Interactions, Distribution of Pinning Energies, and Transport in the Bose Glass Phase of Vortices in Superconductors

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We study the ground state and low energy excitations of vortices pinned to columnar defects in superconductors, taking into account the long–range interaction between the fluxons. We consider the "underfilled" situation in the Bose glass phase, where each flux line is attached to one of the defects, while some pins remain unoccupied. By exploiting an analogy with disordered semiconductors, we calculate the spatial configurations in the ground state, as well as the distribution of pinning energies, using a zero–temperature Monte Carlo algorithm minimizing the total energy with respect to all possible one–vortex transfers. Intervortex repulsion leads to strong correlations whenever the London penetration depth exceeds the fluxon spacing. A pronounced peak appears in the static structure factor $S(q)$ for low filling fractions $f \leq 0.3$. Interactions lead to a broad Coulomb gap in the distribution of pinning energies $g(\epsilon)$ near the chemical potential μ, separating the occupied and empty pins. The vanishing of $g(\epsilon)$ at μ leads to a considerable reduction of variable–range hopping vortex transport by correlated flux line pinning.

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I. INTRODUCTION

Recently, the static and dynamic properties of flux lines in high–temperature superconductors subject to an external magnetic field **B** have been intensively studied both experimentally and theoretically [1]. From a theoretical point of view, the importance of thermal fluctuations [2] and the strong influence of point defects [3,4] pose a variety of interesting problems, leading to strikingly rich and complex phase diagrams [5]. For the purpose of applying (high–temperature) superconductors in external magnetic fields, an effective vortex pinning mechanism is essential, in order to minimize dissipative losses caused by the Lorentz–force induced movement of flux lines across the sample. Therefore, in addition to point defects, the influence of extended or *correlated* disorder, promising stronger pinning effects, on vortex transport properties has been considered. Experimentally, it has been found that by high–energy ion irradiation linear damage tracks are formed in the material. These columnar pins bind the flux lines very strongly, thus significantly increasing the critical current and shifting the irreversibility line upwards [6]. On the theoretical side, the pinning of flux lines to linear defects and the ensuing transport properties have been studied by Lyuksyutov, [7] and in detail by Nelson and Vinokur [8]. In Ref. [8], correlated pinning by twin and grain boundaries is discussed as well, a topic which has been more thoroughly investigated recently by Marchetti and Vinokur [9].

Similar to the physics of flux lines in pure systems, [2] the statistical mechanics of vortices interacting with columnar pinning centers, which are aligned parallel to the magnetic field (Fig. **??**), may be mapped onto the quantum mechanics of bosons in two dimensions [8] (see Sec. II A). At high temperatures, one finds an entangled liquid of delocalized vortices, separated by a sharp second–order phase transition (corresponding to a boson localization transition [10]) from a low–temperature phase of lines strongly attached to the columnar pins. This Bose glass phase is characterized by an infinite tilt modulus, and was shown to be stable over a certain range of tipping angles of **B** away from the direction parallel to the linear pinning centers (transverse Meissner effect) [8]. In addition, the theory also suggests a Mott insulator phase at low temperatures, with both tilt and compressional moduli acquiring infinite values, occurring when the vortex density matches the density of columnar pins [8]. The ensuing phase diagram is sketched schematically in Fig. **??**. Since the Mott insulator phase is predicted to occur at low temperatures deep inside a regime with very slow dynamics, it may actually be difficult to access for kinetic reasons. It is also possible that the Mott insulator vanishes entirely for sufficiently strong disorder. Some of the predictions of Ref. [8] have now been tested and confirmed by experiments on a number of different samples [11–16]. E.g., the scaling behavior near the boson localization transition was studied, [11] and in addition even some properties of the Bose glass itself have been investigated to some detail [14–16].

In the Bose glass phase, the linear resistivity vanishes for low external currents $J \ll J_c$, and the most important mechanism for vortex transport is "tunneling" between different columnar defect sites via the formation of a pair of "superkinks" [8], i.e.: a fluxon forms a tongue-like double kink extending from one pin to another, usually a distant defect with nearly the same energy, such that the tunneling probability between the columnar de-

fects is optimized (Fig. **??**). This is very closely related to variable–range hopping transport of charge carriers in disordered semiconductors, [17] and leads to the highly nonlinear expression [18]

$$\mathcal{E} = \rho_0 J \exp\left[-(E_{\rm K}/T)(J_0/J)^p\right] \qquad (1.1)$$

for the current–induced electric field $\mathcal{E}(J)$ in the limit of very low currents $J \ll J_c$ (Fig. **??**). In Eq. (1.1), ρ_0 denotes the normal–state resistivity, E_K is a typical kink energy, and J_0 sets the current scale, see Eq. (1.2) and Sec. III D. By assuming a short–range repulsive interaction between the flux lines, Nelson and Vinokur could identify p with the Mott variable–range–hopping exponent in two dimensions, $p = 1/3$.

Similarly, for the case of parallel planar defects containing the **B** axis, Marchetti and Vinokur have found an analagous formula, with $p = 1/2$, demonstrating that twin and grain boundaries serve as even better pinning centers for flux lines [9]. Returning to linear defects, Hwa, Le Doussal, Nelson, and Vinokur have suggested that an even more drastic reduction of vortex motion may be obtained by introducing a controlled splay, i.e., dispersion in the orientation of the columnar pins [19]. A new low–temperature phase, the "splayed glass", is found which is related to the Bose glass, but has a finite equilibrium tilt modulus (however, the system is dynamically frozen due to diverging energy barriers preventing the relaxation of small externally induced tilts). Flux–line transport in this phase should be more strongly inhibited than in the Bose glass for two reasons: (i) variable–range hopping must now proceed via a much slower process of selecting both energy *and* angle; as a consequence, the exponent in the nonlinear current–voltage characteristics (1.1) is found to be enhanced to $p = 3/5$ [19]. (ii) Flux lines in the splayed glass ground state are entangled and can only move by cutting through each other, at the expense of a considerable amount of (free) energy [20].

In the above studies, [8,9,19] the intervortex repulsion was only treated approximatively, using order–of–magnitude estimates. Specifically, it was assumed as being essentially short–range. However, in fact the interaction range is set by the London penetration depth λ, which diverges at the critical temperature, and thus the intervortex repulsion may effectively become very long–range, namely when λ is large compared to the average spacing between flux lines, $\lambda \gg a_0 = (4/3)^{1/4}(\phi_0/B)^{1/2}$ (here $\phi_0 = hc/2e$ denotes the magnetic flux quantum). These strong long–range interactions will then determine the width and shape of the distribution of pinning energies, which, as has been emphasized by Gurevich, [21] in turn strongly influences vortex transport properties [8].

Simultaneous observations of both columnar defects and magnetic flux lines, using scanneling tunneling microscopy, have recently been reported for $NbSe_2$ samples in the "underfilled" regime, where the number of columnar pins $N_{\rm D}$ exceeds the number of vortices N [12]. For the high–temperature superconducting material $Bi_2Sr_2CaCu_2O_{8+\delta}$ (BSCCO), a combined chemical etching and magnetic decoration technique has been similarly successful in determining the positions of both the columnar defects and vortices; and indeed cases have been found, in which the strongly correlated spatial distribution of the flux lines suggests that the intervortex repulsion clearly dominates in magnitude over the unavoidable fluctuations in pinning energies stemming from the variations in the ion track diameters [13]. This would already imply a noticeable reduction of vortex transport; for the width of the distribution function $g(\epsilon)$ of the vortex pinning energies ϵ, is then no longer determined by the comparatively small variations in pin diameters, but by the much larger typical interaction scale. Thus the value of $g(\epsilon)$ at the chemical potential has to be reduced, leading to a greater value for the current scale in Eq. (1.1),

$$J_0 = c/\phi_0 g(\mu) d^3 \quad , \qquad (1.2)$$

where d is the average defect distance. Therefore, by a suitable "tailoring" of $g(\epsilon)$, the prefactor of the power law in the exponent of the current–voltage characteristics (1.1) may be adjusted in order to achieve more effective pinning.

Moreover, by further exploiting the analogy to two–dimensional bosons localized at randomly distributed defect sites, one has to at least consider the possibility that these spatial correlations may produce a "Coulomb" gap [17] in the distribution of pinning energies $g(\epsilon)$ near the chemical potential μ, which separates the filled and empty energy levels. In the limit of infinitely long–range interactions, $\lambda \to \infty$, one would expect $g(\epsilon)$ to vanish near μ according to a power law

$$g(\epsilon) \propto |\epsilon - \mu|^s \quad , \qquad (1.3)$$

see Sec. III B. This of course invalidates Eq. (1.1), because there a finite value of $g(\mu) \neq 0$ has been implicitly assumed [8]. Instead, a simple re–analysis of the optimization procedure leading to Eq. (1.1) shows that the exponent $p_0 = 1/3$ is to be replaced by the *larger* value $p = (s+1)/(s+3)$, thus strongly suppressing vortex motion (see Sec. III D; also the current scale J_0 becomes modified).

In order to determine the gap exponent s, and thus p, and also to understand down to which values of λ/d and a_0/d the intervortex repulsion may still be considered as sufficiently long–range to produce a correlation–induced Coulomb gap, one has to resort to numerical simulations in the spirit of previous investigations of the so–called Coulomb glass (localized charge carriers interacting with a $1/r$ potential) [22–24]. Except for a mean–field analysis for the Coulomb potential problem, [17] the only related prior investigation appears to be a scaling analysis of logarithmically interacting vortices in two dimensions by Fisher, Tokuyasu, and Young [25]. We have thus performed an extensive Monte–Carlo study of a two–dimensional system of N localized (i.e., classical) particles subject to a Bessel function potential, representing

the interacting flux lines, as a function of the two parameters λ/d and $f = N/N_{\rm D} = d^2/a_0^2$. We find that for fillings $0.1 \leq f \leq 0.4$ ($d \approx a_0$ in this range), and large values $\lambda/d \geq 5$ a very wide and pronounced gap appears in the distribution of pinning energies, described by an effective gap exponent assuming values up to $s_{\rm eff} \approx 3$, and as a consequence the transport exponent reaches at maximum $p_{\rm eff} \approx 2/3$. This means that correlation effects due to long–range repulsive forces may drastically enhance the correlated pinning of flux lines in the Bose glass phase. However, even for the case $\lambda \approx d \approx a_0$, one still finds a marked minimum in $g(\epsilon)$, with an effective gap exponent $s_{\rm eff} \approx 1$, and hence $p_{\rm eff} \approx 1/2$. Only for $\lambda/d \leq 0.5$ is the non–interacting result $p_0 = 1/3$ recovered. We conclude that the correlation effects due to the intervortex repulsion are surprisingly strong, and may actually be an interesting means to "tailor" the distribution of pinning energies in ion–irradiated samples, such that flux creep is suppressed as effectively as possible. A preliminary account of some of these results and their connection with recent experiments [13] has been given in Ref. [26].

