.20a) of Section II.B)

$$b_H^* = \frac{6\,\epsilon}{n+8} \quad , \qquad v_H^* = 0 \tag{4.15a}$$

$$\Omega_H = \begin{pmatrix} 1 & \frac{2\,(n-1)}{n+8} \\ 0 & \frac{4-n}{n+8} \end{pmatrix} \epsilon \quad , \tag{4.15b}$$

(ii) the Ising fixed point [compare Eq.(3.4a)]

$$b_I^* = v_I^* = \frac{2\,\epsilon}{3} \tag{4.16a}$$

$$\Omega_I = \begin{pmatrix} 1 & 0 \\ -\frac{4}{3} & -\frac{1}{3} \end{pmatrix} \epsilon \tag{4.16b}$$

with saddlepoint-type behavior, and finally (iii) the cubic fixed point

$$b_A^* = \frac{2\,\epsilon}{3}\,\frac{n-1}{n} \quad , \qquad v_A^* = \frac{2\,\epsilon}{3}\,\frac{n-4}{n} \tag{4.17a}$$

$$\Omega_A = \begin{pmatrix} 5\,n-8 & 2\,(n-1) \\ 4\,(4-n) & 4-n \end{pmatrix} \frac{\epsilon}{3\,n} \quad , \tag{4.17b}$$

where Ω_A has the eigenvalues ϵ and $(n-4)\,\epsilon/3\,n$ [20 (and references therein)]. Therefore the situations $n \geq n_c$ and $n < n_c$ have to be distinguished, where to one-loop order $n_c = 4$ (in more elaborate calculations one finds that n_c is between 3 and 4) [25-27]. For $n \geq n_c$ the cubic fixed point (4.17a) is stable, and hence the anisotropy (4.1) constitutes a relevant perturbation of the $O(n)$-symmetric Hamiltonian (2.1); to the contrary, in the case $n < n_c$ the Heisenberg fixed point (4.15) becomes asymptotically stable, which means that the fluctuations dynamically restore the rotational symmetry. (An interpretation for this remarkable difference might be that in high-dimensional spaces most of the "volume" is concentrated in the vicinity of the surface, while for low dimensionality $n < n_c$ the detailed shape of the surface in configuration space is less relevant and the "edges" are effectively smoothed by the fluctuations.) For $n = 4$ Eqs.(4.15) and (4.17) are identical, and of course for $n = 1$ only the Ising fixed point (4.16) remains and becomes stable, compare Section III. In the limit $n \to \infty$ (4.16a) and (4.17a) coincide.

From (4.15)-(4.17) we can deduce the qualitative features of the flow diagrams, which are displayed in Fig.4 for the typical cases $n = 2$ and $n = 8$, respectively. In Fig.4a the attractivity of the Heisenberg fixed point "H" (4.15) is evident, whereas in Fig.4b the anisotropic cubic fixed point "A" (4.17) dominates the flow of the couplings. Those trajectories which are not located in the regime of the respective stable fixed point eventually cross the stability boundary $b = 0$ and the corresponding physical system will then display a first-order phase transition [28-30,20].

We can now turn to the phase with spontaneously broken (discrete cubic) symmetry, i.e.: the case $m \to \infty$. Apart from the (for $\epsilon > 0$) unstable Gaussian fixed point $b_G^* = v_G^* = 0$ there is now the isotropic coexistence fixed point [see (2.21a)]

$$b_C^* = \frac{6\,\epsilon}{n-1} \quad , \qquad v_C^* = 0 \tag{4.18a}$$

$$\Omega_C = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \epsilon \quad , \tag{4.18b}$$

which, however, in contrast to the isotropic model behaves like a saddlepoint in parameter space. As long as $v \neq 0$, we have asymptotically $\overline{m}(\ell) \to \infty$. Hence from (4.10) and (4.11) we conclude

$$m(\ell) \propto \ell^{-1} \tag{4.19a}$$

$$b(\ell) \propto \ell^{-\epsilon} \tag{4.19b}$$

$$v(\ell) \propto \ell^{-\epsilon} \quad , \tag{4.19c}$$

i.e.: the mass parameter and anharmonic couplings diverge in a manner completely analogous to what we have previously discussed for the case $n = 1$. Of course, a crossover to an asymptotically Gaussian theory should in fact be expected because the fluctuations are bound to cease eventually. This can be confirmed by studying the effective couplings

$$b_{eff}(\ell) = \frac{b(\ell)}{[1 + \overline{m}(\ell)^2]^{1+\epsilon/2}} \propto \ell^2 \tag{4.20a}$$

$$v_{eff}(\ell) = \frac{v(\ell)}{[1 + \overline{m}(\ell)^2]^{1+\epsilon/2}} \propto \ell^2 \quad , \tag{4.20b}$$

compare Section III and Fig.5. In particular, we want to point out that in the presence of cubic anharmonic terms the Mermin-Wagner-Hohenberg theorem [22,23] does not apply and hence long-range order is possible in two dimensions, which is reflected in the d-independent divergence of the mass parameter according to (4.19a) instead of the anomalous dimension (2.21b) of the isotropic model.

The fact that (to one-loop order) for $n < 4$ the $O(n)$-symmetry is dynamically restored at the critical point turns out to be most important for the flow of the effective coupling $b_{eff}(\ell)$ and hence the static and dynamic susceptibilities, at least for low initial values of the anisotropy parameter y of Eq.(4.2b). Ideally the system would acquire $v_H^* = 0$ at T_c leading to pure coexistence behavior as described in Section II. However, it is unstable towards any small disturbances in the v-direction followed by a crossover into the asymptotically Gaussian model. We may study this situation quantitatively by changing the initial value $y(1)$ in the numerical solution of the flow equations (4.10), (4.11). In Fig.5 the flow of the effective coupling (4.20a) vs. the scaling variable (2.23) is depicted for several values of $y(1)$; for comparison the crossover from the Heisenberg to Lawrie's coexistence fixed point in the $O(n)$-symmetric model is plotted, too. For very low anisotropy $b_{eff}(\ell)$ rises towards the coexistence fixed point (4.18), but finally approaches the Gaussian limit [see Eq.(3.5)]. The smaller $y(1)$, the higher is the maximum of $b_{eff}(\ell)$, and the later the crossover to $b_{eff}^* = 0$ takes place. In fact the approach to $b_{eff}(\ell) \to 0$ could be universally rescaled by using the "Gaussian scaling variable" $\ell/\overline{m}(1)$ instead of (2.23).

The large attracting region of the isotropic coexistence fixed point (4.18) becomes obvious from the qualitative flow diagram in Fig.6a (with $n = 2$; compare Fig.4a). For $n > 4$, on the other hand, for the allowed values $v < 0$ the Gaussian limit is immediately assumed, without any relics of the coexistence region, see Fig.6b. Even for very low values of $y(1)$ no quasi-Goldstone modes are of importance. We finally remark that the above considerations, especially for the critical region $m = 0$, are inferred from the one-loop approximation only. Thus by taking into account higher orders of the perturbation expansion modifications might be necessary. In particular, the number pf components n_c below which the Heisenberg fixed point is the attractive critical fixed point is shifted. It should be clear, though, how the properties of the critical point imply the asymptotic features for $T < T_c$ and $\vec{q} \to 0$, $\omega \to 0$.

C. Dynamic response and correlation functions

With the help of Eq.(4.12) we can now evaluate the dynamic two-point cumulants to one-loop order from the diagrams listed in the Appendix. We furthermore apply the general matching condition (2.24) designed to connect the low-wavenumber and low-frequency behavior with the asymptotic limit $\ell \to 0$ where the stable fixed points are approached. In the presence of cubic anisotropies the perturbation series cannot be truncated at the one-loop level, however, as opposed to the isotropic case. For this reason, and to avoid somewhat even lengthier expressions, we quote the results for the one-loop vertex functions which may be easily transformed into the cumulants for comparison with the $O(n)$-symmetric models [11]. Specializing (2.24) to $\omega = 0$, using Eq.(4.9) and the abbreviation

$$S(\ell) = \sqrt{1 + \frac{2\,b(\ell) - v(\ell)}{b(\ell)}\, m(\ell)^2 + \frac{[2\,b(\ell) + v(\ell)]^2}{4\,b(\ell)^2}\, m(\ell)^4} \quad , \tag{4.21}$$

the inverse static susceptibilities at $d = 3$ ($\epsilon = 1$) read (see Eq.(5.5) of Ref.[11])

$$\chi_T^{-1}(\vec{q},0) = \mu^2\,\ell^2 \Bigg[1 + \frac{-v(\ell)}{2\,b(\ell)}\, m(\ell)^2\, Z_{\overline{m}}(\ell) + \frac{2\,b(\ell) + v(\ell)}{6} \left(1 - \sqrt{\frac{-v(\ell)}{2\,b(\ell)}} \right) m(\ell) \tag{4.22a}$$
$$- \frac{[b(\ell) - v(\ell)]^2}{6\,b(\ell)} m(\ell)^2 \left(\arcsin \frac{1 - \frac{2\,b(\ell)+v(\ell)}{2\,b(\ell)}\, m(\ell)^2}{S(\ell)} + \arcsin \frac{1 + \frac{2\,b(\ell)+v(\ell)}{2\,b(\ell)}\, m(\ell)^2}{S(\ell)} \right) \Bigg] \Bigg|_{\ell = q/\mu}$$

and

$$\chi_L^{-1}(\vec{q},0) = \mu^2\,\ell^2 \Bigg[1 + m(\ell)^2\, Z_m(\ell) - \frac{3}{2}\, b(\ell)\, m(\ell)^2\, \arcsin \frac{1}{\sqrt{1 + 4\,m(\ell)^2}} \tag{4.22b}$$
$$- \frac{n-1}{6}\, \frac{[b(\ell) - v(\ell)]^2}{b(\ell)}\, m(\ell)^2\, \arcsin \frac{1}{\sqrt{1 + \frac{-2\,v(\ell)}{b(\ell)}\, m(\ell)^2}} \Bigg] \Bigg|_{\ell = q/\mu} \quad .$$

