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Non-equilibrium critical behavior of O(n)-symmetric systems

Effect of reversible mode-coupling terms and dynamical anisotropy

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Dedicated to Franz Schwabl on the occasion of his 60th birthday.

Abstract. Phase transitions in non-equilibrium steady states of $O(n)$-symmetric models with reversible mode couplings are studied using dynamic field theory and the renormalization group. The systems are driven out of equilibrium by dynamical anisotropy in the noise for the conserved quantities, i.e., by constraining their diffusive dynamics to be at different temperatures $T^{\|}$ and $T^{\perp}$ in $d_{\|}$- and $d_{\perp}$-dimensional subspaces, respectively. In the case of the Sasvári-Schwabl-Szépfalusy (SSS) model for planar ferro- and isotropic antiferromagnets, we assume a dynamical anisotropy in the noise for the non-critical conserved quantities that are dynamically coupled to the non-conserved order parameter. We find the equilibrium fixed point (with isotropic noise) to be stable with respect to these non-equilibrium perturbations, and the familiar equilibrium exponents therefore describe the asymptotic static and dynamic critical behavior. Novel critical features are only found in extreme limits, where the ratio of the effective noise temperatures $T^{\|}/T^{\perp}$ is either zero or infinite. On the other hand, for model J for isotropic ferromagnets with a conserved order parameter, the dynamical noise anisotropy induces effective long-range elastic forces, which lead to a softening only of the $d_{\perp}$-dimensional sector in wavevector space with lower noise temperature $T^{\perp} < T^{\|}$. The ensuing static and dynamic critical behavior is described by power laws of a hitherto unidentified universality class, which, however, is not accessible by perturbational means for $d_{\|} \geq 1$. We obtain formal expressions for the novel critical exponents in a double expansion about the static and dynamic upper critical dimensions and $d_{\|}$, i.e., about the equilibrium theory.

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1 Introduction

The equilibrium properties of a generic system in contact with a heat bath are entirely determined by interactions (the dynamics plays a role only in providing the necessary mixing in phase space). In contrast, dynamics is important in non-equilibrium steady states (NESS) where competing dynamics (i.e. contacts with more than one heat baths or driving fields) generate fluxes of energy, mass, etc., or, equivalently, yield non-zero steady-state probability currents in phase space. This difference between equilibrium and NESS has often been illustrated by using an electrodynamical analogy [1]: *equilibrium* ≡ *electrostatics* while *NESS* ≡ *magnetostatics*. The above analogy can actually be developed a bit further by noting that magnetostatics can be described in terms of interactions between currents and thus asking if NESS could also be described in terms of some effective interactions, characterizing the stationary probability distribution.

The search for such effective interactions has been going on for some time. The simplest version of this search consists of taking the logarithm of the steady-state distribution function for small-size systems and looking for the dominant interaction in the emerging effective hamiltonian. Unfortunately, this approach did not turn out to be very useful since, generically, all the interactions, of any range, which are consistent with the symmetries of the system are generated — and the lack of significant differences in the magnitude of the couplings renders the identification of the dominant interactions impossible.

A somewhat more sophisticated approach is based on the extention of universality concepts to phase transitions in NESS. The idea here is that the universality class of a phase transition in NESS provides information about the range of interactions generated by the competing dynamics and, furthermore, also gives the exponent of the long-distance, power-law decay in case of long-range interactions. This approach has yielded some very interesting results in connection with non-equilibrium perturbations imposed on the relaxational models A (non-conserved order parameter dynamics) and B (conserved order parameter dynamics), using the terminology of Ref. [2].

In particular, phase transitions in systems with model-A type of dynamics have been shown to be remarkably robust against the introduction of various competing dynamics which are local and do not conserve the order parameter [3], including competing dynamics which breaks the discrete symmetry of the system [4]. This robustness of the *critical* behavior also persists when the competing dynamics comes from a reversible mode coupling to a non-critical conserved field [5]. Thus there is large class of non-equilibrium steady states where the character of interactions is not modified by the presence of thermodynamic fluxes or, in other words, the probability currents in phase space become irrelevant for the large-scale behavior. Thinking in terms of the electrodynamic analogy, one may say that this corresponds to the magnetostatic problem of a steady electric current along a straight line, which can be reduced to an electrostatic problem after an appropriate coordinate transformation.

Another class of competing dynamics is obtained when model-B type dynamics competes with external drive [1] or with local, *anisotropic*, order-parameter conserving processes [6,7,8]. In these cases, long-range interactions do get generated in the NESS and, furthermore, a common feature of these interactions is that their angular dependence resembles the form of elastic or uniaxial dipolar forces (and for some systems even the power-law decay with distance is that occurring for dipolar interactions [6,7,8]). In terms of the electrodynamics analogy, one might say that the fluxes in the NESS of these systems are equivalent to loops of electric currents which interact, in the first-order multipole expansion, via (pseudo-)dipolar forces. (However, when drawing this analogy, one should be aware that while successive terms in the electrodynamics multipole expansion become more and more suppressed, the effective long-range forces appearing in the NESS of models with conserved order parameter are very dominant.)

There are, of course, examples which do not fit easily into this heuristic straight-line current and loop current classification, which therefore should not be taken too far; e.g., competition of the usual model A with nonlocal dynamics [9] or with *linear* coupling to a conserved field [10] generates isotropic long-range interactions. Nonthermally driven steady states occuring, for example, in the prominent non-equilibrium universality classes of directed percolation [11] and Kardar–Parisi–Zhang surface growth [12] probably cannot be described in terms of such effective interactions at all. Furthermore, there are driven systems where the emerging effective interactions are extremely long-ranged in the sense that the potential is non-integrable and, consequently, the system does not display thermodynamical behavior in the usual sense [13,14].

Nevertheless, since a large variety of competing dynamical processes have yielded a surprisingly small number of effective universality classes, we feel that it is worthwhile to continue the exploration of NESS through studying non-equilibrium phase transitions and thus deducing effective interactions. The investigation of the stability of the known equilibrium dynamic universality classes is important also from another viewpoint. Namely, in many experiments probing critical dynamics, it is by no means a trivial issue to maintain thermodynamic equilibrium, as relaxation times become very long close to a second-order phase transition as a consequence of critical slowing down. For the interpretation of experimental data, it might therefore be important to know if the dynamical system is driven to a NESS that is characterized by scaling behavior which is distinct from its equilibrium counterpart.

In the present work, we follow up a previous investigation [5] of NESS generated in $O(n)$-symmetric systems subject to non-equilibrium perturbations. Here, either a non-conserved order parameter is dynamically coupled to non-critical conserved quantities — this defines the n-component Sasvári-Schwabl-Szépfalusy (SSS) model [15,16], which incorporates model E for planar ferromagnets (with $n = 2$) [17] and model G for isotropic antiferromagnets ($n = 3$) [18] as special cases; or the order parameter and the generators of $O(n)$ are identical, which is realized for $n = 3$ in model J for (idealized) isotropic ferromagnets [19]. (For a review on more realistic dynamics for ferromagnetic systems that includes the effects of dipolar interactions, see Ref. [20].) This field-theoretic renormalization-group (RG) study found that spatially *isotropic* violation of detailed balance generically leaves the equilibrium fixed point stable, thus indicating that the steady-state fluxes involved here do *not* generate any long-range interactions. Since *dynamical anisotropy* appears to be an essential ingredient for the generation of pseudo-dipolar effective interactions in NESS, it is natural to ask if this was the case for the above $O(n)$-symmetric systems as well. More precisely, we allow for dynamical spatially anisotropic noise for the conserved quantities. For the anisotropic non-equilibrium model J, this means that we allow for different effective temperatures $T^{\|} > T^{\perp}$, respectively, governing the order parameter noise in $d_{\|}$- and $d_{\perp}$-dimensional subspaces ($d_{\|} + d_{\perp} = d$). Essentially, this means that there is a non-zero heat current flowing from the "hotter subspace" ($d_{\|}$ dimensions) into the $d_{\perp}$-dimensional "cooler subspace". For the anisotropic non-equilibrium SSS model, a similar distinction applies for the noise in the Langevin equation that describes the dynamics of the purely dynamically coupled conserved field, and we may then explore its influence, and that of the ensuing effective heat current, on the non-conserved order parameter dynamics.

In Sec. 2, we shall discuss our basic model equations for the spatially anisotropic non-equilibrium generalizations of Langevin dynamics appropriate for second-order phase transitions in $O(n)$-symmetric systems. We start our investigations with the anisotropic non-equilibrium version of the SSS model in Sec. 3, and shall find that for this system of coupled Langevin equations with a non-conserved order parameter, the equilibrium dynamic scaling fixed point remains stable, and governs the asymptotic critical behavior. Thus, the ensuing static critical exponents are those of the n-component Heisenberg model, accompanied with the equilibrium SSS strong-scaling dynamic exponent $z = d/2$ (at least to one-loop order). Novel dynamic and static scaling exponents are found in the extreme situations $T^\perp/T^\parallel = 0$ or ∞ only, and may be related to the results for the isotropic non-equilibrium model studied earlier [5]. The latter are of course contained in our present more general study as a special case. These results once more underline previous observations that non-equilibrium generalizations of dynamical models with a *non-conserved* order parameter asymptotically, i.e., in the vicinity of the critical point, display the scaling behavior of the corresponding equilibrium dynamic universality class.

In Sec. 4, we consider the anisotropic non-equilibrium model J with *conserved* order parameter. In contrast to the anisotropic non-equilibrium SSS model, we shall find that only the spatial sector with *lower* noise temperature $T^\perp < T^\parallel$ becomes soft at the transition, while the $d_\parallel$-dimensional sector remains uncritical. As a consequence, long-range elastic (uniaxial pseudo-dipolar for $d_\perp = 1$) effective interactions are generated, as in the two-temperature model B [6,7,8]. The equilibrium dynamic fixed point becomes unstable, and the phase transition is described by a novel universality class, characterized by reduced upper and lower critical dimensions for both statics and dynamics. For general dimensionality of the soft sector, we find runaway renormalization-group flow trajectories, and perturbational methods appear to break down. Presumably, this indicates either strong-coupling scaling behavior, or perhaps even the absence of a non-equilibrium steady state at the critical point. However, we can obtain the new exponents in a double expansion about both the upper critical dimensions, and the dimensionality of the hard sector $d_\parallel$ (which amounts to an expansion about the equilibrium theory). Yet, as this expansion clearly becomes invalid at some critical value of $d_\parallel^c < 1$, at least to one-loop order, this formal expansion should not be taken too seriously for the description of a real physical system. At any rate, this model characterized by a *conserved* order parameter, and *spatially anisotropic* conserved noise, definitely leads to a novel dynamic universality class. Finally, in Sec. 5 we summarize our results again, and discuss their implications. In the Appendix, we present the Ward identities stemming from the $O(n)$ symmetry of the non-equilibrium models under investigation here.

2 Model equations

In this section, we briefly outline the basic model equations for our anisotropic, non-equilibrium generalization of both the SSS model [15] and model J [19]. The equilibrium characteristics of these dynamic models were summarized at length in Ref. [5], and we refer to this paper and the original equilibrium literature (see Refs. [15,16,17,18,19,20,21]) for further details. We shall largely use the notations introduced in Ref. [5], if not explicitly mentioned otherwise.

We consider a second-order phase transition for an n-component vector order parameter S_0^α, $\alpha = 1, \ldots, n$ (we denote unrenormalized quantities by the subscript "0"). As we furthermore assume isotropy in order parameter space, the static critical properties will be described by an $O(n)$-symmetric ϕ^4 Landau–Ginzburg–Wilson free energy in d space dimensions,

$$H[\{S_0^\alpha\}] = \int d^d x \ \left\{ \frac{r_0}{2} \sum_{\alpha=1}^n S_0^\alpha(\mathbf{x})^2 + \frac{1}{2} \sum_{\alpha=1}^n [\nabla S_0^\alpha(\mathbf{x})]^2 + \frac{u_0}{4!} \left[\sum_{\alpha=1}^n S_0^\alpha(\mathbf{x})^2 \right]^2 \right\} , \qquad (2.1)$$

where $r_0 = (T - T_c^0)/T_c^0$ is the relative distance from the mean-field critical temperature T_c^0. This effective free energy determines the equilibrium probability distribution for the vector order parameter S_0^α,

$$P_{\rm eq}[\{S_0^\alpha\}] = \frac{e^{-H[\{S_0^\alpha\}]/k_{\rm B}T}}{\int \mathcal{D}[\{S_0^\alpha\}] e^{-H[\{S_0^\alpha\}]/k_{\rm B}T}} \ . \qquad (2.2)$$

Following standard procedures, one may then compute the two independent critical exponents, e.g., η and ν, by means of perturbation theory with respect to the static non-linear coupling u_0 and by employing the renormalization group procedure, within a systematic expansion in terms of $\epsilon = 4 - d$ about the static upper critical dimension $d_c = 4$. Here, η describes the power-law decay of the order parameter correlation function at criticality, $\langle S^\alpha(\mathbf{x}) S^\beta(\mathbf{x}') \rangle \propto 1/|\mathbf{x} - \mathbf{x}'|^{d-2+\eta}$, or, equivalently, of the static susceptibility, $\chi(\mathbf{q}) \propto 1/q^{2-\eta}$, and the exponent ν characterizes the divergence of the correlation length as T_c is approached, $\xi \propto |T - T_c|^{-\nu}$. Notice that fluctuations also shift the true transition temperature T_c downwards as compared to the mean-field critical temperature T_c^0, i.e., $r_{0c} = T_c - T_c^0 < 0$.

In order to correctly describe the critical dynamics for an $O(n)$-symmetric system, one needs to take into account all the slow modes. Generally, in addition to the order parameter itself, these comprise the diffusive modes associated with the conservation law connected with the rotational symmetry. The SSS model thus consists of dynamically coupled Langevin equations for a *non-conserved* n-component order parameter S_0^α and $n(n-1)/2$ conserved generalized angular momenta $M_0^{\alpha\beta} = -M_0^{\beta\alpha}$ [15]. Physical realizations of this model are the critical dynamics of the XY model ($n = 2$), also called model E [2,17], with the order parameter components S_0^x and S_0^y, and the conserved quantity $M_0^{12} = S_0^z$, which generates rotations in the xy-plane; and the dynamic critical behavior of isotropic antiferromagnets ($n = 3$), known as model G [2,18], with S_0^x, S_0^y, and S_0^z representing the components of the staggered magnetization, and $M_0^{12} = M_0^z$, $M_0^{23} = M_0^x$, and $M_0^{13} = -M_0^y$ denoting the components of the magnetization itself, which are conserved and can be identified with the generators of the symmetry group $O(3)$.

The variables $M_0^{\alpha\beta}$ represent non-critical quantities, and their coupling to the order parameter fluctuations S_0^α is of purely dynamical character. Hence it suffices to simply add a quadratic term to the hamiltonian (2.1),

$$H[\{S_0^\alpha\},\{M_0^{\alpha\beta}\}] = H[\{S_0^\alpha\}] + \frac{1}{2}\int d^dx \sum_{\alpha>\beta} M_0^{\alpha\beta}(\mathbf{x})^2 \ . \tag{2.3}$$

With this free energy funtional H, the coupled non-linear Langevin equations defining the SSS model read [15,16,5]

$$\frac{\partial S_0^\alpha}{\partial t} = g_0 \sum_\beta \frac{\delta H}{\delta M_0^{\alpha\beta}}\, S_0^\beta - \lambda_0 \frac{\delta H}{\delta S_0^\alpha} + \zeta^\alpha \tag{2.4}$$

$$= g_0 \sum_{\beta\neq\alpha} M_0^{\alpha\beta} S_0^\beta - \lambda_0 (r_0 - \nabla^2) S_0^\alpha - -\lambda_0 \frac{u_0}{6} S_0^\alpha \sum_\beta S_0^\beta S_0^\beta + \zeta^\alpha \ , \tag{2.5}$$

and

$$\frac{\partial M_0^{\alpha\beta}}{\partial t} = -g_0 \left(\frac{\delta H}{\delta S_0^\alpha}\, S_0^\beta - \frac{\delta H}{\delta S_0^\beta}\, S_0^\alpha\right) + +D_0 \nabla^2 \frac{\delta H}{\delta M_0^{\alpha\beta}} + \eta^{\alpha\beta} \tag{2.6}$$

$$= -g_0 \left(S_0^\alpha \nabla^2 S_0^\beta - S_0^\beta \nabla^2 S_0^\alpha\right) + +D_0 \nabla^2 M_0^{\alpha\beta} + \eta^{\alpha\beta} \ . \tag{2.7}$$

Here, g_0 denotes the strength of the reversible, so-called mode-coupling terms, and ζ^α and $\eta^{\alpha\beta}$ represent fluctuating forces with zero mean, $\langle \zeta^\alpha(\mathbf{x},t)\rangle = 0$, $\langle \eta^{\alpha\beta}(\mathbf{x},t)\rangle = 0$. In order to fully characterize the dynamics, we furthermore need to specify the correlations of these stochastic forces; for the order parameter, we simply assume a Gaussian distribution for the ζ^α with the second moment

$$\langle \zeta^\alpha(\mathbf{x},t)\zeta^\beta(\mathbf{x}',t')\rangle = 2\widetilde{\lambda}_0\, \delta(\mathbf{x}-\mathbf{x}')\, \delta(t-t')\, \delta^{\alpha\beta} \ , \tag{2.8}$$

corresponding to non-conserved white noise. On the other hand, the conservation law for the generalized angular momenta $M_0^{\alpha\beta}$, and the antisymmetry with respect to the tensor indices α,β implies that $\langle \eta^{\alpha\beta}(\mathbf{x},t)\eta^{\gamma\delta}(\mathbf{x}',t')\rangle \propto -\nabla^2\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\left(\delta^{\alpha\beta}\delta^{\gamma\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma}\right)$. This functional form, prescribed by the Einstein relation ensuring that at long times the equilibrium distribution (2.2) with the free energy (2.3) will be attained, provides us with the possibility to impose a *spatially anisotropic* form of detailed-balance violation through the prescription

$$\langle \eta^{\alpha\beta}(\mathbf{x},t)\eta^{\gamma\delta}(\mathbf{x}',t')\rangle = -2\left(\widetilde{D}_0^\parallel \nabla_\parallel^2 + \widetilde{D}_0^\perp \nabla_\perp^2\right)\delta(\mathbf{x}-\mathbf{x}') \times \times \delta(t-t')\left(\delta^{\alpha\beta}\delta^{\gamma\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma}\right) \ . \tag{2.9}$$

We may interpret this as follows. First, consider the special case $\widetilde{D}_0^\parallel = \widetilde{D}_0^\perp = \widetilde{D}_0$, which is in fact the model investigated in Ref. [5]. We can then identify $\widetilde{\lambda}_0/\lambda_0 = k_{\mathrm{B}} T_S$ and $\widetilde{D}_0/D_0 = k_{\mathrm{B}} T_M$ as the temperatures of the heat baths coupling to the order parameter and conserved quantities, respectively. For $T_S = T_M$, obviously detailed balance holds. More generally, the ratio $T_0 = T_M/T_S$ describes the extent to which this equilibrium condition is violated (in Ref. [5], $\Theta_0 = 1/T_0$ was used instead); for $T_0 < 1$, energy flows from the order parameter heat bath into the conserved-quantities heat bath, and vice versa for $T_0 > 1$. Notice that as we are really interested in the vicinity of the critical point, $T_S \approx T_c$, and T_0 thus gives the heat bath temperature of the generalized angular momenta measured in terms of T_c. While new critical behavior ensues for either $T_0 = 0$ or $T_0 = \infty$, a one-loop renormalization-group analysis shows that for *any* $0 < T_0 < \infty$ the asymptotic critical properties are those of the equilibrium model, i.e., the renormalized $T \to 1$ under scale transformations [5]. In the anisotropic generalization (2.9), we furthermore allow for different temperatures in $d_\parallel$- and $d_\perp$-dimensional sectors in space (with $d_\parallel + d_\perp = d$), with associated distinct temperatures (again, essentially measured with respect to $T_S \approx T_c$),

$$T_0^{\parallel/\perp} = \frac{\widetilde{D}_0^{\parallel/\perp}}{D_0}\, \frac{\lambda_0}{\widetilde{\lambda}_0} \ . \tag{2.10}$$

Therefore there appears an additional degree of freedom here, namely the ratio

$$\sigma_0 = T_0^\perp / T_0^\parallel \tag{2.11}$$

of the conserved-noise heat bath temperatures in the transverse and parallel sectors, respectively, which without loss of generality we may assume to be in the interval $0 \leq \sigma_0 \leq 1$, where $\sigma_0 = 1$ obviously corresponds to the equilibrium situation. The one-loop renormalization-group flow equations for this non-equilibrium SSS model with spatially anisotropic conserved noise will be derived and studied in Sec. 3.