We remark that we do not expect that additional point defects (e.g., oxygen vacancies) could alter our results substantially. First, pinning to point defects is subject to much stronger thermal renormalization than pinning to correlated disorder. Second, for very low currents, these point defects might indeed trap the spreading of the double–kink configuration considered above; however, this becomes effective only on unphysically large length scales (in the km range), [8] and thus is irrelevant for realistic samples. Finally, no strong pinning effects have been reported in superconducting materials *above* the irreversibility line prior to heavy–ion irradiation.

This paper is organized as follows. In the subsequent section we describe how our effective Hamiltonian may be derived from the free energy of interacting flux lines subject to columnar disorder, introduce the relevant parameters and energy scales, and discuss the region in the phase diagram where we expect our model to be valid. Furthermore, we describe our specific Monte Carlo simulation algorithm, also discussing the different possible choices of initial configurations for the energy minimization procedure. In Sec. III, we describe our numerical results and give a number of examples for the spatial configurations we find, as well as for the shape of the ensuing distribution of pinning energies, as we vary the filling f and the interaction range λ/d. We also discuss vortex transport mechanisms in the Bose glass phase, and deduce the form of the current–voltage characteristics in the variable–range hopping regime from the previously determined distribution of pinning energies, identifying the effective Mott exponent $p_{\rm eff}$. A brief summary and discussion of our results concludes this work.

II. DESCRIPTION OF THE MODEL AND MONTE CARLO SIMULATION PROCEDURE

A. Model equations

We start by considering the following model free energy for N flux lines, defined by their trajectories $\mathbf{r}_i(z)$ as they traverse the superconducting sample of thickness L, interacting with each other and with $N_{\rm D}$ columnar defects, parallel to the magnetic field $\mathbf{B}$ which is aligned along the z axis (perpendicular to the copper oxide planes in the case of the anisotropic high–T_c materials), [8] (see Fig. **??**)

$$F_N[\{\mathbf{r}_i\}] = \int_0^L dz \sum_{i=1}^{N} \left\{ \frac{\tilde{\epsilon}_1}{2} \left(\frac{d\mathbf{r}_i(z)}{dz} \right)^2 + \frac{1}{2} \sum_{j \neq i}^{N} V[r_{ij}(z)] + \sum_{k=1}^{N_{\rm D}} V_{\rm D}\left[\mathbf{r}_i(z) - \mathbf{R}_k(z)\right] \right\} . \quad (2.1)$$

The first term describes the elastic line tension, and stems from an expansion with respect to small tipping angles of the line energy of nearly straight vortices [2]. For a dense liquid of flux lines, $\lambda \gg a_0$, the tilt modulus is $\tilde{\epsilon}_1 \approx (M_\perp/M_z)\epsilon_0 \ln(a_0/\xi)$, where the material anisotropy is embodied in the effective–mass ratio $M_\perp/M_z$ ($\ll 1$ for high–temperature superconductors), ξ is the (in–plane) coherent length ($\kappa = \lambda/\xi \approx 100$), and $\epsilon_0 = (\phi_0/4\pi\lambda)^2$ sets the energy scale. On the other hand, for low fields $B \leq \phi_0/\lambda^2$, i.e.: $\lambda \leq a_0$, $\tilde{\epsilon}_1 \approx \epsilon_0$ because of the magnetic coupling between the copper oxide planes [4]. In this situation the vortex lines are therefore considerably less flexible than at high fields.

Furthermore, $r_{ij}(z) = |\mathbf{r}_i(z) - \mathbf{r}_j(z)|$, and

$$V(r) = 2\epsilon_0 K_0\left(r/\lambda\right) \quad (2.2)$$

represents the repulsive interaction potential between the lines; the modified Bessel function $K_0(r/\lambda)$ describes a screened logarithmic interaction, for $K_0(x) \propto -\ln x$ for $x \to 0$, and $K_0(x) \propto x^{-1/2}e^{-x}$ for $x \to \infty$. Thus the (in–plane) London penetration depth λ defines the interaction range. Note that according to the two–fluid model, the penetration depth diverges at the zero–field transition temperature T_c,

$$\lambda(T) = \lambda_0 \left[1 - (T/T_c)^4\right]^{-1/2} \quad , \quad (2.3)$$

and therefore the interaction range becomes considerably longer upon approaching T_c; e.g., for $T/T_c \approx 0.93$ one has $\lambda(T)/\lambda_0 = 2$. Because $\epsilon_0 \propto \lambda^{-2}$, the interactions are both weak and long–range near T_c.

Finally, the columnar pins are described by a sum of $N_{\rm D}$ z–independent potential wells $V_{\rm D}$, with average spacing d, centered on the randomly distributed positions $\{\mathbf{R}_k\}$. The damage track diameters, produced by heavy–ion irradiation, are typically $2c_0 \approx 100\mathring{A}$, with a variation induced by the root mean–square dispersion of the ion beam of $\delta c_k/c_0 \approx 15\%$. The pinning energies U_k are

related to the column diameters via the (interpolation) formula [8]

$$U_k \approx \frac{\epsilon_0}{2} \ln \left[1 + (c_k/\sqrt{2}\xi)^2 \right] \quad , \tag{2.4}$$

the variation of which thus induces a certain distribution of the pinning energies P, with a width $w = \sqrt{\langle \delta U_k^2 \rangle}$ determined from Eq. (2.4),

$$w = \frac{\epsilon_0}{1 + (\sqrt{2}\xi/c_0)^2} \frac{\delta c_k}{c_0} \quad . \tag{2.5}$$

(Notice that here the repulsive vortex interactions have not been accounted for.) Above a certain temperature T_0, however, the effective radius of the normal–conducting region at the defects will be given by the coherence length, which grows with temperature according to

$$\xi(T) = \xi_0 (1 - T/T_c)^{-1/2} \quad ; \tag{2.6}$$

hence $\sqrt{2}\xi(T)$ should be inserted into Eq. (2.4) instead of c_k, whenever $\sqrt{2}\xi(T) \geq c_0$, or $T \geq T_0$, with

$$T_0 = T_c \left(1 - 2\xi_0^2/c_0^2 \right) \quad . \tag{2.7}$$

Consequently, above T_0 the effective fluctuations of the pinning energies P (without interactions) will be smoothened out. For $c_0 \approx 50 \mathring{A}$ and $\xi_0 \approx 10 \mathring{A}$, e.g., this will happen at $T_0 \approx 0.92 T_c$, and the characteristic pinning energies will be $U_0 = \langle U_k \rangle \approx 0.35 \epsilon_0$. On the other hand, at $T = 0$ one finds instead $U_0 \approx 1.3 \epsilon_0$, and, assuming $\delta c_k / c_k \approx 0.15$ one gets $w \approx 0.13 \epsilon_0$, while for $T = 0.8 T_c$ one has $\xi \approx 2.24 \xi_0$, leading to $U_0 \approx 0.63 \epsilon_0$ and $w \approx 0.11 \epsilon_0$.

The thermodynamic properties of the model (2.1) may be derived from the grand–canonical partition function [8]

$$\mathcal{Z}_{\rm gr} = \sum_{N=0}^{\infty} \frac{1}{N!} e^{\mu N L/T} \int \prod_{i=1}^{N} \mathcal{D}[\mathbf{r}_i(z)] e^{-F_N[\{\mathbf{r}_i(z)\}]/T} \quad , \tag{2.8}$$

where μ denotes the chemical potential

$$\mu = \epsilon_0 \ln \kappa - \frac{\phi_0}{4\pi} H = \frac{\phi_0}{4\pi} \left(H_{c_1} - H \right) \quad , \tag{2.9}$$

which changes sign at the lower critical field $H_{c_1} = \phi_0 \ln \kappa / 4\pi\lambda^2$. As is explained in detail in Refs. [2,8], the statistical mechanics of (2.1) and (2.8) can be formally mapped onto a two–dimensional quantum mechanical problem by using a transfer matrix approach, the physical z direction being interpreted as an imaginary time axis. In the thermodynamic limit $L \to \infty$, the properties of our model (2.1) are determined by the lowest energy eigenvalue, i.e., the corresponding ground–state wave function, which is symmetric with respect to exchange of flux lines, and thus of bosonic character; furthermore the effective temperature of this system of interacting bosons is then to be taken as zero. The mapping of the vortex line problem to the quantum mechanics of charged bosons is summarized in Table I (from Ref. [8]). Notice that in this analogy, the real temperature T assumes the role of Planck's constant $\hbar$ in the quantum representation, and the boson electric field and current density map on the superconducting current J and the true electric field $\mathcal{E}$, respectively; hence the roles of conductivity and resistivity become interchanged.

In the following, we shall be interested in the low–temperature properties of flux lines in the Bose glass phase, pinned to columnar defects, with filling fraction

$$f = N/N_{\rm D} = (d/a_0)^2 = B/B_\phi < 1 \quad . \tag{2.10}$$

More specifically, we need $B < B^*(T)$, where vortex interactions become important in determining the localization length, and thus affect the pinning of the flux lines to the columnar defects considerably [8]. In the temperature range of interest here,

$$B^*(T) \approx \begin{cases} B_\phi = (4/3)^{1/2} \phi_0/d^2 & \text{for } T < T_0 \\ B_\phi (c_0/2\xi_0)^2 (1 - T/T_c) & \text{for } T_0 < T < T_1 \end{cases} \quad . \tag{2.11}$$

T_1 denotes the temperature above which the entropy associated with thermal fluctuations becomes relevant for the vortex pinning, by modifying the localization length and thus renormalizing the binding free energies (for the temperature dependence of $B^*(T)$ for $T > T_1$, see Ref. [8], App. D). The estimate of Ref. [8] is

$$T_1 \approx T_c \frac{(c_0/4\xi_0)(\ln \kappa/{\rm Gi})^{1/2}}{1 + (c_0/4\xi_0)(\ln \kappa/{\rm Gi})^{1/2}} \quad , \tag{2.12}$$

where ${\rm Gi} = (M_z/2M_\perp)(T_c/H_c^2 \xi_0^3)^2$ is the so–called Ginzburg number, and the thermodynamic critical field at zero temperature is $H_c = \sqrt{2}\phi_0/4\pi\lambda_0\xi_0$. For YBCO, typically ${\rm Gi} \approx 0.01$, and with $\kappa \approx 100$ we get $T_1/T_c \approx 0.96$ and $B^*(T_1) \approx 0.25 B_\phi$; for the highly anisotropic material BSCCO, upon using ${\rm Gi} \approx 0.1$ instead, one finds $T_1/T_c \approx 0.89$ and $B^*(T_1) \approx 0.69 B_\phi$. For $T < T_1$, each flux line is localized on essentially one columnar defect, and thermal fluctuations play a very minor role; thus all temperatures in this range may be considered as "low". At T_1, entropic effects become important, and at the even higher so–called depinning temperature $T_{\rm dp}$ the localization length of a single vortex is determined by the interplay of a large number of pins (see Fig. **??**). Similarly to T_1, $T_{\rm dp}$ may be estimated as [8]

$$T_{\rm dp} \approx T_c \frac{(c_0\alpha/4\xi_0)(\ln \kappa/{\rm Gi})^{1/2}}{1 + (c_0\alpha/4\xi_0)(\ln \kappa/{\rm Gi})^{1/2}} \quad , \tag{2.13}$$

where $\alpha = \ln^{1/2}[(d/\sqrt{2}\xi_0)(1 - T_1/T_c)^{1/2}]$. With $d \approx 1000 \mathring{A}$, e.g., using the above numbers one finds $T_{\rm dp}/T_c \approx 0.98$ for YBCO, and $T_{\rm dp}/T_c \approx 0.94$ for BSCCO.

