Viability of competing field theories for the driven lattice gas.

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It has recently been suggested that the driven lattice gas should be described by a novel field theory in the limit of infinite drive. We review the original and the new field theory, invoking several well-documented key features of the microscopics. Since the new field theory fails to reproduce these characteristics, we argue that it cannot serve as a viable description of the driven lattice gas. Recent results, for the critical exponents associated with this theory, are re-analyzed and shown to be incorrect.

The critical behavior of the driven lattice gas (DLG) [1] has been the subject of some debate, ever since the first Monte Carlo simulations [2] and field theoretic predictions [3,4] were found to give differing values for the order parameter exponent β. This discrepancy has led to developments in different directions: some researchers [5,6] have modified the simulation data analysis, invoking anisotropic finite size scaling [7], while others [8,9] have suggested that the original field theory might be deficient in the limit of infinite drive, proposing [8,9] and analyzing [10] an alternate coarse-grained theory instead. In this communication, we review both the original [3,4] and the alternate [8–10] field theory, in the light of Monte Carlo simulation data. We first document that the alternate theory is *not* a coarse-grained description of the driven lattice gas, since it fails to exhibit several well-established properties of the microscopic model. In a second step, we re-analyze the proposed theory, assuming that it might describe some other, yet to be determined, microscopics. We show that the renormalization group analysis of Ref. [10] is *seriously flawed*, resulting in incorrect exponents and a proliferation of uncontrolled infrared singularities.

We begin with a brief summary of the background. Microscopically, the DLG is a simple ferromagnetic Ising lattice gas, half-filled and coupled to a heat bath at temperature T, in which particles jump to empty nearest-neighbor sites subject to the usual Ising energetics and a uniform driving force E acting along a particular lattice direction. Thus, the effect of E is identical to adding a locally linear potential. Clearly, $E = 0$ corresponds to the equilibrium Ising model. On the other hand, even $E = \infty$ can be realized if Metropolis rates are used: Simply accept/forbid all forward/backwards jumps. Since large values of E accentuate the nonequilibrium features of this system, most simulations have been performed at $E \gtrsim 50$, in units of the Ising coupling constant.

The driven lattice gas and many of its variants have attracted considerable attention since they evolve into simple nonequilibrium steady states displaying a wealth of counterintuitive characteristics [11]. Two of its most remarkable features are (*i*) the discontinuity singularity of the structure factor $S(\mathbf{k})$ [1,12], which is intimately connected to an r^{-d} decay (in d dimensions) of the two-point correlations [13,14], and (*ii*) the emergence of nontrivial three-point correlations [15] in the disordered phase, corresponding to the violation of the Ising symmetry by E (which drives particles and holes in opposite directions). Such dramatically "non-Ising" characteristics are easily observed in Monte Carlo simulations, at intermediate and large driving fields. They are also confirmed in a high-temperature series expansion, derived directly from the microscopic dynamics [13,16].

These observations from Monte Carlo simulations play a crucial role in identifying the correct field theory. A basic tenet in the study of critical phenomena is that a microscopic model and its coarse-grained field theory should possess the same symmetries, if they are to belong into the same universality class. For the driven lattice gas, the data on the structure factor indicate that the theory is highly *anisotropic*. Moreover, the detailed behavior of the discontinuity singularity, upon approaching the origin in wave vector space from different directions, informs us *precisely how* the familiar Ornstein-Zernike form is modified. Generically, we find [11] that

$$R \equiv \frac{\lim_{|\mathbf{k}_\perp|\to 0} S(\mathbf{k}_\perp, k_\parallel = 0)}{\lim_{k_\parallel \to 0} S(\mathbf{k}_\perp = 0, k_\parallel)} > 1 \tag{1}$$

above criticality, and $R \to \infty$ upon approaching T_c. The subscripts distinguish the parallel ($\parallel$) and transverse ($\perp$) subspaces, with respect to the drive direction. Just as significantly, the non-vanishing three-point functions demonstrate that the usual "up-down" symmetry of the Ising model is broken.

