

**Superfluid Bosons and Flux Liquids: Disorder,  
Thermal Fluctuations, and Finite-Size Effects**  
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— **Erratum** —

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In Sec. 6.6, the text and equations following Eq. (6.107) until Eq. (6.111) should be corrected to read:

...As pointed out in Ref. [44], Eq. (6.106) leads to an estimate for the Bose glass transition temperature from the high-temperature flux liquid phase, complementary to the estimate in Ref. [40]. We note that the average disorder potential for columnar defects is  $\overline{V_D} = -U_0 b^2/d^2$ , where  $U_0 \approx \phi_0^2/2(4\pi\lambda)^2$  denotes the defect potential well strength,  $b$  its width, and  $d$  the average distance between the columns [40]. Accordingly, the defect correlator becomes  $\Delta_0 = U_0^2 b^4/d^2$ . One important modification has to be taken into account near the localization transition (more precisely, for temperatures above the depinning temperature  $T_{dp}$ ), however, namely the effective thermal renormalization of the repulsive flux line interactions. In the boson picture, this can be obtained via a summation of the hitherto neglected vertex corrections, which in two dimensions leads to the replacement [11,51]

$$V_0 \longrightarrow \frac{V_0}{1 + [V_0 \tilde{\epsilon}_1 / 4\pi (k_B T)^2] \ln(1/n\lambda^2)} . \quad (6.108)$$

In the dilute limit  $n\lambda^2 \ll 1$  the scattering length  $a$  is therefore to be replaced with

$$a = \frac{V_0 \tilde{\epsilon}_1}{4\pi} \longrightarrow \frac{(k_B T)^2}{\ln(1/n\lambda^2)} . \quad (6.109)$$

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Upon introducing the characteristic temperature  $k_B T^* = b(\tilde{\epsilon}_1 U_0)^{1/2}$  [40], using  $n \approx n_0 = B/\phi_0$ , and inserting Eq. (6.109) into Eq. (6.106), we find that

$$\overline{c_{44}^{v-1}(T)} = (n\tilde{\epsilon}_1)^{-1} \left[ 1 - (T_{\text{BG}}/T)^4 \right]. \quad (6.110)$$

Thus  $\overline{c_{44}^v}$  diverges as expected at the Bose glass transition temperature (compare Eqs. (D13) and (D6) in Ref. [40]),

$$T_{\text{BG}} = T^* \left[ \frac{\phi_0}{8\pi d^2 B} \ln \frac{\phi_0}{\lambda^2 B} \right]^{1/4}, \quad (6.111)$$

in qualitative agreement (apart from the logarithmic term) with a similar analysis in Ref. [44]. Although one should not trust the exponent for the vanishing of  $\overline{c_{44}^{v-1}}$  predicted by the approximate formula (6.110), this analysis suffices to estimate the transition temperature. ...

The passage following Eq. (6.125), until Eq. (6.129), should be replaced with:

... As for the linear defects, we can estimate a critical temperature from Eq. (6.124). The disorder correlator for planar defects is  $\Delta_0 = U_0^2 b^2/d$ , where  $b$  denotes the diameter of the defect potential, and  $d$  the average defect distance. [Delete Eqs. (6.126) and (6.127).] Upon applying the replacement (6.109) again, Eq. (6.125) yields

$$\overline{c_{xx}^{v-1}(T)} = (n\tilde{\epsilon}_1)^{-1} \left[ 1 - (\tilde{T}_{\text{BG}}/T)^4 \right], \quad (6.128)$$

and the instability occurs at the generalized Bose glass transition temperature marking the onset of localization to the defect planes

$$\tilde{T}_{\text{BG}} = T^* \left[ \frac{1}{16b^2 d} \left( \frac{\phi_0}{\pi B} \ln \frac{\phi_0}{\lambda^2 B} \right)^{3/2} \right]^{1/4}. \quad (6.129)$$

...

We are indebted to Johannes Kötzler for pointing out the errors in our previous estimates to us.

# Superfluid Bosons and Flux Liquids: Disorder, Thermal Fluctuations, and Finite-Size Effects

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## Abstract

The influence of different types of disorder (both uncorrelated and correlated) on the superfluid properties of a weakly interacting or dilute Bose gas, as well as on the corresponding quantities for flux line liquids in high-temperature superconductors at low magnetic fields are reviewed, investigated and compared. We exploit the formal analogy between superfluid bosons and the statistical mechanics of directed lines, and explore the influence of the different "imaginary time" boundary conditions appropriate for a flux line liquid. For superfluids, we discuss the density and momentum correlations, the condensate fraction, and the normal-fluid density as function of temperature for two- and three-dimensional systems subject to a space- and time-dependent random potential as well as conventional point-, line-, and plane-like defects. In the case of vortex liquids subject to point disorder, twin boundaries, screw dislocations, and various configurations of columnar damage tracks, we calculate the corresponding quantities, namely density and tilt correlations, the "boson" order parameter, and the tilt modulus. The finite-size corrections due to periodic vs. open "imaginary time" boundary conditions differ in interesting and important ways. Experimental implications for vortex lines are described briefly.

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## I. INTRODUCTION

The theory of weakly interacting and/or dilute superfluid Bose systems has been studied for several decades, and now constitutes a well-established branch of condensed-matter physics (see, e.g., Refs. [1–3]). More recently, upon utilizing the path integral representation of many-particle quantum mechanics [4,5], considerable progress has been reported for the even more difficult task to quantitatively understand the physical properties of *strongly* interacting boson superfluids such as Helium 4 below the Lambda phase transition line [6]. In addition to the detailed investigation of correlation functions in boson systems with these and other many-particle methods [7–9], field-theoretic and renormalization-group approaches have been put forward in order to analyze these interesting systems [10–16].

While the statistical mechanics of superfluid bosons as such is thus a well-advanced field, historically there has been much less effort to study the *influence of quenched disorder* on superfluid properties; e.g., the effects of point disorder (even to lowest order in the defect concentration) on correlation functions. Characteristic quantities such as the condensate fraction and the superfluid density have been systematically evaluated only in the past few years [17–19], in part motivated by the prospect of boson localization and related quantum phase transitions driven by the interplay of disorder and quantum fluctuations for superfluids in porous media [20–27].

Additional impetus for the investigation of disordered bosons stems from studies of magnetic flux lines in type-II high-temperature superconductors (for recent reviews, see Refs. [28,29]), motivated by the formal analogy of the quantum mechanics of two-dimensional bosons with the statistical mechanics of (2+1)-dimensional directed lines [30–33]. As is readily seen from the path integral formulation of the many-particle quantum mechanics for superfluids [4,5], the corresponding particle world lines behave as directed elastic strings with periodic boundary conditions in the direction of propagation  $z$  (imaginary time), with the boson mass  $m$  representing an effective line tension  $\tilde{\epsilon}_1$ , and quantum fluctuations  $\propto \hbar$  mapping on thermal fluctuations  $\propto T$  (see Sec. II). This mathematical correspondence can be further exploited for the case of magnetic flux lines, as summarized in Table I, with the two-dimensional boson density  $n$  and conjugate chemical potential  $\mu$  translating to the magnetic flux density  $B$  and external magnetic field  $H$ , respectively. Similarly, the (imaginary) velocity  $\mathbf{v}$  and superfluid density  $\rho_s$  correspond to a transverse magnetic field  $\mathbf{h}$  and the inverse tilt modulus  $c_{44}^{-1}$  of the flux line ensemble.

As is depicted in Fig. 1, in a pure system, the superfluid phase of bosons is represented by a highly entangled line liquid<sup>1</sup>, while the normal-fluid phase corresponds to a disentangled liquid of flux lines. Finally, the hexagonal Abrikosov flux lattice is represented by a

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<sup>1</sup>With the appropriate periodic boundary conditions, this reflects the occurrence of macroscopic particle exchange processes; compare the very suggestive pictures and discussions in Ref. [6].

two-dimensional crystalline solid in the boson language [31]. The prospects for finding an intermediate “supersolid” phase with both macroscopic phase coherence and crystalline order (not shown in the figure) may in fact be more promising for flux lines [34,35]. If periodic boundary conditions are applied for the superconductor along the magnetic-field direction, the fact that in two dimensions a second-order Berezinskii–Kosterlitz–Thouless transition separates the normal and superfluid phases implies that there should be a sharp distinction between the entangled and disentangled line liquid states [the Kosterlitz–Thouless transition temperature as function of the boson density  $T_{\text{KT}}(n)$  mapping on a critical sample length  $L(B)$  as function of the magnetic field]. However, if instead one assumes *free* rather than periodic boundary conditions, which should in fact be more realistic for most flux line problems, the correspondence between inverse temperature and sample size along the field direction is only approximate, and there need not be a sharp phase transition between an entangled and disentangled flux liquid [36]. This observation demonstrates that the issue of boundary conditions becomes important when the above correspondence is utilized, and that open boundary conditions may lead to very different physical behavior for  $L < \infty$  from what would “naively” be inferred from the boson picture.

For magnetic flux lines in high-temperature superconductors, disorder changes the phase diagram in essential ways. E.g., the presence of uncorrelated point defects destroys the crystalline order of the Abrikosov lattice [37], and may lead to a new disorder-dominated ground state, termed the vortex glass [38]. For the application of high- $T_c$  superconductors in a magnetic field, an effective mechanism for flux pinning is required, in order to minimize dissipative losses caused by the Lorentz-force induced vortex motion and thus retain a truly superconducting state, defined by a vanishing linear resistivity. For that purpose linear damage tracks have been produced by heavy-ion irradiation leading to columnar defects which turned out to be very effective in localizing the flux lines [28,29]. Theoretically, this corresponds to a two-dimensional boson localization transition; the new ground state of vortices pinned to columnar defects is called the Bose glass [39–41] (see also Ref. [42]). We have recently further exploited the quantum analogy, in this case with disordered semiconductors (the Coulomb glass), in order to show that long-range intervortex repulsions in the Bose glass produce a highly correlated ground state and an additional reduction in vortex transport [43]. Such considerations may eventually allow controlled “tailoring” of defects in such a way that magnetic flux lines are bound most effectively. For example, the introduction of a certain amount of “splay”, i.e., variation in columnar defect tilt angles was suggested to drastically improve flux pinning [44–46]. It was also demonstrated that columnar defects effectively increase the interlayer Josephson coupling in the pancake vortex regime [47]. In addition, the pronounced effect of planar disorder (as is frequently present in the high- $T_c$  ceramics in the form of twin boundaries) was studied [48], and recently also related to the so-called peak effect [49]. The investigation of the effect of transverse magnetic fields trying to tilt the flux lines away from, say, an array of parallel columnar pins, leads to an interesting localization problem in *non-Hermitian* quantum mechanics [50].

In this article, we compare and contrast the behavior of boson superfluids and vortex liquids in the presence of various types of disorder, emphasizing effects due to the different boundary conditions. Coherent-state path integral techniques [4,5] provide a unified framework to discuss both problems. The “open” boundary conditions appropriate to vortex liquids are illustrated in Fig. 1(c), and differ from the periodic boundary conditions relevant

to real bosons by a set of independent integrations over the entry and exit points of the directed lines, see Sec. II B.

Some of the varieties of disorder treated here (to leading order in the disorder strength) are compared for vortex lines and superfluids in Table II. Point disorder in three-dimensional vortex arrays corresponds to a rather unphysical random space- and time-dependent potential applied to, say, superfluid Helium 4 films. However, columnar and planar pins in vortex liquids are directly relevant to boson superfluid films on disordered substrates. Columnar pins, of course, correspond to “point” disorder in two-dimensional superfluids, as would occur on an amorphous substrate, while random planar disorder for vortex liquids is represented by random lines etched (say, via microlithography) onto an otherwise regular substrate. As part of this review, we rederive via the coherent-state method results for point disorder in superfluids [17,18], and produce new results for random line disorder perturbing two- and three-dimensional bosons, and random planes interacting with a three-dimensional superfluid. Among other results, we find that the superfluid density and condensate fraction in a zero-temperature superfluid film can be significantly reduced by depositing a set of lines with random positions and orientations in the substrate.

To first order in the defect concentration, the influence of weak point disorder on correlation functions and thermodynamic properties of the flux liquid was calculated in the limit  $L \rightarrow \infty$  by Nelson and Le Doussal [51,52]; in the present paper, we shall extend these previous investigations, which are based on the path integral representation of the partition sum and the Gaussian approximation, to  $d$ -dimensional generalizations of the disorder displayed in Table II, as well as two families of symmetrically tilted columnar defects and randomly splayed extended disorder. Our calculations of how disorder affects the superfluid density generalize a method developed by Hwa *et al.* to determine the effect of splayed columnar defects on the flux line tilt modulus [44]. Our investigation of the role played by different (namely periodic and free) boundary conditions (first discussed in Ref. [31]) elucidates the subtle differences in the underlying physics of boson superfluids and magnetic flux line liquids<sup>2</sup>. We obtain explicit results for the finite-size corrections for a number of thermodynamic quantities, such as the boson condensate fraction  $n_0$  and the inverse tilt modulus  $c_{44}^{-1}$  (superfluid density  $\rho_s$ ); these predictions should be subject to direct experimental tests. By invoking the boson analogy for a finite system “naively”, i.e., using periodic boundary conditions, one would expect  $n_0$  and  $c_{44}^{-1}$  to decrease as a powers of  $1/L$ , because thermal excitations —  $k_B T \sim 1/L$  — reduce the Bose condensate fraction and the superfluid density at nonzero temperatures (see Sec. III B); for open boundary conditions, however, it turns out that both these quantities *increase* (see Sec. III C)! This behavior arises because at the surfaces the probability distribution for the time-averaged position of a wiggling flux line is determined by the bosonic “wave function” [31,42], and is therefore larger than in the bulk, where this quantity is given by the “wave function” squared (a Gaussian for a free random walker); see Sec. II B and Fig. 2. Near the surfaces, the enhanced thermal wandering facilitates entanglement, and renders the system softer with respect to tilting, effectively *increasing*  $n_0$  and the overall  $c_{44}^{-1}$ .

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<sup>2</sup>In a different context, finite-size effects have also been studied by Marchetti and Nelson, using a hydrodynamic approach [53].

Most of our final results on vortex liquids are summarized for weak, delta–function–like interactions between lines in the direction perpendicular to  $z$ . This restriction is easily relaxed by inserting the full Fourier–transformed pair potential [see Eq. (2.22)] into the appropriate formulas. However, our results for flux liquids are *also* limited to regimes where the interactions between lines are strictly *local* in  $z$ , the magnetic–field direction. Although this restriction facilitates comparisons with nonrelativistic bosons, it does limit straightforward applications of our results to high magnetic fields in superconductors. The most problematic quantity at high fields is the vortex tilt modulus  $c_{44}$ , which has a large compressive contribution due to the interactions nonlocal in  $z$ . As discussed by Larkin and Vinokur [41] for isotropic vortex liquids using a nonlocal generalization of the boson mapping [54], the renormalized tilt modulus may be written as

$$c_{44} = \frac{B^2}{4\pi} \left( 1 + \frac{1}{4\pi\lambda^2 n_s} \right), \quad (1.1)$$

where  $\lambda$  is the London penetration depth and  $n_s$  denotes the superfluid number density. The first term is the compressive contribution due to nonlocality in  $z$ . Nonlocality is expected to be much less important in the second “vortex part” of the tilt modulus  $c_{44}^v$ , which is inversely proportional to the superfluid density of the equivalent boson system<sup>3</sup>. We thus expect that our results may be used as an approximation to this second term of Eq. (1.1) at high fields. The generalization of the Larkin–Vinokur formula to anisotropic superconductors with effective mass ratio  $m_{\perp}/m_z$  has recently been derived by Geshkenbein [55], whose result we quote here for completeness

$$c_{44} = \frac{B^2}{4\pi} \left( 1 + \frac{m_{\perp}/m_z}{4\pi\lambda^2 n_s} \right). \quad (1.2)$$

This article is intended to be self–contained and pedagogical, and we therefore include sufficient introductory material. In the following section, we review the description of both the quantum mechanics of superfluid bosons and the statistical mechanics of magnetic flux lines by coherent–state path integrals. The important role of Galilean invariance (for bosons) and affine transformations (for flux lines) will be discussed. In Sec. III, we study pure systems, first at zero temperature (i.e.,  $L \rightarrow \infty$  for flux lines), and then also the lowest–order corrections for finite temperatures (finite sample thickness for the case of magnetic vortices). In Sec. IV, the disorder contributions to density, momentum, and vorticity correlations, condensate fraction (depletion), and superfluid density are evaluated for bosonic systems (periodic boundary conditions) to lowest order in the defect concentration, and explicit formulas are presented for point, linear, and planar disorder in two and three space dimensions. In Sec. V, we discuss magnetic flux lines in high–temperature superconductors, for which open boundary conditions provide a better description, and determine the corresponding physical

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<sup>3</sup>The correlation function which gives the superfluid density (see Secs. II and III below) depends only on *fluctuations* in the tangents to the vortex lines about configurations parallel to the  $z$  axis. Locality in  $z$  is an excellent approximation for these fluctuations, see App. A of the second part of Ref. [40].



quantities, namely the magnitude of the “boson” order parameter, density and tilt correlations, and the renormalization of the tilt modulus by the disorder. Due to the generality of these computations the resulting formulas are rather lengthy; therefore we present explicit expressions for the cases of point disorder, splayed columnar defects, parallel extended and two families of symmetrically tilted defects in Sec. VI. Our findings for new kinds of correlated disorder in boson superfluids, and the effects of different boundary conditions, are summarized and discussed in Sec. VII. In the Appendices, we derive the fluctuation formula of Hwa *et al.* [44] for the tilt modulus in very thick samples (in the thermodynamic limit) using affine transformations, discuss the “phase-only approximation” and its limitations, present a detailed derivation of the correlation functions and their disorder contributions for the case of open boundary conditions, and list some useful formulas required for performing the Matsubara frequency sums and momentum integrals.

## II. PATH INTEGRAL REPRESENTATION FOR BOSONS AND FLUX LINES WITH DISORDER

In this section, we set the stage for our investigations of weakly interacting superfluid bosons and magnetic flux lines subject to disorder. By writing down the corresponding path integral representations for the partition sum, we elucidate the close formal relationship between the two distinct physical problems, and point out the differences. We also comment on the important role of Galilean invariance and affine transformations, respectively.

### A. Superfluid bosons

Consider the Hamiltonian for  $N$  bosonic particles in  $d$  space dimensions, with mass  $m$ , interacting via the scalar pair potential  $V(r)$ , and subject to an “external” disorder potential  $V_D(\mathbf{r})$ ,

$$\mathcal{H}_N^{\text{bos}} = \sum_{i=1}^N \left[ -\frac{\hbar^2}{2m} \nabla_i^2 + V_D(\mathbf{r}_i) \right] + \frac{1}{2} \sum_{i \neq j} V(|\mathbf{r}_i - \mathbf{r}_j|), \quad (2.1)$$

with the understanding that the corresponding  $N$ -particle wave function is symmetric with respect to all possible permutations. As a consequence, the trace leading to the partition sum is to be restricted to these totally symmetric bosonic states  $\{n\}$  with energy eigenvalues  $E_n^{\text{bos}}$ ,

$$Z_N^{\text{bos}} = \sum_{\text{bosonic states } \{n\}} e^{-\beta E_n^{\text{bos}}}, \quad (2.2)$$

where  $\beta = 1/k_B T$ .

By applying the standard methods of rewriting the Hamiltonian (2.1) in the language of second quantization and transforming to coherent states, the following imaginary-time path integral representation for the grand-canonical partition function is readily derived [4,5],

$$Z_{\text{gr}}^{\text{bos}} = \int \psi^*(r, \beta\hbar) = \psi^*(r, 0) \mathcal{D}[\psi(\mathbf{r}, \tau)] \mathcal{D}[\psi^*(\mathbf{r}, \tau)] e^{-S[\psi^*, \psi]/\hbar}, \quad (2.3)$$

$$\psi(r, \beta\hbar) = \psi(r, 0)$$

with the action

$$S[\psi^*, \psi] = \int_0^{\beta\hbar} d\tau \int d^d r \left\{ \hbar\psi^*(\mathbf{r}, \tau) \frac{\partial\psi(\mathbf{r}, \tau)}{\partial\tau} + \frac{\hbar^2}{2m} |\nabla\psi(\mathbf{r}, \tau)|^2 - [\mu + V_D(\mathbf{r}, \tau)] |\psi(\mathbf{r}, \tau)|^2 + \frac{1}{2} \int d^d r' V(|\mathbf{r} - \mathbf{r}'|) |\psi(\mathbf{r}', \tau)|^2 |\psi(\mathbf{r}, \tau)|^2 \right\}, \quad (2.4)$$

where  $\mu$  is the chemical potential. The constraints for the fields  $\psi$  and  $\psi^*$  in Eq. (2.3) stem from the permutation symmetry requirement and thus reflect the boson statistics; they can be interpreted as periodic boundary conditions for an imaginary-time slab which at finite temperature is of “length”  $\beta\hbar$ .

In the spirit of Landau–Ginzburg mean-field theory, we apply the method of steepest descent and find a nonlinear Schrödinger equation as the stationarity condition [2]

$$0 = \frac{\delta S[\psi^*, \psi]}{\delta\psi^*(\mathbf{r}, \tau)} = \hbar \frac{\partial\psi(\mathbf{r}, \tau)}{\partial\tau} - \frac{\hbar^2}{2m} \nabla^2\psi(\mathbf{r}, \tau) - [\mu + V_D(\mathbf{r}, \tau)]\psi(\mathbf{r}, \tau) + \int d^d r' V(|\mathbf{r} - \mathbf{r}'|) |\psi(\mathbf{r}', \tau)|^2 \psi(\mathbf{r}, \tau). \quad (2.5)$$

Upon specializing to short-range interactions with  $V(\mathbf{r}) = V_0\delta(\mathbf{r})$ , where  $V_0 = V(\mathbf{q} = \mathbf{0})$  and  $V(\mathbf{q}) = \int d^d r V(\mathbf{r})e^{-i\mathbf{q}\mathbf{r}}$  is the Fourier-transformed potential, taking the configurational average (denoted by an overbar), and assuming that  $|\psi(\mathbf{r}, \tau)|^2 \approx n_0$  independent of  $\mathbf{r}$  and  $\tau$ , this yields  $[-(\mu + \overline{V_D}) + n_0V_0]\sqrt{n_0} = 0$ , with solutions

$$\begin{aligned} n_0 &= 0 & \text{for } \mu + \overline{V_D} \leq 0 \\ n_0 &= (\mu + \overline{V_D})/V_0 & \text{for } \mu + \overline{V_D} > 0. \end{aligned} \quad (2.6)$$

The presence of disorder shifts the transition point to a phase with nonzero Bose condensate fraction  $n_0$ , which in a pure system would occur when  $\mu > 0$ . In the above mean-field approximation, the free energy in the superfluid phase becomes

$$F_{\text{gr}}(T, \Omega, \mu) = -k_B T \ln Z_{\text{gr}} = -[\mu + \overline{V_D}]^2 \Omega / 2V_0, \quad (2.7)$$

where  $\Omega = \int d^d r$  denotes the volume of the system. Therefore the mean particle density is

$$n = -\Omega^{-1} (\partial F_{\text{gr}} / \partial \mu)_{T, \Omega} \approx n_0 \quad (2.8)$$

in this approximation, and the mass density  $\rho \approx mn_0$ . Similarly, the isothermal compressibility becomes

$$\kappa_T = n^{-2} (\partial n / \partial \mu)_{T, \Omega} \approx (n_0^2 V_0)^{-1}, \quad (2.9)$$

which implies for the macroscopic sound velocity (with  $\kappa_S \approx \kappa_T$ )

$$c_1 = (nm\kappa_S)^{-1/2} \approx (n_0 V_0 / m)^{1/2}. \quad (2.10)$$

Notice that  $c_1$  vanishes both for  $n_0 \rightarrow 0$  and  $V_0 \rightarrow 0$ .

Eq. (2.8) implies that within mean-field theory, which neglects fluctuations, all particles enter the condensate in the superfluid phase. When density and phase fluctuations are taken into account, however, one finds that actually the condensate fraction is  $n_0 < n$  (see Sec. III and Refs. [1,2,4]). The Bose condensate is formed by those particles which take part in macroscopic ring exchange processes on scales larger than the thermal de Broglie wavelength; the required symmetrization of the many-particle wavefunction then leads to phase coherence. In the path integral picture, superfluidity and  $n_0 \neq 0$  can be related to a macroscopic fraction of world lines entangled and connected by virtue of the periodic boundary conditions, as is illustrated in Ref. [6]. Note that  $n_0$ , the thermodynamic *order parameter*, is conceptually quite distinct from the superfluid density  $\rho_s$ , which is a *transport coefficient* (see Sec. III), defined as an appropriate limit of a dynamical response function [1–4]; in terms of the path integral representation,  $\rho_s$  corresponds to the mean-square world line winding number [6].

If we now allow for an inhomogeneous condensate [including an inhomogeneous chemical potential  $\mu(\mathbf{r}, \tau)$ ], and transform to real time according to  $\tau \rightarrow it$ , Eq. (2.5) becomes

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}, t) - (\mu + \overline{V_D}) \psi(\mathbf{r}, t) + V_0 |\psi(\mathbf{r}, t)|^2 \psi(\mathbf{r}, t) . \quad (2.11)$$

It is then convenient to introduce the “polar” representation in terms of the new fields  $\pi(\mathbf{r}, t)$  and  $\Theta(\mathbf{r}, t)$ ,

$$\psi(\mathbf{r}, t) = \sqrt{n_0 + \pi(\mathbf{r}, t)} e^{i\Theta(\mathbf{r}, t)} , \quad (2.12)$$

implying that the density fluctuations are described by the  $\pi$  field,

$$n(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2 = n_0 + \pi(\mathbf{r}, t) , \quad (2.13)$$

and furthermore assume small fluctuations from the mean condensate density only:  $\psi \approx \sqrt{n_0} (1 + \pi/2n_0) e^{i\Theta}$  [the average of the phase field  $\Theta(\mathbf{r}, t)$  is taken to be zero]. We choose the constant  $n_0$  in Eq. (2.12) to be *exactly* equal to the average boson order parameter squared,

$$n_0 = |\langle \psi(\mathbf{r}, t) \rangle|^2 , \quad (2.14)$$

which is a direct measure of the macroscopic occupation number of the  $\mathbf{p} = \mathbf{0}$  momentum state. Note that in general  $n_0 \neq \langle n(\mathbf{r}, t) \rangle = \langle |\psi(\mathbf{r}, t)|^2 \rangle$ , although we will often set  $n_0 \approx n$ , because  $\langle \pi(\mathbf{r}, t) \rangle \approx 0$  in the Gaussian approximation [see Eqs. (2.13) and (2.8)]. At  $T = 0$ ,  $n_0 = n$  only for the ideal (non-interacting) Bose gas.

Upon defining the superfluid velocity according to

$$\mathbf{v}_s(\mathbf{r}, t) = \hbar \nabla \Theta(\mathbf{r}, t) / m , \quad \nabla \times \mathbf{v}_s = \mathbf{0} , \quad (2.15)$$

the imaginary part of Eq. (2.11) then yields the continuity equation for the irrotational superfluid flow [2],

$$\frac{\partial n(\mathbf{r}, t)}{\partial t} + \nabla \cdot [n(\mathbf{r}, t) \mathbf{v}_s(\mathbf{r}, t)] = 0 . \quad (2.16)$$

On the other hand, from the real part of Eq. (2.11) the characteristic correlation length

$$\xi^2 = \hbar^2/4mn_0V_0 = \hbar^2/4m^2c_1^2 \quad (2.17)$$

is inferred, and if the variation of the density  $\pi(\mathbf{r}, t)$  is slow on a scale  $\xi$ ,  $\xi^2\nabla^2\pi \ll \pi$ , the real part of Eq. (2.11) becomes

$$m \frac{\partial \mathbf{v}_s(\mathbf{r}, t)}{\partial t} + \nabla \left[ \frac{m}{2} \mathbf{v}_s(\mathbf{r}, t)^2 \right] = -\nabla \mu(\mathbf{r}, t) \quad (2.18)$$

(note that  $\pi = \delta n = n^2 \kappa_T \delta \mu \approx \delta \mu / V_0$ ), which is the fundamental equation of superfluid flow [2]. Finally, this may be cast into an alternative form, using the fact that superfluid flow is irrotational,  $\nabla \times \mathbf{v}_s = \mathbf{0}$ ,

$$m \frac{d\mathbf{v}_s(\mathbf{r}, t)}{dt} \equiv m \frac{\partial \mathbf{v}_s(\mathbf{r}, t)}{\partial t} + [\mathbf{v}_s(\mathbf{r}, t) \cdot \nabla] \mathbf{v}_s(\mathbf{r}, t) = -\nabla \mu(\mathbf{r}, t) , \quad (2.19)$$

which can be interpreted as Euler's equation for nonviscous flow, driven by a chemical potential gradient.

In Secs. III and IV, we extend this analysis to incorporate both fluctuations in the fields  $\pi$  and  $\Theta$  (in the Gaussian approximation) and in the disorder potential  $V_D(\mathbf{r})$  (to lowest nontrivial order). First, however, we discuss the boson analogy for the statistical mechanics of magnetic flux lines in some detail.

## B. Magnetic flux lines

In order to describe the large-scale properties of magnetic vortices in high- $T_c$  materials, which are strongly type-II superconductors, we work in the London limit, and define each flux line by its (two-dimensional) trajectory  $\mathbf{r}_i(z)$ , as it traverses the superconducting sample of length  $L$  parallel to the external magnetic field  $\mathbf{H}$ , which is taken to be aligned along the  $z$  direction. The ensuing Gibbs free energy for  $N$  such vortices subject to a disorder potential  $V_D(\mathbf{r})$ , is [31,51]

$$G_N(\mathbf{H}) = \int_0^L dz \left\{ \sum_{i=1}^N \left( \epsilon_1 \left[ 1 + \frac{m_\perp}{m_z} \left| \frac{d\mathbf{r}_i(z)}{dz} \right|^2 \right]^{1/2} + V_D[\mathbf{r}_i(z)] \right) + \frac{1}{2} \sum_{i \neq j} V(|\mathbf{r}_i(z) - \mathbf{r}_j(z)|) \right\} - \frac{\mathbf{H}}{4\pi} \int d^3r \mathbf{b}(\mathbf{r}) ; \quad (2.20)$$

here,  $\epsilon_1 = \epsilon_0 \ln \kappa$  is the effective line tension,  $\epsilon_0 = (\phi_0/4\pi\lambda)^2$  sets the energy scale,  $\phi_0 = hc/2e$  is the magnetic flux quantum,  $\kappa = \lambda/\xi$  is the ratio of the London penetration depth  $\lambda$  and the correlation length  $\xi$ , and the material anisotropy is embodied in the effective mass ratio  $m_\perp/m_z$  ( $\ll 1$  for high-temperature superconductors). Finally, the screened logarithmic intervortex repulsion is given by

$$V(r) = 2\epsilon_0 K_0(r/\lambda) , \quad (2.21)$$

where  $K_0(x)$  is a modified Bessel function, with the asymptotic behavior  $K_0(x) \propto -\ln x$  for  $x \rightarrow 0$ , and  $K_0(x) \propto x^{-1/2}e^{-x}$  for  $x \rightarrow \infty$ . In Fourier space this potential reads

$$V(q) = V_0/(1 + \lambda^2 q^2) , \quad (2.22)$$

where  $V_0 = \phi_0^2/4\pi$ . Note that we have assumed here that the vortex interaction is local in  $z$ , which is a good approximation when the flux lines are essentially straight and parallel to  $z$ , so that  $\langle |d\mathbf{r}/dz|^2 \rangle \ll 1$ . In the same spirit, we can then expand the square root in Eq. (2.20), and reexpress the integral over the magnetic flux density  $\mathbf{b}(\mathbf{r})$  in terms of  $\phi_0$  and  $N$ , which leads to

$$G_N(\mathbf{H}) = \mu N L + F_N[\{\mathbf{r}_i(z)\}] , \quad (2.23)$$

where we have introduced the ‘‘chemical potential’’ [see Eq. (2.26) below]

$$\mu = H \frac{\phi_0}{4\pi} - \epsilon_1 = \frac{\phi_0}{4\pi} (H - H_{c1}) , \quad (2.24)$$

which vanishes at the upper critical field  $H_{c1} = 4\pi\epsilon_1/\phi_0$ , and the vortex free-energy functional

$$F_N[\{\mathbf{r}_i(z)\}] = \int_0^L dz \left\{ \sum_{i=1}^N \left( \frac{\tilde{\epsilon}_1}{2} \left| \frac{d\mathbf{r}_i(z)}{dz} \right|^2 + V_D[\mathbf{r}_i(z)] \right) + \frac{1}{2} \sum_{i \neq j} V(|\mathbf{r}_i(z) - \mathbf{r}_j(z)|) \right\} , \quad (2.25)$$

with the modified line tension  $\tilde{\epsilon}_1 = (m_\perp/m_z)\epsilon_1$ .

A full statistical treatment requires a summation of  $e^{-G_N/k_B T}$  over all possible vortex trajectories  $\{\mathbf{r}_i(z)\}$ ; thus the partition sum becomes

$$Z_{\text{gr}}^{\text{fl}} = \sum_{N=0}^{\infty} \frac{1}{N!} e^{\mu N L/k_B T} Z_N^{\text{fl}} , \quad (2.26)$$

where the canonical partition function for a system of  $N$  lines reads

$$Z_N^{\text{fl}} = \prod_{i=1}^N \int \mathcal{D}[\mathbf{r}_i(z)] e^{-F_N[\{\mathbf{r}_i(z)\}]/k_B T} . \quad (2.27)$$

With Eq. (2.25) this already has the form of a quantum–mechanical partition function in the path integral representation, i.e., for the world lines of  $N$  particles of ‘‘mass’’  $\tilde{\epsilon}_1$  moving through ‘‘imaginary time’’  $z$ , interacting with potential  $V(r)$ , and subject to a disorder potential  $V_D(\mathbf{r})$ . According to Eq. (2.24), the external magnetic field plays the role of a chemical potential, and at  $T = 0$  vortices will start to penetrate the sample when  $H > H_{c1}$ ; furthermore, the areal particle density is related to the magnetic flux via  $n = B/\phi_0$ .

To render this quantum analogy of the classical statistical mechanics problem of directed lines more precise [31,52], we note that the partition sum (2.27) can be rewritten in terms of the transfer matrix  $e^{-\mathcal{H}_N^{\text{fl}} L/k_B T}$  connecting neighboring constant- $z$  slices, i.e., as an integral over a quantum–mechanical matrix element (in ‘‘imaginary time’’),

$$Z_N^{\text{fl}} = \prod_{i=1}^N \int d\mathbf{r}'_i \prod_{i=1}^N \int d\mathbf{r}_i \langle \mathbf{r}'_1 \cdots \mathbf{r}'_N | e^{-\mathcal{H}_N^{\text{fl}} L/k_B T} | \mathbf{r}_1 \cdots \mathbf{r}_N \rangle , \quad (2.28)$$

where the states  $|\mathbf{r}_1 \cdots \mathbf{r}_N\rangle$  and  $\langle \mathbf{r}'_1 \cdots \mathbf{r}'_N|$  describe the entry and exit points of the vortices, respectively, and the Hamiltonian is [compare Eq. (2.1)]

$$\mathcal{H}_N^{\text{fl}} = \sum_{i=1}^N \left[ -\frac{(k_B T)^2}{2\tilde{\epsilon}_1} \nabla_i^2 + V_D(\mathbf{r}_i) \right] + \frac{1}{2} \sum_{i \neq j} V(|\mathbf{r}_i - \mathbf{r}_j|) . \quad (2.29)$$

By inserting a complete set of many-particle energy eigenstates  $|n\rangle$  in Eq. (2.28), and introducing the zero-momentum state

$$|\mathbf{p}_1 = \mathbf{0} \cdots \mathbf{p}_N = \mathbf{0}\rangle = \prod_{i=1}^N \int d\mathbf{r}_i |\mathbf{r}_1 \cdots \mathbf{r}_N\rangle , \quad (2.30)$$

we find

$$\begin{aligned} Z_N^{\text{fl}} &= \langle \mathbf{p}_1 = \mathbf{0} \cdots \mathbf{p}_N = \mathbf{0} | e^{-\mathcal{H}_N^{\text{fl}} L / k_B T} | \mathbf{p}_1 = \mathbf{0} \cdots \mathbf{p}_N = \mathbf{0} \rangle \\ &= \sum_n |\langle n | \mathbf{p}_1 = \mathbf{0} \cdots \mathbf{p}_N = \mathbf{0} \rangle|^2 e^{-E_n^{\text{fl}} L / k_B T} . \end{aligned} \quad (2.31)$$

The “quantum-mechanical” interpretation of this matrix element is that of a system prepared in the ground state  $|\mathbf{p}_1 = \mathbf{0} \cdots \mathbf{p}_N = \mathbf{0}\rangle$  of the *ideal* Bose gas at  $z = 0$ . It is then allowed to evolve in imaginary time under the influence of interactions and disorder. After “time”  $L$ , this interacting superfluid is projected back onto the ideal Bose gas ground state.

To see that only boson states are involved in the statistics of these fictitious quantum particles with “free” boundary conditions, note that the zero-momentum state (2.30) is symmetric under permutations, and therefore only those eigenstates of the permutation-symmetric Hamiltonian (2.29) contribute to the partition sum (2.31) that are permutation-symmetric themselves. Hence the summation effectively includes bosonic many-particle states only,

$$Z_N^{\text{fl}} = \sum_{\text{bosonic states } \{n\}} |\langle n | \mathbf{p}_1 = \mathbf{0} \cdots \mathbf{p}_N = \mathbf{0} \rangle|^2 e^{-E_n^{\text{bos}} L / k_B T} . \quad (2.32)$$

Upon comparing with the “pure” bosonic partition function (2.2), we see that there is a difference arising from the weights involved in the projection onto the zero-momentum ground state. Yet in the limit  $L \rightarrow \infty$ , only the lowest-energy bosonic state contributes to the sum (2.32), and upon identification of the respective quantities in Table I, the quantum mechanics of superfluid bosons at  $T = 0$  becomes completely equivalent to the statistical mechanics of directed lines in the thermodynamic limit. If we impose *periodic* boundary conditions on the vortices, i.e. use a toroidal geometry in the field direction, the exact correspondence of Eqs. (2.32) and (2.2) even survives for finite sample lengths (flux lines) / nonzero temperatures (bosons), and a coherent-state path integral representation for the grand-canonical partition function (2.26) may be derived, which is precisely of the form of Eqs. (2.3),(2.4). Note that thermal fluctuations  $\propto k_B T$  correspond to quantum fluctuations  $\propto \hbar$ , while the torus length  $L$  maps onto the inverse boson temperature  $\beta \hbar$ . In this case, a sharp Berezinskii–Kosterlitz–Thouless transition is to be expected from an entangled (superfluid) phase at large  $L$  to a disentangled (normal fluid) phase at small  $L$  [30,31].