For T less than the characteristic fluctuation temperature T_1, we may thus simply consider the classical limit of the corresponding boson problem ($\hbar \to 0$); furthermore, as the vortices are well separated in this regime [with $B < B^*(T)$], the Bose statistics become irrelevant. Thermal wandering is effectively suppressed, and the flux lines will be essentially straight; hence the tilt energy in Eq. (2.1) can be neglected. Therefore, in the boson representation we eventually have to consider the time–independent problem of N interacting particles located at $N_{\rm D} > N$ available defect positions. Thus we define our model by the two–dimensional effective Hamiltonian

$$H = \frac{1}{2} \sum_{i \neq j}^{N_{\rm D}} n_i n_j V(r_{ij}) + \sum_{i=1}^{N_{\rm D}} n_i t_i \quad , \tag{2.14}$$

and its grand–canonical counterpart

$$\tilde{H} = H - \mu \sum_{i=1}^{N_{\rm D}} n_i \quad . \tag{2.15}$$

Here $i, j = 1, \ldots, N_{\rm D}$ denote the defect sites, randomly distributed on the xy plane; $n_i = 0, 1$ represents the corresponding site occupation number ($n_i = 1$ if a flux line is bound to columnar defect i), $\sum_{i=1}^{N_{\rm D}} n_i = N$. We have also included random site energies t_i, originating in the variation of pin diameters [see Eq. (2.5)]. Their distribution P can be chosen to be centered at $\langle t \rangle = 0$, with width w; this amounts to absorbing the average pinning energy $U_0 = \langle U_k \rangle$ (which may include small thermal renormalizations) into the chemical potential μ. Realistically, the probability distribution of the site energies would likely be Gaussian; [13] however, for simplicity we shall assume a flat distribution

$$P(t_i) = \Theta(w - |t_i|)/2w \tag{2.16}$$

[$\Theta(x)$ denotes the Heaviside step function]. As the total shape and width of the distribution of pinning energies will turn out to be determined by the interactions, the precise form of $P(t_i)$ is of minor importance. Note that P corresponds to a bare density of states.

For the *interacting* system (2.14), we define single–particle site energies ϵ_i as follows: For filled sites ($n_i = 1$), ϵ_i is the energy required to remove the particle at site i to infinity; for empty sites ($n_i = 0$), correspondingly ϵ_i is the energy needed to introduce an *additional* particle from infinity to site i. Equivalently, we may state that ϵ_i constitutes the potential energy on site i, resulting from the disorder *and* the interaction with all other occupied sites j, or

$$\epsilon_i = \frac{\partial H}{\partial n_i} = \sum_{j \neq i}^{N_{\rm D}} n_j V(r_{ij}) + t_i \quad . \tag{2.17}$$

In thermal equilibrium, the chemical potential μ separates the occupied and empty states, $\epsilon_i \leq \mu$ for all i with $n_i = 1$, and $\epsilon_j \geq \mu$ for all j with $n_j = 0$. The distribution of pinning energies $g(\epsilon)$, now with the inter–vortex repulsions taken into account (as opposed to P), can be viewed as an *interacting* single–particle density of states and may be obtained from the statistics of the energy levels ϵ_i; i.e., $g(\epsilon)d\epsilon$ is defined as the number of states per unit area with energies in the interval $[\epsilon, \epsilon + d\epsilon]$. Note that in a system of area $L_\perp^2$ this corresponds to a normalization of the density of states according to

$$\int_{-\infty}^{+\infty} g(\epsilon) d\epsilon = N_{\rm D}/L_\perp^2 = 1/d^2 \quad . \tag{2.18}$$

In the following, the terms "distribution of pinning energies" and "(single–particle) density of states" will be used synonymously. For finite $\lambda > a_0$, the chemical potential may readily be estimated as

$$\mu \approx (\lambda/a_0)^2 V(a_0) \approx \epsilon_0 f (\lambda/d)^2 \ln\left(f\lambda^2/d^2\right) \quad , \tag{2.19}$$

as each line interacts with approximately $(\lambda/a_0)^2$ other vortices. Actually, the prefactor of the logarithm in (2.19) is independent of λ, because $\epsilon_0 \propto 1/\lambda^2$ [27]. Similarly, the typical overall width γ of the energy level distribution is

$$\gamma \approx V(a_0) \approx \epsilon_0 \ln\left(f\lambda^2/d^2\right) \quad , \tag{2.20}$$

and roughly the "bandwidth" for the occupied levels will be $\approx f\gamma$. The relevant energy scales in our problem are thus ordered according to $\mu \geq f\gamma \gg w$.

Another important quantity is the hopping energy $\Delta_{i \to j}$ associated with the transfer of a particle from an occupied site i to an empty site j (conserving the total particle number N). The simplest way to determine this energy cost or gain [17] is to decompose the process into two steps, namely removing a particle from site i to infinity, thus gaining the energy ϵ_i, and then taking it from there back into the system at site j, which costs the amount $\epsilon_j - V(r_{ij})$; the additional contribution $V(r_{ij})$ here stems from the fact that after removing the particle at site i there were only $N-1$ particles left, while ϵ_j was defined for a N–particle system, and thus the (fictitious) interaction with site i has to be accounted for explicitly. Hence one finds

$$\Delta_{i \to j} = \epsilon_j - \epsilon_i - V(r_{ij}) \quad , \tag{2.21}$$

and certainly $\Delta_{i \to j} > 0$ for all pairs of sites with $n_i = 1$ and $n_j = 0$ is a necessary condition for the ground state configuration; in fact, a sufficient condition for a stable ground state may be stated by demanding that the energies associated with any m–particle hops all be positive. Thus a hierarchical scheme for testing specific configurations can be constructed, [24] in which one first tests a putative ground state for stability against one–particle hops, two–particle hops, etc.

With regard to the regime of applicability of the effective Hamiltonian (2.14) or (2.15), we note once more

that for $T > T_1$ entropic corrections become important, and for $T > T_{\rm dp}$ a flux line can certainly be no more associated with a single columnar defect, and our simplified model breaks down. However, the above estimates demonstrate that thermal fluctuations are in fact essentially negligible up to temperatures very near the irreversibility line. Similarly, in order that all the flux lines stay pinned to the columnar defects, the condition $B < B^*(T)$ must be fulfilled, which provides us with upper limits for the filling f, above which the intervortex repulsions will depin some of the flux lines, and move them to interstitial sites between the columnar defects (compare Fig. **??**).

Finally, we remark that upon introducing the spin variables

$$\sigma_i = 2n_i - 1 \quad , \tag{2.22}$$

hence $\sigma_i = +1, -1$ for $n_i = 1, 0$, respectively, the effective Hamiltonian (2.15) is mapped onto a two–dimensional random–site, random–field antiferromagnetic Ising model with long–range exchange interactions, [22]

$$\tilde{H} = \frac{1}{2}\sum_{i\neq j} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i + \tilde{E} \quad , \tag{2.23}$$

with

$$J_{ij} = \frac{1}{4}V(r_{ij}) > 0 \quad , \tag{2.24}$$

$$h_i = \frac{\mu - t_i}{2} - \frac{1}{4}\sum_{j\neq i} V(r_{ij}) \quad , \tag{2.25}$$

and

$$\tilde{E} = \frac{1}{2}\sum_i (t_i - \mu) - \frac{1}{8}\sum_{i\neq j} V(r_{ij}) \quad . \tag{2.26}$$

Finding the density of states for this system by any analytical means beyond mean–field type considerations, [17] or phenomenological scaling arguments, [25] constitutes a very difficult task, and has eluded successful treatment to date. Therefore we have to resort to numerical studies using a suitable Monte Carlo algorithm; fortunately, the Hamiltonian (2.14) is precisely of the form studied in the context of charge carriers localized at random impurities in doped semiconductors (Coulomb glass problem), [17,22,24] and we may utilize the considerable experience already gathered in these previous studies.

B. Simulation algorithm

Our simulation algorithm closely follows the energy–minimization procedure as described in detail by Efros and Shklovskii in Ref. [17], Chap. 14. In a square of linear size $L_\perp$, a random number generator chooses $N_{\rm D}$ defect coordinates (by taking $L_\perp = \sqrt{N_{\rm D}}$, the average defect distance d is held fixed); we have performed simulations for $N_{\rm D} = 100$, 200, 400, and 800. Then each site is assigned a random number t_i drawn from the distribution (2.16), the width of which we have chosen as $w/2\epsilon_0 = 0.1$, and the interaction potentials $V(r_{ij})$ are calculated according to (2.2) for all pairs (ij). Following Ref. [22], we have used periodic boundary conditions, thus closing the initial square to a torus and thereby reducing boundary effects. Here, r_{ij} is defined as the minimum value of the distances between site i and the nine equivalent sites j in the original and repeated "cells". For the interaction range λ, we have studied the values $\lambda/d = 0.5, 1, 2, 5$, and ∞. In order to handle the latter case of infinite–range, purely logarithmic interaction carefully, we have applied the Ewald sum technique, which amounts to including the interaction of each particle with *all* its periodic images; this is achieved by performing the corresponding series in part in direct, and in part in Fourier space. For details, we refer to Ref. [28], App. A. The data t_i and $V(r_{ij})$ can now be stored for later use.