These key features of the microscopics must be reflected in any viable continuum theory for the driven lattice gas. We first consider the original field theory [3,4]. It is based on a Langevin equation, in continuous space and time, which describes the stochastic evolution of the local particle density $\rho(\mathbf{x},t)$. In terms of $\phi \equiv 2\rho - 1$, the equation reads:

$$\partial_t \phi = \lambda \left\{ \left(\tau_\perp - \nabla_\perp^2\right) \nabla_\perp^2 \phi + \tau_\parallel \nabla_\parallel^2 \phi + \mathcal{E} \nabla_\parallel \phi^2 + \frac{g}{3!} \nabla_\perp^2 \phi^3 \right\} - \nabla \xi \ . \tag{2}$$

The Langevin noise term reflects the fast degrees of freedom:

$$\begin{aligned} \langle \xi(\mathbf{x},\mathbf{t}) \rangle &= 0 \\ \langle \nabla \xi(\mathbf{x},t) \nabla' \xi(\mathbf{x}',t') \rangle &= -2 \left(n_\perp \nabla_\perp^2 + n_\parallel \nabla_\parallel^2 \right) \delta(\mathbf{x}-\mathbf{x}') \delta(t-t') \ . \end{aligned}$$

We emphasize that (*i*) all coefficients are strictly positive *except* possibly $\tau_\perp$ and/or $\tau_\parallel$ which control criticality (see below) and (*ii*) independent from one another (i.e., not related by symmetry). The parameter λ sets the time scale.

This theory contains two closely linked key ingredients: First, there is a driving term, $\mathcal{E}\nabla_\parallel \phi^2$, where $\mathcal{E}$ denotes the coarse-grained drive (a naive continuum limit gives $\mathcal{E} \propto \tanh(E/T)$). This term is *required* to break the Ising "up-down" ($\phi \to -\phi$) symmetry. Second, the theory is highly anisotropic, with two *different* diffusion coefficients $\tau_\parallel$ and $\tau_\perp$. In particular, it predicts an equal-time structure factor,

$$S(\mathbf{k}) = \frac{n_\perp k_\perp^2 + n_\parallel k_\parallel^2}{\tau_\perp k_\perp^2 + \tau_\parallel k_\parallel^2 + O(k^4)} \tag{3}$$

in the disordered phase. This S generates a discontinuity singularity $R = (n_\perp \tau_\parallel)/(n_\parallel \tau_\perp)$. To ensure that the *observed* behavior is faithfully reproduced, we demand $n_\perp \tau_\parallel > n_\parallel \tau_\perp$ in the disordered phase. Moreover, criticality *must* be marked by $\tau_\perp = 0$ at *positive* $\tau_\parallel$ if the divergence of S is to be captured correctly. To summarize, the two key features of the original Langevin equation are unambiguously supported by the Monte Carlo data for the microscopic model.

We comment briefly on the issue of *finite* versus *infinite* fields. In all Monte Carlo simulations, the current is observed to saturate as E increases. This saturation is reflected by $\mathcal{E} \propto \tanh(E/T) \to 1$ in the original field theory. Therefore, this theory holds equally well for any nonzero value of the microscopic drive. Furthermore, simulations using *Metropolis rates* with $E = 50$, 100 and ∞ have been performed. The results are (statistically) indistinguishable! Such sensible behavior is entirely consistent with this theory.

The discrepancies arise when critical exponents are measured, specifically the order parameter exponent β, and compared to field theoretic predictions. The original field theory, due to the vanishing of $\tau_\perp$ at positive $\tau_\parallel$, naturally leads to anisotropic scaling of wave vectors: $k_\parallel \sim k_\perp^{1+\Delta}$ in the critical region, with a nontrivial anisotropy exponent Δ. Three important consequences are that, first, the upper critical dimension d_c is shifted to 5, and second, the theory predicts $\Delta = 1 + (5-d)/3$ and $\beta = 1/2$ *exactly*, i.e., to all orders in perturbation theory. The values obtained by simulations differ, depending on the method used to analyze the data. If a careful anisotropic finite size analysis [7] is used, based on system sizes consistent with the field-theoretic scaling, i.e., $L_\parallel / L_\perp^{1+\Delta} = const$, the field-theoretic exponents result in good data collapse for a number of different observables [5,6]. However, data for isotropic systems, $L_\parallel / L_\perp = const$, appear to indicate an order parameter exponent around 0.23 [2]. Since most of the data were taken at very large fields, some authors [8,9] have suggested that the origin of the discrepancies does not reside in the data analysis. Instead, they argue that the standard field theory does not capture the $E \to \infty$ limit correctly and propose an alternate theory. It is based on the Langevin equation:

$$\partial_t \phi = \lambda \left\{ \left(\tau_\perp - \nabla_\perp^2\right) \nabla_\perp^2 \phi - \nabla_\parallel^2 \nabla_\perp^2 \phi + \frac{g}{3!} \nabla_\perp^2 \phi^3 \right\} - \nabla \xi \ . \tag{4}$$

