

Scaling regimes and critical dimensions in the Kardar–Parisi–Zhang problem

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Abstract. – We study the scaling regimes for the Kardar–Parisi–Zhang equation with noise correlator $R(q) \propto (1 + w q^{-2\rho})$ in Fourier space, as a function of ρ and the spatial dimension d . By means of a stochastic Cole–Hopf transformation, the critical and correction-to-scaling exponents at the roughening transition are determined to all orders in a $(d - d_c)$ expansion. We also argue that there is an intriguing possibility that the rough phases above and below the lower critical dimension $d_c = 2(1 + \rho)$ are genuinely different which could lead to a re-interpretation of results in the literature.

The Kardar–Parisi–Zhang (KPZ) equation, which was originally introduced to describe growth of rough surfaces [1], displays generic scale invariance, as well as a non-equilibrium roughening transition separating a smooth from a rough phase above the *lower* critical dimension. As a consequence of its mapping to the noisy Burgers equation [2], to the statistical mechanics of a directed polymer in a random environment [3], as well as to other interesting equilibrium and non-equilibrium systems (for recent reviews, see Refs. [4, 5]), the KPZ problem has emerged as one of the fundamental theoretical models defining possible universality classes for non-equilibrium scaling phenomena and phase transitions.

In one dimension, the roughness and dynamic exponents, χ and z , have long been determined exactly by means of the dynamic renormalization group (RG), utilizing the symmetries of the problem [2]. Furthermore, it has been demonstrated that the associated scaling functions can be computed to high precision by means of the self-consistent one-loop or mode-coupling approximation [6, 7]. For $d > d_c$, a two-loop RG calculation [8] indicated that the critical behavior of the roughening transition might be described by an exact set of exponents as suggested earlier on the basis of scaling arguments [9]. Using a directed-polymer representation, Lässig was able to demonstrate the validity of this statement to all orders in a

$(2 + \varepsilon)$ expansion [10] (see also Refs. [11, 12]). In addition, these results were evidence for a upper critical dimension $d_{\text{uc}} = 4$ of the roughening transition, and suggested that the ensuing strong-coupling rough phase was not accessible within perturbation theory. This is in accord with the divergence of the coupling constant g upon approaching the lower critical dimension from below [8].

The scaling behavior in the strong-coupling rough phase above d_c has been very controversial. Based on very different assumptions and analytic approaches, diverse values for the scaling exponents have been postulated, see, e.g., Refs. [13, 14, 15]. In addition, some authors have claimed $d_{\text{uc}} = 4$ to be the upper critical dimension for the rough phase as well [16, 17, 18], where the scaling exponents assume the values known in infinite dimensions [19, 20]. In contrast, numerical studies observed merely continuously varying exponents as d was increased [21]. In addition, the validity of the continuum Langevin description has been questioned in this regime, and a conceivable breakdown of universality has been conjectured (see, e.g., Refs. [22, 23]).

In order to shed light on some of these open issues, we shall find it useful to add a long-range power-law contribution to the usual spatially local stochastic noise of the KPZ equation, as first introduced by Medina *et al.* [24]. More generally, we investigate the Langevin equation,

$$\partial_t s(\mathbf{x}, t) = D \nabla^2 s(\mathbf{x}, t) + \frac{Dg}{2} [\nabla s(\mathbf{x}, t)]^2 + \zeta(\mathbf{x}, t), \quad (1)$$