In the next step, $N = fN_{\rm D}$ ($f < 1$) of these sites i are assigned the occupation number $n_i = 1$. We have employed two very different schemes to generate this initial configuration: (i) the occupied sites were chosen randomly, and (ii) a suitable triangular lattice, with a number $N_{\rm lat}$ of sites as close as possible to the correct N, was superimposed on the two–dimensional array of defects, and then deformed until the former lattice sites matched $N_{\rm lat}$ of the defect positions; the remaining $N - N_{\rm lat}$ sites were then filled at random. The choices (i) and (ii) naturally correspond to minimal and maximal initial spatial correlations, respectively. Using Eq. (2.17), now the site energies ϵ_i are calculated. Almost certainly, the initial configuration will correspond to a highly non–equilibrium situation, in the sense that many of the occupied sites i will have higher energies ϵ_i than a large number of empty places j. In order to relax this configuration by successive particle–hole transitions, the occupied site p of highest energy, $\epsilon_p = \max_{\{n_i=1\}}\epsilon_i$, and the empty site q of lowest energy, $\epsilon_q = \min_{\{n_i=0\}}\epsilon_j$ are determined. If $\epsilon_q < \epsilon_p$, the particle at site p is moved to site q (the corresponding occupation numbers are interchanged) and the site energies are recalculated. This procedure is repeated until eventually $\epsilon_p \leq \epsilon_q$; now the occupied and empty states are separated, and the chemical potential can be (approximately) obtained from $\mu = (\epsilon_p + \epsilon_q)/2$.

However, this intermediate state is still a very poor approximation to the real ground state, as in general it will be unstable towards single–particle hops. In order to test this, the hopping energies $\Delta_{i\to j}$ are calculated for all possible pairs of occupied sites i and empty sites j, using Eq. (2.21). Among those values of $\Delta_{i\to j}$, which are negative, the minimum is searched for: $\Delta_{p\to q} = \min_{\{\Delta_{i\to j}<0\}}\Delta_{i\to j}$. Next the particle transfer from site p to site q is performed, thus reducing the total energy of the system. Then the new site energies are calculated, and in general the "equilibration step" of the

previous paragraph will have to be repeated. Once more all the $\Delta_{i\to j}$ are determined, and the whole procedure is done again, until finally all the hopping energies acquire positive values. The ensuing configuration is accepted as a fair approximation to the real ground state for the particular distribution of defects, filling fraction f, and interaction range λ/d. The algorithm thus yields the positions of the N flux lines, the corresponding site energies, and the approximative chemical potential μ. By repeating this procedure for a number k of different initial configurations (we took $k = 16000/N_{\rm D}$), one may then obtain an ensemble–average of the chemical potential $\langle\mu\rangle$, of the density of states $\langle g(\epsilon)\rangle$ (from the site energy statistics), and similarly averaged correlation functions, etc.

Of course, to find the correct ground state, one would in principle have to test each configuration against any simultaneous m–particle hops, $m = 2, \ldots, \infty$. However, previous investigations have shown that terminating at $m = 1$ already yields at least qualitatively reliable estimates for the statistics of the energy level distributions [17,22,24]. In fact, the "true" ground state may be very difficult to reach for the corresponding real system as well, and the static and dynamic properties may at least at low temperatures be satisfactorily described by these "pseudo" ground states. The quantitative reliability of the results can furthermore be tested by studying various system sizes, and by different sampling procedures for the site energies. E.g., one can determine $\langle g(\epsilon)\rangle$ and $\langle\mu\rangle$ by directly averaging the energy level statistics of the k different realizations, or by performing the average on the relative energies $\tilde\epsilon = \epsilon - \mu$ calculated in each of the realizations separately. It turns out that these two versions yield practically identical distributions of pinning energies, the tiny differences being confined to the immediate vicinity of $\langle\mu\rangle$, and readily interpreted as finite–size effects (the second procedure always leads to $g(\epsilon) = 0$ for $|\tilde\epsilon| = |\epsilon - \mu| \le V(L_\perp)$, because the lowest possible value of $\tilde\epsilon$ is of the order $V(L_\perp)$, see Ref. [17].) One can also check that the two very different initial conditions described above lead to similar final configurations. In Sec. III D, we shall describe how the essentials of the current–voltage characteristics in the variable–range hopping regime may be obtained from the thus determined single–particle density of states. (We shall drop the explicit notation $\langle\ldots\rangle$ for ensemble averages from now on.)

III. STATIC CORRELATIONS AND TRANSPORT IN THE BOSE GLASS PHASE

A. Spatial correlations

We begin the discussion of our results with a description of the spatial distribution of flux lines among the underlying randomly distributed columnar pins. Fig. **??** presents the final configuration in the xy plane (perpendicular to both the magnetic field and the defect orientation) for three typical simulation runs, performed on the identical defect configuration ($N_{\rm D} = 400$), with infinite–range logarithmic interaction ($\lambda \to \infty$), and using site energies drawn from the flat distribution (2.16) with $w/2\epsilon_0 = 0.1$ (compare the estimates in Sec. II A), but with different filling fractions $f = 0.1$, $f = 0.2$, and $f = 0.4$, respectively, in each case starting from a random initial distribution of vortices (the final states stemming from the other extreme of possible initial conditions, namely a distorted triangular lattice, are not identical but look qualitatively very similar, see below). In contrast to the Poisson distribution of defects (open circles), showing the characteristic voids as well as clusterings, the flux lines (full circles) form a highly correlated structure, trying to keep as distant as possible from each other to minimize the interaction energy. For low fillings [Figs. **??**(a) and (b)] the ensuing spatial distribution of flux lines resembles a distorted triangular lattice. However, for higher filling fractions, $f \ge 0.4$ say, the vortices are increasingly forced to accomodate the underlying random pin distribution. As can be seen in Fig. **??**(c), they then tend to occupy the energetically favorable sites encircling spatial voids in the defect distribution. Upon increasing f even more, beyond $f \ge 0.7$ say, (which may be achieved by applying higher magnetic fields B) the spatial correlations disappear on longer length scales, and the only remaining effect of the intervortex repulsion is to prevent simultaneous occupation of nearest–neighbor sites. One has to keep in mind, however, that in this regime our basic assumption that each flux line be attached to a single defect, which requires $B < B^*(T)$ [Eq. (2.11)], is beginning to break down. As has been explained in Sec. II A, for $B > B^*(T)$ interactions will become important in determining the vortex localization length, and eventually force some of the flux lines to leave the columnar defects and move to interstitial sites. Most of our numerical work is restricted to $f \le 0.4$ for this reason. We note that all of these features do not very strongly depend on the actual value of the parameter λ/d, as long as $\lambda/d \ge 1$, which ensures that the typical interaction energies exceed the fluctuations in pinning potential depths w.

More quantitative conclusions can be drawn from the calculation of the static structure factor $S(\mathbf{q})$,

$$S(\mathbf{q}) = \frac{1}{N}\sum_{i,j=1}^{N} e^{i\mathbf{q}(\mathbf{r}_i - \mathbf{r}_j)} = \frac{1}{N}|n(\mathbf{q})|^2 \quad , \tag{3.1}$$

which is readily obtained from the two–dimensional Fourier transform $n(\mathbf{q}) = \sum_i e^{i\mathbf{q}\mathbf{r}_i}$ of the vortex density $n(\mathbf{r}) = \sum_i \delta(\mathbf{r} - \mathbf{r}_i)$; more precisely, we shall consider the average of (3.1) over the directions in momentum space,

$$S(q) = \int d\Omega_{\mathbf{q}}\, S(\mathbf{q}) = \int_0^{2\pi} d\phi\, S(q,\phi) \quad . \tag{3.2}$$

In Fig. **??**, the calculated $S(q)$ is depicted for both the initial and final vortex configurations, for filling fractions

$f = 0.1$ (a), and $f = 0.2$ (b). These curves were obtained from averaging over $k = 40$ different runs with $N_{\rm D} = 400$, $w/2\epsilon_0 = 0.1$, and $\lambda/d = 5$. Again, the corresponding pictures for any $\lambda/d \geq 1$ (keeping w fixed) look very much alike. Note that the peak in the dashed curve in Fig. **??**(a) at $qa_0 \approx 4\pi/\sqrt{3}$, i.e., at the Bragg peak corresponding to the triangular lattice, is shifted to lower values and diminished in height in Fig. **??**(b); for higher fillings the underlying random pin distribution enforces an average separation of vortices larger than the ideal hexagonal close–packed geometry would permit. For similar reasons, the peak of $S(q)$ for the equilibrated configurations (corresponding to occupied sites in Fig. **??**) is displaced to lower values with respect to the ideal triangular lattice, namely to $qa_0 \approx 0.86(4\pi/\sqrt{3})$, and also broadened considerably due to the appearance of a large number of topological defects (see Fig. **??** below). Interestingly, for $f = 0.1$ there is a double–peak structure. The overall appearance, width and peak height of the dotted and solid curves in Fig. **??** are very similar, suggesting the relative unimportance of the specific choice of initial conditions. In addition to the suppression of fluctuations at low $qa_0 \leq 4$ and the marked peak in $S(q)$ at (roughly) $qa_0 \approx 2\pi$, the shallow dip in the interval $9 \leq qa_0 \leq 12$ should be noted. For these spatial correlation effects, the most sensitive parameter turns out to be the filling fraction f, as illustrated in Fig. **??** for $\lambda/d = 2$. While for $f = 0.1$ and $f = 0.2$ the structure factor is practically indistinguishable from the previous pictures for $\lambda/d = 5$, already for $f = 0.4$ the peak height becomes considerably reduced, and the shallow dip has disappeared. For $f = 0.8$ the peak has vanished, and even the correlation gap at low q is halfway closed, thus rendering $S(q)$ more and more uniform as for a Poisson distribution (cf. the long–dashed curves in Fig. **??**).

Returning to the configuration plots in direct space (Fig. **??**), we can analyze the final structures more thoroughly by performing a Delaunay triangulation, thereby exposing the topological defects (with respect to an ideal six–fold coordinated triangular lattice), simply by determining the ensuing coordination number N_c at each vortex site. The results of this procedure for the configurations of Figs. **??**(a)–(c) are shown in Fig. **??**. Clearly the resulting vortex configuration can only be viewed as a highly distorted "triangular lattice", with its orientational correlations being strongly diminished.