For magnetic flux lines in most superconducting samples, however, periodic boundary conditions are rather artificial, and a more adequate description of finite systems may be obtained using *open* boundary conditions as embodied in Eq. (2.32), where the line exit and entry points are integrated over freely, corresponding to an ideal Bose gas with

$$\psi(\mathbf{r}, 0) = \psi(\mathbf{r}, L) = \sqrt{n} . \quad (2.33)$$

Note that this rigid constraint at the boundaries means that the “depletion”  $n - n_0 = \langle |\psi|^2 \rangle - |\langle \psi \rangle|^2$  (see Sec. III B) vanishes identically at the top and bottom of the sample. As is shown in Ref. [31], within the mean-field approximation the sole effect of the projection amplitude in (2.32) is a shift of the chemical potential according to

$$\tilde{\mu} = \mu - \frac{k_B T}{L} \ln \frac{N}{Z_1} , \quad (2.34)$$

where  $Z_1$  represents the partition function for a single flux line. Physically, this amounts to a downward entropic renormalization of the upper critical field  $H_{c1}$  [see Eq. (2.24)], due to thermal fluctuation of vortices. With the replacement  $\mu \rightarrow \tilde{\mu}$ , and the correspondences of Table I, the results of the previous Sec. II A can then be taken over. Fluctuation effects may furthermore be described by a coherent-state path integral with an action of the form (2.4), provided the additional constraints on the fields  $\psi$  and  $\psi^*$ , or  $\pi$  and  $\Theta$ , respectively, are explicitly taken into account, see Ref. [31] and Sec. III C below. In this geometry, the exact correspondence between  $L$  and  $\beta\hbar$  in a real boson system is lost, and the sharp phase transition from an entangled to a disentangled flux liquid may in fact be replaced by a smooth crossover [36]. We also remark that the bosonic field operators  $\hat{\psi}^\dagger$  and  $\hat{\psi}$  correspond to magnetic monopole and anti-monopole creation operators in the flux line picture.

The vortex free energy (2.25) was derived for a constant magnetic field  $\mathbf{H}$  in the  $z$  direction. The response to “tilt”, i.e., to a small slowly varying magnetic field  $\mathbf{H}_\perp(\mathbf{r}, z)$  perpendicular to  $z$  is also of interest. Such perturbations couple directly to the “tangent field” of the flux lines,

$$\mathbf{t}(\mathbf{r}, z) = \sum_{i=1}^N \frac{d\mathbf{r}_i(z)}{dz} \delta^{(2)}(\mathbf{r} - \mathbf{r}_i(z)) , \quad (2.35)$$

and the free energy (2.25) is modified according to <sup>4</sup>

$$\begin{aligned} F_N &\longrightarrow F_N - \frac{\phi_0}{4\pi} \int d^3r \mathbf{H}(\mathbf{r}, z) \cdot \mathbf{t}(\mathbf{r}, z) \\ &\longrightarrow F_N - \tilde{\epsilon}_1 \sum_{i=1}^N \int_0^L dz \mathbf{v}(\mathbf{r}_i, z) \cdot \frac{d\mathbf{r}_i(z)}{dz} , \end{aligned} \quad (2.36)$$

where

$$\mathbf{v}(\mathbf{r}, z) \equiv \frac{\phi_0}{4\pi\tilde{\epsilon}_1} \mathbf{H}_\perp(\mathbf{r}, z) . \quad (2.37)$$

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<sup>4</sup>To derive this result, note that the perpendicular component of the magnetic field  $\mathbf{b}_\perp(\mathbf{r}, z)$  in Eq. (2.20) satisfies the London equation,  $\mathbf{b}_\perp(\mathbf{r}, z) = (m_z/m_\perp)\lambda^2 \nabla_\perp^2 \mathbf{b}_\perp(\mathbf{r}, z) + \lambda^2 \partial_z^2 \mathbf{b}_\perp(\mathbf{r}, z) + \phi_0 \mathbf{t}(\mathbf{r}, z)$ . Upon substituting for  $\mathbf{b}_\perp(\mathbf{r}, z)$  in Eq. (2.20), the derivative terms on the right-hand side can be eliminated via an integration by parts with negligible error, provided the variations in the external field  $\mathbf{H}_\perp(\mathbf{r}, z)$  are sufficiently slow. For a treatment of the response to more general field perturbations, see Appendix A of Ref. [51].

The field  $\mathbf{v}(\mathbf{r}, z)$  thus couples to the flux lines like a vector potential in this ‘‘Lagrangian’’ formulation of the physics. In the transfer matrix representation of the vortex partition function, the Hamiltonian (2.29) becomes [51]

$$\mathcal{H}_N^{\text{fl}} = \frac{1}{2\tilde{\epsilon}_1} \sum_{j=1}^N \left[ \frac{k_B T}{i} \nabla_j + i\tilde{\epsilon}_1 \mathbf{v}(\mathbf{r}_j, z) \right]^2 + \sum_{i=1}^N V_D(\mathbf{r}_i) + \frac{1}{2} \sum_{i \neq j} V(|\mathbf{r}_i - \mathbf{r}_j|) . \quad (2.38)$$

Note that the tilt field generates an *imaginary* vector potential in this Hamiltonian formulation. If  $\mathbf{H}_\perp$  is constant, then tilting the external field acts like an imaginary Galilean boost with velocity  $i\mathbf{v} = i\phi_0 \mathbf{H}_\perp / 4\pi\tilde{\epsilon}_1$  on the fictitious bosons.

As a simple, but useful application of the mapping of vortex physics onto quantum mechanics, we briefly consider the thermal wandering of a single flux line ( $N = 1$ ) in a sample of length  $L$ , starting at the origin  $\mathbf{r} = \mathbf{0}$ , and ending at the opposite surface at  $\mathbf{r} = \mathbf{r}_\perp$  [40,42]. With Eqs. (2.28) and (2.29), the constrained partition function for this situation reads

$$Z(\mathbf{r}_\perp, \mathbf{0}; L) = \langle \mathbf{r}_\perp | e^{-\mathcal{H}_1^{\text{fl}} L / k_B T} | \mathbf{0} \rangle , \quad (2.39)$$

and the probability distribution for the vortex tip position at the upper surface becomes

$$\mathcal{P}(\mathbf{r}_\perp; L) = Z(\mathbf{r}_\perp, \mathbf{0}; L) \Big/ \int d^d r_\perp Z(\mathbf{r}_\perp, \mathbf{0}; L) = \frac{\sum_n \langle n | \mathbf{0} \rangle \langle \mathbf{r}_\perp | n \rangle e^{-E_n^{\text{fl}} L / k_B T}}{\sum_n \langle n | \mathbf{0} \rangle \int d^d r_\perp \langle \mathbf{r}_\perp | n \rangle e^{-E_n^{\text{fl}} L / k_B T}} , \quad (2.40)$$

where in the second equation we have inserted a complete set of energy eigenstates  $|n\rangle$  with eigenvalues  $E_n^{\text{fl}}$ . For thick samples,  $L \rightarrow \infty$ , the ground state dominates the above sums, and therefore

$$\mathcal{P}(\mathbf{r}_\perp; L) = \frac{\psi_0(\mathbf{r}_\perp)}{\int d^d r_\perp \psi_0(\mathbf{r}_\perp)} \left[ 1 + \mathcal{O}\left(e^{-(E_1^{\text{fl}} - E_0^{\text{fl}})L / k_B T}\right) \right] ; \quad (2.41)$$

here,  $\psi_0(\mathbf{r}_\perp) = \langle \mathbf{r}_\perp | 0 \rangle$  is the ground state ‘‘wave function’’ with eigenvalue  $E_0^{\text{fl}}$ , and  $E_1^{\text{fl}}$  is the energy of the first excited state. The probability distribution for the flux line to have wandered a distance  $r_\perp$  at the surface is thus proportional to the ground state wave function [note that  $\psi_0(\mathbf{r}_\perp)$  is positive and nodeless].

On the other hand, if we are interested in the more general problem of one flux line entering the sample at  $\mathbf{r} = \mathbf{r}_i$  and leaving it at  $\mathbf{r}_f$ , and ask what is its average thermal wandering probability distribution  $\tilde{\mathcal{P}}(\mathbf{r}; L)$  at height  $z$  far away from the two boundaries, the answer is

$$\tilde{\mathcal{P}}(\mathbf{r}; L) = \tilde{Z}(\mathbf{r}; L) \Big/ \int d^d r \tilde{Z}(\mathbf{r}; L) , \quad (2.42)$$

where, using the definition (2.39),

$$\tilde{Z}(\mathbf{r}; L) = \int d^d r_i \int d^d r_f Z(\mathbf{r}_f, \mathbf{r}; L - z) Z(\mathbf{r}, \mathbf{r}_i; z) . \quad (2.43)$$

Upon inserting a complete set of energy eigenstates, for thick samples the final result is



$$\tilde{\mathcal{P}}(\mathbf{r}; L) = \frac{\psi_0(\mathbf{r})^2}{\int d^d r \psi_0(\mathbf{r})^2} \left[ 1 + \mathcal{O} \left( e^{-(E_1^{\text{fl}} - E_0^{\text{fl}})L/k_B T} \right) \right] , \quad (2.44)$$

i.e., in the bulk, this probability distribution is proportional to the square of the ground state wave function. Note that for periodic boundary conditions along the  $z$  direction, which rule out any special surface effects, Eq. (2.44) describes the thermal wandering in the entire sample. On the other hand, for open boundary conditions the surface behavior is different from that in the bulk, and Eq. (2.41) applies. We therefore conclude that near the surfaces, thermal flux line wandering is enhanced for open boundary conditions, as depicted in Fig. 2.

We comment finally on the meaning of the “boson” order parameter for directed lines. On a formal level, superfluidity is usually accompanied by long-range order in the correlation function

$$\tilde{G}(\mathbf{r}, z; \mathbf{r}', z') = \langle \psi(\mathbf{r}, z) \psi^*(\mathbf{r}', z') \rangle , \quad (2.45)$$

specifically

$$\lim_{|\mathbf{r}-\mathbf{r}'| \rightarrow \infty} \tilde{G}(\mathbf{r}, z; \mathbf{r}', z') = \langle \psi(\mathbf{r}, z) \rangle \langle \psi^*(\mathbf{r}', z') \rangle = n_0 \neq 0 , \quad (2.46)$$

where the average is evaluated via coherent-state path integrals. To understand what this long-range order means for vortices [52], it is helpful to consider first a situation where the order is manifestly *short*-ranged, in the Abrikosov flux lattice. Figure 3 shows that the operators  $\hat{\psi}$  and  $\hat{\psi}^\dagger$  create vacancy and interstitial strings in the vortex crystal, terminating in “magnetic monopoles”. The composite operator in Eq. (2.45) creates an extra line at  $(\mathbf{r}', z')$  (i.e., a column of interstitials in the solid), and destroys an existing line at  $(\mathbf{r}, z)$ , creating a column of vacancies. The lowest-energy configuration is then a line of vacancies (for  $z' > z$ ) or interstitials (for  $z' < z$ ) connecting the two points with an energy  $\sigma s$  proportional to the length  $s$  of this “string”. It follows that the correlation function (2.45) decays exponentially to zero (i.e.,  $\propto e^{-\sigma|\mathbf{r}-\mathbf{r}'|/k_B T}$ ) for large separations in the crystalline phase. In an entangled line *liquid*, on the other hand, vacancies and interstitials “melt” away and the asymptotic string tension  $\sigma$  vanishes, implying long-range order (2.46) in  $\tilde{G}(\mathbf{r}, z; \mathbf{r}', z')$ . In this limit,  $\langle \psi(\mathbf{r}, z) \rangle = \langle \psi^*(\mathbf{r}', z') \rangle = e^{-E_{\text{mon}}/k_B T}$ , where  $E_{\text{mon}}$  is the energy of an isolated “magnetic monopole”.

### C. Galilean invariance of the pure system and affine transformations

Before we proceed with the investigation of fluctuation effects in boson superfluids and flux line liquids, we discuss the fundamental symmetries of the pure systems, namely invariance with respect to Galilean transformations and uniform tilts, respectively. We begin with the disorder-free boson Hamiltonian (2.1), and study the effect of a Galilean boost with constant velocity  $\mathbf{v} = \text{const.}$ ,

$$\begin{aligned} \mathbf{r} &\rightarrow \mathbf{r}' = \mathbf{r} + \mathbf{v}t , & t &\rightarrow t' = t , \\ \mathbf{P} &\rightarrow \mathbf{P}' = \mathbf{P} + N m \mathbf{v} , \\ E &\rightarrow E' = E + \mathbf{v} \cdot \mathbf{P} + \frac{1}{2} N m v^2 , \end{aligned} \quad (2.47)$$

where  $\mathbf{P}$  is the total momentum, and  $E$  the total energy of the  $N$ -particle system.

In the coherent-state path integral representation (with imaginary time  $\tau = it$ ), the particle density  $n(\mathbf{r}, \tau)$ , momentum density  $\mathbf{g}(\mathbf{r}, \tau)$ , particle current  $\mathbf{j}(\mathbf{r}, \tau)$ , and energy density  $\tilde{\mathcal{H}}(\mathbf{r}, \tau) = \mathcal{H}(\mathbf{r}, \tau) - \mu n(\mathbf{r}, \tau)$  read

$$n(\mathbf{r}, \tau) = |\psi(\mathbf{r}, \tau)|^2 , \quad (2.48)$$

$$\mathbf{g}(\mathbf{r}, \tau) = m\mathbf{j}(\mathbf{r}, \tau) = \frac{\hbar}{2i} [\psi^*(\mathbf{r}, \tau) \nabla \psi(\mathbf{r}, \tau) - \psi(\mathbf{r}, \tau) \nabla \psi^*(\mathbf{r}, \tau)] , \quad (2.49)$$

$$\tilde{\mathcal{H}}(\mathbf{r}, \tau) = \frac{\hbar^2}{2m} |\nabla \psi(\mathbf{r}, \tau)|^2 - \mu |\psi(\mathbf{r}, \tau)|^2 + \frac{1}{2} \int d^d r' V(\mathbf{r} - \mathbf{r}') |\psi(\mathbf{r}', \tau)|^2 |\psi(\mathbf{r}, \tau)|^2 . \quad (2.50)$$

A Galilean transformation (2.47) is then represented by a unitary transformation acting on the fields,

$$\psi(\mathbf{r}, \tau) \rightarrow \psi'(\mathbf{r}', \tau') = \psi(\mathbf{r}, \tau) e^{im\mathbf{v}\cdot\mathbf{r}/\hbar} ; \quad (2.51)$$

as is easily seen, this leaves the particle density invariant,

$$n'(\mathbf{r}', \tau') = n(\mathbf{r}, \tau) , \quad (2.52)$$

while the transformed momentum and energy densities become

$$\mathbf{g}'(\mathbf{r}', \tau') = \mathbf{g}(\mathbf{r}, \tau) + m\mathbf{v}n(\mathbf{r}, \tau) , \quad (2.53)$$

$$\tilde{\mathcal{H}}'(\mathbf{r}', \tau') = \tilde{\mathcal{H}}(\mathbf{r}, \tau) + \mathbf{v} \cdot \mathbf{g}(\mathbf{r}, \tau) + \frac{1}{2} m v^2 n(\mathbf{r}, \tau) , \quad (2.54)$$

in accord with Eqs. (2.47). Note that the third term in Eq. (2.54) may be absorbed into a shift of the chemical potential,

$$\mathcal{H}'(\mathbf{r}', \tau') = \mathcal{H}(\mathbf{r}, \tau) + \mathbf{v} \cdot \mathbf{g}(\mathbf{r}, \tau) , \quad (2.55)$$

$$\mu' = \mu - m v^2 / 2 . \quad (2.56)$$

In the polar representation

$$\psi(\mathbf{r}, \tau) = \sqrt{n_0 + \pi(\mathbf{r}, \tau)} e^{i\Theta(\mathbf{r}, \tau)} , \quad (2.57)$$

the effect of the Galilean transformation is

$$\pi'(\mathbf{r}', \tau') = \pi(\mathbf{r}, \tau) , \quad (2.58)$$

$$\Theta'(\mathbf{r}', \tau') = \Theta(\mathbf{r}, \tau) + m\mathbf{v} \cdot \mathbf{r} / \hbar . \quad (2.59)$$

Upon introducing the superfluid velocity according to Eq. (2.15),

$$\mathbf{v}_s(\mathbf{r}, \tau) = \hbar \nabla \Theta(\mathbf{r}, \tau) / m , \quad \mathbf{g}(\mathbf{r}, \tau) = m \mathbf{v}_s(\mathbf{r}, \tau) n(\mathbf{r}, \tau) , \quad (2.60)$$

one finds

$$\mathbf{v}'_s(\mathbf{r}', \tau') = \mathbf{v}_s(\mathbf{r}, \tau) + \mathbf{v} . \quad (2.61)$$

For the case of directed line liquids, the Galilean boost (2.47) translates into an affine transformation, namely a uniform tilt away from the  $z$  axis,

$$\mathbf{r} \rightarrow \mathbf{r}' = \mathbf{r} + \mathbf{h}z/\tilde{\epsilon}_1, \quad z \rightarrow z' = z, \quad \frac{d\mathbf{r}}{dz} \rightarrow \frac{d\mathbf{r}'}{dz'} = \frac{d\mathbf{r}}{dz} + \frac{\mathbf{h}}{\tilde{\epsilon}_1}, \quad (2.62)$$

with  $\mathbf{h} = \text{const.}$ , which may be interpreted as a transverse external magnetic field. As discussed above, the uniform tilt  $\mathbf{h}/\tilde{\epsilon}_1 \equiv \phi_0 \mathbf{H}_\perp / 4\pi\tilde{\epsilon}_1$  corresponds to a Galilean transformation with *imaginary* velocity  $i\mathbf{v}$  (see Table I). This imaginary boost velocity means that the phase does *not* change in the simple fashion described by Eq. (2.59). In the corresponding quantum-mechanical problem one has to consider the effect of imaginary vector potentials leading to the appearance of non-Hermitian operators [50]. The transformation formulas for the energy density may be read off from the previous expressions; the shift in the chemical potential becomes

$$\mu' = \mu + h^2/2\tilde{\epsilon}_1. \quad (2.63)$$

We finally remark that under a Galilean boost (2.47) or affine transformation (2.62), the disorder potential is modified according to

$$V_D(\mathbf{r}(\tau), \tau) \rightarrow V_D(\mathbf{r}' + i\mathbf{v}\tau', \tau'), \quad (2.64)$$

and

$$V_D(\mathbf{r}(z), z) \rightarrow V_D(\mathbf{r}' - \mathbf{h}z'/\tilde{\epsilon}_1, z'), \quad (2.65)$$

respectively. Therefore, disorder breaks ‘‘Galilean’’ invariance; however, for uncorrelated defects for which the disorder correlator is a constant, the symmetry is recovered by taking the quenched disorder average. In the Appendix A, we review how these affine transformations can be exploited for the calculation of the disorder contribution to the line tilt modulus in the thermodynamic limit  $L \rightarrow \infty$  [44]. A method which is more convenient for finite  $L$  is discussed in the next section.

### III. FLUCTUATIONS AND CORRELATION FUNCTIONS IN PURE SYSTEMS

We are now ready to evaluate correlation functions for our model action (2.4). We shall henceforth regularly use the boson notation and language, but nevertheless keep the application to flux liquids in mind as well. We start by rederiving and summarizing the known results for the pure system with periodic (bosonic) boundary conditions [4], before we extend our analysis to open boundary conditions (Sec. III C), which are more appropriate to flux line systems, in the presence of uncorrelated and correlated disorder (Secs. IV, V).

#### A. Correlation functions in the Gaussian ensemble with periodic boundary conditions (Bogoliubov approximation)

To separate density and phase fluctuations, it is convenient to use the polar representation (2.57) for the fields  $\psi$  and  $\psi^*$ . Note that the path integral in the partition sum (2.3) is

originally defined by integration over  $\text{Re } \psi = \sqrt{n_0 + \pi} \cos \Theta$  and  $\text{Im } \psi = \sqrt{n_0 + \pi} \sin \Theta$ ; the Jacobian corresponding to the above transformation is  $1/2$ , and being constant may be absorbed into the path integral measure. As we shall only be dealing with small fluctuations, we extend the integration range for the phase field  $\Theta$  from  $[-\pi; \pi]$  to  $[-\infty; +\infty]$ . Thus the grand-canonical partition function becomes a functional integral over the independent variables  $\pi$  and  $\Theta$ ,

$$Z_{\text{gr}}^{\text{bos}} = \int_{\substack{\pi(r, \beta\hbar) = \pi(r, 0) \\ \Theta(r, \beta\hbar) = \Theta(r, 0)}} \mathcal{D}[\pi(\mathbf{r}, \tau)] \mathcal{D}[\Theta(\mathbf{r}, \tau)] e^{-(S_0[\pi, \Theta] + S_{\text{int}}[\pi, \Theta])/\hbar}, \quad (3.1)$$

where, upon splitting the disorder potential into its mean value and the fluctuations around it,  $V_D(\mathbf{r}, \tau) = \overline{V_D} + \delta V_D(\mathbf{r}, \tau)$  and using Eq. (2.6) in the superfluid phase,  $\mu + \overline{V_D} = n_0 V_0$ , the harmonic and ‘‘interacting’’ parts of the action read

$$\begin{aligned} S_0[\pi, \Theta] = \int_0^{\beta\hbar} d\tau \int d^d r \left\{ i\hbar \pi(\mathbf{r}, \tau) \frac{\partial \Theta(\mathbf{r}, \tau)}{\partial \tau} + \frac{\hbar^2}{8mn_0} [\nabla \pi(\mathbf{r}, \tau)]^2 + \frac{\hbar^2 n_0}{2m} [\nabla \Theta(\mathbf{r}, \tau)]^2 \right. \\ \left. - \delta V_D(\mathbf{r}, \tau) [n_0 + \pi(\mathbf{r}, \tau)] + \frac{1}{2} \int d^d r' V(|\mathbf{r} - \mathbf{r}'|) \pi(\mathbf{r}', \tau) \pi(\mathbf{r}, \tau) \right\} \\ - \frac{1}{2} n_0^2 V_0 \beta \hbar \Omega, \end{aligned} \quad (3.2)$$

and

$$S_{\text{int}}[\pi, \Theta] = \int_0^{\beta\hbar} d\tau \int d^d r \left\{ \frac{-\hbar^2}{8mn_0} \frac{\pi(\mathbf{r}, \tau)}{n_0 + \pi(\mathbf{r}, \tau)} [\nabla \pi(\mathbf{r}, \tau)]^2 + \frac{\hbar^2}{2m} \pi(\mathbf{r}, \tau) [\nabla \Theta(\mathbf{r}, \tau)]^2 \right\}, \quad (3.3)$$

respectively. This separation of the action amounts essentially to a splitting into single- and multi-quasiparticle excitations. In this paper, we shall entirely neglect the anharmonic terms (3.3), and restrict ourselves to the Gaussian approximation. Note that taking into account the nonlinear fluctuation terms (3.3) perturbatively would amount to an expansion in powers of  $\hbar^2$  ( $\sim T^2$  for flux lines); this would then include further contributions from the disorder potential which enter via their linear coupling to the density fluctuations  $\pi$ .

The harmonic action (3.2) is of course immediately diagonalized via (discrete) Fourier transformation,

$$\psi(\mathbf{r}, \tau) = \frac{1}{\beta \hbar \Omega} \sum_{\mathbf{q}, \omega_m} \psi(\mathbf{q}, \omega_m) e^{i(\mathbf{q}\mathbf{r} - \omega_m \tau)}, \quad (3.4)$$

with the inverse

$$\psi(\mathbf{q}, \omega_m) = \int d^d r \int_0^{\beta\hbar} d\tau \psi(\mathbf{r}, \tau) e^{-i(\mathbf{q}\mathbf{r} - \omega_m \tau)}. \quad (3.5)$$

In the thermodynamic limit, the discrete wavevector sum is as usual replaced by an integral,  $\sum_{\mathbf{q}} \dots \rightarrow \Omega (2\pi)^{-d} \int d^d q \dots$ ; the periodic boundary conditions in imaginary time on the other hand enforce the discrete bosonic Matsubara frequencies to be

$$\omega_m = m \frac{2\pi}{\beta \hbar}, \quad m = 0, \pm 1, \pm 2, \dots, \quad (3.6)$$

which, similarly to the wavenumbers, approach a continuum as  $\beta \rightarrow \infty$ ,  $\sum_m \dots \rightarrow \beta \hbar / (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega \dots$

For pure systems, i.e., if  $\delta V_D(\mathbf{r}, \tau) = 0$ , the effective action in the Gaussian approximation can be written in the matrix form

$$S_0[\pi, \Theta] = \frac{1}{2\beta \hbar \Omega} \sum_{\mathbf{q}, \omega_m} \left( \Theta(-\mathbf{q}, -\omega_m), \pi(-\mathbf{q}, -\omega_m) \right) \mathbf{A}(\mathbf{q}, \omega_m) \begin{pmatrix} \Theta(\mathbf{q}, \omega_m) \\ \pi(\mathbf{q}, \omega_m) \end{pmatrix} - \frac{1}{2} n_0^2 V_0 \beta \hbar \Omega , \quad (3.7)$$

with

$$\mathbf{A}(\mathbf{q}, \omega_m) = \begin{pmatrix} n_0 \hbar^2 q^2 / m & -\hbar \omega_m \\ \hbar \omega_m & V(q) + \hbar^2 q^2 / 4m n_0 \end{pmatrix} . \quad (3.8)$$

The density and phase correlation functions can therefore be obtained by simple inversion of the matrix (3.8),

$$\hbar \mathbf{A}^{-1}(\mathbf{q}, \omega_m) = \frac{1}{\omega_m^2 + \epsilon_B(q)^2 / \hbar^2} \begin{pmatrix} m \epsilon_B(q)^2 / n_0 \hbar^3 q^2 & \omega_m \\ -\omega_m & n_0 \hbar q^2 / m \end{pmatrix} ; \quad (3.9)$$

here,  $\epsilon_B(q)$  denotes the Bogoliubov quasi-particle spectrum

$$\epsilon_B(q) = \sqrt{n_0 \hbar^2 q^2 V(q) / m + (\hbar^2 q^2 / 2m)^2} , \quad (3.10)$$

which at long wavelengths displays a linear dispersion,

$$\epsilon_B(q) \rightarrow \hbar c_1 q \quad \text{for } q \rightarrow 0 , \quad (3.11)$$

where  $c_1$  is the speed of sound from Eq. (2.10).

We can assign a physical meaning to the Bogoliubov spectrum by investigating the density–density correlation function,

$$S(\mathbf{r}, \tau; \mathbf{r}', \tau') = \langle n(\mathbf{r}, \tau) n(\mathbf{r}', \tau') \rangle - n^2 \approx \langle \pi(\mathbf{r}, \tau) \pi(\mathbf{r}', \tau') \rangle . \quad (3.12)$$

Note that by adding a source term to the action, i.e., by performing a Legendre transformation according to  $S[\delta\mu] = S[\psi^*, \psi] + \int_0^{\beta \hbar} d\tau \int d^d r \delta\mu(\mathbf{r}, \tau) |\psi(\mathbf{r}, \tau)|^2$ , we can also write this in the form of a functional derivative,

$$S(\mathbf{r}, \tau; \mathbf{r}', \tau') = \hbar^2 \frac{\delta^2 \ln Z_{\text{gr}}[\delta\mu]}{\delta[\delta\mu(\mathbf{r}, \tau)] \delta[\delta\mu(\mathbf{r}', \tau')]} \Big|_{\delta\mu=0} = \hbar \frac{\delta \langle \pi(\mathbf{r}, \tau) \rangle_{\delta\mu}}{\delta[\delta\mu(\mathbf{r}', \tau')]} \Big|_{\delta\mu=0} , \quad (3.13)$$

which shows that the imaginary–time correlation function is actually a quantum–mechanical response function. From Eq. (3.9), in Fourier space we immediately find as a consequence of translation invariance in space and time

$$S(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) = S(\mathbf{q}, \omega_m) (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \beta \hbar \delta_{m, -m'} , \quad (3.14)$$

with the reduced density correlation function

$$S(\mathbf{q}, \omega_m) = \frac{n_0 \hbar q^2 / m}{\omega_m^2 + \epsilon_B(q)^2 / \hbar^2} ; \quad (3.15)$$

by performing the simple Matsubara frequency sum [Eq. (D2) in App. D 1] we can readily transform back to imaginary time,

$$S(\mathbf{q}, \tau) = \frac{1}{\beta \hbar} \sum_m S(\mathbf{q}, \omega_m) e^{-i\omega_m \tau} = \frac{n_0 \hbar^2 q^2}{2m\epsilon_B(q)} \frac{e^{(\beta-\tau/\hbar)\epsilon_B(q)} + e^{\epsilon_B(q)\tau/\hbar}}{e^{\beta\epsilon_B(q)} - 1} , \quad (3.16)$$

with the static limit

$$S(\mathbf{q}, 0) = \frac{n_0 \hbar^2 q^2}{2m\epsilon_B(q)} \coth \frac{\beta\epsilon_B(q)}{2} , \quad (3.17)$$

which is precisely the static structure factor in the Bogoliubov approximation for weakly interacting Bose gases [1,2]. Note that  $S(\mathbf{q}, 0)$  is inversely proportional to the excitation spectrum when  $\beta \rightarrow \infty$ . If we now analytically continue to  $t = -i\tau$ , and carry out the usual Fourier integral, we arrive at

$$S(\mathbf{q}, \omega) = \int dt S(\mathbf{q}, t) e^{i\omega t} = \frac{n_0 \hbar^2 q^2}{2m\epsilon_B(q)} \frac{2\pi}{e^{\beta\epsilon_B(q)} - 1} \left[ e^{\beta\epsilon_B(q)} \delta(\omega - \epsilon_B(q)/\hbar) + \delta(\omega + \epsilon_B(q)/\hbar) \right] , \quad (3.18)$$

i.e., the dynamic structure factor directly measured in scattering experiments. One therefore expects quasiparticle peaks with the dispersion relation (3.10), thermally excited with a Bose–Einstein distribution (note the detailed–balance factor  $e^{\beta\epsilon_B(q)}$ ). The Gaussian approximation for the density and phase fluctuations produces exactly the same results as the familiar Bogoliubov approach in many–particle quantum mechanics, where a canonical transformation for the creation and annihilation operators is employed in order to diagonalize the second–quantized quasi–particle Hamiltonian [1,2]. In the long–wavelength limit, the elementary excitations of the Bose condensate are therefore no longer of the free–particle form  $\epsilon_0(q) = \hbar^2 q^2 / 2m$ , but display the linear dispersion (3.11) and may thus be interpreted as phonon modes. Depending on the form of the interaction potential  $V(q)$ , the dispersion relation may display the roton minimum characteristic of strongly interacting Bose liquids like superfluid Helium 4. However, our harmonic approximation is only valid either for weakly interacting systems where  $V(q)$  is sufficiently small, or for dilute Bose gases for which the pair potential may be strong, but where the scattering length  $a = mV_0/4\pi$  is small compared to the average interparticle spacing. In that case, one can exactly sum all the ladder diagrams describing multiple scattering events, with the result that eventually  $mV(q)$  is to be replaced by the effective s–wave scattering amplitude  $f_0 = mV_0$  [1,2]; the dimensionless coupling in the Bogoliubov theory, serving as an expansion parameter in the weak–coupling limit, is in fact  $g^2 \propto n^{(d-2)/d} V_0$  [11]. In both situations the quasiparticle dispersion is a monotonically increasing function, and shows no roton minimum.

We also remark that a second route to arrive at the result (3.18), is to remember that  $S(\mathbf{q}, \omega_m) = \hbar \chi_n(\mathbf{q}, \omega_m)$  is the density *response* function; by analytic continuation in frequency space,  $\omega_m \rightarrow i(\omega + i\eta)$ ,  $\eta \rightarrow 0^+$ , and splitting  $\chi_n(\mathbf{q}, \omega) = \chi'_n(\mathbf{q}, \omega) + i\chi''_n(\mathbf{q}, \omega)$ , where  $\chi'_n(\mathbf{q}, \omega) = [\chi_n(\mathbf{q}, \omega + i\eta) + \chi_n(\mathbf{q}, \omega - i\eta)]/2$  and  $\chi''_n(\mathbf{q}, \omega) = [\chi_n(\mathbf{q}, \omega + i\eta) - \chi_n(\mathbf{q}, \omega - i\eta)]/2i$

denote its components which are continuous and discontinuous across the real axis, respectively, one may then infer the dynamic structure factor (in real time), i.e., the corresponding *correlation function*, from the quantum–mechanical fluctuation–dissipation theorem

$$2\hbar\chi_n''(\mathbf{q}, \omega) = (1 - e^{-\beta\hbar\omega}) S(\mathbf{q}, \omega) . \quad (3.19)$$

As a second important correlation function, we consider the momentum (mass current) correlator in the Gaussian approximation, defined as

$$C_{ij}(\mathbf{r}, \tau; \mathbf{r}', \tau') = \langle g_i(\mathbf{r}, \tau) g_j(\mathbf{r}', \tau') \rangle - \langle g_i(\mathbf{r}, \tau) \rangle \langle g_j(\mathbf{r}', \tau') \rangle . \quad (3.20)$$

According to Eq. (2.60),  $\mathbf{g} = m\mathbf{j} = \hbar(n_0 + \pi)\nabla\Theta$ ; inserting this into Eq. (3.20), we can factorize the four–point correlators into all possible two–point functions (i.e., apply Wick’s theorem), while the three–point correlators vanish in the Gaussian ensemble. In Fourier space, this yields

$$\begin{aligned} C_{ij}(\mathbf{q}, \omega_m; \mathbf{q}', \omega'_m) &= -\hbar^2 n_0^2 q_i q'_j \langle \Theta(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega'_m) \rangle \\ &- \frac{1}{(\beta\Omega)^2} \sum_{\mathbf{p}, \omega_n} \sum_{\mathbf{p}', \omega'_n} p_i p'_j \left[ \langle \pi(\mathbf{q} - \mathbf{p}, \omega_{m-n}) \pi(\mathbf{q}' - \mathbf{p}', \omega_{m'-n'}) \rangle \langle \Theta(\mathbf{p}, \omega_n) \Theta(\mathbf{p}', \omega_{n'}) \rangle \right. \\ &\quad \left. + \langle \pi(\mathbf{q} - \mathbf{p}, \omega_{m-n}) \Theta(\mathbf{p}', \omega_{n'}) \rangle \langle \Theta(\mathbf{p}, \omega_n) \pi(\mathbf{q}' - \mathbf{p}', \omega_{m'-n'}) \rangle \right] . \end{aligned} \quad (3.21)$$

Note that this quantity is *not* simply proportional to the phase correlation function as obtained from Eq. (3.9),

$$\langle \Theta(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega'_m) \rangle = \frac{m\epsilon_B(q)^2/n_0\hbar^3 q^2}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2} (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \beta\hbar \delta_{m, -m'} . \quad (3.22)$$

[In the Appendix B, we discuss the “phase–only approximation” in some detail, which neglects the additional correlators in Eq. (3.21)]. Using the Gaussian correlation functions to be read off from Eq. (3.9), one finds for the transverse current response function

$$\begin{aligned} \chi_\perp(\mathbf{q}, \omega_m) &\equiv \frac{1}{(d-1)\hbar} \sum_{ij} P_{ij}^T(\mathbf{q}) C_{ij}(\mathbf{q}, \omega_m) \\ &= \frac{1}{(d-1)\beta\Omega} \sum_{\mathbf{p}, \omega_n} \frac{q^2 p^2 - (\mathbf{q}\mathbf{p})^2}{q^2 p^2} \frac{(\mathbf{q} - \mathbf{p})^2 \epsilon_B(p)^2/\hbar^2 + p^2 \omega_{m-n} \omega_n}{[\omega_{m-n}^2 + \epsilon_B(|\mathbf{q} - \mathbf{p}|)^2/\hbar^2][\omega_n^2 + \epsilon_B(p)^2/\hbar^2]} . \end{aligned} \quad (3.23)$$

Here, the transverse projector is  $P_{ij}^T(\mathbf{q}) = \delta_{ij} - P_{ij}^L(\mathbf{q})$ , with  $P_{ij}^L(\mathbf{q}) = q_i q_j / q^2$ . Note that  $\chi_\perp(\mathbf{q}, \omega_m)$  would have vanished entirely, had we not taken the nonlinear terms in Eq. (3.21) into account, because the contribution from the phase fluctuations in Eq. (3.21) is purely longitudinal<sup>5</sup>.

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<sup>5</sup>The above factorization of the four–point correlation function actually goes beyond the Gaussian approximation (tree level). In field–theory language, it corresponds to a computation of the one–loop contributions to the composite–operator cumulant  $\langle [\pi\nabla_i\Theta][\pi\nabla_j\Theta] \rangle$ . Therefore, in order to consistently evaluate the longitudinal current response function to this order, one would have to include the one–loop corrections to the phase correlations as well, which would involve the nonlinear vertices of Eq. (3.3). In the limits  $\omega_m \rightarrow 0$  and  $\mathbf{q} \rightarrow \mathbf{0}$ , the loop terms actually cancel, and the ensuing result for the total density is  $\rho = \lim_{q \rightarrow 0} \chi_\parallel(\mathbf{q}, 0) \approx n_0 m$ .

In the limit  $\omega_m \rightarrow 0$  (first), followed by  $\mathbf{q} \rightarrow \mathbf{0}$ , the transverse current response function yields the normal–fluid density, for this quantity is defined precisely as the transport coefficient relating to the system’s response to transverse motion [1–4,6]. This yields

$$\rho_n = \lim_{q \rightarrow 0} \chi_{\perp}(\mathbf{q}, 0) = \frac{1}{d\beta\Omega} \sum_{\mathbf{p}, \omega_n} p^2 \frac{\epsilon_B(p)^2/\hbar^2 - \omega_n^2}{[\epsilon_B(p)^2/\hbar^2 + \omega_n^2]^2}; \quad (3.24)$$

upon performing the Matsubara frequency sum [Eq. (D12) in App. D 1], we find

$$\rho_n = \frac{\beta\hbar^2}{4d} \int \frac{d^d q}{(2\pi)^d} \left( \frac{q}{\sinh[\beta\epsilon_B(q)/2]} \right)^2, \quad (3.25)$$

and therefore  $\rho_n(T = 0) = 0$ ; at finite “temperatures”, Eq. (3.25) will be evaluated explicitly in two and three dimensions in Sec. III B.