Again, their features are most conveniently explored by introducing effective exponents

$$2 - \eta_{eff}^{T/L} = \frac{\partial \ln \chi_{T/L}^{-1}(\vec{q},0)}{\partial \ln q} \quad , \tag{4.23}$$

which are displayed in Fig.7 for $n = 2$ and the same values for the anisotropy parameter $y(1)$ that have been used in Fig.5, including $y(1) = 0$. (In order to allow for a most direct comparison with the isotropic model, here the two-point cumulants are plotted.) The transverse exponent (Fig.7a) shows how the quasi-Goldstone modes for $y(1) \neq 0$ finally acquire a mass and hence the transverse susceptibility becomes finite for $\vec{q} \to 0$. If we had plotted $2 - \eta^T_{eff}$ against $q/\mu\, \overline{m}(1)$ instead of the critical scaling variable following from (2.23), we would have found a universal crossover signature very similar to the one depicted in Fig.2 for the one-component case. For very low anisotropies, on the other hand, coexistence-like anomalies are clearly noticeable in the wavevector-dependence of the longitudinal response function within an intermediate q-region the size of which depends on $y(1)$, if $n < n_c$. For $y(1) < 10^{-4}$, the pronounced minimum of the effective exponent $2 - \eta^L_{eff}$ is then still apparent, while for larger anisotropy the flow becomes much more similar to Fig.2 for the direct crossover to the Gaussian model. Now turning to $n \geq n_c$, according to the preceding Subsection, no remnants of the Goldstone modes are to be expected.

As in the isotropic case, the dynamic response functions may be explicitly written down for arbitrary dimensions $2 < d \leq 4$ in the limit $\vec{q} \to 0$. For model A [1] with non-conserved order parameter one finds (compare Eq.(5.11) of Ref.[11])

$$\begin{aligned}
\chi_T^{-1}(0,\omega) = \mu^2\, \ell^2 \Bigg(& -i + \frac{-v(\ell)}{2\, b(\ell)}\, m(\ell)^2\, Z_{\overline{m}}(\ell) \qquad (4.24a) \\
& + \frac{2\, b(\ell) + v(\ell)}{6\, \epsilon} \left[1 - \left(\frac{-v(\ell)}{2\, b(\ell)} \right)^{1-\epsilon/2} \right] m(\ell)^{2-\epsilon} - \frac{[b(\ell) - v(\ell)]^2}{3\, \epsilon\, b(\ell)} \times \\
\times & \left[\frac{m(\ell)^2}{i - \frac{2\, b(\ell) + v(\ell)}{2\, b(\ell)}\, m(\ell)^2} \left[\left(\frac{-v(\ell)}{2\, b(\ell)}\, m(\ell)^2 \right)^{1-\epsilon/2} - \left(\frac{-i}{2} + \frac{2\, b(\ell) - v(\ell)}{4\, b(\ell)}\, m(\ell)^2 \right)^{1-\epsilon/2} \right] \right. \\
& \left. \left. + \frac{m(\ell)^2}{i + \frac{2\, b(\ell) + v(\ell)}{2\, b(\ell)}\, m(\ell)^2} \left[m(\ell)^{2-\epsilon} - \left(\frac{-i}{2} + \frac{2\, b(\ell) - v(\ell)}{4\, b(\ell)}\, m(\ell)^2 \right)^{1-\epsilon/2} \right] \right] \Bigg) \right|_{\ell = \sqrt{\omega/\lambda\mu^2}}
\end{aligned}$$

$$\begin{aligned}
\chi_L^{-1}(0,\omega) = \mu^2\, \ell^2 \Bigg(& -i + m(\ell)\, Z_m(\ell) + \frac{3\, i}{\epsilon}\, b(\ell)\, m(\ell)^2 \left[m(\ell)^{2-\epsilon} - \left(\frac{-i}{2} + m(\ell)^2 \right)^{1-\epsilon/2} \right] \\
& + \frac{(n-1)\, i}{3\, \epsilon}\, \frac{[b(\ell) - v(\ell)]^2}{b(\ell)}\, m(\ell)^2 \times \qquad (4.24b) \\
& \left. \times \left[\left(\frac{-v(\ell)}{2\, b(\ell)}\, m(\ell)^2 \right)^{1-\epsilon/2} - \left(\frac{-i}{2} + \frac{-v(\ell)}{2\, b(\ell)}\, m(\ell)^2 \right)^{1-\epsilon/2} \right] \Bigg) \right|_{\ell = \sqrt{\omega/\lambda\mu^2}} .
\end{aligned}$$

The logarithmic derivatives of the inverse real part of the longitudinal susceptibility and of the inverse longitudinal correlation function obtained with the help of (2.8) are depicted for $n = 2$ and $\epsilon = 1$ in Figs.8a and 8b, respectively. For the relevant situation

$n < 4$, their general features are similar to that of the static effective exponent $2 - \eta^L_{eff}$; however, the crossover to the asymptotic Gaussian behavior takes place for values of the dynamic scaling variable about an order of magnitude lower than in the static case, and the influence of the cubic anisotropies is considerably reduced.

If the order parameter field is conserved (model B according to Hohenberg and Halperin [1]), due to the diffusion pole the ratio ω/q^2 has to be kept fixed when the limit $\vec{q} \to 0$ is performed. Then merely the zero-loop contributions (to be supplied with the full one-loop flow equations) persist [11]:

$$\chi_T^{-1}(0, \omega/q^2) = \mu^2 \, \ell^2 \left[-i + \frac{-v(\ell)}{2\, b(\ell)} \, m(\ell)^2 \right] \Big|_{\ell = \sqrt{\omega/\lambda\, q^2}} \tag{4.25a}$$

$$\chi_L^{-1}(0, \omega/q^2) = \mu \, \ell^2 \left[-i + m(\ell)^2 \right] \Big|_{\ell = \sqrt{\omega/\lambda\, q^2}} \quad . \tag{4.25b}$$

The ensuing crossover to the Gaussian theory with pronounced remnants of coexistence anomalies again is shown in Fig.9, for the case $n = 2$ and $d = 3$.

For a comparison with the results (5.13) of Ref.[11] we finally quote the complete $\vec{q}$- and ω-dependent response functions for model A at three dimensions

$$\begin{aligned}
\chi_T^{-1}(\vec{q}, \omega) = q^2 - \frac{i\,\omega}{\lambda} + \mu^2 \, \ell^2 \Bigg[& \frac{-v(\ell)}{2\, b(\ell)} \, m(\ell)^2 \, Z_{\overline{m}}(\ell) + \frac{2\, b(\ell) + v(\ell)}{6} \left(1 - \sqrt{\frac{-v(\ell)}{2\, b(\ell)}} \right) m(\ell) \\
& - \frac{[b(\ell) - v(\ell)]^2}{6\, b(\ell)} \, \frac{m(\ell)^2}{q/\mu\, \ell} \Bigg[\arcsin \left(\frac{q^2 - i\,\omega/\lambda}{\mu^2\, \ell^2 \, S_+(\ell)} - \frac{2\, b(\ell) + v(\ell)}{2\, b(\ell)\, S_+(\ell)} \, m(\ell)^2 \right) \\
& + \arcsin \left(\frac{i\omega/\lambda}{\mu^2 \ell^2 S_+(\ell)} + \frac{2b(\ell) + v(\ell)}{2b(\ell) S_+(\ell)} m(\ell)^2 \right) + \arcsin \left(\frac{i\omega/\lambda}{\mu^2 \ell^2 S_-(\ell)} - \frac{2\, b(\ell) + v(\ell)}{2b(\ell) S_-(\ell)} m(\ell)^2 \right) \\
& + \arcsin \left(\frac{q^2 - i\,\omega/\lambda}{\mu^2\, \ell^2 \, S_-(\ell)} + \frac{2\, b(\ell) + v(\ell)}{2\, b(\ell)\, S_-(\ell)} \, m(\ell)^2 \right) \Bigg] \Bigg]
\end{aligned} \tag{4.26a}$$

and

$$\begin{aligned}
\chi_L^{-1}(\vec{q}, \omega) = q^2 - \frac{i\,\omega}{\lambda} + \mu^2 \, \ell^2 \Bigg[& m(\ell)^2 \, Z_m(\ell) \\
& - \frac{n-1}{6} \, \frac{[b(\ell) - v(\ell)]^2}{b(\ell)} \, \frac{m(\ell)^2}{q/\mu\ell} \left(\arcsin \frac{i\omega/\lambda}{\mu^2 \ell^2 S_\perp(\ell)} + \arcsin \frac{q^2 - i\omega/\lambda}{\mu^2 \ell^2 S_\perp(\ell)} \right) \\
& - \frac{3}{2} \, \frac{b(\ell)\, m(\ell)^2}{q/\mu\, \ell} \left(\arcsin \frac{i\,\omega/\lambda}{\mu^2\, \ell^2 \, S_\parallel(\ell)} + \arcsin \frac{q^2 - i\,\omega/\lambda}{\mu^2\, \ell^2\, S_\parallel(\ell)} \right) \Bigg] \quad .
\end{aligned} \tag{4.26b}$$