Model J (in the nomenclature coined in Ref. [2]) corresponds to the situation where the conserved quantities associated with the $O(n)$ symmetry are in fact identical to the order parameter fluctuations themselves. The physical realization, with $n = 3$, is the critical dynamics of isotropic ferromagnets, with S_0^x, S_0^y, and S_0^z denoting the three components of the conserved magnetization vector [19,20]. The three coupled Langevin equations for model J read [19,20,5]

$$\frac{\partial S_0^\alpha}{\partial t} = -g_0 \sum_{\beta,\gamma} \epsilon^{\alpha\beta\gamma} \frac{\delta H}{\delta S_0^\beta}\, S_0^\gamma + \lambda_0 \nabla^2 \frac{\delta H}{\delta S_0^\alpha} + \zeta^\alpha \tag{2.12}$$

$$= -g_0 \sum_{\beta,\gamma} \epsilon^{\alpha\beta\gamma} S_0^\beta \nabla^2 S_0^\gamma + \lambda_0 \nabla^2 (r_0 - \nabla^2) S_0^\alpha + +\lambda_0 \frac{u_0}{6} \nabla^2 S_0^\alpha \sum_\beta S_0^\beta S_0^\beta + \zeta^\alpha \ , \tag{2.13}$$

where g_0 again denotes the reversible mode-coupling constant, and $\langle \zeta^\alpha(\mathbf{x}, t) \rangle = 0$ for the stochastic forces. As now the order parameter is *conserved*, we may introduce spatially anisotropic noise in the correlator of the associated conserved noise, in analogy with Eq. (2.9) for the generalized angular momenta of the SSS model,

$$\langle \zeta^\alpha(\mathbf{x}, t) \zeta^\beta(\mathbf{x}', t') \rangle = -2 \left(\widetilde{\lambda}_0^\parallel \nabla_\parallel^2 + \widetilde{\lambda}_0^\perp \nabla_\perp^2 \right) \delta(\mathbf{x} - \mathbf{x}') \times \times \delta(t - t') \, \delta^{\alpha\beta} \ . \quad (2.14)$$

In the corresponding isotropic variant where $\widetilde{\lambda}_0^\parallel = \widetilde{\lambda}_0^\perp$, a simple rescaling of the non-linear couplings u_0 and g_0 can absorb the effects of detailed-balance violation entirely; this demonstrates immediately that this model is asymptotically governed by the equilibrium critical exponents [5]. Here, on the other hand, we may define different effective temperatures for the longitudinal and transverse sectors, respectively,

$$T_0^{\parallel/\perp} = \widetilde{\lambda}_0^{\parallel/\perp} / \lambda_0 \ , \quad (2.15)$$

and thus once again a novel variable emerges, namely the ratio $\sigma_0 = \widetilde{\lambda}_0^\perp / \widetilde{\lambda}_0^\parallel = T_0^\perp / T_0^\parallel$, with $0 < \sigma_0 < 1$. The dramatic implications of this spatially anisotropic conserved noise will be investigated in Sec. 4.

3 The anisotropic non-equilibrium SSS model

We now proceed by considering the critical behavior of our anisotropic non-equilibrium version of the SSS model [15], as defined by the Langevin equations (2.5) and (2.7), with the noise correlators (2.8) and (2.9), respectively. In Sec. 3.1, we perform the perturbational renormalization to one-loop order, following the procedures that were already employed in Ref. [5]. These in turn constitute the appropriate generalization of the equilibrium renormalization scheme, see Refs. [16,22]. From the renormalization constants (Z factors) that render the field theory finite in the ultraviolet (UV), one may then derive the RG flow functions which enter the Gell-Mann–Low equation. This partial differential equation describes how correlation functions change under scale transformations. In the vicinity of an RG fixed point, the theory becomes scale-invariant, and the information previously gained about the UV behavior can thus be employed to access the physically interesting power laws governing the infrared (IR) regime at the critical point ($\tau \propto T - T_c \to 0$) for long wavelengths (wavevector $\mathbf{q} \to 0$) and low frequencies ($\omega \to 0$).

3.1 Perturbation theory and renormalization

3.1.1 Dynamic field theory

As a first step, we translate the Langevin equations (2.5) and (2.7), with (2.8) and (2.9), into a dynamic field theory, following standard procedures [21,16,5]. This results in a probability distribution for the dynamic fields S_0^α and $M_0^{\alpha\beta}$,

$$P[\{S_0^\alpha, M_0^{\alpha\beta}\}] \propto \int \mathcal{D}[\{i\widetilde{S}_0^\alpha\}] \int \mathcal{D}[\{i\widetilde{M}_0^{\alpha\beta}\}] \times \times e^{J[\{\widetilde{S}_0^\alpha\}, \{S_0^\alpha\}, \{\widetilde{M}_0^{\alpha\beta}\}, \{M_0^{\alpha\beta}\}]} \ , \quad (3.1)$$

with the statistical weight given by Janssen-De Dominicis functional $J = J_{\rm har} + J_{\rm rel} + J_{\rm mc}$. Its harmonic part, in terms of the original dynamic fields $S_0^\alpha(\mathbf{x}, t)$, $M_0^{\alpha\beta}(\mathbf{x}, t)$ and the auxiliary fields $\widetilde{S}_0^\alpha(\mathbf{x}, t)$, $\widetilde{M}_0^{\alpha\beta}(\mathbf{x}, t)$ reads

$$\begin{aligned} J_{\rm har}[\{\widetilde{S}_0^\alpha\}, \{S_0^\alpha\}, \{\widetilde{M}_0^{\alpha\beta}\}, \{M_0^{\alpha\beta}\}] = \\ = \int d^d x \int dt \Bigg\{ \sum_\alpha \widetilde{\lambda}_0 \, \widetilde{S}_0^\alpha \widetilde{S}_0^\alpha - \\ - \sum_\alpha \widetilde{S}_0^\alpha \left[\frac{\partial}{\partial t} + \lambda_0 \left(r_0 - \nabla^2 \right) \right] S_0^\alpha - \\ - \sum_{\alpha > \beta} \widetilde{M}_0^{\alpha\beta} \left(\widetilde{D}_0^\parallel \nabla_\parallel^2 + \widetilde{D}_0^\perp \nabla_\perp^2 \right) \widetilde{M}_0^{\alpha\beta} - \\ - \sum_{\alpha > \beta} \widetilde{M}_0^{\alpha\beta} \left(\frac{\partial}{\partial t} - D_0 \nabla^2 \right) M_0^{\alpha\beta} \Bigg\} \ , \end{aligned} \quad (3.2)$$

while the static non-linearity leads to a relaxation vertex

$$J_{\rm rel}[\{\widetilde{S}_0^\alpha\}, \{S_0^\alpha\}] = -\lambda_0 \frac{u_0}{6} \int d^d x \int dt \sum_{\alpha, \beta} \widetilde{S}_0^\alpha S_0^\alpha S_0^\beta S_0^\beta \ , \quad (3.3)$$

and the purely dynamic couplings generate the mode-coupling vertices

$$\begin{aligned} J_{\rm mc}[\{\widetilde{S}_0^\alpha\}, \{S_0^\alpha\}, \{\widetilde{M}_0^{\alpha\beta}\}, \{M_0^{\alpha\beta}\}] = \\ = g_0 \int d^d x \int dt \sum_{\alpha, \beta} \Bigg\{ \widetilde{S}_0^\alpha M_0^{\alpha\beta} S_0^\beta - \\ - \frac{1}{2} \widetilde{M}_0^{\alpha\beta} \left(S_0^\alpha \nabla^2 S_0^\beta - S_0^\beta \nabla^2 S_0^\alpha \right) \Bigg\} \ . \end{aligned} \quad (3.4)$$

As usual, the harmonic part (3.2) defines the propagators of the field theory, while the perturbation expansion is performed in terms of the non-linear vertices (3.3) and (3.4). Notice that the existence of the reversible forces (3.4) does not show up in dynamic mean-field theory (van Hove theory), which in field-theory language is based on the harmonic action (3.2) only.

We can now construct the perturbation expansion for all possible correlation functions of the dynamic and auxiliary fields, as well as for the associated vertex functions given by the one-particle irreducible Feynman diagrams. A straightforward scaling analysis yields that the upper critical dimension of this model is $d_c = 4$ for both the relaxational *and* the mode-coupling vertices. Therefore, for $d \leq 4$ the perturbation theory will be IR-singular, and non-trivial critical exponents will ensue, while for $d \geq 4$ the perturbation theory contains UV divergences. In order to renormalize the field theory in the ultraviolet, it suffices to render all the non-vanishing two-, three-, and four-point functions finite by introducing multiplicative renormalization constants. This is achieved by demanding the renormalized vertex functions, or appropriate momentum and frequency derivatives thereof, to be finite when the fluctuation integrals are taken at a conveniently chosen normalization point, well outside the IR regime. We shall employ the dimensional regularization scheme in order to compute the emerging momentum integrals, and choose the renormalized mass $\tau = 1$ as our normalization point, or, sufficient to one-loop order, $\tau_0 = r_0 - r_{0c} = \mu^2$. Notice that μ defines an intrinsic momentum scale of the renormalized theory. The Gell-Mann–Low equation can subsequently be used to explore the dependence of the *renormalized* correlation or vertex functions on μ, and thereby obtain information on their scaling behavior.

3.1.2 Dynamics: Vertex function renormalization

The UV-divergent two-, three-, and four-point vertex functions or their derivatives that require multiplicative renormalization are $\partial_\omega \Gamma_{0\,\widetilde{M}M}(\mathbf{q},\omega)$, $\partial_{q^2} \Gamma_{0\,\widetilde{M}M}(\mathbf{q},\omega)$, and $\partial_{q^2} \Gamma_{0\,\widetilde{M}\widetilde{M}}(\mathbf{q},\omega)$; $\Gamma_{0\,\widetilde{S}S}(\mathbf{q},\omega)$, $\partial_\omega \Gamma_{0\,\widetilde{S}S}(\mathbf{q},\omega)$, $\partial_{q^2} \Gamma_{0\,\widetilde{S}S}(\mathbf{q},\omega)$, and $\partial_{q^2} \Gamma_{0\,\widetilde{S}\widetilde{S}}(\mathbf{q},\omega)$; $\Gamma_{0\,\widetilde{S}SM}(-\mathbf{q},-\omega;\frac{\mathbf{q}}{2}-\mathbf{p},\frac{\omega}{2};\frac{\mathbf{q}}{2}+\mathbf{p},\frac{\omega}{2})$ and $\partial_{(\mathbf{q}\cdot\mathbf{p})} \Gamma_{0\,\widetilde{M}SS}(-\mathbf{q},-\omega;\frac{\mathbf{q}}{2}-\mathbf{p},\frac{\omega}{2};\frac{\mathbf{q}}{2}+\mathbf{p},\frac{\omega}{2})$; and, at last, $\Gamma_{0\,\widetilde{S}SSS}(-\mathbf{q},-\omega;\frac{\mathbf{q}}{3},\frac{\omega}{3};\frac{\mathbf{q}}{3},\frac{\omega}{3};\frac{\mathbf{q}}{3},\frac{\omega}{3})$. On the other hand, we have four fluctuating fields ($\widetilde{M}_0^{\alpha\beta}$, $M_0^{\alpha\beta}$, $\widetilde{S}_0^\alpha$, S_0^α) and the seven parameters $\widetilde{D}_0$, D_0, $\widetilde{\lambda}_0$, λ_0, τ_0, g_0, and u_0 available; this leaves us at liberty to choose one of the renormalization constants in a convenient manner.

Starting with the two-point functions $\Gamma_{0\,\widetilde{M}M}(\mathbf{q},\omega)$ and $\Gamma_{0\,\widetilde{S}S}(\mathbf{q},\omega)$ for the conserved quantities and order parameter fluctuations, respectively, we immediately note that as a consequence of the momentum dependence of the mode-coupling vertices

$$\frac{\partial}{\partial(i\omega)} \Gamma_{0\,\widetilde{M}M}(\mathbf{q}=\mathbf{0},\omega) \equiv 1 \tag{3.5}$$

to *all orders* in perturbation theory. Upon defining renormalized fields according to

$$\widetilde{M}^{\alpha\beta} = Z_{\widetilde{M}}^{1/2} \widetilde{M}_0^{\alpha\beta} , \quad M^{\alpha\beta} = Z_M^{1/2} M_0^{\alpha\beta} , \tag{3.6}$$

$$\widetilde{S}^\alpha = Z_{\widetilde{S}}^{1/2} \widetilde{S}_0^\alpha , \quad S^\alpha = Z_S^{1/2} S_0^\alpha , \tag{3.7}$$

and using $\Gamma_{\widetilde{M}M} = (Z_{\widetilde{M}} Z_M)^{-1/2} \Gamma_{0\,\widetilde{M}M}$, we thus obtain the exact relation

$$Z_{\widetilde{M}} Z_M \equiv 1 . \tag{3.8}$$

At this point we utilize our freedom of choice to set

$$Z_{\widetilde{M}} \equiv Z_M \equiv 1 . \tag{3.9}$$

Similarly, for the order parameter fields we demand that $\partial \Gamma_{\widetilde{S}S}(\mathbf{q}=\mathbf{0},\omega)/\partial(i\omega)$ be finite at the normalization point, which yields after evaluating the integrals in dimensional renormalization [23],

$$(Z_{\widetilde{S}} Z_S)^{1/2} = 1 + \frac{n-1}{\epsilon} \left(1 - \frac{d_\parallel}{d} T_0^\parallel - \frac{d_\perp}{d} T_0^\perp \right) \frac{w_0 \widetilde{f}_0 A_d \mu^{-\epsilon}}{(1+w_0)^2} . \tag{3.10}$$

Here, $\epsilon = 4-d$, $A_d = \Gamma(3-d/2)/2^{d-1}\pi^{d/2}$ is a geometric factor (non-singular near $d_c = 4$), $T_0^\parallel$ and $T_0^\perp$ denote the ratios defined in Eq. (2.10), and the effective dynamic couplings are

$$w_0 = \frac{\lambda_0}{D_0} , \tag{3.11}$$

$$\widetilde{f}_0 = \frac{\widetilde{\lambda}_0}{\lambda_0} f_0 = \frac{\widetilde{\lambda}_0}{\lambda_0} \frac{g_0^2}{\lambda_0 D_0} . \tag{3.12}$$

Notice that in equilibrium, when $T_0^\parallel = T_0^\perp = 1$, Eq. (3.10) yields $Z_{\widetilde{S}} Z_S^{1/2} = 1$ to this order. In the isotropic noise case, with $T_0^\parallel = T_0^\perp$, or equivalently, either $d_\parallel$ or $d_\perp = 0$, we recover the result cited in Ref. [5]. In the same way, all the subsequently found Z factors reduce to the results for the isotropic non-equilibrium SSS model when $T_0^\parallel = T_0^\perp = T_0$, and these in turn to the well-established equilibrium expressions for $T_0 = 1$.

As a next step, we compute the three-point vertex function $\Gamma_{0\,\widetilde{S}SM}$ at zero external momenta and frequencies. Upon defining the dimensionless renormalized mode-coupling constant

$$g = Z_g^{1/2} g_0 A_d^{1/2} \mu^{-\epsilon/2} , \tag{3.13}$$

this provides us with the product of Z factors

$$(Z_{\widetilde{S}} Z_S Z_M Z_g)^{1/2} = 1 + \frac{n-1}{\epsilon} \left(1 - \frac{d_\parallel}{d} T_0^\parallel - \frac{d_\perp}{d} T_0^\perp \right) \frac{w_0 \widetilde{f}_0 A_d \mu^{-\epsilon}}{(1+w_0)^2} . \tag{3.14}$$

Direct comparison with Eq. (3.10) implies

$$Z_g = Z_M^{-1} = Z_{\widetilde{M}} = 1 , \tag{3.15}$$

where the choice (3.9) was employed. As shown in the Appendix, as a consequence of the $O(n)$ invariance and the fact that the $M_0^{\alpha\beta}$ are the generators of the rotation symmetry group, one may derive a Ward identity leading to the *exact* relation $Z_g Z_M \equiv 1$. In equilibrium, this result is trivial, and a simple consequence of the fact that the conserved fields $M_0^{\alpha\beta}$ are non-critical. The absence of field renormalization therefore follows directly from the purely quadratic appearance of the fields $M_0^{\alpha\beta}$ in the hamiltonian (2.3), which immediately implies that the static response function for the generalized angular momenta is $X_0(\mathbf{q},\omega=0) = 1$ exactly. While this relation holds even in our variant of the SSS model with dynamic anisotropy, see Eq. (3.26) below, one cannot directly infer the renormalization constant Z_M therefrom, as there is no fluctuation-dissipation theorem to relate this response function with the corresponding correlation function in the general non-equilibrium situation. Therefore Z_g needs to be computed explicitly from the three-point vertex function, or inferred from the above-mentioned Ward identity.