In addition to these gross interaction effects, there emerge subtle correlations in the single–particle energies, namely the spatial clustering of those sites whose energies ϵ_i differ very little from the chemical potential μ. This remarkable effect has previously been observed by Davies, Lee, and Rice for the Coulomb glass case [22]. We illustrate this property for the distribution in Fig. **??**(c), i.e.: $N_{\rm D} = 400$, $w/2\epsilon_0 = 0.01$, $\lambda \to \infty$, and $f = 0.4$. In Fig. **??**, along with the entire configuration, the spatial distribution of those filled and empty sites with energies in the range $|\epsilon_i - \mu| \leq \delta$ are shown, with $\delta/2\epsilon_0 = 0.6$ (a), and $\delta/2\epsilon_0 = 1.2$ (b), respectively [compare Fig. **??**(b) below]. The occupied sites with energies close to μ are strikingly clustered in space, and the corresponding low–energy empty sites seem to occupy the complementary region. The interactions cause long–range spatial fluctuations in the pinning energies, which are apparently yet not understood in detail. Of course, the very fact that a certain occupied site attains a high site energy must mean that there are other occupied sites nearby. It is less clear, however, why the energies of nearby occupied sites for $\delta \ll \mu$ should be so similar, or why the states in the low–energy part of the empty pseudo–band should be similarly correlated in space. Note that the Bose glass is not symmetric with respect to exchange of particles and holes, since the empty sites do not interact. Thus, defining $\tilde{\epsilon}_i = \epsilon_i - \mu$, those states i, j with both $|\tilde{\epsilon}_i|$ and $|\tilde{\epsilon}_j| \leq \delta$, are situated nearby when $\tilde{\epsilon}_i\tilde{\epsilon}_j > 0$, and are widely separated in space when $\tilde{\epsilon}_i\tilde{\epsilon}_j < 0$. Upon increasing the threshold δ, both the occupied and the empty low–$|\tilde{\epsilon}_i|$ clusters appear to percolate through the system, finally becoming increasingly mixed for $\delta \geq \mu$, and the complete configuration is reached.

Recently, the positions of both columnar defects and flux lines have been determined simultaneously in experiments on $NbSe_2$ [12] and BSCCO samples [13], using scanning tunneling microscopy and a combination of chemical etching and magnetic decoration techniques. Indeed, the manifest spatial correlations in Fig. 4 of Ref. [13] strongly suggest the relevance of vortex interactions. In a recent note, [26] we used such experimental data on the distribution of pins and vortices to infer the density of states and the ensuing transport characteristics, which are thus subject to direct verification by measurements (the relevant parameters in that case turned out to be $f \approx 0.24$, $\lambda/d \approx 0.96$, and $w \approx 0.1\epsilon_0$). This possibility of quantitative comparison of the spatial configurations and correlations as obtained from simulations with those measured in actual experiments opens a fascinating new field for detailed studies, which have been far from feasible in the otherwise closely related disordered semiconductor systems [17]. Even the striking clustering in Fig. **??** may possibly be observed experimentally, if the current techniques of manipulating individual vortices with the tip of a magnetic force microscope [29] are further refined, which might eventually permit the measurements of individual pinning energies ϵ_i.

B. Density of states and Coulomb gap

Before we proceed to present our simulation results for the single–particle density of states (i.e.: the distribution of vortex pinning energies, taking their interactions into account), we briefly summarize Efros and Shklovskii's qualitative arguments that long–range repulsive forces lead to a soft gap in $g(\epsilon)$ [17]. Let us consider particles in D dimensions, which are localized on randomly distributed sites $\mathbf{r}_i$, and interact via an arbitrary poten-

tial

$$V(r_{ij}) = 2\epsilon_0 K(r_{ij}/\lambda) \quad , \tag{3.3}$$

where, quite generally, ϵ_0 sets the energy scale, λ defines a characteristic length, and $K(x)$ is a positive, monotonically decreasing function approaching zero for $x \to \infty$. Its inverse therefore exists, and shall be denoted by $\tilde{K}(x) = K^{-1}(x)$.

The simplest argument to determine the interacting density of states now is as follows (see Chap. 10 of Ref. [17]). Consider site energies ϵ_i, ϵ_j located in an interval of width $\tilde{\epsilon}$ around the chemical potential μ, but situated on opposite sides of μ, e.g. with $n_i = 1$ and $n_j = 0$. Then in the ground state a particle transfer from site i to j necessarily costs the positive energy $\Delta_{i\to j} = \epsilon_j - \epsilon_i - V(r_{ij}) > 0$ [Eq. (2.21)], which implies $V(r_{ij}) < \epsilon_j - \epsilon_i \leq \tilde{\epsilon}$. Therefore, hops typically have to occur over a distance

$$r_{ij}(\tilde{\epsilon}) \geq R(\tilde{\epsilon}) \approx \lambda \tilde{K}(\tilde{\epsilon}/2\epsilon_0) \quad . \tag{3.4}$$

The concentration of relevant, i.e.: energetically available, impurities hence is $n(\tilde{\epsilon}) \propto R(\tilde{\epsilon})^{-D}$. Thus for small $\tilde{\epsilon}$ the density of states $g(\tilde{\epsilon}) \propto dn(\tilde{\epsilon})/d\tilde{\epsilon}$ becomes

$$g(\tilde{\epsilon}) = c_D \frac{1}{2\epsilon_0 \lambda^D} \frac{|\tilde{K}'(x)|}{\tilde{K}(x)^{D+1}} \bigg|_{x=\tilde{\epsilon}/2\epsilon_0} \quad , \tag{3.5}$$

where c_D is a proportionality factor of order $O(1)$.

We can now specialize to the true long–range potentials $K(x) = x^{-\sigma}$ with $0 < \sigma < D$ and $K(x) = -\ln x$ (which formally results from the previous case in the limit $\sigma \to 0$). In these situations, the prefactors c_D can actually be determined using the somewhat refined self–consistency argument by Efros [30], which yields $c_D(\sigma) = 2D(1-\sigma/D)\Gamma(D/2+1)\pi^{-D/2}$. Eq. (3.5) then gives for the inverse–power interactions

$$g(\tilde{\epsilon}) = \frac{D\Gamma(D/2+1)}{\sigma \pi^{D/2} \lambda^D \epsilon_0} \left(\frac{\tilde{\epsilon}}{2\epsilon_0} \right)^{D/\sigma - 1} \quad , \tag{3.6}$$

i.e.: the density of states near the chemical potential vanishes as a power law [cf. Eq. (1.3)] with gap exponent $s_0(D) = D/\sigma - 1$; e.g.: for the Coulomb potential ($\sigma = 1$) in $D = 2$ and $D = 3$ dimensions one has $s_0(2) = 1$ and $s_0(3) = 2$, respectively. For the purely logarithmic interaction, one similarly finds

$$g(\tilde{\epsilon}) = \frac{D\Gamma(D/2+1)}{\pi^{D/2} \lambda^D \epsilon_0} \exp\left(\frac{D\tilde{\epsilon}}{2\epsilon_0} \right) \quad ; \tag{3.7}$$

i.e.: although $g(\epsilon)$ does not vanish for $\epsilon = \mu$, there is an exponential dip near the chemical potential. Naturally, there exists a certain upper cut–off for large $\tilde{\epsilon}$, beyond which the density of states assumes its usual shape.

It is important to note that Eqs. (3.5)–(3.7), need not be quantitatively correct. Correlations in the distribution of site energies have been neglected, and the long–range fluctuations embodied in the remarkable clustering described at the end of the previous subsection (Fig. **??**) are not accounted for. Also, only single–particle transfers were considered in the above discussion, and multi–particle hops could possibly modify the results, especially in the case of purely logarithmic interactions [25]. In fact, Monte Carlo simulations using variants of the above–described algorithm [22,24] yield gap exponents somewhat larger than the mean–field estimates $s_0(D)$. E.g., the most detailed study of the Coulomb glass performed by Möbius, Richter, and Drittler found $s(2) = 1.2 \pm 0.1$ and $s(3) = 2.6 \pm 0.2$ in $D = 2$ and $D = 3$ dimensions, respectively, for the best fits to their results on very large systems (up to 40 000 and 125 000 sites, respectively). These authors, moreover, were ultimately unable to decide whether the density of states near the chemical potential is correctly described by a power law at all. Although Eq. (3.7) for logarithmic repulsions may have to be altered as well, a power–law fit in the energy–range accessible to our simulations appears not unreasonable.

In Fig. **??**, we present some of the simulation results for the density of states of our effectively two–dimensional system with the interaction (2.2), in the limit $\lambda \to \infty$, i.e.: for a purely logarithmic potential. They were obtained by sampling the site energies of $k = 40$ runs each with either random initial conditions (open circles) or starting out with a distorted triangular lattice (filled triangles), using a system size of $N_{\rm D} = 400$ defect sites and a random site energy distribution of width $w/2\epsilon_0 = 0.1$. In this case of infinite–range interactions, we have used the Ewald summation procedure in order to obtain reliable results, and before taking the limit $\lambda \to \infty$ a term $\propto \lambda^2$ has been subtracted from the site energies, [31]

$$\epsilon_i' = \epsilon_i - 2\pi f (\lambda/d)^2 \quad . \tag{3.8}$$

The specific choice of the initial conditions for the Monte Carlo algorithm has hardly any influence. Although there is a general tendency for the random initial configurations to yield lower total energies on average, this slight difference lies well in the range of the actual fluctuations in μ over the $k = 40$ runs with different defect distributions. At any rate, the emerging soft "Coulomb" gap near the chemical potential is very distinct and surprisingly broad for both cases $f = 0.2$ [Fig. **??**(a)] and $f = 0.4$ [Fig. **??**(b)]. The overall width of the site–energy distribution is determined by the interactions, and slightly increases with $f = N/N_{\rm D} = B/B_\phi$ [Eq. (2.20)]. As the filling fraction grows, however, the width of the Coulomb gap itself (at half maximum, say) does not change. Its relative width thus decreases sowewhat with larger fillings, as it "sweeps through" the density of states along with the chemical potential. More important, the effective gap exponent apparently depends monotonically on f, attaining its largest values for small fillings.

This becomes more clearly visible in the double–logarithmic plots of Fig. **??**, where both the filled (full symbols) and empty (open symbols) "subbands" have

been folded onto the same side. From the single decade of data showing the increase of the density of states with growing distance from μ, we can infer the effective gap exponents $s_{\rm eff} \approx 2.9$ for $f = 0.1$ [Fig. **??**(a)], and $s_{\rm eff} \approx 2.2$ for $f = 0.4$ [Fig. **??**(b)]. The apparent dependence on f arises because upon increasing the filling the flux lines have to accomodate more and more with the underlying random pin distribution, whereby correlations are destroyed. Of course, we cannot be sure that we have reached an asymptotic f–independent exponent with our simulations. It nevertheless seems clear that the mean–field formula (3.7) is inadequate, and that a power law of the form (1.3) provides a better description of the soft Coulomb gap, with a maximum gap exponent

$$s_{\rm eff}^{\rm max}(f \ll 1) \approx 3 \quad . \tag{3.9}$$

As is implied by our notation, we do not consider these gap exponents as universal quantities; rather, they should be viewed as *effective* exponents describing the shape of the Coulomb gap in some specific energy range.