with Gaussian noise characterized by zero mean and variance $\langle \zeta(\mathbf{x}, t) \zeta(\mathbf{x}', t') \rangle = 2 R_0(\mathbf{x} - \mathbf{x}') \delta(t - t')$, which we assume to be *local* in time, but which may contain spatially long-range power law contributions of the form $R_0(\mathbf{x} - \mathbf{x}') \propto |\mathbf{x} - \mathbf{x}'|^{2\rho-d}$. Thus, typically we have $R_0(\mathbf{q}) = D(1 + w|\mathbf{q}|^{-2\rho})$ in Fourier space. Notice that setting $\mathbf{u} = -\nabla s$ in Eq. (1) leads to the noisy Burgers equation, with the usual locally *conserved* noise for $w = 0$ or $\rho = 0$, but with *non-conserved* noise for $\rho = 1$ (termed model B in Ref. [2]). We find as the most important effects of adding the long-range noise term: (i) the lower critical dimension for the roughening transition is shifted *upwards* to $d_c = 2(1 + \rho)$; (ii) above d_c , there are *two* subtly distinct regimes for the *smooth* phase, characterized by different correction-to-scaling exponents; (iii) below d_c , there are two distinct *rough* regimes, governed by the short-range and long-range noise RG fixed points, respectively, and separated by a line $\rho_*(d)$ in the (ρ, d) plane.

In the following, above d_c , we derive an *exact* integral equation for the noise correlation function by means of a stochastic Cole–Hopf transformation. Based on the ensuing minimally renormalized RG flow functions, we compute critical exponents at the roughening transition to *all* orders in a $(d - d_c)$ expansion, and determine the *exact* scaling exponents in the smooth phase above d_c . We demonstrate that the strong-coupling rough phase above d_c is perturbationally inaccessible, but probably characterized by a spatially local noise correlator. Below the lower critical dimension d_c , we determine the scaling exponents at the long-range noise fixed point *exactly*, provided such a non-trivial finite fixed point exists. We discuss different analytical RG approaches to such a strong-coupling fixed point, and obtain an approximate expression for the separatrix $\rho_*(d)$. Finally, we discuss whether the rough phases above and below d_c , are “continuously” connected as a function of space dimension d and the magnitude of the noise correlation exponent ρ . We shall argue that there is a intriguing possibility that these two rough phases are genuinely different.

A convenient starting point for a systematic analysis of a Langevin equation like Eq. (1) is its reformulation in terms of a dynamic generating functional [8],

$$\begin{aligned} \mathcal{J}[\tilde{s}, s] &= \int d^d x \int dt \tilde{s}(\mathbf{x}, t) \left[\partial_t s(\mathbf{x}, t) - D \nabla^2 s(\mathbf{x}, t) \right. \\ &\quad \left. - \frac{Dg}{2} [\nabla s(\mathbf{x}, t)]^2 - \int d^d x' R_0(\mathbf{x} - \mathbf{x}') \tilde{s}(\mathbf{x}', t) \right], \end{aligned} \quad (2)$$

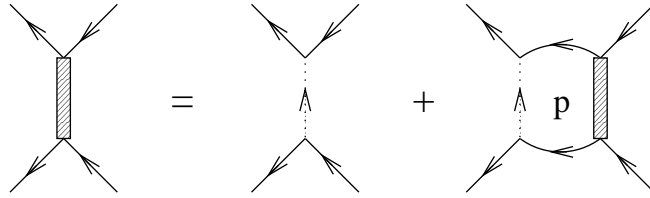


Fig. 1. – Bethe–Salpeter equation for the noise correlator.

which allows the calculation of expectation values by means of path integrals with the statistical weight $\exp(-\mathcal{J}[\tilde{s}, s])$. Upon directly applying a Cole–Hopf transformation to the KPZ equation, Eq. (1), one obtains a diffusion equation subject to multiplicative noise which is often interpreted as a directed polymer in a random potential [3]. If we recast the same idea in terms of the above dynamic functional the corresponding stochastic Cole–Hopf transformation [11] — in the appropriately discretized version of Eq. (2) in the Ito representation — reads

$$\begin{aligned} n(\mathbf{x}, t) &= \frac{2}{g} \exp \left\{ \frac{g}{2} [s(\mathbf{x}, t) + D R_0(\mathbf{0}) t] \right\}, \\ \tilde{n}(\mathbf{x}, t + \tau) &= \tilde{s}(\mathbf{x}, t + \tau) \exp \left\{ -\frac{g}{2} [s(\mathbf{x}, t) + D R_0(\mathbf{0}) t] \right\}. \end{aligned} \quad (3)$$