The other vertex function renormalizing the mode-coupling constant g_0 is $\Gamma_{0\,\widetilde{M}SS}(-\mathbf{q},-\omega;\frac{\mathbf{q}}{2}-\mathbf{p},\frac{\omega}{2};\frac{\mathbf{q}}{2}+\mathbf{p},\frac{\omega}{2}) = 2g_0(\mathbf{q}\cdot\mathbf{p})+O(g_0^3)$. Thus, in order to obtain Z_S, we need to take a derivative with respect to $(\mathbf{q}\cdot\mathbf{p})$; but owing to the dynamical anisotropy, the result depends on whether the components q_i and p_i lie in the longitudinal or transverse sector in momentum space, respectively. This means that we have to introduce *different* field renormalizations $Z_S^{\parallel}$ and $Z_S^{\perp}$ for the longitudinal and transverse field fluctuations, for which one then finds to one-loop order

$$\begin{aligned} Z_S^{\parallel/\perp} &= 1 + \\ &+\frac{n-1}{2\epsilon}\left(1-\frac{d_\parallel}{d}T_0^\parallel-\frac{d_\perp}{d}T_0^\perp\right)\frac{\widetilde{f}_0 A_d\mu^{-\epsilon}}{(1+w_0)^2}\mp \\ &\mp\frac{n-1}{6\epsilon}\frac{d_{\perp/\parallel}}{d}\left(T_0^\parallel-T_0^\perp\right)\frac{1+2w_0}{(1+w_0)^2}\,\widetilde{f}_0 A_d\mu^{-\epsilon}\ . \end{aligned} \tag{3.16}$$

Obviously, for $T_0^\parallel = T_0^\perp$, this novel distinction between $Z_S^\parallel$ and $Z_S^\perp$ disappears, and the isotropic result of Ref. [5] is recovered. Combining Eq. (3.16) with Eq. (3.10) yields

$$\begin{aligned} Z_{\widetilde{S}}^{\parallel/\perp} &= 1 - \\ &-\frac{n-1}{2\epsilon}\left(1-\frac{d_\parallel}{d}T_0^\parallel-\frac{d_\perp}{d}T_0^\perp\right)\frac{1-4w_0}{(1+w_0)^2}\,\widetilde{f}_0 A_d\mu^{-\epsilon}\pm \\ &\pm\frac{n-1}{6\epsilon}\frac{d_{\perp/\parallel}}{d}\left(T_0^\parallel-T_0^\perp\right)\frac{1+2w_0}{(1+w_0)^2}\,\widetilde{f}_0 A_d\mu^{-\epsilon}\ . \end{aligned} \tag{3.17}$$

Next we define renormalized transport coefficients and noise strengths according to [24]

$$\widetilde{D}^{\parallel/\perp} = Z_{\widetilde{D}}^{\parallel/\perp}\,\widetilde{D}_0^{\parallel/\perp}\ ,\quad D^{\parallel/\perp} = Z_D^{\parallel/\perp}\,D_0\ , \tag{3.18}$$

$$\widetilde{\lambda}^{\parallel/\perp} = Z_{\widetilde{\lambda}}^{\parallel/\perp}\,\widetilde{\lambda}_0\ ,\quad \lambda^{\parallel/\perp} = Z_\lambda^{\parallel/\perp}\,\lambda_0\ , \tag{3.19}$$

where we allow for *different* renormalizations in the parallel and transverse sectors of the originally isotropic parameters D_0, $\widetilde{\lambda}_0$, and λ_0. The renormalized noise coefficients can be obtained by demanding that the vertex functions $\partial_{q^2_{\parallel/\perp}}\Gamma_{\widetilde{M}\widetilde{M}}(\mathbf{q}_{\parallel/\perp},\omega=0)|_{\mathbf{q}_{\parallel/\perp}=\mathbf{0}}$ and $\Gamma_{\widetilde{S}\widetilde{S}}(\mathbf{q}=\mathbf{0},\omega=0)$ be UV-finite. This yields, with $Z_{\widetilde{M}}=1$,

$$Z_{\widetilde{D}}^{\parallel/\perp} = 1 + \frac{1}{2\epsilon}\,\frac{\widetilde{f}_0 A_d\mu^{-\epsilon}}{T_0^{\parallel/\perp}}\ , \tag{3.20}$$

and

$$Z_{\widetilde{S}}Z_{\widetilde{\lambda}} = 1 + \frac{n-1}{\epsilon}\left(\frac{d_\parallel}{d}T_0^\parallel+\frac{d_\perp}{d}T_0^\perp\right)\frac{\widetilde{f}_0 A_d\mu^{-\epsilon}}{1+w_0}\ . \tag{3.21}$$

While this product is still isotropic, the anisotropy in the field renormalization (3.17) induces different renormalized order parameter noise strengths in the longitudinal and transverse sectors.

In the same manner, from $\partial_{q^2}\Gamma_{\widetilde{M}M}(\mathbf{q},\omega=0)|_{\mathbf{q}=\mathbf{0}}$ and $\partial_{q^2_{\parallel/\perp}}\Gamma_{\widetilde{S}S}(\mathbf{q}_{\parallel/\perp},\omega=0)|_{\mathbf{q}_{\parallel/\perp}=\mathbf{0}}$ we obtain

$$Z_D = 1 + \frac{1}{2\epsilon}\,\widetilde{f}_0 A_d\mu^{-\epsilon}\ , \tag{3.22}$$

and

$$\begin{aligned} (Z_{\widetilde{S}}Z_S)^{1/2}\,Z_\lambda^{\parallel/\perp} &= 1 + \frac{n-1}{\epsilon}\,\frac{\widetilde{f}_0 A_d\mu^{-\epsilon}}{1+w_0} + \\ &+\frac{n-1}{\epsilon}\left(1-\frac{d_\parallel}{d}T_0^\parallel-\frac{d_\perp}{d}T_0^\perp\right)\frac{w_0\widetilde{f}_0 A_d\mu^{-\epsilon}}{(1+w_0)^3}\mp \\ &\mp\frac{n-1}{3\epsilon}\frac{d_{\perp/\parallel}}{d}\left(T_0^\parallel-T_0^\perp\right)\frac{w_0^2\widetilde{f}_0 A_d\mu^{-\epsilon}}{(1+w_0)^3}\ . \end{aligned} \tag{3.23}$$

Thus, to one-loop order at least, we need not distinguish between the renormalized diffusion constants $D^\parallel$ and $D^\perp$. This concludes our multiplicative vertex function renormalization for the dynamical parameters of the non-equilibrium SSS model with dynamical anisotropy. Notice that the anisotropic parts of the renormalization constants are always proportional to $T_0^\parallel - T_0^\perp$. For $T_0^\parallel = T_0^\perp = 1$, we recover the equilibrium results where $Z_{\widetilde{D}} = Z_D$ and $Z_{\widetilde{\lambda}} = Z_\lambda$, reflecting the Einstein relation.

3.1.3 Statics: Response function renormalization

In order to define the "static" limit of the intrinsically dynamic model under consideration here, we compute the response functions for the generalized angular momenta and the order parameter components, and then take the limit $\omega \to 0$ there. By adding external fields to the hamiltonian (2.3), one may show that the dynamic susceptibilities for the conserved quantities $M_0^{\alpha\beta}$ and the order parameter fluctuations S_0^α are given by

$$\begin{aligned} X_0(\mathbf{q},\omega) &= \Gamma_{0\,\widetilde{M}M}(-\mathbf{q},-\omega)^{-1}\times \\ &\qquad\times\left[D_0q^2+2g_0\,\Gamma_{0\,\widetilde{M}[\widetilde{S}S]}(-\mathbf{q},-\omega)\right], \end{aligned} \tag{3.24}$$

$$\begin{aligned} \chi_0(\mathbf{q},\omega) &= \Gamma_{0\,\widetilde{S}S}(-\mathbf{q},-\omega)^{-1}\times \\ &\qquad\times\left[\lambda_0-g_0\,\Gamma_{0\,\widetilde{S}[\widetilde{M}S]}(-\mathbf{q},-\omega)\right]\ , \end{aligned} \tag{3.25}$$

respectively [21,5]. Notice that composite-operator vertex functions enter these expressions, which in general implies that new renormalization constants are required to remove the UV singularities of the response functions (equivalently, one may utilize the Z factors obtained from the multiplicative renormalization of the vertex functions plus appropriate additive renormalizations [21,5]).

Yet one may show to *all orders* in perturbation theory that

$$\Gamma_{0\,\widetilde{M}M}(\mathbf{q},\omega) = i\omega + D_0q^2 + 2g_0\,\Gamma_{0\,\widetilde{M}[\widetilde{S}S]}(\mathbf{q},\omega)\ , \tag{3.26}$$

and consequently

$$X_0(\mathbf{q},\omega=0) \equiv 1\ , \tag{3.27}$$

which means that there is no additional renormalization required here. On the other hand, the static limit of the order parameter susceptibility is in fact singular, which leads us to define the corresponding renormalized response function via

$$\chi(\mathbf{q},\omega) = Z\,\chi_0(\mathbf{q},\omega)\ . \tag{3.28}$$

The new renormalization constant Z is determined by demanding that $\partial_{q^2_{\parallel/\perp}}\chi(\mathbf{q}_{\parallel/\perp},\omega=0)^{-1}|_{\mathbf{q}_{\parallel/\perp}=\mathbf{0}}$ be UV-finite; once again, the result is different in the longitudinal and transverse momentum space sectors:

$$Z^{\parallel/\perp} = 1 + \frac{n-1}{\epsilon}\left(1 - \frac{d_\parallel}{d}T_0^\parallel - \frac{d_\perp}{d}T_0^\perp\right)\frac{w_0\widetilde{f}_0 A_d\mu^{-\epsilon}}{(1+w_0)^3} \mp \mp\frac{n-1}{3\epsilon}\frac{d_{\perp/\parallel}}{d}\left(T_0^\parallel - T_0^\perp\right)\frac{w_0^2\widetilde{f}_0 A_d\mu^{-\epsilon}}{(1+w_0)^3}\ . \tag{3.29}$$

To one-loop order, $Z^{\parallel/\perp}$ as well as $Z_S^{\parallel/\perp}$ and $Z_{\widetilde{S}}^{\parallel/\perp}$, do not contain the static non-linear coupling u_0.

This is different for the remaining two Z factors needed for the renormalized dimensionless distance from the critical point τ and the static non-linearity u [24],

$$\tau^{\parallel/\perp} = Z_\tau^{\parallel/\perp}\,\tau_0\,\mu^{-2}\ ,\quad \tau_0 = r_0 - r_{0c}\ , \tag{3.30}$$

$$u^{\parallel/\perp} = Z_u^{\parallel/\perp}\,u_0\,A_d\,\mu^{-\epsilon}\ . \tag{3.31}$$

The fluctuation-induced T_c shift is determined from the criticality condition $\chi_0(\mathbf{q}=\mathbf{0},\omega=0)^{-1}=0$ at $r_0=r_{0c}$ ($\tau_0=0$) with the result

$$r_{0c} = -\frac{n+2}{6}\,\widetilde{u}_0\int_k\frac{1}{r_{0c}+k^2} + (n-1)\times \times\left(1 - \frac{d_\parallel}{d}T_0^\parallel - \frac{d_\perp}{d}T_0^\perp\right)\widetilde{f}_0\int_k\frac{1}{w_0(r_{0c}+k^2)+k^2}\ . \tag{3.32}$$

In principle, these momentum integrals should be evaluated with a *finite* upper cutoff, which underlines the non-universality of the T_c shift, i.e., its dependence on short-distance properties. However, if we choose to evaluate the momentum integrals by means of dimensional regularization, we are led to

$$|r_{0c}| = \left(\frac{2A_d}{(d-2)(4-d)}\left[\frac{n+2}{6}\,\widetilde{u}_0 - \frac{n-1}{1+w_0}\times \times\left(\frac{w_0}{1+w_0}\right)^{\frac{d}{2}-1}\left(1 - \frac{d_\parallel}{d}T_0^\parallel - \frac{d_\perp}{d}T_0^\perp\right)\widetilde{f}_0\right]\right)^{\frac{2}{4-d}}\ , \tag{3.33}$$

where we have defined

$$\widetilde{u}_0 = \frac{\widetilde{\lambda}_0}{\lambda_0}\,u_0\ . \tag{3.34}$$

Rendering $\chi(\mathbf{q}=\mathbf{0},\omega=0)^{-1}$ UV-finite, after substituting $r_0=\tau_0+r_{0c}$, then yields the isotropic product

$$Z\,Z_\tau = 1 - \frac{n+2}{6\epsilon}\,\widetilde{u}_0A_d\mu^{-\epsilon} + +\frac{n-1}{\epsilon}\left(1 - \frac{d_\parallel}{d}T_0^\parallel - \frac{d_\perp}{d}T_0^\perp\right)\frac{w_0\widetilde{f}_0A_d\mu^{-\epsilon}}{(1+w_0)^2}\ . \tag{3.35}$$

The anisotropy in the $Z^{\parallel/\perp}$ from Eq. (3.29) then induces the different Z factors for the longitudinal and transverse momentum space sectors anticipated in Eq. (3.30). We note that alternatively we could have used $\Gamma_{\widetilde{S}S}(\mathbf{0},0)$, providing the (isotropic) combination $(Z_{\widetilde{S}}Z_S)^{1/2}Z_\lambda Z_\tau$. (Indeed, the anisotropic contributions to $Z^{\parallel/\perp}$ [Eq. (3.29)] and to $Z_\lambda^{\parallel/\perp}$ [Eq. (3.23)] are identical.)

Finally, we need the renormalization of the static coupling u_0, which we may obtain from the four-point function $\Gamma_{\widetilde{S}SSS}$ at vanishing external wavevectors and frequencies. To one-loop order, there appear ten Feynman diagrams, the contributions of nine of which cancel in equilibrium (and *only* there!). A somewhat tedious calculation eventually gives

$$(Z_{\widetilde{S}}Z_S)^{1/2}\,Z_S\,Z_\lambda\,Z_u = 1 - \frac{n+8}{6\epsilon}\,\widetilde{u}_0A_d\mu^{-\epsilon} + \frac{n-1}{\epsilon}\,\frac{\widetilde{f}_0A_d\mu^{-\epsilon}}{1+w_0} - -\frac{n-1}{\epsilon}\left(1 - \frac{d_\parallel}{d}T_0^\parallel - \frac{d_\perp}{d}T_0^\perp\right)\frac{\widetilde{f}_0A_d\mu^{-\epsilon}}{(1+w_0)^2} - +\frac{n-1}{\epsilon}\left(1 - \frac{d_\parallel}{d}T_0^\parallel - \frac{d_\perp}{d}T_0^\perp\right)\frac{6\widetilde{f}_0^2A_d\mu^{-\epsilon}}{(1+w_0)\widetilde{u}_0}\ . \tag{3.36}$$

The anisotropies in $Z_S^{\parallel/\perp}Z_\lambda^{\parallel/\perp}$ again cause the differences in $Z_u^\parallel$ and $Z_u^\perp$. No further UV renormalizations are required, and we may now turn to the analysis of the ensuing RG flow equations.

3.2 Discussion of the RG flow equations

3.2.1 RG equations for the vertex and response functions

By means of the above renormalization constants, we can now write down the RG (Gell-Mann–Low) equations for the vertex functions and the dynamic susceptibilities. The latter connect the asymptotic theory, where the IR singularities become manifest, with a region in parameter space where the loop integrals are finite and ordinary "naive" perturbation expansion is applicable, and follow from the simple observation that the "bare" vertex functions do not depend on the renormalization scale μ,

$$\mu\frac{d}{d\mu}\bigg|_0\Gamma_{0\,\widetilde{M}^k\widetilde{S}^rM^lS^s}(\{\mathbf{q},\omega\};\{a_0\}) = 0\ , \tag{3.37}$$

with $\{a_0\} = g_0,\widetilde{D}_0^\parallel,\widetilde{D}_0^\perp,\ D_0,\widetilde{\lambda}_0,\lambda_0,\tau_0,u_0$. For the non-equilibrium SSS model with dynamical anisotropy, we have to treat the longitudinal and transverse sectors in momentum space *separately*, i.e., we need to understand the flow of the renormalized set of parameters $\{a^\parallel\} = g,\widetilde{D}^\parallel,D^\parallel,\ \widetilde{\lambda}^\parallel,\lambda^\parallel,\tau^\parallel,\ u^\parallel$, and $\{a^\perp\} = g,\widetilde{D}^\perp,D^\perp,\ \widetilde{\lambda}^\perp,\lambda^\perp,\tau^\perp,u^\perp$, respectively. Replacing the bare parameters and fields in Eq. (3.37) with the renormalized ones, we thus find the following partial differential equations for the renormalized vertex functions in the longitudinal and transverse sectors,

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_{\{a^{\parallel/\perp}\}}\zeta_a^{\parallel/\perp}a^{\parallel/\perp}\frac{\partial}{\partial a^{\parallel/\perp}}+\frac{r}{2}\zeta_{\widetilde{S}}^{\parallel/\perp}+\frac{s}{2}\zeta_S^{\parallel/\perp}\right]\times$$
$$\times\Gamma_{\widetilde{M}^k\widetilde{S}^rM^lS^s}\left(\mu,\{\mathbf{q}_{\parallel/\perp},\omega\};\{a^{\parallel/\perp}\}\right)=0\ . \qquad (3.38)$$

Here, we have introduced Wilson's flow functions

$$\zeta_{\widetilde{S}}^{\parallel/\perp}=\mu\frac{\partial}{\partial\mu}\bigg|_0\ln Z_{\widetilde{S}}^{\parallel/\perp}\ , \qquad (3.39)$$

$$\zeta_S^{\parallel/\perp}=\mu\frac{\partial}{\partial\mu}\bigg|_0\ln Z_S^{\parallel/\perp}\ , \qquad (3.40)$$

and

$$\zeta_a^{\parallel/\perp}=\mu\frac{\partial}{\partial\mu}\bigg|_0\ln\frac{a^{\parallel/\perp}}{a_0} \qquad (3.41)$$

(the index "0" indicates that the renormalized fields and parameters are to be expressed in terms of their bare counterparts prior to performing the derivatives with respect to the momentum scale μ). Note that $\zeta_{\widetilde{M}}=\zeta_M\equiv 0$ and $\zeta_g\equiv-\epsilon/2$ as a consequence of Eqs. (3.15) and (3.13).