With minor renamings of parameters [17], this is Eq. (1) of Ref. [10]. The vanishing of $\tau_\perp$ marks the critical point. The noise satisfies (Eq. (2) of Ref. [10]):

$$\begin{aligned} \langle \xi(\mathbf{x},t) \rangle &= 0 \\ \langle \nabla \xi(\mathbf{x},t) \nabla' \xi(\mathbf{x}',t') \rangle &= -2\lambda \left(\nabla_\perp^2 + \frac{1}{2} \nabla_\parallel^2 \right) \delta(\mathbf{x}-\mathbf{x}') \delta(t-t') \ . \end{aligned}$$

Two key terms appearing in the original field theory are absent in this one, namely,

- the driving term $\nabla_\parallel \phi^2$, and
- a diffusion term $\tau_\parallel \nabla_\parallel^2 \phi$ for the parallel direction.

Since the driving term is absent, the alternate field theory obeys the Ising "up-down" ($\phi \to -\phi$) symmetry. Thus, three-point functions are identically *zero* in this theory, for all $T \geq T_c$. This prediction is in serious disagreement with existing Monte Carlo data! While one may argue that a field theory need not reproduce all of the microscopic detail of the underlying lattice model, one should be very cautious before endowing it with a *higher symmetry*: This is only justified if a high-symmetry fixed point exists and can be shown, via an explicit renormalization group calculation, to be *stable* against perturbations by symmetry-breaking operators. Neither is the case here.

The absence of the parallel diffusion term also has serious consequences. Eq. (4) generates a steady-state structure factor:

$$S(\mathbf{k}) = \frac{k_\perp^2 + \frac{1}{2}k_\parallel^2}{k_\perp^2\,(k^2 + \tau_\perp)} \tag{5}$$

which ought to be a good approximation at high temperatures. Yet, for $k_\parallel \neq 0$ it predicts a *divergence* along the *whole* $k_\perp = 0$ line, at *any* $T > T_c$. This stands in glaring contrast to the Monte Carlo results for the disordered phase, where *all* measured structure factors are found to be finite *everywhere* in k-space.

Since Eq. (4) fails to reproduce the most basic properties of the microscopic model, we conclude that it is not a viable field theory for the driven lattice gas. It may, however, describe some as yet unknown microscopics. Therefore, we now proceed to analyze the field theory, defined by Eq. (4), in its own right.

Following Ref. [10], we recast Eq. (4) as a dynamic functional [18]:

$$\mathcal{L}[\tilde{\phi},\phi] = \int d^dx dt \left\{ \tilde{\phi}\left[\partial_t\phi + \lambda\left(\nabla_\parallel^2\nabla_\perp^2 + (\nabla_\perp^2)^2 - \tau_\perp\nabla_\perp^2\right)\phi - \lambda\frac{g}{3!}\nabla_\perp^2\phi^3\right] - \lambda\tilde{\phi}\left(\nabla_\perp^2 + \frac{1}{2}\nabla_\parallel^2\right)\tilde{\phi}\right\} \tag{6}$$

We first note that Eq. (6) describes a theory with a four-point coupling $\tilde{\phi}\nabla_\perp^2\phi^3$ and anisotropic free propagators as given in Ref. [10]. Therefore, the combinatorics of this theory is identical to that of Model B, which reduces to ϕ^4-theory in the static limit. For such theories, it is well known [19] that the one-loop result for the exponent ν (denoted $\nu_\perp$ in Ref. [10]) is determined by *combinatorics alone*, i.e., the explicit expressions for the Feynman integrals are not required. This is most easily seen by calculating in the critical theory, where $\tau_\perp = 0$, with insertions of $\lambda\tilde{\phi}\nabla_\perp^2\phi$. We denote one-point irreducible vertex functions with $\tilde{n}$ (n) external $\tilde{\phi}$ (ϕ) legs and m insertions by $\Gamma_{\tilde{n}n}^{(m)}$. At one-loop order, there are two primitively divergent vertex functions, namely $\Gamma_{11}^{(1)}$ and $\Gamma_{13}^{(0)}$. Both of these consist of a zero-loop term and a one-loop contribution. Each one-loop contribution consists of a combinatoric factor, the appropriate powers of the coupling constant and the external momentum, and a loop integral. The *key* simplification here is that the loop *integrals* for $\Gamma_{11}^{(1)}$ and $\Gamma_{13}^{(0)}$ are *identical*, independent of the detailed forms of the free propagators. Thus, the two one-loop contributions differ only by a simple factor which is purely combinatoric in origin. As a result, one obtains to first order in $\epsilon \equiv d_c - d$, for *all* of these theories:

$$\nu = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2) \tag{7}$$

Since the authors of Ref. [10] have chosen to calculate at finite $\tau_\perp$, let us illustrate how this result emerges in their case. No insertions are needed here so the upper index of $\Gamma_{\tilde{n}n}^{(m)}$ can be dropped. Keeping track of coupling constants and signs, and taking care of the T_c shift, we can write the two bare vertex functions Γ_{11} and Γ_{13} in the form

$$\begin{aligned} \Gamma_{11} &= i\omega + \lambda k_\perp^2 k^2 + \lambda k_\perp^2 \tau_\perp\left[1 + \frac{1}{2}gI_1\right] \\ \Gamma_{13} &= \lambda g k_\perp^2\left[1 - \frac{3}{2}gI_2\right] \end{aligned} \tag{8}$$

Here, the factors $1/2$ and $-3/2$ arise from combinatorics while the integrals I_1 and I_2 are easily computed in dimensional regularization, resulting in

$$I_1 = \frac{3}{(4\pi)^2\epsilon}\left[1 + O(\epsilon)\right] \quad \text{and} \quad I_2 = \frac{3}{(4\pi)^2\epsilon}\left[1 + O(\epsilon)\right] \tag{9}$$

We notice immediately that the simple ϵ-poles of I_1 and I_2 are *identical.* Thus, their numerical prefactor can be absorbed into the definition of the coupling constant, leaving us with one-loop corrections to ν that are purely combinatoric in origin. Completing the calculation at finite $\tau_\perp$, this provides the key to Eq. (7). Only at *two-loop* order do the detailed forms of the free propagators come into play. Then, of course, exponents are also no longer determined by combinatorics alone.

In Ref. [10], the exponent ν is quoted as $(1+\epsilon/4)/2$, indicating the presence of a computational error. More seriously, however, there are deeper flaws in this theory. Recall that the steady-state structure factor, Eq. (5), diverges along the whole $k_\perp = 0$ line, even for $\tau_\perp > 0$. As a result, the theory is plagued by *infrared* singularities, which are entirely *unrelated* to criticality, and unrenormalizable divergences. We note, for completeness, that one can, of course, regularize such singularities by *re-introducing* the diffusion term $\tau_\parallel \nabla_\parallel^2 \phi$ into Eq. (4). Then, however, one should also reconsider the two *fourth-order* derivative terms. At two-loop order, these will acquire *different* primitive divergences, so that an additional coupling constant ρ^2 is required, appearing in Eq. (4) as $\rho^2(\nabla_\perp^2)^2\phi + \nabla_\parallel^2 \nabla_\perp^2 \phi$.

To summarize, we have shown that the field theory proposed by Garrido, de los Santos and Muñoz [8,9] fails to reproduce the key features of the driven lattice gas. Predicting *infinite* structure factors and *zero* three point correlations (for *all* temperatures above criticality), it cannot be a viable continuum model for the latter. Accepting it as a representation of some other, as yet undetermined, microscopic model, we carry out a standard analysis. First, we find that the one-loop calculation of Ref. [10] is incorrect. Second, beyond one-loop order, uncontrolled infrared singularities proliferate, rendering the field theory unrenormalizable. In contrast, the original field theory [3,4] is consistent with the fundamental symmetries of the driven lattice gas, for any value of the drive. Its predictions for 2- and 3-point functions in the disordered phase are in good agreement with simulation results. Based on the phenomenology of $S(\mathbf{k})$ near criticality, it plumbs the consequences of a highly anisotropic scaling limit, $k_\parallel \sim k_\perp^{1+\Delta} \to 0$. To test its predictions against Monte Carlo simulations, this limit should be respected in the choice of system sizes, i.e., $L_\parallel \sim L_\perp^{1+\Delta}$. If, instead, simulations and finite-size analysis are performed with disregard for such strong anisotropies, complications from extraneous scaling variables [5] or inconsistencies [20] can be expected. In a more exotic scenario, such simulations may indicate a new type of low temperature phase, quite distinct from the ordinary Ising-like one.

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