This leads to a dynamic generating functional,

$$\begin{aligned} \mathcal{J}[\tilde{n}, n] &= \int d^d x \int dt \left\{ \tilde{n}(\mathbf{x}, t) [\partial_t n(\mathbf{x}, t) - D \nabla^2 n(\mathbf{x}, t)] \right. \\ &\quad \left. - \frac{D g^2}{4} \int d^d x' \tilde{n}(\mathbf{x}, t) n(\mathbf{x}, t) R_0(\mathbf{x} - \mathbf{x}') \tilde{n}(\mathbf{x}', t) n(\mathbf{x}', t) \right\}, \end{aligned} \quad (4)$$

very reminiscent of the field theory for diffusion-limited pair annihilation [25]. A remarkable feature of $\mathcal{J}[\tilde{n}, n]$ is that there exist *no* loop diagrams contributing to a renormalization of the diffusion propagator. Hence the dynamic exponent is $z = 2$ whenever standard perturbation theory is applicable. This leaves us with the renormalization of the four-point noise vertex, which is formally achieved to *all* orders in the perturbation expansion via a Bethe–Salpeter equation, as graphically depicted in Fig. 1. Analytically, this leads to an integral equation for the renormalized noise correlator R ,

$$R(\mathbf{k}, \mathbf{k}'; \mu^2) = R_0(\mathbf{k} - \mathbf{k}') + \frac{g^2}{4} \int \frac{d^d p}{(2\pi)^d} \frac{R_0(\mathbf{k} - \mathbf{p})}{p^2 + \mu^2} R(\mathbf{p}, \mathbf{k}'; \mu^2), \quad (5)$$

where $\mu^2 = i\omega/2D + g^2/4$, and \mathbf{q} and ω are the momentum and frequency transfers from right to left in the vertices of Fig. 1, while $\mathbf{k} - \mathbf{k}'$ denotes the momentum transfer from bottom to top. The standard perturbation expansion in terms of g^2 is equivalent to the Neumann series for this Fredholm integral equation. The *exact* relation (5) can be used to determine the scaling function for the renormalized noise self-consistently [26].

In this letter, we focus on the asymptotic scaling exponents. Upon inserting the noise correlator into the Bethe–Salpeter equation (5), one readily obtains the full renormalization of the short-range noise strength D within the minimal-subtraction procedure, which here leads to a formal *double* expansion with respect to $\varepsilon = d - 2$ and ρ . On the other hand, the non-analytic long-range contribution itself is not renormalized perturbationally. In terms of the renormalized counterparts u and v for the bare couplings g^2 and $w g^2$, respectively, the

ensuing *exact* RG β functions are actually precisely those of the one-loop theory,

$$\beta_u = \varepsilon u - (u + v)^2/2, \quad \beta_v = (\varepsilon - 2\rho)v. \quad (6)$$

For $d > d_c = 2(1 + \rho)$, Eq. (6) shows that $v \rightarrow 0$ asymptotically. Therefore, the usual *short-range* noise KPZ scenario applies [2, 1, 24, 8].

In addition to the trivial fixed point $u_* = 0$ and the strong-coupling fixed point $u_* = \infty$, there appears an *unstable* critical fixed point $(u_c, v_c) = (2\varepsilon, 0)$, marking the location of a non-equilibrium *roughening transition* above the *lower* critical dimension d_c . At this $O(\varepsilon)$ fixed point, the general scaling relation $\chi + z = 2$ is valid, and because of $z_c = 2$ we find the marginal roughness exponent $\chi_c = 0$ [9, 10]. Furthermore, the negative eigenvalue of the associated stability matrix defines the crossover or inverse correlation length exponent $\phi_c = \nu_c^{-1} = d - 2$, whereas its positive eigenvalue yields the correction-to-scaling exponent $\omega_c = d - d_c$. Assuming that the Chayes–Fisher bound $\nu > 2/d$ [27] applies for the crossover length scale in this problem (equivalent to a directed polymer in a random environment), one finds that this lower bound is reached in $d_{uc} = 4$ dimensions, which therefore constitutes the *upper* critical dimension for the roughening transition, beyond which $\phi_c = 2$, $\nu_c = 1/2$ [10, 28].