The Gell-Mann–Low equation (3.38) is readily solved with the method of characteristics $\mu\to\mu\ell$; this defines running couplings as the solutions to the first-order differential RG flow equations

$$\ell\,\frac{da^{\parallel/\perp}(\ell)}{d\ell}=\zeta_a^{\parallel/\perp}(\ell)\,a^{\parallel/\perp}(\ell)\,,\ a^{\parallel/\perp}(1)=a^{\parallel/\perp}\ . \qquad (3.42)$$

The solutions of the partial differential equations (3.38) then read

$$\Gamma_{\widetilde{M}^k\widetilde{S}^rM^lS^l}\left(\mu,\{\mathbf{q}_{\parallel/\perp},\omega\};\{a^{\parallel/\perp}\}\right)= \qquad (3.43)$$
$$=\exp\left\{\frac{1}{2}\int_1^\ell\left[r\,\zeta_{\widetilde{S}}^{\parallel/\perp}(\ell')+s\,\zeta_S^{\parallel/\perp}(\ell')\right]\frac{d\ell'}{\ell'}\right\}\times$$
$$\times\Gamma_{\widetilde{M}^k\widetilde{S}^rM^lS^s}\left(\mu\ell,\{\mathbf{q}_{\parallel/\perp},\omega\};\{a^{\parallel/\perp}(\ell)\}\right)\ .$$

In the same manner, one can solve the RG equations for the dynamic susceptibilities, with the results

$$X\left(\mu,\{\mathbf{q}_{\parallel/\perp},\omega\};\{a^{\parallel/\perp}\}\right)= \qquad (3.44)$$
$$=X\left(\mu\ell,\{\mathbf{q}_{\parallel/\perp},\omega\};\{a^{\parallel/\perp}(\ell)\}\right)\ ,$$

and

$$\chi\left(\mu,\{\mathbf{q}_{\parallel/\perp},\omega\};\{a^{\parallel/\perp}\}\right)= \qquad (3.45)$$
$$=\exp\left\{-\int_1^\ell\zeta^{\parallel/\perp}(\ell')\frac{d\ell'}{\ell'}\right\}\times$$
$$\times\chi\left(\mu\ell,\{\mathbf{q}_{\parallel/\perp},\omega\};\{a^{\parallel/\perp}(\ell)\}\right)\ ,$$

where, in analogy with Eq. (3.40),

$$\zeta^{\parallel/\perp}=\mu\frac{\partial}{\partial\mu}\bigg|_0\ln Z^{\parallel/\perp}\ . \qquad (3.46)$$

Upon introducing renormalized anisotropic counterparts for the effective dynamic couplings (3.11), (3.12), and (2.10),

$$w^{\parallel/\perp}=\frac{\lambda^{\parallel/\perp}}{D^{\parallel/\perp}}\ , \qquad (3.47)$$

$$\widetilde{f}^{\parallel/\perp}=\frac{\widetilde{\lambda}^{\parallel/\perp}}{\lambda^{\parallel/\perp}}\,\frac{g^2}{\lambda^{\parallel/\perp}D^{\parallel/\perp}}\ , \qquad (3.48)$$

$$T^{\parallel/\perp}=\frac{\widetilde{D}^{\parallel/\perp}}{D^{\parallel/\perp}}\,\frac{\lambda^{\parallel/\perp}}{\widetilde{\lambda}^{\parallel/\perp}}\ , \qquad (3.49)$$

the zeta functions to one-loop order read explicitly

$$\zeta_{\widetilde{S}}^{\parallel/\perp}=\frac{n-1}{2}\left(1-\frac{d_\parallel}{d}T^\parallel-\frac{d_\perp}{d}T^\perp\right)\frac{(1-4w^{\parallel/\perp})\,\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^2}$$
$$\mp\frac{n-1}{6}\,\frac{d_{\perp/\parallel}}{d}\left(T^\parallel-T^\perp\right)\frac{(1+2w^{\parallel/\perp})\,\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^2}\ , \qquad (3.50)$$

$$\zeta_S^{\parallel/\perp}=-\frac{n-1}{2}\left(1-\frac{d_\parallel}{d}T^\parallel-\frac{d_\perp}{d}T^\perp\right)\frac{\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^2}\pm$$
$$\pm\frac{n-1}{6}\,\frac{d_{\perp/\parallel}}{d}\left(T^\parallel-T^\perp\right)\frac{(1+2w^{\parallel/\perp})\,\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^2}\ , \qquad (3.51)$$

$$\zeta^{\parallel/\perp}=-(n-1)\left(1-\frac{d_\parallel}{d}T^\parallel-\frac{d_\perp}{d}T^\perp\right)\frac{w^{\parallel/\perp}\,\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^3}\mp$$
$$\mp\frac{n-1}{3}\,\frac{d_{\perp/\parallel}}{d}\left(T^\parallel-T^\perp\right)\frac{(w^{\parallel/\perp})^2\,\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^3}\ , \qquad (3.52)$$

$$\zeta_g\equiv-\frac{\epsilon}{2}\ , \qquad (3.53)$$

$$\zeta_{\widetilde{D}}^{\parallel/\perp}=-\frac{1}{2}\,\frac{\widetilde{f}^{\parallel/\perp}}{T^{\parallel/\perp}}\ , \qquad (3.54)$$

$$\zeta_{\widetilde{\lambda}}^{\parallel/\perp}=-(n-1)\frac{\widetilde{f}^{\parallel/\perp}}{1+w^{\parallel/\perp}}+$$
$$+\frac{n-1}{2}\left(1-\frac{d_\parallel}{d}T^\parallel-\frac{d_\perp}{d}T^\perp\right)\frac{(1+6w^{\parallel/\perp})\,\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^2}\mp$$
$$\mp\frac{n-1}{6}\,\frac{d_{\perp/\parallel}}{d}\left(T^\parallel-T^\perp\right)\frac{(1+2w^{\parallel/\perp})\,\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^2}\ , \qquad (3.55)$$

$$\zeta_D^{\parallel/\perp}=-\frac{1}{2}\,\widetilde{f}^{\parallel/\perp}\ , \qquad (3.56)$$

$$\zeta_\lambda^{\parallel/\perp}=-(n-1)\frac{\widetilde{f}^{\parallel/\perp}}{1+w^{\parallel/\perp}}+$$
$$+(n-1)\left(1-\frac{d_\parallel}{d}T^\parallel-\frac{d_\perp}{d}T^\perp\right)\frac{(w^{\parallel/\perp})^2\,\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^3}\mp$$
$$\mp\frac{n-1}{3}\,\frac{d_{\perp/\parallel}}{d}\left(T^\parallel-T^\perp\right)\frac{(w^{\parallel/\perp})^2\,\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^3}\ , \qquad (3.57)$$

$$\zeta_\tau^{\parallel/\perp}=-2+\frac{n+2}{6}\,\widetilde{u}^{\parallel/\perp}-$$
$$-(n-1)\left(1-\frac{d_\parallel}{d}T^\parallel-\frac{d_\perp}{d}T^\perp\right)\frac{{w^{\parallel/\perp}}^2\,\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^3}\mp$$
$$\mp\frac{n-1}{3}\,\frac{d_{\perp/\parallel}}{d}\left(T^\parallel-T^\perp\right)\frac{{w^{\parallel/\perp}}^2\,\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^3}\ , \qquad (3.58)$$

$$\zeta_u^{\parallel/\perp} = -\epsilon + \frac{n+2}{8}\,\widetilde{u}^{\parallel/\perp} + \quad (3.59)$$
$$+\frac{n-1}{2}\left(1-\frac{d_\parallel}{d}T^\parallel - \frac{d_\perp}{d}T^\perp\right)\frac{(3+5w^{\parallel/\perp})\,\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^3} -$$
$$-(n-1)\left(1-\frac{d_\parallel}{d}T^\parallel - \frac{d_\perp}{d}T^\perp\right)\frac{6(\widetilde{f}^{\parallel/\perp})^2}{(1+w^{\parallel/\perp})\,\widetilde{u}^{\parallel/\perp}} \pm$$
$$\pm\frac{n-1}{6}\,\frac{d_{\perp/\parallel}}{d}\left(T^\parallel - T^\perp\right)\frac{(1+3w^{\parallel/\perp}+4w^{\parallel/\perp 2})\widetilde{f}^{\parallel/\perp}}{(1+w^{\parallel/\perp})^3}\;.$$

These results enable us now to study the scaling behavior of the non-equilibrium SSS model with dynamical noise in the vicinity of the different RG fixed points, which are given by the zeros of the appropriate RG beta functions

$$\beta_v = \mu \frac{\partial}{\partial\mu}\bigg|_0 v\;. \quad (3.60)$$

According to

$$\ell\,\frac{dv(\ell)}{d\ell} = \beta_v(\{v(\ell)\})\;, \quad (3.61)$$

these govern the flow of the effective couplings $T^{\parallel/\perp}$, $w^{\parallel/\perp}$, $\widetilde{f}^{\parallel/\perp}$, and $\widetilde{u}^{\parallel/\perp}$, etc. under scale transformations $\mu \to \mu\ell$, and the fixed points $\{v^*\}$ where all $\beta_v(\{v^*\}) = 0$ thus describe scale-invariant regimes.

3.2.2 RG fixed points and critical exponents

We begin with considering the RG flow of the anisotropy parameter

$$\sigma = T^\perp / T^\parallel\;, \quad (3.62)$$

denoting the ratio of the effective conserved noise temperatures (3.49) in the transverse and longitudinal sectors ($0 \leq \sigma \leq 1$). Obviously, $\sigma = 1$ describes the isotropic fixed point. In order to assess its stability against the anisotropic non-equilibrium perturbation, we consider the RG beta function

$$\beta_\sigma = \mu\frac{\partial}{\partial\mu}\bigg|_0 \sigma = \quad (3.63)$$
$$= \sigma\left(\zeta_{\widetilde{D}}^\perp - \zeta_{\widetilde{D}}^\parallel - \zeta_D^\perp + \zeta_D^\parallel + \zeta_\lambda^\perp - \zeta_\lambda^\parallel - \zeta_{\widetilde{\lambda}}^\perp + \zeta_{\widetilde{\lambda}}^\parallel\right)\;.$$

As to first order $w^\parallel = w^\perp = w$ and $\widetilde{f}^\perp = \widetilde{f}^\parallel = \widetilde{f}$ (in the vicinity of the isotropic fixed point, this holds even beyond the one-loop approximation), we may write

$$\beta_\sigma = -\sigma(1-\sigma)\widetilde{f}T^\parallel\left[\frac{1}{2T^\parallel T^\perp} + \frac{n-1}{6}\,\frac{1+3w}{(1+w)^3}\right]\;. \quad (3.64)$$

The expression in square brackets is positive, and thus, as to be expected on physical grounds, there are only two fixed points, namely $\sigma^* = 1$ and $\sigma^* = 0$, realized for $T^\perp = 0$ and $T^\parallel = \infty$. (Of course, if we allow for $\sigma > 1$ as well, there is also the fixed point $\sigma^* = \infty$, realized for $T^\parallel = 0$ and $T^\perp = \infty$; yet clearly the regimes $\sigma < 1$ and $\sigma > 1$ map onto each other through simply relabeling the

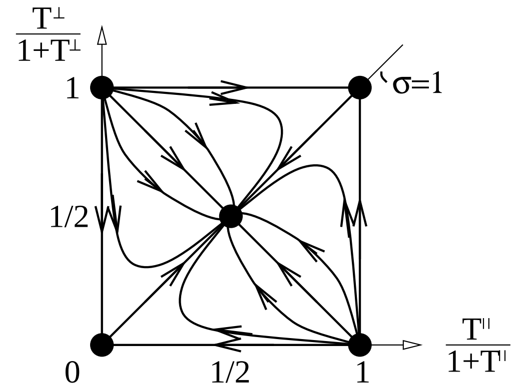

Fig. 3.1. Equilibrium (center) and non-equilibrium (corners) fixed points of the non-equilibrium SSS model with dynamical anisotropy in the $[T^\parallel/(1+T^\parallel), T^\perp/(1+T^\perp)]$–plane.

indices $\parallel\leftrightarrow\perp$.) Furthermore, in the IR regime ($\ell \to 0$), if $0 < \sigma < 1$ initially, $\beta_\sigma < 0$, and thus $\sigma(\ell)$ grows until it reaches the isotropic fixed point $\sigma^* = 1$ (and conversely, if $1 < \sigma < \infty$, then $\beta_\sigma > 0$ and $\sigma(\ell)$ decreases towards $\sigma^* = 1$). Thus, the *isotropic* fixed point is *stable* against the spatially anisotropic perturbations in the noise correlator of the conserved generalized angular momenta. Figure 3.1 depicts these various fixed points in the non-equilibrium SSS model with dynamical anisotropy, and the parameter flow in the $[T^\parallel/(1+T^\parallel), T^\perp/(1+T^\perp)]$–plane. The center of this diagram represents the equilibrium SSS dynamic fixed point ($\sigma = 1$, $T = 1$), and is attractive for the RG flows originating from any point inside the depicted square. Precisely on the edges in parameter space, we find the two isotropic non-equilibrium fixed points with $\sigma = 1$ and $T = 0$ in the lower left, and $T = \infty$ in the upper right corners, respectively, and the two anisotropic fixed points with $\sigma = 0$ and $\sigma = \infty$ in the lower right and upper left corners. Notice that the RG flows on this *critical* surface in parameter space which start in the vicinity of these anisotropic non-equilibrium fixed points tend towards the isotropic non-equilibrium fixed points, but eventually end up at the isotropic equilibrium fixed point provided that initially $0 < T^\parallel < \infty$ and $0 < T^\perp < \infty$.

Before we investigate the properties of the anisotropic fixed point $\sigma^* = 0$ ($T^\perp = 0$, $T^\parallel = \infty$), let us briefly summarize the behavior of the isotropic model with $\sigma^* = 1$. (For more details, and for a graph depicting the various equilibrium and non-equilibrium fixed points in the isotropic parameter subspace, we refer to Ref. [5].) Setting $T^\parallel = T^\perp = T$ in the flow functions (3.50)–(3.59), we find

$$\beta_T = T\left(\zeta_{\widetilde{D}} - \zeta_D + \zeta_\lambda - \zeta_{\widetilde{\lambda}}\right) = \quad (3.65)$$
$$= -T(1-T)\frac{\widetilde{f}}{2}\left[\frac{1}{T} + (n-1)\,\frac{1+7w+4w^2}{(1+w)^3}\right]\;.$$

Clearly, the only possible fixed points here are $T^* = 1$, $T^* = 0$, and $T^* = \infty$. At the equilibrium fixed point $T^* = 1$, one finds $\zeta_\lambda = -(n-1)f/(1+w)$ and $\zeta_D = -f/2$, and the beta functions for the couplings $w = \lambda/D$ and

$f = g^2/\lambda D$ read [15,16]

$$\beta_w = w\,(\zeta_\lambda - \zeta_D) = -wf \left(\frac{n-1}{1+w} - \frac{1}{2} \right) \ , \quad (3.66)$$

$$\beta_f = f\,(2\zeta_g - \zeta_\lambda - \zeta_D) = = f \left[-\epsilon + f \left(\frac{n-1}{1+w} + \frac{1}{2} \right) \right] \ . \quad (3.67)$$

The IR-stable fixed point (to one-loop order at least) turns out to be the strong-scaling SSS fixed point

$$w^* = 2n - 3 \ , \quad f^* = \epsilon + O(\epsilon^2) \quad (3.68)$$

with *equal* time scales governing the critical slowing down of the order parameter ($t_S^c \propto \xi^{z_S} \propto |\tau|^{-z_S \nu}$) and conserved generalized angular momenta fluctuations ($t_M^c \propto \xi^{z_M} \propto |\tau|^{-z_M \nu}$), respectively,

$$z_S = 2 + \zeta_\lambda^* \ , \quad z_M = 2 + \zeta_D^* \ . \quad (3.69)$$

Inserting the fixed-point values (3.68) yields the dynamic critical exponent

$$z = z_S = z_M = 2 - \frac{\epsilon}{2} = \frac{d}{2} \ , \quad (3.70)$$

which is actually an *exact* result, provided $z_S = z_M$ and $0 < f^* < \infty$ is finite, because Eq. (3.67) then requires that $2z = 4 + \zeta_\lambda^* + \zeta_D^* = 4 + 2\zeta_g^* = 4 - \epsilon = d$ [17,16]. The equilibrium static critical behavior is described by the zero of the beta function

$$\beta_u = u\zeta_u = u \left(-\epsilon + \frac{n+8}{6}\, u \right) \ , \quad (3.71)$$

which yields of course the $O(n)$ Heisenberg fixed point

$$u^* = \frac{6}{n+8}\, \epsilon + O(\epsilon^2) \quad (3.72)$$

with the associated two independent critical exponents

$$\nu^{-1} = -\zeta_\tau^* = 2 - \frac{n+2}{n+8}\, \epsilon + O(\epsilon^2) \ , \quad (3.73)$$

$$\eta = -\zeta^* = 0 + O(\epsilon^2) \ . \quad (3.74)$$

For the isotropic non-equilibrium fixed point with temperature ratio $T^* = \infty$, the appropriate effective mode-coupling constant becomes

$$\overline{f} = \frac{T}{w}\, \widetilde{f} = \frac{\widetilde{D}}{\lambda}\, f = \frac{\widetilde{D}}{\lambda}\, \frac{g^2}{\lambda D} \ , \quad (3.75)$$

in terms of which the beta functions read [5]

$$\beta_w = w\,(\zeta_\lambda - \zeta_D) = -(n-1)\, \frac{w^4\, \overline{f}}{(1+w)^3} \ , \quad (3.76)$$

$$\beta_{\overline{f}} = \overline{f} \left(2\zeta_g + \zeta_{\widetilde{D}} - 2\zeta_\lambda - \zeta_D \right) = = \overline{f} \left(-\epsilon + 2(n-1)\, \frac{w^3\, \overline{f}}{(1+w)^3} \right) \ , \quad (3.77)$$

$$\beta_{\widetilde{u}} = \widetilde{u} \left(\zeta_u + \zeta_{\widetilde{\lambda}} - \zeta_\lambda \right) = \quad (3.78)$$

$$= \widetilde{u} \left(-\epsilon + \frac{n+8}{6}\, \widetilde{u} - 2(n-1)\, \frac{w(1+3w+w^2)}{(1+w)^3}\, \overline{f} \right) \ .$$

Thus, the RG fixed points to $O(\epsilon)$ governing this scaling regime are

$$w^* = \infty \ , \quad \overline{f}^* = \frac{\epsilon}{2(n-1)} \ , \quad \widetilde{u}^* = \frac{12}{n+8}\, \epsilon \ . \quad (3.79)$$

Notice that the fixed point of the static coupling takes on twice the Heisenberg value (3.72); correspondingly, the "static" exponents will be modified as compared to the equilibrium results. E.g., the correlation length exponent now becomes

$$\nu^{-1} = 2 - 2\, \frac{n+2}{n+8}\, \epsilon - \frac{\epsilon}{2} + O(\epsilon^2) \quad (3.80)$$

instead of Eq. (3.74), while to one-loop order both the order parameter response and correlation function are characterized by the Wilson-Fisher exponent $\eta = 0 + O(\epsilon^2)$ as in equilibrium [5]. The characteristic time scales for the order parameter and the conserved quantities are now governed by different power laws, namely

$$z_S = 2 - \frac{\epsilon}{2} = \frac{d}{2} \ , \quad z_M = 2 \ . \quad (3.81)$$

As $T^* = \infty$ means that effectively the heat bath for the conserved generalized angular momenta is at infinite temperature, there is effectively an energy current into the order parameter heat bath, but no feedback. This explains why we find the coupled SSS model dynamic exponent for the order parameter fluctuations, while the generalized angular momenta correlations decay *faster* with the purely diffusive exponent $z_M = 2$. Finally, at the critical point there are non-trivial noise correlations $\propto q^\rho$ only for the order parameter noise,

$$\rho_S = 2 + \zeta_{\widetilde{\lambda}}^* = 2 - \frac{3}{2}\, \epsilon + O(\epsilon^2) \ , \quad (3.82)$$

while for the generalized angular momenta

$$\rho_M = 2 + \zeta_{\widetilde{D}}^* = 2 \ , \quad (3.83)$$

as to be expected at infinite temperature [5].

For the other isotropic non-equilibrium fixed point, being characterized by $T^* = 0$, the correct effective mode-coupling constant reads

$$\widetilde{f}' = \frac{\widetilde{f}}{T} = \frac{\widetilde{\lambda}^2 D}{\lambda^2 \widetilde{D}}\, \frac{g^2}{\lambda D} \quad (3.84)$$

(called $\widetilde{f}$ in Ref. [5]), and consequently $\beta_w = 0$, which leaves the fixed point w^* undetermined. The remaining RG beta functions are [5]

$$\beta_{\widetilde{f}'} = \widetilde{f}' \left(2\zeta_g + 2\zeta_{\widetilde{\lambda}} - 3\zeta_\lambda - \zeta_{\widetilde{D}} \right) = \widetilde{f}' \left(-\epsilon + \frac{1}{2}\, \widetilde{f}' \right) \ , \quad (3.85)$$

$$\beta_{\widetilde{u}} = \widetilde{u} \left(\zeta_u + \zeta_{\widetilde{\lambda}} - \zeta_\lambda \right) = \widetilde{u} \left(-\epsilon + \frac{n+8}{6}\, \widetilde{u} \right) \ , \quad (3.86)$$

with the $O(\epsilon)$ fixed points

$$\widetilde{f}'^{*} = 2\epsilon + O(\epsilon^2) \ , \quad \widetilde{u}^* = \frac{6}{n+8}\, \epsilon + O(\epsilon^2) \ . \tag{3.87}$$

As $\widetilde{u}^*$ is identical to the Heisenberg fixed point (3.72), it turns out that the static exponents are indeed those of the equilibrium static theory, (3.73) and (3.74), both for the order parameter response and correlation functions. Now the energy current flows from the order parameter heat bath towards the conserved generalized angular momenta, and thus the mode-coupling effects become negligible for the order parameter fluctuations. The dynamic exponents therefore become model-A like, with

$$z_S = z_M = 2 \tag{3.88}$$

to one-loop order, and with the critical noise exponents

$$\rho_S = 2 \ , \quad \rho_M = 2 - \epsilon = d - 2 \ . \tag{3.89}$$

Again, we observe that these anomalous power laws for the noise correlators apply for those quantities which are governed by the heat bath at *lower* temperature; here, the generalized angular momenta at effectively $T = 0$ [5].