The details of the asymptotic analytical form of $g(\epsilon)$ in the limit $\epsilon \to \mu$ are not our main concern. We are most interested in the striking shape of the density of states, displaying a very marked and wide correlation–induced "Coulomb" gap, because this has a dramatic *qualitative* effect on the current–voltage characteristics (see Sec. III D). One of the most important features of the pseudo–gap of Fig. **??** in fact rests in its remarkable persistence as the interaction range, set by the London penetration depth λ, becomes finite. Figs. **??**–**??** depict a selection of results for the single–particle density of states for decreasing λ, with various fillings f (and the other parameters chosen as before). As can be seen in Fig. **??**, the situation with $\lambda/d = 5$ is practically indistinguishable from the previous case of infinite interaction range, in overall appearance, width and general distribution of pinning energies [cf. Fig. **??**(b) and Fig. **??**(b) for identical $f = 0.4$]. The overall curves are simply shifted along the energy axis according to Eq. (3.8) (see also Fig. **??** below). In Fig. **??** the more extreme cases with (a) $f = 0.1$, with a very broad and flat pseudo-gap, and (c) $f = 0.8$ are shown as well. In the latter case, $g(\mu)$ is manifestly non–zero. This is due in part to the finiteness of λ, thereby subjecting the system more and more to the underlying pinning potential randomness. In addition, the fluctuations in the chemical potential grow considerably as f is increased, thus obscuring the vanishing $g(\mu)$ by sampling over the energies ϵ_i [32]. Nevertheless, it is clear that the Coulomb gap closes: its width becomes narrower and $g(\mu) \propto \lambda^{-2}$ grows, as may be seen by comparing Figs. **??**(a) and **??**(a) for $f = 0.1$, Figs. **??**(a) and **??**(a) for $f = 0.2$, Figs. **??**(b), **??**(b), and **??**(b) for $f = 0.4$, and Figs. **??**(c) and **??**(b) for $f = 0.8$. Even for $\lambda \approx d \leq a_0$ there remains a considerable suppression of $g(\mu)$ itself, accompanied by a strong depletion of the density of states up to $|\tilde{\epsilon}|/2\epsilon_0 \approx 0.2$. In an intermediate range of $\tilde{\epsilon}$, one may still use Eq. (1.3), and the effective gap exponent for $\lambda/d = 1$ and $f = 0.1$, for example, is about $s_{\rm eff} \approx 1.2$. Notice that in accord with our estimate (2.19), the chemical potential roughly scales as $\propto f(\lambda/d)^2$ as filling and interaction range are varied.

C. Finite size effects

We now discuss finite size effects, i.e.: the dependence on the number of defect sites $N_{\rm D}$ used in the Monte Carlo simulations. We have performed extensive studies for the cases $N_{\rm D} = 100, 200, 400$ (for which all the previous plots are depicted), and 800, averaging over $k = 16000/N_{\rm D}$ runs with different initial defect configurations (thus supplying an equal number of data points). The total width and shape of the density of states, as well as the appearance of the Coulomb gap, remained practically identical upon changing $N_{\rm D}$ in the above range. The sole effect is a chemical potential shift, or equivalently, the total energy per particle, for the situations with long–range interactions. For $\lambda/d = 1$, $\mu(N_{\rm D})$ is independent of $N_{\rm D}$ for $N_{\rm D} \geq 200$. In Fig. **??**, the mean values for the obtained chemical potentials are plotted as a function of $1/N_{\rm D}$ for the runs with $\lambda/d = 5$ and several filling fractions $f \leq 0.5$. As is to be expected, upon increasing the number of sites and consequently interacting particles, for this very long–range repulsive forces $\mu(N_{\rm D})$ grows, but the data approach finite values in the limit $N_{\rm D} \to \infty$, close to the open symbols on the $1/N_{\rm D} = 0$ axis. These latter points were obtained by shifting the results for $\mu(N_{\rm D} = 400)$ of the $\lambda \to \infty$ simulations (obtained via the Ewald summation procedure) according to Eq. (3.8), with $\lambda/d = 5$. Thus the size dependence of the chemical potential is easily understood; and there seem to be few other finite–size artifacts except that we cannot sample energies closer to μ than $\tilde{\epsilon} \approx V(L_\perp) \approx 2\tilde{\epsilon}K_0(L_\perp/\lambda)$ (see Sec. II B). Note that $\mu \propto f$ in Fig. **??**, which is equivalent to $B \approx H$ in the flux–line language.

D. Current–voltage characteristics in the variable–range hopping regime

We now determine how the depletion in $g(\epsilon)$ near the chemical potential affects vortex transport properties. An in–plane current $\mathbf{J} \perp \mathbf{B}$ induces a Lorentz force per unit length $\mathbf{f}_{\rm L}$ perpendicular to $\mathbf{J}$, acting on all the flux lines:

$$\mathbf{f}_{\rm L} = \frac{\phi_0}{c}\, \hat{\mathbf{z}} \times \mathbf{J} \quad . \tag{3.10}$$

Accordingly, we add the term

$$\delta F_N[\{\mathbf{r}_i\}] = -\int_0^L dz \sum_{i=1}^{N} \mathbf{f}_{\rm L}\mathbf{r}_i(z) \tag{3.11}$$

to our model free energy (2.1). As stated above, in the boson picture this additional term represents an electric field $\mathbf{E} = \hat{\mathbf{z}} \times \mathbf{J}/c$ acting on the particles carrying charge ϕ_0 (see Table I). This fictitious quantity $\mathbf{E}$ should not be confused with the true electric field $\mathcal{E}(J)$ (the voltage drop induced by the current through the movement of flux lines). In the spirit of the thermally assisted flux–flow (TAFF) model of vortex transport, [33] the superconducting resistivity (i.e.: the conductivity in the boson representation) $\rho \equiv \mathcal{E}/J$ may be written as

$$\rho \approx \rho_0 \exp\left[-U_{\rm B}(J)/T\right] \quad , \tag{3.12}$$

where ρ_0 is a characteristic flux–flow resistivity, and $U_{\rm B}$ represents an effective barrier height. Unlike the original TAFF models, $U_{\rm B}(J)$ actually diverges as $J \to 0$ in the Bose glass phase [see Eq. (1.1)].

Consider first the regime where the motion of a single vortex is unaffected by the other flux lines in the sample. We shall see that this is true only in an intermediate current regime $J_1 < J < J_c$ [8]. Driven by the external current J, a flux line will start to leave its columnar pin by detaching a segment of length z into the defect–free region, thereby forming a half–loop of transverse size r. The free energy price for this process consists of the elastic energy $\approx \tilde{\epsilon}_1 r^2/z$, and the lost pinning energy $\approx U_0 z$, as the first vortex to move will be the most weakly bound one. Note that we have to re–introduce the average pinning potential $U_0 = \langle U_k \rangle$ here. Upon taking into account the work done by the Lorentz force, the free energy of the loop is approximately [8]

$$\delta F_1(r,z) \approx \tilde{\epsilon}_1 r^2/z + U_0 z - f_{\rm L} r z \quad . \tag{3.13}$$

Optimizing $\delta F_1(r,z)$ for $f_{\rm L} = 0$ first, we see that for the saddle–point configuration

$$z^* \approx r^* \sqrt{\tilde{\epsilon}_1/U_0} \quad , \tag{3.14}$$

while for finite currents

$$r^* \approx c U_0/\phi_0 J \quad , \tag{3.15}$$

which yields the saddle–point free energy

$$\delta F_1^* \approx c \tilde{\epsilon}_1^{1/2} U_0^{3/2}/\phi_0 J \quad , \tag{3.16}$$

which we may identify as the energy barrier in Eq. (3.12) for the half–loop nucleation process [8]. For a sufficiently low current J_1, the half–loop will typically extend to the nearest–neighbor pin, namely when $r^* \approx d$, and hence

$$J_1 = c U_0/\phi_0 d \quad . \tag{3.17}$$

The flux line will then form a double–kink of width

$$w_{\rm K} = d\sqrt{\tilde{\epsilon}_1/U_0} \quad , \tag{3.18}$$

which costs a free energy $2E_{\rm K}$, where

$$E_{\rm K} = d\sqrt{\tilde{\epsilon}_1 U_0} \quad . \tag{3.19}$$

These expressions allow us to cast the current–voltage characteristics for $J_1 < J < J_c$ into the form [8]

$$\mathcal{E} \approx \rho_0 J \exp\left[-(E_{\rm K}/T)(J_1/J)\right] \quad . \tag{3.20}$$

In the half–loop regime, the intervortex repulsions thus do not affect the transport exponent $p_{\rm eff} = 1$, cf. Eq. (1.1). In recent experiments on BSCCO samples, exponents $p_{\rm eff} \approx 1$ have been found for intermediate currents, supporting the half–loop nucleation picture [15,16].