It is interesting to note that the transverse momentum response function is actually intimately related to the vorticity correlation function, which (in  $d = 2$  and  $d = 3$  dimensions) is defined as

$$V_{ij}(\mathbf{r}, \tau; \mathbf{r}', \tau') = m^{-2} \langle [\nabla \times \mathbf{g}(\mathbf{r}, \tau)]_i [\nabla' \times \mathbf{g}(\mathbf{r}', \tau')]_j \rangle. \quad (3.26)$$

In Fourier space, and after factorizing according to Wick’s theorem, this becomes in terms of the density and phase fluctuations

$$\begin{aligned} V_{ij}(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) = & \sum_{k,l,k',l'} \epsilon_{ikl} \epsilon_{jk'l'} \frac{1}{(m\beta\Omega)^2} \sum_{\mathbf{p}, \omega_n} \sum_{\mathbf{p}', \omega_{n'}} (q_k - p_k) p_l (q'_{k'} - p'_{k'}) p'_{l'} \times \\ & \times \left[ \langle \pi(\mathbf{q} - \mathbf{p}, \omega_{m-n}) \pi(\mathbf{q}' - \mathbf{p}', \omega_{m'-n'}) \rangle \langle \Theta(\mathbf{p}, \omega_n) \Theta(\mathbf{p}', \omega_{n'}) \rangle \right. \\ & \left. + \langle \pi(\mathbf{q} - \mathbf{p}, \omega_{m-n}) \Theta(\mathbf{p}', \omega_{n'}) \rangle \langle \Theta(\mathbf{p}, \omega_n) \pi(\mathbf{q}' - \mathbf{p}', \omega_{m'-n'}) \rangle \right]. \quad (3.27) \end{aligned}$$

Following the same procedure as above, one may demonstrate explicitly that the vorticity correlations are purely transverse,  $V_{\parallel}(\mathbf{q}, \omega_m) = \sum_{ij} P_{ij}^L(\mathbf{q}) V_{ij}(\mathbf{q}, \omega_m) = 0$ , and after some algebra finds

$$m^2 V_{\perp}(\mathbf{q}, \omega_m) = \hbar q^2 \chi_{\perp}(\mathbf{q}, \omega_m). \quad (3.28)$$

It is therefore the intrinsic vorticity fluctuations that render the transverse current response function and the normal–fluid density (3.25) nonzero at positive temperatures. These fluctuations would be confined to the vortex cores in a “phase–only” Kosterlitz–Thouless picture of finite–temperature superfluid films.

Finally, it is natural for flux lines to introduce the “tilt” correlation function by considering an action perturbed by an imaginary “velocity” field  $\mathbf{v}(\mathbf{r}, \tau)$  given by Eq. (2.37),

$$S[\mathbf{v}] = S[\psi^*, \psi] + \int_0^{\beta\hbar} d\tau \int d^d r [i\mathbf{v}(\mathbf{r}, \tau) \cdot \mathbf{g}(\mathbf{r}, \tau) - m v(\mathbf{r}, \tau)^2 n(\mathbf{r}, \tau)/2] \quad (3.29)$$

[compare Eq. (2.54)], and then defining [51]



$$\begin{aligned}
T_{ij}(\mathbf{r}, \tau; \mathbf{r}', \tau') &= \frac{\hbar^2}{m^2} \frac{\delta^2 \ln Z_{\text{gr}}[\mathbf{v}]}{\delta v_i(\mathbf{r}, \tau) \delta v_j(\mathbf{r}', \tau')} \Big|_{\mathbf{v}=\mathbf{0}} = \langle t_i(\mathbf{r}, \tau) t_j(\mathbf{r}', \tau') \rangle \\
&= (n\hbar/m) \delta_{ij} \delta(\mathbf{r} - \mathbf{r}') \delta(\tau - \tau') - C_{ij}(\mathbf{r}, \tau; \mathbf{r}', \tau') / m^2, \tag{3.30}
\end{aligned}$$

which in Fourier space becomes

$$T_{ij}(\mathbf{q}, \omega_m) = (n\hbar/m) \delta_{ij} - C_{ij}(\mathbf{q}, \omega_m) / m^2. \tag{3.31}$$

Thus, tangent–tangent correlations for flux lines are intimately related to momentum correlations in the equivalent superfluid. To extract the tilt modulus from Eq. (3.31), one has to take  $\mathbf{q} \rightarrow \mathbf{0}$  first, followed by  $\omega_m \rightarrow 0$ . These limits mimic the actual physical situation for open boundary conditions, reflecting the response to an external tilt field which acts on the top and bottom faces (perpendicular to  $z$ ) of a superconducting slab [50,51]. We can again isolate a matrix of “transport” coefficients, namely

$$c_{ij}^{v^{-1}} = (n^2\hbar)^{-1} \lim_{\omega_m \rightarrow 0} T_{ij}(\mathbf{0}, \omega_m) = \frac{1}{nm} \left( \delta_{ij} - \frac{1}{nm\beta\Omega} \sum_{\mathbf{p}, \omega_n} p_i p_j \frac{\epsilon_B(p)^2 / \hbar^2 - \omega_n^2}{[\epsilon_B(p)^2 / \hbar^2 + \omega_n^2]^2} \right). \tag{3.32}$$

For an isotropic system this matrix reduces to the tilt modulus,  $c_{ij}^{v^{-1}} = c_{44}^{v^{-1}} \delta_{ij}$ , with

$$c_{44}^v = (nm)^2 (nm - \rho_n)^{-1} = \rho^2 / \rho_s, \tag{3.33}$$

where  $\rho_s = \rho - \rho_n$  is the superfluid density, and we have set  $\rho = nm$ . In the flux liquid notation,  $\rho_n = 0$  for (disorder–free) thick samples ( $L \rightarrow \infty$ ), and thus  $\rho_s = \rho = nm$  and  $c_{44}^v = nm = n\tilde{\epsilon}_1$  in the vortex language.

At this point we mention that in an Abrikosov flux lattice the transverse tilt correlations assume the form [51]

$$T_{\perp}(\mathbf{q}, \omega_m) = \frac{n^2\hbar\omega_m^2}{c_{66}q^2 + nm\omega_m^2}, \tag{3.34}$$

which reduces to the transverse part of the flux liquid result [cf. Eq. (3.31)] when the shear modulus  $c_{66}$  vanishes. Thus, in the *crystalline* phase,  $c_{44}^v = nm$  as in the liquid, but  $\rho_s = 0$  as the result of the appearance of the shear modulus *and* the opposite order of limits ( $\omega_m \rightarrow 0$  to be taken first, then  $\mathbf{q} \rightarrow \mathbf{0}$ ).

As we shall discuss in detail in Secs. IV and V, the effect of disorder will be twofold. First, for disorder correlated in the  $xy$  plane, the tilt modulus tensor (3.32) will in general be anisotropic (i.e., the normal and superfluid densities, being transport coefficients, will depend on the direction of motion with respect to the correlated defects); second, disorder will lead to a renormalization of  $\rho_n$  and hence  $\rho_s$  and  $c_{ij}^v$ . A phase transition into a localized phase of bosons / flux lines will occur when eventually the superfluid density and therefore the inverse tilt modulus vanish (in a certain direction at least). This “generalized Bose glass” phase then constitutes a true superconductor with zero linear resistivity [40], while the flux liquid with  $\rho_s \neq 0$  behaves like a normal metal [28].

## B. Green's functions, condensate depletion, and normal fluid density for a superfluid Bose gas

It is useful to define the following Green's functions for the Bose system (with periodic boundary conditions in imaginary time),

$$G(\mathbf{r}, \tau; \mathbf{r}', \tau') = \langle \psi(\mathbf{r}, \tau) \psi^*(\mathbf{r}', \tau') \rangle - n_0, \quad (3.35)$$

$$G_{12}(\mathbf{r}, \tau; \mathbf{r}', \tau') = \langle \psi(\mathbf{r}, \tau) \psi(\mathbf{r}', \tau') \rangle - n_0 \quad (3.36)$$

(compare Refs. [1,4]). For a translationally invariant system one deduces from Dyson's equations the general structure

$$G(\mathbf{q}, \omega_m) = [i\omega_m + \hbar q^2/2m - \mu/\hbar + \Sigma_{11}(-\mathbf{q}, -\omega_m)] / D(\mathbf{q}, \omega_m), \quad (3.37)$$

which defines the self-energy  $\Sigma_{11}(\mathbf{q}, \omega_m)$ , while the anomalous Green's function  $G_{12}(\mathbf{q}, \omega_m)$  takes the form

$$G_{12}(\mathbf{q}, \omega_m) = -\Sigma_{12}(\mathbf{q}, \omega_m) / D(\mathbf{q}, \omega_m), \quad (3.38)$$

and the denominator reads

$$D(\mathbf{q}, \omega_m) = - \left[ i\omega_m - \frac{1}{2} (\Sigma_{11}(\mathbf{q}, \omega_m) - \Sigma_{11}(-\mathbf{q}, -\omega_m)) \right]^2 + \left[ \hbar q^2/2m - \mu/\hbar + \frac{1}{2} (\Sigma_{11}(\mathbf{q}, \omega_m) + \Sigma_{11}(-\mathbf{q}, -\omega_m)) \right]^2 - |\Sigma_{12}(\mathbf{q}, \omega_m)|^2. \quad (3.39)$$

For noninteracting bosons,  $\Sigma_{11}(\mathbf{q}, \omega_m) = \Sigma_{12}(\mathbf{q}, \omega_m) = 0$ , and hence the anomalous Green's function vanishes, while the free propagator becomes

$$G_0(\mathbf{q}, \omega_m)^{-1} = -i\omega_m + \hbar q^2/2m - \mu/\hbar. \quad (3.40)$$

Furthermore, it can be shown that  $\Sigma_{11}(\mathbf{0}, 0) - \Sigma_{12}(\mathbf{0}, 0) = \mu/\hbar$  to all orders in perturbation theory [1].

In order to calculate these Green's functions in the harmonic approximation, we expand Eq. (2.57) to first order in the density and phase fluctuations,

$$\psi(\mathbf{r}, \tau) \approx \sqrt{n_0} \left[ 1 + \frac{\pi(\mathbf{r}, \tau)}{2n_0} + i\Theta(\mathbf{r}, \tau) - \frac{\pi(\mathbf{r}, \tau)^2}{8n_0^2} + \frac{i}{2n_0} \pi(\mathbf{r}, \tau) \Theta(\mathbf{r}, \tau) - \frac{1}{2} \Theta(\mathbf{r}, \tau)^2 \right]. \quad (3.41)$$

To leading order, this yields for the Green's functions in Fourier space

$$G(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) \approx \frac{1}{4n_0} \langle \pi(\mathbf{q}, \omega_m) \pi(\mathbf{q}', \omega_{m'}) \rangle - \frac{i}{2} \langle \pi(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega_{m'}) \rangle + \frac{i}{2} \langle \Theta(\mathbf{q}, \omega_m) \pi(\mathbf{q}', \omega_{m'}) \rangle + n_0 \langle \Theta(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega_{m'}) \rangle, \quad (3.42)$$

$$G_{12}(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) \approx \frac{1}{4n_0} \langle \pi(\mathbf{q}, \omega_m) \pi(\mathbf{q}', \omega_{m'}) \rangle + \frac{i}{2} \langle \pi(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega_{m'}) \rangle + \frac{i}{2} \langle \Theta(\mathbf{q}, \omega_m) \pi(\mathbf{q}', \omega_{m'}) \rangle - n_0 \langle \Theta(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega_{m'}) \rangle. \quad (3.43)$$

Upon inserting the results (3.9) from the previous section, one readily finds for the (reduced) Green's functions in Fourier space [see Eq. (3.14)]

$$G(\mathbf{q}, \omega_m) = \frac{i\omega_m + \hbar q^2/2m + n_0 V(q)/\hbar}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2}, \quad (3.44)$$

$$G_{12}(\mathbf{q}, \omega_m) = \frac{-n_0 V(q)/\hbar}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2}, \quad (3.45)$$

and therefore with  $\mu = n_0 V_0$

$$\Sigma_{11}(\mathbf{q}, \omega_m) = n_0[V_0 + V(q)]/\hbar, \quad \Sigma_{12}(\mathbf{q}, \omega_m) = n_0 V(q)/\hbar, \quad (3.46)$$

$$D(\mathbf{q}, \omega_m) = \omega_m^2 + \epsilon_B(q)^2/\hbar^2. \quad (3.47)$$

Here  $\epsilon_B(q)$  denotes the Bogoliubov spectrum, already introduced in Eq. (3.10). The results (3.44)–(3.47), obtained in the Gaussian approximation, are again identical to those from the Bogoliubov theory applicable to either weakly interacting or dilute Bose systems [1,2]. Another useful representation for the Green's functions consists in their separation into single poles and their residues,

$$G(\mathbf{q}, \omega_m) = \frac{u(q)^2}{-i\omega_m + \epsilon_B(q)/\hbar} + \frac{v(q)^2}{i\omega_m + \epsilon_B(q)/\hbar}, \quad (3.48)$$

$$G_{12}(\mathbf{q}, \omega_m) = -u(q)v(q) \left( \frac{1}{-i\omega_m + \epsilon_B(q)/\hbar} + \frac{1}{i\omega_m + \epsilon_B(q)/\hbar} \right), \quad (3.49)$$

with the weight functions

$$\left. \begin{array}{l} u(q)^2 \\ v(q)^2 \end{array} \right\} = \frac{1}{2} \left( \frac{n_0 V(q) + \hbar^2 q^2/2m}{\epsilon_B(q)} \pm 1 \right), \quad (3.50)$$

obeying  $u(q)^2 - v(q)^2 = 1$  and  $u(q)v(q) = n_0 V(q)/2\epsilon_B(q)$ ; these are precisely the coefficients of the Bogoliubov transformation from the original boson to quasiparticle creation and annihilation operators.

We can now use the Green's function to calculate the average particle number in the grand-canonical ensemble; carefully taking the normal ordering of the original Bose field operators into account [4,5], we have to evaluate (with  $\eta \downarrow 0$ )

$$\begin{aligned} N - N_0 &= \int d^d r \left( \langle \psi(\mathbf{r}, \tau - \eta) \psi^*(\mathbf{r}, \tau + \eta) \rangle - n_0 \right) \\ &= \int \frac{d^d q}{(2\pi)^d} \frac{1}{(\beta\hbar)^2} \sum_{m, m'} e^{-i(\omega_m + \omega_{m'})\tau} e^{i(\omega_m - \omega_{m'})\eta} G(\mathbf{q}, \omega_m; -\mathbf{q}, \omega_{m'}). \end{aligned} \quad (3.51)$$

If in addition time translation invariance holds, we find

$$n - n_0 = \int \frac{d^d q}{(2\pi)^d} \frac{1}{\beta\hbar} \sum_m e^{i\omega_m \eta} G(\mathbf{q}, \omega_m) \quad (3.52)$$

(note the crucial exponential factor stemming from the time ordering). Assuming that the particle density  $n$  is known, Eq. (3.52) can be regarded as a formula for the squared

magnitude of the boson order parameter,  $n_0 = |\langle \psi(\mathbf{r}, \tau) \rangle|^2$ . Upon using Eq. (3.48) and Eq. (D1) from App. D 1, we find the condensate depletion in terms of the quasiparticle weights (3.50),

$$n - n_0 = \int \frac{d^d q}{(2\pi)^d} \left[ v(q)^2 + \frac{u(q)^2 + v(q)^2}{e^{\beta \epsilon_B(q)} - 1} \right]. \quad (3.53)$$

Because of the relation  $\mu = n_0 V_0$ , this actually constitutes an implicit equation for the chemical potential. The first term in Eq. (3.53) represents the zero-temperature depletion, i.e., the fraction of particles that are pushed out of the condensate as a consequence of interactions and quantum fluctuations; the second contribution yields the additional depletion caused by thermal excitations<sup>6</sup>.

For short-range interactions,  $V(q) \approx V_0 = 4\pi a/m$ , where  $a$  denotes the (s-wave) scattering length, and Eq. (3.53) can be evaluated explicitly; at  $T = 0$ , we have (with  $x = \hbar q / \sqrt{8\pi n_0 a}$ )

$$\begin{aligned} n(T=0) - n_0 &= \int \frac{d^d q}{(2\pi)^d} v(q)^2 \\ &= (2an_0/\hbar^2)^{d/2} \Gamma(d/2)^{-1} \int_0^\infty x^{d-2} \left[ (2+x^2)^{1/2} - x - (2+x^2)^{-1/2} \right] dx. \end{aligned} \quad (3.54)$$

In  $d = 1$ , this integral diverges logarithmically at the lower limit, which means there is no Bose condensation in one dimension, in accordance with the Hohenberg–Mermin–Wagner theorem. For  $d = 2$ , the integral is equal to  $1/2$ , and hence

$$d = 2 : n(T=0) - n_0 = n_0 a / \hbar^2, \quad (3.55)$$

while in three dimensions [1,2,4]

$$d = 3 : n(T=0) - n_0 = (8/3\sqrt{\pi})(n_0 a / \hbar^2)^{3/2}. \quad (3.56)$$

Note that Eq. (3.55) with  $\hbar \rightarrow k_B T$ ,  $m \rightarrow \tilde{\epsilon}_1$  and  $V_0 = \phi_0^2/4\pi$  yields the “boson order parameter” squared for vortex lines as function of temperature in the limit of infinitely thick samples ( $L \rightarrow \infty$ ),

$$n_0(T) = n \left[ 1 + \tilde{\epsilon}_1 (\phi_0/4\pi k_B T)^2 \right]^{-1}. \quad (3.57)$$

The reduction in this quantity from unity is a measure of the confining effects and diminished entanglement induced by the repulsive vortex interactions.

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<sup>6</sup>The depletion can also be derived somewhat more directly by using the definition (2.14): for, Eq. (3.41) implies that  $n_0 \equiv |\langle \psi \rangle|^2 \approx n_0 + \langle \pi \rangle - \langle \pi^2 \rangle / 4n_0 + i(\langle \pi \Theta \rangle - \langle \Theta \pi \rangle) / 2 - n_0 \langle \Theta^2 \rangle$ , and hence because of  $n \equiv \langle |\psi|^2 \rangle = n_0 + \langle \pi \rangle$  one finds  $n - n_0 = \langle \pi \rangle = \langle \pi^2 \rangle / 4n_0 - i(\langle \pi \Theta \rangle - \langle \Theta \pi \rangle) / 2 + n_0 \langle \Theta^2 \rangle$  [compare Eq. (3.42)], which immediately leads to Eq. (3.53). Note again that formally the contributions to these composite-operator averages amount to loop integrals.

Returning to the boson representation, we now calculate the lowest-order finite-temperature corrections explicitly, approximating the Bogoliubov spectrum by its phonon branch,  $\epsilon_B(q) \approx \hbar c_1 q$ . Then  $1 + 2v(q)^2 \approx mc_1/\hbar q$ , and  $n(T) - n_0 = \Delta n(T)$ , with

$$\Delta n(T) = \int \frac{d^d q}{(2\pi)^d} \frac{1 + 2v(q)^2}{e^{\beta\epsilon_B(q)} - 1} \approx \frac{\Gamma(d-1)\zeta(d-1)m}{2^{d-1}\pi^{d/2}\Gamma(d/2)\hbar^d c_1^{d-2}} (k_B T)^{d-1}, \quad (3.58)$$

where Eq. (D13) from App. D 2 was used. In two dimensions, the  $\zeta$  function diverges, and hence a nonzero boson order parameter is only possible at zero temperature; at finite temperatures, there is still a superfluid phase characterized by a finite superfluid density  $\rho_s$  (see below), and the phase transition will be of the Berezinskii–Kosterlitz–Thouless type. For  $d = 3$ , Eq. (3.58) yields [1,2,4]

$$d = 3 : \Delta n(T) = m(k_B T)^2/12\hbar^3 c_1. \quad (3.59)$$

We can also utilize the Green's function (3.35) to rederive the normal-fluid density (3.25) following a famous argument due to Landau [1,2,4]. Consider a superfluid confined in a cylindrical pipe, whose walls are moved along its axis at constant velocity  $\mathbf{v}$  with respect to the superfluid. In this situation, the superfluid velocity is zero, and the normal fluid velocity is equal to  $\mathbf{v}$ , because according to the two-fluid model only the normal fluid component is dragged along by the moving walls. Thus

$$\langle \mathbf{g}(\mathbf{r}, \tau) \rangle = m \langle \mathbf{j}(\mathbf{r}, \tau) \rangle = \rho_n \mathbf{v}; \quad (3.60)$$

the task now is to calculate the average current to first order in  $\mathbf{v}$ , which then, by comparison with Eq. (3.60), yields  $\rho_n$ . The definition (2.49) leads, for spatially translation-invariant systems, to (with  $\eta \downarrow 0$ )

$$\langle \mathbf{g}(\mathbf{r}, \tau) \rangle = \int \frac{d^d q}{(2\pi)^d} \hbar \mathbf{q} \frac{1}{(\beta \hbar)^2} \sum_{m, m'} e^{-i(\omega_m + \omega_{m'})\tau} e^{-i\omega_{m'}\eta} G(\mathbf{q}, \omega_m; -\mathbf{q}, \omega_{m'})_{-\mathbf{v}}, \quad (3.61)$$

which reduces further to

$$\langle \mathbf{g}(\mathbf{r}, \tau) \rangle = \int \frac{d^d q}{(2\pi)^d} \frac{\mathbf{q}}{\beta} \sum_m e^{i\omega_m \eta} G(\mathbf{q}, \omega_m)_{-\mathbf{v}}, \quad (3.62)$$

if translation invariance in imaginary time holds as well, which is of course the case for a pure system with periodic boundary conditions. The index “ $-\mathbf{v}$ ” serves as a reminder that we need the Green's function for a system with nonzero relative velocity between the walls and the superfluid.

We can readily obtain the Green's function for moving walls from the known quantity  $G(\mathbf{q}, \omega_m)_0$  for fixed walls, Eqs. (3.44),(3.48), by performing a Galilean transformation with constant velocity  $-\mathbf{v}$ . The Matsubara frequencies in the transformed system are [see Eqs. (2.47),(3.4)],

$$\begin{aligned} \psi'(\mathbf{r}', \tau') &= \psi(\mathbf{r} + i\mathbf{v}\tau, \tau) = \frac{1}{\beta \hbar \Omega} \sum_{\mathbf{q}, \omega_m} \psi(\mathbf{q}, \omega_m) e^{i\mathbf{q}\cdot\mathbf{r} - i(\omega_m - i\mathbf{q}\cdot\mathbf{v})\tau} = \sum_{\mathbf{q}', \omega'_m} \psi'(\mathbf{q}', \omega'_m) e^{i\mathbf{q}'\cdot\mathbf{r}' - i\omega'_m \tau'} \\ \implies \quad \mathbf{q} &\rightarrow \mathbf{q}' = \mathbf{q}, \quad \omega_m \rightarrow \omega'_m = \omega_m - i\mathbf{q}\cdot\mathbf{v}; \end{aligned} \quad (3.63)$$

notice that the periodic boundary conditions are also obeyed by the transformed fields,  $\psi'(\mathbf{r}' - i\mathbf{v}\beta\hbar, \beta\hbar) = \psi'(\mathbf{r}', 0)$ .

Upon inserting the shifted Matsubara frequencies in Eq. (3.48), we find

$$G(\mathbf{q}, \omega_m)_{-\mathbf{v}} = \frac{1 + v(q)^2}{-i\omega_m + \epsilon_B(q)/\hbar - \mathbf{q} \cdot \mathbf{v}} + \frac{v(q)^2}{i\omega_m + \epsilon_B(q)/\hbar + \mathbf{q} \cdot \mathbf{v}}, \quad (3.64)$$

with the unchanged Bogoliubov coefficient  $v(q)$ , Eq. (3.50). Performing the Matsubara frequency sum [Eq. (D1)] finally yields Landau's formula

$$\langle \mathbf{g}(\mathbf{r}, \tau) \rangle = \int \frac{d^d q}{(2\pi)^d} \frac{\hbar \mathbf{q}}{e^{\beta[\epsilon_B(q) - \hbar \mathbf{q} \cdot \mathbf{v}]} - 1} \quad (3.65)$$

[the terms  $\propto v(q)^2$  vanish, because their combination is antisymmetric under  $\mathbf{q} \rightarrow -\mathbf{q}$ ]. Eq. (3.65) can be interpreted as a sum over the momenta times a Bose–Einstein quasiparticle occupation number in a system with moving walls (where the energy eigenvalues are reduced by  $\hbar \mathbf{q} \cdot \mathbf{v}$ ). An expansion to linear order in  $\mathbf{v}$  then again leads to the result (3.25).

We now evaluate this expression in the phonon approximation,  $\epsilon_B(q) \approx \hbar c_1 q$ . Via integration by parts, the integral in Eq. (3.25) becomes of the form (D13), and hence

$$\rho_n(T) = \frac{\Gamma(d+2)\zeta(d+1)}{2^{d-1}\pi^{d/2}d\Gamma(d/2)} \frac{(k_B T)^{d+1}}{\hbar^d c_1^{d+2}}. \quad (3.66)$$

Thus, in two dimensions we find

$$d = 2 : \rho_n = 3\zeta(3)(k_B T)^3 / 2\pi\hbar^2 c_1^4, \quad (3.67)$$

with  $\zeta(3) \approx 1.202$ , while for  $d = 3$  [1,2,4]

$$d = 3 : \rho_n = 2\pi^2(k_B T)^4 / 45\hbar^3 c_1^5. \quad (3.68)$$

For a flux liquid with *periodic boundary conditions* in the  $z$  direction along the magnetic field, the leading finite–size corrections to the tilt modulus thus are ( $d = 2$ )

$$\Delta c_{44}^{-1}(L) = -3\zeta(3)k_B T / 2\pi(n\tilde{\epsilon}_1)^2 c_1^4 L^3. \quad (3.69)$$

Thus,  $c_{44}$  *increases* and it is *harder* to tilt vortices in finite–size systems with periodic boundary conditions, as one might expect. In the following section, we determine the corresponding finite–size corrections for the condensate fraction  $n_0$  and the tilt modulus (normal–fluid density) for the more realistic case of open boundary conditions.

We finally remark that a crucial point in the calculation of the normal–fluid density is that both phase *and* density (amplitude) fluctuations are taken into account properly. It is tempting, given that the phase fluctuations are the dominant low–energy modes, to integrate out the  $\pi$  fields in the harmonic action (3.2), and then derive thermodynamic quantities like the depletion and the normal–fluid density from the ensuing effective action. As is already apparent from Eq. (3.23), however, essential parts of the transverse current correlations will then be missing, e.g., there will be no vorticity in the system, unless vortices are explicitly introduced by hand. Hence one finds  $\rho_n = 0$  even for  $T > 0$  in this approximation. This unsatisfactory result arises because it is actually not the leading terms (in a long–wavelength expansion), but the *next-to-leading* contributions containing the amplitude fluctuations, which produce a non–zero vorticity and normal fluid density at finite temperatures. The phase–only approximation (for the disordered boson system) and its limitations are more extensively discussed in App. B.

### C. Finite-size corrections for open boundary conditions

We now proceed to study finite-size corrections for the flux line problem with free boundary conditions. To facilitate comparison with the periodic boundary conditions in the previous section, we retain the boson notation. According to Eq. (2.33), there can be no density and phase fluctuations at the bounding surfaces for all  $\mathbf{r}$ ,

$$\pi(\mathbf{r}, 0) = \pi(\mathbf{r}, \beta\hbar) = 0 = \Theta(\mathbf{r}, 0) = \Theta(\mathbf{r}, \beta\hbar) , \quad (3.70)$$

reflecting “ideal Bose gas” boundary conditions at the top and bottom of the sample. Eq. 3.70 amounts to a constraint for the path integrals as in Eq. (2.3) that has to be taken into account explicitly. Upon introducing the vector notation

$$\Upsilon(\mathbf{r}, \tau) = \begin{pmatrix} \Theta(\mathbf{r}, \tau) \\ \pi(\mathbf{r}, \tau) \end{pmatrix} , \quad (3.71)$$

the two-point correlation functions with the constraints (3.70) requires evaluating the path integral

$$\begin{aligned} \langle \Upsilon(\mathbf{r}, \tau) \Upsilon^T(\mathbf{r}', \tau') \rangle &= \frac{\int \mathcal{D}[\Upsilon(\mathbf{r}, \tau)] \Upsilon(\mathbf{r}, \tau) \Upsilon^T(\mathbf{r}', \tau') e^{-S[\Upsilon]/\hbar} \prod_{\mathbf{r}} \delta(\Upsilon(\mathbf{r}, 0))}{\int \mathcal{D}[\Upsilon(\mathbf{r}, \tau)] e^{-S[\Upsilon]/\hbar} \prod_{\mathbf{r}} \delta(\Upsilon(\mathbf{r}, 0))} \\ &= \frac{\int \mathcal{D}[\lambda(\mathbf{r})] \mathcal{D}[\Upsilon(\mathbf{r}, \tau)] \Upsilon(\mathbf{r}, \tau) \Upsilon^T(\mathbf{r}', \tau') e^{-S[\Upsilon]/\hbar + i \int d^d r \lambda(\mathbf{r}) \Upsilon(\mathbf{r})}}{\int \mathcal{D}[\lambda(\mathbf{r})] \mathcal{D}[\Upsilon(\mathbf{r}, \tau)] e^{-S[\Upsilon]/\hbar + i \int d^d r \lambda(\mathbf{r}) \Upsilon(\mathbf{r})}} , \end{aligned} \quad (3.72)$$

where in the second line a *static* auxiliary field  $\lambda(\mathbf{r})$  has been introduced to enforce the constraints. In the Gaussian approximation, the correlation functions obeying the boundary conditions (3.70) can then be reduced in a straightforward manner to those with periodic boundary conditions that are encoded in the matrix  $\mathbf{A}^{-1}(\mathbf{q}, \omega_m)$  of Eq. (3.9). For the pure case, this has been worked out in detail in Ref. [31], with the result

$$\langle \Upsilon(\mathbf{q}, \omega_m) \Upsilon^T(\mathbf{q}', \omega_{m'}) \rangle = (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \hbar \mathbf{A}^{-1}(\mathbf{q}, \omega_m) \left[ \beta \hbar \delta_{m, -m'} - \mathbf{A}(\mathbf{q}) \mathbf{A}^{-1}(\mathbf{q}, -\omega_{m'}) \right] , \quad (3.73)$$

where  $\mathbf{A}(\mathbf{q})$  denotes the inverse of  $\mathbf{A}^{-1}(\mathbf{q}, \tau = 0) = (\beta\hbar)^{-1} \sum_m \mathbf{A}^{-1}(\mathbf{q}, \omega_m)$ ,

$$\mathbf{A}(\mathbf{q}) = \begin{pmatrix} n_0 \hbar^2 q^2 / m \epsilon_B(q) & 0 \\ 0 & m \epsilon_B(q) / n_0 \hbar^2 q^2 \end{pmatrix} 2\hbar \tanh \frac{\beta \epsilon_B(q)}{2} . \quad (3.74)$$

In App. C, the analogous formula is derived for a general disordered system, and the pure result follows from Eq. (C9) by simply setting the disorder correlator to zero. Note that translation invariance only holds in the  $d$  perpendicular directions, while the second term in Eq. (3.73), representing the corrections from the open-boundary-conditions constraint, explicitly breaks translational invariance along the direction of the lines.

Upon multiplying the matrices (3.74) and (3.9), we find for the density correlation function with open boundary conditions

$$\begin{aligned}
S(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) &= (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \frac{n_0 \hbar q^2 / m}{\omega_m^2 + \epsilon_B(q)^2 / \hbar^2} \times \\
&\times \left( \beta \hbar \delta_{m, -m'} - \frac{2\hbar}{\epsilon_B(q)} \tanh \frac{\beta \epsilon_B(q)}{2} \frac{\omega_m \omega_{m'} + \epsilon_B(q)^2 / \hbar^2}{\omega_{m'}^2 + \epsilon_B(q)^2 / \hbar^2} \right). \quad (3.75)
\end{aligned}$$

The frequency sums required for the equal-time correlation function follow from Eqs. (D5) and (D8), with the result

$$S(\mathbf{q}, \tau; \mathbf{q}', \tau) = \frac{1}{(\beta \hbar)^2} \sum_{m, m'} e^{-i(\omega_m + \omega_{m'})\tau} S(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) = S(\mathbf{q}, 0) (2\pi)^d \delta(\mathbf{q} + \mathbf{q}'), \quad (3.76)$$

$$S(\mathbf{q}, 0) = \frac{n_0 \hbar^2 q^2}{2m \epsilon_B(q)} \coth \frac{\beta \epsilon_B(q)}{2} \left( 1 - \frac{1}{\cosh^2[\beta \epsilon_B(q)/2]} \right) = \frac{n_0 \hbar^2 q^2}{2m \epsilon_B(q)} \tanh \frac{\beta \epsilon_B(q)}{2}. \quad (3.77)$$

Compared to the corresponding result for periodic boundary conditions (3.17), the hyperbolic cotangent function has been replaced by a hyperbolic tangent, as if the excitations on the superfluid ground state were governed by a fermionic rather than a bosonic distribution function [cf. Eq. (D2)]; accordingly, the limit  $S(\mathbf{q}) = n_0 \hbar^2 q^2 / 2m \epsilon_B(q)$  for  $\beta \hbar \rightarrow \infty$  is approached from *below* for open boundary conditions, as opposed to from above for periodic boundary conditions.

Similarly, the Green's function (3.42) reads

$$\begin{aligned}
G(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) &= (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \times \\
&\times \left[ \frac{\hbar q^2 / 2m + n_0 V(q) / \hbar}{\omega_m^2 + \epsilon_B(q)^2 / \hbar^2} \left( \beta \hbar \delta_{m, -m'} - \frac{2\hbar}{\epsilon_B(q)} \tanh \frac{\beta \epsilon_B(q)}{2} \frac{\omega_m \omega_{m'} + \epsilon_B(q)^2 / \hbar^2}{\omega_{m'}^2 + \epsilon_B(q)^2 / \hbar^2} \right) \right. \\
&\left. + \frac{1}{\omega_m^2 + \epsilon_B(q)^2 / \hbar^2} \left( i\omega_m \beta \hbar \delta_{m, -m'} - \frac{2\epsilon_B(q)}{\hbar} \tanh \frac{\beta \epsilon_B(q)}{2} \frac{i(\omega_m - \omega_{m'})}{\omega_{m'}^2 + \epsilon_B(q)^2 / \hbar^2} \right) \right], \quad (3.78)
\end{aligned}$$

from which we can infer the average particle number with Eq. (3.51) using the Matsubara frequency sums in App. D 1; instead of Eq. (3.53) for periodic boundary conditions, the result for the depletion now is

$$\begin{aligned}
n - n_0 &= \int \frac{d^d q}{(2\pi)^d} \left[ v(q)^2 + [u(q)^2 + v(q)^2] \left( \frac{1}{e^{\beta \epsilon_B(q)} - 1} - \frac{1}{\sinh[\beta \epsilon_B(q)]} \right) \right] \\
&= \int \frac{d^d q}{(2\pi)^d} \left[ v(q)^2 - \frac{u(q)^2 + v(q)^2}{e^{\beta \epsilon_B(q)} + 1} \right]; \quad (3.79)
\end{aligned}$$

i.e., the finite-“temperature” (finite-size) corrections change sign, and the corresponding “excitations” are governed by a Fermi-Dirac distribution (with zero “chemical potential”) rather than a Bose-Einstein function as in the case of periodic boundary conditions.

With the aid of Eq. (D14), we can evaluate the “depletion”, which is now actually a condensate *enhancement*, in the phonon approximation. Remarkably, in two dimensions the open boundary conditions lead to a *finite* correction,

$$d = 2 : \Delta n(T) = -(\ln 2) m (k_B T) / 2\pi \hbar^2, \quad (3.80)$$

in contrast to the logarithmic divergence one obtains for the case of periodic boundary conditions. The possibility of long-range order at finite temperatures in two dimensions



arises, due to the static surface “field”  $\lambda(\mathbf{r})$  which enforces the open–boundary–conditions constraints at  $\tau = 0$  and  $\tau = \beta\hbar$ . This field strengthens the order parameter at the free surfaces, which promotes condensation in the bulk, when the thickness in the imaginary–time direction is finite. For higher dimensions we find

$$d \geq 3 : \Delta n(T) = - \left(1 - \frac{1}{2^{d-2}}\right) \frac{\Gamma(d-1)\zeta(d-1)}{2^{d-1}\pi^{d/2}\Gamma(d/2)} \frac{m(k_B T)^{d-1}}{\hbar^d c_1^{d-2}} ; \quad (3.81)$$

thus, in three dimensions,

$$d = 3 : \Delta n(T) = -m(k_B T)^2/24\hbar^3 c_1 , \quad (3.82)$$

which is  $(-1/2)$  times the depletion for periodic boundary conditions, Eq. (3.59). As we have already discussed in the Introduction, the physical reason for the signs in Eqs. (3.80)–(3.82) is in the flux line language that thermal wandering is stronger at the surfaces, and thus line entanglement is facilitated, see Fig. 2.

For the same reason, with open boundary conditions one expects the superfluid density  $\rho_s$  (and the inverse tilt modulus  $c_{44}^{-1}$ ) to be *larger* at finite  $\beta\hbar$  than in the “zero–temperature” limit, and thus the finite–“temperature” / finite–size corrections should have opposite signs compared to the bosonic case with periodic boundary conditions [Eqs.(3.25) and (3.66)]. By repeating Landau’s argument for tilted lines, we obtain the Green’s function from Eq. (3.78) upon replacing  $\omega_m \rightarrow \omega_m - i\mathbf{q} \cdot \mathbf{v}$  and  $\omega_{m'} \rightarrow \omega_{m'} + i\mathbf{q} \cdot \mathbf{v}$  [cf. Eq. (3.63)]. Then expanding to first order in  $\mathbf{q} \cdot \mathbf{v}$  and performing the Matsubara frequency sums with the aid of App. D 1, yields the normal–fluid density, which has now become a function of imaginary time,

$$\rho_n(\tau) = \frac{\hbar}{d} \int \frac{d^d q}{(2\pi)^d} q^2 \left( 2\tau \frac{\sinh[(\beta\hbar - 2\tau)\epsilon_B(q)/\hbar]}{\sinh[\beta\epsilon_B(q)]} - \frac{\beta\hbar}{4 \sinh^2[\beta\epsilon_B(q)/2]} \right) . \quad (3.83)$$

In the center of the bulk ( $\tau = \beta\hbar/2$ ) this reduces to

$$\rho_n(\tau = \beta\hbar/2) = -\frac{\beta\hbar^2}{4d} \int \frac{d^d q}{(2\pi)^d} \left( \frac{q}{\sinh[\beta\epsilon_B(q)/2]} \right)^2 , \quad (3.84)$$

which is just Eq. (3.25) with a *negative* sign. This negative result does not of course have the usual interpretation of two–fluid superfluid hydrodynamics, i.e., of a physical “particle density”, once flux line boundary conditions are applied. In the phonon approximation, we find explicitly

$$\rho_n(T) = -\frac{\Gamma(d+2)\zeta(d+1)}{2^{d-1}\pi^{d/2}d\Gamma(d/2)} \frac{(k_B T)^{d+1}}{\hbar^d c_1^{d+2}} , \quad (3.85)$$

and in two and three dimensions one has

$$d = 2 : \rho_n = -3\zeta(3)(k_B T)^3/2\pi\hbar^2 c_1^4 , \quad (3.86)$$

$$d = 3 : \rho_n = -2\pi^2(k_B T)^4/45\hbar^3 c_1^5 . \quad (3.87)$$

For the finite–size correction of the tilt modulus with open boundary conditions this implies (in the flux line notation,  $d = 2$ )

$$\Delta c_{44}^{-1}(L) = 3\zeta(3)k_B T/2\pi(n\tilde{\epsilon}_1)^2 c_1^4 L^3 . \quad (3.88)$$

As anticipated, the system with open boundary conditions can be more *easily* tilted than one with periodic boundary conditions [Eq. (3.69)].