The above discussion of the isotropic fixed points facilitates the interpretation of the novel fixed points of the SSS model with dynamic anisotropy. From Eq. (3.64) we had already inferred that at the fixed point with $\sigma^* = 0$, one must have ${T^{\|}}^* = \infty$ and ${T^{\perp}}^* = 0$. In analogy with Eqs. (3.75) and (3.84), it is therefore convenient to introduce new effective mode-coupling constants in the longitudinal and transverse sectors, respectively,

$$\overline{f}^{\|} = \frac{T^{\|}}{w^{\|}}\, \widetilde{f}^{\|} = \frac{\widetilde{D}^{\|}}{\lambda^{\|}}\, \frac{g^2}{\lambda^{\|} D^{\|}} \ , \tag{3.90}$$

$$\widetilde{f}'^{\perp} = \frac{\widetilde{f}^{\perp}}{T^{\perp}} = \frac{\widetilde{\lambda^{\perp}}^2 D^{\perp}}{{\lambda^{\perp}}^2 \widetilde{D}^{\perp}}\, \frac{g^2}{\lambda^{\perp} D^{\perp}} \ . \tag{3.91}$$

In terms of these couplings, one finds for $\sigma^* = 0$

$$\begin{aligned} \beta_{T^{\|}} = \frac{\overline{f}^{\|}}{2} \Bigg[& w^{\|} \left(1 - \frac{1}{T^{\|}} \right) - \\ & -(n-1) \left(1 - \frac{d_{\|}}{d}\, T^{\|} \right) \frac{w^{\|} (1 + 7w^{\|} + 4{w^{\|}}^2)}{(1+w^{\|})^3} + \\ & + \frac{n-1}{3}\, \frac{d_{\perp}}{d}\, T^{\|} \frac{w^{\|}(1+3w^{\|})}{(1+w^{\|})^3} \Bigg] \ , \end{aligned} \tag{3.92}$$

$$\begin{aligned} \beta_{T^{\perp}} = -T^{\perp} \frac{\widetilde{f}'^{\perp}}{2} \Bigg[& 1 - T^{\perp} + \\ & +(n-1) T^{\perp} \left(1 - \frac{d_{\|}}{d}\, T^{\|} \right) \frac{1 + 7w^{\perp} + 4{w^{\perp}}^2)}{(1+w^{\perp})^3} + \\ & + \frac{n-1}{3}\, \frac{d_{\|}}{d}\, T^{\|}\, T^{\perp}\, \frac{1+3w^{\perp})}{(1+w^{\perp})^3} \Bigg] \ , \end{aligned} \tag{3.93}$$

which indeed lead to the expected fixed points

$${T^{\|}}^* = \infty \ , \quad {T^{\perp}}^* = 0 \ , \tag{3.94}$$

with ${T^{\|}}^* {T^{\perp}}^* = 0$.

In the longitudinal sector, we may then write the beta functions for $w^{\|}$, $\overline{f}^{\|}$, and $\widetilde{u}^{\|}$ as

$$\beta_{w^{\|}} = -\frac{n-1}{3} \left(1 + 2\frac{d_{\|}}{d} \right) \frac{{w^{\|}}^4\, \overline{f}^{\|}}{(1+w^{\|})^3} \ , \tag{3.95}$$

$$\beta_{\overline{f}^{\|}} = \overline{f}^{\|} \left[-\epsilon + \frac{2(n-1)}{3} \left(1 + 2\frac{d_{\|}}{d} \right) \frac{{w^{\|}}^3\, \overline{f}^{\|}}{(1+w^{\|})^3} \right] \ , \tag{3.96}$$

$$\begin{aligned} \beta_{\widetilde{u}^{\|}} = \widetilde{u}^{\|} \Bigg[& -\epsilon + \frac{n+8}{6}\, \widetilde{u}^{\|} - \\ & -2(n-1)\, \frac{d_{\|}}{d}\, \frac{w^{\|}(1 + 3w^{\|} + {w^{\|}}^2)}{(1+w^{\|})^3}\, \overline{f}^{\|} + \\ & + \frac{2(n-1)}{3}\, \frac{d_{\perp}}{d}\, \frac{{w^{\|}}^3\, \overline{f}^{\|}}{(1+w^{\|})^3} \Bigg] \ , \end{aligned} \tag{3.97}$$

yielding the stable fixed point

$${w^{\|}}^* = \infty \ , \tag{3.98}$$

and consequently

$${\overline{f}^{\|}}^* = \frac{3}{2(n-1)(1+2d_{\|}/d)}\, \epsilon + O(\epsilon^2) \ , \tag{3.99}$$

$$\widetilde{u}^{\|*} = \frac{36\, d_{\|}/d}{(n+8)(1+2d_{\|}/d)}\, \epsilon + O(\epsilon^2) \ . \tag{3.100}$$

For $d_{\perp} = 0$, i.e., $d_{\|} = d$, these expressions reduce to the isotropic fixed point (3.79) with $T^* = \infty$, and in fact, we arrive at very similar results for the dynamic exponents,

$$z_S^{\|} = 2 + \zeta^*_{\lambda^{\|}} = 2 - \frac{\epsilon}{2} = \frac{d}{2} \ , \tag{3.101}$$

$$z_M^{\|} = 2 + \zeta^*_{D^{\|}} = 2 \ , \tag{3.102}$$

as well as for the anomalous noise exponents,

$$\rho_S^{\|} = 2 + \zeta^*_{\widetilde{\lambda}^{\|}} = 2 - \frac{1 + 8d_{\|}/d}{2(1+2d_{\|}/d)}\, \epsilon + O(\epsilon^2) \ , \tag{3.103}$$

$$\rho_M^{\|} = 2 + \zeta^*_{\widetilde{D}^{\|}} = 2 \ , \tag{3.104}$$

compare Eqs. (3.81) and (3.82), (3.83). Moreover, we again find a non-standard correlation length exponent

$$\begin{aligned} {\nu^{\|}}^{-1} = -\zeta^*_{\tau^{\|}} = 2 - & \frac{6(n+2)d_{\|}/d}{(n+8)(1+2d_{\|}/d)}\, \epsilon + \\ & + \frac{1 - 4d_{\|}/d}{2(1+2d_{\|}/d)}\, \epsilon + O(\epsilon^2) \ , \end{aligned} \tag{3.105}$$

but, in addition, non-trivial Wilson-Fisher exponents describing the critical decay of the order parameter response and correlation functions, respectively,

$$\eta^{\|} = -{\zeta^{\|}}^* = \frac{1 - d_{\|}/d}{2(1+2d_{\|}/d)}\, \epsilon + O(\epsilon^2) \ , \tag{3.106}$$

$$\eta_S^{\|} = -{\zeta_S^{\|}}^* = -\frac{1 - d_{\|}/d}{2(1+2d_{\|}/d)}\, \epsilon + O(\epsilon^2) \ ; \tag{3.107}$$

remarkably, $\eta_S^\parallel = -\eta^\parallel$ to $O(\epsilon)$. These anomalous critical exponents at the anisotropic fixed point, appearing in the longitudinal sector with effectively infinite heat bath temperature for the generalized angular momenta, are obviously a consequence of the spatially extremely anisotropic noise correlations in the conserved quantities. They may perhaps be interpreted as remnants of the elastic, pseudo-dipolar interactions generated in a model with conserved order parameter and dynamical anisotropy [6,7,8].

We now turn to the transverse sector, with the conserved quantities being effectively at zero temperature as ${T^\perp}^* = 0$. Quite as at the isotropic non-equilibrium SSS model fixed point with $T^* = 0$, we find that $\beta_{w^\perp} = 0$ and hence ${w^\perp}^*$ has no fixed value, while

$$\beta_{\widetilde{f}'^\perp} = \widetilde{f}'^\perp \left(-\epsilon + \frac{1}{2}\, \widetilde{f}'^\perp \right) \ , \tag{3.108}$$

$$\beta_{\widetilde{u}^\perp} = \widetilde{u}^\perp \left(-\epsilon + \frac{n+8}{6}\, \widetilde{u}^\perp \right) \ , \tag{3.109}$$

which yield the one-loop fixed points

$$\widetilde{f}'^{\perp *} = 2\epsilon + O(\epsilon^2) \ , \quad \widetilde{u}^{\perp *} = \frac{6}{n+8}\, \epsilon + O(\epsilon^2) \ . \tag{3.110}$$

The ensuing critical exponents are precisely those of the $T^* = 0$ isotropic non-equilibrium fixed point:

$$\nu^{\perp -1} = -\zeta_{\tau^{\perp *}} = 2 - \frac{n+2}{n+8}\, \epsilon + O(\epsilon^2) \ , \tag{3.111}$$

$$\eta^\perp = -{\zeta^\perp}^* = \eta_S^\perp = -{\zeta_S^\perp}^* = 0 + O(\epsilon^2) \ , \tag{3.112}$$

and

$$z_S^\perp = 2 + \zeta^*_{\lambda^\perp} = z_M^\perp = 2 + \zeta^*_{D^\perp} = 2 \ , \tag{3.113}$$

$$\rho_S^\perp = 2 + \zeta^*_{\widetilde{\lambda}^\perp} = 2 \ , \tag{3.114}$$

$$\rho_M^\perp = 2 + \zeta^*_{\widetilde{D}^\perp} = 2 - \epsilon = d - 2 \ . \tag{3.115}$$

We may interpret our results for the novel anisotropic fixed point as follows. For $T^\perp = 0$ and $T^\parallel = \infty$, the system breaks up into essentially independent sheets of dimension $d_\parallel$ with infinite heat bath temperature. The associated critical exponents are closely related to the isotropic ones at the fixed point with $T^* = \infty$. However, the additional $d_\perp$ dimensions are reflected in the anomalous Wilson-Fisher exponents (3.106) and (3.107), which are proportional to $d_\perp\, \epsilon$ (while the equilibrium exponent $\eta \propto \epsilon^2$). Fluctuations in the $d_\perp$ transverse direction are effectively at zero temperature for the conserved noise, and consequently are governed by the critical exponents of the isotropic non-equilibrium SSS model with $T^* = 0$. In the converse situation, with $T^\parallel = 0$ and $T^\perp = \infty$, clearly one must simply exchange the roles of the transverse and longitudinal sectors. Yet, we emphasize again that these novel fixed points with their associated rather bizarre critical behavior are *unstable*, and for *any* $0 < \sigma = T^\perp / T^\parallel < \infty$ initially, the static and dynamic critical properties of the system are *asymptotically* described by the *equilibrium* scaling exponents.

4 The anisotropic non-equilibrium model J

In this section, we study the critical behavior of our non-equilibrium version for model J (describing the dynamics of isotropic ferromagnets) with dynamical noise, as defined through Eqs. (2.13) and (2.14). We start by computing the T_c shift from the static susceptibility. As a consequence of the spatially anisotropic conserved noise with $T_0^\perp < T_0^\parallel$, it turns out that the transverse momentum space sector with *lower* noise temperature softens first. Thus, at the critical point, the longitudinal sector remains uncritical (“stiff”), similar to equilibrium anisotropic elastic phase transitions [25]. It is then instructive to switch off the mode-coupling constant g_0, and first recapitulate the properties of the ensuing two-temperature non-equilibrium model B [6,7,8]. In Sec. 4.3, we turn to the perturbational renormalization of the two-temperature non-equilibrium model J to one-loop order, and finally discuss the resulting RG flow equations.

4.1 Dynamic field theory and the anisotropic T_c shift

The probability distribution for the dynamic fields S_0^α ($\alpha = 1, 2, 3$), equivalent to the Langevin equation (2.13) with anisotropic noise correlator (2.14), reads

$$P[\{S_0^\alpha\}] \propto \int \mathcal{D}[\{i\widetilde{S}_0^\alpha\}]\, e^{J[\{\widetilde{S}_0^\alpha\}, \{S_0^\alpha\}]} \ , \tag{4.1}$$

with the Janssen-De Dominicis functional $J = J_{\rm har} + J_{\rm rel} + J_{\rm mc}$, with the harmonic part

$$\begin{aligned} J_{\rm har}[\{\widetilde{S}_0^\alpha\}, \{S_0^\alpha\}] = \\ = \int d^dx \int dt \sum_\alpha \Big\{ -\widetilde{S}_0^\alpha \left(\widetilde{\lambda}_0^\parallel \nabla_\parallel^2 + \widetilde{\lambda}_0^\perp \nabla_\perp^2 \right) \widetilde{S}_0^\alpha - \\ - \widetilde{S}_0^\alpha \left[\frac{\partial}{\partial t} - \lambda_0 \nabla^2 \left(r_0 - \nabla^2 \right) \right] S_0^\alpha \Big\} \ , \end{aligned} \tag{4.2}$$

the non-linear relaxation vertex from model B,

$$J_{\rm rel}[\{\widetilde{S}_0^\alpha\}, \{S_0^\alpha\}] = \lambda_0 \frac{u_0}{6} \int d^dx \int dt \sum_{\alpha,\beta} \widetilde{S}_0^\alpha \nabla^2 S_0^\alpha S_0^\beta S_0^\beta \ , \tag{4.3}$$

and the mode-coupling vertices from the spin precession forces,

$$J_{\rm mc}[\{\widetilde{S}_0^\alpha\}, \{S_0^\alpha\}] = -g_0 \int d^dx \int dt \sum_{\alpha,\beta,\gamma} \epsilon^{\alpha\beta\gamma} \widetilde{S}_0^\alpha S_0^\beta \nabla^2 S_0^\gamma \ . \tag{4.4}$$

In analogy with Eq. (3.25) for the SSS model, the dynamic susceptibility can be expressed as

$$\begin{aligned} \chi_0(\mathbf{q}, \omega) = \Gamma_{0\, \widetilde{S}S}(-\mathbf{q}, -\omega)^{-1} \times \\ \times \left[\lambda_0\, q^2 + g_0\, \Gamma_{0\, \widetilde{S}[\widetilde{S}S]}(-\mathbf{q}, -\omega) \right] \ . \end{aligned} \tag{4.5}$$

From the ensuing expression (to one-loop order), we may determine the fluctuation-induced shift of the critical temperature. Because of the dynamic anisotropy appearing in the noise correlator (2.14), however, the result depends on how the limit $\mathbf{q} \to \mathbf{0}$ is taken; upon defining $q_\parallel = q \cos \Theta$ and $q_\perp = q \sin \Theta$, we find

$$r_{0c}(\Theta) = - \left(\frac{d_\parallel}{d} T_0^\parallel + \frac{d_\perp}{d} T_0^\perp \right) \times \left[\frac{5}{6} u_0 \int_k \frac{1}{r_{0c} + k^2} - \frac{g_0^2}{2(d+2)\lambda_0^2} \int_k \frac{1}{k^2(r_{0c} + k^2)} \right] - \left(T_0^\parallel \cos^2 \Theta + T_0^\perp \sin^2 \Theta \right) \frac{g_0^2}{2(d+2)\lambda_0^2} \int_k \frac{1}{k^2(r_{0c} + k^2)} , \tag{4.6}$$

in contrast with the isotropic Eq. (3.32) for the SSS model. As $T_c = T_c^0 + r_{0c}$, the phase transition will occur at the maximum of the function $r_{0c}(\Theta)$, which for $T_0^\perp < T_0^\parallel$ occurs at $\Theta = \pi/2$. The $d_\perp$-dimensional transverse sector in momentum space thus softens first, and the true T_c shift is given by

$$r_{0c} = - \left(\frac{d_\parallel}{d} T_0^\parallel + \frac{d_\perp}{d} T_0^\perp \right) \frac{5}{6} u_0 \int_k \frac{1}{r_{0c} + k^2} + \frac{d_\parallel}{d} \left(T_0^\parallel - T_0^\perp \right) \frac{g_0^2}{2(d+2)\lambda_0^2} \int_k \frac{1}{k^2(r_{0c} + k^2)} , \tag{4.7}$$

or, after evaluating the integrals in Eq. (4.7) by means of dimensional regularization, as solution of

$$|r_{0c}|^{\frac{6-d}{2}} = \frac{2A_d}{(d-2)(4-d)} \left[\left(\frac{d_\parallel}{d} T_0^\parallel + \frac{d_\perp}{d} T_0^\perp \right) \frac{5}{6} u_0 |r_{0c}| + \frac{d_\parallel}{d} \left(T_0^\parallel - T_0^\perp \right) \frac{g_0^2}{2(d+2)\lambda_0^2} \right] . \tag{4.8}$$

(We remark again, though, that a more physical way to compute this quantity would be by means of cutoff regularization.) For $T_0^\parallel = T_0^\perp = T_0$, we recover the equilibrium result with a rescaled coupling $T_0 u_0 = \widetilde{\lambda}_0 u_0 / \lambda_0$, as to be expected (see Ref. [5] and Eq. (3.33) for the SSS model with $n = 3$). Notice, however, that dynamical anisotropy ($T_0^\parallel \neq T_0^\perp$), *combined* with the reversible mode-coupling terms, has a very drastic effect here: It renders the system soft only in the momentum subspace with lower noise temperature. This effect has a simple physical interpretation: The T_c shift is due to thermal fluctuations, which are reduced in the transverse sector ($T_0^\perp < T_0^\parallel$), and therefore lead to a comparatively stronger downwards shift in the longitudinal sector.