For $J < J_1$ we thus have to take the configurational limitations imposed by the other vortices into account. The most important thermally activated excitation will now be a double superkink (Fig. **??**): the flux line will send out a tongue–like pair of kinks to another possibly very distant defect, which has a favorable pinning energy, thus optimizing the tunneling probability. This is the flux line analogue of variable–range charge transport in disordered semiconductors [17]. The cost in free energy for such a configuration of transverse size R and extension Z along the magnetic–field direction will consist of two terms: (i) the double–superkink energy $2E_{\rm K}(R) = 2E_{\rm K}(d)R/d$ stemming from the elastic term, and (ii) the difference in pinning energies of the highest–energy occupied site, $\epsilon_i \approx \mu$, and the empty site at distance R with $\epsilon_j = \mu + \Delta(R)$. Thus the free energy difference with respect to the situation without kinks and external current is

$$\delta F_{\rm SK} \approx 2E_{\rm K} R/d + Z\Delta(R) - f_L R Z \quad , \tag{3.21}$$

which generalizes Eq. (4.13) of Ref. [8] for the case of non–interacting flux lines. The concentration available states as a function of R with D dimensions transverse to $\mathbf{B}$ (here $D = 2$) on the one hand equals $d^D \int_\mu^{\mu+\Delta(R)} g(\epsilon) d\epsilon$, and on the other hand is simply given by $\approx (d/R)^D$; thus $\Delta(R)$ is to be determined from the equation

$$\int_\mu^{\mu+\Delta(R)} g(\epsilon) d\epsilon = R^{-D} \quad . \tag{3.22}$$

Optimizing now first for vanishing current $J = 0$ gives the longitudinal extent Z^* of the superkink as a function of its transverse size R^*,

$$Z^* \approx -\frac{2E_{\rm K}/d}{(\partial\Delta/\partial R)_{R^*}} \quad . \tag{3.23}$$

Upon balancing the last term in Eq. (3.21) against the optimized sum of the first two, one arrives at

$$J\phi_0/c \approx \Delta(R^*)/R^* \quad , \tag{3.24}$$

which through inversion yields a typical hopping range $R^*(J)$. Inserting back into (3.21) finally yields the result for the optimized free energy barrier for a jump,

$$\delta F^*_{\rm SK}(J) \approx (2E_{\rm K}/d)R^*(J) \quad , \tag{3.25}$$

which we identify with the current–dependent activation energy in Eq. (3.12)

$$\mathcal{E} \approx \rho_0 J \exp\left[-(2E_{\rm K}/Td)R^*(J)\right] \quad . \tag{3.26}$$

Consider now a power–law form for the distribution of pinning energies, $g(\epsilon) = \kappa|\epsilon - \mu|^s$. Then from Eq. (3.22)

$$\Delta(R) = \left(\frac{s+1}{\kappa}\right)^{1/(s+1)} R^{-D/(s+1)} \quad , \tag{3.27}$$

and (3.24) yields

$$R^*(J) = \left(\frac{s+1}{\kappa}\right)^{1/(D+s+1)} \left(\frac{c}{\phi_0 J}\right)^p \quad , \tag{3.28}$$

where

$$p = \frac{s+1}{D+s+1} \quad . \tag{3.29}$$

This immediately leads to

$$\delta F^*_{\rm SK}(J) = 2E_{\rm K}(J_0/J)^p \quad , \tag{3.30}$$

which is of the form (1.1) with the transport exponent (3.29) and the current scale

$$J_0 \approx c/\phi_0 \kappa^{1/(s+1)} d^{1/p} \quad . \tag{3.31}$$

For a constant density of states, $s = 0$ and $\kappa = g(\mu)$, one recovers the non–interacting results, namely the Mott variable–range hopping exponent in D dimensions $p_0 = 1/(D+1)$, and expression (1.2) for J_0 [8]. Using the mean–field result for the gap exponent, $s_0 = D/\sigma - 1$ [Eq. (3.6)], Eq. (3.29) gives $p = 1/(1+\sigma)$, i.e.: $p \to 1$ for purely logarithmic interactions. In the light of our results in Sec. III B, however, we have to expect that $p_{\rm eff}$ will actually be smaller; namely with Eqs. (3.9) and (3.29) we find

$$p_{\rm eff}^{\rm max} \approx 2/3 \tag{3.32}$$

in two dimensions, for $\lambda \to \infty$ and small fillings.

The numerical result (3.32) for the variable–range hopping transport exponent of logarithmically interacting particles in two dimensions is consistent with the analysis by Fisher, Tokuyasu, and Young, [25] although the underlying theoretical arguments appear to be different. One might also question whether the "gauge glass" model studied in Ref. [25], with its quenched random vector potential, correctly captures the physics of a finite density of, say, positive vortices subject to a random pinning potential.

Using the general relations (3.22)–(3.26) and the Monte Carlo results of Sec. III B for the single–particle density of states, we can numerically evaluate the current–voltage characteristics for any form of $g(\epsilon)$. Results derived from the distributions of interacting pinning energies in Figs. **??**–**??** are depicted in Fig. **??** as double–logarithmic plots of the effective hop size $R^*(J) \propto \delta F^*_{\rm SK}(J) = U_{\rm B}(J)$ as function of the current J; according to Eq. (3.26), the slope immediately yields the effective transport exponent $p_{\rm eff}$ [cf. Eq. (3.28)]. Each plot corresponds to a certain filling fraction f, and summarizes the results for various values of the parameter λ/d. In Fig. **??**(a) for $f = 0.1$ the curve corresponding to long–range repulsions ($\lambda/d = 5$, which turns out to be practically indistinguishable from the case $\lambda \to \infty$) yields $p_{\rm eff} \approx 0.70 \pm 0.05$, in accord with (3.32). Upon decreasing the interaction range, $p_{\rm eff}$ becomes smaller, $p_{\rm eff} \approx 0.55 \pm 0.05$ for $\lambda/d = 2$ and $p_{\rm eff} \approx 0.50 \pm 0.05$ for $\lambda/d = 1$, and finally reaches the non–interacting exponent $p_0 = 1/3$ for $\lambda \to 0$. Note that there appears as well a considerable shift in the prefactor for the power law which also enhances the effectiveness of the vortex pinning, when the repulsive interactions are turned on. Long–range repulsive interactions thus reduce vortex voltages $\propto \exp[-2E_{\rm K}R^*(J)/dT]$ by several orders of magnitudes with respect to the non–interacting situation. E.g., for $\log j \approx 1.5$ the exponent in (3.26) is about ten times smaller for $\lambda/d = 5$ as compared to $\lambda \to 0$, and hence the resistivity is reduced by a factor of $\approx 10^{-5}$ [note that μ as function of λ does actually not vary very much, as $\epsilon_0 \propto \lambda^{-2}$, see Eq. (2.19)].

For higher fillings f, this collective pinning mechanism becomes relatively weaker, as the flux lines increasingly have to accomodate the underlying random pin distribution, and correlations are destroyed (see the discussion in Sec. III B). This can be seen in Fig. **??**(b) and (c); for $f = 0.8$ all the curves with different finite λ are essentially parallel to the the one for $\lambda \to 0$, i.e.: $p_{\rm eff} \approx 1/3$. However, there is still a considerable offset due to the change in the prefactor of the power law, and a reduction of the exponential factor in (3.26) by about three, and hence ρ still about twenty times smaller than for the case without interactions. We remark that of course the data for the very lowest currents in Fig. **??** are less reliable, for the finite size of our system becomes important when $R^*(J) \approx L_\perp$, which for $N_{\rm D} = 400$ happens for $\log[R^*(J)/d] \approx 1.3$.

In at least one recent experiment on BSCCO, reported by Konczykowski, Chikumoto, Vinokur, and Feigel'man, [15] an effective transport exponent $p_{\rm eff}$ in the variable–range hopping regime clearly different from the non–interacting $p_0 = 1/3$ was seen. The relevant parameters of the measurement at $T \approx 60$K and $H = 300$Oe depicted for $B_\phi = 0.2$T in Fig. 4 of Ref. [15] are $f = B/B_\phi \approx 0.15$, and, using $\lambda(0) \approx 1400\mathring{A}$, $\lambda(T)/d \approx 1.6$. Across about a half–decade the experimental current–voltage characteristics could be described by an effective exponent $p_{\rm eff} \approx 0.57$, which agrees very well with our result $p_{\rm eff} \approx 0.55$ for $f = 0.1$ and $\lambda/d = 2$ (note that both sets of parameters correspond to the same ratio

$\lambda/a_0 \approx 0.6$). For the experiment with much higher irradiation dose, $B_\phi = 2$T, corresponding to much lower fillings $f \approx 0.015$ and $\lambda/d \approx 5$, however, the variable–range hopping exponent turned out to be *lower*, namely $p_{\rm eff} \approx 0.39$, [16] contrary to the expectations from our above calculations. A possible explanation is that for such low values of f, see also Ref. [14] where $p_{\rm eff} \approx 1/2$ was measured for $f \approx 0.0065$, one has to reconsider the above analysis and take very rare fluctuations in the spatial distributions of rods into account, leading to $p_{\rm eff} = 1/2$, at least for the non–interacting case [8].

Finally, we briefly address the case of samples with finite thickness L. At the very lowest currents $J \to 0$, the current–voltage characteristics will eventually become Ohmic, [8] namely when $L \approx Z^*$, cf. Eq. (3.23); for the power–law density of states (1.3) studied above the typical longitudinal extent of a superkink is

$$Z^*(J) = (2E_{\rm K}/d)(c/\phi_0 J) \quad , \tag{3.33}$$

and hence the crossover current scale to Ohmic resistance becomes

$$J_L = (2E_{\rm K}/d)(c/\phi_0 L) \quad . \tag{3.34}$$

Substituting this into Eq. (3.30), we find

$$\delta F^*_{\rm SK}(L)/T = (L/L_0)^p \tag{3.35}$$

with the characteristic length scale

$$L_0 \approx \kappa^{1/(s+1)} (d/2E_{\rm K})^D T^{1/p} \quad . \tag{3.36}$$

The ensuing resistivity law

$$\rho = \rho_0 \exp\left[-(L/L_0)^p\right] \tag{3.37}$$

with its unusual thickness–dependence is the direct analogue of the variable–range hopping conductivity formula for disordered semiconductors, [17] generalizing Mott's law to the case of long–range interactions. The latter is recovered when $s \to 0$ and hence in $D = 2$ dimensions $p = p_0 = 1/3$ and $L_0 = g(\mu)(d/2E_{\rm K})^2 T^3$ [8].

IV. SUMMARY AND DISCUSSION

We have studied spatial correlations and the interacting density of states of flux lines pinned to columnar defects in the Bose glass phase [8]. Our procedure was to map the vortex problem onto an equivalent two–dimensional disordered boson system at zero temperature, [2,7,8] and examine the latter using an established Monte Carlo simulation algorithm [17,22–24]. We expect our simplified Hamiltonian to apply for $B < B^*$ [Eq. (2.11)] and "low" temperatures $T < T_1$ [Eq. (2.12)], which may in fact extend throughout a large fraction of the Bose glass phase, up to temperatures very near to the irreversibility line. The resulting distribution of pinning energies $g(\epsilon)$, which explicitly takes the possibly long–range interactions into account, was then used to calculate the vortex transport characteristics in the variable–range hopping regime at low currents $J < J_1$ [Eq. (3.17)], by generalizing the optimization procedure of Ref. [8].