Here the abbreviations

$$S_\pm(\ell) = \sqrt{ \left(\frac{i\omega/\lambda}{\mu^2\, \ell^2} \pm \frac{2b(\ell) + v(\ell)}{2b(\ell)} m(\ell)^2 \right)^2 + \left(\frac{q^2 - 2i\omega/\lambda}{\mu^2\, \ell^2} + \frac{2b(\ell) - v(\ell)}{b(\ell)} m(\ell)^2 \right) \frac{q^2}{\mu^2\, \ell^2} } \tag{4.27a}$$

and

$$S_\perp(\ell) = \sqrt{\left(\frac{i\,\omega/\lambda}{\mu^2\,\ell^2}\right)^2 + \left(\frac{q^2 - 2\,i\,\omega/\lambda}{\mu^2\,\ell^2} + \frac{-2\,v(\ell)}{b(\ell)}\,m(\ell)^2\right)\,\frac{q^2}{\mu^2\,\ell^2}} \qquad (4.27b)$$

$$S_\parallel(\ell) = \sqrt{\left(\frac{i\,\omega/\lambda}{\mu^2\,\ell^2}\right)^2 + \left(\frac{q^2 - 2\,i\,\omega/\lambda}{\mu^2\,\ell^2} + 4\,m(\ell)^2\right)\,\frac{q^2}{\mu^2\,\ell^2}} \qquad (4.27c)$$

were used, and the matching condition

$$\ell^2 = \left|\frac{q^2}{\mu^2} - \frac{i\,\omega}{\lambda\,\mu^2}\right| \qquad (4.28)$$

has to be inserted. Investigation of the poles of (4.26) shows that the damping of both the transverse and longitudinal modes asymptotically is independent of q. Similarly, in the case of diffusive relaxation (model B) the dispersion relation becomes $i\,\omega \propto q^2$. The anomalous line shape (2.27) of the longitudinal correlation function still appears in an intermediate range, if $n \leq 4$ and $y(1) \ll 1$; finally it develops into the regular Lorentz form.

The graphs discussed above refer to the situation where the number of components is low, $n < n_c$. For larger n, the cubic anisotropy immediately destroys the massless character of the transverse excitations and hence no coexistence-type anomalies persist. The theory is generally restricted to dimensionality $2 < d \leq 4$. At the upper critical dimension $d_c = 4$, instead of the power laws logarithmic corrections come into play. Although excluded from the scope of the formalism presented here, in the limit $d \to 2$ reasonable statements may be extracted from the general features of the flow equations, in the sense that the results are compatible with exact theorems on the existence of long-range order [22,23]. In Table II we summarize the qualitative behavior under the influence of small cubic anisotropies.

V. THE INFLUENCE OF THE DIPOLAR INTERACTION ON THE DYNAMICS OF MODEL A IN THE ORDERED PHASE

A. Dipolar propagators in the ordered phase and crossover behavior at the critical point

The second very interesting generic anisotropy we want to study is the dipol-dipol interaction. Although the phase transitions, e.g. in magnetic systems, are driven by the short-range exchange interactions, long-range dipolar forces become increasingly important on approaching the critical point where the correlation length diverges. Within the static renormalization group theory, the influence of dipolar terms has been examined by Fisher and Aharony [31], and was systematically treated together with other kinds of anisotropies by Aharony [24]. The interplay between the isotropic Heisenberg fixed point and an additional asympotically stable dipolar fixed point produces remarkable crossover phenomena, which have been intensively investigated in the literature; we quote here the works of Bruce and Aharony [32], Nattermann and Trimper [33], Bruce, Kosterlitz and Nelson [34], and later of Santos [35], Kogon and Bruce [36], which are all based on the renormalization group theory, and expecially the recent papers by Frey and Schwabl [37] who applied the extended renormalization scheme of Amit and Goldschmidt [14] to the dipolar crossover problem.

Taking into account only those terms which are relevant with respect to the renormalization group [31], the starting point of our considerations is the following supplement to the static Ginzburg-Landau functional (2.1)

$$\Delta H_{dip}[\{\phi_0^\alpha\}] = \int_q \frac{g_0}{2} \sum_{\alpha,\beta=1}^{\min(d,n)} \frac{q_\alpha\, q_\beta}{q^2}\, \phi_0^\alpha(\vec{q})\, \phi_0^\beta(-\vec{q}) \quad . \tag{5.1}$$

Hence the dipol-dipol interaction hence leads to a term proportional to the coupling parameter g_0 which is anisotropic in momentum space. We shall treat the general situation $n \neq d$, although in part of our study we shall restrict ourselves to the widely considered case of equal dimensions in momentum and order parameter space. With the aid of the projectors [37]

$$P^T_{\alpha\beta} = \delta_{\alpha\beta} - \frac{q_\alpha\, q_\beta}{q^2} \tag{5.2a}$$

$$P^L_{\alpha\beta} = \frac{q_\alpha\, q_\beta}{q^2} \quad , \tag{5.2b}$$

fulfilling the relations

$$P^T_{\alpha\beta} + P^L_{\alpha\beta} = \delta_{\alpha\beta} \tag{5.3a}$$

$$\sum_{\gamma=1}^{d} P^{T/L}_{\alpha\gamma}\, P^{T/L}_{\gamma\beta} = P^{T/L}_{\alpha\beta} \tag{5.3b}$$

$$\sum_{\gamma=1}^{d} P^T_{\alpha\gamma}\, P^L_{\gamma\beta} = 0 \quad , \tag{5.3c}$$

we may decompose the Hamiltonian into different contributions corresponding to the transverse (T) and longitudinal parts (L) with respect to the wavevector $\vec{q}$

$$H_0[\{\phi_0^\alpha\}] = \int_q \Bigg(\frac{1}{2} \sum_{\alpha,\beta=1}^{\min(d,n)} \Big[(r_0 + q^2)\, P_{\alpha\beta}^T + (r_0 + g_0 + q^2)\, P_{\alpha\beta}^L \Big] \phi_0^\alpha(\vec{q})\, \phi_0^\beta(-\vec{q}) + \frac{1}{2}(r_0 + q^2) \sum_{\alpha=\min(d,n)+1}^{n} \phi_0^\alpha(\vec{q})\, \phi_0^\alpha(-\vec{q}) \Bigg) \quad . \tag{5.4}$$

From (5.4) it is apparent that the fluctuations in the longitudinal sector acquire an additional mass term $\propto g_0$ due to the dipolar forces.

The static critical properties of any dipolar magnet are described by the Hamiltonian (5.1) irrespective of the detailed dynamics. For the latter we remark that in the presence of dipolar forces the magnetization is no more a conserved quantity. Hence the relaxation process would be that of model A [1], Eq.(2.3) with $a = 0$. However, mode-coupling terms are most important for the dynamic properties of ferromagnets and antiferromagnets. Yet for the sake of simplicity, and because the focus of our intention is rather the influence of the anisotropies on the fluctuations, we shall drop the mode-coupling vertex and restrict our investigations to the time-dependent Ginzburg-Landau model A.

In order to avoid unnecessary complications, we start with the important special situation where

$$n = d = 4 - \epsilon \quad , \tag{5.5}$$

and shall come back to the general case later (Subsection C). Below the transition temperature, we may still use the parametrization (2.9), (2.10) for the spontaneous order parameter. The Janssen-De Dominics functional (2.5b) can then be split according to $J = J_0^T + J_0^L + J_{int} + J_{CT}$, with the harmonic contributions

$$J_0^T[\{\tilde{\pi}_0^\alpha\}, \tilde{\sigma}_0, \{\pi_0^\alpha\}, \sigma_0] = \tag{5.6a}$$
$$= \int_q \int_\omega \Bigg[\sum_{\alpha\beta} \Big(\lambda_0\, \tilde{\pi}_0^\alpha(\vec{q},\omega)\, \tilde{\pi}_0^\beta(-\vec{q},-\omega) - \tilde{\pi}_0^\alpha(\vec{q},\omega) \Big[i\,\omega + \lambda_0\, q^2 \Big] \pi_0^\beta(-\vec{q},-\omega) \Big) P_{\alpha\beta}^T + \Big(\lambda_0\, \tilde{\sigma}_0(\vec{q},\omega)\, \tilde{\sigma}_0(-\vec{q},-\omega) - \tilde{\sigma}_0(\vec{q},\omega) \Big[i\,\omega + \lambda_0\, (m_0^2 + q^2) \Big] \sigma_0(-\vec{q},-\omega) \Big) P_{nn}^T \Bigg]$$

and

$$J_0^L[\{\tilde{\pi}_0^\alpha\}, \tilde{\sigma}_0, \{\pi_0^\alpha\}, \sigma_0] = \tag{5.6b}$$

$$= \int_q \int_\omega \Bigg[\sum_{\alpha\beta} \Big(\lambda_0 \, \tilde{\pi}_0^\alpha(\vec{q},\omega) \, \tilde{\pi}_0^\beta(-\vec{q},-\omega) - \tilde{\pi}_0^\alpha(\vec{q},\omega) \left[i\,\omega + \lambda_0 \, (g_0 + q^2) \right] \pi_0^\beta(-\vec{q},-\omega) \Big) \, P_{\alpha\beta}^L$$

$$+ \Big(\lambda_0 \, \tilde{\sigma}_0(\vec{q},\omega) \, \tilde{\sigma}_0(-\vec{q},-\omega) - \tilde{\sigma}_0(\vec{q},\omega) \left[i\omega + \lambda_0 \, (m_0^2 + g_0 + q^2) \right] \sigma_0(-\vec{q},-\omega) \Big) \, P_{nn}^L$$

$$- \sum_\alpha \lambda_0 \, g_0 \left[\tilde{\pi}_0^\alpha(\vec{q},\omega) \, \sigma_0(-\vec{q},-\omega) + \tilde{\sigma}_0(\vec{q},\omega) \, \pi_0^\alpha(-\vec{q},-\omega) \right] P_{\alpha n}^L \Bigg] \quad ,$$

denoting the transverse and longitudinal parts with respect to the wavevector $\vec{q}$. The interaction term J_{int} is identical to that of the isotropic model A and follows from (4.5b) for $v_0 = 0$ and $b_0 = u_0$. Similarly, the counterterm J_{CT} looks like (4.5c), with A replaced by the analogous quantity $\tilde{A}$. In the transverse part (5.6a) of the dynamic functional the π-modes are massless and in harmonic approximation independent of the σ-fluctuation. The longitudinal π-modes, on the other hand, have lost their Goldstone character due to the dipolar interaction, and furthermore couple to the σ-field ($\propto \lambda_0 \, g_0$). We also remark that the operators $P_{\alpha\beta}^T$ and $P_{\alpha\beta}^L$ are no proper projectors in the subspace of the $n-1$ fluctuations transverse with respect to the direction of the order parameter.