In order to characterize the critical properties of our model, we may neglect terms $\propto q_\parallel^4$ in the stiff momentum space sector, because $\tau_0^\parallel = r_0 - r_{0c}(\Theta = 0)$ remains positive at the phase transition where $\tau_0^\perp = r_0 - r_{0c}(\Theta = \pi/2)$ vanishes. In analogy with the situation at anisotropic elastic structural phase transitions [25], or with Lifshitz points in magnetic systems with competing interactions [26], we thus have to scale the soft and stiff wavevector components differently, $[q_\perp] = \mu$, whereas $[q_\parallel] = [q_\perp]^2 = \mu^2$. Consequently, while $[\widetilde{\lambda}_0^\perp] = \mu^0$ if we choose $[\omega] = \mu^4$, we find for the longitudinal scaling dimension $[\widetilde{\lambda}_0^\parallel] = \mu^{-2}$, which implies that the longitudinal noise strength becomes *irrelevant* under scale transformations. Allowing for distinct couplings in the different sectors, one finds in the same manner that the ratios $[\lambda_0^\parallel / \lambda_0^\perp] = [\lambda_0^\parallel u_0^\parallel / \lambda_0^\perp u_0^\perp] = [g_0^\parallel / g_0^\perp] = \mu^{-2}$ all have negative scaling dimension. Thus, for an investigation of the asympotic critical behavior, the longitudinal parameters may be neglected as compared to their transverse counterparts, and can all be set to zero in the effective dynamic functional J. Upon rescaling the fields according to $S_0^\alpha \to (\widetilde{\lambda}_0^\perp / \lambda_0^\perp)^{1/2} S_0^\alpha$, $\widetilde{S}_0^\alpha \to (\lambda_0^\perp / \widetilde{\lambda}_0^\perp)^{1/2} \widetilde{S}_0^\alpha$, defining

$$c_0 = \frac{\lambda_0^\parallel}{\lambda_0^\perp} \tau_0^\parallel , \quad \widetilde{u}_0 = \frac{\widetilde{\lambda}_0^\perp}{\lambda_0^\perp} u_0^\perp , \quad \widetilde{g}_0 = \sqrt{\frac{\widetilde{\lambda}_0^\perp}{\lambda_0^\perp}} g_0^\perp , \tag{4.9}$$

and omitting the labels "$\perp$" again for λ_0 and r_0, the ensuing *effective* Langevin equation of motion becomes

$$\frac{\partial S_0^\alpha}{\partial t} = \lambda_0 \left[c_0 \nabla_\parallel^2 + \nabla_\perp^2 (r_0 - \nabla_\perp^2) \right] S_0^\alpha + \lambda_0 \frac{\widetilde{u}_0}{6} \nabla_\perp^2 S_0^\alpha \sum_\beta S_0^\beta S_0^\beta - \widetilde{g}_0 \sum_{\beta,\gamma} \epsilon^{\alpha\beta\gamma} S_0^\beta \nabla_\perp^2 S_0^\gamma + \zeta^\alpha , \tag{4.10}$$

with the noise correlator

$$\langle \zeta^\alpha(\mathbf{x}, t) \zeta^\beta(\mathbf{x}', t') \rangle = -2\lambda_0 \nabla_\perp^2 \, \delta(\mathbf{x} - \mathbf{x}') \, \delta(t - t') \, \delta^{\alpha\beta} . \tag{4.11}$$

These equations define the *two-temperature non-equilibrium model J*.

It is interesting to note that the anisotropy of the T_c shift in Eq. (4.6) only occurs in the contribution $\propto g_0^2$. In the non-equilibrium model B with dynamical anisotropy, the criticality condition for the response function remains isotropic, at least to one-loop order. Thus, if one does not assume different critical temperatures in the purely diffusive non-linear Langevin equation to begin with, these are *not* generated, and one is not led to the two-temperature model B, which we shall discuss below, as the correct effective theory for the phase transition. Instead, the non-equilibrium perturbations appear to be *irrelevant* to this order, and the model is asymptotically described by the *equilibrium* critical exponents of model B, i.e., the static exponents of the $O(n)$ Heisenberg model, accompanied with the dynamic exponent $z = 4 - \eta$.

4.2 The two-temperature non-equilibrium model B

Before we turn to the analysis of the two-temperature model J, derived above as the *effective critical* theory for the non-equilibrium model J with dynamical anisotropy, we briefly summarize the results for the corresponding two-temperature non-equilibrium model B [6,7,8], which is defined by Eqs. (4.10) and (4.11) with vanishing mode-coupling term $\widetilde{g}_0 = 0$. We may thus generalize to arbitrary number of compenents n again.

With this simplification, the resulting purely relaxational Langevin equation of motion can be written in the form

$$\frac{\partial S_0^\alpha}{\partial t} = \lambda_0 \nabla_\perp^2 \, \frac{\delta H_{\rm eff}[\{S_0^\alpha\}]}{\delta S_0^\alpha} + \zeta^\alpha \ , \tag{4.12}$$

accompanied with the Gaussian noise (4.11). Notice that after the above rescaling, the Einstein relation between the diffusion constant and the noise strength is fulfilled; hence the two-temperature model B is effectively an *equilibrium* system, and describes diffusive relaxation into the stationary state with probability distribution $P_{\rm eq}[\{S_0^\alpha\}] \propto \exp(-H_{\rm eff}[\{S_0^\alpha\}])$. The effective free energy here,

$$\begin{aligned} H_{\rm eff}[\{S_0^\alpha\}] = \\ = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \sum_\alpha \frac{c_0 q_\parallel^2 + q_\perp^2 (r_0 + q_\perp^2)}{q_\perp^2} \, S_0^\alpha(\mathbf{q}) S_0^\alpha(-\mathbf{q}) + \\ + \frac{\widetilde{u}_0}{4!} \int d^d x \sum_{\alpha,\beta} S_0^\alpha(\mathbf{x})^2 S_0^\beta(\mathbf{x})^2 \ , \end{aligned} \tag{4.13}$$

contains long-range elastic interactions (uniaxial pseudo-dipolar for $d_\perp = 1$ [27]), as is evident from the harmonic part of Eq. (4.13). These long-range, non-analytic interactions are generated by the dynamical anisotropy in the original non-equilibrium model [6,7,8]. As critical fluctuations are now restricted to the $d_\perp$-dimensional subsector, one expects that the upper critical dimension of this model is reduced to

$$d_c^{\rm st} = 4 - d_\parallel \ , \tag{4.14}$$

which is confirmed through a direct scaling analysis of the free energy (4.13). E.g., for a one-dimensional hard sector (uniaxial system), the critical dimension above which mean-field exponents become exact, is $d_c^{\rm st} = 3$.

In order to compute the scaling exponents below the upper critical dimension, we have to renormalize the theory in the UV. To this end, we introduce renormalized fields and parameters according to

$$\widetilde{S} = Z_{\widetilde{S}}^{1/2} \, \widetilde{S}_0 \ , \quad S = Z_S^{1/2} \, S_0 \ , \tag{4.15}$$

$$\lambda = Z_\lambda \, \lambda_0 \ , \qquad c = Z_c \, c_0 \ , \tag{4.16}$$

$$\tau = Z_\tau \, \tau_0 \, \mu^{-2} \ , \ \widetilde{u} = Z_{\widetilde{u}} \, \widetilde{u}_0 \, A(d_\parallel, d_\perp) \, \mu^{-\epsilon} \ , \tag{4.17}$$

where $\tau_0 = r_0 - r_{0c}$ as usual, $\epsilon = d_c^{\rm st} - d = 4 - d - d_\parallel = 4 - 2d_\parallel - d_\perp$ denotes the deviation from the upper critical dimension (4.14), and we define the anisotropic geometric factor

$$A(d_\parallel, d_\perp) = \frac{\Gamma(3 - d/2 - d_\parallel/2) \, \Gamma(d/2)}{c_0^{d_\parallel/2} \, 2^{d-1} \pi^{d/2} \, \Gamma(d_\perp/2)} \tag{4.18}$$

with $A(0, d) = A_d$.

Yet, these renormalization constants are not entirely independent of each other. First, as the two-temperature model B in the critical region is equivalent to an equilibrium system, there exists a fluctuation-dissipation theorem that connects the imaginary part of the dynamic susceptibility $\chi_0(\mathbf{q}, \omega)$ with the Fourier transform of the order parameter correlation function $C_0(\mathbf{x}, t; \mathbf{x}', t') \, \delta^{\alpha\beta} = \langle S_0^\alpha(\mathbf{x}, t) \, S_0^\beta(\mathbf{x}', t') \rangle$,

$$C_0(\mathbf{q}, \omega) = \frac{2}{\omega} \, {\rm Im} \, \chi_0(\mathbf{q}, \omega) \ . \tag{4.19}$$

In terms of the two-point vertex functions, $C_0(\mathbf{q}, \omega) = -\Gamma_{0 \, \widetilde{S}\widetilde{S}}(\mathbf{q}, \omega) / |\Gamma_{0 \, \widetilde{S}S}(\mathbf{q}, \omega)|^2$, while for $g_0 = 0$ Eq. (4.5) reduces to $\chi_0(\mathbf{q}, \omega) = \lambda_0 q_\perp^2 / \Gamma_{0 \, \widetilde{S}S}(-\mathbf{q}, -\omega)$; thus the fluctuation-dissipation theorem can equivalently be written as

$$\Gamma_{0 \, \widetilde{S}\widetilde{S}}(\mathbf{q}, \omega) = \frac{2\lambda_0 \, q_\perp^2}{\omega} \, {\rm Im} \, \Gamma_{0 \, \widetilde{S}S}(\mathbf{q}, \omega) \ . \tag{4.20}$$

Precisely the same relation must hold for the corresponding renormalized vertex function, which implies the identity

$$Z_\lambda \equiv \left(Z_S / Z_{\widetilde{S}} \right)^{1/2} \ . \tag{4.21}$$

Second, the equation of motion (4.12) implies that the non-linear relaxation vertices are proportional to the external momentum $q_\perp^2$, and hence the loop contributions to $\Gamma_{0 \, \widetilde{S}S}(\mathbf{q}, \omega)$ must vanish in the limit $\mathbf{q}_\perp \to \mathbf{0}$. Thus, to *all* orders in perturbation theory,

$$\Gamma_{0 \, \widetilde{S}S}(\mathbf{q}_\parallel, \mathbf{q}_\perp = \mathbf{0}, \omega) \equiv i\omega + \lambda_0 \, c_0 \, q_\parallel^2 \ . \tag{4.22}$$

This leads to the additional set of identities

$$Z_{\widetilde{S}} \, Z_S \equiv 1 \ , \quad Z_\lambda \, Z_c \equiv 1 \ , \tag{4.23}$$

and thus, using Eq. (4.21),

$$Z_\lambda \equiv Z_c^{-1} \equiv Z_S \ . \tag{4.24}$$

At last, because of the absence of the composite-operator vertex function in the relation between the dynamic susceptibility and the two-point vertex function, we have

$$\chi(\mathbf{q}, \omega) = Z \, \chi_0(\mathbf{q}, \omega) \tag{4.25}$$

with

$$Z \equiv Z_S \ . \tag{4.26}$$

As in Sec. 3.2 for the non-equilibrium SSS model, we can now define Wilson's zeta functions via logarithmic derivatives of the Z factors with respect to the renormalization scale μ,

$$\zeta_{\widetilde{S}} = \mu \frac{\partial}{\partial \mu} \bigg|_0 \ln Z_{\widetilde{S}} \ , \tag{4.27}$$

$$\zeta_S = \mu \frac{\partial}{\partial \mu} \bigg|_0 \ln Z_S \ , \tag{4.28}$$

$$\zeta_a = \mu \frac{\partial}{\partial \mu} \bigg|_0 \ln \frac{a}{a_0} \ , \tag{4.29}$$

where $\{a\} = \lambda, c, \tau, u$, and write down the RG equations for the vertex and response functions,

$$\Gamma_{\widetilde{S}^r S^l} \left(\mu, \{\mathbf{q}_\parallel, \mathbf{q}_\perp, \omega\}; \{a\} \right) =$$

$$= \exp \left\{ \frac{1}{2} \int_1^\ell \left[r\, \zeta_{\widetilde{S}}(\ell') + s\, \zeta_S(\ell') \right] \frac{d\ell'}{\ell'} \right\} \times$$
$$\times \Gamma_{\widetilde{S}^r S^s} \left(\mu\ell, \{\mathbf{q}_\parallel, \mathbf{q}_\perp, \omega\}; \{a(\ell)\} \right) \ , \qquad (4.30)$$

$$\chi \left(\mu, \{\mathbf{q}_\parallel, \mathbf{q}_\perp, \omega\}; \{a\} \right) = \exp \left\{ - \int_1^\ell \zeta_S(\ell') \frac{d\ell'}{\ell'} \right\} \times$$
$$\times\, \chi \left(\mu\ell, \{\mathbf{q}_\parallel, \mathbf{q}_\perp, \omega\}; \{a(\ell)\} \right) \ . \qquad (4.31)$$

The general scaling form for the renormalized order parameter response and correlation function thus obtained at an IR-stable fixed point $\widetilde{u}^*$ reads [7]

$$\chi(\tau, \mathbf{q}_\parallel, \mathbf{q}_\perp, \omega) = q_\perp^{-2+\eta}\, \hat{\chi} \left(\frac{\tau}{q_\perp^{1/\nu}}, \frac{q_\parallel}{q_\perp^{1+\Delta}}, \frac{\omega}{q_\perp^z} \right) \ , \qquad (4.32)$$

$$C(\tau, \mathbf{q}_\parallel, \mathbf{q}_\perp, \omega) = q_\perp^{-2-z+\eta}\, \hat{C} \left(\frac{\tau}{q_\perp^{1/\nu}}, \frac{q_\parallel}{q_\perp^{1+\Delta}}, \frac{\omega}{q_\perp^z} \right) \ , \qquad (4.33)$$

where in addition to the usual static exponents η and ν, and the dynamic exponent z, we have introduced a scaling exponent Δ originating in the intrinsic anisotropy of the system. Alternatively, we could have defined a set different transverse and longitudinal critical exponents, $\nu_\perp = \nu$, $z_\perp = z$, $\nu_\parallel = \nu/(1+\Delta)$, $z_\parallel = z/(1+\Delta)$ [7]. Notice that as a consequence of the identity (4.26), the exponent η governs the critical decay of the response as well as the correlation function, as is required by the fluctuation-dissipation theorem (4.19). The critical exponents are readily identified with the fixed-point values of the flow functions

$$\eta = -\zeta_S^* \ , \qquad \nu^{-1} = -\zeta_\tau^* \ , \qquad (4.34)$$

$$z = 4 + \zeta_\lambda^* \ , \quad \Delta = 1 - \frac{\zeta_c^*}{2} \ . \qquad (4.35)$$

The exact relation $Z_\lambda \equiv Z_c^{-1}$ implies $\zeta_\lambda \equiv -\zeta_c$, and consequently

$$1 + \Delta \equiv z/2 \ , \qquad (4.36)$$

or $z_\parallel \equiv 2$, which reflects the mean-field character of the fluctuations in the stiff sector, and obviously holds whenever Eq. (4.22) is valid. From the second equation in (4.24) we furthermore infer $\zeta_\lambda \equiv \zeta_S$ for the two-temperature model B, and hence [7]

$$z \equiv 4 - \eta \ , \quad \Delta \equiv 1 - \eta/2 \ . \qquad (4.37)$$

The task is thus to compute the remaining independent static exponents η and ν. To one-loop order, from $\frac{1}{2}(\partial_{q_\perp^2})^2\, \Gamma_{\widetilde{S}S}(\mathbf{q}_\parallel = \mathbf{0}, \mathbf{q}_\perp, \omega = 0)|_{\mathbf{q}_\perp = \mathbf{0}}$ one finds $Z_\lambda \equiv Z_S = 1$. Next, the criticality condition for the static susceptibility yields the T_c shift

$$|r_{0c}| = \left[\frac{n+2}{3}\, \frac{\widetilde{u}_0\, A(d_\parallel, d_\perp)}{(d + d_\parallel - 2)\,(4 - d - d_\parallel)} \right]^{\frac{2}{4-d-d_\parallel}} \ ; \qquad (4.38)$$

the denominator here indicates that in addition to the reduction of the upper critical dimension $d_c^{\rm st}$, the *lower* critical dimension appears to be lowered by $d_\parallel$ as well to $d_{\rm lc} = 2 - d_\parallel$. Thus, in two dimensions, an ordered phase with long-range order may exist at low temperatures and is indeed found [8]; notice that the Mermin-Wagner-Hohenberg theorem is invalidated by the existence of *long-range* elastic or pseudo-dipolar interactions in the system. Upon then replacing $r_0 = \tau_0 + r_{0c}$, rendering $\partial_{q_\perp^2}\, \Gamma_{\widetilde{S}S}(\mathbf{q}_\parallel = \mathbf{0}, \mathbf{q}_\perp, \omega = 0)|_{\mathbf{q}_\perp = \mathbf{0}}$ UV-finite gives

$$Z_\tau = 1 - \frac{n+2}{6\,\epsilon}\, \widetilde{u}_0\, A(d_\parallel, d_\perp)\, \mu^{-\epsilon} \ , \qquad (4.39)$$

while the four-point vertex function $\Gamma_{0\,\widetilde{S}SSS}$ provides us with

$$Z_{\widetilde{u}} = 1 - \frac{n+8}{6\,\epsilon}\, \widetilde{u}_0\, A(d_\parallel, d_\perp)\, \mu^{-\epsilon} \ . \qquad (4.40)$$

Notice that the combinatorics for the Feynman graphs of the anisotropic two-temperature model B is identical to the equilibrium model B, and therefore the above renormalization constants assume the same form as their familiar equilibrium counterparts, apart from the shifted critical dimension and a modified geometric factor $A(d_\parallel, d_\perp)$. The one-loop RG flow functions thus are

$$\zeta_\lambda \equiv -\zeta_c \equiv \zeta_S = 0 \ , \qquad (4.41)$$

$$\zeta_\tau = -2 + \frac{n+2}{6}\, \widetilde{u} \qquad (4.42)$$

$$\beta_{\widetilde{u}} = \widetilde{u}\, \zeta_{\widetilde{u}} = \widetilde{u} \left(-\epsilon + \frac{n+8}{6}\, \widetilde{u} \right) \ , \qquad (4.43)$$

with the stable fixed point [compare Eq. (3.72)]

$$\widetilde{u}^* = \frac{6}{n+8}\, \epsilon + O(\epsilon^2) \ , \qquad (4.44)$$

leading to the $O(\epsilon)$ exponents, with $\epsilon = 4 - d - d_\parallel$,

$$\eta = O(\epsilon^2) \ , \quad \nu^{-1} = 2 - \frac{n+2}{n+8}\, \epsilon + O(\epsilon^2) \ . \qquad (4.45)$$

For the two-loop fixed point and exponent values to $O(\epsilon^2)$ for the case $n = 1$, we refer to Refs. [7,8].

4.3 Renormalization of the two-temperature model J

We now return to the two-temperature model J with non-vanishing mode-coupling term, as defined in Eqs. (4.10) and (4.11). In equilibrium dynamics, the reversible spin precession force of model J constitutes a *relevant* perturbation to the purely diffusive model B with conserved three-component order parameter, and the ensuing non-trivial fixed point (with upper critical dimension $d_c^{\rm dy} = 6$) for the renormalized effective mode-coupling constant $f \propto g^2/\lambda^2$ changes the dynamic critical exponent from $z \equiv 4 - \eta$ to $z \equiv (d + 2 - \eta)/2$, describing *faster* relaxation processes [19,21,20]. As we should expect the reversible mode coupling to be relevant in the two-temperature variant of model J as well, the issue therefore is, does the RG flow again lead to a non-trivial stable fixed point, and what are the values of the ensuing critical exponents?