## IV. DISORDER CONTRIBUTIONS FOR SUPERFLUID BOSONS (PERIODIC BOUNDARY BONDITIONS)

In the preceding section, we studied defect-free systems of superfluid bosons and flux liquids, and thereby established notations, introduced the relevant quantities, explained our approximations, and discussed the relevance of boundary conditions. We now determine the influence of weak disorder on density and phase correlations, the depletion, and the normal-fluid density (tilt modulus). We start by investigating weakly interacting or dilute superfluid Bose gases, or flux liquids with periodic boundary conditions along the magnetic-field direction. The case of open boundary conditions more appropriate to vortices will be treated in Sec. V. We shall derive our results for general disorder first, specializing at the end to specific impurity correlators (Sec. IV C).

### A. Density, current, and vorticity correlations

We retain our approximation to work in the Gaussian ensemble described by the action (3.2) only, neglecting the nonlinear terms (3.3). In Fourier space, the harmonic action, including the disorder terms, reads

$$S_0[\pi, \Theta] = \frac{1}{2\beta\hbar\Omega} \sum_{\mathbf{q}, \omega_m} \left[ \left( \Theta(-\mathbf{q}, -\omega_m), \pi(-\mathbf{q}, -\omega_m) \right) \mathbf{A}(\mathbf{q}, \omega_m) \begin{pmatrix} \Theta(\mathbf{q}, \omega_m) \\ \pi(\mathbf{q}, \omega_m) \end{pmatrix} - 2 \left( \Theta(-\mathbf{q}, -\omega_m), \pi(-\mathbf{q}, -\omega_m) \right) \begin{pmatrix} 0 \\ \delta V_D(\mathbf{q}, \omega_m) \end{pmatrix} \right] - \frac{1}{2} n_0^2 V_0 \beta \hbar \Omega, \quad (4.1)$$

where the harmonic coupling matrix  $\mathbf{A}(\mathbf{q}, \omega_m)$  is given by Eq. (3.8). In order to obtain the two-point correlation functions to first order in the disorder correlator

$$\Delta(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) = \overline{\delta V_D(\mathbf{q}, \omega_m) \delta V_D(\mathbf{q}', \omega_{m'})}, \quad (4.2)$$

we introduce the shifted fields (note that the Jacobian of this transformation is 1)

$$\begin{aligned} \begin{pmatrix} \tilde{\Theta}(\mathbf{q}, \omega_m) \\ \tilde{\pi}(\mathbf{q}, \omega_m) \end{pmatrix} &= \begin{pmatrix} \Theta(\mathbf{q}, \omega_m) \\ \pi(\mathbf{q}, \omega_m) \end{pmatrix} - \mathbf{A}^{-1}(\mathbf{q}, \omega_m) \begin{pmatrix} 0 \\ \delta V_D(\mathbf{q}, \omega_m) \end{pmatrix} \\ &= \begin{pmatrix} \Theta(\mathbf{q}, \omega_m) \\ \pi(\mathbf{q}, \omega_m) \end{pmatrix} - \frac{\delta V_D(\mathbf{q}, \omega_m)}{\omega_m^2 + \epsilon_B(q)^2 / \hbar^2} \begin{pmatrix} \omega_m / \hbar \\ n_0 q^2 / m \end{pmatrix}; \end{aligned} \quad (4.3)$$

upon performing the quenched disorder average we find <sup>7</sup>

$$\begin{aligned} \overline{\langle \Theta(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega_{m'}) \rangle} &= \frac{m \epsilon_B(q)^2 / n_0 \hbar^3 q^2}{\omega_m^2 + \epsilon_B(q)^2 / \hbar^2} (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \beta \hbar \delta_{m, -m'} \\ &\quad + \frac{\omega_m \omega_{m'} / \hbar^2}{[\omega_m^2 + \epsilon_B(q)^2 / \hbar^2][\omega_{m'}^2 + \epsilon_B(q')^2 / \hbar^2]} \Delta(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}), \end{aligned} \quad (4.4)$$

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<sup>7</sup>For a derivation of these results using the replica trick, see Ref. [51], App. B.

$$\begin{aligned} \overline{\langle \Theta(\mathbf{q}, \omega_m) \pi(\mathbf{q}', \omega_{m'}) \rangle} &= \frac{\omega_m}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2} (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \beta \hbar \delta_{m, -m'} \\ &+ \frac{n_0 q'^2 \omega_m / \hbar m}{[\omega_m^2 + \epsilon_B(q)^2/\hbar^2][\omega_{m'}^2 + \epsilon_B(q')^2/\hbar^2]} \Delta(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) , \end{aligned} \quad (4.5)$$

$$\begin{aligned} \overline{\langle \pi(\mathbf{q}, \omega_m) \pi(\mathbf{q}', \omega_{m'}) \rangle} &= \frac{n_0 \hbar q^2 / m}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2} (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \beta \hbar \delta_{m, -m'} \\ &+ \frac{n_0^2 q^2 q'^2 / m^2}{[\omega_m^2 + \epsilon_B(q)^2/\hbar^2][\omega_{m'}^2 + \epsilon_B(q')^2/\hbar^2]} \Delta(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) . \end{aligned} \quad (4.6)$$

Eq. (4.6), of course, gives the dynamic structure factor in harmonic approximation to first order in the defect correlator. In most situations, translational invariance in space and time will be restored statistically, i.e.,

$$\Delta(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) = \Delta(\mathbf{q}, \omega_m) (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \beta \hbar \delta_{m, -m'} ; \quad (4.7)$$

in that case, one finds that the disorder contributions to the structure factor have the typical Lorentzian-squared form,

$$\overline{S(\mathbf{q}, \omega_m)} = \frac{n_0 \hbar q^2 / m}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2} + \left( \frac{n_0 q^2 / m}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2} \right)^2 \Delta(\mathbf{q}, \omega_m) . \quad (4.8)$$

The corresponding static structure factor becomes, using Eqs. (D5) and (D9) of App. D 1,

$$\overline{S(\mathbf{q}, \tau = 0)} = \frac{n_0 \hbar^2 q^2}{2m\epsilon_B(q)} \coth \frac{\beta\epsilon_B(q)}{2} + \left( \frac{n_0 q^2}{m} \right)^2 \frac{1}{\beta \hbar} \sum_n \frac{\Delta(\mathbf{q}, \omega_n)}{[\omega_n^2 + \epsilon_B(q)^2/\hbar^2]^2} . \quad (4.9)$$

With the help of Eqs. (3.21), (3.27) and some tedious algebra, we can also compute the current and vorticity correlations to first order in the defect correlator. Upon assuming that the system is statistically translation-invariant and isotropic, we can define longitudinal and transverse components of the current correlation function, and as in the pure case the transverse current response function is closely related with the (*purely* transverse) vorticity correlations,

$$\begin{aligned} m^2 \overline{V_\perp(\mathbf{q}, \omega_m)} &= \hbar q^2 \overline{\chi_\perp(\mathbf{q}, \omega_m)} \\ &= \frac{\hbar}{(d-1)\beta\Omega} \sum_{\mathbf{p}, \omega_n} \frac{[q^2 p^2 - (\mathbf{q}\mathbf{p})^2][(\mathbf{q} - \mathbf{p})^2 \epsilon_B(p)^2/\hbar^2 + p^2 \omega_{m-n} \omega_n]}{p^2 [\omega_{m-n}^2 + \epsilon_B(|\mathbf{q} - \mathbf{p}|)^2/\hbar^2][\omega_n^2 + \epsilon_B(p)^2/\hbar^2]} \\ &+ \frac{n_0}{(d-1)m\beta\Omega} \sum_{\mathbf{p}, \omega_n} \frac{q^2 p^2 - (\mathbf{q}\mathbf{p})^2}{[\omega_{m-n}^2 + \epsilon_B(|\mathbf{q} - \mathbf{p}|)^2/\hbar^2][\omega_n^2 + \epsilon_B(p)^2/\hbar^2]} \times \\ &\times \left[ \frac{(\mathbf{q} - \mathbf{p})^2}{p^2} \frac{(\mathbf{q} - \mathbf{p})^2 \epsilon_B(p)^2/\hbar^2 + p^2 \omega_{m-n} \omega_n}{\omega_{m-n}^2 + \epsilon_B(|\mathbf{q} - \mathbf{p}|)^2/\hbar^2} \Delta(\mathbf{q} - \mathbf{p}, \omega_{m-n}) \right. \\ &\quad \left. - \frac{(\mathbf{q} - \mathbf{p})^2 \omega_n^2 - p^2 \omega_{m-n} \omega_n}{\omega_n^2 + \epsilon_B(q)^2/\hbar^2} \Delta(\mathbf{p}, \omega_n) \right] . \end{aligned} \quad (4.10)$$

For general, “time-dependent” configurations of disorder, the average mass current does not necessarily vanish; e.g., uniformly moving defects “drag” a certain fraction of the particles along, even at  $T = 0$ . Indeed, upon using Eq. (4.5), one finds

$$\begin{aligned}
\langle \mathbf{g}(\mathbf{q}, \omega_m) \rangle &= \frac{i\hbar}{\beta\Omega} \sum_{\mathbf{p}, \omega_n} \mathbf{p} \langle \pi(\mathbf{q} - \mathbf{p}, \omega_{m-n}) \Theta(\mathbf{p}, \omega_n) \rangle \\
&= i \frac{n_0}{m} \frac{1}{\beta\hbar\Omega} \sum_{\mathbf{p}, \omega_n} \frac{\mathbf{p}(\mathbf{q} - \mathbf{p})^2 \omega_n \Delta(\mathbf{q} - \mathbf{p}, \omega_{m-n}; \mathbf{p}, \omega_n)}{[\omega_{m-n}^2 + \epsilon_B(|\mathbf{q} - \mathbf{p}|)^2/\hbar^2][\omega_n^2 + \epsilon_B(p)^2/\hbar^2]} .
\end{aligned} \tag{4.11}$$

Upon returning to Eq. (4.10), we find for the normal fluid density in an isotropic system

$$\begin{aligned}
\overline{\rho_n} &= \lim_{q \rightarrow 0} \overline{\chi_{\perp}(\mathbf{q}, 0)} = \frac{1}{d\beta\Omega} \sum_{\mathbf{p}, \omega_n} p^2 \frac{\epsilon_B(p)^2/\hbar^2 - \omega_n^2}{[\epsilon_B(p)^2/\hbar^2 + \omega_n^2]^2} \\
&\quad + \frac{n_0}{dm\beta\hbar\Omega} \sum_{\mathbf{p}, \omega_n} p^4 \frac{\epsilon_B(p)^2/\hbar^2 - 3\omega_n^2}{[\epsilon_B(p)^2/\hbar^2 + \omega_n^2]^3} \Delta(\mathbf{p}, \omega_n) ;
\end{aligned} \tag{4.12}$$

we shall evaluate this for interesting special cases in Sec. IV C. For static disorder, i.e.,

$$\Delta(\mathbf{q}, \omega_m) = \Delta(\mathbf{q}) \beta\hbar\delta_{m,0} , \tag{4.13}$$

Eq. (4.12) further simplifies to

$$\overline{\rho_n} = \frac{\beta\hbar^2}{4d} \int \frac{d^d q}{(2\pi)^d} \left( \frac{q}{\sinh[\beta\epsilon_B(q)/2]} \right)^2 + \frac{n_0}{dm} \int \frac{d^d q}{(2\pi)^d} \left( \frac{\hbar q}{\epsilon_B(q)} \right)^4 \Delta(\mathbf{q}) . \tag{4.14}$$

An identical formula for the disorder-induced renormalization of the normal-fluid density was derived by Giorgini, Pitaevskii, and Stringari [18]. Note that the defect contribution to (4.14) is *temperature-independent*, see Sec. V B.

For flux liquids, Eqs. (3.31) and (4.10) lead to the vortex contribution to the tilt modulus

$$\begin{aligned}
\overline{c_{ij}^{v-1}} &= (n^2\hbar)^{-1} \lim_{\omega_m \rightarrow 0} \overline{T_{ij}(\mathbf{0}, \omega_m)} = \frac{1}{nm} \left( \delta_{ij} - \frac{1}{nm\beta\Omega} \sum_{\mathbf{p}, \omega_n} p_i p_j \frac{\epsilon_B(p)^2/\hbar^2 - \omega_n^2}{[\epsilon_B(p)^2/\hbar^2 + \omega_n^2]^2} \right. \\
&\quad \left. - \frac{1}{m^2\beta\hbar\Omega} \sum_{\mathbf{p}, \omega_n} p_i p_j p^2 \frac{\epsilon_B(p)^2/\hbar^2 - 3\omega_n^2}{[\epsilon_B(p)^2/\hbar^2 + \omega_n^2]^3} \Delta(\mathbf{p}, \omega_n) \right) ,
\end{aligned} \tag{4.15}$$

which for an isotropic system agrees with Eq. (3.33). We also remark that the last term in Eq. (4.15) can formally be written as

$$\begin{aligned}
& - \frac{1}{m^2} \int \frac{d^d q}{(2\pi)^d} q_i q_j q^2 \frac{1}{\beta\hbar} \sum_m \frac{\epsilon_B(q)^2/\hbar^2 - 3\omega_m^2}{[\epsilon_B(q)^2/\hbar^2 + \omega_m^2]^3} \Delta(\mathbf{q}, \omega_m) \\
&= \frac{1}{2n_0 m \hbar} \int \frac{d^d q}{(2\pi)^d} q_i q_j \frac{1}{\beta\hbar} \sum_m \Delta(\mathbf{q}, \omega_m) \frac{\partial^2}{\partial \omega_m^2} S(\mathbf{q}, \omega_m) ,
\end{aligned} \tag{4.16}$$

where  $S(\mathbf{q}, \omega_m)$  is the density correlation function of the pure system (3.15). Eq. (4.15) is thus the generalization to finite systems of the tilt modulus formula of Hwa *et al.* [44]. In App. A, we review how the defect contribution to the tilt modulus can be obtained using an affine transformation; it is, however, not straightforward to use the same idea and method to obtain finite-size corrections. In the following subsection, a more general expression for the normal fluid density and the tilt modulus will be derived, valid even when translational invariance along the  $\tau$  direction is not restored statistically.

## B. Condensate fraction and normal–fluid density

Eqs. (4.4)–(4.6) yield the disorder–averaged Green’s function in harmonic approximation (3.42), which reads, to first order in the disorder correlator,

$$\begin{aligned} \overline{G(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'})} &= \frac{i\omega_m + \hbar q^2/2m + n_0 V(q)/\hbar}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2} (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \beta \hbar \delta_{m, -m'} \\ &+ \frac{n_0}{\hbar^2} \frac{(i\omega_m + \hbar q^2/2m)(-i\omega_{m'} + \hbar q'^2/2m)}{[\omega_m^2 + \epsilon_B(q)^2/\hbar^2][\omega_{m'}^2 + \epsilon_B(q')^2/\hbar^2]} \Delta(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) . \end{aligned} \quad (4.17)$$

With Eq. (3.51), we find the following general formula for the depletion in a disordered Bose superfluid, i.e., the reduction in the order parameter magnitude due to disorder, quantum and thermal fluctuations,

$$\begin{aligned} \overline{n(\tau)} - n_0 &= \int \frac{d^d q}{(2\pi)^d} \left\{ \left[ v(q)^2 + \frac{1 + 2v(q)^2}{e^{\beta \epsilon_B(q)} - 1} \right] \right. \\ &+ \left. \frac{n_0}{\hbar^2 \Omega} \frac{1}{(\beta \hbar)^2} \sum_{m, m'} e^{-i(\omega_m + \omega_{m'})\tau} \frac{(i\omega_m + \hbar q^2/2m)(-i\omega_{m'} + \hbar q'^2/2m)}{[\omega_m^2 + \epsilon_B(q)^2/\hbar^2][\omega_{m'}^2 + \epsilon_B(q')^2/\hbar^2]} \Delta(\mathbf{q}, \omega_m; -\mathbf{q}, \omega_{m'}) \right\} \end{aligned} \quad (4.18)$$

This expression depends on imaginary time, when the disorder breaks translation invariance. For a statistically translation–invariant system (4.7), the disorder contribution in Eq. (4.18) becomes

$$\overline{n_\Delta} = \frac{n_0}{\hbar^2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\beta \hbar} \sum_m \frac{(\hbar q^2/2m)^2 - \omega_m^2}{[\epsilon_B(q)^2/\hbar^2 + \omega_m^2]^2} \Delta(\mathbf{q}, \omega_m) , \quad (4.19)$$

which for static defects (4.13) simplifies further to yield the *temperature–independent* result

$$\overline{n_\Delta} = \frac{n_0}{4m^2} \int \frac{d^d q}{(2\pi)^d} \left( \frac{\hbar q}{\epsilon_B(q)} \right)^4 \Delta(\mathbf{q}) . \quad (4.20)$$

These expressions will be evaluated explicitly for interesting cases in Sec. IV C. But we already see that for isotropic static disorder there is an intimate relation between the defect contributions to the depletion and the normal–fluid density, which are both temperature–independent, namely [18,19]

$$m \overline{n_\Delta} = \frac{d}{4} \overline{\rho_n} . \quad (4.21)$$

We conclude this section by deriving the effect of disorder on the normal–fluid density, using Landau’s arguments. This analysis yields the most general expressions for the tilt modulus tensor; note that Eq. (4.15) requires a statistically translation–invariant system (which is the most common situation), while the elegant derivation using affine transformations in App. A [44] is directly applicable only in the zero–temperature limit ( $\beta \hbar \rightarrow \infty$ ).

For a superfluid in motion with respect to the surrounding walls *and* the intrinsic disorder, we have seen that the corresponding Green’s function is simply obtained from the one for the system at rest by shifting the Matsubara frequencies according to  $\omega_m \rightarrow \omega_m - i\mathbf{q} \cdot \mathbf{v}$

and  $\omega_{m'} \rightarrow \omega_{m'} - i\mathbf{q}' \cdot \mathbf{v}$  [cf. Eq. (3.63)]. Because the disorder correlator in the comoving reference frame reads

$$\Delta(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'})_{\mathbf{v}} = \Delta(\mathbf{q}, \omega_m + i\mathbf{q} \cdot \mathbf{v}; \mathbf{q}', \omega_{m'} + i\mathbf{q}' \cdot \mathbf{v}), \quad (4.22)$$

these transformations imply that the disorder correlator to be inserted is effectively the one at rest. Thus,

$$\begin{aligned} \overline{G(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'})_{-\mathbf{v}}} &= \frac{i\omega_m + \hbar q^2/2m + n_0 V(q)/\hbar + \mathbf{q} \cdot \mathbf{v}}{(\omega_m - i\mathbf{q} \cdot \mathbf{v})^2 + \epsilon_B(q)^2/\hbar^2} (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \beta \hbar \delta_{m, -m'} \\ &+ \frac{n_0}{\hbar^2} \frac{(i\omega_m + \hbar q^2/2m + \mathbf{q} \cdot \mathbf{v})(-i\omega_{m'} + \hbar q'^2/2m - \mathbf{q}' \cdot \mathbf{v})}{[(\omega_m - i\mathbf{q} \cdot \mathbf{v})^2 + \epsilon_B(q)^2/\hbar^2][(\omega_{m'} - i\mathbf{q}' \cdot \mathbf{v})^2 + \epsilon_B(q')^2/\hbar^2]} \Delta(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) ; \quad (4.23) \end{aligned}$$

Upon expanding Eq. (4.23) to second order in  $\mathbf{q} \cdot \mathbf{v}$  and  $\mathbf{q}' \cdot \mathbf{v}$ , and making use of Eqs. (3.61) and (3.60), we find for the normal–fluid density, which due to the disorder has become a tensor and in general a function of  $\tau$  as well,

$$\begin{aligned} \overline{\rho_{nij}(\tau)} &= \frac{\beta \hbar^2}{4d} \delta_{ij} \int \frac{d^d q}{(2\pi)^d} \left( \frac{q}{\sinh[\beta \epsilon_B(q)/2]} \right)^2 \\ &+ \frac{n_0}{\hbar \Omega} \int \frac{d^d q}{(2\pi)^d} q_i q_j \frac{1}{(\beta \hbar)^2} \sum_{m, m'} \frac{e^{-i(\omega_m + \omega_{m'})\tau} \Delta(\mathbf{q}, \omega_m; -\mathbf{q}, \omega_{m'})}{[\omega_m^2 + \epsilon_B(q)^2/\hbar^2][\omega_{m'}^2 + \epsilon_B(q)^2/\hbar^2]} \times \\ &\quad \times \left( \frac{(-i\omega_{m'} + \hbar q^2/2m)[i\omega_m \hbar q^2/m - \omega_m^2 + \epsilon_B(q)^2/\hbar^2]}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2} \right. \\ &\quad \left. + \frac{(i\omega_m + \hbar q^2/2m)[-i\omega_{m'} \hbar q^2/m - \omega_{m'}^2 + \epsilon_B(q)^2/\hbar^2]}{\omega_{m'}^2 + \epsilon_B(q)^2/\hbar^2} \right). \quad (4.24) \end{aligned}$$

If translational invariance is statistically restored by averaging over the disorder, the defect contribution to Eq. (4.24) reduces to the equivalent of Eq. (4.15), because all the terms in the numerator of (4.24) which are linear in  $\omega_m$  then vanish. For an isotropic system, the normal–fluid density becomes a scalar, see Eq. (4.12). For possibly anisotropic *static* disorder, the renormalization of the normal fluid density becomes

$$\overline{\rho_{nij}} = \frac{\beta \hbar^2}{4d} \delta_{ij} \int \frac{d^d q}{(2\pi)^d} \left( \frac{q}{\sinh[\beta \epsilon_B(q)/2]} \right)^2 + \frac{n_0}{m} \int \frac{d^d q}{(2\pi)^d} q_i q_j \left( \frac{\hbar^2 q}{\epsilon_B(q)^2} \right)^2 \Delta(\mathbf{q}), \quad (4.25)$$

which generalizes Eq. (4.14).

### C. Results for point–like, linear, and planar defects

We now evaluate the density correlation function and the effective contributions to the depletion and the normal–fluid density caused by pinning to static disorder explicitly for the cases of point and (spatially) extended defects. We work in the boson picture, and use the words “point”, “line”, and “plane” to indicate the dimension of the disorder in the  $d$ –dimensional spacelike direction  $\mathbf{r}$ . According to Eqs. (4.20) and (4.25), the defect contributions for static disorder are independent of boson temperature; i.e., in the case

of periodic boundary conditions there will be no finite-size corrections in the imaginary-time direction for the corresponding quantities in type-II superconductors. For a flux liquid, such “static” disorder translates to defects *correlated* along the magnetic-field direction, and “point” and “linear” defects in  $d = 2$  map on columnar pins and planar disorder, respectively (see Table II for an explicit comparison in  $d = 2$ ). We shall return to these cases, as well as to “true” (uncorrelated in  $\mathbf{r}$  and  $\tau$ ) point disorder and tilted extended defects in Sec. VI, where we shall also compare the results obtained for periodic and open boundary conditions.

For boson point defects, the disorder correlator reads

$$\Delta(\mathbf{r}, \tau; \mathbf{r}', \tau') = \Delta \delta(\mathbf{r} - \mathbf{r}') \quad , \quad \Delta(\mathbf{q}, \omega_m) = \Delta \beta \hbar \delta_{m,0} . \quad (4.26)$$

This immediately yields the density correlation function

$$\overline{S(\mathbf{q}, \omega_m)} = \frac{n_0 \hbar q^2 / m}{\omega_m^2 + \epsilon_B(q)^2 / \hbar^2} + \Delta \left( \frac{n_0 \hbar^2 q^2}{m \epsilon_B(q)^2} \right)^2 \beta \hbar \delta_{m,0} , \quad (4.27)$$

and hence for the static structure factor

$$\overline{S(\mathbf{q}, \tau = 0)} = \frac{n_0 \hbar^2 q^2}{2m \epsilon_B(q)} \coth \frac{\beta \epsilon_B(q)}{2} + \Delta \left( \frac{n_0 \hbar^2 q^2}{m \epsilon_B(q)^2} \right)^2 ; \quad (4.28)$$

see also Eq. (4.9).

Furthermore,

$$m \overline{n_\Delta} = \frac{d}{4} \overline{\rho_{n\Delta}} = \Delta \frac{n_0}{4m} \int \frac{d^d q}{(2\pi)^d} \left( \frac{\hbar q}{\epsilon_B(q)} \right)^4 \approx \Delta \frac{m^3 n_0^{(d/2)-1} a^{(d/2)-2}}{2^{5-d} \pi^2 \Gamma(d/2) \hbar^d} \int_0^\infty \frac{x^{d-1}}{(1+x^2)^2} dx , \quad (4.29)$$

where in the final equation  $V(q) \approx V_0 = 4\pi a/m$  was assumed ( $x = \hbar q/4\sqrt{\pi a n_0}$ ). In two dimensions, one finds

$$d = 2 : m \overline{n_\Delta} = \overline{\rho_{n\Delta}}/2 = \Delta m^3 / 16 \pi^2 \hbar^2 a , \quad (4.30)$$

while in  $d = 3$  [17,18]

$$d = 3 : m \overline{n_\Delta} = 3 \overline{\rho_{n\Delta}}/4 = \Delta m^3 n_0^{1/2} / 8 \pi^{3/2} \hbar^3 a^{1/2} . \quad (4.31)$$

We can readily generalize these results to the case of static parallel defects, extended along a  $d_{\parallel}$ -dimensional subspace  $\mathbf{r}_{\parallel}$ . The disorder is therefore uncorrelated only in the  $d_{\perp}$  transverse spatial directions  $\mathbf{r}_{\perp}$ , i.e.,

$$\Delta(\mathbf{r}, \tau; \mathbf{r}', \tau') = \Delta \delta(\mathbf{r}_{\perp} - \mathbf{r}'_{\perp}) \quad , \quad \Delta(\mathbf{q}, \omega_m) = \Delta (2\pi)^{d_{\parallel}} \delta(\mathbf{q}_{\parallel}) \beta \hbar \delta_{m,0} . \quad (4.32)$$

The dimensions  $d_{\parallel} = 1$  and thus  $d_{\perp} = d - 1$  correspond to line defects, while  $d_{\parallel} = 2$ ,  $d_{\perp} = d - 2$  describe planar disorder, etc. The corresponding dynamic structure factor then reads

$$\overline{S(\mathbf{q}, \omega_m)} = \frac{n_0 \hbar q^2 / m}{\omega_m^2 + \epsilon_B(q)^2 / \hbar^2} + \Delta \left( \frac{n_0 \hbar^2 q_{\perp}^2}{m \epsilon_B(q_{\perp})^2} \right)^2 (2\pi)^{d_{\parallel}} \delta(\mathbf{q}_{\parallel}) \beta \hbar \delta_{m,0} , \quad (4.33)$$

and the static structure factor becomes

$$\overline{S(\mathbf{q}, \tau = 0)} = \frac{n_0 \hbar^2 q^2}{2m\epsilon_B(q)} \coth \frac{\beta\epsilon_B(q)}{2} + \Delta \left( \frac{n_0 \hbar^2 q_\perp^2}{m\epsilon_B(q_\perp)^2} \right)^2 (2\pi)^{d_\parallel} \delta(\mathbf{q}_\parallel) . \quad (4.34)$$

The disorder-induced depletion follows from Eq. (4.29) by replacing  $d$  with  $d_\perp$ ,

$$\overline{n_\Delta} = \Delta \frac{n_0}{4m^2} \int \frac{d^{d_\perp} q_\perp}{(2\pi)^{d_\perp}} \left( \frac{\hbar q_\perp}{\epsilon_B(q_\perp)} \right)^4 = \Delta \frac{m^2 n_0^{(d_\perp/2)-1} a^{(d_\perp/2)-2}}{2^{5-d_\perp} \pi^2 \Gamma(d_\perp/2) \hbar^{d_\perp}} \int_0^\infty \frac{x^{d_\perp-1}}{(1+x^2)^2} dx . \quad (4.35)$$

The normal-fluid density, defined as the transport coefficient characterizing the linear response of the system to motion with respect to its boundaries, will now depend on the direction relative to the disorder. If this motion is enforced along the disorder direction  $\mathbf{r}_\parallel$ , the defects will have no effect [see Eq. (4.25) with  $q_i = q_{\parallel i}$ ],

$$\overline{\rho_{n_\parallel \Delta}} = 0 ; \quad (4.36)$$

on the other hand, if the probe acts transverse to the defects, we find a similar result to Eq. (4.29) (for point disorder, all directions are transverse to the defects), namely

$$m \overline{n_\Delta} = \frac{d_\perp}{4} \overline{\rho_{n_\perp \Delta}} . \quad (4.37)$$

For linear defects in two dimensions or planar disorder in  $d = 3$  ( $d_\perp = 1$ ) this yields

$$d_\perp = 1 : m \overline{n_\Delta} = \overline{\rho_{n_\perp \Delta}}/4 = \Delta m^3 / 64 \pi^{3/2} \hbar n_0^{1/2} a^{3/2} , \quad (4.38)$$

whereas for line defects in three dimension

$$d_\perp = 2 : m \overline{n_\Delta} = \overline{\rho_{n_\perp \Delta}}/2 = \Delta m^3 / 16 \pi^2 \hbar^2 a , \quad (4.39)$$

which is, of course, identical to the result (4.30) for point defects in  $d = 2$ .

In the framework to the Bogoliubov approximation, which amounts to an expansion with respect to both thermal and quantum fluctuations, the disorder-induced depletion for  $d_\perp < 4$  is smaller than the corresponding enhancement of the normal-fluid density, Eq. (4.37). The bosons will become localized when  $\rho_{s_\perp} \rightarrow 0$  (i.e., the corresponding “tilt” modulus  $c_\perp^s$  diverges !). The emerging new low-temperature phase of localized bosons is the Bose glass [20–27] (we use this term in a generalized form which also includes localization by extended defects), and is characterized by a vanishing superfluid density  $\rho_s = 0$ . In the corresponding “Bose glass” phase of localized magnetic flux lines, the tilt modulus transverse to the defect directions diverges, which implies the “transverse Meissner effect”, namely the absence of any response of the superconductor towards transverse magnetic fields (up to some critical tilt angle), see Sec. VI F.

As a final investigation of how disorder affects superfluids, consider a Helium film on a substrate with disorder in the form of lines (deposited, say, via microlithography) with random positions and orientations. Imagine that the lines are arrayed in “families” indexed by a two-dimensional unit vector  $\hat{\mathbf{n}}$  perpendicular to every member of a family. Within a single family, the Fourier-transformed disorder correlator must have the form



$$\Delta_{\hat{n}}(\mathbf{q}, \omega_m) = 2\Delta\delta^{(1)}(\hat{\mathbf{n}} \cdot \mathbf{q}) \beta\hbar\delta_{m,0} , \quad (4.40)$$

where the one-dimensional delta function restricts only the wavevector component parallel to  $\hat{\mathbf{n}}$ . Upon using the integral representation of the delta function,  $\delta(x) = \int_{-\infty}^{\infty} e^{ixs} ds/2\pi$ , we can compute the quenched average over families in different directions by integrating over  $\hat{n} = (\cos\phi, \sin\phi)$  to find

$$\Delta(\mathbf{q}, \omega_m) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi \Delta_{\hat{n}}(\mathbf{q}, \omega_m) = \frac{\Delta}{q} \beta\hbar\delta_{m,0} . \quad (4.41)$$

Results for the density correlations, depletion and normal-fluid density can be obtained by inserting this correlator into Eqs. (4.8), (4.20), and (4.14); thus we find, by deviding the corresponding results (4.33), (4.34) for point defects by  $q$ , the density correlation function

$$\overline{S(\mathbf{q}, \omega_m)} = \frac{n_0\hbar q^2/m}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2} + \frac{\Delta}{q} \left( \frac{n_0\hbar^2 q_{\perp}^2}{m\epsilon_B(q_{\perp})^2} \right)^2 (2\pi)^{d_{\parallel}} \delta(\mathbf{q}_{\parallel}) \beta\hbar\delta_{m,0} , \quad (4.42)$$

and the static structure factor

$$\overline{S(\mathbf{q}, \tau = 0)} = \frac{n_0\hbar^2 q^2}{2m\epsilon_B(q)} \coth \frac{\beta\epsilon_B(q)}{2} + \frac{\Delta}{q} \left( \frac{n_0\hbar^2 q_{\perp}^2}{m\epsilon_B(q_{\perp})^2} \right)^2 (2\pi)^{d_{\parallel}} \delta(\mathbf{q}_{\parallel}) . \quad (4.43)$$

Similarly, we find in analogy with Eq. (4.29)

$$m \overline{n_{\Delta}} = \frac{d}{4} \overline{\rho_{n_{\Delta}}} = \Delta \frac{n_0}{4m} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q} \left( \frac{\hbar q}{\epsilon_B(q)} \right)^4 \approx \Delta \frac{m^3 n_0^{(d-3)/2} a^{(d-5)/2}}{2^{7-d} \pi^{5/2} \Gamma(d/2) \hbar^{d-1}} \int_0^{\infty} \frac{x^{d-2}}{(1+x^2)^2} dx , \quad (4.44)$$

which is logarithmically divergent in  $d = 1$ , but yields a finite result in two dimensions,

$$d = 2 : m \overline{n_{\Delta}} = \overline{\rho_{n_{\Delta}}}/2 = \Delta m^3/128\pi^{3/2} n_0^{1/2} \hbar a^{3/2} , \quad (4.45)$$

whereas in  $d = 3$

$$d = 3 : m \overline{n_{\Delta}} = 3\overline{\rho_{n_{\Delta}}}/4 = \Delta m^3/16\pi^3 \hbar^2 a . \quad (4.46)$$

We see that, although the correlator (4.40) is singular as  $q \rightarrow 0$ , the disorder contributions to Eqs. (4.14) and (4.20) remain infrared-convergent (for  $d > 1$ ). The depletion and normal-fluid density are, however, *increased* due to the correlations, relative to the same amount of disorder scattered about in a ‘‘point-like’’ fashion.

## V. DISORDER CONTRIBUTIONS FOR FLUX LIQUIDS (OPEN BOUNDARY CONDITIONS)

We now investigate the influence of disorder on the correlation functions, ‘‘boson order parameter’’, and the tilt modulus for flux liquids with open boundary conditions along the magnetic-field direction, using the results obtained in App. C. In this and the following section, we employ the flux line language, but retain the boson superfluid notation, in order to facilitate comparison with the results for the case of periodic boundary conditions derived in Sec. IV. We shall discuss the equations for *general* disorder correlators, which lead to rather lengthy and not very transparent formulas. Specific results for a number of interesting types of defects are tabulated in Sec. VI.

### A. Density and tilt correlations

Upon carrying out the matrix products in Eq. (C9), we arrive at the following explicit results for the correlation functions with open boundary conditions in harmonic approximation, to first order in the disorder correlator,

$$\begin{aligned}
& \overline{\langle \Theta(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega_{m'}) \rangle} = \\
& = \frac{m\epsilon_B(q)^2/n_0\hbar^3q^2}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2} \left( \beta\hbar\delta_{m,-m'} - \frac{2\hbar}{\epsilon_B(q)} \tanh \frac{\beta\epsilon_B(q)}{2} \frac{\omega_m\omega_{m'} + \epsilon_B(q)^2/\hbar^2}{\omega_{m'}^2 + \epsilon_B(q)^2/\hbar^2} \right) (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \\
& + \frac{1/\hbar^2}{[\omega_m^2 + \epsilon_B(q)^2/\hbar^2][\omega_{m'}^2 + \epsilon_B(q')^2/\hbar^2]} \sum_{n,n'} \left( \omega_m\delta_{m,n} - \frac{2\epsilon_B(q)}{\beta\hbar^2} \tanh \frac{\beta\epsilon_B(q)}{2} \frac{\omega_m + \omega_n}{\omega_n^2 + \epsilon_B(q)^2/\hbar^2} \right) \times \\
& \quad \times \left( \omega_{m'}\delta_{m',n'} - \frac{2\epsilon_B(q')}{\beta\hbar^2} \tanh \frac{\beta\epsilon_B(q')}{2} \frac{\omega_{m'} + \omega_{n'}}{\omega_{n'}^2 + \epsilon_B(q')^2/\hbar^2} \right) \Delta(\mathbf{q}, \omega_n; \mathbf{q}', \omega_{n'}) , \quad (5.1)
\end{aligned}$$

$$\begin{aligned}
& \overline{\langle \Theta(\mathbf{q}, \omega_m) \pi(\mathbf{q}', \omega_{m'}) \rangle} = \\
& = \frac{\beta\hbar}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2} \left( \omega_m\delta_{m,-m'} - \frac{2\epsilon_B(q)}{\beta\hbar^2} \tanh \frac{\beta\epsilon_B(q)}{2} \frac{\omega_m - \omega_{m'}}{\omega_{m'}^2 + \epsilon_B(q)^2/\hbar^2} \right) (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \\
& + \frac{n_0q^2/\hbar m}{[\omega_m^2 + \epsilon_B(q)^2/\hbar^2][\omega_{m'}^2 + \epsilon_B(q')^2/\hbar^2]} \sum_{n,n'} \left( \omega_m\delta_{m,n} - \frac{2\epsilon_B(q)}{\beta\hbar^2} \tanh \frac{\beta\epsilon_B(q)}{2} \frac{\omega_m + \omega_n}{\omega_n^2 + \epsilon_B(q)^2/\hbar^2} \right) \times \\
& \quad \times \left( \delta_{m',n'} + \frac{2}{\beta\epsilon_B(q')} \tanh \frac{\beta\epsilon_B(q')}{2} \frac{\omega_{m'}\omega_{n'} - \epsilon_B(q')^2/\hbar^2}{\omega_{n'}^2 + \epsilon_B(q')^2/\hbar^2} \right) \Delta(\mathbf{q}, \omega_n; \mathbf{q}', \omega_{n'}) , \quad (5.2)
\end{aligned}$$

$$\begin{aligned}
& \overline{\langle \pi(\mathbf{q}, \omega_m) \pi(\mathbf{q}', \omega_{m'}) \rangle} = \\
& = \frac{n_0\hbar q^2/m}{\omega_m^2 + \epsilon_B(q)^2/\hbar^2} \left( \beta\hbar\delta_{m,-m'} - \frac{2\hbar}{\epsilon_B(q)} \tanh \frac{\beta\epsilon_B(q)}{2} \frac{\omega_m\omega_{m'} + \epsilon_B(q)^2/\hbar^2}{\omega_{m'}^2 + \epsilon_B(q)^2/\hbar^2} \right) (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \\
& + \frac{n_0^2q^2q'^2/m^2}{[\omega_m^2 + \epsilon_B(q)^2/\hbar^2][\omega_{m'}^2 + \epsilon_B(q')^2/\hbar^2]} \sum_{n,n'} \left( \delta_{m,n} + \frac{2}{\beta\epsilon_B(q)} \tanh \frac{\beta\epsilon_B(q)}{2} \frac{\omega_m\omega_n - \epsilon_B(q)^2/\hbar^2}{\omega_n^2 + \epsilon_B(q)^2/\hbar^2} \right) \times \\
& \quad \times \left( \delta_{m',n'} + \frac{2}{\beta\epsilon_B(q')} \tanh \frac{\beta\epsilon_B(q')}{2} \frac{\omega_{m'}\omega_{n'} - \epsilon_B(q')^2/\hbar^2}{\omega_{n'}^2 + \epsilon_B(q')^2/\hbar^2} \right) \Delta(\mathbf{q}, \omega_n; \mathbf{q}', \omega_{n'}) . \quad (5.3)
\end{aligned}$$

These results should be contrasted with Eqs. (4.4)–(4.6), which are valid for periodic boundary conditions, and may be recovered from (5.1)–(5.3) by taking into account only those terms in the brackets containing the Kronecker symbols  $\delta_{m,-m'}$ ,  $\delta_{m,n}$ , and  $\delta_{m',n'}$ .