The evaluation of the harmonic propagators is thus more difficult than in the isotropic case. It is very convenient to collect the response and correlation fields in a $2\,n$-dimensional vector according to $\vec{\psi}_0 = (\tilde{\pi}_0^1, \ldots, \tilde{\pi}_0^{n-1}, \tilde{\sigma}_0, \pi_0^1, \ldots, \pi_0^{n-1}, \sigma_0)$. Then $J_0 = J_0^T + J_0^L$ can be written as

$$J_0[\vec{\psi}_0] = -\frac{1}{2} \int_q \int_\omega \vec{\psi}_0(\vec{q},\omega)^T \, \hat{A}(\vec{q},\omega) \, \vec{\psi}_0(-\vec{q},-\omega) \quad , \tag{5.7a}$$

where the harmonic coupling matrix separates into $n \times n$-submatrices

$$\hat{A}(\vec{q},\omega) = \begin{pmatrix} \hat{D} & \hat{G} \\ \hat{G}^* & 0 \end{pmatrix} \quad . \tag{5.7b}$$

Here $\hat{D}$ is diagonal

$$\hat{D} = \begin{pmatrix} -2\,\lambda_0 & & \\ & \cdots & \\ & & -2\,\lambda_0 \end{pmatrix} \quad , \tag{5.7c}$$

and the symmetric matrix $\hat{G}$ has the form

$$\hat{G} = \begin{pmatrix} B & v \\ v^T & b \end{pmatrix} \quad , \tag{5.7d}$$

with

$$B_{\alpha\beta}(\vec{q},\omega) = (i\,\omega + \lambda_0 \, q^2) \, \delta_{\alpha\beta} + \lambda_0 \, g_0 \, \frac{q_\alpha \, q_\beta}{q^2} \tag{5.8a}$$

$$v_\alpha(\vec{q}) = \lambda_0 \, g_0 \, \frac{q_\alpha \, q_n}{q^2} \tag{5.8b}$$

$$b(\vec{q},\omega) = i\,\omega + \lambda_0 \, (m_0^2 + q^2) + \lambda_0 \, g_0 \, \frac{q_n^2}{q^2} \quad . \tag{5.8c}$$

The $(n-1)\times(n-1)$-matrix B is easily inverted:

$$B^{-1}_{\alpha\beta}(\vec{q},\omega) = \frac{1}{i\,\omega+\lambda_0\,q^2}\left(\delta_{\alpha\beta} - \frac{\lambda_0\,g_0}{i\,\omega+\lambda_0\,(g_0+q^2)-\lambda_0\,g_0\,q_n^2/q^2}\,\frac{q_\alpha\,q_\beta}{q^2}\right) \quad . \tag{5.9}$$

The inverse of $\hat{G}$ then follows as

$$\hat{G}^{-1} = \begin{pmatrix} C & u \\ u^T & c \end{pmatrix} \quad , \tag{5.10a}$$

with

$$c = \frac{1}{b - v^T\,B^{-1}\,v} \tag{5.10b}$$

$$u = -c\,B^{-1}\,v \tag{5.10c}$$

$$C = B^{-1}\,(1 - v\,u^T) \quad . \tag{5.10d}$$

Finally one finds

$$\hat{A}^{-1}(\vec{q},\omega) = \begin{pmatrix} 0 & \hat{G}^{*\,-1} \\ \hat{G}^{-1} & -\hat{G}^{-1}\,\hat{D}\,\hat{G}^{*\,-1} \end{pmatrix} \quad , \tag{5.11}$$

from which the propagators of the theory may be obtained (see Ref.[11,12]). The zero-loop response propagators read

$$G_{0\,\tilde{\pi}^\alpha\pi^\beta}(\vec{q},\omega) = \frac{1}{-i\,\omega+\lambda_0\,q^2}\left[\delta_{\alpha\beta} - \frac{q_\alpha\,q_\beta}{q^2}\,s(\vec{q},\omega)\right] \tag{5.12a}$$

$$G_{0\,\tilde{\pi}^\alpha\sigma}(\vec{q},\omega) = \frac{-1}{-i\,\omega+\lambda_0\,(m_0^2+q^2)}\,\frac{q_\alpha\,q_n}{q^2}\,s(\vec{q},\omega) \tag{5.12b}$$

$$G_{0\,\tilde{\sigma}\sigma}(\vec{q},\omega) = \frac{1}{-i\omega+\lambda_0(m_0^2+q^2)}\left[1 - \frac{q_n^2}{q^2}\,\frac{-i\omega+\lambda_0 q^2}{-i\omega+\lambda_0(m_0^2+q^2)}\,s(\vec{q},\omega)\right]; \tag{5.12c}$$

here the abbreviation

$$s(\vec{q},\omega) = \frac{\lambda_0\,g_0}{-i\,\omega+\lambda_0\,(g_0+q^2)-\lambda_0\,g_0\,\frac{\lambda_0\,m_0^2}{-i\,\omega+\lambda_0\,(m_0^2+q^2)}\,q_n^2/q^2} \tag{5.12d}$$

has been introduced. Note that in the static limit $\omega \to 0$ they acquire the form

$$G_{0\,\alpha\beta}(\vec{q}) = \frac{1}{r_0^\alpha+q^2}\left(\delta_{\alpha\beta} - g_0\,\frac{q_\alpha\,q_\beta}{r_0^\beta+q^2}\,\frac{1}{q^2+g_0\,\tilde{q}^2}\right) \quad , \tag{5.13a}$$

where

$$\tilde{q}^2 = \sum_\alpha \frac{q_\alpha^2}{r_0^\alpha+q^2} \quad , \tag{5.13b}$$

which is precisely the result quite generally derived by Aharony [24] for anisotropic exchange accompanied by dipolar interactions, if $r_0^\alpha = 0$ for $\alpha = 1, \ldots, n-1$ and $r_0^n = m_0^2$ is inserted.

The rather complex structure of the result (5.12) for the propagators of the harmonic theory renders a complete treatment of the crossover regime a very streneous task. Hence we shall restrict our discussion to the coexistence limit $m_0 \to \infty$ only, which suffices for the questions we have in mind. Before embarking on that, however, we are going to discuss very briefly the situation at the critical point.

With $m_0 = 0$, the response and correlation propagators at T_c read

$$G_{0\,\tilde{\phi}^\alpha \phi^\beta}(\vec{q}, \omega) = \frac{1}{-i\,\omega + \lambda_0\, q^2}\, P^T_{\alpha\beta} + \frac{1}{-i\,\omega + \lambda_0\,(g_0 + q^2)}\, P^L_{\alpha\beta} \tag{5.14a}$$

$$G_{0\,\phi^\alpha \phi^\beta}(\vec{q}, \omega) = \frac{2\,\lambda_0}{\omega^2 + \lambda_0^2\, q^4}\, P^T_{\alpha\beta} + \frac{2\,\lambda_0}{\omega^2 + \lambda_0^2\,(g_0 + q^2)^2}\, P^L_{\alpha\beta} \quad . \tag{5.14b}$$

For $T \neq T_c$, one may reintroduce the temperature dependence r_0 (see Ref.[37]). The T_c-shift can be implicitly determined to first order in u_0, e.g. by using the equation of state (2.11), as the solution of the equation

$$g_0 = r_{0c} \left[\left(\frac{12\,\epsilon}{u_0\, A_d}\, r_{0c}^{\epsilon/2} - 2\,n - 3 \right)^{2/(2-\epsilon)} - 1 \right] \quad . \tag{5.15}$$

Compared to the isotropic case, the transition temperature is increased due to the long-range order dipol-dipol interaction which favors condensation [31].