It is essential to note, however, that the two-temperature model J with $\widetilde{g}_0 \neq 0$ *cannot* be recast as the dynamics of an equivalent equilibrium model, with an effective free energy (4.13), because for the reversible forces in Eq. (4.10),

$$\int d^d x \, \frac{\delta}{\delta S_0^\alpha(x)} \left(-\widetilde{g}_0 \sum_{\beta,\gamma} \epsilon^{\alpha\beta\gamma} S_0^\beta \nabla_\perp^2 S_0^\gamma \, e^{-H_{\rm eff}} \right) \neq 0 \ , \tag{4.46}$$

and the necessary Deker–Haake integrability condition [28] that would ensure the stability of the equilibrium probability distribution $\propto \exp(-H_{\rm eff}[\{S_0^\alpha\}])$ is violated, except for $c_0 = 0$ or $d_\parallel = 0$. We remark that this is actually a consequence of the inseparability of "statics" and dynamics in the intrinsically dynamic two-temperature model *with* reversible mode couplings. For, if we could *first* and separately consider the effective *static* free energy, and only *subsequently* introduce the dynamics with the analog of Eq. (2.12), the elastic pseudo-dipolar propagator would appear in the mode-coupling vertex, and the above integrability condition

$$\int d^d x \, \frac{\delta}{\delta S_0^\alpha(x)} \left(-\widetilde{g}_0 \sum_{\beta,\gamma} \epsilon^{\alpha\beta\gamma} \frac{\delta H_{\rm eff}}{\delta S_0^\beta} \, S_0^\gamma \, e^{-H_{\rm eff}} \right) = 0 \tag{4.47}$$

would be satisfied. Following the standard equilibrium procedures [21], the ensuing critical exponents would be given by Eqs. (4.45) and (4.36), with the exact dynamic exponent $z = (d + 2 - \eta)/2$. Yet, such a procedure is not possible here, and the derivation of the effective equation of motion has to proceed in the *dynamic* functional, leading to Eq. (4.10). As opposed to the two-temperature model B, the two-temperature model J thus represents a *genuinely non-equilibrium* dynamical model, for, as we shall see below, the renormalization-group flow does *not* take the renormalized mode coupling $\widetilde{g}$ to zero under scale transformations.

Thus we can invoke no fluctuation-dissipation theorem in order to relate vertex and response function renormalizations, and we have to compute almost all the Z factors, as defined in Eqs. (4.15)–(4.17) and (4.25), independently. Neither is there a Ward identity relating the renormalization of the mode-coupling vertex to simple field renormalizations [29]. Fortunately, though, because of the momentum dependence of the mode-coupling vertex, $\Gamma_{0\,\widetilde{S}^\alpha S^\beta S^\gamma}(-\mathbf{q}_\perp, 0; \frac{\mathbf{q}_\perp}{2} - \mathbf{p}_\perp, 0; \frac{\mathbf{q}_\perp}{2} + \mathbf{p}_\perp, 0) = \widetilde{g}_0 \, (\mathbf{q}_\perp \cdot \mathbf{p}_\perp) \, \epsilon^{\alpha\beta\gamma} + O(\widetilde{g}_0^3)$, at least Eq. (4.22) is valid for the two-temperature model J as well. Consequently the identities (4.23) still hold, leading immediately to Eq. (4.36) or $z_\parallel \equiv 2$, as to be expected. Furthermore, simple power counting yields that the dynamical upper critical dimension, where the mode-coupling constant becomes dimensionless, is

$$d_c^{\rm dy} = 6 - d_\parallel \ , \tag{4.48}$$

i.e., the spatial anisotropy reduces $d_c^{\rm dy}$ from its equilibrium value in exactly the same way by $d_\parallel$ as the static upper critical dimension (4.14). Thus, we define a dimensionless renormalized mode-coupling constant according to

$$\widetilde{g} = Z_{\widetilde{g}}^{1/2} \, \widetilde{g}_0 \, B(d_\parallel, d_\perp)^{1/2} \, \mu^{-\varepsilon/2} \ , \tag{4.49}$$

where $\varepsilon = d_c^{\rm dy} - d = 6 - d - d_\parallel = 6 - 2d_\parallel - d_\perp$, and

$$B(d_\parallel, d_\perp) = \frac{\Gamma(4 - d/2 - d_\parallel/2) \, \Gamma(d/2)}{c_0^{d_\parallel/2} \, 2^d \pi^{d/2} \, \Gamma(d_\perp/2)} \ . \tag{4.50}$$

The appearance of *two different* upper critical dimensions implies that we shall have to compute fixed points and exponents in a *double* expansion with respect to ϵ and ε.

In order to evaluate the renormalization constants, we start with the dynamic susceptibility (4.5), and first compute the fluctuation-induced T_c shift from the condition $\chi_0(\mathbf{q}_\parallel = \mathbf{0}, \mathbf{q}_\perp \to \mathbf{0}, \omega = 0)^{-1} = 0$ for $r_0 = r_{0c}$. Introducing the effective mode-coupling constant

$$\widetilde{f}_0 = \frac{\widetilde{g}_0^2}{2 d_\perp \, \lambda_0^2} \ , \tag{4.51}$$

we find in dimensional regularization

$$|r_{0c}|^{\frac{6-d-d_\parallel}{2}} = \frac{5 \widetilde{u}_0}{3} \, \frac{A(d_\parallel, d_\perp) \, |r_{0c}|}{(d + d_\parallel - 2)(4 - d - d_\parallel)} - \\ - \frac{d_\parallel d_\perp \, \widetilde{f}_0}{(d-2)} \, \frac{B(d_\parallel, d_\perp)}{(4 - d - d_\parallel)\,(6 - d - d_\parallel)} \ , \tag{4.52}$$

to be compared with Eq. (4.38) for the two-temperature model B. Subsequently rendering both $\partial_{q_\perp^2} \chi(\mathbf{q}_\parallel = \mathbf{0}, \mathbf{q}_\perp, \omega = 0)^{-1}|_{\mathbf{q}_\perp = \mathbf{0}}$ and $\chi(\mathbf{q}_\parallel = \mathbf{0}, \mathbf{q}_\perp \to \mathbf{0}, \omega = 0)$ UV-finite yields, after a somewhat tedious calculation, the Z factors

$$Z = 1 - \frac{d_\parallel (6 - d_\parallel)}{2(4 - d_\parallel)} \, \frac{\widetilde{f}_0 \, B(d_\parallel, d_\perp) \, \mu^{-\varepsilon}}{\varepsilon} \ , \tag{4.53}$$

$$Z \, Z_\tau = 1 - \frac{5}{6} \, \frac{\widetilde{u}_0 \, A(d_\parallel, d_\perp) \, \mu^{-\epsilon}}{\epsilon} - \\ - \frac{d_\parallel (3 - d_\parallel)}{4 - d_\parallel} \, \frac{\widetilde{f}_0 \, B(d_\parallel, d_\perp) \, \mu^{-\varepsilon}}{\varepsilon} \ . \tag{4.54}$$

Here, we have employed *minimal* subtraction, where only the residues of the singular ε poles were retained; i.e., in the expressions $\propto \widetilde{f}_0$, d was replaced with $6 - d_\parallel$, and $d_\perp$ with $6 - 2d_\parallel$ to this order. The diffusion constant renormalization is then most conveniently found by considering the composite operator $\partial_{q_\perp^2} \, [\lambda q_\perp^2 + \Gamma_{\widetilde{S}[\widetilde{S}S]}(\mathbf{q}_\parallel = \mathbf{0}, \mathbf{q}_\perp, \omega = 0)]|_{\mathbf{q}_\perp = \mathbf{0}}$, with the result

$$Z^{-1} \, Z_\lambda = 1 + \frac{2(2 - d_\parallel)}{4 - d_\parallel} \, \frac{\widetilde{f}_0 \, B(d_\parallel, d_\perp) \, \mu^{-\varepsilon}}{\varepsilon} \ . \tag{4.55}$$

Equivalently, this combination of Z factors can be establishes by comparing the UV singularities in $\partial_{q_\perp^2} \, \Gamma_{\widetilde{S}S}(\mathbf{q}_\parallel = \mathbf{0}, \mathbf{q}_\perp, \omega = 0)|_{\mathbf{q}_\perp = \mathbf{0}}$ with the previously established ones in $\chi(\mathbf{q}_\parallel = \mathbf{0}, \mathbf{q}_\perp \to \mathbf{0}, \omega = 0)$. From renormalizing the noise vertex function $\partial_{q_\perp^2} \Gamma_{\widetilde{S}\widetilde{S}}(\mathbf{q}_\parallel = \mathbf{0}, \mathbf{q}_\perp, \omega = 0)|_{\mathbf{q}_\perp = \mathbf{0}}$, we obtain

$$Z_S^{-1} Z_\lambda = 1 + \frac{\widetilde{f}_0 \, B(d_\parallel, d_\perp) \, \mu^{-\varepsilon}}{\varepsilon} \ . \qquad (4.56)$$

At last, by means of rather lengthy calculations for the derivatives $\partial_{(\mathbf{q}_\perp \cdot \mathbf{p}_\perp)} \Gamma_{\widetilde{S}^\alpha S^\beta S^\gamma}(-\mathbf{q}_\perp, 0; \frac{\mathbf{q}_\perp}{2} - \mathbf{p}_\perp, 0; \frac{\mathbf{q}_\perp}{2} + \mathbf{p}_\perp, 0)|_{\mathbf{q}_\perp = \mathbf{p}_\perp = \mathbf{0}}$, $\partial_{q_\perp^2} \Gamma_{\widetilde{S}SSS}(-\mathbf{q}_\perp, 0; \frac{\mathbf{q}_\perp}{3}, 0; \frac{\mathbf{q}_\perp}{3}, 0; \frac{\mathbf{q}_\perp}{3}, 0;)|_{\mathbf{q}_\perp = \mathbf{0}}$ we arrive at the coupling constant renormalizations

$$Z_S \, Z_{\widetilde{g}} = 1 - \frac{d_\parallel}{3} \, \frac{\widetilde{f}_0 \, B(d_\parallel, d_\perp) \, \mu^{-\varepsilon}}{\varepsilon} \ , \qquad (4.57)$$

$$Z_S \, Z_\lambda \, Z_{\widetilde{u}} = 1 - \frac{11}{6} \, \frac{\widetilde{u}_0 \, A(d_\parallel, d_\perp) \, \mu^{-\epsilon}}{\epsilon} + + \frac{3 - d_\parallel}{3} \, \frac{\widetilde{f}_0 \, B(d_\parallel, d_\perp) \, \mu^{-\varepsilon}}{\varepsilon} \ . \qquad (4.58)$$

For $\widetilde{f}_0 = 0$, Eqs. (4.54) and (4.58) reduce to the one-loop Z factors (4.39) and (4.40) for the two-temperature model B with $n = 3$, while setting $d_\parallel = 0$ recovers the familiar renormalization constants for the equilibrium model J for isotropic ferromagnets [19,21,20].

4.4 Discussion of the RG flow equations

In terms of the renormalized couplings $\widetilde{u}$ and

$$\widetilde{f} = \frac{\widetilde{g}^2}{2 d_\perp \, \lambda^2} \ , \qquad (4.59)$$

the one-loop zeta functions for the two-temperature model J become

$$\zeta_S \equiv -\zeta_{\widetilde{S}} = \frac{d_\parallel (8 - d_\parallel)}{2(4 - d_\parallel)} \, \widetilde{f} \ , \qquad (4.60)$$

$$\zeta = \frac{d_\parallel (6 - d_\parallel)}{2(4 - d_\parallel)} \, \widetilde{f} \ , \qquad (4.61)$$

$$\zeta_{\widetilde{g}} = -\frac{\varepsilon}{2} - \frac{d_\parallel (16 - d_\parallel)}{12(4 - d_\parallel)} \, \widetilde{f} \ , \qquad (4.62)$$

$$\zeta_\lambda \equiv -\zeta_c = -\frac{d_\parallel^2 - 10 d_\parallel + 8}{2(4 - d_\parallel)} \, \widetilde{f} \ , \qquad (4.63)$$

$$\zeta_\tau = -2 + \frac{5}{6} \, \widetilde{u} - \frac{d_\parallel^2}{2(4 - d_\parallel)} \, \widetilde{f} \ , \qquad (4.64)$$

$$\zeta_{\widetilde{u}} = -\epsilon + \frac{11}{6} \, \widetilde{u} - \frac{2 d_\parallel (10 - d_\parallel)}{3(4 - d_\parallel)} \, \widetilde{f} \ . \qquad (4.65)$$

Notice that the dynamic coupling constant enters the RG flow for the "static" non-linearity $\widetilde{u}$; this is yet another indication that this model is of genuinely dynamical character. In the equilibrium limit $d_\parallel = 0$, the statics and dynamics decouple.

The RG beta functions for $\widetilde{u}$ and the effective mode coupling (4.59) of the two-temperature model J read

$$\beta_{\widetilde{u}} = \widetilde{u} \, \zeta_{\widetilde{u}} = \widetilde{u} \left(-\epsilon + \frac{11}{6} \, \widetilde{u} - \frac{2 d_\parallel (10 - d_\parallel)}{3(4 - d_\parallel)} \, \widetilde{f} \right) \qquad (4.66)$$

$$\beta_{\widetilde{f}} = 2 \widetilde{f} \left(\zeta_{\widetilde{g}} - \zeta_\lambda \right) = = \widetilde{f} \left(-\varepsilon + \frac{7 d_\parallel^2 - 76 d_\parallel + 48}{6(4 - d_\parallel)} \, \widetilde{f} \right) \ . \qquad (4.67)$$

For $d_\parallel = 0$, we thus recover the stable *equilibrium* fixed points

$$\widetilde{f}^* = \frac{\varepsilon}{2} + (\varepsilon^2) \ , \quad \widetilde{u}^* = \frac{6}{11} \, \epsilon + (\epsilon^2) \ , \qquad (4.68)$$

and critical exponents, see Eqs. (4.34) and (4.35),

$$\eta = 0 + O(\epsilon^2) \ , \quad \nu^{-1} = 2 - \frac{5}{11} \, \epsilon + O(\epsilon^2) \ , \qquad (4.69)$$

$$z = 4 - \frac{\varepsilon}{2} + O(\epsilon^2) = \frac{d + 2}{2} + O(\epsilon^2) \ . \qquad (4.70)$$

As can be seen here, the correction to z is merely given by the $O(\epsilon^2)$ contribution to the static exponent η of the three-component Heisenberg model. For, in equilibrium there is an additional identity $Z_g \equiv Z_S$ [29], or $\zeta_g^* \equiv -(\varepsilon + \eta)/2$. The condition for the existence of a non-trivial finite fixed point thus becomes $\zeta_\lambda^* = \zeta_g^* = (d - 6 - \eta)/2$, or

$$z \equiv \frac{d + 2 - \eta}{2} \ . \qquad (4.71)$$

In the full non-equilibrium theory ($d_\parallel > 0$), Eq. (4.67) still implies that the two-temperature model B fixed point with $\widetilde{f}^* = 0$ is *unstable* for $\varepsilon = 6 - d - d_\parallel > 0$. To one-loop order, the finite positive fixed point

$$\widetilde{f}^* = \frac{6(4 - d_\parallel)}{7 d_\parallel^2 - 76 d_\parallel + 48} \, \varepsilon + O(\varepsilon^2, \epsilon^2) \qquad (4.72)$$

exists only in the interval $0 \leq d_\parallel \leq \frac{2}{7}(19 - \sqrt{277}) \approx 0.6733$; already for $d_\parallel = 1$, the RG flow takes the mode coupling to infinity! (According to the one-loop beta function $\beta_{\widetilde{f}}$, there is another regime where $\widetilde{f}^* > 0$, namely $4 \leq d_\parallel \leq \frac{2}{7}(19 + \sqrt{277}) \approx 10.18$; yet in such high dimensions $d_c^{\text{st}} = 4 - d_\parallel \leq 0$.) According to Eq. (4.66), the divergence of $\beta_{\widetilde{f}}(\ell)$ under scale transformations as $\ell \to 0$ furthermore drives the "static" non-linearity to $+\infty$ as well. Apparently, the two-temperature model J asymptotically enters a genuine *strong-coupling* regime, which does not allow for a perturbational calculation of the critical exponents (for $d_\parallel \geq 0.6733$) [30].

Formally though, we may expand about the equilibrium model J, and thus obtain critical exponents in the limit $d_\parallel \ll 1$. To first order in $d_\parallel \varepsilon$, we find

$$\widetilde{f}^* = \frac{\varepsilon}{2} \left(1 + \frac{4}{3} d_\parallel \right) \ , \quad \widetilde{u}^* = \frac{6}{11} \, \epsilon + \frac{5}{11} \, d_\parallel \varepsilon \ , \qquad (4.73)$$

leading to the critical exponents

$$\eta = -\frac{3}{8} \, d_\parallel \varepsilon \ , \quad \eta_S = -\frac{1}{2} \, d_\parallel \varepsilon \ , \qquad (4.74)$$

$$\nu^{-1} = 2 - \frac{5}{11} \, \epsilon - \frac{25}{66} \, d_\parallel \varepsilon \ , \qquad (4.75)$$

$$z = 4 - \frac{\varepsilon}{2} - \frac{1}{6}\, d_\| \varepsilon \ , \tag{4.76}$$

$$\Delta = 1 - \frac{\varepsilon}{4} - \frac{1}{12}\, d_\| \varepsilon \ . \tag{4.77}$$

Notice, however, that this procedure amounts to an expansion with respect to *three* dimensional parameters, namely $\epsilon = 4 - d - d_\|$, $\varepsilon = 6 - d - d_\|$, and $d_\| \varepsilon$. Moreover, the divergence of the non-expanded fixed point $\widetilde{f}^*$ at $d_\| \approx 0.6733$ indicates that an extrapolation of the formal results (4.75) and (4.76) to any physical dimension $d_\| \geq 1$ is unlikely to work. On the other hand, we cannot exclude that this divergence is merely a one-loop artifact, and is cured if one calculates the RG beta functions to higher loop orders. Yet another possibility might well be that the divergence of $\widetilde{f}^*$ and $\widetilde{u}^*$ indicates the absence of a non-equilibrium stationary state of the two-temperature model J in the vicinity of its critical point. A somewhat less drastic implication may be that merely perturbation theory breaks down, and non-perturbative approaches could possibly characterize the scaling behavior at the transition of the two-temperature model J successfully.

At any rate, though, we may draw the following conclusions: (i) As opposed to our non-equilibrium version of the SSS model, model J with dynamical anisotropy remains a genuinely dynamical system, and is very unlikely to be described by any simple effective equilibrium theory at the critical point, certainly not by the equilibrium model J. (ii) Obviously the reversible mode coupling term in the Langevin equation is highly relevant, driving the system away from the well-defined two-temperature model B fixed point towards a strong-coupling regime, which at least to one-loop order cannot be addressed by means of perturbation theory.

5 Summary and final remarks

In summary, we have extended previous studies on the universality classes of non-equilibrium phase transitions by investigating the effect of violating the detailed-balance condition in the diffusive dynamics of a conserved field which is coupled to the order parameter through reversible mode-coupling dynamics.

In a previous work [5], it was established that the universality class of the second-order phase transition in a system with non-conserved order parameter dynamics is not affected by isotropic breaking of detailed balance. Extending this result, we found here that (1) reversible mode coupling apparently remains ineffective in generating a new universality class even if dynamical anisotropy is present in the diffusive dynamics of the conserved field, provided the order parameter itself is *non-conserved* and, (2) dynamical anisotropy does become relevant if the order parameter itself is *conserved*, in which case reversible mode couplings may drive the system towards entirely different critical behavior, which apparently cannot be described by known equilibrium universality classes.

These results give further support to previous observations that the dynamics of a non-conserved order parameter (with model A being the simplest realization) is robust against non-equilibrium perturbations, while the dynamics of a conserved order parameter field (model B, if there are no reversible mode couplings present) is extremely sensitive to detailed-balance breaking through dynamical anisotropy in the system. One of the surprising features of our results is that dynamical anisotropy, even in combination with reversible mode coupling terms, cannot destroy the stability of the equilibrium critical fixed point of the non-equilibrium SSS model with non-conserved order parameter. Only in very extreme cases of effectively zero and infinite conserved noise temperatures can one find new universality classes, which, however, should well influence the crossover behavior and the corrections to scaling at the critical point. In our non-equilibrium model J, the reversible mode coupling terms had a more drastic effect, however. While the two-temperature model B steady state dynamics can be written in the form of an effective equilibrium model with elastic or pseudo-dipolar long-range interactions, such a simple representation is *not* possible for the two-temperature model J with dynamical anisotropy. We were not even able to identify a stable and finite renormalization-group fixed point (to one-loop order) for this model, but were led to RG runaway flows towards a genuinely strong-coupling regime instead (except formally for $d_\| \ll 1$).