In the *smooth* phase, both couplings $u \rightarrow 0$ and $v \rightarrow 0$, and a more careful analysis of the RG flow of the ratio $w = v/u$ is required (details will be presented elsewhere [26]). One finds that actually $w \rightarrow \infty$, and thus the smooth phase, as opposed to the roughening transition, is characterized by the *long-range* algebraic noise, with roughness exponent $\chi_{sm} = 1 + \rho - d/2 \leq 0$, and $z_{sm} = 2$. Furthermore, *two distinct* scaling regimes emerge, distinguished by different correction-to-scaling exponents: For $d_c < d < 2(1 + 2\rho)$, one finds $\omega_1 = d - 2(1 + \rho)$ and $\omega_2 = 2(1 + 2\rho) - d$, whereas for $d \geq 2(1 + 2\rho)$, $\omega_1 = d - 2(1 + 2\rho)$ and $\omega_2 = 2\rho$. Precisely at $d = d_c$, i.e., for $\rho = (d - 2)/2$, there appears a fixed *line*, which is unstable for $w < 1$, and stable for $w > 1$.

Finally, the RG analysis also tells us that the rough phase emerging above u_c is a genuine *strong-coupling* phase in the sense that it remains inaccessible to perturbative methods, even to *all* orders in ε . Numerical solutions of the flow equations also suggest that $w \rightarrow 0$ in the rough phase; hence the algebraic noise correlations appear to be *irrelevant* in the strong-coupling regime.

Up to now we have restricted our discussion to dimensions larger than the lower critical dimension, where there is a kinetic roughening transition from a smooth to a rough phase. If one tries to extend the above analysis to $d < d_c = 2(1 + \rho)$, where there exists *no* roughening transition, Eq. (6) implies that $v \rightarrow \infty$ at long length scales. Thus, the *minimally* renormalized perturbation theory based on the stochastic Cole–Hopf transformation breaks down, and one must resort to other methods. Fortunately, however, we can draw some important *exact* conclusions already from the general structure of the field theory (2). As a consequence of (a) the underlying Galilean invariance, which fixes the renormalization of g , (b) the fact that the non-analytic noise term proportional to Dw cannot renormalize, and (c) the momentum dependence of the non-linear vertex, there are merely *two* independent renormalization constants to be determined, namely for the renormalized fields, $s_R = Z^{1/2} s$ and the renormalized diffusion constant, $D_R = Z_D D$. Upon defining $\gamma = \mu \partial_\mu|_0 \ln Z$ and $\zeta = \mu \partial_\mu|_0 \ln Z_D$, the RG β functions can be expressed as $\beta_u = (d - 2 - \gamma - 2\zeta)u$ and $\beta_v = (d - 2 - 2\rho - 3\zeta)v$. Then by solving the RG equation for the correlation function near a stable RG fixed point, we furthermore identify $\chi = (2 - d + \gamma_*)/2$ and $z = 2 + \zeta_*$. The existence of a *non-zero, finite* RG fixed point u_* then immediately leads to the scaling relation $\chi + z = 2$ [8]. Similarly, at *any* long-range fixed point $0 < v_* < \infty$, ζ_* is fixed, giving the *exact* values