For the renormalization in the critical region conceptually different Z-factors for the transverse and longitudinal field components, respectively, have to be defined [37], namely

$$\tilde{\phi}^\alpha = Z_{\tilde{T}}^{1/2} \sum_\beta P^T_{\alpha\beta}\, \tilde{\phi}_0^\beta + Z_{\tilde{L}}^{1/2} \sum_\beta P^L_{\alpha\beta}\, \tilde{\phi}_0^\beta \tag{5.16a}$$

$$\phi^\alpha = Z_T^{1/2} \sum_\beta P^T_{\alpha\beta}\, \phi_0^\beta + Z_L^{1/2} \sum_\beta P^L_{\alpha\beta}\, \phi_0^\beta \quad . \tag{5.16b}$$

The renormalized counterparts for the timescale λ_0 and the coupling constants u_0, g_0 are conventionally introduced as

$$\lambda = Z_\lambda^{-1}\, \lambda_0 \tag{5.17a}$$

$$u = Z_u^{-1}\, u_0\, A_d\, \mu^{-\epsilon} \tag{5.17b}$$

$$g = Z_g^{-1}\, g_0\, \mu^{-2} \quad . \tag{5.17c}$$

As a consequence of the fluctuation-dissipation theorem (2.8) and the partition of the propagators into transverse and longitudinal components according to (5.14) the following relation between Z_λ and the field renormalizations can be derived

$$Z_\lambda = \frac{Z_T^{-1} + Z_L^{-1}}{(Z_{\tilde{T}}\, Z_T)^{-1/2} + (Z_{\tilde{L}}\, Z_L)^{-1/2}} \quad , \tag{5.18}$$

which for $Z_{\tilde{T}} = Z_{\tilde{L}}$ and $Z_T = Z_L$ (valid for vanishing g_0) reduces to the well-known expression for the isotropic relaxational models [12,11].

In the vicinity of T_c, interesting crossover phenomena occur which are induced by the second lengthscale $1/\sqrt{g_0}$ originating from the dipolar term. An elegant description of these phenomena may be obtained by applying Amit and Goldschmidt's general method [14]; hereby it appears that there are no ultraviolet divergences $\propto g_0$, and hence one concludes [37]

$$Z_g^{-1} = Z_L \quad . \tag{5.19}$$

To one-loop order the field renormalizations vanish

$$Z_{\tilde{T}} = Z_{\tilde{L}} = Z_T = Z_L = 1 \quad , \tag{5.20a}$$

and therefore

$$Z_\lambda = 1 \tag{5.20b}$$

$$Z_g = 1 \quad . \tag{5.20c}$$

For the single non-trivial Z-factor one finds by investigation of the four-point vertex function [37]

$$Z_u = 1 + \frac{n^2 + 18\, n + 116}{144\, \epsilon}\, u_0\, A_d\, \mu^{-\epsilon} - \frac{n^2 - 6\, n - 76}{144\, \epsilon}\, \frac{u_0\, A_d\, \mu^{-\epsilon}}{(1 + g_0/\mu^2)^{\epsilon/2}} \tag{5.20d}$$

(here we have slightly modified the results from Ref.[37] by not explicitly inserting $n = 4$ and refraining from the ϵ-expansion).

Taking advantage of the renormalization group equation we introduce flow-dependent couplings according to

$$\ell\, \frac{dg(\ell)}{d\ell} = g(\ell)\, \zeta_g(\ell) \tag{5.21a}$$

$$\ell\, \frac{du(\ell)}{d\ell} = \beta_u(\ell) \quad , \tag{5.21b}$$

with the usual initial conditions $g(1) = g$ and $u(1) = u$, where

$$\zeta_g = \mu \frac{\partial}{\partial \mu}\bigg|_0 \ln \frac{g^2}{g_0^2} = -2 \tag{5.22a}$$

and

$$\beta_u = \mu \frac{\partial}{\partial \mu}\bigg|_0 u = u \left[-\epsilon + \frac{n^2 + 18\,n + 116}{144}\, u - \frac{n^2 - 6\,n - 76}{144}\, \frac{u}{(1+g)^{1+\epsilon/2}} \right] \tag{5.22b}$$

are Wilson's flow functions to one-loop order. For vanishing dipolar interaction $g = 0$ one rediscovers the isotropic Heisenberg fixed point (2.20a), as the zero of (5.22b). For any finite value of $g(1)$, however, the dipolar strength diverges as

$$g(\ell) = g\,\ell^{-2} \quad , \tag{5.23}$$

which ultimately justifies the corresponding term in (5.20d) and (5.22b). Near the critical point where the correlation length becomes macroscopic, the dipol-dipol interaction hence is a relevant perturbation. The asymptotic behavior is then no more governed by the Heisenberg fixed point but by the dipolar fixed point [31]

$$u_D^* = \frac{144\,\epsilon}{n^2 + 18\,n + 116} \quad . \tag{5.24}$$

The ensuing crossover features have been intensively studied by several authors [32-37]. For the possible values $n = 2, 3, 4$ one finds $u_D^* > u_H^*$, which accounts for the fact that a certain amount of the fluctuations is effectively suppressed by the dipolar forces.

B. Renormalization of the theory in the coexistence limit for $n = d \geq 3$

We shall now come to the properties within the ordered phase, where, as we shall see shortly, the cases $n = 2$ and $n = 3, 4$ have to be distinguished. For $n \geq 3$ the asymptotic theory can be discussed in a manner completely analogous to Section II.B. Substituting the transformed longitudinal fields (2.13) (with $a = 0$) into (5.6) and the interaction terms, and performing the limit $m_0 \to \infty$ one finds for the Janssen-De Dominicis functional

$$J_\infty[\{\tilde{\pi}_0^\alpha\}, \tilde{\sigma}_0, \{\pi_0^\alpha\}, \sigma_0] = \tag{5.25}$$

$$= \int_q \int_\omega \Bigg[\sum_\alpha \Big(\lambda_0\, \tilde{\pi}_0^\alpha(\vec{q}, \omega)\, \tilde{\pi}_0^\alpha(-\vec{q}, -\omega) - \tilde{\pi}_0^\alpha(\vec{q}, \omega) \left[i\,\omega + \lambda_0\, q^2 \right] \pi_0^\alpha(-\vec{q}, -\omega) \Big)$$

$$- \lambda_0\, g_0 \sum_{\alpha\beta} \frac{q_\alpha q_\beta}{q^2}\, \tilde{\pi}_0^\alpha(\vec{q}, \omega)\, \pi_0^\beta(-\vec{q}, -\omega) - \lambda_0\, \tilde{\varphi}_0(\vec{q}, \omega)\, \varphi_0(-\vec{q}, -\omega) \Bigg] \quad ,$$

i.e.: remarkably a harmonic theory again, which can be treated exactly.

One immediate consequence of (5.25) are the relations $\Gamma^{\infty}_{0\,\tilde{\pi}^\alpha\tilde{\pi}^\beta}(\vec{q},\omega) = 2\,\lambda_0\,\delta_{\alpha\beta}$ and $\Gamma^{\infty}_{0\,\tilde{\pi}^\alpha\pi^\beta}(\vec{q},\omega) = B_{\alpha\beta}$ [see Eq.(5.8a)]. Hence the field renormalizations for the π-fluctuations vanish; a similar reasoning applies for the longitudinal fields, and from (5.18) and (5.19) one concludes that Eqs.(5.20) are valid as exact results in the coexistence limit. In particular, for $\ell \to 0$ the parameter $g(\ell)$ diverges as in (5.23), and we simply have to insert $s_\infty(\vec{q},\omega) = q^2/(q^2 - q_n^2)$ into the π-propagators:

$$G^{\infty}_{0\,\tilde{\pi}^\alpha\pi^\beta}(\vec{q},\omega) = \frac{1}{-i\,\omega + \lambda_0\,q^2}\,\overline{P}^T_{\alpha\beta} \tag{5.26a}$$

$$G^{\infty}_{0\,\tilde{\pi}^\alpha\tilde{\pi}^\beta}(\vec{q},\omega) = \frac{2\,\lambda}{\omega^2 + \lambda_0^2\,q^4}\,\overline{P}^T_{\alpha\beta} \quad . \tag{5.26b}$$

Note that of the two proper projectors [compare Eq.(5.3)]

$$\overline{P}^T_{\alpha\beta} = \delta_{\alpha\beta} - \frac{q_\alpha\,q_\beta}{q^2 - q_n^2} \tag{5.27a}$$

$$\overline{P}^L_{\alpha\beta} = \frac{q_\alpha\,q_\beta}{q^2 - q_n^2} \tag{5.27b}$$

only the transversal operator appears in (5.26). (In the case $\vec{q} = (0,\ldots,0,q_n)$ one has $\overline{P}^T_{\alpha\beta} = \delta_{\alpha\beta}$ and $\overline{P}^L_{\alpha\beta} = 0$.) The fluctuations parallel to the wavevector $\vec{q}$ have disappeared, as well as the non-diagonal propagator (5.12b).