From Eq. (5.3), we can deduce the equal-“time” density correlation function,

$$\begin{aligned}
\overline{S(\mathbf{q}, \tau; \mathbf{q}', \tau)} & = \frac{n_0\hbar^2q^2}{2m\epsilon_B(q)} \tanh \frac{\beta\epsilon_B(q)}{2} (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \\
& + \frac{n_0^2\hbar^2q^2q'^2}{m^2\epsilon_B(q)\epsilon_B(q')} \frac{1}{(\beta\hbar)^2} \sum_{n,n'} \frac{\Delta(\mathbf{q}, \omega_n; \mathbf{q}', \omega_{n'})}{[\omega_n^2 + \epsilon_B(q)^2/\hbar^2][\omega_{n'}^2 + \epsilon_B(q')^2/\hbar^2]} \times \\
& \quad \times \left[ \frac{\epsilon_B(q)}{\hbar} \left( e^{-i\omega_n\tau} - \frac{\cosh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right) - i\omega_n \frac{\sinh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right] \times \\
& \quad \times \left[ \frac{\epsilon_B(q')}{\hbar} \left( e^{-i\omega_{n'}\tau} - \frac{\cosh[(\beta\hbar - 2\tau)\epsilon_B(q')/2\hbar]}{\cosh[\beta\epsilon_B(q')/2]} \right) - i\omega_{n'} \frac{\sinh[(\beta\hbar - 2\tau)\epsilon_B(q')/2\hbar]}{\cosh[\beta\epsilon_B(q')/2]} \right] , \quad (5.4)
\end{aligned}$$

generalizing both Eqs. (4.9) and (3.77). The  $d$ -dimensional structure factor is now a function of  $\tau$ , implying that the mean-square variations of the vortex positions depend on the depth of the considered cross-section perpendicular to the magnetic field in the sample. On the surfaces at  $\tau = 0$  and  $\tau = \beta\hbar$ , the fluctuations vanish. Similarly, for the leading term of the current correlation function (i.e., the one originating in the phase fluctuations) we find

$$\begin{aligned} \overline{C_{ij}(\mathbf{q}, \tau; \mathbf{q}', \tau)} &= n_0 m \frac{\epsilon_B(q)}{2} \tanh \frac{\beta\epsilon_B(q)}{2} P_{ij}^L(\mathbf{q}) (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \\ &+ n_0^2 q_i q'_j \frac{1}{(\beta\hbar)^2} \sum_{n, n'} \frac{\Delta(\mathbf{q}, \omega_n; \mathbf{q}', \omega_{n'})}{[\omega_n^2 + \epsilon_B(q)^2/\hbar^2][\omega_{n'}^2 + \epsilon_B(q')^2/\hbar^2]} \times \\ &\times \left[ i\omega_n \left( e^{-i\omega_n \tau} - \frac{\cosh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right) - \frac{\epsilon_B(q)}{\hbar} \frac{\sinh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right] \times \\ &\times \left[ i\omega_{n'} \left( e^{-i\omega_{n'} \tau} - \frac{\cosh[(\beta\hbar - 2\tau)\epsilon_B(q')/2\hbar]}{\cosh[\beta\epsilon_B(q')/2]} \right) - \frac{\epsilon_B(q')}{\hbar} \frac{\sinh[(\beta\hbar - 2\tau)\epsilon_B(q')/2\hbar]}{\cosh[\beta\epsilon_B(q')/2]} \right]. \end{aligned} \quad (5.5)$$

Eq. (3.30) then yields the tilt correlations.

In the center of the sample,  $\tau = \beta\hbar/2$ , these expressions simplify considerably; upon assuming translational invariance in the  $d$  transverse “spatial” directions, the density and current correlators read

$$\begin{aligned} \overline{S(\mathbf{q}, \tau = \beta\hbar/2)} &= \frac{n_0 \hbar^2 q^2}{2m\epsilon_B(q)} \tanh \frac{\beta\epsilon_B(q)}{2} + \frac{1}{(\beta\hbar)^2 \Omega} \sum_{n, n'} \frac{(n_0 q^2/m)^2 \Delta(\mathbf{q}, \omega_n; -\mathbf{q}, \omega_{n'})}{[\omega_n^2 + \epsilon_B(q)^2/\hbar^2][\omega_{n'}^2 + \epsilon_B(q)^2/\hbar^2]} \\ &\times \left( e^{-i\omega_n \beta\hbar/2} - \frac{1}{\cosh[\beta\epsilon_B(q)/2]} \right) \left( e^{-i\omega_{n'} \beta\hbar/2} - \frac{1}{\cosh[\beta\epsilon_B(q)/2]} \right), \end{aligned} \quad (5.6)$$

$$\begin{aligned} \overline{C_{ij}(\mathbf{q}, \tau = \beta\hbar/2)} &= \left[ n_0 m \frac{\epsilon_B(q)}{2} \tanh \frac{\beta\epsilon_B(q)}{2} + \frac{1}{(\beta\hbar)^2 \Omega} \sum_{n, n'} \frac{n_0^2 q^2 \Delta(\mathbf{q}, \omega_n; -\mathbf{q}, \omega_{n'}) \omega_n \omega_{n'}}{[\omega_n^2 + \epsilon_B(q)^2/\hbar^2][\omega_{n'}^2 + \epsilon_B(q)^2/\hbar^2]} \right. \\ &\times \left. \left( e^{-i\omega_n \beta\hbar/2} - \frac{1}{\cosh[\beta\epsilon_B(q)/2]} \right) \left( e^{-i\omega_{n'} \beta\hbar/2} - \frac{1}{\cosh[\beta\epsilon_B(q)/2]} \right) \right] P_{ij}^L(\mathbf{q}). \end{aligned} \quad (5.7)$$

Note that  $e^{-i\omega_n \beta\hbar/2} = (-1)^n$ . For static defects (4.13), the disorder contribution to Eq. (5.7) vanishes, while the static structure factor becomes

$$\overline{S(\mathbf{q}, \tau = \beta\hbar/2)} = \frac{n_0 \hbar^2 q^2}{2m\epsilon_B(q)} \tanh \frac{\beta\epsilon_B(q)}{2} + \Delta(\mathbf{q}) \left( \frac{n_0 \hbar^2 q^2}{m\epsilon_B(q)^2} \right)^2 \left( 1 - \frac{1}{\cosh[\beta\epsilon_B(q)/2]} \right)^2. \quad (5.8)$$

## B. Depletion and vortex tilt modulus

Within the Gaussian approximation (3.42), (5.1)–(5.3) we can readily obtain the disorder contribution to the Green’s function for open boundary conditions, supplementing the pure part (3.78),

$$\begin{aligned}
\overline{G_{\Delta}(\mathbf{q}, \omega_m; \mathbf{q}', \omega'_m)} &= \frac{n_0/\hbar^2}{[\omega_m^2 + \epsilon_B(q)^2/\hbar^2][\omega_{m'}^2 + \epsilon_B(q')^2/\hbar^2]} \sum_{n,n'} \Delta(\mathbf{q}, \omega_n; \mathbf{q}', \omega_{n'}) \times \\
&\left[ \left( \frac{\hbar q^2}{2m} + i\omega_m \right) \delta_{m,n} + \frac{\tanh[\beta\epsilon_B(q)/2]}{\beta\epsilon_B(q)/2} \frac{[\omega_m\omega_n - \epsilon_B(q)^2/\hbar^2]\hbar q^2/2m - i(\omega_m + \omega_n)\epsilon_B(q)^2/\hbar^2}{\omega_n^2 + \epsilon_B(q)^2/\hbar^2} \right] \times \\
&\times \left[ \left( \frac{\hbar q'^2}{2m} - i\omega_{m'} \right) \delta_{m',n'} + \frac{\tanh[\beta\epsilon_B(q')/2]}{\beta\epsilon_B(q')/2} \times \right. \\
&\quad \left. \times \frac{[\omega_{m'}\omega_{n'} - \epsilon_B(q')^2/\hbar^2]\hbar q'^2/2m + i(\omega_{m'} + \omega_{n'})\epsilon_B(q')^2/\hbar^2}{\omega_{n'}^2 + \epsilon_B(q')^2/\hbar^2} \right]. \quad (5.9)
\end{aligned}$$

Upon assuming as usual translational invariance in  $\mathbf{r}$ , this leads to the disorder-induced depletion, i.e., a disentanglement of pinned vortex lines,

$$\begin{aligned}
\overline{n_{\Delta}(\tau)} &= \frac{n_0}{\hbar^2\Omega} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(\beta\hbar)^2} \sum_{n,n'} \frac{\Delta(\mathbf{q}, \omega_n; -\mathbf{q}, \omega'_n)}{[\omega_n^2 + \epsilon_B(q)^2/\hbar^2][\omega_{n'}^2 + \epsilon_B(q)^2/\hbar^2]} \times \\
&\times \left[ \left( \frac{\hbar q^2}{2m} + i\omega_n \right) \left( e^{-i\omega_n\tau} - \frac{\cosh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right) \right. \\
&\quad \left. - \left( \frac{\epsilon_B(q)}{\hbar} + \frac{i\omega_n\hbar^2 q^2}{2m\epsilon_B(q)} \right) \frac{\sinh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right] \times \\
&\times \left[ \left( \frac{\hbar q^2}{2m} - i\omega_{n'} \right) \left( e^{-i\omega_{n'}\tau} - \frac{\cosh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right) \right. \\
&\quad \left. + \left( \frac{\epsilon_B(q)}{\hbar} - \frac{i\omega_{n'}\hbar^2 q^2}{2m\epsilon_B(q)} \right) \frac{\sinh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right], \quad (5.10)
\end{aligned}$$

compare Eq. (4.18) for periodic boundary conditions. It is a straightforward, but rather tedious task to compute the disorder renormalization of the normal-fluid density from Eqs. (3.60) and (3.61), following Landau's arguments (see Sec. IV B); besides expanding to second order in  $\mathbf{q} \cdot \mathbf{v}$ , this involves a number of Matsubara frequency sums for which the list in App. D 1 is helpful. The final result reads

$$\begin{aligned}
\overline{\rho_{nij\Delta}(\tau)} &= -\frac{n_0}{\hbar\Omega} \int \frac{d^d q}{(2\pi)^d} q_i q_j \frac{1}{(\beta\hbar)^2} \sum_{n,n'} \frac{\Delta(\mathbf{q}, \omega_n; -\mathbf{q}, \omega'_n)}{[\omega_n^2 + \epsilon_B(q)^2/\hbar^2][\omega_{n'}^2 + \epsilon_B(q)^2/\hbar^2]} \times \\
&\times \left\{ \left[ \left( \frac{\hbar q^2}{2m} + i\omega_n \right) \left( e^{-i\omega_n\tau} - \frac{\cosh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right) \right. \right. \\
&\quad \left. \left. - \left( \frac{\epsilon_B(q)}{\hbar} + \frac{i\omega_n\hbar^2 q^2}{2m\epsilon_B(q)} \right) \frac{\sinh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right] \times \right. \\
&\times \left[ \frac{\omega_{n'}^2 - \epsilon_B(q)^2/\hbar^2 + i\omega_{n'}\hbar q^2/m}{\omega_{n'}^2 + \epsilon_B(q)^2/\hbar^2} e^{-i\omega_{n'}\tau} \right. \\
&\quad \left. + \left( \frac{\hbar q^2}{2m} - i\omega_{n'} \right) \left( \frac{\beta\hbar \sinh[\epsilon_B(q)\tau/\hbar]}{\sinh[\beta\epsilon_B(q)]} - \frac{\tau \cosh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right) \right. \\
&\quad \left. + \left( \frac{\epsilon_B(q)}{\hbar} - \frac{i\omega_{n'}\hbar^2 q^2}{2m\epsilon_B(q)} \right) \left( \frac{\beta\hbar \cosh[\epsilon_B(q)\tau/\hbar]}{\sinh[\beta\epsilon_B(q)]} - \frac{\tau \sinh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right) \right] \Bigg\}
\end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{\omega_n^2 - \epsilon_B(q)^2/\hbar^2 - i\omega_n\hbar q^2/m}{\omega_n^2 + \epsilon_B(q)^2/\hbar^2} e^{-i\omega_n\tau} \right. \\
& \quad - \left( \frac{\hbar q^2}{2m} + i\omega_n \right) \left( \frac{\beta\hbar \sinh[\epsilon_B(q)\tau/\hbar]}{\sinh[\beta\epsilon_B(q)]} - \frac{\tau \cosh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right) \\
& \quad + \left. \left( \frac{\epsilon_B(q)}{\hbar} + \frac{i\omega_n\hbar^2 q^2}{2m\epsilon_B(q)} \right) \left( \frac{\beta\hbar \cosh[\epsilon_B(q)\tau/\hbar]}{\sinh[\beta\epsilon_B(q)]} - \frac{\tau \sinh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right) \right] \\
& \times \left[ \left( \frac{\hbar q^2}{2m} - i\omega_{n'} \right) \left( e^{-i\omega_{n'}\tau} - \frac{\cosh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right) \right. \\
& \quad \left. + \left( \frac{\epsilon_B(q)}{\hbar} - \frac{i\omega_{n'}\hbar^2 q^2}{2m\epsilon_B(q)} \right) \frac{\sinh[(\beta\hbar - 2\tau)\epsilon_B(q)/2\hbar]}{\cosh[\beta\epsilon_B(q)/2]} \right] \Big\}, \quad (5.11)
\end{aligned}$$

and is to be contrasted with Eq. (4.24). Note that as a consequence of the open boundary conditions (which suppress fluctuations in the boson order parameter at the boundaries), both results (5.10) and (5.11) vanish for  $\tau = 0$  and  $\tau = \beta\hbar$ , respectively. At  $\tau = \beta\hbar/2$ , these expressions simplify, and the depletion and normal-fluid density in the center of the bulk become

$$\begin{aligned}
\overline{n(\tau = \beta\hbar/2)} - n_0 &= \int \frac{d^d q}{(2\pi)^d} \left[ v(q)^2 - \frac{1 + 2v(q)^2}{e^{\beta\epsilon_B(q)} + 1} \right] \\
& + \frac{n_0}{\hbar^2 \Omega} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(\beta\hbar)^2} \sum_{n,n'} \frac{\Delta(\mathbf{q}, \omega_n; -\mathbf{q}, \omega'_n)}{[\omega_n^2 + \epsilon_B(q)^2/\hbar^2][\omega_{n'}^2 + \epsilon_B(q)^2/\hbar^2]} \left( \frac{\hbar q^2}{2m} + i\omega_n \right) \times \\
& \times \left( \frac{\hbar q^2}{2m} - i\omega_{n'} \right) \left( e^{-i\omega_n\beta\hbar/2} - \frac{1}{\cosh[\beta\epsilon_B(q)/2]} \right) \left( e^{-i\omega_{n'}\beta\hbar/2} - \frac{1}{\cosh[\beta\epsilon_B(q)/2]} \right), \quad (5.12)
\end{aligned}$$

$$\begin{aligned}
\overline{\rho_{n_{ij}}(\tau = \beta\hbar/2)} &= -\frac{\beta\hbar^2}{4d} \delta_{ij} \int \frac{d^d q}{(2\pi)^d} \left( \frac{q}{\sinh[\beta\epsilon_B(q)/2]} \right)^2 \\
& - \frac{n_0}{\hbar\Omega} \int \frac{d^d q}{(2\pi)^d} q_i q_j \frac{1}{(\beta\hbar)^2} \sum_{n,n'} \frac{\Delta(\mathbf{q}, \omega_n; -\mathbf{q}, \omega'_n)}{[\omega_n^2 + \epsilon_B(q)^2/\hbar^2][\omega_{n'}^2 + \epsilon_B(q)^2/\hbar^2]} \times \\
& \times \left\{ \left( \frac{\hbar q^2}{2m} + i\omega_n \right) \left( e^{-i\omega_{n'}\beta\hbar/2} - \frac{1}{\cosh[\beta\epsilon_B(q)/2]} \right) \times \right. \\
& \quad \times \left[ \frac{\omega_{n'}^2 - \epsilon_B(q)^2/\hbar^2 + i\omega_{n'}\hbar q^2/m}{\omega_{n'}^2 + \epsilon_B(q)^2/\hbar^2} e^{-i\omega_{n'}\tau} + \frac{\beta\hbar/2}{\sinh[\beta\epsilon_B(q)/2]} \left( \frac{\epsilon_B(q)}{\hbar} - \frac{i\omega_{n'}\hbar^2 q^2}{2m\epsilon_B(q)} \right) \right] \\
& \quad + \left( \frac{\hbar q^2}{2m} - i\omega_{n'} \right) \left( e^{-i\omega_n\beta\hbar/2} - \frac{1}{\cosh[\beta\epsilon_B(q)/2]} \right) \times \\
& \quad \times \left. \left[ \frac{\omega_n^2 - \epsilon_B(q)^2/\hbar^2 - i\omega_n\hbar q^2/m}{\omega_n^2 + \epsilon_B(q)^2/\hbar^2} e^{-i\omega_n\tau} + \frac{\beta\hbar/2}{\sinh[\beta\epsilon_B(q)/2]} \left( \frac{\epsilon_B(q)}{\hbar} + \frac{i\omega_n\hbar^2 q^2}{2m\epsilon_B(q)} \right) \right] \right\}. \quad (5.13)
\end{aligned}$$

For static disorder (4.13), we find the following comparatively simple expressions,

$$\overline{n_\Delta(\tau = \beta\hbar/2)} = \frac{n_0}{4m^2} \int \frac{d^d q}{(2\pi)^d} \left( \frac{\hbar q}{\epsilon_B(q)} \right)^4 \Delta(\mathbf{q}) \left( 1 - \frac{1}{\cosh[\beta\epsilon_B(q)/2]} \right)^2, \quad (5.14)$$

$$\overline{\rho_{n_{ij}\Delta}(\tau = \beta\hbar/2)} = \frac{n_0}{m} \int \frac{d^d q}{(2\pi)^d} q_i q_j \left( \frac{\hbar^2 q}{\epsilon_B(q)^2} \right)^2 \Delta(\mathbf{q}) \times$$

$$\times \left( 1 - \frac{1}{\cosh[\beta\epsilon_B(q)/2]} - \frac{\beta\epsilon_B(q)/2}{\sinh[\beta\epsilon_B(q)/2]} + \frac{\beta\epsilon_B(q)}{\sinh[\beta\epsilon_B(q)]} \right). \quad (5.15)$$

In contrast to Eqs. (4.20) and (4.25), these results do depend on  $\beta\hbar$ . With the defects being “oriented” along the  $\tau$  direction, it is plausible that there are no finite-thickness corrections to the disorder contributions for quantities such as the depletion and the normal-fluid density, provided periodic boundary conditions are applied; for all points along the defect (world) line are then fully equivalent. This explains in physical terms why the effects of static disorder on Bose superfluids turned out to be independent on temperature, see Sec. IV. With open boundary conditions, on the other hand, the enhanced thermal wandering near the surfaces comes into play (compare Sec. II B), which renders the defect influence dependent on the sample thickness. As is apparent from Eqs. (5.6)–(5.8) and (5.12)–(5.15), these finite-size contributions typically decay exponentially with growing sample thickness  $\beta\hbar$ , the characteristic length (the analog of the thermal de Broglie-wavelength for superfluids) being  $\propto \beta\hbar c_1$  at small wavenumbers. Note in addition that there appears no simple relation such as (4.21) or (4.37) between the disorder contributions to the depletion and the normal-fluid density at any  $\beta\hbar < \infty$  any more. We defer the explicit evaluation of Eqs. (5.12) and (5.13) for several types of defects to the following section.

## VI. EXPLICIT RESULTS FOR VARIOUS KINDS OF DISORDER

We now specialize the cumbersome expressions obtained for general disorder in the previous section to specifically interesting types of defects, and in some cases list the corresponding results for both periodic and open boundary conditions along the  $\tau \sim z$  direction. Although we have the application to magnetic flux liquids in high-temperature superconductors in mind, we express our results in the boson superfluid notation in  $d$  “transverse” dimensions. Only in the final subsection VIF, when we summarize a number of experimentally relevant consequences of our investigations, we put  $d = 2$  explicitly and translate to the flux line language.

### A. Uncorrelated disorder – point defects

We start with disorder completely uncorrelated in  $\mathbf{r}$  and  $\tau$ ,

$$\Delta(\mathbf{q}, \omega_m; \mathbf{q}', \omega'_m) = \Delta (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \beta\hbar \delta_{m, -m'}; \quad (6.1)$$

the quenched disorder average restores translational invariance in “space” and “time”. In the flux line picture, Eq. (6.1) describes point defects, e.g. oxygen vacancies in the cuprate planes. (For “true” bosons, a realization of this space- and time-dependent disorder is more difficult to conceive.) For this highly “dynamical” disorder (6.1), we find the following defect contributions to the structure factor and the leading component of the “static” current correlation function (stemming from the phase fluctuations only), first for periodic boundary conditions [see Eqs. (4.9) and (4.4)],

$$\text{p.b.c. : } \overline{S_{\Delta}(\mathbf{q})} = \frac{\Delta}{\hbar\epsilon_{\text{B}}(q)} \left( \frac{n_0 \hbar^2 q^2}{2m\epsilon_{\text{B}}(q)} \right)^2 \coth \frac{\beta\epsilon_{\text{B}}(q)}{2} \left( 1 + \frac{\beta\epsilon_{\text{B}}(q)}{\sinh[\beta\epsilon_{\text{B}}(q)]} \right), \quad (6.2)$$

$$\overline{C_{ij\Delta}(\mathbf{q})} = -\frac{\Delta}{\hbar\epsilon_{\text{B}}(q)} \left( \frac{n_0 \hbar q}{2} \right)^2 \coth \frac{\beta\epsilon_{\text{B}}(q)}{2} \left( 1 - \frac{\beta\epsilon_{\text{B}}(q)}{\sinh[\beta\epsilon_{\text{B}}(q)]} \right) P_{ij}^L(\mathbf{q}), \quad (6.3)$$

while the results for for open boundary conditions [Eqs. (5.6),(5.7)], taken in the center of the sample ( $\tau = \beta\hbar/2$ ), read

$$\text{o.b.c. : } \overline{S_{\Delta}(\mathbf{q})} = \frac{\Delta}{\hbar\epsilon_{\text{B}}(q)} \left( \frac{n_0 \hbar^2 q^2}{2m\epsilon_{\text{B}}(q)} \right)^2 \tanh \frac{\beta\epsilon_{\text{B}}(q)}{2} \left( 1 - \frac{\beta\epsilon_{\text{B}}(q)}{\sinh[\beta\epsilon_{\text{B}}(q)]} \right), \quad (6.4)$$

$$\overline{C_{ij\Delta}(\mathbf{q})} = -\frac{\Delta}{\hbar\epsilon_{\text{B}}(q)} \left( \frac{n_0 \hbar q}{2} \right)^2 \tanh \frac{\beta\epsilon_{\text{B}}(q)}{2} \left( 1 + \frac{\beta\epsilon_{\text{B}}(q)}{\sinh[\beta\epsilon_{\text{B}}(q)]} \right) P_{ij}^L(\mathbf{q}). \quad (6.5)$$

As in the pure case, the limit  $\beta\hbar \rightarrow \infty$  is thus approached from different sides for periodic and open boundary conditions, respectively.

In order to compute the depletion from Eqs. (4.19) and (5.12) for periodic and open boundary conditions, we use the Matsubara frequency sums tabulated in App. D 1. The results are

$$\begin{aligned} \text{p.b.c. : } \overline{n_{\Delta}} &= \frac{n_0}{4} \int \frac{d^d q}{(2\pi)^d} \frac{\Delta}{\hbar\epsilon_{\text{B}}(q)} \coth \frac{\beta\epsilon_{\text{B}}(q)}{2} \times \\ &\quad \times \left[ \left( \frac{\hbar^2 q^2}{2m\epsilon_{\text{B}}(q)} \right)^2 \left( 1 + \frac{\beta\epsilon_{\text{B}}(q)}{\sinh[\beta\epsilon_{\text{B}}(q)]} \right) - \left( 1 - \frac{\beta\epsilon_{\text{B}}(q)}{\sinh[\beta\epsilon_{\text{B}}(q)]} \right) \right], \end{aligned} \quad (6.6)$$

$$\begin{aligned} \text{o.b.c. : } \overline{n_{\Delta}} &= \frac{n_0}{4} \int \frac{d^d q}{(2\pi)^d} \frac{\Delta}{\hbar\epsilon_{\text{B}}(q)} \tanh \frac{\beta\epsilon_{\text{B}}(q)}{2} \times \\ &\quad \times \left[ \left( \frac{\hbar^2 q^2}{2m\epsilon_{\text{B}}(q)} \right)^2 \left( 1 - \frac{\beta\epsilon_{\text{B}}(q)}{\sinh[\beta\epsilon_{\text{B}}(q)]} \right) - \left( 1 + \frac{\beta\epsilon_{\text{B}}(q)}{\sinh[\beta\epsilon_{\text{B}}(q)]} \right) \right]. \end{aligned} \quad (6.7)$$

In the limit  $\beta\hbar \rightarrow \infty$ , these expressions coincide, of course,

$$\overline{n_{\Delta}} = \frac{n_0}{4} \int \frac{d^d q}{(2\pi)^d} \frac{\Delta}{\hbar\epsilon_{\text{B}}(q)} \left[ \left( \frac{\hbar^2 q^2}{2m\epsilon_{\text{B}}(q)} \right)^2 - 1 \right] = -\Delta \frac{n_0^2 \hbar}{4m} \int \frac{d^d q}{(2\pi)^d} \frac{q^2 V(q)}{\epsilon_{\text{B}}(q)^3}. \quad (6.8)$$

With  $x = \hbar q/4\sqrt{\pi n_0 a}$ , we find for a delta–function repulsive potential [ $V(q) \approx V_0 = 4\pi a/m$ ]

$$\overline{n_{\Delta}} = -\Delta \frac{mn_0^{d/2} a^{(d/2)-1}}{2^{4-d}\pi\Gamma(d/2)\hbar^{d+1}} \int_0^{\infty} \frac{x^{d-2}}{(1+x^2)^{3/2}} dx; \quad (6.9)$$

in two dimensions, this yields

$$d = 2 : \quad \overline{n_{\Delta}} = -\Delta mn_0/4\pi\hbar^3, \quad (6.10)$$

and in the three–dimensional case

$$d = 3 : \quad \overline{n_{\Delta}} = -\Delta mn_0^{3/2} a^{1/2}/\pi^{3/2}\hbar^4. \quad (6.11)$$

By exploiting the identities  $\int_0^\infty [x^{d-2}(\coth x - 1) - x^{d-1}/\sinh^2 x]dx = -\delta_{d,2} - [(d-2)/(d-1)] \int_0^\infty x^{d-1}/\sinh^2 x dx$  and  $\int_0^\infty [x^{d-2}(1 - \tanh x) - x^{d-1}/\cosh^2 x]dx = -[(d-2)/(d-1)] \int_0^\infty x^{d-1}/\cosh^2 x dx$  for  $d \geq 2$ , we can derive the leading finite-temperature / sample thickness corrections in the phonon approximation,  $\epsilon_B(q) \approx \hbar c_1 q$  ( $x = \beta \hbar c_1 q/2$ ),

$$\begin{aligned} \text{p.b.c. : } \quad \overline{\Delta n_\Delta(T)} &= -\Delta \frac{n_0}{4\hbar} \int \frac{d^d q}{(2\pi)^d} \left\{ \frac{n_0 \hbar^2 q^2 V(q)}{m \epsilon_B(q)^3} \left( \coth \frac{\beta \epsilon_B(q)}{2} - 1 \right) \right. \\ &\quad \left. - \frac{\beta/2}{\sinh^2[\beta \epsilon_B(q)/2]} \left[ 1 + \left( \frac{\hbar^2 q^2}{2m \epsilon_B(q)} \right)^2 \right] \right\} \quad (6.12) \\ &\approx \frac{\Delta n_0 (k_B T)^{d-1}}{4\pi^{d/2} \Gamma(d/2) \hbar^{d+1} c_1^d} \left[ \delta_{d,2} + \frac{d-2}{d-1} \int_0^\infty \frac{x^{d-1}}{\sinh^2 x} dx + \frac{(k_B T)^2}{m^2 c_1^4} \int_0^\infty \frac{x^{d+1}}{\sinh^2 x} dx \right], \end{aligned}$$

$$\begin{aligned} \text{o.b.c. : } \quad \overline{\Delta n_\Delta(T)} &= \Delta \frac{n_0}{4\hbar} \int \frac{d^d q}{(2\pi)^d} \left\{ \frac{n_0 \hbar^2 q^2 V(q)}{m \epsilon_B(q)^3} \left( 1 - \tanh \frac{\beta \epsilon_B(q)}{2} \right) \right. \\ &\quad \left. - \frac{\beta/2}{\cosh^2[\beta \epsilon_B(q)/2]} \left[ 1 + \left( \frac{\hbar^2 q^2}{2m \epsilon_B(q)} \right)^2 \right] \right\} \quad (6.13) \\ &\approx -\frac{\Delta n_0 (k_B T)^{d-1}}{4\pi^{d/2} \Gamma(d/2) \hbar^{d+1} c_1^d} \left[ \frac{d-2}{d-1} \int_0^\infty \frac{x^{d-1}}{\cosh^2 x} dx + \frac{(k_B T)^2}{m^2 c_1^4} \int_0^\infty \frac{x^{d+1}}{\cosh^2 x} dx \right]; \end{aligned}$$

with Eqs. (D15) and (D16) of App. D 2 we finally find in two dimensions

$$d = 2 : \quad \text{p.b.c. : } \quad \overline{\Delta n_\Delta(T)} = \frac{\Delta n_0 k_B T}{4\pi \hbar^3 c_1^2} + \mathcal{O}(T^3), \quad (6.14)$$

$$\text{o.b.c. : } \quad \overline{\Delta n_\Delta(T)} = -\mathcal{O}(T^3), \quad (6.15)$$

while for  $d > 2$

$$\text{p.b.c. : } \quad \overline{\Delta n_\Delta(T)} = \frac{(d-2)\Gamma(d-1)\zeta(d-1)}{2^d \pi^{d/2} \Gamma(d/2)} \frac{\Delta n_0 (k_B T)^{d-1}}{\hbar^{d+1} c_1^d}, \quad (6.16)$$

$$\text{o.b.c. : } \quad \overline{\Delta n_\Delta(T)} = -\left(1 - \frac{1}{2^{d-2}}\right) \frac{(d-2)\Gamma(d-1)\zeta(d-1)}{2^d \pi^{d/2} \Gamma(d/2)} \frac{\Delta n_0 (k_B T)^{d-1}}{\hbar^{d+1} c_1^d}, \quad (6.17)$$

e.g., in three dimensions,

$$d = 3 : \quad \text{p.b.c. : } \quad \overline{\Delta n_\Delta(T)} = \Delta n_0 (k_B T)^2 / 24 \hbar^4 c_1^3, \quad (6.18)$$

$$\text{o.b.c. : } \quad \overline{\Delta n_\Delta(T)} = -\Delta n_0 (k_B T)^2 / 48 \hbar^4 c_1^3. \quad (6.19)$$

As expected, the sign of the finite-temperature / size corrections is opposite for periodic and open boundary conditions, respectively, again reflecting the fact that the enhanced thermal wandering near the surfaces increases the tendency for entanglement of the (world) lines in the case of open boundary conditions. In  $d = 2$  the leading correction  $\propto k_B T$  vanishes for open boundary conditions.

With somewhat more effort, the disorder renormalization of the normal-fluid density / tilt modulus may be obtained evaluating (4.12) and (5.13), respectively; as the disorder correlator is isotropic, one finds



$$\text{p.b.c. : } \overline{\rho_{n\Delta}} = \frac{n_0\beta^2\hbar^4}{8dm} \int \frac{d^d q}{(2\pi)^d} \frac{\Delta q^4}{\hbar\epsilon_B(q)} \frac{\cosh[\beta\epsilon_B(q)/2]}{\sinh^3[\beta\epsilon_B(q)/2]}, \quad (6.20)$$

$$\text{o.b.c. : } \overline{\rho_{n\Delta}} = -\frac{n_0\beta^2\hbar^4}{8dm} \int \frac{d^d q}{(2\pi)^d} \frac{\Delta q^4}{\hbar\epsilon_B(q)} \frac{1}{\sinh[\beta\epsilon_B(q)]}; \quad (6.21)$$

note that these results for periodic and open boundary conditions are not simply related, because the normal–fluid density arises from the next–to–leading order in a long–wavelength expansion. However, in the limit  $\beta\hbar \rightarrow \infty$  one has

$$\overline{\rho_{n\Delta}(T=0)} = 0, \quad (6.22)$$

implying that mere point disorder cannot alter the tilt modulus of a flux liquid in thick samples; see also App. A. For finite “temperatures”, the leading corrections become for periodic and open boundary conditions, respectively [using  $\int_0^\infty x^{d+2}(\cosh x/\sinh^3 x)dx = [(d+2)/2] \int_0^\infty x^{d+1}/\sinh^2 x$  and Eq. (D15)]

$$\text{p.b.c. : } \overline{\rho_{n\Delta}(T)} = \frac{\Gamma(d+3)\zeta(d+1)}{2^{d+1}\pi^{d/2}\Gamma(1+d/2)} \frac{\Delta n_0(k_B T)^{d+1}}{m\hbar^{d+1}c_1^{d+4}}, \quad (6.23)$$

$$\text{o.b.c. : } \overline{\rho_{n\Delta}(T)} = -\left(1 - \frac{1}{2^{d+3}}\right) \frac{\Gamma(d+3)\zeta(d+3)}{2^{d+2}\pi^{d/2}\Gamma(1+d/2)} \frac{\Delta n_0(k_B T)^{d+1}}{m\hbar^{d+1}c_1^{d+4}}, \quad (6.24)$$

which in two dimensions becomes

$$d=2 : \quad \text{p.b.c. : } \overline{\rho_{n\Delta}(T)} = 3\zeta(3)\Delta n_0(k_B T)^3/\pi m\hbar^3 c_1^6, \quad (6.25)$$

$$\text{o.b.c. : } \overline{\rho_{n\Delta}(T)} = -93\zeta(5)\Delta n_0(k_B T)^3/64\pi m\hbar^3 c_1^6, \quad (6.26)$$

while in three dimensions

$$d=3 : \quad \text{p.b.c. : } \overline{\rho_{n\Delta}(T)} = \pi^2\Delta n_0(k_B T)^4/9m\hbar^4 c_1^7, \quad (6.27)$$

$$\text{o.b.c. : } \overline{\rho_{n\Delta}(T)} = -\pi^4\Delta n_0(k_B T)^4/192m\hbar^4 c_1^7. \quad (6.28)$$

Again, these finite–temperature / sample thickness corrections display opposite signs for the two different boundary conditions.

## B. Correlated random disorder – nearly isotropic splay

For magnetic flux lines in high–temperature superconductors, a deliberate introduction of splayed columnar defects has been suggested as a means to enhance flux pinning [44,45]. In the flux line notation, the disorder correlator for such randomly tilted columnar defects, each described by a trajectory  $\mathbf{r}_i(z) = \mathbf{R}_i + \mathbf{v}_i z$ , with a Gaussian distribution of the tilts  $\mathbf{v}_i$ ,  $P[\mathbf{v}_i] \propto \prod_i e^{-v_i^2/2v_D^2}$ , reads in Fourier space [44,46]

$$\Delta(\mathbf{q}, q_z) = \frac{\Delta_1}{\sqrt{2\pi}v_D q} e^{-q_z^2/2v_D^2 q^2}. \quad (6.29)$$

Upon taking the limit  $v_D \rightarrow \infty$ , with  $\Delta_1/v_D$  held fixed, one has  $\Delta(\mathbf{q}, q_z) \propto 1/q$ , i.e., an “almost” isotropic situation; as this correlator was obtained by starting from a distribution

centered around the  $z$  axis, which thus constitutes a preferred direction, this limit does not quite lead to the “truly” isotropic limit  $\Delta(\mathbf{q}, q_z) \propto 1/(\mathbf{q}^2 + q_z^2)^{-1/2}$  [compare the two-dimensional correlator (4.41)]. The additional  $q_z$  dependence here is not expected to affect the physical implications drastically, but would render the evaluation of the depletion and tilt modulus considerably more cumbersome.

We thus study the simpler case of “nearly isotropic” splay, namely

$$\Delta(\mathbf{q}, \omega_m; \mathbf{q}', \omega'_m) = \frac{\Delta}{q} (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \beta \hbar \delta_{m, -m'} . \quad (6.30)$$

With this correlator, we can immediately take over the results for point defects from the previous subsection, because none of the Matsubara frequency sums is changed; we simply have to divide  $\Delta$  by  $q$  everywhere. Hence

$$\text{p.b.c. : } \overline{S_\Delta(\mathbf{q})} = \frac{\Delta}{\hbar q \epsilon_B(q)} \left( \frac{n_0 \hbar^2 q^2}{2m \epsilon_B(q)} \right)^2 \coth \frac{\beta \epsilon_B(q)}{2} \left( 1 + \frac{\beta \epsilon_B(q)}{\sinh[\beta \epsilon_B(q)]} \right) , \quad (6.31)$$

$$\overline{C_{ij\Delta}(\mathbf{q})} = -\frac{\Delta}{\hbar q \epsilon_B(q)} \left( \frac{n_0 \hbar q}{2} \right)^2 \coth \frac{\beta \epsilon_B(q)}{2} \left( 1 - \frac{\beta \epsilon_B(q)}{\sinh[\beta \epsilon_B(q)]} \right) P_{ij}^L(\mathbf{q}) \quad (6.32)$$

for periodic boundary conditions, while

$$\text{o.b.c. : } \overline{S_\Delta(\mathbf{q})} = \frac{\Delta}{\hbar q \epsilon_B(q)} \left( \frac{n_0 \hbar^2 q^2}{2m \epsilon_B(q)} \right)^2 \tanh \frac{\beta \epsilon_B(q)}{2} \left( 1 - \frac{\beta \epsilon_B(q)}{\sinh[\beta \epsilon_B(q)]} \right) , \quad (6.33)$$

$$\overline{C_{ij\Delta}(\mathbf{q})} = -\frac{\Delta}{\hbar q \epsilon_B(q)} \left( \frac{n_0 \hbar q}{2} \right)^2 \tanh \frac{\beta \epsilon_B(q)}{2} \left( 1 + \frac{\beta \epsilon_B(q)}{\sinh[\beta \epsilon_B(q)]} \right) P_{ij}^L(\mathbf{q}) \quad (6.34)$$

for open boundary conditions, evaluated at  $\tau = \beta \hbar / 2$ .