For any $\lambda \geq a_0$ and low fillings $f \leq 0.3$, we have found that the flux lines form a highly correlated structure, the corresponding $S(q)$ displaying a marked, but broad peak at $qa_0 \approx 2\pi$. This spatial configuration may be understood as a highly distorted triangular lattice, with a large number of topological defects in the vortex coordination shells imposed by the underlying random distribution of columnar pinning sites. The peak in $S(q)$ is destroyed for larger fillings, and/or for weaker interactions $\lambda < a_0$. As for the Coulomb glass, [22] a peculiar spatial clustering of those sites with pinning energies near the chemical potential μ was found, the empty low energy sites and filled high energy sites occupying roughly complementary regions of space. As opposed to the case of localized charge carriers in disordered semiconductors, [17] the above spatial structures can be directly investigated in experiment, and the first such measurements have already been reported [12,13]. Quite unambiguously, highly correlated vortex distributions were found, and thus the flux line repulsions need to be accounted for [13]. Moreover, we have demonstrated how measured data on columnar pin and vortex positions may be utilized to actually predict the corresponding density of states and estimate the current–voltage characteristics [26].

In the distribution of pinning energies (single–particle density of states), for $\lambda \geq a_0$ the interactions lead to the formation of a remarkably prominent and persisting "Coulomb" gap near the chemical potential μ separating the occupied and empty states. This soft pseudo–gap can at least approximately be described by a power law (1.3), characterizing the depletion of states upon approaching μ. With increasing filling f for fixed interaction range λ, the gap exponent becomes smaller and the pseudo–gap eventually closes as the vortices have to accomodate with the underlying randomness, thus losing their spatial correlations. With our simulations, we found effective gap exponents $s_{\rm eff} \approx 1$ for $\lambda \approx d$ up to values $s_{\rm eff} \approx 3$ for $\lambda \to \infty$ and $f \ll 1$, which demonstrates that correlation effects are in fact much stronger than predicted by the qualitative mean–field estimate (3.7).

These equilibrium results were used as a basis to discuss both the half–loop nucleation region at intermediate currents, as well as the variable–range hopping transport regime at very low currents, where the vortices move by forming double–superkinks to favorable, possibly distant sites [8]. For the former case, we found that $p_{\rm eff} = 1$ as in the non–interacting situation. In the variable–range hopping regime, however, the depletion of low–energy states according to Eq. (1.3) implies a change of the effective transport exponent $p_{\rm eff}$ to higher values, rendering the collective pinning of vortices to columnar defects much more effective when $\lambda \geq a_0$. For $\lambda = d$ we have

found $p_{\rm eff} \approx 1/2$, while for very long–range repulsion $\lambda \to \infty$ values up to $p_{\rm eff} \approx 2/3$ may be reached, similar to the scaling analysis by Fisher, Tokuyasu, and Young [25]. In at least one measurement an effective exponent $p_{\rm eff} \approx 0.57$ was found for parameter values well in accord with our simulations [15].

Therefore, we may speculate that the remarkable correlation effects reported in this paper, which are induced by the long–range intervortex repulsions, could in fact be utilized for designing future superconducting materials in order to reduce dissipative flux transport as much as possible. For example, one could specifically "tailor" samples to obtain large values of λ/a_0. The remarkably strong pinning predicted for splayed columnar defects [19] might be even further enhanced by interactions.

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TABLE I. Boson analogy applied to vortex transport.

Charged bosons	Mass	$\hbar$	$\hbar/T$	Pair potential	Charge	Electric field	Current
Superconducting vortices	$\tilde{\epsilon}_1$	T	L	$2\epsilon_0 K_0(r/\lambda)$	ϕ_0	$\hat{\mathbf{z}} \times \mathbf{J}/c$	$\mathcal{E}$

FIG. **??**. Columnar defects and pinned flux lines, with filling fraction $f < 1$.

FIG. **??**. Schematic phase diagram of flux lines interacting with columnar defects that are aligned along the magnetic field direction.

FIG. **??**. Double–superkink configuration, which is the most important excitation relevant for the variable–range hopping transport regime of vortices.

FIG. **??**. Schematic current–voltage characteristics in the Bose glass phase.

FIG. **??**. Spatial positions of unoccupied columnar pins (open circles) and columnar pins occupied by magnetic flux lines (filled circles), as obtained from typical simulations (with randomly chosen initial vortex configuration) using an identical set of underlying defect coordinates ($N_{\rm D} = 400$). The parameters are: $w/2\epsilon_0 = 0.1$, $\lambda \to \infty$ (i.e.: logarithmic interactions), and (a) $f = 0.1$, (b) $f = 0.2$, and (c) $f = 0.4$.

FIG. **??**. Vortex structure function $S(q)$, obtained by averaging over the initial and final states of a series of $k = 40$ different defect configurations with $N_{\rm D} = 400$, $w/2\epsilon_0 = 0.1$, $\lambda/d = 5$, and (a) $f = 0.1$, (b) $f = 0.2$. $S(q)$ for the initial flux line distributions is the long–dashed line for randomly chosen positions, and the dashed line for the distorted triangular lattice; the final vortex distributions are depicted as the dot–dashed line for the configurations stemming from random initial states, and the solid line originating in the distorted triangular lattice positions. Notice the shift in the peak position of the dashed curves in (a) and (b).

FIG. **??**. Dependence of the vortex structure function $S(q)$ on the filling f. The results were obtained by averaging over $k = 40$ different defect configurations (with $N_{\rm D} = 400$ and random initial occupations), where $w/2\epsilon_0 = 0.1$ and $\lambda/d = 2$; solid: $f = 0.1$, dot–dashed: $f = 0.2$, dashed: $f = 0.4$, and long–dashed: $f = 0.8$. A broad peak at $qa_0 \approx 2\pi$ is found only for $f \leq 0.4$.

FIG. **??**. Voronoi analyses (Delaunay triangulations) of the vortex positions shown in Fig. 5; i.e.: $N_{\rm D} = 400$, $w/2\epsilon_0 = 0.1$, $\lambda \to \infty$, and (a) $f = 0.1$, (b) $f = 0.2$, (c) $f = 0.4$. The vortices with sixfold coordination are depicted as full circles, while topological defects with coordination number $N_c = 4$ are shown as open circles, $N_c = 5$ as open diamonds, $N_c = 7$ as encircled asterisks, and $N_c = 8$ as encircled crosses. The histograms depict the relative abundancy $h(N_c)$ for the occurence of the coordination numbers $N_c = 4, 5, 6, 7, 8$, respectively.

FIG. **??**. Spatial clustering of both occupied (filled circles) and empty (open circles) sites with energies near the chemical potential, for the final configuration of Fig. 5(c), i.e.: $N_{\rm D} = 400$, $f = 0.4$, $w/2\epsilon_0 = 0.1$, and $\lambda \to \infty$. The small symbols depict the complete distribution of vortices and pins, while the large circles correspond to those sites whose energies ϵ_i lie in the range (a) $|\epsilon_i - \mu|/2\epsilon_0 \leq 0.6$, and (b) $|\epsilon_i - \mu|/2\epsilon_0 \leq 1.2$, respectively.

FIG. **??**. Normalized distribution of pinning energies $G(E') = 2\epsilon_0 d^2 g(\epsilon)$ as function of the single–particle energies $E' = \epsilon'/2\epsilon_0$, averaged over $k = 2 \times 40$ runs with $N_{\rm D} = 400$, $w/2\epsilon_0 = 0.1$, $\lambda \to \infty$, and (a) $f = 0.2$, (b) $f = 0.4$; $\mu'/2\epsilon_0 \approx -1.3$ (marked by the arrow) in both cases. Open circles: results of the simulations starting from random initial configurations, filled triangles: site–energies from runs where a distorted triangular lattice was used as initial state.

FIG. **??**. Double–logarithmic plots of the normalized distribution of pinning energies $G(E) = 2\epsilon_0 d^2 g(\epsilon)$vs. $|E - E_\mu| = |\epsilon - \mu|/2\epsilon_0$, averaged over $k = 2 \times 40$ runs with $N_{\rm D} = 400$, $w/2\epsilon_0 = 0.1$, $\lambda \to \infty$, and (a) $f = 0.1$, (b) $f = 0.4$. Circles and triangles refer to the choices of random positions or a distorted triangular lattice as initial configurations, as in the previous figure. Filled and open symbols refer to single–particle energies in the occupied ($n_i = 1$) and empty ($n_i = 0$) pseudo–bands, respectively.

FIG. **??**. Normalized distribution of pinning energies $G(E) = d^2 g(\epsilon)$ as function of the single–particle energies $E = \epsilon/2\epsilon_0$, averaged over $k = 2 \times 40$ runs with $N_{\rm D} = 400$, $w/2\epsilon_0 = 0.1$, $\lambda/d = 5$, and (a) $f = 0.1$: $\mu/2\epsilon_0 \approx 10.9$, (b) $f = 0.4$: $\mu/2\epsilon_0 \approx 46.9$, (c) $f = 0.8$: $\mu/2\epsilon_0 \approx 96.8$. The symbols have the same meaning as in Fig. 10.

FIG. **??**. As in Fig. 12, but for $\lambda/d = 2$ and (a) $f = 0.1$: $\mu/2\epsilon_0 \approx 1.74$, (b) $f = 0.4$: $\mu/2\epsilon_0 \approx 9.07$.

FIG. **??**. As in Fig. 12, but for $\lambda/d = 1$ and (a) $f = 0.2$: $\mu/2\epsilon_0 \approx 0.72$, (b) $f = 0.8$: $\mu/2\epsilon_0 \approx 5.33$.

FIG. **??**. Chemical potential $E_\mu = \mu/2\epsilon_0$ as function of the system size ($N_{\rm D} = 100$, 200, 400, 800), averaged over $k = 16000/N_{\rm D}$ runs (with random initial vortex distributions), for $w/2\epsilon_0 = 0.1$ and $\lambda/d = 5$; squares: $f = 0.1$, triangles: $f = 0.2$, circles: $f = 0.3$, inverted triangles: $f = 0.4$, diamonds: $f = 0.5$. The open symbols plotted at $1/N_{\rm D} = 0$ represent the corresponding results for $\tilde{\mu} = \mu'(N_{\rm D} = 400) + 2\pi f(\lambda/d)^2$ obtained with the chemical potential results μ' for the purely logarithmic potential ($\lambda \to \infty$).

FIG. **??**. Double–logarithmic plots of the exponential factor $R^*(J)/d \propto \delta F^*_{\rm SK}(J)$ for variable–range hopping vs. the reduced current $j = J\phi_0 d/2\epsilon_0 c$, derived from the numerically obtained density of states, averaged over $k = 40$ runs with $N_{\rm D} = 400$, random initial configurations, and $w/2\epsilon_0 = 0.1$, for different interaction ranges: diamonds: $\lambda \to 0$, circles: $\lambda/d = 1$, squares: $\lambda/d = 2$, triangles: $\lambda/d = 5$; (a) $f = 0.1$, (b) $f = 0.4$, and (c) $f = 0.8$.