As for the isotropic model, using (2.13), (5.25) and (5.26) one finds the following asymptotically exact results for the σ-correlation functions

$$G^{\infty}_{0\,\tilde{\sigma}\sigma}(\vec{q},\omega) = \frac{1}{\lambda_0 m_0^2}\left(1 + \frac{u_0}{6}\int_k \frac{1}{\left(\frac{\vec{q}}{2} - \vec{k}\right)^2}\,\frac{1}{\frac{-i\,\omega}{2\,\lambda_0} + \frac{q^2}{4} + k^2} \times \right. \tag{5.28a}$$
$$\left. \times \left[n - 2 - \frac{\left(q^2 - q_n^2\right)\,\left(k^2 - k_n^2\right) - \left(\vec{q}\vec{k} - q_n\,k_n\right)^2}{\left(\frac{q^2 - q_n^2}{4} + k^2 - k_n^2\right)^2 - \left(\vec{q}\vec{k} - q_n\,k_n\right)^2}\right]\right)$$

$$G^{\infty}_{0\,\sigma\sigma}(\vec{q},\omega) = \frac{1}{\lambda_0\,m_0^2}\,\frac{u_0}{6}\,\mathrm{Re}\int_k \frac{1}{\left(\frac{q^2}{4} + k^2\right)^2 - \left(\vec{q}\vec{k}\right)^2}\,\frac{1}{\frac{-i\,\omega}{2\,\lambda_0} + \frac{q^2}{4} + k^2} \times \tag{5.28b}$$
$$\times \left[n - 2 - \frac{\left(q^2 - q_n^2\right)\,\left(k^2 - k_n^2\right) - \left(\vec{q}\vec{k} - q_n\,k_n\right)^2}{\left(\frac{q^2 - q_n^2}{4} + k^2 - k_n^2\right)^2 - \left(\vec{q}\vec{k} - q_n\,k_n\right)^2}\right] \quad .$$

Again, these expressions can be identified with the explicit results from first-order perturbation theory for the two-point cumulants in the coexistence limit. As for $g_0 = 0$, the corresponding two-point vertex functions may be represented by a geometric series of the π-loops, see Fig.2 of Ref.[11]. The essential differences between (5.28) and Eq.(3.18) of Ref.[11] are (i) the dependence on the angle between spontaneous order parameter and wavevector as apparent from the second term in the integrands of (5.28), and (ii) the factor $n-2$ within the isotropic term instead of $n-1$.

For the determination of the only non-trivial renormalization constant $Z_m^\infty = Z_u^\infty$ one has to consider $\Gamma^\infty_{0\,\tilde\sigma\sigma}(0,0)$. Hence Z_m is independent of the anisotropies in (5.28), and one finds explicitly that as a consequence of the dipolar interactions the relevant effective number of massless modes is reduced from $n-1$ to $n-2$, because

$$Z_m^\infty = Z_u^\infty = 1 + \frac{n-2}{6\,\epsilon}\, u_0\, A_d\, \mu^{-\epsilon} \quad . \tag{5.29}$$

The remaining non-zero flow functions read

$$\zeta_g^\infty = -2 \tag{5.30a}$$

$$\zeta_m^\infty = -2 + \frac{n-2}{6}\, u \tag{5.30b}$$

$$\beta_u^\infty = u \left[-\epsilon + \frac{n-2}{6}\, u \right] \quad , \tag{5.30c}$$

and as a solution of $\beta_u^\infty(u^*) = 0$ we find the stable dipolar coexistence fixed point

$$u^*_{CD} = \frac{6\,\epsilon}{n-2} \tag{5.31a}$$

replacing (2.21a); however, the anomalous dimension of the mass parameter is not altered

$$\zeta^*_{m\,CD} = -2 + \epsilon \quad . \tag{5.31b}$$

We remark that as a consequence of $Z_m = Z_u$ this is a general result for every non-trivial fixed point value $u^* \neq 0$. Therefore the power laws characteristic of the coexistence anomalies are not changed, either. For example, the dynamic susceptibility will have the asymptotic form

$$\chi_L(\vec q, \omega) \propto \bar\phi^2 \left| \frac{q^2}{\mu^2} - \frac{i\,\omega}{\lambda\,\mu^2} \right|^{-\epsilon/2} \quad , \tag{5.32}$$

leading to the anomalous lineshape of the longitudinal correlation function and the $\vec q$- and ω-divergences summarized in the center column of Table I. But the amplitudes of the scaling functions (5.28) are reduced with respect to the isotropic case, showing certain characteristic angle dependences.

In two special cases, namely if the wavevector is either parallel or perpendicular to the spontaneous order parameter, the $\vec{q}$-dependent contribution in the bracket of Eqs.(5.28a,b) vanishes, and the expressions (5.28) may be directly compared with the isotropic correlation functions for $a = 0$. Due to the renormalization-group invariant (2.22), asymptotically $m(\ell)^2 = m(1)^2\,(u^*/u(1))\,\ell^{-2+\epsilon}$, and hence the amplitude of the longitudinal susceptibility is proportional to $1/u^*$. Inserting the fixed point values (5.31a) and (2.21a), respectively, we therefore we find the following exact amplitude ratio of the longitudinal response functions in the dipolar and isotropic case

$$\frac{\chi_L(\vec{q},\omega)_{dipolar}}{\chi_L(\vec{q},\omega)_{isotropic}} = \frac{n-2}{n-1} \quad . \tag{5.33}$$

For $n = 3$ this universal amplitude ratio is $1/2$, in accord with the results of Pokrovsky [38], and Toh and Gehring [39], obtained in the framework of a spin-wave theory. If the wavevector $\vec{q}$ is neither parallel nor perpendicular to the magnetization, the amplitude of (5.32) will be further reduced.

For $n = d = 4$ there will be logarithmic corrections instead of power-law singularities. As is apparent from (5.29)-(5.33), the situation at $n = d = 2$ requires a separate discussion. Afterwards an immediate generalization to the general case $n \neq d$ will be possible.

C. The cases $n = d = 2$ and $n \neq d$

The form of the dipolar coexistence fixed point (5.31a), in comparison with (2.21a), suggests that the effective number of critical fluctuations is reduced to $n - 2$. Hence for $n = 2$ there are no more massless modes left that may lead to infrared singularities. Indeed, the transverse projector (5.27a) disappears in this case, and inserting $n = 2$ into (5.30) one arrives at the following asymptotic behavior for the ℓ-dependent parameters

$$g(\ell) \propto \ell^{-2} \tag{5.34a}$$

$$m(\ell) \propto \ell^{-1} \tag{5.34b}$$

$$u(\ell) \propto \ell^{-\epsilon} \quad . \tag{5.34c}$$

According to the preceding investigations in Section III and Section IV.B we may readily identify this as the signature of a crossover to a Gaussian theory. The fluctuations die out on leaving the critical region $T \approx T_c$, the σ-mode because of the formation of long-range order, and the π-excitation due to the dipolar coupling. The above discussion of the asymptotic theory thus does not apply for the two-component model, and the effective exponents characterizing the static susceptibilities, for example, will rather look similar to Fig.2.

Remarkably, there appears no inconsistency for the assumption of long-range order in two dimensions, as opposed to the isotropic model. This, again, is no contradiction to the Mermin-Wagner-Hohenberg theorem [22,23] which is based on the assumption of solely short-range isotropic interactions. We would like to emphasize once again that more complicated situations, as inhomogeneous ordering or topological excitations, are not within the scope of the present formalism.

We shall now drop the restriction (5.5) and return to the general situation where the number of components n and the spatial dimension d may be different. Let us begin with the case $d > n$; the upper limits of the sums in (5.1) and (5.4) are then n and the second term in (5.4) is absent. Hence the structure of the asymptotic Janssen-De Dominicis functional (5.25) is not altered. We only have to be careful when inverting matrix B of Eq.(5.8a), resulting in

$$B^{-1}_{\alpha\beta}(\vec{q},\omega) = \frac{1}{i\,\omega + \lambda_0\, q^2} \left(\delta_{\alpha\beta} - \frac{\lambda_0\, g_0}{i\,\omega + \lambda_0\,(g_0 + q^2) - \lambda_0\, g_0\, (1 - Q^2/q^2)}\, \frac{q_\alpha\, q_\beta}{q^2} \right) \quad (5.35a)$$

instead of (5.9); here we have used the notation

$$\vec{Q} = (q_1, \dots, q_{n-1}) \quad . \quad (5.35b)$$

In the coexistence limit the transverse propagators read

$$G^{\infty}_{0\,\tilde{\pi}^\alpha \pi^\beta}(\vec{q},\omega) = \frac{1}{-i\,\omega + \lambda_0\, q^2} \left(\delta_{\alpha\beta} - \frac{q_\alpha\, q_\beta}{Q^2} \right) \quad (5.36a)$$

$$G^{\infty}_{0\,\tilde{\pi}^\alpha \tilde{\pi}^\beta}(\vec{q},\omega) = \frac{2\,\lambda}{\omega^2 + \lambda_0^2\, q^4} \left(\delta_{\alpha\beta} - \frac{q_\alpha\, q_\beta}{Q^2} \right) \quad , \quad (5.36b)$$

and for the longitudinal correlation functions one arrives at

$$G^{\infty}_{0\,\tilde{\sigma}\sigma}(\vec{q},\omega) = \frac{1}{\lambda_0 m_0^2} \left(1 + \frac{u_0}{6} \int_k \frac{1}{\left(\frac{\vec{q}}{2} - \vec{k}\right)^2} \frac{1}{\frac{-i\,\omega}{2\,\lambda_0} + \frac{q^2}{4} + k^2} \times \right. \quad (5.37a)$$

$$\left. \times \left[n - 2 - \frac{Q^2\, K^2 - \left(\vec{Q}\,\vec{K}\right)^2}{\left(\frac{Q^2}{4} + K^2\right)^2 - \left(\vec{Q}\,\vec{K}\right)^2} \right] \right)$$

$$G^{\infty}_{0\,\sigma\sigma}(\vec{q},\omega) = \frac{1}{\lambda_0\, m_0^2}\,\frac{u_0}{6}\,\mathrm{Re}\int_k \frac{1}{\left(\frac{q^2}{4}+k^2\right)^2-\left(\vec{q}\,\vec{k}\right)^2}\,\frac{1}{\frac{-i\,\omega}{2\,\lambda_0}+\frac{q^2}{4}+k^2}\times$$
$$\times\left[n-2-\frac{Q^2\,K^2-\left(\vec{Q}\,\vec{K}\right)^2}{\left(\frac{Q^2}{4}+K^2\right)^2-\left(\vec{Q}\,\vec{K}\right)^2}\right] \quad , \tag{5.37b}$$

which obviously generalizes Eq.(5.28).