In the same manner, dividing the integrands of Eqs. (6.6) and (6.7) by  $q$  yields the defect contribution to the depletion

$$\begin{aligned} \text{p.b.c. : } \overline{n_\Delta} &= \frac{n_0}{4} \int \frac{d^d q}{(2\pi)^d} \frac{\Delta}{\hbar q \epsilon_B(q)} \coth \frac{\beta \epsilon_B(q)}{2} \times \\ &\quad \times \left[ \left( \frac{\hbar^2 q^2}{2m \epsilon_B(q)} \right)^2 \left( 1 + \frac{\beta \epsilon_B(q)}{\sinh[\beta \epsilon_B(q)]} \right) - \left( 1 - \frac{\beta \epsilon_B(q)}{\sinh[\beta \epsilon_B(q)]} \right) \right] , \quad (6.35) \end{aligned}$$

$$\begin{aligned} \text{o.b.c. : } \overline{n_\Delta} &= \frac{n_0}{4} \int \frac{d^d q}{(2\pi)^d} \frac{\Delta}{\hbar q \epsilon_B(q)} \tanh \frac{\beta \epsilon_B(q)}{2} \times \\ &\quad \times \left[ \left( \frac{\hbar^2 q^2}{2m \epsilon_B(q)} \right)^2 \left( 1 - \frac{\beta \epsilon_B(q)}{\sinh[\beta \epsilon_B(q)]} \right) - \left( 1 + \frac{\beta \epsilon_B(q)}{\sinh[\beta \epsilon_B(q)]} \right) \right] . \quad (6.36) \end{aligned}$$

In the limit  $\beta \hbar \rightarrow \infty$ , both expressions lead to

$$\overline{n_\Delta} = \frac{n_0}{4} \int \frac{d^d q}{(2\pi)^d} \frac{\Delta}{\hbar q \epsilon_B(q)} \left[ \left( \frac{\hbar^2 q^2}{2m \epsilon_B(q)} \right)^2 - 1 \right] = -\Delta \frac{n_0^2 \hbar}{4m} \int \frac{d^d q}{(2\pi)^d} \frac{qV(q)}{\epsilon_B(q)^3} . \quad (6.37)$$

For  $V(q) \approx V_0 = 4\pi a/m$  ( $x = \hbar q/4\sqrt{\pi a n_0}$ ) we thus find for the disorder contribution to  $n - \overline{n_0} = n - |\langle \psi \rangle|^2$ ,

$$\overline{n_\Delta} = -\Delta \frac{mn_0^{(d-1)/2} a^{(d-3)/2}}{2^{6-d} \pi^{3/2} \Gamma(d/2) \hbar^d} \int_0^\infty \frac{x^{d-3}}{(1+x^2)^{3/2}} dx, \quad (6.38)$$

which is *negative* and logarithmically *divergent* in two dimensions, suggesting that  $|\overline{\langle \psi \rangle}|^2$  becomes unbounded above! Evidently the flux lines are easily trapped and swept along by the disorder; thermal wandering presumably becomes superdiffusive, leading to the divergence of  $\overline{n_\Delta}$  (see Ref. [32] for a related case). The mapping onto nonrelativistic bosons is inadequate to describe completely this “superentangled” state of interacting lines. In three dimensions, the corrections are finite,

$$\overline{n_\Delta} = -\Delta mn_0 / 4\pi^2 \hbar^3, \quad (6.39)$$

and the wandering remains diffusive. Upon dividing the expressions of the previous subsection by  $q$ , we find for the finite-temperature / sample thickness corrections for  $d > 2$ ,

p.b.c. :

$$\overline{\Delta n_\Delta(T)} \approx \frac{\Delta n_0 (k_B T)^{d-2}}{8\pi^{d/2} \Gamma(d/2) \hbar^d c_1^{d-1}} \left[ \delta_{d,3} + \frac{d-3}{d-2} \int_0^\infty \frac{x^{d-2}}{\sinh^2 x} dx + \frac{(k_B T)^2}{m^2 c_1^4} \int_0^\infty \frac{x^d}{\sinh^2 x} dx \right], \quad (6.40)$$

o.b.c. :

$$\overline{\Delta n_\Delta(T)} \approx -\frac{\Delta n_0 (k_B T)^{d-2}}{8\pi^{d/2} \Gamma(d/2) \hbar^d c_1^{d-1}} \left[ \frac{d-3}{d-2} \int_0^\infty \frac{x^{d-2}}{\cosh^2 x} dx + \frac{(k_B T)^2}{m^2 c_1^4} \int_0^\infty \frac{x^d}{\cosh^2 x} dx \right], \quad (6.41)$$

which for  $d = 3$  yields

$$d = 3 : \quad \text{p.b.c.} : \quad \overline{\Delta n_\Delta(T)} = \frac{\Delta n_0 k_B T}{4\pi^2 \hbar^3 c_1^2} + \mathcal{O}(T^3), \quad (6.42)$$

$$\text{o.b.c.} : \quad \overline{\Delta n_\Delta(T)} = -\mathcal{O}(T^3), \quad (6.43)$$

while for  $d > 3$

$$\text{p.b.c.} : \quad \overline{\Delta n_\Delta(T)} = \frac{(d-3)\Gamma(d-2)\zeta(d-2)}{2^d \pi^{d/2} \Gamma(d/2)} \frac{\Delta n_0 (k_B T)^{d-2}}{\hbar^d c_1^{d-1}}, \quad (6.44)$$

$$\text{o.b.c.} : \quad \overline{\Delta n_\Delta(T)} = -\left(1 - \frac{1}{2^{d-3}}\right) \frac{(d-3)\Gamma(d-2)\zeta(d-2)}{2^d \pi^{d/2} \Gamma(d/2)} \frac{\Delta n_0 (k_B T)^{d-2}}{\hbar^d c_1^{d-1}}. \quad (6.45)$$

In the same manner, we obtain for the normal-fluid density,

$$\text{p.b.c.} : \quad \overline{\rho_{n\Delta}} = \frac{n_0 \beta^2 \hbar^4}{8dm} \int \frac{d^d q}{(2\pi)^d} \frac{\Delta q^3}{\hbar \epsilon_B(q)} \frac{\cosh[\beta \epsilon_B(q)/2]}{\sinh^3[\beta \epsilon_B(q)/2]}, \quad (6.46)$$

$$\text{o.b.c.} : \quad \overline{\rho_{n\Delta}} = -\frac{n_0 \beta^2 \hbar^4}{8dm} \int \frac{d^d q}{(2\pi)^d} \frac{\Delta q^3}{\hbar \epsilon_B(q)} \frac{1}{\sinh[\beta \epsilon_B(q)]}, \quad (6.47)$$

which approaches zero as  $\beta \hbar \rightarrow \infty$ , as is shown to be the case for any disorder correlator  $\Delta(\mathbf{q}, \omega_m)$  independent of  $\omega_m$  in App. A. For finite temperatures or sample thickness,

$$\text{p.b.c.} : \quad \overline{\rho_{n\Delta}(T)} = \frac{\Gamma(d+2)\zeta(d)}{2^{d+1} \pi^{d/2} \Gamma(1+d/2)} \frac{\Delta n_0 (k_B T)^d}{m \hbar^d c_1^{d+3}}, \quad (6.48)$$

$$\text{o.b.c.} : \quad \overline{\rho_{n\Delta}(T)} = -\left(1 - \frac{1}{2^{d+2}}\right) \frac{\Gamma(d+2)\zeta(d+2)}{2^{d+2} \pi^{d/2} \Gamma(1+d/2)} \frac{\Delta n_0 (k_B T)^d}{m \hbar^d c_1^{d+3}}; \quad (6.49)$$

in two dimensions this becomes

$$d = 2 : \quad \text{p.b.c.} : \quad \overline{\rho_{n\Delta}(T)} = \pi \Delta n_0 (k_B T)^2 / 8 m \hbar^2 c_1^5, \quad (6.50)$$

$$\text{o.b.c.} : \quad \overline{\rho_{n\Delta}(T)} = -\pi^3 \Delta n_0 (k_B T)^2 / 256 m \hbar^2 c_1^5, \quad (6.51)$$

and for  $d = 3$

$$d = 3 : \quad \text{p.b.c.} : \quad \overline{\rho_{n\Delta}(T)} = 2\zeta(3) \Delta n_0 (k_B T)^3 / \pi^2 m \hbar^3 c_1^6, \quad (6.52)$$

$$\text{o.b.c.} : \quad \overline{\rho_{n\Delta}(T)} = -31\zeta(5) \Delta n_0 (k_B T)^3 / 32\pi^2 m \hbar^3 c_1^6. \quad (6.53)$$

We see that *only* for  $d = 3$  and with open boundary conditions there exists a simple relation between the finite-size corrections to the disorder renormalization of the depletion and the normal-fluid density, namely

$$d = 3, \text{ o.b.c.} : \quad \overline{\rho_{n\Delta}(T)} = [31\zeta(5)/9\zeta(3)] m \overline{\Delta n_\Delta(T)}. \quad (6.54)$$

Finally, we remark that in the above calculations we have not taken into account the possibility of a *direct* coupling of the defects to the tangent field (i.e., the momentum or current density in the boson language), but have exclusively studied the effects of disorder which couples to the particle density, i.e., changes the chemical potential locally. The latter mechanism should remain the most important effect of the disorder even in the case of splayed correlated defects; however, it is conceivable that in addition a term of the form  $\mathbf{t}_D(\mathbf{r}, \tau) \cdot \mathbf{g}(\mathbf{r}, \tau)$ , see Eq. (2.49), with  $\mathbf{t}_D(\mathbf{r}, \tau)$  representing another quenched random variable, may have to be included at least in an “effective” theory for which one further step of some course-graining procedure has been carried out. We neglect this scenario here, as well as in Secs. VID and VIE below.

### C. Correlated disorder – parallel untilted extended defects

We now turn to the analog of static disorder for bosons, namely untilted extended defects in flux liquids. We can use the correlator (4.32), with the understanding that  $d_{\parallel} = 0$  describes point defects in the boson representation, i.e., linear (columnar) defects in the flux line picture, and similarly  $d_{\parallel} = 1$  represents defect lines / planes, respectively, etc. The results may then simply be expressed in terms of the transverse dimensionality  $d_{\perp} = d - d_{\parallel}$ .

The disorder contribution to the static structure factor with periodic [see Eq. (4.34)] and open boundary conditions [cf. (5.8)] read

$$\text{p.b.c.} : \quad \overline{S_{\Delta}(\mathbf{q})} = \Delta \left( \frac{n_0 \hbar^2 q_{\perp}^2}{m \epsilon_B (q_{\perp})^2} \right)^2 (2\pi)^{d_{\parallel}} \delta(\mathbf{q}_{\parallel}), \quad (6.55)$$

$$\text{o.b.c.} : \quad \overline{S_{\Delta}(\mathbf{q})} = \Delta \left( \frac{n_0 \hbar^2 q_{\perp}^2}{m \epsilon_B (q_{\perp})^2} \right)^2 \left( 1 - \frac{1}{\cosh[\beta \epsilon_B (q_{\perp})/2]} \right)^2 (2\pi)^{d_{\parallel}} \delta(\mathbf{q}_{\parallel}), \quad (6.56)$$

whereas the leading part of the current (tilt) correlators vanish in this case in the static limit.

The results for the depletion and the normal-fluid density in the limit  $\beta \hbar \rightarrow \infty$  have already been explicitly given in Secs. IV B and IV C, where we had also seen that for periodic boundary conditions the finite-temperature / sample thickness corrections vanish entirely,

$$\text{p.b.c. : } \overline{\Delta n_{\Delta}(T)} = 0, \quad \overline{\rho_{n_{ij}\Delta}(T)} = 0. \quad (6.57)$$

However, for open boundary conditions there are finite-size corrections, induced by the stronger thermal fluctuations in the vicinity of the sample surfaces. Eq. (5.14) yields (in the sample center,  $\tau = \beta\hbar/2$ )

$$\text{o.b.c. : } \overline{\Delta n_{\Delta}(T)} = -\Delta \frac{n_0}{2m^2} \int \frac{d^{d_{\perp}} q_{\perp}}{(2\pi)^{d_{\perp}}} \left( \frac{\hbar q_{\perp}}{\epsilon_{\text{B}}(q_{\perp})} \right)^4 \left( \frac{1}{\cosh[\beta\epsilon_{\text{B}}(q)/2]} - \frac{1}{2 \cosh^2[\beta\epsilon_{\text{B}}(q)/2]} \right), \quad (6.58)$$

which can be evaluated in the phonon approximation using Eqs. (D18) and (D16) of App. D 2, where also the function  $\xi(d)$  is defined,

$$\text{o.b.c. : } \overline{\Delta n_{\Delta}(T)} = -\frac{2\Gamma(d_{\perp})}{\pi^{d_{\perp}/2}\Gamma(d_{\perp}/2)} \left[ \xi(d_{\perp}) - 2^{-d_{\perp}} (1 - 2^{2-d_{\perp}}) \zeta(d_{\perp} - 1) \right] \frac{\Delta n_0 (k_{\text{B}}T)^{d_{\perp}}}{m^2 \hbar^{d_{\perp}} c_1^{d_{\perp}+4}}; \quad (6.59)$$

for the relevant transverse dimensionalities this reduces to [ $\xi(2) = G \approx 0.916$  denotes Catalan's constant]

$$d_{\perp} = 1 : \quad \overline{\Delta n_{\Delta}(T)} = -(\pi - 1) \Delta n_0 k_{\text{B}}T / 2\pi m^2 \hbar c_1^5, \quad (6.60)$$

$$d_{\perp} = 2 : \quad \overline{\Delta n_{\Delta}(T)} = -(4G - \ln 2) \Delta n_0 (k_{\text{B}}T)^2 / 2\pi m^2 \hbar^2 c_1^6, \quad (6.61)$$

$$d_{\perp} = 3 : \quad \overline{\Delta n_{\Delta}(T)} = -(3\pi - 1) \Delta n_0 (k_{\text{B}}T)^3 / 12m^2 \hbar^3 c_1^7. \quad (6.62)$$

The normal-fluid density along the direction of the extended defects remains unaffected by the disorder,

$$\text{o.b.c. : } \overline{\rho_{n_{\parallel\Delta}(T)}} = 0. \quad (6.63)$$

In the perpendicular directions, Eq. (5.15) leads to

$$\text{o.b.c. : } \overline{\rho_{n_{\perp\Delta}(T)}} = -\frac{\Delta n_0}{d_{\perp} m} \int \frac{d^{d_{\perp}} q_{\perp}}{(2\pi)^{d_{\perp}}} \left( \frac{\hbar q_{\perp}}{\epsilon_{\text{B}}(q_{\perp})} \right)^4 \times \left( \frac{1}{\cosh[\beta\epsilon_{\text{B}}(q_{\perp})/2]} + \frac{\beta\epsilon_{\text{B}}(q_{\perp})/2}{\sinh[\beta\epsilon_{\text{B}}(q_{\perp})/2]} - \frac{\beta\epsilon_{\text{B}}(q_{\perp})}{\sinh[\beta\epsilon_{\text{B}}(q_{\perp})]} \right), \quad (6.64)$$

which in the phonon approximation is readily evaluated using Eqs. (D17) and (D18),

$$\overline{\rho_{n_{\perp\Delta}(T)}} = -\frac{2\Gamma(d_{\perp})}{\pi^{d_{\perp}/2}\Gamma(1 + d_{\perp}/2)} \left[ \xi(d_{\perp}) + (1 - 2^{-d_{\perp}}) (1 - 2^{-(d_{\perp}+1)}) d_{\perp} \zeta(d_{\perp} + 1) \right] \times \frac{\Delta n_0 (k_{\text{B}}T)^{d_{\perp}}}{m \hbar^{d_{\perp}} c_1^{d_{\perp}+4}}. \quad (6.65)$$

Explicitly one gets

$$d_{\perp} = 1 : \quad \overline{\rho_{n_{\perp\Delta}}(T)} = -(\pi + 4)\Delta n_0 k_B T / 4m\hbar c_1^5 \\ = [\pi(\pi + 4)/2(\pi - 1)] m \overline{\Delta n_{\Delta}(T)} , \quad (6.66)$$

$$d_{\perp} = 2 : \quad \overline{\rho_{n_{\perp\Delta}}(T)} = -[16G + 21\zeta(3)] \Delta n_0 (k_B T)^2 / 8\pi m \hbar^2 c_1^6 \\ = [(16G + 21\zeta(3)) / (16G - 4 \ln 2)] m \overline{\Delta n_{\Delta}(T)} , \quad (6.67)$$

$$d_{\perp} = 3 : \quad \overline{\rho_{n_{\perp\Delta}}(T)} = -\pi(7\pi + 8)\Delta n_0 (k_B T)^3 / 48m\hbar^3 c_1^7 \\ = [\pi(7\pi + 8)/4(3\pi - 1)] m \overline{\Delta n_{\Delta}(T)} ; \quad (6.68)$$

thus the finite-temperature / size corrections to the disorder renormalization of the depletion and the normal-fluid density are proportional to each other for these parallel untilted extended defects, albeit with a complicated numerical factor.

#### D. Correlated disorder – parallel tilted extended defects

More generally, we can also discuss “moving” or tilted parallel extended defects, say, along the  $x$  direction; the corresponding disorder correlator then becomes

$$\Delta(\mathbf{r}, \tau; \mathbf{r}', \tau') = \Delta \delta(\mathbf{r}_{\perp} - \mathbf{r}'_{\perp} + v(\tau - \tau')\mathbf{e}_x) , \\ \Delta(\mathbf{q}, \omega_m) = \Delta (2\pi)^{d_{\parallel}} \delta(\mathbf{q}_{\parallel}) \beta \hbar \delta_{\omega_m, q_x v} , \quad (6.69)$$

generalizing Eq. (4.32). Recall that  $\mathbf{r}_{\perp}$  refers to directions perpendicular to a set of  $d_{\parallel}$ -dimensional defects;  $(d_{\parallel}, d_{\perp}) = (1, d - 1)$  means line defects, while  $(d_{\parallel}, d_{\perp}) = (2, d - 2)$  corresponds to planar disorder. For magnetic flux liquids, the tilt parameter is simply the tangent of the tilt angle,  $v = \tan \alpha$ , see also Eq. (A12) in App. A. We shall restrict ourselves for simplicity to thick samples in this and the following subsection. If one wanted to apply periodic boundary conditions, one would have to keep the transverse momenta discrete as well,  $q_{x,k} = 2\pi k / L_{\perp}$  and ensure that  $\beta \hbar v / L_{\perp} = m/k$  is a rational number.

In the limit  $\beta \hbar \rightarrow \infty$ , Eqs. (4.9) and (5.6) yield the static structure factor

$$\overline{S(\mathbf{q})} = \frac{n_0 \hbar^2 q^2}{2m\epsilon_B(q)} + \Delta \left( \frac{n_0 q_{\perp}^2 / m}{q_x^2 v^2 + \epsilon_B(q_{\perp})^2 / \hbar^2} \right)^2 (2\pi)^{d_{\parallel}} \delta(\mathbf{q}_{\parallel}) , \quad (6.70)$$

while the leading contribution to the static current correlation function [see, e.g., Eq. (5.7)] reads

$$\overline{C_{ij}(\mathbf{q})} = \left[ \frac{n_0 m \epsilon_B(q)}{2} - \Delta \left( \frac{n_0 q_{\perp} q_x v}{q_x^2 v^2 + \epsilon_B(q_{\perp})^2 / \hbar^2} \right)^2 (2\pi)^{d_{\parallel}} \delta(\mathbf{q}_{\parallel}) \right] P_{ij}^L(\mathbf{q}) . \quad (6.71)$$

For  $v \rightarrow 0$ , these results of course reduce to those of the previous subsection. In real time, the density and phase correlation functions display singularities when  $v \rightarrow c_1$ , i.e., when the defects move at the speed of sound of the superfluid.

The disorder contribution to the depletion (4.19) becomes

$$\overline{n_{\Delta}} = \Delta \frac{n_0}{\hbar^2} \int \frac{d^{d_{\perp}} q_{\perp}}{(2\pi)^{d_{\perp}}} \frac{(\hbar q_{\perp}^2 / 2m)^2 - q_x^2 v^2}{[\epsilon_B(q_{\perp})^2 / \hbar^2 + q_x^2 v^2]^2} ; \quad (6.72)$$

this can be easily seen to diverge for any nonzero tilt  $v$  in the cases  $d_{\perp} = 1$  and  $d_{\perp} = 2$ , which we again interpret as indicating that the world lines are “dragged” along by the tilted extended defects, leading to superdiffusive behavior. Only for  $v = 0$  does the depletion remain finite, see Eq. (6.59). For  $d_{\perp} = 3$  (point disorder in a three–dimensional superfluid moving at constant speed  $v$ ) one finds for delta–function interaction with strength  $V_0 = 4\pi a/m$ ,

$$d_{\perp} = 3 : \quad \overline{n_{\Delta}} = \frac{\Delta m^2 n_0^{1/2}}{8\pi^{3/2} \hbar^3 a^{1/2}} \frac{1}{(1 + m^2 v^2 / 4\pi n_0 a)^{1/2}} . \quad (6.73)$$

The disorder renormalization of the normal–fluid density and tilt modulus perpendicular to the extended defects can be inferred from Eq. (4.15), see also App. A,

$$\overline{\rho_{n_{\perp i} \Delta}} = \Delta \frac{n_0}{m} \int \frac{d^{d_{\perp}} q_{\perp}}{(2\pi)^{d_{\perp}}} q_{\perp}^2 q_{\perp i}^2 \frac{\epsilon_{\text{B}}(q_{\perp})^2 / \hbar^2 - 3q_x^2 v^2}{[\epsilon_{\text{B}}(q_{\perp})^2 / \hbar^2 + q_x^2 v^2]^3} , \quad (6.74)$$

while  $\overline{\rho_{n_{\parallel} \Delta}} = 0$ , of course. In the case  $d_{\perp} = 1$ , there is only one perpendicular direction, and

$$d_{\perp} = 1 : \quad \overline{\rho_{n_{\perp} \Delta}} = \frac{\Delta m^3}{16\pi^{3/2} \hbar n_0^{1/2} a^{3/2}} \frac{1 - m^2 v^2 / 2\pi n_0 a}{(1 + m^2 v^2 / 4\pi n_0 a)^{5/2}} ; \quad (6.75)$$

for  $d_{\perp} = 2$ , on the other hand, we have to distinguish between the direction along the tilt and perpendicular to it,

$$d_{\perp} = 2 : \quad \overline{\rho_{n_{\perp x} \Delta}} = \frac{\Delta m n_0}{\pi \hbar^2 v^2} \left[ \frac{1 + m^2 v^2 / 2\pi n_0 a}{(1 + m^2 v^2 / 4\pi n_0 a)^{3/2}} - 1 \right] , \quad (6.76)$$

$$\overline{\rho_{n_{\perp y} \Delta}} = \frac{\Delta m n_0}{\pi \hbar^2 v^2} \left[ 1 - \frac{1}{(1 + m^2 v^2 / 4\pi n_0 a)^{1/2}} \right] , \quad (6.77)$$

and finally, for  $d_{\perp} = 3$  one gets

$$d_{\perp} = 3 : \quad \overline{\rho_{n_{\perp x} \Delta}} = \frac{4\Delta n_0^2 a}{\hbar^3 v^3} \left[ \operatorname{arsinh} \frac{mv}{(4\pi n_0 a)^{1/2}} - \frac{mv / (4\pi n_0 a)^{1/2}}{(1 + m^2 v^2 / 4\pi n_0 a)^{1/2}} \right] , \quad (6.78)$$

$$\overline{\rho_{n_{\perp y} \Delta}} = \frac{2\Delta n_0^2 a}{\hbar^3 v^3} \left[ \frac{mv}{(4\pi n_0 a)^{1/2}} \left( 1 + \frac{m^2 v^2}{4\pi n_0 a} \right)^{1/2} - \operatorname{arsinh} \frac{mv}{(4\pi n_0 a)^{1/2}} \right] . \quad (6.79)$$

These results show that for general nonzero tilts quite complicated formulas arise; and it is only in the very special situation  $v = 0$  that simple relations of the form  $\overline{\rho_{n_{\perp} \Delta}} \propto \overline{n_{\Delta}}$  emerge.

It is interesting to compute the fraction of particles / lines  $\overline{n_v}$  that are actually “dragged” along by the defects,

$$mv \overline{n_v} = \frac{1}{\beta \hbar \Omega} \left| \langle g_x(\mathbf{q} = \mathbf{0}, \omega_m = 0) \rangle \right| . \quad (6.80)$$

With Eq. (4.11) we find

$$\overline{n_v} = \Delta \frac{n_0}{m^2} \int \frac{d^{d_{\perp}} q_{\perp}}{(2\pi)^{d_{\perp}}} \left( \frac{q_{\perp} q_x}{\epsilon_{\text{B}}(q_{\perp})^2 / \hbar^2 + q_x^2 v^2} \right)^2 . \quad (6.81)$$

For  $v \rightarrow 0$  this precisely coincides with the disorder–induced normal–fluid density [Eqs. (4.35),(4.37)]; for static disorder, the superfluid density is thus reduced by exactly the average fraction of particles that are pinned to the defects.

### E. Two families of symmetrically tilted extended defects

As a final step of generalization, we now introduce two families of extended defects, which are symmetrically “moving” (tilted) along the  $x$  direction [44]. The disorder correlator in this situation reads

$$\begin{aligned} \Delta(\mathbf{r}, \tau; \mathbf{r}', \tau') &= \frac{\Delta}{4} \left[ \delta(\mathbf{r}_\perp - \mathbf{r}'_\perp + v(\tau - \tau')\mathbf{e}_x) + \delta(\mathbf{r}_\perp - \mathbf{r}'_\perp - v(\tau - \tau')\mathbf{e}_x) \right. \\ &\quad \left. + \delta(\mathbf{r}_\perp - \mathbf{r}'_\perp + v(\tau + \tau')\mathbf{e}_x) + \delta(\mathbf{r}_\perp - \mathbf{r}'_\perp - v(\tau + \tau')\mathbf{e}_x) \right], \\ \Delta(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) &= \frac{\Delta}{4} (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') (2\pi)^{d_\parallel} \delta(\mathbf{q}_\parallel) \times \\ &\quad \times (\beta\hbar)^2 (\delta_{m, -m'} + \delta_{m, m'}) (\delta_{\omega_m, q_x v} + \delta_{\omega_m, -q_x v}). \end{aligned} \quad (6.82)$$

It is important to realize that translational invariance along the  $\tau$  direction is broken here, which originates in the fact that there is a strong correlation between defects “moving” in the positive and negative  $x$  directions; or, equivalently, of defects “moving” forward and backward in imaginary time. As a consequence, we have to resort to our most general formulas in Secs. IV and V, and Eq. (A11), for example, is not applicable; but again, we shall only discuss the limit  $\beta\hbar \rightarrow \infty$  here. Note that in the limit  $v \rightarrow 0$ , the results of Sec. VIC for untilted parallel defects are recovered.

Interestingly, with this correlator the static structure function, see Eq. (5.4), for example, becomes identical with the result (6.70) for a single family of tilted extended disorder. This must be the case, as in a fixed cross section at  $\tau$ , those two cases look the same. However, in the leading contribution to the current correlation function, the effects of the two oppositely “moving” defect families precisely cancel each other, leaving behind just the pure result (this is actually true for finite  $\beta\hbar$  as well, with any boundary condition applied). This implies that in the Green’s function, from which the depletion and the normal–fluid density are to be determined, we shall face a partial cancellation of the disorder influence, namely in those contributions stemming from the phase correlations, respectively.

Again because of cancellations, the disorder–induced depletion, obtained from Eq. (5.12), turns out to contain only the first term in the numerator of Eq. (6.72),

$$\overline{n_\Delta} = \Delta \frac{n_0}{4m^2} \int \frac{d^{d_\perp} q_\perp}{(2\pi)^{d_\perp}} \left( \frac{q_\perp^2}{\epsilon_B(q_\perp)^2/\hbar^2 + q_x^2 v^2} \right)^2; \quad (6.83)$$

i.e., the leading contribution (in a long–wavelength expansion), which had caused the divergences discussed in the previous subsection, actually has disappeared. Therefore the depletion remains finite even for nonzero tilt. The explicit results read

$$d_\perp = 1 : \quad \overline{n_\Delta} = \frac{\Delta m^2}{64\pi^{3/2} \hbar n_0^{1/2} a^{3/2}} \frac{1}{(1 + m^2 v^2 / 4\pi n_0 a)^{3/2}}, \quad (6.84)$$

$$d_\perp = 2 : \quad \overline{n_\Delta} = \frac{\Delta m^2}{16\pi^2 \hbar^2 a} \frac{1}{(1 + m^2 v^2 / 4\pi n_0 a)^{1/2}}, \quad (6.85)$$

$$d_\perp = 3 : \quad \overline{n_\Delta} = \frac{\Delta m n_0}{4\pi \hbar^3 v} \operatorname{arsinh} \frac{mv}{(4\pi n_0 a)^{1/2}}. \quad (6.86)$$



In a similar manner, the disorder renormalization of the normal–fluid density is reduced, and Eq. (5.13) leads to

$$\overline{\rho_{n \perp i \Delta}} = \Delta \frac{n_0}{m} \int \frac{d^{d_\perp} q_\perp}{(2\pi)^{d_\perp}} q_\perp^2 q_{\perp i}^2 \frac{\epsilon_B(q_\perp)^2/\hbar^2 - q_x^2 v^2}{[\epsilon_B(q_\perp)^2/\hbar^2 + q_x^2 v^2]^3}, \quad (6.87)$$

to be contrasted with Eq. (6.74). For  $d_\perp = 1$  this becomes

$$d_\perp = 1 : \quad \overline{\rho_{n \perp \Delta}} = \frac{\Delta m^3}{16\pi^{3/2} \hbar n_0^{1/2} a^{3/2}} \frac{1 - m^2 v^2/8\pi n_0 a}{(1 + m^2 v^2/4\pi n_0 a)^{5/2}} = 4m\overline{n_\Delta} \frac{1 - m^2 v^2/8\pi n_0 a}{1 + m^2 v^2/4\pi n_0 a}. \quad (6.88)$$

For  $d_\perp = 2$  we find

$$d_\perp = 2 : \quad \overline{\rho_{n \perp x \Delta}} = \frac{\Delta m^3}{8\pi^2 \hbar^2 a} \frac{1}{(1 + m^2 v^2/4\pi n_0 a)^{3/2}} = \frac{2m\overline{n_\Delta}}{1 + m^2 v^2/4\pi n_0 a}, \quad (6.89)$$

$$\overline{\rho_{n \perp y \Delta}} = \frac{\Delta m^3}{8\pi^2 \hbar^2 a} \frac{1}{(1 + m^2 v^2/4\pi n_0 a)^{1/2}} = 2m\overline{n_\Delta}; \quad (6.90)$$

and finally in three “transverse” dimensions,

$$d_\perp = 3 : \quad \overline{\rho_{n \perp x \Delta}} = \frac{\Delta n_0^2 a}{\hbar^3 v^3} \left[ \operatorname{arsinh} \frac{mv}{(4\pi n_0 a)^{1/2}} - \frac{mv}{(4\pi n_0 a)^{1/2}} \frac{1 - m^2 v^2/4\pi n_0 a}{(1 + m^2 v^2/4\pi n_0 a)^{1/2}} \right], \quad (6.91)$$

$$\overline{\rho_{n \perp y \Delta}} = \frac{\Delta n_0^2 a}{4\hbar^3 v^3} \left[ \frac{mv}{(4\pi n_0 a)^{1/2}} \left( 1 + \frac{m^2 v^2}{4\pi n_0 a} \right)^{1/2} - \left( 1 - \frac{m^2 v^2}{2\pi n_0 a} \right) \operatorname{arsinh} \frac{mv}{(4\pi n_0 a)^{1/2}} \right]. \quad (6.92)$$

## F. Experimental consequences for vortex physics

We conclude by translating the results of the previous subsections into the flux line language, specializing to  $d = 2$  dimensions, and discussing some of the experimental consequences of our results. As noted in Sec. II B, we approximate the inter–vortex repulsion by  $V(q) = V_0/(1 + \lambda^2 q^2)$ , with  $V_0 = \phi_0^2/4\pi$  and the magnetic flux quantum  $\phi_0 = hc/2e$ . As shown in Table I, the particle mass is to be replaced by the effective line tension  $\tilde{\epsilon}_1 = (m_\perp/m_z)\epsilon_0 \ln(\lambda/\xi)$ , where the energy scale is set by  $\epsilon_0 = (\phi_0/4\pi\lambda)^2$ ; we shall also need the scattering length

$$a = \tilde{\epsilon}_1 V_0/4\pi = \tilde{\epsilon}_1 (\phi_0/4\pi)^2, \quad (6.93)$$

and the characteristic velocity

$$c_1 = (n_0 V_0/\tilde{\epsilon}_1)^{1/2} = (n_0 \phi_0^2/4\pi \tilde{\epsilon}_1)^{1/2}. \quad (6.94)$$

With these substitutions, and the replacements  $\hbar \rightarrow k_B T$  and  $\beta\hbar \rightarrow L$ , the structure factor and other correlation functions are readily obtained from our results in the previous sections. In the following, we shall review the explicit consequences for the fraction of entangled lines

as well as the renormalization of the tilt modulus by disorder and through modification of the boundary conditions.

For a *pure system*, we found in Sec. III B that the mean “boson” order parameter squared  $n_0 = |\langle\psi\rangle|^2$  as function of temperature in the limit of thick samples  $L \rightarrow \infty$  is

$$n_0(T) = n \left[ 1 + a/(k_B T)^2 \right]^{-1} = n \left[ 1 + \tilde{\epsilon}_1 (\phi_0/4\pi k_B T)^2 \right]^{-1}. \quad (6.95)$$

$\langle\psi\rangle$  measures the degree of entanglement, which increases with increasing temperature. The vortex contribution to the tilt modulus [see Eq. (1.1)] is  $c_{44}^v{}^{-1} = (n_0 \tilde{\epsilon}_1)^{-1}$ . The leading finite-size correction for sample thicknesses  $L < \infty$  to the fraction of disentangled lines diverges logarithmically, if *periodic* boundary conditions are employed along the magnetic-field direction, implying that in fact the “boson” order parameter vanishes in this case. On the other hand, for *open* boundary conditions, which are more realistic for most samples, the reduction in the “boson” order parameter is finite, cf. Eq. (3.80),

$$\Delta n(L) = -(\ln 2) \tilde{\epsilon}_1 / 2\pi k_B T L. \quad (6.96)$$

To leading order in the sample thickness corrections, Eqs. (3.67) and (3.86) yield for the tilt modulus with periodic and open boundary conditions, respectively,

$$c_{44}^v{}^{-1}(L) = (n \tilde{\epsilon}_1)^{-1} \left[ 1 \mp 3\zeta(3) k_B T / 2\pi n \tilde{\epsilon}_1 c_1^4 L^3 \right] \begin{cases} \text{p.b.c.} \\ \text{o.b.c.} \end{cases}, \quad (6.97)$$

with  $\zeta(3) \approx 1.202$ . As already discussed, the different signs for the finite-size corrections reflect the fact that for open boundary conditions thermal flux line wandering is effectively enhanced near the sample surfaces, and therefore both entanglement and response to external tilt are facilitated; the firm restrictions of periodic boundary conditions, on the other hand, reduce the number of entangled lines, and stiffen the system towards external tilting. According to Eq. (1.2), any renormalization of the tilt modulus may also be viewed as a changing the mass anisotropy  $m_\perp/m_z < 1$ , with an increasing  $c_{44}^v{}^{-1}$  implying a smaller effective mass ratio and vice versa. Any enhancement of the normal-fluid density, and correspondingly a reduction of  $c_{44}^v{}^{-1}$ , either through finite-size effects as in Eq. (6.97) for periodic boundary conditions, or through the influence of (correlated) disorder, will therefore render the superconducting system effectively more three-dimensional.