$$z_{lr} = (4 + d - 2\rho)/3, \quad \chi_{lr} = (2 - d + 2\rho)/3 \quad (7)$$

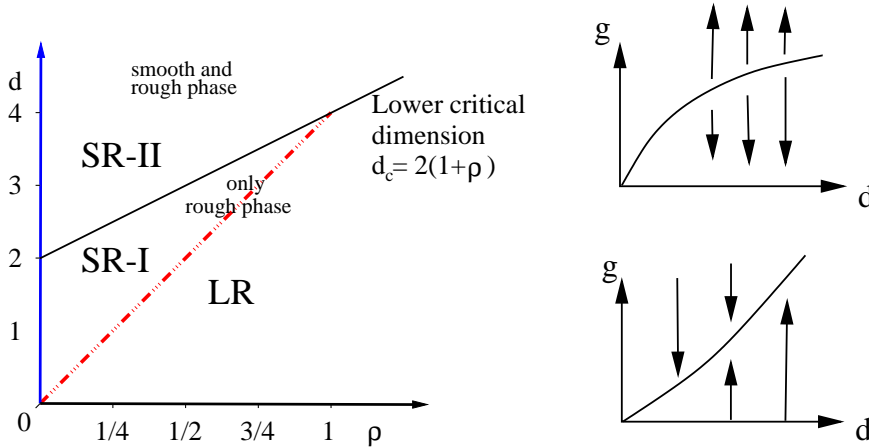


Fig. 2. – Scaling regimes (left) and sketches of the RG flow of the coupling constant g (right) above and below $d_c(\rho)$. The dashed-dotted line indicates a approximate separatrix between the domain of attraction of the short-range and long-range noise fixed point.

for the scaling exponents at the long-range fixed point, *provided* $0 < u_* < \infty$ as well.

In general, these two scaling fixed points compete, and the short-range fixed point must evidently be dominant, if $z_{\text{sr}} < z_{\text{lr}}$ (and vice versa), which indeed implies $\beta_v > 0$ and hence $v \rightarrow 0$. In one dimension, and for purely local noise ($w = 0$), the non-linear reversible force term proportional to g can be shown to fulfill the detailed-balance conditions, guaranteeing that the stationary probability distribution becomes $\mathcal{P}_{\text{st}} \propto \exp(-\int dx [\nabla s(x)]^2/2)$, as for the linear equation. In this Gaussian static theory, there can be no field renormalization ($Z = 1$), hence $\zeta = 0$, leading to the familiar one-dimensional KPZ short-range scaling exponents $\chi_{\text{sr}} = 1/2$ and $z_{\text{sr}} = 3/2$ [2, 1]. Comparing the latter with the long-range dynamic exponent $z_{\text{lr}} = (5 - 2\rho)/3$, we find that the short-range fixed point remains *stable*, provided $\rho < 1/4$ [24]. After some controversy in the literature, this result has been confirmed by simulations for the noisy Burgers equation [29].

Notice that a *minimal* renormalization scheme can never arrive at the exact one-dimensional exponents $\chi_{\text{sr}} = 1/2$ and $z_{\text{sr}} = 3/2$. Instead, in order to address the rough phase below d_c , one may devise a *non-minimal* renormalization procedure at fixed dimension d and ρ [8, 26]. Alternatively, one may utilize the mapping to the Burgers equation and hence to driven diffusive systems, for which a well-defined $(2 - d)$ expansion exists. Adding long-range correlated noise, this actually leads to the identical stability condition $\rho < 1/4$ for the short-range fixed point in $d = 1$ [30]. In the long-range regime, the case $\rho = 1$, corresponding to the Burgers equation with *non-conserved* noise, is accessible through an ε expansion below the upper critical dimension $d_{\text{uc}} = 4$ of this model [2]. The dynamic exponent here is actually obtained to all orders in ε , and reads $z_{\text{lr}} = (2 + d)/3$.

In order to further discuss the implications of the above *exact* results for the KPZ equation with long-range correlated noise it is instructive to consider the scaling regimes in the (d, ρ) landscape; see Fig. 2. One should notice that there are two *qualitatively* distinct regions separated by the lower critical dimension $d_c(\rho) = 2(1 + \rho)$. For $d < d_c(\rho)$ there is only a rough phase and the short-range (SR) and long-range (LR) noise fixed points compete. Above the lower critical dimension there appears a phase transition from a smooth to a rough phase and the RG analysis indicates that LR noise constitutes an irrelevant perturbation in the rough

phase.