In particular the renormalization constants are identical to the previous results; therefore for $d = 4$ and $n = 3$ there will be logarithmic corrections characteristic for the upper critical dimension, and for $d = 3, 4$ and $n = 2$ a crossover to a Gaussian behavior will be found. Thus for $d > n$ there are no specifically interesting modifications with respect to the cases under discussion before.

In the opposite situation, $n > d$, it is crucial whether the spontaneous order is created in one of the directions that are affected by the dipolar interaction, or not. One would expect the former situation to apply, because the dipolar forces favor condensation. Yet if there are further anisotropies present, even transitions between different orientations e.g. of the magnetization of two-dimensional Heisenberg ferromagnets may occur [40]; consequently we shall take both possibilities into account. The modified matrix $B^{-1}_{\alpha\beta}$ keeps the form (5.35a), albeit with

$$\vec{Q} = (q_{n-d+1}, \ldots, q_{n-1}) \quad , \tag{5.38a}$$

if the σ-modes are subject to the dipol-dipol forces, and

$$\vec{Q} = (q_1, \ldots, q_d) \quad , \tag{5.38b}$$

if they are not. In the first case there will be $d-1$ transverse propagators of the type (5.36) with $\alpha, \beta = n-d+1, \ldots, n-1$, and $n-d$ propagators

$$G^{\infty}_{0\,\tilde{\pi}^\alpha\pi^\beta}(\vec{q},\omega) = \frac{1}{-i\,\omega+\lambda_0\,q^2}\,\delta_{\alpha\beta} \tag{5.39a}$$
$$G^{\infty}_{0\,\tilde{\pi}^\alpha\tilde{\pi}^\beta}(\vec{q},\omega) = \frac{2\,\lambda}{\omega^2+\lambda_0^2\,q^4}\,\delta_{\alpha\beta} \tag{5.39b}$$

with $\alpha, \beta = 1, \ldots, n-d$ that are equivalent to those of the isotropic model. If the σ-field is not affected by the dipolar interaction, the first d transverse modes have the form (5.36), while the remaining $n-d-1$ propagators are given by Eq.(5.39).

The σ-correlations and hence the longitudinal (with respect to the order parameter) response functions look like (5.37), where one has to insert (5.38a) or (5.38b), respectively. The relevant asymptotic Z-factor is again (5.29), and for $\epsilon > 0$ the dipolar coexistence fixed point determines the emerging coexistence anomalies. The leading term (in the limit $\vec{q} \to 0$ and $\omega \to 0$) of the longitudinal susceptibility is given by (5.32), accompanied by an angle-dependent scaling function which assumes a maximum if $\vec{Q} = 0$, and is otherwise smaller than unity.

Thus the results of Subsection B can be taken over quite easily to the general case with arbitrary number of components $n \geq 2$ and dimension $2 < d \leq 4$. The qualitative features are summarized in Table III.

VI. SUMMARY AND DISCUSSION

For isotropic models, $n - 1$ massless Goldstone modes appear in the phase with spontaneously broken $O(n)$-symmetry. Their origin is that no special direction of the spontaneous order parameter is preferred by the static Hamiltonian (2.1); hence an infinitesimal rotation of $\bar{\phi}$ costs no (free) energy. These gapless excitations induce so-called coexistence anomalies in the entire ordered phase, see Table I [11]. Within our formalism [11], which is based on the crossover theory of Amit and Goldschmidt [14] and generalizes Lawrie's work [8] to dynamical problems, their analytical form can be traced back to the renormalization group fixed point u_C^* (2.21a), or rather the accompanying anomalous dimension $\zeta_{m\,C}^*$ (2.21b). Remarkably, the coexistence limit ($T < T_c$, $\tilde{h} = 0$, $\vec{q} \to 0$, $\omega \to 0$) can be exactly represented by a one-loop theory for the two-point cumulants or the leading order (spherical limit) of the $1/n$-expansion for the two-point vertex functions.

For the application to real systems, in particular to solids displaying structural or magnetic phase transitions, quite naturally the question arises whether certain anisotropic terms in the free-energy functional will completely destroy the coexistence anomalies, or if, and then under which circumstances, remnants of the Goldstone excitations will still affect the static and dynamic properties. In order to elucidate this issue, we have studied two characteristic cases, namely (i) cubic anharmonicity, and (ii) dipol-dipol interaction. Generically, three possible situations are conceivable: (a) the anisotropy immediately suppresses the transverse fluctuations, as will be the case for uniaxial dipolar forces or any mechanism directly introducing transverse mass terms, and also for the cubic terms provided $n > n_c$. However, (b) there may also be a quasi-Goldstone behavior in an intermediate range within the crossover from the critical to the Gaussian regime, especially if near T_c fluctuations tend to restore the $O(n)$-symmetry, an example of which is the cubic model with $n < n_c$ [25-27]. Even more subtle is the dipolar interaction, where (c) although the model is no more invariant with respect to a continuous symmetry transformation, not all the transverse modes lose their massless character, but only their "effective" number is reduced. Hence we believe that the specific anisotropies we have treated here, besides their obvious importance for many real systems, supply a qualitative picture for what may happen rather generally.

In order to show that by means of our renormalization procedure a coherent description of the crossover from a critical theory to an asymptotically Gaussian model may be obtained, we discussed the time-dependent Ising model (or rather its field-theoretical representation) in the ordered phase. Although we have only pointed out the features of the single-component model for $T < T_c$, it should be quite obvious how generalizations to other situations could be achieved. The effective couplings appearing in Wilson's flow functions vanish in the limit $\ell \to 0$, and hence the power laws are eventually characterized by mean-field exponents. On the other hand, the expansion parameter of the perturbation series for the scaling functions themselves acquires a finite, but non-universal value. Of course, none of the resulting integrals will contain any infrared divergence. This description remains valid until additional contributions come into play which are considered as irrelevant with respect to the critical properties and thus have been omitted in the Hamiltonian.

After these preliminary discussions we have treated the complete crossover region for the relaxational models A and B with cubic anharmonicity. As for the isotropic system, the renormalization constants and flow functions were evaluated to one-loop order, and the ensuing flow equations were solved numerically. An important distinction has to be made between the cases $n \geq n_c = 4$ (to one-loop order), where a direct transition to the uncritical behavior takes place, and $n < n_c$, where in an intermediate wavenumber and frequency range coexistence-type singularities appear. The latter crossover scenario crucially requires that in the vicinity of the critical point the $O(n)$-symmetry is dynamically restored by fluctuation effects [25-27]. Lawrie's coexistence fixed point becomes unstable with respect to cubic distortions. This we have studied by using several values for the renormalized parameter y in Figs.5,7-9, although the latter is merely a qualitative measure of the microscopic anisotropy. If $y < 10^{-4}$, all the characteristic features of Goldstone anomalies in three dimensions are still present in the effective static exponent $2 - \eta^L_{eff}$ (4.23), and they are even more prominent in the frequency dependence of the dynamic response and correlation functions. Therefore experiments investigating the low-frequency behavior of the dynamical susceptibility would be very enlightening: We emphasize that any hint of a minimum of the corresponding effective exponent (compare Figs.7-9) would be a clear signature of the relevance of Goldstone modes. Table II provides an overview for the relevance of coexistence anomalies in the cubic case as the dimension d and the number of components n are varied.

On the other hand, the dipol-dipol interaction induces an anisotropy in momentum space, which also explicitly breaks the $O(n)$-symmetry. Yet the Goldstone modes are not entirely extinguished, but merely their number is in effect reduced by one. Hence while for $n = 2$ a crossover to an asymptotically uncritical theory takes place, for $n \geq 3$ coexistence anomalies persist, governed by the dipolar fixed point (5.31a). Unfortunately the considerably more complex structure of the dipolar propagators (5.12) of model A renders a complete crossover theory a rather cumbersome task. However, the asymptotic model, characterized by diverging mass $m(\ell) \to \infty$ and dipolar coupling $g(\ell) \to \infty$, may be considered on a very similar basis as in the isotropic case. Again, a one-loop theory for the two-point cumulants and the corresponding geometric series for the two-point vertex functions provides an exact representation in the ordered phase for $\vec{q} \to 0$ and $\omega \to 0$. As a consequence of the anomalous dimension (5.31b) remaining unaltered, the asymptotic power laws for $n \geq 3$ are identical to those we have found for $g = 0$, see Table I. The anisotropy appears in the $\vec{q}$-dependence of the relevant scaling functions (5.37), in particular with respect to the angle between order parameter and external wavevector. If the vector $\vec{Q}$ of Eqs.(5.35b) or (5.38a,b), respectively, is not equal to zero, the amplitudes are generally smaller than in the isotropic case [compare (5.33)]. A qualitative summary of the different scenarios is given in Table III.

We finally remark that all our results for the models discussed in this paper are in accord with exact theorems, especially concerning the possibility of long-range order in two dimensions [22,23]. Our formalism also allows extensions to more complicated models containing mode-coupling vertices, which usually constitute a relevant part of the dynamics of $O(n)$-symmetric systems. In this paper we rather intended to study the influence of typical anisotropies than modifications by these reversible forces. Most of the qualitative conclusions based on the relaxational models are expected to hold also for models with mode-coupling terms. For example, in ferromagnets the dipolar coupling $g(\ell)$ diverges faster than the longitudinal mass parameter $m(\ell)$ [compare Eq.(5.30)]. Hence near T_c the influence of the dipolar interaction should dominate the effects due to the formation of a finite order parameter, as has been anticipated by Kötzler, Kaldis, Kamleiter and Weber in the interpretation of their experiments on EuS below T_c [41]. A detailed analysis, however, requires a treatment of model J [1] appropriate for the dynamics of isotropic ferromagnets. Concerning the statics of such dipolar ferromagnets, we emphasize once again that the present theory is already complete in the coexistence limit.