Uncorrelated *point defects*, e.g., oxygen vacancies in the copper-oxide planes, *increase* the fraction of entangled lines (hence the minus sign) by [Eq. (6.10)]

$$\overline{n_\Delta} = -\Delta n_0 \tilde{\epsilon}_1 / 4\pi (k_B T)^3, \quad (6.98)$$

clearly as a result of *enhanced* wandering of each flux line searching for an optimal path through the sample while trying to accommodate most profitably with the uncorrelated pinning centers. In very thick samples, the tilt modulus, however, remains unaffected by this kind of isotropically distributed disorder. The leading finite-thickness corrections to the fraction of pinned vortices remarkably display *different* power-law thickness dependences for periodic and open boundary conditions, respectively, see Eqs. (6.14) and (6.15),

$$\overline{\Delta n_\Delta(L)} = \begin{cases} \Delta n_0 / 4\pi c_1^2 (k_B T)^2 L & \text{p.b.c.} \\ -9\zeta(3) \Delta n_0 / 32\pi \tilde{\epsilon}_1^2 c_1^6 L^3 & \text{o.b.c.} \end{cases}; \quad (6.99)$$

the corresponding expressions for the tilt modulus read

$$\overline{c_{44}^{v-1}(L)} = \begin{cases} (n\tilde{\epsilon}_1)^{-1} [1 - 3\zeta(3)\Delta/\pi\tilde{\epsilon}_1^2 c_1^6 L^3] & \text{p.b.c.} \\ (n\tilde{\epsilon}_1)^{-1} [1 + \mathcal{O}(1/L^3)] & \text{o.b.c.} \end{cases} . \quad (6.100)$$

For an ensemble of *randomly tilted linear defects*, i.e., nearly isotropic splay, we found that the disorder contribution to the fraction of entangled lines actually diverges [Eq. (6.38)], because the thermal flux line wandering becomes superdiffusive as the vortices are “convected” along with the tilted defects. However, in thick samples the tilt modulus is unaffected by this isotropic disorder, and only in a finite sample does one get the following tilt modulus renormalization,

$$\overline{c_{44}^{v-1}(L)} = \begin{cases} (n\tilde{\epsilon}_1)^{-1} [1 - \pi\Delta/8\tilde{\epsilon}_1^2 c_1^5 L^2] & \text{p.b.c.} \\ (n\tilde{\epsilon}_1)^{-1} [1 + \pi^3\Delta/256\tilde{\epsilon}_1^2 c_1^5 L^2] & \text{o.b.c.} \end{cases} . \quad (6.101)$$

The logarithmic divergence of the disorder-induced depletion  $\overline{n_\Delta}$  also occurs for *parallel columnar defects* ( $d_\perp = 2$ ), as long as the tilt angle  $\alpha$  remains nonzero. (For this case, the divergence can presumably be removed by redefining the time-like direction to account for the vortex “drift”.) In the spetial situation  $\alpha = 0$ , Eq. (4.30) becomes

$$\overline{n_\Delta} = \Delta\tilde{\epsilon}_1^2/16\pi^2 a(k_B T)^2 , \quad (6.102)$$

and the corresponding finite-size corrections are nonzero only for open boundary conditions and read, according to (6.61),

$$\overline{\Delta n_\Delta(L)} = -(4G - \ln 2)\Delta n_0/2\pi\tilde{\epsilon}_1^2 c_1^6 L^2 , \quad (6.103)$$

with  $G \approx 0.916$ . For a tilt in the  $x$  direction with angle  $\alpha > 0$ , the non-vanishing tilt modulus tensor components in a thick sample are [Eqs. (6.76), (6.77)]

$$\overline{c_{xx}^{v-1}} = \frac{1}{n\tilde{\epsilon}_1} \left[ 1 - \frac{\Delta}{\pi(k_B T)^2 (\tan \alpha)^2} \left( \frac{1 + \tilde{\epsilon}_1^2 (\tan \alpha)^2 / 2\pi n_0 a}{[1 + \tilde{\epsilon}_1^2 (\tan \alpha)^2 / 4\pi n_0 a]^{3/2}} - 1 \right) \right] , \quad (6.104)$$

$$\overline{c_{yy}^{v-1}} = \frac{1}{n\tilde{\epsilon}_1} \left[ 1 - \frac{\Delta}{\pi(k_B T)^2 (\tan \alpha)^2} \left( 1 - \frac{1}{[1 + \tilde{\epsilon}_1^2 (\tan \alpha)^2 / 4\pi n_0 a]^{1/2}} \right) \right] . \quad (6.105)$$

We remark again that this reduction of the tilt modulus by the correlated linear disorder can be reinterpreted as an increase of the mass anisotropy ratio  $m_\perp/m_z$ ; the vortices are thus effectively rendered more three-dimensional by their interaction with the columnar defects, as should be expected. This mechanism increases the decoupling field in the presence of correlated disorder, above which the vortices change from lines, as they have been treated throughout this paper), to point-like “pancakes”. For  $\alpha \rightarrow 0$ , both the expressions (6.104) and (6.105) reduce to

$$\overline{c_{44}^{v-1}} = (n\tilde{\epsilon}_1)^{-1} [1 - \Delta\tilde{\epsilon}_1^2/8\pi^2 n_0 a(k_B T)^2] , \quad (6.106)$$

which remains unchanged even for finite sample thickness in the case of periodic boundary conditions (toroidal geometry), while we find for open boundary conditions

$$\overline{c_{44}^{v-1}(L)} = (n\tilde{\epsilon}_1)^{-1} \left[ 1 - \Delta\tilde{\epsilon}_1^2/8\pi^2 n_0 a (k_B T)^2 + [16G + 21\zeta(3)]\Delta/8\pi\tilde{\epsilon}_1^2 c_1^6 L^2 \right] . \quad (6.107)$$

For the infinitely thick sample, Eq. (6.106) leads to an estimate for the Bose glass transition temperature from the high-temperature flux liquid phase, complementary to the estimate in Ref. [40]. We note that the average disorder potential for columnar defects is  $\overline{V_D} = -U_0 b^2/d^2$ , where  $U_0 \approx \phi_0^2/2(4\pi\lambda)^2$  denotes the defect potential well strength,  $b$  its width, and  $d$  the average distance between the columns [40]. Accordingly, the defect correlator becomes  $\Delta_0 = U_0^2 b^4/d^2$ . One important modification has to be taken into account near the localization transition (more precisely, for temperatures above the depinning temperature  $T_{dp}$ ), however, namely the effective thermal renormalization of both the disorder strength  $U(T)$  and the replacement of one factor of  $b^2$  by the effective transverse localization length  $l_\perp(T)$ , i.e.:

$$\Delta_0 \longrightarrow \Delta(T) = U(T)^2 l_\perp(T)^2 b^2/d^2 , \quad (6.108)$$

where

$$U(T) = U_0 (T^*/T)^2 \quad , \quad l_\perp(T) = d (T/T^*)^2 , \quad (6.109)$$

and the characteristic temperature is given by  $k_B T^* = b(\tilde{\epsilon}_1 U_0)^{1/2}$  [40]. Upon inserting these results,  $n_0 = B/\phi_0$ , and Eq. (6.93) into Eq. (6.106), and collecting terms, we find that

$$\overline{c_{44}^{v-1}(T)} = (n\tilde{\epsilon}_1)^{-1} \left[ 1 - (T_{BG}/T)^4 \right] . \quad (6.110)$$

So  $c_{44}^v$  diverges as expected at the Bose glass transition temperature [40],

$$T_{BG} = T^* [\phi_0/(4\pi\lambda)^2 B]^{1/4} , \quad (6.111)$$

in agreement with a similar analysis in Ref. [44]. Although one should not trust the exponent for the vanishing of  $c_{44}^{v-1}$  predicted by the approximate formula (6.110), this analysis suffices to estimate the transition temperature. Upon generalizing to tilted linear disorder, and utilizing Eqs. (6.104) and (6.105) we can locate the instabilities for tilt in the  $x$  and  $y$  directions, respectively, as a function of the tilt angle  $\alpha$ , or instead introducing the dimensionless parameter

$$u(\alpha) = \tilde{\epsilon}_1^2 (\tan \alpha)^2 / 4\pi n_0 a , \quad (6.112)$$

and find

$$\left[ \frac{T_x(\alpha)}{T_{BG}} \right]^4 = \frac{2}{u(\alpha)} \left( \frac{1 + 2u(\alpha)}{[1 + u(\alpha)]^{3/2}} - 1 \right) , \quad (6.113)$$

$$\left[ \frac{T_y(\alpha)}{T_{BG}} \right]^4 = \frac{2}{u(\alpha)} \left( 1 - \frac{1}{[1 + u(\alpha)]^{1/2}} \right) . \quad (6.114)$$

The temperatures  $T_x$  and  $T_y$ , where the instabilities occur, are plotted in Fig. 4 as a function of the parameter  $u$ . Notice that while always  $T_y(\alpha) > 0$ ,  $T_x(\alpha_c) = 0$  for  $u_c = (1 + \sqrt{5})/2$ , suggesting that the Bose glass remains stable towards tilt up to this critical tilt angle  $\alpha_c$ . For  $\alpha < \alpha_c$ , the system will thus not respond to transverse magnetic fields, i.e., display a *transverse Meissner effect* [40]. The curve  $T_x(\alpha)$  can therefore be viewed as describing

the phase boundary (localization transition line) in a phase diagram with the thermodynamic variables temperature and transverse magnetic field. However, within the Bogoliubov (Gaussian) approximation used here, we do not find the cusp in  $T_x(\alpha)$  that is predicted by the Bose glass scaling theory [40].

For the case of *two families of symmetrically tilted* (angles  $\pm\alpha$ ) *columnar defects*, the fraction of pinned lines remains finite because the leading logarithmic divergences for both directions precisely cancel; Eq. (6.85) yields

$$\overline{n_\Delta} = \Delta\tilde{\epsilon}_1^2/16\pi^2a(k_B T)^2 \left[1 + \tilde{\epsilon}_1^2(\tan\alpha)^2/4\pi n_0a\right]^{1/2} ; \quad (6.115)$$

and the tilt modulus components (for  $L \rightarrow \infty$ ) deriving from Eqs. (6.89) and (6.90) are

$$\overline{c_{xx}^{v-1}} = \frac{1}{n\tilde{\epsilon}_1} \left(1 - \frac{\Delta\tilde{\epsilon}_1^2}{8\pi^2n_0a(k_B T)^2} \frac{1}{[1 + \tilde{\epsilon}_1^2(\tan\alpha)^2/4\pi n_0a]^{3/2}}\right) , \quad (6.116)$$

$$\overline{c_{yy}^{v-1}} = \frac{1}{n\tilde{\epsilon}_1} \left(1 - \frac{\Delta\tilde{\epsilon}_1^2}{8\pi^2n_0a(k_B T)^2} \frac{1}{[1 + \tilde{\epsilon}_1^2(\tan\alpha)^2/4\pi n_0a]^{1/2}}\right) . \quad (6.117)$$

As before, we can investigate the instabilities of this system towards external tilt, and find

$$[T_x(\alpha)/T_{\text{BG}}]^4 = [1 + u(\alpha)]^{-3/2} , \quad (6.118)$$

$$[T_y(\alpha)/T_{\text{BG}}]^4 = [1 + u(\alpha)]^{-1/2} ; \quad (6.119)$$

note that both  $T_x(\alpha)$  are always  $T_y(\alpha)$  positive (see Fig. 5), and the Bose glass remains stable irrespective of the tilt angle  $\alpha$ .

Turning to *parallel defect planes* ( $d_\perp = 1$ ), oriented in the  $yz$  plane (i.e.,  $\alpha = 0$ ), the reduction in the boson order parameter (4.38) is

$$\overline{n_\Delta} = \Delta\tilde{\epsilon}_1^2/64\pi^{3/2}n_0^{1/2}a^{3/2}k_B T . \quad (6.120)$$

For any nonzero tilt angle, we again find a divergence, as in the case of linear defects. According to Eq. (6.60), the leading finite- $L$  corrections to (6.120) for  $\alpha = 0$  are in the case of open boundary conditions

$$\overline{\Delta n_\Delta(L)} = -(\pi - 1)\Delta n_0/2\pi\tilde{\epsilon}_1^2c_1^5L , \quad (6.121)$$

while no such finite-thickness contributions occur if periodic boundary conditions are applied. Tilting the defect planes in the  $x$  direction with angle  $\alpha$ , the components of the tilt modulus [see Eq. 6.75] read

$$\overline{c_{xx}^{v-1}} = \frac{1}{n\tilde{\epsilon}} \left[1 - \frac{\Delta\tilde{\epsilon}_1^2}{16(\pi n_0a)^{3/2}k_B T} \frac{1 - \tilde{\epsilon}_1^2(\tan\alpha)^2/2\pi n_0a}{[1 + \tilde{\epsilon}_1^2(\tan\alpha)^2/4\pi n_0a]^{5/2}}\right] , \quad (6.122)$$

$$\overline{c_{yy}^{v-1}} = (n\tilde{\epsilon})^{-1} , \quad (6.123)$$

i.e., the tilt modulus in the  $y$  direction transverse to both the tilt and the magnetic field remains unaffected by the disorder. When  $\alpha \rightarrow 0$ , Eq. (6.122) becomes

$$\overline{c_{xx}^{v-1}} = (n\tilde{\epsilon})^{-1} \left(1 - \Delta\tilde{\epsilon}_1^2/16(\pi n_0a)^{3/2}k_B T\right) ; \quad (6.124)$$

only for open boundary conditions will there be corrections for finite sample thickness, which we have worked out in Eq. (6.66) to be

$$\overline{c_{xx}^{v-1}(L)} = (n\tilde{\epsilon})^{-1} \left[ 1 - \Delta\tilde{\epsilon}_1^2/16(\pi n_0 a)^{3/2} k_B T + (\pi + 4)/4\tilde{\epsilon}_1^2 c_1^5 L \right]. \quad (6.125)$$

As for the linear defects, we can estimate a critical temperature from Eq. (6.124). The “bare” disorder correlator for planar defects is  $U_0^2 b^2/d$ , where  $b$  denotes the diameter of the defect potential, and  $d$  the average defect distance. The thermal renormalizations for  $T > T_{dp}$  now read [40]

$$\Delta_0 \longrightarrow \Delta(T) = U(T)^2 b^2/d, \quad (6.126)$$

$$U(T) = U_0(b/d)^{2/3}(T^*/T)^{2/3}, \quad l_\perp(T) = d(b/d)^{2/3}(T/T^*)^{4/3}, \quad (6.127)$$

implying that  $\Delta(T) = U_0^2 b^3/d^2$ , independent of  $T$ . This yields

$$\overline{c_{xx}^{v-1}(T)} = (n\tilde{\epsilon}_1)^{-1} \left[ 1 - \tilde{T}_{BG}/T \right], \quad (6.128)$$

and the instability occurs at the Bose glass transition temperature marking the onset of localization to the defect planes

$$\tilde{T}_{BG} = T^*(b/4d)^2(\phi_0/2\pi\lambda^2 B)^{3/2}. \quad (6.129)$$

As a function of tilt angle  $\alpha$ , or equivalently, the parameter  $u$  introduced in Eq. (6.112), we furthermore find

$$\frac{T_x}{\tilde{T}_{BG}} = \frac{1 - 2u(\alpha)}{[1 + u(\alpha)]^{5/2}}, \quad (6.130)$$

see Fig. 6. In this approximation, the critical angle beyond which the Bose glass phase of localized vortices becomes unstable towards tilt is given by  $u_c = 1/2$ .

Upon generalizing to an ensemble of *two families of symmetrically tilted planar defects* (in thick samples), the divergences present for each of the families with positive and negative tilt angle, respectively, again cancel, and the resulting average density of pinned flux lines (6.84) is

$$\overline{n_\Delta} = \Delta\tilde{\epsilon}_1^2/64\pi^{3/2}n_0^{1/2}a^{3/2}k_B T \left[ 1 + \tilde{\epsilon}_1^2(\tan\alpha)^2/4\pi n_0 a \right]^{3/2}, \quad (6.131)$$

and the tilt modulus [from Eq. (6.88)] along the tilt direction reads

$$\overline{c_{xx}^{v-1}} = \frac{1}{n\tilde{\epsilon}_1} \left( 1 - \frac{\Delta\tilde{\epsilon}_1^2}{16(\pi n_0 a)^{3/2} k_B T} \frac{1 - \tilde{\epsilon}_1^2(\tan\alpha)^2/8\pi n_0 a}{[1 + \tilde{\epsilon}_1^2(\tan\alpha)^2/4\pi n_0 a]^{5/2}} \right), \quad (6.132)$$

while of course  $\overline{c_{yy}^{v-1}} = (n\tilde{\epsilon}_1)^{-1}$ . The angle dependence of the instability in  $\overline{c_{xx}^{v-1}}$  is readily found to be

$$\frac{T_x}{\tilde{T}_{BG}} = \frac{1 - u(\alpha)/2}{[1 + u(\alpha)]^{5/2}}, \quad (6.133)$$

compare Eq. (6.130). As is shown in Fig. 6, in a system with two families of symmetrically tilted planar defects the stability region of the Bose glass in the phase diagram is enhanced, the critical angle now being determined by  $u_c = 2$ .

## VII. SUMMARY AND DISCUSSION

In this paper, we have investigated the influence of weak uncorrelated and correlated disorder on superfluid bosons as well as magnetic flux liquids in high- $T_c$  superconductors, exploiting the formal analogy of the statistical mechanics of directed lines in  $d+1$  dimensions with the path integral representation of the quantum mechanics of  $d$ -dimensional particles. We have specifically addressed the important issue of how different boundary conditions in the directed-polymer case modify the results in systems with finite thickness  $L$ .

For disordered boson superfluids, we have generalized the previous studies by Huang and Meng [17], and Giorgini, Pitaevskii, and Stringari [18] on the effect of point disorder on the depletion as well as the superfluid density to the case of correlated disorder, i.e., linear or planar defects. In Sec. IV, we have derived general results for the density, current, and vorticity correlation functions, as well as the condensate fraction and the normal-fluid density. Explicit formulas for the contributions by pointlike, linear, and planar disorder, as well as randomly positioned and oriented line defects, to the depletion and normal-fluid density in two and three dimensions are summarized in Sec. IV C. We have also found that these results for static disorder are independent of temperature (at least to first order in the defect correlator).

In the case of magnetic flux line liquids in high-temperature superconductors, we saw that different boundary conditions can actually change the effects of disorder qualitatively. Even in the absence of disorder, we found that a homogeneous condensate fraction persists at finite sample thicknesses if open boundary conditions are applied, contrary to the case of periodic (“bosonic”) boundary conditions, where long-range order is destroyed by “thermal” fluctuations in this  $(2+1)$ -dimensional system. We have physically related the qualitatively and quantitatively different results for periodic and open boundary conditions to the enhanced thermal wandering of flux lines near the sample boundaries in the latter case, see Fig. 2. In Sec. V, we have studied the situation for open boundary conditions in some detail, looking at precisely those quantities which are the analogs to those studied for the Bose superfluid, namely density and tilt correlation functions, the “boson order parameter”, which is measure of the fraction of entangled lines, and the tilt modulus. In Sec. VI, we have compared our findings in two and three dimensions for the two kinds of boundary conditions we explored, specializing to (in the flux line language) point defects, nearly isotropically splayed linear disorder, parallel untilted and tilted extended defects (lines and planes), as well as two families of symmetrically tilted extended defects. We have thus significantly generalized earlier work on weak point disorder in infinitely thick systems [51] in two important respects, namely by (i) studying correlated disorder as well, and (ii) taking carefully into account the role of the more realistic open boundary conditions.

In Sec. VI F, we provide explicit results for the average fraction of entangled flux lines and the tilt modulus, and elucidate some of their experimental consequences. E.g., we have commented on how the defect influence on the tilt modulus effected by anisotropic (correlated) disorder may be reinterpreted as a renormalization of the effective mass anisotropy, illustrating how columnar and planar defects render the system effectively more three-dimensional; this renormalization clearly increases the decoupling field, above which the vortices degenerate from three-dimensional objects (lines) to weakly coupled two-dimensional ones (pancakes). Furthermore we have discussed how the renormalized tilt modulus may be used to

estimate the location of the instability towards a localized phase from the high-temperature side of the phase diagram, and to study the stability of this Bose glass with respect to transverse magnetic fields (“transverse Meissner effect”).

Our entire investigation was based on the Gaussian approximation, and nonlinear effects have been neglected. Analysis of these nonlinearities, as well as disorder which couples directly to the tilt field (corresponding to a boson current) are interesting avenues for future investigation. We have restricted our study to local (in  $z$ ) and isotropic repulsive vortex interactions, which is a fair approximation for low magnetic fields. It would be interesting to extend our results to the high-field regime, perhaps along the lines suggested in the Introduction.

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## APPENDIX A: DERIVATION OF THE DISORDER EFFECTS ON THE TILT MODULUS USING AFFINE TRANSFORMATIONS

In this first Appendix, we provide yet another derivation of the disorder-induced renormalization of the tilt modulus of a flux line liquid, utilizing an affine transformation [44], which is applicable in the (thermodynamic) limit of thick samples  $L \rightarrow \infty$ , and for disordered systems in which translational invariance is statistically restored. Upon applying a uniform tilt (2.62), the transformed action in polar representation reads [see Eqs. (2.55),(2.56)]

$$S'[\pi', \Theta'] = \tilde{S}[\pi, \Theta] + \int_0^\infty dz \int d^d r \left[ i \frac{k_B T}{\tilde{\epsilon}_1} \mathbf{h} \cdot \nabla \Theta(\mathbf{r}, z) - \frac{h^2}{\tilde{\epsilon}_1^2} - \delta V_D(\mathbf{r}, z) \right] (n_0 + \pi(\mathbf{r}, z)) , \quad (\text{A1})$$

where  $\tilde{S}[\pi, \Theta]$  denotes the pure part (in the flux line notation) of the action (3.2). This can also be written as

$$S'[\pi', \Theta'] = \tilde{S}'[\pi', \Theta'] - \int dz' \int d^d r' \left[ \delta V_D(\mathbf{r}' - \mathbf{h}z'/\tilde{\epsilon}_1, z') + \frac{h^2}{2\tilde{\epsilon}_1} \right] (n_0 + \pi'(\mathbf{r}', z')) . \quad (\text{A2})$$

The partition function for the tilted system therefore becomes (we henceforth omit the prime)

$$\ln Z_{\text{gr}} = \ln \tilde{Z}_{\text{gr}} + \ln \left\langle \exp \left\{ \frac{1}{k_B T} \int dz \int d^d r \left[ \delta V_D(\mathbf{r} - \mathbf{h}z/\tilde{\epsilon}_1, z) + \frac{h^2}{2\tilde{\epsilon}_1} \right] (n_0 + \pi(\mathbf{r}, z)) \right\} \right\rangle_{\tilde{S}} , \quad (\text{A3})$$

where the thermodynamic average  $\langle \dots \rangle$  is to be performed using the pure action, and the vortex contribution to the tilt modulus [see Eq. (1.1)] can then be inferred from

$$c_{ij}^{v-1} = \frac{k_B T}{n^2 L \Omega} \left. \frac{\delta^2 \ln Z_{\text{gr}}}{\delta h_i \delta h_j} \right|_{\mathbf{h}=\mathbf{0}} . \quad (\text{A4})$$

In order to compute the thermodynamic and quenched disorder averages, we perform a double cumulant expansion, which yields

$$\overline{\ln Z_{\text{gr}}} = \ln \tilde{Z} + \frac{nL\Omega}{\tilde{\epsilon}_1 k_B T} h^2 + \frac{1}{2(k_B T)^2} \int \frac{d^d q}{(2\pi)^d} \int \frac{dq_z}{2\pi} \left[ \Delta(\mathbf{q}, q_z; -\mathbf{q}, -q_z)_{\mathbf{h}} + \left( \frac{h^2}{2\tilde{\epsilon}_1} \right)^2 \right] S(\mathbf{q}, q_z) ; \quad (\text{A5})$$

here,  $S(\mathbf{q}, q_z)$  denotes the density correlation function of the pure system (3.15). Assuming now that the quenched disorder average restores translational invariance statistically,

$$\Delta(\mathbf{q}, q_z; \mathbf{q}', q'_z) = \Delta(\mathbf{q}, q_z) \Omega \delta_{\mathbf{q}, -\mathbf{q}'} L \delta_{q_z, -q'_z} , \quad (\text{A6})$$

and expanding the disorder correlator to second order in the tilt, one obtains with Eq. (A4)

$$\overline{c_{ij}^{v-1}} = \frac{1}{n\tilde{\epsilon}_1} \left[ \delta_{ij} + \frac{1}{2n_0\tilde{\epsilon}_1 k_B T} \int \frac{d^d q}{(2\pi)^d} \int \frac{dq_z}{2\pi} q_i q_j S(\mathbf{q}, q_z) \frac{\partial^2}{\partial q_z^2} \Delta(\mathbf{q}, q_z) \right] \quad (\text{A7})$$

$$= \frac{1}{n\tilde{\epsilon}_1} \left[ \delta_{ij} + \frac{1}{2n_0\tilde{\epsilon}_1 k_B T} \int \frac{d^d q}{(2\pi)^d} \int \frac{dq_z}{2\pi} q_i q_j \Delta(\mathbf{q}, q_z) \frac{\partial^2}{\partial q_z^2} S(\mathbf{q}, q_z) \right]. \quad (\text{A8})$$

If the system is statistically isotropic (with respect to the  $d$  transverse dimensions) as well, Eq. (A8) reduces to [44]

$$\overline{c_{44}^{v-1}} = \frac{1}{n\tilde{\epsilon}_1} \left[ 1 + \frac{1}{2dn_0\tilde{\epsilon}_1 k_B T} \int \frac{d^d q}{(2\pi)^d} \int \frac{dq_z}{2\pi} q^2 \Delta(\mathbf{q}, q_z) \frac{\partial^2}{\partial q_z^2} S(\mathbf{q}, q_z) \right]. \quad (\text{A9})$$

Inserting

$$\frac{\partial^2}{\partial q_z^2} S(\mathbf{q}, q_z) = -\frac{2k_B T n_0 q^2}{\tilde{\epsilon}_1} \frac{\epsilon_B(q)^2 / (k_B T)^2 - 3q_z^2}{[\epsilon_B(q)^2 / (k_B T)^2 + q_z^2]^3}, \quad (\text{A10})$$

we finally arrive at

$$\overline{c_{ij}^{v-1}} = \frac{1}{n\tilde{\epsilon}_1} \left[ \delta_{ij} - \frac{1}{\tilde{\epsilon}_1^2} \int \frac{d^d q}{(2\pi)^d} \int \frac{dq_z}{2\pi} q_i q_j q^2 \frac{\epsilon_B(q)^2 / (k_B T)^2 - 3q_z^2}{[\epsilon_B(q)^2 / (k_B T)^2 + q_z^2]^3} \Delta(\mathbf{q}, q_z) \right]. \quad (\text{A11})$$

compare Eq. (4.15) in Sec. IV A.

Eq. (A7) immediately shows that uncorrelated point defects,  $\Delta(\mathbf{q}, q_z) = \Delta$ , and ‘‘almost isotropically’’ splayed columnar defects,  $\Delta(\mathbf{q}, q_z) = \Delta/q$  [44], have no influence on the tilt modulus in thick samples at all, which is true for any disorder whose correlator is independent of  $q_z$ . For parallel extended defects, tilted in the  $x$  direction,

$$\Delta(\mathbf{q}_{\parallel}, \mathbf{q}_{\perp}, q_z) = \Delta \Omega_{\parallel} \delta_{\mathbf{q}_{\parallel}, \mathbf{0}} L \delta_{q_z, q_x \tan \alpha}, \quad (\text{A12})$$

one sees that the disorder does not change the tilt modulus along the defect directions,

$$\overline{c_{\parallel}^v} = n\tilde{\epsilon}_1, \quad (\text{A13})$$

while in the perpendicular directions, compare Eq. (6.74),

$$\overline{c_{\perp i}^{v-1}} = \frac{1}{n\tilde{\epsilon}_1} \left[ 1 - \frac{\Delta}{\tilde{\epsilon}_1^2} \int \frac{d^{d_{\perp}} q_{\perp}}{(2\pi)^{d_{\perp}}} q_{\perp i}^2 q_{\perp}^2 \frac{\epsilon_B(q_{\perp})^2 / (k_B T)^2 - 3q_x^2 \tan^2 \alpha}{[\epsilon_B(q_{\perp})^2 / (k_B T)^2 + q_x^2 \tan^2 \alpha]^3} \right]. \quad (\text{A14})$$

As discussed in Sec. VI F, a divergence of the transverse tilt modulus in a certain direction,  $\overline{c_{\perp i}^{v-1}} = 0$ , marks a localization transition to a (generalized) Bose glass phase; Eq. (A14) can thus be employed to obtain a ‘‘high-temperature’’ estimate for the Bose glass transition temperature [44].

## APPENDIX B: PHASE-ONLY APPROXIMATION

Considering the two types of fluctuations in our system, described by the harmonic action (3.2), it may be tempting to integrate out the density fluctuations, which acquire a mass  $V_0$ , see Eqs. (4.1) and (3.8), and only retain the massless phase modes; certainly this should correctly describe the leading terms in a long-wavelength expansion. In this Appendix, we study this simple phase-only approximation (with periodic boundary conditions), and explain its deficiencies.

We thus define the effective action  $S_{\text{eff}}[\Theta]$  by

$$e^{-S_{\text{eff}}[\Theta]/\hbar} = \int \mathcal{D}[\pi] e^{-S_0[\pi, \Theta]/\hbar} . \quad (\text{B1})$$

The purely quadratic functional integral is readily evaluated, with the result

$$S_{\text{eff}}[\Theta] = \text{const.} + \frac{1}{\beta \hbar \Omega} \sum_{\mathbf{q}, \omega_m} \left\{ \frac{n_0 \hbar^2}{2m} \frac{\hbar^2 q^2}{\epsilon_B(q)^2} [\omega_m^2 + \epsilon_B(q)^2/\hbar^2] \Theta(-\mathbf{q}, -\omega_m) \Theta(\mathbf{q}, \omega_m) \right. \\ \left. - \frac{n_0}{2m} \frac{\hbar^2 q^2}{\epsilon_B(q)^2} [2\hbar \omega_m \Theta(-\mathbf{q}, -\omega_m) + \delta V_D(-\mathbf{q}, -\omega_m)] \delta V_D(\mathbf{q}, \omega_m) \right\} ; \quad (\text{B2})$$

shifting the  $\Theta$  field as in Eq. (4.3), one then finds for the phase correlation function precisely the result (4.4). However, as density fluctuations have now been neglected entirely, the Green's function is simply

$$\overline{G(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'})} \approx n_0 \overline{\langle \Theta(\mathbf{q}, \omega_m) \Theta(\mathbf{q}', \omega_{m'}) \rangle} , \quad (\text{B3})$$

and similarly the current correlation function is merely given by the first term in Eq. (3.21),

$$\overline{C_{ij}(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'})} = \frac{n_0 m \epsilon_B(q)^2 / \hbar}{\omega_m^2 + \epsilon_B(q)^2 / \hbar^2} P_{ij}^L(\mathbf{q}) (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \beta \hbar \delta_{m, -m'} \\ - \frac{n_0^2 q_i q_j \omega_m \omega_{m'}}{[\omega_m^2 + \epsilon_B(q)^2 / \hbar^2][\omega_{m'}^2 + \epsilon_B(q')^2 / \hbar^2]} \Delta(\mathbf{q}, \omega_m; \mathbf{q}', \omega_{m'}) , \quad (\text{B4})$$

where, in order to be consistent with our long-wavelength expansion, we should use  $\epsilon_B(q) \approx \hbar c_1 q$ . Thus all the nonlinear terms, i.e., the vorticity contributions (3.27) present in the “full” theory (4.1) have disappeared, and as the current correlation function is therefore purely longitudinal, see Eq. (3.28), the normal-fluid density will necessarily be zero at all temperatures in the phase-only approximation,

$$\overline{\rho_n(T)} = 0 . \quad (\text{B5})$$

Using Eq. (3.51), we can also compute the average depletion,

$$\bar{n} - n_0 = \frac{m c_1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q} \left( 1 + \frac{2}{e^{\beta \hbar c_1 q} - 1} \right) \\ + \frac{n_0}{\hbar^2 \Omega} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(\beta \hbar)^2} \sum_{m, m'} \frac{\omega_m \omega_{m'} \Delta(\mathbf{q}, \omega_m; -\mathbf{q}, \omega_{m'})}{(\omega_m^2 + c_1^2 q^2)(\omega_{m'}^2 + c_1^2 q^2)} . \quad (\text{B6})$$

For the pure system, this yields the at least qualitatively correct result that the zero-temperature depletion diverges logarithmically in one dimension; however, for  $d \geq 2$ ,  $n(T = 0)$  cannot be determined from Eq. (B6). On the other hand, the finite-temperature corrections  $\Delta n(T)$  turn out to be identical to the leading contributions (3.58) obtained for the “full” theory in the phonon approximation [i.e., neglect any terms of order  $q^2$  in the integrand of Eq. (4.18)]. In the same manner, the disorder contribution to (B6) yields the leading terms in a long-wavelength expansion for the depletion, whenever these do not vanish. E.g., for uncorrelated (“point”) disorder in space and imaginary time, one finds [see (6.6)]

$$\overline{n_{\Delta}(T)} = -\Delta \frac{n_0}{4\hbar} \int \frac{d^d q}{(2\pi)^d} \frac{\coth[\beta\epsilon_B(q)/2]}{\epsilon_B(q)} \left( 1 - \frac{\beta\epsilon_B(q)}{\sinh[\beta\epsilon_B(q)]} \right), \quad (\text{B7})$$

and for the case of “almost isotropic splay” accordingly, cf. Eq. (6.35),

$$\overline{n_{\Delta}(T)} = -\Delta \frac{n_0}{4\hbar} \int \frac{d^d q}{(2\pi)^d} \frac{\coth[\beta\epsilon_B(q)/2]}{q\epsilon_B(q)} \left( 1 - \frac{\beta\epsilon_B(q)}{\sinh[\beta\epsilon_B(q)]} \right), \quad (\text{B8})$$

while for “tilted extended defects” the result is temperature-independent [see Eq. (6.72)],

$$\overline{n_{\Delta}} = -\Delta \frac{n_0}{\hbar^2} \int \frac{d^d q_{\perp}}{(2\pi)^{d_{\perp}}} \frac{q_x^2 v^2}{[\epsilon_B(q_{\perp})^2/\hbar^2 + q_x^2 v^2]^2}. \quad (\text{B9})$$

All of these expressions correctly represent the leading contributions of the results obtained in the “full” theory where the density fluctuations are taken into as well. However, at  $T = 0$  the integrals (B7) and (B8) become ill-defined when the phonon approximation is employed, as is required for the sake of consistency here. Note in addition that for the case of “untilted” correlated disorder (along the  $\tau$  direction), and for two families of “symmetrically tilted extended defects” the lowest-wavenumber contributions actually cancel, leaving the very poor result  $\overline{n_{\Delta}(T)} = 0$ .

It is actually this very mechanism which leads to a vanishing normal-fluid density in the phase-only approximation, namely a cancellation of the leading-order terms in the long-wavelength expansion. This can be made explicit by actually calculating the average momentum density in a system with walls moving relative to the superfluid; a quick computation shows that even with disorder

$$\overline{\langle \mathbf{g}(\mathbf{r}, \tau) \rangle} = 0 + \mathcal{O}[(\mathbf{q} \cdot \mathbf{v})^2], \quad (\text{B10})$$

quite in accord with the fact that within the framework of the phase-only approximation there is no vorticity in the system, and hence  $\overline{\rho_n(T)} = 0$  for all temperatures.

Summarizing our experiences, we have realized that the phase-only approximation (B2) is sufficient *only* for the evaluation of the longitudinal current correlations, and typically also for the leading finite-temperature corrections to the condensate fraction. It produces quite unsatisfactory results for the transverse current and vorticity correlations, however, which are crucially affected by their coupling to density fluctuations, and hence fails to detect any nonzero normal-fluid density at finite temperatures.

## APPENDIX C: OPEN BOUNDARY CONDITIONS: CORRELATION FUNCTIONS AND DISORDER CONTRIBUTIONS

In this Appendix, we derive the density and phase correlation functions in Gaussian approximation for a disordered flux line system described by an action of the form (3.2), with open boundary conditions. This calculation follows and generalizes Refs. [31] and [51], i.e., the disorder contributions will be treated to first order in the disorder correlator  $\Delta$ , and the open-boundary correlators will be expressed in terms of matrix products of the corresponding correlation functions obtained with periodic boundary conditions. Although the physical application we have in mind is that for magnetic flux liquids, we retain the boson notation here in order to facilitate comparison between the two cases.

Our starting point is Eq. (3.72), where the boundary conditions (3.70) have been explicitly taken into account through the insertion of a static constraint field  $\lambda(\mathbf{r})$ . Introducing the vector

$$u(\mathbf{q}, \omega_m) = \begin{pmatrix} 0 \\ \delta V_D(\mathbf{q}, \omega_m) \end{pmatrix} + i\hbar\lambda(\mathbf{q}) , \quad (\text{C1})$$

we can write the new effective harmonic action as

$$\tilde{S}_0[\lambda, \Upsilon] = \frac{1}{2\beta\hbar\Omega} \sum_{\mathbf{q}, \omega_m} \left[ \Upsilon^T(-\mathbf{q}, -\omega_m) \mathbf{A}(\mathbf{q}, \omega_m) \Upsilon(\mathbf{q}, \omega_m) - 2\Upsilon^T(-\mathbf{q}, -\omega_m) u(\mathbf{q}, \omega_m) \right] , \quad (\text{C2})$$

where  $\mathbf{A}(\mathbf{q}, \omega_m)$  is the matrix (3.8), obeying  $\mathbf{A}^T(-\mathbf{q}, -\omega_m) = \mathbf{A}(\mathbf{q}, \omega_m)$ . Transforming (with Jacobian 1) to new fields  $\tilde{\Upsilon}(\mathbf{q}, \omega_m) = \Upsilon(\mathbf{q}, \omega_m) - \mathbf{A}^{-1}(\mathbf{q}, \omega_m) u(\mathbf{q}, \omega_m)$  [see Eq. (4.3)] , the effective action  $\tilde{S}_0[\lambda, \tilde{\Upsilon}]$  is purely quadratic in  $\tilde{\Upsilon}$ , and hence the functional integral over these variables is readily performed, leading to

$$\begin{aligned} \langle \Upsilon(\mathbf{q}, \omega_m) \Upsilon^T(\mathbf{q}', \omega_{m'}) \rangle &= \hbar \mathbf{A}^{-1}(\mathbf{q}, \omega_m) \beta\hbar\Omega \delta_{\mathbf{q}, -\mathbf{q}'} \delta_{m, -m'} \\ &\quad + \mathbf{A}^{-1}(\mathbf{q}, \omega_m) \langle u(\mathbf{q}, \omega_m) u^T(\mathbf{q}', \omega_{m'}) \rangle_\lambda \mathbf{A}^{-1}(-\mathbf{q}', -\omega_{m'}) , \end{aligned} \quad (\text{C3})$$

with the inverse matrix  $\mathbf{A}^{-1}(\mathbf{q}, \omega_m)$  given in Eq. (3.9), and

$$\langle u(\mathbf{q}, \omega_m) u^T(\mathbf{q}', \omega_{m'}) \rangle_\lambda = \frac{\int \mathcal{D}[\lambda(\mathbf{q})] u(\mathbf{q}, \omega_m) u^T(\mathbf{q}', \omega_{m'}) e^{-S_u[\lambda]/\hbar}}{\int \mathcal{D}[\lambda(\mathbf{q})] e^{-S_u[\lambda]/\hbar}} , \quad (\text{C4})$$

where

$$S_u[\lambda] = -\frac{1}{2\beta\hbar\Omega} \sum_{\mathbf{q}, \omega_m} u^T(-\mathbf{q}, -\omega_m) \mathbf{A}^{-1}(\mathbf{q}, \omega_m) u(\mathbf{q}, \omega_m) . \quad (\text{C5})$$

At this point we caution the reader to note that Eq. (C4) does *not* simply reduce to the negative of the first line in Eq. (C3), for the functional integral is over the *static* field  $\lambda(\mathbf{q})$  only, and not over the dynamic variable  $u(\mathbf{q}, \omega_m)$ ; the fluctuations will only vanish at the special points  $\tau = 0$  and  $\tau = \beta\hbar$ .