The fact that the rather complicated combination of relaxation, diffusion, reversible mode coupling and dynamical anisotropy does not change the nature of the second-order phase transitions in systems with *non-conserved* order parameters, gives us hope that such non-equilibrium phase transitions can be understood in terms of a relatively small number of universality classes. However, for systems with *conserved* order parameters, the situation is obviously quite different since both dynamical anisotropy itself, and the combination of dynamical anisotropy and reversible mode couplings can lead to new universality classes. It is clear that the mapping out of the relevant non-equilibrium perturbations for the case of conserved order parameter remains an open task. It should also be noted that most of the studies of this and related problems have as yet been restricted to systems with *local* currents (heat baths of different temperatures are attached to the system at every point). Problems that are associated with *global* currents, such as, e.g., the driven lattice gas [1], are much richer and more difficult to analyze. Since the effective interactions observed here carry dipole-like angular dependences as well, it may, however, be possible that global currents will not cause a significant increase in the number of non-equilibrium universality classes. A systematic investigation of the effects of global currents, involving the generalization of the models with local-current into global-current models, should clearly be the subject of further studies.

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A Ward identities for the non-equilibrium SSS model

In this appendix we derive the basic Ward identities that were used in Sec. 3.1. In the equilibrium SSS model, one can use the ordinary Ward identities between the non-linear susceptibility and the linear susceptibility [17] to reduce the number of Z factors required to renormalize the theory. However, this procedure is only valid if the response functions renormalize multiplicatively, a property which follows from the fact that in equilibrium the zero-frequency limit of a multi-linear response function is equal to the corresponding static correlation function [16]. Since the non-equilibrium SSS model does not obey detailed balance, these Ward identities are not useful in this case. In this appendix we will thus work directly with the vertex functions, and derive the basic Ward identities which follow from the $O(n)$ symmetry of the dynamic functional, and the fact that the $M_0^{\alpha\beta}$ fields are the generators of this symmetry group.

We consider the following canonical transformation for the fields S_0^α and $M_0^{\alpha\beta}$,

$$\delta S_0^\alpha = \epsilon\, \Lambda^{\mu\nu}\{M_0^{\mu\nu}, S_0^\alpha\} = \epsilon \Lambda^{\alpha\nu} S_0^\nu \tag{A.1}$$

$$\delta M_0^{\alpha\beta} = \epsilon\, \Lambda^{\mu\nu}\{M_0^{\mu\nu}, M_0^{\alpha\beta}\} = \epsilon(\Lambda^{\alpha\nu} M_0^{\nu\beta} - \Lambda^{\beta\nu} M_0^{\nu\alpha}) \ , \tag{A.2}$$

where ϵ is a small parameter and $\Lambda^{\mu\nu}$ is an arbitrary antisymmetric tensor which is constant in space and time. (In this Appendix, we use Einstein's convention of summation over repeated indices.) This transformation preserves the Poisson brackets between S_0^α and $M_0^{\gamma\delta}$ and between $M_0^{\alpha\beta}$ and $M_0^{\gamma\delta}$. If this transformation is supplemented with the transformation laws for the auxiliary fields $\tilde{S}_0^\alpha$ and $\tilde{M}_0^{\alpha\beta}$, as given by

$$\delta \tilde{S}_0^\alpha = \epsilon \Lambda^{\alpha\nu} \tilde{S}_0^\nu \tag{A.3}$$

$$\delta \tilde{M}_0^{\alpha\beta} = \epsilon(\Lambda^{\alpha\nu} \tilde{M}_0^{\nu\beta} - \Lambda^{\beta\nu} \tilde{M}_0^{\nu\alpha}) \ , \tag{A.4}$$

then one can show that the Janssen-de Dominicis functional $J[\{\widetilde{S}_0^\alpha\}, \{S_0^\alpha\}, \{\widetilde{M}_0^{\alpha\beta}\}, \{M_0^{\alpha\beta}\}]$ is invariant with respect to the joint transformation (A.1) to (A.4).

If the tensor $\Lambda^{\mu\nu}(t)$ is now allowed to depend on time, then the dynamic functional is no longer invariant under the transformations (A.1) to (A.4), but picks up the extra terms

$$\delta J = -\epsilon \int d^dx \int dt\, \dot{\Lambda}^{\alpha\beta}(t)\, [\tilde{S}S]^{\alpha\beta}(\mathbf{x}, t) - \epsilon \int d^dx \int dt\, \dot{\Lambda}^{\alpha\beta}(t)\, [\tilde{M}M]^{\alpha\beta}(\mathbf{x}, t) \ , \tag{A.5}$$

where $[\tilde{S}S]_0^{\alpha\beta}(\mathbf{x}, t)$ and $[\tilde{M}M]_0^{\alpha\beta}(\mathbf{x}, t)$ are composite operators which are defined as

$$[\tilde{S}S]^{\alpha\beta}(\mathbf{x}, t) = \frac{1}{2}\left[\, \tilde{S}_0^\alpha(\mathbf{x}, t) S_0^\beta(\mathbf{x}, t) - \tilde{S}_0^\beta(\mathbf{x}, t) S_0^\alpha(\mathbf{x}, t) \,\right] , \tag{A.6}$$

and

$$[\tilde{M}M]^{\alpha\beta}(\mathbf{x}, t) = \frac{1}{2}\left[\, \tilde{M}_0^{\alpha\nu}(\mathbf{x}, t) M^{\beta\nu}(\mathbf{x}, t) - \tilde{M}_0^{\beta\nu}(\mathbf{x}, t) M_0^{\alpha\nu}(\mathbf{x}, t) \,\right] , \tag{A.7}$$

and the "$\cdot$" stands for differentiation with respect to time. In order to proceed, we consider the generating functional

$$\begin{aligned} &Z[h^\alpha, \tilde{h}^\alpha, H^{\alpha\beta}, \tilde{H}^{\alpha\beta}, \mathcal{J}^{\alpha\beta}, J^{\alpha\beta}, \mathcal{L}^\alpha] = \\ &= \int \mathcal{D}[\{S_0^\alpha\}]\mathcal{D}[\{i\widetilde{S}_0^\alpha\}] \int \mathcal{D}[\{M_0^{\alpha\beta}\}]\mathcal{D}[\{i\widetilde{M}_0^{\alpha\beta}\}] \times \\ &\times e^{J[\{\widetilde{S}_0^\alpha\},\{S_0^\alpha\},\{\widetilde{M}_0^{\alpha\beta}\},\{M_0^{\alpha\beta}\}]} \\ &\times e^{\int_{\mathbf{x},t} h^\alpha S_0^\alpha + \tilde{h}^\alpha \tilde{S}_0^\alpha + \frac{1}{2}(H^{\alpha\beta} M_0^{\alpha\beta} + \tilde{H}^{\alpha\beta} \tilde{M}_0^{\alpha\beta})} \\ &\times e^{\frac{1}{2}\int_{\mathbf{x},t}(\mathcal{J}^{\alpha\beta}[\tilde{S}S]_0^{\alpha\beta} + J^{\alpha\beta}[\tilde{M}M]_0^{\alpha\beta} + \mathcal{L}^\alpha[\tilde{M}S]_0^\alpha)} \ , \end{aligned} \tag{A.8}$$

where we have introduced a source term for the fields S_0^α, $\tilde{S}_0^\alpha$, $M_0^{\alpha\beta}$, and $\tilde{M}_0^{\alpha\beta}$, $[\tilde{S}S]_0^{\alpha\beta}$ and $[\tilde{M}M]_0^{\alpha\beta}$ in Z, and also a source term for the composite field $[\tilde{M}S]_0^\alpha = \tilde{M}^{\alpha\nu} S_0^\nu$, since this composite operator also enters in the definition of the linear susceptibility (3.25). One can now obtain the transformation laws for any composite operator from the transformations (A.1) to (A.4).

If one now applies the transformation (A.1) to (A.4) with a time-dependent parameter $\Lambda^{\mu\nu}(t)$ to the dynamic functional Z, then it is easy to see that the two terms generated by the transformation, as given in Eq. (A.5), can be absorbed in the transformation law for the sources of $[\tilde{S}S]_0^{\alpha\beta}$ and $[\tilde{M}M]_0^{\alpha\beta}$.

One ends up with the following identity

$$\begin{aligned} &Z[h^\alpha, \tilde{h}^\alpha, H^{\alpha\beta}, \tilde{H}^{\alpha\beta}, \mathcal{J}^{\alpha\beta}, J^{\alpha\beta}, \mathcal{L}^\alpha] = \\ &Z[h^\alpha - \epsilon\Lambda^{\alpha\beta} h^\beta, \tilde{h}^\alpha - \epsilon\Lambda^{\alpha\beta}\tilde{h}^\beta, \\ &\quad H^{\alpha\beta} - \epsilon(\Lambda^{\alpha\nu} H^{\nu\beta} - \Lambda^{\beta\nu} H^{\nu\alpha}), \\ &\quad \tilde{H}^{\alpha\beta} - \epsilon(\Lambda^{\alpha\nu} \tilde{H}^{\nu\beta} - \Lambda^{\beta\nu} \tilde{H}^{\nu\alpha}), \\ &\quad \mathcal{J}^{\alpha\beta} - \epsilon(\Lambda^{\alpha\nu} \mathcal{J}^{\nu\beta} - \Lambda^{\beta\nu} \mathcal{J}^{\nu\alpha}) - 2\epsilon\dot{\Lambda}^{\alpha\beta}, \\ &\quad J^{\alpha\beta} - \epsilon(\Lambda^{\alpha\nu} J^{\nu\beta} - \Lambda^{\beta\nu} J^{\nu\alpha}) - 2\epsilon\dot{\Lambda}^{\alpha\beta}, \\ &\quad \mathcal{L}^\alpha - \epsilon\Lambda^{\alpha\beta}\mathcal{L}^\beta] \ . \end{aligned} \tag{A.9}$$

Expanding this identity to first order in ϵ, and using the fact that $\Lambda^{\alpha\beta}(t)$ is antisymmetric but otherwise arbitrary, one obtains the following relation for the vertex functions

$$\int d^dx \left\{ \left(s_0^\alpha \frac{\delta\Gamma}{\delta s_0^\beta} - s_0^\beta \frac{\delta\Gamma}{\delta s_0^\alpha} \right) + \left(\tilde{s}_0^\alpha \frac{\delta\Gamma}{\delta \tilde{s}_0^\beta} - \tilde{s}_0^\beta \frac{\delta\Gamma}{\delta \tilde{s}_0^\alpha} \right) \right.$$

$$-\left(m_0^{\alpha\nu} \frac{\delta \Gamma}{\delta m_0^{\nu\beta}} - m_0^{\beta\nu} \frac{\delta \Gamma}{\delta m_0^{\nu\alpha}} \right)$$
$$-\left(\tilde{m}_0^{\alpha\nu} \frac{\delta \Gamma}{\delta \tilde{m}_0^{\nu\beta}} - \tilde{m}_0^{\beta\nu} \frac{\delta \Gamma}{\delta \tilde{m}_0^{\nu\alpha}} \right)$$
$$-\left(\mathcal{J}_0^{\alpha\nu} \frac{\delta \Gamma}{\delta \mathcal{J}_0^{\nu\beta}} - \mathcal{J}_0^{\beta\nu} \frac{\delta \Gamma}{\delta \mathcal{J}_0^{\nu\alpha}} \right)$$
$$-\left(J_0^{\alpha\nu} \frac{\delta \Gamma}{\delta J_0^{\nu\beta}} - J_0^{\beta\nu} \frac{\delta \Gamma}{\delta J_0^{\nu\alpha}} \right)$$
$$+\left(\mathcal{L}_0^{\alpha} \frac{\delta \Gamma}{\delta \mathcal{L}_0^{\beta}} - \mathcal{L}_0^{\beta} \frac{\delta \Gamma}{\delta \mathcal{L}_0^{\alpha}} \right)$$
$$+2\frac{\partial}{\partial t} \left(\frac{\delta \Gamma}{\delta \mathcal{J}_0^{\alpha\beta}} + \frac{\delta \Gamma}{\delta J_0^{\alpha\beta}} \right) \Bigg\} = 0 \ , \qquad \text{(A.10)}$$

where $s_0^\alpha(\mathbf{x}, t) = \langle S_0^\alpha(\mathbf{x}, t) \rangle$, $\tilde{s}_0^\alpha(\mathbf{x}, t) = \langle \tilde{S}_0^\alpha(\mathbf{x}, t) \rangle$, and $m_0^{\alpha\beta}(\mathbf{x}, t) = \langle M_0^{\alpha\beta}(\mathbf{x}, t) \rangle$, $\tilde{m}_0^{\alpha\beta}(\mathbf{x}, t) = \langle \tilde{M}_0^{\alpha\beta}(\mathbf{x}, t) \rangle$.

A similar identity can be obtained for the correlation functions. One can then derive the usual Ward identities for multi-linear response functions from this identity. This establishes the equivalence of the two procedures in the equilibrium case.

Taking the variational derivative of (A.10) with respect to $\tilde{m}^{\gamma\delta}(\mathbf{x}, t)$ and $m^{\nu\zeta}(\mathbf{x}, t)$, and setting the source terms to zero, one obtains, after taking the Fourier transform, the identity

$$-i\tilde{\omega} \left\{ \Gamma^{\gamma\delta\eta\zeta\alpha\beta}_{0\tilde{M}M[\tilde{S}S]}(\mathbf{q}, \omega \,; \mathbf{0}, \tilde{\omega}) + \Gamma^{\gamma\delta\eta\zeta\alpha\beta}_{0\tilde{M}M[\tilde{M}M]}(\mathbf{q}, \omega \,; \mathbf{0}, \tilde{\omega}) \right\} =$$
$$= \frac{1}{2} [\, \delta^{\alpha\eta}(\delta^{\beta\delta}\delta^{\gamma\zeta} - \delta^{\beta\gamma}\delta^{\delta\zeta}) + \delta^{\alpha\zeta}(\delta^{\beta\gamma}\delta^{\delta\eta} - \delta^{\beta\delta}\delta^{\gamma\eta})$$
$$+ \delta^{\alpha\gamma}(\delta^{\beta\eta}\delta^{\delta\zeta} - \delta^{\beta\zeta}\delta^{\delta\eta}) + \delta^{\alpha\delta}(\delta^{\beta\zeta}\delta^{\gamma\eta} - \delta^{\beta\eta}\delta^{\gamma\zeta}) \,]$$
$$\times \{ \Gamma_{0\tilde{M}M}(\mathbf{q}, \omega + \tilde{\omega}) - \Gamma_{0\tilde{M}M}(\mathbf{q}, \omega) \} \ , \qquad \text{(A.11)}$$

where we have used the tensor properties of $\Gamma^{\alpha\beta\gamma\delta}_{0\tilde{M}M}$. This identity relates the vertex functions with one insertion of the composite operators $[\tilde{S}S]^{\alpha\beta}$ and $[\tilde{M}M]^{\alpha\beta}$ to vertex functions with no insertions. Taking the variational derivatives of Eq. (A.10) with respect to the other fields, one can obtain similar Ward identities for other vertex functions.

These Ward identities show that no multiplicative renormalization is needed for the composite operators $[\tilde{S}S]^{\alpha\beta}$ and $[\tilde{M}M]^{\alpha\beta}$, i.e. $Z_{[\tilde{S}S]} = Z_{[\tilde{M}M]} = 1$. However, these identities do not exclude the need of additive renormalization for these operators provided that this renormalization is $\propto q^2$ [21].

We now consider Eq. (3.25), which follows from the identity

$$\left\langle \frac{\delta J}{\delta M_0^{\alpha\beta}(\mathbf{x}, t)} \right\rangle + H^{\alpha\beta}(\mathbf{x}, t) = 0 \ , \qquad \text{(A.12)}$$

which in turn can be proven using the fact that the path integral of a functional derivative vanishes. If one now takes the functional derivative of this equation with respect to $\tilde{m}^{\gamma\delta}(\mathbf{x}', t')$ and performs a Fourier transformation, one obtains Eq. (3.25).

The renormalized version of (3.25) can be obtained if one uses the identity $Z_{[\tilde{S}S]} = 1$ and the definition of the renormalized diffusion constant $\partial_{q^2} \Gamma_{\tilde{M}M}(\mathbf{q}\,\omega) \,|_{NP} = D$ in Eq. (3.25). We thus find

$$\Gamma_{\tilde{M}M}(\mathbf{q}, \omega) = i\omega + Dq^2 +$$
$$+ 2g\mu^{\epsilon/2} A_d^{-1/2} (Z_M Z_g)^{-1/2} \Gamma_{\tilde{M}[\tilde{S}S]}(\mathbf{q}, \omega) \ , \qquad \text{(A.13)}$$

where the renormalized vertex function $\Gamma_{\tilde{M}[\tilde{S}S]}(\mathbf{q}, \omega)$ is defined by

$$\Gamma_{\tilde{M}[\tilde{S}S]}(\mathbf{q}, \omega) \equiv Z_{\tilde{M}}^{-1/2} \qquad \text{(A.14)}$$
$$\times \left(\Gamma_{0\tilde{M}[\tilde{S}S]}(\mathbf{q}, \omega) - q^2 \partial_{q^2} \Gamma_{0\tilde{M}[\tilde{S}S]}(\mathbf{q}, \omega) \,|_{NP} \right) ,$$

and where the second term on the right-hand-side follows from the definition of the renormalized diffusion constant and corresponds to an additive renormalization needed to render $[\tilde{S}S]^{\alpha\beta}$ finite.

Hence, since all the quantities in (A.13) are finite, we conclude that

$$Z_g Z_M = 1 \qquad \text{(A.15)}$$

must hold to *all orders* in perturbation theory. If one takes the ratio of Eqs. (3.14) and (3.10), one can check explicitly that this identity holds to one loop-order.

A similar set of Ward identities can be derived for the equilibrium model J [21,29], but it is unclear if these identities still hold in the non-equilibrium *effective* model J with long-range elastic forces considered in Sec. 4.3.

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27. We remark that the term "dipolar" interactions is slightly misleading here. For truly dipolar forces, there exists a coupling between order parameter and real space of the form $(\mathbf{q} \cdot \mathbf{S})^2$; uniaxial dipolar interactions then do lead to a free energy of the form (4.13), yet for a *single* critical order parameter component only. For any $n \geq 2$, the term "elastic" interactions is therefore more appropriate.
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30. One might hope that perhaps a computation of the Z factors and RG flow functions at *fixed* dimensions d and $d_\parallel$, i.e., without invoking an ε expansion, would lead to a finite, positive fixed point $\widetilde{f}^*$. However, a straightforward one-loop calculation for fixed d and $d_\parallel$ yields that in the physical dimensions $d = 3$, the non-trivial zero of $\beta_{\widetilde{f}}$ is given by $\widetilde{f}^* = -\frac{12}{29}$ for $d_\parallel = 1$, and $\widetilde{f}^* = -\frac{12}{155}$ for $d_\parallel = 2$ (while in any case $\widetilde{f}^* = 0$ for $d = 2$ and $d = 4$). Thus, for any positive initial value of $\widetilde{f}$, the RG flow runs away to ∞, precisely as in the ε expansion.