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APPENDIX: CUBIC TWO-POINT VERTEX FUNCTIONS

In this Appendix we list the zero- and one-loop-diagrams (Fig.10), and the corresponding analytical results for the two-point vertex functions of the relaxational models A ($a=0$) and B ($a=2$) including weak cubic anisotropy. The integration over the internal frequency has already been performed via the residue theorem. (Compare Appendix B of Ref.[11].)

$\Gamma_{0\,\tilde{\pi}\pi}(\vec{q},\omega)$:

$$(a) = i\,\omega + \lambda_0\, q^a \left(\overline{m}_0^2 + q^2 + \frac{2\,b_0+v_0}{6}\,(m_0^2 - \overline{m}_0^2) \int_k \frac{1}{\left(\overline{m}_0^2+k^2\right)\left(m_0^2+k^2\right)} \right)$$

$$(b) = -\lambda_0\, q^a\, \frac{u_0^2}{3\,b_0}\, m_0^2 \int_k \frac{\left(\frac{\vec{q}}{2}+\vec{k}\right)^a}{m_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2}\; \frac{1}{\frac{i\,\omega}{\lambda_0} + \left(\frac{\vec{q}}{2}+\vec{k}\right)^a \left[\overline{m}_0^2+\left(\frac{\vec{q}}{2}+\vec{k}\right)^2\right] + \left(\frac{\vec{q}}{2}-\vec{k}\right)^a \left[m_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2\right]}$$

$$(c) = -\lambda_0\, q^a\, \frac{u_0^2}{3\,b_0}\, m_0^2 \int_k \frac{\left(\frac{\vec{q}}{2}+\vec{k}\right)^a}{\overline{m}_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2}\; \frac{1}{\frac{i\,\omega}{\lambda_0} + \left(\frac{\vec{q}}{2}+\vec{k}\right)^a \left[m_0^2+\left(\frac{\vec{q}}{2}+\vec{k}\right)^2\right] + \left(\frac{\vec{q}}{2}-\vec{k}\right)^a \left[\overline{m}_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2\right]}$$

$\Gamma_{0\,\tilde{\sigma}\sigma}(\vec{q},\omega)$:

$$(d) = i\,\omega + \lambda_0\, q^a \left(m_0^2 + q^2\right)$$

$$(e) = -\lambda_0 q^a\, \frac{n-1}{3}\, \frac{u_0^2}{b_0} m_0^2 \int_k \frac{\left(\frac{\vec{q}}{2}+\vec{k}\right)^a}{\overline{m}_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2}\; \frac{1}{\frac{i\omega}{\lambda_0} + \left(\frac{\vec{q}}{2}+\vec{k}\right)^a \left[\overline{m}_0^2+\left(\frac{\vec{q}}{2}+\vec{k}\right)^2\right] + \left(\frac{\vec{q}}{2}-\vec{k}\right)^a \left[\overline{m}_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2\right]}$$

$$(f) = -\lambda_0\, q^a\, 3\, b_0\, m_0^2 \int_k \frac{\left(\frac{\vec{q}}{2}+\vec{k}\right)^a}{m_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2}\; \frac{1}{\frac{i\,\omega}{\lambda_0} + \left(\frac{\vec{q}}{2}+\vec{k}\right)^a \left[m_0^2+\left(\frac{\vec{q}}{2}+\vec{k}\right)^2\right] + \left(\frac{\vec{q}}{2}-\vec{k}\right)^a \left[m_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2\right]}$$

$\Gamma_{0\,\tilde{\pi}\tilde{\pi}}(\vec{q},\omega)$:

$$(g) = -2\,\lambda_0\, q^a$$

$$(h) = -2\lambda_0 q^{2a}\, \frac{u_0^2}{3\,b_0}\, m_0^2\, \mathrm{Re} \int_k \frac{1}{\left[\overline{m}_0^2+\left(\frac{\vec{q}}{2}+\vec{k}\right)^2\right]\left[m_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2\right]} \times$$
$$\times \frac{1}{\frac{i\,\omega}{\lambda_0} + \left(\frac{\vec{q}}{2}+\vec{k}\right)^a \left[\overline{m}_0^2+\left(\frac{\vec{q}}{2}+\vec{k}\right)^2\right] + \left(\frac{\vec{q}}{2}-\vec{k}\right)^a \left[m_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2\right]}$$

$\Gamma_{0\,\tilde{\sigma}\tilde{\sigma}}(\vec{q},\omega)$:

$$(i) = -2\,\lambda_0\, q^a$$

$$(j) = -2\,\lambda_0\, q^{2a}\, \frac{n-1}{6}\, \frac{u_0^2}{b_0}\, m_0^2\, \mathrm{Re} \int_k \frac{1}{\left[\overline{m}_0^2+\left(\frac{\vec{q}}{2}+\vec{k}\right)^2\right]\left[\overline{m}_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2\right]} \times$$
$$\times \frac{1}{\frac{i\,\omega}{\lambda_0} + \left(\frac{\vec{q}}{2}+\vec{k}\right)^a \left[\overline{m}_0^2+\left(\frac{\vec{q}}{2}+\vec{k}\right)^2\right] + \left(\frac{\vec{q}}{2}-\vec{k}\right)^a \left[\overline{m}_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2\right]}$$

$$(k) = -2\,\lambda_0\, q^{2a}\, \frac{3}{2}\, b_0\, m_0^2\, \mathrm{Re} \int_k \frac{1}{\left[m_0^2+\left(\frac{\vec{q}}{2}+\vec{k}\right)^2\right]\left[m_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2\right]} \times$$
$$\times \frac{1}{\frac{i\,\omega}{\lambda_0} + \left(\frac{\vec{q}}{2}+\vec{k}\right)^a \left[m_0^2+\left(\frac{\vec{q}}{2}+\vec{k}\right)^2\right] + \left(\frac{\vec{q}}{2}-\vec{k}\right)^a \left[m_0^2+\left(\frac{\vec{q}}{2}-\vec{k}\right)^2\right]}$$

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TABLE I. Coexistence anomalies of the isotropic relaxational models.

	Model A	Model B
$\chi_L(\vec{q},0)$	$\propto q^{-\epsilon}$	$\propto q^{-\epsilon}$
$\mathrm{Re}\,\chi_L(0,\omega/q^a)$	$\propto \omega^{-\epsilon/2}$	$\propto (\omega/q^2)^{-\epsilon/2}$
$G_L(0,\omega/q^a)$	$\propto \omega^{-1-\epsilon/2}$	$\propto (\omega/q^2)^{-\epsilon}$

TABLE II. The influence of cubic anisotropy.

	$d=2$	$d=3$	$d=4$
$n=1$	crossover to a	Gaussian theory,	as for $v=0$
$2 \leq n \leq 4$	via $b_C^* = \frac{6\,\epsilon}{n-1}$	$\to$ anomalies	log. corr.
$n > 4$	directly to 0,	no anomalies	log. corr.

TABLE III. The influence of dipolar interactions.

	$d=2$	$d=3$	$d=4$
$n=1$	crossover to a	Gaussian theory,	as for $g=0$
$n=2$	crossover to a	Gaussian theory,	no anomalies
$n \geq 3$	$u_{CD}^* = \frac{6\,\epsilon}{n-2}$	$\to$ anomalies	log. corr.

FIGURE CAPTIONS

FIG.1. Flow of the effective coupling $u_{eff}(\ell)$ for $n = 1$ and $\epsilon = 1$: (a) flow diagram, (b) universal crossover.

FIG.2. Effective exponent $2 - \eta_{eff}$ of the static susceptibility in the case $n = 1$ and $\epsilon = 1$.

FIG.3. Basic elements of the dynamical perturbation theory for the cubic relaxational models below T_c.

FIG.4. Schematic flow diagrams for $m = 0$ ($T = T_c$) in the case $n = 2$ (a) and $n = 8$ (b); $\epsilon = 1$.

FIG.5. Flow of the effective coupling $b_{eff}(\ell)$ for several values of the anisotropy parameter y ($y = 0$ corresponds to the isotropic case); $n = 2$ and $\epsilon = 1$.

FIG.6. Schematic flow diagrams for $m > 0$ ($T < T_c$) and $v < 0$ in the case $n = 2$ (a) and $n = 8$ (b).

FIG.7. Effective exponents $2 - \eta^T_{eff}$ (a) and $2 - \eta^L_{eff}$ (b) of the static susceptibilities for several values of the anisotropy parameter y; $n = 2$ and $\epsilon = 1$.

FIG.8. Frequency dependence of $\mathrm{Re}\,\chi_L(0, \omega)$ (a) and of $G_L(0, \omega)$ (b) for the cubic model A with different values of the anisotropy parameter y; $n = 2$ and $\epsilon = 1$.

FIG.9. Frequency dependence of $\mathrm{Re}\,\chi_L(0, \omega/q^2)$ (a) and of $G_L(0, \omega/q^2)$ (b) for the cubic model B with different values of the anisotropy parameter y; $n = 2$ and $\epsilon = 1$.

FIG.10. Zero- and one-loop diagrams for the two-point vertex functions of the cubic relaxational models A ($a = 0$) and B ($a = 2$). The corresponding explicit analytical expressions are listed in the Appendix.