We thus have to evaluate Eq. (C4) very carefully. Upon defining

$$\mathbf{A}^{-1}(\mathbf{q}) = \mathbf{A}^{-1}(\mathbf{q}, \tau = 0) = \frac{1}{\beta\hbar} \sum_m \mathbf{A}^{-1}(\mathbf{q}, \omega_m) , \quad (\text{C6})$$

the inverse of which is explicitly written down in Eq. (3.74), and furthermore

$$a(\mathbf{q}) = \frac{1}{\beta\hbar} \sum_m \mathbf{A}^{-1}(\mathbf{q}, \omega_m) \begin{pmatrix} 0 \\ \delta V_D(\mathbf{q}, \omega_m) \end{pmatrix}, \quad (\text{C7})$$

and then again shifting (with Jacobian 1) the integration variable  $\tilde{\lambda}(\mathbf{q}) = \lambda(\mathbf{q}) - i\mathbf{A}(\mathbf{q})a(\mathbf{q})/\hbar$ , it becomes a straightforward task to perform the purely quadratic path integral over  $\lambda$ , with the result

$$\begin{aligned} \langle u(\mathbf{q}, \omega_m) u^T(\mathbf{q}', \omega_{m'}) \rangle_\lambda &= -\hbar \mathbf{A}(\mathbf{q}) \Omega \delta_{\mathbf{q}, -\mathbf{q}'} + \delta V_D(\mathbf{q}, \omega_m) \delta V_D(\mathbf{q}', \omega_{m'}) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad - \mathbf{A}(\mathbf{q}) \frac{1}{\beta\hbar} \sum_n \mathbf{A}^{-1}(\mathbf{q}, \omega_n) \delta V_D(\mathbf{q}, \omega_n) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \delta V_D(\mathbf{q}', \omega_{m'}) \\ &\quad - \delta V_D(\mathbf{q}, \omega_n) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\beta\hbar} \sum_{n'} \mathbf{A}^{-1}(-\mathbf{q}', -\omega_{n'}) \delta V_D(\mathbf{q}', \omega_{n'}) \mathbf{A}(\mathbf{q}') \\ &+ \mathbf{A}(\mathbf{q}) \frac{1}{\beta\hbar} \sum_n \mathbf{A}^{-1}(\mathbf{q}, \omega_n) \delta V_D(\mathbf{q}, \omega_n) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\beta\hbar} \sum_{n'} \mathbf{A}^{-1}(-\mathbf{q}', -\omega_{n'}) \delta V_D(\mathbf{q}', \omega_{n'}) \mathbf{A}(\mathbf{q}') . \end{aligned} \quad (\text{C8})$$

Finally performing the quenched-disorder average, and then collecting the various terms, we find for the matrix of the averaged density and phase correlation functions

$$\begin{aligned} \overline{\langle \Upsilon(\mathbf{q}, \omega_m) \Upsilon^T(\mathbf{q}', \omega_{m'}) \rangle} &= (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \hbar \mathbf{A}^{-1}(\mathbf{q}, \omega_m) \left[ \beta\hbar \delta_{m, -m'} - \mathbf{A}(\mathbf{q}) \mathbf{A}^{-1}(\mathbf{q}, -\omega_{m'}) \right] \\ &\quad + \mathbf{A}^{-1}(\mathbf{q}, \omega_m) \frac{1}{(\beta\hbar)^2} \sum_{n, n'} \left\{ \left[ \beta\hbar \delta_{m, n} - \mathbf{A}(\mathbf{q}) \mathbf{A}^{-1}(\mathbf{q}, \omega_n) \right] \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \times \right. \\ &\quad \left. \times \left[ \beta\hbar \delta_{m', n'} - \mathbf{A}(\mathbf{q}') \mathbf{A}^{-1}(\mathbf{q}', \omega_{n'}) \right]^T \Delta(\mathbf{q}, \omega_n; \mathbf{q}', \omega_{n'}) \right\} \mathbf{A}^{-1}(-\mathbf{q}', -\omega_{m'}) \\ &= (2\pi)^d \delta(\mathbf{q} + \mathbf{q}') \left[ \beta\hbar \delta_{m, -m'} - \mathbf{A}^{-1}(\mathbf{q}, \omega_m) \mathbf{A}(\mathbf{q}) \right] \hbar \mathbf{A}^{-1}(\mathbf{q}, -\omega_{m'}) \\ &\quad + \frac{1}{(\beta\hbar)^2} \sum_{n, n'} \left[ \beta\hbar \delta_{m, n} - \mathbf{A}^{-1}(\mathbf{q}, \omega_m) \mathbf{A}(\mathbf{q}) \right] \mathbf{A}^{-1}(\mathbf{q}, \omega_n) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \times \\ &\quad \times \left\{ \left[ \beta\hbar \delta_{m', n'} - \mathbf{A}^{-1}(\mathbf{q}', \omega_{m'}) \mathbf{A}(\mathbf{q}') \right] \mathbf{A}^{-1}(\mathbf{q}', \omega_{n'}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}^T \Delta(\mathbf{q}, \omega_n; \mathbf{q}', \omega_{n'}) . \end{aligned} \quad (\text{C9})$$

It can be easily checked that indeed the fluctuations vanish at the boundaries,

$$\begin{aligned} \overline{\langle \Upsilon(\mathbf{q}, \tau = 0) \Upsilon^T(\mathbf{q}', \tau = 0) \rangle} &= \overline{\langle \Upsilon(\mathbf{q}, \tau = \beta\hbar) \Upsilon^T(\mathbf{q}', \tau = \beta\hbar) \rangle} \\ &= \frac{1}{(\beta\hbar)^2} \sum_{m, m'} \overline{\langle \Upsilon(\mathbf{q}, \omega_m) \Upsilon^T(\mathbf{q}', \omega_{m'}) \rangle} = 0 , \end{aligned} \quad (\text{C10})$$

as required by Eqs. (3.70). Also, if all the second terms in the square brackets, which break translational invariance in  $\tau$ , are disregarded, we recover the two-point correlation functions (4.4)–(4.6) for periodic boundary conditions obtained in Sec. IV A.

## APPENDIX D: MATSUBARA FREQUENCY SUMS AND MOMENTUM INTEGRALS

For the reader's convenience, we finally provide a list of useful explicit formulas for the (bosonic) Matsubara frequency sums required in the above calculations, as well as some definite integrals needed for the momentum integrations.

### 1. Bosonic Matsubara frequency sums

The following fundamental formula for bosonic and fermionic Matsubara frequency sums ( $n = 0, \pm 1, \pm 2, \dots$ ) is readily obtained by performing an appropriate contour integration in the complex plane (for details see Ref. [1], p. 248 *ff.*)

$$\tau \geq 0 : \quad \frac{1}{\beta\hbar} \sum_n \frac{e^{i\omega_n\tau}}{i\omega_n - \epsilon/\hbar} = \mp \frac{e^{\tau\epsilon/\hbar}}{e^{\beta\epsilon} \mp 1} \text{ for } \begin{cases} \text{bosons with } \omega_n = 2n\pi/\beta\hbar, \\ \text{fermions with } \omega_n = (2n+1)\pi/\beta\hbar. \end{cases} \quad (\text{D1})$$

Consequently,

$$\frac{1}{\beta\hbar} \sum_n \frac{e^{i\omega_n\tau}}{\omega_n^2 + \epsilon^2/\hbar^2} = \frac{\hbar}{2\epsilon} \frac{e^{(\beta-\tau/\hbar)\epsilon} \pm e^{\tau\epsilon/\hbar}}{e^{\beta\epsilon} \mp 1} \longrightarrow \frac{\hbar}{2\epsilon} \begin{cases} \coth(\beta\epsilon/2) & (\text{bosons}) \\ \tanh(\beta\epsilon/2) & (\text{fermions}) \end{cases} \text{ for } \tau \rightarrow 0. \quad (\text{D2})$$

Now specializing to bosons, i.e., periodic boundary conditions in imaginary time and therefore Matsubara frequencies with even integers, we may write down the following sequence of formulas (for  $\tau \geq 0$ ):

$$\frac{1}{\beta\hbar} \sum_n \frac{e^{-i\omega_n\tau}}{i\omega_n + \epsilon/\hbar} = \frac{-1}{\beta\hbar} \sum_n \frac{e^{i\omega_n\tau}}{i\omega_n - \epsilon/\hbar} = \frac{e^{\tau\epsilon/\hbar}}{e^{\beta\epsilon} - 1}, \quad (\text{D3})$$

$$\frac{-1}{\beta\hbar} \sum_n \frac{e^{-i\omega_n\tau}}{i\omega_n - \epsilon/\hbar} = \frac{1}{\beta\hbar} \sum_n \frac{e^{i\omega_n\tau}}{i\omega_n + \epsilon/\hbar} = \frac{e^{(\beta-\tau/\hbar)\epsilon}}{e^{\beta\epsilon} - 1}, \quad (\text{D4})$$

$$\frac{1}{\beta\hbar} \sum_n \frac{e^{-i\omega_n\tau}}{\omega_n^2 + \epsilon^2/\hbar^2} = \frac{\hbar \cosh[(\beta\hbar - 2\tau)\epsilon/2\hbar]}{2\epsilon \sinh(\beta\epsilon/2)}, \quad (\text{D5})$$

$$\frac{1}{\beta\hbar} \sum_n \frac{i\omega_n e^{-i\omega_n\tau}}{\omega_n^2 + \epsilon^2/\hbar^2} = \frac{\sinh[(\beta\hbar - 2\tau)\epsilon/2\hbar]}{2 \sinh(\beta\epsilon/2)}. \quad (\text{D6})$$

One then immediately sees that ( $m, n = 0, \pm 1, \pm 2, \dots$ )

$$\frac{1}{(\beta\hbar)^2} \sum_{m,n} \frac{i(\omega_m - \omega_n) e^{-i(\omega_m + \omega_n)\tau}}{(\omega_m^2 + \epsilon^2/\hbar^2)(\omega_n^2 + \epsilon^2/\hbar^2)} = 0, \quad (\text{D7})$$

and hence

$$\frac{1}{(\beta\hbar)^2} \sum_{m,n} \frac{(\omega_m\omega_n + \epsilon^2/\hbar^2) e^{-i(\omega_m + \omega_n)\tau}}{(\omega_m^2 + \epsilon^2/\hbar^2)(\omega_n^2 + \epsilon^2/\hbar^2)} = \frac{e^{\beta\epsilon}}{(e^{\beta\epsilon} - 1)^2} = \frac{1}{4 [\sinh(\beta\epsilon/2)]^2}, \quad (\text{D8})$$

independent of  $\tau$ .

Differentiation of Eqs. (D5) and (D6) with respect to  $\epsilon$  furthermore yields

$$\frac{1}{\beta\hbar} \sum_n \frac{e^{-i\omega_n\tau}}{(\omega_n^2 + \epsilon^2/\hbar^2)^2} = \frac{\hbar^3}{4\epsilon^3 \sinh(\beta\epsilon/2)} \left\{ \cosh \left[ \frac{(\beta\hbar - 2\tau)\epsilon}{2\hbar} \right] - \frac{\tau\epsilon}{\hbar} \sinh \left[ \frac{(\beta\hbar - 2\tau)\epsilon}{2\hbar} \right] + \frac{\beta\epsilon}{2} \frac{\cosh(\tau\epsilon/\hbar)}{\sinh(\beta\epsilon/2)} \right\}, \quad (\text{D9})$$

$$\frac{1}{\beta\hbar} \sum_n \frac{i\omega_n e^{-i\omega_n\tau}}{(\omega_n^2 + \epsilon^2/\hbar^2)^2} = \frac{\hbar}{4\epsilon \sinh(\beta\epsilon/2)} \left\{ \tau \cosh \left[ \frac{(\beta\hbar - 2\tau)\epsilon}{2\hbar} \right] - \frac{\beta\hbar}{2} \frac{\sinh(\tau\epsilon/\hbar)}{\sinh(\beta\epsilon/2)} \right\}, \quad (\text{D10})$$

and combining these with Eq. (D5) gives

$$\frac{1}{\beta\hbar} \sum_n \frac{\omega_n^2 e^{-i\omega_n\tau}}{(\omega_n^2 + \epsilon^2/\hbar^2)^2} = \frac{\hbar}{4\epsilon \sinh(\beta\epsilon/2)} \left\{ \cosh \left[ \frac{(\beta\hbar - 2\tau)\epsilon}{2\hbar} \right] + \frac{\tau\epsilon}{\hbar} \sinh \left[ \frac{(\beta\hbar - 2\tau)\epsilon}{2\hbar} \right] - \frac{\beta\epsilon}{2} \frac{\cosh(\tau\epsilon/\hbar)}{\sinh(\beta\epsilon/2)} \right\}, \quad (\text{D11})$$

$$\frac{1}{\beta\hbar} \sum_n \frac{(\omega_n^2 - \epsilon^2/\hbar^2) e^{-i\omega_n\tau}}{(\omega_n^2 + \epsilon^2/\hbar^2)^2} = \frac{1}{2 \sinh(\beta\epsilon/2)} \left\{ \tau \sinh \left[ \frac{(\beta\hbar - 2\tau)\epsilon}{2\hbar} \right] - \frac{\beta\hbar}{2} \frac{\cosh(\tau\epsilon/\hbar)}{\sinh(\beta\epsilon/2)} \right\}. \quad (\text{D12})$$

## 2. Momentum integrals

We finally list some useful formulas for some of the momentum integrals required for Bose systems in  $d$  dimensions, see Refs. [1,56]:

$$\int_0^\infty \frac{x^{d-2}}{e^x - 1} dx = \Gamma(d-1) \zeta(d-1), \quad (\text{D13})$$

$$\int_0^\infty \frac{x^{d-2}}{e^x + 1} dx = \begin{cases} \ln 2 & \text{for } d = 2, \\ (1 - 2^{-(d-2)}) \Gamma(d-1) \zeta(d-1) & \text{for } d > 2, \end{cases} \quad (\text{D14})$$

$$\int_0^\infty \frac{x^{d-1}}{\sinh^2 x} dx = 2^{-(d-2)} \Gamma(d) \zeta(d-1), \quad (\text{D15})$$

$$\int_0^\infty \frac{x^{d-1}}{\cosh^2 x} dx = 2^{-(d-2)} (1 - 2^{-(d-2)}) \Gamma(d) \zeta(d-1), \quad (\text{D16})$$

$$\int_0^\infty \frac{x^{d-1}}{\sinh x} dx = 2 (1 - 2^{-d}) \Gamma(d) \zeta(d), \quad (\text{D17})$$

$$\int_0^\infty \frac{x^{d-1}}{\cosh x} dx = 2 \Gamma(d) \xi(d). \quad (\text{D18})$$

Here,  $\zeta(n) = \sum_{k=1}^\infty 1/k^n$  and  $\xi(n) = \sum_{k=0}^\infty (-1)^k / (2k+1)^n$ . We note that the  $\zeta$  function at even arguments is related to Bernoulli's numbers,

$$\zeta(2k) = \frac{\pi^{2k} 2^{2k-1}}{(2k)!} B_k, \quad \zeta(0) = -\frac{1}{2}, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945} \dots, \quad (\text{D19})$$

while the  $\xi$  function at odd arguments is connected with Euler's numbers,

$$\xi(2k+1) = \frac{\pi^{2k+1}}{4^{k+1} (2k)!} E_k, \quad \xi(1) = \frac{\pi}{4}, \quad \xi(3) = \frac{\pi^3}{32} \dots, \quad (\text{D20})$$

and  $\zeta(3) \approx 1.202$ ,  $\xi(2) = G \approx 0.915965$  (Catalan's constant [56]).



## REFERENCES

- [1] A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, San Francisco, 1971).
- [2] P. Nozières and D. Pines, *The Theory of Quantum Liquids*, Vol. II (Addison-Wesley, Reading, 1990).
- [3] A. Griffin, *Excitations in a Bose-Condensed Liquid* (Cambridge University, Cambridge, 1993).
- [4] N.V. Popov, *Functional Integrals and Collective Excitations* (Cambridge University Press, New York, 1981).
- [5] J.W. Negele and J. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, New York, 1988).
- [6] D.M. Ceperley, *Rev. Mod. Phys.* **67** (1995) 279.
- [7] A. Griffin, *Phys. Rev. B* **30** (1984) 5057.
- [8] E.L. Pollock and D.M. Ceperley, *Phys. Rev. B* **36** (1987) 8343.
- [9] F. Dalfovo and S. Stringari, *Phys. Rev. B* **46** (1992) 13991.
- [10] M. Rasolt, M.J. Stephen, M.E. Fisher, and P.B. Weichman, *Phys. Rev. Lett.* **53** (1984) 798; P.B. Weichman, M. Rasolt, M.E. Fisher, and M.J. Stephen, *Phys. Rev. B* **33** (1986) 4632.
- [11] D.S. Fisher and P.C. Hohenberg, *Phys. Rev. B* **37** (1988) 4936.
- [12] G.C. Marques, *Phys. Rev. B* **44** (1991) 12485.
- [13] E.B. Kolomeisky and J.P. Straley, *Phys. Rev. B* **46** (1992) 11749; *Phys. Rev. B* **46** (1992) 13942; *Phys. Rev. B* **46** (1992) 13956.
- [14] C. Chang and R. Friedberg, *Phys. Rev. B* **51** (1995) 1117.
- [15] A.S. Alexandrov and W.H. Beere, *Phys. Rev.* **51** (1995) 5887.
- [16] C. Castellani, C. Di Castro, F. Pistolesi, and G.C. Stringari, Preprint cond-mat / 9604076 (1996).
- [17] K. Huang and H.-F. Meng, *Phys. Rev. Lett.* **69** (1992) 644.
- [18] S. Giorgini, L. Pitaevskii, and S. Stringari, *Phys. Rev. B* **49** (1994) 12938.
- [19] A.M.J. Schakel, *Phys. Lett.* **224** A (1997) 287.
- [20] M. Ma, B.I. Halperin, and P.A. Lee, *Phys. Rev. B* **34** (1986) 3136.
- [21] M.P.A. Fisher, P.B. Weichman, G. Grinstein, and D.S. Fisher, *Phys. Rev. B* **40** (1989) 546.
- [22] K.G. Singh and D.S. Rokhsar, *Phys. Rev. B* **46** (1992) 3002.
- [23] L. Zhang, *Phys. Rev. B* **47**, (1993) 14364; P. Nisamaneephong, L. Zhang, and M. Ma, *Phys. Rev. Lett.* **71** (1993) 3830.
- [24] M. Makivić, N. Trivedi, and S. Ullah, *Phys. Rev. Lett.* **71** (1993) 2307.
- [25] M. Wallin, E.S. Sørensen, S.M. Girvin, and A.P. Young, *Phys. Rev. B* **49** (1994) 12115.
- [26] N. Hatano, *J. Phys. Soc. Jpn.* **64** (1995) 1529.
- [27] F. Pázmándi, G. Zimányi, and R. Scalettar, *Phys. Rev. Lett.* **75** (1995) 1356.
- [28] G. Blatter, M.V. Feigel'man, V.B. Geshkenbein, A.I. Larkin, and V.M. Vinokur, *Rev. Mod. Phys.* **66** (1994) 1125.
- [29] *Phenomenology and Applications of High-Temperature Superconductors*, ed. K. Bedell *et al.* (Addison-Wesley, New York, 1991).
- [30] D.R. Nelson, *Phys. Rev. Lett.* **60** (1988) 1973; *J. Stat. Phys.* **57** (1989) 511.
- [31] D.R. Nelson and H.S. Seung, *Phys. Rev. B* **39** (1989) 9153.

- [32] R.D. Kamien, P. Le Doussal, and D.R. Nelson, *Phys. Rev. A* **45** (1992) 8727.
- [33] D.R. Nelson, in: *Observation, Prediction, and Simulation of Phase Transitions in Complex Fluids*, ed. M. Baus *et al.* (Kluwer Academic, Dordrecht, 1995) 293.
- [34] E. Frey, D.R. Nelson, and D.S. Fisher, *Phys. Rev. B* **49** (1994) 9723.
- [35] L. Balents and D.R. Nelson, *Phys. Rev. Lett.* **73** (1994) 2618; *Phys. Rev. B* **52** (1995) 12951; E. Frey and L. Balents, *Phys. Rev. B* **55** (1997) 1050.
- [36] M.P.A. Fisher and D.H. Lee, *Phys. Rev. B* **39** (1989) 2756.
- [37] A.I. Larkin, *Sov. Phys. JETP* **31** (1970) 784 [*Zh. Exsp. Teor. Fiz.* **58** (1970) 1466]; A.I. Larkin and Y.N. Ovchinnikov, *J. Low Temp. Phys.* **34** (1979) 409.
- [38] M.P.A. Fisher, *Phys. Rev. Lett.* **62** (1989) 1415; D.S. Fisher, M.P.A. Fisher, and D.A. Huse, *Phys. Rev. B* **43** (1991) 130.
- [39] I.F. Lyuksyutov, *Europhys. Lett.* **20** (1992) 273.
- [40] D.R. Nelson and V.M. Vinokur, *Phys. Rev. Lett.* **68** (1992) 2398; *Phys. Rev. B* **48** (1993) 13060.
- [41] A.I. Larkin and V.M. Vinokur, *Phys. Rev. Lett.* **75** (1995) 4666.
- [42] D.R. Nelson, in: *Phase Transitions and Relaxation in Systems with Competing Energy Scales*, ed. T. Riste and D. Sherrington (Kluwer Academic, Dordrecht, 1993) 95.
- [43] U.C. Täuber, H. Dai, D.R. Nelson, and C.M. Lieber, *Phys. Rev. Lett.* **74** (1995) 5132; U.C. Täuber and D.R. Nelson, *Phys. Rev. B* **52** (1995) 16106.
- [44] T. Hwa, P. Le Doussal, D.R. Nelson, and V.M. Vinokur, *Phys. Rev. Lett.* **71** (1993) 3545; P. Le Doussal, T. Hwa, D.R. Nelson, and V.M. Vinokur (unpublished).
- [45] P. Le Doussal and D.R. Nelson, *Physica C* **232** (1994) 69.
- [46] T.P. Devereaux, R.T. Scalettar, G.T. Zimanyi, K. Moon, and E. Loh, *Phys. Rev. Lett.* **75** (1995) 4768.
- [47] A.E. Koshelev, P. Le Doussal, and V.M. Vinokur, *Phys. Rev. B* **53** (1996) R8855.
- [48] M.C. Marchetti and V.M. Vinokur, *Phys. Rev. Lett.* **72** (1994) 3409; *Phys. Rev. B* **51** (1995) 16276.
- [49] A.I. Larkin, M.C. Marchetti, and V.M. Vinokur, *Phys. Rev. Lett.* **75** (1995) 2992.
- [50] N. Hatano and D.R. Nelson, *Phys. Rev. Lett.* **77** (1996) 570.
- [51] D.R. Nelson and P. Le Doussal, *Phys. Rev. B* **42** (1990) 10113.
- [52] D.R. Nelson, in: *Phenomenology and Applications of High-Temperature Superconductors*, ed. K. Bedell *et al.* (Addison-Wesley, New York, 1991) 187.
- [53] M.C. Marchetti and D.R. Nelson, *Phys. Rev. B* **47** (1993) 12214; *Phys. Rev. B* **52** (1995) 7720.
- [54] M.V. Feigel'man, V.B. Geshkenbein, and V.M. Vinokur, *JETP Lett.* **52** (1990) 546; M.V. Feigel'man, V.B. Geshkenbein, L.B. Ioffe, and A.I. Larkin, *Phys. Rev. B* **48** (1993) 16641.
- [55] V.B. Geshkenbein, private communication.
- [56] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products*, ed. A. Jeffrey (Academic Press, San Diego, 5th ed. 1994).

## TABLES

TABLE I. Superfluid boson – flux liquid analogy.

Superfluid bosons	$m$	$\hbar$	$\beta\hbar$	$V(r)$	$n$	$\mu$	$i\mathbf{v}$	$\rho_s$
Vortex liquid	$\tilde{\epsilon}_1$	$T$	$L$	$2\epsilon_0 K_0(r/\lambda)$	$B/\phi_0$	$(H - H_{c1})\phi_0/4\pi$	$\mathbf{h}/\tilde{\epsilon}_1$	$\rho^2 c_{44}^{-1}$

TABLE II. Varieties of disorder.

Vortex liquids in three dimensions	Superfluid Helium films
Point disorder (e.g., oxygen vacancies)	Space- and time-dependent random potential
Columnar disorder (e.g., screw dislocations, columnar damage tracks)	Amorphous substrate modelled by uncorrelated “point-like” random potential
Planar disorder (e.g., twin boundaries, grain boundaries)	“Smooth” or crystalline substrate with line disorder deposited microlithographically

## FIGURES

FIG. 1. Schematic representation of the Abrikosov flux lattice (a), a disentangled flux liquid (b), and an entangled flux liquid (c). In the boson picture, the directed lines map onto the imaginary–time particle world lines, and these phases correspond to a crystalline solid (a), a normal liquid (b), and a superfluid (c), respectively.

FIG. 2. Schematic plot of the time–averaged cross section of a thermally fluctuating flux line in a slab with free boundary conditions. The bulk part away from the boundaries has a width determined by the square of the bosonic “wave function”, which is a Gaussian for a free random walker. Near the top and bottom surfaces, thermal wandering is enhanced, reflected by a probability distribution determined just by the “wave function” (not its square).

FIG. 3. Lowest–energy solid phase contribution to the correlation function  $\tilde{G}(\mathbf{r}, z; \mathbf{r}', z')$ , which inserts a flux head and tail into a crystalline vortex array. Dashed lines represent a row of vortices slightly behind the plane of the page. In (a), a vacancy is created at “time”  $z$ , which then propagates and is destroyed at “time”  $z'$ . Interstitial propagation from  $z'$  to  $z$  is shown in (b). The energy of the “string” defect connecting the head to the tail increases linearly with the separation in both cases and leads to the exponential decay of  $\tilde{G}(\mathbf{r}, z; \mathbf{r}', z')$ .

FIG. 4. Critical temperatures  $T_x(\alpha)/T_{\text{BG}}$  (solid line) and  $T_y(\alpha)/T_{\text{BG}}$  (dashed) where the tilt moduli  $\overline{c_{xx}^v}^{-1}$  and  $\overline{c_{yy}^v}^{-1}$  in a system with parallel columnar defects tilted along the  $x$  direction vanish, respectively, as a function of the dimensionless parameter  $u = \tilde{\epsilon}_1^2 (\tan \alpha)^2 / 4\pi n_0 a$ . The curve  $T_x(\alpha)/T_{\text{BG}}$  may be viewed as an estimate for the phase boundary of the Bose glass, which remains stable towards external tilt as long as  $\alpha < \alpha_c$  (transverse Meissner effect).

FIG. 5. Tilt instability temperatures  $T_x(\alpha)/T_{\text{BG}}$  (thick solid line) and  $T_y(\alpha)/T_{\text{BG}}$  (dashed) for a system with two families of symmetrically tilted columnar defects. For comparison,  $T_x(\alpha)/T_{\text{BG}}$  (thin solid line) for a single family of tilted linear disorder is also depicted.

FIG. 6. Tilt instability temperature  $T_x(\alpha)/T_{\text{BG}}$  for both parallel planar defects uniformly tilted along the  $x$  direction (solid line) and two families of symmetrically tilted planes (dashed). The critical tilt angle marking the instability of the Bose glass phase of localized flux lines is shifted upwards in the symmetric–splay case.

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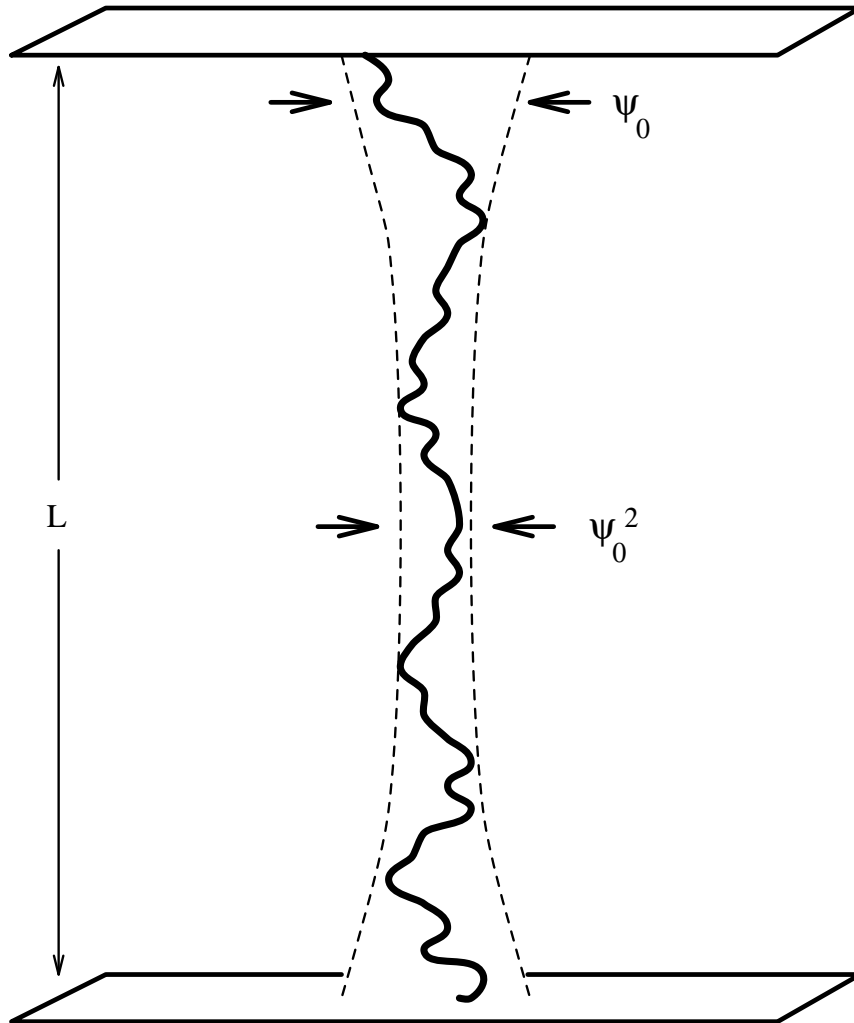


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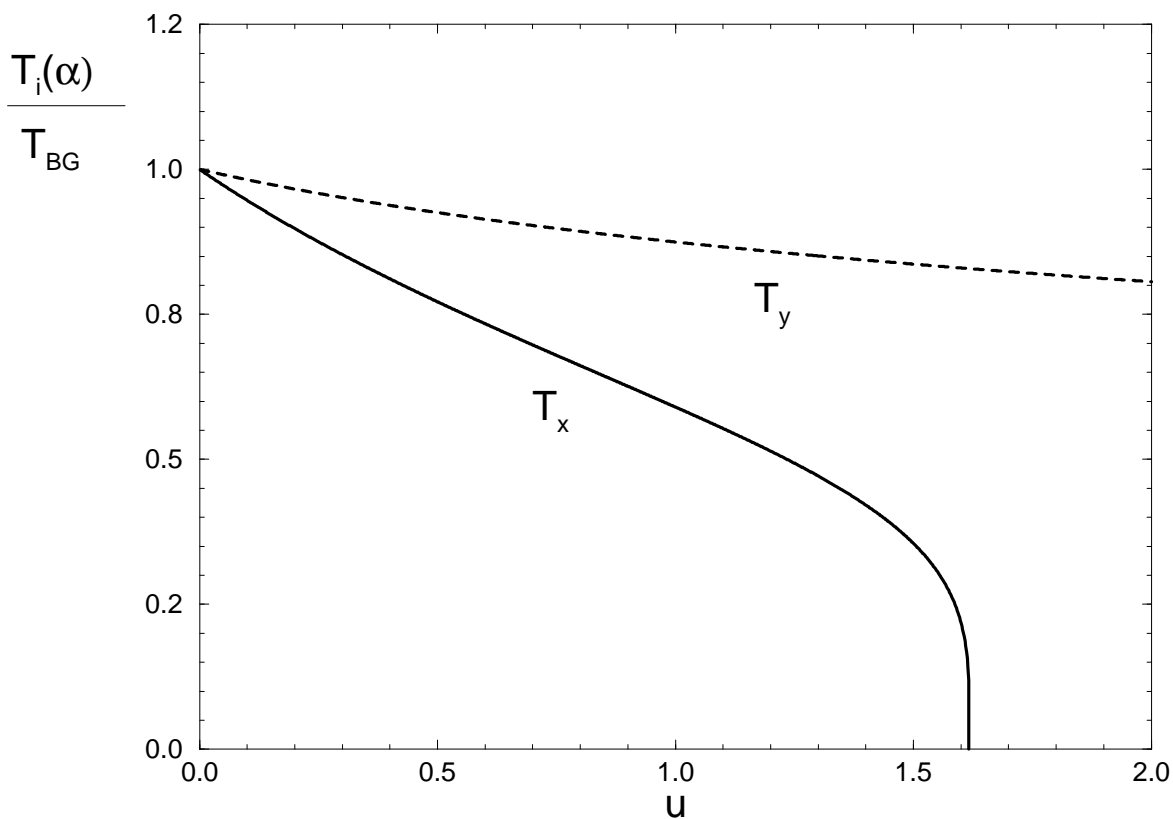


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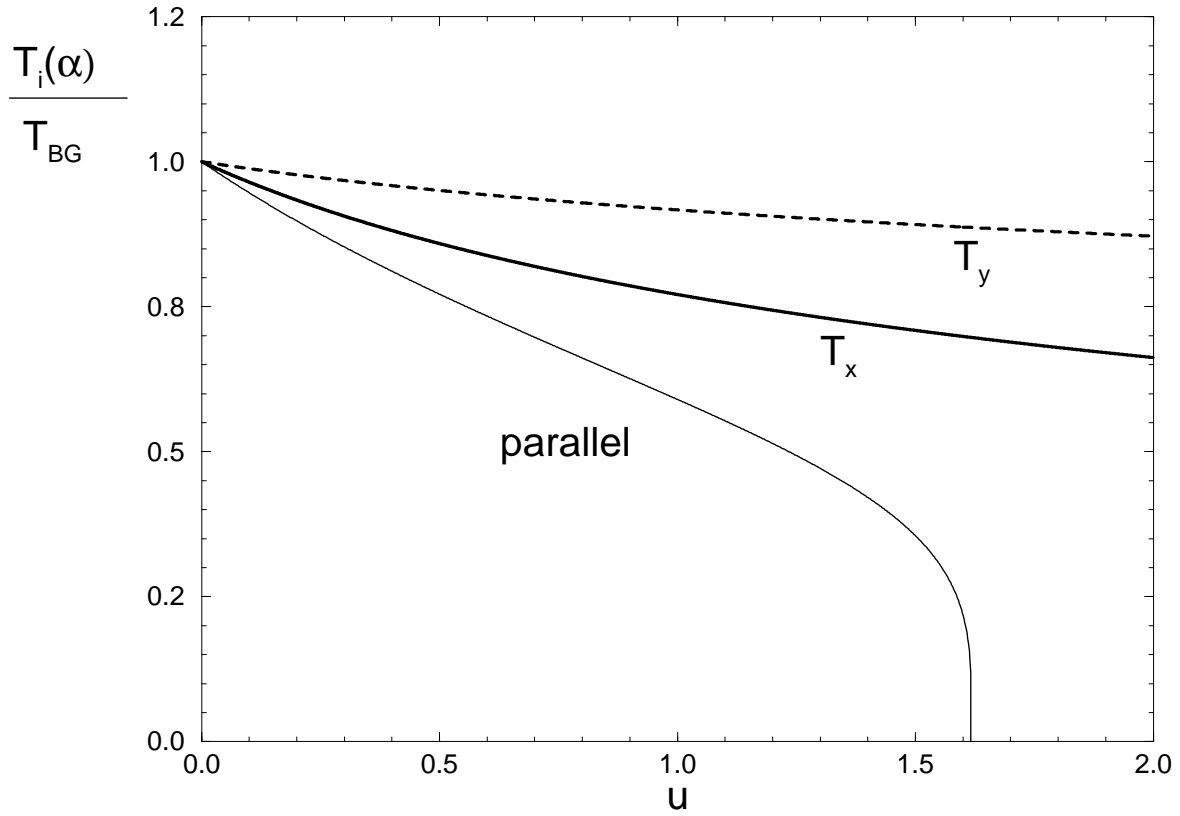


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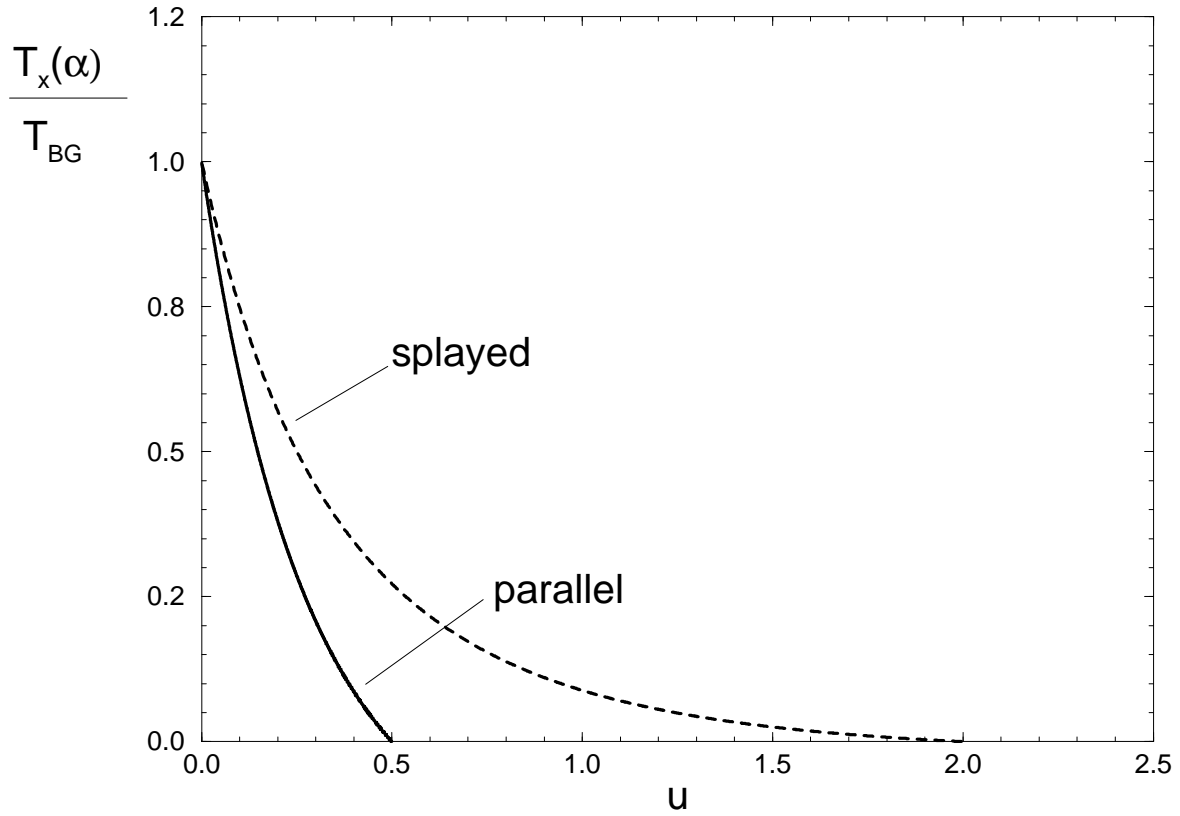


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