From the above analysis we know the values of the scaling exponents in the domain of attraction of the LR fixed point exactly. Since the scaling exponents at the SR fixed point are independent of ρ , the only additional information needed to determine the SR exponents is the separatrix $\rho_*(d)$ between the domains of attraction of the LR and SR fixed points. Of course, for the latter to be true one has to require that the exponents are continuous at $\rho_*(d)$. This has been shown explicitly for $d = 1$, and seems very reasonable in general. From the analysis in this letter we cannot determine the exact form of the separatrix. Yet, we actually can locate some points on this curve. In $d = 1$, the short-range fixed point is stable for $\rho < \rho_* = 1/4$; for $\rho = 1$ (the Burgers equation with non-conserved noise), and below $d_{uc} = 4$, there is only the long-range regime. If we assume that the separatrix $\rho_*(d)$ between the short-range and long-range regimes extends up to four dimensions, then it definitely contains the points $(1, 4)$, $(1, 1/4)$, and probably also $(0, 0)$ in the (ρ, d) plane. A simple linear interpolation yields the function $\rho_*(d) \approx d/4$ (dashed line in Fig. 2); recent computer simulations in fact found $\rho_*(2) \approx 1/2$ [31]. Inserting the linear interpolation approximation (which we do not expect to be exact), $\rho_*(d) \approx d/4$, into Eq. (7) yields

$$z_{sr} \approx (8 + d)/6, \quad \chi_{sr} \approx (4 - d)/6. \quad (8)$$

These values remarkably coincide with Halpin-Healy's results obtained in a functional RG [13] and also with a more recent perturbative mode-coupling study by Bhattacharjee [18].

There are now two possible scenarios. The first one is that all the scaling exponents for the strong-coupling phase are continuous over the whole (ρ, d) plane and in particular upon crossing the line marking the lower critical dimension. This would necessarily imply (if not prove) that $d_{uc} = 4$ is the *upper* critical dimension for the rough phase in agreement with some recent speculations [17] but disagreement with computer simulations [21]. However, one should probably expect to encounter singularities as the lower critical dimension $d_c(\rho)$ is crossed. A second more intriguing scenario is therefore that there are two fundamentally different rough phases, one below and one above the lower critical dimension $d_c(\rho) = 2(1 + \rho)$, which we call type-I (SR-I) and type-II (SR-II). This would then allow for the scaling exponents to be discontinuous at $d_c(\rho)$. Below $d_c(\rho)$ one would have scaling exponents close to those obtained from the linear interpolation of the separatrix in Eq. (8). In particular the upper critical dimension of the type-I rough phase is $d_{uc} = 4$. Above the lower critical dimension $d_c(\rho)$ one would have a different set of exponents (e.g. those given by the numerical simulations) and the upper critical dimension must not necessarily be equal to 4 or any other finite value. At present there is proof for neither of the above scenarios but the following observations are quite indicative. First, from the RG analysis exploiting the Cole–Hopf transformation we have learned that the strong-coupling phase above $d_c(\rho)$ is not accessible by perturbation theory even to infinite order. On the other hand the rough phase at $d = 1$ is accessible by standard perturbation theory using a mapping of the KPZ equation to a driven diffusion model [30]. Second, for $\rho = 0$ an explicit two-loop calculation [8] shows that the fixed point value of the coupling constant g approaches infinity as the lower critical dimension approaches 2 from below. Third, this scenario would provide a coherent picture for most of the available numerical and analytical results for the KPZ equation. Some of the analytical approaches (e.g. functional RG and mode-coupling theory) are self-consistent theories and hence necessarily start out with correlated noise. This in effect shifts the lower critical dimension upwards, and it is well possible that this automatically constrains the results to the SR-I phase as opposed to the SR-II phase these studies were supposedly aiming at. It is difficult to judge on other analytical methods and how they fit into the scheme discussed here. Lastly, there have been suggestions for a breakdown of a continuum description, and perhaps even universality, in the rough phase

[22, 23]. Physically, in a microscopically rough regime, the underlying lattice as well as details of the dynamic rules may well be important; in addition, one might question the existence of a well-defined short-distance expansion [15] in this phase. Clearly, these issues require further clarification through approaches that extend beyond the continuum equation, Eq. (1), and equivalent field theory methods.

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